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Option pricing with perturbation methods

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"Option pricing with perturbation methods"

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Abstract

This thesis discusses the use of perturbation theory in the context of financial mathematics, in particular on the use of matched asymptotic expansions in option pricing.

Our methods are applied to the ordinary Black-Scholes model for illustration. In this simple example of the Black-Scholes model an exact solution is available, so it is in fact not neccessary to apply the method of asymptotic expansions on this model. However, in case we do apply the method, two artificial layers have to be constructed. Making smart choices for the local variables leads to a transformation of the equations into a heat equation, which can easily be solved. Finally, the results are compared to a Taylor expansion of the exact solution to see that this method is very accurate.

After this first instructive model, the method of matched asymptotic expansions is applied to two more advanced models based on papers by Howison [7] and Hagan *et al.* [5]. Here, different choices for the scalings are made.

The former discusses a fast mean-reverting stochastic volatility model that turns out to have many open ends. In Howison's paper [7] quite a lot of assumptions and simplifications are made. Unfortunately, often the motivation for them is not explicitly given in the paper, and in some cases we even think these assumptions and simplifications are incorrect.

The latter examines a new three-parameter stochastic volatility model that successfully prices back the volatility smile as observed in the market nowadays, and that is commonly used. The derivation of this model is the main focus of this thesis. The resulting expression for the implied volatility under the SABR model is obtained by considering the forward and backward Kolmogorov equations per order in ε, making some smart choices for local variables and functions in order to transform them into an equation that looks like a heat equation, which is easier to solve.

Recommendations for further investigation on these models would be to consider several different choices for the scalings and see which one works best.

Preface

This thesis is the result of my Master of Science research project at the Delft University of Technology, The Netherlands. This project has been carried out at the faculty of Electrical Engineering, Mathematics and Computer Science, abbreviated as EEMCS, at the chair of Mathematical Physics and at Rabobank International, at the department of Derivatives Research & Validation.

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Chapter 1

Introduction

Modern financial research depends heavily on mathematics. This thesis focuses on the use of matched asymptotic expansions in option pricing. It presents illustrations of the approach in plain vanilla option valuation using the Black-Scholes model, using a fast mean-reverting (stochastic) volatility model, and in the stochastic α, β, ρ (SABR) model.

We begin in section 2 with explaining how to use the **method of matched asymptotic expansions** to solve a general **singularly perturbed problem**. To explain this method, we will study a simple (physical) example. Four steps have to be accomplished in order to complete the application of this method:

- First, an outer expansion has to be constructed. This leads to an outer solution of the problem.
- After that, the boundary layer has to be analized. For this, an inner expansion has to be made, which leads to an inner solution of the problem.
- Next, these two solutions have to be matched at their boundary, using matching conditions, to determine the remaining constants.
- Finally, we will combine these two solutions to form a composite expansion. This is done by adding the expansions and then subtracting the part that is common to both.

Section 3 is an introduction to the **computational finance subjects** in this thesis. Before presenting a mathematical description, it is necessary to clarify some economical definitions and nomenclature. Next, in section 4 the **financial framework** will be considered. Some formulas and theorems that are commonly used for financial models will be explained here.

In section 5 we will consider the first financial model: the **Black-Scholes model**. We will derive this basic model and show a reduction from the Black-Scholes equation to the heat equation. Because the Black-Scholes approach leads to partial differential equations, many physical applied mathematicians contributed to this field. These people mainly focused on obtaining exact solutions to certain boundary value problems representing the prices of options, or on numerical methods. The purpose of this thesis is to illustrate another technique: asymptotic analysis, with a particular emphasis on the use of matched asymptotic expansions. After showing the exact Black-Scholes solution, perturbation theory will be applied to this model. Finally, we will compare the results of applying perturbation theory with the exact solution.

The next financial model, in section 6, is the **constant elasticity of variance (CEV) model**. This model will be studied following the steps taken in Howison's paper [7]. The CEV model is a more realistic version of the 'ordinary' Black-Scholes model in chapter 5, because studies have shown that relative price variances do change as the stock price changes, while the Black-Scholes model assumes a constant stock price volatility, regardless of the level of the security price.

Boundary-layer techniques can also be applied in the analysis of **fast-mean-reverting stochastic volatility models**. We will look at this third financial model in section 7 and we will show how to construct the boundary layer near expiry for European options. In these models the volatility itself is assumed to follow a stochastic process while the asset price is assumed to follow the lognormal process as before.

European options are often priced and hedged using Black's model, or, equivalently, the Black-Scholes model. In the Black-Scholes model there is a one-to-one correspondence between the price of a European option and the volatility parameter σ_B. Consequently, option prices are often quoted by stating the implied volatility σ_B, the unique value of the volatility which yields the options dollar price when used in the Black-Scholes model. In theory, the volatility σ_B in the Black-Scholes model is a constant. In practice, options with different strikes K require different volatilities σ_B to match their market prices.

To resolve this problem, we derive the **stochastic α, β, ρ (SABR) model** in section 8. The SABR model is very well explained in the paper by Hagan *et al.* [5]. In reality, options with different strikes require different volatilities to match their market prices. So the volatility is assumed to follow a stochastic process again, and the asset price and volatility are correlated. Singular perturbation techniques are used to obtain the prices of European options under the SABR model. From these prices we obtain a closed-form algebraic formula for the implied volatility as a function of todays forward price f and the strike K. This market volatility smile is critical for hedging.

Chapter 2

Perturbation theory

In this chapter we will use the method of **matched asymptotic expansions**[1] to solve a **singularly perturbed problem**. To explain this method, we will study the following example:[2]

$$\varepsilon y'' + 2y' + 2y = 0, \quad \text{for } 0 < x < 1, \tag{2.0.1}$$

with boundary conditions $y(0) = 0$ and $y(1) = 1$. Here $y = y(x)$ and

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x} \quad \text{and} \quad y'' = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}.$$

To construct a first-term approximation of the solution for small ε we will proceed in four steps.[3]

Step 1: Outer solution
To begin, we will assume that the solution of the above problem can be expanded in powers of ε:

$$y_\varepsilon(x) = \sum_{n=0}^{m} \varepsilon^n y_n(x) + O\left(\varepsilon^{m+1}\right) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \dots. \tag{2.0.2}$$

Substituting this into problem (2.0.1), we obtain

$$\varepsilon\left(y_0'' + \varepsilon y_1'' + \dots\right) + 2\left(y_0' + \varepsilon y_1' + \dots\right) + 2\left(y_0 + \varepsilon y_1 + \dots\right) = 0. \tag{2.0.3}$$

By looking at all terms without ε we obtain the $\mathcal{O}(1)$ equation $y_0' + y_0 = 0$. The general solution of this $\mathcal{O}(1)$ equation is

$$y_0(x) = ae^{-x}, \tag{2.0.4}$$

where a is an arbitrary constant. Looking at this solution, we have a dilemma, because there is only one arbitrary constant but we have two boundary conditions. So the solution (2.0.4) and the expansion (2.0.2) are incapable of describing a solution over the entire interval $0 \le x \le 1$. At this moment we have no idea which boundary condition, if any, we should require $y_0(x)$ to satisfy. Based on what is observed in Example 2 in Section 1.7 of [6], it is a reasonable working hypothesis to assume that (2.0.4) describes a solution over most of the interval $0 \le x \le 1$, but there is a boundary layer at either $x = 0$ or at $x = 1$, where a different approximation must be used. Sometimes it's even possible to have several boundary layers at the same time.

[1] To define an asymptotic approximation, first the **order symbols** need to be introduced. See appendix B.
[2] This example is taken from [6], p.48-56.
[3] See page 34 of [12] for a summary of those 4 steps.

For now let's assume that we have a boundary layer at $x = 0$.[4] We will look for a solution in that area in step 2. The solution (2.0.4) is the first term in the expansion of the **outer solution**.

Step 2: Boundary layer
Based on the assumption that there is a boundary layer at $x = x_0 = 0$, we rescale the variable x by introducing a local variable ξ given as

$$\xi = \frac{x - x_0}{\delta(\varepsilon)} = \frac{x}{\delta(\varepsilon)}. \tag{2.0.5}$$

After changing variables from x to ξ, we will take ξ to be fixed when expanding the solution in terms of ε. This has the effect of stretching the area near $x_0 = 0$ when ε becomes small, such that we can analyze the problem locally. At this point we only know that $\delta(\varepsilon) = o(1)$, and we have no a priori knowledge of a suitable choice of $\delta(\varepsilon)$.

The equation with respect to this local variable ξ becomes

$$\frac{\varepsilon}{\delta(\varepsilon)^2}\frac{\partial^2 y^*}{\partial \xi^2} + \frac{2}{\delta(\varepsilon)}\frac{\partial y^*}{\partial \xi} + 2y^* = 0, \tag{2.0.6}$$

with boundary condition $y^*(0) = 0$. The boundary condition at $x = 0$ has been included here because the boundary layer is at the left end of the interval.

Just as with the algebraic equations studied in section 1.4 of [6], it is now neccessary to determine the correct balancing in the following equation:

$$\frac{\varepsilon}{\delta(\varepsilon)^2}\frac{\partial^2}{\partial \xi^2} y^* + \frac{2}{\delta(\varepsilon)}\frac{\partial}{\partial \xi} y^* + 2y^* = 0, \tag{2.0.7}$$

which contains three terms. The balance between the second and third term was already considered in Step 1, so the following two possibilities remain:

- term 1 $\sim$ term 3 and term 2 is of higher order in ε (and becomes smaller as ε becomes small):

  $$\frac{\varepsilon}{\delta(\varepsilon)^2} = 0 \Rightarrow \delta(\varepsilon) = \sqrt{\varepsilon}.$$

  Now term 3 is $\mathcal{O}(1)$ but term 2 is $\mathcal{O}(\varepsilon^{-1/2})$. This violates our original assumption that term 2 is higher order, and so this balance is not possible.

- term 1 $\sim$ term 2 and term 3 is higher order (and becomes smaller as ε becomes small):

  $$\frac{\varepsilon}{\delta(\varepsilon)^2} = \frac{2}{\delta(\varepsilon)} \Rightarrow \delta(\varepsilon) = \varepsilon.$$

  Now term 1 and 2 are $\mathcal{O}\left(\varepsilon^{-1}\right)$ and term 3 is $\mathcal{O}(1)$. In this case, the conclusions are consistent with the original assumptions, and so this is the balancing we are looking for. This is said to be a **distinguished limit**[5] for the equation.

[4] In this case, choosing the boundary layer to be at $x = 1$ gives no solution at all, because the matching in step 3 is not possible. See appendix C.

[5] Also known as **maximum balance** or **significant degeneration**.

Now the differential equation (2.0.7) becomes

$$\frac{\partial^2}{\partial \xi^2} y^* + 2\frac{\partial}{\partial \xi} y^* + 2\varepsilon y^* = 0. \tag{2.0.8}$$

Because $\delta(\varepsilon) = \varepsilon$ the appropriate expansion for the **boundary-layer solution** is given by

$$y^*(\xi) \sim y_0^*(\xi) + \varepsilon y_1^*(\xi) + \dots. \tag{2.0.9}$$

Substituting this expansion (2.0.9) into equation (2.0.8) gives

$$\frac{\partial^2}{\partial \xi^2} \left(y_0^* + \varepsilon y_1^* + \dots\right) + 2\frac{\partial}{\partial \xi} \left(y_0^* + \varepsilon y_1^* + \dots\right) + 2\varepsilon \left(y_0^* + \varepsilon y_1^* + \dots\right) = 0. \tag{2.0.10}$$

By taking a look at all terms of $\mathcal{O}(1)$, we have the following problem to solve:

$$\begin{cases} y_0^{*\prime\prime} + 2y_0^{*\prime} = 0, \quad \text{for } 0 < \xi < \infty, & \text{(2.0.11a)} \\ y_0^*(0) = 0. & \text{(2.0.11b)} \end{cases}$$

The general solution of this problem is $y_0^*(\xi) = A\left(1 - e^{-2\xi}\right)$, where A is an arbitrary constant. In this case, it should be observed that the **boundary-layer equation** (2.0.11) contains at least one term of the **outer-layer equation** (i.e., $y_0' + y_0 = 0$) in Step 1, to have a successful completion of the matching in Step 3. In general, this isn't always the case.

The boundary-layer expansion (2.0.9) is supposed to describe the solution in the immediate neighbourhood of the endpoint $x = 0$. It is therefore not unreasonable to expect that the outer solution (2.0.4) applies over the remainder of the interval (assuming there are no other layers). This means that the outer solution (2.0.4) should satisfy the boundary condition at $x = 1$ (i.e., $y(1) = 1$). So we find that $a = e^1$, which implies $y_0(x) = e^{1-x}$.

Step 3: Matching
It remains to determine the constant A in the first-term approximation of the boundary-layer solution $y_0^*(\xi) = A\left(1 - e^{-2\xi}\right)$. To do this, the approximations we have constructed so far are summarised in figure 2.2 on page 52 of [6]. The important point here is that both the inner (boundary-layer) and outer expansions are approximations of the same function. Therefore, in the transition region between the inner and outer layers we should expect that the two expansions give the same result. This is accomplished by requiring that the value of y_0^* as one comes out of the boundary-layer (as $\xi \to \infty$) is equal to the value of y_0 as one comes into the boundary layer (as $x \to 0$). Imposing the condition $y_0^*(\infty) = y_0(0)$ yields $A = e^1$ and the solution becomes

$$y_0^*(\xi) = e\left(1 - e^{-2\xi}\right) = e^1 - e^{1-2\xi}. \tag{2.0.12}$$

This completes the derivation of the inner and outer approximations of the solution of the example problem (2.0.1). A plot of these solutions can be seen in figure 2.1. The next step is to combine them into a single expression (Step 4).

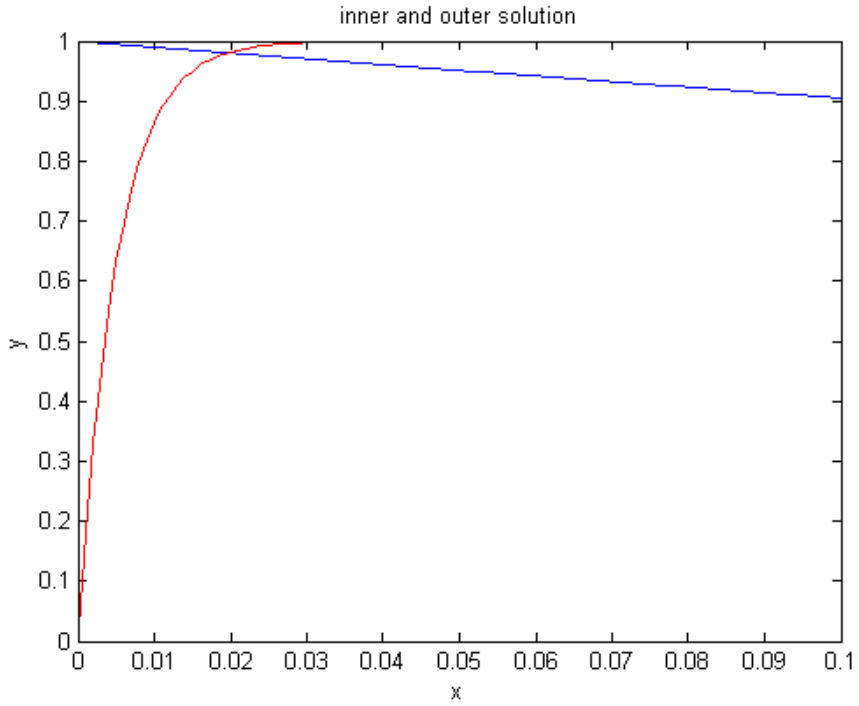

Figure 2.1: The outer (blue) and inner (red) solution of the example problem.

Step 4: Composite expansion
Our description of the solution consists of two pieces, which we will now combine to form a so-called **composite expansion**. This is done by adding the expansions and then subtracting the part that is common to both. Thus,

$$\begin{aligned} y &\sim y_0(x) + y_0^*\left(\frac{x}{\varepsilon}\right) - y_0(0), \\ &\sim e^{1-x} - e^{1-2x/\varepsilon}. \end{aligned} \tag{2.0.13}$$

It may seem strange that it is possible to combine expansions over different intervals and still have an asymptotic approximation of the solution. However, note that the boundary-layer solution $y_0^*(x/\varepsilon)$ is constant to first order inside the outer region. This constant is $y_0(0)$ and to compensate for its contribution, the term $-y_0(0)$ is added to the composite expansion. Similarly, the outer solution $y_0(x)$ is also constant to first order inside the boundary-layer region. However, the term $-y_0(0)$ removes its contribution in this region. The fact that the adjustment in each case involves the constant $y_0(0)$ is not a coincidence, since it is the first term in the inner and outer expansions found from matching: $y_0(0)$ is called the **common part of the expansions**. The composite expansion gives a very good approximation to the solution over the entire interval, see figure 2.4 on page 54 of [6].

In figure 2.2 one can see the composite expansion: this is a combination of the outer (blue) and inner (red) solution.

In the next chapters, we will apply perturbation theory to a couple of different financial models.

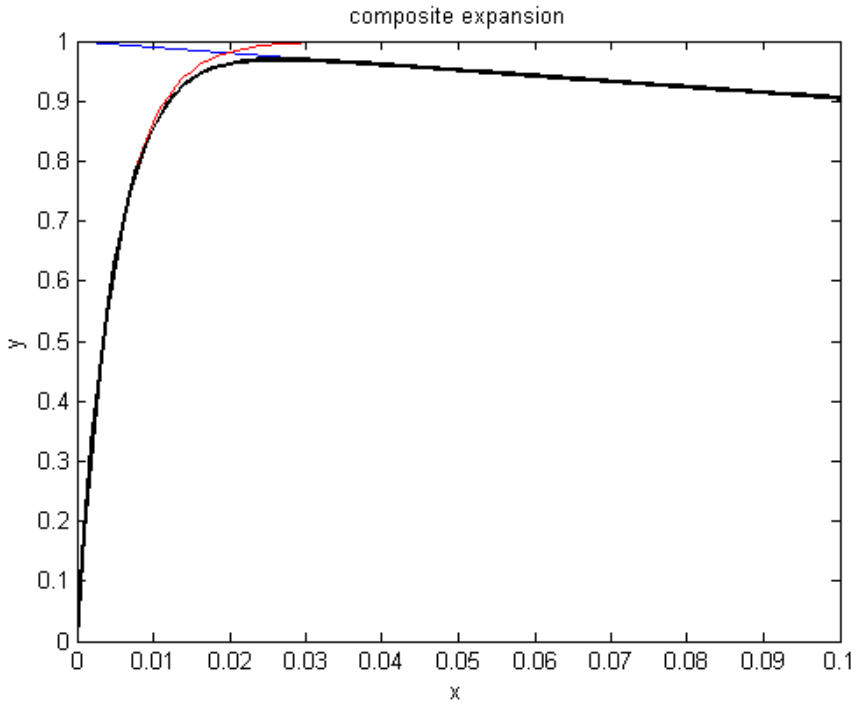

Figure 2.2: Solution of the example problem: combining the outer (blue) and inner (red) solution to a composite expansion (bold black line).

Chapter 3

Introduction to the financial framework

This chapter is an introduction to the computational finance subjects in this thesis. Before presenting a mathematical description, it is necessary to clarify some economical definitions.

Financial market instruments can be divided into two distinct categories. There are the 'underlying' stocks: shares, bonds, commodities, foreign currencies, etcetera, and their 'derivatives': claims that promise some payment or delivery in the future depending on an underlying stock's behaviour. Derivatives can reduce risk, or they can magnify it.

3.1 Bonds

First, we consider a **risk-free interest rate**, r, which represents the growth of money in time. We should be able to lend at that rate, and borrow - and in arbitrary size. To model this, we need something to model the time-value of money: a **zero-coupon bond** B_t, which we can buy or sell at time zero for some price, say B_0. After a small time step $\mathrm{d}t$ it will be worth $B_{\mathrm{d}t} = B_0 + rB_0\ \mathrm{d}t = B_0(1+r\ \mathrm{d}t)$, such that $B_{t+\mathrm{d}t} = B_t + rB_t\ \mathrm{d}t = B_t(1+r\ \mathrm{d}t)$. Consequently, we obtain the following differential equation for the bond:

$$\mathrm{d}B_t = rB_t\ \mathrm{d}t, \tag{3.1.1}$$

such that

$$B_t = B_0 e^{rt}, \tag{3.1.2}$$

see figure 3.1.

3.2 Brownian motion

To model more complicated financial products, a stochastic variable is needed. Brownian motions are often being used for this.

The stochastic process $W = (W_t : t \geq 0)$ is a $\mathbb{P}$**-Brownian motion** if and only if

1. W_t is continuous and $W_0 = 0$,
2. the value of W_t is distributed, under measure $\mathbb{P}$, as a normal random variable $N(0, t)$,

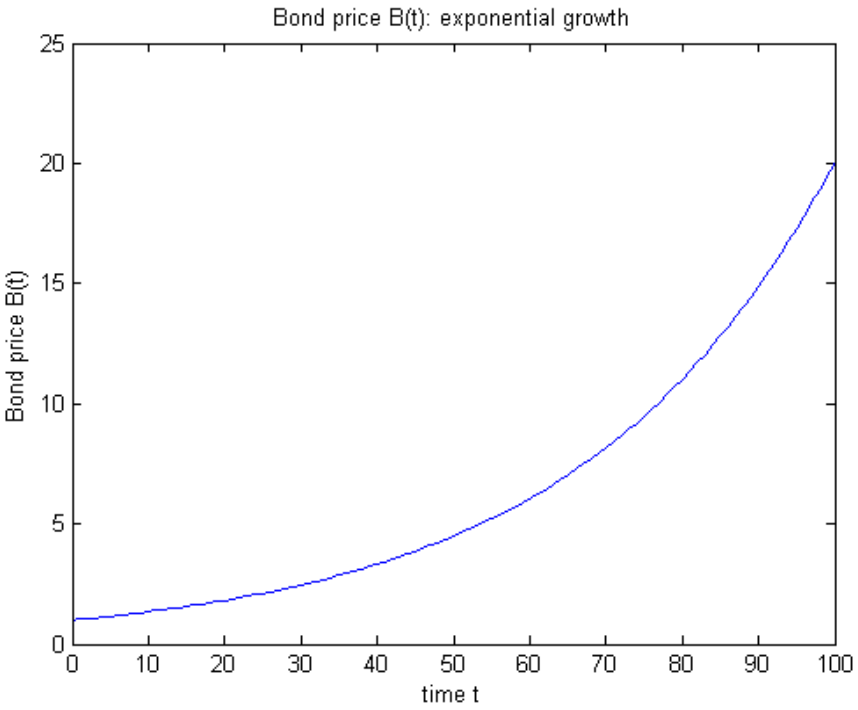

Figure 3.1: Bondprice B_t for different values of t. Here $B_0 = 1$ and $r = 0.03$.

3. the increment $W_{s+t} - W_s$ is distributed as a normal $N(0, t)$, under $\mathbb{P}$, and is independent of $\mathcal{F}_s$, the history of what the process did up to time s.

These are all necessary and sufficient conditions for the process W to be **Brownian motion**. Brownian motion with drift is called **Wiener process**, and is a (one-dimensional) Gaussian process.

We know that if the increment $\mathrm{d}W_t$ is a Wiener process, then $\mathrm{d}W_t^2 := (\mathrm{d}W_t)^2 \to \mathrm{d}t$ as $\mathrm{d}t \to 0$.[1] This can be explained by proving that $\mathrm{E}\left[\mathrm{d}W_1 \, \mathrm{d}W_2\right] = \rho \, \mathrm{d}t$ for $\mathrm{d}W_1$ and $\mathrm{d}W_2$ with correlation ρ.

Proof:
Suppose we have two independent normal distributed random variables $Z_1, Z_2 \sim N(0, t)$. Now we define

$$\begin{aligned} X_1 &= Z_1, \\ X_2 &= \rho Z_1 + \sqrt{1-\rho} Z_2, \end{aligned}$$

such that

$$\begin{aligned} \mathrm{E}\left[X_1 X_2\right] &= \mathrm{E}\left[Z_1(\rho Z_1 + \sqrt{1-\rho} Z_2)\right], \\ &= \mathrm{E}\left[\rho Z_1^2 + \sqrt{1-\rho} Z_1 Z_2)\right], \\ &= \rho \, \mathrm{E}\left[Z_1^2\right] + \sqrt{1-\rho} \, \mathrm{E}\left[Z_1 Z_2\right], \\ &= \rho \mathrm{Var}(Z_1) + \sqrt{1-\rho} \cdot 0, \\ &= \rho t. \end{aligned} \tag{3.2.1}$$

[1] See also page 58-59 of [1].

So X_1 and X_2 are ρ-correlated. In case $X = X_1 = X_2$, we have $\rho = 1$ and $\mathrm{E}\left[X^2\right] = t$.

Substitution of $X = \mathrm{d}W_t$, and $\mathrm{d}W_t \sim N(0, \mathrm{d}t)$, indeed implies that $\mathrm{d}W_t^2 \to \mathrm{d}t$ as $\mathrm{d}t \to 0$. And in the more general case, we have $\mathrm{E}\left[\mathrm{d}W_1\ \mathrm{d}W_2\right] = \rho\ \mathrm{d}t$ for $\mathrm{d}W_1$ and $\mathrm{d}W_2$ with correlation ρ.

□

3.3 Stocks

In business and finance, a share of **stock** (also referred to as equity share) means a share of ownership in a corporation (company). The initial price of a stock, at time $t = 0$, is given by S_0, whereas at time t it is given by S_t. In a small time interval $\mathrm{d}t$, this price will change from S_t to $S_t + \mathrm{d}S_t$. Next, consider the relative change in price: $\frac{\mathrm{d}S_t}{S_t}$. We can split this relative change into two parts:

- A **deterministic** part $\mu_t \mathrm{d}t$, because instead of investing it, one could also store the money (price of the stock) at a bank, and receive interest.
- An **stochastic** part $\sigma_t \mathrm{d}W_t$, where $\mathrm{d}W_t \sim N(0, \mathrm{d}t)$ is known as a **Wiener process**, which we discussed in the previous section.

So we obtain the following stochastic differential equation (SDE):

$$\frac{\mathrm{d}S_t}{S_t} = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}W_t. \tag{3.3.1}$$

The change in the stock price divided by its original value, $\frac{\mathrm{d}S_t}{S_t}$, is called the **return**. Here μ_t is the **drift rate** of the stock S_t. The **volatility** σ_t is related to the standard deviation of the stock price of a share. It is an indication for the random behaviour of the market. The stock price S_t follows the lognormal distribution that arises from SDE (3.3.1). See figure 3.2.

Dividend δ will be received by the owner of a share of some profit making company. Typically, the stock price will decrease when dividend is paid. In all financial models described in this thesis, we will assume absence of dividend: $\delta = 0$.

3.4 Derivatives

Secondly, the main topic of this thesis − option contracts and the pricing of options − will be discussed.

A **stock derivative** is any financial instrument that has a value that is dependent on the price of some underlying stock. Futures and options are the two main types of stock derivatives.

A **forward contract** is an agreement to buy or sell an asset (i.e., a bond, stock or anything else of value that is owned by a person or company) for a certain price at a certain time in the future. The participants in forward contracts are the holder and the writer. The **holder**, who buys the contract, is said to take a **long position** on the asset. The **writer** sells the option and takes a **short position**. A forward contract is binding towards *both* parties: the holder is obliged to buy the asset and the writer is obliged to sell the asset.

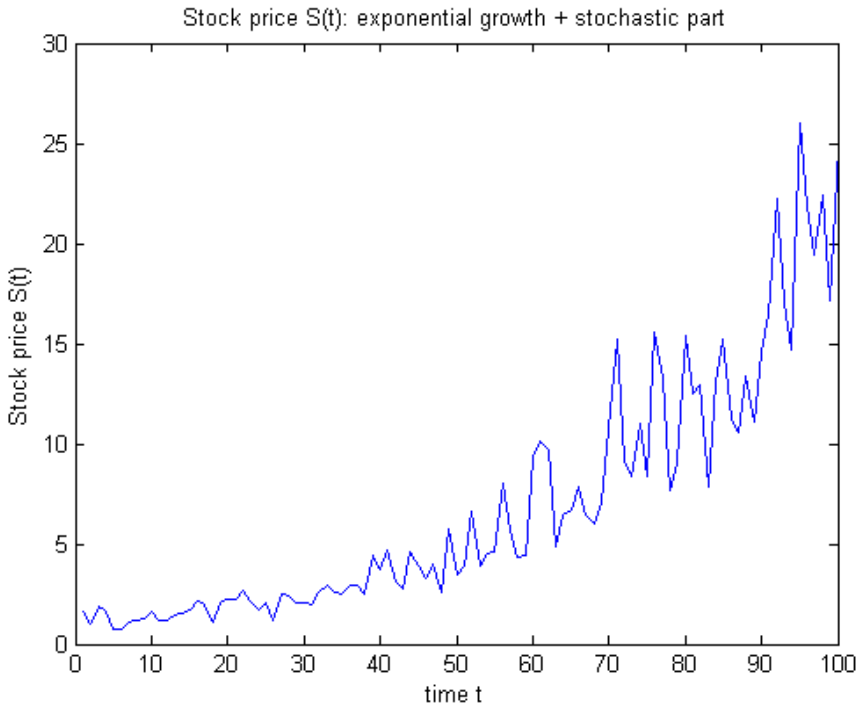

Figure 3.2: Stockprice S_t for different values of t.

This is not the case in **option contracts**. Options give the holder the *right* to exercise the option, so he is *not* obliged to buy the asset. The writer is however *obliged* to sell the asset. There are two basic types of options: the call and the put option. A **call option** is an option that gives the holder the right to *buy* an asset for a certain price. A **put option** gives the holder the right to *sell* an asset for a certain price. The price mentioned in either option contract is called the **exercise price** or **strike** K.

The most commonly traded types of calls and puts are **European** and **American** options, which are often called "*vanilla options*".[2] In the case of European options, the holder may only exercise the option *at* the time of maturity or exercise time, T. American options may also be exercised *in the period before* maturity.

3.5 Speculation and hedging

Options are used for several purposes. The two most important ones are **speculation** and **hedging**.

Speculation is quite easy to understand: if the holder buys a call option V_t^{call} at time t, he expects the stock price to increase. The **strike price** is denoted by K. If the stock price S is greater than the strike K, the call option will be exercised and the net profit of the option will be its **payoff** $P(S_T) := \max(S_T - K, 0)$ minus the option price V_T^{call}:

$$P(S_T) - V_T^{\text{call}} = S_T - K - V_t^{\text{call}} e^{r(T-t)}. \tag{3.5.1}$$

[2]The opposite of a vanilla option is called an **exotic option**: any of a broad category of options that may include complex financial structures.

This is called the **time value of money**.

The second purpose to use options is hedging. The collection of all shares, options and other derivatives owned by a trader is called a **portfolio**. **Hedging** means using options to reduce the risk of this portfolio. A **risk-free portfolio** can sometimes be constructed by choosing your hedging parameters such that all stochastic terms are eliminated in the expression for the value of the portfolio.

Example
Let us take a closer look at hedging and speculating by looking at an example taken from [1]. We have an interest-free bond and a stock, both initially priced at \$1. At the end of the next time interval, the stock is worth either \$2 or \$0.50.

Question: What is the value of a bet that pays \$1 if the stock goes up?

Solution: Let B denote the bond price, S the stock price and X the payoff of the bet. Let us define p as the change that the stock goes up. In this example we take $p = \frac{2}{3}$. The picture in figure 3.3 describes the situation.

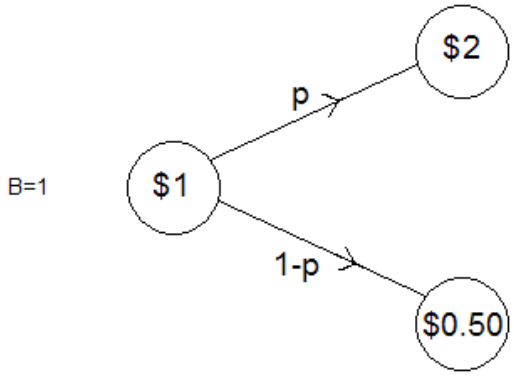

Figure 3.3: Pricing a bet.

Buy a portfolio Π consisting of $\frac{2}{3}$ of a unit of stock and a borrowing of $\frac{1}{3}$ of a unit of bond. The cost of this portfolio Π_0 at time zero is $\Pi_0 = \frac{2}{3} \cdot \$1 - \frac{1}{3} \cdot \$1 = \$0.33$.
After an up-jump, the portfolio becomes worth $\Pi_{\text{up}} = \frac{2}{3} \cdot \$2 - \frac{1}{3} \cdot \$1 = \$1$. And after a down-jump, it is worth $\Pi_{\text{down}} = \frac{2}{3} \cdot \$0.5 - \frac{1}{3} \cdot \$1 = \$0$.

The portfolio exactly simulates the bet's payoff, and must thus be worth exactly the same as the bet X. It must be that the portfolio's initial value of \$0.33 is also the bet's initial value: $X = \$0.33$.

Restrictions
There are some restrictions in hedging, including:

- **Transaction costs**, i.e., a cost incurred in making a financial exchange. For example, most people, when buying or selling a stock, must pay a commission to their broker (a regulated professional who buys and sells shares and other securities through market makers on behalf of investors): that commission is a transaction cost of doing the stock

deal.

- **Liquidity**, i.e., the capacity of a market to withstand an unusual amount of buying or selling without affecting the market substantially. It refers to an asset's ability to be easily converted through an act of buying or selling without causing a significant movement in the price and with minimum impact on its price. Money, or cash on hand, is the most liquid asset.
- **Business days**, which means any day including Monday to Friday and does not include holidays. Trading can only be done on business days. Hence, hedging cannot be done continuously.

Chapter 4

Financial framework

In this chapter the basics of the **financial framework**, with which we can describe the prices of options and other derivatives, will be considered. Here some formulas and theorems will be explained. These are valid for all financial models in the next chapters.

4.1 Notation

Before starting with the formulas and theorems, we list a couple of important variables:

- V_t is the value of an option or other derivative. This value V_t is a function of the stock value S_t and time t.
- Later on we will consider $V^{\text{call}}(S_t, t)$ and $V^{\text{put}}(S_t, t)$ as the values of a call, resp. put option.

The value of the option V_t is dependent of:

- μ_t: the drift of the stock,
- σ_t: the volatility of the stock,
- K: strike of the option,
- r_t: (risk-free) interest rate,
- S_t: the price of the stock,
- T: expiry time, $t = T$,
- t: time.

From now on we will omit the subscripts, and write μ, σ, and r, because in most of the simple financial models they are presumed to be constant. In case they appear to be time dependent, this will explicitly be mentioned, or we will add the subscript again. Also, starting in the next section, we write S instead of S_t and V instead of V_t, for notational convenience.

4.2 Assumptions

Arbitrage indicates that it is possible in a financial market to make risk-free profits beyond the interest gained when placing money in a bank account. We assume that there is no arbitrage and there are no transaction costs. Also we assume the absence of dividends. Furthermore, hedging continuously in time is assumed to be possible.
These assumptions are valid for all financial models, unless explicitly stated otherwise.

4.3 Itô's formula

In order to calculate with stochastic processes we have to consider Itô's Formula. The derivation of Itô's Formula can be found in section 3.3 in the book of Rennie and Baxter ([1]).

Definition of Itô's formula
If X is a stochastic process, satisfying $\mathrm{d}X_t = \mu\ \mathrm{d}t + \sigma\ \mathrm{d}W_t$ and the function f is a deterministic, twice continuously differentiable function, then $f(X_t)$ is also a stochastic process. Letting

$$f'(X_t) = \frac{\partial f}{\partial x}(X_t), \quad \text{and} \quad f''(X_t) = \frac{\partial^2 f}{\partial x^2}(X_t),$$

we obtain

$$\begin{aligned} \mathrm{d}f(X_t) &= f'(X_t)\ \mathrm{d}X_t + \frac{1}{2}f''(X_t)\ \mathrm{d}X_t^2, \\ &= f'(X_t)\,(\mu\ \mathrm{d}t + \sigma\ \mathrm{d}W_t) + \frac{1}{2}\sigma^2 f''(X_t)\ \mathrm{d}t. \end{aligned}$$

Here, all terms up to and including $\mathcal{O}(\mathrm{d}t)$ are taken into account, and we omit all higher order terms. After rearranging terms, we find that the stochastic process $f(X_t)$ is given by

$$\mathrm{d}f(X_t) = \left(\mu f'(X_t) + \frac{1}{2}\sigma^2 f''(X_t)\right)\mathrm{d}t + \sigma f'(X_t)\ \mathrm{d}W_t. \tag{4.3.1}$$

4.4 Feynman-Kac

From the SDE $\mathrm{d}S = \mu S\mathrm{d}t + \sigma S\mathrm{d}W_t$, which was introduced in section 3.3, the Black-Scholes equation

$$\frac{\partial V}{\partial t} + \mu\frac{\partial V}{\partial s} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial s^2} = 0 \tag{4.4.1}$$

can be derived, with $\mu = \mu(s,t)$, $\sigma = \sigma(s,t)$ and $V = V(s,t)$, and terminal condition $V(s,T) = P(s)$. Here, $P(s)$ is the **payoff** received at expiry date $t = T$. This derivation will be considered in section 5.1.

Meanwhile, we will prove that the expectation $V(S_t,t) = E_{\mathbb{Q}}\left[P(S_T)\middle|S_t\right]$ is equivalent to the solution of the Black-Scholes equation, by proving the **Feynman-Kac formulas**[1].

Proof:
Using Itô calculus, first write

$$\mathrm{d}V(S_t,t) = \frac{\partial V(S_t,t)}{\partial t}\mathrm{d}t + \frac{\partial V(S_t,t)}{\partial S_t}\mathrm{d}S_t + \frac{1}{2}\frac{\partial^2 V(S_t,t)}{\partial S_t^2}\mathrm{d}S_t^2 + \ldots. \tag{4.4.2}$$

The next step is to assume the stochastic differential equation (SDE)

$$\mathrm{d}S_t = \mu(S_t,t)S_t\mathrm{d}t + \sigma(S_t,t)S_t\mathrm{d}W_t, \tag{4.4.3}$$

with $\mathrm{d}W_t \sim N(0,\mathrm{d}t)$ a Wiener process, as introduced in section 3.2.

[1] More information on the Feynman-Kac formulas can be found in appendix D.

Substituting this into equation (4.4.2) yields:

$$\begin{aligned} \mathrm{d}V(S_t,t) &= \left(\frac{\partial V(S_t,t)}{\partial t} + \mu(S_t,t)\frac{\partial V(S_t,t)}{\partial S_t} + \frac{1}{2}\sigma^2(S_t,t)\frac{\partial^2 V(S_t,t)}{\partial S_t^2}\right)\mathrm{d}t \\ &+ \sigma(S_t,t)\frac{\partial V(S_t,t)}{\partial S_t}\,\mathrm{d}W_t. \end{aligned} \tag{4.4.4}$$

From the Black-Scholes equation (4.4.1) we know that

$$\frac{\partial V(S_t,t)}{\partial t} + \mu(S_t,t)\frac{\partial V(S_t,t)}{\partial S_t} + \frac{1}{2}\sigma^2(S_t,t)\frac{\partial^2 V(S_t,t)}{\partial S_t^2} = 0, \tag{4.4.5}$$

such that in $\mathrm{d}V(S_t,t)$ only the stochastic term (containing $\mathrm{d}W_t$) remains, i.e.,

$$\mathrm{d}V(S_t,t) = \sigma(S_t,t)\frac{\partial V(S_t,t)}{\partial S_t}\mathrm{d}W_t. \tag{4.4.6}$$

For the price $V(S_t,t)$, we can write

$$\begin{aligned} V(S_t,t) &= V(S_T,T) - \int_{t'=t}^{T} \mathrm{d}V(S_{t'},t'), \\ &= V(S_T,T) - \int_{t'=t}^{T} \sigma(S_{t'},t')\frac{\partial V(S_{t'},t')}{\partial S_{t'}}\,\mathrm{d}W_{t'}. \end{aligned} \tag{4.4.7}$$

Because $f(W_t) = \sigma(S_t,t)\frac{\partial V(S_t,t)}{\partial S_t}$ is an L^2-function and $\mathrm{d}W_t \sim N(0,\mathrm{d}t)$ is a Wiener process, we have[2]

$$\mathrm{E}\left[\int_t^T f(W_t)\,\mathrm{d}W_t \middle| S_t = s\right] = 0. \tag{4.4.8}$$

Hence, using equation (4.4.7), we can write

$$\begin{aligned} V(s,t) &= \mathrm{E}\left[V(S_t,t)\middle|S_t = s\right], \\ &= \mathrm{E}\left[V(S_T,T)\middle|S_t = s\right] - \mathrm{E}\left[\int_t^T f(W_t)\,\mathrm{d}W_t\middle|S_t = s\right], \\ &= \mathrm{E}\left[P(S_T)\middle|S_t = s\right]. \end{aligned} \tag{4.4.9}$$

□

4.5 Self-financing strategies

Now consider a portfolio (Φ_t,Ψ_t) with value Π_t. At time t it contains Φ_t units of security, i.e., stock (stock price is S_t), and also Ψ_t units of bond (bond price B_t). So the value of our portfolio at time t is $\Pi_t = \Phi_t S_t + \Psi_t B_t$. At the next time instance, two things happen: the old portfolio changes value, due to the change in S_t and B_t, and the old portfolio has to be adjusted to give a new portfolio as instructed by the trading strategy (Φ_t,Ψ_t). If the cost of the adjustment is perfectly matched by the profits or losses made by the portfolio then no extra money is required from outside − the portfolio is **self-financing**.

Self-financing property
If (Φ_t,Ψ_t) is a portfolio with stock price S_t and bond price B_t, then:

$$(\Phi_t,\Psi_t) \text{ is self-financing} \quad \Leftrightarrow \quad \mathrm{d}\Pi_t = \Phi_t\mathrm{d}S_t + \Psi_t\mathrm{d}B_t, \quad \text{for all } t \le T. \tag{4.5.1}$$

[2] See appendix E for notes on L^2-functions.

4.6 Put-call parity

In financial mathematics, **put-call parity** defines a relationship between the price of a call option V^{call} and a put option V^{put}, which both have an identical strike price K and expiry time T. For European options the put-call parity relationship will be derived. These options cannot be exercised before expiry time T. Put-call parity can be derived in a manner that is largely model independent.

Suppose we buy a stock S_t and a put option V^{put}, and also we sell a call option V^{call}. At an arbitrary time t our portfolio Π is worth

$$\Pi = S_t + V^{\text{put}} - V^{\text{call}}. \tag{4.6.1}$$

At expiry time T, we make a profit of

$$S_T + \max(K - S_T, 0) - \max(S_T - K, 0) = \begin{cases} S_T + 0 - (S_T - K) = K, & \text{for} \quad S_T > K, \\ S_T + (K - S_T) - 0 = K, & \text{for} \quad S_T \leq K. \end{cases} \tag{4.6.2}$$

Hence, the question is: *"What is the value of a portfolio at an arbitrary time t, if it has a guaranteed profit of K at $t = T$?"*

Again we assume the presence of a risk-free interest rate r. The money that we had to invest in the portfolio Π could also be stored at a bank, such that we would have received the risk-free interest rate r. That's why the portfolio is worth $Ke^{r(T-t)}$; else there would certainly be arbitrage.

Thus we have the following relationship between options and the underlying stock:

$$S_t + V^{\text{put}} - V^{\text{call}} = Ke^{r(T-t)}, \quad \text{for all } t \leq T, \tag{4.6.3}$$

which is called **put-call parity**.

Chapter 5

Black-Scholes model

In this chapter, we will take a look at the first financial model: the Black-Scholes model. First, the model will be derived, and we will show that the Black-Scholes equation can be reduced into a heat equation. After showing the exact solution, perturbation theory will be applied to the model and finally we will compare the result with this exact solution.

5.1 Derivation of the Black-Scholes model

Substituting $V(S,t)$ for f in Itô's Formula (4.3.1) gives:

$$\mathrm{d}V = \sigma S \frac{\partial V}{\partial S}\mathrm{d}W_t + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right)\mathrm{d}t. \tag{5.1.1}$$

Now we first need to construct a *portfolio*.

5.1.1 Hedging: Making a risk-free portfolio

Consider a portfolio Π that contains $-\Phi$ units of stock S at time t, i.e., we are **short** Φ units of stock, which means we have sold them. Also we buy one option V, i.e., we are **long** one option V. The value of our portfolio at time t is $\Pi = V - \Phi S$. If the value of our portfolio Π changes $\mathrm{d}\Pi$ during a small time step $\mathrm{d}t$, we have

$$\mathrm{d}\Pi = \mathrm{d}V - \Phi \mathrm{d}S - S\underbrace{\mathrm{d}\Phi}_{=0} = \mathrm{d}V - \Phi \mathrm{d}S. \tag{5.1.2}$$

Here $\mathrm{d}\Phi = 0$, because the number of stocks in the portfolio can only be changed at the end of every time step $\mathrm{d}t$, not during the time step itself.

