The Asymmetric Brownian Energy Process

Thesis report

by

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to obtain the degree of Master of Science in Applied Mathematics at the Delft University of Technology
to be defended publicly on June 6th

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Abstract

The focus of this thesis is the hydrodynamic limit of the Brownian Energy Process (BEP) and the Asymmetric Brownian Energy Process (ABEP) in infinite volume. The thesis starts by introducing some general theory about Markov processes, after which the topic of Markov duality is introduced. A number of relevant interacting particle systems (IPS) will be introduced, and we will show how the BEP and the ABEP can be related to the simpler Symmetric Inclusion Process (SIP) through Markov duality. Using these tools, the first main result is proven, which states that the hydrodynamic limit of the BEP is a weak solution to the heat equation. As a consequence of this, in the second main result we use the relation between the BEP and the ABEP to prove that the hydrodynamic limit of the ABEP is a weak solution to the viscous Burgers' equation. We then attempt to show a similar result for a newly developed IPS, the Dynamic ABEP. Finally, we prove propagation of chaos for the BEP and the ABEP, where for the latter we argue that this is only possible in a finite volume. As a part of the proof of this we argue that the SIP in infinite volume is very 'similar' to Independent Random Walkers (IRW) when we look at a long enough time-scale, where we give a rough sketch of a proof that significantly improves upon a quantification of this 'similarity' established in the literature.

Acknowledgements

I would first like express my gratitude to my main thesis advisor prof. Frank Redig. I want to thank you not only for providing me with a topic for this thesis and being a helpful guide throughout the project, but also for getting me interested in stochastics through your courses in the first place, back when I began my master's.
Also thanks to my second supervisor Carel Wagenaar. Your feedback has kept me diligent and has enabled me to write this report at a higher level than I otherwise might have achieved.

Contents

1 Introduction

The main goal of this thesis is to prove the hydrodynamic limits of two models for transport of energy, the symmetric Brownian Energy Process (BEP) and its asymmetric version, the Asymmetric Brownian Energy Process (ABEP). The idea behind hydrodynamic limits is that a problem is analyzed on two different levels, a microscopic scale and a macroscopic scale, where we can think of the macroscopic scale as 'infinitely larger' than the microscopic scale. The approach is that we create on a microscopic scale a stochastic transport model of particles or energy, here the BEP/ABEP, and then investigate what behavior emerges on macroscopic scale. Typically we investigate the macroscopic behavior by looking at the density field, and its evolution takes the form of a (stochastic) partial differential equation. This approach of describing the microscopic world (of atoms and molecules) in order to derive macroscopic properties of materials and systems, is one of the core ideas is statistical mechanics. This field of physics emerged in the late 19th - early 20th century with pioneering work by physicists such as Maxwell, Boltzmann, Gibbs and Einstein. Today this field is a cornerstone of modern physics, with notable applications in thermodynamics and quantum field theory.

The BEP and the ABEP, the models of energy transport of interest in this thesis, are examples of interacting particle systems. Interacting particle systems are a class of stochastic models used extensively in statistical physics and probability theory and are designed to capture complex behavior dynamics that emerge from simple, local interactions among particles. Interacting particle systems were first introduced in [19], and have been studied extensively since, with some notable books on the topic being [14], [8] and [6]. There are a number of different types of interacting particle system. Some examples of these are the following: spin systems, where each particle represents a "spin" that can take discrete states, representing for example the magnetic dipole moment of atoms or molecules, voter models / contact processes, where the spread of an opinion or disease through a population over time is modeled, and finally transport models, where particles or a continuous quantity, often representing energy, move through space, interacting with each other along the way. As is evident from these examples, interacting particle systems are often discrete systems, with particles representing binary states (e.g. a particle represent an infection, dipole moment or a unit of energy). The BEP and the ABEP are an exception to this rule, being continuous transport processes with values in $\mathbb{R}_+$, representing energy levels at each site. As we will see, these processes arise as scaling limits of the Symmetric Inclusion Process (SIP), which is an example of a discrete moving particle system. Its characteristic property is that particles attract each other and because of this, the SIP can interpreted in physics as a bosonic system, in contrast to the better known Symmetric Exclusion Process (SEP), which through its repellent nature is interpreted as a fermionic system. In this thesis we will only focus on the mathematical properties of the SIP and not on its physical interpretation. For a more explicitly physical interpretation to the SIP we refer to e.g. [9].

Interacting particle systems (IPS) are used in many different areas of study. The field originates from statistical mechanics, where in the late 1960's it emerged in an attempt to better understand the phenomenon of phase transitions. Nowadays IPS are used in modern physics in different applications, such as thermodynamics, magnetism, quantum systems and superconduction. As it turned out, models with a very similar mathematical structure were derived in other areas of study, and now IPS are used in many different contexts. Some examples of applications are the following: Neural networks, epidemiology, tumor growth, network theory, queuing theroy, and many others.

There are certain mathematical approaches to the study of IPS which allow us to derive conclusions about different aspects of them. For instance, one may be interested in equilibrium systems, where the system is in a stationary state, which does not change as it evolves over time. This is in contrast with a non-equilibrium system, where one may expect there always to be some evolution of the state of the IPS. For instance, in the case of transport models, one may find that a net current of particles/energy will always be present in these systems. In this thesis we will mostly be interested in transient non-equilibrium systems. These are models that start out in a non-equilibrium state, but will eventually settle into an equilibrium. Lastly, another interested mathematical approach is the topic of this thesis, hydrodynamic limits. This involved taking certain scaling limits, e.g. having an infinite number of particles each carrying an infinitesimal amount of energy, and evaluating the resulting dynamics, not of individual particles/sites, but of resulting particle/energy density profiles at macroscopic scale.

There are interesting mathematical properties of IPS which allows us to analyze them. Notably, an IPS as a whole is a Markov process, meaning that the future evolution of the system only depends on its current state. This allows one studying IPS to leverage theory about this widely studied class of stochastic processes. Since the full IPS can be very complicated due to the different interactions of particles, one may be interested in studying individual particles instead of the system as a whole. In this case, the Markovian property is lost, so that often some simplifications have to be made. Mean field theory arises here through the assumption that the interaction between a particle and its surrounding can be approximated through the average interacting in the whole system. In this thesis however, we will always focus our studies on the system as a whole.

Another mathematically interesting decision in creating IPS of moving particles is the space on which the IPS is defined. When dealing with an IPS in a finite 'volume', we have to worry about different kinds of boundary interactions. Notably we can have closed boundaries, where particles are not allowed to cross a point, and open boundaries, where particles can cross from the 'bulk' of the particles into a 'reservoir' at the boundary, and vice versa. Interesting properties emerge as a result of this choice, where for instance the total number of particles is preserved in the former case of closed boundaries, but not in the latter 'reservoir driven' process. When taking hydrodynamic limits, these choices become apparent in the boundary conditions of the resulting PDE (Neumann vs Dirichlet). Another option that is chosen for most of this thesis is to define the IPS on infinite volume, in which case other questions arise, such as whether we know that the IPS can even exist. In this this thesis most processes will be defined in infinite volume, where, like with closed boundaries, we have conservation of energy. This conservation of energy role plays a role in causing the fact that the systems of focus in this thesis will be in transient nonequilibrium state, i.e. initially in a non-equilibrium state but eventually settling to an equilibrium. Without conservation of energy, e.g. reservoirs on the boundaries, we can have a situation where the system never settles into a steady state, and a net current will persist. Finally, although in most of the literature a moving particle system is defined in one dimension, i.e. all particles lie on a line, it is also possible to define the problem in multidimensional space. [20] is an example of a masters thesis in which the hydrodynamic limit is proven for the Symmetric Inclusion Process defined on a d-dimensional lattice, in a very similar approach to this thesis. The possibility of defining particle systems on other structures than a lattice is apparent when dealing with an IPS such as the contact process, where we can think of people being spread out over an arbitrary graph, with their edges representing contacts.

A tool that is central to the study of IPS is Markov duality. Duality allows us to link two different Markov processes via a so-called 'duality function'. This duality function allows us to answer certain questions we may have about one of the processes by investigating its dual process. Often this means we can reduce a difficult problem to a more tractable one involving an easier process. For instance, we can reduce a problem involving infinitely many particles into one involving finitely many particles, or we can turn a problem with a continuous IPS into one involving a discrete IPS. Both of these examples will appear in this thesis. As mentioned, Markov duality is central to the study of interacting particle systems, first introduced to the field in [19]. Nowadays it appears in many applications such as quantum spin chains, population dynamics, and equilibrium analysis.

Some patterns emerge when using Markov duality in the study of IPS. In the study of reservoir driven processes, for example, we see that a current in a non-equilibrium process corresponds to absorption at the boundary of its dual process. Mathematically, the study of Markov duality greatly benefits from the notion of algebraic symmetry, a property of invariance of mathematical structures under certain operations. In a physical interpretation, symmetries correspond to conservation laws. This focus on symmetry sparked a great interest in studying the algebraic structure of interacting particle systems and duality. As a result, a body of literature has emerged in which a general (Lie) algebra approach to duality is being developed. Notably a book on the topic is currently being written [4]. Although not the main focus of this thesis, because of the centrality of duality to the field of IPS and the usefulness of duality in this thesis, we will explain some aspects of this approach throughout this thesis.

The thesis is structured in the following way; In Chapter 2, background information will be given about Markov processes, semigroups and infinitesimal generators. Chapter 3 will explain the concept and usefulness of duality and briefly touch on the underlying theory about the relation between duality and algebras. Chapter 4 will introduce several relevant IPS and their relation to each other, and similarly, Chapter 5 will do the same for the ABEP specifically. Chapter 6 will then give a brief introduction to hydrodynamic limits, and prove the hydrodynamic limit of the BEP, after which in Chapter 7 this will be done for the ABEP and a newly developed generalization of the ABEP, the Dynamic ABEP. After this, in Chapter 8, propagation of chaos under a local equilibrium measure with slowly varying shape parameter will be proven for the BEP and the ABEP. Finally, Chapter 9 provides a conclusion to this thesis and suggests further research questions. Since there is a lot of different notation in this thesis, Appendix A contains a table of relevant symbols.

2 Markov Processes, Semigroups, Generators

2.1 Definition

A Markov process is a stochastic process which satisfies the "Markov property", sometimes referred to as 'memorylessness'. In words, it means that the only information about the process that is relevant to predicting future behavior, is its current state. More formally, we have the following definition.

Definition 2.1 (Markov Process). *Let $\{X_t\}_{t\geq 0}$ be a stochastic process on probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with associated natural filtration $\mathcal{F}_t = \sigma(X_s : s \leq t)$, which is the σ-algebra induced by $(X_s)_{0\leq s\leq t}$. $(X_t)_{t\geq 0}$ is a Markov process if for any bounded, measurable function $f : \Omega \to \mathbb{R}$ and any $t > s \geq 0$ we have*

$$\mathbb{E}[f(X_t)|\mathcal{F}_s] = \mathbb{E}[f(X_t)|X_s],$$

i.e. conditioning on all information of X up to time s is the same as conditioning on the value of X at time s.

For the rest of the Chapter, let $(X_t)_{t\geq 0}$ denote an arbitrary Markov process with state space Ω.

2.2 Semigroups and Generators

2.2.1 Semigroups

Two important and related concepts in the study of Markov processes are semigroups and generators. Both of them describe the time evolution of a Markov process through the effect this evolution has on an arbitrary function. In order to avoid some complications that arise with unrestricted state spaces, we assume for this chapter that Ω is metric compact (or locally compact) so that the Markov process $(X_t)_{t\geq 0}$ is a Feller Process. When dealing with functions, we will assume these to be bounded continuous functions $f : \Omega \to \mathbb{R}$, and we will denote with $C(\Omega)$ the space of these functions. In later Chapters we will deviate from this (e.g. we will see functions on non-compact state spaces with infinite support), but in those cases we will be dealing with specific Markov processes and functions for which it is known that the theory introduced in this chapter applies.

Definition 2.2 (Markov Semigroup). *A (Markov) semigroup $(S_t)_{t\geq 0}$ is a family of operators acting on $f \in C(\Omega)$. For every $t > 0$, $S_t : C(\Omega) \to C(\Omega)$ is defined through*

$$(S_t f)(x) = \mathbb{E}[f(X_t)|X_0 = x] = \mathbb{E}_x[f(X_t)], \tag{1}$$

where $\mathbb{E}_x$ denotes the expectation in Markov process $(X_t)_{t\geq 0}$ starting at $x \in \Omega$.

Proposition 2.1. *For any semigroup $(S_t)_{t\geq 0}$ we have the following properties:*

1. ***Semigroup property:*** *For all $s, t \geq 0$, we have $S_{t+s} = S_t S_s$ and S_0 is the identity operator,*
2. ***Strong right-continuity:*** *For every function $f : \Omega \to \mathbb{R}$ we have $\lim_{t\to 0} S_t f = f$, where convergence is in $C(\Omega)$ with the supremum norm,*
3. ***Positivity:*** *If $f \geq 0$, then $S_t f \geq 0$,*

4. ***Normalization:*** $S_t 1 = 1$,
5. ***Contraction:*** $||S_t f||_\infty \leq ||f||_\infty$.

Proof. See e.g. [2]. $\square$

2.2.2 Generators

Closely related to semigroups are (infinitesimal) generators. What motivates the existence and use of generators are particularly the strong continuity and semigroup property of semigroups. The semigroup property implies that if we wish to know the effect of evolving the Markov process over a time-interval of length t, then we can split the semigroup into multiple copies of itself with time variables adding up to t. Strong continuity implies that we can take the limit case of this, so that we find that if we wish to know the behavior of a Markov process, we only need to know the effect of its semigroup over an infinitesimal time-interval. This effect can be described by a generator, which we could think of as a sort of time-derivative of a semigroup.

Definition 2.3 (Generator)**.** *Let the domain of a generator L be given through*

$$\mathcal{D}(L) = \left\{ f \in C(\Omega) \text{ such that } \lim_{t\to 0} \frac{S_t f - f}{t} \text{ exists} \right\}.$$

Then L is defined as

$$Lf = \lim_{t\downarrow 0} \frac{S_t f - f}{t}. \tag{2}$$

In this definition and throughout the rest of this Chapter, this convergence of function should be read as convergence in $C(\Omega)$, i.e. $f_n \to f$ if and only if $\sup\limits_{x\in\Omega} |f_n(x) - f(x)| \to 0$. We note that the limit in (2) does not necessarily exist for every function $f \in C(\Omega)$. This means that for all $t > 0 : \mathcal{D}(L) \subseteq \mathcal{D}(S_t)$.

2.2.3 Finite state spaces (Markov Chains)

For readers unfamiliar with the subject, semigroups and generators are easier to think of in the special case where Ω is finite. In this case the Markov process is a continuous-time Markov chain (CTMC). In this setting, (1), which gives the definition of a semigroup, then becomes

$$(S_t f)(x) = \mathbb{E}_x[f(X_t)] = \sum_{y\in\Omega} \mathbb{P}(X_t = y | X_0 = x) f(y).$$

From this expression we can see that in this setting with finite state space we can think of a function $f \in C(\Omega)$ as a (column) vector with values representing $f(x)$ for all $x \in \Omega$. Operator S_t can be thought of as a matrix $(S_t)_{x,y\in\Omega}$ with $(S_t)_{x,y} = \mathbb{P}_x[X_t = y]$, which is a transition probability.
CTMCs jump from one state to another randomly, with jumps occurring according to an exponential random variable, whereby the process changes its state to one accessible from the current state. The rate at which a jump occurs from a given state to another state depends on the specific combination of states, which means that we can create a matrix with as elements the jumping rates between states of the process. This matrix if often called the Q-matrix where $Q = (q_{x,y})_{x,y\in\Omega}$ with elements $q_{x,y}$ denoting the jumping rate from state x to y and $q_{x,x} = -\sum\limits_{y\neq x} q_{x,y}$. We may recognize that

the jumping rate $q_{x,y}$ between two states is closely related to the probability of being at the latter state, starting out from the former.
In fact, this Q-matrix of jumping rates is exactly the generator L of the CTMC. In order to see this, we may recognize that as we look at an infinitesimal time-frame, the probability of reaching y from x is dominated by the possibility of jumping directly from x to y, which happens following an exponential distribution at rate $q_{x,y}$. Thus in the limit $t \downarrow 0$, S_t becomes dominated by L, which yields (2). In this setting, the semigroup and generator are also linked via the exponential function,

$$S_t = e^{Lt},$$

where exponentiation of a matrix means taking the Taylor expansion of the exponential function, i.e.

$$S_t = e^{Lt} = \sum_{k=0}^{\infty} \frac{(Lt)^k}{k!},$$

where this sum is absolutely convergent in matrix norm. We can check that this relation is consistent with (2).

2.2.4 Hille-Yosida

In the general setting, where Ω may be infinite, the relation $S_t = e^{Lt}$ via Taylor series isn't always workable due to the potentially unbounded nature of L. There is still a unique correspondence between generators and semigroups, and it's given through the theorem of Hille-Yosida. This provides a more general relation between L and S_t through resolvents, which in the case of a finite Ω is equivalent to the one above.

Theorem 2.1 (Hille-Yosida)**.** *There is a one-to-one correspondence between a Markov semigroup* $(S_t)_{t\geq 0}$ *and a Markov generator* L*, given through*

a. $\mathcal{D}(L) := \{f \in C(\Omega) : \lim\limits_{t\to 0} \frac{S_t f - f}{t} \text{ exists}\}$ *and for* $f \in \mathcal{D}(L)$ *we have* $Lf := \lim\limits_{t\to 0} \frac{S_t f - f}{t}$*,*

b. $S_t = \lim\limits_{n\to\infty} (I - \frac{t}{n} L)^{-n}$*,*

c. Given a generator L *and a function* $f \in \mathcal{D}(L)$*, if we have* $\frac{d}{dt} S_t f = S_t L f = L S_t f$*, then* $S_t f$ *is the unique solution to* $\frac{d\psi_t}{dt} = L\psi_t$ *with initial value* $\psi_0 = f$*.*

Proof. See e.g. [14] □

2.2.5 The Dynkin Martingale

A very useful tool in this thesis is the Dynkin martingale. The Dynkin martingale is related to the Dynkin formula, which is seen as the stochastic analog to the fundamental theorem of calculus. It allows us to express a stochastic problem expressed with generators into one involving a martingale.

Theorem 2.2 (Dynkin Martingale)**.** *If* $\{X_t, t \geq 0\}$ *is a Markov Process with generator* L*, then for any function* $f \in \mathcal{D}(L)$*,*

$$M_t := f(X_t) - f(X_0) - \int_0^t Lf(X_s)ds, \tag{3}$$

is an $\mathcal{F}_t-$martingale which we call a Dynkin martingale.
If additionally $f^2 \in \mathcal{D}(L)$, the (quadratic) variation process of $(M_t)_{t\geq 0}$ is given by

$$[M]_t = \int\limits_0^t \left[Lf^2(X_s) - 2f(X_s)Lf(X_s)\right] ds.$$

Proof. **Proof of Martingale property:**

$$\begin{aligned}\mathbb{E}[M_t - M_s|F_s] &= \mathbb{E}[f(X_t) - f(X_s) - \int\limits_s^t Lf(X_u)du|F_s] \\ &= S_{t-s}f(X_s) - f(X_s) - \int\limits_s^t \mathbb{E}[Lf(X_u)du|F_s] \\ &= S_{t-s}f(X_s) - f(X_s) - \int\limits_s^t S_{u-s}Lf(X_s)du \\ &= S_{t-s}f(X_s) - f(X_s) - \int\limits_s^t \frac{d}{du}S_{u-s}f(X_s)du = 0,\end{aligned}$$

where we used c. from Theorem 2.1 in the second last step.
Proof of variation process
Recall that the (quadratic) variation process of a martingale $([M_t])_{t\geq 0}$ is the unique process such that $M_\cdot^2 - [M]_\cdot$ is a martingale. This means that our goal is to prove that

$$M_t^2 - \int\limits_0^t \left(Lf^2(X_s) - 2f(X_s)Lf(X_s)\right) ds,$$

is an $\mathcal{F}_t$-martingale.
Since adding a constant to a martingale does not change its variation, we can without loss of generality assume that $M_t = f(X_t) - \int\limits_0^t Lf(X_s)ds$, i.e. $f(X_0) = 0$.
Furthermore, the martingale property together with an application of the monotone convergence theorem imply that it suffices to prove that

$$\mathbb{E}\left[M_t^2 - \int_0^t \left(Lf^2(X_s) - 2f(X_s)Lf(X_s)\right) ds \middle| \mathcal{F}_0\right] = o(t), \tag{4}$$

where $o(t)$ means vanishing to 0 faster than t as we take the limit $t \downarrow 0$.
Squaring (3) with $f(X_0) = 0$ yields

$$\begin{aligned}M_t^2 &= f^2(X_t) - 2f(X_t)\int_0^t Lf(X_s)ds + \left(\int_0^t Lf(X_s)ds\right)^2 &(5)\\ &= f^2(X_t) - 2f(X_t)\int_0^t Lf(X_s)ds + o(t). &(6)\end{aligned}$$

Since

$$\mathbb{E}\left[f^2(X_t) - \int_0^t Lf^2(X_s)ds \middle| \mathcal{F}_0 \right] = 0,$$

because this is again a Dynkin martingale (note that $f^2(X_0) = 0$), we find

$$\begin{aligned}
&\mathbb{E}\left[M_t^2 - \int_0^t Lf^2(X_s) - 2f(X_s)Lf(X_s)ds \middle| \mathcal{F}_0 \right] \\
&= \mathbb{E}\left[-2f(X_t)\int_0^t Lf(X_s)ds - \int_0^t 2f(X_s)Lf(X_s)ds + o(t) \middle| \mathcal{F}_0 \right] \\
&= \mathbb{E}\left[\int_0^t \left(f(X_s) - f(X_t)\right) Lf(X_s)ds | \mathcal{F}_0 \right] + o(t) \\
&= o(t),
\end{aligned}$$

as $f(X_s) - f(X_t) = o(1)$, and we integrate it over $[0, t]$. □

Equation (3) appears as well in the so-called "Martingale problem", where it is used in a somewhat converse way as here.

Definition 2.4 (Martingale Problem)**.** *A triple* $((\Omega, \mathcal{F}, \mathbb{P}), (\mathcal{F}_t)_{t\geq 0}, (X(t))_{t\geq 0})$*, where* $(\Omega, \mathcal{F}, \mathbb{P}), (\mathcal{F}_t)_{t\geq 0}$ *is a stochastic basis and* X *an* F_t*-adapted stochastic process, is the solution to the martingale problem posed by operator* A *if for every* $f \in \mathcal{D}(A)$*,*

$$M_t := f(X_t) - f(X_0) - \int_0^t Af(X_s)ds,$$

is an $\mathcal{F}_t$*-martingale.*

We can use martingale problems to prove the unique relation between a Markov process and its generator.

Theorem 2.3. *The unique solution to the martingale problem posed by a Markov generator is its associated Markov process.*

Proof. From Theorem 2.2 follows directly that a Markov process solves the martingale problem posed by its generator, so the only claim that needs to be proven is its uniqueness. This follows from the one-to-one correspondence between a Markov generator and a Markov process. □

We will use Theorem 2.3 in Theorem 5.2 to prove equivalence of two differently constructed Markov processes.

2.3 Markov processes and measures

In this section so far we have seen Markov processes as they relate to the evolution of functions and variables, starting out at some initial condition. This implicitly assumes that this initial condition is known. Let us now change our focus to a situation where this initial condition is also random, and decribed through some probability measure.

Let $\mu : \mathcal{B}(\Omega) \to [0,1]$ be a probability measure and $f \in C(\Omega)$ a continuous function, and suppose we are interested in

$$\int_\Omega S_t f d\mu,$$

i.e. we which to evaluate function f, starting out at a random configuration with its distribution given through μ, and then evolved through semigroup S_t.

One approach to evaluating this integral is the following; first, we take the expected value of $f(x)$ evolved according to the Markov process associated with S_t, after which we integrate the resulting expression of $(S_t f)(x)$ with respect to $\mu(x)$, i.e. we take the expectation with respect to the random variable associated with μ. Alternatively though, we can reframe our problem in such a way both these expectations are taken by integration with respect to one measure, which we call the evolved measure.

Definition 2.5 (Evolved measure)**.** *For semigroup S_t and (probability) measure μ, the evolved measure μS_t is the unique measure such that*

$$\int_\Omega S_t f d\mu = \int_\Omega f d\mu S_t.$$

Although this definition is not restricted to probability measures, these are the only sort of measures for which we will use this notion in this thesis. Therefore for the rest of the chapter let $\mu \in P(\Omega)$ be an arbitrary probability measure on Ω and $P(\Omega)$ the space of probability measures on Ω. The notation μS_t for the evolved measure comes from the setting with finite state spaces, where measures can be seen as row vectors, similar to how functions can be seen as column vectors and semigroups as matrices. In fact in this setting

$$\int_\Omega f d\mu = \mu f = \sum_{x \in \Omega} \mu(x) f(x),$$

so that

$$\int_\Omega S_t f d\mu = \mu(S_t f) = (\mu S_t) f = \int_\Omega f d\mu S_t.$$

This notion of an evolved measure allows us to move the two sources of randomness that are often present in the systems we are working with, i.e. the random initial condition and the evolution of a Markov process, into one measure. At times, this will be a useful tool for describing other objects we are interested in (e.g. in Chapters 6 and 7 where we use an evolved measure to describe an interacting particle system), and at other times deriving this evolved measure will be the goal of our investigation (e.g. in Chapter 8 on propagation of chaos).

In looking at the interaction between the evolution of a Markov process and distribution of the initial condition, two useful properties that the measure describing the initial distribution can have (with respect to the Markov process) are invariance and reversibility. Both properties relate to the measure being 'unaffected' by the evolution of the Markov process, where the exact meaning of this is different between the two.

Definition 2.6 (Invariant measure)**.** *A measure $\mu \in P(\Omega)$ is invariant (or 'stationary') if*

$$\int_\Omega S_t f d\mu = \int_\Omega f d\mu \; \forall f \in C(\Omega). \tag{7}$$

For functions f in the domain of the generator of a Markov process, we can make a similar statement using the generator instead of the semigroup.

