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A mild-slope formulation based on Weyl rule of association with application to coastal wave modelling

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ABSTRACT

Weyl rule of association, proposed by Hermann Weyl for quantum mechanics applications (Weyl, 1931), can be used to associate between the dispersion relation of water waves and a non-local pseudo-differential operator. The central result of this study is that this operator correctly approximates the Dirichlet-to-Neumann operator derived for linear waves over a slowly varying bathymetry. This opens the door to a formal use of Weyl's operational calculus, and consequently, allowing straightforward derivations and generalizations of water waves' models over mild slopes. Specifically, within the framework of linear wave theory, the formulation based on Weyl rule of association provides a generalized mild-slope model which does not impose a limit on the spectral width. Most significantly, the mild-slope formulation based on Weyl rule of association allows to derive a general linear kinetic equation for which the widely used energy balance equation (the central equation of forecasting models such as SWAN and WAVEWATCH) serves as a special case. This result not only provides a formal link between the deterministic description (i.e., Euler equations) and the stochastic description (i.e., the energy balance equation), but also establishes the theoretical foundations for the statistical description of bathymetry-induced wave interferences. Such a statistical description is especially important over coastal waters, where through the interaction with the bathymetry, waves are rapidly scattered and tend to form focal zones and associated interference patterns.

1. Introduction

Wave dynamics in coastal waters are characterized by a rich set of phenomena (e.g., shoaling, refraction, breaking etc.) that are triggered by the wind and through nonlinear wave-wave interactions and the interactions of waves with variable bottom topography and ambient currents. The prediction of these complex dynamics is challenging and highly important to coastal communities and municipalities, as it force nearshore circulation (e.g., [1–4]) and sediment transport processes (e.g. [5,6]), as well as controlling shipping operations and associated downtime, and coastal safety through beach and dune erosion and potential inundation (e.g., [7,8]).

Wave prediction in coastal waters is commonly relies on phase-averaged spectral models (e.g. SWAN model, [9]), which solve the so-called energy balance equation. This central equation is derived based on the premise that over

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relatively small-scales, wave fields can be represented as a superposition of statistically independent plane waves. Weak inhomogeneity is introduced over larger scales through slow variations of the spectral energy density. This study focuses on the statistical inhomogeneity induced by variations in the medium (due to slowly varying currents and bathymetry). Medium-induced wave inhomogeneity is represented by the propagation part of the energy balance equation, written as

$$\partial_t E + \nabla_{\boldsymbol{x}} \cdot (\boldsymbol{C}_x E) + \nabla_{\boldsymbol{k}} \cdot (\boldsymbol{C}_k E) = 0, \tag{1}$$

where the two energy flux terms (written on the left-hand-side of (1)), describing the transport of energy over the spatial and the spectral domains with flux velocities $\boldsymbol{C}_x$ and $\boldsymbol{C}_k$, respectively. Other wave processes (e.g. nonlinear wave evolution, wind generation, wave breaking, etc.) are ignored here and are usually represented by an additional source term on the right-hand-side of (1). Note that the omission of this source term limits the conditions for which (1) is applicable. For example, neglecting nonlinear and breaking effects limits the applicability of (1) to wave fields characterized by rather low wave steepness in deep to intermediate waters, or to waves which are characterized by small height to water depth ratio in shallow waters (see, e.g., Dingemans [10]).

The formulation of (1) is based on the well-known result of the Wentzel–Kramers–Brillouin (WKB) approximation (e.g. [10,11]), which provides a phase-averaged (slowly varying) description of individual wave components over slowly varying media. However, in contrast to the WKB representation, which follows the slow evolution of each wave component separately, the expression given by (1) shows a fully Eulerian representation (often referred to as the phase space representation) based on the independent spatial and spectral variables $\boldsymbol{x}$ and $\boldsymbol{k}$.

The relation between the two representations is obtained in the limit $N \to \infty$, where N is the number of wave components. This relation was argued by Hasselmann [12] based on the wave-particle analogy. Specifically, over large-scales, Hasselmann [12] proposed that wave fields can be seen as a superposition of large number of non-interacting wave packets whose dimension is much smaller than the characteristic scale of medium variations. The elegant result of this representation is that the large-scale dynamics of these wave packets are completely analogous to the dynamics of a non-colliding system of classical particles. Consequently, the so-called collisionless Boltzmann equation (which shows exactly the same structure of (1)) becomes applicable to describe the energy density of such wave packets' systems (see additional details in [13]). Alternatively, (1) can be derived directly, based on the following explicit relation between the energy density, E, and the individual energy components, E_j, of the N wave packets' representation:

$$E(\boldsymbol{x}, \boldsymbol{k}, t) d\boldsymbol{x} d\boldsymbol{k} = \sum_j E_j \Delta(\boldsymbol{x}, \boldsymbol{x}_j, \boldsymbol{k}, \boldsymbol{k}_j) \tag{2}$$

where the function Δ is equal 1 if the phase space location of wave packet j is found within the volume $dV = d\boldsymbol{x}d\boldsymbol{k}$ centred at $(\boldsymbol{x}, \boldsymbol{k})$, and is equal 0 otherwise. Taking the time derivative of both sides of (2) and using the canonical equations governing the phase space trajectories of the wave packets (e.g., [10]) eventually leads to the phase-averaged formulation in (1) (see details in, e.g., Willebrand [14],Hertzog et al. [15],Muraschko et al. [16]). However, the particle-like representation underlying the above derivation alternatives of (1) ignores the wave-like behaviour of the wave packets (note however that [17] suggested a derivation starting with a similar representation of many slowly varying wave components but without invoking a "number density like" relation, e.g. (2), which principally could have allowed him to preserve wave-like effects). Consequently, the contributions of wave interference, which are statistically obtained by the cross-correlations of different wave components, is inherently overlooked by the existing derivations, leading to an incomplete and inconsistent theoretical foundation of the present phase-averaged approach.

Statistically, the importance of the cross-correlations is determined by the ratio between the second-order correlation and medium variation scales (e.g., [18]). In the open ocean, where medium inhomogeneity is typically $O(100)$ km and the wave spectrum is relatively broad, this measure would rarely indicate a significant value and the particle-like representation of the wave components is therefore justified. However, for conditions where the medium is characterized by variations of relatively small-scales (i.e. $O(1-10)$ km) and the wave field is rather narrow-banded, the second-order statistics may be significantly affected by medium-induced interference patterns, leading to non-negligible cross-correlation contributions. Such conditions are rather typical over coastal waters, and thus, call for fundamental modifications of the present phase-averaged formulation (i.e., (1)) for the forecasting of coastal waves over spatial inhomogeneity.

To account for inhomogeneity induced by statistical wave interferences, Smit and Janssen [18] proposed a more general statistical formulation known in other fields of physics as the Wigner–Weyl formulation. This formulation accounts for the generation and transformation of the complete second-order statistics, and effectively reduces to (1) when the statistical correlations between crossing waves are superimposed to a negligible contribution. As such, the Wigner–Weyl formulation has the potential of serving as a consistent statistical formulation for coastal wave transformation over variable medium (e.g. [19,20]). The formulation proposed by Smit and Janssen [18] relies on a Schrödinger-type deterministic equation that is written in terms of the Weyl operator of the linear dispersion relation (i.e., Eq. (3) in [18]). This starting point equation provides a direct and formal derivation of the Wigner–Weyl kinetic equation for water waves. However, the formulation of Smit and Janssen [18] is incomplete since the proposed starting point equation was only verified for specific conditions (i.e., constant depth and slowly evolving monochromatic wave over mild sloping bathymetry), but not derived rigorously. If possible, a formal derivation of this Weyl operator based starting point equation would establish a formal link between the deterministic formulation (e.g. Euler equations) and the stochastic Wigner–Weyl formulation, which includes (1) as a statistically well-defined limiting case.

