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Max-Min-Plus-Scaling Systems in a Discrete-Event Framework with an Application in Urban Railway

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Abstract: In this paper we discuss modelling and control of discrete-event systems using max-min-plus-scaling systems. We analyse how the basic operations max, min, plus, and scaling occur in the modelling phase and we discuss some general forms for the system. Because of the different/deviating character of the signals in a discrete-event MMPS model, we will discuss concepts such as time-invariance and steady-state behavior. In the design of a model predictive controller for MMPS systems we have to revisit the cost functions in light of the discrete-event nature of the signals. We finalize this paper with the an intuitive case study on an urban railway line, performing both modelling and control.

Keywords: Discrete event system, max-min-plus-scaling systems, modeling and control

1. INTRODUCTION

Discrete-event systems form a large class of dynamic systems in which the evolution of the system is driven by the occurrence of certain discrete events. This in contrast to discrete-time systems where the evolution depends on a clock.

Discrete-event systems with only synchronization and no concurrency can be modeled by a max-plus-linear model (Baccelli et al., 1992; Heidergott et al., 2006). This is a model in which the system equations consist of max and plus operations (e.g. paper flow in a printer or scheduling for container terminals). When competition plays a role (e.g. first-come-first-serve mechanisms) we obtain a max-min-plus system (Olsder, 1994; Gunawardena, 1994; Jean-Marie and Olsder, 1996). This is a model in which the system equations consist of max, min, and plus operations (e.g. product flow in a production system with competition). In some occasions a scaling operation will occur. It can happen when the processing times in the system will depend on external parameters or on previous values of the state and input. Such a system can be written as a max-min-plus-scaling (MMPS) system (e.g. traffic management on an urban railway line, see Section 7). MMPS systems also occur when we consider the closed-loop configuration of a max-plus linear systems with a residuation controller or a model predictive controller (Necoara et al., 2008b; Bemporad et al., 2002). Finally, perturbed max-plus linear systems can often be written as max-plus-scaling systems (van den Boom and De Schutter, 2002, 2004).

In Section 2 and 3 of the paper we introduce signals operations and max-min-plus-scaling systems in a discrete-event framework. Because of the deviated nature of the signals we will study time-invarancy in Section 4 and steady-state behavior in Section 5. In Section 6 we elaborate on the cost-function in model predictive control. Finally in Section 7 we consider the modelling and model predictve control of a urban railway line.

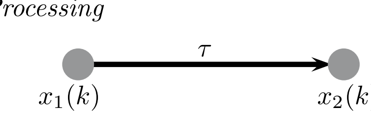
2. SIGNAL OPERATIONS

A dynamic MMPS system in a discrete-event framework will always have states that represent the starting and ending times of the operations for the event cycle k. In the general framework of discrete-event MMPS systems the state may also represent to quantities, such as the number of goods in a production system or the number of people in a train. Also in this case the basic operations will be maximization, minimization, addition, and scaling. In this paper the state of the MMPS system will be

$$x(k) = \begin{bmatrix} x_{\mathrm{t}}(k) \\ x_{\mathrm{q}}(k) \end{bmatrix}$$

where $[x_{\mathrm{t}}(k)]_i$ gives the time instant at which event i will occur for the kth time, and $[x_{\mathrm{q}}]_j(k)$ will represent the value of the jth quantity at event step k. We will now discuss how the basic four operations (max,min,plus,scaling) appear in the system equations.

Addition I: Processing



Let the arrow in the figure represent an operation with processing time τ and let $x_1(k)$ and $x_2(k)$ be the starting and finishing time, respectively, for event cycle k. The relation between $x_1(k)$ and $x_2(k)$ can be represented by the **plus**-operation $x_2(k) = x_1(k) + \tau$.

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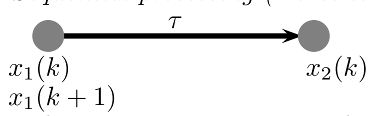
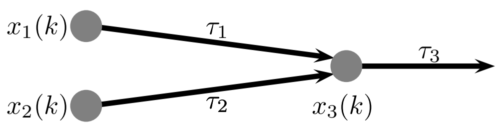
Maximization I: Sequential processing (No concurrency)



Consider two subsequent operations on the same resource in which operation k needs to be finished before operation $k+1$ can take place (no concurrency).

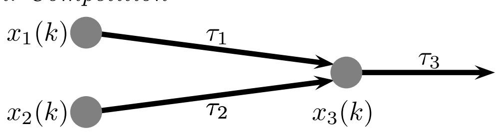
Let $u_1(k+1)$ be the earliest possible starting time of x_1 for cycle $k+1$, then the starting time $x_1(k+1)$ is given by the **max**-operation $x_1(k+1) = \max(x_1(k) + \tau, u_1(k+1))$.