Theorem 2.4. *$\mu \in P(\Omega)$ is an invariant measure if and only if*

$$\int_\Omega Lf d\mu = 0 \; \forall f \in \mathcal{D}(L).$$

Proof. From Chapter 2 of [14]. □

An invariant measure can be thought of as a measure that is unchanged under the evolution of the Markov process. This property is easier to interpret in the finite setting of Markov chain. Although before we talked about finite state spaces when we discussed interpreting functions, measures and operators as vectors and matrices, our results will apply in countably infinite state spaces as well, so we will focus on countable Markov processes. In this setting, the invariance condition is the following.

$$\text{For each } t > 0: \; \mu S_t = \mu \text{ or equivalently } \sum_{y\in\Omega} \mu(y)(S_t)(x,y) = \mu(x) \; \forall x \in \Omega, \tag{8}$$

and Theorem 2.4 yields

$$\sum_{y\in\Omega} \mu(y)L(x,y) = 0 \; \forall x \in \Omega, \tag{9}$$

which can be interpreted as the probability flow in and out of any state being equal.
A stronger property of measures is that of reversibility. Its definition in general setting is the following.

Definition 2.7 (Reversible measure for Markov processes)**.** *$\mu \in P(\Omega)$ is reversible for a Markov process with semigroup S_t if*

$$\int_\Omega (S_t f) g d\mu = \int_\Omega f(S_t g) d\mu \; \forall f, g \in C(\Omega). \tag{10}$$

In the countable setting reversibility can be interpreted as there being an equal probability flow in both directions for each pair of states. For countable Ω, (10) can be shown to be equivalant to

$$\text{for each } t > 0: \mu(x)S_t(x,y) = \mu(y)S_t(y,x) \; \forall x, y \in \Omega,$$

or equivalently

$$\mu(x)q_{x,y} = q_{y,x}\mu(y) \; \forall x, y \in \Omega, \tag{11}$$

where $q_{x,y} = L(x,y)$ denotes the jumping rate between state x and y. (11) is often called the "detailed balance equation". Note that one recovers (8) by summing over j.

Proposition 2.2. *Equation (10) is equivalent to the following two statements*

1. *S_t is self-adjoint on $L^2(\mu)$,*
2. *L is self-adjoint on $L^2(\mu)$.*

Proof. See Chapter 2 of [14] □

3 Markov Duality

3.1 Definition and motivation

(Stochastic) duality is a powerful tool used in interacting particle systems, offering a way to connect two different Markov processes through a so-called duality function. The formal definition of the general form of duality, semigroup duality, is as follows.

Definition 3.1 (Semigroup Duality)**.** *Two Markov processes $(\eta_t)_{t\geq 0}$ and $(\xi_t)_{t\geq 0}$ taking values in Ω and $\hat{\Omega}$ are said to be dual to each other with duality function $D : \Omega \times \hat{\Omega} \to \mathbb{R}$ if for every $\eta \in \Omega, \xi \in \hat{\Omega}$ we have*

$$\mathbb{E}_\eta D(\eta_t, \xi) = \hat{\mathbb{E}}_\xi D(\eta, \xi_t).$$

If $(\eta_t)_{t\geq 0}$ and $(\xi_t)_{t\geq 0}$ are two copies of the same process, we say the process is self-dual and call $D : \Omega \times \Omega \to \mathbb{R}$ the self-duality function.

We can also define duality between Markov processes in terms of their generators.

Definition 3.2 (Generator Duality)**.** *Let L be the generator of Markov process $(x_t)_{t\geq 0}$ and $\hat{L}$ that of $(y_t)_{t\geq 0}$, and let $D : \Omega \times \hat{\Omega}$ be such that $D(\cdot, y)$ is in the domain of L for each $y \in \hat{\Omega}$ and $D(x, \cdot)$ is in the domain of $\hat{L}$ for each $x \in \Omega$. Then we have generator duality between x_t and y_t if for every $x \in \Omega$ and $y \in \hat{\Omega}$ we have*

$$(LD(\cdot, y))(x) = (\hat{L}D(x, \cdot))(y). \tag{12}$$

Reminiscent of Theorem 2.4 we have here that semigroup duality implies generator duality when the duality function is in the domain of the generators. Furthermore we have the convenient finding that if one of the two processes is finite, then semigroup duality implies generator duality. Since in this thesis and practically in every application of interacting particle systems one of the two processes connected via duality is finite, we can treat semigroup duality and generator duality as equivalent. The usefulness of duality depends on which Markov processes we connect via a duality function. Often the power of duality comes from the fact that a complex process can be shown to be dual to a simpler one, allowing us to reduce a difficult problem to an easier problem. This can take many forms. Following the lines of [4], we offer some examples of ways to use duality, many of which are used in this thesis.

1. Continuous $\to$ discrete. We can connect a continuous and a discrete process with each other using duality, which often means that we go from a process with an uncountable state space to one with a finite or countably infinite state space. In this thesis we will see that the many forms of Brownian Energy Process, which are continuous, are dual to the Symmetric Inclusion Process, which is discrete.

2. Many $\to$ few. We can connect a process with an infinite total number of particles or total amount of energy with a process with a finite number of particles. This is also what happens when we connect the BEP and the SIP via duality in this thesis.

3. Non-equilibrium $\to$ absorbing state. This does not appear in this thesis, due to the absence of (open) boundaries. In reservoir driven processes, where particles or energy may flow into or out of the system at the boundaries, a persistent current may arise when the rate at which this happens differs at different boundaries. This is what gives the system its non-equilibrium

property. It has been found that such systems are dual to systems of random walkers where the boundaries are absorbing sites, which are easier systems to analyze.

4. Micro $\to$ macro (hydrodynamic limits). This is the main topic of this thesis. We use duality between different interacting particle systems at micro-scale in order to learn something about the density profile at macro-scale.

5. Structure of measures of infinite interacting systems. In a context with interacting particle systems in infinite volume, it is often difficult to make general statements about infinite-dimensional measures. Here we can use duality in order to reduce a statement about an infinite configuration to one involving finite many dual particles. In this thesis this will be used when we discuss reversible and "local equilibrium" measure of the different versions of the Brownian Energy Process.

6. Proving existence of processes in infinite volume. When working with an infinite volume, it is not always clear that an interacting particle system is well-defined. In such a case, we can prove the existing via the martingale problem, as we saw in Theorem 2.3. In such a proof, duality with an interacting particle system that we know exists in infinite volume can be useful in solving the martingale problem. This application of duality will be briefly mentioned in this thesis in Section 5.3, when we argue why it is not clear whether the ABEP exists in infinite volume with infinite total energy.

3.2 Duality in finite state spaces

At this point duality may still seem somewhat arbitrary, as we don't have a method of arriving at a duality relation, and since we don't yet know anything about the specific duality functions, it is not clear that these are useful for anything. The remainder of this section addresses this first issue by giving a method for how duality relations between Markov processes can be found and expanded. First this will be done for the relatively simply case of Markov processes with finite state spaces, after which we will shift to the more general algebraic approach to duality. The duality relations that are used throughout this thesis have already been established, so this section is mostly to provide background information on how these were derived. For this reason this section will be relatively brief and only cover a small portion of the existing literature.

First let us look at the definition of duality in this finite setting, again taking the approach of treating functions like vectors and operators as matrices. When in the setting of Definition 3.2 with finite Ω, (12) then becomes

$$\begin{aligned}
(LD(\cdot,y))(x) &= (\tilde{L}D(x,\cdot))(y), \\
\left(\sum_{z\in\Omega} L(\cdot,z)D(z,y)\right)(x) &= \left(\sum_{z\in\Omega} \tilde{L}(\cdot,z)D(x,z)\right)(y) \\
\sum_{z\in\Omega} L(x,z)D(z,y) &= \sum_{z\in\Omega} D(x,z)\tilde{L}^T(z,y) \\
(LD)(x,y) &= (D\tilde{L}^T)(x,y).
\end{aligned}$$

Thus in matrix formulation we write as duality condition

$$LD = D\tilde{L}^T. \tag{13}$$

A usual approach to duality is to start out by showing that a process is self-dual. We can find self-duality using the reversible measure introduced in the previous section. This self-duality function is often called a "cheap" self-duality function, because it can be found relatively easy via the following theorem.

Theorem 3.1. *If μ is a reversible measure for Markov process with finite state space $(X_t)_{t\geq 0}$ with generator L, then*

$$D(x,x') = \frac{1}{\mu(x)}\delta_{x,x'},$$

is a self-duality function for L, i.e. we have

$$LD = DL^T.$$

Proof. Will simultaneously be proven with Theorem 3.2. □

In similar fashion we can find a "cheap" duality function in the case where we don't have a reversible measure, but instead a stationary measure. In this case we don't have self-duality anymore, but duality between the process and its time-reversal.

Theorem 3.2. *Suppose M is a stationary measure of $(X_t)_{t\geq 0}$ with generator L and we define $\tilde{L}$ as the generator of the time-reversed process via*

$$\tilde{L}(x,x') = \frac{M(x')L(x',x)}{M(x)}.$$

Then

$$D(x,x') = \frac{1}{M(x)}\delta_{x,x'},$$

is a duality function between X_t and the time reversed version, i.e. $LD = D\tilde{L}^T$.

Proof.

$$\begin{aligned}
(LD)(x,x') - (D\tilde{L}^T)(x,x') &= \sum_{z\in\Omega} L(x,z)D(z,x') - \sum_{z\in\Omega} D(x,z)\tilde{L}^T(z,x') \\
&= \sum_{z\in\Omega} L(x,z)D(z,x') - \sum_{z\in\Omega} D(x,z)\tilde{L}(x',z) \\
&= L(x,x')\frac{1}{M(x')} - \frac{1}{M(x)}\tilde{L}(x',x) \\
&= \frac{L(x,x')}{M(x')} - \frac{L(x,x')M(x)}{M(x)M(x')} \\
&= L(x,x') - L(x,x') = 0,
\end{aligned}$$

which proves Theorem 3.2. Theorem 3.1 then follows as a special case where $L = \tilde{L}$. □

The existence of these so-called cheap (self-)duality functions is useful, because it hands us a starting point in exploring duality relations. Next we will show that from an existing duality function between two (possibly identical) processes we can find more duality functions. To do this we first need the notion of symmetry.

Definition 3.3 (Symmetry)**.** *A matrix S is a symmetry of operator L if S commutes with L, i.e. $LS - SL = 0$.*

This is a general definition of a symmetry. When L is the generator of a Markov chain and we treat it as a matrix, S also takes the form of a matrix. When L is an arbitrary operator, $S : \mathcal{D}(L) \to \mathcal{D}(L)$ is an operator such that

$$(LS)f = (SL)f \text{ for all } f \in \mathcal{D}(L).$$

Applying symmetries to duality functions allows us to find new duality functions.

Theorem 3.3.

1. *If D is a self-duality function for L and S is a symmetry of L, then SD and DS^T are also self-duality functions.*

2. *Suppose D is a duality function for L and $\hat{L}$.*
Then if S is a symmetry of L, then SD is a duality function between L and $\tilde{L}$ as well.
Similarly if $\tilde{S}$ is a symmetry of $\tilde{L}$, then $D\tilde{S}^T$ is a duality function between L and $\tilde{L}$.

Proof. Suppose $LD = D\tilde{L}^T$ and $LS = SL$, then

$$L(SD) = (LS)D = (SL)D = S(LD) = S(D\tilde{L}^T) = (SD)\tilde{L}^T,$$

which proves that (SD) is a duality function between L and $\tilde{L}$.
The proof for $D\tilde{S}^T$ is analogous, which proves 2., and 1. is again the special case where $L = \tilde{L}$. $\square$

These findings allow us to discover new duality functions of the same pair of processes. If we then put certain restrictions on one of the processes we have connected with duality or take certain scaling limits, we can change it to another process, often while retaining the duality relation.

3.3 Duality in uncountable state spaces

On top of being useful themselves, these duality results for finite state spaces are good at offering an intuitive idea of the power and usefulness of duality. Furthermore, the duality results in context of finite state spacs easily translate to countably infinite state spacs. Next, we will focus on uncountable state spacs.
Following the lines of [5] we increase the level of generalization by considering generators which are bounded operators on L^2-spaces. Let $\mathcal{H} = L^2(\Omega, \mathcal{F}, \mathbb{P})$ and $\hat{\mathcal{H}} = L^2(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ be two Hilbert spaces with tensor product Hilbert space $\mathcal{H} \otimes \hat{\mathcal{H}} = L^2(\Omega \otimes \hat{\Omega}, \mathbb{P} \otimes \hat{\mathbb{P}})$, and let L and $\hat{L}$ be two Markov generators on these respective spaces.
First, let us write the definition of duality in (12) purely in terms of operators and functions, similar to (13) in the finite setting. In our context here we have that $D \in \mathcal{H} \otimes \hat{\mathcal{H}}$ is a duality function between L and $\hat{L}$ if

$$(L \otimes I)D = (I \otimes \hat{L})D. \tag{14}$$

In this context, we do not have a straightforward approach for finding a 'cheap' (self-)duality function, as we did in the finite setting in Theorem 3.2. Furthermore, even if we did, finding a stationary or reversible measure is not straightforward either, as we do not have a useful analogue to the detailed balance equation in this setting.

In this setting, we do not have an exact analogue to Theorem 3.3, as the different operators involved may have different domains, and it is not clear that the duality function in question lies in the relevant operator domains. We do have the following theorem, which relates the notion of symmetry to duality.

Theorem 3.4. *The following are equivalent.*

1. *Let L be a generator on $\mathcal{H}$ and assume it has a symmetry, i.e. an operator $S : \mathcal{H} \to \mathcal{H}$ such that $[L, S] = 0$. Then if S can be written in 'kernel operator form', i.e.*

$$Sf(x) = \int_{\Omega} D(x, y) f(y) d\mu(y), \tag{15}$$

then D is a self-duality function for L.

2. *If D is a self-duality function, then S defined via (15) is a symmetry of L.*

Proof. Assume 1. Then

$$LSf(x) = \left(\int_{\Omega} D(\cdot, y) f(y) d\mu(y) \right) = \int_{\Omega} (L \otimes I) D(x, y) f(y) d\mu(y), \text{ and}$$

$$SLf(y) = \int_{\Omega} (D(x, y) Lf(y) d\mu(y) = \int_{\Omega} (I \otimes L^*) D(x, y) f(y) d\mu(y),$$

where L^* denotes the adjoint of L in $\mathcal{H}$, which is equal to L. Thus we find

$$(L \otimes I)D = (I \otimes L)D.$$

Conversely we can go in the other direction to show 2. $\implies$ 1. $\square$

This finding shows that still duality and symmetry are deeply interwoven. However, this theorem does not provide a clear path towards finding duality functions, as it is not evident how to arrive at such a symmetry in kernel operator form. This is where the difficulty of deriving duality functions in uncountable state spaces lies. In fact, when we generalize further by considering unbounded generators, which generally apply for interacting particle systems in infinite volume, even more problems arise regarding domains of operators. For this reason, no general theorems exist for deriving duality functions in an uncountable state space with unbounded operators. Our approach in this thesis will therefore be start with duality in a finite setting, and then translate the findings to another setting with (uncountably) infinite state spaces.

This can be achieved in two different ways. One of these is to take scaling limits of Markov processes defined on countable state spaces, in a way such that duality is preserved for the resulting process on an uncountable state space. In Theorem 4.4 this will yield a duality function between the BEP and the SIP. The second approach requires us to analyze the algebraic properties of the generators that we work with. We then continue our approach of increasing the level of generalization, such that our exploration of duality between Markov generators turns into an exploration of different representations of an algebra. This is the topic of the next section.

3.4 Lie algebraic approach to duality via su(1,1)

In the previous sections we have explored how duality functions are found for one or two given Markov generators. Turning this investigation around, we note that the duality relation between L and $\hat{L}$ with duality function D in the setting of Section 3.3

$$(L \otimes I)D = (I \otimes \hat{L})D,$$

in general does not just hold for L and $\hat{L}$, but for a family of operators on $\mathcal{H}$ and $\hat{\mathcal{H}}$ respectively, both of which can be shown to be an algebra. In this section we give a brief overview of the field of study which provides a method for the construction of Markov generators, which satisfy certain duality relations that we want. Sections 3.2 and 3.3 have made clear the importance of symmetries in the study of duality. As we saw in Theorem 3.3 and Theorem 3.4, symmetries can be used in order to derive new duality functions from existing ones. These symmetries of generators have a natural algebraic structure. These algebraic structures provide guidelines for the construction of new generators with many useful symmetries.

This has led to the emergence of a Lie algebraic approach to duality. It has been found that in many practical cases, duality between two algebras (i.e. families of operators) is actually a duality between two "intertwined" representations of the same Lie algebra. Furthermore a symmetry of a generator is a manifestation of the symmetries that characterize the Lie algebra to which the generator is associated. Because of this, when new generators are now constructed in the study of interacting particle systems, they are often constructed via Lie-algebraic considerations that ensure that they have useful symmetries and duality relations to other processes of interest. The details of much of this lies outside of the scope of this thesis, requiring representation theory and other topics from Lie algebras and Lie groups. For more about the background of the emerging Lie algebraic approach to duality we refer to [4].

The takeaway for this thesis in practice is that each process that will be discussed can be written via a representation of the su(1,1) Lie algebra. This algebra is defined in the following way:

Definition 3.4 (su(1,1))**.** *The su(1,1) Lie algebra is defined through commutation relations*

$$[K^+, K^-] = -2K_0 \qquad [K^0, K^\pm] = \pm K^\pm. \tag{16}$$

The su(1,1) algebra is an important Lie algebra in the study of theoretical physics, especially quantum physics, and (more relevant here) stochastic processes and interacting particle systems. By focusing our study on generators which can be written via a representation of this Lie algebra, we ensure that they are dual to the other processes of this thesis, and that they have useful symmetries from which duality functions are derived. When we introduce generators throughout this thesis, we will show the connection to su(1,1) by writing them as a combination of operators which satisfy the commutation relations in (16). We will then see that if we create a new generator via a systematic transformation of these operators, duality is preserved. In this way duality can be interpreted as a 'change of representation' of the su(1,1) Lie algebra. All of the processes that are used in this thesis have been (or will be) introduced in other studies, so while we allocate some portion of this thesis on the creation of new generators, this just serves as background knowledge about the processes in question.

4 Interacting Particle Systems

4.1 Introduction

The interacting particle systems of interest in this thesis are Markov processes that describe the transport of particles or a continuous quantity which is often interpreted as energy or momentum ('transport models'). They are uniquely defined by their generator via the theory laid out in Chapter 2, in particular the theorem of Hille-Yosida (Theorem 2.1).
We distinguish two main types of interacting particle systems, continuous and discrete. Discrete interacting particle systems are continuous-time Markov jump processes describing the stochastic behavior of particles moving on a lattice, which can be one- or multi-dimensional, and interacting with each other locally. Continuous interacting particle systems are Markov diffusion processes describing the diffusion of a continuous quantity, usually interpreted as energy, over a similar lattice, where again there usually is interdependence between the amount of energy at sites and the stochastic energy flow between them.
Interacting particle systems are often defined on a finite lattice. In this thesis however, most processes are defined on the full integer line $\mathbb{Z}$. We say they are processes in "infinite volume". Since the generators of the processes are often defined as the sum of single-edge generators, each acting on a specific edge in the lattice, these can easily be extended to $\mathbb{Z}$ by increasing the number of single-edge generators to infinity with the number of sites. An advantage of this is that we don't have to deal with conditions at the boundary of the lattice. Problems may arise however when we're dealing with a net flow of energy due to asymmetry, or when we use nonlocal functions (i.e. which 'look' at the whole lattice). Both of these issues will come up in the next chapter about the ABEP. For this chapter we will assume that any function on the space of configurations is local, i.e. only depending on a finite subset of sites, and similarly we will focus on local measures. In the next chapter we will be forced to look at nonlocal functions and measures, and we will discuss the problems that arise as a result.
One of the simplest examples of an interacting particle system is that of Independent Random Walkers (IRW). Calling this an interacting particle system may be a bit misleading, because its main characteristic is that particles don't interact with each other. We define n-IRW(k) as the Markov process where n different particles jump independently from each other from their current site on the integer line towards one of the two neighboring sites at a fixed rate of k.

Definition 4.1 (n-IRW(k))**.** *Let* $|\xi| := \sum\limits_{i=-\infty}^{\infty} \xi_i$ *denote the number of particles in configuration* ξ *and let*

$$\Omega_n := \{\xi \in \mathbb{N}_0^{\mathbb{Z}} : |\xi| = n\},$$

be the space of configurations on $\mathbb{Z}$ *with* n *particles in total. The n-IRW(k) in infinite volume is defined as the Markov process on* Ω_n *with generator defined on local functions* $f : \Omega_n \to \mathbb{R}$ *through*

$$\left(L^{\mathrm{IRW}} f\right)(\xi) = \sum_{i=-\infty}^{\infty} k\xi_i (f(\xi^{i,i-1}) - 2f(\xi) + f(\xi^{i,i+1})),$$

where $\xi^{i,j}$ *denotes the SIP-configuration* ξ *with one particles moved from site* i *to site* j*.*

4.2 Path-space measures

Interacting particle systems are defined on some state space Ω. In the trivial example of n-IRW(k) we saw that $\Omega = \Omega_n$. Elements of this state space are often called configurations and will be called this in the rest of this thesis to distinguish them from other kinds of states. An IPS can be in a deterministic or random state, where in the latter case we usually define some measure to signify its distribution.

As the IPS evolves, the evolution of its configuration will create a trajectory, or path through Ω through time. We write $D([0,T],\Omega)$ to denote the space of càdlàg trajectories through Ω on time-interval $[0,T]$. Starting out with the easier deterministic case, we say that "path-space measure" $\mathbb{P}_\eta^{\mathrm{x}}$ on $D([0,T],\Omega)$ is the probability measure on $D([0,T],\Omega)$ for the IPS (denoted with superscript 'x'), starting at initial configuration η. As an example $\mathbb{P}_\eta^{\mathrm{IRW}}(\eta_s \in B \;\forall s \in [0,t])$ denotes the probability that as we evolve the IRW(k) with initial configuration η, its evolved configuration is in B the whole trajectory up to time t. A similar notation is used for other operators associated with the IPS.

When the initial state is random, we will call the probability measures that quantify its distribution 'configuration measures', and they typically take the form $\mu : \mathcal{B}(\Omega) \to [0,1]$. As described in Definition 2.5, we can define an evolved measure to combine the evolution of the IPS with the distribution of the initial state. The operator of such a path-space measure on $D([0,T],\Omega)$ typically looks like $\mathbb{P}_\mu^{\mathrm{x}}$, where subscript μ denotes the initial distribution. At times however, we want to investigate both sources of randomness (i.e. initial distribution and evolution) separately, and therefore we will write them out separately. To continue the example from before to showcase the two different notations, we may write either the left or right-hand side in the following equation to denote the probability of the IRW(k) being in B on the whole interval $[0,t]$, starting out at a random configuration with distribution μ:

$$\mathbb{P}_\mu^{\mathrm{IRW}}\left(\eta_s \in B \;\forall s \in [0,t]\right) = \int_\Omega \mathbb{P}_\eta^{\mathrm{IRW}}(\eta_s \in B \;\forall s \in [0,t]) d\mu(\eta).$$

4.3 The Symmetric Inclusion Process (SIP)

4.3.1 Definition

The Symmetric Inclusion Process (SIP) is an IPS that is an essential tool in this thesis. It is a relatively simple process of jumping particles, where although in infinite volume its state space is infinite, the number of configurations that can be directly reached from a given configuration is finite. In the SIP a fixed number of particles move randomly on a lattice and attract each other. Particles jump to the sites neighboring their current site at a fixed rate, with additional jumps between neighboring particles towards each other at a rate proportional to the product of the number of particles at both sites. Since jumps only occur between neighboring sites, and this jumping rate only depends on the number of particles on either side, we can write the generator of the SIP as a sum of so-called "single-edge generators", each corresponding to a pair of neighboring sites. The generator of the SIP(k), where parameter k denotes the fixed jumping rate of $2k$, is given through:

Definition 4.2 (SIP(k))**.** *The SIP(k) in infinite volume is defined as the Markov process on* $\mathbb{N}_0^{\mathbb{Z}}$

with generator defined on local functions $f : \mathbb{N}_0^{\mathbb{Z}} \to \mathbb{R}$

$$\left(L^{\mathrm{SIP}} f\right)(\eta) = \sum_{i=-\infty}^{\infty} \left(L^{\mathrm{SIP}}_{i,i+1} f\right)(\eta), \tag{17}$$

with single-edge generators $L^{\mathrm{SIP}}_{i,i+1}$ given through

$$\left(L^{\mathrm{SIP}}_{i,i+1} f\right)(\eta) = \eta_i(2k + \eta_{i+1})(f(\eta^{i,i+1}) - f(\eta)) + \eta_{i+1}(2k + \eta_i)(f(\eta^{i+1,i}) - f(\eta)). \tag{18}$$

Notably the SIP(k) has as fixed jumping rate parameter of 2k, which means that if we were to take away the attraction between particles, the SIP(k) would become the IRW(2k). This parameterization with a fixed rate of $2k$ is common practice, motivated by applications to spin models, where spin k is a half-integer. For the SIP(k) there is no maximum on the amount of particles at a given site and in total, which we see from its state space being $\mathbb{N}_0^{\mathbb{Z}}$. This means the total number of particles may be either finite or grow to infinity, where in the finite case we have conservation of the number of particles over time. The ability for the number of particles to grow indefinitely is one of the two main reasons for our interest in the SIP in this thesis. As we will see later, when we increase the amount of particles in the SIP, we will recover in the limit the Brownian Energy Process (BEP), a finding which will help us to interpret that process.
At times however, the usefulness of working with the SIP comes from it having a finite number of particles. In fact, the second and main reason for our interest in the SIP is its duality to the BEP and the ABEP. When the point of using duality is to reduce a problem involving a difficult process to one involving the SIP, the effectiveness of this is further amplified when the number of particles is explicitly kept fixed at a finite constant. For this reason, we also consider the n-SIP(k), which is the SIP(k) with n particles. Essentially the n-SIP and SIP are the same, but we like to have an explicit name for when the number of particles is finite and fixed under scaling.

