On the Historical Development of the Mathematical Theory of Water Waves

合田良実
Yoshimi Goda

Translated from the original Japanese paper published in 1999
35th Summer Training Course Lecture Collection, Japan Society of Civil Engineers Coastal Engineering Committee
http://library.jsce.or.jp/jsce/open/00027/1999/35-B03.pdf

1. Introduction

The issue of waves on the water surface has long captivated people. Leonardo da Vinci left numerous sketches of whirlpools and wave motion, revealing that waves are the rotational motion of water particles. However, it wasn't until the late 18th century that the problem of water surface waves began to be treated mathematically within the framework of classical mechanics established by Galileo and Newton.

As will be introduced in the following sections, until the birth of coastal engineering in the mid-20th century, wave problems were dealt with by mathematicians and physicists, and were outside the understanding of civil and harbour engineering technicians. This situation was radically changed by World War II. The United States and the UK needed accurate predictions of breaking wave conditions in order to successfully conduct landing operations in the Mediterranean and Pacific theatres.

Therefore, they developed techniques for predicting the development of wind waves, deformation in shallow sea areas, and estimating breaking wave heights. In addition, an artificial harbour was quickly built as a base for landing military supplies during the Normandy landing operation, and the theory of diffraction of electromagnetic waves was applied to the problem of waves. Coastal engineering, a new field born out of these military needs, was established after the war.

One of the driving forces behind the advancement of wave problem research since the 1950s was the rapid increase in oil demand. Drilling for underwater oil was limited to shallow areas with a water depth of a few metres until the 1940s, but by 1967, large platforms were being installed at points with a water depth of 104 metres, and now oil is being pumped from the seabed over 500 metres deep.

The design external force of such deep-sea oil drilling and production equipment is maximum wave power, and the calculation theory of finite amplitude waves has developed. In addition, tankers have become increasingly large to reduce transportation costs, and the sophistication of wave theory has also been pursued for the planning and construction of deep-water ports that can accept them.

Moreover, recent advancements and the miniaturisation of computers have made precise wave propagation and deformation calculations possible, and numerical calculations are now becoming a viable alternative to tank experiments. In this paper, I would like to introduce how these various issues related to surface waves have been studied.

Naturally, research on wave problems spans a vast range, and it is extremely difficult to grasp the whole picture. Here, I want to discuss it as a personal outlook, focusing on what I have studied so far. As for the wave theory itself, it is introduced by Isobe (1999) and others, who are lecturers this year, so I will describe it in a narrative style in this paper.

I believe there are not a few places where I have overlooked important parts of the history of wave research due to my lack of study. I ask for your understanding.

2. The Dawn of Wave Theory

2.1 Wave Speed of Surface Waves

The first theory of wave dynamics involved the relationship between wavelength and wave speed. Already, Newton discussed the speed of surface waves and sound waves using a pendulum analogy in his *Principia* (1687). In the UK, two years before this, Charles II, who had restored the monarchy after Cromwell's Puritan

Revolution, had passed away, and in France, Louis XIV was in the peak of his 44th year of reign. Newcomen's steam engine, which played a leading role in the Industrial Revolution, was invented 28 years later in 1705.

The 18th century was a time when fluid mechanics was established by Bernoulli, Euler, d'Alembert, Lagrange, and others. Regarding surface waves, Lagrange published the long-wave speed formula in 1783 and recorded it in the *Analytical Mechanics* (1788), published in Berlin. However, both Laplace and Poisson pointed out the error of applying this formula to waves of any water depth, and not just long waves.

Laplace (1749 – 1827) was 13 years younger than Lagrange (1736 - 1813), and Poisson was younger still, being born in 1781. Lagrange was active from the reign of Louis XVI to the era of the French Revolution, instructing French mathematics and analytical mechanics as a professor at the Ecole Polytechnique. Laplace also flourished from the revolutionary period through the Napoleonic era, and was active even until the period of the Bourbon Restoration. He is also known as a pioneer of tidal theory.

Poisson presented a theory in 1816 for phenomena such as the propagation of ripples when a stone is thrown onto the water surface, or when a part of the water surface is lifted or pushed down. This phenomenon of surface waves is referred to as the Cauchy-Poisson wave, in reference to Cauchy's presentation of a similar theory the previous year.

2.2 Theory of Trochoidal Waves

Gerstner, of Prague Technical University, was the first to explicitly present the waveform and water particle motion of surface waves in the form of mathematical equations. Although he published this in the Journal of the Bohemian Royal Society of Sciences in 1804[1], his theory remained unknown to mathematicians in France and England for over half a century. According to Stokes (1847), Russell (1844) supposedly sketched the waveform of deep-sea trochoidal waves, but this has not been confirmed.

Rankine (1862) rediscovered trochoidal waves and not only discussed the waveform and water particle motion, but also the underwater pressure. Moreover, in Stokes' (1847) theory of finite amplitude waves, the rotating orbit of water particles advances slightly forward, whereas Rankine pointed out that no such mass transport exists in trochoid waves. Rankine explained this difference as due to the assumption of zero vorticity by Stokes, while in trochoid waves, vorticity remains constant at each depth.

The trochoidal wave theories of Gerstner and Rankine are complete as theories of large wave heights in the deep sea, accurately describing the characteristics of surface waves. However, under the assumption of a perfect fluid, motion with vorticity cannot appear without human intervention. For this reason, as Lamb (1932) pointed out, they have fallen out of the mainstream of wave theory. The only exception is when Sainflou extended the trochoid wave theory to derive the wave pressure theory of duplicate waves.

2.3 Wave Theory Based on Velocity Potential

When Euler's fluid motion equation is transformed, the velocity potential φ of fluid motion is derived. Lagrange presented this, and Laplace introduced it as a fundamental equation for the velocity potential of a perfect fluid:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

The first person to derive the wave theory based on this velocity potential was Airy (1845). He published it in the "Tides and Waves" section which he contributed to the *Metropolitan Encyclopaedia* at that time, which is essentially the theory of infinitesimal amplitude waves as it exists now. Its main parts have been described in many textbooks since then, and it forms the basis of current wave theory. However, the author has not seen the original document. The dispersion relation of wavelength and wave speed, $\omega^2 = gk \tan kh$, is due to Airy.

Two years after this theory of Airy, Stokes (1847) presented the theory of finite amplitude waves by perturbation expansion. At this time, Stokes was a young and promising scholar of 28 years, presumably

[1]According to page 98 of Rouse's *History of Hydraulics*. However, Lamb (1932, p.421) states that it was in 1802. Furthermore, Lamb assesses that Rankine was not aware of Gerstner's paper.

inspired by the theory of Airy, who was 18 years older. While Stokes' formulation is slightly different from the current one, he derived up to the second-order approximation for shallow surface waves and the third-order approximation for deep sea waves. He also noted mass transport in finite amplitude waves and discussed interface waves in stratified densities.