Maximization II: Synchronization



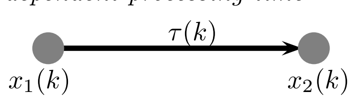
Consider an operation 3 with starting time $x_3(k)$ that will start when both operations 1 and 2 are finished. The starting time $x_3(k)$ is now given by the **max**-operation: $x_3(k) = \max(\, x_1(k) + \tau_1 \,,\, x_2(k) + \tau_2 \,)$.

Minimization: Competition



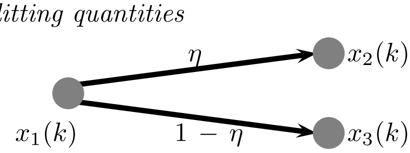
Consider an operation 3 with starting time $x_3(k)$ that will start as soon as either operations A or operation B is finished. The starting time $x_3(k)$ is now given by a **min**-operation: $x_3(k) = \min(x_1(k)+\tau_1 \,, x_2(k)+\tau_2)$.

Scaling I: State-dependent processing time



Consider an operation where the processing time τ is an affine function of the state x, so $\tau(k) = \alpha + \beta^T x(k)$ where $\alpha \in \mathbb{R}_+$ and $\beta \in \mathbb{R}^n_+$ where n is the dimension of the state. The relation between starting time $x_1(k)$ and $x_2(k)$ is now include a **scaling**-operation: $x_2(k) = x_1(k) + \alpha + \beta^T x(k)$.

Scaling II: Splitting quantities



Consider an operation that splits the quantity state $x_1(k)$ into two new quantity states $x_2(k)$ and $x_3(k)$ with ratio η and $(1-\eta)$ respectively, then the quantities are given by a **scaling**-operation: $x_2(k) = \eta\, x_1(k)$, $x_3(k) = (1-\eta)\, x_1(k)$.

3. MAX-MIN-PLUS-SCALING SYSTEMS

Define $\top = \infty$, $\varepsilon = -\infty$, $\mathbb{R}_\top = \mathbb{R} \cup \{\infty\}$, $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{-\infty\}$, and $\mathbb{R}_c = \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$. Further we introduce the conventions $0 \cdot \varepsilon = 0$ and $0 \cdot \top = 0$ and $\top + \varepsilon = 0$. Often we use the set $\mathcal{R}$, which can be either $\mathbb{R}$, $\mathbb{R}_\varepsilon$, $\mathbb{R}_\top$, or $\mathbb{R}_c$.

Definition 1. (De Schutter and van den Boom, 2004) *Max-min-plus-scaling functions.* A max-min-plus-scaling (MMPS) function $f : \mathcal{R}^m \to \mathcal{R}$ of the variables $p_1, \ldots, p_m \in \mathcal{R}$ is defined by the grammar for $i \in \underline{m}$

$$f := p_i | \alpha | \max(f_k, f_l) | \min(f_k, f_l) | f_k + f_l | \beta \cdot f_k \, , \, ,$$

$\alpha \in \mathcal{R}$, $\beta \in \mathbb{R}$, and f_k, f_l are again MMPS functions over the set $\mathcal{R}$. The symbol $|$ stands for "or". The definition is recursive. For vector-valued MMPS functions the above statements hold componentwise.

Definition 2. A max-min-plus-scaling function $f : \mathcal{R}^m \to \mathcal{R}^n$ is well-defined if the following holds:

$$p \in \mathcal{R}^m \quad \Longrightarrow \quad f(p) \in \mathcal{R}^n$$

for $\mathcal{R}$ is $\mathbb{R}$, $\mathbb{R}_\varepsilon$, $\mathbb{R}_\top$, or $\mathbb{R}_c$.

Definition 3. Max-min-plus-scaling system. Consider the vector $p(k) = \left[x^T(k),\, x^T(k-1),\, u^T(k)\right]^T$, where $p \in \mathcal{P} \subseteq \mathcal{R}^{n_\mathrm{p}}$, $x \in \mathcal{R}^n$ is the state, $u \in \mathcal{R}^p$ is the control input, and $w \in \mathcal{R}^z$ is an external signal. A max-min-plus-scaling (MMPS) system is described by a state-space model of the form

$$x(k) = f(p(k)),$$

where f is a vector-valued MMPS function in the variables p.

If the MMPS function f depends on the present state $x(k)$ the system is an implicit MMPS system.

Definition 4. (Bemporad et al., 2002)*Piecewise affine function* A piecewise affine function $f_\mathrm{PWA} : \mathcal{P} \to \mathcal{R}$ is defined by

$$f_\mathrm{PWA}(p) = a_i^T\, p + c_i, \quad p \in \Omega_i$$

where Ω_i, $i = 1, \ldots, n_\Omega$ are convex polyhedra (i.e. given by a finite number of linear inequalities in p), with non-overlapping interiors and $\bigcup_{i=1}^{n_\Omega} \Omega_i = \mathcal{P}$ and $a_i \in \mathbb{R}^{n_p}$, $c_i \in \mathcal{R}$, $i = 1, \ldots, n_\Omega$. For a vector-valued or matrix-valued piecewise affine function the above statements hold componentwise.