Definition 4.3 (n-SIP(k))**.** *The n-SIP(k) in infinite volume is the Markov process on* $\Omega_n = \{\xi \in \{0, \dots, n\}^{\mathbb{Z}} : |\xi| = n\}$ *with generator* L^{SIP} *defined on local functions with compact support* $f : \Omega^n \to \mathbb{R}$ *via* (17) *and* (18)*.*

Closely related to the SIP is the Asymmetric Inclusion Process (ASIP), which as the name suggests is an asymmetric version of the SIP, where particles have a tendency to drift to the right. The ASIP is constructed as a q-analog of the SIP, meaning the SIP is transformed in a way parameterized by an asymmetry parameter q. Since this process is not very important to this thesis, and some of the subtleties of this q-transformation require some work and notation to explain, we refer to Appendix B for more about this process.

4.3.2 Duality

Next, we will use the theory laid out in Chapter 3 in order to find a self-duality function of the SIP. Although we will not use this self-duality in answering the main questions of the thesis (i.e. in proving the hydrodynamic limits of the BEP and ABEP), this helps us to better understand our findings about the BEP and ABEP, which are closely related to the SIP. It will also be a useful showcase for many of the concepts and techniques of the previous Chapter about duality in a relatively simple setting.
We have already established that a configuration of the SIP in infinite volume can only jump to countably many other configurations, but the division of L^{SIP} into single-edge generators $L^{\mathrm{SIP}}_{i,i+1}$ we

saw in (17) and (18), together with the finding that the reversible measure of the SIP is a product measure (as we will see in Theorem 4.1), means that we end up in a setting similar to one with finite state spaces. This means that we can use the theory from Chapter 3.2 concerning duality in finite state spaces.
Before proving self-duality we will motivate our interest in the SIP we start by showing that the generator of SIP can be written via a representation of the su(1,1) algebra.

Proposition 4.1. *L^{SIP}, given in (17) and (18), can be written via a representation of the su(1,1) Lie algebra.*

Proof. For $i \in \mathbb{Z}$ define discrete operators K_i^+, K_i^- and K_i^0, acting on local functions $f : \mathbb{N}_0^{\mathbb{Z}} \to \mathbb{R}$ as follows

$$\begin{aligned} K_i^+ f(\eta) &= (2k + \eta_i) f(\eta + \delta_i), \\ K_i^- f(\eta) &= \eta_i f(\eta - \delta_i), \\ K_i^0 f(\eta) &= (k + \eta_i) f(\eta). \end{aligned} \tag{19}$$

Then one can check that these operators satisfy the su(1,1) commutation relations and

$$L_{i,i+1}^{\mathrm{SIP}} = K_i^+ K_{i+1}^- + K_i^- K_{i+1}^+ - 2K_i^0 K_{i+1}^0 + 2k^2. \tag{20}$$

See [4] for explicit computations. □

Next, we will show how to find a self-duality function for the SIP. As described in Chapter 3, the explorations of duality within an algebra often start with a 'cheap' (self-)duality function, found with the help of a stationary or reversible measure of the process. For the SIP we have the following family of measures that are reversible (and stationary):

Theorem 4.1. *The SIP(k) has a family of reversible and stationary measures with free parameter θ given through*

$$M_\theta^{2k,\infty}(\eta) = \prod_{i=-\infty}^{\infty} \frac{\theta^{\eta_i} \Gamma(\eta_i + 2k)}{\eta_i! \Gamma(2k)}, \tag{21}$$

where Γ is the gamma function defined on a superset of $\mathbb{R}_+$ via

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt.$$

Proof. Since $M_\theta^{2k,\infty}$ is a product measure, we can prove reversibility by looking at each of its marginals seperately. Since we can also note that SIP-particles can only jump to a nearest neighbor, this allows us for each marginal to ignore all but two sites, bringing us to a situation equivalent to one in which the state space is finite. We can therefore interpret L as a matrix of transition rates and use the detailed balance equation. Thus in order to prove reversibility of $M_\theta^{2k,\infty}(\eta) = \prod_{i=-\infty}^{\infty} m_i(\eta)$ it suffices to show for every $i \in \mathbb{Z}$

$$m_i(\eta) m_{i+1}(\eta) L(\eta, \eta^{i,i+1}) = m_i(\eta^{i,i+1}) m_{i+1}(\eta^{i,i+1}) L(\eta^{i,i+1}, \eta),$$

since $m_j(\eta^{i,i+1}) = m_j(\eta)$ for $j \notin \{i, i+1\}$. Taking marginals m_i from (21) and using the known relation of the gamma function $\Gamma(z+1) = z\Gamma(z)$, we find

$$\begin{aligned}
\frac{m_i(\eta)m_{i+1}(\eta)L(\eta,\eta^{i,i+1})}{m_i(\eta^{i,i+1})m_{i+1}(\eta^{i,i+1})L(\eta^{i,i+1},\eta)} &= \frac{m_i(\eta)}{m_i(\eta^{i,i+1})}\frac{m_{i+1}(\eta)}{m_{i+1}(\eta^{i,i+1})}\frac{L(\eta,\eta^{i,i+1})}{L(\eta^{i,i+1},\eta)} \\
&= \frac{\theta(\eta_i - 1 + 2k)}{\eta_i}\frac{\eta_{i+1}+1}{\theta(\eta_{i+1}+2k)}\frac{\eta_i(2k+\eta_{i+1})}{(\eta_{i+1}+1)(2k+\eta_i-1)} \\
&= 1.
\end{aligned}$$

□

This family of reversible measures, like the families of reversible measure for the other interacting particle systems that we will see, benefits from the fact that the SIP has conservation of the total number of particles. This fact makes it such that the SIP(k) has a family of reversible measures (instead of a single or no reversible measure), where the parameter θ relates to the expected amount of energy at a site, and thereby (since $M_\theta^{2k,\infty}$ is a product measure) quantifies the expected total amount of energy if we were on a finite lattice. This latter point does not hold in this setting in infinite volume, as for $\theta > 0$ the total amount of energy is almost surely infinite.
We can use this family of reversible measures to find a useful self-duality function for the SIP.

Theorem 4.2. *The SIP(k) is self-dual with self-duality function*

$$D^{\mathrm{SIP}}(\eta,\xi) = \prod_{i=-\infty}^{\infty} d^{\mathrm{SIP}}(\eta_i,\xi_i), \tag{22}$$

where self-duality polynomials d^{SIP} are given trough

$$d^{\mathrm{SIP}}(m,n) = \frac{n!}{(n-m)!}\frac{\Gamma(2k)}{\Gamma(2k+m)}1_{\{m\le n\}}. \tag{23}$$

Here the dual SIP(k) configuration ξ has a finite number of particles, so calling this number n, $(\xi_t)_{t\ge 0}$ is the n-SIP(k).

Proof. This was proven in [4]. Since the reversible measure in (21) is a product measure, we can again treat each marginal separately and apply Theorem 3.1 to find cheap self-duality function

$$\prod_{i=\infty}^{\infty} \frac{\eta_i!\Gamma(2k)}{\theta^{\eta_i}\Gamma(\eta_i+2k)}.$$

Noting that

$$\prod_{i=-\infty}^{\infty} \theta^{\eta_i} = \theta^{\sum_{i=-\infty}^{\infty}\eta_i} = \theta^{|\eta|},$$

stays constant due to the preservation of the number of particles, we divide out this constant and define cheap self-duality function D^{ch} and polynomials d^{ch} for $i \in \mathbb{Z}$ via

$$D^{ch}(\eta,\xi) := \prod_{i=-\infty}^{\infty} d^{ch}(\eta_i,\xi_i) := \prod_{i=\infty}^{\infty} \frac{\eta_i!\Gamma(2k)}{\Gamma(\eta_i+2k)}.$$

The next step involves noting that $e^{S_i^+} = e^{K_i^+ + K_{i+1}^+}$, with K_j^+ defined in (20), is a symmetry of $L_{i,i+1}^{\mathrm{SIP}}$. Theorem 3.3 then tells us that if we apply $e^{S_i^+}$ to a self-duality function of the SIP(k), the resulting function is a self-duality function as well. The proof is concluded by noting that

$$e^{S_i^+} d^{ch}(\eta_i, \xi_i) = d^{\mathrm{SIP}}(\eta_i, \xi_i).$$

See [4] for more of the technicalities and computations. □

The separation of duality-function D^{SIP} into duality-polynomials d^{SIP} that only take as input the values of η and ξ at a single site, that we see in (22) and (23), is a useful simplification that we will also see when we discuss duality between the BEP and the SIP. It allows us, when evaluating the duality function integrated with respect to a product measure, to simplify this to a product of duality-polynomials, each integrated with respect to a marginal measure.
One application of the self-duality of Theorem 4.2 is that we can connect two different versions of the SIP via duality, where we let the number of particles for one of the two depend on some scaling parameter and take a limit of this parameter, where for the other we keep the number of particles fixed. In e.g. [20] this scaling is done in such a way that is useful in proving hydrodynamic limit of the SIP. Alternatively, we can take a limit in such a way that the scaled SIP becomes the BEP, which we will see in Theorem 4.3.
For a self-duality function of the ASIP we refer to Proposition B.1 in Appendix B.

4.4 The Brownian Energy Process (BEP)

4.4.1 Definition

Central in this thesis is the Brownian Energy Process (BEP), first introduced in [11]. This is a continuous process modeling the stochastic transport of energy between sites on a lattice. It is defined in the following way.

Definition 4.4 (BEP(k))**.** *The BEP(k) in infinite volume is the Markov process on $\mathbb{R}_+^{\mathbb{Z}}$ with generator*

$$\mathcal{L}^{\mathrm{BEP}} = \sum_{i=-\infty}^{\infty} \mathcal{L}_{i,i+1}^{\mathrm{BEP}}, \tag{24}$$

where single-edge generators $\mathcal{L}_{i,i+1}^{\mathrm{BEP}}$ are defined on local $f \in C_c^{\infty}(\mathbb{R}_+^{\mathbb{Z}})$ through

$$\left[\mathcal{L}_{i,i+1}^{\mathrm{BEP}} f\right](y) = 2k(y_{i+1} - y_i)\left(\frac{\partial}{\partial y_i} - \frac{\partial}{\partial y_{i+1}}\right) f(y) + y_i y_{i+1}\left(\frac{\partial}{\partial y_i} - \frac{\partial}{\partial y_{i+1}}\right)^2 f(y). \tag{25}$$

Let us try to interpret this process. In the first term of the right-hand side of (25) we recognize a deterministic flow from the site with more energy to its neighbor with less energy, at a constant rate of 2k. In the second term we recognize a diffusion term, with energy flowing along edge $(i, i+1)$ in a way that is reminiscent of Brownian Motion.
This process makes more sense when we interpret it as a scaling limit of SIP.

Theorem 4.3. *Let $\left\{(\eta_t^N)_{t\geq 0}\right\}_{N\in\mathbb{N}}$ denote a sequence of realizations of the SIP(k) such that*

$$\lim_{N\to\infty} \frac{1}{N}\eta_0^N = y_0 \in \mathbb{R}_+^{\mathbb{Z}}.$$

Then $y_t := \lim_{N\to\infty} \frac{1}{N}\eta_t^N$ *is the BEP(k) starting from* y_0*, where convergence is weak on path space, i.e. for each local* $f \in C_c^\infty(\mathbb{R}_+^{\mathbb{Z}})$

$$\lim_{N\to\infty} \mathbb{P}_{\eta_0}^{\mathrm{SIP}}\left(f\left(\frac{1}{N}\eta_t^N \in B\right)\right) = \mathbb{P}_{y_0}^{\mathrm{BEP}}(f(y_t) \in B).$$

Proof. We follow the lines of [5], which proved a similar statement for the ASIP and the ABEP. The Trotter-Kurtz theorem tells us that it suffices to show that for any such $(\eta_0^N)_{N\in\mathbb{N}}$, y_0 and f we have for every $i \in \mathbb{Z}$,

$$\lim_{N\to\infty} \left(\mathcal{L}_{i,i+1}^{\mathrm{SIP}} f_N\right)(\eta_0^N) = \left(\mathcal{L}_{i,i+1}^{\mathrm{BEP}} f\right)(y_0),$$

where $f_N(\eta_0) = f(\frac{1}{N}\eta_0)$. Omitting the subscript 0, we do this by taking the Taylor expansion of $\mathcal{L}_{i,i+1}^{\mathrm{SIP}} f_N(\eta^N)$ and showing that the terms that don't vanish with $N \to \infty$ form the same expression as $\mathcal{L}_{i,i+1}^{\mathrm{BEP}} f(y)$. Recall that the generator of the SIP is given through

$$(\mathcal{L}_{i,i+1}^{\mathrm{SIP}} f_N)(\eta^N) = \eta_i^N(2k+\eta_{i+1}^N)\left(f_N((\eta^N)^{i,i+1}) - f(\eta^N)\right) + \eta_{i+1}^N(2k+\eta_i^N)\left(f_N((\eta^N)^{i+1,i}) - f_N(\eta^N)\right).$$

The second-order Taylor approximations of these discrete gradients are given through

$$f_N\left(\left(\eta^N\right)^{i,i+1}\right) - f_N(\eta^N) \approx -\frac{1}{N}\left(\frac{\partial}{\partial y_i} - \frac{\partial}{\partial y_{i+1}}\right) f\left(\frac{1}{N}\eta^N\right) + \frac{1}{2N^2}\left(\frac{\partial}{\partial y_i} - \frac{\partial}{\partial y_{i+1}}\right)^2 f\left(\frac{1}{N}\eta^N\right),$$

and

$$f_N\left(\left(\eta^N\right)^{i+1,i}\right) - f_N(\eta^N) \approx \frac{1}{N}\left(\frac{\partial}{\partial y_i} - \frac{\partial}{\partial y_{i+1}}\right) f\left(\frac{1}{N}\eta^N\right) + \frac{1}{2N^2}\left(\frac{\partial}{\partial y_i} - \frac{\partial}{\partial y_{i+1}}\right)^2 f\left(\frac{1}{N}\eta^N\right),$$

where the difference between the left-hand and right-hand sites is $O(N^{-3})$. We therefore find

$$(\mathcal{L}_{i,i+1}^{\mathrm{SIP}} f_N)(\eta^N) = \frac{1}{N}\left(2k(\eta_{i+1}^N - \eta_i^N) + \eta_i^N\eta_{i+1}^N - \eta_i^N\eta_{i+1}^N\right)\left(\frac{\partial}{\partial y_i} - \frac{\partial}{\partial y_{i+1}}\right) f\left(\frac{1}{N}\eta^N\right) \tag{26}$$

$$+ \frac{1}{2N^2}\left(2k(\eta_i^N + \eta_{i+1}^N) + 2\eta_i^N\eta_{i+1}^N\right)\left(\frac{\partial}{\partial y_i} - \frac{\partial}{\partial y_{i+1}}\right)^2 f\left(\frac{1}{N}\eta^N\right) + O(N^{-3}). \tag{27}$$

Taking the limit $N \to \infty$ then yields $(\mathcal{L}_{i,i+1}^{\mathrm{BEP}} f)(y)$. ☐

This finding makes it easier to interpret the BEP. The first term of the single-edge BEP generator $2k(y_i - y_{i-1})\left(\frac{\partial}{\partial y_i} - \frac{\partial}{\partial y_{i+1}}\right)$, corresponding to a linear drift in the direction of the site with lower energy, follows from taking the limit $N \to \infty$ of (26). We see that the jumps resulting from attraction of SIP-particles cancel out (we have both $\pm\eta_i^N\eta_{i+1}^N$), so this drift is the result of a difference in the number of particles jumping symmetrically to their neighbor at fixed rate of 2k. The second term $y_i y_{i+1}\left(\frac{\partial}{\partial y_i} - \frac{\partial}{\partial y_{i+1}}\right)^2$, corresponding to diffusion, follows from taking the limit of (27). Since $\frac{1}{2N^2}(2k(\eta_i^N + \eta_{i+1}^N)) \to 0$ as $N \to \infty$, this diffusion term is entirely the result of the attraction between particles.

Notably in the BEP the total amount of energy is conserved. This makes sense when we view it as a scaling limit of the SIP, where the total number of particles in conserved.

4.4.2 Duality

In this subsection we will establish duality between the BEP and the SIP, by building upon the self-duality for the SIP established in Theorem 4.2. Furthermore, we will give the family of reversible measures for the BEP. These reversible measures will not be used for duality with the SIP or in the hydrodynamic limit of the BEP in general, but a modification of them will appear in Chapter 8, where we prove propagation of chaos for the BEP and the ABEP.
Before giving the duality function between the BEP and SIP, we will first show the relation of the BEP to the su(1,1) Lie algebra.

Proposition 4.2. *$\mathcal{L}^{\mathrm{BEP}}$ can be written via a representation of the su(1,1) Lie algebra.*

Proof. For $i \in \mathbb{Z}$ we define continuous operators $\mathcal{K}_i^+$, $\mathcal{K}_i^-$ and $\mathcal{K}_i^0$ as follows

$$\begin{aligned} \mathcal{K}_i^+ f(y) &= y_i f(y), \\ \mathcal{K}_i^- f(y) &= y_i \frac{\partial^2 f(y)}{\partial y_i^2} + 2k \frac{\partial f(y)}{\partial y_i}, \\ \mathcal{K}_i^0 f(y) &= y_i \frac{\partial f(y)}{\partial y_i} + k f(y). \end{aligned} \tag{28}$$

Then these operators satisfy the su(1,1) commutation relations and

$$\mathcal{L}_{i,i+1}^{\mathrm{BEP}} = \mathcal{K}_i^+ \mathcal{K}_{i+1}^- + \mathcal{K}_i^- \mathcal{K}_{i+1}^+ - 2\mathcal{K}_i^0 \mathcal{K}_{i+1}^0 + 2k^2. \tag{29}$$

See [4] for calculations. □

Next, we have duality between the BEP(k) and the n-SIP(k).

Theorem 4.4. *The BEP(k) is dual to the n-SIP(k) with duality function*

$$D^b(y, \xi) = \prod_{i=-\infty}^{\infty} d^b(y, \xi_i) \ \textit{with} \ d^b(y_i, \xi_i) = \frac{\Gamma(2k)}{\Gamma(2k + \xi_i)} y_i^{\xi_i}. \tag{30}$$

Proof. There are different ways to prove this theorem. In [11] the construction of $\mathcal{L}_{i,i+1}^{\mathrm{BEP}}$ via algebraic representation of su(1,1), given here in (28) and (29), is compared to that of $\mathcal{L}_{i,i+1}^{\mathrm{SIP}}$ in (19) and (20). Functions $C_i(y_i, \xi_i)$ are constructed that satisfy

$$K_i^a C_i = C_i \mathcal{K}_i^a \text{ for } a \in \{-, +, 0\},$$

so that these functions correspond to a change in representation. Then the duality function is contructed via these C_i.
A second approach to proving this theorem is to explicitly calculate $\left(\mathcal{L}_{i,i+1}^{\mathrm{BEP}} D^b(\cdot, \xi)\right)(y)$ and $\left(\mathcal{L}_{i,i+1}^{\mathrm{SIP}} D^b(y, \cdot)\right)(\xi)$ and show that these are equal. This may be the most straightforward approach, but it does not offer any explanation about how the duality function was found and how to extend our findings in the future.
Here we will work out a third approach, which aligns most with the theory discussed so far. This approach is to start with self-duality between two SIP-configurations, and then for one of these take the scaling limit of Theorem 4.3, turning it into the BEP.

We construct the configurations in similar manner to Theorem 4.3 and let $(\eta^N)_{N\in\mathbb{N}}$ and $(y^N)_{N\in\mathbb{N}}$ be sequences of SIP(k) and BEP(k)-configurations respectively such that

$$\lim_{N\to\infty}\left(y^N-\frac{1}{N}\eta^N\right)=0 \text{ and } \lim_{N\to\infty} y^N=y\in\mathbb{R}_+^{\mathbb{Z}}.$$

Then we will show that

$$\lim_{N\to\infty}\frac{D^{\text{SIP}}\left(\eta^N,\xi\right)}{D^b(Ny^N,\xi)}=1,$$

where we take the fraction (and not the difference) to avoid problems that may arise as a result of these duality functions growing to infinity as N grows to infinity.
We note that as $N\to\infty$, $\eta_i^N\to\infty$ for every $i\in\mathbb{Z}$, with the trivial exception of $y_i^N=0$. As a result $\frac{\eta_i^N!}{(\eta_i^N-\xi_i)!}$, which we see in $D^{\text{SIP}}(\eta^N,\xi)$ becomes indistinguishable from $(\eta_i^N)^{\xi_i}\approx(Ny^N)^{\xi_i}$, which we see in $D^b(Ny^N,\xi)$. Furthermore this means that $\lim\limits_{N\to\infty}\eta_i^N>\xi_i$ for every i. Thus we find

$$\lim_{N\to\infty}\frac{D^{\text{SIP}}\left(\eta^N,\xi\right)}{D^b(Ny^N,\xi)}=\lim_{N\to\infty}\prod_{i=-\infty}^{\infty}\frac{\eta_i^N!}{(\eta_i^N-\xi_i)!(Ny_i^N)^{\xi_i}}1_{\{\xi_i\le\eta_i^N\}}=1.$$

□

The fact that ξ, being the n-SIP, only has a finite number of particles, is crucial for using $D^b(y,\xi)$. Had this not been the case, then the infinite product in (30) could potentially result in D^b either converging to 0 or not converging at all. Since $d^b(y_i,0)=1$, $D^b(y,\xi)$ now reduces to a finite product of polynomials, so that we do not have to worry about convergence. Furthermore, when we wish to integrate $D^b(y,\xi)$ over a product measure $\nu=\otimes_{i\in\mathbb{Z}}\nu_i$, then this reduces to

$$\int D^b(y,\xi)d\nu=\prod_{i:\xi_i\neq 0}\int d^b(y_i,\xi_i)d\nu_i.$$

This duality result will be essential throughout this thesis.
Unrelated to this duality between the BEP and the SIP, but important for Chapter 8 on the propagation of chaos, we also have a family of reversible measures of the BEP(k).

Theorem 4.5. *The BEP(k) has reversible and stationary measures in the form of products of Gamma distributions with shape parameter 2k and constant scale parameter θ, which may be chosen freely, i.e.*

$$\nu_\theta^{2k,\infty}(dy)=\prod_{i=-\infty}^{\infty}\nu_\theta^{2k}(dy_i)=\prod_{i=-\infty}^{\infty}\frac{1}{\theta^{2k}}\frac{y_i^{2k-1}}{\Gamma(2k)}e^{-y_i/\theta}dy_i. \tag{31}$$

Proof. We start by showing that $\nu_\theta^{2k,\infty}$ is an invariant measure for the BEP. The key to this is the following finding about the duality-polynomial between the BEP and the SIP from (30),

$$\begin{aligned}\int d(n,y_i)\nu_\theta^{2k}(dy_i) &= \int_{\mathbb{R}} \frac{1}{\theta^{2k}}\frac{y_i^{2k-1}}{\Gamma(2k)}e^{-y_i/\theta}\frac{y_i^n\Gamma(2k)}{\Gamma(2k+n)}dy_i \\ &= \int_{\mathbb{R}} \frac{\theta^{2k-1+n}}{\theta^{2k}}\frac{(y_i/\theta)^{2k-1+n}}{\Gamma(2k+n)}e^{-y_i/\theta}dy_i \\ &= \frac{\theta^n}{\Gamma(2k+n)}\int_{\mathbb{R}}(y_i/\theta)^{2k-1+n}e^{-y_i/\theta}d(y_i/\theta) \\ &= \frac{\theta^n}{\Gamma(2k+n)}\Gamma(2k+n),\end{aligned}$$

so that we find

$$\int d(n,y_i)\nu_\theta^{2k}(dy_i) = \theta^n. \tag{32}$$

This finding allows us to use duality between the BEP and the n-SIP in order to show that

$$\int E_y^{\mathrm{BEP}}[D^b(y_t,\xi)]\nu_\theta^{2k,\infty}(dy) = \int D^b(y,\xi)\nu_\theta^{2k,\infty}(dy),$$

for any BEP-configuration y and n-SIP-configuration ξ. Since through our choice of ξ we can create any polynomial, and the set of all polynomials is dense in the set of local smooth functions on $\mathbb{R}_+^{\mathbb{Z}}$, this means the condition for invariance (7) is satisfied. Indeed we have

$$\begin{aligned}\int D^b(y,\xi)\nu_\theta^{2k,\infty}(dy) &= \prod_{i=-\infty}^{\infty}\int d^b(y_i,\xi_i)\nu_\theta^{2k}(dy_i) \\ &= \prod_{i=-\infty}\theta^{\xi_i} \\ &= \theta^{|\xi|},\end{aligned}$$

and

$$\begin{aligned}\int \mathbb{E}_y^{\mathrm{BEP}}[D^b(y_t,\xi)]\nu_\theta^{2k,\infty}(dy) &= \int \mathbb{E}_y^{\mathrm{BEP}}[D^b(y_t,\xi)]\nu_\theta^{2k,\infty}(dy) \\ &= \int \mathbb{E}_\xi^{\mathrm{SIP}}[D^b(y,\xi_t)]\nu_\theta^{2k,\infty}(dy) \\ &= \mathbb{E}_\xi^{\mathrm{SIP}}\left[\int \theta^{|\xi_t|}\nu_\theta^{2k,\infty}(dy)\right] \\ &= \mathbb{E}_\xi^{\mathrm{SIP}}\left[\theta^{|\xi_t|}\right] \\ &= \theta^{|\xi|},\end{aligned}$$

where in the last step we used the conservation of the number of particles. Via similar arguments we find reversibility of $\nu_\theta^{2k,\infty}$,

$$\begin{aligned}
\int D^b(y,\xi)\mathbb{E}_y^{\mathrm{BEP}}[D^b(y_t,\eta)]d\nu_\theta^{2k,\infty}(y) &= \mathbb{E}_\eta^{\mathrm{SIP}}\left[\int D^b(y,\xi)D^b(y,\eta_t)d\nu_\theta^{2k,\infty}(y)\right] \\
&= \mathbb{E}_\eta^{\mathrm{SIP}}\left[\int D^b(y,\xi+\eta_t)d\nu_\theta^{2k,\infty}(y)\right] \\
&= \mathbb{E}_\eta^{\mathrm{SIP}}\left[\theta^{|\xi|+|\eta_t|}\right] \\
&= \mathbb{E}_\xi^{\mathrm{SIP}}\left[\theta^{|\xi_t|+|\eta|}\right] \\
&= \int \mathbb{E}_\xi^{\mathrm{BEP}}\left[D^b(y_t,\xi)\right]D^b(y,\eta)d\nu_\theta^{2k,\infty}(y).
\end{aligned}$$

□

Like the reversible measure for the SIP(k) $M_\theta^{2k,\infty}$, $\nu_\theta^{2k,\infty}$ is parameterized by θ, which relates to the expected amount of energy under this measure. In fact from (32) we find that the profile of $\nu_\theta^{2k,\infty}$ is given trough

$$\int y_i \nu_\theta^{2k,\infty}(dy) = \int d(y,1)\nu_\theta^{2k}(dy_i) = \theta,$$

so that θ is exactly equal to the expected energy at each site under $\nu_\theta^{2k,\infty}$.
One may note that $\nu_\theta^{2k,\infty}(dy)$, being an infinite product measure, is not a local measure. However, each of its marginals $\nu_\theta^{2k}(dy_i)$ is a local measure, as it only depends on y through y_i. Therefore when we work with local function, i.e. functions only depending on y_i for finitely many sites i, we only need to concern ourselves with finitely many marginal measures, and the marginals working on y_i which do not appear in the function will simply integrate to 1. When we speak of convergence of measures in this infinite setting, this is usually refers to convergence of an arbitrary local test function integrated with respect to these measures. This will come up in Section 6.2, when weak convergence is discussed.

