Delft University of Technology

Document Version
Final published version

Citation (APA)
Wang, J., van Horssen, W. T., Han, Z. J., & Wang, J. M. (2025). Resonances and anti-resonance strategies in movable pulley actuators for bio-mimetic systems subject to transmission point disturbances. *Nonlinear Dynamics*, *113*(21), 28837-28855. https://doi.org/10.1007/s11071-025-11618-x

Nonlinear Dyn
https://doi.org/10.1007/s11071-025-11618-x

RESEARCH

Resonances and anti-resonance strategies in movable pulley actuators for bio-mimetic systems subject to transmission point disturbances

Jing Wang · Wim T. van Horssen · Zhong-Jie Han · Jun-Min Wang

Received: 19 April 2025 / Accepted: 14 July 2025

Abstract In this paper, we develop a vibrating string model to describe the oscillations within a bio-mimetic movable pulley actuator, where transmission point disturbances can induce resonances, and so jeopardise system performance. The dynamics of longitudinal axial vibrations are formulated by a wave equation on a slowly time-varying spatial domain with a moving mass and a small harmonic disturbance at the mass-point. Due to the slow spatial domain variation, a singular perturbation problem arises. Utilizing the semigroup method, we establish the existence and uniqueness of the system's solution. Through an averaging technique and an interior layer analysis, we construct formal asymptotic approximations for the solution. Our findings reveal that, for specific disturbance frequencies, many oscillation modes jump up from small order ε amplitudes to those of order $\sqrt{\varepsilon}$. To suppress the resonances, we introduce viscous damping of varying orders. By employing multiple time-scales perturbation methods, we demonstrate that different orders of the viscous damping produce distinct anti-resonance results. Lastly, numerical simulations validate both the accuracy of our analytical results and the efficacy of the anti-resonance strategies employed.

Keywords Resonance · Longitudinal axial vibrations · Time-varying spatial domain · Semigroup · Perturbation methods

Mathematics Subject Classification 34C05 · 34C15 · 34C30 · 34E05 · 34E10

1 Introduction

Bio-mimetic systems have gained widespread application in robotics, encompassing military operations, assistance for individuals handling heavy objects, and support for those with disabilities [20,28,29]. To address the demands for lightweight, compact designs, operational safety, and everyday portability, researchers have incorporated Series Elastic Actuators (SEAs) into their designs. These actuators enable energy to be stored in a spring, which can then be released as needed [17,22]. For applications requiring both high speed and substantial force, a Dual-Motor

---

J. Wang (✉)
School of Mathematical Sciences, Tianjin Normal University, Tianjin 300387, People's Republic of China
e-mail: jingwang_am@163.com

W. T. van Horssen
Department of Mathematical Physics, Delft Institute of Applied Mathematics, Delft University of Technology, Mekelweg 4, 2628CD Delft, Netherlands
e-mail: W.T.vanHorssen@tudelft.nl

Z.-J. Han
School of Mathematics, Tianjin University, Tianjin 300354, People's Republic of China
e-mail: zjhan@tju.edu.cn

J.-M. Wang
School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, People's Republic of China
e-mail: jmwang@bit.edu.cn

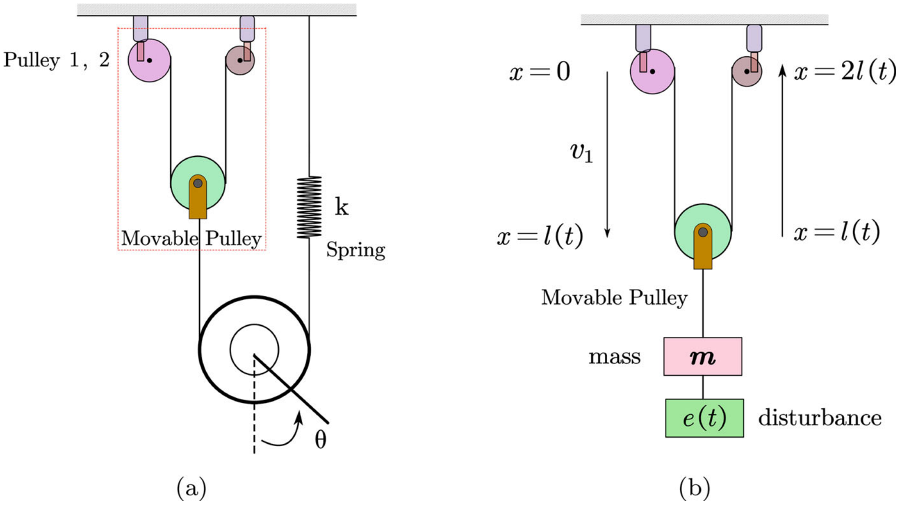


Fig. 1 **a** Operation principle of series elastic actuators in bio-mimetic systems, **b** The longitudinal vibrating string with time-varying string length and a mass in a movable pulley

System utilizing a Movable Pulley with Series Elastic Actuators (DuMPs) has been proposed, as illustrated in Fig. 1a. This system comprises a movable pulley, moving strings, a series of spring connections, and two motors (denoted as pulley 1 and pulley 2), with one motor dedicated to generating large forces and the other to achieving high speeds [18]. Within this configuration, the inherent stretchability and contractility of the moving strings can induce mechanical vibrations. When these strings are excited by unavoidable external disturbances, the vibrations can escalate into resonance, leading to significant deformations and potential catastrophic failures [1,4,19]. To ensure safe working conditions, it is imperative to investigate how external excitations influence the vibration behaviors of the strings in DuMPs, as well as how to develop strategies for mitigating or eliminating the resulting resonances.

The vibration dynamics of the DuMPs, as illustrated in Fig. 1a, can be characterized as a resonance problem triggered by external disturbances in an axially moving system with a time-varying length.

The time-varying length significantly influences the dynamic characteristics of the compliant string systems, complicating the analysis of resonance and vibration suppression. To tackle these challenges, numerous researchers have dedicated their efforts to studying the transversal displacements of axially moving systems subject to classical boundary conditions. Sandilo and van Horssen [21] explored auto-resonance phenomena in a space time-varying mechanical system with a moving Dirichlet boundary condition. Gaiko and van Horssen [7] investigated transverse vibrations of a traveling string subjected to a moving Dirichlet boundary condition with boundary damping, and further discussed resonances and vibrations in an elevator cable system due to boundary sway in [8]. Chen et al. [3] analyzed vibration responses of an axially translating string of fixed length under classical mixed boundary conditions, and in [2], they studied energy dissipation and exchange in a traveling string with fixed boundaries. Wang et al. [16] proposed a vibration suppression mechanism by using a nonlinear energy sink in a cable under harmonic loading. Gugat [9] considered an optimal control problem in a moving finite string, subject to homogeneous Dirichlet boundary conditions. Recently, researchers have begun to study longitudinal vibrations of axially moving systems subjected to moving nonclassical boundary conditions. Wang et al. [27] presented a coupled dynamic model for a flexible guiding hoisting system and numerically simulated the system's response. Crespo et al. [5] introduced a numerical simulation of a stationary high-rise elevator system. Wang et al. [24] investigated axial vibration suppression in a partial differential equation model for an ascending mining elevator cable system.

Note that these studies on longitudinal vibrations mentioned above primarily emphasize numerical simulations rather than mathematical analyses. Compared to systems with classical boundary conditions, the analytical investigation of axially moving systems with moving nonclassical boundary conditions presents a significant research challenge. Traditional analytical methods, such as the method of separation of variables, the (equivalent) Laplace transform method, and perturbation methods, are usually not straightly applicable to these problems. Consequently, there is a need to develop or adapt analytical methods to address these types of problems from a mathematical perspective. Wang and van Horssen [25] studied longitudinal oscillations in a vertically moving string subject to non-classical boundary conditions for a flexible hoisting system, and further considered resonances of transverse and longitudinal oscillations in a hoisting system due to boundary excitations in [26]. Based on these results, this paper focuses on a longitudinal resonance phenomenon in DuMPs with time-varying non-classical boundary conditions, particularly emphasizing the well-posedness of the dynamic solution and anti-resonance strategies, which have received limited attention in the literature to date. While our model was initially motivated by bio-mimetic pulley actuators, its underlying dynamics governed by the wave equation with moving boundaries and harmonic disturbances capture the vibration behavior (resonance) of a broader class of axially moving elastic transmission systems. These include: rope driven cranes, tendon-sheath drive system, linear motor flexible coupling, etc. The key conclusion arises from the relationship between the perturbation energy input, the dynamic behavior of the solution, and the damping dissipation.

The contributions of this paper include:

- The proposed longitudinal vibration dynamics of the DuMPs in bio-mimetic systems can be formulated as a novel coupled axially moving string system with a time-varying length and a moving mass transmission condition. Also, from a practical point of view, the resonance phenomenon which is caused by external impacts in the mass-point, is described and analyzed in detail;
- The well-posedness of the solution for the axially moving string system is proved by a semigroup method. It can also be a supplement to the well-posedness theory for time-varying spatial domain systems;
- We integrate and develop three classic analysis methods (an adapted version of the method of separation of variables, the method of averaging, and singular perturbation techniques) together to perform an interior layer analysis of the proposed axially moving string problem, and construct formal asymptotic approximations of the solutions (resonance results);
- To the best of our knowledge, there are few results reported in the literature on how different orders of viscous damping influence the resonances (and solution behavior) of axially moving systems.

The rest of this paper is organized as follows. Section 2 presents a dynamic model of movable pulley. Section 3 establishes the well-posedness of the problem. Section 4 derives an accurate, analytical approximate solutions of the problem. Section 5 proposes anti-resonance strategies to eliminate the resonance. Section 6 provides numerical approximations. Section 7 concludes the work.

2 Problem formulation and preliminaries

A schematic diagram of DuMPs is illustrated in Fig. 1a. The spring is connected to the lower end of the movable pulley, generating an elastic force. This force propagates through the string to the movable pulley, acting directly on its rotational axis. To focus on longitudinal rope vibrations, we simplify this configuration to the mass-boundary system, i.e., by considering the spring connection as a mass, we obtain a simplified vibration model of an axially moving string with a moving mass, where the small disrurbance $e(t)$ is supposed to be generated in the mass end, and the string in $(0, l(t))$ and $(l(t), 2l(t))$ lets the mass run up and down (see Fig. 1b). We assume that: (1)The string is inextensible. (2) The weight of the string and the movable pulley are negligible compared to the attached mass. Employing Hamilton's principle and through the derivation and simplification, the longitudinal vibration dynamics $u(x, t)$ can be formulated as:

$$\begin{cases} \rho(u_{tt} + 2vu_{xt} + v^2u_{xx}) - EAu_{xx} = 0, \\ \quad 0 < x < l(t),\ l(t) < x < 2l(t), \\ u(0, t) = 0, \quad u(2l(t), t) = 0, \quad t > 0, \end{cases} \tag{1}$$

where $v(x) = \begin{cases} v_1,\ 0 < x < l(t), \\ -v_1,\ l(t) < x < 2\,l(t), \end{cases}$ the transmission conditions at the mass point $x = l(t)$ with disturbance are obtained as:

$$\begin{cases} EA\big[u_x(l(t)^+, t) - u_x(l(t)^-, t)\big] \\ = \big[m(u_{tt} + 2vu_{xt} + v^2u_{xx})\big]_{x=l(t)} + e(t), \\ u(l(t)^-, t) = u(l(t)^+, t),\ t > 0, \end{cases} \tag{2}$$

and the initial conditions $u(x,0) = u_0(x)$, $u_t(x,0) = u_1(x)$, $0 \leq x \leq 2l_0$. $l(t)$ describes the time-varying length of the movable pulley, l_0 is the initial string length, v_1 denotes the longitudinal velocity of the string in $(0, l(t))$, ρ is the mass density of the string, m is the mass of the object, EA is the longitudinal stiffness, where E is Young's elasticity modulus, A is the cross-sectional area of the string, $e(t)$ is the longitudinal fundamental excitation at the middle point of the string. For the parameter v_1,the function $e(t)$ and the material properties, we make the following reasonable assumptions:

- The longitudinal velocity $v_1 = \varepsilon v_0$ is constant and small compared to nominal wave velocity $\sqrt{\frac{EA}{\rho}}$, i.e., $l(t) = l_0 + \varepsilon v_0 t$;
- The oscillation amplitude $e(t)$ is $O(\varepsilon)$. We aim to get the resonance results induced by a small external force, $e(t) = \beta \sin(\alpha t)$ with $\beta = \varepsilon\beta_0$;
- The initial conditions are for simplicity the same, that is, $u_0(x) = O(\varepsilon)$, and $u_1(x) = O(\varepsilon)$ for $0 \leq x \leq l(t)$, and for $l(t) \leq x \leq 2l(t)$;
- ε is a small parameter with $0 < \varepsilon << 1$.

To simplify the formulation of the problem, by using the Buckingham π-theorem [7], the initial-boundary value problem of the longitudinal motion in non-dimensional form then becomes:

$$\begin{cases} u_{tt} - u_{xx} = -2vu_{xt} - v^2 u_{xx}, \quad 0 < x < l(t), \\ \quad l(t) < x < 2l(t), \\ u(l(t)^-, t) = u(l(t)^+, t), \\ \left[u_{tt} + 2vu_{xt} + v^2 u_{xx}\right]_{x=l(t)} - \frac{\rho L}{m}\left[u_x(l(t)^+, t)\right. \\ \quad \left. -u_x(l(t)^-, t)\right] = -e(t), \\ u(0,t) = 0, \; u(2l(t), t) = 0, \quad t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad 0 \leq x \leq l_0, \\ \quad l_0 \leq x \leq 2l_0, \end{cases} \tag{3}$$

where m, ρ, α, β, L and l_0 are positive constants.

To convert the time-varying spatial domain $[0, l(t)]$ to a fixed domain $[0, 1]$, we can set $\xi = \frac{x}{l(t)}$ and $\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{l(t)}$ ($l(t) = \hat{l}(s) = l_0 \mathrm{e}^{\varepsilon v_0 s}$), then

$$u(x,t) = \begin{cases} W_1(\xi, s), & \xi \in [0,1], \\ W_2(2-\xi, s), & \xi \in (1, 2], \end{cases} \tag{4}$$

where W_1 satisfy the following equation:

$$\begin{cases} W_{1,ss} - W_{1,\xi\xi} = \varepsilon\left[v_0 \hat{l}(s) W_{1,s} + 2v_0(\xi - 1) W_{1,\xi s}\right] \\ \qquad -\varepsilon^2\left[v_0^2(\xi-1)^2 W_{1,\xi\xi} + 2v_0^2(\xi-1)W_{1,\xi}\right], \\ \qquad 0 < \xi < 1, \; s > 0, \\ W_1(0,s) = 0, \; s > 0, \\ W_{1,ss}(1,s) + \frac{2\rho L\hat{l}(s)}{m} W_{1,\xi}(1,s) - \varepsilon v_0 \hat{l}(s) W_{1,s}(1,s) \\ \quad = -\varepsilon \hat{l}^2(s)\bar{e}(s), \; s > 0, \\ W_1(\xi, 0) = W_{1,0}(\xi), \quad W_{1,s}(\xi,0) = W_{1,1}(\xi), \\ \quad 0 \leq \xi \leq 1, \end{cases} \tag{5}$$

and $\hat{l}(s) = l_0 \mathrm{e}^{\varepsilon v_0 s}$, $\bar{e}(s) = \beta_0 \sin(\frac{\alpha l_0}{\varepsilon v_0}(\mathrm{e}^{\varepsilon v_0 s} - 1))$. Since the vibration displacements $u(x,t)$ in $(0, l(t))$ and $(l(t), 2l(t))$ are symmetrical, W_1 and W_2 satisfy the same equation.