Substitution of equation (5.1.1) into (5.1.2), using Itô's formula (4.3.1), we find that $\mathrm{d}\Pi$ satisfies:

$$\mathrm{d}\Pi = \sigma S\left(\frac{\partial V}{\partial S} - \Phi\right)\mathrm{d}W_t + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu \Phi S\right)\mathrm{d}t. \tag{5.1.3}$$

To make sure we have a risk-free portfolio, we should eliminate the stochastic term: the one containing $\mathrm{d}W_t$. So we take $\Phi = \frac{\partial V}{\partial S}$, such that the change in value of the portfolio becomes

$$\mathrm{d}\Pi = \left(\mu S\left(\frac{\partial V}{\partial S} - \Phi\right) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right)\mathrm{d}t = \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right)\mathrm{d}t. \tag{5.1.4}$$

By storing your money at a bank, instead of investing it, one can receive a risk-free interest rate r, just like we have seen before, looking at the bond B_t in Chapter 3.1. Then the amount of

money Π (which could have been the value of your portfolio) will produce $r\Pi \mathrm{d}t$ after a small time step $\mathrm{d}t$, without any risk. So substitution of $\mathrm{d}\Pi = r\Pi \mathrm{d}t$ into equation (5.1.4) yields

$$r\Pi \mathrm{d}t = \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right) \mathrm{d}t. \tag{5.1.5}$$

After dividing equation (5.1.5) by $\mathrm{d}t$ and substitution of $\Pi = V - \Phi S$ with $\Phi = \frac{\partial V}{\partial S}$ into equation (5.1.5), we obtain the well-known **Black-Scholes equation**

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{5.1.6}$$

5.2 Heat or diffusion equation

We can reduce the Black-Scholes equation (5.1.6) into an equation of the form

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \tag{5.2.1}$$

which is known as the **heat equation** or **diffusion equation**.

For the call option we have

$$\frac{\partial V^{\text{call}}}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V^{\text{call}}}{\partial S^2} + rS\frac{\partial V^{\text{call}}}{\partial S} - rV^{\text{call}} = 0, \tag{5.2.2}$$

with boundary conditions: $V^{\text{call}}(0,t) = 0$, $V^{\text{call}}(S,t) \sim S$ as $S \to \infty$ and at $t = T$ we have a terminal condition: $V_T^{\text{call}} := V^{\text{call}}(S_T, T) = \max(S - K, 0)$.

This is a backward equation, with non-constant coefficients. To make it a forward equation, we measure time backwards from expiry and scale it with σ, writing[1]

$$t = T - \frac{\tau}{\frac{1}{2}\sigma^2} \quad \Leftrightarrow \quad \tau = \frac{1}{2}\sigma^2(T - t). \tag{5.2.3}$$

Also we substitute $S = Ke^x$ and $V^{\text{call}} = Kv(x,\tau)$. Now the stock price S_t, which followed a log-normal distribution, becomes x, which is normally distributed. Also we scale S and V^{call} with K to make x and $v(x,\tau)$ dimensionless. After these substitutions, we obtain

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv, \tag{5.2.4}$$

with $k = r/\frac{1}{2}\sigma^2$ and initial condition $v(x,0) = \max(e^x - 1, 0)$.

This already looks a lot more like the heat equation (5.2.1). The next step is to substitute $v(x,\tau) = e^{\alpha x + \beta\tau}u(x,\tau)$ for some constants α and β.

Substituting this into differential equation (5.2.4) yields

$$\beta u + \frac{\partial u}{\partial \tau} = \alpha^2 u + 2\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + (k-1)\left(\alpha u + \frac{\partial u}{\partial x}\right) - ku. \tag{5.2.5}$$

[1] Deviding t by σ^2 is done to make τ dimensionless. From the SDE $\mathrm{d}S = \mu S\ \mathrm{d}t + \sigma S\ \mathrm{d}W_t$ it follows that $[\sigma] = \left[\frac{1}{\sqrt{t}}\right]$.

In order to let all terms with u and $\frac{\partial u}{\partial x}$ cancel, we have to solve the following system for α and β:

$$\begin{cases} \beta &= \alpha^2 + (k-1)\alpha - k, \\ 0 &= 2\alpha + (k-1). \end{cases} \tag{5.2.6}$$

The solution of system (5.2.6) is given by

$$\begin{cases} \alpha &= \frac{1}{2}(k-1), \\ \beta &= -\frac{1}{4}(k+1)^2, \end{cases}$$

such that $v(x,\tau) = e^{\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau}u(x,\tau)$ and this indeed gives us the heat equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \quad \text{for } -\infty < x < \infty,\ \tau > 0, \tag{5.2.7}$$

with initial condition $u(x,0) = u_0(x) = \max\left(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0\right)$.

5.3 Model

In [7] Howison first considers an asset (for example: a stock) whose price is S. This stock price S can be modeled as a function of time t by the stochastic differential equation (SDE)

$$\mathrm{d}S = \mu S\ \mathrm{d}t + \sigma S\ \mathrm{d}W_t, \tag{5.3.1}$$

in which $\mathrm{d}W_t$ is the increment of a standard Brownian Motion[2] and μ and σ are, respectively, the drift and volatility of the asset (taken to be constant).

Just like we have seen in the previous sections, we can set up a hedge portfolio $\Pi = V - \Delta S$, where the choice $\Delta = \frac{\partial V}{\partial S}$ makes the portfolio risk-free again.

In absence of arbitrage and transaction costs, the portfolio earns a (constant) risk-free rate r, so that $\mathrm{d}\Pi = r\Pi \mathrm{d}t$. This leads to the well-known **Black-Scholes equation**

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{5.3.2}$$

This backward parabolic equation is to be solved with a terminal condition $V(S,T) = P(S)$, where $P(S)$ is the payoff received at the expiry date $t = T$.

An alternative view of the hedging strategy is that it entails pricing with respect to a probability measure $\mathbb{Q}$ that is risk-neutral, rather than the objective (observed) measure $\mathbb{P}$ associated with the SDE for the asset price cited above in (5.3.1). That is, if the asset is assumed to follow the SDE (5.3.1), then the discount value of the option at time t is given by

$$\frac{V(S_t,t)}{B_t} = \mathrm{E}_{\mathbb{Q}}\left[\frac{P(S_T)}{B_T}\middle| S_t\right], \tag{5.3.3}$$

which is a martingale[3].

[2] See section 3.2.
[3] See appendix F.

Here, we will posit the existence of a deterministic r, μ and σ, such that the bond price B_t and the stock price S_t follow

$$B_t = B_0\, e^{rt}, \tag{5.3.4}$$
$$S_t = S_0\, e^{\mu t+\sigma W_t}, \tag{5.3.5}$$

where r is the deterministic risk-free interest rate, σ is the stock volatility and μ is the stock drift.

For the option price $V(S_t,t)$ this yields

$$\begin{aligned} V(S_t,t) &= B_t\, \mathrm{E}_{\mathbb{Q}}\left[\frac{P(S_T)}{B_T}\middle|S_t\right], \\ &= \frac{B_t}{B_T}\, \mathrm{E}_{\mathbb{Q}}\left[P(S_T)\middle|S_t\right], \\ &= e^{-r(T-t)}\, \mathrm{E}_{\mathbb{Q}}\left[P(S_T)\middle|S_t\right]. \end{aligned} \tag{5.3.6}$$

In practice, the hedging strategy above is impractical and in particular it is impossible to hedge continuously in time, or at each t. For this reason we introduce the **hedge parameter** gamma,

$$\Gamma = \frac{\partial^2 V}{\partial S^2}. \tag{5.3.7}$$

This gamma is a measure of the risk incurred in rehedging at non-infinitesimal time intervals δt. To see this, consider a portfolio $\Pi = V - \Delta_t S$, with $\Delta_t = \frac{\partial V}{\partial S}$ evaluated at (S,t), that is perfectly hedged at time t. Assume that no trading takes place over the interval $(t, t+\delta t)$.

Using Taylor's theorem, the change in the portfolio over this interval is $\delta\Pi = \delta V - \Delta_t \delta S$. In the infinitesimal limit where $\delta t \to \mathrm{d}t$, $\delta\Pi$ is equal to the risk-free return $r\Pi \mathrm{d}t$. Over the finite interval we have a difference between the return on the portfolio and the risk-free interest rate. This difference is called the **hedging error**, and is given by

$$\begin{aligned} \delta\Pi - \Pi\left(e^{r\delta t}-1\right) &= \frac{\partial V}{\partial t}\delta t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\delta S^2 - r\left(V - \frac{\partial V}{\partial S}\delta S\right)\delta t + o\left(\delta t\right), \\ &= \frac{1}{2}\sigma^2 S^2\left(\delta W^2 - \delta t\right)\frac{\partial^2 V}{\partial S^2} + o\left(\delta t\right). \end{aligned} \tag{5.3.8}$$

To obtain this hedging error, we have used the Black-Scholes equation (5.1.6) and the fact that to this order $\delta S^2 = \sigma^2 S^2 \delta W^2$, where $\delta W \sim N(0, \delta t)$ is the small change in a Wiener process W.[4]

The hedging error (5.3.8) seems to be proportional to the random variable $\delta W^2 - \delta t$, whose expectation is zero, multiplied by the option's gamma $\Gamma = \frac{\partial^2 V}{\partial S^2}$.

For call options we have

$$\Gamma(S,T) = \frac{\mathrm{d}^2}{\mathrm{d}S^2}\max\left(S-K,0\right) = \delta\left(S-K\right), \tag{5.3.9}$$

where $\delta(\cdot)$ is the Dirac delta function. For put options we have the same gamma.

For call options without dividends $[\Delta]_{S=0}^{\infty} = 1$ for all t, such that $\int_0^\infty \Gamma(S,t)\ \mathrm{d}S = 1$. As $t \to T$ the gamma of such an option is an approximation of the delta function.

[4] See section 3.2.

5.4 Exact solution

As a first example we consider a call option (or, by **put-call parity**[5], a put option) in the standard Black-Scholes model. The payoff is given by $P(S) = \max(S - K, 0)$ and there is a famous **explicit formula** for the option value,

$$V(S,t) = SN(d_+) - Ke^{-r(T-t)}N(d_-), \tag{5.4.1}$$

where[6]

$$d_\pm = \frac{\log(\frac{S}{K}) + \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \tag{5.4.2}$$

and

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-s^2/2} \mathrm{d}s \tag{5.4.3}$$

is the standard normal cumulative density function.

5.5 Vanilla options near expiry: boundary layers and scalings

In this section a first application of perturbation theory on a financial model will be presented to show the method and find an approximation. Because an **exact solution** is available, we are able to compare the results.

Boundary layers
For small time, we can derive an approximation to the exact solution (5.4.1) as follows. We will construct two 'boundary layers': one when the stock price is near the strike price, i.e. near $S = K$, and one when time t is close to expiry time T, i.e. near $t = T$.

These ***artificial* (boundary) layers** are illustrated in figure 5.1. We have constructed them to show the method of asymptotic expansions applied to this basic financial model.

Note that we will *only* consider the region inside the layer near maturity, i.e. when $T - t$ is small. In this case, the other regions are not taken into account. In particular we will look inside a layer around $S = K$. Here, we have an **inner expansion** and an **inner solution**. In the rest of the layer near $t = T$, i.e., when the spot S is far from the strike K, we have an **outer expansion** and **outer solution**.[7]

Scalings
First of all, we measure time backwards from expiry and scale it with σ^2 to make it dimensionless, writing

$$t = T - \frac{t'}{\sigma^2} \quad \Leftrightarrow \quad t' = (T-t)\sigma^2. \tag{5.5.1}$$

After this scaling the Black-Scholes equation (5.1.6) transforms into

$$\frac{\partial V}{\partial t'} = \frac{1}{2}S^2\frac{\partial^2 V}{\partial S^2} + \alpha S\frac{\partial V}{\partial S} - \alpha V, \tag{5.5.2}$$

[5] See section 4.6.
[6] Note that traditionally d_1 and d_2 are used instead of d_+ and d_-.
[7] Here, Howisons nomenclature is quite unusual, but practical: the outer solution is inside one of the boundary layers. The inner solution is located in the overlap of the two layers.

with $\alpha = \frac{r}{\sigma^2}$ dimensionless. Suppose that $\alpha = \mathcal{O}(1)$ and scaled time is small, such that $\tau = t'/\varepsilon^\eta$ with $0 < \varepsilon \ll 1$. This has the effect of stretching the area near $t = T$. At this point the value of scaling parameter η is not yet known; we will determine it later on.

The scaled Black-Scholes equation (5.5.2) now becomes

$$\frac{1}{\varepsilon^\eta}\frac{\partial V}{\partial \tau} = \frac{1}{2}S^2\frac{\partial^2 V}{\partial S^2} + \alpha S\frac{\partial V}{\partial S} - \alpha V. \tag{5.5.3}$$

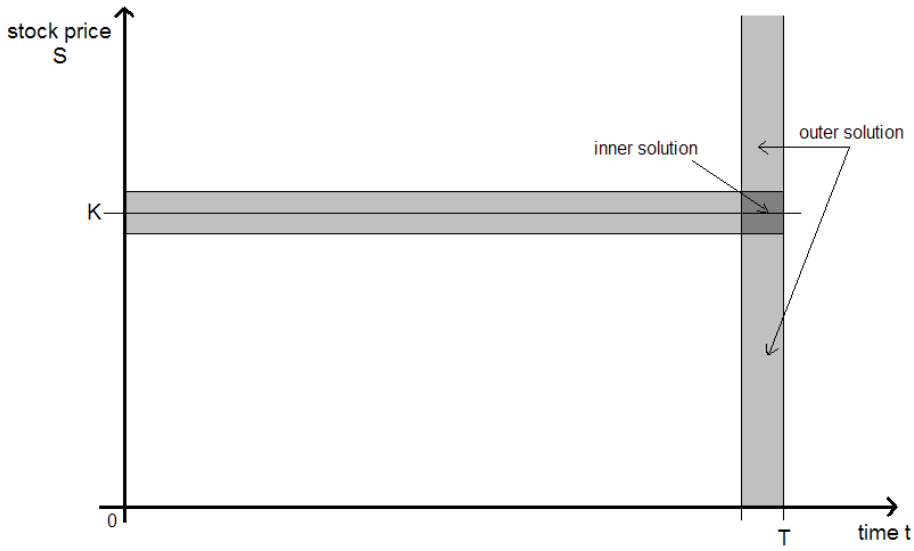

Figure 5.1: In this case there are 2 artificial layers: around $t = T$ and $S = K$.

5.6 Outer problem

In figure 5.1 it is shown that, when the spot S is far from the strike K, we have a **regular outer expansion**

$$V_\varepsilon(S,\tau) = \sum_{n=0}^{m} \varepsilon^{n\eta} V_n(S,\tau) + \mathcal{O}\left(\varepsilon^{\eta(m+1)}\right) = V_0(S,\tau) + \varepsilon^\eta V_1(S,\tau) + \ldots, \tag{5.6.1}$$

which gives

$$\begin{aligned}\frac{\partial}{\partial \tau}\left(V_0(S,\tau) + \varepsilon^\eta V_1(S,\tau) + \ldots\right) &= \varepsilon^\eta \frac{1}{2}S^2\frac{\partial^2}{\partial S^2}\Big(V_0(S,\tau) + \varepsilon^\eta V_1(S,\tau) + \ldots\Big) \\ &+ \varepsilon^\eta \alpha S\frac{\partial}{\partial S}\Big(V_0(S,\tau) + \varepsilon^\eta V_1(S,\tau) + \ldots\Big) \\ &- \varepsilon^\eta \alpha\Big(V_0(S,\tau) + \varepsilon^\eta V_1(S,\tau) + \ldots\Big).\end{aligned}$$

The $\mathcal{O}(1)$ equation becomes $\frac{\partial V_0}{\partial \tau} = 0$, and for $\mathcal{O}(\varepsilon^\eta)$ we have[8]

$$\frac{\partial V_1}{\partial \tau} = \frac{1}{2}S^2\underbrace{\frac{\partial^2 V_0}{\partial S^2}}_{=0} + \alpha\left(S\frac{\partial V_0}{\partial S} - V_0\right) = \alpha\left(S\frac{\partial V_0}{\partial S} - V_0\right),$$

[8] If the spot S is far from the strike K, note that in either region V (hence, also V_0) is linear.

i.e.,

$$\frac{\partial V_1}{\partial \tau} = \begin{cases} \alpha(S-(S-K)) = \alpha K, & \text{for } S \gg K, \\ 0, & \text{for } S \ll K. \end{cases}$$

The term $\frac{1}{2}S^2\frac{\partial^2 V_0}{\partial S^2}$ in the $\mathcal{O}(\varepsilon^\eta)$ equation vanishes, because for $|S-K| \gg \varepsilon^\eta K$ (i.e., if the spot is far from the strike) we have $\Gamma = \frac{\partial^2 V}{\partial S^2} = 0$. This can be seen in figure 5.2.

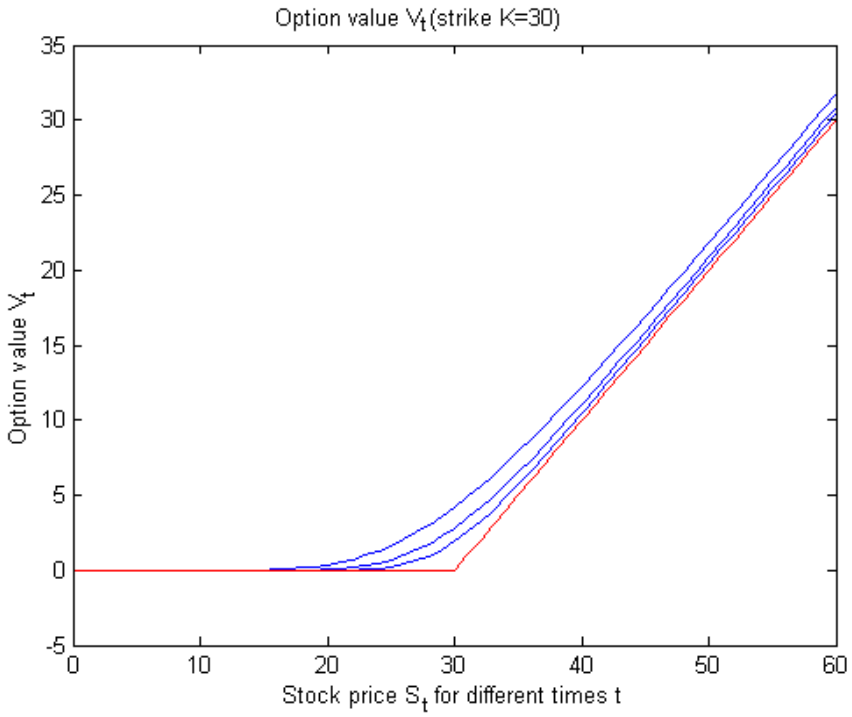

Figure 5.2: European call option payoff $\max(S_t - K, 0)$ at different times $t < T$ before maturity (blue lines) and at maturity (red line) $t = T$ (here: strike $K = 30$).

5.7 Outer solution

The solution of the $\mathcal{O}(1)$ outer equation with final condition $V_0(S,T) = P(S) = \max(S-K,0)$ is given by $V_0(S,t) = \max(S-K,0)$. Next, solving the $\mathcal{O}(\varepsilon^\eta)$ equation with final condition $V_1(S,T) = 0$, yields

$$V_1(S,t) = \begin{cases} \alpha K\tau, & \text{for } S \gg K, \\ 0, & \text{for } S \ll K, \end{cases}$$

such that the expansion becomes

$$V_\varepsilon = V_0 + \varepsilon^\eta V_1 = \begin{cases} S - K + \varepsilon^\eta \alpha K\tau, & \text{for } S \gg K, \\ 0, & \text{for } S \ll K. \end{cases} \tag{5.7.1}$$

which we can rewrite as

$$V_0 + \varepsilon^\eta V_1 = \begin{cases} S - K(1-\alpha\varepsilon^\eta\tau), & \text{for } S - K \gg \varepsilon^\eta K, \text{ far above the strike}, \\ 0, & \text{for } S - K \ll \varepsilon^\eta K, \text{ far below the strike}. \end{cases} \tag{5.7.2}$$

Note that in this case, we have a layer of width ε^η.

Later on, after applying maximum balance, η turns out to be equal to 2. This implies that the previous expression (5.7.2) becomes equivalent to the first two terms in the small time expansion in unscaled variables of the function

$$\begin{cases} S - Ke^{-r(T-t)}, & \text{for} \quad S > Ke^{-r(T-t)}, \\ 0, & \text{for} \quad S < Ke^{-r(T-t)}. \end{cases} \tag{5.7.3}$$

The components of the function (5.7.3) are the value of the forward contract in which the option holder is compelled to buy the asset, corresponding to certain exercise, and zero corresponding to no exercise.

5.8 Inner problem

However, as remarked earlier and as can be seen in figure 5.2, we expect large gamma near the strike K. Hence, the term containing the second derivative with respect to S cannot be ignored. We deal with this by rescaling if S is near the strike price K. So, let us introduce a local inner variable

$$x = \frac{S-K}{\varepsilon^\nu K}, \tag{5.8.1}$$

such that we have scaled $S-K$ with ε^ν and we divide by K to make x dimensionless. The value of ν will be determined later, after applying maximum balance.

Also we introduce a **scaling**

$$v(x,\tau) = \frac{V(S,\tau)}{\varepsilon^\nu K}, \tag{5.8.2}$$

such that we scale $V(S,\tau)$, which is $\mathcal{O}(\varepsilon^\nu)$, with ε^ν to have $v = \mathcal{O}(1)$ and again we devide by K to make $v(x,\tau)$ dimensionless.

After this second scaling, the scaled Black-Scholes equation (5.5.3) becomes the dimensionless equation

$$\frac{1}{\varepsilon^\eta}\frac{\partial v}{\partial \tau} = \frac{1}{2\varepsilon^{2\nu}}(1+\varepsilon^\nu x)^2\frac{\partial^2 v}{\partial x^2} + \frac{\alpha}{\varepsilon^\nu}(1+\varepsilon^\nu x)\frac{\partial v}{\partial x} - \alpha v, \tag{5.8.3}$$

and the payoff is[9]

$$v(x,0) = \frac{V(S,0)}{\varepsilon^\nu K} = \frac{\max(S-K,0)}{\varepsilon^\nu K} = \max(x,0). \tag{5.8.4}$$

Next, we will calculate the solution to this inner problem, and after that we will try to match it with the outer solution (5.7.2).

First, assume ν and μ to be integer, such that the solution can be expanded in integer powers:

$$v_\varepsilon(x,\tau) = v_0(x,\tau) + \varepsilon v_1(x,\tau) + \mathcal{O}\left(\varepsilon^2\right), \tag{5.8.5}$$

which gives

$$\begin{aligned} \frac{1}{\varepsilon^\eta}\frac{\partial}{\partial\tau}\left(v_0(x,\tau)+\varepsilon v_1(x,\tau)+\ldots\right) &= \frac{1}{2\varepsilon^{2\nu}}(1+\varepsilon^\nu x)^2\frac{\partial^2}{\partial x^2}\left(v_0(x,\tau)+\varepsilon v_1(x,\tau)+\ldots\right) \\ &+ \frac{\alpha}{\varepsilon^\nu}(1+\varepsilon^\nu x)\frac{\partial}{\partial x}\left(v_0(x,\tau)+\varepsilon v_1(x,\tau)+\ldots\right) \\ &- \alpha\left(v_0(x,\tau)+\varepsilon v_1(x,\tau)+\ldots\right). \end{aligned} \tag{5.8.6}$$

[9] On page 5 of [7] Howison claims $v(x,0) = \varepsilon\max(x,0)$, which is incorrect.

Maximum balance[10] over the 3 terms of equation (5.8.6) gives $\eta = 2\nu$. A nice integer solution for this constraint is choosing $\eta = 2$ and $\nu = 1$, which yields

$$\begin{aligned}\frac{\partial}{\partial\tau}\left(v_0(x,\tau)+\varepsilon v_1(x,\tau)+\ldots\right) &= \frac{1}{2}(1+\varepsilon x)^2\frac{\partial^2}{\partial x^2}\left(v_0(x,\tau)+\varepsilon v_1(x,\tau)+\ldots\right)\\ &+ \alpha\varepsilon(1+\varepsilon x)\frac{\partial}{\partial x}\left(v_0(x,\tau)+\varepsilon v_1(x,\tau)+\ldots\right)\\ &- \alpha\varepsilon^2\left(v_0(x,\tau)+\varepsilon v_1(x,\tau)+\ldots\right). \qquad (5.8.7)\end{aligned}$$

The $\mathcal{O}(1)$ problem becomes

$$\frac{\partial v_0}{\partial\tau} = \frac{1}{2}\frac{\partial^2 v_0}{\partial x^2}, \quad \text{with } v_0(x,0) = \max(x,0). \qquad (5.8.8)$$

For $x \to \pm\infty$ the conditions for this equation are, consistently with the payoff,

$$\begin{aligned}\lim_{x\to+\infty}\frac{v_0(x,\tau)}{x} &= 1,\\ \lim_{x\to-\infty} v_0(x,\tau) &= 0.\end{aligned}$$

If $x \to -\infty$, S will stay inside the layer, but it will go towards the lower boundary of the layer, because $\frac{S}{K} - 1 \to -\infty$. In that case $v(x,\tau)$ will go to zero.

5.9 Inner solution

The inner problem is much simpler than the original problem. We will use one of the five basic invariance properties of the diffusion equation to choose a local variable correctly.

Five basic invariance properties of the diffusion equation
Suppose we have the initial value problem

$$\begin{cases} u_t = ku_{xx}, & (5.9.1a)\\ u(x,0) = \phi(x). & (5.9.1b)\end{cases}$$

Our method is tantamount to solving the initial value problem (5.9.1a)-(5.9.1b) for a particular $\phi(x)$ and then build the general solution from this particular one. Therefore we will use the following five basic invariance properties:[11]

1. The *translate* $u(x-y,t)$ of any solution $u(x,t)$ of the PDE (5.9.1a) is another solution for any fixed y.
   Proof:
   $$\frac{\partial}{\partial t}u(x-y,t) = \frac{\partial}{\partial t}u(x,t) \quad \text{and} \quad \frac{\partial^2}{\partial x^2}u(x-y,t) = \frac{\partial^2}{\partial x^2}u(x,t).$$
2. Any *derivative* (u_x or u_t, or u_{xx}, etc.) of a solution $u(x,t)$ of the PDE (5.9.1a) is again a solution, because $u(x,t) \in C^\infty$ for $t > 0$.
   Proof: Suppose that $u(x,t)$ satisfies the heat equation $u_t = ku_{xx}$. Define $y := \mathcal{D}u(x,t)$, where $\mathcal{D}$ is an operator that can be any derivative of $u(x,t)$. Then, $y_t = (\mathcal{D}u)_t = \mathcal{D}(u_t) = \mathcal{D}(ku_{xx}) = k(\mathcal{D}u)_{xx} = ky_{xx}$, because derivatives can be interchanged using $u(x,t) \in C^\infty$. Hence, any derivative $y = \mathcal{D}u(x,t)$ of any solution $u(x,t)$ also satisfies the heat equation.

[10] See chapter 2. This is also referred to as **significant degeneration** or **distinguished limit**.
[11] See section 2.4 of [11].

3. A *linear combination* of solutions $u_i(x,t)$ of the heat equation (5.9.1a) is again a solution of (5.9.1a).
   Proof: By linearity:

$$\frac{\partial}{\partial t}\left(\sum_i u_i(x,t)\right) = \sum_i \frac{\partial}{\partial t}u_i(x,t) \quad \text{and} \quad \frac{\partial^2}{\partial x^2}\left(\sum_i u_i(x,t)\right) = \sum_i \frac{\partial^2}{\partial x^2}u_i(x,t).$$

4. An *integral* of solutions of the PDE (5.9.1a) is again a solution.
   Proof: If $S(x,t)$ is a solution of problem (5.9.1a), then so is the translate $S(x-y_i,t)$ for any constant y_i, by property 1. and so is $S(x-y_i,t)g(y_i)$. Multiplication by the constant $\Delta y := y_{i+1} - y_i$ yields that also $S(x-y_i,t)g(y_i)\Delta y_i$ is a solution of the PDE (5.9.1a). Thus, summation over all y_i and letting $\Delta y_i \to 0$ implies that

$$v(x,t) = \int_{-\infty}^{\infty} S(x-y,t)g(y)\ \mathrm{d}y,$$

   is a solution of the heat equation (5.9.1a) for any function $g(y)$, as long as this improper integral converges appropriately. In fact, property 4. is just a limiting form of property 3.

5. If $u(x,t)$ is a solution of the PDE (5.9.1a), so is the *dilated* function $u(\sqrt{a}x, at)$, for any $a > 0$.
   Proof: We can prove this using the chain rule:
   Let $v(x,t) := u(\sqrt{a}x, at)$. Then we have $v_t = \left[\frac{\partial(at)}{\partial t}\right] u_t = au_t$ and $v_x = \left[\frac{\partial(\sqrt{a}x)}{\partial x}\right] u_x$ and $v_{xx} = \sqrt{a}\cdot\sqrt{a}u_{xx} = au_{xx}$.

Using the **dilated function** $u(\sqrt{a}x, at)$ in property 5. we know that the PDE in (5.8.8) has **similarity solution**[12]

$$f\left(\frac{x}{\sqrt{\tau}}\right) := \frac{v_0(x,\tau)}{\sqrt{\tau}}. \tag{5.9.2}$$

Also we introduce a local variable

$$\xi = \frac{x}{\sqrt{\tau}}. \tag{5.9.3}$$

After substituting the transformations (5.9.2) and (5.9.3) into the $\mathcal{O}(1)$ heat equation (5.8.8), and multiplying by $2\sqrt{\tau}$, we obtain the following ODE in $f(\xi)$:

$$f'' + \xi f' - f = 0, \tag{5.9.4}$$

with boundary conditions

$$\begin{aligned} \lim_{\xi\to-\infty} f &= 0, \\ \lim_{\xi\to\infty} \frac{f}{\xi} &= 1. \end{aligned} \tag{5.9.5}$$

The solution can be found by differentiating equation (5.9.4) once with respect to ξ,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\xi}\left(f''(\xi) + \xi f'(\xi) - f(\xi)\right) &= 0, \\ \Rightarrow \frac{\mathrm{d}}{\mathrm{d}\xi}\left(f''(\xi) + \xi f''(\xi)\right) &= 0, \end{aligned} \tag{5.9.6}$$

[12] See appendix G.

which gives us an ordinary differential equation (ODE) in f''. The solution of ODE (5.9.6) is given by

$$f''(\xi) = Ae^{-\frac{1}{2}\xi^2} = c_2\ n(\xi). \tag{5.9.7}$$

This is again a differential equation, which has a solution that is given by

$$f(\xi) = c_2\xi\ N(\xi) + c_2\ n(\xi) + c_1\xi + c_0. \tag{5.9.8}$$

From the boundary conditions (5.9.5) we find that $c_1 = c_0 = 0$ and $c_2 = 1$, such that

$$f(\xi) = \xi\ N(\xi) + n(\xi). \tag{5.9.9}$$

In the original variables, the solution is given by

$$v_0(x,\tau) = xN\left(\frac{x}{\sqrt{\tau}}\right) + \sqrt{\tau}n\left(\frac{x}{\sqrt{\tau}}\right), \tag{5.9.10}$$

where $N(\cdot)$ is as above in (5.4.3) and $n(\cdot)$ is its derivative $e^{-x^2}/\sqrt{2\pi}$.

This solution can also directly be obtained by application of **Green's functions**.[13]

The approximation we found in equation (5.9.10) and equation (H.0.21) is valid in the inner region, while in the outer region we have the outer expansion (5.7.2). In more complicated problems one can often find a uniformly valid expansion, holding in both inner and outer regions, by calculating 'outer+inner−common', in which 'outer' and 'inner' are the expansions we already found, and 'common' is the intermediate limiting behavior of these expansions used in matching.

In our case the outer expansion is so simple that it and the common expansion coincide, and so the inner expansion is in fact uniformly valid and can be used as an approximation for all S and small t'.

So a 1-term approximation is given by

$$\begin{aligned} v_\varepsilon(x,\tau) &= v_0(x,\tau) + \mathcal{O}(\varepsilon), \\ &= x\ N\left(\frac{x}{\sqrt{\tau}}\right) + \sqrt{\tau}\ n\left(\frac{x}{\sqrt{\tau}}\right) \\ &= \frac{S/K-1}{\varepsilon}\ N\left(\frac{S/K-1}{\sigma\sqrt{T-t}}\right) + \frac{\sigma}{\varepsilon}\sqrt{T-t}\ n\left(\frac{S/K-1}{\sigma\sqrt{T-t}}\right). \end{aligned}$$

In the original variables, this expression is

$$\begin{aligned} V(S,t) &\sim V_0(S,t) \sim \varepsilon K\ v_0(x,\tau) \\ &= (S-K)\ N\left(\frac{S/K-1}{\sigma\sqrt{T-t}}\right) + \sigma\sqrt{T-t}K\ n\left(\frac{S/K-1}{\sigma\sqrt{T-t}}\right). \end{aligned} \tag{5.9.11}$$

Note that the parameter ε, which is artificial, does not appear in the expression.

In figure 5.3 one can see the approximation (1-term inner expansion) we found in equation (5.9.11) compared to the exact solution. As can be seen, both approximations are very close to the exact solution.

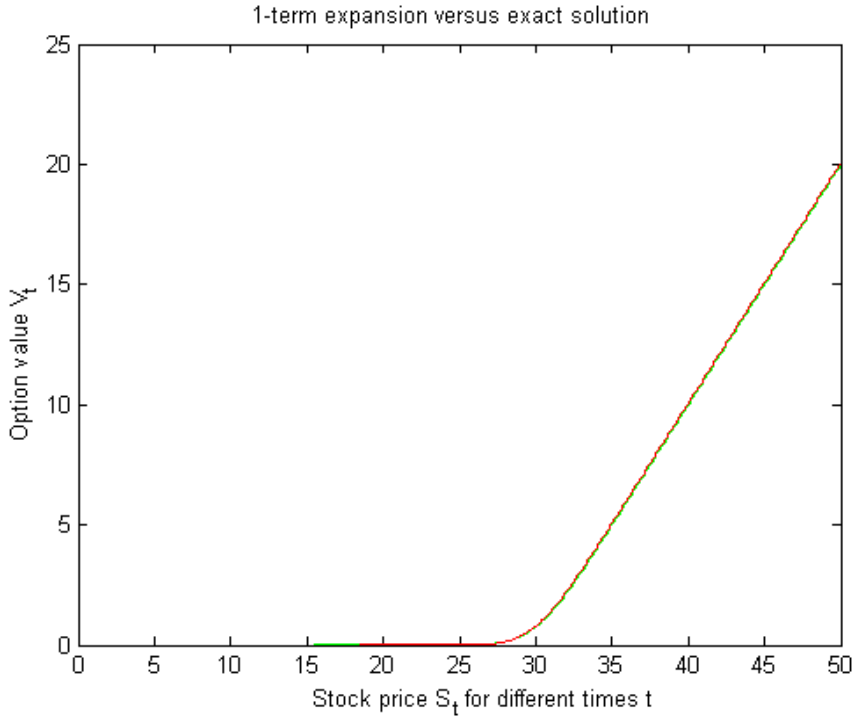

Figure 5.3: The approximation with the 1-term expansion (green) versus the exact solution (red).

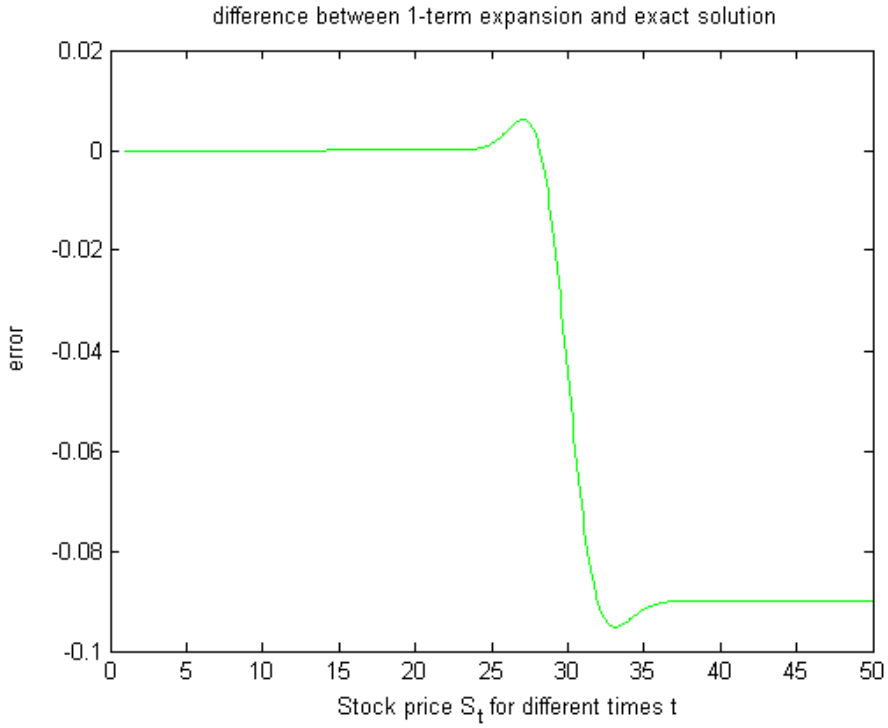

Figure 5.4: The difference between approximation with 1-term expansion and the exact solution.

To see how close this approximation exactly is, we can subtract the exact solution, and make a plot of the discrepancy between the approximation and the exact solution, see figure 5.4.

Using equation (5.8.7) again, the $\mathcal{O}(\varepsilon)$ problem becomes

$$\frac{\partial v_1}{\partial \tau} = \frac{1}{2}\frac{\partial^2 v_1}{\partial x^2} + x\frac{\partial^2 v_0}{\partial x^2} + \alpha\frac{\partial v_0}{\partial x}, \quad \text{with } v_1(x,0) = 0. \tag{5.9.12}$$

As v_0 satisfies the diffusion equation $\partial_t v_0 - \frac{1}{2}\partial_x^2 v_0 = 0$, again we will use some of the the five basic invariance properties of the diffusion equation to find:

- If $u_\tau - \frac{1}{2}u_{xx} = 0$ and $v_\tau - \frac{1}{2}v_{xx} = u$, then a particular solution is $u_p(x,\tau) = \tau u$.
- If u is as above and $v_\tau - \frac{1}{2}v_{xx} = xu$, then a particular solution is $v_p(x,\tau) = x\tau u + \frac{1}{2}\tau^2 u_x$.
- $N''(\xi) + \xi N'(\xi) = 0$.

So a particular solution is readily found and, as it vanishes at $\tau = 0$, it is the solution we need:

$$\begin{aligned} v_1(x,\tau) &= x\tau\frac{\partial^2 v_0}{\partial x^2} + \frac{1}{2}\tau^2\frac{\partial^3 v_0}{\partial x^3} + \alpha\tau\frac{\partial v_0}{\partial x}, \\ &= \frac{1}{2}x\sqrt{\tau}\; n\left(\frac{x}{\sqrt{\tau}}\right) + \alpha\tau\; N\left(\frac{x}{\sqrt{\tau}}\right). \end{aligned} \tag{5.9.13}$$

For the 2-term approximation of v, this gives

$$\begin{aligned} v(x,\tau) &\sim v_0(x,\tau) + \varepsilon v_1(x,\tau), \\ &= x\; N\left(\frac{x}{\sqrt{\tau}}\right) + \sqrt{\tau}\; n\left(\frac{x}{\sqrt{\tau}}\right) + \varepsilon\left(\frac{1}{2}x\sqrt{\tau}\; n\left(\frac{x}{\sqrt{\tau}}\right) + \alpha\tau\; N\left(\frac{x}{\sqrt{\tau}}\right)\right), \\ &= (x + \varepsilon\alpha\tau)\; N\left(\frac{x}{\sqrt{\tau}}\right) + \sqrt{\tau}(1 + \frac{1}{2}\varepsilon x)\; n\left(\frac{x}{\sqrt{\tau}}\right). \end{aligned} \tag{5.9.14}$$

Furthermore the 2-term inner expansion $V(S,\tau) = \varepsilon K v(x,\tau) \sim \varepsilon K(v_0 + \varepsilon v_1)$ is again uniformly valid. In the original variables, the expression reads

$$V(S,t) \sim \big(S - K + rK(T-t)\big)\; N\left(\frac{S/K - 1}{\sigma\sqrt{T-t}}\right) + \sigma\sqrt{T-t}(S+K)\; n\left(\frac{S/K-1}{\sigma\sqrt{T-t}}\right). \tag{5.9.15}$$

Figure 5.5 presents the approximation (with the 2-term inner expansion) we found in (5.9.15) compared to the exact solution and the approximation (1-term inner expansion) we found in (5.9.11). As can be seen, both approximations are very close to the exact solution.

To see how close the approximations exactly are, we can subtract the exact solution, and make a plot of the 'error', see figure 5.6. This figure makes clear that the 2-term expansion (5.9.15) indeed is a better approximation than the 1-term expansion (5.9.11), because the error (blue line) is much closer to zero.

[13] The steps we need to take for this, are being explained in [11], on the pages $47 - 52$. See appendix H.

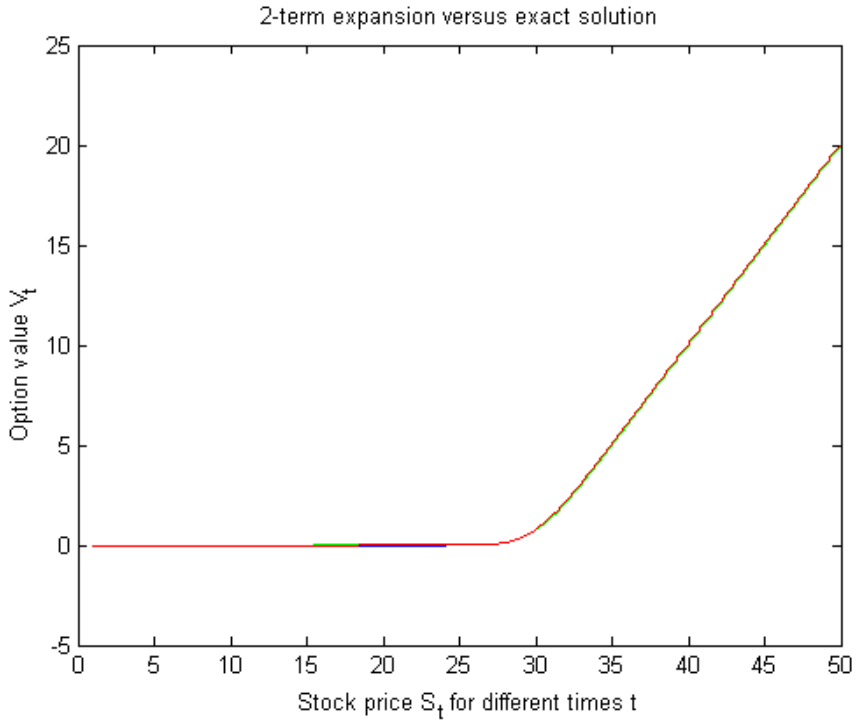

Figure 5.5: The approximation with the 2-term expansion (blue) versus 1-term expansion (green) and the exact solution (red).

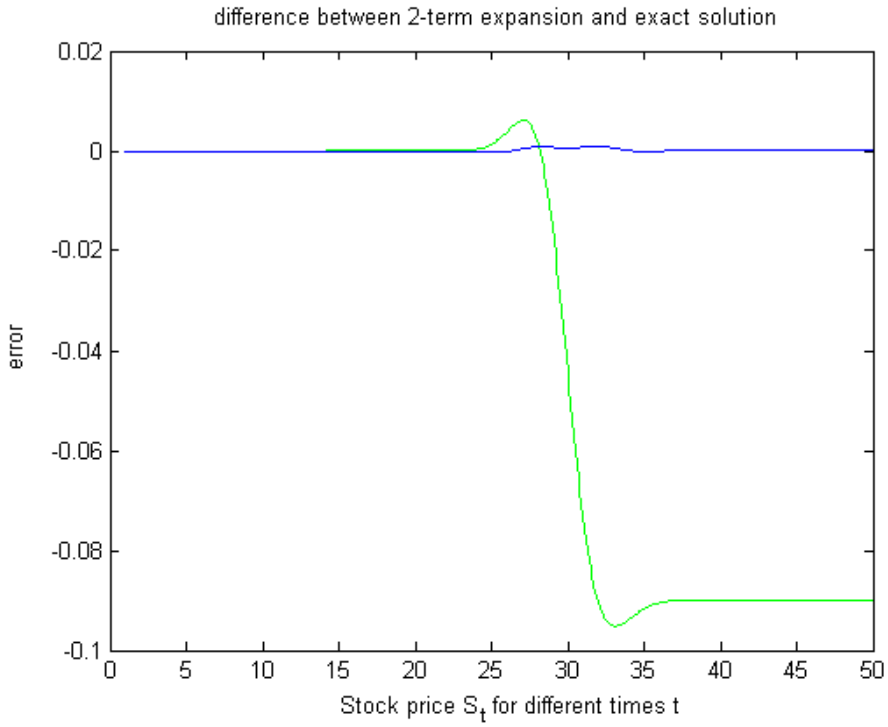

Figure 5.6: The difference between approximation with the 2-term expansion (blue) versus 1-term expansion (green) and the exact solution.

