THE KALMAN-BUCY FILTER
AND
ITS BEHAVIOUR WITH RESPECT TO SMOOTH PERTURBATIONS OF THE INVOLVED WIENER-LÉVY PROCESSES

PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE TECHNISCHE WETENSCHAPPEN AAN DE TECHNISCHE HOGESCHOOL DELFT, OP GEZAG VAN DE RECTOR MAGNIFICUS IR. H. R. VAN NAUTA LEMKE, HOOGLERAAR IN DE AFDELING DER ELEKTROTECHNIEK, VOOR EEN COMMISSIE UIT DE SENAAT TE VERDEDIGEN OP

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PETER ARNOLD RUYMGAART

doctorandus in de wis- en natuurkunde
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PROF. DR. J. G. DIJKMAN

EN

PROF. DR. R. TIMMAN

Samenvatting

Dit proefschrift is voortgekomen uit een aantal rapporten, geschreven in de jaren 1966-1970 . Enkele ervan zijn geschreven in de Verenigde Staten, samen met Dr. T.T.Soong van de staatsuniversiteit van New York te Buffalo.

Behandeld worden lineaire stochastische systemen: Lineaire toestandsvergelijkingen en lineaire schattingen van hun oplossingen.

Het homogene deel van de toestandsvergelijkingen is deterministisch, het niet-homogene deel is een N-dimensionaal Wiener-Lévy proces. Ook de beginvoorwaarden zijn stochastisch. Zoals gebruikelijk zijn de verdere vooronderstellingen dusdanig dat het systeem beschreven kan worden door middel van een lineaire integraalvergelijking in een Hilbert ruimte.

De schatting van de oplossing wordt behandeld in hoofdstuk 6, waar vooral aandacht wordt geschonken aan het Kalman-Bucy filter.

Uitgangspunt is dat de toestandsvergelijkingen een wiskundig model zijn van een technologisch proces. Daarom dienen de resultaten geldig te zijn met betrekking tot de trajectoriën in het model, zie hoofdstuk 1.

Dit in tegenstelling tot de resultaten in hoofdstuk 6, omdat de schattingsmethoden niet de weerspiegeling zijn van enig fysisch gebeuren.

De meest voor de hand liggende en tevens eenvoudigste calculus om mee te werken is voor ons doel de calculus in tweede gemiddelde. Deze wordt uitvoerig behandeld in hoofdstuk 2, en wel voor niet-stationaire stochastische processen. Harmonische analyse wordt in het geheel niet toegepast. De meeste resultaten in dit hoofdstuk zijn bekend. Men vindt ze in technische publicaties, veelal zonder bewijs, of het zijn bijzondere gevallen van algemenere stellingen uit de functionaal analyse. De behoefte aan hoofdstuk 2 is ontstaan, omdat er geen samenhangende en voor ons doel volledige behandeling van deze calculus bestaat. Bovendien ontstond de gelegenheid om van elk resultaat zijn geldigheid aan te tonen met betrekking tot de trajectoriën.

In hoofdstuk 6, paragraaf 6.2, wordt deze calculus nog enigszins uitgebreid, zonder echter te letten op de trajectoriën, omdat zoals we gezien hebben, bij schattingsproblemen de noodzaak daartoe ontbreekt.

Omdat Wiener-Lévy processen niet "fysisch realiseerbaar" zijn, worden deze processen vervangen door perturbaties die wel gerealiseerd kunnen worden. Deze verstoringen worden onderzocht in hoofdstuk 4. Aangetoond wordt dat differentieerbare perturbaties willekeurig dicht in de buurt kunnen komen van de gegeven Wiener-Lévy processen.

In hoofdstuk 5 wordt het effect onderzocht van deze verstoringen op de oplossing van het niet gestoorde systeem. Het blijkt dat de oplossingen van de gestoorde systemen willekeurig dicht in de buurt kunnen komen van de oplossing van het niet gestoorde systeem. Dit betekent dat het ongestoorde systeem een betrouwbaar mathematisch model kan zijn voor een of ander technologisch proces, ondanks de aanwezigheid van de Wiener-Lévy processen. Dit resultaat is een speciaal geval van een algemenere stelling van Wong en Zakai. Deze stelling, waarin gebruik wordt gemaakt van Ito calculus, is voor ons doel echter onnodig ingewikkeld.

In hoofdstuk 6 wordt onderzocht wat de invloed is van deze verstoringen op het Kalman-Bucy filter. Het blijkt dat de relaties en vergelijkingen in het filter geheel komen te vervallen. Het effect is dus niet zodanig dat zekere grootheden in de filtervergelijkingen worden geperturbeerd. Om dit in te zien is een kritische analyse van de rekenwijze van Kalman en Bucy een vereiste. Het blijkt dat hun integraalvoorstelling van de schatting het centrale punt is, zie paragraaf 6.2. De geldigheid hiervan steunt op het Wiener-Lévy proces in de observaties, waaraan bovendien nog een extra voorwaarde moet worden opgelegd. Omdat de observatie-apparatuur zeker geen zuivere Wiener-Lévy processen zal voortbrengen is het verstoren van deze processen in het model alleszins zinvol. In de gestoorde modellen blijkt de geldigheid van de integraalvoorstelling te vervallen, en daarmee het gehele rekenschema van Kalman en Bucy. Zou men desondanks toch een integraalvoorstelling voor de schatting invoeren, dan nog zou er van de filtervergelijkingen vrijwel niets overblijven. Niettemin blijken de schattingen van Kalman en Bucy in zekeren zin bestand tegen de perturbaties, evenals vele andere schattingen, zoals die van Wiener bijvoorbeeld. Het is dus toch zinvol om te werken met het niet gestoorde systeem en met de efficiente rekenwijze van Kalman en Bucy.

1 Introduction

1.1. Motivation of the subject.

Mathematical models of technological processes may contain constants and functions, measured empirically. Values obtained in this way are data with some statistical meaning and are not exact in deterministic sense. Apart from randomness of this type, physical constants and functions are often stochastic in an intrinsic sense as a consequence of uncontrollable influences, present all over in nature. Hence in order to be meaningful, the results derived from deterministic mathematical models should possess a certain stability with respect to small perturbations of the experimentally measured data. The study of stability of this kind is part of the study of deterministic models.

In order to obtain more adequate descriptions of technological or physical processes, also stochastic models are taken into consideration. Strictly speaking, nearly all deterministic models ought to be replaced by stochastic models. However, with regard to the mathematical difficulties that might arise, the application of stochastic models is usually confined to those systems where the random fluctuations have an appriciable impact on the system behaviour. A well known stochastic model is that, used in statistical mechanics. Here randomness is introduced via the stochastic initial conditions of the equations of motion. From probabilistic point of view this model is quite simple, as it is based on a finite number of random variables only. Models in which the degree of randomness is infinite are of more interest to probability theoretical investigations.

Important models of this kind are those, containing white noise, or processes related to it. White noise is a purely mathematical concept. It is the generalized derivative of the Wiener-Lévy process, a mathematical idealization of the phenomenon of Brownian motion. Brownian motion is the origin of a large class of stochastic processes in physics. The most significant property of the Wiener-Lévy process is the stochastic independence of its increments. To a certain extend this property reflects reasonably what one would intuitively expect of a model of the position af a particle in Brownian movement. However, not all properties of the Wiener-Lévy process turn out to be realistically related to its physical counterpart. The assumption of the independence of the increments of the Wiener-Lévy process entails that its trajectories, although continuous in time, are not differentiable and not of bounded variation on any interval with probability 1 . Since a stochastic model of a technological system should be seen as a probability space of which each elementary event represents a possible realization of the technological system in investigation, the mathematical models containing Wiener-Lévy *processes suffer from lack of correspondence* with *realty*. This is one of the reasons why other mathematical idealizations of Brownian movement have been proposed. One of them is the Ornstein-Uhlenbeck process. It is the solution of a linear system, driven by white noise. However, many of the objections made against the Wiener-Lévy process also apply with respect to this process.

Notwithstanding, in mathematical models of systems, influenced by Brownian movement, the Wiener-Lévy processes are widely used. There is not much freedom in adapting random functions to Brownian motion. And Wiener-Lévy processes often give rise to tractable computations, owing to their peculiar stochastic structure. Moreover, the statistical results established in models containing white noise, correspond quite often satisfactorily to engineering practice. So it is worthwhile to investigate when and why it still might make sense to apply Wiener-Lévy processes in mathematical models.

Let be assumed that some information about certain random processes is obtained by means of measurements, in order to design a stochastic model. Then, as in deterministic models, the results derived should be stable with respect to perturbations of the measured data, in order to have some practical meaning. However, in particular when perturbing the trajectories of Wiener-Lévy processes, the whole structure of stochastic interdependences is mutilated and this may have fundamental consequences. On the other hand, because of the poor behaviour of the trajectories of Wiener-Lévy processes, essentially their perturbations might have a realistic meaning, as they may be smooth functions. When perturbing Wiener-Lévy processes, especially the smooth perturbations are significantly related to the physical phenomenon of Brownian motion. The increments of the smoothed Wiener-Lévy processes are "nearly" stochastically independent. They seem to be satisfactory as a model of Brownian motion in all respects.

In order to trace whether a result is stable with respect to perturbations of the involved Wiener-Lévy processes, the following procedure is followed. Apart from investigating the original model, a sequence of models is considered, each model containing a perturbation of the Wiener-Lévy processes. The resulting sequences of perturbations are assumed to tend to the original Wiener-Lévy processes. If it can be shown that the limit of a sequence of results, obtained in the sequence of perturbed models, exists and coincides with the corresponding result in the original, non-perturbed model, the stability of this result is established. As there are smoothly perturbed systems arbitrarily close to the original system, a stable result has a realistic meaning. If the non-perturbed model is easier to treat than its smoothed versions, it makes sense to use it, provided that only the stable results are taken into consideration.

Important stochastic models are systems of stochastic differential or integral equations. Frequently used are Ito equations, ordinary non-linear integral equations, containing Wiener-Lévy processes. In order to obtain unique solutions, Ito introduced a special type of calculus. Here the stochastic integrals exist as limits of Riemann-Stieltjes sums, where the function values are chosen at the lower vertices of the sub-intervals of the partitions of the domain of integration. In this context, sequences of arbitrary Riemann-Stieltjes sums do not converge in general. The Ito solutions have nice statistical properties since they are Markov processes. Originally, Ito studied a class of Markov processes which could be described by the above equations. Later, the equations were used in engineering sciences as models of technological processes. However, in 1965 Wong and Zakai [31] showed that the solutions are not stable in general in the above defined sense. Thus, without more, Ito equations are in general not appropriate as models of technological processes. But it follows also from the theorem of Wong and Zakai that in the special case of linear differential equations whose coefficients are deterministic functions and whose inhomogeneous parts are white noise processes, the solutions obey the above stability principle properly. However, to systems of this kind there is no need for the involved calculus of Ito. They may easier be solved by means of more elementary methods. Then also the stability of the solutions is easily established without using the theorem of Wong and Zakai.

The above sketched way of perturbing has formerly been applied in certain approaches to white noise processes, looked upon as generalized functions, see for instance Urbanik [29].

In accordance with a previous remark, stochastic differential equations as models of technological processes should be considered as probability spaces of which the elementary events are ordinary differential equations, each of them containing one of the realizations of the random elements involved. They should essentially be solved in sample calculus. Results obtained by means of other techniques, as calculus in q.m. for instance, are to be shown to be identical to the corresponding results derived by means of sample calculus.

Of great importance to practical purposes is the estimation of the random variables of stochastic processes. Roughly speaking, until about 1959 the theory of estimation of stochastic processes was confined to stationary processes. Interesting theories, due - among others - to Kolmogorov and Wiener, gave rise to ingenious solutions in closed form, see [32] and [6] for instance.

Some 11 years ago, presumably incited by the demands of space navigation, Kalman and Bucy designed a recursive scheme for determining estimates, also applicable to non-stationary processes. They exploited the new possibilities opened by the development of digital computers and the accompanying adaptation of numerical methods. Their first publications on this subject, see [11], gave rise to an endless stream of engineering literature, "receiving its impetus from the aerospace dollar" according to Jazwinski, [10] . In the so-called Kalman-Bucy filter, the state system is linear. Its homogeneous part is deterministic, its inhomogeneous part is a white noise process and also the initial conditions are random. At any instant, the observation is a linear function of the state and a new white noise process is added to it. The succes of the Kalman-Bucy filter depends entirely on an extra condition, imposed on this latter white noise process. However, according to previous remarks, the observation noise, generated in the observation device, should be modeled as a smooth perturbation of a white noise process. And now it is of interest to investigate the effect of smoothing the white noise processes, figuring in the Kalman-Bucy filter. As thus in particular also the observation noise is smoothed, the whole system of equations and relations in the Kalman-Bucy filter breaks down. And hence the effect of smoothing the noise is not at all the perturbing of some matrices, figuring in the filter. Still it will be possible to establish - to a certain extend - the stability of the Kalman-Bucy estimate, and more general the stability of the estimates of a class, comprising interpolated and extrapolated values, both of Kalman-Bucy type and of the type of Wiener and Kolmogorov.

As estimation is a purely mathematical concept, and not the counterpart of some physical phenomenon, there is no need for using sample calculus in this context. All results here are established by means of calculus in q.m.

Non-linear stochastic differential equations and filters need an entirely different approach. They are not discussed here. In this thesis all methods are related to Hilbert spaces.

1.2. Motivation of the presentation.

This thesis is a compilation of a number of reports, written by the author during the years 1966-1970 , as a staff member of the department of mathematics of the university of technology at Delft, The Netherlands. Some of them are written in the U.S. in collaboration with Dr. T. T. Soong of the state university of New York at Buffalo, see [22- 27].

The tutorial flavour, present in the reports, is not faded out in this thesis. As much as possible, the level of abstraction is adapted to the nature of the subject. Hence the calculus used is simply calculus in q.m., accompanied by sample calculus if necessary in view of the nature of the mathematical model. Many of the topics included - especially in the first chapters - are well known, and the author has gratefully consulted the references cited in this text. However, for lack of a coherent, well-organized source of references, nearly all necessary mathematical tools - whether or not well-known - are included in full detail, hopefully for the benefit of an easier introduction also for those readers who are not an expert in the field. Not included are those definitions and theorems which may be found in the usual introductions to probability theory and treatises on ordinary differential equations.

The extensive literature on the subject is mainly written for engineers. It contains lots of interesting and important examples and applications. In this presentation, no applications are included. There has only been strived for a hopefully correct and complete mathematical exposition.

2 Calculus in q.m.

2.1. Hilbert spaces of second order random variables.

The results of this section may be found in [15] or [18] for instance. Let be given the probability space $\{\Omega, \mathcal{A}, P\}$ with the set of real valued second order random variables or $\mathcal{A}$-measurable functions $\xi(\omega)$, $\omega \in \Omega$, such that

$$E\xi^2 = \int_\Omega \xi^2(\omega)dP < \infty .$$

This set is splitted up into equivalence classes by means of the equivalence relation

$$\xi \sim \eta \quad \text{iff} \quad \xi = \eta \quad \text{a.s.}$$

As usual, the elements of an equivalence class are identified with some representative of that class. Identity is understood to be "equality a.s." . The addition "a.s." will often be omitted.

The class of representatives is denoted by

$H(\mathcal{A})$ or simply H

if no confusion may arise. The elements of H are denoted by $\alpha(\omega), \beta(\omega), \ldots, \xi(\omega), \eta(\omega), \ldots$ with or without sub- or superscripts, or simply by Greek characters $\alpha, \beta, \ldots, \xi, \eta, \ldots$ as conventionally the dependence on ω is suppressed in the notation. Since all degenerate random variables have finite second moments, the real numbers may be seen as elements of H. The real numbers are denoted by $a, b, \ldots, x, y, \ldots$ with or without sub- or superscripts, or by their numerical value.

It follows from measure properties and the inequality of Schwarz

$$(E\,\xi\eta)^2 \leqq E\xi^2 E\eta^2$$

that H is a linear vector space over the real numbers. For, if $\xi \in H$ then $c\xi \in H$ since $c\xi$ is $\mathcal{A}$-measurable and

$$E(c\xi)^2 = c^2 E\xi^2 < \infty ,$$

and if $\xi \in H$, $\eta \in H$, then $\xi + \eta \in H$ since $\xi + \eta$ is $\mathcal{A}$-measurable and

$$E(\xi+\eta)^2 = E\xi^2 + 2E\xi\eta + E\eta^2 \leqq E\xi^2 + 2\sqrt{E\xi^2}\sqrt{E\eta^2} + E\eta^2 =$$
$$= (\sqrt{E\xi^2} + \sqrt{E\eta^2})^2 < \infty .$$

The real number system may be seen as a 1-dimensional linear subspace of H .

H is a real inner product space with inner product

$$(\xi,\eta) = E\xi\eta$$

since

$$E\xi^2 \geqq 0, \quad E\xi^2 = 0 \text{ iff } \xi = 0 \text{ a.s.},$$
$$E\xi\eta = E\eta\xi ,$$
$$Ec\xi\eta = cE\xi\eta \text{ and}$$
$$E(\xi_1+\xi_2)\eta = E\xi_1\eta + E\xi_2\eta.$$

ξ and η are called orthogonal iff $E\xi\eta = 0$, $\xi \perp \eta$.

Necessarily H is a normed linear space with the norm

$$\|\xi\| = \sqrt{E\xi^2} ,$$

endowed with the properties

$$\|\xi\| \geqq 0, \quad \|\xi\| = 0 \text{ iff } \xi = 0 \text{ a.s.},$$
$$\|c\xi\| = |c|.\|\xi\| \text{ and}$$
$$\|\xi+\eta\| \leqq \|\xi\| + \|\eta\| .$$

In H a metric is induced by the distance $\|\xi-\eta\|$. The resulting strong topology will be the only topology of H used. Therefore the addition "strong(ly)" will often be omitted, also in the notation. Or it will be replaced by "in q.m.", i.e. "in quadratic mean".

We recall that H is a complete space: If $\{\xi_n, n=1,2,..\}$ is a Cauchy sequence in H , it has a (unique) limit in H. I.e. if $\|\xi_n - \xi_m\| \to 0$ as $m,n \to \infty$, then there is a unique element $\xi \in H$ such that $\|\xi - \xi_n\| \to 0$ as $n \to \infty$. So H is a Hilbert space. In general, H is not separable.

We shall need the following properties:

(2.1.1) $|E\xi\eta| \leqq \|\xi\|\cdot\|\eta\|$ (Inequality of Schwarz).

(2.1.2) H is a complete space. And if $\xi_n \to \xi$ as $n \to \infty$ then $\|\xi_n\| \to \|\xi\|$, since $\big|\|\xi_n\| - \|\xi\|\big| \leqq \|\xi_n - \xi\|$.

(2.1.3) Continuity of the inner product: If $\xi_n \to \xi$ and $\eta_m \to \eta$ as $n,m \to \infty$, then

$$E\,\xi_n\eta_m \to E\,\xi\eta$$

since $\big|E\,\xi_n\eta_m - E\,\xi\eta\big| = \big|E(\xi_n - \xi)\eta_m + E\,\xi(\eta_m - \eta)\big| \leqq$
$\leqq \|\xi - \xi_n\| \cdot \|\eta_m\| + \|\xi\| \cdot \|\eta_m - \eta\| \to 0$ as $n,m \to \infty$.

(2.1.4) Convergence in q.m. criterion: $\{\xi_n, n=1,2,..\}$ is convergent if and only if $E\,\xi_n\xi_m$ converges to a finite number as $n,m \to \infty$. For,

i) if $E\,\xi_n\xi_m$ converges as $n,m \to \infty$,

$\|\xi_n - \xi_m\|^2 = E\xi_n^2 - 2E\,\xi_n\xi_m + E\,\xi_m^2 \to 0$ as $n,m \to \infty$.

Thus $\{\xi_n, n=1,2,..\}$ is a Cauchy sequence with limit in H since H is complete.

ii) $\xi_n \to \xi$ entails $E\,\xi_n\xi_m \to \|\xi\|^2$ by virtue of (2.1.3).

2.2. Conditional and mathematical expectation.

Given $\{\Omega, \mathcal{A}, P\}$ and $H(\mathcal{A})$, let $\mathcal{B} \subset \mathcal{A}$ be a σ-field and let $\xi \in H(\mathcal{A})$. Then the conditional expectation

$$E^{\mathcal{B}}\xi$$

of ξ with respect to $\mathcal{B}$ is a $\mathcal{B}$-measurable function of $\omega \in \Omega$ such that

$$\int_B E^{\mathcal{B}}\xi\, dP = \int_B \xi\, dP \qquad \text{for all } B \in \mathcal{B}.$$

According to the theorem of Radon-Nikodym, $E^{\mathcal{B}}\xi$ exists and is a.s. unique. The following properties may be found in [15] or [18] for instance.

$EE^{\mathcal{B}}\xi = E\,\xi$.

If η is $\mathcal{B}$-measurable, $E^{\mathcal{B}}\eta = \eta$ and $E^{\mathcal{B}}\eta\xi = \eta E^{\mathcal{B}}\xi$ a.s.

If $g(x)$ is a convex continuous real function of the real variable x, $-\infty < x < \infty$, $g(E^{\mathcal{B}}\xi) \leqq E^{\mathcal{B}}g(\xi)$ a.s. , see also [20] . Therefore, as $|x|$ and x^2 are continuous and convex,

$$|E^{\mathcal{B}}\xi| \leqq E^{\mathcal{B}}|\xi| \text{ and } (E^{\mathcal{B}}|\xi|)^2 \leqq E^{\mathcal{B}}|\xi|^2 = E^{\mathcal{B}}\xi^2 \text{ a.s.}$$

It follows that $E^{\mathcal{B}}\xi \in H(\mathcal{A})$, as $\mathcal{B} \subset \mathcal{A}$ and $\|E^{\mathcal{B}}\xi\| \leqq \|\xi\|$ since
$(E^{\mathcal{B}}\xi)^2 \leqq (E^{\mathcal{B}}|\xi|)^2 \leqq E^{\mathcal{B}}\xi^2$ a.s. and hence $E(E^{\mathcal{B}}\xi)^2 \leqq EE^{\mathcal{B}}\xi^2 = E\xi^2 < \infty$.
So $E^{\mathcal{B}}$ is a mapping of $H(\mathcal{A})$ into $H(\mathcal{A})$.

The $\mathcal{B}$-measurable random variables of $H(\mathcal{A})$ constitute a closed linear subspace $H(\mathcal{B}) \subset H(\mathcal{A})$ and so $H(\mathcal{B})$ is a Hilbert space. For,

$a\eta + b\zeta$ is $\mathcal{B}$-measurable if η and ζ are $\mathcal{B}$-measurable,

and if $\{\eta_n, n=1,2,..\}$ is a Cauchy sequence of $\mathcal{B}$-measurable random variables in $H(\mathcal{A})$, it has an a.s. unique limit $\eta \in H(\mathcal{A})$. As η is also the limit in probability of $\{\eta_n\}$, it is $\mathcal{B}$-measurable .

$E^{\mathcal{B}}$ is a linear operator with domain $H(\mathcal{A})$ and range $H(\mathcal{B})$. For, if $\xi, \eta \in H(\mathcal{A})$, $E^{\mathcal{B}}(a\xi + b\eta) = aE^{\mathcal{B}}\xi + bE^{\mathcal{B}}\eta \in H(\mathcal{B})$. And as $E^{\mathcal{B}}\eta = \eta$ a.s. if $\eta \in H(\mathcal{B})$, $H(\mathcal{B})$ is the range of $E^{\mathcal{B}}$.

$E^{\mathcal{B}}$ is an idempotent operator, since $E^{\mathcal{B}}(E^{\mathcal{B}}\xi) = E^{\mathcal{B}}\xi$ a.s.

$E^{\mathcal{B}}$ is a continuous (bounded) linear operator with $\|E^{\mathcal{B}}\| = 1$ as $\|E^{\mathcal{B}}\xi\| \leqq \|\xi\|$ and $\|E^{\mathcal{B}}(E^{\mathcal{B}}\xi)\| = \|E^{\mathcal{B}}\xi\|$.

$E^{\mathcal{B}}$ is self-adjoint . For, if ξ and η are elements of $H(\mathcal{A})$,

$$E(\xi E^{\mathcal{B}}\eta) = EE^{\mathcal{B}}(\xi E^{\mathcal{B}}\eta) = E(E^{\mathcal{B}}\xi E^{\mathcal{B}}\eta) = EE^{\mathcal{B}}(\eta E^{\mathcal{B}}\xi) = E(\eta E^{\mathcal{B}}\xi).$$

<u>(2.2.1)</u> $E^{\mathcal{B}}$ is the orthogonal projector of $H(\mathcal{A})$ onto $H(\mathcal{B})$. For, if $\xi \in H(\mathcal{A})$ and if $\eta \in H(\mathcal{B})$, then $E^{\mathcal{B}}\eta = \eta$ a.s. and

$$E(\xi - E^{\mathcal{B}}\xi)\eta = E\xi\eta - E\eta E^{\mathcal{B}}\xi = E\xi\eta - E\xi E^{\mathcal{B}}\eta = E\xi\eta - E\xi\eta = 0,$$

i.e. $\xi - E^{\mathcal{B}}\xi \perp \eta$ for all $\eta \in H(\mathcal{B})$.

<u>(2.2.2)</u> If $\xi_n \to \xi$ as $n \to \infty$, then $E^{\mathcal{B}}\xi_n \to E^{\mathcal{B}}\xi$ since

$$\|E^{\mathcal{B}}\xi_n - E^{\mathcal{B}}\xi\| \leqq \|\xi_n - \xi\| .$$

Let us consider the degenerate σ-field $\mathcal{L} = \{\phi, \Omega\}$.
Clearly

$$E^{\mathcal{L}}\xi = E\,\xi \quad \text{a.s.},$$

and so the ordinary mathematical expectation is obtained as a special case of the conditional expectation. It follows that $E^{\mathcal{L}}$ is the orthogonal projector of $H(\mathcal{A})$ onto $H(\mathcal{L})$, the 1-dimensional subspace of degenerate random variables or constants. And each $\xi \in H$ may be a.s. uniquely decomposed as

$$\xi = x + \xi', \quad x = E\xi, \quad \xi' = \xi - x.$$

Then $\xi' \perp x$, and ξ' is centered, i.e. $E\xi' = 0$.

Let $\{\xi_n, n=1,2,..\} \subset H$, $E\,\xi_n = x_n$ and $\xi_n = x_n + \xi'_n$.
Then the following proposition is a special case of (2.2.2). It may also be proved directly, since

$$|E\,\xi| \leqq E|\xi| \leqq \sqrt{E\xi^2} = \|\xi\|.$$

(2.2.3) $\xi_n \to \xi$ as $n \to \infty$ implies $E\,\xi_n \to E\xi$. And so $\xi_n \to \xi$ as $n \to \infty$ if and only if $x_n \to x$ and $\xi'_n \to \xi'$.

2.3. Curves in H. Gaussian processes. Trajectories.

We shall need a calculus in relation to second order random functions or processes $\xi(t)$, $t \in [0,T]$. These functions are mappings of the interval $[0,T]$ of the real line into H. Thus they are characterized by

$$E\,\xi^2(t) < \infty, \text{ or equivalently } \|\xi(t)\| < \infty, \quad t \in [0,T].$$

As only the strong topology of H will be used, the greater part of the theory will consist of calculus in q.m. Frequently used are the mathematical expectation or mean

$$E\,\xi(t), \quad t \in [0,T],$$

of $\xi(t)$ and the autocorrelation or covariance functions and crosscorrelation functions

$$E\,\xi(s)\,\xi(t) \quad \text{and} \quad E\,\xi(s)\,\eta(t), \quad (s,t) \in [0,T]^2.$$

According to (2.1.1) these ordinary real valued functions are defined and finite on the indicated domains. The results of this section may be found in [15], [18], and [2].

A real valued process $\xi(t)$, $t \in [0,T]$, is Gaussian, if all finite systems $\{\xi(t_i),\ t_i \in [0,T],\ i=1,2,..,N\}$ have a Gaussian distribution. (Real valued) Gaussian processes are second order processes: If the above $\xi(t)$ is $\mathcal{A}$-measurable at each $t \in [0,T]$, then $\{\xi(t),\ t \in [0,T]\} \subset H(\mathcal{A})$. If $E\,\xi(t)$, $t \in [0,T]$, and $E\,\xi(s)\,\xi(t)$, $(s,t) \in [0,T]^2$ are given, all finite dimensional distributions may be determined. We recall:

<u>(2.3.1)</u> Given a Gaussian family in H, the closure G in H of the linear subspace, generated by that family, is Gaussian. In other words, linear combinations of elements of a Gaussian family are Gaussian, and limits in q.m. of sequences of elements of a Gaussian family are Gaussian. (If the elements of G have zero expectation, The conditional expectation of an element ξ of G with respect to the σ-field generated by the elements of a subset G' of G, belongs also to G. It is a.s. identical to the orthogonal projection of ξ on the closed linear subspace of G, generated by G'. Then Orthogonality in G is equivalent to stochastic independence.

Sofar, we have not considered the trajectories of stochastic processes. Given a stochastic process $\xi(t)$, $t \in [0,T]$, by means of its (consistent and symmetric) finite dimensional distributions we shall need a representation

$$\xi(\omega,t), \quad (\omega,t) \in \Omega \times [0,T]$$

of this process. Here Ω is the pointset of a suitable probability space $\{\Omega, \mathcal{A}, P\}$. The σ-field $\mathcal{A}$ is always understood to be complete with respect to P. The sections at t of $\xi(\omega,t)$ are the random variables of $\xi(t)$.

There is an infinity of representations of the process $\xi(t)$. For, there is an infinity of probability spaces, suitable for representation purposes. And, given the above $\{\Omega, \mathcal{A}, P\}$ and

$\xi(\omega,t)$, any function $\xi'(\omega,t)$, $(\omega,t) \in \Omega \times [0,T]$, such that at fixed $t \in [0,T]$

$$\xi'(\omega,t) = \xi(\omega,t) \text{ a.s.,}$$

is also a representation of $\xi(t)$, $t \in [0,T]$. With the above property, the representations $\xi(\omega,t)$ and $\xi'(\omega,t)$ are called equivalent.

In order to obtain the trajectories -sections at ω -, as well as to obtain measurable and unique results if the usual operations of analysis, based on the operations "inf" and "sup", are performed on non-denumerable sets of random variables of $\xi(t)$, the representations should be separable in the sense of Doob. According to Doob, to each representation there is a separable representation equivalent to it.

A representation is called sample continuous if its sections at ω are continuous functions of t with probability 1. Sample continuous representations are separable. If one of the separable representations of a process is sample continuous, all the separable representations are sample continuous, according to a criterion of Neveu.

From now on "stochastic process or function" will stand for any separable representation of that process. "Sample continuous process" will stand for any sample continuous representation of a process of which the separable representations are sample continuous.

The trajectories of a sample continuous process are defined as the sections at ω in one of its sample continuous representations. These trajectories may be seen as the elementary events of the probability space, of which the pointset is the Banach space $C[0,T]$, i.e. the normed linear space of real valued continuous functions on $[0,T]$ with the uniform norm, and where the σ-field S is generated by the open sets of $C[0,T]$. The unique probability on S is induced by the sample continuous process. We have shown:

(2.3.2) If a process is sample continuous, its trajectories are a.s. uniquely determined as the sections at ω in any of its separable representations.

2.4. Continuity in q.m.

We recall the conventions of section 2.1. In particular, the addition "in q.m." will often be omitted in appropriate situations. The greater part of the results of this section may be found in [15].

Let $\xi(t)$ be a mapping of $[0,T]$ into H. The values s and t below are always assumed to belong to $[0,T]$.

(2.4.1) Definition: $\xi(t)$ is continuous in q.m. at t iff

$$\xi(s) \to \xi(t), \quad \text{i.e. } \|\xi(s) - \xi(t)\| \to 0 \text{ as } s \to t.$$

$\xi(t)$ is continuous on $[0,T]$ iff it is continuous at each $t \in [0,T]$.

(2.4.2) If $\xi(t)$ is continuous in q.m. on $[0,T]$,

i) $\|\xi(t)\|$ is a continuous real function on $[0,T]$ by virtue of (2.1.2).

ii) $E^{\mathcal{B}}\xi(t)$ is continuous in q.m. on $[0,T]$ by virtue of (2.2.2), and in particular

iii) $E\xi(t)$ is a continuous real function on $[0,T]$.

iv) Continuity in q.m. is not equivalent to sample continuity.

(2.4.3) If $\xi(t)$ and $\eta(t)$ are continuous mappings of $[0,T]$ into H,

$$a\xi(t) + b\eta(t)$$

is continuous in q.m. on $[0,T]$.

(2.4.4) If $\xi(t)$ is a continuous mapping of $[0,T]$ into H and if $f(t)$ is a continuous real valued function, then

$$f(t)\xi(t)$$

is continuous in q.m. on $[0,T]$, since

$$\|f(s)\xi(s) - f(t)\xi(t)\| \leq |f(s)| \cdot \|\xi(s) - \xi(t)\| + |f(s)-f(t)| \cdot \|\xi(t)\|.$$

(2.4.5) If $\{\xi_n(t), n=1,2,..\}$ is a sequence of continuous functions, converging in q.m. to $\xi(t)$ as $n \to \infty$, uniformly in $t \in [0,T]$, then $\xi(t)$ is continuous in q.m. on $[0,T]$. For,

$$\|\xi(s) - \xi(t)\| \leq \|\xi(s) - \xi_n(s)\| + \|\xi_n(s) - \xi_n(t)\| + \|\xi_n(t) - \xi(t)\|.$$

Then the functions below are also continuous and

$E^{\mathcal{B}} \xi_n(t) \to E^{\mathcal{B}} \xi(t)$ uniformly on $[0,T]$ by virtue of (2.2.2), and in particular

$E \xi_n(t) \to E \xi(t)$ uniformly on $[0,T]$.

(2.4.6) Continuity in q.m. criterion: $\xi(t)$ is continuous at t' if and only if $E \xi(s) \xi(t)$ is continuous at (t',t'), and

$\xi(t)$ is continuous on $[0,T]$ if and only if $E \xi(s) \xi(t)$ is continuous on $[0,T]^2$, on account of (2.1.3) and (2.1.4).

(2.4.7) If $\xi(t)$ is continuous in q.m. on $[0,T]$, it is uniformly continuous in q.m. on $[0,T]$.

Proof: Along the same lines as in real analysis. Or with the aid of the covariance function: As the real valued function $E \xi(s) \xi(t)$ is continuous on $[0,T]^2$, it is uniformly continuous on $[0,T]^2$. Therefore, given $\varepsilon > 0$, there is a $\delta > 0$ such that $|E \xi(s) \xi(t) - E \xi(s') \xi(t')| < \varepsilon$ as $d\{(s,t),(s',t')\} < \delta$.
It follows that

$\| \xi(s) - \xi(t) \|^2 \leqq |E \xi(s)^2 - E \xi(s) \xi(t)| + |E \xi(t)^2 - E \xi(s) \xi(t)| \leqq 2\varepsilon$

if $|s - t| < \delta$.

2.5. Differentiability in q.m.

The greater part of the results of this section may be found in [15].

(2.5.1) Definition: $\xi(t)$ is differentiable in q.m. at t iff there is a (necessarily a.s. unique) element $\eta \in H$ such that

$$\frac{\xi(t+h) - \xi(t)}{h} \to \eta \text{ , i.e. } \left\| \frac{\xi(t+h) - \xi(t)}{h} - \eta \right\| \to 0 \text{ as } h \to 0.$$

$\xi(t)$ is differentiable on $[0,T]$ iff it is differentiable at each $t \in [0,T]$.

The derivative in q.m. is denoted by $\frac{d\,\xi(t)}{dt}$.

(2.5.2) If $\xi(t)$ is differentiable in q.m. at t, then

i) $\left\| \frac{d\,\xi(t)}{dt} \right\| = \lim_{h \to 0} \left\| \frac{\xi(t+h) - \xi(t)}{h} \right\|$ on account of (2.1.2),

ii) $\xi(t)$ is continuous in q.m. at t, since $s \to t$ entails
$$\| \xi(s) - \xi(t) \| = |s-t| . \left\| \frac{\xi(s) - \xi(t)}{s - t} \right\| \to 0 . \left\| \frac{d\,\xi(t)}{dt} \right\| = 0.$$

(2.5.3) If $\xi(t)$ is differentiable in q.m., then $E^{\mathcal{B}} \xi(t)$ is differentiable in q.m. with derivative
$$\frac{d}{dt} E^{\mathcal{B}} \xi(t) = E^{\mathcal{B}} \frac{d\,\xi(t)}{dt}$$
on account of (2.2.2). In particular, $E\xi(t)$ is differentiable and
$$\frac{d}{dt} E\,\xi(t) = E\,\frac{d\,\xi(t)}{dt} .$$
As we put $E\,\xi(t) = x(t)$ and $\xi(t) = x(t) + \xi'(t)$, $\xi(t)$ is differentiable in q.m. if and only if $x(t)$ and $\xi'(t)$ are differentiable. Then
$$\frac{d\,\xi(t)}{dt} = \frac{dx(t)}{dt} + \frac{d\xi'(t)}{dt}$$

(2.5.4) If $\xi(t)$ and $\eta(t)$ are differentiable in q.m., then $a\,\xi(t) + b\,\eta(t)$ is differentiable with derivative
$$a\,\frac{d\,\xi(t)}{dt} + b\,\frac{d\,\eta(t)}{dt} .$$

(2.5.5) If $\xi(t)$ is differentiable in q.m. on $[0,T]$, and if $f(t)$ is a differentiable real valued function of $t \in [0,T]$, then $f(t)\,\xi(t)$ is differentiable in q.m. with derivative
$$\frac{d}{dt}\{f(t)\,\xi(t)\} = \frac{df(t)}{dt}\,\xi(t) + f(t)\,\frac{d\,\xi(t)}{dt}$$
since $\left\| \frac{f(t+h)\,\xi(t+h) - f(t)\,\xi(t)}{h} - \frac{df(t)}{dt}\,\xi(t) - f(t)\,\frac{d\,\xi(t)}{dt} \right\| \leq$
$$\left\| \frac{f(t+h)\,\xi(t+h) - f(t)\,\xi(t+h)}{h} - \frac{df(t)}{dt}\,\xi(t) \right\| +$$
$$\left\| \frac{f(t)\,\xi(t+h) - f(t)\,\xi(t)}{h} - f(t)\,\frac{d\,\xi(t)}{dt} \right\| =$$
$$\left\| \frac{f(t+h) - f(t)}{h}\,\xi(t+h) - \frac{df(t)}{dt}\,\xi(t) \right\| + |f(t)| . \left\| \frac{\xi(t+h) - \xi(t)}{h} - \frac{d\,\xi(t)}{dt} \right\|$$
$\to 0$ as $h \to 0$.

(2.5.6) Differentiability in q.m. criterion: Owing to (2.1.4), $\xi(t)$ is differentiable in q.m. at t if and only if
$$\frac{\Delta_h \Delta_k\, E\,\xi(t)\,\xi(t)}{hk} = \frac{E\,\xi(t+h)\,\xi(t+k) - E\,\xi(t+h)\,\xi(t) - E\,\xi(t)\,\xi(t+k) + E\,\xi(t)\,\xi(t)}{h\,k} =$$
$E \left\{ \frac{\xi(t+h) - \xi(t)}{h} \right\} \left\{ \frac{\xi(t+k) - \xi(t)}{k} \right\}$ converges as $h,k \to 0$ independently.

The existence of the above limit is not equivalent to the the existence of

$$\frac{\partial^2}{\partial s \partial s'} E\, \xi(s)\, \xi(s') \quad \text{at} \quad (s,s') = (t,t).$$

(2.5.7) If $\xi(t)$ is differentiable in q.m. on $[0,T]$,

$$\frac{\partial}{\partial s} E\, \xi(s)\, \xi(t), \qquad \frac{\partial}{\partial t} E\, \xi(s)\, \xi(t) \quad \text{and} \quad \frac{\partial^2}{\partial s \partial t} E\, \xi(s)\, \xi(t)$$

exist and are finite on $[0,T]^2$. Then by virtue of (2.1.3) and (2.5.6) the following relations hold:

$$E\, \frac{d\,\xi(s)}{ds}\, \xi(t) = \frac{\partial}{\partial s} E\, \xi(s)\, \xi(t), \qquad E\, \frac{d\,\xi(s)}{ds}\, \frac{d\,\xi(t)}{dt} = \frac{\partial^2}{\partial s \partial t} E\, \xi(s)\, \xi(t).$$

And if also $\eta(t)$ is differentiable in q.m. on $[0,T]$,

$$E\, \frac{d\,\xi(s)}{ds}\, \eta(t) = \frac{\partial}{\partial s} E\, \xi(s)\, \eta(t), \qquad E\, \frac{d\,\xi(s)}{ds}\, \frac{d\,\eta(t)}{dt} = \frac{\partial^2}{\partial s \partial t} E\, \xi(s)\, \eta(t).$$

(2.5.8) If $\xi(t)$ is differentiable in q.m. on $[0,T]$, it is a constant element of H if and only if

$$\frac{d\,\xi(t)}{dt} = 0 \qquad \text{on } [0,T].$$

Proof: If the above relation holds, then on account of (2.5.7)

$$\frac{\partial}{\partial s} E\, \xi(s)\, \xi(t) = E\, \frac{d\,\xi(s)}{ds}\, \xi(t) = 0 \qquad \text{on } [0,T]^2.$$

Therefore $E\, \xi(s)\, \xi(t)$ is independent of s. Because of symmetry it is also independent of t and so it is a constant on $[0,T]^2$, say $E\, \xi(s)\, \xi(t) = c$. Then

$$\|\, \xi(t) - \xi(0)\, \|^2 = E\, \{\xi(t) - \xi(0)\}\, \{\xi(t) - \xi(0)\} = c - c - c + c = 0$$

and hence $\xi(t) = \xi(0)$ at all $t \in [0,T]$.

The "only if" part needs no comment.

(2.5.9) If $\xi(t)$ is differentiable in q.m. and Gaussian on $[0,T]$, $\frac{d\,\xi(t)}{dt}$ is also Gaussian on $[0,T]$ by virtue of (2.3.1). For, the divided differences are Gaussian and so is the limit in q.m. of a sequence of divided differences.

(2.5.10) Let the stochastic process $\xi(t)$, $t \in [0,T]$, be sample continuous and let $\xi(\omega,t)$, $(\omega,t) \in \Omega \times [0,T]$, where Ω is the pointset of a suitable probability space $\{\Omega, \mathcal{A}, P\}$, be a sample continuous representation of $\xi(t)$. Let moreover be assumed that almost all trajectories are differentiable on $[0,T]$. Then

$\xi(t)$ is called sample differentiable. At each $t \in [0,T]$

$$\frac{d\,\xi(t)}{dt} = \lim_{\Delta t \to 0} \frac{\xi(\omega, t+\Delta t) - \xi(\omega, t)}{t}$$

exists as an a.s. limit and is called sample derivative. It is $\mathcal{A}$-measurable as an a.s. limit of a sequence of $\mathcal{A}$-measurable divided differences.

Differentiability in q.m. is not equivalent to sample differentiability of a second order process. If the derivative in q.m. as well as the sample derivative of a second order process exist , they are a.s.identical at fixed t as both may be seen as limits in probability of one and the same sequence of divided differences.

2.6. Riemann-Stieltjes integrals in q.m. I.

In this section, the greater part of the theorems belongs to Riemann-Stieltjes integration theory in Banach spaces with the strong topology, see [9]. Then the methods are analogous to those, used in real analysis, see [21].

From now on a "partition p" of $[0,T]$ will be understood to consist of

i) a set of subdivision points $\{t_k,\ k=0,.,K\}$ such that

$$0 = t_o < t_1 < . < t_K = T ,$$

ii) a set $\{t_k',\ k=1,.,K\}$, such that $t_k' \in [t_{k-1}, t_k]$.

$\Delta(p) = \max\limits_{k=1,.,K}(t_k - t_{k-1})$ is the mesh of p . Let $\{p\}$ denote the set of all partitions of $[0,T]$ of the above type. A partition q is a refinement of a partition p if each subdivision point of p is also a subdivision point of q.

Let $f(t)$ be a mapping of $[0,T]$ into $(-\infty,\infty)$ and $\xi(t)$ a mapping of $[0,T]$ into H. Set

$$\sigma(p) = \sum_{k=1}^{K} \xi(t_k')\{f(t_k) - f(t_{k-1})\} \quad \text{and} \quad \sigma'(p) = \sum_{k=1}^{K} f(t_k')\{\xi(t_k) - \xi(t_{k-1})\} .$$

Clearly $\sigma(p)$ and $\sigma'(p)$ are elements of H.

<u>(2.6.1)</u> Definition: Iff to all sequences $\{p_n, n=1,2,.\} \subset \{p\}$ such that $\Delta(p_n) \to 0$ as $n \to \infty$, the sequences

$$\{\sigma(p_n),\ n=1,2,.\} \qquad \text{and} \qquad \{\sigma'(p_n),\ n=1,2,.\}$$

are Cauchy sequences in H, necessarily with a unique limit, say

σ and σ' respectively,

then σ is the Riemann-Stieltjes integral in q.m. $\int_0^T \xi(t)df(t)$

and σ' is the Riemann-Stieltjes integral in q.m. $\int_0^T f(t)d\,\xi(t)$.

If $f(t)=t$ on $[0,T]$, σ is the Riemann integral in q.m. $\int_0^T \xi(t)dt$.

By virtue of this definition, the integrals are independent of the position of the points $t_k' \in [t_{k-1}, t_k]$ in the partitions p_n. As elements of H, integrals in q.m. have a finite norm.

<u>(2.6.2)</u> Let $f(t)$ be a mapping of $[0,T]$ into $(-\infty,\infty)$ and $\xi(t)$ a mapping of $[0,T]$ into H such that

$$\text{either } \int_0^T \xi(t)df(t) \quad \text{or} \quad \int_0^T f(t)d\,\xi(t)$$

exists as a Riemann-Stieltjes integral in q.m.

i) Then both integrals exist and the following relation holds:

$$\int_0^T \xi(t)df(t) = \big[f(t)\,\xi(t)\big]_0^T - \int_0^T f(t)d\,\xi(t).$$

ii) If $s \in [0,T]$, the integrals in q.m. in the right-hand sides below exist if ~~and only if~~ those in the left-hand side exist. ~~And~~ The

$$\int_0^T \xi(t)df(t) = \int_0^s \xi(t)df(t) + \int_s^T \xi(t)df(t),$$

$$\int_0^T f(t)d\,\xi(t) = \int_0^s f(t)d\,\xi(t) + \int_s^T f(t)d\,\xi(t).$$

iii) If $\eta(t)$ is a mapping of $[0,T]$ into H such that also $\int_0^T \eta(t)df(t)$ exists as a Riemann-Stieltjes integral in q.m., then the integrals below exist and satisfy

$$\int_0^T \{a\,\xi(t) + b\,\eta(t)\}df(t) = a\int_0^T \xi(t)df(t) + b\int_0^T \eta(t)df(t),$$

$$\int_0^T f(t)d\{a\,\xi(t) + b\,\eta(t)\} = a\int_0^T f(t)d\,\xi(t) + b\int_0^T f(t)d\,\eta(t).$$

iv) If $g(t)$ is a mapping of $[0,T]$ into $(-\infty,\infty)$ such that also $\int_0^T \xi(t)dg(t)$ exists as a Riemann-Stieltjes integral in q.m., then the integrals below exist and satisfy

$$\int_0^T \xi(t)d\{af(t) + bg(t)\} = a\int_0^T \xi(t)df(t) + b\int_0^T \xi(t)dg(t),$$

$$\int_0^T \{af(t) + bg(t)\}d\,\xi(t) = a\int_0^T f(t)d\,\xi(t) + b\int_0^T g(t)d\,\xi(t).$$

v) $\int_0^T E^{\mathcal{B}}\xi(t)df(t)$ and $\int_0^T f(t)dE^{\mathcal{B}}\xi(t)$ exist and satisfy

$$\int_0^T E^{\mathcal{B}}\xi(t)df(t) = E^{\mathcal{B}}\int_0^T \xi(t)df(t) \text{ and } \int_0^T f(t)dE^{\mathcal{B}}\xi(t) = E^{\mathcal{B}}\int_0^T f(t)d\,\xi(t).$$

If we put $x(t) = E\,\xi(t)$ and $\xi(t) = x(t) + \xi'(t)$, then $\int_0^T \xi(t)df(t)$ and $\int_0^T f(t)d\,\xi(t)$ exist if and only if $\int_0^T x(t)df(t)$ and $\int_0^T \xi'(t)df(t)$, or $\int_0^T f(t)dx(t)$ and $\int_0^T f(t)d\xi'(t)$ exist. They satisfy

$$\int_0^T \xi(t)df(t) = \int_0^T x(t)df(t) + \int_0^T \xi'(t)df(t),$$

$$\int_0^T f(t)d\,\xi(t) = \int_0^T f(t)dx(t) + \int_0^T f(t)d\xi'(t).$$

vi) If $\xi(t)$ is Gaussian on $[0,T]$,

$$\int_0^s \xi(t)df(t) \text{ and } \int_0^s f(t)d\,\xi(t), \quad s \in [0,T],$$

are Gaussian processes on $[0,T]$ by virtue of (2.3.1), since the above integrals are limits in q.m. of sequences of Gaussian Riemann-Stieltjes sums.

vii) Let be assumed that $\xi(t)$ is sample continuous on $[0,T]$ and that $\xi(\omega,t)$, $(\omega,t)\in\Omega\times[0,T]$, where Ω is the pointset of a suitable probability space $\{\Omega, \mathcal{A}, P\}$, is a sample continuous representation of $\xi(t)$. Then the trajectories are well defined, see (2.3.2). If at almost all $\omega\in\Omega$ the ordinary integral

$$\int_0^T \xi(\omega,t)df(t)$$

exists, then also the ordinary integral

$$\int_0^T f(t)d\,\xi(\omega,t)$$

exists at almost all $\omega\in\Omega$, and reversed, by virtue of the theorem on partial integration. The above integrals are called sample integrals. As they may be evaluated as a.s. limits of the sequences

$$\left\{ \sigma(p_n),\ n=1,2,. \right\} \quad \text{and} \quad \left\{ \sigma'(p_n),\ n=1,2,. \right\},$$

they are $\mathcal{A}$ -measurable.