Now the original problem (3) has been simplified to a more solvable form (5). The presence of a time-varying coefficient in the boundary condition can lead to a singular perturbation problem. In the following sections, our objective is to prove the well-posedness of the problem (5) and to construct analytical approximations of the solution of problem (5) on a time-scale of order $\frac{1}{\varepsilon}$.

3 The well-posedness of the problem

This section is devoted to studying the well-posedness of problem (5), which contains a wave PDE and a second-order derivative in one of the boundary conditions. To facilitate our analysis, let us introduce a new variable: $X(s) = W_{1,s}(1,s)$, $s > 0$. Then, we rewrite the second-order boundary condition at $\xi = 1$ in problem (5) as:

$$\begin{cases} \dot{X}(s) - \varepsilon v_0 \hat{l}(s) X(s) + \frac{2\rho L\hat{l}(s)}{m} W_{1,\xi}(1,s) \\ \qquad = -\varepsilon \hat{l}^2(s)\bar{e}(s), \quad s > 0, \\ W_{1,s}(1,s) = X(s), \; s > 0, \quad X(0) = W_{1,0}(1). \end{cases} \tag{6}$$

Introduce the space $H_*^1(0,1) := \{f \in H^1(0,1) \mid f(0) = 0\}$, where $H^1(0,1)$ is the usual Sobolev space. Let the state space be

$$\mathcal{H} = \left(H_*^1(0,1) \times L^2(0,1) \times R\right) \tag{7}$$

equipped with the inner product: for $U = (\varphi, \psi, z)^T$ and $\tilde{U} = (\tilde{\varphi}, \tilde{\psi}, \tilde{z})^T$ in $\mathcal{H}$,

$$\langle U, \tilde{U} \rangle_{\mathcal{H}} = \int_0^1 \varphi_\xi \tilde{\varphi}_\xi d\xi + \int_0^1 \psi\tilde{\psi} d\xi + \frac{m}{2\rho L\hat{l}(s)} z\tilde{z}. \tag{8}$$

Obviously, $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ is a real Hilbert space. Define the operator $\mathcal{A}(s)$ with $\mathcal{A}(s)U$ equals to

$$\begin{pmatrix} \psi \\ \varphi_{\xi\xi} + \varepsilon\left[v_0\hat{l}(s)\psi + 2v_0(\xi-1)\psi_\xi\right] - \varepsilon^2\left[v_0^2(\xi-1)^2\varphi_{\xi\xi} + 2v_0^2(\xi-1)\varphi_\xi\right] \\ \varepsilon v_0\hat{l}(s)z - \frac{2\rho L\hat{l}(s)}{m}\varphi_\xi(1) \end{pmatrix},$$

and

$$\mathcal{D}(\mathcal{A}(s)) := \left\{ \begin{matrix} (\varphi,\psi,z)^T \in \mathcal{H} \mid \varphi \in H^2(0,1), \\ \psi \in H_*^1(0,1), z \in R, z = \psi(1) \end{matrix} \right\},$$

for any $U = (\varphi,\psi,z)^T \in \mathcal{D}(\mathcal{A}(s))$. Then, system (5) and (6) can be written into an evolution equation in $\mathcal{H}$:

$$\begin{cases} \dot{U}(s) = \mathcal{A}(s)U(s) + F(s), \quad s > 0, \\ U(0) = U_0 = \left(W_{1,0}(\xi), W_{1,1}(\xi), W_{1,0}(1)\right)^T, \end{cases} \tag{9}$$

where $U(s) = (\varphi,\psi,z)^T$ and $F(s) = \Big(0, 0, -\varepsilon\hat{l}^2(s)\,\bar{e}(s)\Big)^T$.

The following lemma gives an important result on the existence and uniqueness of solutions (see [14, Theorem 1.9] and also [15, Theorem 2.13]), which will be employed in this work.

Lemma 1 *Assume that*

1. $Y = \mathcal{D}(\mathcal{A}(0))$ *is a dense subset of* $\mathcal{H}$,
2. $\mathcal{D}(\mathcal{A}(s)) = \mathcal{D}(\mathcal{A}(0)), \quad \forall s > 0$,
3. *for all* $s \in [0,T]$, $\mathcal{A}(s)$ *generates a strongly continuous semigroup on* $\mathcal{H}$ *and the family* $\mathcal{A} = \{\mathcal{A}(s) | s \in [0,T]\}$ *is stable with stability constants* C *and* r *independent of* s *(i.e., the semigroup* $(S_s(\hat{s}))_{\hat{s}\geq 0}$ *generated by* $\mathcal{A}(s)$ *satisfies* $\|S_s(\hat{s})u\|_{\mathcal{H}} \leq Ce^{r\hat{s}}\|u\|_{\mathcal{H}}$, *for all* $u \in \mathcal{H}$ *and* $\hat{s} \geq 0$*),*
4. $\partial_s\mathcal{A}(s)$ *belongs to* $L_*^\infty([0,T], B(Y,\mathcal{H}))$, *which is the space of equivalent classes of essentially bounded, strongly measurable functions from* $[0,T]$ *into the set* $B(Y,\mathcal{H})$ *of bounded linear operators from* Y *into* $\mathcal{H}$.

Then, problem

$$\begin{cases} \dot{U}(s) = \mathcal{A}(s)U(s), \quad s > 0, \\ U(0) = U_0, \end{cases} \tag{10}$$

has a unique $U \in C([0,T],Y) \cap C^1([0,T],\mathcal{H})$ *for any initial data in* Y.

In order to prove the existence and uniqueness of the solution to Eq. (9), we first consider the specific case where $F(s) = 0$. Introduce the following new operator: $\tilde{A}(s) = A(s) - I, \quad s \geq 0$.

Lemma 2 *For any initial data* $U_0 \in \mathcal{H}$, *there exists a unique solution* $\tilde{U}$ *satisfying* $\tilde{U} \in C([0,+\infty),\mathcal{H})$ *for problem*

$$\begin{cases} \dot{\tilde{U}}(s) = \tilde{\mathcal{A}}(s)\tilde{U}(s), \quad s > 0, \\ \tilde{U}(0) = U_0. \end{cases} \tag{11}$$

Moreover, if $U_0 \in \mathcal{D}(\mathcal{A}(0))$, *then* $\tilde{U}(s) \in C([0,+\infty), \mathcal{D}(\tilde{\mathcal{A}}(0))) \cap C^1([0,+\infty),\mathcal{H})$.

Proof It is sufficient to verify the conditions (1)-(4) in Lemma 1 for problem (11). It is worth noting that the conditions (1) and (2) stated in Lemma 1 can be readily verified for the operator $\tilde{\mathcal{A}}(s)$, since the domain of the operator $\mathcal{A}(s)$ is not dependent on time s, i.e., $\mathcal{D}(\tilde{\mathcal{A}}(s)) = \mathcal{D}(\mathcal{A}(s)) = \mathcal{D}(\mathcal{A}(0)), \ \forall s > 0$. In the following, we divide the proof into two parts to verify the conditions (3) and (4), respectively.

Part I. Verification of condition (3) in Lemma 1.

Step 1. $\tilde{\mathcal{A}}(s)$ is dissipative.

Indeed, take $U = (\varphi,\psi,z)^T \in \mathcal{D}(\tilde{\mathcal{A}}(s))$, i.e. $U \in \mathcal{D}(\mathcal{A}(s))$, then

$$\begin{aligned} &\langle \mathcal{A}(s)U, U\rangle_{\mathcal{H}} \\ &\quad= \int_0^1 \psi_\xi\varphi_\xi d\xi \\ &\quad\quad+ \int_0^1 \varphi_{\xi\xi}\psi d\xi - \frac{2\rho L\hat{l}(s)}{m}\varphi_\xi(1)z + O(\varepsilon)\cdot\|U\|_{\mathcal{H}} \\ &\quad= O(\varepsilon)\cdot\|U\|_{\mathcal{H}} < \|U\|_{\mathcal{H}}, \end{aligned} \tag{12}$$

which means that operator $\tilde{\mathcal{A}}(s)$ is dissipative.

Step 2. Operator $\lambda I - \tilde{\mathcal{A}}(s)$ is surjective for any $\lambda > 0$ and $s > 0$.

Indeed, for any given $G = (g_1, g_2, g_3)^T \in \mathcal{H}$, we consider the solvability of equation $(\lambda I - \mathcal{A}(s))U = G, \ U \in \mathcal{D}(\mathcal{A}(s))$, i.e.,

$$\begin{cases} \lambda\varphi - \psi = g_1, \\ \lambda\psi - \varphi_{\xi\xi} - \varepsilon\left[v_0\hat{l}\psi + 2v_0(\xi-1)\psi_\xi\right] \\ \qquad + \varepsilon^2\left[v_0^2(\xi-1)^2\varphi_{\xi\xi} + 2v_0^2(\xi-1)\varphi_\xi\right] = g_2, \\ \lambda z - \varepsilon v_0\hat{l}z + \frac{2\rho L\hat{l}(s)}{m}\varphi_\xi(1) = g_3. \end{cases}$$

By substituting the first equation into the second equation, we obtain the solution for φ, then by substituting the solution for φ into the third equation, we obtain the solution $U \in \mathcal{D}(\mathcal{A}(s))$. Hence, the operator $\lambda I - \mathcal{A}(s)$ is surjective for any $\lambda > 0$ and $s > 0$. Again, we also obtain that $\lambda I - \tilde{\mathcal{A}}(s) = (\lambda + 1)I - \mathcal{A}(s)$ is surjective for any $\lambda > 0$ and $s > 0$.

Step 3. There exists a positive constant c such that

$$\frac{\|U\|_s}{\|U\|_r} \leq e^{c|s-r|}, \quad \forall s, r \in [0, T], \tag{13}$$

where $U = (\varphi, \psi, z)^T$. Indeed, for all $s, r \in [0, T]$, we have

$$\begin{aligned} &\|U\|_s^2 - \|U\|_r^2 e^{2c|s-r|} \\ &= \left(1 - e^{2c|s-r|}\right) \int_0^1 \left(\varphi_\xi^2 + \psi^2\right) d\xi \\ &\quad + \frac{m}{2\rho L l_0} \left(e^{-\varepsilon v_0 s} - e^{-\varepsilon v_0 r + 2c|s-r|}\right) z^2. \end{aligned} \tag{14}$$

Note that when $c > 0$, we get $1 - e^{2c|s-r|} < 0$, if choosing $c = \frac{\varepsilon v_0}{2}$, we have $e^{\varepsilon v_0 (r-s)} \leq e^{2c|s-r|}$, which proves (13).

Part II. Verification of the condition (4) in Lemma 1.

By direct computation, we obtain that

$$\frac{d}{ds}\mathcal{A}(s)U = \begin{pmatrix} 0 \\ \varepsilon^2 v_0^2 \hat{l}(s)\psi \\ \varepsilon^2 v_0^2 \hat{l}(s)z - \frac{2\varepsilon\rho L v_0 \hat{l}(s)}{m}\varphi_\xi(1) \end{pmatrix}. \tag{15}$$

Note that the vector in the right side of equation (15) is bounded on $[0, T]$, therefore, $\frac{d}{ds}\mathcal{A}(s)$ is bounded on $[0, T]$, that is, $\frac{d}{ds}\tilde{\mathcal{A}}(s) \in L_*^\infty([0, T], B(\mathcal{D}(\mathcal{A}(0)), \mathcal{H}))$, in which $L_*^\infty([0, T], B(\mathcal{D}(\mathcal{A}(0)), \mathcal{H}))$ is the space of equivalent classes of essentially bounded and strongly measurable functions from $[0, T]$ into the set $B(\mathcal{D}(\mathcal{A}(0)), \mathcal{H})$ of bounded linear operators from $\mathcal{D}(\mathcal{A}(0))$ into $\mathcal{H}$. Therefore, the conditions (1)-(4) in Theorem 1 for problem (11) are verified. □

Now, we are in the position to prove the existence and uniqueness of solutions to problem (9). We have the following result.

Theorem 1 *[Well-posedness] For any initial condition $U_0 \in \mathcal{H}$, there exists a unique solution U satisfying $U \in C([0, +\infty), \mathcal{H})$ for problem (9). Moreover, if $U_0 \in \mathcal{D}(\mathcal{A}(0))$, then*

$$U(s) \in C([0, +\infty), \mathcal{D}(\mathcal{A}(0))) \cap C^1([0, +\infty), \mathcal{H}).$$

Proof By Lemma 2, for $U_0 \in \mathcal{H}$, problem (11) has a unique solution $\tilde{U}(s) \in C([0, +\infty), \mathcal{D}(\tilde{\mathcal{A}}(0))) \cap C^1([0, +\infty), \mathcal{H})$. Thus, the solution to problem (9) with $F(s) = 0$ is given by

$$U(s) = e^s \tilde{U}(s).$$

Indeed, we have $U_s(s) = e^s \tilde{U}(s) + e^s \tilde{U}_s(s) = e^s (I + \tilde{\mathcal{A}}(s))\tilde{U}(s) = \mathcal{A}(s)U(s)$. Further, since $F(s) \in C[0, T]$, for any initial condition $U_0 \in \mathcal{D}(\mathcal{A}(0))$, there exists a unique solution for problem (9):

$$U(s) = e^{\int_0^s \mathcal{A}(\bar{s})d\bar{s}} U(0) + \int_0^s e^{\int_\zeta^s \mathcal{A}(\bar{s})d\bar{s}} F(\zeta)d\zeta, \tag{16}$$

which belongs to $\in C([0, +\infty), \mathcal{D}(\mathcal{A}(0))) \cap C^1([0, +\infty), \mathcal{H})$. □

4 Approximate solutions

4.1 The adapted method of separation of variables

In order to make the method of separation of variables applicable to problem (5), we reformulate the boundary condition at $\xi = 1$, and use the transformation: $W_1(\xi, s) = \tilde{W}_1(\xi, s) - \varepsilon \frac{m\hat{l}(s)}{2\rho L}\xi\bar{e}(s)$, then the problem for $\tilde{W}_1$ can be written as:

$$\begin{cases} \tilde{W}_{1,ss} - \tilde{W}_{1,\xi\xi} = \varepsilon \left[v_0 \hat{l} \tilde{W}_{1,s} + 2v_0(\xi - 1)\tilde{W}_{1,\xi s}\right] \\ \qquad + \varepsilon \frac{m\alpha^4 \hat{l}^3(s)}{2\rho L}\xi\bar{e}(s) \\ \qquad + O(\varepsilon^2), \quad 0 < \xi < 1, \ s > 0, \\ \tilde{W}_1(0, s) = 0, \ s > 0, \\ \tilde{W}_{1,\xi\xi}(1, s) + \frac{2\rho L\hat{l}(s)}{m}\tilde{W}_{1,\xi}(1, s) = 0, \quad s > 0, \\ \tilde{W}_1(\xi, 0) = \tilde{W}_{1,0}(\xi), \quad \tilde{W}_{1,s}(\xi, 0) = \tilde{W}_{1,1}(\xi), \\ \qquad 0 \leq \xi \leq 1. \end{cases} \tag{17}$$