5.10 Taylor expansion

After substitution of $\bar{\tau} = T - t$ we can show that these expressions (5.9.11) and (5.9.15) agree to $\mathcal{O}\left(\varepsilon^2\right)$ with the small-time expansion of the exact solution below:

$$V(S,\tau) = S\ N(d_+) - Ke^{-r\tau}\ N(d_-), \tag{5.10.1}$$

with

$$d_\pm = \frac{\log(\frac{S}{K}) + \left(r \pm \frac{1}{2}\sigma^2\right)\bar{\tau}}{\sigma\sqrt{\bar{\tau}}}, \tag{5.10.2}$$

and

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-s^2/2} \mathrm{d}s. \tag{5.10.3}$$

The Taylor expansion of $V(S,\bar{\tau})$ around $\bar{\tau} = 0$ is given by

$$\begin{aligned} V^{\text{Taylor}}(S,\bar{\tau}) &= \sum_{n=0}^{\infty} \frac{\bar{\tau}^n}{n!} \frac{\partial^n}{\partial \bar{\tau}^n} V^{\text{Taylor}}_{\bar{\tau}=0}, & (5.10.4)\\ &= V^{\text{Taylor}}_{\bar{\tau}=0} + \bar{\tau}\frac{\partial}{\partial\bar{\tau}} V^{\text{Taylor}}_{\bar{\tau}=0} + \frac{1}{2}\bar{\tau}^2 \frac{\partial^2}{\partial\bar{\tau}^2} V^{\text{Taylor}}_{\bar{\tau}=0} + \mathcal{O}\left(\bar{\tau}^3\right). & (5.10.5) \end{aligned}$$

For the first term, $V^{\text{Taylor}}_{\bar{\tau}=0}$, we have

$$V^{\text{Taylor}}_{\bar{\tau}=0} = \begin{cases} S - K, & \text{for } S > K, \\ 0, & \text{for } S \leq K, \end{cases} \tag{5.10.6}$$

because this is just the payoff function at expiry.

For the second term, $\frac{\partial}{\partial\tau} V^{\text{Taylor}}_{\bar{\tau}=0}$, we find

$$\begin{aligned} \frac{\partial}{\partial\bar{\tau}} V^{\text{Taylor}}_{\bar{\tau}=0} &= \frac{\partial V}{\partial\bar{\tau}}(S,0) = \frac{\partial V}{\partial d_+}\frac{\partial d_+}{\partial\bar{\tau}} + \frac{\partial V}{\partial d_-}\frac{\partial d_-}{\partial\bar{\tau}}, \\ &= \begin{cases} -rK, & \text{for } S > K, \\ 0, & \text{for } S \leq K. \end{cases} \end{aligned}$$

Hence, the Taylor expansion (5.10.5) becomes

$$V^T(S,\bar{\tau}) = V^{\text{Taylor}}_{\bar{\tau}=0} + \bar{\tau}\frac{\partial}{\partial\bar{\tau}} V^{\text{Taylor}}_{\bar{\tau}=0} = \begin{cases} S - K(1-\bar{\tau}r) & \text{for } S > K, \\ 0, & \text{for } S \leq K. \end{cases} \tag{5.10.7}$$

In original variables this is

$$V^{\text{Taylor}}_{\bar{\tau}=0}(S,t) = \begin{cases} S - K + rK(T-t) & \text{for } S > K, \\ 0, & \text{for } S \leq K. \end{cases} \tag{5.10.8}$$

The Taylor expansion (5.10.8) agrees up to $\mathcal{O}(\varepsilon^2)$ with the 1-term and 2-term expansions we found in (5.9.11) and (5.9.15).

5.11 Discussion

In this section a first application of perturbation theory on a financial model has been presented in order to show the techniques and complications of the method of asymptotic expansions in a financial context. In the simple example of the Black-Scholes model an exact solution is available, so that it is possible to compare results.

First, we have constructed two artificial layers: a boundary layer at the option maturity date and an internal layer at the strike price. Next, the method of asymptotic expansions has been applied, in order to find a solution of the Black-Scholes equation (5.3.2). Comparison between the resulting asymptotic expansion and the exact solution is very useful: we can check if the results are consistent. It also indicates how accurate the approximation is.

Here, applying perturbation theory actually gives the same result as making a Taylor series of the exact solution around $\bar{\tau} = T - t = 0$. The more accurate the expansion (i.e., higher order terms are taken into account), the better the approximation will become.

On page 8 of his paper [7] Howison is showing some figures containing this section's results. Here, he plots the approximate call value minus the exact value as a function of moneyness S/K for different times to expiry. In general, the approximation is remarkably good for these practical parameter values, because the error is very small. Near $S = K$ (or $S/K = 1$) the error is a bit larger. This can be explained by the size of the risk factor $\Gamma = \frac{\partial^2 V}{\partial S^2}$, which is a Dirac delta function around $S = K$. Far from the strike, the option price is linear in S if the option is far in the money (ITM), and zero if the option is far out of the money (OTM). In these cases the second derivative is zero. At $S = K$, when the option is at the money (ATM) we have an infinite Γ. In this case, the second derivative $\frac{\partial^2 V}{\partial S^2}$ contributes to the PDE for the option price V.

Our opinion
Generally speaking, we think that the result of applying perturbation theory on the Black-Scholes model is not very useful. For this simple model an exact solution is available, so an asymptotic approximation is not necessary. Besides, the approximation will never be as accurate as the exact solution. However, constructing these artificial layers is an instructive way to show the method and to understand how an asymptotic expansion analysis can be done. Moreover, the results can be compared, because this exact solution is available. Due to this it is possible to observe the accuracy of the method.

This can be seen as a first step in applying perturbation theory on financial models. After understanding this, more complicated models can be considered.

Chapter 6

CEV models

In this chapter, we will consider a second financial model: the **constant elasticity of variance (CEV) model**, which is an improved version of the Black-Scholes model.

Using the CEV functions, we can calculate the theoretical price, sensitivities, and implied volatility of options, by applying a valuation technique based on the constant elasticity of variance option pricing model. With this model we consider the possibility that the volatility of the underlying asset depends on the price of the underlying asset.

This model is more realistic than the 'ordinary' Black-Scholes model, because studies have shown that price variances do indeed change as the stock price changes, while the Black-Scholes model assumes a constant stock price volatility, regardless of the level of the stock price.

6.1 Derivation of the CEV model

Let us change the stochastic differential equation (SDE) in (5.3.1) into

$$\mathrm{d}S = \mu S \ \mathrm{d}t + \tilde{\sigma} S^{\gamma/2} \mathrm{d}W_t, \quad \text{where } \gamma \neq 2. \tag{6.1.1}$$

Here

$$\tilde{\sigma} = S_0^{1-\frac{\gamma}{2}} \sigma, \tag{6.1.2}$$

with $S_0 = S(0)$ the stock price at $t = 0$, to make sure that the dimensions of $S_0^2 \left(\frac{S}{S_0}\right)^\gamma$ are correct (euro^2, instead of euro$^\gamma$).

So, for the change in the option price $\mathrm{d}V$ we have, using Itô's formula (4.3.1),

$$\mathrm{d}V = \tilde{\sigma} S^{\gamma/2} \frac{\partial V}{\partial S} + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2} \tilde{\sigma}^2 S^\gamma \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) \mathrm{d}t. \tag{6.1.3}$$

Again we construct a portfolio $\Pi = V - \phi S$, which - using SDE (6.1.1) and equation (6.1.3) - gives

$$\begin{aligned}
\mathrm{d}\Pi &= \mathrm{d}V - \phi \ \mathrm{d}S, \\
&= \tilde{\sigma} S^{\gamma/2} \frac{\partial V}{\partial S} + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2} \tilde{\sigma}^2 S^\gamma \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) \mathrm{d}t - \phi \mu S \mathrm{d}t - \phi \tilde{\sigma} S^{\gamma/2} \mathrm{d}W_t, \\
&= \tilde{\sigma} S^{\gamma/2} \left(\frac{\partial V}{\partial S} - \phi \right) \mathrm{d}W_t + \left(\mu S \left(\frac{\partial V}{\partial S} - \phi \right) + \frac{1}{2} \tilde{\sigma}^2 S^\gamma \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) \mathrm{d}t.
\end{aligned}$$

Once again we choose $\phi = \frac{\partial V}{\partial S}$ to eliminate the stochastic $\mathrm{d}W_t$ term. This gives

$$\mathrm{d}\Pi = \left(\frac{1}{2}\tilde{\sigma}^2 S^\gamma \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right)\mathrm{d}t. \tag{6.1.4}$$

Storing money at a bank account gives a risk-free interest rate r, so $\mathrm{d}\Pi = r\Pi\mathrm{d}t$. Substituting this into (6.1.4) yields

$$r\Pi\mathrm{d}t = \left(\frac{1}{2}\tilde{\sigma}^2 S^\gamma \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right)\mathrm{d}t. \tag{6.1.5}$$

Using $\Pi = V - \phi S$ with $\phi = \frac{\partial V}{\partial S}$ gives

$$r\left(V - \frac{\partial V}{\partial S}S\right)\mathrm{d}t = \left(\frac{1}{2}\tilde{\sigma}^2 S^\gamma \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right)\mathrm{d}t, \tag{6.1.6}$$

which, after division by $\mathrm{d}t$ and reordering the terms, becomes the CEV equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\tilde{\sigma}^2 S^\gamma \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{6.1.7}$$

In original variables, the CEV equation (6.1.7) is given by

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_0^2 \left(\frac{S}{S_0}\right)^\gamma \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{6.1.8}$$

The difference from the previous Black-Scholes model is that the volatility $\tilde{\sigma}$ is now S-dependent for $\gamma \neq 2$. The model is used to represent a 'leverage' effect whereby the impact of a given stochastic change $\mathrm{d}W_t$ is assumed to be greater when the asset price is small than when it is large, and thus γ is assumed to be less than the Black-Scholes value: $\gamma < 2$.

The parameters S_0 and $\tilde{\sigma}$ are not independent, but writing volatility at a given price level (for example: $S_0 = K$) gives the same at-the-money[1] volatility for options with strike K as γ varies.

Calculating explicit solutions is thus much less straightforward, but the asymptotic procedure is virtually the same as for the case $\gamma = 2$ (the 'ordinary' Black-Scholes model).

6.2 Scalings

As before we measure time backwards from expiry and scale it with σ^2, writing $t = T - t'/\sigma^2$ such that $t' = T - t\sigma^2$. After this scaling again we have

$$\frac{\partial V}{\partial t'} = \frac{1}{2}S^2\frac{\partial^2 V}{\partial S^2} + \alpha S\frac{\partial V}{\partial S} - \alpha V, \tag{6.2.1}$$

with $\alpha = \frac{r}{\sigma^2}$ dimensionless. Suppose that $\alpha = \mathcal{O}(1)$ and scaled time is small, such that $\tau = t'/\varepsilon^2$ with $0 < \varepsilon \ll 1$.

6.3 Outer expansion

For the CEV equation (6.2.1) above, this gives

$$\frac{\partial V}{\partial \tau} = \varepsilon^2\frac{1}{2}S^2\frac{\partial^2 V}{\partial S^2} + \varepsilon^2\alpha S\frac{\partial V}{\partial S} - \varepsilon^2\alpha V, \tag{6.3.1}$$

which is exactly the same as the previously used outer expansion (see section 5.6).

[1] An option is *at the money* if the strike price K of the option is equal to the market price S of the underlying security.

6.4 Inner expansion

For the inner problem, we first introduce a local variable

$$x = \frac{S-K}{\varepsilon K}, \tag{6.4.1}$$

such that x is dimensionless, and $S-K$ is scaled by ε.

Also we introduce a time rescaling

$$v(x,\tau) = \frac{V(S,\tau)}{\varepsilon K}, \tag{6.4.2}$$

such that v is dimensionless, and $V(S,\tau)$ is scaled with ε, where $\tau = \frac{(T-t)\sigma^2}{\varepsilon^2}$.

For the derivatives in the replaced CEV equation (6.1.8) this gives

$$\begin{aligned}
\frac{\partial V}{\partial t} &= \frac{\partial V}{\partial \tau}\frac{\partial \tau}{\partial t} = -\frac{\sigma^2}{\varepsilon^2}\frac{\partial V}{\partial \tau} = -\frac{\sigma^2}{\varepsilon^2}\frac{\partial(\varepsilon K v)}{\partial \tau} = -\frac{\sigma^2 K}{\varepsilon}\frac{\partial v}{\partial \tau},\\
\frac{\partial V}{\partial S} &= \frac{\partial V}{\partial x}\frac{\partial x}{\partial S} = \frac{1}{\varepsilon K}\frac{\partial V}{\partial x} = \frac{1}{\varepsilon K}\frac{\partial(\varepsilon K v)}{\partial x} = \frac{1}{\varepsilon K}\frac{\partial v}{\partial x},\\
\frac{\partial}{\partial S} &= \frac{1}{\varepsilon K}\frac{\partial}{\partial x} \Rightarrow \frac{\partial^2 V}{\partial S^2} = \frac{1}{\varepsilon K}\frac{\partial^2 v}{\partial x^2}.
\end{aligned}$$

Thus the replaced CEV equation (6.1.8) transforms into

$$\begin{aligned}
\frac{\sigma^2 K}{\varepsilon}\frac{\partial v}{\partial \tau} &= \frac{1}{2}\sigma^2 S_0^{2-\gamma}(1+\varepsilon x)^\gamma K^\gamma \frac{1}{\varepsilon K}\frac{\partial^2 v}{\partial x^2} + r(1+\varepsilon x)K\frac{\partial v}{\partial x} - \varepsilon K r v,\\
\Rightarrow \frac{\partial v}{\partial \tau} &= \frac{1}{2}\left(\frac{S_0}{K}\right)^{2-\gamma}(1+\varepsilon x)^\gamma \frac{\partial^2 v}{\partial x^2} + \varepsilon\frac{r}{\sigma^2}(1+\varepsilon x)\frac{\partial v}{\partial x} - \varepsilon^2\frac{r}{\sigma^2}v,\\
\Rightarrow \frac{\partial v}{\partial \tau} &= \frac{1}{2}\left(\frac{S_0}{K}\right)^{2-\gamma}(1+\varepsilon x)^\gamma \frac{\partial^2 v}{\partial x^2} + \varepsilon\alpha(1+\varepsilon x)\frac{\partial v}{\partial x} - \varepsilon^2\alpha v,
\end{aligned}$$

such that the inner equation becomes

$$\frac{\partial v}{\partial \tau} = \frac{1}{2}\kappa^2(1+\varepsilon x)^\gamma \frac{\partial^2 v}{\partial x^2} + \varepsilon\alpha(1+\varepsilon x)\frac{\partial v}{\partial x} - \varepsilon^2\alpha v. \tag{6.4.3}$$

Here $\kappa^2 = (S_0/K)^{2-\gamma}$, and we have the same payoff as before: $v(x,0) = \max(S-K,0)$.

First we rescale time by setting $\tau_\gamma = \tau\kappa^2$, and again we expand

$$v_\varepsilon(x,\tau_\gamma) = v_0(x,\tau_\gamma) + \varepsilon v_1(x,\tau_\gamma) + \mathcal{O}\left(\varepsilon^2\right), \tag{6.4.4}$$

which gives

$$\begin{aligned}
\kappa^2\frac{\partial}{\partial \tau_\gamma}\Big(v_0(x,\tau_\gamma) + \varepsilon v_1(x,\tau_\gamma) + \dots\Big) &= \frac{1}{2}\kappa^2(1+\varepsilon x)^\gamma\frac{\partial^2}{\partial x^2}\Big(v_0(x,\tau_\gamma) + \varepsilon v_1(x,\tau_\gamma) + \dots\Big)\\
&+ \varepsilon\alpha(1+\varepsilon x)\frac{\partial}{\partial x}\Big(v_0(x,\tau_\gamma) + \varepsilon v_1(x,\tau_\gamma) + \dots\Big)\\
&- \varepsilon^2\alpha\Big(v_0(x,\tau_\gamma) + \varepsilon v_1(x,\tau_\gamma) + \dots\Big),
\end{aligned} \tag{6.4.5}$$

Because this appears to be the same problem as the previous one for v_0, in (5.9.12), we will obtain an inner solution that is similar to the one in (5.9.13), namely,

$$v(x,\tau) \sim v_0(x,\tau) + \varepsilon v_1(x,\tau) = (x + \varepsilon\alpha\tau)\ N\left(\frac{\kappa x}{\sqrt{\tau}}\right) + \kappa\sqrt{\tau}\left(1 + \frac{1}{4}\varepsilon\gamma x\right)\ n\left(\frac{\kappa x}{\sqrt{\tau}}\right). \quad (6.4.6)$$

When $\gamma = 2$, the approximation (6.4.6) reduces to the previous expression (5.9.14).

In original variables, we have the approximation

$$\begin{aligned} V(S,t) &= \varepsilon K v(x,\tau) \sim \varepsilon K\Big(v_0(x,\tau) + \varepsilon v_1(x,\tau)\Big), \\ &= \varepsilon K\left((x + \varepsilon\alpha\tau)\ N\left(\frac{\kappa x}{\sqrt{\tau}}\right) + \kappa\sqrt{\tau}\left(1 + \frac{1}{4}\varepsilon\gamma x\right)\ n\left(\frac{\kappa x}{\sqrt{\tau}}\right)\right), \\ &= \big(S - K + rK(T-t)\big)\ N\left(\frac{\kappa(S/K-1)}{\sigma\sqrt{T-t}}\right), \\ &+ \kappa\sigma\sqrt{T-t}\left(\frac{\gamma S + (4-\gamma)K}{4}\right)\ n\left(\frac{\kappa(S/K-1)}{\sigma\sqrt{T-t}}\right). \end{aligned} \quad (6.4.7)$$

In figure 6.1 the CEV values for a call option are presented for different values of $\gamma < 2$. The absolute and relative differences between the $\gamma = 2$ case ('ordinary' Black-Scholes model) and CEV values with $\gamma < 2$ can be seen in figure 6.2 and 6.3 respectively.

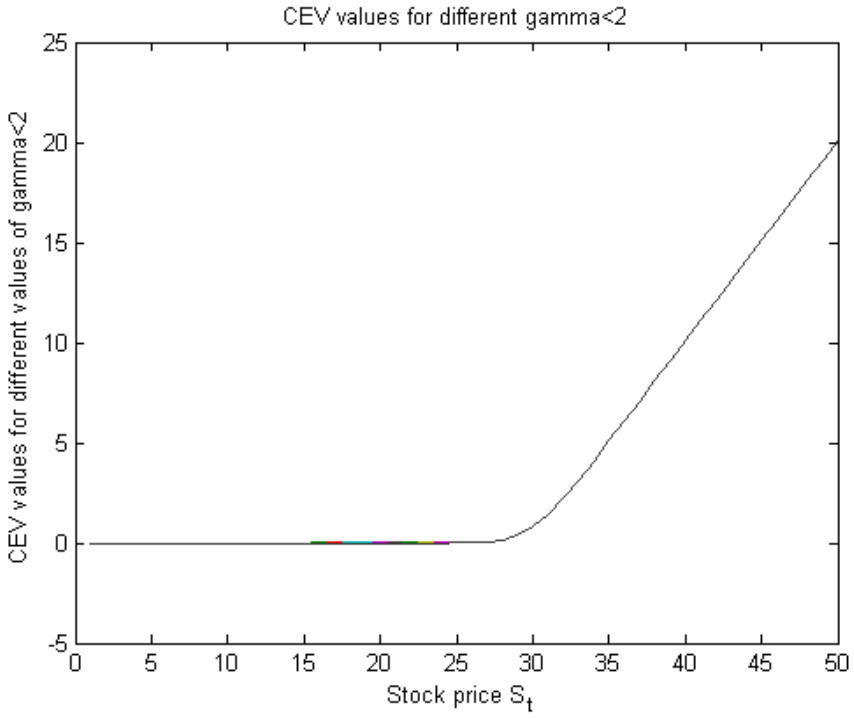

Figure 6.1: CEV values (call option) for different $\gamma < 2$

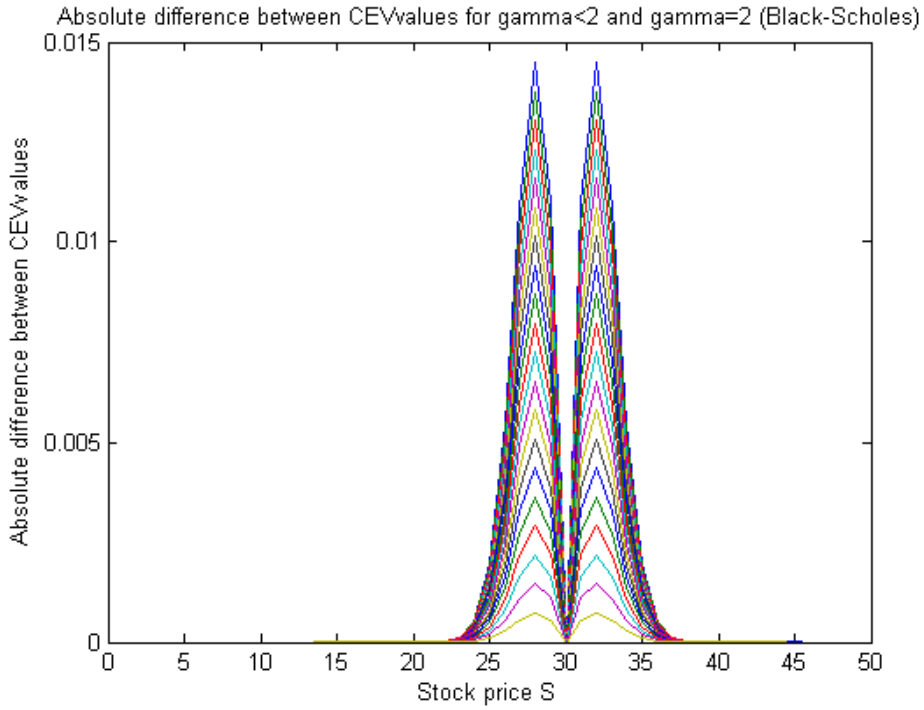

Figure 6.2: Absolute difference between the $\gamma = 2$ case ('ordinary' Black-Scholes model) and CEV values with $\gamma < 2$.

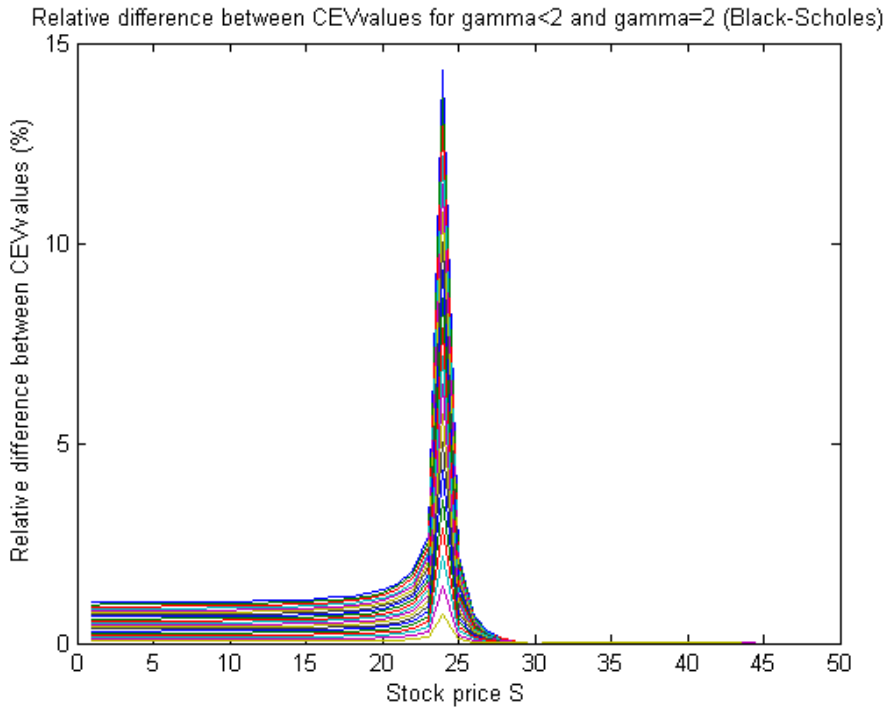

Figure 6.3: Relative difference (%) between the $\gamma = 2$ case ('ordinary' Black-Scholes model) and CEV values with $\gamma < 2$.

Chapter 7

Fast mean-reverting volatility

Boundary-layer techniques can be applied in the analysis of **fast mean-reverting stochastic volatility models**. In these models the volatility itself is assumed to follow a stochastic process while the asset price is assumed to follow the lognormal process as before.[1]

$$\begin{cases} \mathrm{d}S_t = \mu_t S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}W_t, & (7.0.1a) \\ \mathrm{d}\sigma_t = M_t \mathrm{d}t + \Sigma_t \mathrm{d}\widetilde{W}_t. & (7.0.1b) \end{cases}$$

Here M_t and Σ_t are the drift and the volatility of the volatility σ_t respectively. The instantaneous correlation between the Brownian motions W_t and $\widetilde{W}_t$ is denoted by ρ, i.e.,[2]

$$\mathrm{E}\left[\mathrm{d}W_t \; \mathrm{d}\widetilde{W}_t\right] = \rho \; \mathrm{d}t.$$

A well-known example of a stochastic volatility model is the **Heston model**, for which

$$\mathrm{d}(\sigma_t^2) = -\kappa \left(\sigma_t - \bar{\sigma}_\infty\right) \mathrm{d}t + \theta \sigma_t \mathrm{d}\widetilde{W}_t, \tag{7.0.2}$$

for constants $\kappa > 0$, θ and $\bar{\sigma}_\infty$.

Leaving out the stochastic part (containing $\mathrm{d}\widetilde{W}_t$), we have

$$\frac{\mathrm{d}\sigma_t^2}{\mathrm{d}t} = \begin{cases} -\kappa \left(\sigma_t - \bar{\sigma}_\infty\right) & \leq 0, \quad \text{for} \;\; \sigma_t \geq \bar{\sigma}_\infty, \\ ,, & > 0, \quad \text{for} \;\; \sigma_t < \bar{\sigma}_\infty, \end{cases} \tag{7.0.3}$$

such that in the long term the volatility σ_t will always return to its equilibrium value $\bar{\sigma}_\infty$. That is why this model is called **mean-reverting**. The *fast* mean-reversion is caused by the parameter κ, which we will assume to be quite large. This will be explained in terms of time scales in section 7.2.

7.1 Derivation of the fast mean-reverting stochastic volatility model

Suppose we have the following model:[3]

$$\begin{cases} \mathrm{d}S = rS \; \mathrm{d}t + \sigma S \; \mathrm{d}W_t, & (7.1.1a) \\ \mathrm{d}\sigma = (M - \lambda\Sigma) \, \mathrm{d}t + \Sigma \; \mathrm{d}\widetilde{W}_t, & (7.1.1b) \end{cases}$$

[1] Therefore we write σ_t instead of σ in this case.
[2] See chapter 3.2.
[3] Note that we omit the subscripts from here on, for notational convenience.

where λ is the market price of volatility risk[4], and we know that $\mathrm{E}\left[\mathrm{d}W_t\ \mathrm{d}\widetilde{W}_t\right] = \rho\ \mathrm{d}t$.

The value V of an option is now a function of the stock price S, time t and volatility σ: $V = V(S, \sigma, t)$. Using Itô's formula (4.3.1) we obtain up to and including $\mathcal{O}(\mathrm{d}t)$ the differential

$$\mathrm{d}V = \frac{\partial V}{\partial t}\mathrm{d}t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\mathrm{d}S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}\mathrm{d}\sigma^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\mathrm{d}\sigma\mathrm{d}S + \frac{\partial V}{\partial \sigma}\mathrm{d}\sigma + \frac{\partial V}{\partial S}\mathrm{d}S. \tag{7.1.2}$$

Substituting $\mathrm{d}S$ and $\mathrm{d}\sigma$ as given in definition (7.1.1a) and (7.1.1b) resp. into equation (7.1.2), yields

$$\begin{aligned} \mathrm{d}V &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\Sigma^2\frac{\partial^2 V}{\partial \sigma^2} + \sigma\Sigma S\rho\frac{\partial^2 V}{\partial S\partial\sigma} + (M - \lambda\Sigma)\frac{\partial V}{\partial \sigma} + rS\frac{\partial V}{\partial S}\right)\mathrm{d}t \\ &+ \frac{\partial V}{\partial \sigma}\Sigma\mathrm{d}\widetilde{W}_t + \frac{\partial V}{\partial S}\sigma S\mathrm{d}W_t. \end{aligned} \tag{7.1.3}$$

Again we will set up a portfolio Π.[5] Because now we have two stochastic parts (due to two Brownian motions $\mathrm{d}W_t$ and $\mathrm{d}\widetilde{W}_t$), we have two hedge parameters Δ and Δ_1, such that

$$\Pi = V - \Delta S - \Delta_1 V_1. \tag{7.1.4}$$

So our portfolio Π contains the option whose value is $V(S, \sigma, t)$, a quantity $-\Delta$ of the stock S, and a quantity $-\Delta_1$ of another asset whose value V_1 depends on the stock price S, volatility σ and time t. Note that the quantities Δ and Δ_1 can also become negative in case you sell stock (*going short*).

The change $\mathrm{d}\Pi$ in this portfolio in a time $\mathrm{d}t$ is given by

$$\begin{aligned} \mathrm{d}\Pi &= \mathrm{d}V - \Delta\mathrm{d}S - \Delta_1\mathrm{d}V_1, \qquad (7.1.5)\\ &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\Sigma^2\frac{\partial^2 V}{\partial \sigma^2} + \sigma\Sigma S\rho\frac{\partial^2 V}{\partial S\partial\sigma} + (M - \lambda\Sigma)\frac{\partial V}{\partial \sigma} + rS\frac{\partial V}{\partial S}\right)\mathrm{d}t \\ &+ \frac{\partial V}{\partial \sigma}\Sigma\mathrm{d}\widetilde{W}_t + \frac{\partial V}{\partial S}\sigma S\mathrm{d}W_t - \Delta\left(rS\mathrm{d}t + \sigma S\mathrm{d}W_t\right) - \Delta_1\left(\frac{\partial V_1}{\partial \sigma}\Sigma\mathrm{d}\widetilde{W}_t + \frac{\partial V_1}{\partial S}\sigma S\mathrm{d}W_t\right) \\ &- \Delta_1\left(\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V_1}{\partial S^2} + \frac{1}{2}\Sigma^2\frac{\partial^2 V_1}{\partial \sigma^2} + \sigma\Sigma S\rho\frac{\partial^2 V_1}{\partial S\partial\sigma} + (M - \lambda\Sigma)\frac{\partial V_1}{\partial \sigma} + rS\frac{\partial V_1}{\partial S}\right)\mathrm{d}t. \end{aligned}$$

To make the portfolio instantaneously risk-free, all terms containing $\mathrm{d}W_t$ and $\mathrm{d}\widetilde{W}_t$ must vanish. That is true if

$$\frac{\partial V}{\partial S} - \Delta_1\frac{\partial V_1}{\partial S} - \Delta = 0, \quad \text{and}$$
$$\frac{\partial V}{\partial \sigma} - \Delta_1\frac{\partial V_1}{\partial \sigma} = 0,$$

i.e.,

$$\Delta_1 = \frac{\left(\frac{\partial V}{\partial \sigma}\right)}{\left(\frac{\partial V_1}{\partial \sigma}\right)}, \quad \text{and} \tag{7.1.6a}$$

$$\Delta = \frac{\partial V}{\partial S} - \frac{\left(\frac{\partial V}{\partial \sigma}\right)}{\left(\frac{\partial V_1}{\partial \sigma}\right)}\frac{\partial V_1}{\partial S}. \tag{7.1.6b}$$

[4] A definition and explanation of this concept will be given later on in this chapter.
[5] Based on the method used in [3] on p. 4-6.

Substituting Δ_1 and Δ as above into equation (7.1.5), yields

$$\begin{aligned} \mathrm{d}\Pi &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\Sigma^2 \frac{\partial^2 V}{\partial \sigma^2} + \sigma\Sigma S\rho \frac{\partial^2 V}{\partial S \partial \sigma}\right) \mathrm{d}t \\ &- \frac{\left(\frac{\partial V}{\partial \sigma}\right)}{\left(\frac{\partial V_1}{\partial \sigma}\right)} \left(\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + \frac{1}{2}\Sigma^2 \frac{\partial^2 V_1}{\partial \sigma^2} + \sigma\Sigma S\rho \frac{\partial^2 V_1}{\partial S \partial \sigma}\right) \mathrm{d}t. \end{aligned}$$

The fact that the return on a risk-free portfolio must equal the risk free rate r, which we assume to be deterministic, implies that $\mathrm{d}\Pi = r\Pi \mathrm{d}t$. Or explicitely, following the steps taken in [3], and collecting all V-dependent terms on the left-hand side and all V_1-dependent terms on the right-hand side, we obtain

$$\begin{aligned} &\frac{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\Sigma^2 \frac{\partial^2 V}{\partial \sigma^2} + \sigma\Sigma S\rho \frac{\partial^2 V}{\partial S \partial \sigma} + rS\frac{\partial V}{\partial S} - rV}{\frac{\partial V}{\partial \sigma}} \\ &= \frac{\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + \frac{1}{2}\Sigma^2 \frac{\partial^2 V_1}{\partial \sigma^2} + \sigma\Sigma S\rho \frac{\partial^2 V_1}{\partial S \partial \sigma} + rS\frac{\partial V_1}{\partial S} - rV_1}{\frac{\partial V_1}{\partial \sigma}}. \end{aligned} \tag{7.1.7}$$

Because the left-hand side of equation (7.1.7) is explicitly independent of V_1 and the right-hand side is explicitly independent of V, either side must be independent of both V and V_1. The only way that this can be true is for both sides to be equal to some function of the independent variables S, σ and t. We deduce that

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\Sigma^2 \frac{\partial^2 V}{\partial \sigma^2} + \sigma\Sigma S\rho \frac{\partial^2 V}{\partial S \partial \sigma} + rS\frac{\partial V}{\partial S} - rV \\ = -\Big((M - \lambda\Sigma) - \lambda\Sigma\Big)\frac{\partial V}{\partial \sigma}, \end{aligned} \tag{7.1.8}$$

where, without loss of generality, we have written the arbitrary function of S, σ and t as $\Big((M - \lambda\Sigma) - \lambda\Sigma\Big)$, where $(M - \lambda\Sigma)$ and Σ are the drift and volatility functions from the SDE (7.1.1b) for instantaneous variance. This approach is analougos to the one in [3]. Here, $\lambda = \lambda(S, \sigma, t)$ is an arbitrary function.

The market price of volatility risk

To see why λ is called the **market price of volatility risk**, we will consider the following portfolio, consisting of a Δ-hedged (but not Δ_1-hedged) option V:

$$\Pi_1 = V - \frac{\partial V}{\partial S} S. \tag{7.1.9}$$

After applying Itô's formula (4.3.1) we obtain

$$\begin{aligned} \mathrm{d}\Pi_1 &= \mathrm{d}V - \frac{\partial V}{\partial S}\mathrm{d}S, \\ &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\Sigma^2 \frac{\partial^2 V}{\partial \sigma^2} + \sigma\Sigma S\rho \frac{\partial^2 V}{\partial S \partial \sigma} + (M - \lambda\Sigma)\frac{\partial V}{\partial \sigma}\right) \mathrm{d}t \\ &+ \frac{\partial V}{\partial \sigma}\Sigma \mathrm{d}\widetilde{W}_t. \end{aligned} \tag{7.1.10}$$

Note that all terms with $\mathrm{d}W_t$ vanish, because the option is Δ-hedged.

Again we expect to have $\mathrm{d}\Pi_1 = r\Pi_1 \mathrm{d}t$, such that $\mathrm{d}\Pi_1 - r\Pi_1 \mathrm{d}t = 0$. However, because we did not Δ_1-hedge the portfolio, we have

$$\begin{aligned}
\mathrm{d}\Pi_1 - r\Pi_1 \mathrm{d}t &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\Sigma^2 \frac{\partial^2 V}{\partial \sigma^2} + \sigma\Sigma S\rho \frac{\partial^2 V}{\partial S \partial \sigma} + (M - \lambda\Sigma)\frac{\partial V}{\partial \sigma}\right)\mathrm{d}t \\
&+ \frac{\partial V}{\partial \sigma}\Sigma \mathrm{d}\widetilde{W}_t - r\left(V - \frac{\partial V}{\partial S}S\right)\mathrm{d}t, \\
&= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\Sigma^2 \frac{\partial^2 V}{\partial \sigma^2} + \sigma\Sigma S\rho \frac{\partial^2 V}{\partial S \partial \sigma} + (M - \lambda\Sigma)\frac{\partial V}{\partial \sigma}\right. \\
&\quad \left. + rS\frac{\partial V}{\partial S} - rV\right)\mathrm{d}t + \frac{\partial V}{\partial \sigma}\Sigma \mathrm{d}\widetilde{W}_t. \qquad (7.1.11)
\end{aligned}$$

Using equation (7.1.8) we find

$$\begin{aligned}
\mathrm{d}\Pi_1 - r\Pi_1 \mathrm{d}t &= -\Big((M - \lambda\Sigma) - \lambda\Sigma\Big)\frac{\partial V}{\partial \sigma}\mathrm{d}t + (M - \lambda\Sigma)\frac{\partial V}{\partial \sigma}\mathrm{d}t + \Sigma\frac{\partial V}{\partial \sigma}\mathrm{d}\widetilde{W}_t, \\
&= \lambda\Sigma\frac{\partial V}{\partial \sigma}\mathrm{d}t + \Sigma\frac{\partial V}{\partial \sigma}\mathrm{d}\widetilde{W}_t = \Sigma\frac{\partial V}{\partial \sigma}\left(\lambda \mathrm{d}t + \mathrm{d}\widetilde{W}_t\right) \neq 0. \qquad (7.1.12)
\end{aligned}$$

Taking the expectation to get rid of the $\mathrm{d}\widetilde{W}_t$ term yields

$$\mathrm{E}\left(\mathrm{d}\Pi_1 - r\Pi_1 \mathrm{d}t\right) = \sum \frac{\partial V}{\partial \sigma}\lambda \mathrm{d}t. \qquad (7.1.13)$$

We see that the extra return per unit of volatility risk $\mathrm{d}\widetilde{W}_t$ scales linearly with λ, which is known as the market price of volatility risk.

Hence, the price $V(S, \sigma, t)$ satisfies the backward parabolic equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho S\sigma\Sigma \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2}\Sigma^2 \frac{\partial^2 V}{\partial \sigma^2} + rS\frac{\partial V}{\partial S} + (M - \lambda\Sigma)\frac{\partial V}{\partial \sigma} - rV = 0, \qquad (7.1.14)$$

of which the Black-Scholes equation is a special case.

7.2 Scalings

We assume for simplicity that the drift and volatility of the volatility σ are functions of only σ, not of S or t. Let us consider the commonly occurring situation in which the volatility process is fast mean-reverting, which means that the timescale for mean-reversion is much shorter than that for the evolution of the asset price, their ratio $\frac{r}{M - \lambda\Sigma}$ being small, such that we can apply perturbation theory.

We now introduce the scaled variables

$$m := \varepsilon M \quad \Leftrightarrow \quad M = \frac{m}{\varepsilon} \qquad \text{and} \qquad \zeta := \sqrt{\varepsilon}\,\Sigma \quad \Leftrightarrow \quad \Sigma = \frac{\zeta}{\sqrt{\varepsilon}}, \qquad (7.2.1)$$

such that $\frac{\partial \sigma}{\partial t} = M - \lambda\Sigma$ is large when $M = \frac{m}{\varepsilon}$ is large.[6] This causes the fast mean-reversion.

[6] Note that $M = \mathcal{O}(\frac{1}{\varepsilon})$, which implies $m = \mathcal{O}(1)$, and note that $\Sigma = \mathcal{O}(\frac{1}{\sqrt{\varepsilon}})$, which implies $\zeta = \mathcal{O}(1)$.

The relative sizes of these coefficients are chosen such that σ has a nontrivial invariant distribution. We denote this time-independent invariant distribution by

$$p_\infty(\sigma) := \lim_{t\to\infty} p(\sigma, t|\sigma_0, 0),$$

where $p(\sigma, t|\sigma_0, 0)$ is the transition density function for σ at time t, conditional on σ_0 at time zero, which satisfies the forward Kolmogorov equation[7]

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial \sigma}\Big[(M - \lambda\Sigma)p\Big] + \frac{1}{2}\frac{\partial^2}{\partial \sigma^2}\left(\Sigma^2 p\right). \tag{7.2.2}$$

In terms of the original variables, this reads

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial \sigma}\left[\left(\frac{m}{\varepsilon} - \lambda\frac{\zeta}{\sqrt{\varepsilon}}\right)p\right] + \frac{1}{2}\frac{\partial^2}{\partial \sigma^2}\left(\frac{\zeta^2}{\varepsilon}p\right). \tag{7.2.3}$$

After applying the above scalings, the pricing equation becomes

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{\rho\zeta}{\sqrt{\varepsilon}}\sigma S\frac{\partial^2 V}{\partial S\partial\sigma} + \frac{1}{2}\frac{\zeta^2}{\varepsilon}\frac{\partial^2 V}{\partial\sigma^2} + rS\frac{\partial V}{\partial S} + \left(\frac{m}{\varepsilon} - \frac{\lambda\zeta}{\sqrt{\varepsilon}}\right)\frac{\partial V}{\partial\sigma_t} - rV = 0. \tag{7.2.4}$$

7.3 Outer expansion

First, we write the pricing equation (7.2.4) for the fast mean-reverting process in the form

$$\left(\mathcal{L}_0 + \sqrt{\varepsilon}\mathcal{L}_{\frac{1}{2}} + \varepsilon\mathcal{L}_1\right)V = 0, \tag{7.3.1}$$

where

$$\begin{aligned}
\mathcal{L}_0 &= \frac{1}{2}\zeta^2\frac{\partial^2}{\partial\sigma^2} + m\frac{\partial}{\partial\sigma},\\
\mathcal{L}_{\frac{1}{2}} &= \rho\zeta\sigma S\frac{\partial^2}{\partial S\partial\sigma} - \lambda\zeta\frac{\partial}{\partial\sigma},\\
\mathcal{L}_1 &= \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2}{\partial S^2} + rS\frac{\partial}{\partial S} - r.
\end{aligned}$$

Note that $\mathcal{L}_1$ is the Black-Scholes operator[8] with volatility σ. Because $\mathcal{L}_0$ is the generator of the backward Kolmogorov equation for σ, its adjoint $\mathcal{L}_0^*$ is the generator of the forward equation. These Kolmogorov equations govern the evolution in time of the transition density function p.

Hence, $p_\infty(\sigma)$ satisfies the forward Kolmogorov equation, with $\frac{\partial p_\infty}{\partial t} = 0$, because $p_\infty(\sigma)$ does not depend on t. Therefore, $p_\infty(\sigma)$ satisfies

$$\mathcal{L}_0^* p_\infty = \frac{\mathrm{d}^2}{\mathrm{d}\sigma^2}\left(\frac{1}{2}\zeta^2 p_\infty\right) - \frac{\mathrm{d}}{\mathrm{d}\sigma}(mp_\infty) = 0. \tag{7.3.2}$$

Integrating once yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\zeta^2 p_\infty(\sigma)\right) - mp_\infty(\sigma) = c_0. \tag{7.3.3}$$

[7] We found this forward Kolmogorov equation using the method on p. 291 of [10]. This is very different from the equation Howison writes down on page 14 of [7].

[8] The Black-Scholes equation reads $\mathcal{L}_1 V = 0$.

The homogeneous version of this equation is given by

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\sigma_t}\left(\zeta^2 p_\infty^{\mathrm{hom}}(\sigma_t)\right) - m p_\infty^{\mathrm{hom}}(\sigma_t) = 0, \tag{7.3.4}$$

which can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\zeta^2 p_\infty^{\mathrm{hom}}(\sigma)\right) - \frac{2m}{\zeta^2}\left(\zeta^2 p_\infty^{\mathrm{hom}}(\sigma)\right) = 0, \tag{7.3.5}$$

such that

$$p_\infty^{\mathrm{hom}}(\sigma) = \frac{2c_1}{\zeta^2}\exp\left\{\int_0^\sigma \frac{2m(s)}{\zeta^2(s)}\,\mathrm{d}s\right\}. \tag{7.3.6}$$

As a particular solution we take p_∞ to be constant:

$$p_\infty^{\mathrm{part}}(\sigma) := c_2, \tag{7.3.7}$$

because it has to look like the inhomogeneous part of the integrated Kolmogorov equation (7.3.3). After substituting this particular solution into equation (7.3.3), we find $c_2 = \frac{c_0}{m}$.

The general solution is thus given by

$$p_\infty(\sigma) = p_\infty^{\mathrm{hom}}(\sigma) + p_\infty^{\mathrm{part}}(\sigma) = \frac{2c_1}{\zeta^2}\exp\left\{\int_0^\sigma \frac{2m(s)}{\zeta^2(s)}\,\mathrm{d}s\right\} + \frac{c_0}{m}, \tag{7.3.8a}$$

with $p_\infty(\sigma)$ satisfying the property

$$\int_{-\infty}^{\infty} p_\infty(\sigma)\,\mathrm{d}\sigma = 1, \tag{7.3.8b}$$

because $p_\infty(\sigma)$ is a transition density function.

Howison does not explicitly show his solution in [7], but he only remarks that it is proportional to

$$\frac{1}{\zeta^2(\sigma)}\exp\left(-2\int^\sigma \frac{m(s)}{\zeta^2(s)}\,\mathrm{d}s\right),$$

assuming that ζ^2 and m are such that p_∞ exists. This is very different, because we did not find the minus sign in the exponent, and also the integration 'constant' $\frac{c_0}{m}$ is missing here.