This study tackles this derivation problem by attempting to show the equivalence between the Weyl operator of the dispersion relation and the formal definition of the Dirichlet-to-Neumann (DtN) operator of waves over variable bathymetry. This attempt is discussed in Section 2. Based on this equivalence and using Weyl's operational calculus (e.g., [21]), the evolution of linear waves over slowly varying depth can be described through a compact Schrödinger-type equation as demonstrated in Section 3. In fact, this is the same Schrödinger equation assumed by Smit and Janssen [18], which therefore leads to the desired formal justification of the Wigner–Weyl formulation to statistically describe the evolution of coastal waves. This statistical framework is presented in details in Section 4. Finally, concluding remarks are drawn in Section 5.

2. The equivalence between Weyl and DtN operators over mild slopes

2.1. The mild-slope DtN operator

Under the framework of the potential theory, which considers the flow to be incompressible, inviscid and irrotational, the water wave problem is formulated as a Laplace problem in terms of the velocity potential Φ in a three dimensional domain $\mathcal{D}$. However, it is well known that the interior solution, Φ, of the Laplace problem is fully determined by the flow values given on the boundaries, and therefore, the original problem can be potentially reduced to a two-dimensional one. Zakharov [22] showed that it is possible to formulate the potential problem for water waves in terms of the two canonical surface variables, ϕ and η. The former denotes the surface potential and the latter is the elevation function. Through the assumption of small surface fluctuations of $O(\epsilon)$ (relative to the typical wave length of the fluctuations in deep/intermediate water or to the water depth in shallow water) and by ignoring surface-tension effect, Zakharov's formulation can be written as

$$\mathcal{H} = \int \left(\frac{1}{2} g \eta^2 + \frac{1}{2} \phi W_0 \right) d\boldsymbol{x}, \tag{3}$$

where $\mathcal{H}$ is the Hamiltonian (the sum of potential and kinetic energy), g is the gravitational acceleration and W_0 is the free surface vertical velocity defined as $W_0 = (\partial_z \Phi)_0$. Additionally, z denotes the vertical coordinate and $\boldsymbol{x} = (x, y)$ denote the horizontal coordinates. Finally, the subscript $()_0$ represents terms that are evaluated on $z = 0$.

The linear Zakharov's formulation (3) leaves the vertical velocity W_0 as the only non-free surface component, as it is defined through Φ. Therefore, in order to obtain self-contained free-surface equations (provided by the canonical equation $\partial_t \eta = \delta_\phi \mathcal{H}$ and $\partial_t \phi = -\delta_\eta \mathcal{H}$), it is required to relate between W_0 and the free-surface variables. Such a relation can be obtained through the constraint posed by the following Laplace problem:

$$\begin{aligned} &\Delta \Phi = 0, \quad \text{in } \mathcal{D}, \\ &\Phi = \phi, \quad \text{on } z = 0, \\ &\partial_z \Phi + \nabla_{\boldsymbol{x}} h \cdot \nabla_{\boldsymbol{x}} \Phi = 0, \quad \text{on } z = -h, \end{aligned} \tag{4}$$

where $\nabla_{\boldsymbol{x}} = (\partial_x, \partial_y)$, $\nabla = (\partial_x, \partial_y, \partial_z)$, $\Delta \equiv \nabla \cdot \nabla$ and h is the still water depth. Therefore, given a solution, Φ, of the Laplace problem (4), one can formulate the following relation:

$$W_0 = \mathcal{G}_0 \phi. \tag{5}$$

This relation introduces the so-called DtN operator, $\mathcal{G}_0$, that maps between the Dirichlet value ϕ and the Neumann value W_0. Consequently, if an explicit solution Φ is found, the corresponding DtN map (5) leads to the desired dimensional reduction of the potential problem for linear water waves.

A general and explicit solution form of the linear Laplace problem (4) is achieved through the Boussinesq approach. This amounts to the expansion of the velocity potential Φ around some arbitrary level z_a, allowing to express its solution using only two unknowns functions. If the level around which the expansion is performed is $z_a = 0$, then the two unknowns are the surface potential ϕ and the vertical velocity W_0, and thus, the general solution receives the following form (see [23] for further details):

$$\Phi = \mathcal{C}(z|D_{\boldsymbol{x}}|)\phi + \mathcal{S}(z|D_{\boldsymbol{x}}|)|D_{\boldsymbol{x}}|^{-1} W_0, \tag{6}$$

where $D_{\boldsymbol{x}} \equiv -i\nabla_{\boldsymbol{x}}$. Note that the above formulation differs from the formulation of Agnon et al. [23] in terms of notation only. Here $\mathcal{C}$ and $\mathcal{S}$ indicate the pseudo-differential operators $\cosh(z|D_{\boldsymbol{x}}|)$ and $\sinh(z|D_{\boldsymbol{x}}|)$ (these types of operators can be interpreted, for example, through their power series). In addition, the operator $|D_{\boldsymbol{x}}|$ is defined here as $|D_{\boldsymbol{x}}| = (-\Delta_{\boldsymbol{x}})^{1/2}$ (e.g, [24]), where $\Delta_{\boldsymbol{x}} \equiv \nabla_{\boldsymbol{x}} \cdot \nabla_{\boldsymbol{x}}$. The equivalence between the formulation using functions of $(z\nabla_{\boldsymbol{x}})$ (e.g., [23]) and the formulation using functions of $(z|D_{\boldsymbol{x}}|)$ (e.g, [24]) is understood due to the functional symmetry of the operators involved (e.g., $\cosh(z|D_{\boldsymbol{x}}|) \equiv \cos(z\nabla_{\boldsymbol{x}})$). The advantage of using $|D_{\boldsymbol{x}}|$ relies on its direct correspondence with the wavenumber magnitude $|\boldsymbol{k}| = (k_x^2 + k_y^2)^{1/2}$, where $\boldsymbol{k} = (k_x, k_y)$ defines the wavenumber space. Also note that these types of solution forms were already proposed by earlier studies (e.g., [25,26]).

Ultimately, the desired DtN relation between ϕ and W_0, (5), is obtained through the bottom boundary condition of the considered Laplace problem (4). Accordingly, the expression for Φ, (6), is substituted into the bottom boundary condition (specified in (4)) which results in

$$\mathcal{C}W_0 = \mathcal{S}\ |D_{\boldsymbol{x}}|\phi + D_{\boldsymbol{x}}h \cdot \Big(\mathcal{C}\ D_{\boldsymbol{x}}\phi - \mathcal{S}|D_{\boldsymbol{x}}|^{-1}D_{\boldsymbol{x}}W_0\Big). \tag{7}$$

Note that through the substitution of $z = -h$ (as is required by the bottom boundary condition), the above expression does not dependent on the vertical coordinates, z (the operators $\mathcal{C}$ and $\mathcal{S}$ are now functions of $h|D_{\boldsymbol{x}}|$). In order to proceed, it should be recalled that the aim here is to derive the leading contributions of the DtN relation for mild slopes. To this end, W_0 is written asymptotically as

$$W_0 = W_0^{(0)} + W_0^{(1)} + \cdots, \tag{8}$$

where the superscript $()^{(m)}$ represents contribution of $O(\beta^m)$, and β stands for the ratio between the characteristic wave length and the characteristic variation length of h, which for mild slopes, is assumed to be small. By substituting the expansion in (8) into the expression in (7) one obtains

$$\mathcal{C}\Big(W_0^{(0)} + W_0^{(1)} + \cdots\Big) = \mathcal{S}\ |D_{\boldsymbol{x}}|\phi + D_{\boldsymbol{x}}h \cdot \Big(\mathcal{C}\ D_{\boldsymbol{x}}\phi - \mathcal{S}|D_{\boldsymbol{x}}|^{-1}D_{\boldsymbol{x}}\Big(W_0^{(0)} + \cdots\Big)\Big). \tag{9}$$