It should be noted that $\hat{l}(s) = l_0 \mathrm{e}^{\varepsilon v_0 s}$. By defining a slow time variable $\tau = \varepsilon s$, and by letting $T(s, \tau)X(\xi, \tau)$ be a nontrivial solution of (17), we have the following lemma:

Lemma 3 *Let $\tilde{W}_1(\xi, s)$ be the solution of problem (17), then we have*

$$\tilde{W}_1(\xi, s) = \sum_{n=1}^{\infty} \tilde{T}_n(s)\sin(\lambda_n(\tau)\xi), \tag{18}$$

where $\tau = \varepsilon s$, $\lambda_n(\tau)$ is the n-th positive root of

$$\tan(\lambda_n(\tau)) = -\frac{2\rho L\bar{l}(\tau)}{m}\frac{1}{\lambda_n(\tau)}, \qquad \bar{l}(\tau) = l_0 \mathrm{e}^{v_0 \tau}, \tag{19}$$

and the following initial value problem must be satisfied with respect to $\tilde{T}_k(s)$*,*

$$\begin{cases} \tilde{T}_{k,ss} + \lambda_k^2(\tau)\tilde{T}_k = \varepsilon v_0 \bar{l} \tilde{T}_{k,s} \\ \qquad -2\sum_{n=1}^{\infty} \varepsilon\Big(c_{n,k}^1(\tau)\frac{\mathrm{d}\lambda_n(\tau)}{\mathrm{d}\tau} - v_0 c_{n,k}^2(\tau)\Big)\tilde{T}_{n,s} \\ \qquad -\varepsilon\beta_0 d_k(\tau)\sin\left(\frac{\alpha l_0}{\varepsilon v_0}\left(\mathrm{e}^{\varepsilon v_0 s} - 1\right)\right) + O\left(\varepsilon^2\right), \\ \tilde{T}_k(0) = F_k, \quad \tilde{T}_{k,s}(0) = G_k, \end{cases} \tag{20}$$

where $F_k = O(\varepsilon)$, $G_k = O(\varepsilon)$, *and*

$$\begin{aligned} c_{n,k}^1(\tau) &= \frac{\int_0^1 \sigma(\tau,\xi)\xi\cos(\lambda_n(\tau)\xi)\sin(\lambda_k(\tau)\xi)\mathrm{d}\xi}{\int_0^1 \sigma(\tau,\xi)\sin^2(\lambda_k(\tau)\xi)\mathrm{d}\xi}, \\ c_{n,k}^2(\tau) &= \frac{\lambda_n(\tau)\int_0^1 \sigma(\tau,\xi)(\xi-1)\cos(\lambda_n(\tau)\xi)\sin(\lambda_k(\tau)\xi)\mathrm{d}\xi}{\int_0^1 \sigma(\tau,\xi)\sin^2(\lambda_k(\tau)\xi)\mathrm{d}\xi}, \\ d_k(\tau) &= \frac{\int_0^1 \sigma(\tau,\xi)\left(1 + \frac{\xi^2\alpha^2\bar{l}^2}{2}\right)\sin(\lambda_k(\tau)\xi)\mathrm{d}\xi}{\int_0^1 \sigma(\tau,\xi)\sin^2(\lambda_k(\tau)\xi)\mathrm{d}\xi}, \end{aligned} \tag{21}$$

and $\sigma(\tau,\xi) = 1 - \frac{m}{2\rho L\bar{l}(\tau)}\delta(\xi-1)$ *(with* $\delta(\xi-1) = 0$ *for* $\xi \neq 1$*, and* $\int_0^1 \delta(\xi-1)\mathrm{d}\xi = \frac{1}{2}$*).*

The solution of the linear ordinary differential equation (20) with the slowly varying frequencies $\lambda_k(\tau)$ as given by (19) can be approximated by using the averaging method in the following subsections. For the motion resonance described by the ordinary differential equation subject to periodic excitations, Kandil in [11] provides the approximate solutions to the coupled blade's equations of motion by the multiple time-scales perturbation method, based on which, he analyses response curves of the system. In [10] he further explores the effects of beam parameters on the nonlinear dynamical behavior of such beams, and plots different response curves by the multiple time-scales perturbation method and Routh–Hurwitz criterion. For more coupled or nonlinear modeling with external excitations we refer to [6,12,13]. The above literature focuses on constant frequencies in time, whereas time-dependent frequencies in (19) pose greater challenges. By an interior layer analysis (including a rescaling and balancing procedure), the slowly varying frequencies $\lambda_k(\tau)$ turn out to lead to many resonance manifolds.

4.2 Resonance time

Let us introduce the following standard Liouville-Green transformations: $\phi_k(s) = \int_0^s \lambda_k(\varepsilon\bar{s})\mathrm{d}\bar{s}$ and $\Phi = \frac{\alpha l_0}{\varepsilon v_0}(\mathrm{e}^{v_0\tau} - 1)$, and according to an adapted version of the Lagrange variation of constants method, we assume that $\tilde{T}_k(s)$ and the derivative $\tilde{T}_{k,s}(s)$ can be described by $A_k(s)$, $B_k(s)$ in the following way:

$$\begin{aligned} \tilde{T}_k(s) &= A_k(s)\sin(\phi_k(s)) + B_k(s)\cos(\phi_k(s)), \\ \tilde{T}_{k,s}(s) &= \lambda_k(\tau)A_k(s)\cos(\phi_k(s)) \\ &\qquad -\lambda_k(\tau)B_k(s)\sin(\phi_k(s)). \end{aligned} \tag{22}$$

Then, by substituting (22) into problem (20), we obtain the following problem (where the dot "·" represents the differentiation with respect to s):

$$\begin{cases} \dot{A}_k = \tilde{A}_k(s) + \varepsilon\frac{\beta_0 d_k(\tau)}{2\lambda_k(\tau)} \\ \quad (\sin(\Phi + \phi_k) + \sin(\Phi - \phi_k)), \\ \dot{B}_k = \tilde{B}_k(s) + \varepsilon\frac{\beta_0 d_k(\tau)}{2\lambda_k(\tau)}(\cos(\Phi + \phi_k) \\ \quad - \cos(\Phi - \phi_k)), \\ \dot{\tau} = \varepsilon, \quad \dot{\Phi} = \alpha l_0 \mathrm{e}^{v_0\tau}, \quad \dot{\phi}_k = \lambda_k(\tau), \end{cases} \tag{23}$$

where

$$\begin{aligned} \tilde{A}_k(s) = {} & \frac{1}{2}\varepsilon v_0 \bar{l} \\ & [A_k(s)(\cos(2\phi_k(s)) + 1) - B_k(s)\sin(2\phi_k(s))] \\ + \varepsilon & \frac{\mathrm{d}\lambda_k(\tau)}{\mathrm{d}\tau}\frac{1}{2\lambda_k(\tau)} \\ & [B_k(s)\sin(2\phi_k(s)) - A_k(s)(\cos(2\phi_k(s)) + 1)] \\ & -\varepsilon\eta_{k,k}(\tau) \\ & [A_k(s)(\cos(2\phi_n(s)) + 1) - B_k(s)\sin(2\phi_k(s))] \\ & -2\varepsilon\sum_{n\neq k}\frac{\lambda_n(\tau)}{\lambda_k(\tau)}\eta_{n,k}(\tau) \\ & [A_n(s)\cos(\phi_n(s))\cos(\phi_k(s)) \\ & -B_n(s)\sin(\phi_n(s))\cos(\phi_k(s))], \\ \tilde{B}_k(s) = {} & -\frac{1}{2}\varepsilon v_0 \bar{l} \\ & [A_k(s)\sin(2\phi_k(s)) - B_k(s)(1 - \cos(2\phi_k(s)))] \\ & +\varepsilon\frac{\mathrm{d}\lambda_k(\tau)}{\mathrm{d}\tau}\frac{1}{2\lambda_k(\tau)} \\ & [A_k(s)\sin(2\phi_k(s)) - B_k(s)(1 - \cos(2\phi_k(s)))] \\ & +\varepsilon\eta_{k,k}(\tau) \\ & [A_k(s)\sin(2\phi_k(s)) - B_k(s)(1 - \cos(2\phi_n(s)))] \\ & +2\varepsilon\sum_{n\neq k}\frac{\lambda_n(\tau)}{\lambda_k(\tau)}\eta_{n,k}(\tau) \\ & [A_n(s)\cos(\phi_n(s))\sin(\phi_k(s)) \\ & -B_n(s)\sin(\phi_n(s))\sin(\phi_k(s))], \end{aligned}$$

and $\eta_{n,k}(\tau) = c_{n,k}^1(\tau)\frac{\mathrm{d}\lambda_n(\tau)}{\mathrm{d}\tau} - v_0 c_{n,k}^2(\tau)$. Resonance in (23), can be expected when $\dot{\Phi} - \dot{\phi}_k \approx 0$, or $\dot{\Phi} + \dot{\phi}_k \approx 0$.

But, since $\alpha l_0 e^{v_0\tau}$ and $\lambda_k(\tau) > 0$, resonance only will occur when

$$\alpha l_0 \mathrm{e}^{v_0\tau} \approx \lambda_k(\tau), \quad \tau > 0. \tag{24}$$

Since $\lambda_k(\tau)$ satisfies (19), that is, $\tan(\lambda_k(\tau)) = -\frac{2\rho L l_0 \mathrm{e}^{v_0\tau}}{m}\frac{1}{\lambda_k(\tau)}$, it follows that resonance occurs when

$$\lambda_k \approx \arctan\left(-\frac{2\rho L}{\alpha m}\right) + (k-1)\pi, \quad k = 1, 2, \ldots, \tag{25}$$

corresponding to the manifold τ around τ_k with

$$\tau_k = \frac{1}{v_0}\ln\left(\frac{\arctan(-\frac{2\rho L}{\alpha m}) + (k-1)\pi}{\alpha l_0}\right),$$
$$\lambda_k \geq \alpha l_0, \quad k = 1, 2, \ldots \tag{26}$$

Thus, from (26) and $\tau_k = \varepsilon s_k$, we can conclude that the resonance time of problem (20) is $\frac{\tau_k}{\varepsilon}$, that is, resonance occurs around the time $\frac{\tau_k}{\varepsilon}$, and according to the expression for τ_k, we also can conclude that no matter what the frequency of external force is, there will be many resonance manifolds.

4.3 Resonance zones and approximate solutions

Theorem 2 *[Asymptotic approximations] Let $W_1(\xi, s)$ be the solution of problem (5), then the explicit approximations of $W_1(\xi, s)$ on long time-scales can be constructed as follows:*

$$\begin{aligned}
W_1(\xi, s) = \sum_{k=1}^{\infty} \sqrt{\varepsilon} M_k
&\Big[\Big(\sin\Big(\frac{\alpha l_0}{\varepsilon v_0}(\mathrm{e}^{\varepsilon v_0 s_k} - 1)\Big)C_{Fr}(s, s_k) \\
&+ \cos\Big(\frac{\alpha l_0}{\varepsilon v_0}(\mathrm{e}^{\varepsilon v_0 s_k} - 1)\Big)S_{Fr}(s, s_k)\Big) \\
&\sin\Big(\int_{s_k}^{s} \lambda_k(\varepsilon \bar{s})\mathrm{d}\bar{s}\Big) \\
&\Big(-\cos\Big(\frac{\alpha l_0}{\varepsilon v_0}(\mathrm{e}^{\varepsilon v_0 s_k} - 1)\Big)C_{Fr}(s, s_k) \\
&+ \sin\Big(\frac{\alpha l_0}{\varepsilon v_0}(\mathrm{e}^{\varepsilon v_0 s_k} - 1)\Big)S_{Fr}(s, s_k)\Big) \\
&\cos\Big(\int_{s_k}^{s} \lambda_k(\varepsilon \bar{s})\mathrm{d}\bar{s}\Big)\Big] \sin(\lambda_k(\varepsilon s)\xi) \\
&+ O(\varepsilon), \ 0 < \xi < 1,
\end{aligned} \tag{27}$$

where $s_k = \dfrac{\tau_k}{\varepsilon}$, τ_k is given by (26), $\lambda_k(\tau)$ satisfies (19), and

$$M_k = \frac{\bar{\alpha}\beta_0 d_k(\tau_k)}{2\lambda_k(\tau_k)}, \quad \bar{\alpha} = \sqrt{\frac{2}{\mu}}, \quad \bar{\beta} = \sqrt{\frac{\mu}{2}},$$

$$\mu = \frac{m v_0 \lambda_k^3(\tau_k)}{m\lambda_k^2(\tau_k) - \rho L \bar{l}(\tau_k)\cos^2(\lambda_k(\tau_k))} \neq 0,$$

$$C_{Fr}(s, s_k) = \int_{-\sqrt{\varepsilon}\bar{\beta}s_k}^{\sqrt{\varepsilon}\bar{\beta}(s-s_k)} \cos(u^2)\mathrm{d}u, \ S_{Fr}(s, s_k) = \int_{-\sqrt{\varepsilon}\bar{\beta}s_k}^{\sqrt{\varepsilon}\bar{\beta}(s-s_k)} \sin(u^2)\mathrm{d}u. \tag{28}$$

Proof Outside the resonance manifold, we can average the right-hand side of the equations in (23) over ϕ_k and Φ while keeping A_k and B_k fixed [23]. Note that $\tilde{A}_k(s)$ and $\tilde{B}_k(s)$ are slowly varying, therefore they will not average out. The last terms of the first and second equations in (23) are fast varying terms outside the resonance manifolds, they will average out. Thus, the averaged equation for A_k now becomes

$$\begin{aligned}
\dot{A}_k^a = \Big[&\frac{1}{2}\varepsilon v_0 \bar{l} - \varepsilon c_{k,k}^1(\tau)\frac{d\lambda_k(\tau)}{d\tau} \\
&+ \varepsilon v_0 c_{k,k}^2(\tau) - \varepsilon\frac{\mathrm{d}\lambda_k(\tau)}{\mathrm{d}\tau}\frac{1}{2\lambda_k}\Big]A_k^a,
\end{aligned} \tag{29}$$

where the upper index a indicates that this is the averaged function, and B_k^a satisfies the same equation. We then obtain

$$\begin{aligned}
&A_k^a(s) = \frac{G_k}{\lambda_k(0)}e^{-\int_0^\tau \zeta(\varrho)\mathrm{d}\varrho}, \quad B_k^a(s) = F_k e^{-\int_0^\tau \zeta(\varrho)\mathrm{d}\varrho}, \\
&\zeta(\tau) = \frac{1}{2}v_0\bar{l} - c_{k,k}^1(\tau)\frac{d\lambda_k(\tau)}{d\tau} \\
&\qquad + v_0 c_{k,k}^2(\tau) - \frac{\mathrm{d}\lambda_k(\tau)}{\mathrm{d}\tau}\frac{1}{2\lambda_k},
\end{aligned} \tag{30}$$

where the initial conditions $G_k = O(\varepsilon)$, $F_k = O(\varepsilon)$ are given in (20). Hence, outside the resonance manifold the solution of system (20) can be obtained as:

$$\begin{aligned}
\tilde{T}_k(s) = &\frac{G_k}{\lambda_k(0)}\mathrm{e}^{-\int_0^{\varepsilon s} \zeta(\varrho)\mathrm{d}\varrho}\sin(\phi_k(s)) \\
&+ F_k \mathrm{e}^{-\int_0^{\varepsilon s} \zeta(\varrho)\mathrm{d}\varrho}\cos(\phi_k(s)),
\end{aligned} \tag{31}$$

where $s = O(\frac{1}{\varepsilon})$, based on which, we observe that outside the resonance zone $\tilde{T}_k(s)$ remains order ε.