Furthermore, in 1876, Stokes introduced the theory of group velocity as the superposition of two wave groups. At that time, the problem of group velocity seems to have attracted the attention of mathematicians and physicists (Levi, 1995 p.467), with Rayleigh (1877) also developing a general theory of group velocity.

3. Transforming Waves and Stationary Waveforms

In Stokes' 1847 paper, it is stated that if a/h is smaller than h/L^2, the wave will not deform. Conversely, it would have been naturally accepted from the observation of actual waves that long waves with large amplitudes change.

Russell (1844) pointed out that there are cases where waves propagate in a uniform cross-sectional water channel without changing shape. In 1834, when he was a 26-year-old teacher at the University of Edinburgh, the Scottish Canal Society requested an investigation into the possibility of using a paddle steam tug instead of towing barges with horses (Levi, 1995, p.180).

Russell continued to observe the canal several times for this reason, and at one point, a barge drawn by two horses stopped suddenly, then began to move forward with great momentum. The barge was lifted by a solitary wave coming from the opposite side within the canal, and was released from the flow resistance of the solitary wave with the passing of the wave peak. Russell recorded the propagation speed and wave height decay of such solitary waves along the canal.

The company that asked Russell for an investigation was likely seeking strategies to compete with railway transportation, as this was the 1830s in Britain, a time when railways were becoming more prominent than canals.

The phenomenon of solitary waves was reported earlier by Bidone (1826) in Turin, but it seems that researchers in Britain and the Netherlands did not notice it. The solitary wave phenomenon reported by Russell was seriously studied 27 years later by Boussinesq (1871) and five years later by Rayleigh (1876). Rayleigh's paper provided an overview of wave theory at that time, dealing with deep sea waves and surface vibrations in cylindrical containers.

The theory of solitary waves was later refined by McCowan (1891), who formalized waveforms, underwater pressure, effective wavelength, water particle motion, and wave energy. McCowan also mentioned the maximum wave height of solitary waves in this paper and scrutinized this issue in a paper four years later in 1894, concluding that the limit wave height was 0.78 times the water depth. This theory of solitary waves by McCowan was used as a guideline when Munk (1949) summarized the breaking wave index after World War II.

Furthermore, when calculating the wave force acting on underwater structures, it was necessary to have the vertical distribution of horizontal particle velocity, and for this purpose, Munk's computation chart for solitary waves was used until the 1960s.

On the other hand, Korteweg and de Vries (1895) derived the theory of stationary waveforms with periodicity in shallow water areas. As the waveform is represented using Jacobi's elliptic function (*cn*), they named it cnoidal wave. In developing the theory, they performed a kind of perturbation expansion for horizontal-vertical particle velocities and velocity potential. They substituted this into the flow function and found the equation of the wave shape that satisfies the condition of a steady wave form.

After deriving the condition of a higher-order approximate solution, they finally presented a second-order approximate solution of the wave form. They also proved that McCowan's solitary wave is the limiting form of the cnoidal wave. Recently, the theory of Korteweg and de Vries has been rewritten into the following equation and is used as a basic equation in numerical analysis of the behaviour of nonlinear dispersive waves such as electromagnetic waves, plasmas, and solitons (Zabusky and Galvin, 1971):

$$\frac{\partial(h+\eta)}{\partial t}+(h+\eta)\frac{\partial(h+\eta)}{\partial x}+\delta^2\frac{\partial^3(h+\eta)}{\partial x^3}=0$$

Boussinesq's (1871) research included both solitary waves and periodic waves, and it was possible to handle waves that deform as they propagate. However, as far as searching for subsequent literature is concerned, no papers appear to have inherited and developed this research in the field of wave motion. It was not until the 1970s, about 100 years later, that the Boussinesq equation became popular as the darling of numerical calculation of waves.

Research on waves from the 19th century to the early 20th century, including the search for limit wave heights in the next section, was an academic interest of mathematicians and physicists, and except for Russell's observation of solitary waves in a canal, it seems to have been detached from real engineering problems. There were limited papers dealing with waves, and not a few described their own research results after citing papers from decades ago. Compared to the current situation where many researchers are constantly publishing new papers, there was a laid-back atmosphere.

4. Approach to Limit Wave Height

The year before McCowan calculated the limit wave height of the solitary wave, Mitchell (1893) sought the breaking limit of deep-sea waves. He showed that by accurately calculating the perturbation expansion according to the definition that Stokes formed, a convex angle of 120° at the top of the limit wave, the maximum value of the wave shape gradient is 0.142, and the instantaneous wave speed is 1.2 times the small amplitude wave.

This series representation by such perturbation expansion was continued by Rayleigh (1917), Havelock (1918), and others. However, there remained doubts about the convergence of such series expansions. The person who solved this problem was Levi-Civita of the University of Rome (1925).

Discussions of limit wave heights were mostly for deep sea waves. Havelock's paper seems to have dealt with finite water depths, but it has been cited very little since then. In the field of coastal engineering, Miche (1944) discussed the breaking limit in dealing extensively with the wave problem in shallow areas and presented the following breaking limit equation.

$$(H/L)_b = 0.142 \tanh 2\pi\, (h/L)_b$$

Miche was a professor at the French civil engineering school (École des Ponts et Chaussees) founded in 1747. The school made significant contributions to the development of wave theory in the first half of the 20th century.

In the first half of the 20th century, this university greatly contributed to the development of wave theory, with Sainflou publishing a theory on overlapping wave pressure in the institution's journal (Annales des Ponts et Chaussees). Notably, the civil engineering school was repositioned as a higher-level institution following Napoleon's reform of the engineering school (École Polytechnique) in 1804 and remains so to this day (Kurita 1992, p.30).

The discussion on the limit of breaking wave height was further refined by Hikoji Yamada of the Institute of Applied Mechanics at Kyushu University (Yamada 1957 a,b; Yamada et al. 1986 a,b). Through his theoretical work and numerical calculations, he examined the limit of the steady wave form at a fixed water depth. It is safe to say that the wave height has been confirmed.

Yamada's research is occasionally cited by applied mathematicians in the West, but it remains unnoticed in the field of coastal engineering. Consequently, despite Yamada's accurate value of $0.8261h$ for the breaking wave height limit of solitary waves, the value of 0.78 by McCowan is still frequently cited.

5. Various Problems with Wave Forces Acting on Structures

For harbour engineers, the size of the wave force acting on lighthouses and breakwaters has been of interest for a long time. Stevenson, who constructed lighthouses on islands with harsh wave conditions in Scotland, had been working on wave force observations using a maximum wave pressure recorder with a panel system since 1842 (Stevenson, 1886). In the United States, Gaillard (1905) proposed a wave pressure formula in the form of a pressure head of water flow colliding with a vertical wall, based on observations along the Great Lakes.