The contribution of the DtN relation at each order is consistently extracted by a careful consideration of the operator compositions $(\mathcal{C}^{-1}\mathcal{C})$ and $(\mathcal{C}^{-1}\mathcal{S})$. Such compositions arise as a result of isolating the contribution of W_0 at each order on the left-hand-side of (9). $O(1)$ contribution of the composition $(\mathcal{C}^{-1}\mathcal{C})$ is simply the unit operator associated to 1, while $O(1)$ contribution of the composition $(\mathcal{C}^{-1}\mathcal{S})$ is $\mathcal{T}$, where $\mathcal{T} \equiv \tanh(h|D_{\boldsymbol{x}}|)$ (see details in Appendix A). In addition, $O(\beta)$ contributions of these compositions can be written as

$$\begin{aligned}(\mathcal{C}^{-1}\mathcal{C})_{O(\beta)} &= -h(D_{\boldsymbol{x}}h)\cdot\mathcal{T}^2 D_{\boldsymbol{x}},\\ (\mathcal{C}^{-1}\mathcal{S})_{O(\beta)} &= -h(D_{\boldsymbol{x}}h)\cdot\mathcal{T} D_{\boldsymbol{x}}.\end{aligned} \tag{10}$$

Following (9) and based on the above observations, the expression obtained for $W_0^{(0)}$ is given by

$$W_0^{(0)} = \mathcal{T}\ |D_{\boldsymbol{x}}|\phi, \tag{11}$$

whereas the expression for $W_0^{(1)}$ is provided by

$$W_0^{(1)} = (\mathcal{C}^{-1}\mathcal{S})_{O(\beta)}|D_{\boldsymbol{x}}|\phi - (\mathcal{C}^{-1}\mathcal{C})_{O(\beta)}W_0^{(0)} + (D_{\boldsymbol{x}}h)\cdot\Big(D_{\boldsymbol{x}}\phi - \mathcal{T}|D_{\boldsymbol{x}}|^{-1}D_{\boldsymbol{x}}W_0^{(0)}\Big). \tag{12}$$

The expression for $W_0^{(1)}$ can be written more explicitly by substituting the expressions for the $O(\beta)$ contributions of the operator compositions, (10), and the expression for $W_0^{(0)}$, (11), into (12). Finally, summing the contributions due to $W_0^{(0)}$ and $W_0^{(1)}$ leads to the following formal mild-slope derivation of the DtN operator $\mathcal{G}_0$:

$$\mathcal{G}_0 = \mathcal{T}\ |D_{\boldsymbol{x}}| + (D_{\boldsymbol{x}}h)\cdot(1-\mathcal{T}^2)(1-h\mathcal{T}|D_{\boldsymbol{x}}|)D_{\boldsymbol{x}}. \tag{13}$$

Next, it is aimed to show that the Weyl operator reduces, up to $O(\beta)$, to the same expression for $\mathcal{G}_0$.

2.2. Weyl operator and its asymptotic form

The definition of the Weyl operator is based on the Weyl rule of association, namely, the association between a "phase space" symbol (a function which is defined in $(\boldsymbol{x}, \boldsymbol{k})$ space) and a pseudo-differential operator. For the considered linear Laplace problem (4) and under the mild-slope assumption, the symbol $G(\boldsymbol{x}, \boldsymbol{k}) = \sigma^2/g$ naturally arises as the "phase space" symbol (as is implied by the usual WKB analysis, e.g., Dingemans [10]; also refer to [27]), where σ is defined through the linear dispersion relation, $\sigma(\boldsymbol{x}, \boldsymbol{k}) = \sqrt{|\boldsymbol{k}|g\tanh(|\boldsymbol{k}|h)}$. Given a "phase space" symbol, the corresponding operator in the physical space can be defined through the association between $\boldsymbol{k}$ and $D_{\boldsymbol{x}}$. However, because $\boldsymbol{x}$ and $D_{\boldsymbol{x}}$ do not commute, one must follow an association rule. Hermann Weyl [28] suggested a rule of association that is defined through the following Fourier transform of $G(\boldsymbol{x}, \boldsymbol{k})$ (see, e.g., Cohen [21]):

$$G(\boldsymbol{x}, \boldsymbol{k}) = \int \hat{G}(\boldsymbol{q}, \boldsymbol{p})\exp(i\boldsymbol{q}\cdot\boldsymbol{x} + i\boldsymbol{p}\cdot\boldsymbol{k})d\boldsymbol{q}d\boldsymbol{p}. \tag{14}$$

The Weyl operator is then defined by substituting the operator $D_{\boldsymbol{x}}$ instead of $\boldsymbol{k}$, which provides the following expression:

$$\mathcal{G}_w(\boldsymbol{x}, D_{\boldsymbol{x}}) = \int \hat{G}(\boldsymbol{q}, \boldsymbol{p})\exp(i\boldsymbol{q}\cdot\boldsymbol{x} + i\boldsymbol{p}\cdot D_{\boldsymbol{x}})d\boldsymbol{q}d\boldsymbol{p}, \tag{15}$$

and which can be simplified using the Baker–Campbell–Hausdorff formula and through the fact that the commutator, $[i\boldsymbol{q}\cdot\boldsymbol{x}, i\boldsymbol{p}\cdot D_{\boldsymbol{x}}] = -i\boldsymbol{q}\cdot\boldsymbol{p}$, is constant, to obtain

$$\mathcal{G}_w(\boldsymbol{x}, D_{\boldsymbol{x}}) = \int \hat{G}(\boldsymbol{q}, \boldsymbol{p})\exp(\frac{i}{2}\boldsymbol{q}\cdot\boldsymbol{p})\exp(i\boldsymbol{q}\cdot\boldsymbol{x})\exp(i\boldsymbol{p}\cdot D_{\boldsymbol{x}})d\boldsymbol{q}d\boldsymbol{p}, \tag{16}$$

where the subscript $()_w$ of the operator indicates that it is a Weyl operator. In order to show the equivalence between the Weyl operator $\mathcal{G}_w(\boldsymbol{x}, D_{\boldsymbol{x}})$ and the formal DtN operator up to $O(\beta)$, an explicit asymptotic form of the Weyl operator is required. This explicit asymptotic form is formally derived in Appendix A and is given by (45). As an alternative of the formal derivation, the asymptotic form can be also observed directly from (16). Where by associating between $D_{\boldsymbol{x}}$ and $\boldsymbol{k}$, the corresponding symbol that is obtained is expressed as follows:

$$R(\boldsymbol{x}, \boldsymbol{k}) = \int \hat{G}(\boldsymbol{q}, \boldsymbol{p}) \exp(\frac{i}{2}\boldsymbol{q} \cdot \boldsymbol{p}) \exp(i\boldsymbol{q} \cdot \boldsymbol{x} + i\boldsymbol{p} \cdot \boldsymbol{k}) d\boldsymbol{q} d\boldsymbol{p}. \tag{17}$$