A continuous piecewise affine (C-PWA) system is described by a state-space model of the form

$$x(k) = f_\mathrm{PWA}(p(k)),$$

where f_PWA is a continuous vector-valued PWA function in the variables p.

Lemma 5. (De Schutter and van den Boom, 2004) A C-PWA system is equivalent to an MMPS system.

We already discussed that we can divide the state $x(k)$ in two substates, namely x_t denoting states related to the timing of discrete events, and x_q denoting quantities. In a similar way we can split $p(k)$ into $p_\mathrm{t}(k)$ and $p_\mathrm{q}(k)$, so we obtain

$$x(k) = \begin{bmatrix} x_\mathrm{t}(k) \\ x_\mathrm{q}(k) \end{bmatrix} \quad \text{and} \quad p(k) = \begin{bmatrix} p_\mathrm{t}(k) \\ p_\mathrm{q}(k) \end{bmatrix}$$

with

$$p_\mathrm{t}(k) = \left[x_\mathrm{t}^T(k),\, x_\mathrm{t}^T(k-1),\, u_t^T(k)\right]^T,$$
$$p_\mathrm{q}(k) = \left[x_\mathrm{q}^T(k),\, x_\mathrm{q}^T(k-1),\, u_q^T(k)\right]^T,$$

with $p_\mathrm{t} \in \mathcal{P}_t$ and $p_\mathrm{q} \in \mathcal{P}_q$. With these definitions we can rewrite the MMPS system as

$$\begin{aligned} x_\mathrm{t}(k) &= f_\mathrm{t}(p_\mathrm{t}(k), p_\mathrm{q}(k)) \\ x_\mathrm{q}(k) &= f_\mathrm{q}(p_\mathrm{t}(k), p_\mathrm{q}(k)) \end{aligned} \qquad (1)$$

4. TIME INVARIANCE

Consider an MMPS system

$$x(k) = f(p(k))$$

To discuss time invariancy we start with introducing the property of partly homogeneous systems:

Definition 6. **Partly additive homogeneous system** Consider an MMPS system with time signal p_t and quantity signal p_q such that the system is given by 1. The MMPS system is partly additive homogeneous if

$$\begin{aligned} f_t(p_t+\lambda, p_q) &= f_t(p_t, p_q) + \lambda \\ f_q(p_t+\lambda, p_q) &= f_q(p_t, p_q) \end{aligned} \tag{2}$$

for any $\lambda \in \mathbb{R}$.

The intuition of the additive homogeneity can be found in the concept of time invariance. Consider a MMPS system with only time-signals $x_t(k)$, given by

$$x_t(k) = f_t(p_t(k))$$

Time invariancy for the system f_t means that if we shift the signal p_t in time ($p_t(k) \longrightarrow p_t(k) + \tau$) then the state x_t will shift in time as well ($x_t(k) \longrightarrow x_t(k) + \tau$). This means that system f_t will be time-invariant if it is additive homogeneous.

Time invariancy for an MMPS system (1) with both time-signals and quantity signals means that if ($x_t(k)$, $x_q(k)$, $p_t(k)$, $p_q(k)$) will be a valid trajectory of the MMPS system f, then ($x_t(k)+\tau, x_q(k), p_t(k)+\tau, p_q(k)$) will also be a valid trajectory of f. In other words the system is time-invariant if it is partly additive homogeneous.

5. STEADY-STATE BEHAVIOR

Consider the time-invariant MMPS system

$$\begin{aligned} x_t(k) &= f_t(p_t(k), p_q(k)) \\ x_q(k) &= f_q(p_t(k), p_q(k)) \end{aligned} \tag{3}$$

A first observation is that the two signals x_t and x_q have different nature, and that their steady-state behavior will therefore be different. The time signal will usually be non-decreasing and so in general the time signal will not reach an equilibrium. Instead we consider steady-state behavior for the time signal and study stationary regimes which means that the growth of x_t becomes constant.

For (x_t, p_t) a steady-state is reached if for a certain k_{ss} the growth of x_t and p_t becomes constant, so

$$p_t(k) = p_t(k-1) + \tau_{t,ss}\ , \quad \text{for } k \geq k_{ss}$$

where $\tau_{t,ss}$ is a scalar constant. For the quantity variables an steady-state or equilibrium means that p_q becomes constant, so

$$p_t(k) = p_q(k-1)\ , \quad \text{for } k \geq k_{ss}$$

We obtain the steady-state conditions

$$\begin{bmatrix} p_t(k) \\ p_q(k) \end{bmatrix} = \begin{bmatrix} p_{ss,t} + k\,\tau_{ss,t} \\ p_{ss,q} \end{bmatrix}\ , \quad \text{for } k \geq k_{ss}$$