In Japan, Hiroi (1919, 1920) proposed the famous *Hiroi formula* based on his observations of wave pressure at Otaru Harbour, considering the pressure of the falling water flow accompanied by breaking waves. This formula has been trusted for over 60 years as the basic formula for breakwater design in our country.

However, in the West, interest in vertical breakwaters has been increasing recently, and before the advancement of research on wave pressure, it was almost unknown. These wave pressure formulas were empirically derived independent of wave theory. The first to theoretically derive the problem of wave pressure was Sainflou (1928).

In Europe, where there were numerous cases of vertical breakwater damage in the early 20th century and a return to rubble mound breakwaters, Sainflou's theory, which theoretically clarified the wave pressure acting on the wall surface, garnered overwhelming support from harbour engineers (Ito, 1969). The theoretical formula was quite complicated as it was based on the trochoidal wave in shallow water. However, it seems that it gained a lot of support because Sainflou himself, a graduate of the School of Civil Engineering, also presented a simple formula convenient for use.

Sainflou's simplified wave pressure formula was approved as the basic formula at the 1935 PIANC conference, held every four years, and has since been listed in Western textbooks. In Japan, Matsuo (1941) introduced it in the form of a translation in the magazine *"Harbor"*.

The problem with Sainflou wave pressure is that the vorticity is not zero because it is based on the trochoidal wave theory, but it introduced finite amplitude effects, such as showing that the median wave height is higher than the average water surface. If one dares to apply a perturbation expansion order, it will correspond to a 1.5 order approximation.

Based on Stokes' wave theory, the finite amplitude theory of the velocity potential and the wall wave pressure were derived by Gourret (1935). According to Tanaka (1958), this is a second-order approximation theory of overlapping wave pressure. Tanaka also introduced the second-order overlapping wave pressure theory by Lagrange coordinates derived by Missie (1944). The expansion of such overlapping wave pressure theory from the second order to the third order was done by Penney and Price (1952) for deep-sea waves, and Tadjkash & Keller (1960) provided a theoretical solution for shallow surface waves.

Aida and Kakizaki (1966) further expanded the theoretical formula to the fourth order from the perspective of design wave pressure on breakwaters. However, this level of approximation accuracy includes convergence errors in the surface boundary conditions. Therefore, they calculated the wave pressure by adding a forced numerical convergence condition and obtained results that almost match the experimental results.

On the other hand, for the wave force acting on the columnar members that form the basis of the design of offshore oil drilling equipment, Morison et al. (1950) and Morison (1951) proposed a method to obtain it as the sum of drag force and inertia force. This was when offshore oil drilling was gradually moving offshore from shallow water depths, and a method for calculating wave force was sought from a practical point of view.

The theoretical basis for adding drag force and inertia force is weak, but it is presumed that the background of the proposal for the practical formula was that experimental data on drag coefficients had been accumulated from the development of wing theory for aircraft since the 1930s.

Theoretically, it is easier to derive the inertia force, and three years after Morison's announcement, MacCamy and Fuchs (1954) derived the inertia force acting on large-diameter circular columns from diffraction theory.

In addition, Aida and Yoshimura (1971)[2] developed the diffraction theory of waves by vertical elliptical cylinders and presented the inertia force coefficient when expressing wave force as inertia force.

Because marine structures are generally complex in shape, it is difficult to theoretically obtain the velocity potential. Therefore, in recent years, it has become common to solve the velocity potential by numerical analysis using various techniques.

6. Wave Generation Theory for Hydraulic Model Experiments
Creating waves in a laboratory tank to act on a model has been done for quite some time. Though I have not specifically researched this, it was done at the old Ministry of the Interior's Civil Engineering Testing Station (now the Ministry of Construction's Civil Engineering Research Institute), established in 1923, at least from the early 1930s for experiments such as harbour shielding. Froude in the UK is the pioneer of ship model testing, and there is a possibility that the first wave generation experiment was done in relation to ships.

The design and manufacture of these experimental wave-making devices was entirely based on experience. The period was set at a specified value by adjusting the rotation speed of the DC motor or the speed of the continuously variable transmission connected to the AC motor, and the wave height was adjusted by changing the amplitude of the wave-making board through trial and error. The first to present a theory for such wave-making devices, according to Biesel (1951), was Havelock (1929), who published in the *Philosophical Magazine*. I haven't researched the literature, but it's likely for a cylindrical plunger-type wave-making device.

Biesel (1951) and Biesel and Suquet (Biesel-Suquet 1951) determined the velocity potential of the waves generated by the piston type, flap type, and their mixed-type wave-making boards. The solution is a superposition of a progressive wave and a stationary attenuating wave near the wave-making board, and not only the wave height of the generated wave but also the wave force acting on the wave-making board and the required horsepower are calculated.

The latter paper (in French) was soon translated into English in the United States, which was brought to Japan and used in the design of the 105-metre large wave-making flume at the Port and Harbour Research Department of the then Transport Technology Research Institute (now the Ministry of Transport's Port and Harbour Technology Research Institute) (Tsuruta and Hisada, 1957).

When large waves are generated in the wave-making flume in shallow water, small secondary wave crests may appear between the crests of the waves. This secondary wave crest moves slower than the main crest, so it is overtaken and absorbed by the main crest as the wave propagates, and then reappears behind the main crest after a short time. This phenomenon is due to the nonlinearity of the waves. That is, strong nonlinear waves, as shown in the Stokes wave theory, have a constrained wave of twice the frequency.

This constrained wave induces a horizontal motion of twice the frequency in the water particles. However, the wave-making board only makes a sinusoidal back-and-forth motion of the fundamental frequency. Therefore, a mismatch occurs in the water particle motion at the wave-maker position. In reality, a free wave of twice the frequency in reverse phase to the constrained wave is excited, resulting in only the sinusoidal back-and-forth motion of the water particles at the wave-maker position. The free wave of twice the frequency moves slower than the constrained wave, which results in the secondary wave crest appearing to move slowly.

Furthermore, the free wave of twice the frequency interferes with the progressing wave of the fundamental frequency, exciting a wave of the third order. Therefore, when the waveform records at each point in the flume are Fourier analysed, the amplitudes of each frequency increase and decrease at a constant interval. The double and triple frequency components change in phase, and the fundamental frequency component is out of phase with these.

As the Fourier amplitudes vary spatially in this way, it is easy to think that energy exchange is occurring due to resonance between the frequency components. However, the progressing wave of the fundamental frequency, the accompanying secondary and tertiary constrained waves, the free wave of twice the frequency,

[2] Takayama is the current reference.

and the interfering waves of the fundamental and triple frequencies each propagate while retaining their own amplitude, and no energy exchange occurs between them. It only appears that the Fourier amplitudes analysed at fixed positions in the flume are changing.