5 The (Dynamic) Asymmetric Brownian Energy Process (ABEP / DABEP)

5.1 Definition

The ABEP is the asymmetric version of the BEP, and the topic of much of the new research in this thesis. It was first introduced in [5], a paper that is of central importance in this thesis. The ABEP(σ, k) with asymmetry parameter σ is defined as follows.

Definition 5.1 (ABEP(σ,k))**.** *Let* $\Omega_f := \left\{ x \in \mathbb{R}_+^{\mathbb{Z}} : \sum_{i=-\infty}^{\infty} x_i < \infty \right\}$ *be the subspace of configuration space* $\mathbb{R}_+^{\mathbb{Z}}$ *consisting of the configurations with finite energy. Then the ABEP(σ, k) is the Markov process on* Ω_f *with generator*

$$\mathcal{L}^{\mathrm{ABEP}} = \sum_{i=-\infty}^{\infty} \mathcal{L}_{i,i+1}^{\mathrm{ABEP}} \text{ with for local } f \in C_c^{\infty}(\Omega_f)$$

$$\begin{aligned}\left[\mathcal{L}_{i,i+1}^{\mathrm{ABEP}} f\right](x) =& \frac{1}{4\sigma^2}(1 - e^{-2\sigma x_i})(e^{2\sigma x_{i+1}} - 1)\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right)^2 f(x) \\ &- \frac{1}{2\sigma}\left\{(1 - e^{-2\sigma x_i})(e^{2\sigma x_{i+1}} - 1) + 2k(2 - e^{-2\sigma x_i} - e^{2\sigma x_{i+1}})\right\}\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right) f(x).\end{aligned}$$

Similar to how the BEP arises as a scaling limit of the SIP, we can show that the ABEP arises as an analogous scaling limit of the ASIP. For this, we refer to Proposition B.2 in Appendix B. We can think of the ABEP as a version of the BEP where there will be a tendency for a current of energy to arise to the right side due to the asymmetry. In fact, one can prove the following proposition.

Proposition 5.1. *Let* x_t *be the ABEP starting out at* $x \in \Omega_f$ *and denote by* $J_i(t) = E_i(x_t) - E_i(x)$ *the net current through site* i*. Then*

$$\mathbb{E}_x^{\mathrm{ABEP}}\left[e^{-2\sigma J_i(t)}\right] = \sum_{n=-\infty}^{\infty} e^{-2\sigma(E_n(x) - E_i(x))}\mathbb{P}^{\mathrm{IRW}}\left(l(t) = n | l(0) = i\right),$$

where $l(t)$ *denotes the location of a single random walker at time* t*.*

Proof. See [5]. □

This current is consequential to the space on which we can define the ABEP. Notably in Definition 5.1 we defined the ABEP on Ω_f, a subspace of $\mathbb{R}_+^{\mathbb{Z}}$ with the additional constraint the total energy of an ABEP-configuration x is finite. The reason for this is that it has not been proven that the ABEP in infinite volume exists without this constraint. As we can see from Proposition 5.1, with an infinite amount of energy there may arise an infinitely big current spread out over the integer line. It is conceivable that an infinitely big part of this current will accumulate to an increasingly small interval, eventually creating a singularity where there is an infinite amount of energy on a single site, making the process ill-defined. Until the existence of the ABEP in infinite volume with infinite energy is proven, likely via duality with well-defined processes on infinite volume such as the SIP, the exploration of the ABEP will focus on configurations with finite energy. One can easily see that if we let asymmetry parameter σ decrease to 0, the generator of the ABEP(σ, k) reduces to that of the BEP(k).

5.2 Mapping from the ABEP to the BEP

We ended the last paragraph by making the claim that if we let the asymmetry disappear by taking the limit $\sigma \downarrow 0$, the ABEP turns into the BEP. There is another, interesting way in which the ABEP can be reduced to the more workable BEP. Given that x is a configuration of ABEP(σ, k), we introduce transformation g given through

$$g(x) = (g_i(x), i \in \mathbb{Z}) \text{ with } g_i(x) = \frac{e^{-2\sigma E_{i+1}(x)} - e^{2\sigma E_i(x)}}{2\sigma}, \tag{33}$$

where the function

$$E_i : \mathbb{R}_+^{\mathbb{Z}} \to (\mathbb{R}_+ \cup \{\infty\}), \text{ given through } E_i(x) := \sum_{l=i}^{\infty} x_l,$$

denotes the partial energy to the right of site i. Note that for each i, g_i is a nonlocal function, as it depends on the value of x at infinitely many sites. We then claim that if $(x_t)_{t\geq 0}$ is the ABEP(σ, k) starting out from x, then $(g(x_t))_{t\geq 0}$ is the BEP(k) starting out from $g(x)$. This finding is similar to the approach in [10], where the author applied a similar transformation to the Asymmetric Symmetric Exclusion Process (ASEP), which produced the symmetric version (SEP). The type of transformation in (33) can be seen as a microscopic version of the Cole-Hopf transformation, introduced in [13], a transformation that was used in order to solve the viscous Burgers equation. In the proof of the hydrodynamic limit of the ABEP in Chapter 7, we will see that this microscopic version (i.e. looking at energy levels at individual sites) of the Cole-Hopf transformation will lead to the classic Cole-Hopf transformation of the emerging density field at macroscopic scale, which we then use in order to derive the PDE for the density field.
The main part of the proof of the claim that $(g(x_t))_{t\geq 0}$ is the BEP consists of showing that applying the ABEP-generator on a nested function $(f \circ g)(x)$ is the same as applying the BEP-generator directly on $f(g(x))$. Then, by using this equality to show that $g(x_t)$ solves the martingale problem associated with the generator of BEP(k), we find that $g(x_t)$ is the BEP(k).

Lemma 5.1. for every local $f \in C_c^\infty(\mathbb{R}_+^{\mathbb{Z}})$ we have

$$\left[\mathcal{L}^{\text{ABEP}} f \circ g\right](x) = \left[\mathcal{L}^{\text{BEP}} f\right](g(x)). \tag{34}$$

Proof. Straightforward calculation on $g_i(x) = \frac{e^{-2\sigma E_{i+1}(x)} - e^{-2\sigma E_i(x)}}{2\sigma}$ yields

$$\frac{\partial}{\partial x_i} g_j(x) = \begin{cases} e^{-2\sigma E_i(x)} - e^{-2\sigma E_{i+1}(x)} & \text{for } j < i, \\ e^{-2\sigma E_i(x)} & \text{for } j = i, \\ 0 & \text{for } j > i, \end{cases} \tag{35}$$

so that

$$\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right) g_j(x) = \begin{cases} e^{-2\sigma E_{i+1}(x)} & \text{for } j = i, \\ -e^{-2\sigma E_{i+1}(x)} & \text{for } j = i+1, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, the second derivative is:

$$\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right)^2 g_j(x) = \begin{cases} 2\sigma e^{-2\sigma E_{i+1}(x)} & \text{for } j = i, \\ -2\sigma e^{-2\sigma E_{i+1}(x)} & \text{for } j = i+1, \\ 0 & \text{otherwise.} \end{cases}$$

Denote $\partial_i := \frac{\partial}{\partial x_i}$. Then

$$(\partial_i - \partial_{i+1})(f\circ g)(x) = \sum_j \frac{\partial f(g)}{\partial g_j}(\partial_i - \partial_{i+1})g_j(x) = \frac{\partial f(g)}{\partial g_i}e^{-2\sigma E_{i+1}(x)} - \frac{\partial f(g)}{\partial g_{i+1}}e^{-2\sigma E_{i+1}(x)}.$$

If we split the ABEP generator into two via $\mathcal{L}^{\mathrm{ABEP}}_{i,i+1} = \mathcal{L}^1_{i,i+1} + \mathcal{L}^2_{i,i+1}$ with

$$\mathcal{L}^1_{i,i+1}g(x) := -\frac{k}{\sigma}(2 - e^{-2\sigma x_i} - e^{2\sigma x_{i+1}})(\partial_i - \partial_{i+1})g(x),$$

then we get

$$\begin{aligned}
[\mathcal{L}^1_{i,i+1}(f\circ g)](x) &= -\frac{k}{\sigma}(2 - e^{-2\sigma x_i} - e^{2\sigma x_{i+1}})e^{-2\sigma E_{i+1}(x)}\left[\left(\frac{\partial}{\partial g_i} - \frac{\partial}{\partial g_{i+1}}\right)f\right]\circ g(x) \\
&= 2k(-e^{-2\sigma E_i(x)} + 2e^{-2\sigma E_{i+1}(x)} - e^{-2\sigma E_{i+2}(x)})\left(\frac{\partial}{\partial g_i} - \frac{\partial}{\partial g_{i+1}}\right)f(g(x)) \\
&= 2k(g_i(x) - g_{i+1}(x))\left(\frac{\partial}{\partial g_i(x)} - \frac{\partial}{\partial g_{i+1}(x)}\right)f(g(x)), \qquad (36)
\end{aligned}$$

where we recognize part of the BEP generator.
Similarly

$$\begin{aligned}
(\partial_i - \partial_{i+1})^2(f\circ g)(x) &= (\partial_i - \partial_{i+1})\left(\sum_j \frac{\partial f}{\partial g_j}(\partial_i - \partial_{i+1})g_j(x)\right) \\
&= \sum_j\sum_k \frac{\partial^2 f}{\partial g_j \partial g_k}\left[(\partial_i - \partial_{i+1})g_j\right]\left[(\partial_i - \partial_{i+1})g_k\right] + \sum_j \frac{\partial f}{\partial g_j}(\partial_i - \partial_{i+1})^2 g_j(x) \\
&= \frac{\partial^2 f}{\partial g_i^2}[(\partial_i - \partial_{i+1})g_i]^2 + 2\frac{\partial^2 f}{\partial g_i \partial g_{i+1}}[(\partial_i - \partial_{i+1})g_i][(\partial_i - \partial_{i+1})g_{i+1}] \\
&+ \frac{\partial^2 f}{\partial g_{i+1}^2}[(\partial_i - \partial_{i+1})g_{i+1}]^2 + \frac{\partial f}{\partial g_i}(\partial_i - \partial_{i+1})^2 g_i + \frac{\partial f}{\partial_{i+1}}(\partial_i - \partial_{i+1})g_{i+1}.
\end{aligned}$$

This means that the second part of the generator, given by

$$\begin{aligned}
[\mathcal{L}^2_{i,i+1}f](x) : = &\frac{1}{4\sigma^2}(1 - e^{-2\sigma x_i})(e^{2\sigma x_{i+1}} - 1)(\partial_i - \partial_{i+1})^2 f(x) \\
&- \frac{1}{2\sigma}(1 - e^{-2\sigma x_i})(e^{2\sigma x_{i+1}} - 1)(1 - e^{-2\sigma x_i})(\partial_i - \partial_{i+1})f(x),
\end{aligned}$$

yields

$$
\begin{aligned}
\left[\mathcal{L}^2_{i,i+1}(f\circ g)\right](x) &= \frac{1}{4\sigma^2}(1-e^{-2\sigma x_i})(e^{2\sigma x_{i+1}}-1)\left[(e^{-2\sigma E_{i+1}}(x))^2\frac{\partial^2 f}{\partial g_i^2}\right.\\
&\left.+2(e^{-2\sigma E_{i+1}})(-e^{-2\sigma E_{i+1}(x)})\frac{\partial^2 f}{\partial g_i \partial g_{i+1}}+(e^{-2\sigma E_{i+1}(x)}(x))^2\frac{\partial^2 f}{\partial g_{i+1}^2}\right]\\
&+\frac{1}{4\sigma^2}(1-e^{-2\sigma x_i})(e^{2\sigma x_{i+1}}-1)\left[2\sigma e^{-2\sigma E_{i+1}(x)}\frac{\partial f}{\partial g_i}-2\sigma e^{-2\sigma E_{i+1}(x)}\frac{\partial f}{\partial g_{i+1}}\right]\\
&-\frac{1}{2\sigma}(e^{-2\sigma x_{i+1}}-1)e^{-2\sigma E_{i+1}(x)}\left(\frac{\partial f}{\partial g_i}-\frac{\partial f}{\partial g_{i+1}}\right)\\
&=\frac{1}{4\sigma^2}(e^{-2\sigma E_{i+1}(x)}-e^{-2\sigma E_i(x)})(e^{-2\sigma E_{i+2}(x)}-e^{-2\sigma E_{i+1}(x)})\left(\frac{\partial}{\partial g_i}-\frac{\partial}{\partial g_{i+1}}\right)^2 f(g(x))\\
&=g_i g_{i+1}\left(\frac{\partial}{\partial g_i}-\frac{\partial}{\partial g_{i+1}}\right)^2 f(g(x)). \qquad (37)
\end{aligned}
$$

Combining (36) and (37) gives us

$$
\begin{aligned}
[\mathcal{L}^{\mathrm{ABEP}}_{i,i+1}(f\circ g)](x) &= [\mathcal{L}^1_{i,i+1}(f\circ g)](x)+[\mathcal{L}^2_{i,i+1}(f\circ g)](x)\\
&=2k(g_i-g_{i+1})\left(\frac{\partial}{\partial g_i}-\frac{\partial}{\partial g_{i+1}}\right)f(g(x))+g_i g_{i+1}\left(\frac{\partial}{\partial g_i}-\frac{\partial}{\partial g_{i+1}}\right)^2 f(g(x))\\
&=[\mathcal{L}^{\mathrm{BEP}}_{i,i+1}f](g(x)).
\end{aligned}
$$

This holds for all $i\in\mathbb{Z}$ so that we can conclude the proof of Lemma 5.1 via

$$
[\mathcal{L}^{\mathrm{ABEP}} f\circ g](x) = \sum_{i=-\infty}^{\infty}[\mathcal{L}^{\mathrm{ABEP}}_{i,i+1} f\circ g](x) = \sum_{i=-\infty}^{\infty}[\mathcal{L}^{\mathrm{BEP}}_{i,i+1} f](g(x)) = [\mathcal{L}^{\mathrm{BEP}} f](g(x)).
$$

□

In the use of map g we see again a need for the requirement of finite energy ($\sum_i x_i < \infty$). If this requirement is relaxed, then we can end up with a configuration x such that for a pair of sites we have $E_i(x) = E_{i+1}(x) = \infty$, and as a result $g_i(x) = 0$. Although strictly speaking Lemma 5.1 still holds, this is trivially because both sides of (34) are equal to 0. Using map g would mean mapping the ABEP-configuration to a BEP-configuration consisting of just zero's, which loses all information about the process. We continue by concluding from Lemma 5.1 that $g(x_t)$ is the ABEP.

Theorem 5.2. *If $(x_t)_{t\geq 0}$ is the ABEP(σ, k) starting out at x, then $(g(x_t))_{t\geq 0}$ is the BEP(k) starting out at $g(x)$.*

Proof. We prove Theorem 5.2 by showing that showing that $g(x_t)$ solves the martingale problem associated with $\mathcal{L}^{\mathrm{ABEP}}$. We know that since x_t is the ABEP(σ, k), it must solve the martingale problem associated to $\mathcal{L}^{\mathrm{ABEP}}$, which means that

$$
M_t := \phi(x_t) - \phi(x_0) - \int_0^t (\mathcal{L}^{\mathrm{ABEP}}\phi)(x_s)ds,
$$

is a martingale for any suitable function ϕ. Taking $\phi := f \circ g$ for any suitable f tells us

$$\begin{aligned} M_t &= (f \circ g)(x_t) - (f \circ g)(x_0) - \int_0^t (\mathcal{L}^{\text{ABEP}} f \circ g)(x_s) ds \\ &= f(g(x_t)) - f(g(x_0)) - \int_0^t (\mathcal{L}^{\text{BEP}} f)(g(x_s)) ds \\ &= f(g_t) - f(g_0) - \int_0^t (\mathcal{L}^{\text{BEP}} f)(g_s) ds. \end{aligned}$$

This means that $g_t := g(x_t)$ is the solution to the martingale problem associated to $\mathcal{L}^{\text{BEP}}$. By uniqueness of this solution (Theorem 2.3), $g(x_t)$ is the BEP. □

We can similarly map in the other direction, transforming the BEP(k) into the ABEP(σ,k).

Corollary 5.1. *The inverse of map g from* (33) *exists and maps a BEP(k) to an ABEP(σ, k). It is given through*

$$g^{-1}(y) = \frac{1}{2\sigma} \ln \left\{ \frac{1 - 2\sigma E_{i+1}(y)}{1 - 2\sigma E_i(y)} \right\}. \tag{38}$$

Proof. Let

$$y_i = g_i(x) = \frac{e^{-2\sigma E_{i+1}(x)} - e^{-2\sigma E_i(x)}}{2\sigma}.$$

Then

$$2\sigma y_i = e^{-2\sigma E_{i+1}(x)} - e^{-2\sigma E_i(x)},$$

so that summing terms of y_i yields

$$2\sigma \sum_{j=i}^{\infty} y_j = \sum_{j=i}^{\infty} e^{-2\sigma E_{j+1}(x)} - e^{-2\sigma E_j(x)} = 1 - e^{-2\sigma E_i(x)},$$

which means

$$1 - 2\sigma E_i(y) = e^{-2\sigma E_i(x)},$$

and

$$-\frac{\ln(1 - 2\sigma E_i(y))}{2\sigma} = E_i(x).$$

Similarly

$$E_{i+1}(x) = -\frac{\ln(1 - 2\sigma E_{i+1}(y)}{2\sigma},$$

so that

$$x_i = E_i(x) - E_{i+1}(x) = \frac{1}{2\sigma} \ln \left\{ \frac{1 - 2\sigma E_{i+1}(y)}{1 - 2\sigma E_i(y)} \right\}.$$

□

One may note that the method of this proof breaks down when $E_i(y) \geq \frac{1}{2\sigma}$, which tells us g^{-1} does not have full domain, meaning we can't map every BEP-configuration to a corresponding ABEP-configuration using this map. This is not a problem for our purposes in this thesis, as we only wish to convert the ABEP into the more workable BEP using g, which is defined for all ABEP configurations. Thus whenever we need the inverse map g^{-1}, we are in a situation where the BEP-configuration we are working with was created from an ABEP-configuration, ensuring that the BEP-configuration is in $\mathcal{D}(g^{-1})$.

5.3 Duality

The maps g and g^{-1} between the BEP and ABEP provide the key to duality and reversibility for the ABEP. Applying these maps will allow us to derive results for the ABEP directly from those of the BEP. In this section we will explain how g and g^{-1} allow us to derive a duality function between the ABEP and the SIP from the duality function of the BEP and the SIP.
Before we give the duality function between the ABEP and the SIP, we will first show that with these maps we can construct operators C_g and $C_{g^{-1}}$ corresponding to a change of representation of the su(1,1) Lie algebra, allowing us to write $\mathcal{L}^{\text{ABEP}}$ via a representation of su(1,1).

Proposition 5.2. *$\mathcal{L}^{\text{ABEP}}$ can be written via a representation of the su(1,1) Lie algebra.*

Proof. We use the fact that the ABEP and BEP can be related to each other via transformation g to write $\mathcal{L}^{\text{ABEP}}$ via a representation of su(1,1). Let $\mathcal{K}_i^+$, $\mathcal{K}_i^-$ and $\mathcal{K}_i^0$ be the operators from (28) to define $\mathcal{L}^{\text{BEP}}$ via (29). Then

$$\tilde{\mathcal{K}}_i^a = C_g \circ \mathcal{K}_i^a \circ C_{g^{-1}},$$

with $a \in \{+, -, 0\}$ and

$$\begin{aligned}(C_{g^{-1}} f)(y) &= (f \circ g^{-1})(y),\\ (C_g f)(x) &= (f \circ g)(x),\end{aligned}$$

satisfy the su(1,1) commutation relations and

$$\mathcal{L}^{\text{ABEP}} = \sum_{i=-\infty}^{\infty} \left(\tilde{\mathcal{K}}_i^+ \tilde{\mathcal{K}}_{i+1}^- + \tilde{\mathcal{K}}_i^- \tilde{\mathcal{K}}_{i+1}^+ - 2\tilde{\mathcal{K}}_i^0 \tilde{\mathcal{K}}_{i+1}^0 + 2k^2 \right).$$

See [5] for further elaboration. □

Similarly we use g in order to derive a duality function of the ABEP and the SIP from D^b.

Theorem 5.3. *The ABEP(σ, k) is dual to the n-SIP(k) with duality function*

$$D^a(x, \xi) = \prod_{i=-\infty}^{\infty} \frac{\Gamma(2k)}{\Gamma(2k + \xi_i)} \left(\frac{e^{-2\sigma E_{i+1}(x)} - e^{2\sigma E_i(x)}}{2\sigma} \right)^{\xi_i}.$$

Proof. We recognize that $D^a(x, \xi) = D^b(g(x), \xi)$. Then duality follows from the duality of BEP and SIP through

$$\begin{aligned}\left[\mathcal{L}^{\text{ABEP}} D^a(\cdot, \xi)\right](x) &= \left[\mathcal{L}^{\text{ABEP}} D^b(g(\cdot), \xi)\right](x)\\ &= \left[\mathcal{L}^{\text{BEP}} D^b(\cdot, \xi)\right](g(x))\\ &= \left[\mathcal{L}^{\text{SIP}} D^b(g(x), \cdot)\right](\xi)\\ &= \left[\mathcal{L}^{\text{SIP}} D^a(x, \cdot)\right](\xi).\end{aligned}$$

$\square$

Again we can split duality-function D into an infinite product of duality polynomials, but this time this takes a different form. We have

$$D^a(x,\xi) = \prod_{i=-\infty}^{\infty} d_i^a(x,\xi_i) \text{ with } d_i^a(x,\xi_i) = \frac{\Gamma(2k)}{\Gamma(2k+\xi_i)} \left(\frac{e^{-2\sigma E_{i+1}(x)} - e^{-2\sigma E_i(x)}}{2\sigma} \right)^{\xi_i},$$

where the nonlocal nature of D^a is clear from the presence of $E_i(x)$ and $E_{i+1}(x)$ in this expression. As a result of this nonlocality, we have for configurations x with an infinite amount of energy to the right side, i.e. for any $i \in \mathbb{Z}$

$$E_i(x) = E_{i+1}(x) = \infty,$$

that any duality polynomial is equal to 0, i.e.

$$d_i^a(x,\xi_i) = 0.$$

As a results for any $\xi \in \Omega_n$, $D^a(x,\xi) = 0$ no matter how this infinite amount of energy in x is distributed along the sites, making the duality function meaningless. Thus in practice we can only use this duality function when the total amount of energy is finite. This makes sense in light of our claim in Section 5.1 that it is not clear that the ABEP exists in infinite volume with infinite energy. Had we not had the issue that $g(x) = 0$ for $x \in \mathbb{R}_+^{\mathbb{Z}}$ such that $|x| = \infty$, then the existence of the ABEP with infinite energy may be proven via the martingale problem (Theorem 2.3) facilitated by this duality function.

5.4 Pushforward and reversible measure

In this section we will show how the distribution of an ABEP(σ,k) configuration x_t relates to a BEP(k) configuration y_t, constructed via this map g from (33) by introducing the so-called 'pushforward measure'. This pushforward measure is derived from a measure on a finite state space and a function with that maps from and to finite-dimensional state spaces. We will therefore start in this context, and only later investigate whether we can generalize our results to the infinitely-dimensional, nonlocal setting.

Theorem 5.4. *If $X \in \mathbb{R}^n$ is a random vector distributed according to measure μ, then for a function $f : \mathbb{R}^n \to \mathbb{R}^m$ whose inverse f^{-1} exists, $f(X)$ is distributed according to*

$$\nu := \left(\mu \cdot \det(\mathcal{J}(f^{-1}))\right) \circ f^{-1},$$

where $\mathcal{J}(\cdot)$ denotes the Jacobian matrix. ν is called the pushforward measure of μ by f.

Proof. Let $Y = f(X)$. Our goal is to show that

$$\nu(Y \in B) = \mu(f(X) \in B) = \mu(X \in f^{-1}(B)).$$

Let the density of μ be given by $\psi(x)dx = \mu(dx)$. We find by change of variable $x = f^{-1}(y)$,

$$\begin{aligned}\mu(X \in f^{-1}(B)) &= \int_{f^{-1}(B)} \psi(x)dx \\ &= \int_B \psi(f^{-1}(y)) \left|\mathcal{J}(f^{-1}(y))\right| dy \\ &= \left(\left(\mu \cdot \left|\mathcal{J}(f^{-1})\right|\right) \circ f^{-1}\right)(Y \in B) \\ &= \nu(Y \in B).\end{aligned}$$

□

With this pushforward measure we can find a family of reversible measures for the ABEP(σ, k) in finite volume. We do this by defining $x = (x_i, i \in [-L, L] \cap \mathbb{Z})$ to be the ABEP(σ, k) on $[-L, L]$, with closed boundaries, meaning sites $-L$ and L are only connected to $-L+1$ and $L-1$ respectively. This choice of closed boundaries has been made because this produces a situation most similar to that of infinite volume, in the sense that the process is only driven by energy transport between neighboring sites and we have conservation of total energy. For more about the ABEP defined on an interval with closed boundaries we again refer to [5], where this approach was taken. The following theorem gives the family of reversible measures for the ABEP on finite volume with closed boundaries. The proof follows the lines of [3], where the same was proven, except in a setting with open boundaries, i.e. reservoirs at the boundaries with equal fixed energy level ('temperature'). The existence of these reservoirs in general leads to much different dynamics, but for the reversible measure the proof works out in almost exactly the same way. The only difference is that in our setting we end up with a parameter θ, which we can freely choose, representing the expected mean temperature of the measure of the BEP from which the measure of the ABEP is constructed, where in the setting of [3] this value is determined by the temperature of the reservoirs.