We now expand

$$V(S,\sigma,t) \sim V_0(S,\sigma,t) + \varepsilon^{\frac{1}{2}} V_{\frac{1}{2}}(S,\sigma,t) + \varepsilon V_1(S,\sigma,t) + \varepsilon^{\frac{3}{2}} V_{\frac{3}{2}}(S,\sigma,t) + \varepsilon^2 V_2(S,\sigma,t) + \ldots.$$

After substituting this into the pricing equation (7.3.1), and collecting coefficients of equal powers of ε together, we obtain

$$\begin{aligned}
\mathcal{O}(1): &\quad \mathcal{L}_0 V_0 &&= 0,\\
\mathcal{O}(\sqrt{\varepsilon}): &\quad \mathcal{L}_0 V_{\frac{1}{2}} + \mathcal{L}_{\frac{1}{2}} V_0 &&= 0,\\
\mathcal{O}(\varepsilon): &\quad \mathcal{L}_0 V_1 + \mathcal{L}_{\frac{1}{2}} V_{\frac{1}{2}} + \mathcal{L}_1 V_0 &&= 0,\\
\mathcal{O}\left(\varepsilon^{\frac{3}{2}}\right): &\quad \mathcal{L}_0 V_{\frac{3}{2}} + \mathcal{L}_{\frac{1}{2}} V_1 + \mathcal{L}_1 V_{\frac{1}{2}} &&= 0,\\
\mathcal{O}\left(\varepsilon^2\right): &\quad \mathcal{L}_0 V_2 + \mathcal{L}_{\frac{1}{2}} V_{\frac{3}{2}} + \mathcal{L}_1 V_1 &&= 0.
\end{aligned}$$

We will first solve the $\mathcal{O}(1)$ equation $\mathcal{L}_0 V_0 = 0$ for V_0, i.e.,

$$\frac{1}{2}\zeta^2 \frac{\partial^2 V_0}{\partial \sigma^2} + m\frac{\partial V_0}{\partial \sigma} = 0, \tag{7.3.9}$$

which we can rewrite as

$$\frac{\partial^2 V_0}{\partial \sigma^2} = -\frac{2m}{\zeta^2}\frac{\partial V_0}{\partial \sigma}. \tag{7.3.10}$$

Integrating once yields

$$\frac{\partial V_0}{\partial \sigma} = c(S,t)\exp\left\{\int_0^\sigma \frac{2m(s)}{\zeta^2(s)}\mathrm{d}s\right\}. \tag{7.3.11}$$

After another integration we find

$$\begin{aligned} V_0(S,\sigma,t) &= \int_0^\sigma c(S,t)\exp\left\{\int_0^{\bar\sigma}\frac{2m(s)}{\zeta^2(s)}\mathrm{d}s\right\}\mathrm{d}\bar\sigma + c_0(S,t), \\ &= c(S,t)\int_0^\sigma \exp\left\{\int_0^{\bar\sigma}\frac{2m(s)}{\zeta^2(s)}\mathrm{d}s\right\}\mathrm{d}\bar\sigma + c_0(S,t), \end{aligned} \tag{7.3.12}$$

such that we can write this solution V_0 as

$$V_0(S,\sigma,t) = c(S,t)g(\sigma) + c_0(S,t). \tag{7.3.13}$$

Here, $V_0(S,\sigma,t)$ satisfies the following terminal condition at $t = T$:

$$V_0(S,\sigma,T) = \max(S-K,0). \tag{7.3.14}$$

Also, we know that the solution V_0 satisfies the following limits[9]

$$\lim_{\frac{S}{K}\to\infty}\frac{V_0}{S} = 1, \quad \text{and} \quad \lim_{\frac{S}{K}\to 0}\frac{V_0}{S} = 0. \tag{7.3.15}$$

Next, Howison uses the disputable[10] assumption that V_0 is independent of σ, to obtain $c(S,t) = 0$ and thus $V_0(S,\sigma,t) = c_0(S,t)$.

The $\mathcal{O}(\sqrt{\varepsilon})$ equation is given by

$$\mathcal{L}_0 V_{\frac{1}{2}} + \mathcal{L}_{\frac{1}{2}} V_0 = 0, \tag{7.3.16}$$

with initial condition $V_{\frac{1}{2}}(S,\sigma,0) = 0$.

Since $V_0 = c_0(S,t)$ is a function of S and t only, we have $\mathcal{L}_{\frac{1}{2}} V_0 = 0$, such that the $\mathcal{O}(\sqrt{\varepsilon})$ equation (7.3.16) becomes

$$\mathcal{L}_0 V_{\frac{1}{2}} = 0. \tag{7.3.17}$$

This implies that $V_{\frac{1}{2}}$ is independent of σ, and therefore

$$V_{\frac{1}{2}} = c_{\frac{1}{2}}(S,t). \tag{7.3.18}$$

Note that if Howison would not have done the above assumption about the σ-independence of V_0, the $\mathcal{O}(\sqrt{\varepsilon})$ equation would be completely different, because then the term $\mathcal{L}_{\frac{1}{2}} V_0$ would not be zero. Consequently, it would have been a lot more complicated to obtain the $\mathcal{O}(\sqrt{\varepsilon})$ solution

[9] See [10], p. 289.
[10] See discussion in section 7.6.

$V_{\frac{1}{2}}(S,\sigma,t)$.

Because $\mathcal{L}_{\frac{1}{2}}$ contains derivatives with respect to σ only, the $\mathcal{O}(\varepsilon)$ equation

$$\mathcal{L}_0 V_1 + \mathcal{L}_{\frac{1}{2}} V_{\frac{1}{2}} + \mathcal{L}_1 V_0 = 0, \tag{7.3.19}$$

thus becomes

$$\mathcal{L}_0 V_1 = -\mathcal{L}_1 V_0, \tag{7.3.20}$$

with initial condition $V_1(S,\sigma,0) = 0$.

To solve the $\mathcal{O}(\varepsilon)$ equation (7.3.20) we use the inner product with some arbitrary function f that is not yet determined:

$$< \mathcal{L}_0 V_1, f >= - < \mathcal{L}_1 V_0, f > . \tag{7.3.21}$$

This can also be written as

$$< V_1, \mathcal{L}_0^* f >= - < \mathcal{L}_1 V_0, f >, \tag{7.3.22}$$

where $\mathcal{L}_0^*$ is the **Hermitian adjoint**[11] of $\mathcal{L}_0$.

Because we know the dynamics of p_∞ and the fact that $\mathcal{L}_0^* p_\infty = 0$, we choose $f = p_\infty$ to obtain

$$< V_1, 0 >= - < \mathcal{L}_1 V_0, f >, \tag{7.3.23}$$

i.e.,

$$< \mathcal{L}_1 V_0, p_\infty >= 0. \tag{7.3.24}$$

Since $\mathcal{L}_1$ is an operator, we can write down the eigenvalue equation

$$\mathcal{L}_1 \Phi_i = \lambda_i \Phi_i, \tag{7.3.25}$$

where Φ_i is an eigenfunction, and λ_i is its corresponding eigenvalue. In this case we know that there is an eigenvalue $\lambda_i = 0$, because we have $< \mathcal{L}_1 V_0, p_\infty >= 0$. By the **Fredholm alternative**[12], we know that there is either no solution V_0 possible, or there are infinitely many solutions, which implies that V_0 cannot be determined uniquely.

Define $\overline{\mathcal{L}_i} V_j :=< \mathcal{L}_i V_j, p_\infty >$ for $i = 0, \frac{1}{2}, 1$ and $j = 0, 1, 2, \ldots$. For $i = 1$ and $j = 0$ and using equation (7.3.24) we obtain

$$\begin{aligned} \overline{\mathcal{L}_1} V_0 &= \int_{-\infty}^{\infty} \mathcal{L}_1 V_0 p_\infty(\sigma)\, \mathrm{d}\sigma, \\ &= \int_{-\infty}^{\infty} \left(\frac{\partial V_0}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_0}{\partial S^2} + rS\frac{\partial V_0}{\partial S} - rV_0 \right) p_\infty(\sigma)\, \mathrm{d}\sigma = 0. \end{aligned} \tag{7.3.26}$$

Here, the only σ-dependence is in the volatility coefficient σ^2, so in all other terms the integral vanishes:

$$< 1, p_\infty >= \int_{-\infty}^{\infty} p_\infty(\sigma)\, \mathrm{d}\sigma = 1, \tag{7.3.27}$$

because p_∞ is a probability distribution function. Hence,

$$\overline{\mathcal{L}_1} V_0 = \frac{\partial c_0}{\partial t} + \frac{1}{2}\overline{\sigma^2} S^2 \frac{\partial^2 c_0}{\partial S^2} + rS\frac{\partial c_0}{\partial S} - rc_0 = 0, \tag{7.3.28}$$

[11] For determining this Hermitian adjoint $\mathcal{L}_0^*$, see appendix I.

[12] See appendix J.

with

$$\overline{\sigma^2} := < \sigma^2, p_\infty > = \int_{-\infty}^{\infty} \sigma^2 p_\infty(\sigma)\ \mathrm{d}\sigma, \tag{7.3.29}$$

representing the average of σ^2 with respect to the invariant distribution p_∞.

Equation (7.3.28) is the Black-Scholes equation with averaged variance $\overline{\sigma^2}$. The solution can be calculated by standard Black-Scholes techniques once a final condition is given.

First of all, if the volatility in the Black-Scholes equation is a given deterministic function $\sigma(t)$ of time, the option value can be calculated by replacing σ in the relevant constant-volatility formula by

$$\left(\frac{1}{T-t}\int_t^T \sigma^2(s)\ \mathrm{d}s\right)^{\frac{1}{2}}. \tag{7.3.30}$$

Next, we can calculate V_1. Because $\overline{\mathcal{L}_1}V_0 = 0$, we have, by subtraction,

$$\mathcal{L}_1 V_0 = \mathcal{L}_1 V_0 - \overline{\mathcal{L}_1} V_0 = \frac{1}{2}\left(\sigma^2 - \overline{\sigma^2}\right) S^2 \frac{\partial^2 c_0}{\partial S^2}. \tag{7.3.31}$$

Here we can eliminate the majority of the terms in the right-hand side of equation (7.3.20), leaving

$$\mathcal{L}_0 V_1 = \frac{1}{2}\left(\overline{\sigma^2} - \sigma^2\right) S^2 \frac{\partial^2 c_0}{\partial S^2}. \tag{7.3.32}$$

Because $S^2\frac{\partial^2 c_0}{\partial S^2}$ and eigenfunction $c_1(S,t)$ are independent of σ, and $\mathcal{L}_0$ contains only derivatives with respect to σ, this can be written as

$$\mathcal{L}_0\left(\frac{V_1 - c_1}{S^2\frac{\partial^2 c_0}{\partial S^2}}\right) = \frac{1}{2}\left(\overline{\sigma^2} - \sigma^2\right), \tag{7.3.33}$$

Hence, we know that the solution has the form

$$V_1(S,\sigma,t) = g_1(\sigma) S^2 \frac{\partial c_0}{\partial S^2} + c_1(S,t), \tag{7.3.34}$$

where $g_1(\sigma)$ satisfies

$$\mathcal{L}_0 g_1 = \frac{1}{2}\left(\overline{\sigma^2} - \sigma^2\right), \tag{7.3.35}$$

i.e.,

$$\frac{1}{2}\zeta^2 \frac{\partial^2 g_1}{\partial \sigma^2} + m\frac{\partial g_1}{\partial \sigma} = \frac{1}{2}\left(\overline{\sigma^2} - \sigma^2\right). \tag{7.3.36}$$

This can be rewritten as

$$\frac{\partial^2 g_1}{\partial \sigma^2} + h(\sigma)\frac{\partial g_1}{\partial \sigma} = j(\sigma), \tag{7.3.37}$$

with

$$h(\sigma) := \frac{2m}{\zeta^2}, \quad \text{and} \quad j(\sigma) := \frac{\overline{\sigma^2} - \sigma^2}{\zeta^2}. \tag{7.3.38}$$

Writing $G_1 := \frac{\partial g_1}{\partial \sigma}$, we have the following differential equation in terms of G_1:

$$\frac{\partial G_1}{\partial \sigma} + h(\sigma) G_1 = j(\sigma). \tag{7.3.39}$$

Multiplying both sides by $e^{\int_0^\sigma h(s)\ \mathrm{d}s}$ yields

$$e^{\int_0^\sigma h(s)\ \mathrm{d}s}\ \frac{\partial G_1}{\partial \sigma} + h(\sigma) G_1 e^{\int_0^\sigma h(s)\ \mathrm{d}s} = j(\sigma)\ e^{\int_0^\sigma h(s)\ \mathrm{d}s}. \tag{7.3.40}$$

This can be rewritten as

$$\frac{\partial}{\partial \sigma}\left(e^{\int_0^\sigma h(s)\ \mathrm{d}s} \cdot G_1\right) = j(\sigma)\ e^{\int_0^\sigma h(s)\ \mathrm{d}s}. \tag{7.3.41}$$

Solving this differential equation for $e^{\int_0^\sigma h(s)\ \mathrm{d}s} \cdot G_1$, gives

$$e^{\int_0^\sigma h(s)\ \mathrm{d}s} \cdot G_1 = \int_0^\sigma j(\tilde{\sigma})\ e^{\int_0^{\tilde{\sigma}} h(s)\ \mathrm{d}s}\ \mathrm{d}\tilde{\sigma} + H_1(S,t), \tag{7.3.42}$$

such that

$$\frac{\partial V_1}{\partial \sigma} = G_1 = e^{-\int_0^\sigma h(s)\ \mathrm{d}s} \int_0^\sigma j(\tilde{\sigma})\ e^{\int_0^{\tilde{\sigma}} h(s)\ \mathrm{d}s}\ \mathrm{d}\tilde{\sigma} + H_1(S,t) e^{-\int_0^\sigma h(s)\ \mathrm{d}s}, \tag{7.3.43}$$

After integrating once we find

$$V_1(S,\sigma,t) = \int_0^\sigma \left(e^{-\int_0^{\hat{\sigma}} h(s)\ \mathrm{d}s} \int_0^{\hat{\sigma}} j(\tilde{\sigma})\ e^{\int_0^{\tilde{\sigma}} h(s)\ \mathrm{d}s}\ \mathrm{d}\tilde{\sigma} + H_1(S,t) e^{-\int_0^{\hat{\sigma}} h(s)\ \mathrm{d}s}\right) \mathrm{d}\hat{\sigma} + H_2(S,t). \tag{7.3.44}$$

This is the integral form of the solution. One of these two "*complementary solutions*", namely $H_2(S,t)$, does not depend on σ and can be absorbed into eigenfunction $c_1(S,t)$. The other one is unbounded at infinity. Hence, we choose the particular solution $v_1(S,\sigma,t)$ to be equal to zero.

Before proceeding any further, we outline the pattern followed by succesive iterations of the solution procedure. For all $n \geq 0$ we have the n^{th} equation

$$\mathcal{L}_0 V_n = -\mathcal{L}_{\frac{1}{2}} V_{n+\frac{1}{2}} - \mathcal{L}_1 V_{n-1} \quad \text{for } V_n(S,\sigma,t). \tag{7.3.45}$$

The right-hand side is assumed to be known from earlier stages. Solving this, we obtain a particular solution $v_n(S,\sigma,t)$ and an eigenfunction $c_n(S,t)$. For the case $n = 0$ the former is zero and the latter is $c_0(S,t)$.

Next, we repeat this process for $V_{n+\frac{1}{2}}$, obtaining a further particular solution $v_{n+\frac{1}{2}}$ and a further eigenfunction $c_{n+\frac{1}{2}}$. Note that the eigenfunction $c_n(S,t)$ does not depend on σ and is annihilated by the operator $\mathcal{L}_0$. Finally, we substitute the functions just found into the right-hand side of the equation for V_{n+1}:

$$\mathcal{L}_0 V_{n+1} = -\mathcal{L}_{n+\frac{1}{2}} V_{n+\frac{1}{2}} - \mathcal{L}_1 V_n \quad \text{for } V_{n+1}(S,\sigma,t). \tag{7.3.46}$$

Because the eigenfunction $c_{n+\frac{1}{2}}(S,t)$ does not depend on σ, it is annihilated by the operator $\mathcal{L}_{\frac{1}{2}}$. Therefore, the right-hand side is known in terms of the particular solutions just obtained.

Here, the solvability condition $\overline{\mathcal{L}_0 V_{n+1}} =< \mathcal{L}_0 V_{n+1}, p_\infty >= 0$ for existence of a solution can be applied. From equation (7.3.46) it follows that

$$\overline{\mathcal{L}_0 V_{n+1}} = \overline{-\mathcal{L}_{n+\frac{1}{2}} V_{n+\frac{1}{2}} - \mathcal{L}_1 V_n}, \tag{7.3.47}$$

i.e.,

$$0 = -\overline{\mathcal{L}_{n+\frac{1}{2}} v_{n+\frac{1}{2}}} - \overline{\mathcal{L}_1 v_n} - \overline{\mathcal{L}_1} c_n. \tag{7.3.48}$$

This can be written as

$$\overline{\mathcal{L}_1} c_n = -\overline{\mathcal{L}_{n+\frac{1}{2}} v_{n+\frac{1}{2}}} - \overline{\mathcal{L}_1 v_n}. \tag{7.3.49}$$

We now apply this procedure to the case $n = \frac{1}{2}$. We have already found $V_{\frac{1}{2}}$ and V_1 up to eigenfunctions $c_{\frac{1}{2}}(S,t)$ resp. $c_1(S,t)$, so we need only apply the solvability condition to the equation

$$\mathcal{L}_0 V_{\frac{3}{2}} = -\mathcal{L}_{\frac{1}{2}} V_1 - \mathcal{L}_1 V_{\frac{1}{2}} \quad \text{for } V_{\frac{1}{2}}(S,\sigma,t). \tag{7.3.50}$$

This yields

$$\overline{\mathcal{L}_0 V_{\frac{3}{2}}} = \overline{-\mathcal{L}_{\frac{1}{2}} V_1 - \mathcal{L}_1 V_{\frac{1}{2}}}, \tag{7.3.51}$$

i.e.,

$$0 = -\overline{\mathcal{L}_{\frac{1}{2}} V_1} - \overline{\mathcal{L}_1 V_{\frac{1}{2}}}. \tag{7.3.52}$$

Using the fact that eigenfunction $c_1(S,t)$ is annihilated by operator $\mathcal{L}_{\frac{1}{2}}$, we obtain

$$\overline{\mathcal{L}_1} V_{\frac{1}{2}} = -\overline{\mathcal{L}_{\frac{1}{2}} v_1}. \tag{7.3.53}$$

Substituting $v_1(S,\sigma,t) = g_1(\sigma) S^2 \frac{\partial^2 c_0}{\partial S^2}$, we obtain

$$\begin{aligned} \overline{\mathcal{L}_1} V_{\frac{1}{2}} &= -\overline{\mathcal{L}_{\frac{1}{2}} g_1(\sigma) S^2 \frac{\partial^2 c_0}{\partial S^2}}, \\ &= -\overline{\rho\zeta\sigma S \frac{\mathrm{d}g_1}{\mathrm{d}\sigma}} \frac{\partial}{\partial S}\left(S^2 \frac{\partial^2 c_0}{\partial S^2} \right) + \overline{\lambda\zeta \frac{\mathrm{d}g_1}{\mathrm{d}\sigma}} S^2 \frac{\partial^2 c_0}{\partial S^2}. \end{aligned} \tag{7.3.54}$$

For ease of notation, we set

$$\mathcal{D} := S \frac{\partial}{\partial S} \tag{7.3.55}$$

equivalent to using a logarithmic price variable.[13] Defining

$$A_{\frac{1}{2},1} := \overline{\lambda\zeta \frac{\mathrm{d}g_1}{\mathrm{d}\sigma}} \quad \text{and} \quad A_{\frac{1}{2},2} := -\overline{\rho\zeta\sigma \frac{\mathrm{d}g_1}{\mathrm{d}\sigma}}, \tag{7.3.56}$$

we obtain

$$\overline{\mathcal{L}_1} V_{\frac{1}{2}} = \left(A_{\frac{1}{2},1} + A_{\frac{1}{2},2} \mathcal{D} \right) \left(\mathcal{D}^2 - \mathcal{D} \right) c_0. \tag{7.3.57}$$

Note that this is a partial differential equation with constant coefficients. Next, we use that $\overline{\mathcal{L}_1} c_0 = 0$ and the fact that the time derivative of a solution of the homogeneous equation is again a solution of the homogeneous equation. Also, we assume *pro tem* that the correct final condition (from matching) is $V_{\frac{1}{2}}(S,T) = 0$. We find[14]

$$V_{\frac{1}{2}}(S,\sigma,t) = -(T-t) \left(A_{\frac{1}{2},1} + A_{\frac{1}{2},2} \mathcal{D} \right) \left(\mathcal{D}^2 - \mathcal{D} \right) c_0. \tag{7.3.58}$$

where c_0 is already known. As noted earlier, $V_{\frac{1}{2}}$ is independent of σ.

This result, and elaborations thereof, is an important practical consequence of the method, since it allows calibration of the three constants $\overline{\sigma^2}$, $\overline{\rho\zeta\sigma \frac{\mathrm{d}g_1}{\mathrm{d}\sigma}}$ and $\overline{\lambda\zeta \frac{\mathrm{d}g_1}{\mathrm{d}\sigma}}$ to market prices of options (as represented by an implied volatility surface) in a simple manner: the key point is that only these directly deducable constants are needed, rather than the unobservable functions $M(\sigma,t)$

[13] This can be explained by setting $x := \log S$. Then, $\frac{\partial}{\partial x} = \frac{\partial S}{\partial x}\frac{\partial}{\partial S} = e^x \frac{\partial}{\partial S} = S\frac{\partial}{\partial S}$.

[14] Howison made a small mistake here, forgetting the minus sign.

and $\Sigma(\sigma,t)$.

We first calculate

$$\mathcal{L}_0 V_{\frac{3}{2}} = -\mathcal{L}_{\frac{1}{2}} V_1 - \mathcal{L}_1 V_{\frac{1}{2}}, \tag{7.3.59a}$$

using

$$\begin{aligned} -\mathcal{L}_{\frac{1}{2}} V_1 &= -\zeta\Big(\rho\sigma\mathcal{D} - \lambda\Big)\frac{\partial}{\partial\sigma}\left(g_1(\sigma)\left(\mathcal{D}^2-\mathcal{D}\right)c_0(S,t) + c_1(S,t)\right), \\ &= \zeta\frac{\mathrm{d}g_1}{\mathrm{d}\sigma}(\lambda-\rho\sigma\mathcal{D})\left(\mathcal{D}^2-\mathcal{D}\right)c_0(S,t). \end{aligned} \tag{7.3.59b}$$

and

$$\begin{aligned} -\mathcal{L}_1 V_{\frac{1}{2}} &= -\overline{\mathcal{L}_1} V_{\frac{1}{2}} + \frac{1}{2}\left(\overline{\sigma^2}-\sigma^2\right)\left(\mathcal{D}^2-\mathcal{D}\right)V_{\frac{1}{2}} \\ &= -\left(A_{\frac{1}{2},1} + A_{\frac{1}{2},2}\mathcal{D}\right)\left(\mathcal{D}^2-\mathcal{D}\right)c_0 + \frac{1}{2}\left(\overline{\sigma^2}-\sigma^2\right)\left(\mathcal{D}^2-\mathcal{D}\right)c_0. \end{aligned} \tag{7.3.59c}$$

From this, it follows that

$$\begin{aligned} \mathcal{L}_0 V_{\frac{3}{2}} = \Bigg[\zeta\frac{\mathrm{d}g_1}{\mathrm{d}\sigma}(\lambda-\rho\sigma\mathcal{D}) + \frac{1}{2}\left(\overline{\sigma^2}-\sigma^2\right)(T-t)\left(A_{\frac{1}{2},1}+A_{\frac{1}{2},2}\mathcal{D}\right)\left(\mathcal{D}^2-\mathcal{D}\right) \\ -A_{\frac{1}{2},1} - A_{\frac{1}{2},2}\mathcal{D}\Bigg]\left(\mathcal{D}^2-\mathcal{D}\right)c_0(S,t), \end{aligned} \tag{7.3.60}$$

such that $V_{\frac{3}{2}}$ has the form

$$V_{\frac{3}{2}}(S,\sigma,t) = \Big[g_2(\sigma) + g_3(\sigma)\mathcal{D} + g_1(\sigma)(T-t)\left(A_{\frac{1}{2},1}+A_{\frac{1}{2},2}\mathcal{D}\right)\left(\mathcal{D}^2-\mathcal{D}\right)\Big]\left(\mathcal{D}^2-\mathcal{D}\right)c_0. \tag{7.3.61}$$

Here, $g_1(\sigma)$, $g_2(\sigma)$ and $g_3(\sigma)$ satisfy

$$\frac{1}{2}\zeta^2\frac{\mathrm{d}^2 g_1}{\mathrm{d}\sigma^2} + m\frac{\mathrm{d}g_1}{\mathrm{d}\sigma} = \frac{1}{2}\left(\overline{\sigma^2}-\sigma^2\right), \tag{7.3.62a}$$

$$\frac{1}{2}\zeta^2\frac{\mathrm{d}^2 g_2}{\mathrm{d}\sigma^2} + m\frac{\mathrm{d}g_2}{\mathrm{d}\sigma} = \frac{1}{2}\left(\overline{\sigma^2}-\sigma^2\right) + \lambda\zeta\frac{\mathrm{d}g_1}{\mathrm{d}\sigma} - A_{\frac{1}{2},1}, \tag{7.3.62b}$$

$$\frac{1}{2}\zeta^2\frac{\mathrm{d}^2 g_3}{\mathrm{d}\sigma^2} + m\frac{\mathrm{d}g_3}{\mathrm{d}\sigma} = \frac{1}{2}\left(\overline{\sigma^2}-\sigma^2\right) - \rho\sigma\frac{\mathrm{d}g_1}{\mathrm{d}\sigma} - A_{\frac{1}{2},2}. \tag{7.3.62c}$$

Then, the solvability condition for V_2, applied to the equation

$$\mathcal{L}_0 V_2 = -\mathcal{L}_{\frac{1}{2}} V_{\frac{3}{2}} - \mathcal{L}_1 V_1, \tag{7.3.63}$$

gives

$$\overline{\mathcal{L}_0 V_2} = -\overline{\mathcal{L}_{\frac{1}{2}} V_{\frac{3}{2}} - \mathcal{L}_1 V_1}, \tag{7.3.64}$$

i.e.,

$$0 = -\overline{\mathcal{L}_{\frac{1}{2}} V_{\frac{3}{2}}} - \overline{\mathcal{L}_1 V_1}, \tag{7.3.65}$$

We know that V_1 can be written as a particular solution $v_1(S,\sigma,t)$, plus an eigenfunction $c_1(S,t)$. From this, it follows that

$$\overline{\mathcal{L}_1 c_1} = -\overline{\mathcal{L}_{\frac{1}{2}} V_{\frac{3}{2}}} - \overline{\mathcal{L}_1 v_1} = -\overline{\mathcal{L}_{\frac{1}{2}} V_{\frac{3}{2}}}, \tag{7.3.66}$$

recalling that $v_1(S,\sigma,t) = 0$.

Hence,

$$\begin{aligned}\overline{\mathcal{L}_1}c_1 &= -\overline{\mathcal{L}_{\frac{1}{2}}V_{\frac{3}{2}}} \\ &= -\overline{\zeta}\left(\overline{\rho\sigma}\mathcal{D}-\overline{\lambda}\right)\left[\overline{\frac{\mathrm{d}g_2}{\mathrm{d}\sigma}}+\overline{\frac{\mathrm{d}g_3}{\mathrm{d}\sigma}}\mathcal{D}+\overline{\frac{\mathrm{d}g_1}{\mathrm{d}\sigma}}(T-t)\left(A_{\frac{1}{2},1}+A_{\frac{1}{2},2}\mathcal{D}\right)\left(\mathcal{D}^2-\mathcal{D}\right)\right]\left(\mathcal{D}^2-\mathcal{D}\right)c_0 \\ &= \left[A_{1,1}+A_{1,2}\mathcal{D}+\left(A_{\frac{1}{2},1}+A_{\frac{1}{2},2}\mathcal{D}\right)(T-t)\left(A_{\frac{1}{2},1}+A_{\frac{1}{2},2}\mathcal{D}\right)\left(\mathcal{D}^2-\mathcal{D}\right)\right]\left(\mathcal{D}^2-\mathcal{D}\right)c_0,\end{aligned}$$

where[15]

$$A_{1,1} := \overline{\zeta}\left(\overline{\lambda}-\overline{\rho\sigma}\mathcal{D}\right)\overline{\frac{\mathrm{d}g_2}{\mathrm{d}\sigma}} \quad \text{and} \quad A_{1,2} := \overline{\zeta}\left(\overline{\lambda}-\overline{\rho\sigma}\mathcal{D}\right)\overline{\frac{\mathrm{d}g_3}{\mathrm{d}\sigma}}. \tag{7.3.67}$$

The relevant particular solution $c_1(S,t)$ with zero payoff $c_1(S,T)=0$ can be obtained in a similarly way. Again, we use that $\overline{\mathcal{L}_1}V_1=0$ and the fact that the time derivative of a solution of the homogeneous equation is again a solution of the homogeneous equation. This yields that the particular solution $c_1(S,t)$ is equal to

$$-\left[(T-t)\left(A_{1,1}+A_{1,2}\mathcal{D}\right)+\frac{1}{2}(T-t)^2\left(A_{\frac{1}{2},1}+A_{\frac{1}{2},2}\mathcal{D}\left(\mathcal{D}^2-\mathcal{D}\right)\right)\left(A_{\frac{1}{2},1}+A_{\frac{1}{2},2}\mathcal{D}\right)\right]\left(\mathcal{D}^2-\mathcal{D}\right)c_0.$$

However, we leave open the possibility of adding a further solution $c_1'(S,t)$, if the payoff, determined by matching into the boundary layer, dictates that we should do so. Similarly, the σ-dependence of the solution can only be resolved by matching.

7.4 Boundary layer analysis

Reasoning by means of analogies to chapter 5, let us introduce a boundary layer in t near $t=T$ of size $\mathcal{O}(\varepsilon)$, defining the inner time variable τ via

$$t = T+\varepsilon\tau \quad \Leftrightarrow \quad \tau = \frac{t-T}{\varepsilon}, \quad \text{for} \quad \tau<0. \tag{7.4.1}$$

Note that here $\tau<0$, opposite the $t=T-\varepsilon\tau$ in the Black-Scholes calculations in chapter 5, where $\tau>0$.

The pricing equation for $V(S,\sigma,\tau)$ reads

$$\frac{1}{\varepsilon}\frac{\partial V}{\partial\tau}+\frac{1}{2}\sigma^2S^2\frac{\partial^2V}{\partial S^2}+\frac{\rho\zeta}{\sqrt{\varepsilon}}\sigma S\frac{\partial^2V}{\partial S\partial\sigma}+\frac{1}{2}\frac{\zeta^2}{\varepsilon}\frac{\partial^2V}{\partial\sigma^2}+rS\frac{\partial V}{\partial S}+\left(\frac{m}{\varepsilon}-\frac{\lambda\zeta}{\sqrt{\varepsilon}}\right)\frac{\partial V}{\partial\sigma}-rV=0. \tag{7.4.2}$$

Again we can write this in the form

$$\left(\widetilde{\mathcal{L}}_0+\sqrt{\varepsilon}\widetilde{\mathcal{L}}_{\frac{1}{2}}+\varepsilon\widetilde{\mathcal{L}}_1\right)V(S,\sigma,\tau)=0, \tag{7.4.3}$$

using the operators

$$\begin{aligned}\widetilde{\mathcal{L}}_0 &= \frac{\partial}{\partial\tau}+\mathcal{L}_0 = \frac{\partial}{\partial\tau}+\frac{1}{2}\zeta^2\frac{\partial^2}{\partial\sigma^2}+m\frac{\partial}{\partial\sigma}, \\ \widetilde{\mathcal{L}}_{\frac{1}{2}} &= \mathcal{L}_{\frac{1}{2}} = \rho\zeta\sigma S\frac{\partial^2}{\partial S\partial\sigma}-\lambda\zeta\frac{\partial}{\partial\sigma}, \\ \widetilde{\mathcal{L}}_1 &= \frac{1}{2}\sigma^2S^2\frac{\partial^2}{\partial S^2}+rS\frac{\partial}{\partial S}-r.\end{aligned}$$

[15] Howison claims that the definitions should be $A_{1,1} := \overline{\zeta g_2'(\sigma)(\lambda-\rho\sigma)}$ and $A_{1,2} := \overline{\zeta g_3'(\sigma)(\lambda-\rho\sigma)}$.

Note that $\widetilde{\mathcal{L}}_0$, unlike $\mathcal{L}_0$, contains the time derivative $\frac{\partial}{\partial\tau}$. Next, we expand

$$V_\varepsilon(S,\sigma,\tau) = V_0 + \sqrt{\varepsilon}V_{\frac{1}{2}} + \varepsilon V_1 + \dots. \tag{7.4.4}$$

From this equation, the following $\mathcal{O}(1)$ equation can be determined:

$$\widetilde{\mathcal{L}}_0 V_0 = 0, \tag{7.4.5a}$$

i.e.,

$$\frac{\partial V_0}{\partial\tau} + \frac{1}{2}\zeta^2\frac{\partial^2 V_0}{\partial\sigma^2} + m\frac{\partial V_0}{\partial\sigma} = 0, \tag{7.4.5b}$$

with initial condition $V_0(S,\sigma,0) = P(S)$.

Following the disputable assumption[16] in Howison's paper [7], that V_0 is independent of σ, only

$$\frac{\partial V_0}{\partial\tau} = 0 \tag{7.4.6}$$

is left, which yields that V_0 is a function of S only. The initial condition implies that this function is the payoff. Hence,

$$V_0(S,\sigma,\tau) = P(S). \tag{7.4.7}$$

Note that for $\tau \to -\infty$ this first order inner approximation matches automatically with the one-term outer solution $V_0(S,t)$ as $t \to T$. The procedure of matching will be discussed in the next section.

The $\mathcal{O}(\sqrt{\varepsilon})$ equation is given by

$$\widetilde{\mathcal{L}}_0 V_{\frac{1}{2}} + \widetilde{\mathcal{L}}_{\frac{1}{2}} V_0 = 0, \tag{7.4.8}$$

with initial condition $V_{\frac{1}{2}}(S,\sigma,0) = 0$.[17]

Using the fact that V_0 does not depend on σ yields

$$\widetilde{\mathcal{L}}_{\frac{1}{2}} V_0 = \rho\zeta\sigma S\frac{\partial^2 V_0}{\partial S\partial\sigma} - \lambda\zeta\frac{\partial V_0}{\partial\sigma} = 0, \tag{7.4.9}$$

such that

$$\widetilde{\mathcal{L}}_0 V_{\frac{1}{2}} = 0, \tag{7.4.10}$$

which is the same differential equation as the $\mathcal{O}(1)$ equation. This time the initial condition $V_{\frac{1}{2}}(S,\sigma,0) = 0$ gives

$$V_{\frac{1}{2}}(S,\sigma,t) = 0. \tag{7.4.11}$$

The $\mathcal{O}(\varepsilon)$ equation is given by

$$\widetilde{\mathcal{L}}_0 V_1 + \widetilde{\mathcal{L}}_{\frac{1}{2}} V_{\frac{1}{2}} + \widetilde{\mathcal{L}}_1 V_0 = 0, \tag{7.4.12}$$

with initial condition $V_1(S,\sigma,0) = 0$. Using the solutions we found above, i.e., $V_0 = P(S)$ and $V_{\frac{1}{2}} = 0$, this reduces to

$$\widetilde{\mathcal{L}}_0 V_1 = -\widetilde{\mathcal{L}}_1 P(S). \tag{7.4.13}$$

[16] This issue was already discussed in the previous section 7.3 of this thesis.

[17] On page 20 of [7] Howison states that the initial condition is given by $V_{\frac{1}{2}}(S,T) = 0$, which must be a typographical error, because we are working in (S,σ,τ)-coordinates now.

Writing

$$\widetilde{\mathcal{L}}_0 V_1 = \frac{\partial V_1}{\partial \tau} + \mathcal{L}_0 V_1, \tag{7.4.14}$$

gives

$$\frac{\partial V_1}{\partial \tau} + \mathcal{L}_0 V_1 = -\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P(S)}{\partial S^2} - rS\frac{\partial P(S)}{\partial S} + rP(S). \tag{7.4.15}$$

Because $\frac{\partial P}{\partial \tau} = 0$, we can write

$$-\frac{\partial P}{\partial \tau} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P(S)}{\partial S^2} - rS\frac{\partial P(S)}{\partial S} + rP(S) \tag{7.4.16}$$

for the right hand side. This can also be written as[18]

$$\frac{1}{2}\left(\overline{\sigma^2} - \sigma^2\right) S^2 \frac{\partial^2 P}{\partial S^2} - \overline{\mathcal{L}_1} P. \tag{7.4.18}$$

Equation (7.4.15) thus becomes

$$\frac{\partial V_1}{\partial \tau} + \mathcal{L}_0 V_1 = \frac{1}{2}\left(\overline{\sigma^2} - \sigma^2\right) S^2 \frac{\partial^2 P}{\partial S^2} - \overline{\mathcal{L}_1} P. \tag{7.4.19}$$

Although it is not immediately clear which steps are taken to obtain this, according to Howison, a particular solution is

$$V_1^{\infty} = g_1(\sigma) S^2 \frac{\partial^2 P}{\partial S^2} - \tau \overline{\mathcal{L}_1} P + \mathcal{V}_1(S), \tag{7.4.20}$$

where $\mathcal{V}_1$ is an arbitrary function. This indeed is a solution of problem (7.4.19) with initial condition $V_1(S, \sigma, 0) = 0$. In fact, this is the correct form for the asymptotic behaviour of V_1 as $\tau \to -\infty$, see figure 7.1. To see this, first note that applying the inner product to equation (7.4.19) yields

$$< \frac{\partial V_1}{\partial \tau} + \mathcal{L}_0 V_1, p_\infty > = < \frac{1}{2}\left(\overline{\sigma^2} - \sigma^2\right) S^2 \frac{\partial^2 P}{\partial S^2} - \overline{\mathcal{L}_1} P, p_\infty > . \tag{7.4.21}$$

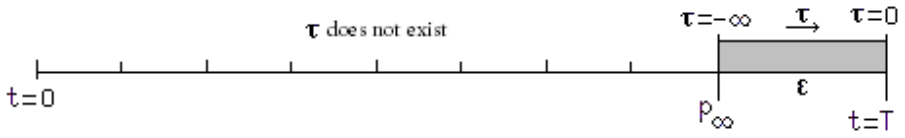

Figure 7.1: As $\tau \to -\infty$, it goes towards the left boundary of the layer (away from maturity T).

This gives

$$\frac{\partial}{\partial \tau} < V_1, p_\infty > = -\overline{\mathcal{L}_1} P, \tag{7.4.22}$$

[18] Recall that when computing the outer solution, we defined

$$\overline{\mathcal{L}_1} V := < \mathcal{L}_1 V, p_\infty > = \frac{\partial V}{\partial \tau} + \frac{1}{2}\overline{\sigma^2} S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV, \text{ with } \overline{\sigma^2} := < \sigma^2, p_\infty > . \tag{7.4.17}$$

because $< \mathcal{L}_0 V_1, p_\infty >= 0$, $< \overline{\mathcal{L}_1} P, p_\infty >= \overline{\mathcal{L}_1} P$ and

$$\int_{-\infty}^{\infty} \frac{1}{2}\left(\overline{\sigma^2} - \sigma^2\right) S^2 \frac{\partial^2 P}{\partial S^2} \mathrm{d}\sigma = \frac{1}{2} S^2 \frac{\partial^2 P}{\partial S^2} \left(\overline{\sigma^2} \underbrace{\int_{-\infty}^{\infty} p_\infty \mathrm{d}\sigma}_{=1} - \underbrace{\int_{-\infty}^{\infty} \sigma^2 p_\infty \mathrm{d}\sigma}_{=:\overline{\sigma^2}} \right) = 0. \tag{7.4.23}$$

And so, integrating equation (7.4.22) and using the initial condition $V_1(S, \sigma, 0) = 0$,

$$< V_1, p_\infty >= -\tau \overline{\mathcal{L}_1} P. \tag{7.4.24}$$

Using the solution (7.4.20) found for V_1^∞, we also have that

$$\begin{aligned} < V_1^\infty, p_\infty > &= < g_1(\sigma) S^2 \frac{\partial^2 P}{\partial S^2} - \tau \overline{\mathcal{L}_1} P + \mathcal{V}_1(S), p_\infty >, \\ &= < g_1(\sigma), p_\infty > S^2 \frac{\partial^2 P}{\partial S^2} - \tau \overline{\mathcal{L}_1} P + \mathcal{V}_1(S), \\ &= \overline{g_1(\sigma)} S^2 \frac{\partial^2 P}{\partial S^2} - \tau \overline{\mathcal{L}_1} P + \mathcal{V}_1(S). \end{aligned} \tag{7.4.25}$$

Comparing these two expressions (7.4.24) and (7.4.25) for the inner product, we conclude that

$$\mathcal{V}_1(S) = -\overline{g_1(\sigma)} S^2 \frac{\partial^2 P}{\partial S^2}. \tag{7.4.26}$$

Thus, as $\tau \to -\infty$,

$$V_1 \sim V_1^\infty = \left(g_1(\sigma) - \overline{g_1(\sigma)}\right) S^2 \frac{\partial^2 P}{\partial S^2} - \tau \overline{\mathcal{L}_1} P, \tag{7.4.27}$$

since what is left after subtracting the particular solution, $V_1 - V_1^\infty$, satisfies the homogeneous version of the parabolic equation, has initial data that vanishes at large and small σ, and therefore vanishes as $\tau \to -\infty$. This is the first step in the matching, because here we consider what happens if we go to the boundary of the layer ($\tau \to -\infty$). Further mathching will be done in the next section 7.5.

7.5 Matching

We can now complete the matching. From the outer expansion, written in inner variables, we have

$$\begin{aligned} V_\varepsilon := V_0(S, \sigma, T + \varepsilon\tau) &+ \varepsilon^{\frac{1}{2}} V_{\frac{1}{2}}(S, \sigma, T + \varepsilon\tau) + \varepsilon V_1(S, \sigma, T + \varepsilon\tau) + \dots \\ &\sim V_0(S, \sigma, T) + \varepsilon\tau \frac{\partial V_0}{\partial t}(S, \sigma, T) + \varepsilon^{\frac{1}{2}} \cdot 0 + \varepsilon V_1(S, \sigma, T), \\ &= V_0(S, \sigma, T) + \varepsilon \left(\tau \frac{\partial V_0}{\partial t}(S, \sigma, T) + V_1(S, \sigma, T) \right), \\ &= c_0(S, T) + \varepsilon \left(\tau \frac{\partial c_0}{\partial t}(S, T) + g_1(\sigma) S^2 \frac{\partial^2 c_0}{\partial S^2} + c_1(S, T) \right). \end{aligned} \tag{7.5.1}$$

Note that the particular solutions in $V_{\frac{1}{2}}$ and V_1 that are multiplied by $T - t$ do not contribute to this, because they are $\mathcal{O}\left(\varepsilon^{\frac{3}{2}}\right)$.