As for these nonlinear wave-making phenomena, Fontanet (1961) was the first to derive a second-order theory. Using Lagrangian coordinates, he solved the velocity potential, focusing on the constrained wave of twice the frequency. Takayama made explicit the generation of the constrained and free waves, suggesting that the modulation interval of the Fourier amplitude at twice the frequency is given by the reciprocal of the difference in wave numbers between the constrained and free waves. However, as this paper was published in French, it was not well known to researchers in the English-speaking world.

The nonlinear wave-making theory in Eulerian coordinates was given by Flick and Guza (1980) along with Bendykowska and Massel (1988) among others, providing a second-order solution, and the author (Goda, 1997) presented a theoretical solution that included third-order interference. This solution to the third-order interference clarified the mechanism of the movement of the secondary wave peak, which had long remained unexplained.

7. Birth of Coastal Engineering and Introduction of Wave Deformation Calculations

Rewinding back to the first half of the 20th century, the mathematical theory of waves had been significantly developed, a state that can be overviewed in Lamb's textbook (Lamb 1932). However, such wave theories were irrelevant to harbour engineers, and coastal and port structures were designed based on empirical formulas. The only exception was the overlapping wave pressure theory by Sainflou (1928).

The question of how large waves would occur in actual seas could only be answered by empirical formulas derived from visual observations at various locations. People like Stevenson, Hiroi, and Molitor were displaying the relationship between wave height and wind speed with cross-shore distance as a parameter. There were also empirical formulas given for the wave height within the harbour when its entrance is narrowed by breakwaters, which were cited in textbooks on harbour engineering (Masaji Suzuki 1932, pages 51-52). However, concepts of wave diffraction or refraction were not known in engineering.

The outbreak of World War II dramatically changed this situation. As many books introduce, the war spread across the Pacific, Atlantic, and the Mediterranean, and it was inevitable to carry out landing operations on coasts occupied by the enemy. When soldiers were sent into enemy territory on landing craft, rough seas could capsize the boats and cause unnecessary loss of life. Therefore, accurate forecasts of weather and wave conditions on the day of the landing operation were essential, and meteorologists and oceanographers were mobilised.

Research of this type was initiated in Japan, but it was America that succeeded in establishing a wave prediction method. Under the guidance of Sverdrup, the director of the Scripps Institution of Oceanography at the University of California, Munk, a member of the institute, analysed a large amount of wave data from the sea as a member of the Army Air Corps Meteorological Section's Ocean Division. Combining this with experimental data from wind tunnels and waterways, he developed a new scientific wave calculation method by devising a model of energy transfer from wind to waves.

This was the beginning of the significant wave method, which was later expanded and reorganized with the addition of more data, and widely disseminated as the SMB method, named after the initials of Sverdrup, Munk, and Bretschneider.

Bates (1949) introduces the contribution of wave forecasting for landing operations during World War II. The wave forecasting method of Sverdrup and Munk was a military secret during the war, and it was not until 1946 that it was published at the American Geophysical Union in a co-authored paper. They also issued a manual for practitioners as publication No. 601 of the U.S. Navy Hydrographic Office in 1947, enabling many people to use the new wave forecasting method.

For landing operations, not only ocean wave information but also information on the surf zone is necessary. Therefore, charts calculating the refractive effect of waves and surf indices were compiled and delivered as manuals to weather officers (U.S. Hydrographic Office 1944). Munk's paper on surf published later (Munk

1949) appears to be a reorganisation of the data from that time using revised solitary wave theory. Regarding the creation of refraction charts, Johnson et al. (1948) compiled early methods, and later a more convenient wave-ray method was published by Arthur et al. (1949).

On the other hand, for the Normandy landing operations in June 1944, it was necessary to secure a long sandy beachhead for unloading military supplies after the successful landing. Therefore, many floating breakwaters and simple breakwaters were made in the UK and towed to be sunk offshore, allowing the rapid construction of an artificial harbour. For this plan, it was necessary to appropriately evaluate the shielding effect of breakwaters.

Someone came up with the idea to apply Sommerfeld's electromagnetic wave diffraction theory, established in physics, to surface waves. The calculation results of the diffracted wave height were compiled and published in a paper by Penney and Price (1944). Later, Johnson (1951) created diffraction charts for various openings of breakwaters, which were cited in textbooks and technical guidelines for a long time.

With the rapid development of wave calculation methods out of wartime necessity and the aim of having many engineers utilise them, a symposium on the new interdisciplinary field of "coastal engineering" was held in Long Beach, California, USA in 1951. The instigator was Professor O'Brien from the University of California, Berkeley, who, reaching out not just to civil engineers but also to experts in meteorology, geology, and other fields, organized the Wave Research Council and hosted the Coastal Engineering Conference.

As is well known, this was the origin of coastal engineering. Initially only involving the U.S. and the UK, it soon drew attention from around the world and grew into a biennial international conference.

8. Advances in Wave Theory through Offshore Oil Production Projects

The next catalyst for the development of wave theory was the advancement into deep-sea areas of offshore oil drilling projects. Oil fields are located several thousand meters below the seafloor, and after several exploratory drillings, one may be fortunate enough to succeed in finding an oil field. In order to find and extract oil from these fields, structures that can withstand high waves of 10 to 30 meters must be built in the sea.

Many of these marine structures for oil drilling are made up of large-diameter steel pipes assembled in a truss shape (platforms), with the main force against wave load being drag. Since drag is proportional to the square of the particle speed of the water, a highly accurate theory against high waves was required. As a result, since the late 1950s, there have been successive presentations on higher order finite amplitude wave theory.

First, Chappelear (1959) considered particle motion of shallow sea surface waves from deep-sea areas to a water depth wavelength ratio of about 0.1 and their breaking limits and vicinity. However, the approximation accuracy was slightly inferior compared to Yamada (1957a). Although it was easy to extend Stokes' (1847) second-order finite amplitude wave theory to a third-order approximation, Skjelbreia and Hendrikson (1960) presented a fifth-order theory using perturbation expansion.

Meanwhile, Laiton (1960) targeted extremely shallow sea areas and derived a second-order approximate solution for cnoidal and solitary waves. Compared to Stokes waves, the waves in extremely shallow sea areas require a fairly complex mathematical expansion even for a second-order approximation.

As the finite amplitude wave theory became more accurate, a large amount of numerical calculation was required to specifically determine waveforms and water particle motion. For example, Skjelbreia and Hendrikson conducted computer calculations for a wide range of water depth wavelength ratios and wave slope gradients, and compiled the results into a numerical table exceeding 400 pages for the convenience of users. However, actual design conditions do not perfectly match the conditions of such numerical tables. Therefore, research started leaning towards direct numerical calculation of wave motion in high waves, assuming the use of a computer.

Chappelear (1961) was one of the pioneers in this regard. Eventually, Dean (1965)[3], who was then employed by an oil company, developed a numerical calculation method that automatically expands the stream function to any order on a computer, and this made it possible to determine wave motion with extremely high accuracy. It seems that this stream function method is still being used as the basis for design calculations of marine structures.