Theorem 5.5. *Let $\tilde{E}_i(x) := \sum_{j=i}^{L} x_j$ denote the partial energy in finite volume. Then*

$$\mu_\theta^{2k,L}(dx) = \prod_{i=-L}^{L} \frac{\left(e^{2\sigma x_i} - 1\right)^{2k-1} e^{-4k\sigma x_i (i+L+1)}}{\theta^{2k} 2\sigma \Gamma(2k)} \exp\left(\frac{e^{-2\sigma \tilde{E}_i(x)} - e^{-2\sigma \tilde{E}_{i+1}(x)}}{2\sigma\theta}\right) dx_i, \tag{39}$$

is a reversible measure for the ABEP in finite volume.

Proof. Let $\tilde{g}$ denote the finite-volume analogue to map g in (33) with $\tilde{E}_i$ instead of E_i, and let $\nu_\theta^{2k,L}$ be the reversible measure of the BEP with the same marginals as $\nu_\theta^{2k,\infty}$ in (31), but defined on $[-L, L]$. Taking the pushforward measure of $\nu_\theta^{2k,L}$ by $\tilde{g}$ yields

$$\mu_\theta^{2k,L} = \left(\nu_\theta^{2k,L} \cdot |\mathcal{J}|\right) \circ \tilde{g}.$$

From (35) we see that the Jacobian is an upper triangular matrix, which means that its determinant is equal to the product of the values on the diagonal, i.e.

$$\left|\mathcal{J}(\tilde{g}(\cdot))\right|(x) = \prod_{i \in S} e^{-2\sigma E_i(x)}.$$

Denoting $f_Y(y)dy := \nu_\theta^{2k,L}(dy)$ and $f_X(x)dx := \mu_\theta^{2k,L}(dx)$ we find

$$\begin{aligned}
f_X(x) &= ((f_Y(\cdot)\,|\mathcal{J}(\cdot)|) \circ \tilde{g})\,(x) \\
&= f_Y(\tilde{g}(x))\,|\mathcal{J}(\tilde{g}(\cdot))|\,(x) \\
&= \prod_{i\in S} \frac{1}{\theta^{2k}} \frac{g_i(x)^{2k-1}}{\Gamma(2k)} e^{-g_i(x)/\theta} e^{-2\sigma \tilde{E}_i(x)} \\
&= \prod_{i\in S} \frac{\left(e^{-2\sigma E_{i+1}(x)} - e^{-2\sigma \tilde{E}_i(x)}\right)^{2k-1} e^{-2\sigma \tilde{E}_i(x)}}{\theta^{2k} 2\sigma \Gamma(2k)} \exp\left(\frac{e^{-2\sigma \tilde{E}_i(x)} - e^{-2\sigma \tilde{E}_{i+1}(x)}}{2\sigma\theta}\right) \\
&= \prod_{i\in S} \frac{\left(e^{2\sigma x_i} - 1\right)^{2k-1} e^{-4k\sigma \tilde{E}_i(x)}}{\theta^{2k} 2\sigma \Gamma(2k)} \exp\left(\frac{e^{-2\sigma \tilde{E}_i(x)} - e^{-2\sigma \tilde{E}_{i+1}(x)}}{2\sigma\theta}\right).
\end{aligned}$$

Noting that

$$\prod_{i=-L}^{L} \exp\left(-4k\sigma \tilde{E}_i(x)\right) = \exp\left(-4k\sigma \sum_{i=-L}^{L} \sum_{j=i}^{L} x_j\right) = \exp\left(-4k\sigma \sum_{i=-L}^{L} x_i (L+i+1)\right),$$

we arrive at (39).
Next, we want to show that $\mu_\theta^{2k,L}$ is a reversible measure for the ABEP(σ, k) on $[-L, L]$. Following the lines of [3] we find the following. For every $f \in \mathcal{D}(\mathcal{L}^{\mathrm{BEP}})$,

$$\begin{aligned}
\int \left(\mathcal{L}^{\mathrm{ABEP}}(f\circ\tilde{g})\right)(x)(h\circ\tilde{g})(x)\mu_\theta^{2k,L}(dx) &= \int \left((\mathcal{L}^{\mathrm{BEP}} f)\circ\tilde{g}\right)(x)(h\circ\tilde{g})(x)\mu_\theta^{2k,L}(dx) \\
&= \int \left(\mathcal{L}^{\mathrm{BEP}} f\right)(\tilde{g}(x)) h(\tilde{g}(x))\,|\mathcal{J}(\tilde{g}(x))|\,(\nu_\theta^{2k,L}\circ\tilde{g})(dx) \\
&= \int \left(\mathcal{L}^{\mathrm{BEP}} f\right)(y) h(y) \nu_\theta^{2k,L}(dy). \qquad (40)
\end{aligned}$$

If we interchange f and h, we find

$$\int (f\circ\tilde{g})(x)\left(\mathcal{L}^{\mathrm{ABEP}}(h\circ\tilde{g})\right)(x)\mu_\theta^{2k,L}(dx) = \int f(y)(\mathcal{L}^{\mathrm{BEP}} h)(y)\nu_\theta^{2k,L}(dy). \qquad (41)$$

By reversibility of $\nu_\theta^{2k,L}$ for the BEP, (40) and (41) are equal, hence $\mu_\theta^{2k,L}(dx)$ is reversible for the ABEP. □

$\mu_\theta^{2k,L}$ is not a product measure due to the nonlocal nature of the factors

$$\exp\left(\frac{e^{-2\sigma \tilde{E}_i(x)} - e^{-2\sigma \tilde{E}_{i+1}}(x)}{2\sigma\theta}\right),$$

which for every i depend on every x_j with $j \geq i$. Due to the structure of these factors, with an exponential function with nested exponential arguments, we are unable to split them into the product of two functions, with one of them only depending on x_i and one only depending on $x_{i+1}, \ldots, x_L$. Because of this, we are not able to turn this measure into a product measure.

As a result, whenever we integrate a function with respect to $\mu_\theta^{2k,L}(dx)$, we need to evaluate the integral with respect to each marginal $\mu_\theta^{2k}(dx_j)$ for j which is greater than the lowest value of i for which x_i appears in the function. This seems to makes it impossible to find a closed-form formula for the profile of the measure, i.e. the expected amounts of energy at each site $\mathbb{E}^{\mu_\theta^{2k,L}}[x_i]$. Furthermore, the fact that $\mu_\theta^{2k,L}$ can not be turned into a product measure prevents us from bringing this measure into a setting with infinite volume.
In [3] the authors note that

$$\prod_{i=-L}^{L} \exp\left(\frac{e^{-2\sigma\tilde{E}_i(x)} - e^{-2\sigma\tilde{E}_{i+1}(x)}}{2\sigma\theta}\right) = \exp\left(\frac{\sum_{i=-L}^{L} e^{-2\sigma\tilde{E}_i(x)} - e^{-2\sigma\tilde{E}_{i+1}(x)}}{2\sigma\theta}\right)$$
$$= \exp\left(\frac{e^{-2\sigma\tilde{E}_{-L}(x)} - e^{-2\sigma\tilde{E}_L(x)}}{2\sigma\theta}\right), \tag{42}$$

so that we find

$$\mu_\theta^{2k,L}(dx) = \exp\left(\frac{e^{-2\sigma\tilde{E}_{-L}(x)} - e^{-2\sigma\tilde{E}_L(x)}}{2\sigma\theta}\right) \prod_{i=-L}^{L} \frac{\left(e^{2\sigma x_i} - 1\right)^{2k-1} e^{-4k\sigma x(i+L+1)}}{\theta^{2k} 2\sigma\Gamma(2k)} dx_i. \tag{43}$$

This may seem to solve the nonlocality problem, because we have a product measure on the right, and on the left a prefactor only depending on the total amount of energy and the energy at the rightmost site (which is 0 in infinite volume). However, both in infinite and finite volume, this total amount of energy $E_{-L}(x)$ or $E_{-\infty}(x) := \lim_{i\to-\infty} E_i(x)$ is random with its distribution depending on every marginal $\mu_\theta^{2k,L}(dx_i)$, so we still have the same nonlocality issues.

5.5 The Dynamic ABEP

The dynamic ABEP (DABEP) is a new process, not yet introduced in a publication. It was constructed as part of the exploration of the su(1,1) Lie algebra, as a more general version of the ABEP. The construction of this 'dynamic' version of the ABEP is analogous to the construction of the dynamic ASEP from the ASEP as described in [12]. Not much research has yet been done on the DABEP, but this is no problem for our purposes here, because a map from the DABEP to the BEP is known, similar to (33) for the ABEP. We start by giving the generator of the process, although we will not use this generator directly anywhere in the thesis.

Definition 5.2 (DABEP(σ, k, λ))**.** *The DABEP(σ, k, λ) is the Markov process with generator $\mathcal{L}^{\mathrm{DABEP}}$ acting on functions $f \in C_c^\infty(\Omega_f)$ via*

$$\mathcal{L}^{\mathrm{DABEP}} = \sum_{i=-\infty}^{\infty} \mathcal{L}_{i,i+1}^{\mathrm{DABEP}},$$

with single-edge generators given through

$$(\mathcal{L}_{i,i+1}^{\mathrm{DABEP}} f)(x) = A(x,\sigma,k,\lambda)\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right)^2 f(x) + B(x,\sigma,k,\lambda)\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right) f(x),$$

where

$$
\begin{aligned}
A =& \frac{1}{4\sigma^2}\big(1-e^{2\sigma x_{i+1}}\big)\big(e^{-2\sigma x_i}-1\big)\frac{\big(1-e^{2\sigma(\lambda+2E_{i+1}-x_{i+1})}\big)\big(1-e^{2\sigma(\lambda+2E_{i+1}+x_i)}\big)}{\big(1-e^{2\sigma(\lambda+2E_{i+1})}\big)^2},\\
B =& \frac{1}{2\sigma}\Bigg[2k\big(1-e^{-2\sigma x_i}\big)\left(\frac{1-e^{2\sigma(\lambda+2E_{i+1}+x_i)}}{1-e^{2\sigma(\lambda+2E_{i+1})}}\right)+2k\big(1-e^{2\sigma x_{i+1}}\big)\left(\frac{1-e^{2\sigma(\lambda+2E_{i+1}-x_{i+1})}}{1-e^{2\sigma(\lambda+2E_{i+1})}}\right)\\
&+\big(1-e^{2\sigma x_{i+1}}\big)\big(e^{-2\sigma x_i}-1\big)\frac{\big(1+e^{2\sigma(\lambda+2E_{i+1})}\big)\big(1-e^{2\sigma(\lambda+2E_{i+1}-x_{i+1})}\big)\big(1-e^{2\sigma(\lambda+2E_{i+1}+x_i)}\big)}{\big(1-e^{2\sigma(\lambda+2E_{i+1})}\big)^3}\Bigg].
\end{aligned}
$$

Interpreting this generator directly is difficult so instead we will give the map from the DABEP to the BEP, and then interpret the process from that.

Theorem 5.6. *If $\lambda \le -2\sigma E_{-\infty}(x)$ or $\lambda \ge 0$, the map $\hat{g} = (\hat{g}_i,\ i \in \mathbb{Z})$ maps from the DABEP(σ, k, λ) to the BEP(k), i.e. if $(x_t)_{t\ge 0}$ is the DABEP(σ, k, λ) starting out at $x \in \Omega_f$, then $(\hat{g}(x))_{t\ge 0}$ is the BEP(k) starting out at $\hat{g}(x) \in \mathbb{R}_+^{\mathbb{Z}}$. Functions $\hat{g}_i : \Omega_f \to \mathbb{R}_+^{\mathbb{Z}}$ are given through*

$$
\hat{g}_i(x) = \alpha\frac{\cosh(\sigma\lambda + 2\sigma E_{i+1}(x)) - \cosh(\sigma\lambda + 2\sigma E_i(x))}{\sigma}, \tag{44}
$$

where α is an $\mathbb{R}$-valued parameter, which can be freely chosen under the constraint that

$$
\begin{cases} \alpha < 0 & \text{if } \lambda \ge 0,\\ \alpha > 0 & \text{if } \lambda \le -2\sigma E_{-\infty}(x), \end{cases}
$$

in order to ensure that $\hat{g}_i(x) \ge 0$.

Proof. The proof of this will be given in a future publication to formally introduce the DABEP. This can be done e.g. via direct calculation similar to Lemma 5.1. Since we will not use the generator of the DABEP as introduced in Definition 5.2 anywhere directly, we can for our purposes define the DABEP as the process for which Theorem 5.6 holds. □

Note that in Theorem 5.6 we require λ not to be in $(-2\sigma E_{-\infty}(x), 0)$. This is because for $\lambda \in (-2\sigma E_{-\infty}(x), 0)$, there will be values of j for which $\sigma\lambda + 2\sigma E_j(x)$ is positive and values for which it is negative. As a result, $\hat{g}_i(x)$ will be positive for some values of i and negative for others for any choice of α, leading to an invalid BEP-configuration with negative energy levels.
We see from (44) that the DABEP relates to the BEP in a similar way to the ABEP, except that the expression contains hyperbolic cosines instead of exponentials. Futhermore we have an additional constant $\sigma\lambda$. This constant can be thought of as representing a sort of reservoir of energy at the right boundary in a finite setting (in our case at $x = \infty$). Because of this, this version of the DABEP is referred to as the right-DABEP (ABEP$_R$), in constrast to the left-DABEP (ABEP$_L$). Since we will only focus on the right-DABEP here, as it is consistent with our definition of the ABEP (where for the partial energy we look to the right of a site), we will simply refer to it as the DABEP. Furthermore we have free constant α which is not very important, but ensures that $\hat{g}_i(x) \in \mathbb{R}$ through our choices of parameters, where we may take certain limits. For instance it can be shown that if we take $\alpha = \frac{1}{2}\exp(\sigma\lambda)$ and then $\lambda \to -\infty$, we recover the ABEP(σ, k). From Theorem 5.6 immediately follows a duality function between the DABEP and the SIP.

Corollary 5.2. *The DABEP(σ, k, λ) and the n-SIP(k) are dual with duality function*

$$D^d(x, \xi) = \prod_{i=-\infty}^{\infty} \frac{\Gamma(2k)}{\Gamma(2k + \xi_i)} \hat{g}_i(x)^{\xi_i}. \tag{45}$$

Proof. Analogously to the proof of Theorem 5.3 we can use Theorem 5.6, so that we have the following.

$$\begin{aligned}
\left[\mathcal{L}^{\text{DABEP}} D^d(\cdot, \xi)\right](x) &= \left[\mathcal{L}^{\text{DABEP}} D^b(\hat{g}(\cdot), \xi)\right](x) \\
&= \left[\mathcal{L}^{\text{BEP}} D^b(\cdot, \xi)\right](\hat{g}(x)) \\
&= \left[\mathcal{L}^{\text{SIP}} D^b(\hat{g}(x), \cdot)\right](\xi) \\
&= \left[\mathcal{L}^{\text{SIP}} D^d(x, \cdot)\right](\xi).
\end{aligned}$$

$\square$

One can see that if we take again $\alpha = \frac{1}{2}\exp(\sigma\lambda)$ and let $\lambda \to -\infty$, the duality function D^a between the ABEP(σ, k) and the BEP(k) is recovered. Note that in Theorem 5.6 we had to choose α in such a way that $\hat{g}_i(x) \geq 0$, and when the specific combinations of λ and x were such that this was not possible, we could not map the DABEP to the BEP. In Corollary 5.2 we are under no such restrictions, as the duality function in (45) is allowed to have a negative value.
We will end this section by providing an overview of the different interacting particle systems introduced throughout this and previous section, and how they relate to each other. This ends the first part of the thesis, which aimed to provide background information about the topic. The next chapters will provide the results of this thesis, which consist of the hydrodynamic limit of the BEP in Chapter 6, the hydrodynamic limit of the ABEP in Chapter 7 and propagation of chaos for the BEP and the ABEP in Chapter 8.

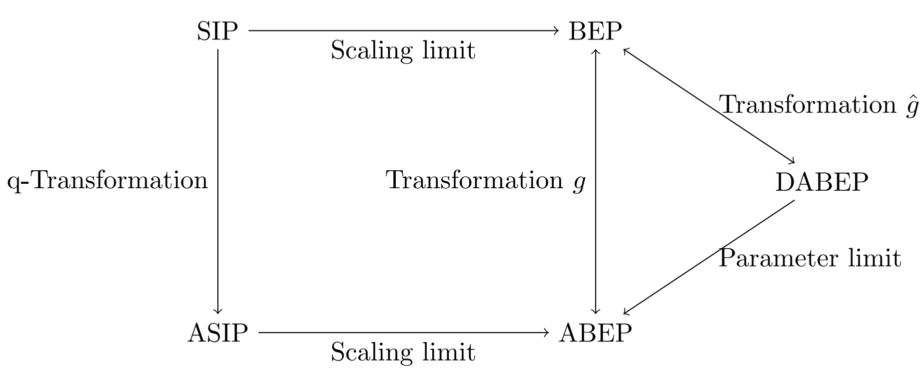

Figure 5.1. Overview of relevant interacting particle systems.

6 Hydrodynamic limit of the BEP

6.1 Introduction to Hydrodynamic limits

The idea behind hydrodynamic limits is that we describe the stochastic transport of particles/energy on microscopic level, in order to learn something about the emergent behavior of the density of particles/energy at a macroscopic scale. They are a claim about the evolution of the distribution of energy over time, motivated by the conservation of the total amount of energy. This conservation of energy allows us to draw meaningful conclusions about the behaviour of average energy levels at the macroscopic scale, despite random deviations at the microscopic scale. In order to distinguish between micro- and macro-scale, we define a scaling parameter N. As we then move from micro to macro, we rescale the space-dimension with factor N (i.e. macroscopic x corresponds to microscopic $\lfloor xN \rfloor$) and the time-dimension with factor N^2 (macro t corresponds to micro N^2t). The idea behind this is that a random walk in time N^2 typically ends up at a distance of order N away from its starting location, a scaling law closely related to the central limit theorem. For asymmetric processes, this scaling law doesn't apply in this way (due to an expected linear drift), so we need to be more thoughtful in our transition from micro to macro.

This thesis follows in broad terms the lines of argument presented in [10]. This is a seminal paper written in 1987 by Gärtner, proving the hydrodynamic limit of another IPS, the so-called Asymmetric Exclusion Process (ASEP). The resulting density turns out to solve a version of Burgers' equation with a nonlinear term. As we will see in our hydrodynamic limit of ABEP, a nonlinear term is a typical result for an asymmetric interacting particle system.

An important observation in this proof by Gärtner is that we can find a transformation of the PDE into a simpler, linear PDE. The crucial idea is then that we can perform a similar transformation to the particle system, so that the hydrodynamic limit of this transformed particle system corresponds to the transformed PDE. Because this PDE is simpler, and most importantly, linear, we are able to prove this hydrodynamic limit, where we may not have been able to do this directly on the non-transformed system.

This approach provides a nice framework for the upcoming proofs. The structure here is different from how it is presented in [10] however. In this chapter, we will prove that the hydrodynamic limit density of the BEP is the solution to the heat equation. This finding does not come as a surprise, as it was already suggested in [17], but not rigorously proven.

Though this is a new result in of itself, we can also think of this as a prerequisite for the more challenging task of finding the hydrodynamic limit of the ABEP. In fact, finding the hydrodynamic limit of the BEP neatly corresponds to the second part of the proof of Gärtner, namely finding the hydrodynamic limit of a relatively simple problem, with a linear corresponding PDE. In the Chapter 7, we will use this finding to prove the hydrodynamic limit of the ABEP. We will use the fact that the BEP arises as a transformation of the ABEP, and as in the case of the paper of Gärtner, we will work out the PDE that results from carrying this transformation into macroscopic scale.

6.2 Weak topology and empirical (trajectory) measure

When making the transition from micro- to macro-level we stop looking at energy at individual sites, but instead look at the distribution of energy along the lattice. We define a so-called empirical measure as a way of quantifying the amount of energy in a given region of the lattice on which the process is defined. Formally we have the following.

Definition 6.1 (Empirical measure Λ_N). *Let M_+ be the space of non-negative locally finite measures on $\mathbb{R}$. The empirical measure of a configuration $y \in \mathbb{R}_+^{\mathbb{Z}}$ is then given by the function $\Lambda_N : \mathbb{R}_+^{\mathbb{Z}} \to M_+$, which is defined through*

$$\Lambda_N(y) = \frac{1}{N} \sum_{i=-\infty}^{\infty} y_i \delta_{i/N}, \tag{46}$$

where δ_x denotes the Diraq measure at point $x \in \mathbb{R}$.

In this definition we have defined the state space of the configurations on which Λ_N acts to be $\mathbb{R}_+^{\mathbb{Z}}$, implicitly assuming we are dealing with continuous processes in infinite volume. If we were interested in other discrete processes or processes in finite volume, we would still use Λ_N as expressed in (46) but define its input space differently.
These empirical measures lie in M_+, which endowed with the weak topology. One way this can be achieved is by equipping the measure space with the Prokhorov metric, given by

$$r(\mu, \nu) = \inf\{\epsilon > 0 : \nu(F) \leq \mu(F^{(\epsilon)}) + \epsilon \text{ and } \mu(F) \leq \nu(F^{(\epsilon)}) + \epsilon \text{ for every } F \in \mathcal{B}(\mathbb{R})\},$$

where $F^{(\epsilon)}$ is th ϵ-neighborhood of F. One can show that under this metric M_+ is completely metrizable, making it a Polish space. Convergence under this metric is equivalent to weak convergence [18], which is defined in the following way.

Definition 6.2 (Weak convergence). *We say a sequence $(\mu_n)_{n\in\mathbb{N}}$ of measures on $\mathbb{R}$ (with standard metric) convergences weakly to μ if for all smooth test functions with compact support ϕ taking values in $\mathbb{R}$ (i.e. $\phi \in C_c^{\infty}(\mathbb{R})$), we have*

$$\langle \mu_n, \phi \rangle := \int \phi(x)\mu_n(dx) \to \int \phi(x)\mu(dx) = \langle \mu, \phi \rangle . \tag{47}$$

The empirical measures that we will encounter in this thesis will always be random and as a result we will focus on weak convergence in probability and the convergence $\langle \mu_n, \phi \rangle \to \langle \mu, \phi \rangle$ in (47) will mean convergence in probability.
This shift in focus from configuration to empirical measure trajectories results in our interest now being in trajectories through M_+. Such a trajectory will typically be denoted $(\beta_t)_{t\geq 0}$ with $\beta_t \in M_+$ for each fixed t or $\alpha : [0, T] \to M_+$ and lies in $D([0, T], M_+)$, the space of right-continuous path measures.
Convergence in this space is quite subtle. What follows is a short summary of some of the challenges of proving the hydrodynamic limits, which amounts to proving convergence of the trajectories of the empirical measures of the IPS of interest.
Again we are interested in weak convergence in probability, only now for the whole trajectory, which means the weak convergence in probability is uniform in $t \in [0, T]$. What this means is that for a sequence of empirical measure trajectories $(\alpha_n)_{n\in\mathbb{N}}$ with for each $n \in \mathbb{N}$, $\alpha_n \in D([0, T], M_+)$, we have convergence in probability

$$\sup_{t\in[0,T]} \left| \int \phi d\alpha_n(t) - \int \phi d\alpha(t) \right| \to 0, \tag{48}$$

towards some limiting trajectory $\alpha \in D([0, T], M_+)$. Not only do we need to find what this limit trajectory α would look like, but we need to establish the existence of the limit in the first place.

The approach of choice for the latter question of existence of this limit, is to show that the sequence of probability measures with respect to which the trajectories $(\alpha_n)_{n\in\mathbb{N}}$ are defined, is "tight", meaning that for every n the vast majority of α_n concentrates on a compact subspace of $D([0,T],M_+)$ under the relevant probability measure.
Tightness can be proven via Aldous' tightness criterion [1], a two-part criterion where one of these is showing convergence of the so-called 'modulus of continuity'. This modulus of continuity is a function we construct, representing the difference between measures corresponding to the trajectory evaluated at different times, in order to give meaning to the notion of continuity when dealing with trajectories of measures. Mitoma's theorem [15] shows us how we can create this modulus of continuity by pairing the measures with a test function and constructing a metric consistent with the weak topology.
Once tightness is established, Prokhorov's Theorem tells us that this sequence of probability measures is sequentially compact, meaning it has convergent subsequences. Then by showing that the limits of these subsequences are concurrent, we can conclude the proof of (48). Finally, in order for this line of reasoning to work, we note that since M_+ endowed with the weak topology is a Polish space, $D([0,T],M_+)$ endowed with the Skorokhod topology must be a Polish space as well (See appendix A.2.2 of [18]). $D([0,T],M_+)$ being a Polish space is required for e.g. Prokhorov's theorem.
Many of the steps in this section are quite technical, and not really the focus of this thesis. Furthermore, in many parts they are exactly or almost equivalent to the steps used in the hydrodynamic limits of other processes presented in different papers. For this reason only the proof of tightness via Aldous' tightness criterion and Mitoma's theorem will be worked out in detail, both for the BEP and the ABEP. For the other parts of the proof we will refer to the book by Timo Seppäläinen about the hydrodynamic limit of the SEP [18].

6.3 Setting and main result

We start out with random configurations of the BEP which depend on some scaling parameter N. These configurations are chosen in such a way that their empirical measures as defined in (46) (i.e. how they appear on macroscopic scale) converge as we pass N to infinity. Let $(y^{(N)})_{N\in\mathbb{N}}$ be a sequence of BEP(k)-configuration in infinite volume, i.e. for $N\in\mathbb{N}:\ y^{(N)}\in\mathbb{R}_+^{\mathbb{Z}}$. In a setting where N remains constant we will often omit superscript (N) from notation. Let for each N the measure $\nu_N:\mathcal{B}(\mathbb{R}_+^{\mathbb{Z}})\to[0,1]$ denote the distribution of $y^{(N)}$ and $\mathbb{E}^{\nu_N}$ the expectation with respect to this measure.
To prevent explosive behavior we put a bound on the amount of energy at a single site. Since $y^{(N)}$ are random it suffices bound y_i in expectation.