As the above sequences are also Cauchy sequences in H, it follows that the sample integrals are a.s. identical to the corresponding integrals in q.m.

Proof of (2.6.2): The proofs of i - iv are analogous to the proofs of the corresponding theorems of real analysis. It remains to show v :

Let $\left\{ p_n,\ n=1,2,. \right\} \subset \left\{ p \right\}$ be such that $\Delta(p_n) \rightarrow 0$ as $n \rightarrow \infty$.

$$E^{\mathcal{B}}\sigma(p_n) = \sum_{k=1}^{K} E^{\mathcal{B}}\xi(t_k') \left\{ f(t_k) - f(t_{k-1}) \right\} .$$

As $n \rightarrow \infty$, the left-hand side tends to $E^{\mathcal{B}}\int_0^T \xi(t)df(t)$ by virtue of (2.2.2). Then necessarily also the right-hand side is convergent as $n \rightarrow \infty$. According to definition (2.6.1) its limit in q.m. is $\int_0^T E^{\mathcal{B}}\xi(t)df(t)$.

In the other statements of v , also statement iii is used.

2.7. Riemann-Stieltjes integrals in q.m. II.

The greater part of this section consists of immediate generalizations of results of real analysis. Covariance function techniques may be found in lots of books, for instance in [19].

(2.7.1) If the mapping $\xi(t)$ of $[0,T]$ into H is continuous in q.m. and if the mapping $f(t)$ of $[0,T]$ into $(-\infty,\infty)$ is of bounded variation on $[0,T]$ with total variation V, then

$$\int_0^T \xi(t)df(t)$$

exists as a Riemann-Stieltjes integral in q.m. and the assertions of (2.6.2) are valid.

Proof: Let $\{p_n\}$ be any sequence of partitions of $[0,T]$ such that $\Delta(p_n) \to 0$ as $n \to \infty$. Being continuous in q.m. on the compact set $[0,T]$, $\xi(t)$ is uniformly continuous in q.m. on $[0,T]$, according to (2.4.7). So, given $\varepsilon > 0$, there is a $\delta > 0$ such that $[t,t'] \subset [0,T]$ and $|t-t'| < \delta$ imply $\|\xi(t) - \xi(t')\| < \varepsilon$. If n and m are sufficiently large and if p is a refinement of both p_n and p_m,

$$\|\sigma(p_m) - \sigma(p_n)\| \leqq \|\sigma(p_m) - \sigma(p)\| + \|\sigma(p) - \sigma(p_n)\| \leqq \varepsilon V + \varepsilon V.$$

The statement is shown by virtue of definition (2.6.1).

<u>(2.7.2)</u> Under the same conditions as in (2.7.1),

i) $\left\| \int_0^T \xi(t)df(t) \right\| \leqq M V$ as $M = \max_{t \in [0,T]} \|\xi(t)\|$.

ii) $\left\| \int_0^T \xi(t)\,dt \right\| \leqq \int_0^T \|\xi(t)\|\,dt \leqq M T$.

iii) $\int_0^t \xi(t')dt'$ is continuous in q.m. as a function of $t \in [0,T]$ and continuously differentiable in q.m. with derivative

$$\frac{d}{dt}\int_0^t \xi(t')dt' = \xi(t)$$

iv) If $\frac{d\,\xi(t)}{dt}$ is continuous in q.m. on $[0,T]$, then if $t \in [0,T]$,

$$\int_0^t \frac{d\,\xi(t')}{dt'}\,dt' = \xi(t) - \xi(0).$$

Proof: i) Let $\{p_n\}$ be a sequence of partitions of $[0,T]$ such that $\Delta(p_n) \to 0$ as $n \to \infty$. As

$$\sigma(p_n) = \sum_{k=1}^{K} \xi(t_k') \cdot \{ f(t_k) - f(t_{k-1}) \},$$

$$\|\sigma(p_n)\| \leqq \sum_{k=1}^{K} \|\xi(t_k')\| \cdot |f(t_k) - f(t_{k-1})| \leqq M V.$$

As $n \to \infty$, the left-hand side of the inequality tends to $\left\| \int_0^T \xi(t)df(t) \right\|$, cf. (2.1.2), and the right-hand sides remain constant.

Concerning ii, $\|\xi(t)\|$ is a continuous real valued function of $t \in [0,T]$ according to (2.4.2). So both integrals exist.

$$\|\sigma(p_n)\| = \left\| \sum_{k=1}^{K} \xi(t_k')(t_k - t_{k-1}) \right\| \leqq \sum_{k=1}^{K} \|\xi(t_k')\| \cdot (t_k - t_{k-1}) \leqq M T, \text{ and}$$

the above sums tend to the corresponding integrals as $n \to \infty$.

Concerning iii, it is sufficient to establish the formula for the derivative in q.m. Provided that $[t,t+h] \subset [0,T]$, and given $\varepsilon > 0$,

$$\left\| \frac{\int_t^{t+h} \xi(t')dt' - \int_0^t \xi(t')dt'}{h} - \xi(t) \right\| = \left|\frac{1}{h}\right| \cdot \left\| \int_t^{t+h} \xi(t')dt' - h\xi(t) \right\| =$$

$$\left|\frac{1}{h}\right| \cdot \left\| \int_t^{t+h} \{\xi(t') - \xi(t)\} dt' \right\| \leqq \left|\frac{1}{h}\right| \cdot \varepsilon \cdot |h| = \varepsilon$$

if $|h|$ is sufficiently small.

Concerning iv, on account of iii,

$$\frac{d}{dt} \left\{ \int_0^t \frac{d\xi(t')}{dt'} dt' - \xi(t) \right\} = \frac{d\xi(t)}{dt} - \frac{d\xi(t)}{dt} = 0.$$

Therefore, $\int_0^t \frac{d\xi(t')}{dt'} dt' - \xi(t)$ is a constant element of H according to (2.5.8). Putting $t = 0$, this constant is seen to be $0 - \xi(0)$ and so

$$\int_0^t \frac{d\xi(t')}{dt'} dt' - \xi(t) = -\xi(0).$$

(2.7.3) If $\xi(t)$ and $\eta(t)$ are continuous mappings of $[0,T]$ into H and if $f(t)$ and $g(t)$ are real valued functions of bounded variation on $[0,T]$,

$$E\left\{\int_0^s \xi(s')df(s') \int_0^t \eta(t')dg(t')\right\} = \int_0^s \int_0^t E\, \xi(s')\eta(t')ddf(s')g(t'),$$

$(s,t) \in [0,T]^2$, and in particular,

$$\left\| \int_0^T \xi(t)df(t) \right\|^2 = \int_0^T \int_0^T E\, \xi(s)\xi(t)\, dd\, f(s)f(t)$$

Proof: Let $\{p_m\}$ and $\{q_n\}$ be sequences of partitions of $[0,s]$ and $[0,t]$ respectively, such that $\Delta(p_m)$ and $\Delta(q_n)$ tend to 0 as $m,n \to \infty$, and let

$$\sigma(p_m) = \sum_{i=1}^{I} \xi(s_i')\{f(s_i)-f(s_{i-1})\}, \quad \sigma(q_n) = \sum_{j=1}^{J} \eta(t_j')\{g(t_j)-g(t_{j-1})\}.$$

Then

$$E\sigma(p_m)\sigma(q_n) = \sum_{i=1}^{I} \sum_{j=1}^{J} E\, \xi(s_i')\eta(t_j')\{f(s_i)-f(s_{i-1})\}\{g(t_j)-g(t_{j-1})\}.$$

As $m,n \to \infty$, the left-hand side tends to

$$E\left\{\int_0^s \xi(s')df(s') \int_0^t \eta(t')dg(t')\right\}$$

by virtue of (2.1.3). The right-hand side is a Riemann-Stieltjes sum belonging to

$$\int_0^s \int_0^t E\,\xi(s')\,\eta(t')\,dd\,f(s')g(t').$$

This ordinary Riemann-Stieltjes integral of real analysis exists, since $E\,\xi(s')\,\eta(t')$ is continuous on $[0,s]\times[0,t]$ on account of (2.1.3), and since $f(s')g(t')$ is of bounded variation on $[0,s]\times[0,t]$, cf. section 2.9.

<u>(2.7.4)</u> If $f(t)$ is a continuously differentiable mapping of $[0,T]$ into $(-\infty,\infty)$ and if $\xi(t)$ is a continuous mapping of $[0,T]$ into H, then

$$\int_0^T \xi(t)df(t) = \int_0^T \xi(t)\,\frac{df(t)}{dt}\,dt \qquad \text{in q.m.}$$

Proof: As $f(t)$ is continuously differentiable on $[0,T]$, it is also of bounded variation on $[0,T]$. So both integrals exist as Riemann (-Stieltjes) integrals in q.m. Let $\{p_n\}$ be a sequence of partitions of $[0,T]$ such that $\Delta(p_n)\to 0$ as $n\to\infty$. Since by virtue of the ordinary mean value theorem

$$f(t_k) - f(t_{k-1}) = (t_k - t_{k-1})\,\frac{df(t'_{k-1})}{dt}\;, \quad t'_{k-1}\in(t_{k-1},t_k),$$

$$\sigma(p_n) = \sum_{k=1}^{K} \xi(t'_{k-1})\{f(t_k)-f(t_{k-1})\} = \sum_{k=1}^{K}\xi(t'_{k-1})\,\frac{df(t'_{k-1})}{dt}\,(t_k-t_{k-1})$$

where the values of $\xi(t)$ may be taken at the same points t'_{k-1} as those of $\frac{df(t)}{dt}$. As $n\to\infty$, the statement follows on account of definition (2.6.1).

<u>(2.7.5)</u> If $f(t)$ is a real valued function of bounded variation on $[0,T]$ and if $\{\xi_n(t),\ n=1,2,.\}$ is a sequence of continuous mappings of $[0,T]$ into H, converging in q.m. to $\xi(t)$ as $n\to\infty$, uniformly in $t\in[0,T]$, then as $n\to\infty$,

$$\int_0^t \xi_n(s)df(s) \to \int_0^t \xi(s)df(s) \quad \text{in q.m., uniformly in } t\in[0,T].$$

Proof: Since by virtue of (2.4.5) $\xi(t)$ is continuous in q.m. on $[0,T]$, the existence of all integrals figuring in the assertion is ensured. If $\varepsilon>0$ and n sufficiently large, then for all $t\in[0,T]$,

$$\left\|\int_0^t \xi_n(s)df(s) - \int_0^t \xi(s)df(s)\right\| = \left\|\int_0^t\{\xi_n(s) - \xi(s)\}\,df(s)\right\| \leqq \varepsilon V\,,$$

where V is the total variation of $f(t)$ on $[0,T]$.

By means of partial integration, a number of results concerning Riemann-Stieltjes integrals of the type

$$\int_0^T f(t)d\xi(t)$$

may be derived from the above statements. For example

(2.7.6) If $\{\xi_n(t), n=1,2,.\}$ is a sequence of continuous mappings of $[0,T]$ into H, converging in q.m. to $\xi(t)$ as $n \to \infty$, uniformly in $t \in [0,T]$, and if $f(t)$ is a real valued continuously differentiable function of $t \in [0,T]$, then as $n \to \infty$,

$\int_0^t f(s)d\xi_n(s) \to \int_0^t f(s)d\xi(s)$ in q.m., uniformly in $t \in [0,T]$.

Proof: By virtue of (2.4.5), $\xi(t)$ is continuous in q.m. on $[0,T]$. According to (2.6.2) and (2.7.4),

$$\int_0^t f(s)d\xi_n(s) = \left[f(s)\xi_n(s)\right]_0^t - \int_0^t \xi_n(s)\frac{df(s)}{ds} \quad \text{and}$$

$$\int_0^t f(s)d\xi(s) = \left[f(s)\xi(s)\right]_0^t - \int_0^t \xi(s)\frac{df(s)}{ds} .$$

Apparently all above integrals in q.m. exist. Application of (2.7.5) completes the proof.

2.8. Riemann-Stieltjes integrals in q.m. III.

The existence of integrals in q.m. of the type

$$\int_0^T f(t)d\xi(t)$$

cannot always be reduced by means of partial integration to the existence of

$$\int_0^T \xi(t)df(t) .$$

In certain circumstances the following method might be useful.

Let $\xi(t)$ be a mapping of $[0,T]$ into H. Let p be the partition of $[0,T]$, defined in section 2.6, and let $\{p\}$ be the set of all partitions of $[0,T]$.

(2.8.1) Definition:

$$V(\xi(t), p, [0,T]) = \sum_{k=1}^{K} \|\xi(t_k) - \xi(t_{k-1})\|,$$

$$V(\xi(t), [0,T]) = \sup_{p \in \{p\}} V(\xi(t), p, [0,T]).$$

$\xi(t)$ is of bounded variation in the strong sense on $[0,T]$ iff

$$V(\xi(t), [0,T]) < \infty .$$

(2.8.2) If $f(t)$ is a continuous mapping of $[0,T]$ into $(-\infty, \infty)$, and if $\xi(t)$ is a mapping of $[0,T]$ into H of bounded variation on $[0,T]$ in the strong sense, the Riemann-Stieltjes integral in q.m.

$$\int_0^T f(t) d\,\xi(t)$$

exists and the assertions of (2.6.2) are valid.

Proof: Let $\{p_n\}$ be a sequence of partitions of $[0,T]$ such that $\Delta(p_n) \to 0$ as $n \to \infty$. As $[0,T]$ is compact, $f(t)$ is uniformly continuous on $[0,T]$. So, given $\varepsilon > 0$ there is a $\delta > 0$ such that $[s,t] \subset [0,T]$ and $|s - t| < \delta$ imply $|f(s) - f(t)| < \varepsilon$. If n and m are sufficiently large and if p is a refinement of both p_n and p_m,

$$\|\sigma'(p_n) - \sigma'(p_m)\| \leq \|\sigma'(p_n) - \sigma'(p)\| + \|\sigma'(p) - \sigma'(p_m)\| \leq \varepsilon V + \varepsilon V.$$

Therefore $\{\sigma'(p_n)\}$ is a Cauchy sequence in H and the assertion is proved according to definition (2.6.1).

A number of statements analogous to those of section 2.7 may now be proved for integrals of the type

$$\int_0^T f(t) d\,\xi(t) .$$

This, however, will be done in the next section under less stringent conditions. Here we shall confine ourselves to the following theorem.

(2.8.3) If $\xi(t)$ is a mapping of $[0,T]$ into H, continuously differentiable in q.m., and if $f(t)$ is a continuous mapping of $[0,T]$ into $(-\infty, \infty)$, then

i) $$V(\xi(t), [0,T]) < \infty ,$$

ii) $$\int_0^T f(t) d\,\xi(t) = \int_0^T f(t) \frac{d\,\xi(t)}{dt}\, dt.$$

Proof: Let p be the partition of $[0,T]$, defined in section 2.6.
Concerning i, it follows from (2.7.2) that

$$\sum_{k=1}^{K} \|\xi(t_k)-\xi(t_{k-1})\| = \sum_{k=1}^{K} \left\| \int_{t_{k-1}}^{t_k} \frac{d\,\xi(t)}{dt}\,dt \right\| \leqq M\,T$$

where $M = \max_{t\in[0,T]} \left\| \frac{d\,\xi(t)}{dt} \right\|$. And so $V(\xi(t), [0,T]) \leqq M\,T$.

Concerning ii, according to (2.8.2) and (2.7.1) both integrals exist as Riemann (-Stieltjes) integrals in q.m. Let

$$\sigma'(p) = \sum_{k=1}^{K} f(t_{k-1}) \left\{ \xi(t_k) - \xi(t_{k-1}) \right\},$$

$$\sigma(p) = \sum_{k=1}^{K} f(t_{k-1}) \frac{d\,\xi(t_{k-1})}{dt} (t_k - t_{k-1}).$$

If p passes through a sequence $\{p_n\}$ such that $\Delta(p_n) \rightarrow 0$ as $n \rightarrow \infty$, then

$$\sigma'(p_n) \rightarrow \int_0^T f(t)d\,\xi(t) \quad \text{and} \quad \sigma(p_n) \rightarrow \int_0^T f(t) \frac{d\,\xi(t)}{dt}\,dt.$$

On account of the definition and the existence of the integrals, figuring in ii, we were alowed to choose $t_k' = t_{k-1}$, $k=1,.,K$.
By virtue of (2.7.2),

$$\|\sigma'(p)-\sigma(p)\| \leqq \sum_{k=1}^{K} |f(t_{k-1})| \cdot \left\| \frac{d\,\xi(t_{k-1})}{dt}(t_k-t_{k-1}) - \left\{\xi(t_k)-\xi(t_{k-1})\right\} \right\| =$$

$$\sum_{k=1}^{K} |f(t_{k-1})| \cdot \left\| \int_{t_{k-1}}^{t_k} \left\{ \frac{d\,\xi(t_{k-1})}{dt} - \frac{d\,\xi(t)}{dt} \right\} dt \right\| \leqq$$

$$\sum_{k=1}^{K} |f(t_{k-1})| \cdot \int_{t_{k-1}}^{t_k} \left\| \frac{d\,\xi(t_{k-1})}{dt} - \frac{d\,\xi(t)}{dt} \right\| dt.$$

As $M = \max_{t\in[0,T]} |f(t)|$, given $\varepsilon > 0$ it follows that

$$\| \sigma'(p_n) - \sigma(p_n) \| \leqq \varepsilon M\,T$$

if n is sufficiently large, since $\frac{d\,\xi(t)}{dt}$ is uniformly continuous in q.m. on the compact set $[0,T]$. Therefore, as $n \rightarrow \infty$,

$$\left\| \int_0^T f(t)d\,\xi(t) - \int_0^T f(t) \frac{d\,\xi(t)}{dt}\,dt \right\| \leqq$$

$$\left\| \int_0^T f(t)d\,\xi(t) - \sigma'(p_n) \right\| + \| \sigma'(p_n) - \sigma(p_n) \| + \left\| \sigma(p_n) - \int_0^T f(t) \frac{d\,\xi(t)}{dt}\,dt \right\| \rightarrow 0.$$

2.9. Riemann-Stieltjes integrals in q.m. IV .

The first statements in this section belong to standard calculus in q.m., see [15], or [19] for instance.

Let us first introduce and recall several notations and notions of real analysis. If $D = [0,S]\times[0,T]$ is the Cartesian product of two intervals of the real line, the finite set $p(D)$ of of rectangles $d = [s,s']\times[t,t']$ with union D, such that the intersection of every two rectangles d consists at most of an edge, is a partition of D. The mesh of $p(D)$ is defined as

$$\Delta(p(D)) = \max_{d\in p(D)} |s'-s|, \ |t'-t| .$$

A partition $q(D)$ of D is a refinement of $p(D)$ if each element of $p(D)$ is the union of some elements of $q(D)$.

If p and q are partitions of $[0,S]$ and $[0,T]$ defined by $0 = s_0 < s_1 < . < s_I = S$ and $0 = t_0 < t_1 < . < t_J = T$ respectively,

$$p(D) = p\times q = \left\{ [s_{i-1},s_i]\times[t_{j-1},t_j] \ , \ i=1,.,I, \quad j=1,.,J \right\}$$

is a product partition of D. Every partition of D can be refined by product partitions. If $D = [0,T]^2$, every partition of D can be refined by product partitions of the type $p^2 = p\times p$.

If $G(s,t)$ is a mapping of D into $(-\infty,\infty)$, we define, as $\{p(D)\}$ is the set of all partitions of D, and if

$$d = [s,s']\times[t,t'] \in p(D),$$

$$\underset{d}{\Delta\Delta} G(s,t) = G(s',t') - G(s,t') - G(s',t) + G(s,t),$$

$$V(G(s,t),p(D),D) = \sum_{d\in p(D)} \left| \underset{d}{\Delta\Delta} G(s,t) \right| ,$$

and

$$V(G(s,t),D) = \sup_{\{p(D)\}} V(G(s,t),p(D),D) .$$

$V(G(s,t),D)$ is the total variation of $G(s,t)$ on D.

If $q(D)$ is a refinement of $p(D)$,

$$V(G(s,t),q(D),D) \geqq V(G(s,t),p(D),D) .$$

Since each partition may be refined by a product partition, it follows that $V(G(s,t),D)$ may be evaluated by means of the product partitions alone, and by means of the partitions of type p^2 if D is a square. If the total variation $V(G(s,t),D)$ of $G(s,t)$ on D is finite, $G(s,t)$ is called of bounded variation on D.

$V(G(s,t),D)$ is a non-negative σ-additive set function with respect to D.

If $D = [0,S] \times [0,T]$ and $G(s,t) = g_1(s)g_2(t)$,

then $V(G(s,t),D) = V(g_1(s),[0,S]).V(g_2(t),[0,T])$

where $V(g_1(s),[0,S])$ and $V(g_2(t),[0,T])$

are the total variations of $g_1(s)$ and $g_2(t)$ on $[0,S]$ and $[0,T]$ respectively.

Let the notion of partition $p(D)$ be extended in this sense that to each $d \in p(D)$ there is an arbitrary point $(s_d,t_d) \in d$. Let $F(s,t)$ and $G(s,t)$ be mappings of D into $(-\infty,\infty)$. If to all sequences $\{p_n(D)\}$, such that $\Delta(p_n(D)) \rightarrow 0$ as $n \rightarrow \infty$, the corresponding sequences

$$\left\{ S(p_n(D)) = \sum_{d \in p_n(D)} F(s_d,t_d) \Delta\underset{d}{\Delta} G(s,t) \right\}$$

are convergent, necessarily with one and the same limit, say S, then

$$S = \iint_D F(s,t)ddG(s,t)$$

is the ordinary Riemann-Stieltjes integral of $F(s,t)$ with respect to $G(s,t)$ on D.

This integral may be shown to exist if $F(s,t)$ is continuous and $G(s,t)$ of bounded variation on D. Then

$$|S| \leqq M V$$

where $M = \max_{(s,t) \in D} |F(s,t)|$ and V is the total variation of $G(s,t)$ on D.

Now we shall establish the existence of the integral in (2.8.2) under a less stringent condition:

(2.9.1) If $f(t)$ is a continuous mapping of $[0,T]$ into $(-\infty,\infty)$, and if $\xi(t)$ is a mapping of $[0,T]$ into H such that $E\,\xi(s)\,\xi(t)$ is of bouded variation on $[0,T]^2$,

$$\int_0^T f(t)d\,\xi(t)$$

exists as a Riemann-Stieltjes integral in q.m., and the assertions of (2.6.2) are valid. Then

$$\left\|\int_0^T f(t)d\,\xi(t)\right\|^2 = \int_0^T\int_0^T f(s)f(t)ddE\,\xi(s)\,\xi(t) \leqq M^2V$$

if $M = \max_{t\in[0,T]} |f(t)|$ and $V = V(E\,\xi(s)\,\xi(t), [0,T]^2)$.

Proof: Let $\{p_n\}$ be a sequence of partitions of $[0,T]$ such that $\Delta(p_n)\to 0$ as $n\to\infty$. In regard to definition (2.6.1), it is to be established that $\{\sigma'(p_n)\}$ is a Cauchy sequence in H or, equivalently on account of the convergence in q.m. criterion (2.1.4), that

$$E\sigma'(p_m)\sigma'(p_n)$$

converges as $m,n\to\infty$. If p_m and p_n are defined by $0 = s_0 < s_1 < . < s_I = T$ and $0 = t_0 < t_1 < . < t_J = T$ respectively,

$$E\sigma'(p_m)\sigma'(p_n) = \sum_{i=1}^{I}\sum_{j=1}^{J} f(s_i')f(t_j')E\left\{\xi(s_i)-\xi(s_{i-1})\right\}\left\{\xi(t_j)-\xi(t_{j-1})\right\} =$$

$$\sum_{i=1}^{I}\sum_{j=1}^{J} f(s_i')f(t_j') \underset{[s_{i-1},s_i]\times[t_{j-1},t_j]}{\Delta\Delta} E\,\xi(s)\,\xi(t).$$

The right-hand side may be seen as a Riemann-Stieltjes sum belonging to the ordinary integral

$$\int_0^T\int_0^T f(s)f(t)ddE\,\xi(s)\,\xi(t).$$

This integral exists as $f(s)f(t)$ is continuous and $E\,\xi(s)\,\xi(t)$ of bounded variation on $[0,T]^2$. And so $E\sigma'(p_m)\sigma'(p_n)$ converges as $n,m\to\infty$.

(2.9.2) Corollary: If also $g(t)$ is a continuous mapping of $[0,T]$ into $(-\infty,\infty)$ and $\eta(t)$ a mapping of $[0,T]$ into H such that also $E\,\xi(s)\,\eta(t)$ and $E\,\eta(s)\,\eta(t)$ are of bounded variation on $[0,T]^2$, then

$$E\left\{\int_0^s f(s')d\,\xi(s')\int_0^t g(t')d\,\eta(t')\right\} = \int_0^s\int_0^t f(s')g(t')ddE\,\xi(s')\,\xi(t').$$

The condition in (2.9.1)

" $E\,\xi(s)\,\xi(t)$ is of bounded variation on $[0,T]^2$ "

is less stringent than the condition in (2.8.2)

" $\xi(t)$ is of bounded variation on $[0,T]$ in the strong sense "

since

(2.9.3) $$V(E\,\xi(s)\,\xi(t),\ [0,T]^2) \leqq \left\{V(\ \xi(t),\ [0,T])\right\}^2$$

For, if p is a partition of $[0,T]$, defined by $0=t_0 < t_1 < . < t_N = T$, $p \times p$ is a partition of $[0,T]^2$. On account of the inequality of Schwarz (2.1.1),

$$V(E\,\xi(s)\,\xi(t), p\times p,\ [0,T]^2) = \sum_{i=1}^{N}\sum_{j=1}^{N}\left|E\left\{\xi(t_i)-\xi(t_{i-1})\right\}\left\{\xi(t_j)-\xi(t_{j-1})\right\}\right| \leqq$$

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\left\|\xi(t_i)-\xi(t_{i-1})\right\|\cdot\left\|\xi(t_j)-\xi(t_{j-1})\right\| =$$

$$\left\{\sum_{i=1}^{N}\left\|\xi(t_i)-\xi(t_{i-1})\right\|\right\}^2 = \left\{V(\ \xi(t), p, [0,T])\right\}^2 .$$

As we have seen that total variations may be evaluated by means of the partitions $p \times p$ alone, the statement follows as p passes through a sequence of partitions $\{p_n\}$ of $[0,T]$ such that $\Delta(p_n) \to 0$ as $n \to \infty$.

Application to the Wiener-Lévy process in chapter 4 will show that the reverse is not true. Also in chapter 4, the existence of Riemann-Stieltjes integrals in q.m. with respect to Wiener-Lévy processes may be established, owing to theorem (2.9.2) or (2.7.1) but not to (2.8.2).

(2.9.4) Let $\xi(t)$ be a mapping of $[0,T]$ into H such that $E\,\xi(s)\,\xi(t)$ is of bounded variation on $[0,T]^2$, and let $f(s)$ and $g(s)$ be continuous mappings of $[0,T]$ into $(-\infty,\infty)$.

i) Then, if

$$\eta(t) = \int_0^t g(s)d\,\xi(s),$$

$E\,\eta(s)\,\eta(t)$ is of bounded variation on $[0,T]^2$.

ii) $$\int_0^T f(t)d\left\{\int_0^t g(s)d\,\xi(s)\right\} = \int_0^T f(t)g(t)d\,\xi(t).$$

Proof: Concerning i , on account of (2.9.2),

$$E\eta(s)\eta(t) = \int_0^s \int_0^t g(s')g(t')ddE\xi(s')\xi(t') .$$

If $d = [s_1,s_2]\times[t_1,t_2] \subset [0,T]^2$ and $m = \max_{t\in[0,T]} |g(t)|$, then

$$\left|\Delta_d \Delta E\eta(s)\eta(t)\right| = \left|\int_{s_1}^{s_2}\int_{t_1}^{t_2} g(s)g(t)ddE\xi(s)\xi(t)\right| \leq m^2 V(E\xi(s)\xi(t),d)$$

according to (2.9.1). And so

$$V(E\eta(s)\eta(t), [0,T]^2) \leq m^2 V(E\xi(s)\xi(t), [0,T]^2) < \infty .$$

Concerning ii, both integrals exist, owing to (2.9.1). Let p be the partition in section 2.6. It gives rise to the following Riemann-Stieltjes sum, belonging to the integral in the left-hand side of ii :

$$\sigma'(p) = \sum_{k=1}^{K} f(t_k')\left\{\int_0^{t_k} g(s)d\xi(s) - \int_0^{t_{k-1}} g(s)d\xi(s)\right\} =$$

$$\sum_{k=1}^{K} f(t_k')\int_{t_{k-1}}^{t_k} g(s)d\xi(s) = \sum_{k=1}^{K} \int_{t_{k-1}}^{t_k} f(t_k')g(s)d\xi(s) =$$

$$\int_0^T h_p(t)g(t)d\xi(t), \quad \text{if } h_p(t)=f(t_k') \text{ as } t_{k-1}\leq t<t_k, \quad k=1,.,K.$$

The above calculations are valid by virtue of (2.6.2) and (2.9.1). It follows also from (2.9.1) that

$$\left\|\int_0^T f(t)g(t)d\xi(t) - \sigma'(p)\right\|^2 = \left\|\int_0^T \left\{f(t)g(t)-h_p(t)g(t)\right\}d\xi(t)\right\|^2 \leq m_p^2 V,$$

where

$$m_p = \sup_{t\in[0,T]} \left|f(t)-h_p(t)\right|.\left|g(t)\right| \quad \text{and} \quad V = V(E\xi(s)\xi(t),[0,T]^2).$$

Since $f(t)$ is uniformly continuous on the compact set $[0,T]$, $m_p \to 0$ if p passes through a sequence $\{p_n\}$ of partitions of $[0,T]$, such that $\Delta(p_n) \to 0$ as $n\to\infty$.

3 Ordinary linear systems, driven by random functions

3.1. The Banach space H^N.

If H is a Hilbert space of the type defined in section 2.1 and if N is a natural number, the ordered N-tuples or N-vectors, written as column vectors

$$\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_N \end{pmatrix}, \qquad \xi_i \in H, \quad i=1,.,N,$$

constitute a linear space under the natural rules of addition and scalar multiplication (here throughout with the real numbers). Obviously

$$\| \xi \|_N = \max_{i=1,.,N} \| \xi_i \|$$

is a norm on this space. With this norm it will be called the space H^N. Since H is complete, H^N is complete with respect to this norm. So it is a Banach space.

If

$$A = \begin{pmatrix} a_{11} & \dots & a_{1N} \\ \vdots & & \vdots \\ a_{N1} & \dots & a_{NN} \end{pmatrix}$$

is an $N \times N$-matrix with real valued entries a_{ij}, $i,j=1,.,N$, then

$$A\xi \in H^N,$$

where $A\xi$ stands for the usual matrix-vector multiplication.
If

$$|A| = \max_{i=1,.,N} \sum_{j=1}^{N} |a_{ij}|,$$

then

$$\| A\xi \|_N \leqq |A| \cdot \| \xi \|_N .$$

The following definitions are alowed, owing to the properties of the elements of H :

$$E^{\mathcal{B}}\xi = \begin{pmatrix} E^{\mathcal{B}}\xi_1 \\ \vdots \\ E^{\mathcal{B}}\xi_N \end{pmatrix} \quad \text{and in particular} \quad E\xi = \begin{pmatrix} E\xi_1 \\ \vdots \\ E\xi_N \end{pmatrix},$$

$\xi^{T} = (\xi_1 \dots \xi_N)$ is the transpose of ξ and the covariance matrix of ξ is

$$E\xi\xi^{T} = \begin{pmatrix} E\xi_1\xi_1 & \dots\dots & E\xi_1\xi_N \\ \vdots & & \vdots \\ E\xi_N\xi_1 & \dots\dots & E\xi_N\xi_N \end{pmatrix}$$

Let $\xi(t)$, $t \in [0,T]$ be a mapping of $[0,T]$ into H^N. Continuity of $\xi(t)$ in the strong sense, again called continuity in q.m., is defined in the natural way:

$\xi(t)$ is continuous in q.m. at $t \in [0,T]$ iff
$\|\xi(s) - \xi(t)\|_N \to 0$ as $s \to t$, $s \in [0,T]$.

It follows that $\xi(t)$ is continuous in q.m. as a mapping of $[0,T]$ into H^N if and only if each component of $\xi(t)$ is continuous in q.m. as a mapping of $[0,T]$ into H.

In analogous way, sample continuity and the diverse types of differential quotients and integrals are defined. In all relevant cases below, assertions about $\xi(t)$ are true if and only if the corresponding assertions are true with respect to each of the separate components of $\xi(t)$. Thus many notions and assertions in this chapter have their 1-dimensional counterpart in chapter 2. They will often be used without further comment and without introducing new names and symbols as the context will make clear what is meant.

So $\xi(t)$ is differentiable in q.m. on $[0,T]$ iff

$$\frac{d}{dt}\xi(t) = \begin{pmatrix} \frac{d}{dt}\xi_1(t) \\ \vdots \\ \frac{d}{dt}\xi_N(t) \end{pmatrix} \quad \text{exists on } [0,T].$$

If $\xi(t)$ is continuous in q.m. on $[0,T]$, its Riemann integral in q.m.

$$\int_0^t \xi(s)ds = \begin{pmatrix} \int_0^t \xi_1(s)ds \\ \vdots \\ \int_0^t \xi_N(s)ds \end{pmatrix}, \quad t \in [0,T],$$

exists, is continuous in q.m. and differentiable in q.m. on $[0,T]$, and satisfies $\frac{d}{dt}\int_0^t \xi(s)ds = \xi(t)$. Then also

$E^{\mathcal{B}}\int_0^t \xi(s)ds = \int_0^t E^{\mathcal{B}}\xi(s)ds$ and in particular $E\int_0^t \xi(s)ds = \int_0^t E\xi(s)ds$.

If $\xi(t)$ is differentiable in q.m., then also $E^{\mathcal{B}}\xi(t)$, and $E^{\mathcal{B}}\frac{d}{dt}\xi(t) = \frac{d}{dt}E^{\mathcal{B}}\xi(t)$, in particular $E\frac{d}{dt}\xi(t) = \frac{d}{dt}E\xi(t)$.

3.2. The homogeneous system.

In this section are used several classical fixed point theorems for Banach spaces, see [8] for instance.

(3.2.1) Let $A(t)$ be an $N \times N$-matrix, whose entries $a_{ij}(t)$, $i,j=1,..,N$, are continuous mappings of $[0,T]$ into $(-\infty,\infty)$. Let $\xi(t)$ be any continuous mapping of $[0,T]$ into H^N, satisfying in q.m. sense the system of differential equations

(3.2.1a) $$\frac{d}{dt}\xi(t) = A(t)\xi(t), \qquad t \in [0,T],$$

with initial condition

(3.2.1b) $$\xi(0) = 0,$$

or, equivalently, the system of intgral equations

(3.2.1c) $$\xi(t) = \int_0^t A(s)\xi(s)ds, \quad t \in [0,T].$$

Then $\xi(t) = 0$, $t \in [0,T]$.

In other words, the above system is uniquely solvable with solution identical to $0 \in H^N$.

Proof: Clearly $\xi(t) = 0$ is a solution. In order to show that it is the unique solution, let be assumed that also the arbitrary continuous mapping $\xi(t)$ of $[0,T]$ into H^N satisfies (3,2,1c). As we write

$$\mathcal{T}\xi(t) = \int_0^t A(s)\xi(s)ds , \qquad t \in [0,T],$$

then (3.2.1c) reads

$$\xi(t) = \mathcal{T}\xi(t) .$$

It follows that at $t \in [0,T]$,

$$\|\xi(t)\|_N = \|\mathcal{T}\xi(t)\|_N = \left\| \int_0^t A(s)\xi(s)ds \right\|_N \leqq \int_0^t |A(s)| \cdot \|\xi(s)\|_N ds \leqq Mmt,$$

where $M = \max_{s\in[0,T]} |A(s)|$ and $m = \max_{s\in[0,T]} \|\xi(s)\|_N$.

Then $\|\xi(t)\|_N = \|\mathcal{T}^2\xi(t)\|_N = \|\mathcal{T}(\mathcal{T}\xi(t))\|_N = \left\| \int_0^t A(s)\mathcal{T}\xi(s)ds \right\|_N \leqq$ $\int_0^t |A(s)| \cdot \|\mathcal{T}\xi(s)\|_N ds \leqq \int_0^t M\, Mms\, ds = m\frac{(Mt)^2}{2!}$,

and by induction,

$$\|\xi(t)\|_N = \|\mathcal{T}^n\xi(t)\|_N \leqq m\frac{(Mt)^n}{n!} \leqq m\frac{(MT)^n}{n!}$$

for all natural numbers n and for all $t \in [0,T]$. Necessarily $\xi(t)$ is identical to 0 on $[0,T]$.

<u>(3.2.2)</u> If $A(t)$ is the matrix in (3.2.1), and if $\gamma \in H^N$, then there is a unique continuous mapping $\xi(t)$ of $[0,T]$ into H^N, satisfying in q.m. sense the system of differential equations

<u>(3.2.2a)</u> $$\frac{d}{dt}\xi(t) = A(t)\xi(t) , \qquad t \in [0,T],$$

with initial condition

<u>(3.2.2b)</u> $$\xi(0) = \gamma,$$

or, equivalently, the system of integral equations

<u>(3.2.2c)</u> $$\xi(t) = \gamma + \int_0^t A(s)\xi(s)ds, \quad t \in [0,T].$$

Proof: If there is a continuous mapping $\xi(t)$ of $[0,T]$ into H^N, satisfying the above system, it is unique. For, if also the continuous mapping $\eta(t)$ of $[0,T]$ into H^N satisfies the above system, then $\xi(t) - \eta(t)$ satisfies the system in (3.2.1) and so $\xi(t) - \eta(t)$ is identical to 0 .

In order to show the existence of a solution to (3.2.2c), let $\xi(t)$ be a contiuous mapping of $[0,T]$ into H^N. We shall use several results of (3.2.1) and write

$$\mathcal{S}\xi(t) = \gamma + \mathcal{T}\xi(t), \quad t \in [0,T].$$

$\mathcal{S}\xi(t)$ is a continuous mapping of $[0,T]$ into H^N and

$$\mathcal{S}^2\xi(t) = \mathcal{S}(\mathcal{S}\xi(t)) = \mathcal{S}(\gamma + \mathcal{T}\xi(t)) = \gamma + \mathcal{T}(\gamma + \mathcal{T}\xi(t))$$

whereas

$$\mathcal{T}(\gamma + \mathcal{T}\xi(t)) = \int_0^t A(s)\left\{\gamma + \mathcal{T}\xi(s)\right\}ds = \mathcal{T}\gamma + \mathcal{T}^2\xi(t).$$

So

$$\mathcal{S}^2\xi(t) = \gamma + \mathcal{T}\gamma + \mathcal{T}^2\xi(t)$$

and by induction, since

(3.2.2d) $$\mathcal{S}^n\xi(t) = \gamma + \mathcal{T}\mathcal{S}^{n-1}\xi(t),$$

$$\mathcal{S}^n\xi(t) = \gamma + \mathcal{T}\gamma + \ldots + \mathcal{T}^{n-1}\gamma + \mathcal{T}^n\xi(t), \quad n=1,2,\ldots$$

It is seen that $\mathcal{S}^n\xi(t)$ is a continuous mapping of $[0,T]$ into H^N. If $c = \|\gamma\|_N$ and if k is a fixed natural number, then on account of the results in (3.2.1),

$$\|\mathcal{S}^{n+k}\xi(t) - \mathcal{S}^n\xi(t)\|_N = \|\mathcal{T}^n\gamma + \ldots + \mathcal{T}^{n+k-1}\gamma + \mathcal{T}^{n+k}\xi(t) - \mathcal{T}^n\xi(t)\|_N \leqq$$

$$\|\mathcal{T}^n\gamma\|_N + \ldots + \|\mathcal{T}^{n+k-1}\gamma\|_N + \|\mathcal{T}^{n+k}\xi(t)\|_N + \|\mathcal{T}^n\xi(t)\|_N \leqq$$

$$c\,\frac{(MT)^n}{n!} + \ldots + c\,\frac{(MT)^{n+k-1}}{(n+k-1)!} + m\,\frac{(MT)^{n+k}}{(n+k)!} + m\,\frac{(MT)^n}{n!} \longrightarrow 0 \text{ as } n \to \infty,$$

uniformly in $t \in [0,T]$. And so, since H^N is a complete space, there is a mapping $\lambda(t)$ of $[0,T]$ into H^N such that

$$\mathcal{S}^n\xi(t) \to \lambda(t) \text{ as } n \to \infty, \text{ uniformly in } t \in [0,T].$$

By virtue of (2.4.5), $\lambda(t)$ is continuous on $[0,T]$. Finally, according to (3.2.2d) and (2.7.5),

$$\lambda(t) = \gamma + \int_0^t A(s)\,\lambda(s)ds,$$

showing the existence of a solution to the given system.

If the N components of γ in (3.2.2b) are degenerate random variables, a deterministic homogeneous linear system is obtained as a special case of (3.2.2). We recall the following result of the deterministic theory:

(3.2.3) The class of deterministic N-vector valued functions, satisfying (3.2.2a), is a linear vector space of dimension N. The fundamental matrix associated with $A(t)$ is the unique $N \times N$-matrix

$$F(t)\ , \qquad t \in [0,T],$$

whose entries are deterministic (in this case real valued) differentiable functions on $[0,T]$, such that its column vectors are a base of the above linear space, and such that

$$F(0) = I_N\ ,$$

I_N being the $N \times N$ identity matrix. Obviously

$$\frac{d}{dt} F(t) = A(t)F(t)\ , \qquad t \in [0,T],$$

and

$$F(t) = I_N + \int_0^t A(s)F(s)ds\ , \qquad t \in [0,T].$$

$F(t)$ is non-singular on $[0,T]$, $F^{-1}(t)F(t) = I_N$.
Also $F^{-1}(t)$ is continuously differentiable on $[0,T]$ and satisfies

$$0 = \frac{d}{dt} I_N = \frac{d}{dt}\left\{F^{-1}(t)F(t)\right\} = \frac{d}{dt}\left\{F^{-1}(t)\right\}F(t) + F^{-1}(t)\frac{d}{dt}F(t)$$

or

$$\frac{d}{dt} F^{-1}(t) = -F^{-1}(t)A(t)F(t)F^{-1}(t) = -F^{-1}(t)A(t)\ .$$

Now the unique solution in q.m. $\xi(t)$ of the system in (3.2.2) may be represented as

$$\xi(t) = F(t)\gamma\ ,$$

since by virtue of (2.5.5) and the properties of $F(t)$,

$$\frac{d\,\xi(t)}{dt} = \frac{d}{dt}\left\{F(t)\gamma\right\} = A(t)F(t)\gamma = A(t)\,\xi(t)\ ,$$

where the differential quotients are derivatives in q.m., and since

$$\xi(0) = F(0)\gamma = \gamma\ .$$

If we consider $\gamma = \gamma(\omega)$ at fixed ω, it follows from the above deterministic theory that the system in (3.2.2) has a unique sample solution if the derivative in (3.2.2a) or the integral in (3.2.2c) are assumed to be a sample derivative or sample integral. Also this sample solution may be represented as

$$\xi(\omega,t) = F(t)\gamma(\omega) .$$

So we have obtained the following theorem:

(3.2.4) The system in (3.2.2) has a unique solution in q.m. as well as in sample sense. If $F(t)$ is the fundamental matrix associated with $A(t)$, see (3.2.3), the solutions of both types may be represented as

$$\xi(t) = F(t)\gamma , \qquad t \in [0,T].$$

They coincide a.s. at each $t \in [0,T]$.

3.3. Inhomogeneous systems.

Let $A(t)$ be an $N\times N$-matrix whose entries are continuous mappings of $[0,T]$ into $(-\infty,\infty)$, let $\gamma \in H^N$ and let $\alpha(t)$ be a continuous mapping of $[0,T]$ into H^N.

Let be given the formal systems

(3.3a) $$\frac{d}{dt}\xi(t) = A(t)\xi(t) + \alpha(t) , \qquad t \in [0,T],$$

(3.3b) $$\xi(0) = \gamma$$

and

(3.3c) $$\xi(t) = \gamma + \int_0^t \left\{A(s)\xi(s) + \alpha(s)\right\} ds , \ t \in [0,T].$$

(3.3.1) If $\xi(t)$ is a continuous mapping of $[0,T]$ into H^N, the system (3.3a),(3.3b) is equivalent to (3.3c) in calculus in q.m. And

(3.3.1a) $$\xi(t) = F(t)\gamma + F(t)\int_0^t F^{-1}(s)\alpha(s)ds$$

is the unique solution in q.m. to (3.3a,b,c). Here $F(t)$ is the fundamental matrix, associated with $A(t)$, see (3.2.3), and the integral is a Riemann integral in q.m.

Proof: If also $\eta(t)$ satisfies (3.3a,b) in q.m. sense, then $\xi(t) - \eta(t)$ is a solution in q.m. to the system in (3.2.1) and so $\xi(t) - \eta(t) = 0$, $t \in [0,T]$, showing the uniqueness.

The solution in q.m. to (3.3a,b) will be constructed by means of the method of Lagrange. Let be assumed that $\varphi(t)$ is a mapping of $[0,T]$ into H^N, such that

$$\zeta(t) = F(t)\varphi(t)$$

satisfies (3.3a). Substitution into (3.3a) yields owing to (2.5.5),

$$\frac{dF(t)}{dt}\varphi(t) + F(t)\frac{d\varphi(t)}{dt} = A(t)F(t)\varphi(t) + \alpha(t).$$

Since $\frac{dF(t)}{dt} = A(t)F(t)$, see (3.2.3), it follows that

$$F(t)\frac{d\varphi(t)}{dt} = \alpha(t).$$

So we may put

$$\varphi(t) = \int_0^t F^{-1}(s)\alpha(s)ds .$$

Then $\varphi(0) = 0$, and $\xi(t) = F(t)\gamma + \zeta(t)$ satisfies (3.3a,b). All above calculations in q.m. are valid on account of the assertions in chapter 2 and by virtue of (3.2.4).

<u>(3.3.2)</u> If $\alpha(t)$ is sample continuous, (3.3a,b) is equivalent to (3.3c) for sample continuous processes $\xi(t)$ on $[0,T]$. Then (3.3a,b) has a unique solution in the sense of sample calculus, which may also be represented as (3.3.1a), provided that the integral is interpreted as a Riemann sample integral.

If $\alpha(t)$ is continuous in q.m. as well as in sample sense, both types of solution exist uniquely and coincide.

<u>(3.3.3)</u> If the system $\{\gamma;\ \alpha(t), t \in [0,T]\}$ is Gaussian, the solution (3.3.1a) to (3.3a,b) is Gaussian on $[0,T]$ by virtue of (2.3.1) and vi in (2.6.2).