Theories of regular finite amplitude waves continue to be researched, and achievements have been made in cnoidal waves and others. For these, please refer to Isobe (1999) and others at this year's summer training session.

9. Development of Numerical Analysis Method for Wave Transformation

The advancement and high-performance of computers have dramatically accelerated the numerical analysis of various wave issues. Initially, many performed numerical calculations according to theoretical solutions that had already been provided, such as the diffraction problem of waves by breakwaters or the fifth-order approximate solution of Stokes waves.

Gradually, a method was developed to solve the equations that determine wave transformation according to the seabed topography at each step, like creating refraction diagrams by computers, essentially numerical simulation of wave transformation. For the theory and methods concerning this, please refer to Nadaoka (1999) in this workshop.

The need for improved work efficiency and corresponding cost reduction seems to have stimulated the development of numerical analysis methods for refraction diagrams. Speaking of wave estimation methods for wind wave development, Wilson (1955) developed a schematic solution for wind fields that vary spatially and temporally in deep sea areas.

Furthermore, Sakamoto et al. (1960) proposed a schematic solution developed from Wilson's method for fluctuating wind fields in shallow sea areas. As these schematic solutions were time-consuming and complex, automatic analysis programs were eventually created by computers and used in actual problems.

These numerical analyses are based on linear theory and belong to this category, including the response of bays to tsunamis and resonance analysis of harbor water surfaces. Moreover, in complex seabed topography such as on a spherical shoal, the wave direction lines of the refraction diagram may intersect. In such places, diffraction waves occur due to rapid changes in the surface gradient of the wave shape.

This was known experimentally, but it was Berkhoff (1972) who made its numerical analysis possible. He presented the mild slope equation as a basic equation of wave transformation, and clarified that the wave height distribution can be obtained by its numerical analysis. Furthermore, Radder (1979) presented an elliptical equation as its approximate formula. Since then, many papers have been published on the transformation problem of linear waves.

However, they mostly remain objects of researcher interest, and their practical use is not very commonplace. This is because in practice, the main focus is on analyzing the wave height distribution in ports, for which methods such as Barailler and Gaillard (1967) based on the Green's function method and others have been developed. However, these were for regular waves and may not necessarily be practical for actual irregular waves.

In contrast, the Boussinesq equations make it possible to perform numerical calculations incorporating the nonlinearity of waves. In particular, phenomena such as the forward inclination of the wave shape just before wave breaking, or wave splitting phenomena on a submerged breakwater, can be reproduced fairly accurately by the Boussinesq equations.

The first to handle the transformation problem of nonlinear waves was Biesel (1951). He incorporated the spatial variation of the wave number f_c for the propagation of small amplitude waves in a field where the water depth changes, into the integral form, and demonstrated the change in wave shape up to just before

[3] Subsequently, Dr. Robert G. Dean moved to a university where he has been conducting research primarily on sediment transport problems, including the design of artificial beach nourishment.

wave breaking on the slope. However, this paper does not seem to have had much influence on subsequent research.

Eventually, Peregrine (1966, 1967) discussed the transformation problem of long waves on a slope and the generation problem of bores using the Korteweg & de Vries equation. Perhaps inspired by this, Byatt-Smith (1971) has discussed the application of the Boussinesq equation.

Those who promoted the development of numerical calculation models paying attention to the potential of this Boussinesq equation seem to be like Professor Abbott of the International Institute for Hydraulic and Environmental Engineering in the Netherlands. The author is not familiar with these numerical calculation models, so I cannot state when the first model was put into practical use, but I believe one of the early ones in the literature is Abbott et al. (1978). The Boussinesq equation was originally targeted for long waves, so the calculation accuracy for the propagation speed of short period waves is low. For this reason, various modifications have been made to improve the approximation accuracy of the equation, and it is now applicable over a wide range of water depth to wavelength ratios.

Calculation programs for three-dimensional topography, including the mooring area of the port, have been developed and are used in actual problems. In particular, by giving the time-space wave shape of irregular waves with a direction spectrum at the offshore boundary as input, it is possible to calculate the transformation problem of irregular waves at once (for example: Nwogu and Mansard, 1994). However, it takes quite a long time for the calculation, so we may need to wait for further improvements in computer performance to use it in practical calculations.

10. Development of Irregular Wave Theory

Actual waves are irregular waves represented by directional spectra, and in applying wave theory to practical problems, we must incorporate the irregularity of the waves. The flow of irregular wave research has been mentioned by the author in the 1992 summer training session (Goda 1992), and although it overlaps somewhat with that, it will be briefly introduced below.

It could be said that it was common sense among ocean physicists that sea waves are irregular waves and their structure is only revealed by the spectrum, from the time when waveforms of sea surface waves could be obtained. Therefore, in 1961, a workshop titled "Ocean Wave Spectrum" was held, gathering leading scientists and engineers, and the proceedings of the meeting, complete with discussions, were published by Prentice-Hall in 1963.

This record included many basic documents, such as Barber's (1963) discussion of the basic formula of directional spectrum analysis, Longuet-Higgins et al.'s (1963) introduction of directional spectrum observation by two-way inclined buoys, and Tick's (1963) presentation of the theory of secondary interference of component waves in the frequency spectrum structure. Also, the author (Goda 1970) simulated the time waveform by providing a frequency vector and analyzed the statistical properties of wave height and period, which was inspired by the discussions at this conference.

The directional spectrum is based on the recognition that ocean waves are linear superpositions of countless component waves with different frequencies and directions. The first to analyze the shallow water-refraction deformation of waves based on this concept was Pierson et al. (1952). This paper was presented at the 3rd Coastal Engineering Conference, but it was ignored by researchers and practitioners in the field of coastal engineering. At that time, the field of coastal engineering was just reaching a stage where the drawing method of refraction diagrams based on the concept of meaningful waves was established, and the importance of cutting-edge research may not have been understood. Also, at that point, it was cumbersome to draw refraction diagrams for each spectral component and it may have been considered impractical.

The next to introduce the directional spectrum to wave deformation calculation was Karlsson (1969). This research is about solving the transport equation of directional spectrum energy density, which is in the same manner as wave prediction based on the concept of the spectrum. However, since refraction changes the direction of each component wave, a 3-dimensional differential equation is solved by adding wave direction as a variable in addition to the two orthogonal axes.

In wave prediction, time is a variable, but in wave deformation, a steady state is assumed and the time term is removed. Also, the calculation proceeds from offshore to the shore, ignoring reflected waves heading offshore. Karlsson's energy balance equation was also not discussed among overseas coastal scholars for a while. The first to adopt this in practical calculation was a Japanese harbour group (Nagai et al. 1974, Goda and Suzuki 1975). It wasn't until the late 1980s that the energy balance equation started to be addressed in Europe and the US.