Here the association is unique, since the order of the factors $\boldsymbol{x}$ and $D_{\boldsymbol{x}}$ is explicitly given in (16) such that all the factors $D_{\boldsymbol{x}}$ are placed to right of the $\boldsymbol{x}$ factors. Therefore, the asymptotic form of the operator can be obtained by writing the symbol $R(\boldsymbol{x}, \boldsymbol{k})$ asymptotically and then associate back to the operator representation, preserving the rule that all the factors $D_{\boldsymbol{x}}$ should be placed to right of the $\boldsymbol{x}$ factors. The asymptotic form of $R(\boldsymbol{x}, \boldsymbol{k})$ is directly obtained in terms of the original symbol $G(\boldsymbol{x}, \boldsymbol{k})$ from the expression written in (17) as

$$R(\boldsymbol{x}, \boldsymbol{k}) = \exp(\frac{i}{2} D_{\boldsymbol{x}} \cdot D_{\boldsymbol{k}}) G(\boldsymbol{x}, \boldsymbol{k}), \tag{18}$$

which exactly represents the operation of the first exponent in (45). Whereas the back association is defined by the second exponent in (45). Neglecting $O(\beta^2)$ contributions, the approximation of $R(\boldsymbol{x}, \boldsymbol{k})$ reads

$$R(\boldsymbol{x}, \boldsymbol{k}) \sim (1 + \frac{i}{2} D_{\boldsymbol{x}} \cdot D_{\boldsymbol{k}}) G(\boldsymbol{x}, \boldsymbol{k}), \tag{19}$$

which may be expressed as

$$R(\boldsymbol{x}, \boldsymbol{k}) \sim T|\boldsymbol{k}| + (D_{\boldsymbol{x}} h) \cdot (1 - T^2)(1 - hT|\boldsymbol{k}|)\boldsymbol{k}, \tag{20}$$

where $T = \tanh(h|\boldsymbol{k}|)$ and recall that $G(\boldsymbol{x}, \boldsymbol{k}) = \sigma^2/g$, which means that $G(\boldsymbol{x}, \boldsymbol{k}) = T|\boldsymbol{k}|$. Ultimately, the association back to the operator representation recovers the formal mild-slope definition of the DtN, (13), and therefore, shows the equivalence between Weyl and DtN operators over mild slopes.

Note that the expression in (20) could be written using a more recognizable form using the relation $\nabla_{\boldsymbol{k}} G = 2CC_g \boldsymbol{k}/g$, where C and C_g are the phase and group velocity, defined as $C = \sigma/|\boldsymbol{k}|$ and $C_g = \partial_{|\boldsymbol{k}|}\sigma$. Then, one can write the approximation for the symbol $R(\boldsymbol{x}, \boldsymbol{k})$ as follows:

$$R(\boldsymbol{x}, \boldsymbol{k}) \sim \frac{1}{g}\Big(\sigma^2 + D_{\boldsymbol{x}}(CC_g) \cdot \boldsymbol{k}\Big), \tag{21}$$

which seems to relate to the classical mild-slope operator [29]. Appendix B presents a closer look on the relation between the Weyl operator and the classical mild-slope operator and confirms that the former indeed reduces to the latter for quasi-periodic wave fields.

The equivalence between Weyl and DtN operators is the principle results of this study since it opens the door to the formal use of the Weyl operator and Weyl calculus for application in deterministic and stochastic modelling of water waves. Most significantly, this result leads to the establishment of the connection between Euler equations (and the associated linear wave theory) and the widely used energy balance equation as discussed next.

3. A Schrödinger-type model for linear waves over variable bathymetry

The energy balance equation can be formally derived starting with the following Schrödinger equation:

$$\partial_t \zeta = -i\Sigma(\boldsymbol{x}, D_{\boldsymbol{x}})\zeta, \tag{22}$$

where Σ is the Weyl operator that is associated with the dispersion relation σ and ζ is a complex variable which should be directly related to the energy density of the wave field. Specifically, the complex variable ζ should satisfy the following:

$$\rho\langle|\zeta|^2\rangle = m_0 + O(\beta), \tag{23}$$

where the angular parentheses, $\langle\ldots\rangle$, should be read as ensemble average, the variable m_0 provides a leading order measure (in β) of the mean energy density and ρ is the water mass density. The specific definitions required for Σ and ζ provide a direct path to the formulation of the energy balance equation as recently shown by Smit and Janssen [18]. However, formal derivation of (22) is unavailable. This derivation is made possible based on Weyl's operational calculus. The starting point is the linearized Hamiltonian given by (3). Then, the linear DtN map, (5), is employed to reduce the dimension of the problem,

$$W_0 = \mathcal{G}_w \phi, \tag{24}$$

where instead of $\mathcal{G}_0$, the equivalent Weyl operator, $\mathcal{G}_w$, is now being used. Finally, the evolution equations for η and ϕ are obtained through the canonical equations, $\partial_t \eta = \delta_\phi \mathcal{H}$ and $\partial_t \phi = -\delta_\eta \mathcal{H}$ as

$$\partial_t \begin{bmatrix} \eta \\ \phi \end{bmatrix} = \boldsymbol{A} \begin{bmatrix} \eta \\ \phi \end{bmatrix}, \tag{25}$$

where the matrix $\boldsymbol{A}$ is defined as

$$\boldsymbol{A} = \begin{bmatrix} 0 & \mathcal{G}_w \\ -g & 0 \end{bmatrix}. \tag{26}$$

In order to derive the Schrödinger equation, (22), based on the above linear system, it is required that the matrix $\boldsymbol{A}$ is diagonalizable. Namely, $\boldsymbol{A}$ is required to satisfy the following expression:

$$\boldsymbol{A} = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1}. \tag{27}$$

It is further required that the diagonal matrix $\boldsymbol{\Lambda}$ is composed of the Weyl operator Σ and its complex conjugate on the main diagonal. Additionally, $\boldsymbol{P}$ is required to define the following transformation:

$$\begin{bmatrix} \zeta \\ \zeta^* \end{bmatrix} = \boldsymbol{P}^{-1} \begin{bmatrix} \eta \\ \phi \end{bmatrix}, \tag{28}$$

such that the eigenvectors appearing along its columns provide the necessary relation between the complex variable ζ (or ζ^* where $()^*$ denotes complex conjugate) and the energy density.

The required result for $\boldsymbol{\Lambda}$ is obtained through the convenient formulas of operator composition (54). In a certain sense, these formulas allow one to manipulate operators as if they are simple functions. The "eigenvalues" of $\boldsymbol{A}$, which are found on the main diagonal of $\boldsymbol{\Lambda}$ and indicated by $\lambda_{1,2}$, can be obtained by setting the determinant $|\boldsymbol{A} - \lambda_{1,2}\boldsymbol{I}|$ to zero. This results in the following equation:

$$\lambda_{1,2} + \Sigma^2 = O(\beta^2). \tag{29}$$

By neglecting $O(\beta^2)$ terms, the desired results that $\lambda_1 = -i\Sigma$ and $\lambda_2 = i\Sigma^*$ are obtained. The corresponding definition of $\boldsymbol{P}^{-1}$ can be written as

$$\boldsymbol{P}^{-1} = \frac{1}{\sqrt{2g}} \begin{bmatrix} g & i\Sigma \\ g & -i\Sigma \end{bmatrix}. \tag{30}$$

This leads to the following definition for ζ:

$$\zeta = \frac{1}{\sqrt{2g}} \left(g\eta + i\Sigma\phi \right), \tag{31}$$

which by substituting in the required relation, (23), reads,

$$\rho\langle|\zeta|^2\rangle = \left\langle \frac{1}{2}\rho g \eta^2 \right\rangle + \left\langle \frac{1}{2g}\rho(\Sigma\phi)^2 \right\rangle. \tag{32}$$

This expression indeed equals to the leading order ($O(\beta^0)$) contribution of m_0 (see detailed explanation in [20], Appendix B), which consists of the mean potential energy density (the first term on the right-hand-side of (32)) and the mean kinetic energy density (the second term on the right-hand-side of (32)). This completes the verification that the system consisting of the Schrödinger equation, (22) and its complex conjugate is equivalent to the system given by (25)–(26). As a consequence, the Schrödinger equation, (22), is now made formally available as a mild-slope equation for linear water waves which can be conveniently used for the derivation of the energy balance equation.