To study the behavior of the solution in the resonance zone, we introduce $\psi = \Phi(t) - \phi_k(t)$ and rescale $\tau - \tau_k = \delta(\varepsilon)\bar{\tau}$ with $\bar{\tau} = O(1)$, and τ_k is given by (26). System (23) then becomes:

$$\begin{aligned}
&\dot{A}_k^a = \tilde{A}_k(s) + \varepsilon\frac{\beta_0 d_k(\tau)}{2\lambda_k(\tau)}(\sin(\Phi + \phi_k) + \sin(\psi)), \\
&\dot{B}_k^a = \tilde{B}_k(s) + \varepsilon\frac{\beta_0 d_k(\tau)}{2\lambda_k(\tau)}(\cos(\Phi + \phi_k) - \cos(\psi)),
\end{aligned} \tag{32}$$

combined with the slow/fast variables

$$\begin{cases} \dot{\tau} = \varepsilon,\ \dot{\bar{\tau}} = \frac{\varepsilon}{\delta(\varepsilon)},\ \dot{\Phi} = \lambda_k(\tau_k)\mathrm{e}^{v_0\delta(\varepsilon)\bar{\tau}}, \\ \quad \dot{\phi}_k = \lambda_k(\tau_k + \delta(\varepsilon)\bar{\tau}), \\ \dot{\psi} = \lambda_k(\tau_k)\mathrm{e}^{v_0\delta(\varepsilon)\bar{\tau}} - \lambda_k(\tau_k + \delta(\varepsilon)\bar{\tau}) \\ \quad = \left(v_0\lambda_k(\tau_k) - \frac{\mathrm{d}\lambda_k}{\mathrm{d}\tau}|_{\tau=\tau_k}\right)\delta(\varepsilon)\bar{\tau} + O(\delta^2(\varepsilon)). \end{cases} \tag{33}$$

By differentiating (19) with respect to τ, we obtain

$$\frac{\mathrm{d}\lambda_k(\tau)}{\mathrm{d}\tau} = \frac{-\rho L v_0 \bar{l}(\tau)\lambda_k(\tau)\cos^2(\lambda_k(\tau))}{m\lambda_k^2(\tau) - \rho L\bar{l}(\tau)\cos^2(\lambda_k(\tau))}. \tag{34}$$

This implies for $\dot{\psi}$ (see (33)) that $\dot{\psi} = \mu\delta(\varepsilon)\bar{\tau} + O(\delta^2(\varepsilon))$, where μ is given by (28). It now follows from (33) that a balance occurs by choosing $\frac{\varepsilon}{\delta(\varepsilon)} = \delta(\varepsilon)$, that is, $\delta(\varepsilon) = \frac{\sqrt{\varepsilon}}{\varepsilon}$. This is the size of the resonance layer. So, together with $\tau - \tau_k = \delta(\varepsilon)\bar{\tau}$, it follows from (33) that $\bar{\tau} = \sqrt{\varepsilon}(s - s_k)$, $s_k = \frac{\tau_k}{\varepsilon}$, and so $\psi(s) = \psi(s_k) + \frac{1}{2}\mu\varepsilon(s - s_k)^2$. Hence, in the resonance zone, A_k^a can be written as

$$\begin{aligned} A_k^a(s) &= \frac{G_k}{\lambda_k(0)}\mathrm{e}^{-\int_0^{\varepsilon s}\zeta(\varrho)\mathrm{d}\varrho} \\ &+ \frac{\varepsilon\beta_0 d_k(\tau_k)}{2\lambda_k(\tau_k)} \\ &\int_0^s \sin\left[\frac{1}{2}\mu\varepsilon(\bar{s} - s_k)^2 + \frac{\alpha l_0}{\varepsilon v_0}\left(\mathrm{e}^{\varepsilon v_0 s_k} - 1\right) - \phi_k(s_k)\right]\mathrm{d}\bar{s} \\ &+ O(\varepsilon) \\ &= \sqrt{\varepsilon}M_k \sin\left(\frac{\alpha l_0}{\varepsilon v_0}\left(\mathrm{e}^{\varepsilon v_0 s_k} - 1\right) - \phi_k(s_k)\right)C_{Fr}(s, s_k) \\ &+ \sqrt{\varepsilon}M_k \cos\left(\frac{\alpha l_0}{\varepsilon v_0}\left(\mathrm{e}^{\varepsilon v_0 s_k} - 1\right) - \phi_k(s_k)\right)S_{Fr}(s, s_k) \\ &+ O(\varepsilon), \end{aligned} \tag{35}$$

where μ, M_k, $C_{Fr}(s, s_k)$ and $S_{Fr}(s, s_k)$ are given by (28). B_k^a can also be approximated by a similar expression as for A_k^a. So, in the resonance zone, by using the transformations (17), (18), (22), and (35), the solution $W_1(\xi, s)$ of problem (5) can be derived and is given by (27). □

It follows from (26) and Theorem 2, we can conclude that:

Remark 1 [Resonance results] For a fixed external disturbance frequency α, the resonance always occurs for time s near resonance time s_k (26). At s_k, the k-th resonance mode jumps up from $O(\varepsilon)$ to $O(\sqrt{\varepsilon})$, and the size of the resonance zone in s is of $O(\frac{1}{\sqrt{\varepsilon}})$.

5 Anti-resonance strategies

In this section, we aim to eliminate the longitudinal resonances present in problem (1)-(2) by introducing the damping terms $c_1 u_t(x, t)$ in equation (1) and $c_2 u_t(l(t), t)$ in the transmission condition (2) with possibly different orders in magnitude in the damping coefficient. Then, the problem can be reformulated as follows:

$$\begin{aligned} &\rho(u_{tt} + 2vu_{xt} + v^2 u_{xx}) - EAu_{xx} \\ &\quad + c_1 u_t = 0,\ 0 < x < l(t),\ l(t) < x < 2l(t). \end{aligned} \tag{36}$$

The transmission conditions at the disturbance point $x = l(t)$ can be written as:

$$\begin{aligned} &EA\left[u_x(l(t)^+, t) - u_x(l(t)^-, t)\right] \\ &\quad = \left[m(u_{tt} + 2vu_{xt} + v^2 u_{xx}) + c_2 u_t\right]|_{x=l(t)} + e(t). \end{aligned}$$

Theorem 3 *[Anti-resonance results] Assuming that the damping parameters c_1 and c_2 are of the order $O(\varepsilon^{\hat{\alpha}})$ with $0 < \varepsilon << 1$, $0 \le \hat{\alpha} < 1$, and $c_1 < c_2$, along with the conditions that the string mass ρL and movable pulley mass m satisfy $m < 2\rho L\bar{l}(\tau)$, we have the following two anti-resonance results:*

Case 1: when $\frac{1}{2} \le \hat{\alpha} < 1$, resonance suppression is not achievable;

Case 2: when $0 \le \hat{\alpha} < \frac{1}{2}$, resonance suppression is achievable.

Proof In the following, we divide the proof into two steps. In step 1, we transform the original problem into different order problems by identifying feasible timescales. In step 2, we analyze the order problems for two cases in detail.

Step 1 (Obtain equations up to $O(\varepsilon\sqrt{\varepsilon})$). Using the same hypothesis and following the analysis as given in (3)-(5) and (17)-(20), that is, $u \rightarrow u^* \rightarrow W_i \rightarrow \tilde{W}_1(\xi, s) \rightarrow \tilde{T}_k$, we can derive the problem for $\tilde{T}_k(s)$:

$$\begin{aligned} &\tilde{T}_{k,ss} + \lambda_k^2(\tau)\tilde{T}_k + c_1\bar{l}(\tau)\tilde{T}_{k,s} + \sum_{n=1}^{\infty} c_{n,k}^3(\tau)\tilde{T}_{n,s} \\ &= \varepsilon v_0\bar{l}\tilde{T}_{k,s} - 2\varepsilon\sum_{n=1}^{\infty}\left(c_{n,k}^1(\tau)\frac{\mathrm{d}\lambda_n(\tau)}{\mathrm{d}\tau} - v_0 c_{n,k}^2(\tau)\right)\tilde{T}_{n,s} \\ &\quad - \varepsilon\beta_0 d_k(\tau)\sin\left(\frac{\alpha l_0}{\varepsilon v_0}(\mathrm{e}^{\varepsilon v_0 s} - 1)\right), t, \tau > 0, \end{aligned} \tag{37}$$

where

$$
\begin{aligned}
&c_{n,k}^{3}(\tau)\\
&=\frac{(c_2-c_1)\sin(\lambda_n(\tau))[\int_0^1\sigma(\tau,\xi)\sin(\lambda_k(\tau)\xi)(\lambda_n^2(\tau)\xi^2+2)\mathrm{d}\xi]}{2\bar{l}(\tau)\int_0^1\sigma(\tau,\xi)\sin^2(\lambda_k(\tau)\xi)\mathrm{d}\xi},\\
&c_{k,k}^{3}(\tau)\\
&=\frac{(c_2-c_1)\sin^2(\lambda_k(\tau))}{\bar{l}(\tau)}\left(1-\frac{m}{2\rho L\bar{l}(\tau)}\right).
\end{aligned}
$$

In the previous section, it was shown that (under certain conditions on the fundamental excitation frequency α) resonances can occur around times s_k, for $k = 1, 2, ...$, where s_k is given by (26). In order to construct accurate approximations in the neighborhood of s_k, we rescale s with $s = \tilde{s} + s_k$, $\tau = \varepsilon\tilde{s} + \tau_k$, and set $\phi_k(s) = \int_0^s \lambda_k(\varepsilon\bar{s})d\bar{s}$, $\phi_k(s) = \phi_k(\tilde{s} + s_k) = \tilde{\phi}_k(\tilde{s}) = \int_{-s_k}^{\tilde{s}} \lambda_k(\tau_k + \varepsilon\bar{s})\mathrm{d}\bar{s}$. Introduce three timescales $\tilde{s}_0 = \tilde{s}$, $\tilde{s}_1 = \sqrt{\varepsilon}\tilde{s}$, $\tilde{s}_2 = \varepsilon\tilde{s}$, $\tau = \tilde{s}_2 + \tau_k$, and so

$$
\begin{aligned}
\tilde{\phi}_{k,0} &= \int_a^{\tilde{s}_0} \lambda_k(\tau_k + \varepsilon\bar{s})\mathrm{d}\bar{s},\\
\tilde{\phi}_{k,1} &= \int_b^{\tilde{s}_1} \lambda_k(\tau_k + \sqrt{\varepsilon}\bar{s})\mathrm{d}\bar{s},\\
\tilde{\phi}_{k,2} &= \int_c^{\tilde{s}_2} \lambda_k(\tau_k + \bar{s})\mathrm{d}\bar{s},
\end{aligned}
$$

where $a = -s_k$, $b = -\sqrt{\varepsilon}s_k$, $c = -\varepsilon s_k$. These scalings are based on the size of the resonance zone $O(\frac{1}{\sqrt{\varepsilon}})$ (which has been found in the previous section), and on the natural scalings for weakly nonlinear equations. By using the three timescales perturbation method, the function $\tilde{T}_k(s)$ is supposed to be a function of time variable $\tilde{\phi}_{k,0}, \tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}$, that is, $\tilde{T}_k(s) = w_k(\tilde{\phi}_{k,0}, \tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}; \sqrt{\varepsilon})$. By substituting it into (37), we obtain equations up to $O(\varepsilon\sqrt{\varepsilon})$ (see(63)).

Step 2 (Analyze the order problems for two cases). By using the three-timescales perturbation method, the function $w_k(\tilde{\phi}_{k,0}, \tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}; \sqrt{\varepsilon})$ is approximated by the formal asymptotic expansion:

$$
\begin{aligned}
&w_k(\tilde{\phi}_{k,0}, \tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}; \sqrt{\varepsilon})\\
&\quad= \sqrt{\varepsilon}w_{k,0}(\tilde{\phi}_{k,0}, \tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}; \sqrt{\varepsilon})\\
&\quad+\varepsilon w_{k,1}(\tilde{\phi}_{k,0}, \tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}; \sqrt{\varepsilon})\\
&\quad+\varepsilon\sqrt{\varepsilon}w_{k,2}(\tilde{\phi}_{k,0}, \tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}; \sqrt{\varepsilon}) + O(\varepsilon^2).
\end{aligned}
\tag{38}
$$

Based on the different orders of damping coefficients, we analyze the order problems for two cases.

Case 1: $c_1 = O(\varepsilon^{\hat{\alpha}})$, $c_2 = O(\varepsilon^{\hat{\alpha}})$, and $\frac{1}{2} \leq \hat{\alpha} < 1$.