One-term matching of the outer and inner expansion yields

$$\lim_{t\to T} c_0(S,t) = \lim_{\tau\to-\infty} V_0(S,\sigma,\tau) \quad \Rightarrow \quad c_0(S,T) = P(S) \tag{7.5.2}$$

and

$$\begin{aligned} \overline{\mathcal{L}}_1\ c_0 = 0 \quad &\Rightarrow \quad \frac{\partial c_0}{\partial t} + \frac{1}{2}\overline{\sigma^2}S^2\frac{\partial^2 c_0}{\partial S^2} + rS\frac{\partial c_0}{\partial S} - rc_0 = 0, \\ &\Rightarrow \quad \frac{\partial c_0}{\partial t} = \frac{1}{2}\overline{\sigma^2}S^2\frac{\partial^2 c_0}{\partial S^2} + rS\frac{\partial c_0}{\partial S} - rc_0. \end{aligned} \tag{7.5.3}$$

The inner expansion yields $V_0(S,\sigma,t) = P(S)$, such that one-term matching this with the outer expansion gives $c_0(S,t) = P(S)$. This implies that

$$\frac{\partial c_0}{\partial t} = -\widehat{\mathcal{L}}_1 P(S), \tag{7.5.4}$$

where $\widehat{\mathcal{L}}_1$ is the inner product of the operator as it was defined formulating the inner expansion:

$$\widehat{\mathcal{L}}_1 P(S) := \overline{\widetilde{\mathcal{L}}}_1 P(S) =< \widetilde{\mathcal{L}}_1 P(S), p_\infty(\sigma) >= \frac{1}{2}\overline{\sigma^2}S^2\frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP. \tag{7.5.5}$$

Replacing $c_0(S,T)$ by $P(S)$ and $\frac{\partial c_0}{\partial t}(S,T)$ by $-\widehat{\mathcal{L}}_1 P$, the three-term outer expansion written in inner variables (7.5.1) can be rewritten as

$$V_\varepsilon = P(S) + \varepsilon\left(-\tau\widehat{\mathcal{L}}_1 P + g_1(\sigma)S^2\frac{\partial^2 c_0}{\partial S^2} + c_1(S,T)\right). \tag{7.5.6}$$

As demonstrated in section 7.4, the large-τ behaviour of the three-term inner expansion is

$$P(S) + \varepsilon\left(-\tau\widehat{\mathcal{L}_1 P} + \left(g_1(\sigma) - \overline{g_1(\sigma)}\right)S^2\frac{\partial^2 P}{\partial S^2}\right). \tag{7.5.7}$$

Matching these two expressions, the missing final condition for $c_1(S,t)$ is

$$c_1(S,T) = -\overline{g_1(\sigma)}S^2\frac{\partial^2 P}{\partial S^2}. \tag{7.5.8}$$

Let us define $w := S^2\frac{\partial^2 c_0}{\partial S^2}$. We can rewrite $\widehat{\mathcal{L}_1 c_0} = 0$ as

$$\frac{1}{2}\overline{\sigma^2}S^2\frac{\partial^2 c_0}{\partial S^2} + rS\frac{\partial c_0}{\partial S} - rc_0 = 0, \tag{7.5.9}$$

i.e.,

$$S^2\frac{\partial^2 c_0}{\partial S^2} = -\frac{2r}{\overline{\sigma^2}}\left(rS\frac{\partial c_0}{\partial S} - rc_0\right). \tag{7.5.10}$$

So we can replace w by $-\frac{2r}{\overline{\sigma^2}}\left(rS\frac{\partial c_0}{\partial S} - rc_0\right)$, such that

$$\begin{aligned} \widehat{\mathcal{L}_1 w} &= \frac{1}{2}\overline{\sigma^2}S^2\frac{\partial^2 w}{\partial S^2} + rS\frac{\partial w}{\partial S} - rw, \\ &= \frac{1}{2}\overline{\sigma^2}S^2\frac{\partial^2}{\partial S^2}\left[-\frac{2r}{\overline{\sigma^2}}\left(rS\frac{\partial c_0}{\partial S} - rc_0\right)\right] + rS\frac{\partial w}{\partial S} - rw, \\ &= -rS^2\frac{\partial}{\partial S}\left(S\frac{\partial^2 c_0}{\partial S^2} + \frac{\partial c_0}{\partial S} - \frac{\partial c_0}{\partial S}\right) + rS\frac{\partial w}{\partial S} - rw, \\ &= -rS^2\frac{\partial}{\partial S}\left(\frac{w}{S}\right) + rS\frac{\partial w}{\partial S} - rw, \\ &= -rS^2\frac{-1}{S^2}w - rS^2\frac{1}{S}\frac{\partial w}{\partial S} + rS\frac{\partial w}{\partial S} - rw = 0. \end{aligned}$$

Hence, as $w = S^2 \frac{\partial^2 c_0}{\partial S^2}$ itself appears to be a solution of $\widehat{\mathcal{L}_1} V = 0$, we have

$$c_1(S,t) = -\overline{g_1(\sigma)} S^2 \frac{\partial^2 c_0}{\partial S^2}, \tag{7.5.11}$$

and the complete outer expansion to the three lowest orders is

$$\begin{aligned} V(S,\sigma,t) \;\sim\;& c_0(S,t) + \varepsilon^{\frac{1}{2}}(T-t)\left(A_{\frac{1}{2},1} + A_{\frac{1}{2},2}\mathcal{D}\right) S^2\left(\mathcal{D}^2 - \mathcal{D}\right) c_0 \\ +\;& \varepsilon\Big[g_1(\sigma) - \overline{g_1(\sigma)} - (T-t)\left(A_{1,1} + A_{1,2}\mathcal{D}\right) \\ & - \frac{1}{2}(T-t)^2\left(A_{\frac{1}{2},1} + A_{\frac{1}{2},2}\mathcal{D}\left(\mathcal{D}^2 - \mathcal{D}\right)\right)\left(A_{\frac{1}{2},1} + A_{\frac{1}{2},2}\mathcal{D}\right)\Big]\left(\mathcal{D}^2 - \mathcal{D}\right) c_0, \end{aligned}$$

recalling that $\mathcal{D} := S\frac{\partial}{\partial S}$.

This seems to be a nice solution, however there are several occurences of $c_0(S,t)$ in the above expression for $V(S,\sigma,t)$, which is an undetermined function.

7.6 Discussion

In Howison's paper [7], some assumptions, simplifications and remarks are presented below. After that, the method and solution will be discussed.

Method and assumptions
As a final example in his paper [7], Howison shows the applicability of boundary-layer techniques in the analysis of fast mean-reverting stochastic volatility models. A well-known example of a stochastic volatility model is the Heston model.

In these models the volatility itself is assumed to follow a stochastic process while the asset price is assumed to follow the lognormal process as before in section 5, considering the Black-Scholes model. That is,

$$\begin{cases} \mathrm{d}S_t = \mu_t S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}W_t, & \text{(7.6.1a)} \\ \mathrm{d}\sigma_t = \; M_t \mathrm{d}t + \Sigma_t \mathrm{d}\widetilde{W}_t. & \text{(7.6.1b)} \end{cases}$$

The two Wiener processes are assumed to be correlated by

$$\mathrm{E}\left[\mathrm{d}W_t \; \mathrm{d}\widetilde{W}_t\right] = \rho \; \mathrm{d}t. \tag{7.6.2}$$

Here, the timescale for mean-reversion is assumed to be much shorter than that for the evolution of the asset price. Therefore, their ratio $\frac{r}{M-\lambda\Sigma}$ is small, such that we can apply perturbation theory. This results in a scaling of M and Σ by ε resp. $\sqrt{\varepsilon}$, such that $\frac{\partial\sigma}{\partial t} = M - \lambda\Sigma$ is large when $M = \frac{m}{\varepsilon}$ is large.

We do know that the scaled variables m and ζ depend on σ. Unfortunately, we do not know their functional form.

First, we have made an outer expansion, where we have assumed *pro tem* that the correct final condition (from matching) for the $\mathcal{O}\left(\varepsilon^{\frac{1}{2}}\right)$ problem is $V_{\frac{1}{2}}(S,T) = 0$. Secondly, we have investigated the boundary layer near $t = T$. At the end of the boundary layer analysis, Howison starts

matching already, by introducing a particular solution V_1^∞, which vanishes as $\tau \to -\infty$. After that, the matching is completed by one-term matching of the outer and inner expansion.

Finally, an expression for the option price V is obained. This solution will be discussed at the end of this section.

Simplifications

- On page 16 of [7] Howison claims that the solution V_0 of the $\mathcal{O}(1)$ equation is a function of the stock price S and time t only (i.e., $V_0 = c_0(S,t)$, which is independent of the volatility σ). He argues that this choice follows naturally from the behaviour of V_0 in the limits for small and large S. We, on the other hand, have found a general solution of the form

$$V_0(S,\sigma,t) = c(S,t)g(\sigma) + c_0(S,t). \tag{7.6.3}$$

  However, this solution can satisfy the following two limits:

$$\lim_{\frac{S}{K}\to\infty} \frac{V_0}{S} = 1, \quad \text{and} \quad \lim_{\frac{S}{K}\to 0} \frac{V_0}{S} = 0 \tag{7.6.4}$$

  and exhibit σ-dependence simultaneously. For example, if we would take $c(S,t) = S^2 e^{-S}$, both limits would be satisfied, and V_0 still depends of σ in the middle part, because these are only conditions for very large and very small stock prices S. So, for intermediate S, the solution V_0 can still depend on σ. We have asked Howison if there is a model assumption which says it cannot, but he did not answer that question.

- Note that if Howison would not have done the above assumption about the σ-independence of V_0, the higher order equations would be completely different, because then the operator $\mathcal{L}_{\frac{1}{2}}$ would contribute and give some non-zero terms. Consequently, it would have been a lot more complicated to obtain the higher order solutions $V_{\frac{1}{2}}(S,\sigma,t)$, $V_1(S,\sigma,t)$, etcetera.

Remarks

- On page 14 of [7] Howison presents a forward Kolmogorov equation

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial \sigma}\left(\frac{1}{2}\Sigma^2 \frac{\partial p}{\partial \sigma}\right) - \frac{\partial}{\partial \sigma}(Mp),$$

  while we have found

$$\begin{aligned} \frac{\partial p}{\partial t} &= -\frac{\partial}{\partial \sigma}\Big((M - \lambda\Sigma)p\Big) + \frac{1}{2}\frac{\partial^2}{\partial \sigma^2}\left(\Sigma^2 p\right), \\ \text{i.e., } \frac{\partial p}{\partial t} &= \frac{\partial}{\partial \sigma}\left(\frac{1}{2}\Sigma^2 \frac{\partial p}{\partial \sigma}\right) - \frac{\partial}{\partial \sigma}\left(Mp\right) + \frac{\partial}{\partial \sigma}\left(p\Sigma\left(\frac{\partial \Sigma}{\partial \sigma} + \lambda\right)\right). \end{aligned} \tag{7.6.5}$$

  using p. 291 of [10]. After asking Howison for explanation of why this extra term vanishes, he answered that we are right that there is a missing term in the FKE:
  "*I forgot to include the market price of risk in that term (I think the first version of the paper incorporated it into M). As a result I think that $p_\infty(\sigma)$ should have an additional term of $O\left(\epsilon^{\frac{1}{2}}\right)$ and this will have consequences further on (i.e., will make the subsequent formulae worse). Thank you for pointing out this error.*"

However, if we assume it to be incorporated into M, we would still have an extra term left, namely,

$$\frac{\partial}{\partial\sigma}\left(p\Sigma\frac{\partial\Sigma}{\partial\sigma}\right). \tag{7.6.6}$$

We have asked Howison to explain this, but unfortunately he did not reply to any of our subsequent e-mails.

- The solution of this forward Kolmogorov equation is also different from Howisons answer. On page 15 of [7] Howison does not explicitly show his solution p_∞, but he only remarks that if it exists, then it is proportional to

$$\frac{1}{\zeta^2(\sigma)}e^{-2\int^\sigma m(s)/\zeta^2(s)\ \mathrm{d}s}.$$

This is very different from the expression (7.3.8) we obtained, because we did not find the minus sign in the exponent, and also the integration 'constant' $\frac{c_0}{m}$ is missing here.

- In the expression for $V_{\frac{1}{2}}$ on page 18 of [7] Howison lacks a minus sign.

- On page 19 of [7] the definitions of $A_{1,1}$ and $A_{1,2}$ are probably incorrect: the operator $\mathcal{D}$ is missing from both $A_{1,1}$ and $A_{1,2}$.

- The particular solution $\widetilde{V_1}^\infty = g_1(\sigma)S^2\frac{\partial^2 P}{\partial S^2} - \tau\overline{\mathcal{L}}_1 P + \widetilde{\mathcal{V}_1}(S)$ on page 20 of [7] is indeed a solution of the initial value problem

$$\begin{aligned}\frac{\partial\widetilde{V_1}}{\partial\tau} + \mathcal{L}_0\widetilde{V_1} &= \frac{1}{2}\left(\overline{\sigma^2} - \sigma^2\right)S^2\frac{\partial^2 P}{\partial^2 S} - \overline{\mathcal{L}}_1 P, &\text{(7.6.7a)}\\ V_1(S,\sigma,0) &= 0. &\text{(7.6.7b)}\end{aligned}$$

It is, however, not clear which steps are taken to obtain this solution.

Method and solution

Howison made certain choices for the scalings. However, often the motivation for them was not explicitly given. There are many other ways to scale your parameters. For instance, an other combination of parameters could be chosen in the maximum balance and/or the order of the scaling in ε might be different. Due to lack of information, for example about how m and ζ depend on σ, we cannot determine an $\mathcal{O}(1)$ outer solution V_0 that depends on σ. Howison is actually 'throwing away' some information at this point by taking $c(S,t) = 0$.

After applying the method of matched asymptotic expansions to the fast mean-reverting stochastic volatility model and following Howison's assumptions, we have obtained the following expression for the three lowest order terms of the solution:

$$\begin{aligned}V(S,\sigma,t) \sim\ & c_0(S,t) + \varepsilon^{\frac{1}{2}}(T-t)\left(A_{\frac{1}{2},1} + A_{\frac{1}{2},2}\mathcal{D}\right)S^2\left(\mathcal{D}^2-\mathcal{D}\right)c_0\\ +\ & \varepsilon\Big[g_1(\sigma) - \overline{g_1(\sigma)} - (T-t)\left(A_{1,1} + A_{1,2}\mathcal{D}\right)\\ & - \frac{1}{2}(T-t)^2\left(A_{\frac{1}{2},1} + A_{\frac{1}{2},2}\mathcal{D}\left(\mathcal{D}^2-\mathcal{D}\right)\right)\left(A_{\frac{1}{2},1} + A_{\frac{1}{2},2}\mathcal{D}\right)\Big]\left(\mathcal{D}^2-\mathcal{D}\right)c_0,\end{aligned}$$

where $\mathcal{D} := S\frac{\partial}{\partial S}$. However, since the function $c_0(S,t)$ is still undetermined, this result doesn't seem to be very useful.

Our opinion
In the paper written by Howison [7], quite a lot of assumptions and simplifications are made. Unfortunately, often the motivation for them is not explicitly given in the paper, and in some cases we even think these assumptions and simplifications are incorrect.

Also, the choice of the scalings has not been explained very well. Howison presumably attempts to apply the same procedure as he used for the Black-Scholes model, as discussed in section 5 of this thesis. However, this is not the only way to scale the parameters in this model. He has chosen the derivative with respect to time to be $\mathcal{O}(\varepsilon)$, instead of $\mathcal{O}(1)$. So in the first order approach in equation (7.3.9), the time derivative has been left out of the problem, because $\frac{\partial}{\partial t}$ only occurs in the operator $\mathcal{L}_1$. The reason for this might be that in case the time derivative would occur in the $\mathcal{O}(1)$ problem, it is likely that one would obtain a more complicated solution V that does not only depend on t, but also on εt.

Multiple-scale analysis is a global perturbation scheme that is useful in systems characterized by disparate time scales. A two-scale expansion might work well for this model. The trick is to introduce a new variable $\vartheta = \varepsilon t$. This variable is called the *slow time* because it does not become significant until $t \sim \frac{1}{\varepsilon}$. Then take an expansion of the form

$$V(S, \sigma, t) = V_0(S, \sigma, \vartheta) + \varepsilon V_1(S, \sigma, \vartheta) + \dots. \tag{7.6.8}$$

So, a recommendation for further investigation on this model would be to use multiple-scale methods.

Chapter 8

The SABR model

In the classical Black-Scholes model, the volatility σ is assumed to be constant. But in reality, options with different strikes require different volatilities for the underlyings to match their market prices. This is called the **market volatility smile**. Handling these market smiles correctly is critical for hedging.

The **SABR model** is a stochastic volatility model that attempts to capture this volatility smile in derivatives markets. The name is an abbreviation of "**Stochastic Alpha, Beta, Rho**", referring to the three key parameters of the model.

The SABR model describes a single forward F, such as a forward interest rate, a forward swap rate, or a forward stock price. The volatility of the forward F is described by a parameter α. Here the volatility α is not constant, but is itself a random function of time. SABR is a dynamic model in which both F and α are stochastic state variables whose time evolution is given by the following system of stochastic differential equations:

$$\begin{cases} \mathrm{d}F = \hat{\alpha} F^{\beta} \mathrm{d}W_1, & F(0) = f \quad (8.0.1a) \\ \mathrm{d}\hat{\alpha} = \nu \hat{\alpha}\ \mathrm{d}W_2, & \hat{\alpha}(0) = \alpha. \quad (8.0.1b) \end{cases}$$

Here, f and α are the forward and volatility resp. at time $t = 0$. On the exercise date t_{ex} the decision whether or not to exercise the option is made. On the settlement date t_{set} all payments are made and the forward contract matures. The period $t_{\text{set}} - t_{\text{ex}}$ is called the settlement delay. W_1 and W_2 are two correlated Wiener processes with correlation coefficient $-1 < \rho < 1$. The constant parameter β satisfies the condition $0 \leq \beta \leq 1$. The above dynamics are a stochastic version of the CEV model with the skewness parameter β, because this CEV model is augmented by stochastic volatility. In fact, it reduces to the CEV model if $\nu = 0$.

Under the SABR model (8.0.1), the price of European options is given by **Black's formula**[1]

$$V_{\text{call}} = D(t_{\text{set}}) \left\{ f\mathcal{N}(d_+) - K\mathcal{N}(d_-) \right\}, \quad (8.0.2)$$

$$V_{\text{put}} = V_{\text{call}} + D(t_{\text{set}}) \left[K - f \right], \quad (8.0.3)$$

[1] Note that there is a difference between *Black's formula* and the *Black-Scholes formula*:
In case we are considering a spot price as we did before, the spot measure has to be used. In this case, the Black-Scholes formula can be derived. Here, we have a forward, which is a martingale in the forward measure because its expectation is equal to its present value. This leads to Black's formula. These two formulas are equivalent for deterministic interest rates. When the interest rate is stochastic, an extra term occurs in Black's formula.

with $D(t)$ the discount factor for date t, i.e., the value today of \$1 to be delivered on date t, and

$$d_{\pm} = \frac{\log\left(f/K\right) \pm \frac{1}{2}\sigma_B^2 t_{\text{ex}}}{\sigma_B\sqrt{t_{\text{ex}}}}, \tag{8.0.4}$$

where the implied volatility $\sigma_B(f,K)$ is given by

$$\sigma_B(f,K) = \alpha(fK)^{(\beta-1)/2}\left\{1 + \frac{(1-\beta)^2}{24}\log^2\left(\frac{f}{K}\right) + \frac{(1-\beta)^4}{1920}\log^4\left(\frac{f}{K}\right) + \ldots\right\}^{-1} \cdot \left(\frac{z}{x(z)}\right) \cdot \left\{1 + \left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(fK)^{1-\beta}} + \frac{1}{4}\frac{\rho\beta\nu\alpha}{(fK)^{(1-\beta)/2}} + \frac{2-3\rho^2}{24}\nu^2\right]t_{\text{ex}} + \ldots\right\}.$$

Here,

$$z = \frac{\nu}{\alpha}(fK)^{(1-\beta)/2}\log\left(\frac{f}{K}\right), \tag{8.0.5}$$

and $x(z)$ is defined by

$$x(z) = \log\left\{\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right\}. \tag{8.0.6}$$

In section 8.6 we will derive this result.

For the special case of at-the-money options, options struck at $K = f$, this formula reduces to

$$\sigma_{ATM} = \sigma_B(f,f) = \frac{\alpha}{f^{(1-\beta)}}\left\{1 + \left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{f^{2-2\beta}} + \frac{1}{4}\frac{\rho\beta\alpha\nu}{f^{(1-\beta)}} + \frac{2-3\rho^2}{24}\nu^2\right]t_{\text{ex}} + \ldots\right\}. \tag{8.0.7}$$

8.1 Derivation of the differential equation

After applying Itô's formula (4.3.1), we find the following expression for the infinitesimal increment in the option price:

$$\begin{aligned}
\mathrm{d}V &= \frac{\partial V}{\partial t}\mathrm{d}t + \frac{\partial V}{\partial F}\mathrm{d}F + \frac{\partial V}{\partial \alpha}\mathrm{d}\alpha + \frac{1}{2}\frac{\partial^2 V}{\partial F^2}\mathrm{d}F^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \alpha}\mathrm{d}\alpha^2 + \frac{\partial^2 V}{\partial F\partial\alpha}\mathrm{d}F\mathrm{d}\alpha, \\
&= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\alpha^2 F^{2\beta}\frac{\partial^2 V}{\partial F^2} + \frac{1}{2}\nu^2\alpha^2\frac{\partial^2 V}{\partial \alpha^2} + \rho\nu\alpha^2 F^{\beta}\frac{\partial^2 V}{\partial F\partial\alpha}\right)\mathrm{d}t + \alpha F^{\beta}\frac{\partial V}{\partial F}\mathrm{d}W_1 + \nu\alpha\frac{\partial V}{\partial\alpha}\mathrm{d}W_2.
\end{aligned}$$

Next we construct a portfolio $\Pi = V - \Delta F - \Delta_1 V_1$, such that $\mathrm{d}\Pi = \mathrm{d}V - \Delta\mathrm{d}F - \Delta_1\mathrm{d}V_1$. The observation that $\mathrm{d}\Pi = r\Pi\mathrm{d}t$, leads to

$$r\left(V - \Delta F - \Delta_1 V_1\right)\mathrm{d}t = \mathrm{d}V - \Delta\mathrm{d}F - \Delta_1\mathrm{d}V_1. \tag{8.1.1}$$

Substituting the formulas for $\mathrm{d}V$, $\mathrm{d}F$ and $\mathrm{d}V_1$ into equation (8.1.1) yields

$$\begin{aligned}
&\left(\frac{\partial V}{\partial t} + \frac{1}{2}\alpha^2 F^{2\beta}\frac{\partial^2 V}{\partial F^2} + \frac{1}{2}\nu^2\alpha^2\frac{\partial^2 V}{\partial\alpha^2} + \rho\nu\alpha^2 F^{\beta}\frac{\partial^2 V}{\partial F\partial\alpha} - rV\right)\mathrm{d}t \\
+ \quad &\alpha F^{\beta}\frac{\partial V}{\partial F}\mathrm{d}W_1 + \nu\alpha\frac{\partial V}{\partial\alpha}\mathrm{d}W_2 - \Delta\alpha F^{\beta}\mathrm{d}W_1 - \Delta F\mathrm{d}t \\
= \quad &\Delta_1\left(\frac{\partial V_1}{\partial t} + \frac{1}{2}\alpha^2 F^{2\beta}\frac{\partial^2 V_1}{\partial F^2} + \frac{1}{2}\nu^2\alpha^2\frac{\partial^2 V_1}{\partial\alpha^2} + \rho\nu\alpha^2 F^{\beta}\frac{\partial^2 V_1}{\partial F\partial\alpha} - rV_1\right)\mathrm{d}t.
\end{aligned} \tag{8.1.2}$$

Collecting all stochastic terms containing $\mathrm{d}W_1$ and $\mathrm{d}W_2$ together, this equation (8.1.2) transforms into

$$\begin{aligned}
&\left(\frac{\partial V}{\partial t}+\frac{1}{2}\alpha^2F^{2\beta}\frac{\partial^2V}{\partial F^2}+\frac{1}{2}\nu^2\alpha^2\frac{\partial^2V}{\partial\alpha^2}+\rho\nu\alpha^2F^{\beta}\frac{\partial^2V}{\partial F\partial\alpha}-rV\right)\mathrm{d}t-\Delta F\mathrm{d}t\\
=\;&\Delta_1\left(\frac{\partial V_1}{\partial t}+\frac{1}{2}\alpha^2F^{2\beta}\frac{\partial^2V_1}{\partial F^2}+\frac{1}{2}\nu^2\alpha^2\frac{\partial^2V_1}{\partial\alpha^2}+\rho\nu\alpha^2F^{\beta}\frac{\partial^2V_1}{\partial F\partial\alpha}-rV_1\right)\mathrm{d}t\\
+\;&\left(\Delta_1\alpha F^{\beta}\frac{\partial V_1}{\partial F}-\alpha F^{\beta}\frac{\partial V}{\partial F}+\Delta\alpha F^{\beta}\right)\mathrm{d}W_1+\left(\Delta_1\nu\alpha\frac{\partial V_1}{\partial\alpha}-\nu\alpha\frac{\partial V}{\partial\alpha}\right)\mathrm{d}W_2. \qquad (8.1.3)
\end{aligned}$$

To obtain a risk-free portfolio, these stochastic terms have to be eliminated, as we did before in section 7.1. Hence, we choose

$$\begin{cases}
\Delta_1=\dfrac{\left(\frac{\partial V}{\partial\alpha}\right)}{\left(\frac{\partial V_1}{\partial\alpha}\right)}, & (8.1.4a)\\[2ex]
\Delta=\dfrac{\partial V}{\partial F}-\Delta_1\dfrac{\partial V_1}{\partial F}=\dfrac{\partial V}{\partial F}-\dfrac{\left(\frac{\partial V}{\partial\alpha}\right)}{\left(\frac{\partial V_1}{\partial\alpha}\right)}\dfrac{\partial V_1}{\partial F}. & (8.1.4b)
\end{cases}$$

Following the steps taken in [3], substituting Δ_1 and Δ as above into equation (8.1.3), and collecting all V-dependent terms on the left-hand side and all V_1-dependent terms on the right-hand side, we obtain

$$\begin{aligned}
&\frac{\frac{\partial V}{\partial t}+\frac{1}{2}\alpha^2F^{2\beta}\frac{\partial^2V}{\partial F^2}+\frac{1}{2}\nu^2\alpha^2\frac{\partial^2V}{\partial\alpha^2}+\rho\nu\alpha^2F^{\beta}\frac{\partial^2V}{\partial F\partial\alpha}-F\frac{\partial V}{\partial F}-rV}{\left(\frac{\partial V}{\partial\alpha}\right)}\\
=\;&\frac{\frac{\partial V_1}{\partial t}+\frac{1}{2}\alpha^2F^{2\beta}\frac{\partial^2V_1}{\partial F^2}+\frac{1}{2}\nu^2\alpha^2\frac{\partial^2V_1}{\partial\alpha^2}+\rho\nu\alpha^2F^{\beta}\frac{\partial^2V_1}{\partial F\partial\alpha}-F\frac{\partial V_1}{\partial F}-rV_1}{\left(\frac{\partial V_1}{\partial\alpha}\right)}. \qquad (8.1.5)
\end{aligned}$$

Because the left-hand side of equation (8.1.5) is explicitly independent of V_1 and the right-hand side is explicitly independent of V, either sides must be independent of both V and V_1. The only way that this can be is for both sides to be equal to some function of the independent variables F, α and t. We deduce that

$$\frac{\partial V}{\partial t}+\frac{1}{2}\alpha^2F^{2\beta}\frac{\partial^2V}{\partial F^2}+\frac{1}{2}\nu^2\alpha^2\frac{\partial^2V}{\partial\alpha^2}+\rho\nu\alpha^2F^{\beta}\frac{\partial^2V}{\partial F\partial\alpha}-F\frac{\partial V}{\partial F}-rV=\lambda\nu\alpha\frac{\partial V}{\partial\alpha}, \qquad (8.1.6)$$

where, without loss of generality, we have written the arbitrary function of F, α and t as $\Big(0-\lambda\nu\alpha\Big)$, where 0 and α are the drift and volatility functions from the SDE (8.0.1) for instantaneous variance. Note that here $\lambda=\lambda(F,\alpha,t)$.

Hence, the price $V(F,\alpha,t)$ satisfies the following partial differential equation (PDE):

$$\frac{\partial V}{\partial t}+\frac{1}{2}\alpha^2F^{2\beta}\frac{\partial^2V}{\partial F^2}+\frac{1}{2}\nu^2\alpha^2\frac{\partial^2V}{\partial\alpha^2}+\rho\nu\alpha^2F^{\beta}\frac{\partial^2V}{\partial F\partial\alpha}-F\frac{\partial V}{\partial F}-\lambda\nu\alpha\frac{\partial V}{\partial\alpha}-rV=0. \qquad (8.1.7)$$

Scaling parameters in the above PDE, using maximum balance, does not seem to work very well, since after scaling we obtain PDEs that are still difficult to solve. Hence, for the SABR model we will follow the procedure of Hagan *et al.*, applying the scalings on the system of stochastic differential equations and using Kolmogorov equations, see section 8.2 of this thesis.

8.2 Scalings

Consider the following model in which both F and α are represented by stochastic state variables whose time evolution is given by the following system of stochastic differential equations:

$$\begin{cases} \mathrm{d}F = \hat{\alpha} C(F) \mathrm{d}W_1, & F(0) = f, \quad (8.2.1a) \\ \mathrm{d}\hat{\alpha} = \hat{\nu}\hat{\alpha}\ \mathrm{d}W_2, & \hat{\alpha}(0) = \alpha, \quad (8.2.1b) \end{cases}$$

under the forward measure, where the two Wiener processes are correlated by

$$\mathrm{E}\left[\mathrm{d}W_1\ \mathrm{d}W_2\right] = \rho\ \mathrm{d}t. \tag{8.2.2}$$

8.3 Application of perturbation theory to SABR model

Initially, we will analyze the model with a general function $C(F)$, for notational convenience. After that, the results are specialized to the power law F^β.

Our analysis is based on a small volatility expansion, where we take both the volatility α and the volatility of volatility[2] ν to be small. To carry out this analysis systematically, we first scale them by ε as follows:

$$\sigma = \frac{\hat{\alpha}}{\varepsilon}, \qquad \Leftrightarrow \qquad \hat{\alpha} = \varepsilon\sigma, \tag{8.3.1a}$$

$$\nu = \frac{\hat{\nu}}{\varepsilon}, \qquad \Leftrightarrow \qquad \hat{\nu} = \varepsilon\nu, \tag{8.3.1b}$$

such that σ and α are $\mathcal{O}_s(1)$. This yields

$$\begin{cases} \mathrm{d}F = \varepsilon\sigma C(F)\ \mathrm{d}W_1, & F(0) = f, \quad (8.3.2a) \\ \mathrm{d}\sigma = \varepsilon\nu\sigma\ \mathrm{d}W_2, & \sigma(0) = \alpha, \quad (8.3.2b) \end{cases}$$

Suppose that the market is in state $F(t) = f$, $\sigma(t) = \alpha$ at date t. Define the transition density function p by

$$p(t, f, \alpha; T, F, A)\ \mathrm{d}F\mathrm{d}A = \mathrm{prob}\left(F < F(T) < F + \mathrm{d}F, A < \sigma(T) < A + \mathrm{d}A \middle| F(t) = f, \sigma(t) = \alpha\right).$$

According to [10], p. 291, the unique **Forward Kolmogorov Equation** (FKE) of a stochastic differential equation (SDE)

$$\mathrm{d}f = D^1\mathrm{d}t + D^2\mathrm{d}W, \tag{8.3.3}$$

where D^1 and D^2 are matrices, and f, $\mathrm{d}t$ and $\mathrm{d}W$ are vectors, is given by

$$\frac{\partial f}{\partial T} = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(D_i^1(x_1, \ldots, x_N) f\right) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial^2}{\partial x_i \partial x_j}\left(D_{ij}^2(x_1, \ldots, x_N) f\right). \tag{8.3.4}$$

Next, we can construct the FKE for the transition density function p, which is given by

$$\begin{aligned} \frac{\partial}{\partial T} p(t, f, \alpha; T, F, A) &= \frac{1}{2}\frac{\partial^2}{\partial F^2}\left(\varepsilon^2 A^2 C^2(F)\ p\right) + \frac{1}{2}\frac{\partial^2}{\partial A^2}\left(\varepsilon^2\nu^2 A^2\ p\right) \\ &+ \frac{1}{2}\frac{\partial^2}{\partial F \partial A}\left(\rho\varepsilon^2\nu A^2\ p\right) + \frac{1}{2}\frac{\partial^2}{\partial A \partial F}\left(\rho\varepsilon^2\nu A^2\ p\right), \\ &= \frac{1}{2}\varepsilon^2 A^2 \frac{\partial^2}{\partial F^2}\left(C^2(F)\ p\right) + \rho\varepsilon^2\nu\frac{\partial^2}{\partial F \partial A}\left(A^2 C(F)\ p\right) + \frac{1}{2}\varepsilon^2\nu^2\frac{\partial^2}{\partial A^2}\left(A^2\ p\right), \end{aligned}$$

[2] This is commonly abbreviated as "**volvol**".

for $t < T$, with $p = \delta(F - f)\delta(A - \alpha)$ at $t = T$.[3]

Let $V(t, f, \alpha)$ be the value of a European call option at date t, when the economy is in state $F(t) = f$, $\sigma(t) = \alpha$. Let t_{ex} be the option's exercise date, and let K be its strike. Omitting the discount factor $D(t_{\text{ex}})$, because we do a current valuation of the final payoff, the value of the option is

$$\begin{aligned} V(t, f, \alpha) &= \mathrm{E}\left[\max(F(t_{\text{ex}}) - K, 0) | F(t) = f, \sigma(t) = \alpha\right], \\ &= \int_{A=-\infty}^{\infty} \int_{F=K}^{\infty} (F - K)\ p(t, f, \alpha; t_{\text{ex}}, F, A)\ \mathrm{d}F\ \mathrm{d}A. \end{aligned} \tag{8.3.5}$$

Since

$$p(t, f, \alpha; t_{\text{ex}}, F, A) = \delta(F - f)\delta(A - \alpha) + \int_t^{t_{\text{ex}}} \frac{\partial}{\partial T} p(t, f, \alpha; T, F, A)\ \mathrm{d}T, \tag{8.3.6}$$

we can rewrite equation (8.3.5) as

$$V(t, f, \alpha) = \max(f - K, 0) + \int_{T=t}^{t_{\text{ex}}} \int_{F=K}^{\infty} \int_{A=-\infty}^{\infty} (F - K)\ \frac{\partial}{\partial T} p(t, f, \alpha; T, F, A)\ \mathrm{d}A\ \mathrm{d}F\ \mathrm{d}T. \tag{8.3.7}$$

Next, substitute the FKE

$$\frac{\partial}{\partial T} p(t, f, \alpha; T, F, A) = \frac{1}{2}\varepsilon^2 A^2 \frac{\partial^2}{\partial F^2}\left(C^2(F)\ p\right) + \rho\varepsilon^2\nu \frac{\partial^2}{\partial F \partial A}\left(A^2 C(F)\ p\right) + \frac{1}{2}\varepsilon^2\nu^2 \frac{\partial^2}{\partial A^2}\left(A^2\ p\right)$$

into equation (8.3.7), where integrating the A derivatives, i.e.,

$$\varepsilon^2\rho\nu \frac{\partial^2}{\partial F \partial A}\left(A^2 C(F) p\right) \quad \text{and} \quad \frac{1}{2}\varepsilon^2\nu^2 \frac{\partial^2}{\partial A^2}\left(A^2 p\right) \tag{8.3.8}$$

over all A yields zero. This is caused by the fact that p and it's derivatives with respect to A resp. F go faster to zero than A resp. F and it's powers go to $\pm\infty$, see [2]. Therefore our option price reduces to

$$\begin{aligned} V(t, f, \alpha) &= \max(f - K, 0) \\ &+ \frac{1}{2}\varepsilon^2 \int_{T=t}^{t_{\text{ex}}} \int_{A=-\infty}^{\infty} \int_{F=K}^{\infty} A^2 (F - K)\ \frac{\partial^2}{\partial F^2}\left(C^2(F)\ p\right)\ \mathrm{d}F\ \mathrm{d}A\ \mathrm{d}T, \end{aligned} \tag{8.3.9}$$

where we have changed the order of integration. Integration by parts twice with respect to F yields

$$\begin{aligned} \int_K^{\infty} A^2 (F - K)\ \frac{\partial^2}{\partial F^2}\left(C^2(F)\ p\right)\ \mathrm{d}F &= \left[A^2 (F - K)\ \frac{\partial}{\partial F}\left(C^2(F)\ p\right)\right]_K^{\infty} \\ &- \int_K^{\infty} A^2\ \frac{\partial}{\partial F}\left(C^2(F)\ p\right)\ \mathrm{d}F, \\ &= -A^2\left[C^2(F)\ p\right]_K^{\infty} = A^2 C^2(K) p, \end{aligned}$$

Again, this is caused by the property that p goes faster to zero than $C(F)$ and its powers go to $\pm\infty$ [2] such that

$$V(t, f, \alpha) = \max(f - K, 0) + \frac{1}{2}\varepsilon^2 C^2(K) \int_{T=t}^{t_{\text{ex}}} \int_{A=-\infty}^{\infty} A^2\ p(t, f, \alpha; T, K, A)\ \mathrm{d}A\ \mathrm{d}T. \tag{8.3.10}$$

[3] This terminal condition follows directly from the definition of the transition density function p, as given above.

According to [10], p. 291, the unique **Backward Kolmogorov Equation** (BKE) of a stochastic differential equation (SDE)

$$\mathrm{d}f = D^1 \mathrm{d}t + D^2 \mathrm{d}W, \tag{8.3.11}$$

where D^1 and D^2 are matrices, and f, $\mathrm{d}t$ and $\mathrm{d}W$ are vectors, is given by

$$-\frac{\partial f}{\partial t} = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(D_i^1(x_1, \ldots, x_N) f \right) + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_i \partial x_j} \left(D_{ij}^2(x_1, \ldots, x_N) f \right). \tag{8.3.12}$$

Before we will construct the BKE for the transition density function p, let us define

$$P(t, f, \alpha; T, K) := \int_{-\infty}^{\infty} A^2 p(t, f, \alpha; T, K, A) \, \mathrm{d}A, \tag{8.3.13}$$

to simplify the problem further. Then P satisfies the following **Backward Kolmogorov Equation** (BKE):

$$\frac{\partial P}{\partial t} + \frac{1}{2}\varepsilon^2 \alpha^2 C^2(f) \frac{\partial^2 P}{\partial f^2} + \rho \varepsilon^2 \nu \alpha^2 C(f) \frac{\partial^2 P}{\partial f \partial \alpha} + \frac{1}{2}\varepsilon^2 \nu^2 \alpha^2 \frac{\partial^2 P}{\partial \alpha^2} = 0, \tag{8.3.14}$$

for $t < T$ and $P = \alpha^2 \ \delta(f - K)$ for $t = T$.

Since t does not appear explicitly in this equation, P depends only on the combination $T - t$, and not on t and T separately. So, define

$$\tau := T - t \quad \text{and} \quad \tau_{\text{ex}} := t_{\text{ex}} - t. \tag{8.3.15}$$

Then our pricing equation becomes

$$V(t, f, \alpha) = \max(f - K, 0) + \frac{1}{2}\varepsilon^2 C^2(K) \int_0^{\tau_{\text{ex}}} P(\tau, f, \alpha; K) \mathrm{d}\tau, \tag{8.3.16}$$

where P is the solution of the problem

$$\begin{cases} \dfrac{\partial P}{\partial \tau} = \dfrac{1}{2}\varepsilon^2 \alpha^2 C^2(f) \dfrac{\partial^2 P}{\partial f^2} + \rho \varepsilon^2 \nu \alpha^2 C(f) \dfrac{\partial^2 P}{\partial f \partial \alpha} + \dfrac{1}{2}\varepsilon^2 \nu^2 \alpha^2 \dfrac{\partial^2 P}{\partial \alpha^2}, \quad \text{for } \tau > 0, & \text{(8.3.17a)} \\ P = \alpha^2 \ \delta(f - K), \quad \text{at } \tau = 0. & \text{(8.3.17b)} \end{cases}$$

Since P starts out as a delta function, initially its derivatives will be large enough so that the size of the $\frac{\partial^2 P}{\partial f^2}$ term offsets the smallness of ε.[4]

To capture this limit, let us define the local variable

$$\xi := \frac{f - K}{\varepsilon}, \tag{8.3.18}$$

such that f can be replaced by $K + \varepsilon\xi$.

Then,

$$\frac{\partial}{\partial f} \to \frac{1}{\varepsilon} \frac{\partial}{\partial \xi}, \qquad \frac{\partial^2}{\partial f^2} \to \frac{1}{\varepsilon^2} \frac{\partial^2}{\partial \xi^2}, \tag{8.3.19}$$

[4]This means, the product $\varepsilon^2 \frac{\partial^2 P}{\partial f^2}$ is still large, even if ε is small, because $\frac{\partial^2 P}{\partial f^2}$ is the second derivative of the delta function, which has a very steep slope.

and we have the Taylor expansion

$$C(f) = C(K + \varepsilon\xi) = C_0 \left\{ 1 + \varepsilon\gamma_1\xi + \frac{1}{2}\varepsilon^2\gamma_2\xi^2 + \dots \right\}, \tag{8.3.20}$$

where

$$C_0 = C(K), \qquad \gamma_1 = \frac{C'(K)}{C(K)}, \qquad \gamma_2 = \frac{C''(K)}{C(K)}. \tag{8.3.21}$$

Substituting this into the PDE yields,

$$\begin{aligned} \frac{\partial P}{\partial \tau} &= \frac{1}{2}\alpha^2 C_0^2 \left\{ 1 + 2\varepsilon\gamma_1\xi + \varepsilon^2\left(\gamma_2 + \gamma_1^2\right)\xi^2 + \dots \right\} \frac{\partial^2 P}{\partial \xi^2} \\ &+ \varepsilon^2\rho\nu\alpha^2 C_0 \left\{ 1 + \varepsilon\gamma_1\xi + \dots \right\} \frac{\partial^2 P}{\partial \xi \partial \alpha} + \frac{1}{2}\varepsilon^2\nu^2\alpha^2 \frac{\partial^2 P}{\partial \alpha^2}, \end{aligned} \tag{8.3.22a}$$

for $\tau > 0$ with the initial condition[5]

$$P = \alpha^2\delta(\varepsilon\xi) = \frac{\alpha^2}{\varepsilon}\delta(\xi) \qquad \text{as} \quad \tau \to 0. \tag{8.3.22b}$$

Expanding

$$P_\varepsilon(t, \xi, \alpha) = \frac{1}{\varepsilon}P_0(t, \xi, \alpha) + P_1(t, \xi, \alpha) + \varepsilon P_2(t, \xi, \alpha) + \dots, \tag{8.3.23}$$

substituting this expansion into problem (8.3.22) and equating like powers of ε leads to the following $\mathcal{O}(\frac{1}{\varepsilon})$ problem:

$$\begin{cases} \dfrac{\partial P_0}{\partial \tau} = \dfrac{1}{2}\alpha^2 C_0^2 \dfrac{\partial^2 P_0}{\partial \xi^2}, & \text{for} \quad \tau > 0, \qquad (8.3.24a) \\ P_0 = \alpha^2\delta(\xi), & \text{as} \quad \tau \to 0. \qquad (8.3.24b) \end{cases}$$

The solution of the above problem can be computed by applying the following transformations:

$$\begin{cases} y := \dfrac{\xi}{\alpha^2}, & (8.3.25a) \\ k := \dfrac{C_0^2}{2\alpha^2}, & (8.3.25b) \\ t := \tau, & (8.3.25c) \\ u(t, y) := P_0(\tau, \xi). & (8.3.25d) \end{cases}$$

Then,

$$\frac{\partial^2}{\partial y^2} = \alpha^4 \frac{\partial^2}{\partial \xi^2}, \tag{8.3.26}$$

and the problem (8.3.31) transforms into

$$\begin{cases} \dfrac{\partial u}{\partial t} - k\dfrac{\partial^2 u}{\partial y^2} = 0, & (8.3.27a) \\ u(t = 0, y) = \delta(y), & (8.3.27b) \end{cases}$$

which is the heat or diffusion equation, subject to a Dirac delta in the initial condition.

[5]Here, $\delta(\cdot)$ is the Dirac delta function. For a proof of the equality $\delta(\varepsilon\xi) = \frac{\delta(\xi)}{\varepsilon}$, see Appendix L.

The solution to this problem (8.3.27) is given by the fundamental solution or heat kernel

$$u(t,y) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{y^2}{4kt}}. \tag{8.3.28}$$

In terms of the original variables, this equals

$$P_0(\tau,\xi) = \frac{1}{\sqrt{4\pi \frac{C_0^2}{2\alpha^2}\tau}} \exp\left(-\frac{\frac{\xi^2}{\alpha^4}}{4\frac{C_0^2}{2\alpha^2}\tau}\right) = \frac{|\alpha|}{\sqrt{2\pi C_0^2 \tau}} \exp\left(-\frac{\xi^2}{2\alpha^2 C_0^2 \tau}\right), \tag{8.3.29}$$

i.e.,

$$P_0(\tau,\xi) = \frac{|\alpha|}{\sqrt{2\pi C_0^2 \tau}} \exp\left(-\frac{(f-K)^2}{2\varepsilon^2\alpha^2 C_0^2 \tau}\right). \tag{8.3.30}$$

The $\mathcal{O}(1)$ problem is given by

$$\begin{cases} \dfrac{\partial P_1}{\partial \tau} - \dfrac{1}{2}\alpha^2 C_0^2 \dfrac{\partial^2 P_1}{\partial \xi^2} = \gamma_1 \alpha^2 C_0^2 \xi \dfrac{\partial^2 P_0}{\partial \xi^2}, & \text{for } \tau > 0, \quad (8.3.31a) \\ P_1 = 0, & \text{as } \tau \to 0. \quad (8.3.31b) \end{cases}$$

whereas the $\mathcal{O}(\varepsilon)$ problem is given by

$$\begin{cases} \dfrac{\partial P_2}{\partial \tau} - \dfrac{1}{2}\alpha^2 C_0^2 \dfrac{\partial^2 P_2}{\partial \xi^2} = \gamma_1 \alpha^2 C_0^2 \xi \dfrac{\partial^2 P_1}{\partial \xi^2} + \rho\nu C_0 \dfrac{\partial^2 P_0}{\partial \xi \partial \alpha}, & \text{for } \tau > 0, \quad (8.3.32a) \\ P_2 = 0, & \text{as } \tau \to 0. \quad (8.3.32b) \end{cases}$$

Because the differential equations and initial conditions are independent of ε, all P_i $(i = 1, 2, \ldots)$ are $\mathcal{O}(1)$ so we can conclude that [4]

$$P = \frac{1}{\varepsilon}P_0 + P_1 + \varepsilon P_2 + \ldots = \frac{\alpha}{\sqrt{2\pi\varepsilon^2\alpha^2 C_0^2 \tau}} \exp\left(-\frac{(f-k)^2}{2\varepsilon^2\alpha^2 C_0^2 \tau}\right)\{1 + \ldots\}. \tag{8.3.33}$$

The expansion

$$C(f) = \sum_{n=0}^{\infty} C^{(n)}(K)\frac{(\varepsilon\xi)^n}{n!} = C(K) + \varepsilon C'(K)\xi + \mathcal{O}\left(\varepsilon^2\right) \tag{8.3.34}$$

can be rewritten as

$$C(f) = C(K)\left(1 + \frac{C'(K)}{C(K)}\varepsilon\xi\right) = C(K)\left(1 + \frac{C'(K)}{C(K)}(f-K)\right), \tag{8.3.35}$$

such that the difference $C(f) = C(K)$ becomes

$$C(f) - C(K) = C'(K)(f-K). \tag{8.3.36}$$

Since the "$+\ldots$" in equation (8.3.33) involves powers of $(f-K)/\varepsilon\alpha C(K)$, this expansion would become inaccurate as soon as $(f-K)C'(K)/C(K)$ becomes a significant fraction of 1, i.e., as soon as $C(f)$ and $C(K)$ are significantly different.