4. The Wigner–Weyl formulation as a statistical framework for water waves

The energy balance equation is the central equation underlying the widely used operational forecasting wave models, e.g. WAM model [30], WAVEWATCH model [31] and SWAN model [9]. This important equation can be written in the following form (which is equivalent to the one in (1)):

$$\partial_t E = \{\sigma, E\}, \tag{33}$$

where E represents the spectrum of the energy density and the brackets {} are the so-called Poisson brackets which define the following operation:

$$\{\sigma, E\} \equiv \sigma \left(\overleftarrow{\nabla}_{\boldsymbol{x}} \cdot \overrightarrow{\nabla}_{\boldsymbol{k}} - \overleftarrow{\nabla}_{\boldsymbol{k}} \cdot \overrightarrow{\nabla}_{\boldsymbol{x}} \right) E, \tag{34}$$

where the arrows indicate the function on which the differential operator should operate, i.e., σ or E.

The existing theoretical justification for the energy balance equation is based on the heuristic analogy between wave packets and classical particles [12]. The main aim of this section is to present an alternative formal derivation of the energy balance equation that is obtained as a well defined statistical limit of the more general Wigner–Weyl formulation.

4.1. A formal derivation of the energy balance equation and its generalization

Based on (22), the energy balance equation is formally obtained as a special case of the general stochastic description provided by the Wigner–Weyl formulation (also referred to here as Wigner–Weyl kinetic equation). This formulation essentially accounts for the generation and transformation of the complete second-order statistics including the contribution of the cross-correlations. The detailed derivation of the Wigner–Weyl formulation, starting with (22), is provided by numerous studies in other fields of physics (e.g., [32–37]). In the context of water waves, the derivation is given by Smit and Janssen [18]. Concisely, the Wigner–Weyl formulation is derived by considering the time derivative of the correlation function $\Gamma(\boldsymbol{x}_1, \boldsymbol{x}_2, t) = \langle \zeta(\boldsymbol{x}_1, t)\zeta^*(\boldsymbol{x}_2, t)\rangle$. Then, using the Schrödinger equation, (22), and the variable transformation, $\boldsymbol{x}_1 = \boldsymbol{x} + \boldsymbol{x}'/2$ and $x_2 = \boldsymbol{x} - \boldsymbol{x}'/2$, one obtains

$$\partial_t \Gamma(\boldsymbol{x}, \boldsymbol{x}', t) = -i[\Sigma(\boldsymbol{x} + \boldsymbol{x}'/2, -i\nabla_{\boldsymbol{x}'} - i\nabla_{\boldsymbol{x}}/2) - \Sigma(\boldsymbol{x} - \boldsymbol{x}'/2, -i\nabla_{\boldsymbol{x}'} + i\nabla_{\boldsymbol{x}}/2)]\Gamma(\boldsymbol{x}, \boldsymbol{x}', t). \tag{35}$$

The corresponding Wigner–Weyl kinetic equation, formulated in phase space, is derived based on the definition of the Wigner distribution,

$$\mathcal{W}(\boldsymbol{x}, \boldsymbol{k}, t) = \int \Gamma(\boldsymbol{x}, \boldsymbol{x}', t)\exp(-i\boldsymbol{k}\cdot\boldsymbol{x}')d\boldsymbol{x}', \tag{36}$$

and associating the factor $\boldsymbol{x}'$ with $i\nabla_{\boldsymbol{k}}$ and the operator $-i\nabla_{\boldsymbol{x}}$ with $\boldsymbol{k}$. Ultimately, one arrives (also through the definition of the Weyl operator Σ as detailed in [20], Appendix D) at the correct expression for the Wigner–Weyl kinetic equation, which can be written as

$$\partial_t \mathcal{W} = \{\{\sigma, \mathcal{W}\}\}, \tag{37}$$

As suggested by (36), the Wigner distribution captures the same information as the correlation function, Γ, and as a consequence, generalizes the concept of the energy density spectrum by including the cross-correlation terms that correspond to wave interferences (see further details in [20]). As such, the Wigner distribution, in contrast to the energy density spectrum, provides a complete spectral description of the second order statistics of a given wave field. In addition, the brackets $\{\{\}\}$ are now the so-called Moyal brackets [32] defined as

$$\{\{\sigma, \mathcal{W}\}\} \equiv 2\sigma \sin\left(\overleftarrow{\nabla}_{\boldsymbol{x}}\cdot\overrightarrow{\nabla}_{\boldsymbol{k}}/2 - \overleftarrow{\nabla}_{\boldsymbol{k}}\cdot\overrightarrow{\nabla}_{\boldsymbol{x}}/2\right)\mathcal{W}. \tag{38}$$

Following the asymptotic relation $\sin(x) \sim x$ that applies for small values of x, one can immediately see that $\{\{\}\} \sim \{\}$ when the products $\left(\sigma\overleftarrow{\nabla}_{\boldsymbol{x}}\cdot\overrightarrow{\nabla}_{\boldsymbol{k}}\mathcal{W}\right)$ and $\left(\sigma\overleftarrow{\nabla}_{\boldsymbol{k}}\cdot\overrightarrow{\nabla}_{\boldsymbol{x}}\mathcal{W}\right)$ are small. Furthermore, under these conditions, cross-correlations that may develop due to variations in the medium are negligible and $\mathcal{W}$ becomes asymptotically equal to E, which ultimately leads to the reduction from (37) to (33). This modelling reduction is explained rigorously by Smit and Janssen [18]. Briefly speaking, the reduction is based on two parameters. The first controls the magnitude of $\left(\sigma\overleftarrow{\nabla}_{\boldsymbol{x}}\cdot\overrightarrow{\nabla}_{\boldsymbol{k}}\mathcal{W}\right)$ and represents the ratio between the correlation length scale and the medium variation scale. The second controls the magnitude of $\left(\sigma\overleftarrow{\nabla}_{\boldsymbol{k}}\cdot\overrightarrow{\nabla}_{\boldsymbol{x}}\mathcal{W}\right)$ and represents the ratio between the wave length that corresponds to $\boldsymbol{k}$ and the characteristic length scale of the interference structures stored in $\boldsymbol{k}$. The assumption that the first parameter is small essentially means that the wave field typically decorrelates over a smaller spatial scale than the scale of variation in the medium, namely, correlations of medium-induced crossing waves are negligible. The assumption that the second parameter is small essentially means that contributions due to wave interferences of relatively small length scale are neglected.

In the open ocean, the typical scale of medium variation may roughly evaluated as $O(100)$ km and the wave spectrum is relatively broad. Under such conditions, both of the above discussed parameters are small, since the typical correlation scale would be smaller than the characteristic length scale of the medium, and thus, there is no mechanism that would generate cross-correlation contributions and associated interference patterns. Therefore, under such conditions, the asymptotic relations $\{\{\}\} \sim \{\}$ and $\mathcal{W} \sim E$ are valid and the reduction from the Wigner–Weyl formulation to the energy balance equation is justified. However, in coastal waters, where the medium is characterized by variations of relatively small-scales (i.e. $O(1-10)$ km) and the wave field is rather narrow-banded, the first parameter may become equal to or greater than $O(1)$, indicating that medium-induced cross-correlation contributions and associated interference patterns may become significant.