Since (3) is a time-invariant system we have

$$\begin{bmatrix} f_t(p_t+\lambda, p_q) \\ f_q(p_t+\lambda, p_q) \end{bmatrix} = \begin{bmatrix} f_t(p_t, p_q) + \lambda \\ f_q(p_t, p_q) \end{bmatrix} \tag{4}$$

Note that f_t and f_q are MMPS functions and also be written as C-PWA functions. This means we can find matrices $E_{i,tt}$, $E_{i,tq}$, $E_{i,qt}$, $E_{i,qq}$, and vectors e_i (all with the appropriate dimensions), and non-overlapping convex polyhedra $\mathcal{S}_i$, $i = 1, \ldots, n_{\mathcal{S}}$ such that for $p(k) \in \mathcal{S}_i$ we have

$$\begin{bmatrix} f_t(p_t, p_q) \\ f_q(p_t, p_q) \end{bmatrix} = \begin{bmatrix} E_{i,tt} \\ E_{i,qt} \end{bmatrix} p_t + \begin{bmatrix} E_{i,tq} \\ E_{i,qq} \end{bmatrix} p_q + \begin{bmatrix} e_{i,t} \\ e_{i,q} \end{bmatrix}$$

We assume the system to be time-invariant, so it must satisfy condition (4). Therefore we derive for all $i = 1, \ldots, n_{\mathcal{S}}$:

$$\sum_{\ell,j} [E_{i,tt}]_{\ell,j} = 1, \forall \ell, \quad \sum_{\ell,j} [E_{i,qt}]_{\ell,j} = 0, \forall \ell$$

This means that if there exist values ($x_{ss,t}, x_{ss,q}, p_{ss,t}, p_{ss,q}$, $\tau_{ss,t}$) such that

$$\begin{aligned} x_{ss,t} &= f_t(p_{ss,t}, p_{ss,q}) \\ x_{ss,q} &= f_q(p_{ss,t}, p_{ss,q}) \end{aligned}$$

then ($x_{s,t}, x_{ss,q}, p_{ss,t}, p_{ss,q}, \tau_{ss,t}$) is a steady-state with

$$\begin{aligned} x_{ss,t} + k\,\tau_{ss,t} &= f_t(p_{ss,t} + k\,\tau_{ss,t}, p_{ss,q}) \\ x_{ss,q} &= f_q(p_{ss,t} + k\,\tau_{ss,t}, p_{ss,q}) \end{aligned}$$

This result can be summarized in the following lemma:

Proposition 7. Consider a time-invariant MMPS system. If there is an index $i \in \{1, \ldots, n_s\}$ such that there are values ($x_{ss,t}, x_{ss,q}, p_{ss,t}, p_{ss,q}, \tau_{ss,t}$) satifying

$$\begin{bmatrix} x_{ss,t} + \tau_{ss,t} \\ x_{ss,q} \end{bmatrix} = \begin{bmatrix} E_{i,tt} \\ E_{i,qt} \end{bmatrix} p_{ss,t} + \begin{bmatrix} E_{i,tq} \\ E_{i,qq} \end{bmatrix} p_{ss,q} + \begin{bmatrix} e_{i,t} \\ e_{i,q} \end{bmatrix}$$

for $\begin{bmatrix} p_{ss,t} \\ p_{ss,q} \end{bmatrix} \in \mathcal{S}_i$, then ($x_{ss,t}, x_{ss,q}, p_{ss,t}, p_{ss,q}, \tau_{ss}$) is a steady-state.

6. MODEL PREDICTIVE CONTROL

This section shortly discusses the Model Predictive Control (MPC) technique for MMPS systems in a discrete-event framework. MPC is a control strategy that makes use of a receding horizon N (De Schutter and van den Boom, 2004; Necoara et al., 2008a). At each event step k the controller predicts the optimal control inputs by minimizing a cost function over the finite horizon N: $\bar{u}(k) = \{u(k),\ u(k+1),\ \ldots,\ u(k+N-1)\}$. The inputs related to the time signals will be denoted by u_t and inputs related to the quantity signals will be denoted by u_q.

Similar to the observation we made in the computation of a steady-state we have to take into account that the two state signals x_t and x_q and their input signals u_t and u_q have different natures, and so we use different measures in the cost-function. The measure in the cost function related to the time signals x_t and u_t are usually associated with the buffer levels, which are defined as the time delay between the occurrences of different events in either the same event cycle k or the consecutive ones (De Schutter and van den Boom, 2001). Examples of state cost functions are

$$\text{Regime}: J_{x,1}(k,\tilde{u}) = \sum_{j=1}^{N} \| x_t(k+j) \|_{\mathbb{P}}$$

$$\text{Makespan}: J_{x,2}(k,\tilde{u}) = \| x_{t,i}(k+N) \|_{\infty}$$

$$\text{Tracking}: J_{x,3}(k,\tilde{u}) = \sum_{j=1}^{N} \| x_q(k+j) - r_q(k+j) \|_1$$