Note that small changes in the exponent cause much greater changes in the probability density. Therefore, a better approach is to re-cast the series as

$$P = \frac{\alpha}{\sqrt{2\pi\varepsilon^2 C^2(K)\tau}} \exp\left(-\frac{(f-K)^2}{2\varepsilon^2\alpha^2 C^2(K)\tau}\right)\{1 + \ldots\} \tag{8.3.37}$$

and expand the exponent, since one expects that only small changes to the exponent will be needed to effect the much larger changes in the density.

We can refine this approach by noting that the exponential can be written in terms of an integral as follows:

$$\frac{(f-K)^2}{2\varepsilon^2\alpha^2C^2(K)\tau}\{1+\ldots\} = \frac{1}{2\tau}\left(\frac{1}{\varepsilon\alpha}\int_K^f \frac{\mathrm{d}f'}{C(f')}\right)^2\{1+\ldots\}. \tag{8.3.38}$$

This can be explained by writing out the integral in the right-hand side as follows:

$$\begin{aligned}
\int_K^f \frac{\mathrm{d}f'}{C(f')} &= \varepsilon\int_0^\xi \frac{\mathrm{d}\xi'}{C(K+\varepsilon\xi')}, && \text{(8.3.39a)}\\
&= \varepsilon\int_0^\xi \frac{\mathrm{d}\xi'}{C(K)+\varepsilon C'(K)\xi'+\mathcal{O}\left(\varepsilon^2\right)}, && \text{(8.3.39b)}\\
&= \frac{\varepsilon}{C(K)}\int_0^\xi \frac{\mathrm{d}\xi'}{1+\varepsilon\frac{C'(K)}{C(K)}\xi'+\mathcal{O}\left(\varepsilon^2\right)}, && \text{(8.3.39c)}\\
&= \frac{\varepsilon}{C(K)}\int_0^\xi \frac{\mathrm{d}\xi'}{1+\varepsilon\gamma_1\xi'+\mathcal{O}\left(\varepsilon^2\right)}, && \text{(8.3.39d)}\\
&= \frac{\varepsilon}{C(K)}\int_0^\xi \left(1-\varepsilon\gamma_1\xi'+\mathcal{O}\left(\varepsilon^2\right)\right)\mathrm{d}\xi', && \text{(8.3.39e)}\\
&= \frac{\varepsilon}{C(K)}\left(\xi-\frac{1}{2}\varepsilon\gamma_1\xi^2+\mathcal{O}\left(\varepsilon^2\right)\right), && \text{(8.3.39f)}\\
&= \frac{\varepsilon\xi}{C(K)} = \frac{f-K}{C(K)}. && \text{(8.3.39g)}
\end{aligned}$$

Here, the Mercator series[6] are used to perform the step from equation (8.3.39d) to equation (8.3.39e), i.e.,

$$\frac{1}{1+\varepsilon+\mathcal{O}\left(\varepsilon^2\right)} = 1+-1\cdot\varepsilon+\mathcal{O}\left(\varepsilon^2\right) = 1-\varepsilon+\mathcal{O}\left(\varepsilon^2\right). \tag{8.3.41}$$

Having found that

$$\frac{f-K}{C(K)} = \int_K^f \frac{\mathrm{d}f'}{C(f')} \tag{8.3.42}$$

implies that

$$\frac{(f-K)^2}{C^2(K)} = \left(\int_K^f \frac{\mathrm{d}f'}{C(f')}\right)^2 \tag{8.3.43}$$

and thus we can maken the substitution

$$\frac{(f-K)^2}{2\varepsilon^2\alpha^2C^2(K)\tau}\{1+\ldots\} = \frac{1}{2\tau}\left(\frac{1}{\varepsilon\alpha}\int_K^f \frac{\mathrm{d}f'}{C(f')}\right)^2\{1+\ldots\}, \tag{8.3.44}$$

which concludes the derivation of equation (8.3.38).

[6] In mathematics, the **Mercator series** or Newton-Mercator series is the series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^n = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots. \tag{8.3.40}$$

This is the Taylor series for the natural logarithm shifted by 1, i.e., $\log(1+x)$.

Here, it is a 'natural' choice to change variables from f to

$$z := \frac{1}{\varepsilon\alpha}\int_K^f \frac{\mathrm{d}f'}{C(f')}, \tag{8.3.45}$$

since f only occurs in combination with this integral. Also, we define

$$B(\varepsilon\alpha z) := C(f). \tag{8.3.46}$$

Then,

$$\frac{\partial}{\partial f} \to \frac{1}{\varepsilon\alpha C(f)}\frac{\partial}{\partial z} = \frac{1}{\varepsilon\alpha B(\varepsilon\alpha z)}\frac{\partial}{\partial z} \qquad \text{and} \qquad \frac{\partial}{\partial \alpha} \to \frac{\partial}{\partial \alpha} - \frac{z}{\alpha}\frac{\partial}{\partial z}, \tag{8.3.47}$$

such that

$$\frac{\partial^2}{\partial f^2} \to \frac{1}{\varepsilon^2\alpha^2 B^2(\varepsilon\alpha z)}\left\{\frac{\partial^2}{\partial z^2} - \varepsilon\alpha\frac{B'(\varepsilon\alpha z)}{B(\varepsilon\alpha z)}\frac{\partial}{\partial z}\right\}, \tag{8.3.48a}$$

$$\frac{\partial^2}{\partial f\partial \alpha} \to \frac{1}{\varepsilon\alpha B(\varepsilon\alpha z)}\left\{\frac{\partial^2}{\partial z\partial \alpha} - \frac{z}{\alpha}\frac{\partial^2}{\partial z^2} - \frac{1}{\alpha}\frac{\partial}{\partial z}\right\}, \tag{8.3.48b}$$

$$\frac{\partial^2}{\partial \alpha^2} \to \frac{\partial^2}{\partial \alpha^2} - \frac{2z}{\alpha}\frac{\partial^2}{\partial z\partial \alpha} + \frac{z^2}{\alpha^2}\frac{\partial^2}{\partial z^2} + \frac{2z}{\alpha^2}\frac{\partial}{\partial z}. \tag{8.3.48c}$$

Also,[7]

$$\delta(f-K) = \delta(\varepsilon\alpha z C(K)) = \frac{1}{\varepsilon\alpha C(K)}\delta(z). \tag{8.3.49}$$

Therefore, (8.3.16) through (8.3.17) become

$$V(t,f,\alpha) = \max(f-K,0) + \frac{1}{2}\varepsilon^2 C^2(K)\int_0^{\tau_{\mathrm{ex}}} P(\tau,z,\alpha)\mathrm{d}\tau, \tag{8.3.50}$$

where $P(\tau,z,\alpha)$ is the solution of the boundary value problem

$$\begin{cases} \dfrac{\partial P}{\partial \tau} = \dfrac{1}{2}\dfrac{\partial^2 P}{\partial z^2} - \dfrac{1}{2}\varepsilon\alpha\dfrac{B'}{B}\dfrac{\partial P}{\partial z} + \rho\varepsilon\nu\alpha\dfrac{\partial^2 P}{\partial z\partial\alpha} - \varepsilon\rho\nu z\dfrac{\partial^2 P}{\partial z^2} - \rho\varepsilon\nu\dfrac{\partial P}{\partial z} \\ \qquad + \dfrac{1}{2}\varepsilon^2\nu^2\alpha^2\dfrac{\partial^2 P}{\partial \alpha^2} - \varepsilon^2\nu^2\alpha z^2\dfrac{\partial^2 P}{\partial z\partial\alpha} + \dfrac{1}{2}\varepsilon^2\nu^2 z^2\dfrac{\partial^2 P}{\partial z^2} + \varepsilon^2\nu^2 z\dfrac{\partial P}{\partial z}, \quad \text{for } \tau > 0, & (8.3.51a) \\ P = \dfrac{\alpha}{\varepsilon C(K)}\delta(z), \quad \text{at } \tau = 0, & (8.3.51b) \end{cases}$$

i.e.,

$$\begin{cases} \dfrac{\partial P}{\partial \tau} = \dfrac{1}{2}\left(1 - 2\varepsilon\rho\nu z + \varepsilon^2\nu^2 z^2\right)\dfrac{\partial^2 P}{\partial z^2} - \dfrac{1}{2}\varepsilon\alpha\dfrac{B'}{B}\dfrac{\partial P}{\partial z} \\ \qquad + \left(\varepsilon\rho\nu - \varepsilon^2\nu^2 z\right)\left(\alpha\dfrac{\partial^2 P}{\partial z\partial\alpha} - \dfrac{\partial P}{\partial z}\right) + \dfrac{1}{2}\varepsilon^2\nu^2\alpha^2\dfrac{\partial^2 P}{\partial \alpha^2}, \quad \text{for } \tau > 0, & (8.3.52a) \\ P = \dfrac{\alpha}{\varepsilon C(K)}\delta(z), \quad \text{at } \tau = 0. & (8.3.52b) \end{cases}$$

Accordingly, let us define[8]

$$\hat{P}(\tau,z,\alpha) := \frac{\varepsilon}{\alpha}C(K)P, \tag{8.3.53}$$

[7] In integral form.

[8] Here, we have adopted the notation of Hagan *et al.*, containing $C(K)$. Actually, in this case it would be better to write $\hat{P}(\tau,z,\alpha) := \frac{\varepsilon}{\alpha}B(0)P$. Recall that $B(\varepsilon\alpha z) = C(f)$ and $B(0) = C(K)$.

such that P can be replaced by $\frac{\alpha}{\varepsilon C(K)}\hat{P}$.

Then,

$$\frac{\varepsilon C(K)}{\alpha}\frac{\partial P}{\partial f} \quad \rightarrow \quad \frac{\varepsilon C(K)}{\alpha}\frac{\partial}{\partial \alpha}\left(\frac{\alpha}{\varepsilon C(K)}\hat{P}\right) = \frac{1}{\alpha}\hat{P} + \frac{\partial \hat{P}}{\partial \alpha}, \qquad \text{and} \tag{8.3.54a}$$

$$\begin{aligned}\frac{\varepsilon C(K)}{\alpha}\frac{\partial^2 P}{\partial \alpha^2} \quad \rightarrow \quad & \frac{\varepsilon C(K)}{\alpha}\frac{\partial^2}{\partial \alpha^2}\left(\frac{\alpha}{\varepsilon C(K)}\hat{P}\right) = \frac{1}{\alpha}\frac{\partial}{\partial \alpha}\left(\hat{P} + \alpha\frac{\partial \hat{P}}{\partial \alpha}\right), \\ & = \frac{1}{\alpha}\left(\frac{\partial \hat{P}}{\partial \alpha} + \frac{\partial \hat{P}}{\partial \alpha} + \alpha\frac{\partial^2 \hat{P}}{\partial \alpha^2}\right) = \frac{2}{\alpha}\frac{\partial \hat{P}}{\partial \alpha} + \frac{\partial^2 \hat{P}}{\partial \alpha^2}.\end{aligned} \tag{8.3.54b}$$

In terms of $\hat{P}$ we obtain

$$V(t, f, \alpha) = \max(f - K, 0) + \frac{1}{2}\varepsilon\alpha C(K)\int_0^{\tau_{\text{ex}}} \hat{P}(\tau, z, \alpha)\mathrm{d}\tau, \tag{8.3.55}$$

where $\hat{P}(\tau, z, \alpha)$ is the solution of

$$\begin{cases} \dfrac{\partial \hat{P}}{\partial \tau} = \dfrac{1}{2}\left(1 - 2\varepsilon\rho\nu z + \varepsilon^2\nu^2 z^2\right)\dfrac{\partial^2 \hat{P}}{\partial z^2} - \dfrac{1}{2}\varepsilon\alpha\dfrac{B'}{B}\dfrac{\partial \hat{P}}{\partial z} \\ \qquad + \left(\varepsilon\rho\nu - \varepsilon^2\nu^2 z\right)\alpha\dfrac{\partial^2 \hat{P}}{\partial z \partial \alpha} + \dfrac{1}{2}\varepsilon^2\nu^2\left(\alpha^2\dfrac{\partial^2 \hat{P}}{\partial \alpha^2} + 2\alpha\dfrac{\partial \hat{P}}{\partial \alpha}\right), \quad \text{for } \tau > 0, & (8.3.56a) \\ \hat{P} = \delta(z), \qquad \text{at } \tau = 0. & (8.3.56b) \end{cases}$$

To leading order, $\hat{P}$ is the solution of the standard diffusion problem

$$\frac{\partial \hat{P}}{\partial \tau} = \frac{1}{2}\frac{\partial^2 \hat{P}}{\partial z^2}, \tag{8.3.57}$$

with $\hat{P} = \delta(z)$ at $\tau = 0$. So it is a Gaussian to leading order. The next stage is to transform the problem into the standard diffusion problem through $\mathcal{O}(\varepsilon)$, and then through $\mathcal{O}\left(\varepsilon^2\right)$, etc. This is the near identify transform method[9] which has proven so powerful in near-Hamiltonian systems[10].

Note that the variable α does not enter the problem for $\hat{P}$ until $\mathcal{O}(\varepsilon)$, so

$$\hat{P}(\tau, z, \alpha) = P_0(\tau, z) + \varepsilon P_1(\tau, z, \alpha) + \ldots. \tag{8.3.59}$$

Consequently, the derivatives $\frac{\partial^2 \hat{P}}{\partial z \partial \alpha}$, $\frac{\partial^2 \hat{P}}{\partial \alpha^2}$ and $\frac{\partial \hat{P}}{\partial \alpha}$ are all $\mathcal{O}(\varepsilon)$. Recall that we are only solving for $\hat{P}$ through $\mathcal{O}\left(\varepsilon^2\right)$.

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[9] See appendix M.

[10] A near-Hamiltonian system is of the form

$$\begin{cases} \dfrac{\mathrm{d}p}{\mathrm{d}t} = \dfrac{\partial H}{\partial q} + \varepsilon(\ldots) \quad =: F, & (8.3.58a) \\ \dfrac{\mathrm{d}q}{\mathrm{d}t} = -\dfrac{\partial H}{\partial p} + \varepsilon(\ldots) =: G, & (8.3.58b) \end{cases}$$

with

$$\frac{\partial F}{\partial p} = -\frac{\partial G}{\partial q}.$$

So, through this order, we can re-write our boundary value problem as

$$\begin{cases} \dfrac{\partial \hat{P}}{\partial \tau} = \dfrac{1}{2}\left(1 - 2\varepsilon\rho\nu z + \varepsilon^2\nu^2 z^2\right)\dfrac{\partial^2 \hat{P}}{\partial z^2} - \dfrac{1}{2}\varepsilon\alpha\dfrac{B'}{B}\dfrac{\partial \hat{P}}{\partial z} + \varepsilon\rho\nu\alpha\dfrac{\partial^2 \hat{P}}{\partial z \partial \alpha}, & \text{for } \tau > 0, \quad (8.3.60a) \\ \hat{P} = \delta(z), & \text{at } \tau = 0. \quad (8.3.60b) \end{cases}$$

Let us now eliminate the $\frac{1}{2}\varepsilon\alpha\frac{B'}{B}\frac{\partial \hat{P}}{\partial z}$ term. Define $H(\tau, z, \alpha)$ by

$$\hat{P} = \left(\frac{C(f)}{C(K)}\right)^n H = \left(\frac{B(\varepsilon\alpha z)}{B(0)}\right)^n H, \tag{8.3.61}$$

where n is an $\mathcal{O}(1)$ constant that will be determined in the next steps.

Then,

$$\begin{aligned} \frac{\partial \hat{P}}{\partial z} &= \left(\frac{B(\varepsilon\alpha z)}{B(0)}\right)^n \left(\frac{\partial H}{\partial z} + \varepsilon n\alpha\frac{B'}{B}H\right), & (8.3.62a) \\ \frac{\partial^2 \hat{P}}{\partial z^2} &= \left(\frac{B(\varepsilon\alpha z)}{B(0)}\right)^n \left(\frac{\partial^2 H}{\partial z^2} + 2\varepsilon n\alpha\frac{B'}{B}\frac{\partial H}{\partial z} + \varepsilon^2\alpha^2\left[n\frac{B''}{B} + (n^2 - n)\frac{B'^2}{B^2}\right]H\right) & (8.3.62b) \\ \frac{\partial^2 \hat{P}}{\partial z \partial \alpha} &= \left(\frac{B(\varepsilon\alpha z)}{B(0)}\right)^n \left(\frac{\partial^2 H}{\partial z \partial \alpha} + \varepsilon n z\frac{B'}{B}\frac{\partial H}{\partial z} + \varepsilon n\alpha\frac{B'}{B}\frac{\partial H}{\partial \alpha} + \varepsilon n\frac{B'}{B}H + \mathcal{O}\left(\varepsilon^2\right)\right) & (8.3.62c) \end{aligned}$$

The option price now becomes

$$V(t, f, \alpha) = \max(f - K, 0) + \varepsilon n\alpha B(0)\left(\frac{B(\varepsilon\alpha z)}{B(0)}\right)^n \int_0^{\tau_{\text{ex}}} H(\tau, z, \alpha)\, \mathrm{d}\tau, \tag{8.3.63}$$

where $H(\tau, z, \alpha)$ is the solution of

$$\begin{aligned} \frac{\partial H}{\partial \tau} &= \frac{1}{2}\left(1 - 2\varepsilon\rho\nu z + \varepsilon^2\nu^2 z^2\right)\left(\frac{\partial^2 H}{\partial z^2} + 2\varepsilon n\alpha\frac{B'}{B}\frac{\partial H}{\partial z} + \varepsilon^2\alpha^2\left[n\frac{B''}{B} + (n^2 - n)\frac{B'^2}{B^2}\right]H\right) \\ &+ \varepsilon\rho\nu\alpha\left(\frac{\partial^2 H}{\partial z \partial \alpha} + \varepsilon n z\frac{B'}{B}\frac{\partial H}{\partial z} + \varepsilon n\alpha\frac{B'}{B}\frac{\partial H}{\partial \alpha} + \varepsilon n\frac{B'}{B}H + \mathcal{O}\left(\varepsilon^2\right)\right) \\ &- \frac{1}{2}\varepsilon\alpha\frac{B'}{B}\left(\frac{\partial H}{\partial z} + \varepsilon n\alpha\frac{B'}{B}H\right) \end{aligned}$$

for $\tau > 0$, with initial condition $H = \delta(z)$ at $\tau = 0$. Omitting all $\mathcal{O}\left(\varepsilon^3\right)$ terms, and combining some terms yields

$$\begin{aligned} \frac{\partial H}{\partial \tau} &= \frac{1}{2}\left(1 - 2\varepsilon\rho\nu z + \varepsilon^2\nu^2 z^2\right)\frac{\partial^2 H}{\partial z^2} + \varepsilon n\alpha\frac{B'}{B}\frac{\partial H}{\partial z} \\ &+ \frac{1}{2}\varepsilon^2\alpha^2\left[n\frac{B''}{B} + (n^2 - n)\frac{B'^2}{B^2}\right]H - \varepsilon^2 n\rho\nu\alpha\frac{B'}{B}\left(z\frac{\partial H}{\partial z} - H\right) - \frac{1}{2}\varepsilon\alpha\frac{B'}{B}\frac{\partial H}{\partial z} \\ &- \frac{1}{2}\varepsilon^2 n\alpha^2\frac{B'^2}{B^2}H + \varepsilon\rho\nu\alpha\left(\frac{\partial^2 H}{\partial z \partial \alpha} + \varepsilon n\alpha\frac{B'}{B}\frac{\partial H}{\partial \alpha}\right). \end{aligned} \tag{8.3.64}$$

Next, we combine the third and sixth term of equation (8.3.64) and reorder terms to obtain

$$\begin{aligned} \frac{\partial H}{\partial \tau} &= \frac{1}{2}\left(1 - 2\varepsilon\rho\nu z + \varepsilon^2\nu^2 z^2\right)\frac{\partial^2 H}{\partial z^2} + \varepsilon\alpha\left(n - \frac{1}{2}\right)\frac{B'}{B}\frac{\partial H}{\partial z} \\ &+ \frac{1}{2}\varepsilon^2\alpha^2\left[n\frac{B''}{B} + (n^2 - 2n)\frac{B'^2}{B^2}\right]H - \varepsilon^2 n\rho\nu\alpha\frac{B'}{B}\left(z\frac{\partial H}{\partial z} - H\right) \\ &+ \varepsilon\rho\nu\alpha\left(\frac{\partial^2 H}{\partial z \partial \alpha} + \varepsilon n\alpha\frac{B'}{B}\frac{\partial H}{\partial \alpha}\right). \end{aligned} \tag{8.3.65}$$

The choice $n = \frac{1}{2}$, eliminates the term containing $\frac{\partial H}{\partial z}$. This yields

$$V(t,f,\alpha) = \max(f-K,0) + \frac{1}{2}\varepsilon\alpha B(0)\left(\frac{B(\varepsilon\alpha z)}{B(0)}\right)^{\frac{1}{2}} \int_0^{\tau_{\text{ex}}} H(\tau,z,\alpha)\ \mathrm{d}\tau, \tag{8.3.66}$$

for the option price, where $H(\tau,z,\alpha)$ is the solution of

$$\begin{aligned}\frac{\partial H}{\partial \tau} &= \frac{1}{2}\left(1-2\varepsilon\rho\nu z+\varepsilon^2\nu^2 z^2\right)\frac{\partial^2 H}{\partial z^2} - \frac{1}{2}\varepsilon^2\rho\nu\alpha\frac{B'}{B}\left(z\frac{\partial H}{\partial z}-H\right)\\ &+ \varepsilon^2\alpha^2\left[\frac{1}{4}\frac{B''}{B}-\frac{3}{8}\frac{B'^2}{B^2}\right]H + \varepsilon\rho\nu\alpha\left(\frac{\partial^2 H}{\partial z\partial\alpha}+\frac{1}{2}\varepsilon\alpha\frac{B'}{B}\frac{\partial H}{\partial\alpha}\right),\end{aligned} \tag{8.3.67}$$

for $\tau > 0$, with initial condition $H = \delta(z)$ at $\tau = 0$.

Since this equation (8.3.67) is independent of α to leading order, i.e.,

$$\begin{cases}\dfrac{\partial H}{\partial\tau} = \dfrac{1}{2}\dfrac{\partial^2 H}{\partial z^2}, & \text{for } \tau > 0, \quad (8.3.68a)\\ H = \delta(z), & \text{at } \tau = 0, \quad (8.3.68b)\end{cases}$$

we can conclude that the α-derivatives $\frac{\partial H}{\partial\alpha}$ and $\frac{\partial^2 H}{\partial\alpha^2}$ are no larger than $\mathcal{O}(\varepsilon)$.

At $\mathcal{O}(\varepsilon)$ equation (8.3.67) depends on α only through the last term

$$\varepsilon\rho\nu\alpha\left(\frac{\partial^2 H}{\partial z\partial\alpha}+\frac{1}{2}\varepsilon\alpha\frac{B'}{B}\frac{\partial H}{\partial\alpha}\right). \tag{8.3.69}$$

Because the α-derivatives are no larger than $\mathcal{O}(\varepsilon)$, this last term is actually only $\mathcal{O}\left(\varepsilon^3\right)$, and can thus be neglected, since only the terms through $\mathcal{O}_s\left(\varepsilon^2\right)$ are taken into account.

This yields

$$\begin{cases}\dfrac{\partial H}{\partial\tau} = \dfrac{1}{2}\left(1-2\varepsilon\rho\nu z+\varepsilon^2\nu^2 z^2\right)\dfrac{\partial^2 H}{\partial z^2} - \dfrac{1}{2}\varepsilon^2\rho\nu\alpha\dfrac{B'}{B}\left(z\dfrac{\partial H}{\partial z}-H\right)\\ \qquad + \varepsilon^2\alpha^2\left[\dfrac{1}{4}\dfrac{B''}{B}-\dfrac{3}{8}\dfrac{B'^2}{B^2}\right]H, & \text{for } \tau > 0, \quad (8.3.70a)\\ H = \delta(z), & \text{at } \tau = 0. \quad (8.3.70b)\end{cases}$$

There are no longer any α-derivatives, so now α can be treated as a parameter instead of as an independent variable. This means we have succesfully reduced the problem to one dimension.

Let us now remove the $\frac{\partial H}{\partial z}$ term through $\mathcal{O}\left(\varepsilon^2\right)$. To leading order the ratios

$$\frac{B'(\varepsilon\alpha z)}{B(\varepsilon\alpha z)} \quad \text{and} \quad \frac{B''(\varepsilon\alpha z)}{B(\varepsilon\alpha z)} \tag{8.3.71}$$

are constant. We can replace them by

$$b_1 := \frac{B'(\varepsilon\alpha z_0)}{B(\varepsilon\alpha z_0)} \quad \text{and} \quad b_2 := \frac{B''(\varepsilon\alpha z_0)}{B(\varepsilon\alpha z_0)}, \tag{8.3.72}$$

committing only an $\mathcal{O}(\varepsilon)$ error, where the constant z_0 will be chosen later on. This can be seen by expanding $z = z_0 + \varepsilon z_1 + \ldots$.

We now define $\hat{H}$ by

$$H = \exp\left(\frac{1}{4}\varepsilon^2\rho\nu\alpha b_1 z^2\right)\hat{H}, \tag{8.3.73}$$

such that

$$\frac{\partial H}{\partial \tau} \rightarrow \exp\left(\frac{1}{4}\varepsilon^2\rho\nu\alpha b_1 z^2\right)\frac{\partial \hat{H}}{\partial \tau}, \tag{8.3.74a}$$

$$\frac{\partial H}{\partial z} \rightarrow \exp\left(\frac{1}{4}\varepsilon^2\rho\nu\alpha b_1 z^2\right)\left(\frac{\partial \hat{H}}{\partial z} + \frac{1}{2}\varepsilon^2\rho\nu\alpha b_1 z\hat{H}\right), \tag{8.3.74b}$$

$$\frac{\partial^2 H}{\partial z^2} \rightarrow \exp\left(\frac{1}{4}\varepsilon^2\rho\nu\alpha b_1 z^2\right)\left(\frac{\partial^2 \hat{H}}{\partial z^2} + \varepsilon^2\rho\nu\alpha b_1 z\frac{\partial \hat{H}}{\partial z} + \frac{1}{2}\varepsilon^2\rho\nu\alpha b_1\hat{H}\right). \tag{8.3.74c}$$

Again, only terms up to and including $\mathcal{O}\left(\varepsilon^2\right)$ have to be taken into account.

Then our option price becomes

$$V(t,f,\alpha) = \max(f-K,0) + \frac{1}{2}\varepsilon\alpha\sqrt{B(0)B(\varepsilon\alpha z)}e^{\varepsilon^2\rho\nu\alpha b_1 z^2/4}\int_0^{\tau_{\text{ex}}}\hat{H}(\tau,z,\alpha)\,\mathrm{d}\tau, \tag{8.3.75}$$

and equation (8.3.70a) transforms into

$$\begin{aligned}\frac{\partial \hat{H}}{\partial \tau} &= \frac{1}{2}\left(1-2\varepsilon\rho\nu z+\varepsilon^2\nu^2 z^2\right)\left(\frac{\partial^2 \hat{H}}{\partial z^2} + \varepsilon^2\rho\nu\alpha b_1 z\frac{\partial \hat{H}}{\partial z} + \frac{1}{2}\varepsilon^2\rho\nu\alpha b_1\hat{H}\right)\\ &- \frac{1}{2}\varepsilon^2\rho\nu\alpha b_1\left[z\left(\frac{\partial \hat{H}}{\partial z} + \frac{1}{2}\varepsilon^2\rho\nu\alpha b_1 z\hat{H}\right) - \hat{H}\right] + \varepsilon^2\alpha^2\left[\frac{1}{4}b_2 - \frac{3}{8}b_1^2\right]\hat{H}.\end{aligned} \tag{8.3.76}$$

Neglecting all terms that are $\mathcal{O}\left(\varepsilon^3\right)$, this can be rewritten as

$$\begin{aligned}\frac{\partial \hat{H}}{\partial \tau} &= \frac{1}{2}\left(1-2\varepsilon\rho\nu z+\varepsilon^2\nu^2 z^2\right)\frac{\partial^2 \hat{H}}{\partial z^2} + \frac{1}{2}\varepsilon^2\rho\nu\alpha b_1 z\frac{\partial \hat{H}}{\partial z} + \frac{1}{4}\varepsilon^2\rho\nu\alpha b_1\hat{H}\\ &- \frac{1}{2}\varepsilon^2\rho\nu\alpha b_1 z\frac{\partial \hat{H}}{\partial z} + \frac{1}{2}\varepsilon^2\rho\nu\alpha b_1\hat{H} + \varepsilon^2\alpha^2\left[\frac{1}{4}b_2 - \frac{3}{8}b_1^2\right]\hat{H},\end{aligned} \tag{8.3.77}$$

i.e.,

$$\frac{\partial \hat{H}}{\partial \tau} = \frac{1}{2}\left(1-2\varepsilon\rho\nu z+\varepsilon^2\nu^2 z^2\right)\frac{\partial^2 \hat{H}}{\partial z^2} + \varepsilon^2\alpha^2\left[\frac{1}{4}b_2 - \frac{3}{8}b_1^2\right]\hat{H} + \frac{3}{4}\varepsilon^2\rho\nu\alpha b_1\hat{H}, \tag{8.3.78}$$

for $\tau > 0$, with initial condition $H(0,z,\alpha) = \delta(z)$.

Next, we define

$$x := \frac{1}{\varepsilon\nu}\int_0^{\varepsilon\nu z}\frac{\mathrm{d}\zeta}{\sqrt{1-2\rho\zeta+\zeta^2}}, \tag{8.3.79}$$

i.e.,

$$x = \frac{1}{\varepsilon\nu}\int_0^{\varepsilon\nu z}\frac{\mathrm{d}\zeta}{\sqrt{1-\rho^2+(\zeta-\rho)^2}} = \frac{1}{\varepsilon\nu}\int_0^{\varepsilon\nu z}\frac{\mathrm{d}\zeta}{\sqrt{1-\rho^2}\sqrt{1+\frac{(\zeta-\rho)^2}{1-\rho^2}}}. \tag{8.3.80}$$

Define

$$\omega := \frac{\zeta - \rho}{\sqrt{1-\rho^2}}, \tag{8.3.81}$$

such that x can be written as

$$x = \frac{1}{\varepsilon\nu}\int_{\zeta=0}^{\varepsilon\nu z} \frac{\mathrm{d}\omega}{\sqrt{1+\omega^2}}. \tag{8.3.82}$$

Pythagoras' theorem states that in a right triangle with sides of length 1 and ω, the hypotenuse (the side opposite the right angle) is equal to $\sqrt{1+\omega^2}$. Also we define the angle θ as illustrated in figure 8.1.

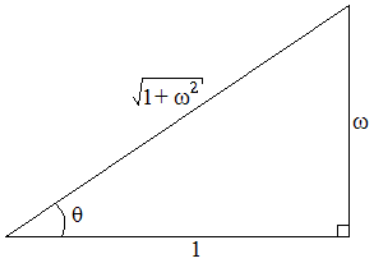

Figure 8.1: Right triangle with sides of length 1 and ω, and hypotenuse $\sqrt{1+\omega^2}$.

From this we can derive that

$$\frac{1}{\sqrt{1+\omega^2}} = \cos\theta, \tag{8.3.83a}$$

$$\omega = \tan(\theta) \Rightarrow \mathrm{d}\omega = \mathrm{d}\tan\theta = \frac{\mathrm{d}\theta}{\cos^2\theta}. \tag{8.3.83b}$$

Substituting this into the expressions for x (8.3.82) yields

$$x = \frac{1}{\varepsilon\nu}\int_{\zeta=0}^{\varepsilon\nu z} \frac{\cos\theta}{\cos^2\theta}\mathrm{d}\theta = \frac{1}{\varepsilon\nu}\int_{\zeta=0}^{\varepsilon\nu z} \frac{\mathrm{d}\theta}{\cos\theta}. \tag{8.3.84}$$

After computing this integral, we obtain

$$x = \frac{1}{\varepsilon\nu}\left[\log\left|\frac{1}{\cos\theta} + \tan\theta\right|\right]_{\zeta=0}^{\varepsilon\nu z}. \tag{8.3.85}$$

Using the expressions of ω in terms of θ and the fact that $\sqrt{1+\omega^2} > \omega$, this can be written as

$$x = \frac{1}{\varepsilon\nu}\left[\log\left(\sqrt{1+\omega^2}+\omega\right)\right]_{\zeta=0}^{\varepsilon\nu z}. \tag{8.3.86}$$

The last step is to substitute the definition of ω back into this expression, to obtain

$$\begin{aligned} x &= \frac{1}{\varepsilon\nu}\left[\log\left(\frac{\sqrt{1-2\rho\zeta+\zeta^2}}{\sqrt{1-\rho^2}} + \frac{\zeta-\rho}{\sqrt{1-\rho^2}}\right)\right]_{\zeta=0}^{\varepsilon\nu z}, \\ &= \frac{1}{\varepsilon\nu}\log\left(\frac{\sqrt{1-2\rho\varepsilon\nu z+\varepsilon^2\nu^2 z^2}+\varepsilon\nu z-\rho}{\sqrt{1-\rho^2}}\right) - \frac{1}{\varepsilon\nu}\log\left(\frac{1-\rho}{\sqrt{1-\rho^2}}\right). \end{aligned} \tag{8.3.87}$$

Hence, the definition of x (8.3.79) can also be written as

$$x := \frac{1}{\varepsilon\nu}\log\left(\frac{\sqrt{1-2\rho\varepsilon\nu z+\varepsilon^2\nu^2 z^2}-\rho+\varepsilon\nu z}{1-\rho}\right). \tag{8.3.88}$$

It is also possible to find an implicit expression for z in terms of x, by defining

$$y := \varepsilon\nu z - \rho. \tag{8.3.89}$$

Then,

$$y^2 = (\varepsilon\nu z-\rho)^2 = \varepsilon^2\nu^2 z^2 - 2\rho\varepsilon\nu z + \rho^2, \tag{8.3.90}$$

such that equation (8.3.88) can be written as

$$x = \frac{1}{\varepsilon\nu}\log\left(\frac{\sqrt{1-\rho^2+y^2}+y}{1-\rho}\right). \tag{8.3.91}$$

From this, it follows that

$$(1-\rho)e^{\varepsilon\nu x} = \sqrt{1-\rho^2+y^2}+y. \tag{8.3.92}$$

We can also write

$$\begin{aligned}\sqrt{1-\rho^2+y^2}-y &= \left(\sqrt{1-\rho^2+y^2}-y\right)\frac{\sqrt{1-\rho^2+y^2}+y}{\sqrt{1-\rho^2+y^2}+y} = \frac{1-\rho^2+y^2-y^2}{\sqrt{1-\rho^2+y^2}+y}\\ &= \frac{1-\rho^2}{(1-\rho)e^{\varepsilon\nu x}} = \frac{(1-\rho)(1+\rho)}{(1-\rho)e^{\varepsilon\nu x}} = (1+\rho)e^{-\varepsilon\nu x}.\end{aligned} \tag{8.3.93}$$

Using these two expressions (8.3.92) and (8.3.93), we can write y as

$$\begin{aligned}y &= \frac{1}{2}\left[\left(\sqrt{1-\rho^2+y^2}+y\right)-\left(\sqrt{1-\rho^2+y^2}-y\right)\right],\\ &= \frac{1}{2}\left[(1-\rho)\,e^{\varepsilon\nu x}-(1+\rho)\,e^{-\varepsilon\nu x}\right],\\ &= \frac{1}{2}\left[\left(e^{\varepsilon\nu x}-e^{-\varepsilon\nu x}\right)-\rho\left(e^{\varepsilon\nu x}+e^{-\varepsilon\nu x}\right)\right],\\ &= \sinh(\varepsilon\nu x)-\rho\cosh(\varepsilon\nu x).\end{aligned} \tag{8.3.94}$$

Substituting this into the definition of y (8.3.89) and reordering yields

$$\varepsilon\nu z = y+\rho = \sinh(\varepsilon\nu x)-\rho\cosh(\varepsilon\nu x)+\rho. \tag{8.3.95}$$

Hence, expression (8.3.88) can be written implicitly as

$$\varepsilon\nu z = \sinh(\varepsilon\nu x)-\rho\left(\cosh(\varepsilon\nu x)-1\right). \tag{8.3.96}$$

After the transformation from z to x, we have

$$\frac{\partial\hat{H}}{\partial z} \rightarrow \frac{1}{\sqrt{1-2\varepsilon\rho\nu z+\varepsilon^2\nu^2 z^2}}\frac{\partial\hat{H}}{\partial x} = \frac{1}{I(\varepsilon\nu z)}\frac{\partial\hat{H}}{\partial x}, \tag{8.3.97a}$$

$$\begin{aligned}\frac{\partial^2\hat{H}}{\partial z^2} &\rightarrow \frac{1}{I^2(\varepsilon\nu z)}\left(\frac{\partial^2\hat{H}}{\partial x^2}-\varepsilon\nu I'(\varepsilon\nu z)\frac{\partial\hat{H}}{\partial x}\right),\\ &= \frac{1}{1-2\varepsilon\rho\nu z+\varepsilon^2\nu^2 z^2}\left(\frac{\partial^2\hat{H}}{\partial x^2}-\varepsilon\nu I'(\varepsilon\nu z)\frac{\partial\hat{H}}{\partial x}\right).\end{aligned} \tag{8.3.97b}$$

Here,

$$I(\zeta) = \sqrt{1 - 2\rho\zeta + \zeta^2}. \tag{8.3.98}$$

In terms of x, we have

$$V(t, f, \alpha) = \max(f - K, 0) + \frac{1}{2}\varepsilon\alpha\sqrt{B(0)B(\varepsilon\alpha z)}\exp\left(\frac{1}{4}\varepsilon^2\rho\nu\alpha b_1 z^2\right)\int_0^{\tau_{\text{ex}}} \hat{H}(\tau, x)\ \mathrm{d}\tau, \tag{8.3.99}$$

with

$$\frac{\partial\hat{H}}{\partial\tau} = \frac{1}{2}\frac{\partial^2\hat{H}}{\partial x^2} - \frac{1}{2}\varepsilon\nu I'(\varepsilon\nu z)\frac{\partial\hat{H}}{\partial x} + \varepsilon^2\alpha^2\left[\frac{1}{4}b_2 - \frac{3}{8}b_1^2\right]\hat{H} + \frac{3}{4}\varepsilon^2\rho\nu\alpha b_1\hat{H}, \tag{8.3.100}$$

for $\tau > 0$, with initial condition $H = \delta(x)$, at $\tau = 0$.

The final step is to define $Q(\tau, x)$ by

$$\hat{H} = I^{\frac{1}{2}}(\varepsilon\nu z(x))\, Q = \left(1 - 2\rho\varepsilon\nu z + \varepsilon^2\nu^2 z^2\right)^{\frac{1}{4}} Q. \tag{8.3.101}$$

Then,[11]

$$\begin{aligned}
\frac{\partial\hat{H}}{\partial\tau} &\rightarrow I^{\frac{1}{2}}(\varepsilon\nu z)\frac{\partial Q}{\partial\tau}, && \text{(8.3.102a)}\\
\frac{\partial\hat{H}}{\partial x} &\rightarrow I^{\frac{1}{2}}(\varepsilon\nu z)\left(\frac{\partial Q}{\partial x} + \frac{1}{2}\varepsilon\nu I'(\varepsilon\nu z)Q\right), && \text{(8.3.102b)}\\
\frac{\partial^2\hat{H}}{\partial x^2} &\rightarrow I^{\frac{1}{2}}\left(\frac{\partial^2 Q}{\partial x^2} + \varepsilon\nu I'(\varepsilon\nu z)\frac{\partial Q}{\partial x} + \varepsilon^2\nu^2\left(\frac{1}{2}I''I + \frac{1}{4}I'^2\right)Q\right). && \text{(8.3.102c)}
\end{aligned}$$

Hence, the option price becomes

$$V(t, f, \alpha) = \max(f - K, 0) + \frac{1}{2}\varepsilon\alpha\sqrt{B(0)B(\varepsilon\alpha z)}I^{\frac{1}{2}}(\varepsilon\nu z)e^{\frac{1}{4}\varepsilon^2\rho\nu\alpha b_1 z^2}\int_0^{\tau_{\text{ex}}} Q(\tau, x)\ \mathrm{d}\tau, \tag{8.3.103}$$

where Q is the solution of

$$\begin{aligned}
\frac{\partial Q}{\partial\tau} &= \frac{1}{2}\frac{\partial^2 Q}{\partial x^2} + \frac{1}{2}\varepsilon\nu I'\frac{\partial Q}{\partial x} + \frac{1}{2}\varepsilon^2\nu^2\left(\frac{1}{2}I''I + \frac{1}{4}I'^2\right)Q - \frac{1}{2}\varepsilon\nu I'\frac{\partial Q}{\partial x}\\
&- \frac{1}{4}\varepsilon^2\nu^2 I'^2 Q + \varepsilon^2\alpha^2\left[\frac{1}{4}b_2 - \frac{3}{8}b_1^2\right]Q + \frac{3}{4}\varepsilon^2\rho\nu\alpha b_1 Q,
\end{aligned} \tag{8.3.104}$$

i.e.,

$$\frac{\partial Q}{\partial\tau} = \frac{1}{2}\frac{\partial^2 Q}{\partial x^2} + \varepsilon^2\nu^2\left(\frac{1}{4}I''I - \frac{1}{8}I'^2\right)Q + \varepsilon^2\alpha^2\left[\frac{1}{4}b_2 - \frac{3}{8}b_1^2\right]Q + \frac{3}{4}\varepsilon^2\rho\nu\alpha b_1 Q, \tag{8.3.105}$$

for $\tau > 0$, with initial condition $Q = \delta(x)$, at $\tau = 0$.

As above, we can replace $I(\varepsilon\nu z)$, $I'(\varepsilon\nu z)$, $I''(\varepsilon\nu z)$ by the constants $I(\varepsilon\nu z_0)$, $I'(\varepsilon\nu z_0)$, $I''(\varepsilon\nu z_0)$, and commit only $\mathcal{O}(\varepsilon)$ errors.

[11] Here, the fact that $\frac{\partial z}{\partial x} = I(\varepsilon\nu z)$ is used to compute the derivatives with respect to x.

Define the constant κ by

$$\kappa := \nu^2 \left(\frac{1}{4} I''(\varepsilon\nu z_0) I(\varepsilon\nu z_0) - \frac{1}{8} \left(I'(\varepsilon\nu z_0) \right)^2 \right) + \alpha^2 \left[\frac{1}{4} b_2 - \frac{3}{8} b_1^2 \right] + \frac{3}{4} \rho\nu\alpha b_1, \tag{8.3.106}$$

where z_0 will be chosen later.

Then, through $\mathcal{O}\left(\varepsilon^2\right)$, we can simplify our problem to

$$\begin{cases} \dfrac{\partial Q}{\partial \tau} = \dfrac{1}{2} \dfrac{\partial^2 Q}{\partial x^2} + \varepsilon^2 \kappa Q, & \text{for } \tau > 0, \quad (8.3.107a) \\ Q = \delta(x), & \text{at } \tau = 0. \quad (8.3.107b) \end{cases}$$

The solution of system (8.3.107) is

$$Q(\tau, x) = \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{x^2}{2\tau}} e^{\varepsilon^2 \kappa \tau}. \tag{8.3.108}$$

Expanding the last exponential $e^{\varepsilon^2\kappa\tau}$ in a Taylor series around $\varepsilon = 0$ yields $1 + \varepsilon^2\kappa\tau + \mathcal{O}\left(\varepsilon^4\right)$. Note that this expanding $\frac{1}{\left(1 - \frac{2}{3}\kappa\varepsilon^2\tau + \ldots\right)^{\frac{3}{2}}}$ in a Taylor series around 0 gives the same result. Hence,

$$Q(\tau, x) = \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{x^2}{2\tau}} \frac{1}{\left(1 - \frac{2}{3}\kappa\varepsilon^2\tau + \ldots\right)^{\frac{3}{2}}}, \tag{8.3.109}$$

through $\mathcal{O}\left(\varepsilon^2\right)$.