At the same time in Japan, a method was introduced into practical calculation to calculate the diffraction of waves by the superposition of directional spectral component waves, and in 1979 it was officially adopted in the guidance of the director of the Ministry of Transport's Port Bureau on technical standards for port facilities. However, in Europe and the US, more effort has been put into research on the deformation calculation of irregular waves in recent years, and various programs operating on PCs have been developed, aiming to spread them to developing countries.

Regarding the statistical properties of waves, the theory of wave height Rayleigh distribution by Longuet-Higgins (1952) is best known. However, his 1957 paper is more important as a basic document. He discusses not only wave height but also the statistical quantities of period and water surface slope in relation to the directional spectrum.

However, because it is somewhat difficult to understand and is rarely cited by other researchers, Longuet-Higgins published a paper in 1975 that made the section on the joint distribution of wave height and period a little easier to understand. This may be one example where even an excellent paper is not fully appreciated if the gap is large between those who digest and use it.

For irregular waves, there are many important practical themes, such as the theory of nonlinear interference between component waves. However, as this deviates somewhat from the main theme of this paper, "Wave Problems", I would like readers to refer to appropriate literature and other resources.

11. Conclusion

The wave problem is one of the few fields in civil engineering where advanced solutions can be found through mathematical theory. As initially stated, the founders of modern science such as Newton, Lagrange, and Laplace also had an interest in wave problems. In this paper, I have tried to understand the history of wave problem research under the backdrop of the time.

As is the case in any field, research is not isolated from the demands of the times. Particularly in the field of engineering, theory matures and develops with its application to real problems. Overly advanced theoretical research may be ignored if its value is not fully understood. Gerstner's trochoidal wave theory is an example of this, and the refraction calculation method of directional spectrum waves by Pearson and others is an example of theory being shelved by practitioners. There are also cases where it has not been used for more than 100 years, like Boussinesq's (1871) research.

Research themes have trends, and when one excellent study is published, many follow-up studies inspired by it ensue. The calculation of the limit wave height by Mitchell (1893) and McCowan (1894) is an example of this. However, following up on a theme often yields little result for the effort involved. It is not easy to achieve results that surpass previously published research. In the peer review of overseas journals, repeat studies are seldom highlighted.

Where we should focus the most in research is not on refining theory or experimental techniques, but on uncovering fresh and widely applicable excellent themes. One must heighten their problem consciousness and search for what kind of resolution is demanded.

If a research theme is borrowed from conference proceedings or collections of papers, the research may have few opportunities to be highlighted even after it is completed. The excavation of research themes is always a troubling matter, but recognising its importance could be said to be the first step towards excellent research results.

References Published in Japanese:

- 磯部雅彦（1999):波の基礎理論，第 35 回水工学に関する夏期研修会講義集，土木学会海岸工学委員会.
(Isobe, M., 1999. Fundamental Theory of Waves. In: 35th Summer Training Course Lecture Collection, Japan Society of Civil Engineers Coastal Engineering Committee.)

- 伊藤喜行（1969):防波堤構造論史，港湾技研資料 No.69, 78p.
(Ito, Y., 1969. History of Breakwater Structure Theory, Port and Airport Research Institute Document No.69. 78pp.)

- 栗田啓子（1992) :「エンジニア.エコノミストーフランス公共経済学の成立一 J ,東京大学出版会，305p.
(Kurita, K., 1992. Engineer. Economist- Establishment of French Public Economics. Tokyo: University of Tokyo Press. 305pp.)

- 合田良実（1992):不規則波の研究の流れと展望，第 28 回水工学に関する夏期研修会講義集，土木学会水理工学委員会， pp.B-5-l-20.
(Goda, Y., 1992. Trends and Prospects in Irregular Wave Research. In: 28th Summer Training Course Lecture Collection, Japan Society of Civil Engineers Hydraulic Engineering Committee. pp.B-5-l-20.)

- 合田良実・柿崎秀作（1966):有限振幅重複波ならびにその波圧に関する研究，港湾技術研究所報告，第 5 巻, 第 10 号，57p.
(Goda, Y. and Kakizaki, S., 1966. Research on Finite Amplitude Duplicated Waves and Their Wave Pressure. Port and Airport Research Institute Report, Vol. 5, No.10. 57pp.)

- 合田良実・鈴木康正（1975):光易型方向スペクトルによる不規則波の屈折・回折計算，港湾技研資料，No.230, 45p.
(Goda, Y. and Suzuki, Y., 1975. Refraction and Diffraction Calculation of Irregular Waves by Koshi Easy Type Direction Spectrum. Port and Airport Research Institute Document, No.230. 45pp.)

- 合田良実・吉村友司（1971):海中に孤立した巨大構造物に働く波力の計算，港湾技術研究所報告，第 10 巻, 第 4 号， pp.3-52.
(Goda, Y. and Yoshimura, T., 1971. Calculation of Wave Forces Acting on Isolated Large Structures in the Sea. Port and Airport Research Institute Report, Vol. 10, No.4. pp.3-52.)

- 坂本信雄・井島武士 ・佐藤昭二・青野.尚（1960):浅海における風波の図式計算法 t 第 7 回海岸工学講演会講演集， pp.137-147.
(Sakamoto, N., Ijima, T., Sato, S. and Aono, N., 1960. Schematic Calculation Method of Wind Waves in Shallow Sea. In: 7th Coastal Engineering Conference Proceedings. pp.137-147.)

- 鈴木雅次(1932) :「港工学」，風間書房（1952 年増補改訂版)
(Suzuki, M., 1932. Harbor Engineering. Revised and enlarged edition 1952, Kazama Shobo.)

- 田中清・室田明（ 1958):海岸と港湾の問題，「応用水理学中！！ 応用水理学（U)(石原籐次郎・本間仁編）」，丸善， pp.568-569.
(Tanaka, K. and Murota, A., 1958. Issues of Coast and Ports. In: Applied Hydrodynamics!! Applied Hydrodynamics(U) (Ishihara Tojigoro and Honma Hitoshi, eds). Maruzen. pp.568-569.)

- 鶴田千里・久田安夫（1957):大型造波水路の設計について，運輸技術研究所報告，第 7 巻，第 11 号，pp.253-325.
(Tsuruta, C. and Hisada, Y., 1957. On the Design of Large Wave Channels. Institute of Transport Technology Report, Vol. 7, No.11. pp.253-325.)

- 永井康平（1972):不規則な海の波の屈折および回折計算，港湾技術研究所報告，第 11 巻，第 1 号， pp.3-37.
(Nagatomo, Y., 1972. Calculation of Refraction and Diffraction of Irregular Waves in the Sea. Port and Airport Research Institute Report, Vol. 11, No.1. pp.3-37.)

- 永井康平・堀口孝男・高井俊郎（1974):方向スペクトルを持つ沖波の浅海域における伝播の計算について，第 21 回海岸工学講演会論文集， pp.437-448.