$$\text{Tardiness}: J_{x,4}(k,\tilde{u}) = \sum_{j}^{N}\sum_{i=1}^{n_t} \max(x_{t,i}(k+j) - r_{t,i}(k+j), 0)$$

where $\|z\|_{\mathbb{P}} = \max_{i\in\{1,\ldots,n\}} z_i - \min_{j\in\{1,\ldots,n\}} z_j$ is the max-plus Hilbert projective norm (Heidergott et al., 2006). In tardiness criterion $J_{s,1}$ the state $x_t(k)$ is to follow a due date reference signal $r_t(k)$, in $J_{s,2}$ the makespan is minimized, and in $J_{s,3}$ we aim for a steady regime. The last criterion $J_{s,4}$ aims at the quantity state x_q to track a reference quantity r_q. Likewise we can define input cost functions:

$$\text{Regime}: J_{u,2}(k) = \sum_{j=1}^{N} \| u_q \|_1$$

$$\text{Just}-\text{in}-\text{time}: J_{u,1}(k) = \sum_{j=1}^{N}\sum_{i=1}^{n_t} r_{t,i}(k+j) - u_{t,i}(k+j)$$

The input criterion $J_{u,1}$ maximixes the input u_t leading to just-in-time operation. The last input criterion measures the cost of the quantity input.

The final cost function in MPC is chosen as follows:

$$J_{\text{tot}}(k,\tilde{u}) = \sum_{j=1}^{4} \lambda_j J_{x,j} + \sum_{\ell=1}^{2} \mu_\ell J_{u,\ell} \qquad (5)$$

where $\lambda_j \in [0,1]$ and $\mu_\ell \in [0,1]$ are trade-off parameters. This total cost function J_{tot} represents a trade-off between the different cost. distance of the state from the origin and the cost of the control input. By choosing the parameters λ_i and μ_i, we can balance the rate of performance with the cost of the control.

The optimization problem now becomes

$$\min_{\tilde{u}(k)} J_{\text{tot}}(k,\tilde{u})$$

subj. to

$$A_t x_t(k) + A_q x_q(k) + B_t u_t(k) + B_q u_q(k) \leq M$$

where $A_t x_t(k) + A_q x_q(k) + B_t u_t(k) + B_q u_q(k) \leq M$ reflect general linear inequality constraints on the inputs and states of the system. We now apply the first input $u(k)$ to the system and shifts the horizon one event step, such that it now runs from $k+1$ to $k+N+1$.

7. APPLICATION: AN URBAN RAILWAY LINE

Consider an urban railway line as given in Figure 1 with J station and K trains. We assume there is no timetable, but

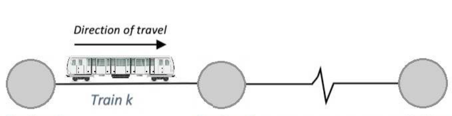


Fig. 1. Urban railway line

trains $k = 1,\ldots,K$ depart from station 1 with a headway interval τ_0, they stop at each station $j = 1,\ldots,J$, and they depart if all passengers have disembarked and boarded the train. We denote the arrival and departure time of train k at station j by $a_j(k)$ and $d_j(k)$, respectively. Denote the number of passengers in the train k when leaving station j by $\rho_j(k)$, and denote the number of passengers at station j when train k is leaving the station by $\sigma_j(k)$. In the example we assume every trains has a limited capacity of $\rho_{\max}$ passengers. Consider the running times $\tau_{r,j}$ from station $j-1$ to station j to be fixed. Let e_j denote the number of passengers entering the platform at station j per second. Let b be the number of passengers that can board the train per time unit and let f denote the number of passengers that can disembark the train per time unit. (We assume $b > e_j$ for all j.) We assume that the number of passengers leaving train k at a particular station j is a fixed fraction β_j of the number of the passengers in train k when entering station j.

The arrival time of train k at station j is the maximum of the departure time of train k at station $j-1$ plus the running time, and the departure time of train $k-1$ at station j plus the headway time, so

$$a_j(k) = \max\Big(d_{j-1}(k) + \tau_{r,j}\ ,\ d_j(k-1) + \tau_H\Big)$$

The dwell time at each station is the sum of the time for disembarking ($\tau_{d,j}(k)$) and boarding the train ($\tau_{b,j}(k)$). If we assume there is no additional waiting time the train will depart at

$$d_j(k) = a_j(k) + \tau_{d,j}(k) + \tau_{b,j}(k)$$

The number of passengers in train k when leaving station j is equal to the number of passengers in the train when leaving station $j-1$ minus the passengers disembarking the train at station j plus the passengers boarding the train at station j, so

$$\rho_j(k) = \rho_{j-1}(k) - f\tau_{d,j}(k) + b\tau_{b,j}(k)$$

The number of passengers that are still on the platform when train k leaves station j is equal to the number of passengers that were still on the platform when train $k-1$ left station j plus the number of passengers that enter the station between the departures of train $j-1$ and train j minus the passengers boarding the train at station j, so

$$\sigma_j(k) = \sigma_j(k-1) + e_j(d_j(k) - d_j(k-1)) - e_j\tau_{b,j}(k)$$

(We assume that the passengers that disembark the train immediately leave the station).