(3.3.4) If $\xi(t)$ is the solution in q.m. to (3.3a,b) or (3.3c), then by virtue of (2.5.3) and v in (2.6.2),

$$\varphi(t) = E^{\mathcal{B}}\xi(t) = F(t)E^{\mathcal{B}}\gamma + F(t)\int_0^t F^{-1}(t')E^{\mathcal{B}}\alpha(t')dt'$$

is the unique solution in q.m. to

$$\frac{d}{dt}\varphi(t) = A(t)\varphi(t) + E^{\mathcal{B}}\alpha(t), \qquad t \in [0,T],$$
$$\varphi(0) = E^{\mathcal{B}}\gamma,$$

or to

$$\varphi(t) = E^{\mathcal{B}}\gamma + \int_0^t \left\{A(s)\varphi(s) + E^{\mathcal{B}}\alpha(s)\right\}ds, \qquad t \in [0,T].$$

In particular, as we set

$$E\xi(t) = x(t), \qquad \xi(t) = x(t) + \xi'(t),$$
$$E\alpha(t) = a(t), \qquad \alpha(t) = a(t) + \alpha'(t),$$
$$E\gamma = c, \qquad \gamma = c + \gamma',$$

$x(t)$ is the unique solution of the deterministic system

$$\frac{d}{dt}x(t) = A(t)x(t) + a(t), \qquad t \in [0,T],$$
$$x(0) = c,$$

and $\xi'(t)$ is the unique solution in q.m. of

$$\frac{d}{dt}\xi'(t) = A(t)\xi'(t) + \alpha'(t), \qquad t \in [0,T],$$
$$\xi'(0) = \gamma'.$$

The above two systems together are equivalent to (3,3a,b), interpreted in q.m. sense. On account of (2.4.2), a(t) and $\alpha'(t)$ are continuous (in q.m.)

(3.3.5) If γ and $\alpha(t)$, $t \in [0,T]$, are stochastically independent, and if $E\gamma = 0$ and (or) $E\alpha(t) = 0$, $t \in [0,T]$, the covariance function matrix

$$E\xi(s)\xi^T(t), \qquad (s,t) \in [0,T]^2,$$

of

$$\xi(t) = F(t)\gamma + F(t)\int_0^t F^{-1}(v)\alpha(v)dv$$

is equal to

(3.3.5a) $$F(s)\left[E\gamma\gamma^T + \int_0^s\int_0^t F^{-1}(u)\left\{E\alpha(u)\alpha^T(v)\right\}(F^{-1}(v))^T du dv\right]F^T(t).$$

This is seen as follows:

$$E\,\xi_i(s)\,\xi_j(t) = E\left[\sum_{h=1}^{N} f_{ih}(s)\left\{\gamma_h + \sum_{k=1}^{N}\int_0^s f_{hk}^{-1}(u)\alpha_k(u)du\right\}\right] \cdot$$

$$\cdot\left[\sum_{h'=1}^{N} f_{jh'}(t)\left\{\gamma_{h'} + \sum_{k'=1}^{N}\int_0^t f_{h'k'}^{-1}(v)\alpha_{k'}(v)dv\right\}\right],$$

where the $f_{hk}^{-1}(u)$ are the elements of $F^{-1}(u)$.

According to (2.7.3), (2.4.4) and (3.2.3),

$$E\int_0^s f_{hk}^{-1}(u)\alpha_k(u)du \,.\, \int_0^t f_{h'k'}^{-1}(v)\alpha_{k'}(v)dv =$$

$$\int_0^s\int_0^t f_{hk}^{-1}(u)\left\{E\,\alpha_k(u)\,\alpha_{k'}(v)\right\} f_{h'k'}^{-1}(v)dudv \,.$$

Since $\gamma_i \perp \alpha_j(t)$, $i,j=1,.,N$, $t \in [0,T]$,

$$E\,\gamma_h\sigma = 0, \qquad \text{where} \quad \sigma = \int_0^t f_{h'k'}^{-1}(v)\alpha_k(v)dv \,.$$

For, if $\{\sigma_n,\ n=1,2,..\}$ is a sequence of Riemann sums, converging in q.m. to σ as $n \to \infty$, then by virtue of (2.1.3),

$$0 = E\,\gamma_h\sigma_n = \lim_{n\to\infty} E\,\gamma_h\sigma_n = E\,\gamma_k\sigma \,.$$

And hence

$$E\,\xi_i(s)\,\xi_j(t) = \sum_{h,h'=1}^{N} f_{ih}(s)\,E\,\gamma_h\gamma_{h'}\,f_{jh'}(t) +$$

$$\sum_{h,k,k',h'=1}^{N} f_{ih}(s).\int_0^s\int_0^t f_{hk}^{-1}(u)\left\{E\,\alpha_k(u)\,\alpha_{k'}(v)\right\} f_{h'k'}^{-1}(v)dudv.f_{jh'}(t)\,,$$

showing (3.3.5a) .

4 Wiener-Lévy processes and some of their smooth perturbations

4.1. The 1-dimensional Wiener-Lévy process and calculus.

A detailed account of the 1-dimensional Wiener-Lévy process may be found in [15]. See also [6] and [14].

Let $\beta(t)$ be a stochastic function of $t \in [0,T]$. It will be assumed that s and t, with or without subscripts belong to $[0,T]$ and that $s_i < s_j$, $t_i < t_j$ as $i < j$.

We recall, if

i) $\beta(t)$ is real valued on $[0,T]$,

ii) $\beta(0) = 0$ a.s.,

iii) $\beta(t)$ is sample continuous on $[0,T]$,

and if

iv) the increments of $\beta(t)$ are stochastically independent, i.e.

$$\{\beta(t_2) - \beta(t_1)\} \quad \text{and} \quad \{\beta(t_4) - \beta(t_3)\}$$

are stochastically independent if

$$[t_1, t_2) \cap [t_3, t_4) = \emptyset ,$$

then $\beta(t)$ is necessarily Gaussian and so of second order, and continuous in q.m. on $[0,T]$. Then also $E\beta(t)$ exists and is continuous on $[0,T]$, see (2.4.2).

Moreover, if

v) $E\beta(t) = 0$, $t \in [0,T]$,

then the increments of $\beta(t)$ are orthogonal. For, if $[t_1, t_2) \cap [t_3 . t_4) = \emptyset$,

$$E\{\beta(t_2) - \beta(t_1)\}\{\beta(t_4) - \beta(t_3)\} = E\{\beta(t_2) - \beta(t_1)\}E\{\beta(t_4) - \beta(t_3)\} = 0.$$

Finally, if

vi) the increments of $\beta(t)$ are stationary in the sense that

$$E\{\beta(s) - \beta(t)\}^2 \text{ depends only on } |s - t| ,$$

say $E\{\beta(s) - \beta(t)\}^2 = f(|s-t|)$, then

$$f(t_3 - t_1) = E\{\beta(t_3) - \beta(t_1)\}^2 = E[\{\beta(t_3) - \beta(t_2)\} + \{\beta(t_2) - \beta(t_1)\}]^2 =$$

$$E\{\beta(t_3) - \beta(t_2)\}^2 + E\{\beta(t_2) - \beta(t_1)\}^2 = f(t_3 - t_2) + f(t_2 - t_1) ,$$

and it can be shown that

$$E\left\{\beta(s)-\beta(t)\right\}^2=\sigma^2|s-t|,\ \sigma^2 \text{ being any positive number.}$$

Now the choice of σ^2 is the only freedom remained. Unless stated otherwise, it will be assumed that

vii) $\sigma^2 = 1$.

(4.1.1) Definition: Endowed with the properties i-vii , $\beta(t)$ is the 1-dimensional Wiener-Lévy process on $[0,T]$. Unless stated otherwise, the symbol $\beta(t)$, without further comment, will stand for the process with the above properties.

We shall recall and discuss some more properties of $\beta(t)$.

With probability 1, the trajectories of $\beta(t)$ are not of bounded variation and not differentiable on any sub-interval of $[0,T]$. However, they are continuous because of assumption iii.

$\beta(t)$ is not of bounded variation on $[0,T]$ in the strong sense, cf. (2.8.1). For, if p_n is the partition of $[0,T]$ defined by the subdivision points

$$0,\ \frac{1}{n}T,\ \frac{2}{n}T,\ \ldots,\ \frac{n-1}{n}T,\ T\ ,$$

then

$$V(\beta(t),p_n,[0,T]) = \sum_{k=1}^{n}\left\|\beta(\tfrac{k}{n}T)-\beta(\tfrac{k-1}{n}T)\right\| = \sum_{k=1}^{n}\sqrt{\frac{T}{n}} = n\sqrt{\frac{T}{n}} \to \infty \quad \text{as} \quad n\to\infty .$$

It follows from the differentiability in q.m. criterion (2.5.6) that $\beta(t)$ is nowhere differentiable in q.m.

Since $\beta(t)$ is continuous in q.m. on $[0,T]$, $E\beta(s)\beta(t)$ is continuous on $[0,T]^2$ by virtue of (2.4.6).

If $0 \leq s \leq t \leq T$,

$$E\beta(s)\beta(t)=E\left\{\beta(s)-\beta(0)\right\}\left[\left\{\beta(t)-\beta(s)\right\}+\left\{\beta(s)-\beta(0)\right\}\right]=E\left\{\beta(s)-\beta(0)\right\}^2=s.$$

Let us use the notions and notations in section 2.9. Then, if

$$d=[s_1,s_2]\times[t_1,t_2]\subset[0,T]^2\ ,$$

it is analogously shown that

$$\Delta_d\Delta\, E\beta(s)\beta(t) = \begin{cases} M-m & \text{if } [s_1,s_2)\cap[t_1.t_2)=[m,M)\ , \\ 0 & \text{if } [s_1,s_2)\cap[t_1,t_2)=\emptyset\ . \end{cases}$$

In this way, $E\beta(s)\beta(t)$ induces a non-negative measure on the rectangles d of $[0,T]^2$. If the intersection of d with the diagonal $s=t$ of $[0,T]^2$ contains at most one point, the above measure of d is equal to 0. It follows that for each partition $p([0,T]^2)$ of $[0,T]^2$,

$V(E\beta(s)\beta(t),\ p([0,T]^2),\ [0,T]^2) = T$ and so $V(E\beta(s)\beta(t), [0,T]^2) = T$, i.e:

(4.1.2) $E\beta(s)\beta(t)$ is of bounded variation on $[0,T]^2$ with total variation T.

(4.1.3) If $f(t)$ is a continuous mapping of $[0,T]$ into $(-\infty,\infty)$, then

i)
$$\int_0^t f(s)d\beta(s)\ , \quad t \in [0,T]$$

exists as a Riemann-Stieltjes integral in q.m. on account of (4.1.2) and (2.9.1),

ii) the assertions of (2.6.2) are valid,

iii) and if also $g(t)$ is a continuous mapping of $[0,T]$ into $(-\infty,\infty)$,

(4.1.3a) $$E\int_0^s f(s')d\beta(s')\int_0^t g(t')d\beta(t') = \int_0^m f(s')g(s')ds'\ , \quad m=\min(s,t)$$

and in particular

(4.1.3b) $$\left\|\int_0^t f(s)d\beta(s)\right\|^2 = \int_0^t f^2(s)ds\ .$$

Proof: We only have to show (4.1.3a). According to (2.9.2),

$$I = E\int_0^s f(s')d\beta(s')\int_0^t g(t')d\beta(t') = \int_0^s\int_0^t f(s')g(t')ddE\beta(s')\beta(t'),$$

where the latter integral exists as an ordinary Riemann-Stieltjes integral since $f(s')g(t')$ is continuous and $E\beta(s')\beta(t')$ of bounded variation on $D = [0,s]\times[0,t]$. Hence it is the limit of a sequence of Riemann-Stieltjes sums, constructed on some sequence of partitions of D of which the mesh tends to 0. Assume $s \leqq t$. Then $m = s$. We may use partitions $p(D) = p\times q$, where the partitions p and q of $[0,s]$ and $[0,t]$ are defined by subdivision points s_k, t_m, satisfying

$0 = s_0 < s_1 < \ldots < s_K = s$ and $0 = s_0 < s_1 < \ldots < s_K = s \leqq t_1 \leqq \ldots \leqq t_M = t$

respectively. Owing to the properties of $\beta(t)$ discussed above,

$E\beta(s)\beta(t)$ induces a measure unequal to 0 only on the squares $d_k = [s_{k-1}, s_k]^2$ of $p(D)$. Therefore, since

$$\underset{d_k}{\Delta\Delta} E\beta(s)\beta(t) = s_k - s_{k-1} ,$$

a Riemann-Stieltjes sum belonging to $p(D)$ is

$$S(p(D)) = \sum_{k=1}^{K} f(s_k')g(s_k'')\{s_k - s_{k-1}\}$$

where (s_k' , s_k'') is an arbitrary point in d_k , $k=1,..,K$.

Also the integral $J = \int_0^{m=s} f(s')g(s')ds'$ exists and may be evaluated as the limit of a sequence of Riemann sums, constructed on any sequence of partitions of $[0,s]$, such that the mesh tends to 0 . Choosing the above p , we obtain the Riemann sum

$$S(p) = \sum_{k=1}^{K} f(s_k''')g(s_k''')\{s_k - s_{k-1}\}$$

where s_k''' is any point in $[s_{k-1}, s_k]$, $k=1,..,K$. Now in

$$\|I - J\| \leqq \|I - S(p(D))\| + \|S(p(D)) - S(p)\| + \|S(p) - J\| ,$$

$$S(p(D)) - S(p) = \sum_{k=1}^{K} \{f(s_k')g(s_k'') - f(s_k''')g(s_k''')\}\{s_k - s_{k-1}\}.$$

If $p(D)$ passes through a sequence $\{p_n(D) , n=1,2,.\}$ such that $\Delta(p_n(D)) \to 0$ as $n \to \infty$, the first and the last term in the above inequality tend to 0 as $n \to \infty$ on account of the definition and the existence of the integrals I and J . The middle term tends to 0 by virtue of the uniform continuity of $f(s')g(t')$ on $[0,s]\times[0,t]$. And hence I = J .

4.2. A class of smooth perturbations of the 1-dimensional Wiener-Lévy process.

The approximation of $\beta(t)$ by smooth functions, treated in this section, corresponds to the operation of smoothing, used in the theory of generalized functions, see [29] . Cf. also [33] . And so there will be need for the testing functions of distribution theory.

(4.2.1) The testing functions $p(t)$ used in this section are assumed to be endowed with the following properties:

i) $p(t)$ is a non-negative, symmetric mapping of $(-\infty,\infty)$ into itself, with compact support $[-1,+1]$. I.e: $p(-t) = p(t)$, $p(t) \geqq 0$ if $t \in [-1,+1]$ and $p(t) = 0$ outside $[-1,+1]$.

ii) $p(t)$ is smooth, i.e. infinitely often differentiable on $(-\infty,\infty)$.

iii) $\int_{-\infty}^{+\infty} p(t)dt = \int_{-1}^{+1} p(t)dt = 1$.

For instance : $p(t) = \begin{cases} \left\{ e^{1/(t^2-1)} \right\} / \int_{-1}^{+1} e^{1/(s^2-1)}\,ds & \text{if } |t| < 1, \\ 0 & \text{if } |t| \geqq 1. \end{cases}$

(4.2.2) As we define $p_n(t) = n\,p(nt)$, $n=1,2,..$, $t \in (-\infty,\infty)$, then $p_1(t) = p(t)$, and $p_n(t)$ enjoys the above properties i, ii and iii with the exception that its support is $\left[-\frac{1}{n}, +\frac{1}{n}\right]$. It shrinks to $\{0\}$ as $n \to \infty$.

(4.2.3) As we define $q_n(t) = \int_{-\infty}^{t} p_n(s)ds$,

$q_n(t) \downarrow 0$ as $t \to -\infty$ and $q_n(t) \uparrow 1$ as $t \to \infty$,
$q_n(t) = 0$ if $t \leqq -\frac{1}{n}$ and $q_n(t) = 1$ if $t \geqq \frac{1}{n}$.
And since $q_n(t) = \int_{-\infty}^{t} n\,p(ns)ds = \int_{-\infty}^{nt} p(x)dx$,

$$q_n(t) \begin{cases} \downarrow 0 & \text{if } t < 0 \\ = \frac{1}{2} & \text{if } t = 0 \\ \uparrow 1 & \text{if } t > 0 \end{cases} \qquad \text{as } n \to \infty .$$

The limit function of the sequence $\{q_n(t),\ n=1,2,..\}$ is the unit step function of Heaviside.

We shall need the following easily verified result of real analysis. It should be noted that the function $f(t)$ below may be non-differentiable and not of bounded variation on $[0,T]$.

(4.2.4) If $f(t)$ is a continuous mapping of $[0,T]$ into $(-\infty,\infty)$ such that $f(0) = 0$, then as $t \in [0,T]$,

i) $f_n(t) = \int_0^T q_n(t-s)df(s) = q_n(t-T)f(T) + \int_0^T f(s)p_n(t-s)ds$ exists,

ii) $f_n(t)$ is smooth on $[0,T]$, $f_n^{(k)}(t) = \int_0^T p_n^{(k-1)}(t-s)df(s)$,

iii) and as $n \to \infty$, the sequence $\{f_n(t),\ n=1,2,..\}$ converges to $f(t)$, uniformly in $t \in [0,T]$.

We shall need the following lemma:

(4.2.5) Let $\{\xi_n(t),\ t \in [0,T],\ n=1,2,..\}$ be a sequence of real valued sample continuous stochastic processes with sample continuous representations

$$\xi_n(\omega,t)\ ,\quad (\omega,t)\in \Omega\times[0,T],\quad n=1,2,..\ ,$$

where Ω is the point set of a suitable probability space $\{\Omega, \mathcal{A}, P\}$.

If at a.a. $\omega \in \Omega$

$$\xi_n(\omega,t) \to \xi(\omega,t) \text{ as } n\to\infty \text{, uniformly in } t\in[0,T],$$

then $\xi_n \to \xi$ a.s. in the sense that for any $\varepsilon > 0$,

(4.2.5a) $$P\Big[\bigcup_{n\geqq n'}\big[\omega:\ \sup_{t\in[0,T]}\big|\xi_n(\omega,t)-\xi(\omega,t)\big|\geqq\varepsilon\big]\Big]\downarrow 0 \text{ as } n'\to\infty .$$

Proof: Because of the uniform convergence of a.a. sequences of trajectories, also the limit process $\xi(\omega,t)$ is sample continuous on $[0,T]$ and so is the process

$$\big|\xi_n(\omega,t) - \xi(\omega,t)\big|\ ,\quad (\omega,t)\in\Omega\times[0,T] .$$

It is separable in the sense of Doob, cf. section 2.3. Hence

(4.2.5b) $$\alpha_n(\omega) = \sup_{t\in[0,T]}\big|\xi_n(\omega,t) - \xi(\omega,t)\big|$$

is an $\mathcal{A}$-measurable function of $\omega\in\Omega$. As we set

(4.2.5c) $$A_{n'} = \bigcup_{n\geqq n'}\big[\omega:\alpha_n(\omega)\geqq\varepsilon\big],$$

then $A_{n'}\in\mathcal{A}$, and $\{A_{n'},\ n'=1,2,..\}$ is a shrinking sequence of $\mathcal{A}$-measurable sets. So it converges to an $\mathcal{A}$-measurable set, say $A_{n'}\downarrow A\in\mathcal{A}$ as $n'\to\infty$ and hence

(4.2.5d) $$P(A_{n'})\downarrow P(A) \text{ as } n'\to\infty .$$

If $\omega_A\in A$, then $\omega_A\in A_{n'}$ for all n'. This means, according to (4.2.5c), that at each n', there is an $n''\geqq n'$ such that

$$\alpha_{n''}(\omega_A)\geqq\varepsilon .$$

Therefore, according to (4.2.5b), $\xi_n(\omega_A,t)$ does not converge uniformly in $t\in[0,T]$ as $n\to\infty$, and hence

$$P(A) = 0 ,$$

showing (4.2.5a) on account of (4.2.5d) and (4.2.5c).

Corollary: We have shown $\alpha_n(\omega) \to 0$ a.s. as $n \to \infty$. Hence, owing to a theorem of Egorov, see [7], given $\varepsilon > 0$, there is a set $A_\varepsilon \in \mathcal{A}$ with $P(A_\varepsilon) < \varepsilon$, such that $\alpha_n(\omega)$ converges uniformly in $\Omega \setminus A_\varepsilon$, as $n \to \infty$. This means

$$\xi_n(\omega,t) \to \xi(\omega,t) \quad \text{as } n \to \infty\text{, uniformly in}$$
$$(\omega,t) \in (\Omega \setminus A_\varepsilon) \times [0,T], \quad P(A_\varepsilon) < \varepsilon \quad \text{for any } \varepsilon > 0.$$

Remark: The convergence

$$\xi_n \to \xi \quad \text{a.s. as } n \to \infty$$

in the sense of (4.2.5a) implies

$$\xi_n \to \xi \quad \text{in } P \text{ as } n \to \infty$$

in the sense that for any $\varepsilon > 0$,

$$P\left[\omega : \sup_{t \in [0,T]} \left| \xi_n(\omega,t) - \xi(\omega,t) \right| \geq \varepsilon \right] \to 0 \quad \text{as } n \to \infty .$$

In turn, this type of convergence implies

$$\xi_n \to \xi \quad \text{in distribution as } n \to \infty$$

in the sense that the probability measures, induced by the processes ξ_n onto the σ-field S of $C[0,T]$, see section 2.3, converge weakly to the probability measure, induced onto S by ξ, cf. [2].

Let $\beta(t)$ be the 1-dimensional Wiener-Lévy process on $[0,T]$, and $q_n(t)$ the function in (4.2.3). Then

$$\int_0^T q_n(t-s)d\beta(s)$$

exists as a Riemann-Stieltjes integral in q.m. by virtue of (4.1.3). According to i in (2.6.2) and to (2.7.4),

$$\int_0^T q_n(t-s)d\beta(s) = \left[q_n(t-s)\beta(s)\right]_0^T - \int_0^T \beta(s)dq_n(t-s) =$$
$$q_n(t-T)\beta(T) + \int_0^T \beta(s)p_n(t-s)ds .$$

It is seen that the rigt-hand side also exists as a Riemann sample integral, and again by partial integration, so does the left-hand side. On account of vii in (2.6.2) the sample integrals coincide with the integrals in q.m. We recall that with probability 1, the trajectories of $\beta(s)$ are neither differentiable, nor of bounded variation on $[0,T]$.

Now the following definition is admissible:

(4.2.6) $\beta_n(t) = \int_o^T q_n(t-s)d\beta(s), \quad t \in [0,T], \quad n=1,2,..$,

to be interpreted as a sample integral, as well as an integral in q.m., is a smooth perturbation of the 1-dimensional Wiener-Lévy process $\beta(t)$ in (4.1.1). It may also be represented as

(4.2.6a) $\beta_n(t) = q_n(t-T)\beta(T) + \int_o^T \beta(s)p_n(t-s)ds$.

We shall derive a number of properties of $\beta_n(t)$.

i) With probability 1 the trajectories of $\beta_n(t)$ are smooth functions of t on $[0,T]$, on account of (4.2.4). The sample derivatives may be represented as

(4.2.6b) $\beta_n^{(k)}(t) = \int_o^T p_n^{(k-1)}(t-s)d\beta(s), \quad k=1,2,..$.

ii) Now we shall consider $\beta_n(t)$ as a mapping of $[0,T]$ into H . Since $E\beta(t) = 0$ on $[0,T]$, it follows from v in (2.6.2) that $E\beta_n(t)$ exists and is identical to 0 on $[0,T]$.

$\beta_n(t)$ is smooth in q.m. Its derivatives in q.m. may also be represented as (4.2.6b) and coincide with the sample derivatives.

Let us first show

$$\frac{d}{dt}\beta_n(t) = \int_o^T p_n(t-s)d\beta(s) \quad \text{in q.m.}$$

If t and $t+h$ are fixed values in $[0,T]$, $\{q_n(t+h-s)-q_n(t-s)\}/h$ is a continuous function of s. On account of the mean value theorem of real analysis, it may be written as $p_n(t-s+\theta(s))$, where $\theta(s)$ is a value between 0 and h . Now, given $\varepsilon > 0$ and $|h|$ sufficiently small, it follows on account of (4.1.3b) and by virtue of the uniform continuity of $p_n(s)$ on the compact set $[0,T]$, that

$$\left\| \frac{1}{h}\left\{ \int_o^T q_n(t+h-s)d\beta(s) - \int_o^T q(t-s)d\beta(s)\right\} - \int_o^T p_n(t-s)d\beta(s) \right\|^2 =$$

$$\left\| \int_o^T \left[\frac{1}{h}\left\{ q_n(t+h-s) - q_n(t-s)\right\} - p_n(t-s)\right] d\beta(s) \right\|^2 =$$

$$\left\| \int_o^T \{p_n(t+\theta(s)-s)-p_n(t-s)\} d\beta(s)\right\|^2 = \int_o^T \{p_n(t+\theta(s)-s)-p_n(t-s)\}^2 dt < \varepsilon^2 T.$$

The differentiability in q.m. of (4.2.6b) is shown analogously, and it follows by induction that $\beta_n(t)$ is smooth in q.m.

iii) $\beta_n(t)$ is Gaussian on $[0,T]$, by virtue of vi in (2.6.2), since $\beta(t)$ is Gaussian on $[0,T]$.

iv) If the intervals $[t_1,t_2]$ and $[t_3,t_4]$ are disjunct, then the increments

$$\{\beta_n(t_2)-\beta_n(t_1)\} \text{ and } \{\beta_n(t_4)-\beta_n(t_3)\}$$

of $\beta_n(t)$ are stochastically independent and orthogonal in H , see (2.3.1), if n is sufficiently large.

For, according to (4.1.3a),

$$E\{\beta_n(t_2)-\beta_n(t_1)\}\{\beta_n(t_4)-\beta_n(t_3)\} =$$
$$E\int_o^T\{q_n(t_2-s) - q_n(t_1-s)\}d\beta(s)\int_o^T\{q_n(t_4-s) - q_n(t_3-s)\}d\beta(s) =$$
$$\int_o^T\{q_n(t_2-s) - q_n(t_1-s)\}\{q_n(t_4-s) - q_n(t_3-s)\}ds .$$

Since the support of $p_n(t)$ tends to $\{0\}$, the support of $q_n(t_{k+1}-s) - q_n(t_k-s)$ tends to $[t_k,t_{k+1}]$ as $n\to\infty$. Hence, since $[t_1,t_2]$ and $[t_3,t_4]$ are disjunct, the function

$$\{q_n(t_2-s) - q_n(t_1-s)\}\{q_n(t_4-s) - q_n(t_3-s)\}$$

is identical to 0 if n is sufficiently large. Then the above integral is equal to 0 , showing the orthogonality of the increments.

v) As $n\to\infty$, $\beta_n\to\beta$ a.s. in the sense of (4.2.5a). For, if $\{\Omega,\mathcal{A},P\}$ is a suitable probability space and

$$\{\beta_n(\omega,t),\quad (\omega,t)\in\Omega\times[0,T],\quad n=1,2,..\},$$

a sequence of sample continuous representations of the processes $\beta_n(t)$, then at a.a. ω the assertion iii of (4.2.4) is applicable, since

$$\beta_n(\omega,t) = \int_o^T q_n(t-s)d\beta(\omega,s) .$$

As also the remark and the corollary to (4.2.5) are applicable: $\beta_n(\omega,t)\to\beta(\omega,t)$ as $n\to\infty$, uniformly in $(\omega,t)\in(\Omega\setminus A_\varepsilon)\times[0,T]$, $A_\varepsilon\in\mathcal{A}$ and $P(A_\varepsilon)<\varepsilon$ for any $\varepsilon>0$, and

$\beta_n\to\beta$ in P and in distribution as $n\to\infty$.

vi) As $n \to \infty$, $\beta_n(t) \to \beta(t)$ in q.m., uniformly in $t \in [0,T]$.
For, on account of (4.1.3a),

$$\|\beta_n(t) - \beta(t)\|^2 = \left\| \int_0^T q_n(t-s)d\beta(s) - \beta(t)\right\|^2 =$$

$$E\left\{\int_0^T q_n(t-u)d\beta(u) - \beta(t)\right\}\left\{\int_0^T q_n(t-v)d\beta(v) - \beta(t)\right\} =$$

$$E\int_0^T q_n(t-u)d\beta(u)\int_0^T q_n(t-v)d\beta(v) - 2E\int_0^t 1\, d\beta(u)\int_0^T q_n(t-v)d\beta(v) + E\beta^2(t) =$$

$$\int_0^T q_n(t-s)^2 ds - 2\int_0^t q_n(t-s)ds + t =$$

$$\int_0^T \left\{q_n(t-s) - h(t-s)\right\}^2 ds \leqq \int_{-T}^{+T} \left\{q_n(x) - h(x)\right\}^2 dx ,$$

where $h(x)$ is the unit step function of Heaviside. Because of the properties of $q_n(t)$, exposed in (4.2.3),

$$q_n(x) - h(x)$$

tends to 0 on $(-\infty, \infty)$ as $n \to \infty$. As it is also uniformly bounded in n,

$$\int_{-T}^{+T} \left\{q_n(x) - h(x)\right\}^2 \to 0 \quad \text{as} \quad n \to \infty \text{ , uniformly in } t\in[0,T],$$

by virtue of the dominated convergence theorem of Lebesgue.

Now $E\beta_n(s)\beta_n(t) \to E\beta(s)\beta(t) = \min(s,t)$ as $n \to \infty$, uniformly in $(s,t) \in [0,T]^2$ by virtue of the following lemma:

<u>(4.2.6c)</u> If $\{\xi_n(t),\ n=1,2,..\}$ and $\{\eta_n(t),\ n=1,2,..\}$ are sequences of mappings of $[0,T]$ into H , converging in q.m. as $n \to \infty$, uniformly in $t \in [0,T]$, to $\xi(t)$ and $\eta(t)$ respectively, then the sequence of crosscorrelation functions $\{E\,\xi_n(s)\eta_n(t),\ n=1,2,..\}$ tends to $E\,\xi(s)\eta(t)$ as $n \to \infty$, uniformly in $(s,t) \in [0,T]^2$.

For,

$$\left|E\,\xi_n(s)\eta_n(t) - E\,\xi(s)\eta(t)\right| \leqq \left|E\,\xi_n(s)\left\{\eta_n(t) - \eta(t)\right\}\right| + \left|E\left\{\xi_n(s) - \xi(s)\right\}\eta(t)\right| \leqq$$

$$\|\xi_n(s)\|\cdot\|\eta_n(t) - \eta(t)\| + \|\xi_n(s) - \xi(s)\|\cdot\|\eta(t)\| ,$$

on account of the inequality of Schwarz.

vii) Using (4.1.2) and the notation, introduced in section 2.9,
$V(E\beta_n(s)\beta_n(t), [0,T]^2) \uparrow T = V(E\beta(s)\beta(t), [0,T]^2)$ as $n \to \infty$.

For, according to (4.1.3a),

$$E\beta_n(s)\beta_n(t) = E\int_o^T q_n(s-u)d\beta(u)\int_o^T q_n(t-v)d\beta(v) =$$
$$\int_o^T q_n(s-x)q_n(t-x)dx \ .$$

Let $d = [s_1,s_2]\times[t_1,t_2]\subset[0,T]^2$. Then

$$\underset{d}{\Delta\Delta} E\beta_n(s)\beta_n(t) = \underset{d}{\Delta\Delta}\int_o^T q_n(s-x)q_n(t-x)dx =$$
$$\int_o^T\{q_n(s_2-x) - q_n(s_1-x)\}\{q_n(t_2-x) - q_n(t_1-x)\}dx \geqq 0 ,$$

the latter inequality being true since $q_n(t)$ is non-decreasing, see (4.2.3). Therefore

$$\left|\underset{d}{\Delta\Delta} E\beta_n(s)\beta_n(t)\right| = \underset{d}{\Delta\Delta} E\beta_n(s)\beta_n(t)$$

and so

$$V(E\beta_n(s)\beta_n(t), [0,T]^2) = \underset{[0,T]^2}{\Delta\Delta} E\beta_n(s)\beta_n(t) =$$
$$\underset{[0,T]^2}{\Delta\Delta}\int_o^T q_n(s-x)q_n(t-x)dx = \int_o^T\left\{q_n(T-x) - q_n(-x)\right\}^2dx \ .$$

It follows from the properties of $q_n(t)$ in (4.2.3) that

$$\left\{q_n(T-x) - q_n(-x)\right\} \uparrow 1 \quad \text{as } n \to \infty , \quad 0 < x < T .$$

Hence, by virtue of the monotone convergence theorem of Lebesgue,

$$\int_o^T\left\{q_n(T-x) - q_n(-x)\right\}^2dx \uparrow \int_o^T 1\, dx = T , \text{ as } n \to \infty .$$

As it may be convenient that also the perturbed Wiener-Lévy processes start with the value 0 at $t = 0$, we may introduce the smooth perturbations

<u>(4.2.7)</u> $\left\{\beta_n(t) - \beta_n(0)\right\}$, $n=1,2,..$, $t \in [0,T]$.

Since $\beta_n(0) \to 0$ in q.m. and a.s. as $n \to \infty$, and since $\beta_n(0)$ is independent of t , the processes (4.2.7) possess all above properties i - vii , and moreover they are a.s. identical to 0 at $t = 0$.

The above results are gathered in the following statement:

(4.2.8) To the 1-dimensional Wiener-Lévy process $\beta(t)$, $t \in [0,T]$, exists a sequence

$$\{ \beta_n(t),\ t \in [0,T],\ n=1,2,.. \}$$

of smooth perturbations with the properties

i) $\beta_n(t)$ has smooth trajectories with probability 1 ,

ii) $\beta_n(t)$ is a smooth function of second order, with $E\beta_n(t) = 0$ on $[0,T]$,

iii) $\beta_n(t)$ is Gaussian on $[0,T]$, and may be defined on $\beta(t)$,

iv) the increments of $\beta_n(t)$ on disjunct closed intervals are stochastically independent and orthogonal as n is large,

v) as $n \rightarrow \infty$, $\beta_n \rightarrow \beta$ a.s., in P and in distribution in the sense of (4.2.5), its corollary and remark,

vi) $\beta_n(t) \rightarrow \beta(t)$ in q.m., uniformly in $t \in [0,T]$ as $n \rightarrow \infty$, implying
$E\beta_n(s)\beta_n(t) \rightarrow E\beta(s)\beta(t) = \min(s,t)$, uniformly in $(s,t) \in [0,T]^2$,

vii) the total variation of $E\beta_n(s)\beta_n(t)$ on $[0,T]^2$ increases to T, the total variation of $E\beta(s)\beta(t)$ on $[0,T]^2$, as $n \rightarrow \infty$,

viii) the perturbations $\beta_n(t)$ may be assumed to start with the value 0 at $t = 0$.

In (4.2.8), the first 4 properties describe $\beta_n(t)$ as a reasonable mathematical model of the coordinates of a particle in Brownian motion. The properties v - vii show that $\beta_n(t)$ converges satisfactorily to $\beta(t)$ as $n \rightarrow \infty$.

4.3. Smooth perturbations of a finite degree of randomness.

It will be shown that there are smooth perturbations of $\beta(t)$ in the sense of (4.2.8), based on a finite number of random variables $\beta(t_i)$, cf. [22]. Perturbations of this kind have been used by Wong and Zakai, see [31].

(4.3.1) Let $r(x)$ be a mapping of $[0,1]$ into itself with the properties

i) $r(0) = 0$, $r(1) = 1$, and $r(x)$ is non-decreasing on $[0,1]$, and

ii) $r(x)$ is smooth on $[0,1]$ and moreover such that

$$r^{(k)}(0) = r^{(k)}(1) = 0, \quad k=1,2,.. \quad .$$

For instance,

$$r(x) = q_2(x-\tfrac{1}{2})$$

where $q_2(t)$ is one of the functions defined in (4.2.3).

(4.3.2) If p is the partition of $[0,T]$, defined by the subdivision points t_i satisfying $0=t_o < t_1 < .. < t_I = T$, we define

$$r_i(t) = r\left(\frac{t - t_{i-1}}{t_i - t_{i-1}}\right) \quad \text{if} \quad t_{i-1} \leqq t \leqq t_i \ , \quad i=1,..,I \ .$$

The behavior of $r_i(t)$ on $[t_{i-1},t_i]$ corresponds with the behavior of $r(x)$ on $[0,1]$.

Let $\beta(t)$, $t \in [0,T]$, be the 1-dimensional Wiener-Lévy process, see (4.1.1). Let $\{\Omega, \mathcal{A}, P\}$ be a suitable probability space, and $\beta(\omega,t)$, $(\omega,t) \in \Omega \times [0,T]$, a sample continuous representation of $\beta(t)$.

(4.3.3) Given (4.3.1) and (4.3.2), we define

(4.3.3a) $$\beta_p(t) = \beta(t_{i-1}) + r_i(t)\left\{\beta(t_i) - \beta(t_{i-1})\right\} \quad \text{if} \quad t \in [t_{i-1},t_i], \quad i=1,..,I.$$

Obviously, there is no ambiguity at the subdivision points t_i.

If $\{p_n, n=1,2,..\}$ is any sequence of partitions of $[0,T]$, such that $\Delta(p_n) \to 0$ as $n \to \infty$, and if we set

$$\beta_n(t) = \beta_{p_n}(t) ,$$

then

(4.3.3b) $\left\{ \beta_n(t) , \quad n=1,2,.. \right\}$

is a sequence of smooth perturbations of the 1-dimensional Wiener-Lévy process.

It is endowed with the properties i - viii in (4.2.8).

Proof:

i) It is an immediate result of the definition that the trajectories $\beta_p(\omega,t)$ are smooth on the open intervals (t_{i-1},t_i) and that the right- and left-hand derivatives exist and are equal to 0 at the subdivision points t_i . Hence the derivatives of all orders exist also at the subdivision points and so the trajectories are smooth on $[0,T]$.

ii) $\beta_p(t)$ is of second order, since at any $t \in [0,T]$ it is a linear combination of some of the random variables $\beta(t_i)$. Clearly $E\beta_p(t) = 0$ on $[0,T]$. It is smooth in q.m. by virtue of arguments analogous to those, used in i and by applying a simple version of (2.5.5).

iii) $\beta_p(t)$ is Gaussian on $[0,T]$ on account of (2.3.1), as at any $t \in [0,T]$ it is a linear combination of random variables, belonging to the Gaussian family $\{\beta(t), t \in [0,T]\}$.

iv) The increments of $\beta_n(t)$ on disjunct closed intervals *of* $[0,T]$ *are stochastically independent and hence orthogonal* as $E\beta(t) = 0$, if n is sufficiently large. For,if

$$0 \leqq s_1 \leqq s_2 < s_3 \leqq s_4 \leqq T,$$

there is an N such that $\Delta(p_n) < s_3 - s_2$ as $n > N$. So, to any partition p_n with $n > N$, there is a subdivision point $t_{i(n)}$ in $[s_2,s_3]$. And hence

$\beta_n(s_2)-\beta_n(s_1)$ is defined on some increments of $\beta(t)$, $t \in [0,t_{i(n)}]$,

whereas

$\beta_n(s_4)-\beta_n(s_3)$ is defined on some increments of $\beta(t)$, $t \in [t_{i(n)},T]$.

The assertion is shown, owing to the independence and orthogonality of the increments of $\beta(t)$, see (4.1.1) .

v) At a.a. $\omega \in \Omega$, $\beta_n(\omega,t) \to \beta(\omega,t)$ uniformly in $t \in [0,T]$ as $n \to \infty$. For, being continuous on the compact set $[0,T]$, $\beta(\omega,t)$ is uniformly continuous on $[0,T]$. So, given $\varepsilon > 0$, there is a $\delta > 0$ such that $[s,t] \subset [0,T]$ and $|s-t| < \delta$ imply $|\beta(s) - \beta(t)| < \varepsilon$. Since $\Delta(p_n) \to 0$ as $n \to \infty$, there is an N such that $n > N$ implies $\Delta(p_n) < \delta$. Let us assume that partition p in (4.3.2) satisfies $\Delta(p) < \delta$. Given $t \in [0,T]$, there is an index i such that $t \in [t_{i-1}, t_i]$. Then $|t_i - t_{i-1}| < \delta$, $|t - t_{i-1}| < \delta$,

$$\beta_n(\omega,t) = \beta(\omega,t_{i-1}) + r_i(t)\{\beta(\omega,t_i) - \beta(\omega,t_{i-1})\},$$

and so

$$|\beta_n(\omega,t) - \beta(\omega,t_{i-1})| = |r_i(t)| \cdot |\beta(\omega,t_i) - \beta(\omega,t_{i-1})| < 1.\varepsilon .$$

Therefore

$$|\beta_n(\omega,t) - \beta(\omega,t)| \leqq |\beta_n(\omega,t) - \beta(\omega,t_{i-1})| + |\beta(\omega,t_{i-1}) - \beta(\omega,t)| < 2\varepsilon .$$

The index N is obtained independently of t and hence

$$\beta_n(\omega,t) \to \beta(\omega,t)$$

uniformly in $t \in [0,T]$ as $n \to \infty$. Theorem (4.2.5) is applicable, yielding

$$\beta_n \to \beta \text{ a.s. as } n \to \infty$$

in the sense of (4.2.5a). Also the corollary and the remark to (4.2.5) apply.

vi) $\beta_n(t) \to \beta(t)$ in q.m., uniformly in $t \in [0,T]$ as $n \to \infty$. This may be shown by arguments analogous to those in v since $\beta(t)$ is also uniformly continuous in q.m. on $[0,T]$, according to (2.4.7). So, given $\varepsilon > 0$, there is a $\delta > 0$ such that $[s,t] \subset [0,T]$ and $|s-t| < \delta$ imply $\|\beta(s) - \beta(t)\| < \varepsilon$. Again there is an N such that $n > N$ implies $\Delta(p_n) < \delta$. Let us assume that the partition p in (4.3.2) satisfies $\Delta(p) < \delta$. If $t \in [0,T]$, there is an index i such that $t \in [t_{i-1}, t_i]$. Now we obtain, see v,

$$\|\beta_n(t) - \beta(t_{i-1})\| = |r_i(t)| \cdot \|\beta(t_i) - \beta(t_{i-1})\| < \varepsilon$$

and therefore

$$\|\beta_n(t) - \beta(t)\| \leqq \|\beta_n(t) - \beta(t_{i-1})\| + \|\beta(t_{i-1}) - \beta(t)\| < 2\varepsilon .$$

Again the index N is obtained independently of $t \in [0,T]$ and hence the convergence in q.m. of $\beta_n(t)$ to $\beta(t)$ is uniform in $t \in [0,T]$.

On account of lemma (4.2.6c), $E\beta_n(s)\beta_n(t) \to E\beta(s)\beta(t)$ as $n \to \infty$, uniformly in $(s,t) \in [0,T]^2$.

vii) Also this assertion is true since we may show that for any partition p of $[0,T]$,

$$V(E\beta_p(s)\beta_p(t),\ [0,T]^2) = T = V(E\beta(s)\beta(t),\ [0,T]^2)\ .$$

Let p be the partition in (4.3.2) and let

$$d_{ij} = [t_{i-1},t_i] \times [t_{j-1},t_j].$$

Then, as the total variation is an additive set-function,

$$V(E\beta_p(s)\beta_p(t),\ [0,T]^2) = \sum_{i,j=1,.,I} V(E\beta_p(s)\beta_p(t),d_{ij}).$$

If $i=j$ and $(s,t) \in d_{ii}$,

$$\beta_p(s) = \beta(t_{i-1}) + r_i(s)\{\beta(t_i) - \beta(t_{i-1})\}\ ,$$

$$\beta_p(t) = \beta(t_{i-1}) + r_i(t)\{\beta(t_i) - \beta(t_{i-1})\} \quad \text{and}$$

$$E\beta_p(s)\beta_p(t) = t_{i-1} + (t_i - t_{i-1})r_i(s)r_i(t)\ .$$

Since $r_i(t)$ is non-decreasing on $[t_{i-1},t_i]$ with total variation equal to 1 ,

$$V(E\beta_p(s)\beta_p(t),\ d_{ii}) = t_i - t_{i-1}\ .$$

If $i < j$ and $(s,t) \in d_{ij}$,

$$\beta_p(s) = \beta(t_{i-1}) + r_i(s)\{\beta(t_i) - \beta(t_{i-1})\}$$

and $\beta_p(t)$ may be written as

$$\beta_p(t) = \beta(t_{i-1}) + \{\beta(t_i) - \beta(t_{i-1})\} + \{\beta(t_{j-1}) - \beta(t_i)\} + r_j(t)\{\beta(t_j) - \beta(t_{j-1})\}.$$

And hence

$$E\beta_p(s)\beta_p(t) = t_{i-1} + (t_i - t_{i-1})r_i(s)\ .$$

And now, since $E\beta_p(s)\beta_p(t)$ depends on s alone,

$$V(E\beta_p(s)\beta_p(t),\ d_{ij}) = 0\ .$$

If $i > j$ and $(s,t) \in d_{ij}$, a similar result is obtained.
And so we obtain

$$V(E\beta_p(s)\beta_p(t), [0,T]^2) = \sum_{i=1,.,I} V(E\beta_p(s)\beta_p(t), d_{ii}) =$$

$$\sum_{i=1,.,I} (t_i - t_{i-1}) = t_I - t_o = T .$$

viii) $\beta_n(0) = \beta(0) = 0$, by virtue of the definition of $\beta(t)$ and the definition of $\beta_n(t)$ in this section.

The perturbations $\beta_n(t)$ of $\beta(t)$ in this section are defined on a finite number of random variables.

The mappings $\beta_n(t)$ of $[0,T]$ into H are polygons. The derivatives in q.m. at the vertices are identical to 0 .

4.4. The N-dimensional Wiener-Lévy process and some of its differentiable perturbations.

(4.4.1) Let $\beta_{o1}(t),.., \beta_{oN}(t)$ be N mutually stochastically independent (and orthogonal) 1-dimensional Wiener-Lévy processes on $[0,T]$ of type (4.1.1). Then

(4.4.1a) $E\beta_{oi}(s)\beta_{oj}(t) = \delta_{ij}\min(s,t)$, $i,j=1,.,N$, $s,t \in [0,T]$,

where the Kronecker δ_{ij} is equal to 0 if $i \neq j$ and equal to 1 if $i=j$.

The column N-vector

(4.4.1b) $$\beta_o(t) = \begin{pmatrix} \beta_{o1}(t) \\ \vdots \\ \beta_{oN}(t) \end{pmatrix} , \quad t \in [0,T],$$

is the standard N-dimensional Wiener-Lévy process. It follows from (4.4.1a) that

(4.4.1c) $E\beta_o(s)\beta_o^T(t) = \min(s,t)\ I_N$,

where I_N is the $N\times N$-identity matrix.

(4.4.2) Let $G(t)$ be an $N\times N$-matrix whose entries $g_{ij}(t)$ are continuously differentiable mappings of $[0,T]$ into $(-\infty,\infty)$.+)
The N-dimensional Wiener-Lévy process on $[0,T]$ is the mapping $\beta(t)$ of $[0,T]$ into H^N, defined by

(4.4.2a) $$\beta(t) = \int_0^t G(s)d\beta_o(s) \quad , \quad t\in[0,T] .$$

Here $\beta_o(s)$ is the standard N-dimensional Wiener-Lévy process of (4.4.1b). From now on $\beta(t)$ will stand for N-dimensional Wiener-Lévy process. The components of $\beta(t)$ are

(4.4.2b) $$\beta_i(t) = \int_0^t \sum_{k=1}^N g_{ik}(s)d\beta_{ok}(s) = \sum_{k=1}^N \int_0^t g_{ik}(s)d\beta_{ok}(s) \ , \quad i=1,.,N.$$

The integrals in (4.4.2b) exist as Riemann-Stieltjes integrals in q.m. on account of (4.1.3).

By virtue of i in (2.6.2) and by (2.7.4),

(4.4.2c) $$\beta_i(t) = \sum_{k=1}^N g_{ik}(t)\beta_{ok}(t) - \sum_{k=1}^N \int_0^t \frac{d}{ds} g_{ik}(s)\beta_{ok}(s)ds, \quad i=1,.,N.$$

Now it is seen that the integrals also exist as ordinary Riemann sample integrals +).

Hence (4.4.2a) is meaningful as an integral in q.m. as well as a sample integral. Both types of integrals coincide, owing to vii in (2.6.2). And the following relation holds:

(4.4.2d) $$\beta(t)=\int_0^t G(s)d\beta_o(s) = G(t)\beta_o(t) - \int_0^t \frac{d}{ds} G(s)\beta_o(s)ds.$$

+) Usually the elements of $G(t)$ are only demanded to be continuous functions. In order to establish the meaning of (4.4.2a) or (4.4.2c) as a sample integral, continuity of these elements is not sufficient. It would be sufficient that they were moreover of bounded variation on $[0,T]$. However, in order to subject $G(t)$ to a more realistic condition, we demanded that the elements $g_{ij}(t)$ should be continuously differentiable on $[0,T]$.