By substituting (38) into problem (63), and after equating the coefficients of like powers in $\sqrt{\varepsilon}$, we obtain as the $O(\sqrt{\varepsilon})$-problem:

$$
\begin{aligned}
&\frac{\partial^2 w_{k,0}}{\partial\tilde{\phi}_{k,0}^2} + w_{k,0} = 0, \quad w_{k,0}(0,0,0) = 0,\\
&\frac{\partial w_{k,0}}{\partial\tilde{\phi}_{k,0}}(0,0,0) = 0,
\end{aligned}
\tag{39}
$$

as the $O(\varepsilon)$-problem:

$$
\begin{aligned}
&\frac{\partial^2 w_{k,1}}{\partial\tilde{\phi}_{k,0}^2} + w_{k,1}\\
&= -P_k(\tau)\frac{\partial w_{k,0}}{\partial\tilde{\phi}_{k,0}}\\
&-\sum_{n\neq k} c_{n,k}^3(\tau)\frac{\lambda_n(\tau)}{\lambda_k^2(\tau)}\frac{\partial w_{n,0}}{\partial\tilde{\phi}_{n,0}}\\
&-2\frac{\partial^2 w_{k,0}}{\partial\tilde{\phi}_{k,0}\partial\tilde{\phi}_{k,1}} - \beta_0\bar{l}(\tau)d_k(\tau)\sin\left(\frac{\alpha l_0}{\varepsilon v_0}(\mathrm{e}^{v_0\tau}-1)\right),\\
&w_{k,1}(0,0,0) = \bar{F}_k,\\
&\frac{\partial w_{k,1}}{\partial\tilde{\phi}_{k,0}}(0,0,0) = -\frac{\partial w_{k,0}}{\partial\tilde{\phi}_{k,1}}(0,0,0) + \frac{\bar{G}_k}{\lambda_k(0)},
\end{aligned}
\tag{40}
$$

where $P_k(\tau) = \frac{\bar{c}_1\bar{l}(\tau)}{\lambda_k(\tau)} + \frac{\bar{c}_{k,k}^3(\tau)}{\lambda_k(\tau)} > 0$, $\bar{c}_1$, $\bar{c}_{k,k}^3 = O\left(\varepsilon^{\alpha-\frac{1}{2}}\right)$,

and as the $O(\varepsilon\sqrt{\varepsilon})$-problem:

$$
\begin{aligned}
&\frac{\partial^2 w_{k,2}}{\partial\tilde{\phi}_{k,0}^2} + w_{k,2}\\
&= -\frac{\bar{c}_1\bar{l}(\tau)}{\lambda_k(\tau)}\frac{\partial w_{k,1}}{\partial\tilde{\phi}_{k,0}} - \frac{\bar{c}_1\bar{l}(\tau)}{\lambda_k(\tau)}\frac{\partial w_{k,0}}{\partial\tilde{\phi}_{k,1}}\\
&-2\frac{\partial^2 w_{k,1}}{\partial\tilde{\phi}_{k,0}\partial\tilde{\phi}_{k,1}}\\
&-2\frac{\partial^2 w_{k,0}}{\partial\tilde{\phi}_{k,0}\partial\tilde{\phi}_{k,2}} - \frac{\partial^2 w_{k,0}}{\partial\tilde{\phi}_{k,1}^2}\\
&+\Bigg[v_0\bar{l}(\tau)\lambda_k(\tau) - \sum_{n=1}^{\infty} c_{n,k}^3(\tau)\frac{\lambda_n(\tau)}{\lambda_k^2(\tau)}\frac{\partial w_{n,0}}{\partial\tilde{\phi}_{n,1}}\\
&-\sum_{n=1}^{\infty} c_{n,k}^3(\tau)\frac{\lambda_n(\tau)}{\lambda_k^2(\tau)}\frac{\partial w_{n,1}}{\partial\tilde{\phi}_{n,0}}\\
&-\frac{\mathrm{d}\lambda_k(\tau)}{\mathrm{d}\tau}\Bigg]\frac{1}{\lambda_k^2(\tau)}\frac{\partial w_{k,0}}{\partial\tilde{\phi}_{k,0}}
\end{aligned}
$$

$$-2\sum_{n=1}^{\infty}\left[c^1_{n,k}(\tau)\frac{\mathrm{d}\lambda_n(\tau)}{\mathrm{d}\tau} - v_0 c^2_{n,k}(\tau)\right]\frac{1}{\lambda_n(\tau)}\frac{\partial w_{n,0}}{\partial\tilde{\phi}_{n,0}},$$

$$w_{k,2}(0,0,0) = 0, \quad \frac{\partial w_{k,2}}{\partial\tilde{\phi}_{k,0}}(0,0,0) = -\frac{\partial w_{k,0}}{\partial\tilde{\phi}_{k,2}}(0,0,0) - \frac{\partial w_{k,1}}{\partial\tilde{\phi}_{k,1}}(0,0,0), \tag{41}$$

Firstly, the $O(\sqrt{\varepsilon})-$ problem has solutions

$$\begin{aligned} &w_{k,0}(\tilde{\phi}_{k,0}, \tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}; \sqrt{\varepsilon}) \\ &= C_{k,1}(\tilde{\phi}_{k,1}, \tilde{\phi}_{k,2})\sin(\tilde{\phi}_{k,0}) \\ &\quad + C_{k,2}(\tilde{\phi}_{k,1}, \tilde{\phi}_{k,2})\cos(\tilde{\phi}_{k,0}), \end{aligned} \tag{42}$$

where $C_{k,1}$ and $C_{k,2}$ are still unknown functions depending on the slow variables $\tilde{\phi}_{k,1}$ and $\tilde{\phi}_{k,2}$, and they can be determined by avoiding secular terms in the solutions of the $O(\varepsilon)-$ and the $O(\varepsilon\sqrt{\varepsilon})-$ problems. By using the initial conditions in Eq.(39), it follows that $C_{k,1}(0,0) = C_{k,2}(0,0) = 0$. Then, the $O(\varepsilon)-$ problem (40) (outside as well as inside the resonance manifold) can be written as

$$\begin{aligned} &\frac{\partial^2 w_{k,1}}{\partial\tilde{\phi}^2_{k,0}} + w_{k,1} \\ &= -2\left[\frac{\partial C_{k,1}}{\partial\tilde{\phi}_{k,1}}\cos(\tilde{\phi}_{k,0}) - \frac{\partial C_{k,2}}{\partial\tilde{\phi}_{k,1}}\sin(\tilde{\phi}_{k,0})\right] \\ &\quad - P_k(\tau)\left[C_{k,1}\cos(\tilde{\phi}_{k,0}) - C_{k,2}\sin(\tilde{\phi}_{k,0})\right] \\ &\quad - \sum_{n\neq k} c^3_{n,k}(\tau)\frac{\lambda_n(\tau)}{\lambda^2_k(\tau)} \\ &\quad \left[C_{n,1}\cos(\tilde{\phi}_{n,0}) - C_{n,2}\sin(\tilde{\phi}_{n,0})\right] \\ &\quad - \beta_0\bar{l}(\tau)d_k(\tau)\sin\left(\frac{\alpha l_0}{\varepsilon v_0}(\mathrm{e}^{v_0\tau} - 1)\right), \\ &w_{k,1}(0,0,0) = \bar{F}_k, \quad \frac{\partial w_{k,1}}{\partial\tilde{\phi}_{k,0}}(0,0,0) \\ &= -\frac{\partial w_{k,0}}{\partial\tilde{\phi}_{k,1}}(0,0,0) + \bar{G}_k. \end{aligned} \tag{43}$$

Next, we will calculate $C_{k,1}$ and $C_{k,2}$ in (42). In order to make the solution results more clear, we analyze the solution for two parts: outside the resonance zone and inside the resonance zone.

Outside the resonance zone, it should be observed that the last term in Eq. (43) does not give rise to secular terms in $w_{k,1}$. To avoid secular terms, it follows from (43) that $C_{k,1}$ and $C_{k,2}$ have to satisfy the following conditions

$$\frac{\partial C_{k,i}}{\partial\tilde{\phi}_{k,1}} + \frac{P_k(\tau)}{2}C_{k,i} = 0, \quad i = 1, 2, \tag{44}$$

which has solutions

$$C_{k,i}(\tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}) = \bar{C}_{k,i}(\tilde{\phi}_{k,2})e^{-\frac{P_k(\tau)}{2}\tilde{\phi}_{k,1}}, \quad i = 1, 2, \tag{45}$$

where $\bar{C}_{k,1}$ and $\bar{C}_{k,2}$ are still unknown functions of the slow variable $\tilde{\phi}_{k,2}$, which can be deduced by solving the $O(\varepsilon\sqrt{\varepsilon})-$ problem (41) later. Since $C_{k,1}(0,0) = C_{k,2}(0,0) = 0$, this implies that $\bar{C}_{k,1}(0) = \bar{C}_{k,2}(0) = 0$.

Inside the resonance zone, we observe that the last term in Eq. (43) gives rise to secular terms in $w_{k,1}$. According to (35), we can write

$$\begin{aligned} &\sin\left(\frac{\alpha l_0}{\varepsilon v_0}(\mathrm{e}^{v_0\tau} - 1)\right) \\ &= \sin\left(\frac{1}{2}\mu\varepsilon\tilde{s}^2_1 + \frac{\alpha l_0}{\varepsilon v_0}(\mathrm{e}^{\varepsilon v_0 s_k} - 1) - \phi_k(s_k) + \tilde{\phi}_{k,0}\right). \end{aligned}$$

Then, it follows from Eq. (43) that inside the resonance zone $C_{k,1}$ and $C_{k,2}$ have to satisfy

$$\begin{aligned} &-2\frac{\partial C_{k,1}}{\partial\tilde{\phi}_{k,1}} - P_k(\tau)C_{k,1} - \beta_0\bar{l}(\tau)d_k(\tau) \\ &\quad \sin\left(\frac{1}{2}\mu\varepsilon\tilde{s}^2_1 + \vartheta(s_k)\right) = 0, \\ &2\frac{\partial C_{k,2}}{\partial\tilde{\phi}_{k,1}} + P_k(\tau)C_{k,2} - \beta_0\bar{l}(\tau)d_k(\tau) \\ &\quad \cos\left(\frac{1}{2}\mu\varepsilon\tilde{s}^2_1 + \vartheta(s_k)\right) = 0, \end{aligned}$$

where $\vartheta(s_k) = \frac{\alpha l_0}{\varepsilon v_0}(\mathrm{e}^{\varepsilon v_0 s_k} - 1) - \phi_k(s_k)$, which yields that

$$\begin{aligned} &C_{k,1}(\tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}) \\ &\quad = \bar{C}_{k,1}(\tilde{\phi}_{k,2})e^{-P_k(\tau)\tilde{\phi}_{k,1}} - \tilde{F}(\tilde{s}_1, \tau) + h.o.t., \\ &C_{k,2}(\tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}) \\ &\quad = \bar{C}_{k,2}(\tilde{\phi}_{k,2})e^{-P_k(\tau)\tilde{\phi}_{k,1}} + \tilde{G}(\tilde{s}_1, \tau) + h.o.t., \end{aligned} \tag{46}$$

where

$$\begin{aligned} &\tilde{F}(\tilde{s}_1, \tau) = \beta_0\bar{l}(\tau)d_k(\tau) \\ &\quad [\sin(\vartheta(s_k))\bar{C}_{Fr}(\tilde{s}_1) + \cos(\vartheta(s_k))\bar{S}_{Fr}(\tilde{s}_1)], \\ &\tilde{G}(\tilde{s}_1, \tau) = \beta_0\bar{l}(\tau)d_k(\tau) \\ &\quad [\cos(\vartheta(s_k))\bar{C}_{Fr}(\tilde{s}_1) - \sin(\vartheta(s_k))\bar{S}_{Fr}(\tilde{s}_1)], \\ &\bar{C}_{Fr}(\tilde{s}_1) = \int_{\bar{\beta}b}^{\bar{\beta}\tilde{s}_1}\cos(u^2)\mathrm{d}u, \\ &\qquad \bar{S}_{Fr}(\tilde{s}_1) = \int_{\bar{\beta}b}^{\bar{\beta}\tilde{s}_1}\sin(u^2)\mathrm{d}u, \end{aligned}$$

$\bar{C}_{k,1}$ and $\bar{C}_{k,2}$ are still unknown functions of the slow variables $\tilde{\phi}_{k,2}$, which can be determined by solving the $O\left(\varepsilon\sqrt{\varepsilon}\right)-$ problem (41).

Now, we will calculate the functions $\bar{C}_{k,1}$ and $\bar{C}_{k,2}$. Taking into account the secularity conditions, the general solution $w_{k,1}$ of the $O(\varepsilon)-$ problem is given by

$$\begin{aligned} &w_{k,1}(\tilde{\phi}_{k,0},\tilde{\phi}_{k,1},\tilde{\phi}_{k,2};\sqrt{\varepsilon}) \\ &\quad= D_{k,1}(\tilde{\phi}_{k,0},\tilde{\phi}_{k,1},\tilde{\phi}_{k,2})\sin(\tilde{\phi}_{k,0}) \\ &\quad+D_{k,2}(\tilde{\phi}_{k,0},\tilde{\phi}_{k,1},\tilde{\phi}_{k,2})\cos(\tilde{\phi}_{k,0}). \end{aligned}$$

By substituting $w_{k,0}$ and $w_{k,1}$ into $O\left(\varepsilon\sqrt{\varepsilon}\right)-$ problem (41), and by avoiding secular terms in the solution of $w_{k,2}$, it follows that $\bar{C}_{k,1}$, and $\bar{C}_{k,2}$ have to satisfy the following conditions inside and outside the resonance zone:

$$\begin{aligned} &\frac{\partial\bar{C}_{k,1}}{\partial\tilde{\phi}_{k,2}}-\frac{P_k^2(\tau)}{4}\bar{C}_{k,2}-\frac{1}{2}\bar{\zeta}(\tau)\frac{1}{\lambda_k(\tau)}\bar{C}_{k,1}=0, \\ &\frac{\partial\bar{C}_{k,2}}{\partial\tilde{\phi}_{k,2}}+\frac{P_k^2(\tau)}{4}\bar{C}_{k,1}-\frac{1}{2}\bar{\zeta}(\tau)\frac{1}{\lambda_k(\tau)}\bar{C}_{k,2}=0, \end{aligned}\tag{47}$$

where $\bar{\zeta}(\tau)=v_0\bar{l}(\tau)-\frac{\mathrm{d}\lambda_k(\tau)}{\mathrm{d}\tau}\frac{1}{\lambda_k(\tau)}-2c_{k,k}^1(\tau)\frac{\mathrm{d}\lambda_k(\tau)}{\mathrm{d}\tau}+2v_0c_{k,k}^2(\tau)$.

Since $\bar{C}_{k,1}(0)=\bar{C}_{k,2}(0)=0$, and $\lambda_k(\tau)$ and $\bar{l}(\tau)$ are bounded and change slowly, it follows from Eq. (45) and Eq. (47) that outside the resonance zone

$$C_{k,1}(\tilde{\phi}_{k,1},\tilde{\phi}_{k,2})=0,\quad C_{k,2}(\tilde{\phi}_{k,1},\tilde{\phi}_{k,2})=0.\tag{48}$$

It follows from Eq. (46) and Eq. (47) that inside the resonance zone,

$$\begin{aligned} &C_{k,1}(\tilde{\phi}_{k,1},\tilde{\phi}_{k,2}) \\ &=-\beta_0\bar{l}(\tau)d_k(\tau)[\sin(\vartheta(s_k))\bar{C}_{Fr}(\tilde{s}_1) \\ &\quad+\cos(\vartheta(s_k))\bar{S}_{Fr}(\tilde{s}_1)],\ C_{k,2}(\tilde{\phi}_{k,1},\tilde{\phi}_{k,2}) \\ &=\beta_0\bar{l}(\tau)d_k(\tau)[\cos(\vartheta(s_k))\bar{C}_{Fr}(\tilde{s}_1) \\ &\quad-\sin(\vartheta(s_k))\bar{S}_{Fr}(\tilde{s}_1)]. \end{aligned}\tag{49}$$

To obtain more accurate approximations for the solution of problem (37), the $O(\varepsilon\sqrt{\varepsilon})-$ problem and the $O(\varepsilon^2)-$ problem can also be solved by using a similar analysis as for the $O(\sqrt{\varepsilon})-$ problem. At this moment, only the first term in the expansion of the solution for the string problem is important from the physical point of view. So, to shorten the paper, we are not interested in the high-order approximations.