The disembark time $\tau_{d,j}(k)$ is proportional to the number of passengers that disembark, or

$$\tau_{d,j}(k) = \frac{\beta_j}{f}\,\rho_{j-1}(k)$$

Next we consider the boarding time $\tau_{b,j}(k) = d_j(k) - a_j(k) - \tau_{d,j}(k) = d_j(k) - a_j(k) - \frac{\beta_j}{f}\,\rho_{j-1}(k)$. The departure time $d_j(k)$ depends on the number of passengers who want to board the train. However if the number of passengers in the train reaches its maximum $\rho_{\max}$, some passengers will be left on the platform.

Now we consider two cases. In the first case the number of passengers that want to board the train fits in the train (so $\rho_j(k) \leq \rho_{\max}$). In the second case there are too many passengers who want to enter the train and we get $\rho_j(k) = \rho_{\max}$.

In the first case the number of passengers that actually board train k at station j (so $b(d_j(k) - a_j(k) - \frac{\beta_j}{f}\,\rho_{j-1}(k))$) is equal to the number of passengers that want to board train k at station j (so $\sigma_j(k-1)+e_j(d_j(k)-d_j(k-1))$), or

$$b(d_j(k)-a_j(k)-\frac{\beta_j}{f}\,\rho_{j-1}(k)) = \sigma_j(k-1)+e_j(d_j(k)-d_j(k-1))$$

so in case 1 we derive departure time

$$d_j(k)=\mu_1 a_j(k)+\mu_2\rho_{j-1}(k) + \mu_3\sigma_j(k-1)+(1-\mu_1)d_j(k-1)$$

where $\mu_1=\dfrac{b}{b-e_j}$, $\mu_2=\dfrac{b}{b-e_j}\dfrac{\beta_j}{f}$, and $\mu_3=\dfrac{1}{b-e_j}$.

In the second case the train leaves station k as soon as the train is full, so the number of passengers in train k after disembarking at station j plus the number of passengers boarding train k at station j is equal to the maximum capacity of the train:

$$(1-\beta_j)\rho_{j-1}(k)+b(d_j(k)-a_j(k)-\frac{\beta_j}{f}\,\rho_{j-1}(k))=\rho_{\max}$$

so in case 2 we derive the departure time

$$d_j(k)=\gamma_1+a_j(k)+\gamma_2\rho_{j-1}(k)$$

where $\gamma_1=\dfrac{1}{b}\rho_{\max}$ and $\gamma_2=\dfrac{\beta_j}{f}-\dfrac{1-\beta_j}{b}$.

Combining case 1 and case 2 gives actual departure time $d_j(k)$ which is the minimum of the values computed in case 1 and case 2:

$$d_j(k)=\min\Big(\mu_1 a_j(k)+\mu_2\rho_{j-1}(k)+\mu_3\sigma_j(k-1) + (1-\mu_1)d_j(k-1)\;,\;\gamma_1+a_j(k)+\gamma_2\rho_{j-1}(k)\Big)$$

Now the final system equations can be derived. For $j>1$ and $k>0$ we obtain the following MMPS model:

$$\begin{aligned}
a_j(k) &= \max\Big(d_{j-1}(k)+\tau_{r,j}\;,\;d_j(k-1)+\tau_H\Big)\\
d_j(k) &= \min\Big(\mu_1 a_j(k)+\mu_2\rho_{j-1}(k)+\mu_3\sigma_j(k-1)\\
&\qquad +(1-\mu_1)d_j(k-1)\;,\;\gamma_1+a_j(k)+\gamma_2\rho_{j-1}(k)\Big)\\
\rho_j(k) &= (1-\beta_j)\rho_{j\text{-}1}(k)+b(d_j(k)-a_j(k)-\frac{\beta_j}{f}\rho_j(k-1))\\
\sigma_j(k) &= \sigma_j(k-1)+e_j(d_j(k)-d_j(k-1))\\
&\qquad -b(d_j(k)-a_j(k)-\frac{\beta_j}{f}\,\rho_{j-1}(k)))
\end{aligned} \tag{6}$$

We initialize

$$\begin{aligned}
&\text{For } k=0:\;\rho_j(0)=\bar\rho_j,\;\;\sigma_j(0)=0,\;\;d_j(0)=\bar d_j,\;\;\forall j\\
&\text{For } j=1:\;d_1(k)=\bar d_1+k\,\bar\tau,\;\;\rho_1(k)=\bar\rho_1,\;\;\forall k
\end{aligned} \tag{7}$$