Assumption 6.1.

$$\sup_{i\in\mathbb{Z},N\in\mathbb{N}}\mathbb{E}^{\nu_N}\left[\left(y_i^{(N)}\right)^2\right]\le C\text{ for some }C>0. \tag{49}$$

Initially we have convergence of the empirical measure of $(y^{(N)})_{N\in\mathbb{N}}$ to a measure with a known density.

Assumption 6.2. $\Lambda_N(y^{(N)})$ *converges weakly in probability to a measure with density* $\rho:\mathbb{R}\to\mathbb{R}_+$

(meaning $\lim_{N\to\infty}\left(\Lambda_N(y^{(N)})\right)(dx) = \rho(x)dx$*), i.e. for every* $\phi \in C_c^\infty(\mathbb{R})$ *and* $\epsilon > 0$

$$\nu_N\left(\left|\left\langle \Lambda_N(y^{(N)})\delta_{i/N}, \phi\right\rangle - \int_{\mathbb{R}} \rho(x)\phi(x)dx\right| > \epsilon\right) \to 0.$$

In this setting, the central claim is that as we evolve $y^{(N)}$ over time $t \in [0,T]$ as the BEP(k), then uniformly in t we still have weak convergence in probability of the empirical density to a new limit measure with density function ρ_t, where ρ_t is the unique solution to the heat equation with initial condition ρ.

Theorem 6.1. *Let* $\mathbb{P}_{\nu_N}^{\mathrm{BEP}_N}$ *denote the path-space measure of the accelerated BEP(k) with generator* $\mathcal{L}^{\mathrm{BEP}_N} := N^2\mathcal{L}^{\mathrm{BEP}}$*, with evolved configuration* $y_t^{(N)}$ *starting out at* $y^{(N)}$ *with initial distribution* ν_N*.*
For any smooth test function with compact support $\phi \in C_c^\infty(\mathbb{R})$*, we have for every* $\epsilon > 0$

$$\lim_{N\to\infty} \mathbb{P}_{\nu_N}^{\mathrm{BEP}_N}\left(\sup_{t\in[0,T]}\left|\left\langle \Lambda_N(y_t^{(N)}), \phi\right\rangle - \int_{\mathbb{R}} \rho_t(x)\phi(x)dx\right| > \epsilon\right) = 0$$

where ρ_t *is the solution to the heat equation,*

$$\frac{\partial \rho_t(x)}{\partial t} = 2k\frac{\partial^2 \rho_t(x)}{\partial x^2}, \tag{50}$$

with initial condition $\rho_0 = \rho$*.*

For the rest of the section, assume that when unspecified, $y_t^{(N)}$ refers to configuration $y^{(N)}$ evolved with respect to $\mathbb{P}_{\nu_N}^{\mathrm{BEP}_N}$ for time t.

6.4 Proof

As explained near the end of Section 6.2, there are two main challenges in proving the convergence of a sequence of trajectory measures in $D([0,T], M_+)$; showing that the limit exists, and finding an expression for it. In this proof of Theorem 6.1 we will do this second part first. In Section 6.4.1 we will first introduce a Dynkin martingale that will be central in this proof. After this in Section 6.4.2 and 6.4.3 we will find the expression (50) for the hydrodynamic limit of the BEP, assuming that this limit exists. In Section 6.4.4 we will use the tools introduced throughout the proof to show that this hydrodynamic exists, using tightness of sequences of measures. Then in Section 6.4.5 we will finish the proof and conclude that Theorem 6.1 holds.

6.4.1 Dynkin Martingale representation of empirical density

We define a Dynkin Martingale to capture the deviation of the empirical measure from its expected evolution under $\mathcal{L}^{\mathrm{BEP}_N}$. For every $N \in \mathbb{N}$ let $(M_t^N)_{t\geq 0}$ be the Dynkin martingale with respect to $\mathbb{P}_{\nu_N}^{\mathrm{BEP}_N}$ created through function $\langle \Lambda_N(\cdot), \phi\rangle$, i.e.

$$M_t^N := \left\langle \Lambda_N(y_t^{(N)}), \phi\right\rangle - \left\langle \Lambda_N(y^{(N)}), \phi\right\rangle - \int_0^t \left[\mathcal{L}^{\mathrm{BEP}_N}\left\langle \Lambda_N(\cdot), \phi\right\rangle\right](y_s^{(N)})ds. \tag{51}$$

The proof of the theorem consists of three parts. First we show that

$$\lim_{N\to\infty}\left[\mathcal{L}^{\mathrm{BEP}_N}\left\langle \Lambda_N(\cdot),\phi\right\rangle\right](y_s^{(N)}) = \lim_{N\to\infty}\left\langle \Lambda_N(y_s^{(N)}),\phi''\right\rangle,$$

i.e. applying the BEP(k) generator to the empirical measure is the same as taking the second space-derivative of the test function that is integrated over that measure.
Second we will show that $M_t^N \to 0$ uniformly in t in probability as $N \to \infty$.

Third, we will show that for a subsequence $\{N_j\}_{j\in\mathbb{N}}$ of the natural numbers the limit $\lim_{j\to\infty}\left(\Lambda_N(y_t^{(N_j)})\right)_{t\geq 0}$ exists in $D([0,T],M_+)$ using tightness of trajectories.
Together these two findings and the concurrence of limit points between different subsequences $\{N_j\}_{j\in\mathbb{N}}$ will allow us to conclude that the density of $\lim_{N\to\infty}\left(\Lambda_N(y_t^{(N)})\right)_{t\geq 0}$ is deterministically a weak solution to the heat equation, and by uniqueness of the weak solution of the heat equation, the strong solution as well.

6.4.2 The effect of microscopic BEP-dynamics on the macroscopic density field

Definition 6.3. *We define the hydrodynamic limit density ρ_t as the density of the limit in N of the empirical measures of the evolved configuration $y_t^{(N)}$ under $\mathbb{P}_{\nu_N}^{\mathrm{BEP}_N}$, i.e. for every $\phi \in C_c^\infty(\mathbb{R})$ and $\epsilon > 0$,*

$$\mathbb{P}_{\nu_N}^{\mathrm{BEP}_N}\left(\sup_{t\in[0,T]}\left|\left\langle \Lambda_N(y_t^{(N)}),\phi\right\rangle - \int\limits_{\mathbb{R}} \rho_t(x)\phi(x)dx\right| > \epsilon\right) \to 0,$$

as N goes to infinity. For now we assume that this limit exists. In Chapter 6.4.4 on tightness we will show existence of this limit using the tools that will be introduced throughout the proof.

The central claim of this subsection concerns the effect of applying the generator to the coupling $\langle \Lambda_N(\cdot),\phi\rangle\,(y_s^{(N)})$. We will first do this for initial configuration $y^{(N)}$ instead of evolved configuration $y_s^{(N)}$, and after this we will lay the necessary groundwork to apply this result to $y_s^{(N)}$.
This groundwork consists of boundedness of $\mathbb{E}_{\nu_N}^{\mathrm{BEP}_N}[(y_s^{(N)})_i]$, which we will show in Proposition 6.1 follows from boundedness of $\mathbb{E}^{\nu_N}[y_i]$, and convergence of $\Lambda_N(y_t^{(N)})$ to some limit measure, which we will prove in Section 6.4.4.

Lemma 6.2. *For each $\phi \in C_c^\infty(\mathbb{R})$ we have the following convergence in probability.*

$$\left[\mathcal{L}^{\mathrm{BEP}_N}\left\langle \Lambda_N(\cdot),\phi\right\rangle\right](y^{(N)}) \to 2k\int \rho(x)\frac{d^2\phi}{dx^2}(x)dx.$$

Proof. For ease of notation we write $\phi_i := \phi(\frac{i}{N})$, $\mathcal{L} := \mathcal{L}^{\mathrm{BEP}}$ and $y := y^{(N)}$.

$$\begin{aligned}\left[\mathcal{L}^{\mathrm{BEP}_N}\left\langle\Lambda_N(\cdot),\phi\right\rangle\right](y) &= N^2\mathcal{L}\left\langle\frac{1}{N}\sum_{i=-\infty}^{\infty}(\cdot)_i\delta_{i/N},\phi\right\rangle(y)\\ &= N\sum_{j=-\infty}^{\infty}\mathcal{L}_{j,j+1}\left[\sum_{i=-\infty}^{\infty}(\cdot)_i\phi_i\right](y)\\ &= N\sum_{i=-\infty}^{\infty}\phi_i\sum_{j=-\infty}^{\infty}[\mathcal{L}_{j,j+1}(\cdot)_i](y).\end{aligned}$$

Applying $\mathcal{L}^{\mathrm{BEP}}_{j,j+1}$ to $f(y) = y_i$ yields $\mathcal{L}_{j,j+1}y_i = \begin{cases}2k(y_{i+1}-y_i) \text{ if } j=i,\\ 2k(y_{i-1}-y_i) \text{ if } j=i-1,\\ \quad 0 \text{ otherwise,}\end{cases}$ so that

$$\sum_{j=-\infty}^{\infty}\mathcal{L}^{\mathrm{BEP}}_{j,j+1}y_i = 2k(y_{i+1}-2y_i+y_{i-1}).$$

Thus we get

$$\begin{aligned}N\sum_{i=-\infty}^{\infty}\phi_i\sum_{j=-\infty}^{\infty}[\mathcal{L}_{j,j+1}(\cdot)_i](y) &= N\sum_{i=-\infty}^{\infty}2k\phi_i(y_{i+1}-2y_i+y_{i-1})\\ &= N\sum_{i=-\infty}^{\infty}2ky_i(\phi_{i+1}-2\phi_i+\phi_{i-1})\\ &= N\sum_{i=-\infty}^{\infty}2ky_i\left[\frac{1}{N^2}\phi_i''+\frac{1}{N^4}(\phi^{(4)}(x_1^i)+\phi^{(4)}(x_2^i))\right]\\ &= \frac{2k}{N}\sum_{i=-\infty}^{\infty}y_i\phi_i''+\frac{2k}{N^3}\sum_{i=-\infty}^{\infty}y_i\left[\phi^{(4)}(x_1^i)+\phi^{(4)}(x_2^i)\right]. \qquad (52)\end{aligned}$$

Here we used Taylor approximations with Lagrange remainders evaluated at $x_1^i \in [\frac{i}{N},\frac{i+1}{N}]$ and $x_2^i \in [\frac{i-1}{N},\frac{i}{N}]$ in the third step.
To see that the second term in (52) we note that ϕ_i and its derivatives are nonzero for $O(N)$ values of i. More precisely, since $\mathrm{supp}(\phi)$ is compact, there must exist $M_1, M_2 \in \mathbb{R}$ such that $\mathrm{supp}(\phi) \subseteq [M_1, M_2]$. Define $M := M_1 - M_2$ as the length of this interval, then for any N there can be at most $(M+1)N$ values of i for which $\frac{i}{N} \in [M_1, M_2] \supseteq \mathrm{supp}(\phi)$.
Denote

$$a_N := \left[\mathcal{L}^{\mathrm{BEP}_N}\left\langle\Lambda_N,\phi\right\rangle\right](y) - 2k\int\rho(x)\phi''(x)dx.$$

$$
\begin{aligned}
&\nu_N\left(|a_N|>2\epsilon\right)\\
&\leq \nu_N\left(\left|\frac{2k}{N}\sum_{i=-\infty}^{\infty} y_i\phi_i'' - 2k\int\rho(x)\phi''(x)dx\right|>\epsilon\right)+\nu_N\left(\left|\frac{2k}{N^3}\sum_{i=-\infty}^{\infty} y_i\left(\phi^{(4)}(x_1^i)+\phi^{(4)}(x_2^i)\right)\right|>\epsilon\right)\\
&\leq \nu_N\left(2k\left|\langle\Lambda_N(y),\phi''\rangle-\int\rho(x)\phi''(x)dx\right|>\epsilon\right)+\frac{1}{\epsilon}\mathbb{E}^{\nu_N}\left|\frac{2k}{N^3}\sum_{i=-\infty}^{\infty} y_i\left(\phi^{(4)}(x_1^i)+\phi^{(4)}(x_2^i)\right)\right|\\
&\leq \nu_N\left(2k\left|\langle\Lambda_N(y),\phi''\rangle-\int\rho(x)\phi''(x)dx\right|>\epsilon\right)+\frac{2k}{N^3}\sum_{i=\lfloor M_1N\rfloor}^{\lceil NM_2\rceil}\left(\phi^{(4)}(x_1^i)+\phi^{(4)}(x_2^i)\right)\mathbb{E}^{\nu_N}|y_i|\,.
\end{aligned}
$$

Assumption 6.2 of initial convergence of $\Lambda_N(y^{(N)})$ tells us that the first term converges to 0. Note that for $t>0$, convergence of $\Lambda_N(y_t^{(N)})$ has yet to be proven.
For the latter term we note that from assumption 6.1 follows that $\mathbb{E}^{\nu_N}|y_i|\leq\sqrt{C}$, so that the sum over i has at most $(M+1)N$ nonzero terms which are all bounded, so that we can conclude that

$$
2k\frac{1}{N^3}\sum_{i:\frac{i}{N}\in[M_1,M_2]}\left(\phi^{(4)}(x_1^i)+\phi^{(4)}(x_2^i)\right)\mathbb{E}^{\nu_N}|y_i|\leq\frac{2k(M+1)2\left|\left|\phi^{(4)}\right|\right|_\infty\sqrt{C}}{N^2}.
$$

Thus we finish the proof of the Lemma by concluding that

$$
\nu_N\left(\left|a^N\right|>\epsilon\right)\to 0 \text{ as } N\to\infty.
$$

□

Proposition 6.1. *For every $t>0$, $i,j\in\mathbb{Z}$, $N\in\mathbb{N}$, $\mathbb{E}_{\nu_N}^{\mathrm{BEP}_N}[(y_t)_i]$ and $\mathbb{E}_{\nu_N}^{\mathrm{BEP}_N}[(y_t)_i(y_t)_j]$ are bounded.*

Proof. We start by proving boundedness of $\mathbb{E}_{\nu_N}^{\mathrm{BEP}_N}[(y_t)_i(y_t)_j]$. To show this boundedness, we note that $(y_t)_i(y_t)_j$ corresponds to the duality function D^b (30) between BEP(k) configuration y_t and 2-SIP(k) configuration $\xi=\delta_i+\delta_j$.

$$
\begin{aligned}
(y_t)_i(y_t)_{i+1}&=c_k\prod_{v\in\{i,j\}}\frac{\Gamma(2k)}{\Gamma(2k+\xi_v)}(y_t)_j^{\xi_v}\\
&=c_kD^b(y_t,\xi),
\end{aligned}
$$

where

$$
c_k=\begin{cases}(2k)^2 & \text{if } i\neq j,\\ 2k(2k+1) & \text{if } i=j,\end{cases}
$$

so c_k is finite and only depending on k.
This means that we can use duality between BEP(k) and n-SIP(k) to interchange the expectation in y evolved through BEP(k) dynamics, with the expectation in ξ evolved through SIP(k) dynamics. This is useful, because then our problem can be expressed through initial configuration y, which we know is bounded in expectation. For this, let $X(t)$ and $Y(t)$ denote the location of evolved

SIP(k)-particles, starting out from sites i and j respectively, and let $\mathbb{E}^{\text{SIP}}_{i,j}$ denote the expectation with respect to this process.

$$\begin{aligned}
\mathbb{E}^{\text{BEP}_N}_{\nu_N}\left[(y_t)_i(y_t)_j\right] &= c_k\mathbb{E}^{\text{BEP}}_{\nu_N}\left[D^b(y_{tN^2},\delta_i+\delta_j)\right]\\
&= c_k\mathbb{E}^{\nu_N}\left[\mathbb{E}^{\text{BEP}}_{y}\left[D^b(y_{tN^2},\delta_i+\delta_j)\right]\right]\\
&= c_k\mathbb{E}^{\nu_N}\left[\mathbb{E}^{\text{SIP}}_{i,j}\left[D^b(y,\delta_{X(tN^2)}+\delta_{Y(tN^2)})\right]\right]\\
&\leq \frac{2k+1}{2k}\mathbb{E}^{\nu_N}\left[\mathbb{E}^{\text{SIP}}_{i,j}[y_{X(tN^2)}y_{Y(tN^2)}]\right]\\
&= \frac{2k+1}{2k}\mathbb{E}^{\text{SIP}}_{i,j}\left[\mathbb{E}^{\nu_N}[y_{X(tN^2)}y_{Y(tN^2)}]\right]\\
&\leq \frac{2k+1}{2k}\sup_{i,j\in\mathbb{Z}}\mathbb{E}^{\nu_N}[y_iy_j]\\
&\leq \frac{2k+1}{2k}\sup_{i,j\in\mathbb{Z}}\sqrt{\mathbb{E}^{\nu_N}[y_i^2]\mathbb{E}^{\nu_N}[y_j^2]}\\
&\leq \frac{2k+1}{2k}C =: \tilde{C}.
\end{aligned}$$

Here C is the bound from (49) and $\hat{c}_k$ is a constant similar to c_k depending on whether $X(tN^2)$ is equal to $Y(tN^2)$.

Boundedness of $\mathbb{E}^{\text{BEP}_N}_{\nu_N}[(y_t)_i]$ follows then from Jensen's inequality by taking $j=i$, i.e.

$$\mathbb{E}^{\text{BEP}_N}_{\nu_N}[(y_t)_i] \leq \sqrt{\mathbb{E}^{\text{BEP}_N}_{\nu_N}\left[(y_t)_i^2\right]} \leq \hat{C}^2.$$

□

Corollary 6.1. *If assumption 6.2 holds for evolved BEP-configuration $y_t^{(N)}$ and evolved density $\rho_t:\mathbb{R}\to\mathbb{R}_+$, i.e. for every $\phi\in C_c^\infty(\mathbb{R})$, $t>0$ and $\epsilon>0$,*

$$\mathbb{P}^{\text{BEP}_N}_{\nu_N}\left(\left|\left\langle\Lambda_N(y_t^{(N)})\delta_{i/N},\phi\right\rangle-\int\limits_{\mathbb{R}}\rho_t(x)\phi(x)dx\right|>\epsilon\right)\to 0,$$

then

$$\left[\mathcal{L}^{\text{BEP}_N}\left\langle\Lambda_N(\cdot),\phi\right\rangle\right](y_t^{(N)}) \to 2k\int\rho_t(x)\frac{d^2\phi}{dx^2}(x)dx, \tag{53}$$

in probability for $t\in[0,T]$.

Proof. Proposition 6.1 and the assumption of the corollary imply that we can apply Lemma 6.2 to find (53). □

6.4.3 Vanishing of quadratic variation at macroscopic scale

In this section we will prove that M_t^N converges to 0 in probability uniformly in t. The proof resolves around showing that the quadratic variation process of M_t^N converges in expectation to 0. Since from the definition of M_t^N it's clear that $M_0^N=0$ for each N, we can then use Doob's inequality we can show that $\lim\limits_{N\to\infty}M_t^N=0$ in probability uniformly in t.

We start by giving an expression of $[M^N]_t$.

Corollary 6.2. *The quadratic variation process of* $(M_t^N)_{t\geq 0}$ *is given through*

$$[M^N]_t = \int\limits_0^t \left[\mathcal{L}^{\mathrm{BEP}_N} \left\langle \Lambda_N(\cdot),\phi\right\rangle^2\right](y_s^{(N)}) - 2\left\langle \Lambda_N(y_s^{(N)}),\phi\right\rangle \mathcal{L}^{\mathrm{BEP}_N} \left\langle \Lambda_N(\cdot),\phi\right\rangle (y_s^{(N)})ds.$$

Proof. Direct application of Theorem 2.2. □

Next we will show that this expression converges in expectation to 0.

Lemma 6.3. *We have the following convergence,*

$$\lim_{N\to\infty} \mathbb{E}_{\nu_N}^{\mathrm{BEP}_N}\left[[M^N]_t\right] = 0.$$

Proof. For ease of notation we omit superscript BEP(k) and (N) from notation so that we write $\mathcal{L}$ instead of $\mathcal{L}^{\mathrm{BEP}}$ and y_s instead of $y_s^{(N)}$, and we write $\mathcal{L}y_i$ to denote $\mathcal{L}f(y)$ with $f(y) = y_i$, and $\mathcal{L}y_iy_j$ to denote $\mathcal{L}f(y)$ with $f(y) = y_iy_j$. By applying the accelerated BEP(k) generator we find the following,

$$\begin{aligned}
[M^N]_t &= \int_0^t \left[\mathcal{L}^{\mathrm{BEP}_N} \left\langle \Lambda_N(\cdot),\phi\right\rangle^2\right](y_s) - 2\left\langle \Lambda_N(y_s),\phi\right\rangle \left[\mathcal{L}_N \left\langle \Lambda_N(\cdot),\phi\right\rangle\right](y_s)ds \\
&= \int_0^t \left[\mathcal{L}^{\mathrm{BEP}_N}\left(\frac{1}{N}\sum_{i=-\infty}^{\infty}(\cdot)_i\phi_i\right)^2\right](y_s) - \frac{2}{N}\sum_{i=-\infty}^{\infty}(y_s)_i\phi_i\left[\mathcal{L}^{\mathrm{BEP}_N}\frac{1}{N}\sum_{j=-\infty}^{\infty}(\cdot)_i\phi_i\right](y_s)ds \\
&= \int_0^t \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\phi_i\phi_j\left(\mathcal{L}(y_s)_i(y_s)_j - 2(y_s)_i\mathcal{L}y_j\right)ds \\
&= \int_0^t \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\phi_i\phi_j\left(\sum_{p=-\infty}\mathcal{L}_{p,p+1}(y_s)_i(y_s)_j - 2(y_s)_i\mathcal{L}_{p,p+1}(y_s)_j\right)ds \\
&= \int_0^t \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\phi_i\phi_j\sum_{p=-\infty}^{\infty}\left(\mathcal{L}_{p,p+1}(y_s)_i(y_s)_j - (y_s)_i\mathcal{L}_{p,p+1}(y_s)_j - (y_s)_j\mathcal{L}_{p,p+1}(y_s)_i\right)ds.
\end{aligned}$$

We proceed by straightforwardly applying the BEP single-edge generators $\mathcal{L}_{k,k+1}$ to these functions $y \mapsto y_i$, $y \mapsto y_j$ and $y \mapsto y_iy_j$. We do this by first considering the case where $k \in \{i-1, i\}$ and $j \in \{k, k+1\}$ and then showing that every other combination of indices yields 0.
Recall that BEP generator $\mathcal{L}_{i,i+1}$ is given through

$$(\mathcal{L}_{i,i+1}f)(z) = 2k(y_i - y_{i+1})(\partial_i - \partial_{i+1}) + y_iy_{i+1}(\partial_i - \partial_{i+1})^2.$$

We find the following.

$$\mathcal{L}_{i,i+1}y_i^2 = 4k(y_i - y_{i+1})y_i + 2y_iy_{i+1}$$
$$2y_i\mathcal{L}_{i,i+1}y_i = 4k(y_i - y_{i+1})y_i,$$

so that

$$\mathcal{L}_{i,i+1}y_i^2 - 2y_i\mathcal{L}_{i,i+1}y_i = 2y_iy_{i+1},$$

and

$$\mathcal{L}_{i,i+1}y_iy_{i+1} = 2k(y_i - y_{i+1})(y_{i+1} - y_i) - 2y_iy_{i+1},$$
$$y_i\mathcal{L}_{i,i+1}y_{i+1} = -2k(y_i - y_{i+1})y_i,$$
$$y_{i+1}\mathcal{L}_{i,i+1}y_i = 2k(y_i - y_{i+1})y_{i+1},$$

so that

$$\mathcal{L}_{i,i+1}y_iy_{i+1} - y_i\mathcal{L}_{i,i+1}y_{i+1} - y_{i+1}\mathcal{L}_{i,i+1}y_i = -2y_iy_{i+1}.$$

In similar manner we can find

$$\mathcal{L}_{i-1,i}y_i^2 - 2y_i\mathcal{L}_{i-1,i}y_i = 2y_{i-1}y_i,$$

and

$$\mathcal{L}_{i-1,i}y_{i-1}y_i - y_{i-1}\mathcal{L}_{i-1,i}y_i - y_i\mathcal{L}_{i-1,i}y_{i-1} = -2y_{i-1}y_i.$$

So that summing these terms, corresponding to $k \in \{i-1, i\}, j \in \{k, k+1\}$, yields

$$\begin{aligned}
&\sum_{i=-\infty}^{\infty} \sum_{k\in\{i-1,i\}} \sum_{j\in\{k,k+1\}} \phi_i\phi_j \left(\mathcal{L}_{k,k+1}y_iy_j - y_i\mathcal{L}_{k,k+1}y_j - y_j\mathcal{L}_{k,k+1}y_i\right) \\
&= \sum_{i=-\infty}^{\infty} \phi_i^2(2y_iy_{i+1} + 2y_{i-1}y_i) - 2\phi_i\phi_{i+1}y_iy_{i+1} - 2\phi_{i-1}\phi_iy_iy_{i-1} \\
&= \sum_{i=-\infty}^{\infty} 2y_iy_{i+1}(\phi_i^2 + \phi_{i+1}^2 - 2\phi_i\phi_{i+1}),
\end{aligned}$$

where we used the fact that i is summed from $-\infty$ to ∞ to shift the indices of the latter terms in the second line.
By smoothness of g we can approximate g_{i+1} using Taylor series with Lagrange remainders at some $a_1^i, a_2^i \in [\frac{i}{N}, \frac{i+1}{N}]$,

$$\begin{aligned}
\phi_{i+1} &= \phi_i + \frac{1}{N}\phi_i' + \frac{1}{2N^2}\phi_i'' + \frac{1}{3!N^3}\phi'''(a_1^i), \\
\phi_{i+1}\phi_i &= \phi_i^2 + \frac{1}{N}\phi_i'\phi_i + \frac{1}{2N^2}\phi_i''\phi_i + \frac{1}{3!N^3}\phi_i\phi'''(a_1^i), \\
\phi_{i+1}^2 &= \phi_i^2 + \frac{2}{N}\phi_i'\phi_i + \frac{2}{2N^2}\phi_i\phi_i'' + \frac{1}{N^2}(\phi_i')^2 + \frac{1}{3!N^3}(\phi^2)'''(a_2^i),
\end{aligned}$$

so that

$$\phi_i^2 + \phi_{i+1}^2 - 2\phi_i\phi_{i+1} = \frac{1}{N^2}(\phi_i')^2 + \frac{b_i}{N^3}.$$