This solution yields the option price

$$V(t, f, \alpha) = \max(f - K, 0) + \frac{1}{2} \varepsilon\alpha \sqrt{B(0)B(\varepsilon\alpha z)} I^{\frac{1}{2}}(\varepsilon\nu z) e^{\frac{1}{4}\varepsilon^2\rho\nu\alpha b_1 z^2} \int_0^{\tau_{\text{ex}}} \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{x^2}{2\tau}} e^{\varepsilon^2\kappa\tau} \, \mathrm{d}\tau.$$

Observe that this can be written as

$$V(t, f, \alpha) = \max(f - K, 0) + \frac{1}{2} \frac{f - K}{x} \int_0^{\tau_{\text{ex}}} \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{x^2}{2\tau}} e^{\varepsilon^2\theta} e^{\varepsilon^2\kappa\tau} \, \mathrm{d}\tau, \tag{8.3.110a}$$

where

$$\varepsilon^2\theta := \log\left(\frac{\varepsilon\alpha z}{f - K} \sqrt{B(0)B(\varepsilon\alpha z)} \right) + \log\left(\frac{x I^{\frac{1}{2}}(\varepsilon\nu z)}{z} \right) + \frac{1}{4}\varepsilon^2\rho\nu\alpha b_1 z^2. \tag{8.3.110b}$$

Expanding $\varepsilon^2\theta$ through $\mathcal{O}\left(\varepsilon^2\right)$ yields [4]

$$\varepsilon^2\theta \sim \left(\frac{1}{12} b_2 - \frac{1}{8} b_1^2 \right) \varepsilon^2\alpha^2 z^2 + \left[\frac{1}{12} \frac{I''(\frac{1}{2}\varepsilon\nu z)}{I(\frac{1}{2}\varepsilon\nu z)} - \frac{1}{24} \left(\frac{I'(\frac{1}{2}\varepsilon\nu z)}{I_0(\frac{1}{2}\varepsilon\nu z)} \right)^2 \right] \varepsilon^2\nu^2 z^2 + \frac{1}{4}\varepsilon^2\rho\nu\alpha b_1 z^2. \tag{8.3.111}$$

Using $\frac{z}{x} = 1 + \mathcal{O}(\varepsilon)$ and $I(\varepsilon\nu z) = 1 + \ldots$, we note that $\frac{\theta}{x^2}$ matches

$$\frac{\kappa}{3} = \frac{1}{3}\nu^2 \left(\frac{1}{4} I''(\varepsilon\nu z_0) I(\varepsilon\nu z_0) - \frac{1}{8} \left(I'(\varepsilon\nu z_0) \right)^2 \right) + \frac{1}{3}\alpha^2 \left[\frac{1}{4} b_2 - \frac{3}{8} b_1^2 \right] + \frac{1}{4} \rho\nu\alpha b_1, \tag{8.3.112}$$

and thus

$$e^{\varepsilon^2\kappa\tau} = \frac{1}{\left(1-\frac{2}{3}\kappa\varepsilon^2\tau\right)^{\frac{3}{2}}} = \frac{1}{\left(1-2\varepsilon^2\tau\frac{\theta}{x^2}\right)^{\frac{3}{2}}} + \mathcal{O}\left(\varepsilon^4\right), \tag{8.3.113}$$

through $\mathcal{O}\left(\varepsilon^2\right)$.

Therefore, our option price is

$$V(t,f,\alpha) = \max(f-K,0) + \frac{1}{2}\frac{f-K}{x}\int_0^{\tau_{\text{ex}}} \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{x^2}{2\tau}} e^{\varepsilon^2\theta} \frac{1}{\left(1-2\varepsilon^2\tau\frac{\theta}{x^2}\right)^{\frac{3}{2}}}\,\mathrm{d}\tau. \tag{8.3.114}$$

Changing integration variables to $q := \frac{x^2}{2\tau}$ reduces this to

$$V(t,f,\alpha) = \max(f-K,0) + \frac{|f-K|}{4\sqrt{\pi}}\int_{\frac{x^2}{2\tau_{\text{ex}}}}^{\infty} \frac{e^{-q+\varepsilon^2\theta}}{(q-\varepsilon^2\theta)^{\frac{3}{2}}}\,\mathrm{d}q. \tag{8.3.115}$$

That is, the value of a European call option is given by

$$V(t,f,\alpha) = \max(f-K,0) + \frac{|f-K|}{4\sqrt{\pi}}\int_{\frac{x^2}{2\tau_{\text{ex}}}-\varepsilon^2\theta}^{\infty} q^{-\frac{3}{2}}e^{-q}\mathrm{d}q, \tag{8.3.116a}$$

with

$$\varepsilon^2\theta := \log\left(\frac{\varepsilon\alpha z}{f-K}\sqrt{B(0)B(\varepsilon\alpha z)}\right) + \log\left(\frac{xI^{\frac{1}{2}}(\varepsilon\nu z)}{z}\right) + \frac{1}{4}\varepsilon^2\rho\nu\alpha b_1 z^2, \tag{8.3.116b}$$

through $\mathcal{O}\left(\varepsilon^2\right)$.

Solving problem (8.3.17) for $P(\tau,f,\alpha;K)$ and then substituting this into the pricing formula (8.3.16) to obtain the option value $V(t,f,\alpha)$ under the SABR model, the resulting formulas (8.3.116) we find are awkward and not very useful.

First comparison: Normal model
To cast the results in a more usable form, we re-compute the option price under the normal model

$$\mathrm{d}F = \sigma_N \mathrm{d}W, \tag{8.3.117}$$

and then equate the two prices to determine which normal volatility σ_N needs to be used to reproduce the option price under the SABR model. That is, we find the **implied normal volatility** of the option under the SABR model.

Second comparison: Lognormal model
By doing a second comparison between option prices under the lognormal model

$$\mathrm{d}F = \sigma_B F \mathrm{d}W, \tag{8.3.118}$$

and the normal model (8.4.1), we then convert the implied normal volatlity to the usual implied lognormal (Black-Scholes) volatility, i.e., we quote the option price predicted by the SABR model in terms of the option's implied volatility.

8.4 Normal model

Equations (8.3.116a)-(8.3.116b) form a general formula for the dollar price of the call option under the SABR model. The utility and beauty of this formula is not overwhelmingly apparent. To obtain a useful formula, we convert this dollar price into the equivalent implied volatilities.

To cast the results in a more usable form, we re-compute the option price under the normal model

$$\mathrm{d}F = \sigma_N \mathrm{d}W, \tag{8.4.1}$$

and then equate the two prices to determine which normal volatility σ_N needs to be used to reproduce the option price under the SABR model. That is, we find the **implied normal volatility** of the option under the SABR model.

Suppose we repeated the above analysis for the ordinary **normal model**

$$\mathrm{d}F = \hat{\alpha} C(F) \mathrm{d}W_1, \quad F(0) = f. \tag{8.4.2}$$

Here, we set

$$C(f) = 1 \quad \Rightarrow \quad B(\varepsilon\nu z) = 1, B(0) = 1, b_1 = \frac{B'}{B} = 0 \text{ and } b_2 = \frac{B''}{B} = 0. \tag{8.4.3a}$$

$$\varepsilon\alpha = \sigma_N, \tag{8.4.3b}$$

$$\nu = 0 \quad \Rightarrow \quad \sigma_N \text{ is constant, not stochastic.} \tag{8.4.3c}$$

As $\nu \to 0$ we have to apply L'Hôpital's rule for x to avoid '$\frac{0}{0}$'. This yields

$$x = \lim_{\nu\to 0} \frac{1}{\varepsilon\nu} \frac{\varepsilon\nu z}{\sqrt{1 - 2\rho\varepsilon\nu z + \varepsilon^2\nu^2 z^2}} = \lim_{\nu\to 0} \frac{z}{\sqrt{1 - 2\rho\varepsilon\nu z + \varepsilon^2\nu^2 z^2}} = z. \tag{8.4.4a}$$

Furthermore,

$$\begin{aligned} \varepsilon^2\theta &:= \log\left(\frac{\varepsilon\alpha z}{f-K}\sqrt{B(0)B(\varepsilon\alpha z)}\right) + \log\left(\frac{xI^{\frac{1}{2}}}{z}\right) + \frac{1}{4}\varepsilon^2\rho\nu\alpha b_1 z^2, \\ &= \log\left(\frac{\sigma_N}{f-K}z\right) + \log\left(I^{\frac{1}{2}}\right) + 0 = \log\left(\frac{\sigma_N}{f-K}zI^{\frac{1}{2}}\right) = \log\left(I^{\frac{1}{2}}\right) \\ &= \log\left((1 - 2\rho\varepsilon\nu z + \varepsilon^2\nu^2 z^2)^{\frac{1}{4}}\right) = \frac{1}{4}\log(1) = 0, \end{aligned} \tag{8.4.4b}$$

for $\nu = 0$.

Hence, the lower limit of the integral in formula (8.3.116a)

$$\frac{x^2}{2\tau_{\text{ex}}} - \varepsilon^2\theta \tag{8.4.5a}$$

can be replaced by

$$\frac{(f-K)^2}{2\sigma_N^2\tau_{\text{ex}}}, \tag{8.4.5b}$$

such that the option value for the normal model is exactly

$$V(t, f, \alpha) = \max(f - K, 0) + \frac{|f-K|}{4\sqrt{\pi}} \int_{\frac{(f-K)^2}{2\sigma_N^2\tau_{\text{ex}}}}^{\infty} q^{-\frac{3}{2}} e^{-q} \mathrm{d}q. \tag{8.4.6}$$

Working out the integral and applying the property $N(-a) = 1 - N(a)$ for all a then yields the exact European option price

$$V(t, f, \alpha) = (f - K)N\left(\frac{f-K}{\sigma_N\sqrt{\tau_{\text{ex}}}}\right) + \sigma_N\sqrt{\tau_{\text{ex}}}n\left(\frac{f-K}{\sigma_N\sqrt{\tau_{\text{ex}}}}\right) \tag{8.4.7}$$

for the normal model.

From equation (8.4.6) it is clear that the expression for the option price under the normal model matches the general expression for the option price under the SABR model (8.3.116a)-(8.3.116b) if and only if the lower integral limits are the same, i.e.,

$$\frac{x^2}{2\tau_{\text{ex}}} - \varepsilon^2\theta = \frac{(f-K)^2}{2\sigma_N^2\tau_{\text{ex}}}. \tag{8.4.8}$$

This implies that the normal volatility σ_N satisfies

$$\frac{1}{\sigma_N^2} = \left(\frac{x^2}{2\tau_{\text{ex}}} - 2\varepsilon^2\theta\right)\frac{2\tau_{\text{ex}}}{(f-K)^2} = \frac{x^2}{(f-K)^2}\left(1 - \varepsilon^2\frac{\theta}{x^2}\tau_{\text{ex}}\right). \tag{8.4.9}$$

Taking the square root and expanding

$$\sqrt{\frac{1}{1 - \varepsilon^2\frac{\theta}{x^2}\tau_{\text{ex}}}} \tag{8.4.10}$$

in a Taylor series around $\varepsilon = 0$ shows that the option's **implied normal** (absolute) **volatility** is given by

$$\sigma_N = \frac{f-K}{x}\left(1 + \varepsilon^2\frac{\theta}{x^2}\tau_{\text{ex}} + \ldots\right). \tag{8.4.11}$$

Since $x = z\,(1 + \mathcal{O}(\varepsilon))$, we can rewrite the answer as[12]

$$\sigma_N = \left(\frac{f-K}{z}\right)\left(\frac{z}{x(z)}\right)\left(1 + \varepsilon^2\left(\phi_1 + \phi_2 + \phi_3\right)\tau_{\text{ex}} + \ldots\right), \tag{8.4.12}$$

where

$$\frac{f-K}{z} = \frac{\varepsilon\alpha(f-K)}{\int_K^f \frac{\mathrm{d}f'}{C(f')}} = \left(\frac{1}{f-K}\int_K^f \frac{\mathrm{d}f'}{\varepsilon\alpha C(f')}\right)^{-1}. \tag{8.4.13}$$

On page 28 of their paper [5] Hagan and his coauthors state: "*This factor represents the average difficulty in diffusing from today's forward f to the strike K, and would be present even if the volatility were not stochastic.*" This probably means that this factor is the main multiplier for the volatility and its appearence does not depend on the value of the volvol ν.

The next factor is

$$\frac{z}{x(z)} = \frac{z}{\frac{1}{\varepsilon\nu}\log\left(\frac{\sqrt{1-2\rho\varepsilon\nu z+\varepsilon^2\nu^2z^2}-\rho+\varepsilon\nu z}{1-\rho}\right)} = \frac{\zeta}{\log\left(\frac{\sqrt{1-2\rho\zeta+\zeta^2}-\rho+\zeta}{1-\rho}\right)}, \tag{8.4.14}$$

[12] This can be seen by making a Taylor expansion of the definition of x (8.3.88) around $\varepsilon = 0$ up to and including $\mathcal{O}(\varepsilon)$. This yields $x = z\left(1 + \frac{1}{2}\rho\nu z\varepsilon\right) = z\,(1 + \mathcal{O}(\varepsilon))$.

where

$$\zeta := \varepsilon\nu z = \varepsilon\nu \frac{1}{\varepsilon\alpha}\int_K^f \frac{\mathrm{d}f'}{C(f')} = \frac{\nu}{\alpha}\int_K^f \frac{\mathrm{d}f'}{C(f')} = \frac{\nu}{\alpha}\frac{f-K}{C\left(f_{\text{avg}}\right)}\left(1+\mathcal{O}\left(\varepsilon^2\right)\right). \tag{8.4.15}$$

Here, $f_{\text{avg}} := \sqrt{fK}$ is the geometric average of f and K.[13] On page 28 of [5] Hagan and colleagues claim: "*This factor represents the main effect of the stochastic volatility.*" With this they presumably mean that in the expression for the normal volatility (8.4.12) the lowest order term in ν is driven by f_{avg}.

The coefficients ϕ_1, ϕ_2 and ϕ_3 provide relatively minor corrections. Through $\mathcal{O}\left(\varepsilon^2\right)$ these corrections are

$$\varepsilon\phi_1 = \frac{1}{x^2}\log\left(\frac{\varepsilon\alpha z}{f-K}\sqrt{B(0)B(\varepsilon\alpha z)}\right), \tag{8.4.16a}$$

$$\varepsilon\phi_2 = \frac{1}{x^2}\log\left(\frac{xI^{\frac{1}{2}}}{z}\right), \tag{8.4.16b}$$

$$\varepsilon\phi_3 = \frac{1}{x^2}\varepsilon^2\rho\alpha\nu\frac{B'}{4B}. \tag{8.4.16c}$$

i.e., using $x = z\left(1+\mathcal{O}\left(\varepsilon\right)\right)$ as we did before,

$$\varepsilon\phi_1 = \frac{1}{z^2}\log\left(\frac{\varepsilon\alpha z}{f-K}\sqrt{C(f)C(K)}\right) = \frac{2\gamma_2-\gamma_1^2}{24}\varepsilon^2\alpha^2C^2\left(f_{\text{avg}}\right)+\ldots, \tag{8.4.17a}$$

$$\varepsilon\phi_2 = \frac{1}{z^2}\log\left(\frac{x}{z}\left(1-2\varepsilon\rho\nu z+\varepsilon^2\nu^2z^2\right)^{\frac{1}{4}}\right) = \frac{2-3\rho^2}{24}\varepsilon^2\nu^2+\ldots, \tag{8.4.17b}$$

$$\varepsilon\phi_3 = \frac{1}{z^2}\varepsilon^2\rho\alpha\nu\frac{B'}{4B} = \frac{1}{4z^2}\rho\nu\alpha\gamma_1C\left(f_{\text{avg}}\right)+\ldots, \tag{8.4.17c}$$

where

$$\gamma_1 := \frac{C'\left(f_{\text{avg}}\right)}{C(f_{\text{avg}})} \quad \text{and} \quad \gamma_2 := \frac{C''\left(f_{\text{avg}}\right)}{C\left(f_{\text{avg}}\right)}. \tag{8.4.18}$$

Let us briefly summarize before continuing. Under the **normal model**, the value of a European call option with strike K and exercise date τ_{ex} is given by

$$V(t,f,\alpha) = (f-K)N\left(\frac{f-K}{\sigma_N\sqrt{\tau_{\text{ex}}}}\right)+\sigma_N\sqrt{\tau_{\text{ex}}}\,\varphi\left(\frac{f-K}{\sigma_N\sqrt{\tau_{\text{ex}}}}\right). \tag{8.4.19}$$

For the SABR model,

$$\begin{cases}\mathrm{d}\hat{F} = \varepsilon\hat{\alpha}C(\hat{F})\,\mathrm{d}W_1, & \hat{F}(0)=f, \qquad (8.4.20a)\\ \mathrm{d}\hat{\alpha} = \varepsilon\nu\hat{\alpha}\,\mathrm{d}W_2, & \hat{\alpha}(0)=\alpha, \qquad (8.4.20b)\end{cases}$$

where

$$\mathrm{E}\left[\mathrm{d}W_1\mathrm{d}W_2\right] = \rho\,\mathrm{d}t. \tag{8.4.20c}$$

The value of the call option is given by the same formula, at least through $\mathcal{O}\left(\varepsilon^2\right)$, provided we use the implied normal volatility

$$\sigma_N(K) = \frac{\varepsilon\alpha(f-K)}{\int_K^f\frac{\mathrm{d}f'}{C(f')}}\cdot\left(\frac{\zeta}{\hat{x}(\zeta)}\right)\cdot \tag{8.4.21a}$$
$$\left\{1+\left(\frac{2\gamma_2-\gamma_1^2}{24}\alpha^2C^2\left(f_{\text{avg}}\right)+\frac{2-3\rho^2}{24}\nu^2+\frac{1}{4}\rho\nu\alpha\gamma_1C\left(f_{\text{avg}}\right)\right)\varepsilon^2\tau_{\text{ex}}+\ldots\right\}.$$

[13] Note that the arithmetic average $\frac{f+K}{2}$ could have been used equally well at this order of accuracy.

Here,

$$f_{\rm avg} = \sqrt{fK}, \quad \gamma_1 = \frac{C'\left(f_{\rm avg}\right)}{C\left(f_{\rm avg}\right)}, \quad \gamma_2 = \frac{C''\left(f_{\rm avg}\right)}{C\left(f_{\rm avg}\right)}, \tag{8.4.21b}$$

$$\zeta = \frac{\nu}{\alpha}\frac{f-K}{C\left(f_{\rm avg}\right)}, \quad \hat{x}(\zeta) = \log\left(\frac{\sqrt{1-\rho\zeta+\zeta^2}-\rho+\zeta}{1-\rho}\right). \tag{8.4.21c}$$

The first two factors provide the dominant behaviour, with the remaining factor $1 + [\ldots]\varepsilon^2\tau_{\rm ex}$ usually providing small corrections[14].

One can repeat the analysis for a European put option, or simply use put-call parity as we have seen before in section 4.6. Because in this case we consider a forward f (instead of a stock S), there is no discount factor $e^{-r(T-t)}$ needed. Hence, the put-call parity is given by

$$V^{\rm put} = V^{\rm call} + K - f, \tag{8.4.22}$$

Using the fact that $N(-a) = 1 - N(a)$ and $n(-a) = n(a)$ for all values of a, we obtain the following expression for the value of a European put option with strike K and exercise date $\tau_{\rm ex}$:

$$\begin{aligned} V^{\rm put} &= (f-K)N\left(\frac{f-K}{\sigma_N\sqrt{\tau_{\rm ex}}}\right) + \sigma_N\sqrt{\tau_{\rm ex}}\left(\frac{f-K}{\sigma_N\sqrt{\tau_{\rm ex}}}\right) + K - f, \\ &= (K-f)N\left(\frac{K-f}{\sigma_N\sqrt{\tau_{\rm ex}}}\right) + \sigma_N\sqrt{\tau_{\rm ex}}\left(\frac{K-f}{\sigma_N\sqrt{\tau_{\rm ex}}}\right), \end{aligned} \tag{8.4.23}$$

where the implied volatility σ_N is given by the same formulas (8.4.21a)-(8.4.21c) as the call.

8.5 Lognormal model

By doing a second comparison between option prices under the lognormal model

$$\mathrm{d}F = \sigma_B F \mathrm{d}W, \tag{8.5.1}$$

and the normal model (8.4.1), we then convert the implied normal volatlity to the usual implied lognormal (Black-Scholes) volatility, i.e., we quote the option price predicted by the SABR model in terms of the option's implied volatility. Here $C(F) = F$.

To derive the implied Black volatility, consider Black's model

$$\mathrm{d}F = \varepsilon\sigma_B F \mathrm{d}W, \tag{8.5.2}$$

where we have written the volatility as $\varepsilon\sigma_B$ to stay consistent with the preceding analysis. For Black's model, the value of a European call with strike K and exercise date $\tau_{\rm ex}$ is

$$V_{\rm call} = fN(d_+) - KN(d_-), \tag{8.5.3a}$$

$$V_{\rm put} = V_{\rm call} + D(t_{\rm set})(K-f), \tag{8.5.3b}$$

with

$$d_\pm = \frac{\log(\frac{S}{K}) \pm \frac{1}{2}\varepsilon^2\sigma_B^2\tau_{\rm ex}}{\varepsilon\sigma_B\sqrt{\tau_{\rm ex}}}, \tag{8.5.3c}$$

where we are omitting the overall factor $D(t_{\rm set})$ as before.

[14] Hagan *et al.* claim: "These are corrections of around 1% or so."

We can obtain the implied normal volatility for Black's model by repeating the preceding analysis for the SABR model with $C(f) = f$ and $\nu = 0$.

Setting $C(f) = f$ and $\nu = 0$ in equation (8.4.21a)-(8.4.21c) shows that

$$\sigma_N(K) = \frac{\varepsilon\alpha(f-K)}{\log(f/K)}\left(1 - \frac{1}{24}\varepsilon^2\sigma_B^2\tau_{\text{ex}} + \dots\right) \tag{8.5.4}$$

through $\mathcal{O}\left(\varepsilon^2\right)$.

When we equate the two formulas for $\sigma_N(K)$ (8.5.4) and (8.4.21a), and multiply through by

$$\left[\frac{\log(f/K)}{f-K}\right]\frac{1}{\varepsilon}\left(1 + \frac{1}{24}\varepsilon^2\sigma_B^2\tau_{\text{ex}}\right) \tag{8.5.5}$$

to clear up the left-hand side, we obtain

$$\begin{aligned}\sigma_B &= \frac{\alpha\log(f/K)}{\int_K^f \frac{\mathrm{d}f'}{C(f')}}\cdot\left(\frac{\zeta}{\hat{x}(\zeta)}\right)\cdot\left(1 + \frac{1}{24}\varepsilon^2\sigma_B^2\tau_{\text{ex}}\right)\cdot \\ &\quad \left\{1 + \left(\frac{2\gamma_2 - \gamma_1^2}{24}\alpha^2 C^2\left(f_{\text{avg}}\right) + \frac{2-3\rho^2}{24}\nu^2 + \frac{1}{4}\rho\nu\alpha\gamma_1 C\left(f_{\text{avg}}\right)\right)\varepsilon^2\tau_{\text{ex}} + \dots\right\}.\end{aligned} \tag{8.5.6}$$

Remember that we are only working through $\mathcal{O}\left(\varepsilon^2\right)$, so we can neglect any higher order terms that arise. The final answer, which is an expression for σ_B is obtained by substituting for σ_B its first order approximation [4]

$$\sigma_B = \alpha\frac{C(f_{\text{avg}})}{f_{\text{avg}}}\left(1 + \mathcal{O}(\varepsilon)\right), \tag{8.5.7}$$

so we can replace

$$\left[1 + \varepsilon^2\sigma_B^2\tau_{\text{ex}}\right] \tag{8.5.8}$$

on the right-hand side by

$$\left[1 + \varepsilon^2\left(\frac{1}{24}\alpha^2 C(f_{\text{avg}})^2\frac{1}{f_{\text{avg}}^2}\right)\tau_{\text{ex}}\right] \tag{8.5.9}$$

which can be combined with the other $\varepsilon^2\tau_{\text{ex}}$ terms. Through $\mathcal{O}\left(\varepsilon^2\right)$ this yields

$$\begin{aligned}\sigma_B(K) &= \frac{\alpha\log(f/K)}{\int_K^f \frac{\mathrm{d}f'}{C(f')}}\cdot\left(\frac{\zeta}{\hat{x}(\zeta)}\right)\cdot \\ &\quad \left\{1 + \left(\frac{2\gamma_2 - \gamma_1^2 + \frac{1}{f_{av}^2}}{24}\alpha^2 C^2\left(f_{\text{avg}}\right) + \frac{2-3\rho^2}{24}\nu^2 + \frac{1}{4}\rho\nu\alpha\gamma_1 C\left(f_{\text{avg}}\right)\right)\varepsilon^2\tau_{\text{ex}} + \dots\right\}.\end{aligned} \tag{8.5.10}$$

This is the main result of the paper written by Hagan and colleagues [5].

8.6 Stochastic β model

As originally stated, the SABR model consists of the special case $C(f) = f^\beta$. The model then becomes

$$\begin{cases}\mathrm{d}\hat{F} = \varepsilon\hat{\alpha}\hat{F}^\beta\,\mathrm{d}W_1, & \hat{F}(0) = f, \qquad (8.6.1a)\\ \mathrm{d}\hat{\alpha} = \varepsilon\nu\hat{\alpha}\,\mathrm{d}W_2, & \hat{\alpha}(0) = \alpha, \qquad (8.6.1b)\end{cases}$$

where

$$\mathrm{E}\left[\mathrm{d}W_1\mathrm{d}W_2\right] = \rho\ \mathrm{d}t. \tag{8.6.1c}$$

For $C(f) = f^\beta$, we have

$$\gamma_1 = \frac{C'(f_{\text{avg}})}{C(f_{\text{avg}})} = \frac{\beta f_{\text{avg}}^{\beta-1}}{f_{\text{avg}}^\beta} = \beta f_{\text{avg}}^{-1} \quad \text{and} \tag{8.6.2a}$$

$$\gamma_2 = \frac{C''(f_{\text{avg}})}{C(f_{\text{avg}})} = \beta(\beta-1)\frac{\beta f_{\text{avg}}^{\beta-2}}{f_{\text{avg}}^\beta} = \beta(\beta-1) f_{\text{avg}}^{-2}. \tag{8.6.2b}$$

Substituting this into equation (8.4.21a)-(8.4.21c) shows that the normal volatility for this model is

$$\begin{aligned}
\sigma_N(K) &= \frac{\varepsilon\alpha(f-K)}{\int_K^f f^{-\beta}\mathrm{d}f'}\cdot\left(\frac{\zeta}{\hat{x}(\zeta)}\right)\cdot \\
&\quad \left\{1+\left(\frac{2\beta(\beta-1)f_{\text{avg}}^{-2}-\beta^2 f_{\text{avg}}^{-2}}{24}\alpha^2 f_{\text{avg}}^{2\beta}+\frac{2-3\rho^2}{24}\nu^2+\frac{1}{4}\rho\nu\alpha\beta f_{\text{avg}}^{-1}f_{\text{avg}}^{\beta}\right)\varepsilon^2\tau_{\text{ex}}+\ldots\right\}, \\
&= \frac{\varepsilon\alpha(1-\beta)(f-K)}{f^{1-\beta}-K^{1-\beta}}\cdot\left(\frac{\zeta}{\hat{x}(\zeta)}\right)\cdot \\
&\quad \left\{1+\left(\frac{-\beta(2-\beta)\alpha^2}{24 f_{av}^{2-2\beta}}+\frac{2-3\rho^2}{24}\nu^2+\frac{1}{4}\frac{\rho\nu\alpha\beta}{4f_{av}^{1-\beta}}\right)\varepsilon^2\tau_{\text{ex}}+\ldots\right\},
\end{aligned} \tag{8.6.3a}$$

through $\mathcal{O}\left(\varepsilon^2\right)$ with $f_{\text{avg}} = \sqrt{fK}$ as before and

$$\zeta = \frac{\nu}{\alpha}\frac{f-K}{f_{av}^\beta} \quad \text{and} \quad \hat{x}(\zeta) = \log\left(\frac{\sqrt{1-2\rho\zeta}-\rho+\zeta}{1-\rho}\right). \tag{8.6.3b}$$

This can be simplified by expanding[15]

$$\begin{aligned}
f-K &= \sqrt{fK}\log\left(\frac{f}{K}\right)\left(1+\frac{1}{24}\log^2\left(\frac{f}{K}\right)+\frac{1}{1920}\log^4\left(\frac{f}{K}\right)+\ldots\right), \\
f^{1-\beta}-K^{1-\beta} &= (1-\beta)(fK)^{(1-\beta)/2}\log\left(\frac{f}{K}\right) \\
&\quad \cdot\left(1+\frac{(1-\beta)^2}{24}\log^2\left(\frac{f}{K}\right)+\frac{(1-\beta)^4}{1920}\log^4\left(\frac{f}{K}\right)+\ldots\right).
\end{aligned}$$

Here, terms higher than fourth order can be neglected, because $f-K = \mathcal{O}(\varepsilon)$ and therefore $\log\left(\frac{f}{K}\right)$ is small.

[15] Define $x := \log(f/K)$ and write

$$\begin{aligned}
f-K &= \sqrt{fK}\left(\sqrt{\frac{f}{K}}-\sqrt{\frac{K}{f}}\right) = 2\sqrt{fK}\frac{\left(e^{\frac{1}{2}x}-e^{-\frac{1}{2}x}\right)}{2} = 2\sqrt{fK}\sinh\left(\frac{x}{2}\right), \\
&= \sqrt{fK}x\left(1+\frac{x^2}{3!\cdot 2^3}+\frac{x^4}{5!\cdot 2^4}+\ldots\right) = \sqrt{fK}\log\left(\frac{f}{K}\right)\left(1+\frac{1}{24}\log^2\left(\frac{f}{K}\right)+\frac{1}{1920}\log^4\left(\frac{f}{K}\right)+\ldots\right).
\end{aligned}$$

A similar approach can be used to expand $f^{1-\beta}-K^{1-\beta}$.

Then, the implied normal volatility (8.6.3a) reduces to

$$\begin{aligned}\sigma_N(K) &= \varepsilon\alpha(fK)^{\beta/2}\frac{1+\frac{1}{24}\log^2\left(\frac{f}{K}\right)+\dots}{1+\frac{(1-\beta)^2}{24}\log^2\left(\frac{f}{K}\right)+\dots}\cdot\left(\frac{\zeta}{\hat{x}(\zeta)}\right)\cdot \\ &\quad\left\{1+\left(\frac{-\beta(2-\beta)\alpha^2}{24(fK)^{1-\beta}}+\frac{2-3\rho^2}{24}\nu^2+\frac{1}{4}\frac{\rho\nu\alpha\beta}{(fK)^{(1-\beta)/2}}\right)\varepsilon^2\tau_{\text{ex}}+\dots\right\},\end{aligned} \tag{8.6.5a}$$

where

$$\zeta=\frac{\nu}{\alpha}(fK)^{(1-\beta)/2}\log\left(\frac{f}{K}\right)\quad\text{and}\quad\hat{x}(\zeta)=\log\left(\frac{\sqrt{1-2\rho\zeta+\zeta^2}-\rho+\zeta}{1-\rho}\right). \tag{8.6.5b}$$

Equating the above formula (8.6.5a) for $\sigma_N(K)$ for the SABR model and the expression for $\sigma_N(K)$ for Black's model (8.5.4) and multiplying through by

$$\left[\frac{\log(f/K)}{f-K}\right]\frac{1}{\varepsilon}\left(1+\frac{1}{24}\varepsilon^2\sigma_B^2\tau_{\text{ex}}\right) \tag{8.6.6}$$

yields

$$\begin{aligned}\sigma_B &= \frac{\alpha}{(fK)^{(1-\beta)/2}}\frac{1+\frac{1}{24}\varepsilon^2\sigma_B^2\tau_{\text{ex}}}{1+\frac{(1-\beta)^2}{24}\log^2\left(\frac{f}{K}\right)+\dots}\cdot\frac{\zeta}{\hat{x}(\zeta)}\cdot \\ &\quad\left\{1+\left(\frac{-\beta(2-\beta)\alpha^2}{24(fK)^{1-\beta}}+\frac{2-3\rho^2}{24}\nu^2+\frac{1}{4}\frac{\rho\nu\alpha\beta}{(fK)^{(1-\beta)/2}}\right)\varepsilon^2\tau_{\text{ex}}+\dots\right\}.\end{aligned} \tag{8.6.7}$$

Next σ_B has to be replaced by

$$\sigma_B=\alpha\frac{C(f_{\text{avg}})}{f_{\text{avg}}}\left(1+\mathcal{O}(\varepsilon)\right) \tag{8.6.8}$$

as we did before, such that we can replace

$$1+\frac{1}{24}\varepsilon^2\sigma_B^2\tau_{\text{ex}} \tag{8.6.9}$$

on the right-hand side by

$$1+\frac{1}{24}\varepsilon^2\alpha^2 f_{\text{avg}}^{2\beta-1}\tau_{\text{ex}} \tag{8.6.10}$$

Hence, the expression for the implied Black volatility for the SABR model (8.6.8) becomes[16]

$$\begin{aligned}\sigma_B &= \frac{\alpha}{(fK)^{(1-\beta)/2}}\frac{1}{1+\frac{(1-\beta)^2}{24}\log^2\left(\frac{f}{K}\right)+\dots}\cdot\frac{\zeta}{\hat{x}(\zeta)}\cdot \\ &\quad\left\{1+\left(\frac{(1-\beta)^2\alpha^2}{24(fK)^{1-\beta}}+\frac{2-3\rho^2}{24}\nu^2+\frac{1}{4}\frac{\rho\nu\alpha\beta}{(fK)^{(1-\beta)/2}}\right)\varepsilon^2\tau_{\text{ex}}+\dots\right\}.\end{aligned} \tag{8.6.11}$$

through $\mathcal{O}\left(\varepsilon^2\right)$, where ζ and $\hat{x}(\zeta)$ are given by equation (8.6.5b) as before.

[16] In equation (B.69c) on page 31 of [5], Hagan *et al.* claim that there should also be an ε in the numerator of the first fraction. This must be a typographical mistake, because in equation (2.17a) on page 9 of the same paper [5] no ε occurs.

8.7 Special cases: $\beta = 0$ and $\beta = 1$

There are two special cases we will take a special look at: the stochastic normal model ($\beta = 0$) and the stochastic lognormal model ($\beta = 1$).

Normal model: $\beta = 0$

For the stochastic *normal* model ($\beta = 0$), the implied volatilities of European call and put options are

$$\sigma_N(K) = \varepsilon\alpha \left(1 + \frac{2-3\rho^2}{24}\varepsilon^2\nu^2\tau_{\text{ex}} + \dots\right), \tag{8.7.1a}$$

$$\sigma_B(K) = \varepsilon\alpha \frac{\log\left(\frac{f}{K}\right)}{f-K} \left(\frac{\zeta}{\hat{x}(\zeta)}\right) \left(1 + \left[\frac{\alpha^2}{24fK} + \frac{2-3\rho^2}{24}\nu^2\right]\varepsilon^2\tau_{\text{ex}} + \dots\right), \tag{8.7.1b}$$

through $\mathcal{O}\left(\varepsilon^4\right)$, where

$$\zeta = \frac{\nu}{\alpha}\sqrt{fK}\log\left(\frac{f}{K}\right) \quad \text{and} \quad \hat{x}(\zeta) = \log\left(\frac{\sqrt{1-2\rho\zeta+\zeta^2}-\rho+\zeta}{1-\rho}\right). \tag{8.7.1c}$$

Lognormal model: $\beta = 1$

For the stochastic *lognormal* model ($\beta = 1$), the implied volatilities are

$$\sigma_N(K) = \varepsilon\alpha \frac{f-K}{\log\left(\frac{f}{K}\right)} \left(\frac{\zeta}{\hat{x}(\zeta)}\right) \left(1 + \left[-\frac{\alpha^2}{24} + \frac{\rho\alpha\nu}{4} + \frac{2-3\rho^2}{24}\nu^2\right]\varepsilon^2\tau_{\text{ex}} + \dots\right), \tag{8.7.2a}$$

$$\sigma_B(K) = \varepsilon\alpha \left(\frac{\zeta}{\hat{x}(\zeta)}\right) \left(1 + \left[\frac{\rho\alpha\nu}{4} + \frac{2-3\rho^2}{24}\nu^2\right]\varepsilon^2\tau_{\text{ex}} + \dots\right), \tag{8.7.2b}$$

through $\mathcal{O}\left(\varepsilon^4\right)$, where

$$\zeta = \frac{\nu}{\alpha}\log\left(\frac{f}{K}\right) \quad \text{and} \quad \hat{x}(\zeta) = \log\left(\frac{\sqrt{1-2\rho\zeta+\zeta^2}-\rho+\zeta}{1-\rho}\right). \tag{8.7.2c}$$

8.8 Discussion

The most important assumptions that are made in the paper by Hagan *et al.* [5] are presented below. Furthermore, the method and solution will be discussed in this section.

Method and assumptions
Hagan and his colleagues use singular perturbation techniques to obtain the plain-vanilla option prices implied by the SABR model, and from these the associated implied volatilities. In their book [9] Rebonato, McKay and White explain that "*Implied volatilities are just 'the wrong number to put in the wrong formula to get the right price', so there is no great fundamental meaning in obtaining implied volatilities rather than prices. However, for very good reasons, these 'wrong numbers' have become the common metric in the market place to communicate the prices of options.*"

The SABR model describes a single forward F. The volatility of this forward F is described by the parameter α, which itself follows a stochastic process, while the forward is assumed to follow the CEV process as before in section 6. That is,

$$\begin{cases} \mathrm{d}F = \hat{\alpha} C(F) \mathrm{d}W_1, & F(0) = f, \quad (8.8.1a) \\ \mathrm{d}\hat{\alpha} = \hat{\nu}\hat{\alpha}\ \mathrm{d}W_2, & \hat{\alpha}(0) = \alpha. \quad (8.8.1b) \end{cases}$$

The two Wiener processes are assumed to be correlated by

$$\mathrm{E}\left[\mathrm{d}W_1\ \mathrm{d}W_2\right] = \rho\ \mathrm{d}t. \tag{8.8.2}$$

All the parameters of the model, ν, β and ρ, are assumed to be constants, not functions of time, and there is no mean-reversion in the stochastic process for volatility.

Both the volatility $\hat{\alpha}$ and the volvol $\hat{\nu}$ are assumed to be small. This results in the following scalings:

$$\sigma = \frac{\hat{\alpha}}{\varepsilon}, \qquad \Leftrightarrow \qquad \hat{\alpha} = \varepsilon\sigma, \tag{8.8.3a}$$

$$\nu = \frac{\hat{\nu}}{\varepsilon}, \qquad \Leftrightarrow \qquad \hat{\nu} = \varepsilon\nu. \tag{8.8.3b}$$

The PDE approach does not seem to work very well in this case, because after scaling we obtain PDEs that are difficult to solve. Hence, for the SABR model it is better to follow the procedure of Hagan *et al.*, who directly apply the scalings on the system of stochastic differential equations (8.8.1) and use Kolmogorov equations, as we did in section 8.2 of this thesis.

The resulting expression for the implied volatility under the SABR model is obtained by considering the forward and backward Kolmogorov equations per order in ε, making smart choices for local variables and functions in order to attempt to transform them into an equation that looks like a heat equation, which is easier to solve.

Solution
The SABR model is very well explained in the paper by Hagan *et al.* [5]. Singular perturbation techniques are used to obtain the prices of European options under the SABR model. From these prices we obtain the following closed-form algebraic formula for the implied volatility as a function of todays forward price f and the strike K:

$$\begin{aligned} \sigma_B \quad = \quad & \frac{\alpha}{(fK)^{(1-\beta)/2}} \frac{1}{1 + \frac{(1-\beta)^2}{24}\log^2\left(\frac{f}{K}\right) + \ldots} \cdot \frac{\zeta}{\hat{x}(\zeta)} \cdot \quad (8.8.4a) \\ & \left\{ 1 + \left(\frac{(1-\beta)^2\alpha^2}{24(fK)^{1-\beta}} + \frac{2-3\rho^2}{24}\nu^2 + \frac{1}{4}\frac{\rho\nu\alpha\beta}{(fK)^{(1-\beta)/2}} \right) \varepsilon^2 \tau_{\mathrm{ex}} + \ldots \right\} \end{aligned}$$

through $\mathcal{O}\left(\varepsilon^2\right)$, where ζ and $\hat{x}(\zeta)$ are given by

$$\zeta = \frac{\nu}{\alpha}(fK)^{(1-\beta)/2} \log\left(\frac{f}{K}\right) \quad \text{and} \quad \hat{x}(\zeta) = \log\left(\frac{\sqrt{1 - 2\rho\zeta + \zeta^2} - \rho + \zeta}{1 - \rho} \right). \tag{8.8.4b}$$

Our opinion
Though the SABR model is very well explained in the paper by Hagan *et al.* [5], it took us quite some time and effort to understand all derivations. Fortunately, Patrick Hagan was very helpful in answering our questions via e-mail and explaining the steps we did not immediately understand. This helped us a lot in comprehensing the details of this particular application of perturbation theory.

Chapter 9

Conclusions

This section summarizes the main results and conclusions that we have obtained in this thesis. Also some recommendations for further investigation will be presented below.

This thesis discusses the use of perturbation theory in the context of financial mathematics, in particular on the use of matched asymptotic expansions in option pricing. Our methods are applied to the ordinary Black-Scholes model for illustration, and two more advanced models based on papers by Howison [7] and Hagan *et al.* [5].

Black-Scholes model
A first application of perturbation theory on a financial model has been presented in section 5 in order to show the techniques and complications of the method of asymptotic expansions in a financial context. In this simple example of the Black-Scholes model an exact solution is available, so it is in fact not neccessary to apply the method of asymptotic expansions on this model. However, in case we do apply the method, we can construct two artificial layers, and make smart choices for the local variables, in order to attempt to transform the equations into a heat equation, which can be solved. A nice property of this model is that it is possible to compare the results with the exact solution, to see that it is a very accurate method. Note that this exact solution can be obtained by transforming the Black-Scholes equation into a heat equation (as we have shown in section 5.2).

Fast mean-reverting stochastic volatility model
Howison's paper [7] discusses a fast mean-reverting stochastic volatility model that turns out to have many open ends. In this paper quite a lot of assumptions and simplifications are made. Unfortunately, often the motivation for them is not explicitly given in the paper, and in some cases we even think these assumptions and simplifications are incorrect.

An important simplification is that Howison makes some assumptions about σ-independence of the $\mathcal{O}(1)$ solution V_0. He argues that this choice follows naturally from the behaviour of V_0 in the limits for small and large S, but he doesn't explain this. This simplification has large consequences for the higher order equations and their solutions. If the $\mathcal{O}(1)$ solution would depend on σ, these higher order equations would be completely different and it would make solving them a lot more complicated.

Another important point of discussion is that Howison has chosen to make the derivative with respect to time to be $\mathcal{O}(\varepsilon)$, instead of $\mathcal{O}(1)$. As a consequence, the remaining $\mathcal{O}(1)$ equation does not look like a heat equation anymore, because in the first order approach in equation (7.3.9),

the time derivative has been left out of the problem. The similarities with the application of perturbation theory on the Black-Scholes model have thus dissappeared here. Howison does not explicitely motivate his choice. A reason for this choice might be that it is likely that taking the time derivative into account in the $\mathcal{O}(1)$ problem leads to a more complicated first order solution V_0, which does not only depend on t, but also on εt. In that case, a totally different approach, using multiple timescales, would be needed.

After applying the method of matched asymptotic expansions to the fast mean-reverting stochastic volatility model and following Howison's assumptions, we have obtained the following expression for the three lowest order terms of the solution:

$$\begin{aligned} V(S,\sigma,t) &\sim c_0(S,t) + \varepsilon^{\frac{1}{2}}(T-t)\left(A_{\frac{1}{2},1} + A_{\frac{1}{2},2}\mathcal{D}\right) S^2\left(\mathcal{D}^2-\mathcal{D}\right)c_0 \\ &+ \varepsilon\Big[g_1(\sigma) - \overline{g_1(\sigma)} - (T-t)\left(A_{1,1}+A_{1,2}\mathcal{D}\right) \\ &\quad - \frac{1}{2}(T-t)^2\left(A_{\frac{1}{2},1} + A_{\frac{1}{2},2}\mathcal{D}\left(\mathcal{D}^2-\mathcal{D}\right)\right)\left(A_{\frac{1}{2},1} + A_{\frac{1}{2},2}\mathcal{D}\right)\Big]\left(\mathcal{D}^2-\mathcal{D}\right)c_0, \end{aligned} \tag{9.0.1}$$

where $\mathcal{D} := S\frac{\partial}{\partial S}$. However, since the function $c_0(S,t)$ is still undetermined, this result doesn't seem to be very useful.

Lognormal-normal model

The next model that has been considered, is a lognormal stock process with a normal volatility process, also known as the Schöbel-Zhu model, given by

$$\begin{cases} \mathrm{d}S = \mu S\ \mathrm{d}t + \sigma S\ \mathrm{d}W, & \mathrm{d}W \sim N(0,\mathrm{d}t), \qquad (9.0.2a) \\ \mathrm{d}\sigma = a\ \mathrm{d}t + b\ \mathrm{d}\widetilde{W}, & \mathrm{d}\widetilde{W} \sim N(0,\mathrm{d}t). \qquad (9.0.2b) \end{cases}$$

Here, $a, b \in \mathbb{R}$ are constants and the stochastic processes W and $\widetilde{W}$ have correlation ρ.

This model was supposed to be a first step to the SABR model. We expected it to be easier than SABR, because the drift and volatility of volatility were chosen to be constant. However, after trying to apply the method of asymptotic expansions to this model, unfortunately this appeared to be not that easy. In particular, the inner equations were hard to solve, because these partial differential equations contained coefficients that depend on the time and space parameters. Therefore, we have decided to abandon this model and directly continue with the SABR model.