On account of (4.4.1a), (4.4.2b) and (4.1.3a),

(4.4.2e) $$E\beta_i(s)\beta_j(t) = \sum_{k=1}^{N}\sum_{h=1}^{N} E\int_o^s g_{ik}(u)d\beta_{ok}(u)\int_o^t g_{jh}(v)d\beta_{oh}(v) =$$
$$\sum_{k=1}^{N}\int_o^m g_{ik}(u)g_{jk}(u)du\ ,\quad m = \min(s,t)\ ,\qquad (s,t)\in[0,T]^2\ ,$$

and hence the covariance function matrix satisfies

(4.4.2f) $$E\beta(s)\beta^T(t) = \int_o^m G(u)G^T(u)du,\ m = \min(s,t),\ (s,t)\in[0,T]^2\ .$$

As we set

(4.4.2g) $$B(t) = G(t)G^T(t)\ ,\qquad t\in[0,T]\ ,$$

then

(4.4.2h) $$\frac{d}{dt}E\beta(t)\beta^T(t) = B(t),\qquad t\in[0,T]\ .$$

It follows by the method of (2.9.4) and the result in (4.4.2e) that the elements of the covariance function matrix $E\beta(s)\beta(t)$ are of bounded variation on $[0,T]^2$:

(4.4.2i) $$V(\ E\beta_i(s)\beta_j(t),\ [0,T]^2\ ) \leqq \sum_{k=1}^{N}\int_o^T |g_{ik}(u)g_{kj}(u)|du \leqq$$
$$\leqq N\,T\,M^2 \quad \text{as}\quad M = \max_{i,j=1,..,N,\quad t\in[0,T]} |g_{ij}(t)|\ .$$

The separate components $\beta_i(t)$ of $\beta(t)$, i=1,.,N, possess the properties i-v of the 1-dimensional Wiener-Lévy process (4.1.1). If, for instance, the matrix $G(t)$ is independent of t on $[0,T]$, the components $\beta_i(t)$ also possess the property vi, i.e. they have stationary increments. If $G(t)$ is identical to I_N on $[0,T]$, $\beta(t)$ is identical to $\beta_o(t)$, the standard N-dimensional Wiener-Lévy process, whose components possess all properties i - vii of the 1-dimensional Wiener-Lévy process (4.1.1).

In non-trivial cases, the components of $\beta(t)$ are not differentiable in q.m., and their trajectories are with probability 1 not differentiable and not of bounded variation on the sub-intervals of $[0,T]$.

In order to obtain continuously differentiable perturbations of the N-dimensional Wiener-Lévy process (4.4.2a), we shall start with perturbing the N-dimensional standard Wiener-Lévy process $\beta_o(t)$.

Its respective components $\beta_{oi}(t)$, $i=1,.,N$, are replaced by processes $\beta_{oi}(n,t)$, $n=1,2,..$. Here $\beta_{oi}(n,t)$ is related to $\beta_{oi}(t)$ as $\beta_n(t)$ to $\beta(t)$ in (4.2.8). And so

(4.4.3) $$\beta_o(n,t) = \begin{pmatrix} \beta_{o1}(n,t) \\ \vdots \\ \beta_{oN}(n,t) \end{pmatrix}, \quad t \in [0,T], \quad n=1,2,.. ,$$

is a smoothly perturbed standard N-dimensional Wiener-Lévy process. Its components are endowed with all properties i - viii in (4.2.8). Hence, if $i \neq j$, $\beta_{oi}(n,s)$ and $\beta_{oj}(n',t)$ are stochastically independent and orthogonal as they have zero expectation, and as they are defined on $\beta_{oi}(u)$, $u \in [0,T]$ and on $\beta_{oj}(v)$, $v \in [0,T]$ respectively.

(4.4.4) Definition: If $G(t)$ is the matrix in (4.4.2) and if $\beta_o(n,t)$ is the N-vector (4.4.3),

$$\beta(n,t) = \begin{pmatrix} \beta_1(n,t) \\ \vdots \\ \beta_N(n,t) \end{pmatrix} = \int_0^t G(s)d\beta_o(n,s) \quad , \quad t \in [0,T],$$

is a continuously differentiable perturbation of the N-dimensional Wiener-Lévy process (4.4.2).

By virtue of (2.8.3), (2.6.2) and the properties of $\beta_o(n,s)$,

(4.4.4a) $$\beta(n,t) = \int_0^t G(s) \frac{d}{ds}\beta_o(n,s)ds =$$

$$G(t)\beta_o(n,t) - \int_0^t \frac{d}{ds} G(s)\, \beta_o(n,s)ds.$$

The components $\beta_i(n,t)$ of $\beta(n,t)$ satisfy

(4.4.4b) $$\beta_i(n,t) = \sum_{k=1}^{N} \int_0^t g_{ik}(s)d\beta_{ok}(n,s) =$$

$$= \sum_{k=1}^{N} \int_0^t g_{ik}(s) \frac{d}{ds}\beta_{ok}(n,s)ds = \sum_{k=1}^{N} \left\{ g_{ik}(t)\beta_{ok}(n,t) - \int_0^t \frac{d}{ds} g_{ik}(s)\beta_{ok}(n,s)ds \right. .$$

The above derivatives and integrals exist in sample sense as well as in q.m. In both senses the results coincide by virtue of (2.6.2).

Owing to (4.4.4b), it is a result of real analysis and of (2.7.2) respectively, that the components $\beta_i(n,t)$ are differentiable in sample sense, as well as in q.m. with derivatives

$$\frac{d}{dt}\beta_i(n,t) = \sum_{k=1}^{N} g_{ik}(t) \frac{d}{dt}\beta_{ok}(n,t) , \qquad i=1,.,N .$$

Since $\beta_{ok}(n,t)$, k=1,.,N, is infinitely often differentiable in sample sense and in q.m., and as $g_{ik}(t)$ is assumed to be continuously differentiable,

$$\frac{d}{dt}\beta_i(n,t)$$

is again continuously differentiable in sample sense and in q.m., see (2.5.5). If the matrix $G(t)$ possesses a derivative of order K, the components $\beta_i(n,t)$ possess derivatives of order K+1, both in sample sense and in q.m. The results in both senses coincide.

Since $E\beta_o(n,t) = 0$ on account of (4.4.3) and (4.2.8), also

$$E\beta(n,t) = 0$$

on account of (4.4.4) and of v in (2.6.2).

And as $\beta_o(n,0) = 0$, also $\beta(n,0) = 0$, see (4.4.4a).

By assumption in (4.4.1), the components $\beta_{oi}(t)$ of $\beta_o(t)$ are stochastically independent and orthogonal processes on $[0,T]$. Hence the random variables figuring in $\beta_o(t)$, $t \in [0,T]$, belong to a centered Gaussian subspace of H, see section 2.3. It follows from (4.4.4), (4.4.3) with (4.2.8) and from (2.3.1) that the random variables figuring in $\beta(n,t)$ also belong to that Gaussian subspace.

If $0 \leqq t_1 \leqq t_2 < t_3 \leqq t_4 \leqq T$,

$$\beta_i(n,t_2) - \beta_i(n,t_1) = \sum_{k=1}^{N} \int_{t_1}^{t_2} g_{ik}(t)d\beta_{ok}(n,t)$$

and

$$\beta_j(n,t_4) - \beta_j(n,t_3) = \sum_{k=1}^{N} \int_{t_3}^{t_4} g_{jk}(t)d\beta_{ok}(n,t) .$$

So the above increments are defined on increments of $\beta_{ok}(n,t)$, k=1,.,N,

in $[t_1,t_2]$ and $[t_3,t_4]$ respectively.

Owing to (4.2.8) these respective increments are stochastically independent and orthogonal if n is sufficiently large, and hence the same is true with respect to the above increments of the components of $\beta(n,t)$.

Let us consider the difference of the components of the N-dimensional Wiener-Lévy process (4.4.2) and its perturbation (4.4.4):

$$\beta_i(t)-\beta_i(n,t)=\sum_{k=1}^{N}\left[g_{ik}(t)\left\{\beta_{ok}(t)-\beta_{ok}(n,t)\right\}-\int_o^t \frac{d}{ds}\,g_{ik}(s)\left\{\beta_{ok}(s)-\beta_{ok}(n,s)\right\}ds\right]$$

$$t\in[0,T],\quad i=1,.,N\ .$$

The integrals may be seen as sample integrals, as well as integrals in q.m. The integrals of both types coincide.

By virtue of (4.2.8), the trajectories of $\beta_{ok}(n,s)$ tend to the corresponding trajectories of $\beta_{ok}(s)$ as $n\to\infty$, uniformly in $s\in[0,T]$, $k=1,.,N$. As by assumption $\frac{d}{ds}\,g_{ik}(s)$ is continuous on $[0,T]$, the trajectories of

$$\frac{d}{ds}\,g_{ik}(s)\left\{\beta_{ok}(s)-\beta_{ok}(n,s)\right\}$$

tend to 0 as $n\to\infty$, uniformly in $s\in[0,T]$. Hence, on account of the rules of ordinary real analysis, the trajectories of $\beta_i(t)-\beta_i(n,t)$ tend to 0, uniformly in $t\in[0,T]$. Lemma (4.2.5) applies and so the trajectories of $\beta_i(n,t)$ tend as $n\to\infty$ to the corresponding trajectories of $\beta_i(t)$ a.s. in the sense of (4.2.5a). This type of convergence implies convergence in probability and in distribution in the sense of the remark to (4.2.5).

By virtue of (4.2.8), $\beta_{ok}(n,s)$ tends to $\beta_{ok}(s)$ in q.m. as $n\to\infty$, uniformly in $s\in[0,T]$. Since $\frac{d}{ds}\,g_{ik}(s)$ is continuous on $[0,T]$,

$$\left\|\frac{d}{ds}\,g_{ik}(s)\left\{\beta_{ok}(s)-\beta_{ok}(n,s)\right\}\right\|\to 0\quad\text{as}\quad n\to\infty\,,$$

uniformly in $s\in[0,T]$. Hence, by virtue of (2.7.5), $\beta_i(n,t)\to\beta_i(t)$ in q.m. as $n\to\infty$, uniformly in $t\in[0,T]$.

Because of lemma (4.2.6c), this latter result entails

$$E\,\beta_i(n,s)\,\beta_j(n,t)\to E\,\beta_i(s)\,\beta_j(t)\quad\text{as}\quad n\to\infty\,,$$

uniformly in $(s,t)\in[0,T]^2$, $i,j=1,.,N$.

Finally, let us consider the total variation on $[0,T]^2$ of the elements of the covariance function matrix $E\beta(n,s)\beta^T(n,t)$. On account of (2.9.2),

$$E\beta_i(n,s)\beta_j(n,t) = E\sum_{k=1}^{N}\int_o^s g_{ik}(u)d\beta_{ok}(n,u)\sum_{h=1}^{N}\int_o^t g_{jh}(v)d\beta_{oh}(n,v) =$$

$$= \sum_{k=1}^{N}\int_o^s\int_o^t g_{ik}(u)g_{jk}(v)\,ddE\beta_{ok}(n,u)\beta_{ok}(n,v)\ , \quad i,j=1,..,N\ ,$$

since $\beta_{ok}(n,u)$ and $\beta_{oh}(n,v)$ are orthogonal if $h \neq k$, see (4.4.3). Obviously,

$$V(\ E\beta_i(n,s)\beta_j(n,t)\ ,\ [0,T]^2\ ) \leqq$$

$$\sum_{k=1}^{N} V(\ \int_o^s\int_o^t g_{ik}(u)g_{jk}(v)\,ddE\beta_{ok}(n,u)\beta_{ok}(n,v)\ ,\ [0,T]^2\ ).$$

Because of i in (2.9.4) and vii in (4.2.8),

$$V(\ \int_o^s\int_o^t g_{ik}(u)g_{jk}(v)\,ddE\beta_{ok}(n,u)\beta_{ok}(n,v)\ ,\ [0,T]^2\ ) \leqq M^2\ T\ ,$$

if $M = \max_{i,j=1,..,N,\quad t\in[0,T]} |\ g_{ij}(t)|$.

And hence

$$V(\ E\beta_i(n,s)\beta_j(n,t)\ ,\ [0,T]^2\ ) \leqq N\ M^2\ T\ , \quad n=1,2,..\ , \quad i,j=1,..,N.$$

We recall (4.4.2i), where we established

$$V(\ E\beta_i(s)\beta_j(t)\ ,\ [0,T]^2\ ) \leqq N\ M^2\ T\ .$$

And so the elements of the covariance function matrices

$$E\beta(n,s)\beta^T(n,t) \quad \text{and} \quad E\beta(s)\beta^T(t)$$

are of bounded variation on $[0,T]^2$, uniform in $n=1,2,..$.

If the components of $\beta_o(n,t)$ are of the type, defined in section 4.3, then $\beta(n,t)$, $t\in[0,T]$ is defined on a finite number of random variables.

We have shown that $\beta(n,t)$ is a reasonable mathematical model of the position of a particle in Brownian motion. And we have shown the measure in which $\beta(n,t)$ approaches $\beta(t)$ if n is large.

The above results are gathered together in the following statement:

(4.4.5) Given the N-dimensional Wiener-Lévy process $\beta(t)$, $t \in [0,T]$, in (4.4.2), there is a sequence

$$\{ \beta(n,t) , \quad t \in [0,T], \qquad n=1,2,.. \}$$

of perturbations of $\beta(t)$ with the following properties.

As $i,j=1,.,N$,

i) the components $\beta_i(n,t)$ have continuously differentiable trajectories with probability 1 ,

ii) the components $\beta_i(n,t)$ are continuously differentiable in q.m. on $[0,T]$,

iii) $E\beta(n,t) = 0$, $t \in [0,T]$, and the random variables, figuring in $\beta(n,t)$ and $\beta(t)$, $t \in [0,T]$, belong to the centered Gaussian system, generated by $\beta_0(t)$, $t \in [0,T]$, for all $n=1,2,..$,

iv) the increments of $\beta_i(n,t)$ and $\beta_j(n,t)$ on disjunct closed intervals of $[0,T]$ are orthogonal if n is sufficiently large,

v) as $n \to \infty$, the trajectories of $\beta_i(n,t)$ tend to the corresponding trajectories of $\beta_i(t)$ a.s. in the sense of (4.2.5a) and hence also in probability and in distribution in the sense of the remark to (4.2.5),

vi) as $n \to \infty$, $\beta_i(n,t) \to \beta_i(t)$ in q.m., uniformly in $t \in [0,T]$,
implying that $E\,\beta_i(n,s)\,\beta_j(n,t) \to E\,\beta_i(s)\,\beta_j(t)$ as $n \to \infty$, uniformly in $(s,t) \in [0,T]^2$,

vii) the elements of the covariance function matrices $E\beta(n,s)\beta^T(n,t)$ and $E\beta(s)\beta^T(t)$ are of bounded variation on $[0,T]^2$, uniform in $n=1,2,..$,

viii) $\beta(n,0) = 0$, $n=1,2,..$.

At each n , $\beta(n,t)$, $t \in [0,T]$, may be based on a finite set of random variables.

5 Ordinary linear systems, driven by white noise and the behaviour of their solutions with respect to differentiable perturbations of the involved noise processes

5.1. The solution of an ordinary linear system, driven by an N-dimensional Wiener-Lévy process.

We shall continue chapter 3.

(5.1.1) Let be given the system

(5.1.1a)
$$d\,\xi(t) = A(t)\,\xi(t)dt + d\,\beta(t)^{+)}, \quad t \in [0,T],$$
$$\xi(0) = \gamma$$

whose meaning is

(5.1.1b)
$$\xi(t) = \gamma + \int_0^t A(s)\,\xi(s)ds + \beta(t)^{+)}, \quad t \in [0,T].$$

We recall that

i) $A(t)$ is an $N \times N$-matrix whose entries $a_{ij}(t)$, $i,j=1,.,N$, are continuous mappings of $[0,T]$ into $(-\infty,\infty)$,

ii) γ, with components γ_i, $i=1,.,N$, is a centered Gaussian N-vector, $E\,\gamma = 0$,

iii) $\beta(t)$ is the N-dimensional Wiener-Lévy process (4.4.2a) with components $\beta_i(t)$, $t \in [0,T]$, $i=1,.,N$.

And it is assumed that moreover

iv) γ_i and $\beta_j(t)$, $t \in [0,T]$, $i,j=1,.,N$, are stochastically independent.

+) Usually, $d\,\beta(t)$ in (5.1.1a) is replaced by $M(t)d\,\beta(t)$, where the matrix $M(t)$ is endowed with the same properties as $G(t)$ in (4.4.2). This means that $\beta(t) = \int_0^t G(s)d\,\beta_0(s)$ in (5.1.1b) is replaced by $\int_0^t M(s)d\beta(s)$. According to (2.9.4),
$$\int_0^t M(s)d\beta(s) = \int_0^t M(s)d\int_0^s G(u)d\beta_0(u) = \int_0^t M(s)G(s)d\,\beta_0(s) = \int_0^t \tilde{G}(s)d\,\beta_0(s).$$
Since the matrix $\tilde{G}(s) = M(s)G(s)$ is also continuously differentiable on $[0,T]$, it is not a restriction to treat systems like (5.1.1a).

Because of the above assumptions,

$$\left\{ \gamma_i , \beta_j(t), \quad i,j=1,.,N, \quad t \in [0,T] \right\}$$

is a centered Gaussian system, and

$$\gamma_i \perp \beta_j(t) , \quad i,j=1,.,N, \quad t \in [0,T] .$$

(5.1.1c) $\xi(t)$ stands for a mapping of $[0,T]$ into H^N, continuous in q.m. and sample continuous.

Since $\beta(t)$ is not differentiable on $[0,T]$, neither in q.m., nor in sample sense, there is not a system of differential equations of type (3.3a), equivalent to (5.1.1b).

(5.1.2) There is a unique $\xi(t)$ of type (5.1.1c), satisfying (5.1.1b) in sample sense and in q.m. It may be represented as

(5.1.2a) $$\xi(t) = F(t)\gamma + F(t) \int_0^t F^{-1}(s)d\beta(s) =$$
$$F(t)\gamma + \beta(t) + F(t) \int_0^t F^{-1}(s)A(s)\beta(s)ds ,$$

where $F(t)$ is the fundamental matrix, associated with $A(t)$, see (3.2.3). The integrals may be seen as sample integrals as well as integrals in q.m.

Proof: If there is a solution to (5.1.1b), it is unique. For, if both $\xi(t)$ and $\eta(t)$ satisfy (5.1.1b) then $\xi(t) - \eta(t)$ satisfies (3.2.1c) and hence $\xi(t) - \eta(t) = 0, \quad t \in [0,T]$.

We shall show that substitution of (5.1.2a) into

$$\xi(t) - \int_0^t A(s)\xi(s)ds$$

yields

$$\gamma + \beta(t) .$$

On account of (3.2.2) and (3.2.4) substitution of $\xi(t)=F(t)\gamma$ yields γ. It remains to show that substitution of

$$\xi(t) = F(t) \int_0^t F^{-1}(s)d\beta(s) \text{ yields } \beta(t) .$$

Because of $A(s)F(s) = \frac{d}{ds} F(s)$, see (3.2.3), it is by means of partial integration (2.6.2) and by virtue of (2.7.4) and (2.9.4) that we obtain

$$\xi(t)-\int_0^t A(s)F(s)\left\{\int_0^s F^{-1}(u)d\beta(u)\right\}ds = \xi(t)-\int_0^t \frac{dF(s)}{ds}\left\{\int_0^s F^{-1}(u)d\beta(u)\right\}ds =$$
$$\xi(t) - \left[F(s) \int_0^s F^{-1}(u)d\beta(u)\right]_0^t + \int_0^t F(s)d\left\{\int_0^s F^{-1}(u)d\beta(u)\right\} =$$
$$\xi(t)-F(t)\int_0^t F^{-1}(s)d\beta(s)+\int_0^t F(s)F^{-1}(s)d\beta(s)= \xi(t)- \xi(t)+\int_0^t d\beta(s)= \beta(t).$$

By virtue of foregoing results, all above operations are valid in q.m. as well as in sample sense. The results in both senses coincide.

Owing to v in (2.6.2),

$$\varphi(t) = F(t)E^{\mathcal{B}}\gamma + F(t)\int_0^t F^{-1}(s)dE^{\mathcal{B}}\beta(s)$$

is the unique solution in q.m. to

$$\varphi(t) = E^{\mathcal{B}}\gamma + \int_0^t A(s)\varphi(s)ds + E^{\mathcal{B}}\beta(t).$$

Here $E\,\gamma = 0$ and $E\beta(t) = 0$, $t \in [0,T]$. If γ and $\beta(t)$ in (5.1.1b) were replaced by $c + \gamma$ and $b(t) + \beta(t)$ respectively, where $b(t)$ is assumed to be continuously differentiable on $[0,T]$, the solution to the thus modified system could be written as $x(t) + \xi(t)$, with $E\,\xi(t) = 0$ on $[0,T]$. Analogous to the results in (3.3.4), $x(t)$ is governed by a deterministic system and $\xi(t)$ by a stochastic system.

Below we shall discuss several properties of $\xi(t)$, the solution (5.1.2a) to (5.1.1b).

<u>(5.1.2b)</u> The components $\xi_i(t)$, $i=1,.,N$, of $\xi(t)$ are continuous in q.m. and sample continuous on $[0,T]$. These properties were imposed on $\xi(t)$ in (5.1.1c). The representation (5.1.2a) shows that $\xi(t)$ is endowed with these properties.

However, in non-trivial cases the components $\xi_i(t)$ are not differentiable in q.m. on $[0,T]$. And with probability 1 , the trajectories are neither differentiable, nor of bounded variation on the sub-intervals of $[0,T]$.

<u>(5.1.2c)</u> $\left\{ \xi_i(t),\ i=1,.,N,\ t \in [0,T] \right\}$

is a centered Gaussian system. It is Gaussian by virtue of (2.3.1) and representation (5.1.2a), whereas $E\,\xi(t) = 0$ on $[0,T]$ on account of v in (2.6.2) since $E\,\gamma = 0$ and $E\beta(t) = 0$ on $[0,T]$, see also the discussion on the top of this page.

On account of the definition of $\beta(t)$ in (4.4.2), the above Gaussian system is a subset of the Gaussian linear space, generated by the Gaussian system

$$\left\{ \gamma_i,\ i=1,.,N,\ \beta_{oj}(t),\ j=1,.,N,\ t \in [0,T] \right\}.$$

(5.1.2d) $$E\,\xi(s)\,\xi^{T}(t) = F(s)\left\{E\gamma\gamma^{T} + \int_0^m F^{-1}(u)B(u)\left[F^{-1}(u)\right]^{T}du\right\}F^{T}(t),$$

where $m = \min(s,t)$ and $B(t)$ is the matrix (4.4.2g).

For, the elements of matrix $E\,\xi(s)\,\xi^{T}(t)$ may be computed as follows:

$$E\,\xi_i(s)\,\xi_j(t) = E\left[\left(\sum_{k=1}^{N} f_{ik}(s)\left\{\gamma_k + \int_0^s \sum_{h=1}^{N} f_{kh}^{-1}(u)d\beta_h(u)\right\}\right)\times\right.$$
$$\left.\left(\sum_{k'=1}^{N} f_{jk'}(t)\left\{\gamma_{k'} + \int_0^t \sum_{h'=1}^{N} f_{k'h'}^{-1}(v)d\beta_{h'}(v)\right\}\right)\right],$$

where $f_{ij}(s)$ and $f_{ij}^{-1}(t)$, $i,j=1,.,N$, are the elements of the matrices $F(s)$ and $F^{-1}(t)$ respectively.

According to (4.4.2a) and (2.9.4),

$$\int_0^s f_{kh}^{-1}(u)d\beta_h(u) = \int_0^s f_{kh}^{-1}(u)\sum_{r=1}^{N} d\int_0^u g_{hr}(u')d\beta_{or}(u') =$$
$$\sum_{r=1}^{N}\int_0^s f_{kh}^{-1}(u)g_{hr}(u)d\beta_{or}(u),$$

and hence, on account of (4.4.2e),

$$E\int_0^s f_{kh}^{-1}(u)d\beta_h(u)\int_0^t f_{k'h'}^{-1}(v)d\beta_{h'}(v) =$$
$$\sum_{r=1}^{N}\sum_{r'=1}^{N} E\int_0^s f_{kh}^{-1}(u)g_{hr}(u)d\beta_{or}(u)\int_0^t f_{k'h'}^{-1}(v)g_{h'r'}(v)d\beta_{or'}(v) =$$
$$\sum_{r=1}^{N}\int_0^m f_{kh}^{-1}(u)g_{hr}(u)f_{k'h'}^{-1}(u)g_{h'r}(u)du =$$
$$\int_0^m f_{kh}^{-1}(u)\left\{\sum_{r=1}^{N} g_{hr}(u)g_{h'r}(u)\right\} f_{k'h'}^{-1}(u)du = \int_0^m f_{kh}^{-1}(u)b_{hh'}(u)f_{k'h'}^{-1}(u)du,$$

if $b_{ij}(t)$, $i,j=1,.,N$, are the elements of $B(t)$.

As $\gamma_i \perp \beta_j(t)$, $i,j=1,.,N$, $t\in[0,T]$,

$$E\gamma_k\sigma = 0\,, \quad\text{where}\quad \sigma = \int_0^t f_{k'h'}^{-1}(v)d\beta_{h'}(v)\,.$$

For, if $\{\sigma_n,\ n=1,2,..\}$ is a sequence of Riemann-Stieltjes sums, converging in q.m. as $n\to\infty$ to σ, then on account of (2.1.3),

$$0 = E\gamma_k\sigma_n = \lim_{n\to\infty} E\gamma_k\sigma_n = E\gamma_k\sigma\,.$$

And hence

$$E\,\xi_i(s)\,\xi_j(t) = \sum_{k,k'=1}^{N} f_{ik}(s) E\,\gamma_k \gamma_{k'} f_{jk'}(t) +$$

$$\sum_{k,h,h',k'=1}^{N} f_{ik}(s) \left\{ \int_0^m f_{kh}^{-1}(u) b_{hh'}(u) f_{k'h'}^{-1}(u)\,du \right\} f_{jk'}(t) \;,$$

showing (5.1.2d).

<u>(5.1.2e)</u> $\xi(t)$ is an N-dimensional Markov process.

Proof: Let $\{\Omega\;, \mathcal{A}\;, P\}$ be a probability space, suitable for representing all events below.

Let $0 \leqq s \leqq t \leqq T$.

Let $\mathcal{B}[s]$ be the minimal σ-field generated by $\xi(s)$.

Let $\mathcal{B}[0,s]$ be the minimal σ-field generated by $\xi(u)$, $u \in [0,s]$.

We have to establish the Markov property

$$E^{\mathcal{B}[0,s]}\xi(t) = E^{\mathcal{B}[s]}\xi(t) \quad \text{a.s.} \;,$$ see [15].

Since

$$\xi(s) = F(s)\gamma + F(s)\int_0^s F^{-1}(u)\,d\,\beta(u),$$

it follows that

$$\xi(t) = F(t)F^{-1}(s)\,\xi(s) + F(t)\int_s^t F^{-1}(v)\,d\,\beta(v) =$$

$$\eta[s] + \zeta[s,t] \;,$$

as we write

$$\eta[s] = F(t)F^{-1}(s)\,\xi(s), \qquad \zeta[s,t] = F(t)\int_s^t F^{-1}(v)\,d\,\beta(v) \;.$$

$\zeta[s,t]$ is defined on the increments of $\beta(v)$ in $[s,t]$ and $\xi(u)$, $u \in [0,s]$ is defined on the increments of $\beta(v)$ in $[0,u]$ and on γ . Hence, $\zeta[s,t]$ is stochastically independent of $\xi(u)$, $u \in [0,s]$, and so

$$E^{\mathcal{B}[0,s]}\zeta[s,t] = E^{\mathcal{B}[s]}\zeta[s,t] = E\,\zeta[s,t] = 0 \text{ a.s.}$$

$\eta[s]$ is $\mathcal{B}[s]$- and $\mathcal{B}[0,s]$-measurable, entailing

$$E^{\mathcal{B}[0,s]}\eta[s] = E^{\mathcal{B}[s]}\eta[s] = \eta[s] \text{ a.s.}$$

Therefore

$$E^{\mathcal{B}[0,s]}\,\xi(t) = E^{\mathcal{B}[0,s]}\{\eta[s] + \zeta[s,t]\} = E^{\mathcal{B}[0,s]}\,\eta[s] + E^{\mathcal{B}[0,s]}\zeta[s,t] =$$

$$\eta[s] = E^{\mathcal{B}[s]}\{\eta[s] + \zeta[s,t]\} = E^{\mathcal{B}[s]}\,\xi(t) \quad \text{a.s.}$$

It is contained in the above lines that

$$E^{\mathcal{B}[0,s]}\,\xi(t) = \eta[s] = F(t)F^{-1}(s)\,\xi(s) .$$

Hence $\xi(t)$, $t \in [0,T]$ is in general not an N-dimensional martingale. It is also seen that $\xi(t)$ is in general not a process with independent increments.

The 1-dimensional wide-sense stationary version of the solution $\xi(t)$ is called Ornstein-Uhlenbeck process. Ornstein and Uhlenbeck proposed to use this process as a mathematical model of the velocity of a particle in Brownian motion, see [3] and [30] for instance. However, as we remarked in (5.1.2b), it is not differentiable in q.m., and with probability 1 its trajectories are neither differentiable, nor of bounded variation on the sub-intervals of $[0,T]$.

5.2. The behaviour of the solution with respect to differentiable perturbations of the N-dimensional Wiener-Lévy process.

As we explained in chapter 1 , it is meaningful to investigate whether solution $\xi(t)$,(5.1.2a), to system (5.1.1b) is stable with respect to differentiable perturbations of the N-dimensional Wiener-Lévy process $\beta(t)$ in (5.1.1b). And in particular, attention should be paid to the sample behaviour.

To this purpose, $\beta(t)$ in (5.1.1b) will be replaced by an element of the sequence

$$\{ \beta(n,t), \quad t \in [0,T], \qquad n=1,2,.. \}$$

in (4.4.5) of continuously differentiable perturbations of $\beta(t)$. Thus at each n, n=1,2,.. , we obtain a system

$$\xi(n,t) = \gamma + \int_0^t A(s)\,\xi(n,s)ds + \beta(n,t) \quad , \quad t \in [0,T] .$$

Since $\beta(n,t)$ is continuously differentiable in sample sense as well as in q.m., this system is equivalent to the system of differential equations

$$\frac{d}{dt}\xi(n,t) = A(t)\,\xi(n,t) + \frac{d}{dt}\beta(n,t), \quad t\in[0,T],$$

with initial condition

$$\xi(n,0) = \gamma ,$$

on account and in the sense of (3.3.1) and (3.3.2). So there is a unique solution $\xi(n,t)$ which may be interpreted in q.m. as well as in sample sense. It may be represented as

$$\xi(n,t) = F(t)\gamma + F(t)\int_0^t F^{-1}(s)\,\frac{d}{ds}\beta(n,s)ds =$$

$$F(t)\gamma + \beta(n,t) + F(t)\int_0^t F^{-1}(s)A(s)\beta(n,s)ds ,$$

where $F(t)$ is the fundamental matrix associated with $A(t)$, see (3.2.3). The second equality is a consequence of partial integration (2.6.2) and of the properties of $F(t)$. All theorems of section 3.3 are applicable to $\xi(n,t)$.

If $\xi(n,t)$ tends to $\xi(t)$ in (5.1.2) as $n\to\infty$, $\xi(t)$ is "stable". The exact meaning of this kind of stability depends on the way in which $\beta(n,t)$ tends to $\beta(t)$ and $\xi(n,t)$ tends to $\xi(t)$ as $n\to\infty$. A detailed account of the mode of stability will be given in each relevant situation separately.

(5.2.1) Summarizing,

$$\xi(t) = F(t)\gamma + \beta(t) + F(t)\int_0^t F^{-1}(s)A(s)\beta(s)ds$$

is the solution (5.1.2a) to the system

$$\xi(t) = \gamma + \int_0^t A(s)\,\xi(s)ds + \beta(t), \quad t\in[0,T],$$

in (5.1.1). All conditions in section 5.1 are assumed to be fulfilled and so all results in 5.1 are valid.

$\beta(n,t)$ is an element of the sequence in (4.4.5), endowed with all properties i - viii listed there.

$$\xi(n,t) = F(t)\gamma + \beta(n,t) + F(t)\int_0^t F^{-1}(s)A(s)\beta(n,s)ds$$

is the solution to the system

$$\frac{d}{dt}\xi(n,t) = A(t)\,\xi(n,t) + \frac{d}{dt}\beta(n,t), \quad t\in[0,T], \qquad \xi(n,0) = \gamma ,$$

and hence $\xi(n,t)$ is endowed with all properties established in section 3.3.

We shall show:

$\xi(t)$, $t \in [0,T]$ is stable with respect to continuously differentiable perturbations of the N-dimensional Wiener-Lévy process $\beta(t)$, $t \in [0,T]$, in the following sense: If $n \to \infty$, then

i) $\xi_i(n,t) \to \xi_i(t)$ in q.m., uniformly in $t \in [0,T]$, i=1,.,N, entailing
$E\,\xi_i(n,s)\,\xi_j(n,t) \to E\,\xi_i(s)\,\xi_j(t)$ uniformly in $(s,t) \in [0,T]^2$, i,j=1,.,N, owing to (4.2.6c),

ii) $\xi_i(n) \to \xi_i$ a.s. in the sense of (4.2.5a), i=1,.,N, entailing also convergence in probability and in distribution, see the remark to (4.2.5).

Proof: According to the above exposition,

$$\xi(t) - \xi(n,t) = \beta(t) - \beta(n,t) + F(t)\int_0^t F^{-1}(s)A(s)\{\beta(s) - \beta(n,s)\}ds$$

where the integral may be interpreted as a Riemann integral in q.m. as well as a Riemann sample integral.

Concerning assertion i, let us first note that
$\beta_i(n,t) \to \beta_i(t)$ in q.m. as $n \to \infty$, uniformly in $t \in [0,T]$, i=1,.,N,
on account of vi in (4.4.5), or, equivalently

$$\max_{t\in[0,T]} \|\beta(t) - \beta(n,t)\|_N \to 0 \quad \text{as} \quad n \to \infty ,$$

see chapter 3. Hence it follows that

$$\max_{t\in[0,T]} \|\xi(t) - \xi(n,t)\|_N \leqq \max_{t\in[0,T]} \|\beta(t) - \beta(n,t)\|_N +$$
$$(\max_{t\in[0,T]} |F(t)|).(\max_{t\in[0,T]} |F^{-1}(t)A(t)|).T.(\max_{t\in[0,T]} \|\beta(t)-\beta(n,t)\|_N) \to 0$$

as $n \to \infty$, showing i .

Concerning ii, let $\{\Omega, \mathcal{A}, P\}$ be a probability space, suitable for representing the above stochastic processes. As $\omega \in \Omega$, and in obvious notation,

$$|\xi_i(\omega,t) - \xi_i(n,\omega,t)| \leqq |\beta_i(\omega,t) - \beta_i(n,\omega,t)| +$$
$$\sum_{j,k,h=1}^{N} |f_{ij}(t)| \int_0^T |f_{jk}^{-1}(s)|.|a_{kh}(s)|.|\beta_h(\omega,s) - \beta_h(n,\omega,s)|\,ds .$$

At a.a. fixed $\omega \in \Omega$, $|\beta_h(\omega,s) - \beta_h(n,\omega,s)| \to 0$ as $n \to \infty$, uniformly in $s \in [0,T]$, h=1,.,N. Hence, at a.a. $\omega \in \Omega$,

$\left| \xi_i(\omega,t) - \xi_i(n,\omega,t) \right| \to 0$ as $n \to \infty$, uniformly in $t \in [0,T]$, according to the results of ordinary real analysis. Theorem (4.2.5) applies, and ii is established.

If in the above systems

$$\gamma, \quad \beta(t) \quad \text{and} \quad \beta(n,t)$$

are replaced by

$$c + \gamma, \; b(t) + \beta(t) \;\text{ and }\; b(t) + \beta(n,t) \quad \text{respectively,}$$

the deterministic parts of the solutions coincide and the above theorem on stability is applicable, see (3.3.4) and (5.1.2).

Also $E^{\mathcal{B}}\xi(n,t) \to E^{\mathcal{B}}\xi(t)$ in q.m. as $n \to \infty$, uniformly in $t \in [0,T]$, see (2.2.2).

The 1-dimensional version of the result in (5.2.1) may be seen as a special case of a general theorem of Wong and Zakai, see [31]. Wong and Zakai investigated the above type of stability with respect to the 1-dimensional Ito-equation

$$d\,\xi(t) = f(\xi(t),t)dt + g(\xi(t),t)d\beta(t), \qquad t \in [0,T],$$
$$\xi(0) = \gamma.$$

In general, it is only by means of Ito-calculus or related calculi that this equation is (uniquely) solvable, provided that certain conditions are satisfied. It should be stressed that in general the results of Ito-calculus cannot be obtained by means of the ordinary calculus, exposed in chapter 2. Essentially, Ito-integrals are limits of sequences of Riemann-Stieltjes sums, where the values $t_k' \in [t_{k-1}, t_k]$ in the intervals of the partitions are fixed at $t_k' = t_{k-1}$. Wong and Zakai showed that the sequence of ordinary sample solutions of the ordinary stochastic differential equations

$$\frac{d}{dt}\xi(n,t) = f(\xi(n,t),t) + g(\xi(n,t),t)\,\frac{d}{dt}\beta(n,t), \qquad t \in [0,T],$$
$$\xi(n,0) = \gamma, \qquad n=1,2,.. ,$$

tends to the Ito-solution of

$$d\,\xi(t) = f(\xi(t),t)dt + \tfrac{1}{2}g(\xi(t),t).\frac{\partial}{\partial \xi}g(\xi(t),t)dt + g(\xi(t),t)d\beta(t),$$
$$t \in [0,T], \qquad \xi(0) = \gamma,$$

if certain conditions, imposed on the functions involved are fulfilled. And so the Ito-solution of the Ito-equation is in general not stable in the sense of (5.2.1).

In particular, the equation in (5.2.1), i.e. equation (5.1.1a), may be seen as an Ito-equation. It may be solved by means of Ito-calculus. Then, however, the Ito-solution coincides with the ordinary sample solution and solution in q.m. in the sense of section 5.1 . The theorem of Wong and Zakai is applicable, yielding that here the solution is stable as in this particular situation the coefficient of $d\beta(t)$ is independent of $\xi(t)$. However, we prefered to avoid the involved Ito-calculus and theorem of Wong and Zakai, and to establish the stability directly.

A result, similar to that of (5.2.1) , has been derived in [27]. There an N^{th} order linear differential equation with deterministic coefficients and a stochastic right-hand side is treated along the lines of this chapter.

The stability of the solution $\xi(t)$ in (5.2.1) with respect to perturbations of the other involved data may be shown by methods analogous to those in treatises on ordinary differential equations, see for instance [8].

6 Kalman-Bucy and related estimates, and their behaviour with respect to differentiable perturbations of the involved white noise processes

6.1. Linear minimum variance estimates of Wiener and Kalman-Bucy.

Throughout we shall use the conventions of section 3.1.

(6.1.1) Let be given the N-dimensional linear system of (5.1.1),

(6.1.1a) $$\begin{cases} d\,\xi(t) = A(t)\,\xi(t)dt + d\,\beta(t), & t \in [0,T], \\ \xi(0) = \gamma\,, \end{cases}$$

with the properties i-iv described there.

Then the solution (5.1.2a),

(6.1.1b) $$\xi(t) = F(t)\gamma + F(t)\int_0^t F^{-1}(s)d\beta(s), \quad t \in [0,T],$$

is endowed with all properties listed in (5.1.2).

Given $t \in [0,T]$, $\xi(t)$ is "observed" as the column M-vector $\zeta(s)$

(6.1.1c) $$\begin{cases} \zeta(s) = \eta(s) + \tilde{\beta}(s), \text{ where the column M-vector} \\ \eta(s) = \int_0^s H(u)\,\xi(u)du\,, \\ \text{at all } s \in S_t\,. \end{cases}$$

Here S_t is a subset of $[0,T]$. It may vary with t .

$H(u)$ is an $M \times N$-matrix, whose entries are continuous mappings of $[0,T]$ into $(-\infty,\infty)$. Since $\xi(u)$ is continuous in q.m. on $[0,T]$, $H(u)\,\xi(u)$ is also continuous in q.m. on $[0,T]$, see (2.4.4). And so $\eta(s)$ exists as a Riemann integral in q.m. on account of (2.7.1). By virtue of (2.7.2), $\eta(s)$ is continuously differentiable in q.m. and satisfies

(6.1.1d) $$\frac{d}{ds}\,\eta(s) = H(s)\,\xi(s)\,.$$

Since $E\,\xi(u) = 0$, $u \in [0,T]$, $E\,\eta(s) = 0$, see v in (2.6.2).

Finally, $\widetilde{\beta}(s)$, $s \in [0,T]$, is an M-dimensional Wiener-Lévy process of type (4.4.2). So it may be written as

$$\widetilde{\beta}(s) = \int_0^s \widetilde{G}(u) d\widetilde{\beta}_0(u) ,$$

where $\widetilde{G}(u)$ is endowed with properties similar to those of $G(t)$ in (4.4.2), and where $\widetilde{\beta}_0(u)$, $u \in [0,T]$, is an M-dimensional standard Wiener-Lévy process. It is moreover assumed that at each $u \in [0,T]$, $\widetilde{\beta}_0(u)$ is stochastically independent of γ and $\beta(t)$, $t \in [0,T]$. And hence, since all involved random elements are centered,

$$\widetilde{\beta}_j(s) \perp \gamma \ , \quad \widetilde{\beta}_j(s) \perp \beta_i(t), \quad j=1,.,M, \quad i=1,.,N, \quad s,t \in [0,T],$$

and so

<u>(6.1.1e)</u> $$\widetilde{\beta}_j(s) \perp \xi_i(t), \quad \widetilde{\beta}_j(s) \perp \eta_k(u),$$
$$s,t,u \in [0,T], \quad i=1,.,N, \quad j,k=1,.,M.$$

As we write

$$\widetilde{B}(u) = \widetilde{G}(u)\widetilde{G}^T(u),$$

then according to (4.4.2f),

<u>(6.1.1f)</u> $$\begin{cases} E\,\widetilde{\beta}(s)\widetilde{\beta}^T(t) = \int_0^m \widetilde{B}(u)du , \quad m=\min(s,t) , \quad s,t \in [0,T], \\ \frac{d}{ds} E\,\widetilde{\beta}(s)\widetilde{\beta}^T(s) = \widetilde{B}(s) . \end{cases}$$

Later, an additional condition will be imposed on $\widetilde{B}(s)$.

The problem to be treated is to describe the conditional expectation $\hat{\xi}(t|S_t)$ of $\xi(t)$, given the class of observations

$$C(S_t) = \left\{ \zeta_j(s), \quad j=1,.,M, \quad s \in S_t \right\} .$$

Let H be the Hilbert space, generated by the centered Gaussian system

$$\left\{ \gamma_k,\ k=1,.,N,\ \beta_{oi}(t),\ i=1,.,N,\ t \in [0,T],\ \widetilde{\beta}_{oj}(s),\ j=1,.,M,\ s \in [0,T] \right\}.$$

Then also H is a centered Gaussian system, containing all random elements described above, see (2.3.1). Let

$$H\left[C(S_t)\right]$$

be the closed linear subspace of H , generated by the elements of $C(S_t)$. Then according to (2.3.1),

<u>(6.1.1g)</u> $$\hat{\xi}(t \mid S_t) = \begin{pmatrix} \hat{\xi}_1(t \mid S_t) \\ \vdots \\ \hat{\xi}_N(t \mid S_t) \end{pmatrix}$$

may be seen as the column N-vector, whose components $\hat{\xi}_i(t \mid S_t)$ are the orthogonal projections of $\xi_i(t)$ onto $H[C(S_t)]$, i=1,.,N. Or, the conditional expectation of an element $\xi_i(t)$ coincides a.s. with its linear minimum variance estimate (linear least squares approximation). Ignoring (2.3.1), estimates of the latter type exist uniquely as orthogonal projections onto the closed subspace $H[C(S_t)]$, by virtue of the projection theorem for Hilbert spaces, see [1] e.g.

As $\hat{\xi}_i(t \mid S_t)$ is the orthogonal projection of $\xi_i(t)$ onto $H[C(S_t)]$,

$$\xi_i(t) - \hat{\xi}_i(t \mid S_t)$$

is orthogonal to all elements of $H[C(S_t)]$, or equivalently, to all elements of $C(S_t)$ on account of the continuity of the inner product in H , see (6.1.4). Hence each $\hat{\xi}_i(t \mid S_t)$ is characterized as the unique element of $H[C(S_t)]$ with the property

$$E\,\xi_i(t)\,\zeta_j(s) = E\,\hat{\xi}_i(t \mid S_t)\,\zeta_j(s)\ , \qquad j=1,.,M, \qquad s \in S_t\ .$$

Or, $\hat{\xi}(t \mid S_t)$ is the unique solution in $H^N[C(S_t)]$ to the ~~N-dimensional~~ Wiener-Hopf system

<u>(6.1.1h)</u> $$E\,\xi(t)\zeta^T(s) = E\,\hat{\xi}(t \mid S_t)\zeta^T(s)\ , \qquad s \in S_t\ .$$

Depending on the position of t and S_t in $[0,T]$, $\hat{\xi}(t \mid S_t)$ is a predicted (extrapolated) or interpolated random vector.

If the involved stochastic processes are wide-sense stationary, the estimates are of the type of Wiener and Kolmogorov, see [32].

<u>(6.1.2)</u> If $S_t = [0,t]$, then $\hat{\xi}(t \mid S_t)$ in (6.1.1) , in this case usually denoted as $\hat{\xi}(t \mid t)$, is the Kalman-Bucy filter estimate, see [4], [5], [12], [17] for instance.

In this case an extra condition is imposed on the process $\tilde{\beta}(t)$ in regard to the feasibility of certain computations. This will be discussed in the following sections.

In (6.1.1) all stochastic elements are centered at zero expectation. If

$$\gamma \quad , \quad \beta(t) \quad \text{and} \quad \tilde{\beta}(s)$$

are replaced by

$$c + \gamma \quad , \quad b(t) + \beta(t) \text{ and } \tilde{b}(s) + \tilde{\beta}(s)$$

respectively, with

$$E\,\gamma = 0 \ , \quad E\,\beta(t) = 0 \quad \text{and} \quad E\,\tilde{\beta}(s) = 0 \ ,$$

where $\frac{d}{dt}\, b(t) = a(t)$ is assumed to exist and to be continuous on $[0,T]$, then the solution to the thus modified system (6.1.1a) may be written as

$$x(t) + \xi(t), \quad E\,\xi(t) = 0.$$

Here $x(t)$ is the solution to the deterministic system in (3.3.4) and $\xi(t)$ is the solution to the original system (6.1.1a).

It follows from (6.1.1c) that also the observations $\zeta(s)$ may be splitted up into a deterministic part and a stochastic part with zero expectation. The deterministic part can be determined and the stochastic part, being of type (6.1.1c), may be treated as in (6.1.1).

So it is not a restriction to assume that the stochastic elements, figuring in (6.1.1) have zero expectation.

Since the estimation of $\xi(t)$ in (6.1.1) is exclusively a mathematical operation in the Hilbert space H, the sample behaviour of the stochastic processes is not relevant.

In later sections we shall need several properties of orthogonal projectors of H onto closed linear subspaces $H[C(S)]$, generated by the elements of a class $C(S) = \{\zeta_j(s), \ j=1,..,M, \ s \in S\}$.

If $\mathcal{O}$ is an operator, $R(\mathcal{O})$ will stand for its range. If Σ is a subset of H, $\Sigma^{\perp}$ will stand for its orthogonal complement in H.

We recall, see [1],

<u>(6.1.3)</u> Given the closed linear subspace $H[C(S)]$, there is a unique orthogonal projector $\mathcal{P}$ defined on H, with range $H[C(S)]$. If $\mathcal{J}$ stands for the identity operator on H, also $\mathcal{J} - \mathcal{P}$ is an orthogonal projector with domain H. The ranges of $\mathcal{P}$ and $\mathcal{J} - \mathcal{P}$ are each others orthogonal complement in H. They are closed.