Here $b_i = \frac{1}{3!}\left[(\phi^2)'''(a_2^i) - 2\phi_i\phi'''(a_1^i)\right]$ is bounded by smoothness of ϕ (boundedness of ϕ and its derivatives), and equal to 0 for i such that $\frac{i}{N}$ and $\frac{i+1}{N}$ are outside of supp(ϕ).
For the other pairs of j and k we make the following claim:

Claim 6.1. *for* $k \notin \{i-1, i\}$ or $j \notin \{k, k+1\}$ we have

$$\mathcal{L}_{k,k+1}y_iy_j - y_i\mathcal{L}_{k,k+1}y_j - y_j\mathcal{L}_{k,k+1}y_i = 0.$$

For these combinations of j and k we note that for $k \notin \{i-1, i\}$, we have

$$(\partial_k - \partial_{k+1}) y_i = (\partial_k - \partial_{k+1})^2 y_i = 0,$$

so that

$$\mathcal{L}_{k,k+1} y_i = 0,$$

i.e. both $\mathcal{L}_{k,k+1}$ as a whole and the derivatives that compose it produce zero when applied to y_i. This means that applying these operators to $y_i y_j$ yields the following equations.

$$\begin{aligned}
&(\partial_k - \partial_{k+1}) y_i y_j = y_i (\partial_k - \partial_{k+1}) y_i, \\
&(\partial_k - \partial_{k+1})^2 y_i y_j = y_i (\partial_k - \partial_{k+1})^2 y_j, \\
&\mathcal{L}_{k,k+1} y_i y_j = y_i \mathcal{L}_{k,k+1} y_j,
\end{aligned}$$

from which we conclude

$$\mathcal{L}_{k,k+1} y_i y_j - y_i \mathcal{L}_{k,k+1} y_j - y_j \mathcal{L}_{k,k+1} y_i = y_i \mathcal{L}_{k,k+1} y_j - y_i \mathcal{L}_{k,k+1} y_j - 0 = 0.$$

For $j \notin \{k, k+1\}$ we note $(\partial_k - \partial_{k+1}) y_j = 0$ and the proof is similar.
We can now finish the proof of the Lemma. Bringing back the dependency on time, we find

$$\begin{aligned}
[M^N]_t &= \int_0^t \left[\mathcal{L}^{\mathrm{BEP}_N} \langle \Lambda_N(\cdot), \phi \rangle\right](y_s) - 2 \left\langle \nu_s^N, \phi \right\rangle \left[\mathcal{L}^{\mathrm{BEP}_N} \langle \Lambda_N(\cdot), \phi \rangle\right](y_s) ds \\
&= \int_0^t \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \phi_i \phi_j (\mathcal{L}^{\mathrm{BEP}} (y_s)_i (y_s)_j - (y_s)_i \mathcal{L}^{\mathrm{BEP}} (y_s)_j - (y_s)_j \mathcal{L}^{\mathrm{BEP}} (y_s)_i) \\
&= \int_0^t \frac{1}{N^2} \sum_{i=\lfloor M_1 N \rfloor}^{\lceil M_2 N \rceil + 1} 2 (y_s)_i (y_s)_{i+1} \left((\phi_i')^2 + \frac{b_i}{N} \right) ds.
\end{aligned}$$

Where again we can use the fact that this sum has $(M+2)N$ terms. By combining boundedness of $\left((\phi_i')^2 + \frac{b_i}{N}\right)$ and $\mathbb{E}_{\nu_N}^{\mathrm{BEP}_N}[(y_s)_i (y_s)_{i+1}]$ (see Prop. 6.1), we can now finish the proof of the Lemma 6.3.

$$\begin{aligned}
\mathbb{E}_{\nu_N}^{\mathrm{BEP}_N} \left[M^N\right]_t &= \int_0^t \frac{1}{N^2} \sum_{i=\lfloor M_1 N \rfloor}^{\lceil M_2 N \rceil + 1} 2 \left((\phi_i')^2 + \frac{b_i}{N} \right) \mathbb{E}_{\nu_N}^{\mathrm{BEP}_N} \left[(y_s)_i (y_s)_{i+1}\right] ds \\
&\le \int_0^t \frac{1}{N^2} \sum_{i=\lfloor M_1 N \rfloor}^{\lceil M_2 N \rceil + 1} 2 \tilde{C} \left((\phi_i')^2 + \frac{b_i}{N} \right) ds \\
&\le \frac{2t(M+2)\tilde{C} \sup_i \left|(\phi_i')^2 + \frac{b_i}{N}\right|}{N} \to 0 \text{ as } N \to \infty.
\end{aligned}$$

This concludes the proof of the Lemma ☐

Corollary 6.3. *Dynkin martingale $(M_t^N)_{t \ge 0}$ converges in probability to 0 uniformly in $t \in [0, T]$.*

Proof. We have found in Lemma 6.3 that $\mathbb{E}_{\nu_N}^{\mathrm{BEP}_N}[(M_t^N)^2] = \mathbb{E}_{\nu_N}^{\mathrm{BEP}_N}[M^N]_t \to 0$ as $N \to \infty$.
Via Doob's martingale inequality, which states that for a submartingale $(X_t)_{t\geq 0}$ with probability measure $\mathbb{P}$, we have

$$\mathbb{P}\left(\sup_{0\leq s\leq t} X_s \geq C\right) \leq \frac{\mathbb{E}[X_s^+]}{C},$$

we find that for each $\epsilon > 0$ inserting $X_s = (M_s^N)^2$ and $C = \epsilon^2$ yields

$$\mathbb{P}_{\mu_N}^{\mathrm{BEP}_N}\left(\sup_{0\leq s\leq t} |M_t^N| \geq \epsilon\right) \leq \frac{\mathbb{E}_{\nu_N}^{\mathrm{BEP}_N}[M^N]_t}{\epsilon^2} \to 0 \text{ as } N \to \infty.$$

□

6.4.4 Tightness

The main goal of this section is to prove that the sequence of path-space measures $(\mathbb{P}_{\nu_N}^{\mathrm{BEP}_N})_{N\in\mathbb{N}}$ is tight in $D([0,T], P(\Omega))$. Mitoma's theorem tells us that in order to prove this, it suffices to prove tightness of the sequence of empirical measure trajectories $\alpha_N \in D([0,T], M_+)$ paired with appropriate test functions, under $\mathbb{P}_{\nu_N}^{\mathrm{BEP}_N}$ [15]. Tightness of these pairings in $D([0,T],\mathbb{R})$ can be proven with Aldous' tightness criterion [1]. Combining these two important findings we find the following two criteria for tightness of $\{\mathbb{P}_{\nu_N}^{\mathrm{BEP}_N}\}_{N\in\mathbb{N}}$, in line with the literature (see lemma 8.5, page 121 of [18]).
After showing tightness of $\{\mathbb{P}_{\nu_N}^{\mathrm{BEP}_N}\}_{N\in\mathbb{N}}$, we will be able to conclude that $\Lambda_N(y^{(N)})$ converges.

Lemma 6.4. *The sequence of path-space measures $\{\mathbb{P}_{\nu_N}^{\mathrm{BEP}_N}\}_{N\in\mathbb{N}}$ satisfies the following two criteria, making it tight on $D([0,T], M_+)$:*

1. ***Compact Containment:*** *there exists a compact set of measures B such that for every $t > 0$ and $\epsilon > 0$,*
$$\lim_{N\to\infty} \mathbb{P}_{\nu_N}^{\mathrm{BEP}_N}\left(\Lambda_N(y_t^{(N)}) \in B\right) > 1 - \epsilon.$$
2. ***Modulus of Continuity:*** *We define metric d_M, which is consistent with the weak topology, to quantify the distance between metrics via*
$$d_M(\mu,\nu) := \sum_{j=0}^{\infty} \frac{\left|\int \phi^j\, d\mu - \int \phi^j\, d\nu\right|}{1 + \left|\int \phi^j\, d\mu - \int \phi^j\, d\nu\right|} 2^{-j}, \tag{54}$$
where $\phi^j \in C_c^\infty(\mathbb{R})$. Then for every $\epsilon > 0$ and $0 < T < \infty$ there is a $\delta > 0$ such that
$$\limsup_{N\to\infty} \mathbb{P}_{\nu_N}^{\mathrm{BEP}_N}\left(\sup_{\substack{|s-t|\leq\delta \\ s,t\in[0,T]}} d_M(\Lambda_N(y_t^{(N)}), \Lambda_N(y_s^{(N)})) \geq \epsilon\right) \leq \epsilon.$$

Proof.

1. **Compact Containment:** Let

$$B := \{\mu \in M : \mu([-b,b]) < (2b+1)b \ \forall b \in \mathbb{R}_+\}.$$

Then we know $\bar{K}$ is compact because for every compact $S \subset \mathbb{R}^{\mathbb{Z}}$ we have $\sup_{\mu \in \bar{K}}(\mu(S)) < \infty$. Furthermore, using Proposition 6.1, we find for every $t > 0$ and $\epsilon > 0$ the following.

$$\begin{aligned}
\mathbb{P}^{\mathrm{BEP}_N}_{\nu_N}(\Lambda_N(y_t) \notin B) &= \mathbb{P}^{\mathrm{BEP}_N}_{\nu_N}(\Lambda_N(y_t)([-b,b]) \geq (2b+1)b \ \forall b \in \mathbb{R}_+) \\
&\leq \inf_{b \in \mathbb{R}_+} \mathbb{P}^{\mathrm{BEP}_N}_{\nu_N}\left(\frac{1}{N}\sum_{i=\lfloor bN \rfloor}^{\lceil bN \rceil}(y_t)_i \geq (2b+1)b\right) \\
&\leq \inf_{b \in \mathbb{R}_+} \frac{1}{(2b+1)b}\mathbb{E}^{\mathrm{BEP}}_{\nu_N}\left[\frac{1}{N}\sum_{i=\lfloor bN \rfloor}^{\lceil bN \rceil}(y_{tN^2})_i\right] \\
&\leq \inf_{b \in \mathbb{R}_+} \frac{C}{b} \\
&\leq \epsilon.
\end{aligned}$$

2. **Modulus of continuity:** We have

$$\begin{aligned}
\sup_{\substack{t,s \in [0,T] \\ |t-s| \leq \delta}} d_M(\Lambda_N(y_t), \Lambda_N(y_s)) &= \sum_{j=0}^{\infty} \sup_{\substack{t,s \in [0,T] \\ |t-s| \leq \delta}} 2^{-j} \frac{\left|\left\langle \Lambda_N(y_t) - \Lambda_N(y_s), \phi^j \right\rangle\right|}{1 + |\langle \Lambda_N(y_t) - \Lambda_N(y_s), \phi^j \rangle|} \\
&\leq 2^{-m} + \sum_{j=0}^{m} \sup_{\substack{t,s \in [0,T] \\ |t-s| \leq \delta}} 2^{-j} \frac{\left|\left\langle \Lambda_N(y_t) - \Lambda_N(y_s), \phi^j \right\rangle\right|}{1 + |\langle \Lambda_N(y_t) - \Lambda_N(y_s), \phi^j \rangle|} \\
&\leq 2^{-m} + \sum_{j=0}^{m} \sup_{\substack{t,s \in [0,T] \\ |t-s| \leq \delta}} 2^{-j} \left|\left\langle \Lambda_N(y_t) - \Lambda_N(y_s), \phi^j \right\rangle\right| \\
&\leq 2^{-m} + \sum_{j=0}^{m} \sup_{\substack{t,s \in [0,T] \\ |t-s| \leq \delta}} \left|\left\langle \Lambda_N(y_t) - \Lambda_N(y_s), \phi^j \right\rangle\right|.
\end{aligned}$$

By using the Dynkin Martingale from (51) we can bound the terms in this sum. From the definition of M_t^N given in (51) it follows that

$$\left\langle \Lambda_N(y_t) - \Lambda_N(y_s), \phi^j \right\rangle = M_t^N - M_s^N - \int_s^t \left[\mathcal{L}^{\mathrm{BEP}_N}\left\langle \Lambda_N(\cdot), \phi^j \right\rangle\right](y_u)du.$$

Therefore taking the absolute value, supremum with respect to s and t, and then expectation yields the following.

$$\mathbb{E}_{\nu_N}^{\mathrm{BEP}_N}\left[\sup_{\substack{t,s\in[0,T]\\|t-s|\leq\delta}}\left|\left\langle\Lambda_N(y_t)-\Lambda_N(y_s),\phi^j\right\rangle\right|\right]\leq\mathbb{E}_{\nu_N}^{\mathrm{BEP}_N}\left[\sup_{\substack{t,s\in[0,T]\\|t-s|\leq\delta}}\left|M_t^N-M_s^N\right|\right]$$
$$+\mathbb{E}_{\nu_N}^{\mathrm{BEP}_N}\left[\sup_{\substack{t,s\in[0,T]\\|t-s|\leq\delta}}\int_s^t\left[\mathcal{L}^{\mathrm{BEP}_N}\left\langle\Lambda_N(\cdot),\phi^j\right\rangle\right](y_u)\right].$$

This expression is potentially confusing because this expectation is with respect to the accelerated (macroscopic) measure with subscript N. If we would take the expectation with respect to this measure of e.g. $(y_t)_i$, which is defined in microscopic scale, we would have to first apply the transformation to macroscopic scale via $\mathbb{E}_{\nu_N}^{\mathrm{BEP}_N}[(y_t)_i]=\mathbb{E}_{\nu_N}^{\mathrm{BEP}}[(y_{tN^2})_i]$.
However, the expression we have here follows from our definition of Dynkin Martingale M_t^N, which is defined with respect to the accelerated measure and should be interpreted on macroscopic level. Most notably, this mans that when taking the expectation of the second term of the RHS of the previous equation, we should not rescale the bounds of the integral to microscopic scale.
To make our calculations relatively short we denote

$$\Xi\phi_i^j:=\partial_{xx}\phi_i^j+\frac{1}{N^2}\left(\phi^{(4)}(x_1^i)+\phi^{(4)}(x_2^i)\right).$$

By smoothness of ϕ^j there must be a constant $K>0$ such that $\Xi\phi_i^j<K$ for every $i\in\mathbb{Z}$ and $j\in\mathbb{N}$. Using the fact ϕ^j has compact support to assume $\mathrm{supp}(\phi^j)\subseteq[-MN,MN]$, we find

$$\begin{aligned}&\mathbb{E}_{\nu_N}^{\mathrm{BEP}_N}\left[\sup_{\substack{t,s\in[0,T]\\|t-s|\leq\delta}}\left|\int_s^t\mathcal{L}^{\mathrm{BEP}_N}\left\langle\Lambda_N(\cdot),\phi^j\right\rangle(y_u)du\right|\right]\\
&=\mathbb{E}_{\nu_N}^{\mathrm{BEP}_N}\left[\sup_{\substack{t,s\in[0,T]\\|t-s|\leq\delta}}\left|\int_s^t 2k\left\langle\Lambda_N(\cdot),\partial_{xx}\phi^j\right\rangle(y_u)+a_u^N du\right|\right]\\
&\leq\mathbb{E}_{\nu_N}^{\mathrm{BEP}_N}\left[\sup_{\substack{t,s\in[0,T]\\|t-s|\leq\delta}}\int_s^t 2k\frac{1}{N}\sum_{i=-\lfloor MN\rfloor}^{\lceil MN\rceil}\left|(y_u)_i\Xi\phi_i^j\right|du\right]\\
&=\mathbb{E}_{\nu_N}^{\mathrm{BEP}_N}\left[2k\sup_{\substack{t,s\in[0,T]\\|t-s|\leq\delta}}\frac{1}{N}\sum_{i=-\lfloor MN\rfloor}^{\lceil MN\rceil}\int_s^t\left|(y_u)_i\Xi\phi_i^j\right|du\right],\end{aligned}$$

Where we used the Fubini-Tonelli theorem to interchange summation and integration between lines 3 and 4. Continuing by applying Cauchy-Swarz, we find

$$\begin{aligned}
&\mathbb{E}_{\nu_N}^{\mathrm{BEP}_N}\left[2k \sup_{\substack{t,s\in[0,T]\\|t-s|\le\delta}} \frac{1}{N}\sum_{i=-\lfloor MN\rfloor}^{\lceil MN\rceil}\int_s^t \left|(y_u)_i \Xi\phi_i^j\right| du\right]\\
&\le \mathbb{E}_{\nu_N}^{\mathrm{BEP}_N}\left[2k \sup_{\substack{t,s\in[0,T]\\|t-s|\le\delta}} \frac{1}{N}\sum_{i=-\lfloor MN\rfloor}^{\lceil MN\rceil}\left(\int_s^t (y_u)_i^2 du\right)^{1/2}\left(\int_s^t (\Xi\phi_i^j)^2 du\right)^{1/2}\right]\\
&\le \mathbb{E}_{\nu_N}^{\mathrm{BEP}_N}\left[2k \sup_{\substack{t,s\in[0,T]\\|t-s|\le\delta}} \frac{1}{N}\sum_{i=-\lfloor MN\rfloor}^{\lceil MN\rceil}\left(\int_s^t (y_u)_i^2 du\right)^{1/2}\sqrt{\delta K^2}\right]\\
&\le 2kK\sqrt{\delta}\frac{1}{N}\sum_{i=-\lfloor MN\rfloor}^{\lceil MN\rceil}\left(\mathbb{E}_{\nu_N}^{\mathrm{BEP}_N}\left[\sup_{\substack{t,s\in[0,T]\\|t-s|\le\delta}}\int_s^t (y_u)_i^2 du\right]\right)^{1/2}\\
&\le 2kK\sqrt{\delta}\frac{1}{N}\sum_{i=-\lfloor MN\rfloor}^{\lceil MN\rceil}\left(\mathbb{E}_{\nu_N}^{\mathrm{BEP}_N}\left[\int_0^T (y_u)_i^2 du\right]\right)^{1/2}\\
&\le 2kK\sqrt{\delta}\frac{1}{N}\sum_{i=-\lfloor MN\rfloor}^{\lceil MN\rceil}\left(\int_0^T \mathbb{E}_{\nu_N}^{\mathrm{BEP}_N}\left[(y_u)_i^2\right] du\right)^{1/2}\\
&\le 2kK\sqrt{\delta}\frac{1}{N}\sum_{i=-\lfloor MN\rfloor}^{\lceil MN\rceil}\sqrt{TC^2}\\
&\le 2kKC(2M+1)\sqrt{\delta T},
\end{aligned}$$

where we used boundedness of $\Xi\phi_i^j$ by K (see proof of Lemma 6.2) between lines 5 and 6, Jensen's inequality on the concave square-root function between lines 6 and 7 and boundedness of $\mathbb{E}_{\nu_N}^{\mathrm{BEP}_N}[(y_u)_i^2]$ by C from Proposition 6.1 in the last step.
Furthermore we can use the fact that a variation process is nondecreasing to bound

$$\mathbb{E}_{\nu_N}^{\mathrm{BEP}_N}\left[\sup_{\substack{t,s\in[0,T]\\|t-s|\le\delta}}|M_t^N - M_s^N|\right] \le \mathbb{E}_{\nu_N}^{\mathrm{BEP}_N}[(M_T^N)^2],$$

which we have seen converges to 0 as we increase N for finite T. This means that we can conclude the proof via

$$\begin{aligned}
\mathbb{P}_{\nu_N}^{\mathrm{BEP}_N}\left(\sup_{\substack{|s-t|\le\delta\\s,t\in[0,T]}} d_M(\Lambda_N(y_t),\Lambda_N(y_s))\ge\epsilon\right) &\le \mathbb{E}_{\nu_N}^{\mathrm{BEP}_N}\left[\sup_{\substack{|s-t|\le\delta\\s,t\in[0,T]}} d_M(\Lambda_N(y_t),\Lambda_N(y_s))\right]\epsilon^{-1}\\
&\le \frac{2^{-m}+2kKC\sqrt{\delta T}+\mathbb{E}_{\nu_N}^{\mathrm{BEP}_N}[(M_T^N)^2]}{\epsilon},
\end{aligned}$$

which can be made smaller than ϵ as we take $N \to \infty$ and let arbitrary variables m and δ go however far needed in the directions $m \to \infty$ and $\delta \downarrow 0$.

$\square$

As described in Section 6.2, Lemma 6.4 tells via Prokhorov's theorem [7] that $\left\{\mathbb{P}^{\mathrm{BEP}_N}_{\nu_N}\right\}_{N\in\mathbb{N}}$ has convergent subsequences. This means that $\left\{\Lambda_N(y_t^{(N)})\right\}_{N\in\mathbb{N}}$ has subsequences that converge in probability, to possibly different limits. Since from the earlier sections of this chapter we know such limiting trajectories $\alpha : [0,T] \to M_+$ satisfy

$$\langle \alpha(t), \phi\rangle - \langle \alpha(0), \phi\rangle = \int_0^t \langle \alpha_s, \partial_{xx}\phi\rangle \, ds,$$

we find convergence of $\left\{\Lambda_N(y_t^{(N)})\right\}_{N\in\mathbb{N}}$ by uniqueness of the solution to the heat equation. This result is formalized in the next Lemma. Since the details of the proof of this Lemma are quite technical and very similar to an established proof for a different IPS, we leave the full proof outside the scope of this thesis and refer to the relevant literature instead.

Lemma 6.5. $\lim_{N\to\infty} \left(\Lambda_N(y_t^{(N)})\right)_{t\geq 0}$ *exists in* $D([0,T], M_+)$ *equipped with Skorokhod topology in probability with respect to* $\mathbb{P}^{\mathrm{BEP}_N}_{\nu_N}$.

Proof. The proof of this is analogous to the proof for the Symmetric Exclusion Process, presented in section 8.1 of [18]. The main idea is that because of tightness of $\left\{\mathbb{P}^{\mathrm{BEP}_N}_{\nu_N}\right\}_{N\in\mathbb{N}}$, which we have proven in Lemma 6.4, we only need to show that the limit points of $\left\{\left(\Lambda_N(y_t^{(N)})\right)_{t\geq 0}\right\}_{N\in\mathbb{N}}$ are concurrent. Since the details of this are quite technical and very similar to pages 123-126 of [18], we refer to this book for further elaboration. $\square$

6.4.5 Conclusion

The findings in previous paragraphs in combination with the definition of $(M^N)_{t\geq 0}$ tell us that

$$\left\langle \Lambda_N(y_t^{(N)}), \phi\right\rangle - \left\langle \Lambda_N(y^{(N)}), \phi\right\rangle - \int_0^T \left\langle \Lambda_N(y_s^{(N)}), \partial_{xx}\phi\right\rangle ds \to 0,$$

in probability, uniform in $t \in [0,T]$.
Since per definition of ρ_s

$$\left\langle \Lambda_N(y_s^{(N)}), \phi\right\rangle \to \int_{\mathbb{R}} \rho_s(x)\phi(x)dx,$$

in probability, this means that

$$\left\langle \Lambda_N(y_t^{(N)}), \phi \right\rangle - \left\langle \Lambda_N(y^{(N)}), \phi \right\rangle - \int_0^t \left\langle \Lambda_N(y_s^{(N)}), \partial_{xx}\phi \right\rangle ds$$

$$\to \int_{\mathbb{R}} \rho_t(x)\phi(x) - \rho_0(x)\phi(x)dx - \int_0^t \int_{\mathbb{R}} \rho_s(x)\partial_{xx}\phi(x)dxds$$

$$= \int_{\mathbb{R}} \int_0^t \partial_s\rho_s(x)\phi(x)dsdx - \int_{\mathbb{R}} \int_0^t \phi(x)\partial_{xx}\rho_s(x)dsdx$$

$$= \int_{\mathbb{R}} \int_0^t \phi(x)\left(\partial_s - 2k\partial_{xx}\right)\rho_s(x)dxds.$$

From these two findings we can conclude that ρ_t is a weak solution to the heat equation with initial condition $\rho_0 = \rho$. By uniqueness of the solution to the heat equation ρ_t must therefore also be the strong solution to the heat equation. Thus $\left\{ \left(\Lambda_N(y_t^{(N)})\right)_{t\geq 0} \right\}_{N\in\mathbb{N}}$ converges weakly in probability to the trajectory solving the heat equation $(\rho_t(dx))_{t\geq 0}$ in $D([0,T], M_+)$ with the Skorokhod toplogy. Since the paths $\left(\Lambda_N(y_t^{(N)})\right)_{t\geq 0}$ can be shown to be continuous through the modulus of continuity from Lemma 6.4, convergence in the Skorokhod topology is equivalent to the uniform convergence in Theorem 6.1. $\square$

7 Hydrodynamic Limit of ABEP in infinite Volume with Finite Energy

7.1 Setting and main result

The setting of the hydrodynamic limit op the ABEP is very similar to that of the BEP from the previous Chapter. We start with random initial configurations of ABEP(σ,k) dependent on scale-parameter N, with initial convergence of their empirical measures.
As described in Chapter 6.1, the desired rescaling of space by N and time by N^2 becomes problematic in the presence of asymmetry. This is because as we scale time by a factor N^2, the expected net current will increase with a factor N^2 as well. Because of this, we let asymmetry parameter $\sigma = \sigma^{(N)}$ depend on N in such a way that it is $O(N^{-1})$, so that the expected flow resulting from asymmetry remains of the same order as we move from micro- to macro-scale. As $\sigma^{(N)}$ is vanishing in N, we cal this process ABEP($\sigma^{(N)}, k$) weakly asymmetric.
Let $x^{(N)}$ be a sequence of ABEP($\sigma^{(N)}$,k) configurations in infinite volume, i.e. for every $N \in \mathbb{N}$, $x^{(N)} \in \Omega_f$.
The assymmetry parameter $\sigma = \sigma^{(N)}$ is given through $\sigma^{(N)} = \frac{\gamma}{2N}$ where $\gamma > 0$ is a constant.
For each N let measure $\mu_N : \mathcal{B}(\Omega_f) \to [0,1]$ denote the distribution of $y^{(N)}$ and $\mathbb{E}^{\mu_N}$ the expectation with respect to this measure.
In this setting we put the same assumptions about the initial configurations as in the case for BEP, i.e. boundedness of energy at a given site and convergence of the empirical measure.