Finally, the solution of problem (37) is given by

$$\tilde{T}_k(s)=\sqrt{\varepsilon}w_{k,0}(\tilde{\phi}_{k,0},\tilde{\phi}_{k,1},\tilde{\phi}_{k,2};\sqrt{\varepsilon})+O(\varepsilon),\tag{50}$$

where $w_{k,0}$ is given by (42), (48) and (49). It turns out that by introducing the damping terms $c_1u_t(x,t)$ and $c_2u_t(l(t),t)$ with $\left(c_1,c_2=O(\varepsilon^{\hat{\alpha}}),\frac{1}{2}\leq\hat{\alpha}<1\right)$, the resonances can not be eliminated. Outside the resonance zone the oscillation amplitudes may decrease (it depends on the solution of $w_{n,1}$ in the $O(\varepsilon)$-problem (40)), but within the resonance zone the oscillation amplitudes still increase from initial conditions $O(\varepsilon)$ to $O(\sqrt{\varepsilon})$, and the size of the resonance zone in s is of $O\left(\frac{1}{\sqrt{\varepsilon}}\right)$.

Case 2: $c_1=O(\varepsilon^{\hat{\alpha}})$, $c_2=O(\varepsilon^{\hat{\alpha}})$, and $0\leq\hat{\alpha}<\frac{1}{2}$.

By substituting the power series expansion (38) of the form w_k into the problem (63) and equating the coefficients of like powers of $\sqrt{\varepsilon}$, we obtain the following $O\left((\sqrt{\varepsilon})^n\right)-$ problems to solve for $n\in\mathbb{N}^+$.

The $O(\sqrt{\varepsilon})$-problem is given by:

$$\frac{\partial^2 w_{k,0}}{\partial\tilde{\phi}_{k,0}^2}+w_{k,0}+Q_k(\tau)\frac{\partial w_{k,0}}{\partial\tilde{\phi}_{k,0}}=0,\tag{51}$$

the $O(\varepsilon)$-problem:

$$\begin{aligned} &\frac{\partial^2 w_{k,1}}{\partial\tilde{\phi}_{k,0}^2}+w_{k,1}+Q_k(\tau)\frac{\partial w_{k,1}}{\partial\tilde{\phi}_{k,0}} \\ &\quad+\sum_{n\neq k}c_{n,k}^3(\tau)\frac{\lambda_n(\tau)}{\lambda_k^2(\tau)}\frac{\partial w_{n,1}}{\partial\tilde{\phi}_{n,0}} \\ &=-Q_k(\tau)\frac{\partial w_{k,0}}{\partial\tilde{\phi}_{k,1}} \\ &\quad-2\frac{\partial^2 w_{k,0}}{\partial\tilde{\phi}_{k,0}\partial\tilde{\phi}_{k,1}}-\beta_0\bar{l}(\tau)d_k^i(\tau) \\ &\quad\sin\left(\frac{\alpha l_0}{\varepsilon v_0}(\mathrm{e}^{v_0\tau}-1)\right), \end{aligned}\tag{52}$$

and the $O(\varepsilon\sqrt{\varepsilon})$-problem:

$$\begin{aligned} &\frac{\partial^2 w_{k,2}}{\partial\tilde{\phi}_{k,0}^2}+w_{k,2}+Q_k(\tau)\frac{\partial w_{k,2}}{\partial\tilde{\phi}_{k,0}} \\ &\quad+\sum_{n\neq k}c_{n,k}^3(\tau)\frac{\lambda_n(\tau)}{\lambda_k^2(\tau)}\frac{\partial w_{n,2}}{\partial\tilde{\phi}_{n,0}} \\ &\quad=-Q_k(\tau)\frac{\partial w_{k,1}}{\partial\tilde{\phi}_{k,1}}-Q_k(\tau)\frac{\partial w_{k,0}}{\partial\tilde{\phi}_{k,2}} \\ &\quad-2\frac{\partial^2 w_{k,1}}{\partial\tilde{\phi}_{k,0}\partial\tilde{\phi}_{k,1}}-2\frac{\partial^2 w_{k,0}}{\partial\tilde{\phi}_{k,0}\partial\tilde{\phi}_{k,2}}-\frac{\partial^2 w_{k,0}}{\partial\tilde{\phi}_{k,1}^2} \end{aligned}$$

$$-\sum_{n\neq k} c_{n,k}^3(\tau)\frac{\lambda_n(\tau)}{\lambda_k^2(\tau)}\frac{\partial w_{n,1}}{\partial\tilde{\phi}_{n,1}}$$
$$-\sum_{n\neq k} c_{n,k}^3(\tau)\frac{\lambda_n(\tau)}{\lambda_k^2(\tau)}\frac{\partial w_{n,0}}{\partial\tilde{\phi}_{n,2}}, \qquad (53)$$

where $Q_k(\tau)=\dfrac{c_1\bar{l}(\tau)+c_{k,k}^3(\tau)}{\lambda_k(\tau)}>0.$

The objective of this subsection is to suppress the large vibration inside the resonance zone. In the previous section, it was shown that the size of resonance zone is $O\left(\frac{1}{\sqrt{\varepsilon}}\right)$, and that $\bar{l}$, λ_k change slowly (19). So, for a certain time $\tilde{s}^*$ inside the resonance zone, within its $O\left(\frac{1}{\sqrt{\varepsilon}}\right)$ resonance domain, we have $\bar{l}=\bar{l}(\varepsilon\tilde{\phi}_{k,0}^*)+O(\sqrt{\varepsilon})$ and $\lambda_k=\lambda_k(\varepsilon\tilde{\phi}_{k,0}^*)+O(\sqrt{\varepsilon})$, where the relationship between $\tilde{s}^*$ and $\tilde{\phi}_{k,0}^*$ is given by (38). Thus, in the resonance zone involving $\tilde{s}^*$, the $O(\sqrt{\varepsilon})-$ problem (51) has solution

$$\begin{aligned} &w_{k,0}(\tilde{\phi}_{k,0},\tilde{\phi}_{k,1},\tilde{\phi}_{k,2};\sqrt{\varepsilon})\\ &\quad=C_{k,1}(\tilde{\phi}_{k,1},\tilde{\phi}_{k,2})e^{\alpha_1\tilde{\phi}_{k,0}}\\ &\qquad+C_{k,2}(\tilde{\phi}_{k,1},\tilde{\phi}_{k,2})e^{\alpha_2\tilde{\phi}_{k,0}}, \end{aligned} \qquad (54)$$

where

$$\alpha_1=-\frac{Q_k(\tau^*)}{2}+\sqrt{\frac{Q_k^2(\tau^*)}{4}-1}+O(\sqrt{\varepsilon}),$$
$$\alpha_2=-\frac{Q_k(\tau^*)}{2}-\sqrt{\frac{Q_k^2(\tau^*)}{4}-1}+O(\sqrt{\varepsilon}),$$

and $C_{k,1}$, $C_{k,2}$ are still unknown functions depending on the slow variables $\tilde{\phi}_{k,1}$, $\tilde{\phi}_{k,2}$, they can be determined by solving the $O(\varepsilon)-$ problem and $O(\varepsilon\sqrt{\varepsilon})-$ problem. Since the term $-\beta_0\bar{l}(\tau)d_k(\tau)\sin(\frac{\alpha l_0}{\varepsilon v_0}(\mathrm{e}^{v_0\tau}-1))$ can not produce unbounded terms in the solutions of the $O(\varepsilon)-$ problem and the $O(\varepsilon\sqrt{\varepsilon})-$ problem, we have to set

$$\frac{\partial C_{k,1}}{\partial\tilde{\phi}_{k,1}}=\frac{\partial C_{k,2}}{\partial\tilde{\phi}_{k,1}}=\frac{\partial C_{k,1}}{\partial\tilde{\phi}_{k,2}}=\frac{\partial C_{k,2}}{\partial\tilde{\phi}_{k,2}}=0, \qquad (55)$$

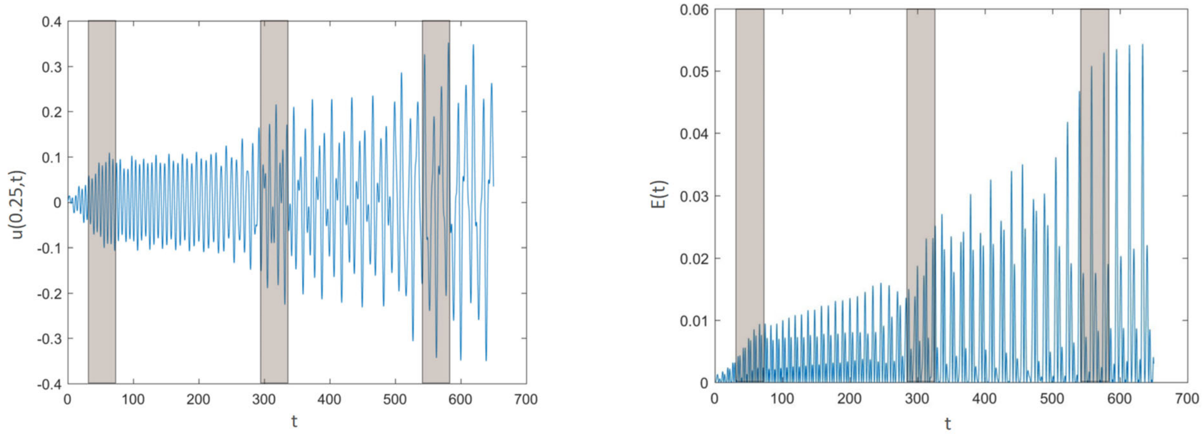
which leads to bounded functions $w_{k,0}$, $w_{k,1}$, and $w_{k,2}$. By substituting the initial conditions in (54) and (55), we find the solution of the $O(\sqrt{\varepsilon})$-problem: $w_{k,0}=0$. It turns out that by introducing the damping terms $c_1u_t(x,t)$ and $c_2u_t(l(t),t)$ with $\big(c_1,c_2=O(\varepsilon^{\hat{\alpha}}),0\leq\hat{\alpha}<\dfrac{1}{2}\big)$, the resonances can be eliminated on long timescales. □

Remark 2 [The threshold for anti-resonance] To suppress the resonance, viscous dampings are adopted in the system. The core function of damping is to dissipate energy, thereby directly reducing the amplitude of the resonance peak, rather than altering the natural frequency of the system. It is shown that when the order of damping coefficient is too small, that is, smaller than or equal to $O(\sqrt{\varepsilon})$, the resonance can not be eliminated; when the order of damping coefficient is larger than $O(\sqrt{\varepsilon})$, the resonances can be eliminated. The effectiveness of damping strategies is subject to a theoretical lower threshold for the damping coefficient $O(\sqrt{\varepsilon})$, below which resonance induced by transmission point perturbations cannot be effectively eliminated. This discovery reveals inherent limitations in damping coefficient selection. However, this threshold $O(\sqrt{\varepsilon})$ is not an arbitrary constraint, rather, it is a mathematically rigorous result derived from multi-scale perturbation analysis, rooted in the intrinsic dynamic properties of the system. Physically, this threshold reflects an energy balance: under the initial set of assumptions, when damping coefficient $(0<c_1,c_2\leq O(\sqrt{\varepsilon}))$, the energy input from perturbations exceeds the energy dissipation capacity of the damping mechanism during oscillation cycles. This energy imbalance leads to sustained amplitude growth rather than effective resonance elimination. The threshold's direct dependence on the perturbation parameter ε indicates that stronger perturbations require proportionally greater minimum damping to counteract their effects. This provides engineers with clear quantitative design objectives, enabling them to select or design structures with sufficient damping.

6 Numerical simulations

In this section, we show that the analytical approximations of the solution to the problem agree very well with numerically obtained approximations.

Since the original problem (1)-(2) has a time dependent space domain, we proceed to transform it into the following non-dimensional initial boundary value problem ($\xi=\frac{x}{l(t)}$), and integrate the so-obtained problem with a numerical method. The problem becomes:



(a) (b)

Fig. 2 Without damping term: **a** Displacements at $\xi = 0.25$ of the cable for times t up to t = 650, **b** The energy of the cable. The shadowed bands represent the resonance zones

$$\begin{cases} \tilde{u}_{tt} - \frac{1}{l^2}\tilde{u}_{\xi\xi} = \frac{2v}{l}(\xi - 1)\tilde{u}_{\xi t} \\ \qquad -\frac{v^2}{l^2}(\xi - 1)^2\tilde{u}_{\xi\xi} + \frac{2v^2}{l^2}(1-\xi)\tilde{u}_{\xi}, \\ \qquad\quad 0 < \xi < 1,\ 1 < \xi < 2,\ t > 0, \\ \tilde{u}(1^-, t) = \tilde{u}(1^+, t), \\ [\tilde{u}_{tt}(1,t) + \frac{\rho L}{ml}[u_\xi(1^+, t) - u_\xi(1^-, t)] \\ \qquad = -e(t) + O(\varepsilon^2), \quad t > 0, \\ \tilde{u}(0,t) = 0,\ \tilde{u}(2,t) = 0, \quad t > 0, \\ \tilde{u}(\xi, 0) = \tilde{u}_0(\xi), \quad \tilde{u}_t(\xi, 0) = \tilde{u}_1(\xi), \quad 0 \le \xi \le 2, \end{cases} \tag{56}$$

where $l = l(t)$, $\tilde{u}_0(\xi) = u_0(\xi l_0)$, and $\tilde{u}_1(\xi) = u_1(\xi l_0) - \varepsilon \frac{\xi v_0}{l_0} u_\xi(\xi, 0)$. We first discretize the partial differential equation in (56) in the $\xi-$ coordinate by using a central finite difference scheme. Then, we rewrite the so-obtained discretized equation in a matrix form, and use the numerical time integration method of Crank-Nicolson [25]. The numerical approach used (central finite difference + Crank-Nicolson) is a standard, reliable method for problems of this nature, and its implementation was rigorously verified through convergence studies. The parameter values utilized in this process are specified as follows:

$$\begin{aligned} &\varepsilon = 0.01,\ l_0 = 1,\ v_0 = 1,\ \rho = 1,\ m = 10, \\ &L = 10,\ \beta_0 = 1,\ \alpha = 0.4\pi,\ \gamma = 0.25. \end{aligned} \tag{57}$$

For simplicity, assume that only the initial displacement is prescribed, so that

$$\tilde{u}_0(\xi) = \varepsilon \sin^2(\pi\xi),\ \tilde{u}_1(\xi) = \varepsilon \cos^2(\pi\xi),\ 0 \le \xi \le 2.$$