<u>(6.1.4)</u> If S is any subset of $[0,T]$, $H[C(S)]^{\perp} = C(S)^{\perp}$.

Proof: i) Since $C(S) \subset H[C(S)]$, $C(S)^{\perp} \supset H[C(S)]^{\perp}$.

ii) Assume $\xi \perp C(S)$. If $\zeta \in H[C(S)]$, it may be represented as

<u>(6.1.4a)</u> $$\zeta = \sum_{j=1}^{M} \sum_{k=1}^{K} a_{jk}\, \zeta_j(s_k), \quad s_k \in S, \quad a_{jk} \text{ real},$$

or as

<u>(6.1.4b)</u> $$\zeta = \lim_{n \to \infty} \text{ in q.m. } \zeta_n, \quad \zeta_n \text{ being of type (6.1.4a).}$$

Hence, if ζ is represented as in (6.1.4a), $E\xi\zeta = 0$. If ζ is represented as in (6.1.4b), $E\xi\zeta_n = 0$ for all n. So, by virtue of the continuity of the inner product, see (2.1.3),

$$0 = E\xi\zeta_n \to E\xi\zeta = 0 \quad \text{as } n \to \infty .$$

This means $\xi \in H[C(S)]^{\perp}$ and hence $C(S)^{\perp} \subset H[C(S)]^{\perp}$.

In the next assertion, the continuity in q.m. of $\zeta_j(s)$, $j=1,..,M$, on $[0,T]$ is essentially needed.

<u>(6.1.5)</u> If $S_o \subset S \subset \overline{S} \subset [0,T]$, where S_o is dense in S and $\overline{S}$ is the closure of S in $[0,T]$, then

$$H[C(S_o)] = H[C(\overline{S})] .$$

Proof: i) Since $S_o \subset \overline{S}$, $H[C(S_o)] \subset H[C(\overline{S})]$.

ii) It remains to show $H[C(\overline{S})] \subset H[C(S_o)]$. The elements of $H[C(\overline{S})]$ may be represented as

<u>(6.1.5a)</u> $$\zeta = \sum_{j=1}^{M} \sum_{k=1}^{K} a_{jk}\, \zeta_j(s_k), \quad s_k \in \overline{S}, \quad a_{jk} \text{ real},$$

or as

<u>(6.1.5b)</u> $$\zeta = \lim_{n \to \infty} \text{ in q.m. } \zeta_n, \quad \zeta_n \text{ being of type (6.1.5a).}$$

To each $s_k \in \overline{S}$, there is a sequence $\{ s_{km}, m=1,2,.. \}$ in S_o, tending to s_k as $m \to \infty$. By virtue of the continuity in q.m. on $[0,T]$ of $\zeta_j(s)$, $j=1,..,M$, $\zeta_j(s_{km}) \to \zeta_j(s_k)$ in q.m. as $m \to \infty$. Hence ζ in (6.1.5a) is the limit in q.m. of some sequence in $H[C(S_o)]$ and so it belongs to $H[C(S_o)]$. The same is true of the elements ζ_n in (6.1.5b) and hence also ζ in (6.1.5b) belongs to $H[C(S_o)]$, showing the assertion.

The assertions in (6.1.4) and (6.1.5) together give

(6.1.6) If $S_o \subset S \subset \bar{S} \subset [0,T]$, where S_o is dense in S and $\bar{S}$ the closure of S in $[0,T]$, then

$$C(S_o)^{\perp} = H[C(S_o)]^{\perp} = H[C(\bar{S})]^{\perp},$$

owing to the continuity in q.m. of $\zeta_j(s)$ on $[0,T]$, $j=1,.,M$.

If $S_n \subset [0,T]$, $n=1,2,..$, let $H[C(S_n)]$ be the closed linear subspace of H, generated by the elements of $C(S_n) = \{\zeta_j(s),\ j=1,.,M,\ s \in S_n\}$ and let $\mathcal{P}_n$ denote the orthogonal projector of H onto $H[C(S_n)]$. Let $\mathcal{P}_m (\leqq) \mathcal{P}_n$ mean $E\xi\,\mathcal{P}_m\xi\ (\leqq)\ E\xi\,\mathcal{P}_m\xi$ for all $\xi \in H$.

We recall, see [1]:

(6.1.7) a) The following three statements are equivalent:

i) $\mathcal{P}_m (\leq) \mathcal{P}_n$,

ii) $\|\mathcal{P}_m\xi\| (\leqq) \|\mathcal{P}_n\xi\|$ for all $\xi \in H$,

iii) $H[C(S_m)] \subset H[C(S_n)]$.

b) The above assertions are implied by $S_m \subset S_n$.

c) If $\{\mathcal{P}_n,\ n=1,2,..\}$ is an increasing, or a decreasing sequence of orthogonal projectors of H, there is a unique orthogonal projector $\mathcal{P}$ of H, such that $\mathcal{P}_n \uparrow \mathcal{P}$, or $\mathcal{P}_n \downarrow \mathcal{P}$ respectively, in the sense that

$$\|\mathcal{P}_n\xi - \mathcal{P}\xi\| \to 0$$

and

$$\|\mathcal{P}_n\xi\| \uparrow \|\mathcal{P}\xi\| \text{ , or } \|\mathcal{P}_n\xi\| \downarrow \|\mathcal{P}\xi\| \text{ respectively}$$

as $n \to \infty$, for all $\xi \in H$.

(6.1.8) Let $S_o \subset S \subset [0,T]$, where S_o is dense in S, and let $\{S_n,\ n=1,2,..\}$ be a sequence of subsets of $[0,T]$. If

$$S_n \uparrow S_o, \text{ or } S_n \downarrow S_o \text{ as } n \to \infty,$$

then

$$\mathcal{P}_n \uparrow \mathcal{P}, \text{ or } \mathcal{P}_n \downarrow \mathcal{P} \text{ respectively,}$$

where $\mathcal{P}$ is the unique orthogonal projector of H onto $H[C(S)]$.
We recall that $\zeta_j(s)$ in $C(S)$ is continuous in q.m. on $[0,T]$, $j=1,.,M$.

Proof: Because of (6.1.3), (6.1.5) and (6.1.7) it is sufficient to establish that the range $R(\mathcal{P})$ of $\mathcal{P}$ coincides with $H[C(S)]$.

i) Assume $S_n \uparrow S_o$. Then $H[C(S_n)] \uparrow$ ~~$H[C(S)]$~~ and $\mathcal{P}_n \uparrow \mathcal{P}$. Since $\mathcal{P} \geqq \mathcal{P}_n$ for all n , $R(\mathcal{P}) \supset H[C(S_n)]$ for all n , see (6.1.7), and hence , as $R(\mathcal{P})$ is closed,

$$R(\mathcal{P}) \supset H[C(S)] .$$

If $\varphi \in R(\mathcal{P})$, there is a $\xi \in H$, such that

$$\varphi = \mathcal{P}\xi = \lim_{n \to \infty} \text{ in q.m. } \mathcal{P}_n \xi .$$

Since $\mathcal{P}_n \xi \in H[C(S_n)] \subset H[C(S)]$ for all n , $\mathcal{P}\xi \in H[C(S)]$ as $H[C(S)]$ is closed. And so

$$R(\mathcal{P}) \subset H[C(S)] .$$

ii) Assume $S_n \downarrow S_o$. Then $H[C(S_n)] \downarrow H[C(S)]$ and $\mathcal{P}_n \downarrow \mathcal{P}$. Since $\mathcal{P} \leqq \mathcal{P}_n$ for all n , $R(\mathcal{P}) \subset H[C(S_n)]$ for all n , see (6.1.7), and hence

$$R(\mathcal{P}) \subset H[C(S)] .$$

If $\varphi \in H[C(S)]$, $\varphi \in H[C(S_n)]$ for all n . This means

$$E(\xi - \mathcal{P}_n \xi)\varphi = 0$$

for all n and for all $\xi \in H$. Since

$$\mathcal{P}_n \xi \to \mathcal{P}\xi \quad \text{in q.m. as } n \to \infty ,$$

it follows on account of the continuity of the inner product in H , see (2.1.3) that $E(\xi - \mathcal{P}\xi)\varphi = 0$ for all $\xi \in H$. So $\varphi \in R(\mathcal{P})$ on account of (6.1.3), and hence

$$R(\mathcal{P}) \supset H[C(S)] .$$

6.2. The integral representation of the Kalman-Bucy estimate. A generalization of the Riemann-Stieltjes integral in q.m. A generalization of a theorem of Karhunen.

(6.2.1) Concerning the Kalman-Bucy estimate (6.1.2), the Wiener-Hopf system (6.1.1h) may be written as

(6.2.1a) $$E\, \xi(t)\zeta^T(s) = E\, \hat{\xi}(t \mid t)\zeta^T(s) , \qquad s \in [0,t] .$$

From now on it is moreover assumed that

(6.2.1b) $\tilde{B}(s) > 0, \quad s \in [0,T]$.

Then the unique solution in $H^N[C([0,t])]$ to (6.2.1a) can be specified, owing to the remarkable circumstance that under condition (6.2.1b) the elements of $H^N[C([0,t])]$ may uniquely be represented as integrals of a type, to be described below, cf. [23].

Let us first consider the symmetric $M \times M$-matrix $\tilde{B}(s) = \tilde{G}(s)\tilde{G}^T(s)$. On account of (4.4.2), the elements $\tilde{g}_{jk}(s)$, $j,k=1,.,M$, of $\tilde{G}(s)$ are continuously differentiable mappings of $[0,T]$ into $(-\infty,\infty)$. Here however, it is already sufficient that the $\tilde{g}_{jk}(s)$ are continuous on $[0,T]$, as we are not interested in sample calculus, cf. (4.4.2c). Anyhow, it follows that the elements $\tilde{b}_{jk}(s)$, $j,k=1,.,M$, of $\tilde{B}(s)$ are continuous mappings of $[0,T]$ into $(-\infty,\infty)$.

If A_1 and A_2 are $M\times M$-matrices, we recall that by definition

$$A_1 \,(\geqq)\, A_2 \quad \text{iff} \quad x^T(A_1-A_2)x \,(\geqq)\, 0 \quad \text{for all} \quad x \in X ,$$

where X is the sphere of column M-vectors x with real valued components such that $x^T x = 1$. Then

$$x^T\tilde{B}(s)x = x^T\tilde{G}(s)\tilde{G}^T(s)x \geqq 0 \text{ , i.e. } \tilde{B}(s) \geqq 0$$

and hence (6.2.1b) is a restriction.

(6.2.1c) In order to avoid certain difficulties, also in section 6.4, we shall define $\tilde{B}(s)$ in a slightly different way, cf. (4.4.2).

Let

$$D(s) = \begin{pmatrix} d_{11}(s) & & \\ & \ddots & \\ & & d_{MM}(s) \end{pmatrix} \text{ and hence also } D^{-1}(s) = \begin{pmatrix} 1/d_{11}(s) & & \\ & \ddots & \\ & & 1/d_{MM}(s) \end{pmatrix}$$

be a diagonal $M\times M$-matrix , whose diagonal elements are continuous, positive mappings of $[0,T]$ into $(-\infty,\infty)$. Then there are positive numbers e and e' such that

$$0 < e = \min_{j=1,.,M,\ s\in[0,T]} d_{jj}(s) \leqq \max_{j=1,.,M,\ s\in[0,T]} d_{jj}(s) = e' < \infty .$$

Let $O(s)$, $s \in [0,T]$, be a real, continuous $M\times M$-matrix such that it is orthogonal at each $s \in [0,T]$. Now we define

$$\tilde{B}(s) = O^T(s)D(s)O(s) \text{ , and hence } \tilde{B}^{-1}(s) = O^T(s)D^{-1}(s)O(s).$$

$\tilde{B}(s)$ and $\tilde{B}^{-1}(s)$ are continuous on $[0,T]$, symmetric and satisfy

$$e\, I_M \leqq \tilde{B}(s) \leqq e' I_M \quad , \quad \frac{1}{e'} I_M \leqq \tilde{B}^{-1}(s) \leqq \frac{1}{e} I_M \quad , \quad s \in [0,T] .$$

In the chapters 3 - 5 it looked the best to work with Riemann (-Stieltjes) integrals in q.m., because of the continuity of the functions involved, the need for differentiation with respect to the upper limit of the domain of integration, the sample behaviour and the perturbations.

In this chapter, there will be an essential need for stochastic integrals of other types. For example, in section 6.4 we shall incidentally meet the stochastic integral

$$\int_0^t f(s)\varphi(s)ds ,$$

where $\varphi(s)$ is continuous in q.m. on $[0,t]$, and where $f(s)$ is an $L_2[0,t]$-function. As $f(s)$ is only defined almost everywhere on $[0,t]$, the above integral can in general not be evaluated as a Riemann integral in q.m. Still we may assign a well defined meaning to it. Since $f(s) \in L_2[0,t]$, it may be approximated by the elements of a sequence

$$\{ f_n(s), n=1,2,.. \}$$

of stepfunctions constant on intervals, and converging to $f(s)$ as $n \to \infty$ in the sense of $L_2[0,t]$. It is seen that at each n

$$\int_0^t f_n(s)\varphi(s)ds$$

exists as a Riemann integral in q.m., since $\varphi(s)$ is continuous in q.m. on $[0,t]$. Now we define

$$\int_0^t f(s)\varphi(s)ds = \underset{n\to\infty}{\text{lim in q.m.}} \int_0^t f_n(s)\varphi(s)ds .$$

This limit exists, as is seen with the aid of the convergence in q.m. criterion (2.1.4), since as $m,n \to \infty$,

$$E\int_0^t f_m(u)\varphi(u)du\int_0^t f_n(v)\varphi(v)dv = \int_0^t\int_0^t f_m(u)f_n(v)\ E\varphi(u)\varphi(v)dudv \to$$
$$\int_0^t\int_0^t f(u)f(v)\ E\varphi(u)\varphi(v)dudv .$$

This follows from the convergence of $f_m(u)f_n(v)\ E\varphi(u)\varphi(v)$ to $f(u)f(v)\ E\varphi(u)\varphi(v)$ in $L_2[0,t]^2$ as $m,n \to \infty$. The value of the integral is independent of the sequence $\{ f_n(s) , n=1,2,.. \}$ as it may be similarly shown that $\int_0^t f_n(s)\varphi(s)ds \to 0$ in q.m. as $n \to \infty$, if $f_n(s) \to 0$ in $L_2[0,t]$.

If $f(s)$ is continuous on $[0,t]$, the above integral may be evaluated as a Riemann integral in q.m., and also in the above sense. It is easily shown that the results are identical. Now the following question arises: When may integrals be evaluated by different principles, and

and when do the thus obtained values coincide? Of course, in this form the question is far too general. Below it will be answered in detail in case of

$$\int_0^t f(s)d\,\zeta_j(s) ,$$

where $\zeta_j(s)$, j=1,.,M, is a component of the observation $\zeta(s)$, defined in (6.1.1). We shall need the above integral as a generalization of the Riemann-Stieltjes integral in q.m.

Let $\mathcal{L}_2[0,t]$ be the linear space of Lebesgue-measurable mappings of $[0,t]$ into $(-\infty,\infty)$ such that their squares have a finite Lebesgue integral on $[0,t]$. If $\{0\}$ is the subspace of functions, equal to 0 a.e. in $[0,t]$, then $L_2[0,t]=\mathcal{L}_2[0,t]/\{0\}$ with the usual inner product is a Hilbert space. It is stressed that here we shall deal with $\mathcal{L}_2[0,t]$. Its topology is assumed to be induced by that of $L_2[0,t]$. Hence a sequence $\{f_n(s),\ n=1,2,..\}\subset\mathcal{L}_2[0,t]$ is a Cauchy sequence iff $\int_0^t \{f_m(s)-f_n(s)\}^2 ds\to 0$ as $m,n\to\infty$. It has a limit in $\mathcal{L}_2[0,t]$ which is unique mod $\{0\}$. As only the strong topology of $L_2[0,t]$ will be induced into $\mathcal{L}_2[0,t]$, the addition "strong (ly)" will be omitted in relevant situations.

Let us recall the inequality of Schwarz: If $f_j(s)\in\mathcal{L}_2[0,t]$, j=1,2, then

(6.2.1d) $$\left\{\int_0^t |f_1(s)f_2(s)|\,ds\right\}^2 \leqq \int_0^t f_1^2(s)ds\int_0^t f_2^2(s)ds ,$$

entailing

(6.2.1e) $$\left\{\int_0^t |f_1(s)|\,ds\right\}^2 \leqq t\int_0^t f_1^2(s)ds .$$

Let $\mathcal{J}[0,t]\subset\mathcal{L}_2[0,t]$ be the linear subspace generated by the indicator functions of the intervals of $[0,t]$. We recall that $\mathcal{J}[0,t]$ is dense in $\mathcal{L}_2[0,t]$. If $i(s)\in\mathcal{J}[0,t]$, there is a set of numbers $\{i_k,\ k=1,.,K\}$ and a set $\{t_k,\ k=o,.,K\}\subset[0,t]$, $0=t_o<t_1<..<t_K=t$, such that

$$i(s)=i_k \quad \text{if} \quad s\in(t_{k-1},t_k) ,\ k=1,.,K ,$$

the values $i(t_k)$, k=o,.,K, being immaterial as we shall see. So the elements of $\mathcal{J}[0,t]$ are step functions with a finite set of discontinuities. It is easily seen that

(6.2.1f) $\int_0^t i(s)d\,\zeta_j(s)$ exists as a Riemann-Stieltjes integral in q.m. owing to the continuity in q.m. of $\zeta_j(s)$ on $[0,t]$, and

$$\int_0^t i(s)d\,\zeta_j(s) = \sum_{k=1}^{K} i_k\left\{\zeta_j(t_k)-\zeta_j(t_{k-1})\right\} , \quad j=1,.,M .$$

Let us consider the arbitrary mapping $f(s)$ of $[0,t]$ into $(-\infty,\infty)$ and the partition p of $[0,t]$, consisting of the subdivision points t_k, $k=o,.,K$, such that $0 = t_o < .. < t_K = t$, and the numbers s_k, $k=1,.,K$, such that $s_k \in [t_{k-1},t_k]$, cf. section 2.6. Let P be the set of all partitions of this kind.
Define

(6.2.1g) $\quad f(p,s) = f(s_k)$ if $s \in (t_{k-1},t_k)$, $k=1,.,K$,

the values $f(p,t_k)$, $k=o,.,K$, being immaterial.

Then $f(p,s) \in \mathcal{J}[0,t]$ and $\int_o^t f(p,s)d\zeta_j(s) = \sum_{k=1}^{K} f(s_k)\{\zeta_j(t_k) - \zeta_j(t_{k-1})\}$ according to (6.2.1f). Owing to definition (2.6.1),

(6.2.1h) $\int_o^t f(s)d\zeta_j(s)$ exists as a Riemann-Stieltjes integral in q.m. if and only if

$$\left\{ \int_o^t f(p_n,s)d\zeta_j(s) \;,\; n=1,2,.. \right\}$$

is a Cauchy sequence in H for all sequences $\{p_n \,,\; n=1,2,..\} \subset P$ such that $\Delta(p_n) \to 0$ as $n \to \infty$.

Let us recall the following results of (6.1.1):

i) $\quad \zeta_j(s) = \eta_j(s) + \tilde{\beta}_j(s)$,

ii) $\quad \eta_j(u) \perp \tilde{\beta}_k(v)$,

iii) $\quad \eta_j(s)$ is continuously differentiable in q.m.

Let $i(s)$, $i_1(s)$ and $i_2(s)$ be elements of $\mathcal{J}[0,t]$. Then

$$\int_o^t i_1(s)d\zeta_j(s) = \int_o^t i_1(s)\,\frac{d}{ds}\eta_j(s)ds + \int_o^t i_1(s)d\tilde{\beta}_j(s)$$

on account of (6.2.1f), and

(6.2.1i) $\quad E\int_o^t i_1(u)d\zeta_j(u)\int_o^t i_2(v)d\zeta_k(v) =$

$$\int_o^t\int_o^t i_1(u)i_2(v)\,\frac{\partial^2}{\partial u\partial v}\,E\,\eta_j(u)\,\eta_k(v)dudv + \int_o^t i_1(u)i_2(u)\tilde{b}_{jk}(u)du$$

on account of (2.5.7). The derivation is similar to that in (5.1.2d). The integrals in the right-hand side are ordinary Riemann integrals.

Since $\frac{d}{ds}\eta_j(s)$ is continuous in q.m. on $[0,t]$,

$$\frac{\partial^2}{\partial u\partial v}\,E\,\eta_j(u)\eta_k(v) = E\,\frac{d}{du}\eta_j(u)\,\frac{d}{dv}\eta_k(v)$$

is continuous on $[0,t]^2$ by virtue of the continuity of the inner product in H, see (2.1.3).

And as $\tilde{b}_{jk}(u)$ is continuous on $[0,t]$, there is a constant $A > 0$ such that

$$\left| \frac{\partial^2}{\partial u \partial v} E\,\eta_j(u)\eta_k(v) \right| \leqq A, \qquad (u,v) \in [0,t]^2, \qquad j,k=1,..,M$$
$$\left| \tilde{b}_{jk}(u) \right| \leqq A, \qquad u \in [0,t].$$

Hence, also by virtue of (6.2.1e),

(6.2.1j)
$$\left| \int_0^t\int_0^t i_1(u)i_2(v) \frac{\partial^2}{\partial u \partial v} E\,\eta_j(u)\eta_k(v)dudv \right| \leqq$$
$$A \int_0^t |i_1(u)|\,du \int_0^t |i_2(u)|\,du \leqq A\,t\sqrt{\int_0^t i_1^2(u)du \int_0^t i_2^2(u)du},$$

and on account of (6.2.1d),

(6.2.1k)
$$\left| \int_0^t i_1(u)i_2(u)\tilde{b}_{jk}(u)du \right| \leqq A \int_0^t |i_1(u)i_2(u)|\,du \leqq$$
$$A\sqrt{\int_0^t i_1^2(u)du \int_0^t i_2^2(u)du}.$$

In particular, if $i(s) = i_1(s) = i_2(s)$, (6.2.1i) - (6.2.1k) give

$$E\left\{\int_0^t i(s)d\,\zeta_j(s)\right\}^2 = \int_0^t\int_0^t i(u)i(v) \frac{\partial^2}{\partial u \partial v} E\,\eta_j(u)\eta_j(v)dudv + \int_0^t i^2(s)\tilde{b}_{jj}(s)ds,$$

(6.2.1m)
$$\left\| \int_0^t i(s)d\,\zeta_j(s) \right\|^2 = \left\| \int_0^t i(s)d\,\eta_j(s) \right\|^2 + \left\| \int_0^t i(s)d\tilde{\beta}_j(s) \right\|^2,$$

and a number $A > 0$ such that for $j=1,..,M$,

(6.2.1n)
$$\left\| \int_0^t i(s)d\,\eta_j(s) \right\|^2 = \int_0^t\int_0^t i(u)i(v) \frac{\partial^2}{\partial u \partial v} E\,\eta_j(u)\eta_j(v)dudv \leqq$$
$$A\,t \int_0^t i^2(s)ds$$

and

(6.2.1p)
$$\left\| \int_0^t i(s)d\tilde{\beta}_j(s) \right\|^2 = \int_0^t i^2(s)\tilde{b}_{jj}(s)ds \leqq A \int_0^t i^2(s)ds.$$

Finally, since by assumption $\tilde{B}(s) > 0$ on $[0,t]$, see (6.2.1b), the following assertion is valid, owing to (6.2.1c):

(6.2.1q) There is a number $e > 0$ such that for $j=1,..,M$,
$$0 \leqq e \int_0^t i^2(s)ds \leqq \int_0^t i^2(s)\tilde{b}_{jj}(s)ds = \left\| \int_0^t i(s)d\tilde{\beta}_j(s) \right\|^2.$$

(6.2.2) If $j_n(s) \in \mathcal{J}[0,t]$, $n=1,2,..$, and if $\zeta_j(s)$ is any component of the observation $\zeta(s)$ in (6.1.1), then

$$\left\{ \int_0^t j_n(s) d\, \zeta_j(s), \quad n=1,2,.. \right\} \text{ is a Cauchy sequence in } H$$

if and only if

$$\left\{ j_n(s), \quad n=1,2,.. \right\} \text{ is a Cauchy sequence in } \mathcal{L}_2[0,t].$$

Proof: It is to be shown, if $m,n \to \infty$ that

$$\left\| \int_0^t \left\{ j_m(s)-j_n(s) \right\} d\, \zeta_j(s) \right\|^2 \to 0 \text{ if and only if } \int_0^t \left\{ j_m(s)-j_n(s) \right\}^2 ds \to 0.$$

Or equivalently, since $j_m(s)-j_n(s) \in \mathcal{J}[0,t]$, $m,n=1,2,..$, it is to be shown, if $i_n(s) \in \mathcal{J}[0,t]$, $n=1,2,..$ and as $n \to \infty$ that

$$\left\| \int_0^t i_n(s) d\, \zeta_j(s) \right\|^2 \to 0 \text{ if and only if } \int_0^t i_n^2(s) ds \to 0 .$$

Now on account of (6.2.1m) and (6.2.1q), and as $n \to \infty$,

$$\left\| \int_0^t i_n(s) d\, \zeta_j(s) \right\|^2 \to 0 \text{ implies } \int_0^t i_n^2(s) ds \to 0,$$

and by virtue of (6.2.1n), (6.2.1p) and (6.2.1m), and as $n \to \infty$,

$$\int_0^t i_n^2(s) ds \to 0 \text{ implies } \left\| \int_0^t i_n(s) d\, \zeta_j(s) \right\|^2 \to 0.$$

(6.2.3) If $f(s)$ is an arbitrary mapping of $[0,t]$ into $(-\infty,\infty)$, and if $\zeta_j(s)$ is any component of the observation $\zeta(s)$ in (6.1.1),

$$\int_0^t f(s) d\, \zeta_j(s) \text{ exists as a Riemann-Stieltjes integral in q.m.}$$

if and only if

$$\int_0^t \left\{ f(p_m,s)-f(p_n,s) \right\}^2 ds \to 0 \text{ as } m,n \to \infty$$

for all sequences $\left\{ p_n, \ n=1,2,.. \right\} \subset P$ with $\Delta(p_n) \to 0$ as $n \to \infty$.

Proof: On account of definition (6.2.1g), the functions $f(p_m,s)$ are elements of $\mathcal{J}[0,t]$. Hence the above equivalence follows from the equivalences in (6.2.2) and (6.2.1h).

(6.2.4) Let $f(s)$ be any mapping of $[0,t]$ into $(-\infty,\infty)$, and $\zeta_j(s)$, $j=1,..,M$, a component of the observation $\zeta(s)$ in (6.1.1). Then

$\int_0^t f(s)d\,\zeta_j(s)$ exists as a Riemann-Stieltjes integral in q.m.

if and only if

(6.2.4a) $f(s)$ is bounded and Riemann- integrable on $[0,t]$.

(Or, equivalently, see [21],

$f(s)$ is bounded and continuous a.e. on $[0,t]$).

(6.2.4b) If the above conditions are fulfilled, then $f(s)\in \mathcal{L}_2[0,t]$ and $f(p_n,s)\to f(s)$ in $\mathcal{L}_2[0,t]$ as $n\to\infty$, for any sequence $\{p_n,\ n=1,2,..\}\subset P$ such that $\lim\limits_{n\to\infty}\Delta(p_n)=0$.

Proof: By virtue of (6.2.3) it is sufficient to show that the condition

(6.2.4c) " to any $\varepsilon>0$ exists a $\delta>0$ such that

$$\int_0^t \{f(p_m,s)-f(p_n,s)\}^2 ds<\varepsilon \quad \text{if } \Delta(p_n),\Delta(p_m)<\delta \text{ "}$$

is equivalent to (6.2.4a).

i) Assume that condition (6.2.4c) is satisfied. Then according to (6.2.1e), if $\Delta(p_m)<\delta$ and $\Delta(p_n)<\delta$,

$$\left|\int_0^t \{f(p_m,s)-f(p_n,s)\}\,ds\right| \leqq \int_0^t \left|\,f(p_m,s)-f(p_n,s)\right| ds \leqq$$
$$\sqrt{t\int_0^t \{f(p_m,s)-f(p_n,s)\}^2 ds} \leqq \sqrt{t\,\varepsilon}\ .$$

So, for all sequences $\{p_n,\ n=1,2,..\}\subset P$ with $\Delta(p_n)\to 0$ as $n\to\infty$,

$$\left\{\int_0^t f(p_n,s)ds\ ,\ n=1,2,..\ \right\}$$

is a Cauchy sequence in the real line. And hence, owing to the definition of $f(p_n,s)$ in (6.2.1g),

$$\int_0^t f(s)ds$$

exists as a Riemann integral in the sense of definition (2.6.1), specialized to degenerate random functions.

The above integral of $f(s)$ on $[0,t]$ is a Riemann integral in the sense of the usual definitions in real analysis if moreover $f(t)$ is bounded on $[0,t]$. This however, is also implied by (6.2.4c): Let p_m and p_n be fixed partitions of $[0,t]$ such that $\Delta(p_m) < \delta$, $\Delta(p_n) < \delta$, and let be assumed that $f(s)$ is unbounded on $[0,t]$. Then there is an interval in the partition p_m, say $[t_{k-1}, t_k]$, where $f(s)$ is unbounded. And so there is a sequence

$$\{s_{ki},\ i=1,2,..\} \subset [t_{k-1}, t_k]$$

such that $f(s_{ki}) \to \infty$ as $i \to \infty$. Let us consider the sequence

$$\{p_{mi},\ i=1,2,..\} \subset P,$$

where partition p_{mi} is identical to p_m except for the value $s_k \in [t_{k-1}, t_k]$ in p_m, which is replaced by s_{ki}.

Hence on the one hand $\Delta(p_n) < \delta$, $\Delta(p_{mi}) < \delta$, $i=1,2,..$, whereas on the other hand $\int_0^t \{f(p_{mi},s) - f(p_n,s)\}^2 ds \to \infty$ as $i \to \infty$. This is absurd since we assumed (6.2.4c). And hence $f(s)$ is bounded on $[0,t]$.

ii) Assume that condition (6.2.4a) is satisfied. Then to any $\varepsilon > 0$ there is a $\delta > 0$ such that for all $p \in P$ with $\Delta(p) < \delta$,

$$\left| \int_0^t f(s)ds - \int_0^t f(p,s)ds \right| < \varepsilon .$$

If p is defined by the subdivision points t_k, $k=o,.,K$, such that $0 = t_o < .. < t_K = t$ and by the values $s_k \in [t_{k-1}, t_k]$, set

$$\left.\begin{aligned} \overline{f}(p,s) &= \sup_{u \in [t_{k-1}, t_k]} f(u) \\ \underline{f}(p,s) &= \inf_{u \in [t_{k-1}, t_k]} f(u) \end{aligned}\right\} \quad \text{if } s \in (t_{k-1}, t_k),\ k=1,.,K,$$

the values $\overline{f}(p,t_k)$ and $\underline{f}(p,t_k)$, $k=o,.,K$, being immaterial. Since $f(s)$ is bounded on $[0,t]$, say $|f(s)| \leqq B$,

$$\text{(6.2.4d)} \qquad -B \leqq \underline{f}(p,s) \leqq \begin{matrix} f(p,s) \\ f(s) \end{matrix} \leqq \overline{f}(p,s) \leqq B$$

for all $p \in P$ and $s \in [0,t]$, possibly with the exception of those s, coinciding with the subdivision points of the partition. Then to fixed p, there are partitions p' and p" in P , containing the same set $\{t_k\ ,\ k=0,..,K\}$ as p, but where the values s_k' and s_k'' in the intervals $[t_{k-1}, t_k]$ are respectively chosen such that

$$\begin{aligned} 0 &\leqq \bar{f}(p,s) - f(p',s) < \varepsilon \\ 0 &\leqq f(p'',s) - \underline{f}(p,s) < \varepsilon \end{aligned} \qquad \text{for} \quad s \in [0,t] \setminus \{t_0,\dots,t_K\}.$$

Since $\Delta(p') = \Delta(p'') = \Delta(p) < \delta$,

$$\int_0^t \left| \bar{f}(p,s) - \underline{f}(p,s) \right| ds = \left| \int_0^t \left\{ \bar{f}(p,s) - \underline{f}(p,s) \right\} ds \right| \leqq$$

$$\int_0^t \left\{ \bar{f}(p,s) - f(p',s) \right\} ds + \left| \int_0^t \left\{ f(p',s) - f(s) \right\} ds \right| +$$
$$+ \left| \int_0^t \left\{ f(s) - f(p'',s) \right\} ds \right| + \int_0^t \left\{ f(p'',s) - \underline{f}(p,s) \right\} ds \leqq$$

$$\varepsilon t + \varepsilon + \varepsilon + \varepsilon t = 2(1+t)\varepsilon \ , \quad \text{i.e:}$$

$\left| \bar{f}(p,s) - \underline{f}(p,s) \right| \rightarrow 0$ in measure on $[0,t]$ as $\Delta(p) \rightarrow 0$.

Hence, by virtue of (6.2.4d),

$\left| f(p,s) - f(s) \right| \rightarrow 0$ in measure on $[0,t]$ as $\Delta(p) \rightarrow 0$,

and so

$\left\{ f(p,s) - f(s) \right\}^2 \rightarrow 0$ in measure on $[0,t]$ as $\Delta(p) \rightarrow 0$.

On account of (6.2.4d), $\left\{ f(p,s) - f(s) \right\}^2$ is uniformly bounded on $[0,t]$ by $4B^2$. The dominated convergence theorem of Lebesgue applies, see [15] , and establishes (6.2.4b)

$$\int_0^t \left\{ f(p,s) - f(s) \right\}^2 ds \rightarrow 0 \quad \text{as} \quad \Delta(p) \rightarrow 0 \ ,$$

and therefore (6.2.4c),

completing the proof of (6.2.4).

If $\mathcal{R}[0,t]$ denotes the linear space of (bounded) Riemann-integrable mappings of $[0,t]$ into the real line, it is shown in the above theorem that $\mathcal{R}[0,t]$ coincides with the set of real valued functions $f(s)$, $s \in [0,t]$, to which

$$\int_0^t f(s) d\, \zeta_j(s), \quad j=1,..,M,$$

exists as a Riemann-Stieltjes integral in q.m.

It is shown that $\mathcal{J}[0,t] \subset \mathcal{R}[0,t] \subset \mathcal{L}_2[0,t]$ and we recall that $\mathcal{J}[0,t]$ is dense in $\mathcal{L}_2[0,t]$. $\int_0^t f(s)d\,\zeta_j(s)$ may be seen as a mapping $\mathcal{F}_j$ of the linear subspace $\mathcal{J}[0,t]$ of $\mathcal{L}_2[0,t]$ into H . It is seen in iv of (2.6.2) that $\mathcal{F}_j$ is linear. It follows from (6.2.2) that $\mathcal{F}_j$ is continuous on $\mathcal{J}[0,t]$, and that $\mathcal{F}_j$ can be extended from the domain $\mathcal{J}[0,t]$ to $\mathcal{L}_2[0,t]$ as a continuous linear operator, entailing the following generalization of the notion of integral in q.m. with respect to $\zeta_j(s)$:

(6.2.5) Definition: If $f(s) \in \mathcal{L}_2[0,t]$, and if $\zeta_j(s)$ is a component of the observation $\zeta(s)$ in (6.1.1), then

$$\int_0^t f(s)d\,\zeta_j(s)\ , \quad j=1,.,M,$$

is the limit in H of any sequence

(6.2.5a) $$\left\{ \int_0^t i_n(s)d\,\zeta_j(s),\ n=1,2,.. \right\} \subset H$$

of Riemann-Stieltjes integrals, such that

(6.2.5b) $$\left\{ i_n(s),\ n=1,2,.. \right\} \subset \mathcal{J}[0,t]$$

is a Cauchy sequence in $\mathcal{L}_2[0,t]$, tending to $f(s)$ as $n \to \infty$.

(6.2.5c) The above definition is admissible.

Proof: Since $\mathcal{J}[0,t]$ is dense in $\mathcal{L}_2[0,t]$, there are always sequences (6.2.5b) tending to $f(s)$ in $\mathcal{L}_2[0,t]$. According to (6.2.2), the corresponding sequences (6.2.5a) have one and the same limit in the complete space H.

And given $f(s) \in \mathcal{L}_2[0,t]$, both definitions (2.6.1) and (6.2.5) are applicable in order to compute $\int_0^t f(s)d\,\zeta_j(s)$, if and only if $f(s) \in \mathcal{R}[0,t]$. The the integrals of both types coincide.

Since there is no ambiguity when integrating $f(s) \in \mathcal{L}_2[0,t]$ with respect to $\zeta_j(s)$ in the sense of definition (6.2.5) or, if possible, in the sense of definition (2.6.1), no new symbols will be introduced and here the figuring stochastic integrals may simply be refered to as integrals in q.m.

It follows directly from definition (6.2.5) that the assertions of (2.6.2), specialized to $\xi = \zeta_j$ are valid if f and g are elements of $\mathcal{L}_2$, with the exception of the rule of partial integration i in (2.6.2).

(6.2.6) If $f_1(s)$ and $f_2(s)$ are elements of $\mathcal{L}_2[0,t]$, then

(6.2.6a)
$$E \int_0^t f_1(s)d\zeta_j(s) \int_0^t f_2(s)d\zeta_k(s) = \int_0^t\int_0^t f_1(u)f_2(v) \frac{\partial^2}{\partial u \partial v} E\,\eta_j(u)\eta_k(v)dudv + \int_0^t f_1(s)f_2(s)\tilde{b}_{jk}(s)ds,$$

where the integrals in the right-hand side exist as ordinary Lebesgue integrals, $j,k=1,.,M$.

Proof: Let $\{i_{1n}(s),\ n=1,2,..\}$ and $\{i_{2m}(s),\ m=1,2,..\}$ be sequences in $\mathcal{J}[0,t]$, such that

(6.2.6b)
$$\int_0^t \{f_1(s) - i_{1n}(s)\}^2 ds \to 0 \text{ as } n \to \infty ,$$
$$\int_0^t \{f_2(s) - i_{2m}(s)\}^2 ds \to 0 \text{ as } m \to \infty .$$

On account of (6.2.1i), for all $n,m=1,2,..$, and as $j,k=1,.,M$,

(6.2.6c)
$$E \int_0^t i_{1n}(s)d\zeta_j(s) \int_0^t i_{2m}(s)d\zeta_k(s) = \int_0^t\int_0^t i_{1n}(u)i_{2m}(v) \frac{\partial^2}{\partial u \partial v} E\,\eta_j(u)\eta_k(v)dudv + \int_0^t i_{1n}(s)i_{2m}(s)\tilde{b}_{jk}(s)ds.$$

According to (6.2.2),

$$\left\{ \int_0^t i_{1n}(s)d\zeta_j(s),\ n=1,2,.. \right\} \text{ and } \left\{ \int_0^t i_{2m}(s)d\zeta_k(s),\ m=1,2,.. \right\}$$

are Cauchy sequences in H , converging as $n,m \to \infty$ to

$$\int_0^t f_1(s)d\zeta_j(s) \quad \text{and} \quad \int_0^t f_2(s)d\zeta_k(s)$$

respectively on account of definition (6.2.5). Owing to the continuity of the inner product in H, see (2.1.3), the left-hand side of (6.2.6c) tends to the left-hand side of (6.2.6a) as $n,m \to \infty$.

In the right-hand side of (6.2.6a) the integrands are Lebesgue-measurable on $[0,t]^2$ and $[0,t]$ respectively, since $f_1(s)$ and $f_2(s)$ belong to $\mathcal{L}_2[0,t]$, $\frac{\partial^2}{\partial u \partial v} E\,\eta_j(u)\eta_k(v)$ is continuous on $[0,t]^2$ and $\tilde{b}_{jk}(s)$ is continuous on $[0,t]$. The latter two functions are bounded on their respective domains by

$A > 0$, $j,k=1,.,M$, see (6.2.1i). Hence the integrals below exist as ordinary Lebesgue integrals. They satisfy

$$\int_o^t\int_o^t \left| \frac{\partial^2}{\partial u \partial v} E\, \eta_j(u)\, \eta_k(v) \left\{ f_1(u)f_2(v) - i_{1n}(u)i_{2m}(v) \right\} \right| dudv \leqq$$

$$A \int_o^t\int_o^t \left\{ |f_1(u)| . |f_2(v)-i_{2m}(v)| + |i_{2m}(v)| . |f_1(u)-i_{1n}(u)| \right\} dudv =$$

$$A \int_o^t |f_1(u)| du . \int_o^t |f_2(v)-i_{2m}(v)| dv + A \int_o^t |i_{2m}(v)| dv . \int_o^t |f_1(u)-i_{1n}(u)| du$$

$\rightarrow 0$ as $n,m \rightarrow \infty$ by virtue of (6.2.1e) and (6.2.6b).

Similarly,

$$\int_o^t | \tilde{b}_{jk}(s) \left\{ f_1(s)f_2(s)-i_{1n}(s)i_{2m}(s) \right\} | ds \leqq$$

$$A \int_o^t | f_1(s)f_2(s)-i_{1n}(s)i_{2m}(s) | ds \rightarrow 0 \quad \text{as} \quad n,m \rightarrow \infty .$$

Hence, the right-hand side of (6.2.6c) tends to the right-hand side of (6.2.6a) as $n,m \rightarrow \infty$, and (6.2.6a) is shown to be true.

(6.2.7) It follows from the above result that the identities and inequalities in (6.2.1i) - (6.2.1q) remain valid if the elements $i(s)$, $i_1(s)$ and $i_2(s)$ of $\mathcal{J}[0,t]$ are replaced by arbitrary elements $f(s)$, $f_1(s)$ and $f_2(s)$ respectively of $\mathcal{L}_2[0,t]$, provided that the integrals of the real functions are interpreted as Lebesgue integrals.

Let us again consider the Hilbert space $L_2[0,t] = \mathcal{L}_2[0,t] / \{0\}$. If $\bar{f}(s) \in L_2[0,t]$ and if $f(s) \in \mathcal{L}_2[0,t]$ is a representative of $\bar{f}(s)$, we may unambiguously define

$$\int_o^t \bar{f}(s) d\, \zeta_j(s) = \int_o^t f(s) d\, \zeta_j(s) , \quad j=1,.,M.$$

Then the mapping $\mathcal{F}_j$ induces a continuous linear mapping F_j of $L_2[0,t]$ into H . Owing to (6.2.1q) - essentially to (6.2.1b) - it can be shown that F_j is a (1-1)-correspondence between $L_2[0,t]$ and the Hilbert space, generated by the elements $\zeta_j(s)$, $s \in [0,t]$. Thus F_j is a generalization of the isomorphy between $L_2[0,t]$ and the Hilbert space, generated by the elements $\beta(s)$, $s \in [0,t]$, of the 1-dimensinal Wiener-Lévy process in (4.1.1). This isomorphy was first shown by Karhunen, see [13] . See also [6] . The above property of F_j follows as a special case from a more general theorem below.

(6.2.8) Let us consider the closed linear subspace $H\left[C([0,t])\right]$ of H , generated by the elements of the class

$$C([0,t]) = \left\{ \zeta_j(s), \quad j=1,.,M, \quad s \in [0,t] \right\} .$$

$H\left[C([0,t])\right]$ is a Hilbert space. (And it is separable since $\zeta(s)$ is continuous in q.m. on $[0,t]$). Its elements ζ may be represented as

(6.2.8a) $$\zeta = \sum_{j=1}^{M} \left\{ \sum_{k=1}^{K} i_{jk} \, \zeta_j(s_k) \right\} ,$$

where the coefficients i_{jk} are real numbers and $\{s_1,..,s_K\} \subset [0,t]$, or as

(6.2.8b) $$\zeta = \lim_{n \to \infty} \zeta_n \quad \text{in q.m.} ,$$

where ζ_n is of type (6.2.8a) and $\{\zeta_n, \, n=1,2,.. \}$ is a Cauchy sequence in H .

It may be assumed in (6.2.8a) that $0 < s_1 < .. \; s_K \leqq t$. The value $s=0$ may be omitted since $\zeta(0) = \eta(0) + \tilde{\beta}(0) = 0$, see (6.1.1). At each j , $j=1,.,M$, we define the function $i_j(s) \in \mathcal{J}[0,t]$ as follows :

$$i_j(s) = \begin{cases} \sum_{k=1}^{K} i_{jk} & \text{if } 0 \leqq s < s_1 \\ \sum_{k=2}^{K} i_{jk} & \text{if } s_1 \leqq s < s_2 \\ \ldots\ldots & \ldots\ldots\ldots\ldots\ldots \\ i_{jK} & \text{if } s_{K-1} \leqq s < s_K \\ 0 & \text{if } s_K \leqq s \leqq t \end{cases}$$

Then ζ in (6.2.8a) may be written as

(6.2.8c) $$\zeta = \sum_{j=1}^{M} \int_0^t i_j(s) d\, \zeta_j(s) .$$

Let $\mathcal{J}^M[0,t]$ be the space of column M-vectors $i(s)$ whose components are elements of $\mathcal{J}[0,t]$.

Let $\mathcal{L}_2^M[0,t]$ be the space of column M-vectors $f(s)$ whose components are elements of $\mathcal{L}_2[0,t]$.

Let $\zeta(s)$ be the column M-vector, defined in (6.1.1), with components $\zeta_j(s)$, $j=1,.,M$.

(6.2.9) $$\zeta \in H\left[C([0,t])\right]$$

if and only if ζ may be represented as

(6.2.9a) $$\zeta = \int_0^t f^T(s)d\,\zeta(s)\ , \qquad f(s) \in \mathcal{L}_2^M[0,t]\ .$$

Given $\zeta \in H\left[C([0,t])\right]$, the components $f_j(s)$ of $f(s)$ in (6.2.9a) are unique mod $\{0\}$, i.e. unique as elements of $L_2[0,t]$.

Proof: i) Assume ζ is represented as in (6.2.9a). The integral exists according to (6.2.5), and clearly $\zeta \in H\left[C([0,t])\right]$.

ii) Assume $\zeta \in H\left[C([0,t])\right]$. Then on account of (6.2.8a) and (6.2.8b) there is a Cauchy sequence

$$\left\{ \zeta_n = \int_0^t i_n^T(s)d\,\zeta(s)\ ,\quad i_n(s) \in \mathcal{J}^M[0,t],\quad n=1,2,..\right\}$$

tending in q.m. to ζ as $n \to \infty$. And so

$$\|\zeta_m - \zeta_n\|^2 \to 0 \quad \text{as} \quad m,n \to \infty\ .$$

If $i_{mn}(s)$ is the column M-vector with components $i_{jm}(s) - i_{jn}(s)$, $j=1,.,M$, then by virtue of (6.2.1i) and (6.2.1c), i.e. (6.2.1b),

$$\|\zeta_m - \zeta_n\|^2 = \left\|\int_0^t i_{mn}^T(s)d\,\zeta(s)\right\|^2 = \left\|\int_0^t i_{mn}^T(s)d\,\eta(s) + \int_0^t i_{mn}^T(s)d\tilde{\beta}(s)\right\|^2 =$$

$$\left\|\int_0^t i_{mn}^T(s)d\,\eta(s)\right\|^2 + \left\|\int_0^t i_{mn}^T(s)d\tilde{\beta}(s)\right\|^2 \geqq \left\|\int_0^t i_{mn}^T(s)d\tilde{\beta}(s)\right\|^2 =$$

$$\int_0^t i_{mn}^T(s)\tilde{B}(s)i_{mn}(s)ds \geqq e\int_0^t i_{mn}^T(s)i_{mn}(s)ds =$$

$$e\sum_{j=1}^{M}\int_0^t \left\{i_{jm}(s) - i_{jn}(s)\right\}^2 ds\ , \qquad e > 0\ .$$

Hence, $\|\zeta_m - \zeta_n\|^2 \to 0$ implies that at each j, $j=1,.,M$, $\{ i_{jn}(s)\ ,\ n=1,2,..\}$ is a Cauchy sequence in $\mathcal{L}_2[0,t]$, necessarily with a limit, say $f_j(s) \in \mathcal{L}_2[0,t]$ since $\mathcal{L}_2[0,t]$ is a complete space. Since ζ_n is assumed to converge to ζ, by virtue of (6.2.2) and (6.2.5)

$$\zeta = \lim_{n\to\infty} \text{in q.m.} \int_0^t i_n^T(s)d\,\zeta(s) = \int_0^t f^T(s)d\,\zeta(s)\ ,$$

showing (6.2.9a).

iii) Assume that f(s) in (6.2.9a) is not unique in the stated sense. Then there is an element $g(s) \in \mathcal{L}_2^M[0,t]$ such that

$$\zeta = \int_0^t f^T(s)d\zeta(s) = \int_0^t g^T(s)d\zeta(s) \ .$$

Or, as we set $h(s) = f(s) - g(s)$, then by virtue of (6.2.7) and (6.2.1c), i.e. (6.2.1b),

$$0 = \left\| \int_0^t h^T(s)d\zeta(s) \right\|^2 = \left\| \int_0^t h^T(s)d\eta(s) \right\|^2 + \left\| \int_0^t h^T(s)d\tilde{\beta}(s) \right\|^2 \geqq$$

$$\left\| \int_0^t h^T(s)d\tilde{\beta}(s) \right\|^2 = \int_0^t h^T(s)\tilde{B}(s)h(s)ds \geqq e\int_0^t h^T(s)h(s)ds =$$

$$e\sum_{j=1}^{M}\int_0^t h_j^2(s)ds \ , \qquad e > 0 \ .$$

And hence $h_j(s) = 0$ a.e. on $[0,t]$, i.e. $f_j(s) = g_j(s)$ a.e. on $[0,t]$, $j=1,..,M$.