(Nagai, Y., Horiguchi, T. and Takai, T., 1974. On the Calculation of Propagation of Offshore Waves with Directional Spectrum in Shallow Sea Area. In: 21st Coastal Engineering Conference Proceedings. pp.437-448.)

- 灘岡和夫（1999):波動方程式一理論と数値シミュレーションー，第 35 回水工学に関する夏期研修会講義集，土木学会海岸工学委員会.
(Nadaoka, K., 1999. Theory and Numerical Simulation of Wave Equations. In: 35th Summer Training Course Lecture Collection, Japan Society of Civil Engineers Coastal Engineering Committee.)

- 広井勇(1920) :「築港（上)」，丸善.
(Hiroi, I., 1920. Construction of Ports (Volume I). Maruzen.)

- 松尾春雄（1941):サンフルー公式及その適用，港湾第 19 巻第 1 号， pp：67-77,第 2 号， pp.50-61.
(Matsuo, H., 1941. The Sainflou Formula and its Application, Harbour, Vol. 19, No.1. pp.67-77, No.2. pp.50-61.)

- H.ラウス《S.イ ン ス （ 高 橋 裕 ・鈴木高明訳）： 「水理学史」，鹿島出版会， 1974 年.
(Lamb, H. and Ince, S., translated by Takahashi, Y. and Suzuki, T., 1974. A History of Hydrodynamics. Kashima Publishing.)

Other References:

Abbott, M.B., Petersen, H.M., and Skovgaard, O., 1978. Computation of short waves in shallow water. In: Proceedings of the 16th International Conference on Coastal Engineering, Hamburg. American Society of Civil Engineers, pp. 414-433.

Airy, G.B., 1845. Tides and Waves. In: Encyclopaedia Metropolitana. London.*

Arthur, R.S., Munk, W.H., and Isaacs, J.D., 1952. The direct construction of wave rays. Transactions of the American Geophysical Union, 33(6).

Barailler, L. and Gaillard, P., 1967. Recent evolution of mathematical models of agitation due to swell: Calculation of diffraction in non-uniform depth. La Houille Blanche, 22(8), pp.267-275.

Barber, N.F., 1963. The directional resolving power of an array of wave detectors. In: Ocean Wave Spectra. Prentice-Hall Inc., pp.137-150.

Bates, C.C., 1949. Utilisation of wave forecasting in the invasions of Normandy, Burma, and Japan. Annals of the New York Academy of Sciences, 51(3), pp.545-572.

Bednarek, G. and Massel, S.R., 1988. On theory and experiment of mechanically generated waves. In: Proceedings of the 2nd International Symposium on Wave Research and Coastal Engineering, Hannover, pp.401-413.

Berkhoff, J.C.W., 1972. Computation of combined refraction-diffraction. In: Proceedings of the 13th International Conference on Coastal Engineering. American Society of Civil Engineers, pp.479-490.

Biesel, B., 1951. Study of wave propagation in water of gradually varying depth. In: Gravity Waves. National Bureau of Standard Circular 521, pp.243-253.

Biesel, F., 1951. Étude Theoretique d’un certain type d’appraeil a houle. La Houlle Blanche, pp.156-165. (Theoretical study of a certain type of wave device).

Biesel, F. and Suquet, F., 1951. Wave devices in the laboratory. La Houlle Blanche, 6(2, 4, & 5). Translated by St. Anthony Falls Hyd. Lab. University of Minnesota, Report No. 39.

Boussinesq, J., 1871. Theory of the liquid swell called a solitary wave or translation propagating in a rectangular channel. Comptes Rendus, 72, p.755.*

Byatt-Smith, J.G.B., 1971. An integral equation for unsteady surface waves and a comment on the Boussinesq equation. Journal of Fluid Mechanics, 49, pp.625-633.

Chappelear, J.H., 1959. On the theory of the highest waves. U.S. Army Corps of Engineers, Beach Erosion Board, Technical Memorandum, No. 116, 37p.

Chappelear, J.H., 1961. Direct numerical calculation of wave properties. Journal of Geophysical Research, 66, pp.501-508.

Deacon, G.E.R., 1949. Recent studies of waves and swell. Annals of the New York Academy of Sciences, 51(3), pp.475-482.

Dean, R.G., 1965. Stream function representation of nonlinear ocean waves. Journal of Geophysical Research, 70, pp.4561-4572.

Flick, R.E. and Guza, R.T., 1980. Paddle generated waves in laboratory channels. Journal of Waterway, Port, Coastal, and Ocean Engineering, 106(WW1), pp.79-97.

Fontanet, P., 1961. Theory of the generation of cylindrical waves by a flat beater. La Houille Blanche, 16(1), pp.3-31, (2), pp.174-197.

Gaillard, B., 1905. Wave action in relation to engineering structures. Engineering News, No. 188, 23 Feb.*

Goda, Y., 1970. Numerical experiments on wave statistics with spectral simulation. Port and Harbour Research Institute, 9(4), pp.3-57.

Goda, Y., 1997. Recurring evolution of water waves through nonresonant interactions. In: Proceedings of the 3rd International Symposium on Ocean Wave Measurement and Analysis (WAVES ’97). American Society of Civil Engineers, pp.1-23.

Gourret, M., 1935. On the approximate movement of waves: Application to the calculation of vertical dikes. Annales des Ponts et Chaussees, 105(16).

Havelock, T.H., 1918. Periodic irrotational waves of finite height. Proceedings of the Royal Society, Series A, 95, pp.38-51.*

Hiroi, I., 1919. On a method of estimating the force of waves. Memoirs of Engineering Faculty, Imperial University of Tokyo, 10(1), p.19.

Johnson, J.W., 1951. Generalized wave diffraction diagrams. Proceedings of the 1st Conference on Coastal Engineering, Long Beach.*

Johnson, J.W., O'Brien, M.O., and Isaacs, J.D., 1948. Graphical construction of wave diffraction diagrams. U.S. Hydrographic Office, U.S. Navy Department, Publication No. 605.*

Karlsson, T., 1969. Refraction of continuous ocean wave spectra. Proceedings of the American Society of Civil Engineers, 95(WW4), pp.471-490.

Korteweg, D.J. and de Vries, G., 1895. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, Series 5, 39(240), pp.422-443.

Laiton, E.V., 1960. The second approximation to cnoidal and solitary waves. Journal of Fluid Mechanics, 9, pp.430-444.

Lamb, H., 1932. Hydrodynamics (6th ed.). Cambridge: Cambridge University Press.

Levi, E., 1995. The Science of Water. New York: ASCE.

Levi-Civita, T., 1925. Determination rigoureuse des ondes permanentes d’ampleur finie. Mathematische Annalen, 93, pp.264-314. (Exact determination of finite amplitude permanent waves).