As discussed by Smit and Janssen [18], if the first parameter is not small, then the use of Taylor expansion for the interpretation of the Moyal brackets, $\{\{\}\}$, is no longer valid (this also implies that the conventional energy balance equation, (33), loses validity). Alternatively, the operation of $\{\{\}\}$ can be partially defined using a Fourier integral, leading to an integro-differential form, which remains valid also for cases in which the correlation length is larger then the characteristic scale of medium variation, but retains the assumption of weak spatial variability of the field statistics through the Taylor interpretation based on the second parameter. This new interpretation of the Moyal brackets, $\{\{\}\}$, leads to a generalized energy balance equation that can be written in the following form (see [19]):

$$\partial_t \mathcal{W} = \{\sigma, \mathcal{W}\} + S_{QC}, \tag{39}$$

where S_{QC} is a scattering term that forces the generation of statistical wave interferences induced by variable bathymetry, and the subscript QC stands for 'quasicoherent' approximation [18]. The added value introduced by S_{QC} for the statistical description of coastal waves is demonstrated through several representative cases of wave-bottom interactions [18,19] and has recently been generalized and demonstrated for cases of wave–current interactions [20]. A classical example that clearly highlights this added value is the case of wave propagation over a submerged shoal as considered by Smit and Janssen [18]. Smit and Janssen [18] showed that for relatively narrow-banded incoming wave fields, omission of the statistical contribution of wave interferences may lead to significant deviations of wave prediction. Specifically, Fig. 6 of the study by Smit and Janssen [18] demonstrates that neglecting the statistical contribution of wave interferences may result in an error of about 40% in wave height at the focusing point behind the shoal and to even larger discrepancies downwave of the focusing area.

5. Conclusions

The principle result of this study is the equivalence between a formal definition of the linear Dirichlet-to-Neumann operator for mild slopes and the Weyl operator that is associated with the linear dispersion relation of water waves. This allows formal use of Weyl rule of association and Weyl operational calculus for deterministic and stochastic applications in water waves. In this study, Weyl rule of association is used to statistically formulate the evolution of linear water waves in inhomogeneous media. To this end, the formulation based on the Weyl operator is written in the form of a Schrödinger-type equation, which allows a direct derivation of a general statistical description known as the Wigner–Weyl formulation. The first consequence of the formal availability of the Wigner–Weyl formulation for water waves is that it leads to a formal theoretical foundation for the widely used energy balanced equation. This result is demonstrated based on a multiple-scale analysis through which the energy balanced equation is derived as a statistically well-defined limiting case of the Wigner–Weyl formulation. The fact that the Wigner–Weyl formulation accounts for the complete second-order statistics leads to the second consequence of the formal availability of the Wigner–Weyl formulation. Specifically, the Wigner–Weyl formulation allows the derivation of a generalized energy balanced equation which accounts for medium-induced cross-correlation contributions and associated wave interference patterns. These cross-correlation contributions are neglected by the presently used energy balance equation, but are typically necessary for a reliable statistical description of waves in the coastal environment.

CRediT authorship contribution statement

Gal Akrish: Conceptualization, Methodology, Writing – original draft. **Pieter Smit:** Conceptualization, Methodology, Writing – review & editing, Results discussion. **Marcel Zijlema:** Supervision, Writing – review & editing, Results discussion. **Ad Reniers:** Supervision, Writing – review & editing, Results discussion.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix A. Weyl calculus

This appendix summarizes the main tools required to work with the Weyl operator and its generalizations. The derivations here follow closely after the book by Leon Cohen, "The Weyl operator and its generalization" [21]. The starting point of this appendix is the definition of the association between the phase space symbol $G(\boldsymbol{x}, \boldsymbol{k})$ and the pseudo-differential operator $\mathcal{G}(\boldsymbol{x}, D_{\boldsymbol{x}})$. Such an association is not uniquely defined because $\boldsymbol{x}$ and $D_{\boldsymbol{x}}$ do not commute. However, the infinitely different possible associations can be analysed in a unified manner through a generalization of Weyl's definition [21]. The generalized Weyl operator is defined as

$$\mathcal{G}_g(\boldsymbol{x}, D_{\boldsymbol{x}}) = \int \hat{G}(\boldsymbol{q}, \boldsymbol{p})\mathcal{K}(\boldsymbol{q}, \boldsymbol{p}) \exp(i\boldsymbol{q} \cdot \boldsymbol{x} + i\boldsymbol{p} \cdot D_{\boldsymbol{x}}) d\boldsymbol{q} d\boldsymbol{p}, \tag{40}$$

where $\hat{G}(\boldsymbol{q},\boldsymbol{p})$ is the Fourier transform of $G(\boldsymbol{x},\boldsymbol{k})$, the subscript $()_g$ of the operator indicates that it is a generalized operator and the kernel $\mathcal{K}(\boldsymbol{q},\boldsymbol{p})$ defines different rules of associations. Using the Baker–Campbell–Hausdorff formula and utilizing the fact that the commutator, $[i\boldsymbol{q}\cdot\boldsymbol{x}, i\boldsymbol{p}\cdot D_{\boldsymbol{x}}] = -i\boldsymbol{q}\cdot\boldsymbol{p}$, is a constant, the operator definition can be simplified as follows:

$$\mathcal{G}_g(\boldsymbol{x}, D_{\boldsymbol{x}}) = \int \hat{G}(\boldsymbol{q},\boldsymbol{p})\mathcal{K}(\boldsymbol{q},\boldsymbol{p})\exp(\frac{i}{2}\boldsymbol{q}\cdot\boldsymbol{p})\exp(i\boldsymbol{q}\cdot\boldsymbol{x})\exp(i\boldsymbol{p}\cdot D_{\boldsymbol{x}})d\boldsymbol{q}d\boldsymbol{p}. \tag{41}$$

The generalization expressed in the generalized operator definition amounts to the inclusion of the kernel $\mathcal{K}(\boldsymbol{q},\boldsymbol{p})$, as can be understood by substituting $\mathcal{K}(\boldsymbol{q},\boldsymbol{p}) = 1$, for which the original definition of the Weyl association is recovered. For the purposes of this study, besides the Weyl rule of association, the definition of the so-called Standard rule of association, for which $\mathcal{K}(\boldsymbol{q},\boldsymbol{p}) = \exp(-i\boldsymbol{q}\cdot\boldsymbol{p}/2)$, is needed as well. Therefore, it will be easier to summarize the following definitions using the generalized operator definition.

A.1. Asymptotic operational form

The asymptotic operational form that follows from the operator definition (41) is required in order to present its operation explicitly and to extract its leading order contributions. The derivation of the asymptotic operational form depends on a Taylor expansion of the dispersion relation, and therefore (at least conceptually), should be defined around $\boldsymbol{k} \neq 0$, since derivatives of the dispersion relation at $\boldsymbol{k} = 0$ are singular. The starting point of the derivation expresses the Fourier function $\hat{G}(\boldsymbol{q},\boldsymbol{p})$ in term of its inverse Fourier transform around $\boldsymbol{k}$ as,

$$\hat{G}(\boldsymbol{q},\boldsymbol{p}) = \int \exp(i\overline{\boldsymbol{k}}\cdot D_{\boldsymbol{k}})\hat{G}(\boldsymbol{q},\boldsymbol{k})\exp(-i\boldsymbol{p}\cdot(\overline{\boldsymbol{k}}+\boldsymbol{k}))d\overline{\boldsymbol{k}}, \tag{42}$$

where the expansion $\exp(i\overline{\boldsymbol{k}}\cdot D_{\boldsymbol{k}})\hat{G}(\boldsymbol{q},\boldsymbol{k})$ essentially represents the shifted function $\hat{G}(\boldsymbol{q},\overline{\boldsymbol{k}}+\boldsymbol{k})$. By substituting the relation (42) into (41) the following operational form is obtained:

$$\begin{aligned}\mathcal{G}_g(\boldsymbol{x}, D_{\boldsymbol{x}}) = &\int \exp(i\overline{\boldsymbol{k}}\cdot D_{\boldsymbol{k}})\hat{G}(\boldsymbol{q},\boldsymbol{k})\exp(-i\boldsymbol{p}\cdot\overline{\boldsymbol{k}})\mathcal{K}(\boldsymbol{q},\boldsymbol{p})\exp(\frac{i}{2}\boldsymbol{q}\cdot\boldsymbol{p})\\ &\exp(i\boldsymbol{q}\cdot\boldsymbol{x})\exp(i\boldsymbol{p}\cdot(D_{\boldsymbol{x}}-\boldsymbol{k}))d\overline{\boldsymbol{k}}d\boldsymbol{q}d\boldsymbol{p},\end{aligned} \tag{43}$$