<u>(6.2.10)</u> Let $L_2^M[0,t]$ be the space of column M-vectors $\bar{f}(s),\bar{g}(s),..$ whose components $\bar{f}_j(s),\bar{g}_j(s),..$, $j=1,..,M$, are elements of $L_2[0,t]$. Under the natural rules of addition and scalar multiplication, at each $t \in T$ $L_2^M[0,t]$ is a linear vector space over the real numbers. Obviously

<u>(6.2.10a)</u> $$(\bar{f}(s),\bar{g}(s))_t = \int_0^t \bar{f}^T(s)\bar{g}(s)ds = \sum_{j=1}^{M}\int_0^t \bar{f}_j(s)\bar{g}_j(s)ds$$

is an inner product in $L_2^M[0,t]$. Then

<u>(6.2.10b)</u> $$\|\bar{f}(s)\|_t = (\bar{f}(s),\bar{f}(s))_t^{\frac{1}{2}}$$

is a norm on $L_2^M[0,t]$. Since the components $\bar{f}_j(s)$ are elements of the complete space $L_2[0,t]$, also $L_2^M[0,t]$ is complete in the sense of the topology induced by the above norm. Hence $L_2^M[0,t]$ is a real Hilbert space .

<u>(6.2.10c)</u> As we define to any $\bar{f} \in L_2^M[0,t]$

$$F\bar{f} = \chi = \int_0^t \bar{f}^T(s)d\zeta(s) \in H\left[C(\,[0,t]\,)\right] \ ,$$

F is a linear (1-1) and bicontinuous mapping of $L_2^M[0,t]$ onto $H[C([0,t])]$.

Proof: F is obviously linear. It is (1-1) by virtue of (6.2.9). It remains to show that it is bicontinuous. We need the identity

$$\left\| \int_0^t \bar{f}^T(s)d\zeta(s) \right\|^2 = \left\| \int_0^t \bar{f}^T(s)d\eta(s) \right\|^2 + \left\| \int_0^t \bar{f}^T(s)d\tilde{\beta}(s) \right\|^2 =$$

$$\int_0^t\int_0^t \bar{f}^T(u)\, E\,\eta(u)\eta^T(v)\, \bar{f}(v)\, dudv + \int_0^t \bar{f}^T(s)\tilde{B}(s)\bar{f}(s)ds \ .$$

Since $E\eta(u)\eta^T(v)$ and $\widetilde{B}(s)$ are continuous on their respective domains, there is a positive constant C such that

$$\left\| \int_0^t \bar{f}^T(s)d\zeta(s) \right\|^2 \leqq C \int_0^t \bar{f}^T(u)du \int_0^t \bar{f}(v)dv + C \int_0^t \bar{f}^T(s)\bar{f}(s)ds \quad .$$

Since $B(s) \geqq e\, I_M$ on $[0,t]$, see (6.2.1c), where e is a positive constant,

$$\left\| \int_0^t \bar{f}^T(s)d\zeta(s) \right\|^2 \geqq e \int_0^t \bar{f}^T(s)\bar{f}(s)ds$$

Let χ_n, n=1,2,.. and χ be elements of $H[C([0,t])]$. By means of the above defined mapping F they correspond (1-1) to elements $\bar{g}_n$, n=1,2,.. and $\bar{g}$ respectively of $L_2^M[0,t]$. It follows from the above inequalities that if $n \to \infty$,

$$\chi_n \to \chi \quad \text{in} \quad H[C([0,t])]$$

if and only if

$$\bar{g}_n \to g \quad \text{in the Hilbert space} \quad L_2^M[0,t] \ ,$$

showing that both F and F^{-1} are continuous.

Thus F generalizes the previously discussed mapping F_j, and both F and F_j generalize the afore-mentioned theorem of Karhunen.

It should be observed, if $\zeta(s)$ were an arbitrary mapping of $[0,t]$ into H^M, nothing could be said about the possibility of representations of the kind (6.2.9a) and their uniqueness. Representation (6.2.9a) is obtained, owing to the structure of the observation $\zeta(s)$ and owing to the property $\widetilde{B}(s) > 0$, $s \in [0,t]$, of $\widetilde{\beta}(s)$.

Let us return to (6.1.2). Since the components of $\hat{\xi}(t \mid t)$ are elements of $H[C([0,t])]$, we obtain by virtue of (6.2.9) and (6.1.1):

(6.2.11) Provided that condition (6.2.1b,c) is satisfied, the Kalman-Bucy estimate may uniquely represented at each $t \in [0,T]$ as

$$\hat{\xi}(t|t) = \int_0^t K(t,u)\,d\zeta(u)$$

where $K(t,u)$ is an $N \times M$-matrix, whose entries $k_{ij}(t,u)$, $i=1,.,N$, $j=1,.,M$, are unique elements of $L_2[0,t]$ as functions of u.

Given $\xi(t)$ and $\zeta(s)$, $s \in [0,t]$, the system

(6.2.11a) $$E\,\xi(t)\zeta^T(s) = E\left\{\int_0^t K(t,u)\,d\zeta(u)\right\}\zeta^T(s), \qquad s \in [0,t],$$

is necessarily solvable with unique solution matrix $K(t,u)$, $u \in [0,t]$, whose entries are elements of $L_2[0,t]$.

Let us observe that we may write

$$\hat{\xi}_i(t|t) = \sum_{j=1}^{M} \int_0^t k_{ij}(t,u)\,d\zeta_j(u), \qquad i=1,.,N,$$

and that (6.2.11a) may be splitted up into N systems of M equations, whereas the i^{th} system may be written as

(6.2.11b) $$\left\{\begin{array}{l} E\,\zeta_1(s)\,\xi_i(t) = \sum_{j=1}^{M} E\,\zeta_1(s)\int_0^t k_{ij}(t,u)\,d\zeta_j(u) \\ \quad\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ E\,\zeta_M(s)\,\xi_i(t) = \sum_{j=1}^{M} E\,\zeta_M(s)\int_0^t k_{ij}(t,u)\,d\zeta_j(u) \end{array}\right. \qquad s \in [0,t].$$

At each i the M elements $k_{ij}(t,u)$, $j=1,.,M$, occur jointly in (6.2.11b). The N systems (6.2.11b), $i=1,.,N$, may be treated separately.

6.3. Further properties of the integral representation of the Kalman-Bucy estimate and its Wiener-Hopf system.

If not explained in this section, the meaning of the symbols used may be found in the sections 6.1 and 6.2.

Let us reconsider (6.2.11a).

(6.3.1) As we define

$$\varrho(s) = H(s)\,\xi(s)\ ,$$

then according to (6.1.1d),

$$\varrho(s) = \frac{d}{ds}\eta(s) \text{ in q.m.} \quad \text{and} \quad \eta(s) = \int_0^s \varrho(v)dv \text{ in q.m.}$$

Since the components of $\xi(t)$ are orthogonal to the components of $\widetilde{\beta}(s)$,

$$E\,\xi(t)\zeta^T(s) = E\,\xi(t)\left\{\eta^T(s) + \widetilde{\beta}^T(s)\right\} = E\,\xi(t)\eta^T(s)\ .$$

And hence, on account of the continuity of the inner product in H , see (2.1.3),

(6.3.1a) $$E\,\xi(t)\zeta^T(s) = E\,\xi(t)\int_0^s \varrho^T(v)dv = \int_0^s E\,\xi(t)\varrho^T(v)dv\ .$$

Since the components of $\widetilde{\beta}(s)$ are orthogonal to the components of $\eta(s)$, and by virtue of (2.1.3) and (6.1.1f),

$$E\,\zeta(u)\zeta^T(s) = E\left\{\eta(u) + \widetilde{\beta}(u)\right\}\left\{\eta^T(s) + \widetilde{\beta}^T(s)\right\} = E\,\eta(u)\eta^T(s) + \int_0^m \widetilde{B}(v)dv,$$

where $m = \min(u,s)$. Hence

$$E\,\zeta(u)\zeta^T(s) = \int_0^u dv\int_0^s dw\ E\,\varrho(v)\varrho^T(w) + \int_0^u \widetilde{B}(v)dv \quad \text{if } 0 \leqq u \leqq s$$

and

$$E\,\zeta(u)\zeta^T(s) = \int_0^u dv\int_0^s dw\ E\,\varrho(v)\varrho^T(w) + \int_0^s \widetilde{B}(v)dv \quad \text{if } s \leqq u \leqq t\ .$$

And so $E\,\zeta(u)\zeta^T(s)$ is differentiable if $u \neq s$ with derivative

(6.3.1b) $$\begin{cases} \frac{\partial}{\partial u} E\,\zeta(u)\zeta^T(s) = \int_0^s E\varrho(u)\varrho^T(w)dw + \widetilde{B}(u) & \text{if } 0 \leqq u < s\ , \\ \frac{\partial}{\partial u} E\,\zeta(u)\zeta^T(s) = \int_0^s E\,\varrho(u)\varrho^T(w)dw & \text{if } s < u \leqq t\ . \end{cases}$$

On account of these formulae $\frac{\partial}{\partial u} E\,\zeta(u)\zeta^T(s)$ is continuous in (u,s) and continuously differentiable in s, if $u \neq s$. If $u \uparrow s$ and $u \downarrow s$, $\frac{\partial}{\partial u} E\,\zeta(u{=}s^-)\zeta^T(s)$ and $\frac{\partial}{\partial u} E\,\zeta(u{=}s^+)\zeta^T(s)$ are continuous in s.

We shall show

(6.3.1c) $$E\left\{\int_0^t K(t,u)d\zeta(u)\right\}\zeta^T(s) = \int_0^t K(t,u)\frac{\partial}{\partial u}\left[E\,\zeta(u)\zeta^T(s)\right]du\ .$$

For that purpose, let $\left\{K_n(t,u)\ ,\ n=1,2,..\right\}$ be a sequence of $N \times M$-matrices whose entries as functions of u are elements of

$\mathcal{J}[0,t]$, such that they tend in $L_2[0,t]$ to the corresponding elements of $K(t,u)$ as $n \to \infty$. Then on account of (2.1.3), (6.3.1b) and (6.2.5),

$$E\left\{\int_0^t K(t,u)d\,\zeta(u)\right\}\zeta^T(s) = E\left\{\underset{n\to\infty}{\text{lim in q.m.}}\left[\int_0^t K_n(t,u)d\,\zeta(u)\right]\zeta^T(s)\right\} =$$

$$\lim_{n\to\infty} E\int_0^t K_n(t,u)d\,\zeta(u)\,\zeta^T(s) = \lim_{n\to\infty}\int_0^t K_n(t,u)\,d_u\left[E\,\zeta(u)\,\zeta^T(s)\right] =$$

$$\lim_{n\to\infty}\int_0^t K_n(t,u)\,\frac{\partial}{\partial u}\left[E\,\zeta(u)\,\zeta^T(s)\right]du = \int_0^t K(t,u)\,\frac{\partial}{\partial u}\left[E\,\zeta(u)\,\zeta^T(s)\right]du\ .$$

The last equality holds since $K_n(t,u)\frac{\partial}{\partial u}\left[E\,\zeta(u)\zeta^T(s)\right]$ tends to $K(t,u)\frac{\partial}{\partial u}\left[E\,\zeta(u)\zeta^T(s)\right]$ in $L_2[0,t]$, owing to the convergence of $K_n(t,u)$ as $n \to \infty$ and owing to the behaviour of $\frac{\partial}{\partial u}\left[E\,\zeta(u)\zeta^T(s)\right]$ as a function of u , see (6.3.1b).

Substitution of (6.3.1a) and (6.3.1c) into (6.2.11a), utilizing the results in (6.3.1b) , yields

$$\int_0^s E\,\xi(t)\varrho^T(v)dv = \int_0^t K(t,u)\int_0^s E\,\varrho(u)\varrho^T(w)dw\,du + \int_0^s K(t,u)\widetilde{B}(u)du\ .$$

Since the entries of $K(t,u)$ as functions of u belong to $L_2[0,t]$, and since $E\varrho(u)\varrho^T(w)$ is continuous in (u,w) , the elements of the matrix $K(t,u)E\varrho(u)\varrho^T(w)$ are Lebesgue-measurable with finite Lebesgue integral on $[0,t]\times[0,s]$. Hence in the above expression the order of integration may be changed, yielding

$$\int_0^s du\left\{\int_0^t K(t,w)E\,\varrho(w)\varrho^T(u)dw + K(t,u)\widetilde{B}(u) - E\,\xi(t)\varrho^T(u)\right\} = 0$$

at all $s \in [0,t]$.

Or equivalently,

$$\text{(6.3.1d)}\quad\begin{cases}\int_0^t K(t,w)E\,\varrho(w)\varrho^T(u)dw + K(t,u)\widetilde{B}(u) - E\,\xi(t)\varrho^T(u) = 0\\ \text{at } u \in [0,t]\text{ , possibly with exception of a set of}\\ \text{Lebesgue measure } 0\ .\end{cases}$$

By virtue of (6.2.11), at each fixed $t \in [0,T]$ there is one and only one matrix $K(t,w)$ satisfying (6.3.1d). As functions of w , the entries of $K(t,w)$ are elements of $L_2[0,t]$.

(6.3.1e) If the system in (6.3.1d) is multiplied on the right by $B^{-1}(u)$, the separate equations become

$$k_{ij}(t,u) = \sum_{k=1}^{M} E\,\xi_i(t)\,\varrho_k(u)\tilde{b}_{kj}^{-1}(u) - \sum_{h,k=1}^{M} \int_0^t k_{ih}(t,w)E\,\varrho_h(w)\,\varrho_k(u)\tilde{b}_{kj}^{-1}(u)dw\ ,$$

$i=1,.,N,\quad j=1,.,M$.

Since the functions $E\,\xi_i(t)\,\varrho_k(u)\tilde{b}_{kj}^{-1}(u)$ and $E\,\varrho_h(w)\,\varrho_k(u)\tilde{b}_{kj}^{-1}(u)$ are continuous in $u \in [0,t]$, and on account of the inequality of Schwarz,

$$\left|k_{ij}(t,u) - k_{ij}(t,v)\right| \leqq \sum_{k=1}^{M} \left|E\,\xi_i(t)\left\{\varrho_k(u)\tilde{b}_{kj}^{-1}(u) - \varrho_k(v)\tilde{b}_{kj}^{-1}(v)\right\}\right| +$$

$$\sum_{h,k=1}^{M} \int_0^t \left|k_{ih}(t,w)E\,\varrho_h(w)\left\{\varrho_k(u)\tilde{b}_{kj}^{-1}(u) - \varrho_k(v)\tilde{b}_{kj}^{-1}(v)\right\}\right| dw \rightarrow 0 \quad \text{as} \quad v \rightarrow u\ .$$

Hence the elements of $K(t,u)$ are continuous functions of $u \in [0,t]$ and the equality in (6.3.1d) holds at all $u \in [0,t]$.

Combining the above results with (6.1.11) we obtain:

(6.3.2) Provided that condition (6.2.1b,c) is satisfied, the Kalman-Bucy estimate may uniquely be represented at each $t \in [0,T]$ as

$$\hat{\xi}(t\,|\,t) = \int_0^t K(t,s)d\,\zeta(s)\ .$$

$K(t,s)$ is continuous as a function of $s \in [0,t]$. It is necessarily the unique solution to the $N \times M$-system

(6.3.2a) $$K(t,s)\tilde{B}(s) + \int_0^t K(t,u)E\,\varrho(u)\varrho^T(s)du = E\,\xi(t)\varrho^T(s)\ ,\quad s \in [0,t].$$

(6.3.2b) The elements of the kernel, i.e. the elements of the $M \times M$-matrix

$$E\,\varrho(u)\varrho^T(s)$$

are continuous functions of $(u,s) \in [0,t]^2$ by virtue of the continuity of the inner product, see (2.1.3), since $\varrho(s)$ is continuous in q.m. on $[0,t]$, see (6.3.1).

(6.3.2c) Let us consider the inhomogeneous part of (6.3.2a), i.e. the $N \times M$-matrix

$$E\,\xi(t)\varrho^T(s)\ ,\quad \text{where} \quad 0 \leqq s \leqq t \leqq T\ .$$

Since $\varrho(s) = H(s)\,\xi(s)$, see (6.3.1), it follows on account of (5.1.2d) that

$$E\,\xi(t)\,\varrho^T(s) = E\,\xi(t)\,\xi^T(s)H^T(s) =$$
$$F(t)\left\{E\gamma\gamma^T + \int_0^s F^{-1}(u)B(u)\left[F^{-1}(u)\right]^T du\right\} F^T(s)H^T(s) \ ,$$

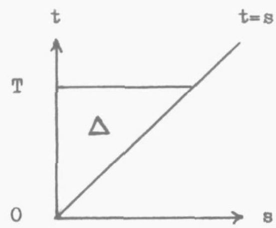
where $F(t)$ is the fundamental matrix, associated with the matrix $A(t)$, see (3.2.3). Since $F(t)$ is continuously differentiable with derivative

$$\frac{d}{dt} F(t) = A(t)F(t) \ ,$$

it follows that $E\,\xi(t)\varrho^T(s)$ is differentiable with respect to t if $0 \leqq s \leqq t \leqq T$, with partial derivative

$$\frac{\partial}{\partial t} E\,\xi(t)\varrho^T(s) = A(t)E\,\xi(t)\varrho^T(s) \ ,$$

whereas the above formula represents the derivative from the right if $s=t$, and from the left if $t=T$.



Let the compact set Δ be defined by

$$0 \leqq s \leqq t \ , \quad t \in [0,T] \ .$$

Since $\xi(t)$ and $\varrho(s)$ are continuous in q.m., it follows from the continuity of the inner product, see (2.1.3), that $E\,\xi(t)\varrho^T(s)$ and $\frac{\partial}{\partial t} E\,\xi(t)\varrho^T(s) = A(t)E\,\xi(t)\varrho^T(s)$ are continuous, and hence uniformly continuous and bounded functions of (t,s) on the compact domain Δ , provided that the value of $\frac{\partial}{\partial t} E\,\xi(t)\varrho^T(s)$ at (T,T) is suitably defined.

Remark: With respect to the above partial derivative, we recall that $\xi(t)$ is not differentiable in q.m., see (5.1.2b).

<u>(6.3.3)</u> Multiplication from the right with $\tilde{B}^{-1}(s)$ and transposition of (6.3.2a) yields

<u>(6.3.3a)</u> $K^T(t,s) + \int_0^t \tilde{B}^{-1}(s)E\varrho(s)\varrho^T(u)K^T(t,u)du = \tilde{B}^{-1}(s)E\varrho(s)\xi^T(t) \ , \ s \in [0,t]$

This $M \times N$-system may be splitted up into N uncoupled systems of M coupled equations

<u>(6.3.3b)</u> $K_i^T(t,s) + \int_0^t \tilde{B}^{-1}(s)E\varrho(s)\varrho^T(u)K_i^T(t,u)du = \tilde{B}^{-1}(s)E\varrho(s)\,\xi_i(t),$
$s \in [0,t] \ , \quad i=1,..,N,$

where $K_i(t,s)$ represents the i^{th} column M-vector of the matrix $K^T(t,s)$ and $\xi_i(t)$ the i^{th} component of the N-vector $\xi(t)$, cf. (6.2.11b).

By virtue of (6.2.11), there is a unique solution $K_i^T(t,s)$ to to each of the M-systems in (6.3.3b).

If t is a "small" fixed value in $[0,T]$, $K_i^T(t,s)$ may be expressed by means of a Neumann expansion, giving some "local" information about $K(t,s)$.

It is of more interest to observe that at fixed $t \in [0,T]$,

$$\int_0^t \tilde{B}^{-1}(s)E\,\varrho(s)\varrho^T(u)K_i^T(t,u)du$$

defines a compact mapping of $L_2^M[0,t]$ into itself. Then, owing to the a priori knowledge of existence and uniqueness of a solution to (6.3.3b), some properties of the "global" solution $K(t,s)$ might be derived, cf. [28], page 281. However, with the aid of these ideas it looks hard to establish all qualities of $K(t,s)$, needed in the Kalman-Bucy filter.

Therefore, in the next section we shall transform (6.3.3b), in order to yield an equivalent system from which all necessary information about $K(t,s)$ can be obtained.

Finally, let us notice that the 1-dimensional version of (6.3.3b) is not an integral equation of Volterra-type, in spite of the variable upper limit t of the domain of integration. At each fixed $t \in [0,T]$, the 1-dimensional version of (6.3.3b) is an integral equation of Fredholm.

6.4. A system of integral equations, related to the Kalman-Bucy estimate.

If not defined in this section, the meaning of the symbols used, may be found in the previous sections. If no confusion may arise, a vector a or matrix A with real valued entries will be called continuous or differentiable e.g., iff all elements enjoy the assigned property. The components of a and the elements of A will be denoted by a_j and a_{jk} respectively.

(6.4.1) Throughout this section, $\tilde{B}(s)$ stands for the matrix, defined in (6.2.1c).

Let $d^{\frac{1}{2}}_{jj}(s)$ represent the positive square root of $d_{jj}(s)$, j=1,.,M, and define the diagonal matrices

$$D^{\frac{1}{2}}(s) = \begin{pmatrix} d^{\frac{1}{2}}_{11}(s) & & \\ & \ddots & \\ & & d^{\frac{1}{2}}_{MM}(s) \end{pmatrix}, \quad D^{-\frac{1}{2}}(s) = \begin{pmatrix} 1/d^{\frac{1}{2}}_{11}(s) & & \\ & \ddots & \\ & & 1/d^{\frac{1}{2}}_{MM}(s) \end{pmatrix}, \quad s \in [0,T],$$

see (6.2.1c). Let $O(s)$ be as in (6.2.1c) and set

$$\tilde{B}^{\frac{1}{2}}(s) = O^{T}(s)D^{\frac{1}{2}}(s)O(s) \quad \text{and} \quad \tilde{B}^{-\frac{1}{2}}(s) = O^{T}(s)D^{-\frac{1}{2}}(s)O(s) .$$

In this way, $\tilde{B}^{\frac{1}{2}}(s)$ and $\tilde{B}^{-\frac{1}{2}}(s)$ are (uniquely) defined continuous, definite positive $M \times M$-matrices, $s \in [0,T]$, symmetric and such that

$$\tilde{B}^{\frac{1}{2}}(s)\tilde{B}^{\frac{1}{2}}(s) = \tilde{B}(s), \quad \tilde{B}^{-\frac{1}{2}}(s)\tilde{B}^{-\frac{1}{2}}(s) = \tilde{B}^{-1}(s) \quad \text{and} \quad \tilde{B}^{\frac{1}{2}}(s)\tilde{B}^{-\frac{1}{2}}(s) = I_M .$$

(6.4.2) Let us multiply the M-dimensional system (6.3.3b) at the left-hand side with $\tilde{B}^{\frac{1}{2}}(s)$. Then we obtain

(6.4.2a) $$\tilde{B}^{\frac{1}{2}}(s)K_i^{T}(t,s) + \int_0^t \left\{ B^{-\frac{1}{2}}(s)E\varrho(s)\varrho^{T}(u)\tilde{B}^{-\frac{1}{2}}(u)\right\}\left\{\tilde{B}^{\frac{1}{2}}(u)K_i^{T}(t,u)\right\}du =$$
$$= \tilde{B}^{-\frac{1}{2}}(s)E\varrho(s)\,\xi_i(t), \quad s \in [0,t], \quad t \in [0,T], \quad i=1,.,N.$$

As we set

(6.4.2b) $\tilde{B}^{\frac{1}{2}}(s)K_i^{T}(t,s) = \tilde{K}_i(s,t)$ - hence $K_i^{T}(t,s) = \tilde{B}^{-\frac{1}{2}}(s)\tilde{K}_i(s,t)$ -

and

(6.4.2c) $$\tilde{B}^{-\frac{1}{2}}(s)E\varrho(s)\varrho^{T}(u)\tilde{B}^{-\frac{1}{2}}(u) = C(s,u) ,$$

then system (6.4.2a) is transformed into

(6.4.2d) $$\tilde{K}_i(s,t) + \int_0^t C(s,u)\tilde{K}_i(u,t)du = \tilde{B}^{-\frac{1}{2}}(s)E\varrho(s)\,\xi_i(t), \quad s \in [0,t],$$
$$t \in [0,T], \quad i=1,.,N .$$

It should be noted that the kernel $C(s,u)$ does not depend on i. At each i system (6.4.2d) corresponds with (6.3.3b) or (6.2.11b).

(6.4.3) We shall investigate the M-dimensional system of integral equations

(6.4.3a) $$\lambda x(s,t) - \int_0^t C(s,u)x(u,t)du = y(s,t) \ , \quad s \in [0,t], \quad t \in [0,T].$$

Here λ is a parameter, $C(s,u)$ is the $M \times M$-matrix in (6.4.2c), $y(s,t)$ is a given element of $L_2^M[0,t]$ and $x(s,t) \in L_2^M[0,t]$ is to be solved.

$L_2^M[0,t]$ is the real Hilbert space, defined in (6.2.10).

As we substitute

$$\lambda = -1 \quad \text{and} \quad y(s,t) = -B^{-\frac{1}{2}}(s)E\varrho(s)\,\xi_i(t)$$

into (6.4.3a), system (6.4.2d) is obtained, with $x(s,t) = \tilde{K}_i(s,t)$.

(6.4.3b) Let us first recall several properties of $L_2^M[0,t]$.

If $u(s,t)$ and $v(s,t)$ are elements of $L_2^M[0,t]$, the inner product is defined as follows:

$$(u(s,t),v(s,t))_t = \int_0^t u^T(s,t)v(s,t)ds = \sum_{j=1}^{M} \int_0^t u_j(s,t)v_j(s,t)ds \ .$$

Hence

$$\|v(s,t)\|_t^2 = \int_0^t v^T(s,t)v(s,t)ds = \sum_{j=1}^{M} \int_0^t v_j^2(s,t)ds \ .$$

The inequality of Schwarz reads

$$\left| \int_0^t u^T(s,t)v(s,t)ds \right| = \left|(u(s,t),v(s,t))_t\right| \leqq \|u(s,t)\|_t \cdot \|v(s,t)\|_t \ .$$

If the components of $u(s,t)$ are essentially bounded in s on $[0,t]$ (e.g. if $u(s,t)$ is continuous in s on $[0,t]$), then there is a finite number $u(t)$ such that

$$\operatorname*{ess\,sup}_{s\in[0,t],\ j=1,..,M} \left|u_j(s,t)\right| = u(t)$$

and hence

$$\|u(s,t)\|_t^2 = \sum_{j=1}^{M} \int_0^t u_j^2(s,t)ds \leqq Mtu^2(t)$$

Then

$$\left| \sum_{j=1}^{M} \int_0^t u_j(s,t)v_j(s,t)ds \right| = \left| \int_0^t u^T(s,t)v(s,t)ds \right| =$$

$$\left|(u(s,t),v(s,t))_t\right| \leqq \|u(s,t)\|_t \cdot \|v(s,t)\|_t \leqq u(t)\sqrt{Mt}\,\|v(s,t)\|_t \ .$$

(6.4.3c) The kernel $C(s,u)$ of (6.4.3a), i.e. the $M\times M$-matrix

$$C(s,u) = B^{-\frac{1}{2}}(s)E\,\varrho(s)\varrho^T(u)B^{-\frac{1}{2}}(u)$$

enjoys the properties

i) $C(s,u)$ is continuous in (s,u) on $[0,T]^2$, owing to (6.3.2b) and (6.4.1). Hence its elements are uniformly continuous on $[0,T]^2$ and bounded. So there is a finite number c such that

$$c = \max_{(s,u)\in[0,T]^2,\ j,k=1,..,M} |c_{jk}(s,u)| .$$

ii) $C^T(s,u) = C(u,s)$.

(6.4.3d) The operation

$$z(s,t) = \int_0^t C(s,u)x(u,t)du$$

defines a mapping $\mathcal{C}_t$ of $L_2^M[0,t]$ into itself with the properties :

i) $\mathcal{C}_t$ is a linear operator .

ii) $\mathcal{C}_t$ is a continuous operator with norm $\|\mathcal{C}_t\|_t$ satisfying

$$\|\mathcal{C}_t\|_t \leqq cMt , \quad t\in[0,T] .$$

For,

$$\mathcal{C}_t\, x(s,t) = z(s,t) = \begin{pmatrix} z_1(s,t) \\ \vdots \\ z_M(s,t) \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^M \int_0^t c_{1k}(s,u)x_k(u,t)du \\ \vdots \\ \sum_{k=1}^M \int_0^t c_{Mk}(s,u)x_k(u,t)du \end{pmatrix}$$

And on account of (6.4.3b) and (6.4.3c),

$$|z_j(s,t)| = \left|\sum_{k=1}^M \int_0^t c_{jk}(s,u)x_k(u,t)du\right| \leqq c\sqrt{Mt}\, \|x(s,t)\|_t , \quad j=1,..,M .$$

Hence

$$\|\mathcal{C}_t\, x(s,t)\|_t^2 = \|z(s,t)\|_t^2 = \sum_{j=1}^M \int_0^t z_j^2(s,t)ds \leqq M\int_0^t c^2Mt\,\|x(s,t)\|_t^2\, ds =$$

$$= c^2M^2t^2\,\|x(s,t)\|_t^2 .$$

iii) $\mathscr{C}_t$ is a compact operator.

Proof: Let

$$\{x(n,s,t),\ n=1,2,..\}$$

be any bounded sequence in $L_2^M[0,t]$. This means, there is a value $a(t)$ such that

$$\|x(n,s,t)\|_t \leqq a(t),\quad n=1,2,.. \quad .$$

By definition, $\mathscr{C}_t$ is compact at $t\in[0,T]$ iff the sequence

(1) $$\{z(n,s,t) = \mathscr{C}_t\, x(n,s,t)\ ,\quad n=1,2,..\}$$

contains a subsequence, converging in $L_2^M[0,t]$.

If $s,s' \in [0,t]$,

$$\left|z_j(n,s,t) - z_j(n,s',t)\right| = \left|\sum_{k=1}^{M} \int_0^t \{c_{jk}(s,u) - c_{jk}(s',u)\}\, x_k(n,u,t)du\right| \ .$$

The elements $c_{jk}(s,u)$ are uniformly continuous on $[0,T]^2$. Hence given $\varepsilon > 0$, there is a number $\delta > 0$ such that uniformly in $u\in[0,T]$

$$\left|c_{jk}(s,u) - c_{jk}(s',u)\right| < \varepsilon \qquad \text{if} \qquad |s-s'| < \delta \quad .$$

Then by virtue of (6.4.3b),

$$\left|\sum_{k=1}^{M} \int_0^t \{c_{jk}(s,u)-c_{jk}(s',u)\}x_k(n,u,t)du\right| \leqq \varepsilon\sqrt{Mt}\ \|x(n,u,t)\|_t \leqq \varepsilon\sqrt{Mt}\ a(t) \ .$$

And hence for all n and $j=1,.,M$, if $|s-s'| < \delta$ then

$$\left|z_j(n,s,t) - z_j(n,s',t)\right| \leqq \varepsilon\sqrt{Mt}\ a(t) \ .$$

This means that at each j the elements of the sequence

(2) $$\{z_j(n,s,t)\ ,\quad n=1,2,..\}$$

constitute an equicontinuous system of functions of s on $[0,t]$.

The elements of (2) are also uniformly bounded, as also on account of (6.4.3b),

$$\left|z_j(n,s,t)\right| = \left|\sum_{k=1}^{M} \int_0^t c_{jk}(s,u)x_k(n,u,t)du\right| \leqq c\sqrt{Mt}\ \|x(n,u,t)\|_t \leqq c\sqrt{Mt}\ a(t) \ .$$

Therefore, owing to Ascoli's theorem, at each j there is a subsequence of (2)

$$\{z_j(n',s,t),\ n' = ..\} \ ,$$

converging uniformly on $[0,t]$ as $n' \to \infty$. And hence, there is also a subsequence of (1) ,

$$\{z(n'',s,t),\ n''= ..\},$$

such that at each j the corresponding sequence of components

$$\{z_j(n'',s,t),\ n''= ..\}$$

converges uniformly on $[0,t]$ as $n'' \to \infty$, say to $z_j(s,t)$.

Then also

$$\|z(n'',s,t) - z(s,t)\|_t^2 = \sum_{j=1}^{M} \int_0^t \{z_j(n'',s,t) - z_j(s,t)\}^2 ds \to 0 \quad \text{as} \quad n \to \infty ,$$

showing the asserted compactness of $\mathcal{C}_t$.

iv) $\mathcal{C}_t$ is a symmetric operator.

For, if $f(s,t)$ and $g(s,t)$ are elements of $L_2^M[0,t]$,

$$(f(s,t),\ \mathcal{C}_t\, g(s,t))_t = \int_0^t f^T(s,t)\left[\int_0^t C(s,u)g(u,t)du\right] ds =$$

$$= \int_0^t \int_0^t f^T(s,t)C(s,u)g(u,t)dsdu,$$

whereas on account of ii in (6.4.3c),

$$(\mathcal{C}_t\, f(s,t),g(s,t))_t = \int_0^t \left[\int_0^t C(s,u)f(u,t)du\right]^T g(s,t)ds =$$

$$= \int_0^t \int_0^t f^T(u,t)C^T(s,u)g(s,t)dsdu = \int_0^t \int_0^t f^T(u,t)C(u,s)g(s,t)dsdu$$

Hence $(f(s,t),\ \mathcal{C}_t\, g(s,t))_t = (\mathcal{C}_t\, f(s,t),g(s,t))_t$.

v) $\mathcal{C}_t$ is a non-negative definite operator.

For, if $f(s,t) \in L_2^M[0,t]$, then

$$(f(s,t),\ \mathcal{C}_t\, f(s,t))_t = \int_0^t \int_0^t f^T(s,t)\tilde{B}^{-\frac{1}{2}}(s)E\varrho(s)\varrho^T(u)\tilde{B}^{-\frac{1}{2}}(u)f(u,t)dsdu .$$

Set

$$\varphi(u,t) = \varrho^T(u)\tilde{B}^{-\frac{1}{2}}(u)f(u,t)$$

Then $\varphi(u,t) = \varphi^T(u,t)$, as it is 1-dimensional. It is the sum of a number of second order random functions of the type, discussed in (6.2.1).

Hence

$$\int_0^t \varphi(u,t)du$$

is well defined as a stochastic integral in q.m. It has a finite second moment and satisfies

$$E\left[\int_0^t \varphi(u,t)du\right]^2 = \int_0^t \int_0^t E\,\varphi(s,t)\,\varphi(u,t)\,dsdu =$$
$$= \int_0^t \int_0^t f^T(s,t)\tilde{B}^{-\frac{1}{2}}(s)E\,\varrho(s)\,\varrho^T(u)\tilde{B}^{-\frac{1}{2}}(u)f(u,t)dsdu,$$

see (6.2.1). So we obtain

$$(f(s,t),\ \mathcal{C}_t\, f(s,t))_t = E\left[\int_0^t \varphi(u,t)du\right]^2 \geqq 0\ .$$

<u>(6.4.4)</u> Let $\mathcal{J}_t$ represent the identity operator in $L_2^M[0,t]$. Then system (6.4.3a) may be written as

<u>(6.4.4a)</u> $$(\lambda\,\mathcal{J}_t - \mathcal{C}_t\,)\ x(s,t) = y(s,t)\ , \qquad t\in[0,T]\ ,$$

where λ is a parameter, $y(s,t)$ a given element of $L_2^M[0,t]$, and where on account of (6.4.3) $\mathcal{C}_t$ is at each $t\in[0,T]$ a linear compact (and continuous) , symmetric and non-negative definite mapping of $L_2^M[0,t]$ into itself. The element $x(s,t)$ is to be solved.

Since at $t\in[0,T]$ $\mathcal{C}_t$ is linear and compact, its spectrum consists - possibly with the exception of the value 0 - of a bounded denumerable set of eigenvalues $\lambda_n(t)$, n=1,2,.. alone, and only the number 0 may be a condensation point. Since $\mathcal{C}_t$ is symmetric, the eigenvalues are real, and since $\mathcal{C}_t$ is non-negative definite, they are non-negative.

Hence $\lambda = -1$ is not in the spectrum of $\mathcal{C}_t$, entailing that at each $t\in[0,T]$, $-\mathcal{J}_t - \mathcal{C}_t$ is invertible, with inverse

<u>(6.4.4b)</u> $$(-\mathcal{J}_t - \mathcal{C}_t\,)^{-1}\ .$$

$(-\mathcal{J}_t - \mathcal{C}_t\,)^{-1}$ is a linear, continuous operator, defined on the whole space $L_2^M[0,t]$, since $\mathcal{C}_t$ is compact and symmetric, see [28] , page 337.

As a consequence, at each $t\in[0,T]$

<u>(6.4.4c)</u> $$(-\mathcal{J}_t - \mathcal{C}_t\,)\ x(s,t) = 0$$

is uniquely solvable in $L_2^M[0,t]$, with solution

$$x(s,t) = 0 \in L_2^M[0,t] ,$$

since $x(s,t) = 0$ satisfies (6.4.4c).

It follows also that at each $t \in [0,T]$, system (6.4.3a) with $\lambda = -1$, and hence system (6.4.2d) has a unique solution. This is in accordance with the result in (6.3.2) or (6.2.11), and with the fact that the Wiener-Hopf equations of the Kalman-Bucy estimate have a unique solution.

Since $\mathcal{C}_t$ is compact, symmetric and non-negative, there is at each $t \in [0,T]$ a maximum eigenvalue, satisfying

$$\max_{n=1,2,..} \lambda_n(t) = \|\mathcal{C}_t\|_t .$$

Hence according to ii in (6.4.3d),

(6.4.4d) $$0 \leqq \max_{n=1,2,..} \lambda_n(t) \leqq cMt , \qquad t \in [0,T] .$$

As we define

(6.4.4e) $$\alpha(\lambda,t) = \sup_{n=1,2,..} \left| \frac{\lambda_n(t)}{\lambda - \lambda_n(t)} \right| , \quad t \in [0,T] , \lambda \neq 0 , \lambda \neq \lambda_n(t), \; n=1,2,..$$

it can be shown that

(6.4.4f) $$\|(\lambda \mathcal{I}_t - \mathcal{C}_t)^{-1}\|_t \leqq \frac{1}{|\lambda|} \left\{ 1 + \alpha(\lambda,t) \right\} ,$$

see [28] page 338.

In the context of the Kalman-Bucy filter, we are only interested in the case that $\lambda = -1$, see (6.4.4b) and (6.4.3a). Then, as all $\lambda_n(t)$ are non-negative, (6.4.4e) shows

$$0 \leqq \alpha(-1,t) \leqq \max_{n=1,2,..} \lambda_n(t)$$

and hence according to (6.4.4d),

$$0 \leqq \alpha(-1,t) \leqq cMt , \qquad t \in [0,T] .$$

Hence, owing to (6.4.4f) with $\lambda = -1$, we obtain the important result

(6.4.4g) $$\|(- \mathcal{I}_t - \mathcal{C}_t)^{-1}\|_t \leqq 1 + cMt , \qquad t \in [0,T] .$$

This means, if t varies in $[0,T]$, then $\|(- \mathcal{I}_t - \mathcal{C}_t)^{-1}\|_t$ is uniformly bounded.

Let us return to (6.4.3a) with $\lambda = -1$. We have shown :

<u>(6.4.5)</u> Let be given the M-dimensional system

<u>(6.4.5a)</u> $$x(s,t) = d(s,t) - \int_0^t C(s,u)x(u,t)du \ , \qquad s \in [0,t] \ ,$$

where $C(s,u)$ is the $M \times M$-matrix in (6.4.3c) and where $d(s,t)$ is any element of $L_2^M[0,t]$.

Then at each $t \in [0,T]$:

i) There is a unique solution $x(s,t) \in L_2^M[0,t]$.

ii) The solution $x(s,t)$ is a linear transformation of $d(s,t)$, and a continuous transformation in the sense of the topology of $L_2^M[0,t]$.

iii) $\|x(s,t)\|_t \leqq (1 + cMt).\|d(s,t)\|_t \ , \qquad t \in [0,T]$.

We shall frequently use the following consequence of i and ii :

iv) If $d(s,t) = 0$, then $x(s,t) = 0$.

And at each fixed $t \in [0,T]$,

$$d(s,t) \to 0 \ \text{ in } \ L_2^M[0,t] \quad \text{entails} \quad x(s,t) \to 0 \ \text{ in } \ L_2^M[0,t] \ .$$

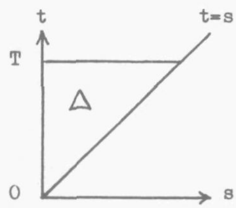
We recall, if $d(s,t) = \tilde{B}^{-\frac{1}{2}}(s)E\varrho(s)\xi_i(t)$, then the unique solution to (6.4.5a) is

$$x(s,t) = \tilde{B}^{\frac{1}{2}}(s)K_i^T(t,s) \ .$$

<u>(6.4.6)</u> Let us recall the compact domain Δ , defined by

$$0 \leqq s \leqq t \ , \qquad t \in [0,T] \ ,$$

see (6.3.2).



In this subsection it is assumed that

<u>(6.4.6a)</u> the components of $d(s,t)$ are bounded on Δ .

Hence there is a finite number d such that

$$d = \sup_{(s,t) \in \Delta \ , \ j=1,..,M} |d_j(s,t)|$$

Then

(6.4.6b) $$\|d(s,t)\|_t^2 = \sum_{j=1}^{M} \int_0^t d_j^2(s,t)ds \leqq d^2Mt\ , \qquad t\in[0,T]\ .$$

Since

$$x(s,t) = d(s,t) - \int_0^t C(s,u)x(u,t)du\ , \qquad s\in[0,t]\ ,$$

then by virtue of (6.4.3b)

$$|x_j(s,t)| \leqq |d_j(s,t)| + \left|\sum_{k=1}^{M}\int_0^t c_{jk}(s,u)x_k(u,t)du\right| \leqq$$
$$\leqq d + c\sqrt{Mt}\,\|x(u,t)\|_t\ .$$

And as on account of (6.4.6b) and of iii in (6.4.5)

$$\|x(u,t)\|_t \leqq (1 + cMt).\|d(s,t)\|_t \leqq (1 + cMt)d\sqrt{Mt}\ , \qquad t\in[0,T]\ ,$$

it follows that

(6.4.6c) $$|x_j(s,t)| \leqq d + dcMt(1 + cMt) \leqq d + dcMT + dc^2M^2T^2\ ,$$
uniformly in $(s,t)\in\Delta$, $j=1,.,M$.

I.e.: If the components of $d(s,t)$ are bounded on Δ, and if $x(s,t)$ is the solution to (6.4.5a), then also the components of $x(s,t)$ are bounded on Δ .

(6.4.7) In iii of (6.4.3d) we established among other things that

$$\int_0^t C(s,u)x(u,t)du$$

is a continuous function of s on $[0,t]$, if $x(u,t)\in L_2^M[0,t]$. Hence, if $d(s,t)$ is continuous in s on $[0,t]$, the solution $x(s,t)$ to (6.4.5a) is also continuous in s on $[0,t]$, cf. (6.3.1e).

However, we need to establish the continuity of $x(s,t)$ in (s,t) on Δ , under the assumption that

(6.4.7a) $d(s,t)$ is continuous in (s,t) on Δ .

Then the components $d_j(s,t)$ are uniformly continuous on Δ , and bounded on Δ , say by d, and (6.4.6c) applies. I.e. :

(6.4.7b) Under the assumption (6.4.7a), the components $x_j(s,t)$ of the solution $x(s,t)$ to (6.4.5a) are bounded on Δ .

In this subsection condition (6.4.7a) is assumed to be fulfilled. Hence (6.4.7b) is valid.

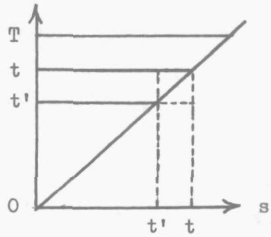
Let t be a fixed value in $[0,T]$.

Assume $0 \leqq t' < t$.

Consider the unique solutions

$x(s,t)$ and $x(s,t')$

to system (6.4.5a) at t and at t' respectively.



Define
$$\underline{x}(s,t') = \begin{cases} x(s,t') & \text{if } 0 \leqq s \leqq t' \\ 0 & \text{if } t' < s \leqq t \end{cases}$$

Then
$$\underline{x}(s,t') = \begin{cases} d(s,t') - \int_0^t C(s,u)\underline{x}(u,t')du & \text{if } 0 \leqq s \leqq t' \\ 0 = \int_0^{t'} C(s,u)x(u,t')du - \int_0^t C(s,u)\underline{x}(u,t')du & \text{if } t' < s \leqq t \end{cases}$$

Since
$$x(s,t) = d(s,t) - \int_0^t C(s,u)x(u,t)du \quad , \qquad s \in [0,t] ,$$

we obtain at each $t \in [0,T]$ the identity

(6.4.7c) $$x(s,t)-\underline{x}(s,t') = u(s,t,t') - \int_0^t C(s,u)\left\{x(u,t)-\underline{x}(u,t')\right\}du, \quad s \in [0,t],$$

where

(6.4.7d) $$u(s,t,t') = \begin{cases} d(s,t)-d(s,t') & \text{if } 0 \leqq s \leqq t' \\ d(s,t)-\int_0^{t'} C(s,u)x(u,t')du & \text{if } t' < s \leqq t \end{cases}$$

Now consider at $t \in [0,T]$ the system

(6.4.7e) $$y(s,t,t') = u(s,t,t') - \int_0^t C(s,u)y(u,t,t')du , \quad s \in [0,t] .$$

As it is of type (6.4.5a), it is uniquely solvable with solution
$$y(s,t,t') = x(s,t) - \underline{x}(s,t') ,$$
according to identity (6.4.7c).

We shall show
$$u(s,t,t') \to 0 \quad \text{in} \quad L_2^M[0,t] \quad \text{as} \quad t' \uparrow t ,$$
or equivalently

(6.4.7f) $$\|u(s,t,t')\|_t^2 \to 0 \quad \text{as} \quad t' \uparrow t .$$

By virtue of (6.4.7d),

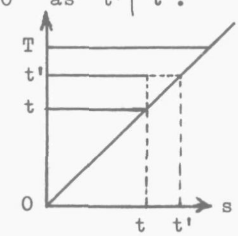
$$\|u(s,t,t')\|_t^2 = \sum_{j=1}^{M} \int_o^t u_j^2(s,t,t')ds =$$

$$\sum_{j=1}^{M} \int_o^{t'} \{d_j(s,t)-d_j(s,t')\}^2 ds + \sum_{j=1}^{M} \int_{t'}^{t} \{d_j(s,t) - \sum_{k=1}^{M} \int_o^{t'} c_{jk}(s,u)x_k(u,t')du\}^2 ds .$$

The first term in the right-hand side tends to 0 as $t' \uparrow t$, because of the uniform continuity of $d_j(s,t)$ in (s,t) on Δ. In the second term, the integrand is bounded on account of the continuity of $d(s,t)$ on Δ, and by virtue of the boundedness on Δ of $c_{jk}(s,u)$ - see (6.4.3c) - and of $x_k(u,t')$ - see (6.4.7b) - . Then also the second term tends to 0, since the measure $t-t'$ of the domain of integration tends to 0 as $t' \uparrow t$.

Hence (6.4.7f) is established. And so, by virtue of (6.4.7e) and of iv in (6.4.5),

<u>(6.4.7g)</u> $\|y(s,t,t')\|_t = \|x(s,t) - \underline{x}(s,t')\|_t \to 0$ as $t' \uparrow t$.



Now we assume $0 \leqq t < t' \leqq T$, t fixed.

We shall similarly show that

<u>(6.4.7h)</u> $\|x(s,t) - x(s,t')\|_t \to 0$ as $t' \downarrow t$.