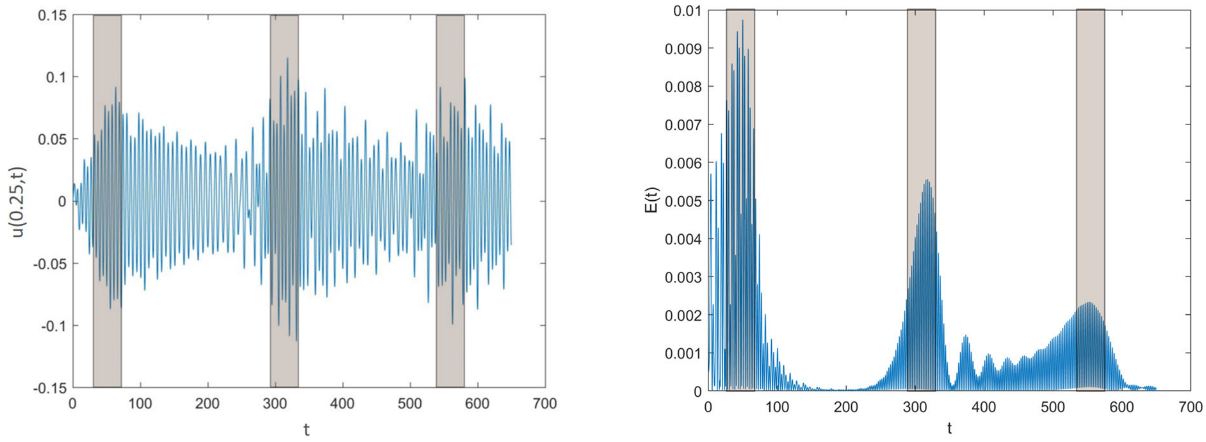
By using (24), we observe that the resonance occurs around time instants s_k satisfying $\alpha l_0 \mathrm{e}^{v_0\tau} = \lambda_k,\ \tau = \varepsilon s$, and by using the Liouville-Green transformation with $\frac{ds}{dt} = \frac{1}{l(t)}$, we obtain $\alpha l(t) = \lambda_k,\ l(t) = l_0 + \varepsilon v_0 t$, which implies that resonance occurs in time at time instants

$$t_k = \frac{\lambda_k - \alpha l_0}{\varepsilon \alpha v_0}, \quad k \in \mathbb{N}, \tag{58}$$

where λ_k is given by (25). From the analysis in section 4, the resonance times depend on the mode numbers k. Under the aforementioned parameter settings, for the first, second and third oscillation modes, resonances emerge for times $t_1 \approx 70$, $t_2 \approx 320$, $t_3 \approx 570$, respectively.

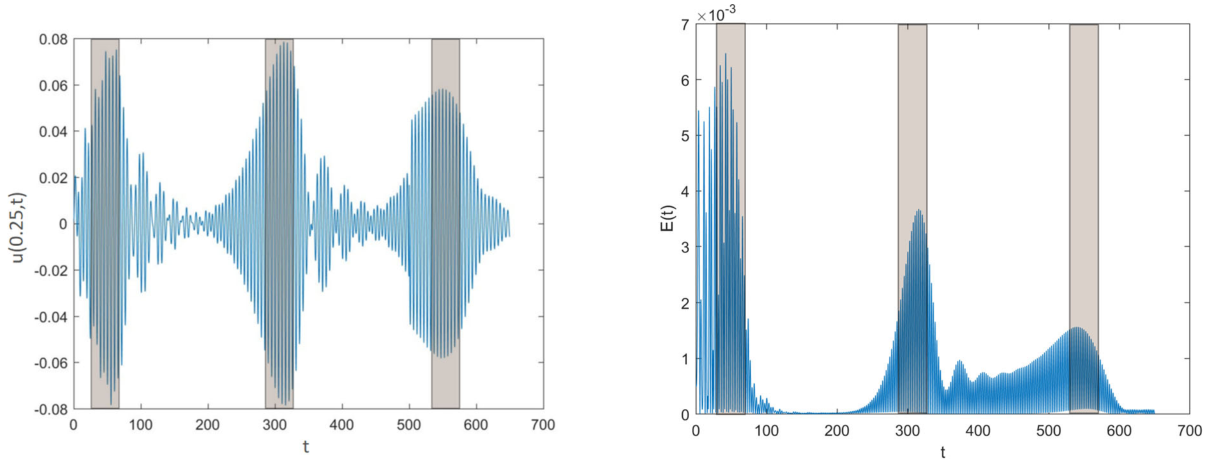
Figure 2 displays the longitudinal displacements $\tilde{u}(\xi, t)$ at $\xi = 0.25$ and the vibratory energy of the axially string without damping term over a time period up to t=650. From the figure one can see that in the resonance zones the longitudinal displacements and the vibratory energy increase and, between these zones, stay phase-locked. Also, the resonance displacements jump from $O(\varepsilon)$ to $O(\sqrt{\varepsilon})$. The shadowed bands represent the resonance layers which have the size of $O\left(\frac{1}{\sqrt{\varepsilon}}\right)$ as was obtained in Sect. 4 analytically. It is clearly evident that the numerical simulations depicted in Fig. 2 align remarkably well with the analytical results presented in the previous section.

After introducing damping terms $c_1 u_t(x,t), c_2 u_t(l(t), t)$ $(c_1 = O(\varepsilon^{\hat{\alpha}}), c_2 = O(\varepsilon^{\hat{\alpha}}))$, Figure 3 depicts the longitudinal displacements $\tilde{u}$ at $\xi = 0.25$ and the vibratory energy with damping parameter $\hat{\alpha} = \dfrac{2}{3}$



(a) (b)

Fig. 3 Damping coefficient $c = O(\varepsilon^{\hat{\alpha}})$, $\hat{\alpha} = \frac{2}{3}$: **a** Displacements at $\xi = 0.25$ of the cable for times t up to t = 650, **b** The energy of the cable. The shadowed bands represent the resonance zones

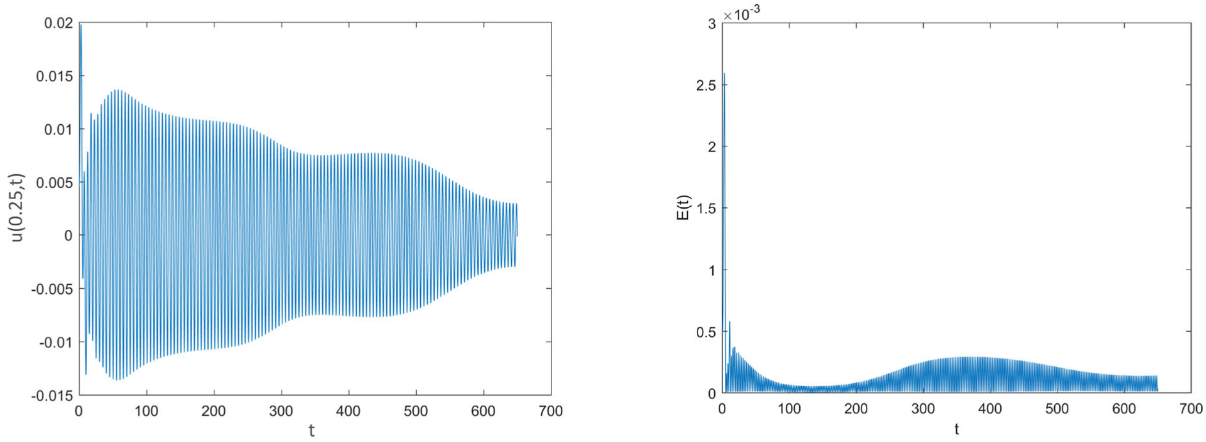


(a) (b)

Fig. 4 Damping coefficient $c = O(\varepsilon^{\hat{\alpha}})$, $\hat{\alpha} = \frac{1}{2}$: **a** Displacements at $\xi = 0.25$ of the cable for times t up to t = 650, **b** The energy of the cable. The shadowed bands represent the resonance zones

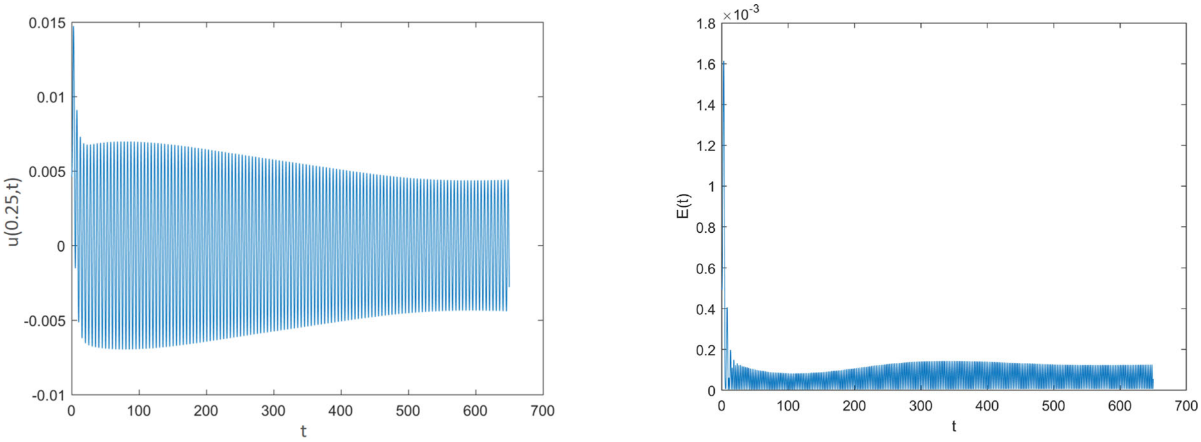
over a time period up to $t = 650$. From the figure one can see that with the damping coefficients $c_1, c_2 = O(\varepsilon^{\frac{2}{3}}) < O(\sqrt{\varepsilon})$, the longitudinal displacements and the energy still increase, and the displacements jump from $O(\varepsilon)$ to $O(\sqrt{\varepsilon})$ at times $t \approx 70$, $t \approx 320$, $t \approx 570$. This confirms that resonance persists under the damping condition $\hat{\alpha} = \dfrac{2}{3}$. Figure 4 shows the longitudinal displacements $\tilde{u}$ at $\xi = 0.25$ and the vibratory energy for $\hat{\alpha} = \dfrac{1}{2}$, corresponding to damping coefficients $c_1, c_2 = O(\sqrt{\varepsilon})$ over $t \leq 650$. The figure reveal the growth at times $t \approx 70$, $t \approx 320$, $t \approx 570$ in both longitudinal displacements and energy. This confirms that resonance persists under the damping condition $\hat{\alpha} = \dfrac{1}{2}$. Furthermore, the numerical simulations in Figs. 3 and 4 show excellent agreement with the analytical predictions for Case 1 in Section 5.

Figure 5 displays the longitudinal displacements $\tilde{u}$ at $\xi = 0.25$ and the vibratory energy with damping parameter $\hat{\alpha} = \dfrac{1}{4}$ over a time period up to t=650. Under



(a) (b)

Fig. 5 Damping coefficient $c = O(\varepsilon^{\hat{\alpha}})$, $\hat{\alpha} = \frac{1}{4}$: **a** Displacements at $\xi = 0.25$ of the cable for times t up to t = 650, **b** The energy of the cable



(a) (b)

Fig. 6 Damping coefficient $c = O(\varepsilon^{\hat{\alpha}})$, $\hat{\alpha} = 0$: **a** Displacements at $\xi = 0.25$ of the cable for times t up to t = 650, **b** The energy of the cable

the damping parameter condition, damping coefficients satisfy $c_1, c_2 = O(\varepsilon^{\frac{1}{4}}) > O(\sqrt{\varepsilon})$. From the figure one can see that both the longitudinal displacements and the energy exhibit temporal decay without resonant response. This demonstrates successful resonance elimination under the damping condition $\hat{\alpha} = \frac{1}{4}$. Figure 6 shows the longitudinal displacements $\tilde{u}$ at $\xi = 0.25$ and the energy for damping parameter $\hat{\alpha} = 0$, and so $c_1, c_2 = O(1) > O(\sqrt{\varepsilon})$ over $t \leq 650$. From the figure one can see that the longitudinal displacements and vibratory energy both decay over time without resonant response, demonstrating effective resonance suppression under the damping condition $\hat{\alpha} = 0$. Furthermore, numerical simulations in Figs. 5 and 6 exhibit excellent agreement with the analytical predictions for Case 2 in Sect. 5.

7 Concluding remarks

In this paper, the longitudinal vibrations and associated resonances in a movable pulley string actuator due to a harmonic excitation have been studied. The problem is described by a partial differential equation (PDE) on a time-varying spatial interval with a small harmonic disturbance and a moving nonclassical boundary condition. By assuming that the small harmonic disturbance is of order ε and by assuming that the initial values are also small and of order ε, it is shown in this paper that the initial-boundary value problem is well-posed, and that for a given arbitrary boundary disturbance frequency, many oscillation modes jump up from $O(\varepsilon)$ to $O(\sqrt{\varepsilon})$. Furthermore, explicit, and accurate approximations of the solution are constructed. These approximations are valid on time-scales of order ε^{-1}.

To suppress the resonance, viscous dampings are adopted in the system. It is shown that when the order of damping coefficient is too small, that is, smaller than or equal to $O(\sqrt{\varepsilon})$, the resonance can not be avoided; when the order of damping coefficient is larger than $O(\sqrt{\varepsilon})$, the resonances can be eliminated. Finally, numerical approximations of the problem are computed and are in full agreement with the analytically obtained results.

This study focuses on resonance responses induced by small amplitude perturbations $O(\varepsilon)$. For disturbances with amplitudes significantly exceeding $O(\varepsilon)$, we recommend further analysis through higher-order models or numerical simulations. Nevertheless, the revealed coupling mechanisms between damping and perturbation characteristics provide a fundamental theoretical basis for suppressing energy within resonance bands. Future work will explore: higher-order perturbation models and nonlinear control strategies for large disturbances.

Acknowledgements The authors would like to acknowledge the financial support from the National Natural Science Foundation of China grants NSFC-12401587, 62473281, 92471108 and 12131008.

Author contributions Jing Wang and Wim T. van Horssen wrote the Resonance and anti-resonance section. Jing Wang, Zhong-Jie Han and Jun-Min Wang wrote the well-posedness section . All authors reviewed the manuscript.

Funding This work was supported by the Natural Science Foundation of China grant NSFC-12401587, 62473281, 92471108, 12131008.

Data Availability datasets were generated or analysed during the current study.

Declarations

Conflict of interests The authors declare that thay have no Conflict of interests.