SABR model

The paper written by Hagan and colleagues [5] examines a new three-parameter stochastic volatility model (the SABR model) that successfully prices back the volatility smile as observed in the market nowadays, and that is commonly used. This resulting expression for the implied volatility under the SABR model is obtained by considering the forward and backward Kolmogorov equations per order in ε, making some smart choices for local variables and functions in order to attempt to transform them into an equation that looks like a heat equation, which is easier to solve. Note that, contrary to Howison's approach, Hagan *et al.* do take into account the first derivative with respect to time in their $\mathcal{O}(1)$ equation.

The main result of this section is the following closed-form algebraic formula for the implied

volatility as a function of todays forward price f and the strike K:

$$\begin{aligned}\sigma_B &= \frac{\alpha}{(fK)^{(1-\beta)/2}} \frac{1}{1+\frac{(1-\beta)^2}{24}\log^2\left(\frac{f}{K}\right)+\dots} \cdot \frac{\zeta}{\hat{x}(\zeta)} \cdot \\ &\quad \left\{1+\left(\frac{(1-\beta)^2\alpha^2}{24(fK)^{1-\beta}}+\frac{2-3\rho^2}{24}\nu^2+\frac{1}{4}\frac{\rho\nu\alpha\beta}{(fK)^{(1-\beta)/2}}\right)\varepsilon^2\tau_{\text{ex}}+\dots\right\}\end{aligned} \tag{9.0.3a}$$

through $\mathcal{O}\left(\varepsilon^2\right)$, where ζ and $\hat{x}(\zeta)$ are given by

$$\zeta = \frac{\nu}{\alpha}(fK)^{(1-\beta)/2}\log\left(\frac{f}{K}\right) \quad \text{and} \quad \hat{x}(\zeta) = \log\left(\frac{\sqrt{1-2\rho\zeta+\zeta^2}-\rho+\zeta}{1-\rho}\right). \tag{9.0.3b}$$

Recommendations
Recommendations for further investigation on these models would be to consider several different scalings and see which one works best.

For example, using multiple-scale methods to find the correct $\mathcal{O}(1)$ equation and solution for Howison's fast mean-reverting volatility model. Also, the σ-independence of the $\mathcal{O}(1)$ solution V_0 should be reconsidered to see if it is possible to obtain a first order approach that does depend on σ.

For the SABR model, also some choices are made for the scalings. Here, a nice suggestion for further investigation would be to see if any other small parameters occur in this model. For example, to examine what happens if (also) $(1-\rho)$ and/or $(\beta-\frac{1}{2}$ are small.

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Appendix A

List of symbols

Greek symbols

Γ: Second order derivative of the option price, one of the Greeks.
γ: Parameter in the CEV model.
Δ: First order derivative of the option price, one of the Greeks.
Used for heding, to determine the amount of stock that has to be in the portfolio.
δ: Dividend.
ε: Stretching parameter.
μ_t: Drift.
σ_t: Volatility.

Latin symbols

B_t: Bond price.
$D(t)$: Discount factor for time t.
K: Strike price.
$N(\cdot)$: Standard normal cumulative distribution function, $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{z^2}{2}} \mathrm{d}z$.
$N(\mu, \sigma)$: Normal distribution with mean μ and variance σ^2.
$n(\cdot)$: Standard normal probability density function, $n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$.
$P(S)$: Payoff.
r_t: Risk-free interest rate.
S_t: Stock price.
T: Expiry time.
t: Time.
V: General option price.
V^{call}: Price of a call option.
V^{put}: Price of a put option.
W_t: Wiener Process.

Appendix B

Order symbols

To define an asymptotic approximation, first the **order symbols**[1] need to be introduced. The reason for this is that we will be interested in how functions behave as a parameter, typically ε, becomes small.

For example, the function $f(\varepsilon) = \varepsilon$ does not converge to zero as fast as $g(\varepsilon) = \varepsilon^2$ when $\varepsilon \to 0$. Therefore, a notation to denote this fact is needed.

Definitions

- $g = \mathcal{O}(f)$ as $\varepsilon \downarrow \varepsilon_0$ means that there are constants k_0 and ε_1 (independend of ε), such that

$$|g(\varepsilon)| \leq k_0 |f(\varepsilon)| \quad \text{for } \varepsilon_0 < \varepsilon < \varepsilon_1. \tag{B.0.1}$$

  We say that g is "**big Oh**" of f as $\varepsilon \downarrow \varepsilon_0$.

- $g = o(f)$ as $\varepsilon \downarrow \varepsilon_0$ means that for every positive δ, there is an ε_2 (independent of ε), such that

$$|g(\varepsilon)| \leq \delta |f(\varepsilon)| \quad \text{for } \varepsilon_0 < \varepsilon < \varepsilon_2. \tag{B.0.2}$$

  We say that g is "**little oh**" of f as $\varepsilon \downarrow \varepsilon_0$.

- $f(\varepsilon) = \mathcal{O}_s\left(g(\varepsilon)\right)$ if $f = \mathcal{O}(g)$ and $f \neq o(g)$ for $\varepsilon \to 0$.

Another useful way to make this determination involves the limit

$$l := \lim_{\varepsilon \downarrow \varepsilon_0} \frac{g(\varepsilon)}{f(\varepsilon)}. \tag{B.0.3}$$

- If this limit l exists and is finite, then $g = \mathcal{O}(f)$ as $\varepsilon \downarrow \varepsilon_0$.
- Similarly, if $l = 0$, then $g = o(f)$ as $\varepsilon \downarrow \varepsilon_0$.

[1] Also referred to as **Landau symbols**.

Appendix C

Example problem in perturbation theory: layer at $x = 1$

In chapter 2 the following example problem is being considered:

$$\varepsilon y'' + 2y' + 2y = 0, \quad \text{for } 0 < x < 1, \tag{C.0.1}$$

with boundary conditions $y(0) = 0$ and $y(1) = 1$.

Step 1: Outer solution
If the layer is chosen to be at $x = 1$, the $\mathcal{O}(1)$ outer solution $y_0(x) = ae^{-x}$ should satisfy the boundary condition at $x = 0$, i.e., $y(0) = 0$. This is only the case if we choose the arbitrary constant a to be equal to zero, such that the outer solution is given by $y_0(x) = 0$.

Step 2: Boundary layer analysis
Inside the layer at $x = 1$ we have a local variable

$$\xi = \frac{x - 1}{\delta(\varepsilon)}. \tag{C.0.2}$$

After substitution of this variable ξ in problem (C.0.1), we obtain

$$\frac{\varepsilon}{\delta(\varepsilon)^2}\frac{\partial^2 y^*}{\partial \xi^2} + \frac{2}{\delta(\varepsilon)}\frac{\partial y^*}{\partial \xi} + 2y^* = 0. \tag{C.0.3}$$

Maximum balance yields $\delta(\varepsilon) = \varepsilon$, such that equation (C.0.3) transforms into

$$\frac{\partial^2 y^*}{\partial \xi^2} + 2\frac{\partial y^*}{\partial \xi} + 2\varepsilon y^* = 0. \tag{C.0.4}$$

Expanding $y^* = y_0^* + \varepsilon y_1^* + \ldots$ gives the following $\mathcal{O}(1)$ equation:

$$\frac{\partial^2 y^*}{\partial \xi^2} + 2\frac{\partial y^*}{\partial \xi} = 0. \tag{C.0.5}$$

The general solution of the $\mathcal{O}(1)$ equation is $y_0^*(\xi) = Ae^{-2\xi} + B$. Here, the boundary condition at $x = 1$ should be satisfied, i.e., at $\xi = \frac{x-1}{\varepsilon} = 0$ we have $y_0^*(0) = 1$. This implies that $A+B = 1$, such that the inner solution becomes $y_0^*(\xi) = A\left(e^{-2\xi} - 1\right) + 1$.

Step 3: Matching
To determine the arbitrary constant A in the inner solution, a matching condition is needed. Letting $\xi \to -\infty$, which means going towards the boundary of the layer, yields $A = 0$, to avoid that the inner solution would become infinitely large. The inner solution thus becomes $y_0^*(\xi) = 1$. Unfortunately, in this case we are not able to match this with the outer solution $y(x) = 0$.

Conclusion
The boundary layer cannot be located at $x = 1$. Therefore, let us asume it to be at $x = 0$.

The importance of matching cannot be overemphasized: It is one of the essential steps. If the inner and outer solution do not match, it is necessary to go back and determine where the error was made. The possibilities where this happens are almost endless. On page 56 of the book written by Holmes [6] there's a list of useful places to start to look.

Appendix D

Feynman-Kac

Richard Feynman and Mark Kac have established a link between partial differential equations (PDEs) and stochastic processes. It offers a method of solving certain PDEs by simulating random paths of a stochastic process. Suppose we are given the PDE:

$$\frac{\partial f}{\partial t} + g(x,t)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 f}{\partial x^2} = 0, \tag{D.0.1}$$

subject to the boundary condition $f(x,T) = \eta(x)$, then the **Feynman-Kac formula** reads:

$$f(x,t) = E\left[\eta(X_T)|\mathcal{F}_t\right]. \tag{D.0.2}$$

Here, X is an Itô process driven by the equation

$$\mathrm{d}X = g(X,t)\ \mathrm{d}t + \sigma(X,t)\ \mathrm{d}W_t, \tag{D.0.3}$$

with W_t a Wiener process and the initial value for $X(t)$ is $X(0) = x$.

Proof of the Feynman-Kac formula[1]
The PDE for $f(x,t)$ is given, so using Itô's formula (4.3.1) on f we obtain:

$$\begin{aligned}
\mathrm{d}f &= \frac{\partial f}{\partial t}\mathrm{d}t + \frac{\partial f}{\partial x}\mathrm{d}x + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\mathrm{d}x^2, \\
&= \left(\frac{\partial f}{\partial t} + g(x,t)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 f}{\partial x^2}\right)\mathrm{d}t + \sigma(x,t)\ \mathrm{d}W_t, \\
&= \sigma(x,t)\ \mathrm{d}W_t.
\end{aligned}$$

Integrating both sides gives

$$\int_t^T \mathrm{d}f = f(X_T,T) - f(x,t) = \int_t^T \sigma(x,t)\frac{\partial f}{\partial x}\ \mathrm{d}W_t. \tag{D.0.4}$$

Taking the expectation, we find

$$f(x,t) = E\left[f(X_T,T)\right] = E\left[\eta(X_T)\right]. \tag{D.0.5}$$

[1] Source: Lecture notes of the TU Delft course "Computational Finance" (WI4154), lecture 04, by Lech A. Grzelak and C.W. Oosterlee.

Appendix E

Notes on L^2 functions

Consider an L^2 function f, and the expectation value of its Itô integral

$$\mathrm{E}\left[\int_t^T f(W_t)\,\mathrm{d}W_t\right], \tag{E.0.1}$$

which is named after Kiyoshi Itô. See figure E.1 below.

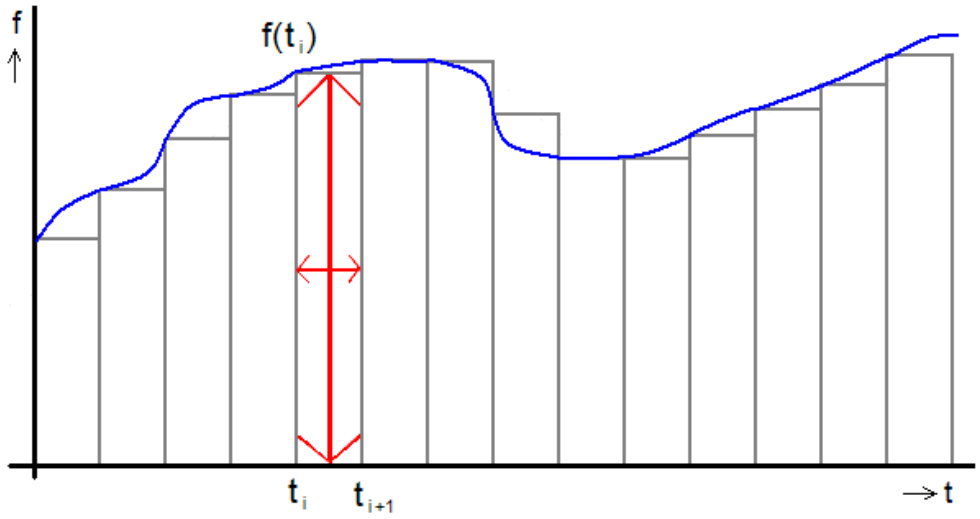

Figure E.1: Itô integral

Using the fact that the increment $W_{t_{i+1}} - W_{t_i}$ is independent of W_{t_i}, such that we can split up the expectation, yields

$$\begin{aligned}\mathrm{E}\left[\int_t^T f(W_t)\,\mathrm{d}W_t\right] &= \mathrm{E}\left[\sum_{t_i} f(W_{t_i})\left(W_{t_{i+1}} - W_{t_i}\right)\right], \\ &= \sum_{t_i} \mathrm{E}\left[f(W_{t_i})\right]\mathrm{E}\left[W_{t_{i+1}} - W_{t_i}\right].\end{aligned}$$

Because $\mathrm{E}\left[f(W_{t_i})\right] < \infty$ and $\mathrm{E}\left[W_{t_{i+1}} - W_{t_i}\right] = 0$, we conclude that

$$\mathrm{E}\left[\int_t^T f(W_t)\,\mathrm{d}W_t\right] = 0. \tag{E.0.2}$$

Appendix F

Martingales

In probability theory, a **martingale** is a stochastic process[1], such that the conditional expected value of an observation at some time t, given all observations up to some earlier time $s < t$, is equal to the observation at that earlier time s. The concept of martingale in probability theory was introduced by Paul Pierre Lévy.

Definition of a martingale
A discrete-time martingale is a discrete-time stochastic process $X_1, X_2, X_3, \ldots$ that satisfies

$$\mathrm{E}(|X_n|) < \infty, \tag{F.0.1}$$

$$\mathrm{E}(X_{n+1}|X_1, \ldots, X_n) = X_n, \tag{F.0.2}$$

for all $n \geq 0$.

Example
Suppose X_n is a gambler's fortune after n tosses of a fair coin, where the gambler wins \$1 if the coin comes up heads and loses \$1 if the coin comes up tails. The gambler's conditional expected fortune after the next trial, given the history, is equal to his present fortune. So this sequence is a martingale.

Properties
Perfectly **tradable** goods, like shares of stock, are subject to the law of one price: they should cost the same amount wherever they are bought. This law requires an efficient and liquid market. Any discrepancy that may exist in pricing perfectly tradable goods, will lead to an *arbitrage* opportunity. Goods that cannot be costlessly traded are not subject to this law.

Here, a nice property is that *martingales* are *tradables*, and *non-martingales* are *non-tradables*. More information on this statement can be found on page 116-118 of the book writen by Baxter and Rennie ([1]).

[1] I.e., a sequence of random variables $X_1, X_2, X_3, \ldots$.

Appendix G

Similarity solutions for PDEs

The technique of similarity solutions is one of the various techniques for reducing a partial differential equation (PDE) into an ordinary differential equation (ODE), or at least turn the original PDE into another PDE, reducing the number of independent variables.

This is an approach that identifies equations for which the solution depends on certain groupings of the independent variables rather than depending on each of the independent variables separately.

First consider the heat equation

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = 0. \tag{G.0.1}$$

We introduce the so-called **dilation transformation**

$$\begin{aligned} z &= \varepsilon^a x, \\ s &= \varepsilon^b t, \\ v(z,s) &= \varepsilon^c u\left(\varepsilon^{-a} z, \varepsilon^{-b} s\right). \end{aligned} \tag{G.0.2}$$

So the heat equation (G.0.1) transforms into

$$\varepsilon^{b-c}\frac{\partial v}{\partial s} - D\varepsilon^{2a-c}\frac{\partial^2 v}{\partial z^2} = 0. \tag{G.0.3}$$

Hence, for $b-c = 2a-c$ (i.e., $b = 2a$), this equation is invariant under this transformation. So if $u(x,t)$ solves the heat equation in the variables x, t, then for z, s, $v(z,s)$ as given $v(z,s)$ solves the heat equation in the variables z, s.
Note that

$$\begin{aligned} vs^{-c/b} &= (\varepsilon^c u)\left(\varepsilon^b t\right)^{-c/b} = ut^{-c/b}, \text{ and} \\ \frac{z}{s^{a/b}} &= \frac{\varepsilon^a x}{(\varepsilon^b t)^{a/b}} = \frac{x}{t^{a/b}}, \end{aligned}$$

such that both groupings of variables are invariant under the transformation (G.0.2) for all choices of a, b, c. This suggests that we look for a solution of the heat equation (G.0.1) that is of the form

$$u = t^{c/b} y(\xi) \quad \text{for} \quad \xi = \frac{x}{t^{a/b}} = \frac{x}{\sqrt{t}}, \quad \text{since} \quad \frac{a}{b} = \frac{1}{2}, \tag{G.0.4}$$

which gives

$$\begin{aligned}\frac{\partial u}{\partial t} &= t^{c/2a-1}\left(\frac{c}{2a}y(\xi)-\frac{\xi}{2}y'(\xi)\right),\\ \frac{\partial u}{\partial x} &= t^{c/2a-1/2}y'(\xi),\\ \frac{\partial^2 u}{\partial x^2} &= t^{c/2a-1}y''(\xi),\end{aligned}$$

such that the heat equation (G.0.1) transforms into

$$t^{c/2a-1}\left[Dy''(\xi)-\frac{c}{2a}y(\xi)+\frac{\xi}{2}y'(\xi)\right]=0. \tag{G.0.5}$$

So the PDE (G.0.1) has indeed been reduced to the ODE (G.0.5). A non-zero solution $y(\xi)$ satisfying (G.0.5) is called a **similarity solution** of the heat equation (G.0.1).

If the heat equation (G.0.1) is satisfied for $x>0$, $t>0$ and if $u(x,t)$ satisfies $u(x,0)=0$ for $x>0$, $u(x,t)\to\infty$ for $x\to\infty$ and $\frac{\partial u}{\partial x}(0,t)=Q$ for $t>0$, then it follows that $\xi\to\infty$ for $x\to\infty$ or $t\to 0$ and $\xi=0$ if $x=0$, because $y(\xi)=t^{-c/2a}u(x,t)$ and $\xi=x/\sqrt{t}$.

For y this gives $y(\infty)=0$ and $\frac{\partial u}{\partial x}(0,t)=t^{-c/2a-1/2}y'(0)=Q$ if and only if $c=a$, because Q is a constant. So the initial boundary value problem for $u(x,t)$ reduces to the following problem for $y(\xi)$:

$$\left\{\begin{array}{rcl} Dy''(\xi)-\frac{1}{2}y(\xi)+\frac{\xi}{2}y'(\xi) &=& 0,\\ y'(0) &=& Q,\\ y(\xi)\to 0 \text{ as } \xi &\to& \infty.\end{array}\right. \tag{G.0.6}$$

If the boundary condition at $x=0$ is given by $u(0,t)=u_0$ then $u(0,t)=t^{c/2a}y(0)=u_0$ if and only if $c=0$, because u_0 is a constant. In this case the initial boundary value problem for $u(x,t)$ reduces to

$$\left\{\begin{array}{rcl} Dy''(\xi)+\frac{\xi}{2}y'(\xi) &=& 0,\\ y(0) &=& u_0,\\ y(\xi)\to 0 \text{ as } \xi &\to& \infty.\end{array}\right. \tag{G.0.7}$$

We can integrate the equation $Dy''(\xi)+\frac{\xi}{2}y'(\xi)=0$ once to obtain $y'(\xi)=c_1e^{-\xi^2/4D}$, such that

$$y(\xi)=c_1\int_0^\xi e^{-\lambda^2/4D}d\lambda+c_2=c_3\ \mathrm{erf}\left(\frac{\xi}{\sqrt{4D}}\right)+c_2, \tag{G.0.8}$$

where $\mathrm{erf}(x):=\frac{2}{\sqrt{\pi}}\int_0^x e^{-\eta^2}d\eta$. This has the property that $\mathrm{erf}(\infty)=1$.

The boundary conditions now lead to

$$\begin{aligned} y(\xi) &= u_0-u_0\ \mathrm{erf}\left(\frac{\xi}{\sqrt{4D}}\right),\\ u(x,t) &= u_0-u_0\ \mathrm{erf}\left(\frac{x}{\sqrt{4Dt}}\right)=u_0\ \mathrm{erfc}\left(\frac{x}{\sqrt{4Dt}}\right),\end{aligned}$$

where $\mathrm{erfc}(s)=1-\mathrm{erf}(s)$.

The solution of problem (G.0.6) is given by

$$y(\xi) = C_1\xi + C_2\left[2\sqrt{\pi D}e^{-\xi^2/4D} + \pi\xi \text{ erf}\left(\frac{\xi}{\sqrt{4D}}\right)\right]. \tag{G.0.9}$$

And since $y'(\xi) = C_1 + C_2\left[\pi \text{ erf}\left(\frac{\xi}{\sqrt{4D}}\right)\right]$, the other conditions are satisfied by

$$\begin{aligned} y(\xi) &= Q\xi - \frac{Q}{\pi}\left[2\sqrt{\pi D}e^{-\xi^2/4D} + \pi\xi \text{ erf}\left(\frac{\xi}{\sqrt{4D}}\right)\right], \\ &= Q\xi \text{ erfc}\left(\frac{\xi}{\sqrt{4D}}\right) - 2Q\sqrt{\frac{D}{\pi}}e^{-\xi^2/4D}. \end{aligned}$$

Appendix H

Green's functions

Green's functions are named after the British mathematician George Green. They are the basic solutions to linear differential equations. A Green's function is a building block that can be used to construct many useful solutions.

Consider differential equations of the form

$$Lu(x) = f(x), \tag{H.0.1}$$

where $L = L(x)$ is a linear differential operator acting on distributions over a subset of the Euclidean space $\mathbb{R}^n$, at a point s.

Definition
A **Green's function**[1] $G(x, s)$ of a linear differential operator $L(x)$ is any solution of

$$LG(x, s) = \delta(x - s), \tag{H.0.2}$$

where $\delta(\cdot)$ is the Dirac delta function.

Motivation
If a function $G(x, s)$ can be found for an operator L, then we multiply equation (H.0.2) by $f(s)$ and integrate with respect to s to obtain

$$\int LG(x, s)f(s)\mathrm{d}s = \int \delta(x - s)f(s)\ \mathrm{d}s = f(x). \tag{H.0.3}$$

By equation (H.0.1), the right hand side of equation (H.0.3) is equal to $Lu(x)$, so (H.0.3) becomes

$$Lu(x) = \int LG(x, s)f(s)\ \mathrm{d}s. \tag{H.0.4}$$

Because operator $L(x)$ is linear and acts on the variable x only, we can take the operator L out of the integration on the right hand side, and obtain

$$Lu(x) = L\left(\int G(x, s)f(s)\ \mathrm{d}s\right), \tag{H.0.5}$$

which implies that the solution $u(x)$ of differential equation (H.0.1) is given by

$$u(x) = \int G(x, s)f(s)\ \mathrm{d}s. \tag{H.0.6}$$

[1] Also referred to as source functions, fundamental solutions, gaussians or propagators of the diffusion equation, or simply diffusion kernels.

Application
Our goal is to find a particular solution of the heat equation

$$u_\tau = \frac{1}{2}u_{xx}, \tag{H.0.7}$$

with initial condition $u(x,0) = \phi(x) = \max(x,0)$, and then to construct all the other solutions, using property 4. in section 5.9, which says that an *integral* of solutions of the heat equation is again a solution.

The particular solution we will look for is the one, denoted $Q(x,\tau)$, that satisfies the special initial condition

$$Q(x,0) = \begin{cases} 1 \text{ for } x > 0, \\ 0 \text{ for } x < 0. \end{cases} \tag{H.0.8}$$

This function for the initial condition is known as the **Heaviside function**. The reason for this choice is that this initial condition does not change under dilation[2].

First we will look for $Q(x,\tau)$ of the form

$$Q(x,\tau) = g(p) \quad \text{where } p = \frac{x}{\sqrt{2\tau}}, \tag{H.0.9}$$

and g is a function of one variable (to be determined).

Clearly, the initial condition (H.0.8) does not change at all under the dilation $x \to \sqrt{a}x$, $t \to at$. So we look for a Q that satisfies the heat equation (H.0.7) and initial condition (H.0.8) and has the form (H.0.9).

Using (H.0.9), we can convert the heat equation (H.0.7) into an ODE for g, by use of the chain rule. This gives

$$\begin{aligned} 0 &= Q_\tau - \frac{1}{2}Q_{xx}, \\ &= -\frac{1}{2\tau}\frac{x}{\sqrt{2\tau}}g'(p) - \frac{1}{2}\frac{1}{2\tau}g''(p), \\ &= -\frac{1}{4\tau}\Big(2pg'(p) + g''(p)\Big), \end{aligned} \tag{H.0.10}$$

such that the ODE becomes

$$g''(p) + 2pg'(p) = 0. \tag{H.0.11}$$

The ODE (H.0.11) is solved using the integrating factor $e^{\int 2p \, \mathrm{d}p} = e^{p^2}$, such that we obtain

$$g'(p) = c_1 e^{-p^2} \tag{H.0.12}$$

and

$$Q(x,\tau) = g(p) = c_1 \int_0^{x/\sqrt{2\tau}} e^{-p^2}\mathrm{d}p + c_2, \quad \text{for } \tau > 0. \tag{H.0.13}$$

Using initial condition (H.0.8), we can express the limits as follows.

$$\text{If} \quad x > 0, \quad 1 = \lim_{t\downarrow 0} Q(x,\tau) = c_1 \int_0^\infty e^{-p^2}\mathrm{d}p + c_2 = c_1\frac{\sqrt{\pi}}{2} + c_2. \tag{H.0.14}$$

$$\text{If} \quad x < 0, \quad 0 = \lim_{t\downarrow 0} Q(x,\tau) = -c_1 \int_{-\infty}^0 e^{-p^2}\mathrm{d}p + c_2 = -c_1\frac{\sqrt{\pi}}{2} + c_2. \tag{H.0.15}$$

[2]See appendix G for the definition of a dilation transformation.

This determines the coefficients

$$\begin{cases} c_1 = \frac{1}{\sqrt{\pi}}, \\ c_2 = \frac{1}{2}. \end{cases}$$

Therefore $Q(x,\tau)$ is the function

$$Q(x,\tau) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{2\tau}} e^{-p^2} \mathrm{d}p, \quad \text{for } \tau > 0. \tag{H.0.16}$$

After checking, we find that solution (H.0.16) does indeed satisfy the heat equation (H.0.7), initial condition (H.0.8) and has the form (H.0.9).

Now define[3]

$$G(x,\tau) = \frac{\partial Q}{\partial x} = \frac{1}{2\sqrt{\pi k \tau}} e^{-x^2/2\tau}, \quad \text{for } \tau > 0. \tag{H.0.17}$$

and

$$u(x,\tau) = \int_{-\infty}^{\infty} G(x-y,\tau)\phi(y)\ \mathrm{d}y, \quad \text{for } \tau > 0. \tag{H.0.18}$$

By property 4. in section 5.9, the integral $u(x,\tau)$ of solution $G(x,\tau)$ is a different solution of the heat equation (H.0.7). We even claim that $u(x,\tau)$ is the unique solution of (H.0.7) satisfying the initial condition $u(x,0) = \phi(x)$.

Hence, after substituting (H.0.17) and the initial condition $\phi(x) = \max(x,0)$ into equation (H.0.18) we find that the solution is given by

$$\begin{aligned} u(x,\tau) &= \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} e^{-(x-y)^2/2\tau} \max(y,0)\ \mathrm{d}y, \quad \text{for } \tau > 0. \\ &= \frac{1}{\sqrt{2\pi\tau}} \int_0^{\infty} y e^{-(x-y)^2/2\tau}\ \mathrm{d}y, \quad \text{for } \tau > 0. \end{aligned}$$

Now transform (x,y,τ) into (x,z,τ) by using

$$z = \frac{x-y}{\sqrt{\tau}}, \tag{H.0.19}$$

such that we can replace y by $y = x - \sqrt{\tau}z$. Assuming x and τ to be constant with respect to z, we have $\mathrm{d}z = -\frac{1}{\sqrt{\tau}}\mathrm{d}y$. This gives

$$\begin{aligned} u(x,\tau) &= -\frac{1}{\sqrt{2\pi}} \int_{x/\sqrt{\tau}}^{-\infty} (x - \sqrt{\tau}z) e^{-z^2/2}\ \mathrm{d}z, \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x/\sqrt{\tau}} (x - \sqrt{\tau}z) e^{-z^2/2}\ \mathrm{d}z, \\ &= \frac{x}{\sqrt{2\pi}} \int_{-\infty}^{x/\sqrt{\tau}} e^{-z^2/2}\ \mathrm{d}z - \frac{\sqrt{\tau}}{\sqrt{2\pi}} \int_{-\infty}^{x/\sqrt{\tau}} z e^{-z^2/2}\ \mathrm{d}z, \\ &= \frac{x}{\sqrt{2\pi}} \int_{-\infty}^{x/\sqrt{\tau}} e^{-z^2/2}\ \mathrm{d}z - \frac{\sqrt{\tau}}{\sqrt{2\pi}} \int_{-\infty}^{x/\sqrt{\tau}} z e^{-z^2/2}\ \mathrm{d}z, \quad \text{for } \tau > 0. \end{aligned} \tag{H.0.20}$$

[3]Note that $\frac{\partial Q}{\partial x}$ is the **Dirac delta function**, because $Q(x,0)$ is equal to the Heaviside function.

The second part of $u(x,\tau)$ we can compute by partial integration.

$$\int_{-\infty}^{x/\sqrt{\tau}} z e^{-z^2/2}\,\mathrm{d}z = \int_{-\infty}^{x/\sqrt{\tau}} e^{-z^2/2}\,\mathrm{d}\left(\frac{z^2}{2}\right) = \left[-e^{z^2/2}\right]_{-\infty}^{x/\sqrt{\tau}}.$$

So the solution (H.0.20) indeed becomes

$$\begin{aligned} u(x,\tau) &= \frac{x}{\sqrt{2\pi}}\int_{-\infty}^{x/\sqrt{\tau}} e^{-z^2/2}\,\mathrm{d}z - \frac{1}{\sqrt{2\pi}}\left[-e^{z^2/2}\right]_{-\infty}^{x/\sqrt{\tau}}, \\ &= \frac{x}{\sqrt{2\pi}}\int_{-\infty}^{x/\sqrt{\tau}} e^{-z^2/2}\,\mathrm{d}z + \sqrt{\tau}e^{x^2/2\tau}, \\ &= x\ N\left(\frac{x}{\sqrt{\tau}}\right) + \sqrt{\tau}\ n\left(\frac{x}{\sqrt{\tau}}\right), \quad \text{for } \tau > 0. \end{aligned} \tag{H.0.21}$$

Thus, using Green's functions we find exactly the same solution as we already found in (5.9.10).

Appendix I

Hermitian adjoint

In mathematics, specifically in functional analysis, each linear operator on a Hilbert space has a corresponding **adjoint operator**. The adjoint of an operator A is also sometimes called the **Hermitian adjoint** (after Charles Hermite) of A and is denoted by A^*.

One can show that there exists a unique continuous linear operator $A^* : H \to H$ with the following property:

$$< Ax, y > \;\; = \;\; < x, A^*y > \quad \text{for all } x, y \in H. \tag{I.0.1}$$

Here $< \cdot, \cdot >$ denotes the inner product, which is defined as follows:

$$< f, g > := \int_a^b f(t)g(t)\ \mathrm{d}t. \tag{I.0.2}$$

Immediate properties:

- $A^{**} = A$.
- If A is invertible, so is A^*. Then, $(A^*)^{-1} = (A^{-1})^*$.
- $(A + B)^* = A^* + B^*$.
- $(\lambda A)^* = \lambda^* A^*$, where λ^* denotes the complex conjugate of the complex number λ.
- $(AB)^* = B^* A^*$.

Application
Consider the operator

$$\mathcal{L}_0 = \frac{1}{2}\zeta^2 \frac{\partial^2}{\partial \sigma^2} + m\frac{\partial}{\partial \sigma}. \tag{I.0.3}$$

We will compute its adjoint by using the definition of the inner product

$$< \mathcal{L}_0 u, v > \;\; = \int_{-\infty}^{\infty} \left(\frac{1}{2}\zeta^2 \frac{\partial^2 u}{\partial \sigma^2} + m\frac{\partial u}{\partial \sigma} \right) v\ \mathrm{d}x. \tag{I.0.4}$$

After applying integration by parts twice, this gives

$$< \mathcal{L}_0 u, v > \;\; = \int_{-\infty}^{\infty} u \left(\frac{1}{2}\zeta^2 \frac{\partial^2 v}{\partial \sigma^2} - m\frac{\partial v}{\partial \sigma} \right)\ \mathrm{d}x = \;\; < u, \mathcal{L}_0^* v > . \tag{I.0.5}$$

Here,

$$\mathcal{L}_0^* = \frac{1}{2}\zeta^2 \frac{\partial^2}{\partial \sigma^2} - m\frac{\partial}{\partial \sigma} \tag{I.0.6}$$

is called the adjoint operator of $\mathcal{L}_0$.

Appendix J

Fredholm Alternative

Consider a differential equation

$$\mathcal{L}(u) = f, \tag{J.0.1}$$

where $\mathcal{L}$ is an operator that operates on u.

First we expand

$$u = \sum_n a_n \phi_n, \tag{J.0.2}$$

where ϕ_n are the eigenvectors of u and a_n are coefficients that are yet undetermined.

Because ϕ_n are the eigenvectors of u, we have

$$\mathcal{L}(\phi_n) = -\lambda_n \phi_n, \tag{J.0.3}$$

with λ_n are the corresponding eigenvalues.

Substitution into the differential equation yields

$$\mathcal{L}(\sum_n a_n \phi_n) = f, \tag{J.0.4}$$

which gives

$$-\sum_n a_n \lambda_n \phi_n = f. \tag{J.0.5}$$

Multiplication by a function ϕ_m, which is orthogonal to ϕ_n, and integration over a domain D results in

$$-a_m \lambda_m \int_D \phi_m^2 \mathrm{d}V = \int_D f \phi_m \mathrm{d}V. \tag{J.0.6}$$

Now the **Fredholm Alternative** (named after the Swedish mathematician Ivar Fredholm) states that the following 3 situations are possible:

- If $\lambda_m \neq 0$ for all m, then the coefficients a_m are unique, and the solution can also be determined uniquely.

In case there is a $\lambda_m = 0$, then:

- If $\int_D f \phi_m \mathrm{d}V \neq 0$, there is no solution.
- If $\int_D f \phi_m \mathrm{d}V = 0$, then the coefficients a_m are not determined, and there are infinitely many solutions.

Appendix K

Dimensional analysis

Dimensional analysis is a conceptual tool often applied to understand physical situations involving certain (physical) quantities. It is routinely used by mathematicians, statisticians, physical scientists and engineers to check the plausibility of derived equations and computations.

The **Buckingham** Π **theorem** is a key theorem in dimensional analysis. The theorem loosely states that if we have a physically meaningful equation involving a certain number, n, of physical variables, and these variables are expressible in terms of k independent fundamental physical quantities, then the original expression is equivalent to an equation involving a set of $p = n - k$ dimensionless variables constructed from the original variables: it is a scheme for nondimensionalization. This provides a method for computing sets of dimensionless parameters from the given variables, even if the form of the equation is still unknown. However, the choice of dimensionless parameters is not unique: Buckingham's theorem only provides a way of generating sets of dimensionless parameters, and will not choose the most 'physically meaningful'.

Application
On page 11 of [7] Howison claims that the value of an option satisfies the following nonlinear equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV + \lambda\sigma^2 S^2 \left(\frac{\partial^2 V}{\partial S^2}\right)^2 + \frac{1}{2}\lambda^2\beta^2\sigma^2 S^4 \left(\frac{\partial^2 V}{\partial S^2}\right)^3 = 0. \tag{K.0.1}$$

If we define $\mathcal{L}_{BS}$ as the Black-Scholes differential operator, given by

$$\mathcal{L}_{BS} = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + rS\frac{\partial}{\partial S} - r, \tag{K.0.2}$$

we can write (K.0.1) as

$$\mathcal{L}_{BS}V + \lambda\sigma^2 S^2 \left(\frac{\partial^2 V}{\partial S^2}\right)^2 + \frac{1}{2}\lambda^2\beta^2\sigma^2 S^4 \left(\frac{\partial^2 V}{\partial S^2}\right)^3 = 0. \tag{K.0.3}$$

Let us express all dimensions in the problem in terms of the dimensions of K and t, as follows:

$$\begin{cases} [V] &= [K], \\ [S] &= [K], \\ [r] &= \left[t^{-1}\right], \\ [\sigma] &= \left[t^{-\frac{1}{2}}\right], \\ [\beta] &= 1, \\ [\lambda] &= 1, \end{cases} \tag{K.0.4}$$

we find that

$$[\mathcal{L}_{BS}V] = \left[\frac{K}{t}\right] + \left[\frac{1}{t}K^3\frac{1}{K^2}\right] + \left[\frac{1}{t}K\frac{1}{K}\right] - \left[\frac{K}{t}\right] = \left[\frac{K}{t}\right], \quad \text{(K.0.5)}$$

$$\left[\lambda\sigma^2 S^2\left(\frac{\partial^2 V}{\partial S^2}\right)^2\right] = \left[1\cdot\frac{1}{t}K^2\left(\frac{K}{K^2}\right)^2\right] = \left[\frac{1}{t}\right], \quad \text{(K.0.6)}$$

$$\left[\frac{1}{2}\lambda^2\beta^2\sigma^2 S^4\left(\frac{\partial^2 V}{\partial S^2}\right)^3\right] = \left[1\cdot\frac{1}{t}K^4\left(\frac{K}{K^2}\right)^3\right] = \left[\frac{K}{t}\right]. \quad \text{(K.0.7)}$$

So the terms of (K.0.3) have the following dimensions

$$\left[\frac{K}{t}\right] + \left[\frac{1}{t}\right] + \left[\frac{K}{t}\right]. \quad \text{(K.0.8)}$$

Dimensionally, this is incorrect, we cannot sum up terms that have different dimensions.

Suggestion
If we replace equation (K.0.1) by

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV + \lambda\sigma^2\mathbf{S^3}\left(\frac{\partial^2 V}{\partial S^2}\right)^2 + \frac{1}{2}\lambda^2\beta^2\sigma^2 S^4\left(\frac{\partial^2 V}{\partial S^2}\right)^3 = 0, \quad \text{(K.0.9)}$$

the dimensions are correct. The S^2 indeed turns out to be a typographical error when it was copied from page 24 of [8]. If we replace it by S^3, we obtain equation (K.0.9), that is correct.

Appendix L

The Dirac Delta

The **Dirac delta** is a mathematical construct introduced by theoretical physicist Paul Dirac. Informally, it is a generalized function representing an infinitely sharp peak bounding unit area: a 'function'[1] $\delta(x)$ that has the value zero everywhere, except at $x = 0$ where its value is infinitely large in such a way that its total integral is 1:

$$\forall_{x\neq 0}\ \delta(x) = 0 \qquad \text{and} \qquad \int_{-\infty}^{\infty} \delta(x)\ \mathrm{d}x = 1. \tag{L.0.1}$$

In section 8.3 we use the fact that

$$\delta(\varepsilon x) = \frac{\delta(x)}{\varepsilon}, \tag{L.0.2}$$

which will be proven in this section.

Proof:
Using the property $\forall_{x\neq 0}\ \delta(x) = 0$, also

$$\forall_{x\neq 0} |\varepsilon| \delta(\varepsilon x) = 0. \tag{L.0.3}$$

From this, it follows that

$$\int_{-\infty}^{\infty} |\varepsilon| \delta(x)\ \mathrm{d}x = \int_{-\infty}^{\infty} \delta(\varepsilon x)\ \mathrm{d}(\varepsilon x) = \int_{-\infty}^{\infty} |\varepsilon| \delta(y)\ \mathrm{d}y = 1. \tag{L.0.4}$$

Conclusion:

$$\delta(x) = |\varepsilon| \delta(\varepsilon x), \tag{L.0.5}$$

and thus

$$\delta(\varepsilon x) = \frac{\delta(x)}{|\varepsilon|}. \tag{L.0.6}$$

In section 8.3 we will use this property without taking the absolute value of ε in the denominator. This is allowed, because we have $\varepsilon > 0$.

□

[1] The Dirac delta is not strictly a function. While for many purposes it can be manipulated as such, formally it can be defined as a distribution that is also a measure.

Appendix M

Near-identity transformation

Definition
Suppose that a solution $x_\varepsilon(t)$ can be approximated by $y_\varepsilon(t)$ for $t \geq 0$. If for $t \geq 0$

$$x_\varepsilon(t) - y_\varepsilon(t) = \mathcal{O}\left(\varepsilon^m\right), \quad 0 \leq t\varepsilon^n \leq C, \tag{M.0.1}$$

with m, n, C constants independent of ε, we call $y_\varepsilon(t)$ an $\mathcal{O}\left(\varepsilon^m\right)$ approximation of $x_\varepsilon(t)$ on the **timescale** $\frac{1}{\varepsilon^n}$.

Application
Consider the n-dimensional equation in the standard form

$$\dot{x} = \varepsilon f(t,x) + \varepsilon^2 g(t,x) + \varepsilon^3 R(t,x,\varepsilon), \tag{M.0.2}$$

in which the vector fields f and g are T-periodic in t with averages f^0 and g^0.

Next, define the vectorfield

$$u(t,x) := \int_0^t \left(f(s,x) - f^0(x)\right) \mathrm{d}s - a(x), \tag{M.0.3}$$

where $f(s,x) - f^0(x)$ has average zero, but this does not hold necessarily for the integral. The function $a(x)$ is chosen such that $u^0(x)$ (i.e., the average of $u(t,x)$) vanishes.

We now introduce the **near-identity transformation**

$$x(t) := w(t) + \varepsilon u(t, w(t)). \tag{M.0.4}$$

We call this "near identity" as $x(t) - w(t) = \mathcal{O}(\varepsilon)$ for $t \geq 0$ This will be used to simplify equation (M.0.2). The near identity transformation is also called the *averaging* or *normalising transformation*. Substituting this into the equation for x (M.0.2), we obtain

$$\begin{aligned} & \dot{w}(t) + \varepsilon \frac{\partial u}{\partial t}(t, w(t)) + \varepsilon \frac{\partial u}{\partial w}(t, w(t))\dot{w}(t) \\ = \;& \varepsilon f\left((t, w(t) + \varepsilon u(t, w(t))\right) + \varepsilon^2 g\left((t, w(t) + \varepsilon u(t, w(t))\right) + \varepsilon^3 \ldots . \end{aligned} \tag{M.0.5}$$

Using the definition of $u^1(t,x)$, the left-hand side of equation (M.0.5) becomes

$$\begin{aligned} \text{LHS} &= \dot{w}(t) + \varepsilon \nabla \cdot u(t, w(t))\dot{w}(t) + \varepsilon \left(f(t,w) - f^0(w)\right), \\ &= \left(I + \varepsilon \nabla \cdot u(t, w(t))\right) \dot{w}(t) + \varepsilon f(t,w) - \varepsilon f^0(w). \end{aligned} \tag{M.0.6}$$

Here, I is the $n \times n$ identity matrix, and with $\nabla \cdot f(t,x)$ we denote the derivative with respect to x only, this is an $n \times n$ matrix.

Because u is uniformly bounded, so is $\nabla \cdot u(t, w(t))$, and we may invert the matrix $(I + \varepsilon \nabla \cdot u(t, w(t)))$ to obtain

$$(I + \varepsilon \nabla \cdot u(t, w(t)))^{-1} = I - \varepsilon \nabla \cdot u(t, w(t)) + \mathcal{O}\left(\varepsilon^2\right), \quad t \geq 0. \tag{M.0.7}$$

Expanding f and g, we obtain

$$\begin{aligned} f\left((t, w(t) + \varepsilon u(t, w(t))\right) &= f(t,w) + \varepsilon \nabla \cdot f(t,w) u(t,w) + \ldots, & \text{(M.0.8a)} \\ g\left((t, w(t) + \varepsilon u(t, w(t))\right) &= g(t,w) + \varepsilon \nabla \cdot g(t,w) u(t,w) + \ldots, & \text{(M.0.8b)} \end{aligned}$$

such that

$$\dot{w}(t) = \varepsilon f^0(w) + \varepsilon^2 \nabla \cdot f(t,w) u(t,w) + \varepsilon^2 g(t,w) + \varepsilon^2 f^0(w) \nabla \cdot u(t,w) + \mathcal{O}\left(\varepsilon^3\right). \tag{M.0.9}$$

Put $f_1(t,w) := \nabla \cdot f(t,w) u(t,w) - f^0(w) \nabla \cdot u(t,w)$, with average f_1^0. This yields

$$\dot{w}(t) = \varepsilon f^0(w) + \varepsilon^2 f_1(t,w) + \varepsilon^2 g(t,w) + \mathcal{O}\left(\varepsilon^3\right), \tag{M.0.10}$$

such that after averaging we have

$$\dot{v}(t) = \varepsilon f^0(v) + \varepsilon^2 f_1^0(v) + \varepsilon^2 g^0(v) + \mathcal{O}\left(\varepsilon^3\right). \tag{M.0.11}$$

The average of $\dot{w}(t)$ is denoted as $\dot{v}(t)$, for notational convenience.

We can prove that

$$x(t) = v(t) + \varepsilon u(t, v(t)) + \mathcal{O}\left(\varepsilon^2\right) \tag{M.0.12}$$

on the timescale $\frac{1}{\varepsilon}$.