Instead of identity (6.4.7c) we now obtain

<u>(6.4.7i)</u> $x(s,t')-x(s,t) = v(s,t,t')-\int_o^t C(s,u)\{x(u,t')-x(u,t)\}du, \quad s \in [0,t],$

where

<u>(6.4.7j)</u> $v(s,t,t') = d(s,t')-d(s,t)-\int_t^{t'} C(s,u)x(u,t')du$ at each $s \in [0,t]$.

Then

$$\|v(s,t,t')\|_t^2 = \sum_{j=1}^{M} \int_o^t v_j^2(s,t,t')ds =$$

$$= \sum_{j=1}^{M} \int_o^t \{d_j(s,t') - d_j(s,t) - \sum_{k=1}^{M} \int_t^{t'} c_{jk}(s,u)x_k(u,t')du\}^2 ds .$$

As $t' \downarrow t$, $d_j(s,t') - d_j(s,t) \to 0$, uniformly in $s \in [0,t]$, because of the uniform continuity of $d(s,t)$ on Δ. Since $c_{jk}(s,u)x_k(u,t')$ is bounded on Δ by virtue of (6.4.3c) and of (6.4.7b) respectively, $\int_t^{t'} c_{jk}(s,u)x_k(u,t')du$ tends to 0, uniformly in $s \in [0,t]$ since the measure $t'-t$ of the domain of integration tends to 0 as $t' \downarrow t$.

Hence, according to (6.4.7g,h),

(6.4.7k) Given $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$\| x(s,t) - \underline{x}(s,t') \|_t < \varepsilon \qquad \text{if} \quad |t-t'| < \delta \ .$$

The bar in $\underline{x}(s,t')$ may be omitted if $t' \geqq t$.

(6.4.7m) If the components of the M-vector $d(s,t)$ are continuous functions of (s,t) on Δ, then also the components of the solution $x(s,t)$ to (6.4.5a) are continuous functions of (s,t) on Δ .

Proof : Let (s,t) and (s',t') in Δ, (s,t) fixed. Necessarily $0 \leqq s \leqq t$, and $0 \leqq s' \leqq t'$. We may write

$$x(s,t) - x(s',t') = x(s,t) - \underline{x}(s',t') =$$

$$= d(s,t) - d(s',t') - \left\{ \int_0^t C(s,u)x(u,t)du - \int_0^{t'} C(s',u)\underline{x}(u,t')du \right\} =$$

$$= d(s,t) - d(s',t') - \int_0^t C(s,u) \left\{ x(u,t) - \underline{x}(u,t') \right\} du$$
$$- \int_0^t \left\{ C(s,u) - C(s',u) \right\} \underline{x}(u,t')du + \int_t^{t'} C(s',u)\underline{x}(u,t')du ,$$

where the bar in $\underline{x}(u,t')$ may be omitted if $t' \geqq t$.

Hence

$$\left| x_j(s,t) - x_j(s',t') \right| = \left| x_j(s,t) - \underline{x}_j(s',t') \right| \leqq$$

$$\left| d_j(s,t)-d_j(s',t') \right| + \left| \sum_{k=1}^{M} \int_0^t c_{jk}(s,u) \left\{ x_k(u,t)-\underline{x}_k(u,t') \right\} du \right| +$$
$$+ \sum_{k=1}^{M} \int_0^t \left| c_{jk}(s,u)-c_{jk}(s',u) \right| . \left| \underline{x}_k(u,t') \right| du + \sum_{k=1}^{M} \int_t^{t'} \left| c_{jk}(s',u)\underline{x}_k(u,t') \right| du \ .$$

Let $\varepsilon > 0$.

i) Since $d(s,t)$ is uniformly continuous on Δ, there is a neighbourhood $\mathcal{N}$ of (s,t), such that

$$\left| d_j(s,t) - d_j(s',t') \right| < \varepsilon \qquad \text{if} \quad (s',t') \in \mathcal{N} \cap \Delta \ .$$

ii) On account of (6.4.3b,c) and (6.4.7k), $|t-t'| < \delta$ entails

$$\left| \sum_{j=1}^{M} \int_0^t c_{jk}(s,u) \left\{ x_k(u,t) - \underline{x}_k(u,t') \right\} du \right| < c\sqrt{Mt}\ \varepsilon \ .$$

iii) Since $c_{jk}(s,u)$ is uniformly continuous on Δ , there is a number δ' such that

$$\left|c_{jk}(s,u) - c_{jk}(s',u)\right| < \varepsilon \quad \text{if} \quad |s-s'| < \delta' \ , \quad k=1,.,M \ .$$

According to (6.4.7b), $\underline{x}_k(u,t')$ is uniformly bounded on Δ , say

$$\left|\underline{x}_k(u,t')\right| < x \ , \quad (u,t') \in \Delta \ , \quad k=1,.,M \ .$$

Hence

$$\sum_{k=1}^{M} \int_0^t \left|c_{jk}(s,u)-c_{jk}(s',u)\right| \cdot \left|\underline{x}_k(u,t')\right| du \leqq M \varepsilon x t \quad \text{if} \quad \left|s-s'\right| < \delta' \ .$$

iv) If $t' \leqq t$,

$$\sum_{k=1}^{M} \int_t^{t'} \left|c_{jk}(s',u)\underline{x}_k(u,t')\right| du = 0 \ .$$

And if $t' > t$,

$$\sum_{k=1}^{M} \int_t^{t'} \left|c_{jk}(s',u)\underline{x}_k(u,t')\right| du < Mcx.(t'- t) \ .$$

The asserted continuity of $x(s,t)$ as a vector function of (s,t) on Δ is shown in i - iv .

<u>(6.4.7n)</u> If $d(s,t) = \tilde{B}^{-\frac{1}{2}}(s)E\varrho(s)\xi_i(t)$, $(s,t) \in \Delta$, it satisfies condition (6.4.7a), see (6.3.2c). Then the solution to system (6.4.5a) is

$$x(s,t) = \tilde{B}^{\frac{1}{2}}(s)K_i(t,s)$$

Hence the components of $\tilde{B}^{\frac{1}{2}}(s)K_i^T(t,s)$ and therefore those of $K_i^T(t,s) = \tilde{B}^{-\frac{1}{2}}(s)x(s,t)$, and finally the elements of the Kalman matrix $K(t,s)$ in (6.2.11) are continuous functions of (s,t) on Δ . It is observed that in (6.3.2) only the continuity of the elements of $K(t,s)$ as functions of s was shown.

It is seen that the conditions, imposed on the covariance matrix $\tilde{B}(s)$ of the Wiener-Lévy process in the observations, see (6.2.1c), ~~is~~ are fully exploited.

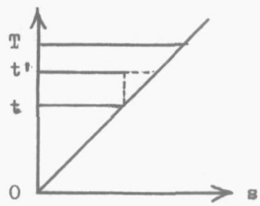
<u>(6.4.8)</u> We have seen in (6.3.2c) that if $d(s,t) = \tilde{B}^{-\frac{1}{2}}(s) E \varrho(s) \xi_i(t)$,

<u>(6.4.8a)</u> $d(s,t)$ is partial differentiable on $\triangle$ with respect to t , and

$$d(s,t) \quad \text{and} \quad \frac{\partial}{\partial t} d(s,t)$$

are continuous vector functions of (s,t) on $\triangle$.

In this subsection we shall show that under condition (6.4.8a), the solution $x(s,t)$ to (6.4.5a) is also partial differentiable with respect to t and that it satisfies an appropriate system of integral equations.



Let condition (6.4.8a) be fulfilled.

Let t be a fixed value in $[0,T)$ and assume

$$0 \leqq t < t' \leqq T .$$

<u>(6.4.8b)</u> According to the mean value theorem of calculus,

$$\frac{d_j(s,t') - d_j(s,t)}{t' - t} = \frac{\partial}{\partial t''(s)} d_j(s,t''(s))$$

where $\qquad 0 < t''(s)-t < t'-t$.

And since on account of (6.4.8a) $\frac{\partial}{\partial t''} d_j(s,t'')$ is uniformly continuous on $\triangle$,

$$\frac{d_j(s,t') - d_j(s,t)}{t' - t} \longrightarrow \frac{\partial}{\partial t} d_j(s,t) \quad \text{as} \quad t' \downarrow t \text{ , uniformly in } s \in [0,t] .$$

Let $x(s,t)$ and $x(s,t')$ be the solutions to (6.4.5a) at t and t' respectively. Then owing to (6.4.7i,j) we obtain the identity

<u>(6.4.8c)</u> $$\frac{x(s,t')-x(s,t)}{t'-t} = w(s,t,t') - \int_0^t C(s,u) \frac{x(u,t')-x(u,t)}{t'-t} du \text{ , } s \in [0,t] ,$$

where

<u>(6.4.8d)</u> $$w(s,t,t') = \frac{d(s,t')-d(s,t)}{t'-t} - \frac{1}{t'-t} \int_t^{t'} C(s,u)x(u,t')du \text{ , } s \in [0,t] .$$

By virtue of (6.4.8a), also condition (6.4.7a) is fulfilled. And so (6.4.7m) applies. Hence, also on account of (6.4.3c), $C(s,u)$, $x(u,t)$ and $C(s,u)x(u,t)$ are continuous on Δ. By virtue of the mean value theorem for integrals of continuous functions,

$$\frac{1}{t'-t}\int_t^{t'} c_{jk}(s,u)x_k(u,t')du = c_{jk}(s,t'')x_k(t'',t) \ , \quad t < t'' < t' \ ,$$

where t'' varies with s. We may write

$$\left| c_{jk}(s,t)x_k(t,t) - \frac{1}{t'-t}\int_t^{t'} c_{jk}(s,u)x_k(u,t')du \right| =$$

$$= \left| c_{jk}(s,t)x_k(t,t) - c_{jk}(s,t'')x_k(t'',t') \right| \leqq$$

$$\leqq \left| c_{jk}(s,t)-c_{jk}(s,t'') \right| \cdot \left| x_k(t,t) \right| + \left| c_{jk}(s,t'') \right| \cdot \left| x_k(t,t)-x_k(t'',t') \right| .$$

As $c_{jk}(s,t)$ and $x_k(t'',t')$ are uniformly continuous on Δ and bounded,

$$\left| c_{jk}(s,t)x_k(t,t) - \frac{1}{t'-t}\int_o^t c_{jk}(s,u)x_k(u,t')du \right| \to 0 \quad \text{as} \quad t' \downarrow t \ ,$$

uniformly in $s \in [0,t]$.

Hence the components of $\frac{1}{t'-t}\int_t^{t'} C(s,u)x(u,t')du$ tend to the corresponding components of $C(s,t)x(t,t)$ as $t' \downarrow t$, uniformly in $s \in [0,t]$.

Owing to the above result and to (6.4.8b),

<u>(6.4.8e)</u> $$w_j(s,t,t') = \frac{d_j(s,t')-d_j(s,t)}{t'-t} - \frac{1}{t'-t}\int_t^{t'} \left\{ \sum_{k=1}^{M} c_{jk}(s,u)x_k(u,t') \right\} du$$

$$\to \frac{\partial}{\partial t} d_j(s,t) - \sum_{k=1}^{M} c_{jk}(s,t)x_k(t,t) \quad \text{as} \quad t' \downarrow t \text{ , uniformly in } s \in [0,t].$$

According to (6.4.8a), (6.4.3c) and (6.4.7m),

<u>(6.4.8f)</u> $\frac{\partial}{\partial t} d(s,t) - C(s,t)x(t,t)$ is continuous in (s,t) on Δ.

Consider the system

<u>(6.4.8g)</u> $$y(s,t) = \left\{ \frac{\partial}{\partial t} d(s,t) - C(s,t)x(t,t) \right\} - \int_o^t C(s,u)y(u,t)du \ , \quad s \in [0,t],$$

where $x(t,t)$ is the value at $s=t$ of the solution $x(s,t)$ to the system

$$x(s,t) = d(s,t) - \int_0^t C(s,u)x(u,t)du \ .$$

As system (6.4.8g) is of type (6.4.5a), it is endowed with a unique solution. Moreover, since according to (6.4.8f) $\frac{\partial}{\partial t} d(s,t) - C(s,t)x(t,t)$ is continuous in (s,t) on $\triangle$, (6.4.7m) applies and hence

(6.4.8h) The solution $y(s,t)$ to system (6.4.8g) is continuous in (s,t) on $\triangle$, provided that condition (6.4.8a) is fulfilled.

Now consider the identity

(6.4.8i) $$\frac{x(s,t') - x(s,t)}{t' - t} - y(s,t) =$$

$$\left\{w(s,t,t') - \frac{\partial}{\partial t} d(s,t) + C(s,t)x(t,t)\right\} - \int_0^t C(s,u)\left\{\frac{x(u,t')-x(u,t)}{t'-t} - y(u,t)\right\}du, \quad s \in [0,t],$$

where $w(s,t,t')$ is defined in (6.4.8d), where $y(s,t)$ is the solution to (6.4.8g) and $x(s,t)$ the solution to (6.4.5a),

and consider the system

(6.4.8j) $$z(s,t,t') = r(s,t,t') - \int_0^t C(s,u)z(u,t,t')du \ , \quad s \in [0,t] \ ,$$

where

(6.4.8k) $$r(s,t,t') = w(s,t,t') - \frac{\partial}{\partial t} d(s,t) + C(s,t)x(t,t) \ , \quad s \in [0,t] \ ,$$

$x(t,t)$ being the solution to (6.4.5a) at $s=t$.

Since (6.4.8j) is of type (6.4.5a), it is endowed with a unique solution. On account of the identity (6.4.8i) this solution is

(6.4.8m) $$z(s,t,t') = \frac{x(s,t') - x(s,t)}{t' - t} - y(s,t) \ ,$$

$x(s,t)$ and $y(s,t)$ being the solutions to (6.4.5a) and (6.4.8g) respectively.

According to (6.4.8e) and (6.4.8k),

$$\| r(s,t,t') \|_t^2 = \sum_{j=1}^{M} \int_0^t r_j^2(s,t,t')ds \to 0 \quad \text{as} \quad t' \downarrow t \ ,$$

entailing, because of (6.4.8j), (6.4.8m) and iv in (6.4.5),

(6.4.8n) $$\| z(s,t,t') \|_t = \left\| \frac{x(s,t')-x(s,t)}{t'-t} - y(s,t) \right\|_t \to 0 \quad \text{as} \quad t' \downarrow t \ .$$

(6.4.8p) I.e.: If condition (6.4.8a) is satisfied, then the solution $x(s,t)$ to

$$x(s,t) = d(s,t) - \int_0^t C(s,u)x(u,t)du \quad , \qquad s \in [0,t] \ ,$$

is differentiable from the right with respect to t in the topology of $L_2^M[0,t]$. The derivative in this sense will be denoted by $D_2^{t+}x(s,t)$. It satisfies

$$D_2^{t+}x(s,t) = y(s,t) \ ,$$

where $y(s,t)$ is the solution to system (6.4.8g). Hence

$$D_2^{t+}x(s,t) = \left\{\frac{\partial}{\partial t} d(s,t) - C(s,t)x(t,t)\right\} - \int_0^t C(s,u)D_2^{t+}x(s,t)du \ , \ s \in [0,t].$$

A similar approach in case that $t' \uparrow t$ looks not feasible in view of (6.4.7d), cf. (6.4.7j).

(6.4.8q) On account of (6.4.8h), $D_2^{t+}x(s,t)$ is continuous in (s,t) on Δ , provided that it is appropriately defined at $t = T$.

Denoting the components of $D_2^{t+}x(s,t)$ by $\left[D_2^{t+}x(s,t)\right]_j$, it follows from (6.4.8i) and (6.4.8k) since $D_2^{t+}x(s,t) = y(s,t)$ that

$$\frac{x_j(s,t') - x_j(s,t)}{t' - t} - \left[D_2^{t+}x(s,t)\right]_j =$$

$$= r_j(s,t,t') - \int_0^t \sum_{k=1}^{M} c_{jk}(s,u) \left\{ \frac{x_k(u,t') - x_k(u,t)}{t' - t} - \left[D_2^{t+}x(u,t)\right]_k \right\} du, \ s \in [0,t].$$

Hence on account of (6.4.8j,k), (6.4.8e), (6.4.8n) where $y(s,t) = D_2^{t+}x(s,t)$, and (6.4.3b),

$$\frac{x_j(s,t') - x_j(s,t)}{t' - t} - \left[D_2^{t+}x(s,t)\right]_j \to 0 \quad \text{as} \quad t' \downarrow t \ , \quad s \in [0,t] \ , \ t \in [0,T],$$

i.e. $x(s,t)$ is partial differentiable from the right on Δ with respect to t in the ordinary sense. Its right-hand side partial derivative is denoted by $D^{t+}x(s,t)$. It satisfies

$$D^{t+}x(s,t) = D_2^{t+}x(s,t) \ ,$$

and hence it is continuous in (s,t) on Δ on account of (6.4.8q).

Among other things, we have established that under condition (6.4.8a) the M-vector $x(s,t)$ is continuous in (s,t) on $\triangle$, and partial differentiable from the right with respect to t, and that also the derivative $D^{t+}x(s,t)$ is continuous in (s,t) on $\triangle$. Hence, owing to lemma (6.4.8t) below,

> $x(s,t)$ is partial differentiable with respect to t from both sides in the ordinary sense and
>
> $$\frac{\partial}{\partial t}x(s,t) = D^{t+}x(s,t) = D_2^{t+}x(s,t) \quad , \quad (s,t) \in \triangle .$$

Hence, according to (6.4.8p) :

<u>(6.4.8r)</u> If condition (6.4.8a) is satisfied, the solution $x(s,t)$ to

$$x(s,t) = d(s,t) - \int_0^t C(s,u)x(u,t)du \quad , \quad s \in [0,t] \; , \quad t \in [0,T] \; ,$$

is partial differentiable with respect to t and $\frac{\partial}{\partial t}x(s,t)$ satisfies

$$\frac{\partial}{\partial t}x(s,t) = \frac{\partial}{\partial t}d(s,t) - C(s,t)x(t,t) - \int_0^t C(s,u)\frac{\partial}{\partial t}x(u,t)du \; , \quad (s,t) \in \triangle .$$

<u>(6.4.8s)</u> At the beginning of this subsection we recalled that

$$d(s,t) = \tilde{B}^{-\frac{1}{2}}(s)E\varrho(s)\xi_i(t)$$

satisfies condition (6.4.8a). So, the components of $\tilde{B}^{\frac{1}{2}}(s)K_i^T(t,s)$, hence those of $\tilde{B}^{-\frac{1}{2}}(s)K_i^T(s,t)$ and therefore the elements of the Kalman-matrix $K(t,s)$ in (6.2.11) are partial differentiable with respect to t on $\triangle$ (provided that condition (6.2.1b,c) is satisfied).

$$K(t,s) \quad \text{and} \quad \frac{\partial}{\partial t}K(t,s)$$

are continuous on $\triangle$ and the partial derivative satisfies

$$\frac{\partial}{\partial t}K(t,s)B(s) = A(t)E\xi(t)\varrho^T(s) - K(t,t)E\varrho(t)\varrho^T(s) - \int_0^t \frac{\partial}{\partial t}K(t,u)E\varrho(u)\varrho^T(s)du \; ,$$

$(s,t) \in \triangle$, see (6.4.7n), (6.4.2) and (6.3.2a,c).

It is observed that the properties of $d(s,t)$ are reflected in the solution $x(s,t)$ to system (6.4.5a).

It remains to show the lemma, announced below (6.4.8q).

<u>(6.4.8t)</u> Let $f(x)$ be a mapping of $[a,b]$ into $(-\infty,\infty)$. Provided that they exist, let the ordinary derivative, the derivative from the right and the derivative from the left of $f(x)$ be represented by $Df(x)$, $D^+f(x)$ and $D^-f(x)$ respectively.

Lemma: Assume

i) $f(x)$ is continuous on $[a,b]$.

ii) $D^+f(x)$ exists on $[a,b)$.

iii) $D^+f(x)$ is continuous on $[a,b)$, $\lim_{x\uparrow b} D^+f(x)$ exists, and

by definition : $D^+f(b) = \lim_{x\uparrow b} D^+f(x)$, cf. (6.4.8q).

Then $f(x)$ is differentiable on $[a,b]$ and $Df(x) = D^+f(x)$, $x \in [a,b]$.

Proof: Consider

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a) \quad , \quad x \in [a,b] .$$

It is seen that also $g(x)$ enjoys the above properties i , ii and iii . And moreover,

iv) $g(a) = g(b)$.

A) We shall first establish the assertion

(1) there is a value $x_o \in [a,b]$ such that $D^+g(x_o) = 0$.

Proof: Assume the contrary,

(2) at all $x \in [a,b]$, $D^+g(x) \neq 0$.

Since $D^+g(x)$ is continuous on $[a,b]$, it attains a minimum m^+ and a maximum M^+ on $[a,b]$. Moreover

$$\text{either} \quad M^+ \geqq m^+ > 0 \qquad \text{or} \qquad m^+ \leqq M^+ < 0 \quad ,$$

otherwise (2) would be false because of the continuity of $D^+g(x)$ on $[a,b]$.

Assume $m^+ > 0$.

As also $g(x)$ is continuous on $[a,b]$, it attains a maximum value M on $[a,b]$, say $M = g(x_M)$. Suppose $x_M \in [a,b)$. Since

$D^+g(x_M) \geqq m^+ > 0$, there is a value $x' \in (x_M, b]$ such that

$$\frac{g(x') - g(x_M)}{x' - x_M} > \frac{m^+}{2} , \quad \text{i.e.} \quad g(x') > g(x_M) + \frac{m^+}{2}(x' - x_M) > g(x_M) = M .$$

This is absurd and hence $x_M = b$, and $M = g(b)$.

There is also a value $x'' \in (a,b]$ such that

$$\frac{g(x'') - g(a)}{x'' - a} > \frac{m^+}{2} , \quad \text{i.e.} \quad g(x'') > g(a) + \frac{m^+}{2}(x'' - a) > g(a) .$$

Hence, combining the above results,

(3) $M = g(b) \geqq g(x'') > g(a)$ in conflict with iv .

The other alternative, $M^+ < 0$, is treated similarly with result

(4) $g(a) > g(b)$.

Both (3) and (4) are in conflict with iv . Hence (2) is false and (1) is true. In other words:

(5) If the conditions i , ii and iii are fulfilled, then there is a value $x_o \in [a,b]$ such that

$$\frac{f(b) - f(a)}{b - a} = D^+f(x_o)$$

B) Under the conditions i , ii and iii , $f(x)$ is differentiable on $[a,b]$.

Proof: At $x=a$, $Df(a) = D^+f(a)$ by definition. Assume

$$\underline{x} \in (a,b] \quad \text{and} \quad a \leqq x < \underline{x} .$$

Then according to (5) there is a value x_o such that

$$\frac{f(\underline{x}) - f(x)}{\underline{x} - x} = D^+f(x_o) \quad \text{and} \quad x \leqq x_o \leqq \underline{x} .$$

If $x \uparrow \underline{x}$, then $x_o \to \underline{x}$ and hence $D^+f(x_o) \to D^+f(\underline{x})$ because of the continuity of $D^+f(x)$ on $[a,b]$. Therefore

$$D^-f(\underline{x}) = \lim_{x \uparrow \underline{x}} \frac{f(\underline{x}) - f(x)}{\underline{x} - x} = \lim_{x_o \to \underline{x}} D^+f(x_o) = D^+f(\underline{x}) ,$$

establishing the lemma .

6.5. The Kalman-Bucy filter.

For sake of completeness we shall establish the computation scheme of Kalman and Bucy, solving the Wiener-Hopf system concerning $\hat{\xi}(t|t)$, see for instance [16].

If not defined here, the meaning of the symbols used may be found in previous sections, as well as the proofs of the validity of a number of relations and formulae, used by Kalman and Bucy in the computations below. Many details are omitted here as they are amply discussed before.

Let be given the N-dimensional system (6.1.1),

(6.5.1a)
$$\begin{cases} \xi(t) = \gamma + \int_0^t A(s)\,\xi(s)ds + \beta(t)\ , \quad t \in [0,T]\ , \\ \text{with the M-dimensional observations} \\ \zeta(s) = \eta(s) + \tilde{\beta}(s) \quad \text{at all} \quad s \in [0,t]\ , \text{ where} \\ \eta(s) = \int_0^s \varrho(u)du\ , \qquad \varrho(u) = H(u)\,\xi(u)\ , \\ \frac{d}{ds} E\tilde{\beta}(s)\tilde{\beta}^T(s) = \tilde{B}(s) = O^T(s)D(s)O(s) > 0\ , \text{ see (6.2.1c)}, \\ E\beta(t)\gamma^T = 0\ ,\ E\tilde{\beta}(t)\gamma^T = 0\ ,\ E\beta(t)\tilde{\beta}^T(s) = 0 \text{ and hence} \\ E\,\xi(t)\tilde{\beta}^T(s) = 0\ ,\ E\tilde{\beta}(u)\varrho^T(t) = 0 \text{ and } E\,\eta(u)\tilde{\beta}^T(s) = 0\ , \\ \text{see (6.1.1).} \end{cases}$$

All above random elements are centered, and may be embedded in a Gaussian Hilbert space H . The components of $\zeta(s)$, $s \in [0,t]$, generate a closed subspace $H[C([0,t])]$ of H . The Kalman-Bucy estimate $\hat{\xi}(t|t)$ is characterized by

(6.5.1b)
$$\begin{cases} \hat{\xi}_i(t|t) \in H[C([0,t])\ , \quad i=1,..,N, \text{ and} \\ E\{\xi(t) - \hat{\xi}(t|t)\}\,\zeta = 0 \text{ for all } \zeta \in H[C([0,t])]\ . \\ \text{In particular,} \\ E\{\xi(t) - \hat{\xi}(t|t)\}\,\zeta^T(s) = 0\ ,\ E\{\xi(t) - \hat{\xi}(t|t)\}\,\hat{\xi}^T(t|t) = 0, \\ \text{see (6.1.1).} \end{cases}$$

The Kalman-Bucy estimate may uniquely be represented as

(6.5.2) $$\hat{\xi}(t|t) = \int_0^t K(t,u)\,d\zeta(u)\ , \qquad t \in [0,T]\ ,$$

where $K(t,u)$ is an $N \times M$-matrix whose entries at fixed t are elements of $L_2[0,t]$, owing to the fact that $\tilde{B}(s) > 0$, see (6.2.11).

At each $t \in [0,T]$, $K(t,s)$ is the unique solution to the system

(6.5.3) $$K(t,s)\tilde{B}(s) + \int_0^t K(t,u)E\varrho(u)\varrho^T(s)du = E\,\xi(t)\varrho^T(s)\ , \qquad s \in [0,t]\ ,$$

see (6.3.2) and (6.4.5).

$E\,\varrho(u)\varrho^T(s)$ is continuous in (u,s) on $[0,T]^2$,

$E\,\xi(t)\,\varrho^T(s)$ is continuous in (t,s) on $\triangle$,

$\frac{\partial}{\partial t} E\,\xi(t)\varrho^T(s) = A(t)\,\xi(t)\varrho^T(s)$ exists and is continuous in (t,s) on $\triangle$, see (6.3.2).

It follows that $K(t,s)$ is continuous in (t,s) on $\triangle$, see (6.4.7n), and that

$\frac{\partial}{\partial t} K(t,s)$ exists, is continuous in (t,s) on $\triangle$ and satisfies

(6.5.4) $$\frac{\partial}{\partial t} K(t,s)\tilde{B}(s) + K(t,t)E\varrho(t)\varrho^T(s) + \int_0^t \frac{\partial}{\partial t} K(t,u)E\,\varrho(u)\,\varrho^T(s)du = A(t)E\,\xi(t)\varrho^T(s)\ , \quad (s,t) \in \triangle\ ,$$ see (6.4.8s).

The estimation error is $\bar{\xi}(t) = \xi(t) - \hat{\xi}(t|t)$.
$P(t)$ represents the error covariance matrix

$$P(t) = E\,\bar{\xi}(t)\,\bar{\xi}^T(t)\ , \qquad t \in [0,T]\ .$$

In the Kalman-Bucy filter, $P(t)$ is not only of importance as a measure of the magnitude of the estimation error. It is also an essential tool in the calculations. With the aid of $P(t)$, Kalman and Bucy arranged the computations in such a way that $K(t,s)$ is needed only at the diagonal $s=t$.

Since $E\,\bar{\xi}(t)\,\hat{\xi}^T(t|t) = 0$, see (6.5.1b),

$$P(t) = E\,\bar{\xi}(t)\left\{\xi^T(t) - \hat{\xi}^T(t|t)\right\} = E\,\bar{\xi}(t)\xi^T(t) .$$

Hence, since $\varrho(t) = H(t)\,\xi(t)$,

(6.5.5a) $$P(t)H^T(t) = E\,\bar{\xi}(t)\xi^T(t)H^T(t) = E\,\bar{\xi}(t)\varrho^T(t) =$$
$$= E\,\xi(t)\varrho^T(t) - E\,\hat{\xi}(t|t)\varrho^T(t) =$$
$$= E\,\xi(t)\varrho^T(t) - E\left\{\int_0^t K(t,u)d\,\zeta(u)\right\}\varrho^T(t) .$$

As $\zeta(u) = \eta(u) + \tilde{\beta}(u)$,

(6.5.5b) $$\int_0^t K(t,u)d\,\zeta(u) = \int_0^t K(t,u)d\,\eta(u) + \int_0^t K(t,u)d\tilde{\beta}(u) .$$

Since $\eta(u)$ is continuously differentiable in q.m. with derivative in q.m. $\varrho(u)$, see (2.7.2), and since $K(t,u)$ is continuous in $u \in [0,t]$, we may apply (2.8.3), owing to the results in (6.2.5) where it is shown that the ordinary Riemann-Stieltjes integral in q.m. - if possible - may be used in this context. Hence

(6.5.5c) $$\int_0^t K(t,u)d\,\eta(u) = \int_0^t K(t,u)\,\frac{d}{du}\eta(u)du = \int_0^t K(t,u)\,\varrho(u)du .$$

Then on account of the continuity of the inner product, see (2.1.3), and as $E\tilde{\beta}(u)\varrho^T(t) = 0$, we obtain according to (6.5.5a,b,c),

$$P(t)H^T(t) = E\,\xi(t)\,\varrho^T(t)-E\left\{\int_0^t K(t,u)\,\varrho(u)du\right\}\varrho^T(t)-E\left\{\int_0^t K(t,u)d\tilde{\beta}(u)\right\}\varrho^T(t) =$$
$$= E\,\xi(t)\,\varrho^T(t) - \int_0^t K(t,u)E\,\varrho(u)\,\varrho^T(t)du .$$

Hence according to (6.5.3) at $s=t$,

(6.5.6) $$P(t)H^T(t) = K(t,t)\tilde{B}(t) \quad , \quad t \in [0,T] .$$

Consider (6.5.4), i.e.

$$\frac{\partial}{\partial t}K(t,s)\tilde{B}(s) + K(t,t)E\,\varrho(t)\varrho^T(s) + \int_0^t \frac{\partial}{\partial t}K(t,u)E\,\varrho(u)\varrho^T(s)du =$$
$$= A(t)E\,\xi(t)\varrho^T(s) , \quad (s,t) \in \triangle .$$

Since $\varrho(t) = H(t)\,\xi(t)$ and according to (6.5.3) it follows that

$$K(t,t)E\,\varrho(t)\varrho^T(s) = K(t,t)H(t)E\,\xi(t)\varrho^T(s) =$$
$$= K(t,t)H(t)\left\{K(t,s)\tilde{B}(s) + \int_0^t K(t,u)E\,\varrho(u)\varrho^T(s)du\right\} .$$

Substitution of this result in the left-hand side, and of (6.5.3) in the right-hand side of (6.5.4) yields

$$\frac{\partial}{\partial t} K(t,s)\widetilde{B}(s) + K(t,t)H(t)\left\{K(t,s)\widetilde{B}(s) + \int_0^t K(t,u)E\varrho(u)\varrho^T(s)du\right\} +$$

$$+ \int_0^t \frac{\partial}{\partial t} K(t,u)E\varrho(u)\varrho^T(s)du = A(t)\left\{K(t,s)\widetilde{B}(s) + \int_0^t K(t,u)E\varrho(u)\varrho^T(s)du\right\},$$

i.e.

$$\left\{\frac{\partial}{\partial t} K(t,s) + K(t,t)H(t)K(t,s) - A(t)K(t,s)\right\}\widetilde{B}(s) +$$

$$\int_0^t \left\{\frac{\partial}{\partial t} K(t,u) + K(t,t)H(t)K(t,u) - A(t)K(t,u)\right\} E\varrho(u)\varrho^T(s)du = 0 ,$$

$s \in [0,t]$, $t \in [0,T]$, and hence on account of (6.4.5),

(6.5.7) $$\frac{\partial}{\partial t} K(t,s) + K(t,t)H(t)K(t,s) - A(t)K(t,s) = 0 , \quad (s,t) \in \Delta .$$

Owing to (6.4.8s) we may write

$$K(t,s) = K(s,s) + \int_s^t \frac{\partial}{\partial u} K(u,s)du .$$

Hence

$$\hat{\xi}(t|t) = \int_0^t K(t,s)d\zeta(s) = \int_0^t K(s,s)d\zeta(s) + \int_0^t\left\{\int_s^t \frac{\partial}{\partial u} K(u,s)du\right\}d\zeta(s) =$$

(6.5.7a) $$= \int_0^t K(s,s)d\zeta(s) + \int_0^t\left\{\int_0^u \frac{\partial}{\partial u} K(u,s)d\zeta(s)\right\}du ,$$

since the order of integration may be changed. This will be shown at the end of this section.

Substitution of (6.5.7) yields

$$\hat{\xi}(t|t) = \int_0^t K(s,s)d\zeta(s) + \int_0^t\left\{\int_0^u \left[A(u)K(u,s) - K(u,u)H(u)K(u,s)\right] d\zeta(s)\right\}du ,$$

and as $\int_0^u K(u,s)d\zeta(s) = \hat{\xi}(u|u)$,

(6.5.8) $$\hat{\xi}(t|t) = \int_0^t K(s,s)d\zeta(s) + \int_0^t \left\{A(u) - K(u,u)H(u)\right\}\hat{\xi}(u|u)du , \quad t \in [0,T].$$

This is a system of stochastic linear integral equations for $\hat{\xi}(t|t)$, involving $K(t,s)$ only at the diagonal $s=t$, whereas (6.5.6) gives the relation between $K(t,t)$ and $P(t)$.

We shall derive a system of integral equations for $\bar{\xi}(t) = \xi(t) - \hat{\xi}(t|t)$.
On account of (6.5.5b,c) and since $\varrho(s) = H(s)\xi(s)$,

$$\int_0^t K(s,s)d\zeta(s) = \int_0^t K(s,s)H(s)\xi(s)ds + \int_0^t K(s,s)d\widetilde{\beta}(s) .$$

Hence (6.5.8) may be rewritten as

$$\hat{\xi}(t|t) = \int_0^t A(s)\,\hat{\xi}(s|s)ds + \int_0^t K(s,s)H(s)\left\{\xi(s)-\hat{\xi}(s|s)\right\}ds + \int_0^t K(s,s)d\tilde{\beta}(s)$$

Subtracting this result from (6.5.1), i.e. from

$$\xi(t) = \gamma + \int_0^t A(s)\,\xi(s)ds + \beta(t) ,$$

delivers , as $\bar{\xi}(t) = \xi(t) - \hat{\xi}(t|t)$,

(6.5.9) $$\bar{\xi}(t) = \gamma + \int_0^t \left\{A(s)-K(s,s)H(s)\right\}\bar{\xi}(s)ds + \beta(t) - \int_0^t K(s,s)d\tilde{\beta}(s) ,$$
$t \in [0,T]$,

where according to (6.5.6), $K(s,s) = P(s)H^{\mathsf{T}}(s)\tilde{B}^{-1}(s)$.

Owing to the footnote in (5.1.1) and to (4.4.2),

$$\int_0^t K(s,s)d\tilde{\beta}(s)$$

is an N-dimensional Wiener-Lévy process. Hence also

$$\bar{\beta}(t) = \beta(t) - \int_0^t K(s,s)d\tilde{\beta}(s)$$

is an N-dimensional Wiener-Lévy process, since $E\beta(t)\tilde{\beta}^{\mathsf{T}}(s) = 0$.
It is seen similarly as in (4.4.2e,f,g,h) that

$$\bar{B}(t) = \frac{d}{dt}\,E\bar{\beta}(t)\bar{\beta}^{\mathsf{T}}(t)$$

satisfies - also on account of $E\beta(t)\tilde{\beta}^{\mathsf{T}}(s) = 0$ -

(6.5.10) $$\bar{B}(t) = \frac{d}{dt}\,E\left\{\beta(t)-\int_0^t K(s,s)d\tilde{\beta}(s)\right\}\left\{\beta(t)-\int_0^t K(s,s)d\tilde{\beta}(s)\right\}^{\mathsf{T}} =$$
$$= \frac{d}{dt}\left\{\int_0^t B(s)ds + \int_0^t K(s,s)\tilde{B}(s)K^{\mathsf{T}}(s,s)ds\right\} =$$
$$= B(t) + K(t,t)\tilde{B}(t)K^{\mathsf{T}}(t,t) .$$

Since moreover

$$E\bar{\beta}(s)\gamma^{\mathsf{T}} = E\beta(s)\gamma^{\mathsf{T}} - E\left\{\int_0^t K(s,s)d\tilde{\beta}(s)\right\}\gamma^{\mathsf{T}} = 0 ,$$

system (6.5.9) is of the type, treated in section 5.1. Set

(6.5.11) $$C(t) = E\,\gamma\gamma^{\mathsf{T}} , \qquad \bar{A}(t) = A(t) - K(t,t)H(t)$$

and let $\bar{F}(t)$ be the fundamental matrix associated with $\bar{A}(t)$.
I.e. $\bar{F}(t)$ is the $N\times N$-matrix with the properties

(6.5.12) $$\bar{F}(0) = I_N , \qquad \frac{d}{dt}\bar{F}(t) = \bar{A}(t)\bar{F}(t) ,$$

where I_N is the $N\times N$-identity matrix, see (3.2.3).

Now system (6.5.9) may be written as follows:

$$\bar{\xi}(t) = \gamma + \int_0^t \bar{A}(s)\bar{\xi}(s)ds + \bar{\beta}(t) .$$

On account of the above comments, (5.1.2d) applies and yields

$$P(t) = E\,\bar{\xi}(t)\bar{\xi}^T(t) = \bar{F}(t)\left\{C + \int_0^t \bar{F}^{-1}(s)\bar{B}(s)\left[\bar{F}^{-1}(s)\right]^T ds\right\}\bar{F}^T(t) \quad , \quad t \in [0,T].$$

And according to (6.5.12),

(6.5.13) $$P(0) = \bar{F}(0)C\bar{F}^T(0) = I_N \cdot C \cdot I_N = C ,$$

and as $P(t)$ is apparently differentiable,

(6.5.14) $$\frac{d}{dt}P(t) = \bar{A}(t)\bar{F}(t)\left\{C + \int_0^t \bar{F}^{-1}(s)\bar{B}(s)\left[F^{-1}(s)\right]^T ds\right\}\bar{F}^T(t) + \\ + \bar{F}(t)\bar{F}^{-1}(t)\bar{B}(t)\left[\bar{F}^{-1}(t)\right]^T\bar{F}^T(t) + \\ + \bar{F}(t)\left\{C + \int_0^t \bar{F}^{-1}(s)\bar{B}(s)\left[\bar{F}^{-1}(s)\right]^T ds\right\}\bar{F}^T(t)\bar{A}(t) = \\ = \bar{A}(t)P(t) + \bar{B}(t) + P(t)\bar{A}^T(t) \quad , \quad t \in [0,T] .$$

Substitution of (6.5.6), (6.5.10) and (6.5.11) into (6.5.14) yields

$$\frac{d}{dt}P(t) = \left\{A(t) - P(t)H^T(t)\tilde{B}^{-1}(t)H(t)\right\}P(t) + \\ + B(t) + P(t)H^T(t)\tilde{B}^{-1}(t)\tilde{B}(t)\tilde{B}^{-1}(t)H(t)P^T(t) + \\ + P(t)\left\{A^T(t) - H^T(t)\tilde{B}^{-1}(t)H(t)P^T(t)\right\} ,$$

i.e.

(6.5.15) $$\frac{d}{dt}P(t) = A(t)P(t) + P(t)A^T(t) - P(t)H^T(t)\tilde{B}^{-1}(t)H(t)P(t) + B(t) , \\ \text{if} \quad t \in [0,T] ,$$

whereas according to (6.5.13)

$$P(0) = C .$$

Except for $P(t)$, all matrices in (6.5.15) are assumed to be known. Hence $P(t)$ is the solution of an N-dimensional Riccati-system with a given initial condition. This solution is calculated numerically.

With (6.5.6), system (6.5.8) delivers

(6.5.16) $$\hat{\xi}(t|t) = \int_0^t \left\{A(s) - P(s)H^T(s)\tilde{B}^{-1}(s)H(s)\right\}\hat{\xi}(s|s)ds + \\ + \int_0^t P(s)H^T(s)\tilde{B}^{-1}(s)d\,\zeta(s) \quad , \quad t \in [0,T] .$$

This is a system of stochastic linear integral equations. Given the matrices $A(s)$, $B(s)$, $H(s)$, the calculated matrix $P(s)$ and the sampled observations $\zeta(s)$, (6.5.16) is treated as a deterministic system and the corresponding sample of $\hat{\xi}(t|t)$ is determined in accordance with the theory of linear minimum variance estimation.

Also in this section it is seen that the condition $\tilde{B}(s) > 0$ is essentially needed, as $\tilde{B}^{-1}(s)$ figures in the computations.

The matrix $P(t)$ gives information about the accuracy of the estimation. A discussion on the numerical evaluation of $P(t)$, $t \in [0,T]$, may be found in [4].

In practice, the above computations are systematically discretized. The validity of the continuous counterpart shows the validity of the discrete data processing, and its relative independence of the size of the then introduced time-differences Δt, see also theorem (6.1.8).

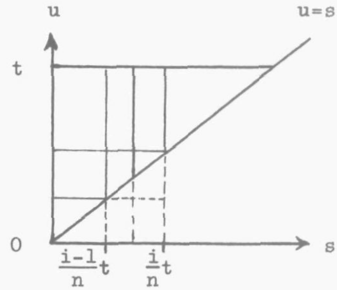
An alternative method of finding the estimate $\hat{\xi}(t|t)$ would consist of numerical computation of the matrix $K(t,s)$, see for instance system (6.3.3a). If $K(t,s)$ is known, the estimate

$$\hat{\xi}(t|t) = \int_0^t K(t,s)d\,\zeta(s)$$

is simply evaluated by numerical integration.

However, the matrix $K(t,s)$ depends on two variables. And the numerical evaluation of $K(t,s)$, $(s,t) \in \Delta$, looks rather hard in comparison with numerically solving the Riccati system of $P(t)$, $t \in [0,T]$, in the Kalman-Bucy filter, as $P(t)$ depends on one variable only.

It remains to show that the order of integration in (6.5.7a) may be changed.



Let $\triangle$ be the domain defined by

$0 \leqq s \leqq u \ , \quad 0 \leqq u \leqq t \ .$

Let $f(u,s)$ represent one of the entries $\frac{\partial}{\partial u} K_{ij}(u,s)$ of $\frac{\partial}{\partial u} K(u,s)$. Then $f(u,s)$ is continuous on $\triangle$ according to (6.4.8s). Let $\varphi(s)$ represent one of the components of $\zeta(s) = \eta(s) + \tilde{\beta}(s) = \int_0^s \varrho(w)dw + \tilde{\beta}(s)$. Then $\varphi(s)$ is continuous in q.m. on $[0,t]$. If $0 \leqq u \leqq v \leqq t$,

$$E\,\zeta(u)\zeta^T(v) = \int_0^u \int_0^v E\,\varrho(u')\varrho^T(v')du'dv' + \int_0^u \tilde{B}(u')du' \ .$$

Here $E\,\varrho(u')\varrho^T(v')$ and $\tilde{B}(u')$ are continuous in (u',v') on $[0,t]^2$ and in u' on $[0,t]$ respectively. So there are functions $g(u',v')$ and $h(u')$, continuous in (u',v') on $[0,t]^2$ and in u' on $[0,t]$ respectively, such that

$$E\,\varphi(u)\varphi(v) = \int_0^u \int_0^v g(u',v')du'dv' + \int_0^u h(u')du' \ , \quad u \leqq v \ .$$

Hence, if $[x,y] \subset [0,t]$, the total variation $V(E\,\varphi(u)\varphi(v) \ , \ [x,y]^2)$ of $E\,\varphi(u)\varphi(v)$ on $[x,y]^2$ satisfies

$$V(E\,\varphi(u)\varphi(v) \ , \ [x,y]^2) \leqq \int_x^y \int_x^y \left|g(u',v')\right|du'dv' + \int_x^y \left|h(u')\right|du' \ ,$$

see also section 2.9. Hence, if

$$M = \max \left\{ \max_{(u',v') \in [0,t]^2} \left|g(u',v')\right| \ , \ \max_{u' \in [0,t]} \left|h(u')\right| \right\} \ ,$$

then

(1) $$V(E\,\varphi(u)\varphi(v) \ , \ [x,y]^2) \leqq M\left|y-x\right|^2 + M\left|y-x\right| \ .$$

Define

$$\alpha = \int_0^t \left\{ \int_s^t f(u,s)du \right\} d\,\varphi(s) \quad , \quad \beta = \int_0^t \left\{ \int_0^u f(u,s)d\varphi(s) \right\} du \ .$$

Since $f(u,s)$ is continuous on $\triangle$, $\int_s^t f(u,s)du$ is a continuous function of s on $[0,t]$. Hence α exists on account of (1) and (2.9.1), see also (6.2.5).

Since $f(u,s)$ is uniformly continuous on the compact domain Δ, if $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$|f(u,s) - f(u',s')| < \varepsilon \quad \text{if} \quad |u-u'| < \delta \quad \text{and} \quad |s-s'| < \delta .$$

Let

$$A = \max_{(u,s)\in\Delta} |f(u,s)| .$$

Hence, if $|u-v| < \delta$ and $0 \leqq u \leqq v \leqq s$, according to (2.9.1) we obtain

$$\left\| \int_0^u f(u,s)d\varphi(s) - \int_0^v f(v,s)d\varphi(s) \right\| \leqq$$

$$\leqq \left\| \int_0^u \{f(u,s) - f(v,s)\} d\varphi(s) \right\| + \left\| \int_u^v f(v,s)d\varphi(s) \right\| \leqq$$

$$\leqq \varepsilon \sqrt{V(E\varphi(u')\varphi(v') , [0,u]^2)} + A\sqrt{V(E\varphi(u')\varphi(v') , [u,v]^2)} .$$

Now because of (1) it is seen that $\int_0^u f(u,s)d\varphi(s)$ is continuous in q.m. as a function of u on $[0,t]$. Hence β exists according to (2.7.1), see also (6.2.5).

Set
$$\bar{f}(u,s) = \begin{cases} f(u,s) & \text{if } (u,s) \in \Delta , \\ 0 & \text{otherwise} \end{cases}$$

Then, since $\varphi(s)$ is continuous in q.m. on $[0,t]$, we may write

$$\alpha = \int_0^t \left\{ \int_0^t \bar{f}(u,s)du \right\} d\varphi(s) \quad , \quad \beta = \int_0^t \left\{ \int_0^t \bar{f}(u,s)d\varphi(s) \right\} du .$$

Let n, i and j be natural numbers. We define

$$f_n(u,s) = \bar{f}\left(\frac{i-1}{n}t , \frac{j-1}{n}t\right) \quad \text{if} \quad \frac{i-1}{n}t \leqq u < \frac{i}{n}t , \quad \frac{j-1}{n}t \leqq s < \frac{j}{n}t ,$$
$$i,j = 1,.,n ,$$

$$f_n(t,s) = \bar{f}(t,s) .$$

Since $f(u,s)$ is uniformly continuous on Δ, it follows, if n is sufficiently large, that

$$|\bar{f}(u,s) - f_n(u,s)| < \varepsilon \quad , \quad (u,s) \in [0,t]^2 ,$$

possibly with exception of the points (u,s) satisfying

$$\frac{i-1}{n}t \leqq u < \frac{i}{n}t \quad , \quad \frac{i-1}{n}t \leqq s < \frac{i}{n}t \quad , \quad i=1,.,n .$$

Define

$$\alpha_n = \int_0^t \left\{ \int_0^t f_n(u,s)du \right\} d\varphi(s) \; , \; \beta_n = \int_0^t \left\{ \int_0^t f_n(u,s)d\varphi(s) \right\} du .$$

Then

$$\left| \int_o^t \bar{f}(u,s)du - \int_o^t f_n(u,s)du \right| \leqq \int_o^t \left| \bar{f}(u,s) - f_n(u,s) \right| du < \varepsilon t + A \frac{t}{n} ,$$

uniformly in $s \in [0,t]$. And hence, according to (2.9.1),

$$\| \alpha - \alpha_n \| \leqq (\varepsilon t + A \frac{t}{n}) \sqrt{V(E\varphi(u')\varphi(v') , [0,t]^2)} .$$

Therefore, also on account of (1) ,

(2) $\alpha_n \to \alpha$ in q.m. as $n \to \infty$.