Time-invariancy in the urban railway line model The states of the system are now recognized as

$$x(k)=\begin{bmatrix}x_1(k)\\ \vdots\\ x_J(k)\end{bmatrix},\;\;x_{t,j}(k)=\begin{bmatrix}a_j(k)\\ d_j(k)\end{bmatrix},\;\;x_{q,j}(k)=\begin{bmatrix}\rho_j(k)\\ \sigma_j(k)\end{bmatrix}$$

$$p_t(k)=\begin{bmatrix}x_t(k)\\ x_t(k-1)\end{bmatrix}\;,\;\;p_q(k)=\begin{bmatrix}x_q(k)\\ x_q(k-1)\end{bmatrix}$$

To check time-invariancy we compute $f_{t,j}(p_{\rm t}+\lambda,p_{\rm q})$ for all j and find:

$$\begin{aligned}
&[f_{t,j}(p_{\rm t}+\lambda,p_{\rm q})]_1 =\\
&\quad= \max\Big(d_{j-1}(k)+\lambda+\tau_{r,j}\;,\;d_j(k-1)+\lambda+\tau_H\Big)\\
&\quad= \max\Big(d_{j-1}(k)+\tau_{r,j}\;,\;d_j(k-1)+\tau_H\Big)+\lambda\\
&\quad= [f_{t,j}(p_{\rm t},p_{\rm q})]_1+\lambda\\
&\quad= [f_{t,j}(p_{\rm t},p_{\rm q})]_2+\lambda
\end{aligned}$$

Simililarly we compute

$$\begin{aligned}
[f_{t,j}(p_{\rm t}+\lambda,p_{\rm q})]_2 &= [f_{t,j}(p_{\rm t},p_{\rm q})]_2+\lambda\\
[f_{q,j}(p_{\rm t}+\lambda,p_{\rm q})]_1 &= [f_{q,j}(p_{\rm t},p_{\rm q})]_1\\
[f_{q,j}(p_{\rm t}+\lambda,p_{\rm q})]_2 &= [f_{q,j}(p_{\rm t},p_{\rm q})]_2
\end{aligned}$$

We see that the system equations satisfy the time-invariancy condition, therefore the urban railway line model is time-invariant.

Steady-state for the urban railway line model Now we want a steady-state for the model such that the trains are never completely full (so no people are left on the platform, or $\sigma_j(k)=0$), and that there is always enough headway between the trains, so $d_{e,j-1}+\tau_{r,j}\geq d_{e,j}+\tau_H$. This leads to the piecewise-linear train equations in the equilibrium (for $\sigma_j(k)=0$):

$$\begin{aligned}
a_j(k) &= d_{j-1}(k)+\tau_{r,j}\\
d_j(k) &= \mu_1 a_j(k)+\mu_2\rho_{j-1}(k)\\
\rho_j(k) &= (1-\beta_j)\rho_{j\text{-}1}(k)+b(d_j(k)-a_j(k)-\frac{\beta_j}{f}\,\rho_j(k-1))\\
\sigma_j(k) &= 0
\end{aligned}$$

Consider the steady-state $(a_{\rm e,j},d_{\rm e,j},\rho_{\rm e,j},\sigma_{\rm e,j},\tau_{\rm e})$ with

$$\begin{aligned}
a_{\rm e,j}+\tau_{\rm e} &= d_{\rm e,j-1}+\tau_{\rm e}+\tau_{r,j}\\
d_{\rm e,j}+\tau_{\rm e} &= \mu_1(a_{\rm e,j}+\tau_{\rm e})+\mu_2\rho_{\rm e,j-1}\\
\rho_{\rm e,j} &= (1-\beta_j)\rho_{\rm e,j-1}+b(d_{\rm e,j}+\tau_{\rm e}-(a_{\rm e,j}+\tau_{\rm e})-\frac{\beta_j}{f}\,\rho_{\rm e,j-1}\\
\sigma_j(k) &= \sigma_{\rm e,j}=0
\end{aligned}$$

Starting with the initial conditions (7) we derive

$$\begin{aligned}
a_{\rm e,j} &= d_{\rm e,j-1}+\tau_{r,j}\\
d_{\rm e,j}+\tau_{\rm e} &= \mu_1(d_{\rm e,j-1}+\tau_{r,j}+\tau_{\rm e})+\mu_2\rho_{\rm e,j-1}\\
\rho_{\rm e,j} &= (1-\beta_j)\rho_{\rm e,j-1}+b(d_{\rm e,j}+\tau_{\rm e}-(a_{\rm e,j}+\tau_{\rm e})-\frac{\beta_j}{f}\,\rho_{\rm e,j-1}\\
\sigma_j(k) &= \sigma_{\rm e,j}=0
\end{aligned}$$

Consider the model of (6) with the following parameters: $\tau_r = 180$ s, $\tau_H = 30$ s, $\rho_{\max} = 150$ passengers, $b = f_j = 2$ passengers/s, $e_j = 0.5$ passengers/s, $\beta_j = 0.5\;\forall j$. The intitial conditions ar given by (7) with $\bar\tau = 120$ sec, $\bar\rho_j = 120$passengers, $\bar d_j = (j{-}1)120$s, $\forall j$. From Proposition 7 we find that $(a_{\rm e,j},d_{\rm e,j},\rho_{\rm e,j},\sigma_{\rm e,j},\tau_{\rm e})$ is indeed a steady-state.