which, after Fourier transform with respect to $\overline{\boldsymbol{k}}$, reduces to

$$\mathcal{G}_g(\boldsymbol{x}, D_{\boldsymbol{x}}) = \int \delta(\boldsymbol{p})\Big[\hat{G}(\boldsymbol{q},\boldsymbol{k})\exp(i\boldsymbol{q}\cdot\boldsymbol{x})\exp(i\overleftarrow{D}_{\boldsymbol{k}}\cdot\overrightarrow{D}_{\boldsymbol{p}})\mathcal{K}(\boldsymbol{q},\boldsymbol{p})\exp[i\boldsymbol{p}\cdot(D_{\boldsymbol{x}}-\boldsymbol{k}+\frac{\boldsymbol{q}}{2})]\Big]d\boldsymbol{q}d\boldsymbol{p}, \tag{44}$$

and eventually, leading to the following asymptotic form for the Weyl operator by setting $\mathcal{K}(\boldsymbol{q},\boldsymbol{p}) = 1$:

$$\mathcal{G}_w(\boldsymbol{x}, D_{\boldsymbol{x}}) = \Big[G(\boldsymbol{x},\boldsymbol{k})\exp(\frac{i}{2}\overleftarrow{D}_{\boldsymbol{x}}\cdot\overleftarrow{D}_{\boldsymbol{k}})\exp[i\overleftarrow{D}_{\boldsymbol{k}}\cdot(D_{\boldsymbol{x}}-\boldsymbol{k})]\Big], \tag{45}$$

or to the asymptotic form for the Standard operator by setting $\mathcal{K}(\boldsymbol{q},\boldsymbol{p}) = \exp(-i\boldsymbol{q}\cdot\boldsymbol{p}/2)$:

$$\mathcal{G}_s(\boldsymbol{x}, D_{\boldsymbol{x}}) = \Big[G(\boldsymbol{x},\boldsymbol{k})\exp[i\overleftarrow{D}_{\boldsymbol{k}}\cdot(D_{\boldsymbol{x}}-\boldsymbol{k})]\Big], \tag{46}$$

where the arrows indicate the function on which the differential operator should operate, and the subscripts $()_w$ and $()_s$ indicate on a Weyl or a Standard operator, respectively. These expressions can be interpreted as a combination of two steps. First, define the symbol that corresponds to the operator for which all the factors $D_{\boldsymbol{x}}$ are placed to right of the $\boldsymbol{x}$ factors, such that the former does not operate on the latter. Secondly, replace all the $\boldsymbol{k}$ factors with $D_{\boldsymbol{x}}$. Note that the Standard rule already defines the original symbol such that the factors $D_{\boldsymbol{x}}$ are placed to right of the $\boldsymbol{x}$, and therefore, the first step is not included in its asymptotic form (46).

A.2. Operator composition

Operator composition is defined symbolically as follows:

$$\mathcal{A}_g(\boldsymbol{x}, D_{\boldsymbol{x}}) = (\mathcal{U}_g \circ \mathcal{L}_g)(\boldsymbol{x}, D_{\boldsymbol{x}}), \tag{47}$$

where $\mathcal{A}_g$, $\mathcal{U}_g$ and $\mathcal{L}_g$ are some generalized operators. A general formula of the above generalized composition can be derived by substituting in (47) the definition of the generalized operator (41) for $\mathcal{U}_g$ and $\mathcal{L}_g$ and by using the Baker–Campbell–Hausdorff formula. This cumbersome derivation is detailed in [21]. Here only the end result is given, written in terms of the corresponding symbols as

$$A(\boldsymbol{x},\boldsymbol{k}) = U(\boldsymbol{x},\boldsymbol{k})\mathcal{J}\exp(-i\overleftarrow{D}_{\boldsymbol{x}}\cdot\overrightarrow{D}_{\boldsymbol{k}}/2 + i\overleftarrow{D}_{\boldsymbol{k}}\cdot\overrightarrow{D}_{\boldsymbol{x}}/2)L(\boldsymbol{x},\boldsymbol{k}), \tag{48}$$

where $\mathcal{J}$ is defined in terms of the kernel $\mathcal{K}$ (see also [21]) as

$$\mathcal{J} = \frac{\mathcal{K}(\overleftarrow{D}_{\boldsymbol{x}}, \overleftarrow{D}_{\boldsymbol{k}})\mathcal{K}(\overrightarrow{D}_{\boldsymbol{x}}, \overrightarrow{D}_{\boldsymbol{k}})}{\mathcal{K}(\overleftarrow{D}_{\boldsymbol{x}}+\overrightarrow{D}_{\boldsymbol{x}}, \overleftarrow{D}_{\boldsymbol{k}}+\overrightarrow{D}_{\boldsymbol{k}})}. \tag{49}$$

This result leads to the formulas for operator compositions (in terms of the corresponding symbols) of two Weyl operators,

$$A(\boldsymbol{x}, \boldsymbol{k}) = U(\boldsymbol{x}, \boldsymbol{k}) \exp(-i \overleftarrow{D}_{\boldsymbol{x}} \cdot \overrightarrow{D}_{\boldsymbol{k}}/2 + i \overleftarrow{D}_{\boldsymbol{k}} \cdot \overrightarrow{D}_{\boldsymbol{x}}/2) L(\boldsymbol{x}, \boldsymbol{k}), \tag{50}$$

or two Standard operators,

$$A(\boldsymbol{x}, \boldsymbol{k}) = U(\boldsymbol{x}, \boldsymbol{k}) \exp(i \overleftarrow{D}_{\boldsymbol{k}} \cdot \overrightarrow{D}_{\boldsymbol{x}}) L(\boldsymbol{x}, \boldsymbol{k}). \tag{51}$$

The above formulas together with the tools that were summarized in this appendix allow to significantly simplify necessary operator manipulations. In particular, under the mild-slope assumption for which $O(\beta^2)$ terms are neglected, these tools allow to define and interpret the operation of operators in a straightforward manner. In the following, several examples of operator compositions which arise in the main text are considered. The first is the mild-slope composition of two Standard operators. For this case, the formula in (51) provides the following approximation for $A(\boldsymbol{x}, \boldsymbol{k})$:

$$A(\boldsymbol{x}, \boldsymbol{k}) \sim U(\boldsymbol{x}, \boldsymbol{k})(1 + i \overleftarrow{D}_{\boldsymbol{k}} \cdot \overrightarrow{D}_{\boldsymbol{x}}) L(\boldsymbol{x}, \boldsymbol{k}), \tag{52}$$

for which an approximation for $\mathcal{A}_s(\boldsymbol{x}, D_{\boldsymbol{x}})$ is obtained by associating between $\boldsymbol{k}$ and $D_{\boldsymbol{x}}$, recalling that all the factors $D_{\boldsymbol{x}}$ should be placed to right of the $\boldsymbol{x}$ factors. The second example is the mild-slope composition of two Weyl operators. The composition formula for Weyl operators, (50), generates the following approximation for $A(\boldsymbol{x}, \boldsymbol{k})$:

$$A(\boldsymbol{x}, \boldsymbol{k}) \sim U(\boldsymbol{x}, \boldsymbol{k})(1 - i \overleftarrow{D}_{\boldsymbol{x}} \cdot \overrightarrow{D}_{\boldsymbol{k}}/2 + i \overleftarrow{D}_{\boldsymbol{k}} \cdot \overrightarrow{D}_{\boldsymbol{x}}/2) L(\boldsymbol{x}, \boldsymbol{k}). \tag{53}$$

This approximation reveals the following useful mild-slope results:

$$\begin{cases} (\mathcal{L}_w \circ \mathcal{L}_w)(\boldsymbol{x}, D_{\boldsymbol{x}}) \leftrightarrow L^2(\boldsymbol{x}, \boldsymbol{k}), \\ (\mathcal{L}_w^{-1} \circ \mathcal{L}_w)(\boldsymbol{x}, D_{\boldsymbol{x}}) \leftrightarrow 1, \end{cases} \tag{54}$$

where $\leftrightarrow$ means "associated with" and $\mathcal{L}_w^{-1}(\boldsymbol{x}, D_{\boldsymbol{x}})$ is the Weyl operator that is associated with the symbol $L^{-1}(\boldsymbol{x}, \boldsymbol{k})$.