It is similarly shown that

(3) $\beta_n \to \beta$ in q.m. as $n \to \infty$.

Finally, since $\varphi(s)$ is continuous in q.m. on $[0,t]$,

$$\alpha_n = \frac{t}{n} \sum_{j=1}^{n} \sum_{i=j}^{n} f(\frac{i-1}{n} t , \frac{j-1}{n} t) \left\{ \varphi(\frac{i}{n} t) - \varphi(\frac{j-1}{n} t) \right\} =$$

(4)

$$= \frac{t}{n} \sum_{i=1}^{n} \sum_{j=1}^{i} f(\frac{i-1}{n} t , \frac{j-1}{n} t) \left\{ \varphi(\frac{i}{n} t) - \varphi(\frac{j-1}{n} t) \right\} = \beta_n .$$

Hence, as by virtue of (4)

$$\| \alpha - \beta \| = \| \alpha - \alpha_n + \beta_n - \beta \| \leqq \| \alpha - \alpha_n \| + \| \beta - \beta_n \| ,$$

$\alpha = \beta$ on account of (2) and (3) .

6.6. The behaviour of Kalman-Bucy and related estimates with respect to differentiable perturbations of the involved white noise processes.

If not explained in this section, the meaning of the symbols used may be found in previous sections. If a character represents a vector or a matrix, the same character with one ore more subscripts represents a component or an entry of that vector or matrix.

Since the linear minimum variance estimation in the previous section is merely a mathematical processing of information, and not the counterpart of any physical phenomenon, there is no need for a critical study of the sample behaviour of the estimate $\hat{\xi}(t|t)$. In accordance with the theory of linear minimum variance estimation, we were interested in just one trajectory of $\hat{\xi}(t|t)$, corresponding to the registered trajectory of the observation $\zeta(s)$.

However, in the state equations and in the observations are figuring the Wiener-Lévy processes $\beta(t)$ and $\tilde{\beta}(s)$ respectively. In accordance with the comments in section 1.1 we are interested in the behaviour of the estimates in case $\beta(t)$ and $\tilde{\beta}(s)$ are perturbed.

Especially perturbing $\tilde{\beta}(s)$ has deep consequences. In section 6.2 we have seen that the validity of the integral representation

$$\int_0^t K(t,s)d\,\zeta(s) \quad \text{of} \quad \hat{\xi}(t|t)$$

hinges entirely on the properties of $\tilde{B}(s)$, exposed in (6.2.1bc). In case $\tilde{\beta}(s)$ is replaced by $\tilde{\beta}(n,s)$, $\tilde{\beta}(n,s)$ being a perturbation defined in (4.4.4),

$$E\,\tilde{\beta}(n,u)\tilde{\beta}^T(n,v)$$

is a function of u and v , and not a perturbation of the matrix

$$E\,\tilde{\beta}(u)\tilde{\beta}^T(v) = \int_0^m \tilde{B}(w)dw \quad , \quad m=\min(u,v).$$

Hence the argumentation in section 6.2 is not applicable to the perturbed estimate.

Even, if one would "try" an integral representation

$$\int_0^t K_n(t,s)d\,\zeta_n(s)$$

to the perturbed version of $\hat{\xi}(t|t)$, it is because of the structure of $E\tilde{\beta}(n,u)\tilde{\beta}^T(n,v)$ that the computation scheme in section 6.5 breaks down completely.

So in this case perturbing the Wiener-Lévy processes is not simply reflected in the perturbation of some matrices in the formulae of the previous section. And the investigation of the effect of the perturbation of the Wiener-Lévy processes on the estimate will be performed without using the formulae of section 6.5. Then, as we shall see, the results are not confined to the Kalman-Bucy estimate alone.

(6.6.1) We shall be concerned with the situation depicted in subsection (6.1.1). Omitting many details, we recall

(6.6.1a)

the N-dimensional system

$$\xi(t) = \gamma + \int_0^t A(s)\,\xi(s)ds + \beta(t)\ , \quad t \in [0,T]\ ,$$

and the M-dimensional observations

$$\zeta(s) = \eta(s) + \tilde{\beta}(s) \quad \text{at all} \quad s \in S_t \subset [0,T]\ , \text{ where}$$

$$\eta(s) = \int_0^t \varrho(u)du\ ,$$

$$\varrho(u) = H(u)\,\xi(u)\ ,$$

$$\left.\begin{aligned}\beta(t) &= \int_0^t G(u)d\beta_0(u)\\ \tilde{\beta}(s) &= \int_0^s \tilde{G}(v)d\tilde{\beta}_0(v)\end{aligned}\right\}\quad \text{see section } 4.4\ ,$$

$$E\gamma\beta_0^T(u) = 0\ , \quad E\gamma\tilde{\beta}_0^T(v) = 0\ , \quad E\beta_0(u)\tilde{\beta}_0^T(v) = 0\ ,$$

and hence at $s,t,u,v \in [0,T]$,

$$E\gamma\beta^T(t) = 0\ , \quad E\gamma\tilde{\beta}^T(s) = 0\ , \quad E\beta(t)\tilde{\beta}^T(s) = 0\ ,$$

$$E\,\xi(t)\,\tilde{\beta}^T(s) = 0, \quad E\varrho(u)\tilde{\beta}^T(s) = 0, \quad E\eta(u)\tilde{\beta}^T(s) = 0\ .$$

All random elements are centered and Gaussian and may be embedded in the Gaussian Hilbert space H , generated by the components of γ , $\beta_0(t)$ and $\tilde{\beta}_0(s)$, $s,t \in [0,T]$.

We recall

$$\tilde{B}(s) = \frac{d}{ds}\,E\tilde{\beta}(s)\tilde{\beta}^T(s)\ .$$

Also here condition (6.2.1c) is assumed to be satisfied,

(6.6.1b)

$\tilde{B}(s) = O^T(s)D(s)O(s) \geqq e\,I_M$, $e > 0$, $s \in [0,T]$, where $O(s)$ and $D(s)$ are continuous orthogonal and diagonal respectively.

(6.6.1c)

$H[C(S_t)]$ is the closed subspace of H , generated by the elements of the class

$$C(S_t) = \{\zeta_j(s),\ j=1,..,M\ ,\ s \in S_t\}\ .$$

$\hat{\xi}(t|S_t)$ is characterized by $\hat{\xi}_i(t|S_t) \in H[C(S_t)]$, $i=1,..,N$ and $E\{\xi - \hat{\xi}(t|S_t)\}\zeta = 0$ for all $\zeta \in H[C(S_t)]$, or equivalently $E\{\xi - \hat{\xi}(t|S_t)\}\zeta^T(s) = 0$, $s \in S_t$,

see (6.1.4).

S_t may vary with t. Dependent on the structure of $S_t \subset [0,T]$, $\hat{\xi}(t|S_t)$ is an interpolated (smoothed), filtered or extrapolated (predicted) random N-vector. If the involved stochastic processes are wide sense stationary, $\hat{\xi}(t|S_t)$ is of the type of Wiener and Kolmogorov. If $S_t = [0,t]$, $\hat{\xi}(t|[0,t]) = \hat{\xi}(t|t)$ is the Kalman-Bucy estimate.

(6.6.2) Let

(6.6.2a) $\{\tilde{\beta}(m,s),\ s \in [0,T],\ m=1,2,..\}$ and $\{\beta(n,t),\ t \in [0,T],\ n=1,2,..\}$ be sequences of differentiable perturbations of $\tilde{\beta}(s)$ and $\beta(t)$ respectively, endowed with the properties exposed in (4.4.5).

According to the comments in section 1.1, at each n, m we shall consider

(6.6.2b)

the N-dimensional system

$$\xi(n,t) = \gamma + \int_0^t A(s)\,\xi(n,s)ds + \beta(n,t), \quad t \in [0,T],$$

and the M-dimensional observations

$\zeta(n,m,s) = \eta(n,s) + \tilde{\beta}(m,s)$ at all $s \in S_t$, where

$\eta(n,s) = \int_0^s \varrho(n,u)du$ and hence $\eta(n,0) = 0$, and

$\varrho(n,u) = H(u)\,\xi(n,u)$.

We recall $\tilde{\beta}(m,0) = 0$. Hence

$\zeta(n,m,0) = \eta(n,0) + \tilde{\beta}(m,0) = 0$.

On account of (6.6.1a) and (6.6.2a),

$E\,\xi(n,t)\tilde{\beta}^T(m,s) = 0$ and $E\,\eta(n,u)\tilde{\beta}^T(m,s) = 0$.

It is seen that also all random elements here, are centered and Gaussian. All random variables are elements of H in (6.6.1a).

Now theorem (5.2.1) is applicable:

(6.6.2c) $\xi_i(n,t) \to \xi_i(t)$ in q.m., uniformly in $t \in [0,T]$, $i=1,..,N$, as $n \to \infty$.

Hence according to (2.7.5),

$$\eta_j(n,s) = \sum_{k=1}^{N} \int_0^s H_{jk}(u)\xi_k(n,u)du \longrightarrow \eta_j(s) \quad \text{in q.m. as} \quad n \longrightarrow \infty ,$$

uniformly in $s \in [0,T]$, $j=1,.,M$.

And since $\zeta(n,m,s) = \eta(n,s) + \tilde{\beta}(m,s)$, it follows - also by virtue of (6.6.2a) - that

(6.6.2d) $\zeta_j(n,m,s) \to \zeta_j(s)$ uniformly in $s \in [0,T]$ as $n,m \to \infty$.

Moreover, we notice that $\zeta_j(n,m,s)$ is continuously differentiable in q.m. and $\zeta_j(s)$ continuous in q.m. on $[0,T]$, $j=1,.,M$.

Let

$$H[C(n,m,S_t)]$$

be the closed linear subspace of H , generated by the elements of the class

$$C(n,m,S_t) = \left\{\zeta_j(n,m,s),\ j=1,.,M,\ \ s \in S_t\right\}$$

Let $\hat{\xi}(n,m,t|S_t)$ be the conditional expectation of $\xi(n,t)$, given $C(n,m,S_t)$. Since also here all random elements are centered Gaussian, the components of $\hat{\xi}(n,m,t|S_t)$ are the linear minimum variance estimates of the corresponding components of $\xi(n,t)$, given $C(n,m,S_t)$, i.e.

(6.6.2e)
$$\begin{cases} \hat{\xi}_i(n,m,t|S_t) \in H[C(n,m,S_t)] \ , \ i=1,.,N, \ \text{and} \\ E\left\{\xi(n,t) - \hat{\xi}(n,m,t|S_t)\right\}\zeta(n,m) = 0 \ \text{for all} \\ \zeta(n,m) \in H[C(n,m,S_t)], \ \text{or equivalently} \\ E\left\{\xi(n,t) - \hat{\xi}(n,m,t|S_t)\right\}\zeta^T(n,m,s) = 0 \ \text{at all} \ s \in S_t, \end{cases}$$

see (6.1.4).

(6.6.3) Definition: The estimate $\hat{\xi}(t|S_t)$ in (6.6.1) is stable with respect to differentiable perturbations of the involved Wiener-Lévy processes iff

(6.6.3a) $\hat{\xi}_i(n,m,t|S_t) \to \hat{\xi}_i(t|S_t)$ in q.m. as $n,m \to \infty$, $i=1,.,N$,

cf. section 1.1.

Hence in investigations on stability in the sense of this definition, the value $t \in [0,T]$ and the set $S_t \subset [0,T]$ are kept fixed.

<u>(6.6.4)</u> Let t and S_t be a fixed value and a fixed set in $[0,T]$. Let us introduce the orthogonal projectors

$$\mathcal{P} \quad \text{and} \quad \mathcal{P}(n,m)$$

of H onto

$$H[C(S_t)] \quad \text{and} \quad H[C(n,m,S_t)]$$

respectively. Then

$$\hat{\xi}_i(t|S_t) = \mathcal{P}\,\xi_i(t) \quad \text{and} \quad \hat{\xi}_i(n,m,t|S_t) = \mathcal{P}(n,m)\,\xi_i(n,t)$$

and stability condition (6.6.3a) reads

<u>(6.6.4a)</u> $\mathcal{P}(n,m)\,\xi_i(n,t) \rightarrow \mathcal{P}\,\xi_i(t)$ in q.m. as $n,m \rightarrow \infty$, $i=1,.,N$.

Since $\mathcal{P}$ and $\mathcal{P}(n,m)$ are orthogonal projectors in H,

$$\|\mathcal{P}\| = \|\mathcal{P}(n,m)\| = 1 \text{ , uniformly in } m \text{ and } n \text{ ,}$$

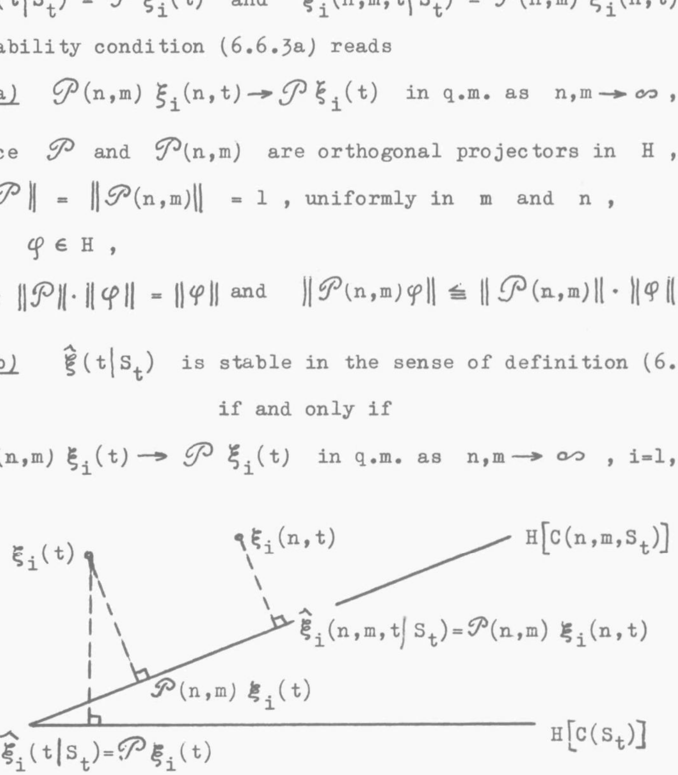
and if $\varphi \in H$,

$$\|\mathcal{P}\varphi\| \leqq \|\mathcal{P}\|\cdot\|\varphi\| = \|\varphi\| \text{ and } \|\mathcal{P}(n,m)\varphi\| \leqq \|\mathcal{P}(n,m)\|\cdot\|\varphi\| = \|\varphi\| .$$

<u>(6.6.4b)</u> $\hat{\xi}(t|S_t)$ is stable in the sense of definition (6.6.3)

if and only if

$$\mathcal{P}(n,m)\,\xi_i(t) \rightarrow \mathcal{P}\,\xi_i(t) \text{ in q.m. as } n,m \rightarrow \infty \text{ , } i=1,.,N .$$



Proof: The asserted equivalence follows from the identity

$$\mathcal{P}(n,m)\,\xi_i(n,t)-\mathcal{P}\,\xi_i(t) = \mathcal{P}(n,m)\left\{\xi_i(n,t)-\xi_i(t)\right\} + \left\{\mathcal{P}(n,m)\,\xi_i(t)-\mathcal{P}\,\xi_i(t)\right\} ,$$

and from the result below, being valid on account of (6.6.2c):

$$\left\|\mathcal{P}(n,m)\left\{\xi_i(n,t)-\xi_i(t)\right\}\right\| \leqq \|\xi_i(n,t)-\xi_i(t)\| \rightarrow 0 \text{ as } n \rightarrow \infty ,$$

$i=1,.,N$.

And hence

(6.6.4c) In order that $\hat{\xi}(t|S_t)$ be stable in the sense of (6.6.3), it is sufficient that

$$\mathcal{P}(n,m)\varphi \to \mathcal{P}\varphi \quad \text{in q.m. as } n,m \to \infty, \text{ for all } \varphi \in H .$$

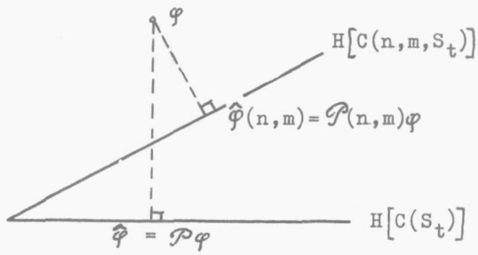
If $\varphi \in H$, let us write

$$\hat{\varphi} = \mathcal{P}\varphi$$

and

$$\hat{\varphi}(n,m) = \mathcal{P}(n,m)\varphi$$

Then (6.6.4c) reads:



(6.6.4d) In order that $\hat{\xi}(t|S_t)$ be stable in the sense of (6.6.3), it is sufficient that

$$\hat{\varphi}(n,m) \to \hat{\varphi} \quad \text{in q.m. as } n,m \to \infty, \text{ for all } \varphi \in H ,$$

where $\hat{\varphi}$ and $\hat{\varphi}(n,m)$ are characterized by

$$\hat{\varphi} \in H[C(S_t)] , \qquad \hat{\varphi}(n,m) \in H[C(n,m,S_t)],$$

and

$$\varphi - \hat{\varphi} \perp H[C(S_t)] , \qquad \varphi - \hat{\varphi}(n,m) \perp H[C(n,m,S_t)] \text{ or equivalently}$$

$$E\{\varphi - \hat{\varphi}\}\zeta_j(s) = 0 \quad \text{and} \quad E\{\varphi - \hat{\varphi}(n,m)\}\zeta_j(n,m,s) = 0 ,$$

$s \in S_t$, $j=1,..,M$, $n,m=1,2,..$, see (6.1.4).

(6.6.4e) i) $E\{\hat{\varphi}(n,m) - \hat{\varphi}\}\zeta_j(s) \to 0$ and ii) $E\{\hat{\varphi}(n,m) - \hat{\varphi}\}\zeta_j(n,m,s) \to 0$

if $n,m \to \infty$, $s \in S_t$, $j=1,..,M$.

Proof: i) On account of (6.6.4d),

$$E\{\hat{\varphi}(n,m) - \hat{\varphi}\}\zeta_j(s) = E\{\varphi - \hat{\varphi}(n,m)\}\{\zeta_j(n,m,s) - \zeta_j(s)\} .$$

Because of

$$\|\varphi - \hat{\varphi}(n,m)\| \leq \|\varphi\| + \|\mathcal{P}(n,m)\varphi\| \leq 2\|\varphi\|$$

and, according to (6.6.2d),

$$\| \zeta_j(n,m,s) - \zeta_j(s) \| \rightarrow 0 \text{ as } n,m \rightarrow \infty ,$$

it follows from the inequality of Schwarz, as $n,m \rightarrow \infty$, that

$$\left| E\left\{ \hat{\varphi}(n,m) - \hat{\varphi} \right\} \zeta_j(s) \right| \leqq \| \varphi - \hat{\varphi}(n,m) \| . \| \zeta_j(n,m,s) - \zeta_j(s) \| \rightarrow 0 .$$

ii) Similarly,

$$\left| E\left\{ \hat{\varphi}(n,m) - \varphi \right\} \zeta_j(n,m,s) \right| = \left| E\left\{ \varphi - \hat{\varphi} \right\} \left\{ \zeta_j(n,m,s) - \zeta_j(s) \right\} \right| \leqq$$

$$\leqq 2 \| \varphi \| . \| \zeta_j(n,m,s) - \zeta_j(s) \| \rightarrow 0 \text{ as } n,m \rightarrow \infty .$$

(6.6.4f) The condition

$$\hat{\varphi}(n,m) \rightarrow \hat{\varphi} \text{ in q.m. as } n,m \rightarrow \infty , \quad \varphi \in H ,$$

in (6.6.4d) is equivalent to

$$E\left\{ \hat{\varphi}(n,m) - \hat{\varphi} \right\} \hat{\varphi}(n,m) \rightarrow 0 \text{ as } n,m \rightarrow \infty , \quad \varphi \in H .$$

Proof: The first condition is equivalent to

$$E\left\{ \hat{\varphi}(n,m) - \hat{\varphi} \right\}^2 \rightarrow 0 \text{ as } n,m \rightarrow \infty ,$$

i.e.

$$E\left\{ \hat{\varphi}(n,m) - \hat{\varphi} \right\} \hat{\varphi}(n,m) - E\left\{ \hat{\varphi}(n,m) - \hat{\varphi} \right\} \hat{\varphi} \rightarrow 0 \text{ as } n,m \rightarrow \infty .$$

So, in order to proof the statement it is sufficient to show

$$E\left\{ \hat{\varphi}(n,m) - \hat{\varphi} \right\} \hat{\varphi} \rightarrow 0 \text{ as } n,m \rightarrow \infty .$$

Since $\hat{\varphi} \in H[C(S_t)]$, $\hat{\varphi}$ is the strong limit of a sequence, whose members are finite linear combinations of elements of $C(S_t)$. Hence, given $\varepsilon > 0$, there is a decomposition

$$\hat{\varphi} = \zeta + \psi$$

such that

$$\zeta = \sum_{j=1}^{M} \sum_{k=1}^{K} a_{jk} \zeta_j(s_k) \quad \text{and} \quad \| \psi \| < \varepsilon ,$$

where the coefficients a_{jk} are real numbers and $s_k \in S_t$, $k=1,..,K$. So

$$\left| E\left\{ \hat{\varphi}(n,m) - \hat{\varphi} \right\} \hat{\varphi} \right| \leqq \left| E\left\{ \hat{\varphi}(n,m) - \hat{\varphi} \right\} \zeta \right| + \left| E\left\{ \hat{\varphi}(n,m) - \hat{\varphi} \right\} \psi \right| \leqq$$

$$\sum_{j=1}^{M} \sum_{k=1}^{K} |a_{jk}| . \left| E\left\{ \hat{\varphi}(n,m) - \hat{\varphi} \right\} \zeta_j(s_k) \right| + \left| E\left\{ \mathcal{P}(n,m)\varphi - \mathcal{P}\varphi \right\} \psi \right| .$$

The first term in the right-hand side tends to 0 as $n,m \to \infty$ on account of i in (6.6.4e). And by virtue of the inequality of Schwarz,

$$\left| E\{\mathscr{P}(n,m)\varphi - \mathscr{P}\varphi\}\psi \right| \leq \|\mathscr{P}(n,m)\varphi - \mathscr{P}\varphi\| \cdot \|\psi\| \leq 2\|\varphi\|\varepsilon \ .$$

Hence, on account of (6.6.4d) and (6.6.4f):

(6.6.4g) In order that $\hat{\xi}(t|S_t)$ be stable in the sense of (6.6.3), it is sufficient that

$$E\{\hat{\varphi}(n,m) - \hat{\varphi}\}\hat{\varphi}(n,m) \to 0 \quad \text{as} \quad n,m \to \infty \ , \text{ for all } \varphi \in H \ .$$

In spite of ii in (6.6.4e), an approach to $E\{\hat{\varphi}(n,m) - \hat{\varphi}\}\hat{\varphi}(n,m)$ similar to the method in (6.6.4f) breaks down since $\hat{\varphi}(n,m)$ may vary with n and m. The result in subsection (6.6.6) shows that the condition in (6.6.4g) is not fulfilled in general.

(6.6.5) In this subsection we shall show the stability of $\hat{\xi}(t|S_t)$ in the sense of definition (6.6.3), in case S_t is a finite set.

Let t be an arbitrary value in $[0,T]$, and let

$$S_t = \{s_k \ , \ k=0,1,..,K\} \ , \quad \text{such that} \quad 0 = s_o < s_1 < .. < s_K \leq T \ .$$

It is for sake of convenience that we put $0 \in S$. This may be done without loss of generality as according to (6.6.1a) and (6.6.2b)

$$\zeta_j(0) = \zeta_j(n,m,0) = 0 \ , \quad j=1,..,M \ , \quad n,m=1,2,.. \quad .$$

Now also the classes of observations

$$C(S_t) = \{\zeta_j(s_k) \ , \ j=1,..,M \ , \ k=0,..,K\}$$

and

$$C(n,m,S_t) = \{\zeta_j(n,m,s_k) \ , \ j=1,..,M \ , \ k=0,..,K\} \ , \quad n,m=1,2,.. \quad ,$$

are finite. Hence the Hilbert spaces

$$H[C(S_t)] \quad \text{and} \quad H[C(n,m,S_t)] \ , \quad n,m=1,2,.. \quad ,$$

are finite dimensional - Euclidean - in this case. They may also be generated by the elements of the classes of differences

$$D(S_t) = \left\{ \zeta_{jk} = \zeta_j(s_k) - \zeta_j(s_{k-1}) \ , \ j=1,..,M \ , \ k=1,..,K \right\}$$

and

$$D(n,m,S_t) = \left\{ \zeta_{jk}(n,m) = \zeta_j(n,m,s_k) - \zeta_j(n,m,s_{k-1}), \ j=1,..,M, \ k=1,..,K \right\}$$

respectively. As we also introduce

$$\eta_{jk}(n) = \eta_j(n,s_k) - \eta_j(n,s_{k-1})$$

and

$$\tilde{\beta}_{jk}(m) = \tilde{\beta}_j(m,s_k) - \tilde{\beta}_j(m,s_{k-1}) \ ,$$

then , since $\zeta_j(n,m,s) = \eta_j(n,s) + \tilde{\beta}_j(m,s)$, it follows that

$$\zeta_{jk}(n,m) = \eta_{jk}(n) + \tilde{\beta}_{jk}(m) \ , \ j=1,..,M \ , \ k=1,..,K \ , \ n,m=1,2,..$$

So, on account of (6.6.4e),

<u>(6.6.5a)</u>

$$\begin{cases} E\left\{\hat{\varphi}(n,m) - \hat{\varphi}\right\} \zeta_{jk} \rightarrow 0 \quad \text{and} \\ E\left\{\hat{\varphi}(n,m) - \hat{\varphi}\right\} \zeta_{jk}(n,m) \rightarrow 0 \\ \text{as} \quad n,m \rightarrow \infty \ , \quad j,k=1,..,M \ . \end{cases}$$

Also here, condition (6.2.1b,c), see (6.6.1b), i.e.
$\tilde{B}(s) \geqq e\, I_M$, $e > 0$, $s \in [0,T]$, and $\tilde{B}(s)$ continuous on $[0,T]$,
is essentially needed.

<u>(6.6.5b)</u>

$$\int_{s_{k-1}}^{s_k} \tilde{B}(s)ds \geqq e(s_k - s_{k-1})\, I_M \ .$$

For, if X is the sphere of column M-vectors x with real valued components such that $x^T x = 1$, see (6.2.1), then for all $x \in X$,

$$x^T\left[\int_{s_{k-1}}^{s_k} \tilde{B}(s)ds\right]x = \int_{s_{k-1}}^{s_k}\left[x^T\tilde{B}(s)x\right]ds \geqq \int_{s_{k-1}}^{s_k} e\, ds = e(s_k - s_{k-1}) \ .$$

<u>(6.6.5c)</u> If

$$A > 0$$

and if

$$\left\{ A(m) \ , \ m=1,2,.. \right\}$$

is a sequence of $L \times L$-matrices with real valued entries $A_{ij}(m)$ such that

$$A_{ij}(m) \rightarrow A_{ij} \quad \text{as} \quad m \rightarrow \infty \ , \quad i,j=1,..,L \ ,$$

then there is a number r such that

$$A(m) > 0 \quad \text{if} \quad m > r \ .$$

For, as $A > 0$, there is a number $a > 0$ such that $A \geqq a I_L$. So, if Y is the sphere of column L-vectors y with real valued components such that $y^T y = 1$, then if $y \in Y$ (hence also $|y_i| \leqq 1$, $i=1,.,L$) ,

$$\left| y^T \{A-A(m)\} y \right| = \left| \sum_{i,j=1}^{L} \{A_{ij}(m)-A_{ij}\} y_i y_j \right| \leqq \sum_{i,j=1}^{L} \left| A_{ij}(m)-A_{ij} \right| < \tfrac{1}{2}a$$

if m is sufficiently large, say $m > r$, i.e.

$$A - \tfrac{1}{2}aI_L < A(m) < A + \tfrac{1}{2}aI_L .$$

And as $A \geqq aI_L > 0$,

$$A(m) \geqq A - \tfrac{1}{2}aI_L \geqq \tfrac{1}{2}aI_L > 0 \quad \text{if } m > r .$$

In order to show the stability of $\hat{\xi}(t|S_t)$, according to (6.6.4g) it is sufficient to establish

$$E\{\hat{\varphi}(n,m) - \hat{\varphi}\} \hat{\varphi}(n,m) \to 0 \quad \text{as } n,m \to \infty , \quad \varphi \in H .$$

Since $\hat{\varphi}(n,m) \in H[C(n,m,S_t)] = H[D(n,m,S_t)]$, $n,m=1,2,..$, $H[D(n,m,S_t)]$ being the Euclidean space generated by the elements of $D(n,m,S_t)$, $\hat{\varphi}(n,m)$ may be decomposed as

(6.6.5d) $$\hat{\varphi}(n,m) = \sum_{j=1}^{M} \sum_{k=1}^{K} a_{jk}(n,m)\, \zeta_{jk}(n,m),$$

where the coefficients $a_{jk}(n,m)$ are real numbers.

Given $\hat{\varphi}(n,m)$, the coefficients $a_{jk}(n,m)$ are not necessarily unique, since the elements $\zeta_j(n,m,s_k)$ and hence the elements $\zeta_{jk}(n,m)$ might not be linearly independent (if m is small) .

(6.6.5e) Given $\varphi \in H$, there is a number r , such that the coefficients $a_{jk}(n,m)$ in (6.6.5d) are uniformly bounded in n and m , provided that $m > r$.

Proof: According to (6.6.5d),

$$\mathcal{P}(n,m)\varphi = \hat{\varphi}(n,m) = \sum_{j=1}^{M} \sum_{k=1}^{K} a_{jk}(n,m)\, \eta_{jk}(n) + \sum_{j=1}^{M} \sum_{k=1}^{K} a_{jk}(n,m)\, \tilde{\beta}_{jk}(m) .$$

By virtue of (6.6.2b) the two terms in the right-hand side are orthogonal . Hence

$$\|\varphi\|^2 \geqq \|\mathcal{P}(n,m)\varphi\|^2 =$$

(6.6.5f) $$= \Big\|\sum_{j=1}^{M}\sum_{k=1}^{K} a_{jk}(n,m)\,\eta_{jk}(n)\Big\|^2 + \Big\|\sum_{j=1}^{M}\sum_{k=1}^{K} a_{jk}(n,m)\,\tilde{\beta}_{jk}(m)\Big\|^2 \geqq \Big\|\sum_{j=1}^{M}\sum_{k=1}^{K} a_{jk}(n,m)\,\tilde{\beta}_{jk}(m)\Big\|^2 .$$

Let us introduce the row MK-vector

$$a^T(n,m) = (\, a_{11}(n,m) \,..\, a_{M1}(n,m), \;\; .. \;\; , a_{1K}(n,m) \,..\, a_{MK}(n,m) \;)$$

and the covariance $MK \times MK$-matrix

$$A(m) = \begin{pmatrix} (A_{11})(m) & \dots\dots & (A_{1K})(m) \\ \vdots & & \vdots \\ (A_{K1})(m) & \dots\dots & (A_{KK})(m) \end{pmatrix}$$

whose $M \times M$-submatrices (A_{kh}) are defined as

$$(A_{kh})(m) = E \begin{pmatrix} \tilde{\beta}_{1k}(m) \\ \vdots \\ \tilde{\beta}_{Mk}(m) \end{pmatrix} \cdot \Big(\tilde{\beta}_{1h}(m) \;\dots\; \tilde{\beta}_{Mh}(m) \Big) =$$

$$= E\Big[\tilde{\beta}(m,s_k) - \tilde{\beta}(m,s_{k-1})\Big] \cdot \Big[\tilde{\beta}(m,s_h) - \tilde{\beta}(m,s_{h-1})\Big]^T , \quad k,h=1,.,K \;,\; n,m=1,2,..$$

Then (6.6.5f) reads

(6.6.5g) $$\|\varphi\|^2 \geq a^T(n,m)A(m)a(n,m) .$$

By virtue of the properties of $\tilde{\beta}(m,s)$ and the Wiener-Lévy process $\tilde{\beta}(s)$ it follows as $m \to \infty$, that the elements of $(A_{kh})(m)$ tend to 0 if $k \neq h$, and to the corresponding elements of the $M \times M$-matrix $\int_{s_{k-1}}^{s_k} \tilde{B}(s)ds$ if $k = h$. Hence as $m \to \infty$, the elements of $A(m)$ tend to the corresponding elements of the covariance $MK \times MK$-matrix A, defined as

$$A = \begin{pmatrix} \int_0^{s_1} \tilde{B}(s)ds & \dots & (0) \\ \vdots & & \vdots \\ (0) & \dots & \int_{s_{K-1}}^{s_K} \tilde{B}(s)ds \end{pmatrix}$$

On account of (6.6.5b), $\int_{s_{k-1}}^{s_k} \tilde{B}(s)ds \geqq e(s_k - s_{k-1})I_M$, k=1,.,K , and hence

$$A \geqq edI_{MK} > 0 \quad , \quad d = \min_{k=1,.,K} (s_k - s_{k-1}) ,$$

where I_{MK} is the MK×MK-identity matrix.

Owing to (6.6.5c) it follows from the convergence of the elements of $A(m)$ to the corresponding elements of A that there is a number r such that

$$A(m) \geqq \tfrac{1}{2}edI_{MK} \quad \text{as} \quad m > r ,$$

i.e.

$$\tfrac{1}{2}ed \sum_{j=1}^{M} \sum_{k=1}^{K} a_{jk}^2(n,m) \leqq a^T(n,m)A(m)a(n,m) .$$

Finally, according to (6.6.5g),

$$\sum_{j=1}^{M} \sum_{k=1}^{K} a_{jk}^2(n,m) \leqq 2 \frac{\|\varphi\|}{ed}$$

if $m > r$ and for all n , showing the uniform boundedness of the coefficients $a_{jk}(n,m)$, asserted in (6.6.5e).

(6.6.5h) Theorem: If the number of observations is finite, i.e. if S_t is a finite set, then $\hat{\xi}(t|S_t)$ is stable with respect to differentiable perturbations of the involved Wiener-Lévy processes, independently of the position of t and S_t in $[0,T]$.

Proof: Owing to (6.6.4g) it is sufficient to show that

$$E\{\hat{\varphi}(n,m) - \hat{\varphi}\}\, \hat{\varphi}(n,m) \to 0 \quad \text{as} \quad n,m \to \infty \text{ , for all } \varphi \in H .$$

Let a be the bound of the coefficients $a_{jk}(n,m)$, n=1,2,.. , $m > r$, established in (6.6.5e). Then on account of (6.6.5d) and (6.6.5a),

$$\left| E\{\hat{\varphi}(n,m) - \hat{\varphi}\}\, \hat{\varphi}(n,m) \right| = \left| \sum_{j=1}^{M} \sum_{k=1}^{K} a_{jk}(n,m) E\{\hat{\varphi}(n,m) - \hat{\varphi}\}\, \zeta_{jk}(n,m) \right| \leqq$$

$$\leqq a \sum_{j=1}^{M} \sum_{k=1}^{K} \left| E\{\hat{\varphi}(n,m) - \hat{\varphi}\}\, \zeta_{jk}(n,m) \right| \to 0 \quad \text{as} \quad n,m \to \infty .$$

(6.6.6) We shall show that the estimate $\hat{\xi}(t|S_t)$ does not need to be stable in the sense of definition (6.6.3) in case S_t is an infinite set.

Consider the Kalman-Bucy filter. There $S_t = [0,t]$. Assume $M=N=1$. This restriction is not essential and serves merely for simplifying the notation.

Assume moreover that the 1-dimensional perturbed Wiener-Lévy processes $\beta(n,t)$ and $\tilde{\beta}(m,s)$ are of the kind, discussed in section 4.3. Then they have a finite degree of randomness. Let $\beta(n,t)$ and $\tilde{\beta}(m,s)$ be constructed by means of the partitions p and q respectively, where

p is defined by $0 = t_o < t_1 < .. < t_P = T$

and q by $0 = s_o < s_1 < .. < s_Q = T$.

Assume that the position of t in $[0,T]$ is such that

$$t_{P'-1} < t \leq t_{P'} \quad , \quad s_{Q'-1} < t \leq s_{Q'} \quad ,$$

$t_{P'}$ and $s_{Q'}$ being subdivision points of p and q respectively.

(6.6.6a) Obviously, the (1-dimensional) solution $\xi(n,t)$ to the state equation is some linear combination of the elements

$$\gamma \ , \ \beta(t_1) \ , \ .. \ , \ \beta(t_{P'}) \ .$$

Here $\beta(t_o)$ may be omitted since $\beta(t_o) = \beta(0) = 0$.

(6.6.6b) Obviously, the (1-dimensional) observations $\zeta(n,m,s)$, $s \in [0,t]$, are linear combinations of the elements of the class

(6.6.6c) $$\{\gamma, \beta(t_1) , .. , \beta(t_{P'}) , \tilde{\beta}(s_1) , .. , \tilde{\beta}(s_{Q'})\}$$

In (6.6.6c), $\beta(t_o)$ and $\tilde{\beta}(s_o)$ may be omitted since they are identical to 0 . Now the elements in (6.6.6c) are linearly independent. Hence there is a unique decomposition

(6.6.6d) $$\zeta(n,m,s) = u(s)\gamma + v_1(s)\beta(t_1) + .. + v_{P'}(s)\beta(t_{P'}) + \\ + w_1(s)\tilde{\beta}(s_1) + .. + w_{Q'}(s)\tilde{\beta}(s_{Q'}) \text{ at each } s \in [0,t] .$$

The coefficient functions are mappings of $[0,t]$ into $(-\infty,\infty)$. In non-trivial systems they are linearly independent (hence also non-0) on $[0,t]$. Hence there is a set of values

$$\{ x_1 , \dots , x_{1+P'+Q'} \} \subset [0,t]$$

such that the $(1+P'+Q')^2$-matrix

$$X = \begin{pmatrix} u(x_1) & v_1(x_1) & \cdot & v_{P'}(x_1) & w_1(x_1) & \cdot & w_{Q'}(x_1) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ u(x_{1+P'+Q'}) & v_1(x_{1+P'+Q'}) & \cdot & v_{P'}(x_{1+P'+Q'}) & w_1(x_{1+P'+Q'}) & \cdot & w_{Q'}(x_{1+P'+Q'}) \end{pmatrix}$$

is non-singular.

Then by means of the $1+P'+Q'$ relations, obtained by substituting successively

$$s = x_1 , \quad \dots \quad , s = x_{1+P'+Q'}$$

into (6.6.6d), the elements of class (6.6.6c) may be linearly expressed in

$$\zeta(n,m,x_1) , \quad \dots \quad , \zeta(n,m,x_{1+P'+Q'})$$

as follows:

(6.6.6e)
$$\begin{pmatrix} \gamma \\ (t_1) \\ \cdot \\ \cdot \\ (t_{P'}) \\ (s_1) \\ \cdot \\ \cdot \\ (s_{Q'}) \end{pmatrix} = X^{-1} \cdot \begin{pmatrix} \zeta(n,m,x_1) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \zeta(n,m,x_{1+P'+Q'}) \end{pmatrix}$$

Let $E(n,m)$ be the Eclidean space, generated by the elements of class (6.6.6c), and - as always - let $H[C(n,m, [0,t])]$ be the Hilbert (Euclidean) space, generated by the elements $\zeta(n,m,s)$, $s \in [0,t]$.

It follows from (6.6.6b) that $H[C(n,m, [0,t])] \subset E(n,m)$ and from (6.6.6e) that $H[C(n,m, [0,t])] \supset E(n,m)$.

Hence

(6.6.6f)
$$H[C(n,m, [0,t])] = E(n,m) .$$

According to (6.6.6a), $\xi(n,t) \in E(n,m)$ and hence owing to (6.6.6f) $\xi(n,t) \in H[C(n,m, [0,t])]$. Then necessarily $\hat{\xi}(n,t | [0,t]) = \xi(n,t)$.

By virtue of (5.2.1), $\xi(n,t) \to \xi(t)$ in q.m. as $n \to \infty$.
Hence

$$\hat{\xi}(n,t \mid [0,t]) = \xi(n,t) \to \xi(t) \text{ in q.m. as } n \to \infty .$$

Since $\hat{\xi}(t \mid [0,t]) = \hat{\xi}(t \mid t)$ satisfies equation (6.5.16), it is seen that in non-trivial cases $\hat{\xi}(t \mid [0,t]) \neq \xi(t)$. Hence in the above situation $\hat{\xi}(t \mid [0,t]) = \hat{\xi}(t \mid t)$ may be not stable in the sense of definition (6.6.3).

<u>(6.6.7)</u> Let S be an arbitrary subset of $[0,T]$, and t an arbitrary value in $[0,T]$.

Let

$$\{ S_k \, , \, k=1,2,.. \}$$

be a sequence of finite subsets of S, increasing to S or to any set dense in S, as $k \to \infty$.

We recall, see (6.6.2d), that $\zeta_j(n,m,s)$ and $\zeta_j(s)$, $j=1,.,M$, $n,m = 1,2,..$, are continuous in q.m. on $[0,T]$. Hence (6.1.8) applies.

Let n_i and m_j pass to infinity through sequences

$$\{ n_1 \, , \, n_2 \, , \, .. \} \quad \text{and} \quad \{ m_1 \, , \, m_2 \, , \, .. \}$$

respectively.

Then owing to (6.1.8) and (6.6.5h) we may present the following diagram. The arrow means "converges in q.m. to" .

$$\begin{array}{ccccc}
\hat{\xi}(n_1,m_1,t \mid S_1) & \hat{\xi}(n_2,m_2,t \mid S_1) & \cdots\cdots & \longrightarrow & \hat{\xi}(t \mid S_1) \\
\hat{\xi}(n_1,m_1,t \mid S_2) & \hat{\xi}(n_2,m_2,t \mid S_2) & \cdots\cdots & \longrightarrow & \hat{\xi}(t \mid S_2) \\
\cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots & & \cdots\cdots \\
\downarrow & \downarrow & & & \downarrow \\
\hat{\xi}(n_1,m_1,t \mid S) & \hat{\xi}(n_2,m_2,t \mid S) & \cdots\cdots & \not\longrightarrow & \hat{\xi}(t \mid S)
\end{array}$$

The convergence in q.m. along the vertical lines is shown in (6.1.8).

The possible behaviour along the last horizontal line is illustrated in the previous subsection (6.6.6).

The convergence along the other horizontal lines is established in theorem (6.6.5h).

Some conclusions and remarks:

There is no hope for better estimates when increasing the number of observations in a given interval indefinitely, because of the possible instability of $\hat{\xi}(t|S)$. The question, how many observations would be the best, is unanswered.

The (discretized) computation schemes and formulae, belonging to $\hat{\xi}(t|S)$ may be used if it is convenient, as in the Kalman-Bucy filter e.g. This is seen in the last vertical line. Then the result is stable with respect to small perturbations of (the) finitely many observations (used).

The above diagram is applicable to a large class of estimates, comprising predicted, filtered and smoothed estimates of an endless variety, both of Kalman-Bucy type and of the type of Wiener and Kolmogorov.

We have explained at the beginning of this section, that the Kalman-Bucy filter breaks down if the involved Wiener-Lévy processes are differentiably perturbed. It might be hard to design a mathematically exact computation scheme to the estimates of the last horizontal line in the diagram.

The convergence in the other horizontal lines of the diagram depends entirely on the assumption (6.2.1b,c),

$$\tilde{B}(s) > 0 \text{ and continuous on } [0,T] .$$

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S T E L L I N G E N

1. Het is weinig zinvol om Ito-calculus te gebruiken bij de behandeling van de stochastische lineaire systemen die voorkomen in dit proefschrift.

   Overigens bieden de hier gebezigde methoden geen uitzicht op generalisatie met betrekking tot algemenere (Ito-)systemen.

2. Heuristische beschouwingen die dienen om het Kalman-Bucy filter met continue tijd-parameter geloofwaardig te maken zijn weinig overtuigend of onhoudbaar.

3. Het "colored noise filter" van Bucy is in wezen een gewoon Kalman-Bucy filter. Zie
   R.S. Bucy, Optimal filtering for correlated noise, J. of mathematical analysis and applications 20, 1967
   en
   Bucy and Joseph, Filtering for stochastic processes with applications to guidance, Interscience publishers, 1968.

   "Augmented state" methoden en "noise whitening" technieken voeren tot gewone Kalman-Bucy filters. Zie
   Bryson and Johansen, Linear filtering for time-varying systems using measurements containing colored noise, IEEE Trans. Automatic Control, AC-10, 1, 1965
   en
   Nahi, Estimation theory and applications, Wiley, 1969.

   De geperturbeerde filters in § 6.6 van dit proefschrift zijn van fundamenteel andere aard dan Kalman-Bucy filters.

4. Aan de gevoeligheid van het Kalman-Bucy filter voor onnauwkeurigheden in de gemeten of geschatte grootheden in de filtervergelijkingen wordt de nodige aandacht besteed, zie bijvoorbeeld
   Jazwinski, Stochastic processes and filtering theory, Academic Press, 1970.
   De gevoeligheid voor verstoringen van de alles beheersende, en fysisch niet eens realiseerbare Wiener-Lévy processen wordt echter buiten beschouwing gelaten.

5. In
   P.L. Falb, The Kalman-Bucy filter in Hilbert space, Information and control 11, 1967,
   wordt de mathematische behandeling van het Kalman-Bucy filter nodeloos ingewikkeld gemaakt, terwijl de kern van de zaak niet wordt geraakt. De bewering van de schrijver, dat een "fully rigourus theory" verkregen is, is aanvechtbaar. Zie ook
   Kalman, Falb, Arbib, Topics in mathematical system theory, McGraw-Hill, 1969.

6. De meerderheid van de publicaties in de Engelse taal op het gebied van stochastische systeem analyse is van een onvoldoende mathematisch gehalte.

Dit kan in verband gebracht worden met de invloed die is uitgegaan van het ministerie van defensie van de Verenigde Staten van Amerika, getuige bijvoorbeeld het dankwoord van Richard Bellman, uitgesproken ter gelegenheid van het in ontvangst nemen van de "Norbert Wiener Prize in Applied Mathematics".

7. Als $\xi(t)$ een reëel stochastisch tweede orde proces is op $[0,T]$, dan is het kwadraat van de totale variatie van $E\,\xi(t)^2$ op $[0,T]$ kleiner dan, of gelijk aan de totale variatie van $E\,\xi(s)\xi(t)$ op $[0,T]^2$.

8. Van een Hilbert ruimte X wordt de sterke topologie beschouwd. Gegeven is de continue afbeelding $\xi(t)$ van $(-\infty,\infty)$ in X. Het gedrag van $\xi(t)$ in de buurt van $+\infty$ en $-\infty$ is zodanig dat de afbeeldingen

$$\xi_n(t) = \int_{-\infty}^{\infty} e^{-\frac{n^2 s^2}{2}}\, \xi(t-s)\, ds\,, \quad t \in (-\infty,\infty)\,, \quad n = 1,2,\ldots,$$

van $(-\infty,\infty)$ in X gedefinieerd zijn.

De vectoren $\xi(t)$ spannen een lineaire deelruimte van X op, waarvan de afsluiting in X de Hilbert ruimte H zij. Op analoge wijze brengt $\xi_n(t)$ de Hilbert ruimte H_n voort.

Als $\mathcal{P}$ de orthogonale projector is van X op H, en $\mathcal{P}_n$ die van X op H_n, dan is

$$\lim_{n\to\infty} \mathcal{P}_n \xi = \mathcal{P}\xi \quad \text{voor iedere } \xi \in X\,.$$

9. De openheid die thans wordt betracht inzake de adoptie van pleegkinderen is bevorderlijk voor de geestelijke gezondheid, de ontplooiing en de inpassing in de maatschappij van het geadopteerde kind.

The Kalman-Bucy filter and its behaviour with respect to smooth perturbations of the involved Wiener-Lévy processes