Longuet-Higgins, M.S., 1952. On the statistical distribution of the heights of sea waves. Journal of Marine Research, 9(3), pp.245-266.

Longuet-Higgins, M.S., 1957. The statistical analysis of a random, moving surface. Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences, 249(966), pp.321-387.

Longuet-Higgins, M.S., 1975. On the joint distribution of the periods and amplitudes of sea waves. Journal of Geophysical Research, 80(18), pp.2688-2694.

Longuet-Higgins, M.S., Cartwright, D.E., and Smith, N.D., 1963. Observation of the directional spectrum of sea waves using the motions of a floating buoy. In: Ocean Wave Spectra. Englewood Cliffs, NJ: Prentice-Hall, pp.111-132.

MacCamy, R.C. and Fuchs, R.A., 1954. Wave forces on piles: A diffraction theory. U.S. Army Corps of Engineers, Beach Erosion Board, Technical Memorandum, No. 69.

McCowan, J., 1891. On the solitary wave. The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, Series 5, 32(197), pp.45-58.

McCowan, J., 1894. On the highest wave of permanent type. The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, Series 5, 38(231), pp.351-358.

Miche, M., 1944. Mouvements ondulatoires de la mer en profondeur constante ou décroissante. Annales des Ponts et Chaussées, 121(3).* (Wave movements of the sea in constant or decreasing depth).

Mitchell, J.H., 1893. The highest waves in water. The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, Series 5, 36(220), pp.430-437.

Morison, J.R., 1951. Design of piling. Proceedings of the 1st Conference on Coastal Engineering, Long Beach, pp.149-154.

Morison, J.R., O'Brien, M.P., Johnson, J.W., and Schaaf, S.A., 1950. The force exerted by surface waves on piles. Transactions of the American Institute of Mining and Metallurgical Engineers, 189, pp.149-154.*

Munk, W.H., 1949. The solitary wave theory and its application to surf problems. Annals of the New York Academy of Sciences, 51(3), pp.376-424.

Nwogu, O. and Mansard, E.P.D., 1994. Time-domain simulation of directional wave propagation into harbours. Proceedings of HYDRO-PORT ’94, Port and Harbour Research Institute, Yokosuka, pp.243-265.

Penney, W.G. and Price, A.T., 1944. Diffraction of sea waves by breakwaters. Directorate of Miscellaneous Weapons Development, Technical History No. 26, Artificial Harbours, Section 3-D.*

Penney, W.G. and Price, A.T., 1952. Finite periodic stationary gravity waves in a perfect liquid. Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences, 224(582-593), pp.254-284.

Peregrine, D.H., 1966. Calculations of the development of an undular bore. Journal of Fluid Mechanics, 25(2), pp.321-330.

Peregrine, D.H., 1967. Long waves on a beach. Journal of Fluid Mechanics, 27(4), pp.815-827.

Pierson, W.J., Tuttell, J.J., and Woolley, J.A., 1952. The theory of the refraction of a short crested Gaussian sea surface with application to the northern New Jersey coast. Proceedings of the 3rd Conference on Coastal Engineering, pp.86-108.

Radder, A.C., 1979. On the parabolic equation method for water-wave propagation. Journal of Fluid Mechanics, 95(1), pp.159-176.

Rankine, W.J.M., 1863. On the exact form of waves near the surface of deep water. Philosophical Transactions of the Royal Society of London, 153, pp.127-138.

Rayleigh, Lord, 1876. On waves. The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, Series 5, 1(4), pp.257-279.

Rayleigh, Lord, 1877. On progressive waves. Proceedings of the London Mathematical Society, Series 1, 1, pp.21-26.

Rayleigh, Lord, 1917. On periodical irrotational waves at the surface of deep water. The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, Series 6, 32(191), pp.381-389.

Russell, J.S., 1844. Report on waves. Report on the XIV Meetings of the British Association for the Advancement of Science, York.*

Sainflou, G., 1928. Essai sur les digues maritimes verticales. Annales des Ponts et Chaussées, 98(4).* (Essay on vertical sea dikes).

Skjelbreia, L. and Hendrickson, J., 1960. Fifth order gravity wave theory. Proceedings of the 7th Conference on Coastal Engineering, The Hague, pp.184-196.

Skjelbreia, L. and Hendrickson, J. (1960) Fifth Order Gravity Wave Theory with Tables of Functions, National Engineering Science Co., Pasadena, California.

Stokes, G. G. (1847) 'On the theory of oscillatory waves', Cambridge Transactions, vol. 8, no. 4, pp. 442-455.

Stevenson, T. (1886) The Design and Construction of Harbours, 3rd edn, Adam & Charles Black.*

Sverdrup, H.U, and Munk, W.H. (1946) 'Empirical and theoretical relations between wind, sea, and swell', Transactions of the American Geophysical Union, vol. 27, no. 6, pp. 823-827.

Sverdrup, H.U. and Munk, W.H. (1947) Wind, Sea and Swell; Principles for Forecasting, U.S. Hydrographic Office, U.S. Navy Department, Pub. No. 601.

Tadjibaksh, I. and Keller, J.B. (1960) 'Standing surface waves of finite amplitude', Journal of Fluid Mechanics, vol. 8, pp. 442-451.

Tick, L.J. (1963) 'Nonlinear probability models of ocean waves', Ocean Wave Spectra, Prentice-Hall Inc., pp. 163-169.

U.S. Hydrographic Office, U.S. Navy Department (1944) Breakers and Surf: Principles in Forecasting, H.O. Pub. No. 234.*

Wilson, B.W. (1955) 'Graphical approach to the forecasting of waves in moving fetches', U.S. Army Corps of Engineers, Beach Erosion Board, Technical Memorandum, No. 73.

Yamada, H. (1957a) 'Highest waves of permanent type on the surface of deep water', Report of Research Institute of Applied Mechanics, Kyushu University, vol. 5, no. 18, pp. 37-52.

Yamada, H. (1957b) 'On the highest solitary wave', Report of Research Institute of Applied Mechanics, Kyushu University, vol. 5, no. 18, pp. 53-67.

Yamada, H., Kimura, G., and Okabe, J. (1968) 'Precise determination of the solitary wave of extreme height on water of a uniform depth', Report of Research Institute of Applied Mechanics, Kyushu University, vol. 16, no. 52, pp. 15-32.

Yamada, H. and Shiotani, T. (1968) 'On the highest water waves of permanent type', Bulletin of the Disaster Prevention Research Institute, Kyoto University, vol. 18, part 2, no. 135, pp. 1-22.

Zabusky, N. J. and Galvin, C. J. (1971) 'Shallow-water waves, the Korteweg-deVries equation and solitons', Journal of Fluid Mechanics, vol. 47, pp. 811-824.

*Note: The documents marked with this symbol have been cited from other sources, and their contents have not been verified. It should be noted that a considerable amount of literature from the 19th to the early 20th century can be obtained through inter-university library services.