Appendix B. The relation with the classical mild-slope equation

The formulation of linear water waves over bathymetry, as given by (25)–(26), provides a convenient starting-point for the derivation of the Schrödinger-type model discussed in Section 3. Here however, it is aimed to demonstrate the relation of this linear formulation with the classical mild-slope equation, for which, a convenient starting point is the following combined form:

$$\partial_t^2 \phi + g \mathcal{G}_w \phi = 0. \tag{55}$$

This combined form is derived by time differentiating the second equation in the system (25) and substituting the first equation accordingly. This combined formulation is reduced to the classical mild-slope equation [38] provided that the following relation holds:

$$g \mathcal{G}_w \phi \sim \Big[D_{\boldsymbol{x}} \cdot (C C_g D_{\boldsymbol{x}}) + (\sigma_0^2 - |\boldsymbol{k}_0|^2 C C_g) \Big] \phi, \tag{56}$$

where σ_0^2, C_0 and $C_{g,0}$ are defined as $\sigma_0^2 = gG(\boldsymbol{x}, \boldsymbol{k}_0)$, $C_0 = \sigma/|\boldsymbol{k}_0|$ and $C_{g,0} = \partial_{|\boldsymbol{k}_0|}\sigma$, respectively.

The asymptotic equivalence written above can be understood through the fundamental assumption underlying the derivation of the classical mild-slope equation, that is, the assumption of quasi-periodic motion in time at any spatial point. Equivalently, this assumption means that the spectrum of the field is narrowly supported in the direction of $|\boldsymbol{k}|$ around $|\boldsymbol{k}_0|$. In order to see how this fundamental assumption leads to the asymptotic relation, (56), it may be useful to demonstrate the effective operation of a pseudo-differential operator operating on a function with narrow-banded spectrum. To this end, consider a narrow-banded wave field propagating over a constant depth. In such a case, the linear DtN relation (24) can be written as a simple function multiplication in wavenumber space as,

$$\mathcal{G}_w(D_{\boldsymbol{x}})\phi(\boldsymbol{x}) = \int G(\boldsymbol{k})\hat{\phi}(\boldsymbol{k}) \exp(i\boldsymbol{k} \cdot \boldsymbol{x}) d\boldsymbol{k}, \tag{57}$$

where the Fourier transform $\hat{\phi}$ is assumed to be narrowly supported around $\boldsymbol{k}_0$, say between $[\boldsymbol{k}_0 - \Delta\boldsymbol{k}, \boldsymbol{k}_0 + \Delta\boldsymbol{k}]$. Using the change of variable $\bar{\boldsymbol{k}} = \boldsymbol{k} - \boldsymbol{k}_0$, the operation of $\mathcal{G}_w$ is effectively given by

$$\mathcal{G}_w(D_{\boldsymbol{x}})\phi(\boldsymbol{x}) = \exp(i\boldsymbol{k}_0 \cdot \boldsymbol{x}) \int_{-\Delta\boldsymbol{k}}^{\Delta\boldsymbol{k}} G(\boldsymbol{k}_0 + \bar{\boldsymbol{k}})\hat{A}(\bar{\boldsymbol{k}}) \exp(i\bar{\boldsymbol{k}} \cdot \boldsymbol{x}) d\bar{\boldsymbol{k}} + c.c., \tag{58}$$

where $\hat{A}(\bar{\boldsymbol{k}}) = \hat{\phi}(\boldsymbol{k}_0 + \bar{\boldsymbol{k}})$ is the Fourier transform of the slowly varying complex amplitude $A(\boldsymbol{x})$ and $c.c.$ stands for complex conjugate. This representation clearly shows that for a function with narrow spectrum the operation of $\mathcal{G}_w$ requires only limited information of G around $\boldsymbol{k}_0$. Consequently, G can be efficiently expanded around $\boldsymbol{k}_0$, leading to the following representation:

$$\mathcal{G}_w(D_{\boldsymbol{x}})\phi(\boldsymbol{x}) = \exp(i\boldsymbol{k}_0 \cdot \boldsymbol{x}) \Big[G \exp(i \overleftarrow{D}_{\boldsymbol{k}} \cdot \overrightarrow{D}_{\boldsymbol{x}}) A \Big]_{\boldsymbol{k}=\boldsymbol{k}_0}, \tag{59}$$

where the arrows indicate the function on which the differential operator should operate, i.e., G or A. To summarize, this example demonstrates the interpretation of a pseudo-differential operation (e.g., $\mathcal{G}_w$) on a narrow-banded function. Where in the limit given by $\hat{\phi} = A\delta(\boldsymbol{k} - \boldsymbol{k}_0)$ (for which A is a constant) the operation becomes a multiplication by $G(\boldsymbol{k}_0)$, while for a narrow spectrum of finite width, this operation can be approximated as a polynomial in $D_{\boldsymbol{x}}$, as described by (59).

These observations point out the expansion of G around $\boldsymbol{k}_0$ as the key to derive the approximation (56) that relates the Weyl operator with the operator of the classical mild-slope equation. However, in order to obtain a valuable model, the expansion of G should admit some constraints. Most important, the approximated operator should be self adjoint and should allow wave propagation in all directions (recall that the fundamental assumption of the classical mild-slope equation does not prioritize any direction of propagation). This means that the expansion of G should preserve the symmetry characterizes the original G, and therefore, requires a symmetrical expansion, namely an expansion in terms of $|\boldsymbol{k}|$. Additionally, it is also beneficial to preserve the symmetrical structure of G, which means that the approximation should only consist of terms such as $|\boldsymbol{k}|^n$ where n is even number. This requirement avoids terms like $|D_{\boldsymbol{x}}|$ which are difficult to interpret. Accordingly, an appropriate expansion is given as follows [39]:

$$G(\boldsymbol{x}, \boldsymbol{k}) \sim \frac{1}{g}\left[\sigma_0^2 + C_0 C_{g,0}(|\boldsymbol{k}|^2 - |\boldsymbol{k}_0|^2)\right], \tag{60}$$

where $C_0 C_{g,0}/g = (\partial_{|\boldsymbol{k}|^2} G)_{\boldsymbol{k}=\boldsymbol{k}_0}$ and recall that $\sigma_0^2/g = G(\boldsymbol{x}, \boldsymbol{k}_0)$. The Weyl operator of this approximation is obtained by calculating first the corresponding $R(\boldsymbol{x}, \boldsymbol{k})$ symbol using (19), which reads,

$$R(\boldsymbol{x}, \boldsymbol{k}) \sim \frac{1}{g}\left[\sigma_0^2 - |\boldsymbol{k}_0|^2 C_0 C_{g,0} + C_0 C_{g,0}|\boldsymbol{k}|^2 + (D_{\boldsymbol{x}} C_0 C_{g,0}) \cdot \boldsymbol{k}\right], \tag{61}$$

then, by the subsequent back association from $\boldsymbol{k}$ to $D_{\boldsymbol{x}}$, the classical mild-slope operator, as given by (56), is derived. This result shows that the Weyl operator of the full symbol G is equivalent to the classical mild-slope operator for quasi-periodic wave fields, and implies that the Weyl formulation as given by either of the formulations, namely (25) or (55), provides a generalized mild-slope model for wave fields of arbitrary spectral width.

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