Appendix A: The derivation of motion (1)

To illustrate more clearly, we only consider the longitudial vibration in the interval $(0, l(t))$. Following the explanation of the parameters underneath Eq.(2), the partial differential equation (PDE) can be derived by Hamilton's principle:

$$\int_{t_1}^{t_2}(\delta E_k(t)-\delta E_p(t))\mathrm{d}t=0. \tag{59}$$

The Kinetic energy $E_k(t)$ can be represented as $E_k(t)=\frac{1}{2}\rho\int_0^{l(t)}(\frac{\mathrm{D}u}{\mathrm{D}t}+v)^2\mathrm{d}x+\frac{1}{2}m(\frac{\mathrm{D}u}{\mathrm{D}t}+v)^2|_{x=l(t)}$, the Potential energy $E_p(t)$ can be expressed as $E_p(t)=\frac{1}{2}EA\int_0^{l(t)}u_x^2\mathrm{d}x+\int_0^{l(t)}Tu_x\mathrm{d}x+E_{gs}-\int_0^{l(t)}\rho gu\mathrm{d}x-mgu|_{x=l(t)}$, where the spatiotemporally varying tension $T(x,t)$ is given by $T(x,t)=[m+\rho(l(t)-x)]g$, $0\le x\le l(t)$, g is gravity, and

$$\begin{aligned}&\delta E_k(t)-\delta E_p(t)\\&\quad=\rho\int_0^{l(t)}(\frac{\mathrm{D}u}{\mathrm{D}t}+v)\delta\frac{\mathrm{D}u}{\mathrm{D}t}\mathrm{d}x\\&\quad+m(\frac{\mathrm{D}u}{\mathrm{D}t}+v)\delta\frac{\mathrm{D}u}{\mathrm{D}t}|_{x=l(t)}\\&\quad-[EA\int_0^{l(t)}u_x\delta u_x\mathrm{d}x+\int_0^{l(t)}T\delta u_x\mathrm{d}x\\&\quad-\int_0^{l(t)}\rho g\delta u\mathrm{d}x-mg\delta u|_{x=l(t)}],\end{aligned}\tag{60}$$

where the operator $\frac{\mathrm{D}u}{\mathrm{D}t}$ is defined as $\frac{\mathrm{D}u}{\mathrm{D}t}=\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial x}=u_t+vu_x$.

By substituting the equations (60) into (59), we obtain

$$\begin{aligned}&\int_{t_1}^{t_2}\int_0^{l(t)}\rho(\frac{\mathrm{D}u}{\mathrm{D}t}+v)\delta\frac{\mathrm{D}u}{\mathrm{D}t}\mathrm{d}x\mathrm{d}t\\&\quad+\int_{t_1}^{t_2}m(\frac{\mathrm{D}u}{\mathrm{D}t}+v)\delta\frac{\mathrm{D}u}{\mathrm{D}t}|_{x=l(t)}\mathrm{d}t\\&\quad-EA\int_{t_1}^{t_2}\int_0^{l(t)}u_x\delta u_x\mathrm{d}x\mathrm{d}t-\int_{t_1}^{t_2}\int_0^{l(t)}T\delta u_x\mathrm{d}x\mathrm{d}t\\&\quad+\int_{t_1}^{t_2}\int_0^{l(t)}\rho g\delta u\mathrm{d}x\mathrm{d}t\\&\quad+\int_{t_1}^{t_2}mg\delta u|_{x=l(t)}\mathrm{d}t=0.\end{aligned}\tag{61}$$

By integrating by parts, it then follows that (61) can be rewritten in:

$$\begin{aligned}&\int_{t_1}^{t_2}\int_0^{l(t)}[-\rho(u_{tt}+2vu_{xt}+v^2u_{xx})\\&+EAu_{xx}+T_x+\rho g]\delta u\mathrm{d}x\mathrm{d}t\\&+\int_{t_1}^{t_2}[-m(u_{tt}+2vu_{xt}+v^2u_{xx})\\&-EAu_x-T+mg]\delta u|_{x=l(t)}\mathrm{d}t\\&+\int_{t_1}^{t_2}[\rho v(u_t+vu_x+v)+EAu_x+T]\\&\delta u|_{x=0}\mathrm{d}t=0.\end{aligned} \tag{62}$$

So, the governing equation of the system can be obtained from (62) as

$$\rho(u_{tt}+2vu_{xt}+v^2u_{xx})-EAu_{xx}-T_x-\rho g=0.$$

$T(x,t)$ is given by (60), it then follows that the governing equation is given by

$$\rho(u_{tt}+2vu_{xt}+v^2u_{xx})-EAu_{xx}=0,$$

where $0\le x\le l(t)$ and $t>0$.

Appendix B: Equations up to $O(\varepsilon\sqrt{\varepsilon})$

In section 5, by substituting time variable $\tilde{\phi}_{k,0}$, $\tilde{\phi}_{k,1}$, $\tilde{\phi}_{k,2}$ into (37), we obtain the following equations up to $O(\varepsilon\sqrt{\varepsilon})$:

$$\begin{aligned}&\frac{\partial^2 w_k}{\partial\tilde{\phi}_{k,0}^2}+w_k+2\sqrt{\varepsilon}\frac{\partial^2 w_k}{\partial\tilde{\phi}_{k,0}\partial\tilde{\phi}_{k,1}}\\&\quad+\varepsilon(2\frac{\partial^2 w_k}{\partial\tilde{\phi}_{k,0}\partial\tilde{\phi}_{k,2}}+\frac{\partial^2 w_k}{\partial\tilde{\phi}_{k,1}^2})\\&\quad+2\varepsilon\sqrt{\varepsilon}\frac{\partial^2 w_k}{\partial\tilde{\phi}_{k,1}\partial\tilde{\phi}_{k,2}}+\frac{c_1\bar{l}(\tau)}{\lambda_k(\tau)}\frac{\partial w_k}{\partial\tilde{\phi}_{k,0}}\\&\quad+\sum_{n=1}^{\infty}c_{n,k}^3(\tau)\frac{\lambda_n(\tau)}{\lambda_k^2(\tau)}\frac{\partial w_n}{\partial\tilde{\phi}_{n,0}}\\&=-\sqrt{\varepsilon}\Big[\frac{c_1\bar{l}(\tau)}{\lambda_k(\tau)}\frac{\partial w_k}{\partial\tilde{\phi}_{k,1}}+\sum_{n=1}^{\infty}c_{n,k}^3(\tau)\frac{\lambda_n(\tau)}{\lambda_k^2(\tau)}\frac{\partial w_n}{\partial\tilde{\phi}_{n,1}}\Big]\\&\quad+\varepsilon\Big[-\frac{c_1\bar{l}(\tau)}{\lambda_k(\tau)}\frac{\partial w_k}{\partial\tilde{\phi}_{k,2}}-\sum_{n=1}^{\infty}c_{n,k}^3(\tau)\frac{\lambda_n(\tau)}{\lambda_k^2(\tau)}\frac{\partial w_n}{\partial\tilde{\phi}_{n,2}}\\&\quad-\frac{\mathrm{d}\lambda_k(\tau)}{\mathrm{d}\tau}\frac{1}{\lambda_k^2(\tau)}\frac{\partial w_k}{\partial\tilde{\phi}_{k,0}}+\frac{v_0\bar{l}(\tau)}{\lambda_k(\tau)}\frac{\partial w_k}{\partial\tilde{\phi}_{k,0}}\\&\quad-\beta_0 d_k^i(\tau)\sin(\frac{\alpha l_0}{\varepsilon v_0}(\mathrm{e}^{v_0\tau}-1))\\&\quad-2\sum_{n=1}^{\infty}(c_{n,k}^1(\tau)\frac{\mathrm{d}\lambda_n(\tau)}{\mathrm{d}\tau}-v_0c_{n,k}^2(\tau))\frac{\lambda_n(\tau)}{\lambda_k^2(\tau)}\frac{\partial w_n}{\partial\tilde{\phi}_{n,0}}\Big]\\&\quad+\varepsilon\sqrt{\varepsilon}\Big[-\frac{\mathrm{d}\lambda_k(\tau)}{\mathrm{d}\tau}\frac{1}{\lambda_k^2(\tau)}\frac{\partial w_k}{\partial\tilde{\phi}_{k,1}}+\frac{v_0\bar{l}(\tau)}{\lambda_k(\tau)}\frac{\partial w_k}{\partial\tilde{\phi}_{k,1}}\\&\quad-2\sum_{n=1}^{\infty}(c_{n,k}^1(\tau)\frac{\mathrm{d}\lambda_n(\tau)}{\mathrm{d}\tau}-v_0c_{n,k}^2(\tau))\frac{\lambda_n(\tau)}{\lambda_k^2(\tau)}\frac{\partial w_n}{\partial\tilde{\phi}_{n,1}}\Big],\end{aligned} \tag{63}$$

where $w_k(0,0,0;\sqrt{\varepsilon})=F_k=\varepsilon\bar{F}_k$ and $\frac{\partial w_k}{\partial\tilde{\phi}_{k,0}}(0,0,0;\sqrt{\varepsilon})+\sqrt{\varepsilon}\frac{\partial w_k}{\partial\tilde{\phi}_{k,1}}(0,0,0;\sqrt{\varepsilon})+\varepsilon\frac{\partial w_k}{\partial\tilde{\phi}_{k,2}}(0,0,0;\sqrt{\varepsilon})=\frac{G_k}{\lambda_k(0)}=\varepsilon\bar{G}_k$ are $O(\varepsilon)$, and where τ is the function of $\tilde{\phi}_{k,2}$.

References

1. Andrianov, I.V., Awrejcewicz, J., Danishevskyy, V.V., Markert, B.: Influence of geometric and physical nonlinearities on the internal resonances of a finite continuous rod with a microstructure. J. Sound Vib. **386**, 359–371 (2017)
2. Chen, E.W., Yuan, J.F., Ferguson, N.S., Zhang, K., Zhu, W.D., Lu, Y.M., Wei, H.Z.: A wave solution for energy dissipation and exchange at nonclassical boundaries of a traveling string. Mech. Syst. Signal Process. **150**, 107272 (2021)
3. Chen, E.W., Zhang, K., Ferguson, N.S., Wang, J., Lu, Y.M.: On the reflected wave superposition method for a travelling string with mixed boundary supports. J. Sound Vib. **440**, 129–146 (2019)
4. Chen, J., Hu, W.H., Li, Q.S.: Nonlinear dynamics of a foldable multibeam structure with one to two internal resonances. Int. J. Mech. Sci. **150**, 369–378 (2019)
5. Crespo, R.S., Kaczmarczyk, S., Picton, P., Su, H.: Modelling and simulation of a nonstationary high-rise elevator system to predict the dynamic interactions between its components. Int. J. Mech. Sci. **137**, 24–45 (2018)
6. Desoky, E., Kamel, M., Kandil, A.: Primary, superharmonic and subharmonic resonances control of an oscillatory cantilever beam excited transversely at its free end. Afr. Mat. **36**(91), 1–24 (2025)
7. Gaiko, N.V., van Horssen, W.T.: On the transverse, low frequency vibrations of a traveling string with boundary damping. J. Vib. Acoust. **137**(4), 041004.1-041004.10 (2015)
8. Gaiko, N.V., van Horssen, W.T.: Resonances and vibrations in an elevator cable system due to boundary sway. J. Sound Vib. **424**, 272–292 (2018)
9. Gugat, M.: Optimal energy control in finite time by varying the length of the string. SIAM J. Control. Optim. **46**(5), 1705–1725 (2007)
10. Kandil, A.: Internal resonances among the first three modes of a hinged-hinged beam with cubic and quintic nonlinearities. Int. J. Non-Linear Mech. **127**, 103592 (2020)
11. Kandil, A.: Study of Hopf curves in the time delayed active control of a 2dof nonlinear dynamical system. SN Appl. Sci. **2**(11), 1–21 (2020)

12. Kandil, A., Hamed, Y., Alsharif, A., Awrejcewicz, J.: 2d and 3d visualizations of the mass-damper-spring model dynamics controlled by a servo-controlled linear actuator. IEEE Access **9**, 153012–153026 (2021)
13. Kandil, A., Hamed, Y.S.: Integral resonant controller for suppressing car's oscillations and eliminating its inherent jump phenomenon. Eur. J. Pure Appl. Math. **16**(4), 2729–2750 (2023)
14. Kato, T.: Linear and quasilinear equations of evolution of hyperbolic type. Springer, Heidelberg (1976)
15. Kato, T.: Abstract Differential Equations and Nonlinear Mixed Problems. Scuola Normale Superiore, Pisa (1985)
16. Kim, T.H., Noe, S.H., Choi, Y.M., Lee, H.J.: Vibration-damped tool holders in industrial robotic machining systems. Int. J. Precis. Eng. Manuf. **24**(9), 1535–1548 (2023)
17. Lee, H., Choi, Y.: A new actuator system using dual-motors and a planetary gear. IEEE ASME Trans. Mechatron. **17**(1), 192–197 (2012)
18. Lee, H., Lee, C., Kim, S., Choi, Y.: New actuator system using movable pulley for bio-mimetic system and wearable robot applications. In: 2010 IEEE International Conference on Robotics and Automation, Anchorage, AK, USA, pp. 2183–2188 (2010)
19. Li, H., Liu, H., Zou, J.: Minnaert resonances for bubbles in soft elastic materials. SIAM J. Appl. Math. **82**(1), 119–141 (2022)
20. Nasr, A., Hunter, J., Dickerson, C.R., Mcphee, J.: Evaluation of a machine-learning-driven active-passive upper-limb exoskeleton robot: Experimental human-in-the-loop study. Wearable Technol. **4**(e13), 1–20 (2023)
21. Sandilo, S.H., van Horssen, W.T.: On a cascade of autoresonances in an elevator cable system. Nonlinear Dyn. **80**(3), 1613–1630 (2015)
22. Song, Y., Song, Z., Chen, H., Chen, X., Guo, H., Wu, H.: Wearable stretchable double-sided micro-supercapacitors with porous conductive elastomers. In: 2018 IEEE Micro Electro Mechanical Systems (MEMS), Belfast, UK, pp. 608–611 (2018)
23. Verhulst, F.: Nonlinear Differential Equations and Dynamical Systems. Springer, Heidelberg (2006)
24. Wang, J., Koga, S., Pi, Y., Krstic, M.: Axial vibration suppression in a partial differential equation model of ascending mining cable elevator. J. Dyn. Sys. Meas., Control **140**(11), 111003 (2018)
25. Wang, J., Van Horssen, W.T.: Analysis of longitudinal oscillations in a vertically moving cable subject to nonclassical boundary conditions. Appl. Math. Model. **111**, 44–62 (2022)
26. Wang, J., Van Horssen, W.T.: On resonances and transverse and longitudinal oscillations in a hoisting system due to boundary excitations. Nonlinear Dyn. **111**, 5079–5106 (2022)
27. Wang, N.G., Cao, G.H., Yan, L., Wang, L.: Modelling and passive control of flexible guiding hoisting system with time-varying length. Math. Comput. Model. Dyn. Syst. **26**(1), 31–54 (2020)
28. Yang, J., Moon, J., Ryu, J., Kim, J., Nam, K., Park, S., Kim, Y., Lee, G.: Design of a quasi-direct drive actuator with embedded pulley for a compact, lightweight, and high-bandwidth exosuit. Actuators **12**(1), 21 (2023)
29. Yin, P., Yang, L., Qu, S.G.: Development of an ergonomic wearable robotic device for assisting manual workers. Int. J. Adv. Robot. Syst. **18**(5), 1–11 (2021)