Model predictive control of the urban railway line (Beek, 2022). A control input $u_j(k)$, $j=2,\ldots,N$ is introduced to increase or descrease the running time (additional to the nominal running time) of a train running from station $j-1$ to station j and so

$$a_j(k)=\max\Big(d_{j-1}(k)+\tau_{r,j}+u_j(k)\;,\;d_j(k-1)+\tau_H\Big)$$

We define the performance signal $p_j^{\rm wait}(k) = e_j(a_j(k) - d_j(k-1)) + \sigma_j(k-1)$, which respresents the number of people waiting on train k at station j at the moment of arrival of train k and define the cost function

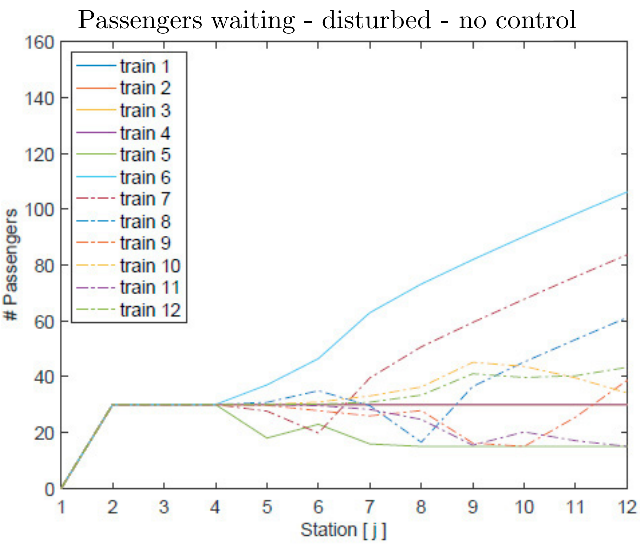


Fig. 2. Number of passengers waiting of the disturbed urban railway line.

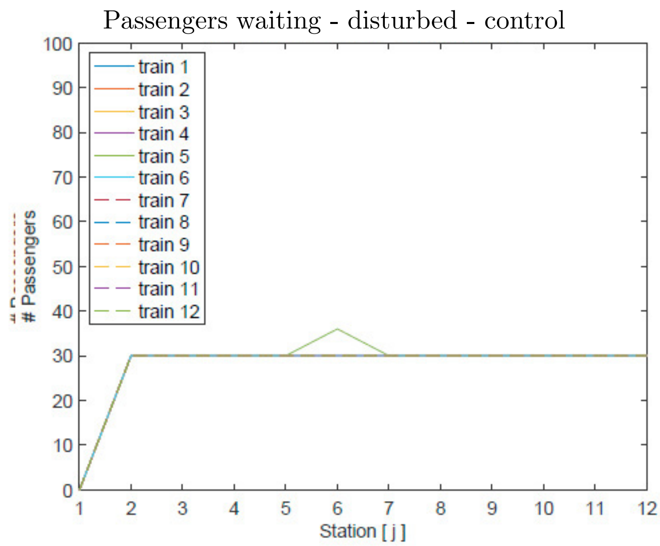


Fig. 3. Number of passengers waiting of the urban railway line with control.

$$J(k) = \sum_{i=0}^{N-1} \sum_{i=1}^{M} |p_j^{\text{wait}}(k+i) - p_{\text{ref}}^{\text{wait}}| + \lambda\, |u_j(k+i)|$$

where $\lambda = 0.1$ is a trade-off weight and $p_{\text{ref}}^{\text{wait}} = 30$ is a reference value. We introduce the constraint

$$-20 \le u_j(k) \le 70$$

If there is no disturbance or model error in the system the trains will all run with a perfect interval of 120 seconds. The number of passengers on the platform at the time of a train arriving is constant at 30. If we introduce a disturbance in the form of a decrease in the number of people entering the station, i.e. the parameter e changes from 0.5 passengers/s to 0.3 passengers/s for the 5th train at station 5.The boarding time of fifth train is decreased and so it will slowly catch with the fourth train, while the sixth train will be delayed. The number of passengers on the platforms will increase for later trains, see Figure 2. If we use MPC we see that the trains run regular again and the disturbance in the number of passengers on the platforms is limited to the fifth train on station 5, as can been seen in Figure 3.

8. DISCUSSION

In this paper max-min-plus-scaling systems are discussed in a discrete-event framework for the first time. We observed that the state will consist of time signals and possibly also of quantity signals. We studied time-invariancy for the discrete-event MMPS systems, and derived an expression for an equilibrium. An urban railway line has been studied and we derived a model predictive controller that can stabilize this system in the case of disturbances.

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