Assumption 7.1. *The amount of energy at each site initially is bounded in expectation via*

$$\sup_{i\in\mathbb{Z}, N\in\mathbb{N}} \mathbb{E}^{\mu_N}\left[\left(x_i^{(N)}\right)^2\right] \leq C \text{ for some } C > 0.$$

Assumption 7.2. $\Lambda_N(x^{(N)})$ *converges weakly in probability to a measure with with density* $\chi : \mathbb{R} \to \mathbb{R}_+$, *i.e. for every* $\phi \in C_c^\infty(\mathbb{R})$ *and* $\epsilon > 0$,

$$\mu_N\left(\left|\left\langle \Lambda_N(x_i^{(N)}), \phi\right\rangle - \int \chi(x)\phi(x)dx\right| > \epsilon\right) \to 0. \tag{55}$$

Furthermore we have convergence of the total measure

$$\mu_N\left(\left|\Lambda_N(x^{(N)})(\mathbb{R}) - \int_{-\infty}^{\infty} \chi(x)dx\right| > \epsilon\right) \to 0. \tag{56}$$

(56) seems to follow roughly from (55), but is required to be explicitly assumed due to the constraint of ϕ in (55) to functions with compact support, which prevents us from taking $\phi \equiv 1$ in order to arrive at (56). As discussed when we first defined the ABEP in Definition 5.1, for fixed $\sigma > 0$ we have to limit the state space of the ABEP to Ω_f due to problems resulting from the current that arises. Here we are not dealing with a single configuration and σ, but with a sequence of both. Since the asymmetry parameter $\sigma^{(N)}$ is decreasing in N, the problematic nature of this current becomes smaller as N increases, which results in the following assumption being sufficient for the ABEP($\sigma^{(N)}, k$) to be well-defined as we increase N indefinitely.

Assumption 7.3. *for every $N \in \mathbb{N}$ the total amount of energy is bounded, i.e.*

$$\frac{1}{N}\sum_{i=-\infty}^{\infty} x_i^{(N)} \leq E < \infty.$$

Assumption 7.3 ensures that for every fixed N, $x^{(N)} \in \Omega_f$, but as we take the limit $N \to \infty$, $\sum\limits_{i=-\infty}^{\infty} x_i^{(N)}$ does not necessarily remain bounded. In fact, we can see from upcoming Proposition 7.1 that Assumption 7.2 implies that $\lim\limits_{N\to\infty} \sum\limits_{i=-\infty}^{\infty} x_i^{(N)} = \infty$ for every χ except $\chi \equiv 0$.
In this setting, we apply the same rescaling of the space- and time-dimension as we did for BEP in order to find the hydrodynamic limit. We claim that as a result of the asymmetry, as described by asymmetry parameter $\sigma^{(N)} = \frac{\gamma}{2N}$, we gain an additional nonlinear term in the resulting PDE. We thus end up with a hydrodynamic limit density, which solves the viscous Burgers' equation.

Theorem 7.1. *For each N let $\mathbb{P}_{\mu_N}^{\mathrm{ABEP}_N}$ denote the path-space measure of the accelerated ABEP($\sigma^{(N)}, k$) with generator $\mathcal{L}^{\mathrm{ABEP}_N} = N^2\mathcal{L}^{\mathrm{ABEP}}$, with evolved configuration $x_t^{(N)}$ starting out at $x^{(N)}$ with initial distribution μ_N.*
We have for every $\phi \in C_c^\infty(\mathbb{R})$ and $\delta > 0$:

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N}^{\mathrm{ABEP}_N}\left(\sup_{t\in[0,T]}\left|\left\langle \Lambda_N(x_t^{(N)}), \phi\right\rangle - \int \chi(x)\phi(x)dx\right| > \delta\right) = 0,$$

where χ_t is the unique solution to the viscous Burgers' equation, i.e.

$$\frac{\partial \chi_t(x)}{\partial t} = 2k\frac{\partial^2\chi_t(x)}{\partial x^2} + 2k\gamma\frac{\partial \chi_t(x)^2}{\partial x}, \tag{57}$$

with initial condition $\chi_0 = \chi$.

7.2 Proof

7.2.1 Transformation from the ABEP to the BEP

A challenge for this proof is the nonlinearity that arises as a result of the asymmetry. If we were to proceed in the same way as we did for BEP, then we would find that our expression of $\left[\mathcal{L}^{\mathrm{ABEP}}\left\langle\Lambda_N(\cdot), \phi\right\rangle\right](x)$ would contain quadratic terms of x_i, which would prevent us from finding results analogous to Lemma 6.2.
A tool that is therefore essential to this proof is the transformation from the ABEP to the BEP via map g in (33). This map allows us to transform the ABEP x_t into the BEP, allowing us to use the result from the previous Chapter to find the hydrodynamic limit of the ABEP. The main challenges will then be to show that the hydrodynamic limit of the non-transformed ABEP exists, and to figure out what the hydrodynamic limit of the transformed ABEP tells us about it.
Let $y^{(N)}(x^{(N)}) := (y_i^{(N)}(x^{(N)}), i \in \mathbb{Z})$ be a tranformation of ABEP-configuration $x^{(N)}$ with $y_i : \mathbb{R}_+^{\mathbb{Z}} \to \mathbb{R}_+^{\mathbb{Z}}$ defined through map g from (33), i.e.

$$y_i^{(N)}(x) := \frac{e^{-2\sigma^{(N)}E_{i+1}(x)} - e^{-2\sigma^{(N)}E_i(x)}}{2\sigma^{(N)}}. \tag{58}$$

Corollary 7.1. *For each $N \in \mathbb{N}$ we have that if x_t is the* ABEP*($\sigma^{(N)}$, k) starting out at x, then $y^{(N)}(x_t)$ is the BEP(k) starting out from $y^{(N)}(x)$.*

Proof. Follows from Theorem 5.2. $\square$

Using the fact that $y(x_t)$ is the BEP(k) allows us to use the hydrodynamic limit that we found in Theorem 6.1 for the BEP here, in order to find the hydrodynamic limit of the ABEP. The proof is structured in the following way.
In Section 7.2.2, we will show how the convergence of the initial empirical measure $\Lambda_N(x^{(N)})$ to a limiting measure in Assumption 7.2 implies convergence of $\Lambda_N(y(x^{(N)})$, and we will show how the densities of these respective measures relate to each other. In Section 7.2.3 we will apply Theorem 6.1 to $y(x_t^{(N)})$ in order to show that as x_t is evolved as the ABEP, the trajectory $(\Lambda_N(y(x_t^{(N)})))_{t\geq 0}$ converges to the solution to the heat equation. Then in Section 7.2.4 we will use tightness to prove the existence of a similar limit of the trajectory $(\Lambda_N(x_t^{(N)}))_{t\geq 0}$. Then finally in Section 7.2.5 we will use the relation between the densities of the limiting measures in order to derive (57).

7.2.2 Relation between macroscopic density of y and x

In this section we will show how from the convergence of the initial empirical measure of the ABEP, $\Lambda_N(x^{(N)})$ (Assumption 7.2), we find that the initial empirical measure of the BEP we constructed via (58) converges. Furthermore, we will give the relation between the densities of the respective limiting measures. The central claim of this section is that there exists a limiting measure with density $\rho : \mathbb{R} \to \mathbb{R}_+$ such that

$$\Lambda_N(x^{(N)})(dx) \to \rho(x)dx,$$

where convergence is weak convergence in probability, and the relation between ρ and χ is given through

$$\rho(x) = \chi(x)e^{-\gamma \int_x^\infty \chi(y)dy}. \tag{59}$$

We will prove this is in two parts. First we show convergence of $\sigma^{(N)}E_{i(N)}$ as $N \to \infty$, where $i = i(N)$ is chosen in such a way that the site of interest is dependent on N in such a way that it corresponds to the same macro-coordinate as we take $N \to \infty$. Next we express $y_i^{(N)}$ in $x^{(N)}$ in such a way that taking the limit of $\langle \Lambda_N(y(x^{(N)}), \phi \rangle$ yields (59).

Proposition 7.1. *We let $i = i(N)$ depend on N in such a way that $\lim_{N\to\infty} \frac{i(N)}{N}$ exists and set this limit equal to w, so that microscopic site $i(N) \in \mathbb{Z}$ corresponds to macroscopic coordinate $w \in \mathbb{R}$. Then*

$$\lim_{N\to\infty} \sigma^{(N)}E_{i(N)}(x) = \int_w^\infty 2\gamma\chi(x)dx,$$

where the convergence is in probability.

Proof. The statement in the proposition is equivalent to

$$\lim_{N\to\infty} \Lambda_N(x^{(N)}) \left(\left[\frac{i(N)}{N}, \infty \right) \right) = \int_w^\infty \chi(x)dx. \tag{60}$$

This may seem like a trivial statement, as we have by assumption convergence of $\Lambda_N(x^{(N)})$ to a measure with density χ. This is however weak convergence in probability, which means that we only have the following convergence for any $\phi \in C_c^\infty(\mathbb{R})$,

$$\left\langle \Lambda_N(x^{(N)}), \phi \right\rangle \to \int \chi(x)\phi(x)dx.$$

In order to prove the Proposition we would need ϕ to be the step function at $\frac{i(N)}{N}$, which does not lie in $C_c^\infty(\mathbb{R})$.
Luckily the discontinuity only occurs at the left boundary of interval $B = [w, \infty)$, and we have $\int\limits_{\partial B} \chi(x)dx = 0$ (where ∂B denotes the boundary of B). We have not defined $\Lambda_N(x^{(N)})$ and its limit in N as probability measures, but since $\Lambda_N(x^{(N)})(\mathbb{R})$ is finite for all N and so is $\lim\limits_{N\to\infty} \Lambda_N(x^{(N)})(\mathbb{R})$ as a result of Assumption 7.3 of finite energy and Assumption 7.2 of convergence of the total measure, we can turn them into probability measures by rescaling them so that these values are 1. To this end, let

$$C_N = \Lambda_N(x^{(N)})(\mathbb{R}),$$

denote the total measure of $\Lambda_N(x^{(N)})$, and note that Assumption 7.2 implies that C_N converges in probability to

$$C = \int\limits_{-\infty}^{\infty} \chi(x)dx.$$

We then derive probability measures from the measures we are working with through

$$m_N = \frac{\Lambda_N(x^{(N)})}{C_N} \qquad m = \frac{\chi(x)dx}{C}$$

Via the continuous mapping theorem we know that weak convergence in probability of $\Lambda_N(x^{(N)})$ to $\chi(x)dx$ and convergence in probability of C_N to C imply that $m_N \to m$ weakly in probability, where now m_N and m are probability measures. The Portmanteau theorem (See e.g. [21]) then states that in this setting for any Borel set A with $m(\partial A) = 0$ we have $m_N(A) \to m(A)$, i.e. weak convergence has implies convergence in distribution.
This allows us to finish the proof. One more thing we need to be careful about is the fact that in the left-hand side of (60) we have $i(N)$ as microscopic location of evaluation and on the right-hand side we have w as macroscopic location. For this we note that $i(N)/N \to w$ implies that for every $\delta > 0$ we have for N large enough

$$m_N([w+\delta,\infty)) \leq m_N\left(\left[\frac{i(N)}{N},\infty\right)\right) \leq m_N([w-\delta,\infty)).$$

Applying the Portmanteau then yields

$$m([w+\delta,\infty)) \leq m_N\left(\left[\frac{i(N)}{N},\infty\right)\right) \leq m([w-\delta,\infty)),$$

which after taking the limits $N \to \infty$ and $\delta \downarrow 0$ yields the convergence

$$m_N\left(\left[\frac{i(N)}{N},\infty\right)\right) \to m([w,\infty]). \tag{61}$$

Finally we return from probability measures m_N and m to the original measures by multiplying the left and right-hand side of (61) with C_N and C respectively, which results in (60) $\square$

Next we will rewrite our map $x \mapsto y(x)$ in such a way that we can use the initial convergence of $\Lambda_N(x^{(N)})$ from assumption 7.2 in order to find ρ as a function of χ.

Lemma 7.2. *The empirical measure of $y(x^{(N)})$ converges to a limiting measure with density equal to the right-hand side of (59), i.e. for each $\phi \in C_c^\infty$ and $\epsilon > 0$ we have*

$$\mu_N\left(\left|\left\langle \Lambda_N\left(y(x^{(N)})\right), \phi\right\rangle - \int_{\mathbb{R}} \chi(x) e^{-\gamma \int_x^\infty \chi(y)dy} \phi(x)dx\right| > \epsilon\right) \to 0.$$

Proof.

$$\begin{aligned} y_i(x)e^{2\sigma E_{i+1}(x)} &= \frac{1 - e^{-2\sigma(E_i(x) - E_{i+1}(x))}}{2\sigma} \\ &= \frac{1 - e^{-2\sigma x_i}}{2\sigma} \\ &= x_i + \sum_{p=1}^\infty (-2\sigma)^p \frac{x_i^{p+1}}{(p+1)!}, \end{aligned}$$

which implies

$$y_i(x) = x_i e^{-2\sigma E_{i+1}(x)} + a_{i,N},$$

where clearly the absolute value of

$$a_{i,N} := -e^{-2\sigma E_{i+1}(x)} \sum_{p=1}^\infty (-2\sigma)^p \frac{x_i^{p+1}}{(p+1)!},$$

converges to 0 in expectation due to the $(-2\sigma)^p$ factor, where we recall that $\sigma = \frac{\gamma}{2N}$ vanishes as $N \to \infty$. Using this expression of $y_i(x_t)$ and with

$$d := \langle \Lambda_N(y(x(N)), g\rangle - \int \chi(x) e^{-\gamma \int_x^\infty \chi(y)dy} \phi(x)dx,$$

we find

$$\begin{aligned} |d| =& \left|\frac{1}{N}\sum_{i=-\infty}^\infty \left(x_i e^{-\frac{\gamma}{N}E_{i+1}(x)} + a_{i,N}\right)\phi_i - \int \chi(x) e^{-\gamma \int_x^\infty \chi(y)dy} \phi(x)dx\right| \\ \leq& \left|\frac{1}{N}\sum_{i=-\infty}^\infty \left(x_i e^{-\frac{\gamma}{N}E_{i+1}(x)} + a_{i,N}\right)\phi_i - \frac{1}{N}\sum_{i=-\infty}^\infty x_i e^{-\frac{\gamma}{N}E_i(x)}\phi_i\right| \\ &+ \left|\frac{1}{N}\sum_{i=-\infty}^\infty x_i e^{-\gamma \int_{i/N}^\infty \chi(y)dy}\phi_i - \frac{1}{N}\sum_{i=-\infty}^\infty x_i e^{-\frac{\gamma}{N}E_i(x)}\phi_i\right| \\ &+ \left|\frac{1}{N}\sum_{i=-\infty}^\infty x_i e^{-\gamma \int_{i/N}^\infty \chi(y)dy}\phi_i - \int \chi(x) e^{-\gamma \int_x^\infty \chi(y)dy}\phi(x)dx\right| \\ :=& |d_1| + |d_2| + |d_3|. \end{aligned}$$

We finish the proof of the lemma by showing that each of these 3 terms converges in probability to 0. The expectation of the first term is

$$\mathbb{E}^{\mu_N}|d_1| = \mathbb{E}^{\mu_N}\left|\frac{1}{N}\sum_{i=-\infty}^{\infty}\left(a_{i,N} + x_i e^{-\frac{\gamma}{N}E_{i+1}(x)} - x_i e^{-\frac{\gamma}{N}E_i(x)}\right)\phi_i\right|$$
$$\leq \frac{1}{N}||\phi||_\infty \sum_{i=\lceil NM_1\rceil}^{\lfloor NM_2\rfloor}\mathbb{E}^{\mu_N}\left|a_{i,N} + \frac{x_i^2\gamma}{N}\right|,$$

which converges to 0, where we used the observation that for $x, y < 0$ we have $|e^x - e^y| \leq |x - y|$. Via Markov's inequality we then find convergence in probability.
For the second term we find

$$\mathbb{E}^{\mu_N}|d_2| \leq \frac{1}{N}\sum_{i=-\infty}^{\infty}\phi_i\mathbb{E}^{\mu_N}\left|x_i\left(e^{-\gamma\int\limits_{i/N}^{\infty}\chi(y)dy} - e^{-\frac{\gamma}{N}E_i(x)}\right)\right|$$
$$\leq \frac{1}{N}\sum_{i=\lceil NM_1\rceil}^{\lfloor NM_2\rfloor} g_i\mathbb{E}^{\mu_N}\left[x_i\left|-\gamma\int\limits_{i/N}^{\infty}\chi(y)dy + \frac{\gamma}{N}E_i(x)\right|\right].$$

Convergence of this expectation to 0 clearly follows from boundedness of $\mathbb{E}^{\mu_N}[x_i]$ and convergence in probability to 0 of $-\gamma\int\limits_{i/N}^{\infty}\chi(y)dy + \frac{\gamma}{N}E_i(x)$, but since x_i and $\frac{\gamma}{N}E_i(x)$ are not independent, this requires some work to prove. We do this by truncating $J_N(x) := -\gamma\int\limits_{i/N}^{\infty}\chi(y)dy + \frac{\gamma}{N}E_i(x)$ into

$$J_N(x) = J_N^{\epsilon}(x) + J_N^{\text{tail}}(x),$$

with

$$J_N^{\epsilon}(x) = J_N(x)1_{\{|J_N(x)|\leq\epsilon\}} \text{ and } J_N^{\text{tail}}(x) = J_N(x)1_{\{|J_N(x)|>\epsilon\}}.$$

Then for every $i \in \mathbb{Z}$ and $N \in \mathbb{N}$,

$$\mathbb{E}^{\mu_N}\left[x_i|J_N^{\epsilon}(x)|\right] \leq \mathbb{E}^{\mu_N}[x_i\sup_x|J_N^{\epsilon}(x)|] = \mathbb{E}^{\mu_N}[x_i\epsilon] \leq \sqrt{C}\epsilon,$$

and

$$\mathbb{E}^{\mu_N}[x_i|J_N^{\text{tail}}|] \leq \mathbb{E}^{\mu_N}\left[x_i|J_N^{\text{tail}}|\,\middle|\,J_N > \epsilon\right]\mu_N(J_N > \epsilon) \leq \mathbb{E}^{\mu_N}[x_i 2E]\mu_N(J_N > \epsilon),$$

where we used boundedness of both terms in $J_N(x)$ by total energy E (assumption 7.3). Thus

$$\lim_{N\to\infty}[x_i|J_N(x)|] \leq \lim_{N\to\infty}\sqrt{C}\epsilon + 2E\sqrt{C}\mu_N(J_N > \epsilon) = \sqrt{C}\epsilon,$$

by convergence in probability of $J(x)$ to 0. Letting $\epsilon \downarrow 0$ we find that $\mathbb{E}^{\mu_N}|d_2|\to 0$, where again we use Markov's inequality to conclude that we have convergence in probability.
Convergence of the third term to 0 follows from weak convergence in probability of $\Lambda_N(x^{(N)})$ by taking $e^{-\gamma\int\limits_{x}^{\infty}\chi(y)dy}\phi(x)$ as test function. □

7.2.3 Application of the hydrodynamic limit of the BEP

Now that we have established weak convergence of $\Lambda_N(y(x^{(N)}))$ to a limiting measure with density

$$\rho(x) = \chi(x)e^{-\gamma \int\limits_x^\infty \chi(y)dy},$$

we wish to know whether convergence of the empirical measure remains as we evolve $x^{(N)}$ as the ABEP, and how the limiting measure evolves accordingly. Since we have constructed $y(x^{(N)})$ to be the BEP when $x^{(N)}$ is evolved as the ABEP, we can apply Theorem 6.1 to $y(x^{(N)})$. Before we can do this, we must ensure that each of the assumptions that we had on $y^{(N)}$ in the setting of Theorem 6.1 is satisfied for $y(x^{(N)})$ in this setting. This means we have to prove boundedness of initial configuration $\mathbb{E}^{\mu_N}[(y_i^{(N)}(x))^2]$. Proving this requires us to relate boundedness of x_i and $y_i(x)$ to each other in such a way that also allows us to show that for every $t > 0$, $\mathbb{E}^{\text{ABEP}}_{\mu_N}[(x_t)_i]$ is bounded, which we will need later.

Lemma 7.3. *The following two bounds hold:*

1.
$$\sup_{N\in\mathbb{N}, i\in\mathbb{Z}} \mathbb{E}^{\mu_N}\left[\left(y_i^{(N)}(x)\right)^2\right] \leq C.$$

2. *There is a constant $V > 0$ such that*
$$\sup_{N\in\mathbb{N}, i\in\mathbb{Z}} \mathbb{E}^{ABEP}_{\mu_N}[(x_t)_i] \leq V.$$

Proof. Note that for each i

$$y_i(x) = \int_{E_{i+1}(x)}^{E_i(x)} e^{-2\sigma z}dz. \tag{62}$$

This means $y_i(x) \leq \int_{E_{i+1}(x)}^{E_i(x)} dz \leq x_i$, so that for 1. we find

$$\mathbb{E}^{\mu_N}\left[\left(y_i^{(N)}(x)\right)^2\right] \leq \mathbb{E}^{\mu_N}\left[\left(x_i^{(N)}\right)^2\right] \leq C.$$

For 2. we note that (62) implies

$$\begin{aligned} y_i(x) &\geq (E_i - E_{i+1})e^{-2\sigma E_i} \\ &= x_i e^{-\frac{\gamma}{N}E_i} \\ &\geq x_i e^{-\gamma E}, \end{aligned}$$

so that applying Proposition 6.1 to the BEP $y(x_t)$ yields

$$\mathbb{E}^{\text{ABEP}}_{\mu_N}[(x_t)_i] \leq e^{\gamma E}\mathbb{E}^{\text{ABEP}}_{\mu_N}[y_i(x_t)] = e^{\gamma E}\mathbb{E}^{\text{BEP}}_{\nu_N}[(y_t)_i] \leq e^{\gamma E}\sqrt{C},$$

where ν_N is the pushforward measure of μ_N by g. $\square$

Next we use the results from the previous Chapter to find an expression of the hydrodynamic limit of the BEP we created by transforming the ABEP in (58).

Corollary 7.2. *The sequence of trajectories* $\left\{\left(\Lambda_N(y(x_t^{(N)}))\right)_{t\geq 0}\right\}_{N\in\mathbb{N}}$ *converges in* $D([0,T],M_+)$ *with respect to* $\mathbb{P}^{\mathrm{ABEP}_N}_{\mu_N}$ *to a trajectory that we denote* $(\rho_t)_{t\geq 0}$*, i.e. for each* $\phi \in C_c^\infty(\mathbb{R})$ *and* $\epsilon > 0$*, we have*

$$\lim_{N\to\infty} \mathbb{P}^{\mathrm{ABEP}_N}_{\mu_N}\left(\sup_{t\in[0,T]}\left|\left\langle \Lambda_N(y_i(x_t^{(N)}))\delta_{i/N},\phi\right\rangle - \int_{\mathbb{R}} \rho_t(x)\phi(x)dx\right| > \epsilon\right) = 0. \tag{63}$$

This trajectory $(\rho_t)_{t\geq 0}$ *the unique solution to the heat equation*

$$\frac{\partial \rho_t(x)}{\partial t} = 2k\frac{\partial^2 \rho_t(x)}{\partial x^2},$$

with initial condition $\rho_0 = \rho = \chi(x)e^{-\gamma\int_x^\infty \chi(y)dy}$.

Proof. Corollary 7.1, Lemma 7.2 and Lemma 7.3 imply that we are in the same situation as in the setting of the hydrodynamic limit of the BEP. Thus this result follows from Theorem 6.1. □

7.2.4 Tightness

Now that we have shown that for the BEP $y(x_t^{(N)})$ the hydrodynamic limit is the solution to the heat equation, we want to find the hydrodynamic limit of the ABEP $x_t^{(N)}$. In this section we will prove that such a limit exists, by showing that $\left\{\left(\Lambda_N(x_t^{(N)})\right)_{t\geq 0}\right\}_{N\in\mathbb{N}}$ is tight. This will be done in a similar manner to Lemma 6.4 for the BEP.

Lemma 7.4. *For* $\phi \in C_c^\infty(\mathbb{R})$ *an arbitrary test function, the following two criteria hold, making the sequence* $\{\mathbb{P}^{\mathrm{ABEP}_N}_{\mu_N}\}_{N\in\mathbb{N}}$ *tight.*

1. ***Compact Containment:*** *For every* $t > 0$ *and* $\epsilon > 0$ *there exists a compact set of measures* B *such that*
$$\lim_{N\to\infty} \mathbb{P}^{\mathrm{ABEP}_N}_{\mu_N}\left(\Lambda_N(x_t^{(N)}) \in B\right) > 1-\epsilon.$$

2. ***Modulus of Continuity:*** *Let* d_M *be the metric from* (54)*. Then for each* $\epsilon > 0$ *and* $T < \infty$ *there is a* $\delta > 0$ *such that*
$$\limsup_{N\to\infty} \mathbb{P}^{\mathrm{ABEP}_N}_{\mu_N}\left(\sup_{\substack{|s-t|\leq\delta\\ s,t\in[0,T]}} d_M(\Lambda_N(x_t^{(N)}),\Lambda_N(x_s^{(N)})) \geq \epsilon\right) \leq \epsilon.$$

Proof.

1. **Compact Containment:** We can use the same set of measures as for BEP, and since $\mathbb{E}^{ABEP_N}_{\mu_N}[(x_t)_i]$ has been shown to be bounded in Lemma 7.3, the proof works in the same way. Let
$$B := \{\mu \in M : \mu\left([-b,b]\right) < (2b+1)b\ \forall b \in \mathbb{R}_+.$$

Then $\bar{B}$ is compact and for every $t > 0$ and $\epsilon > 0$, we have

$$\begin{aligned}
\mathbb{P}_{\mu_N}^{\mathrm{ABEP}_N}\left(\Lambda_N(x_t^{(N)}) \notin B\right) &\leq \inf_{b\in\mathbb{R}_+} \mathbb{P}_{\mu_N}^{\mathrm{ABEP}_N}\left(\frac{1}{N}\sum_{i=\lfloor bN\rfloor}^{\lceil bN\rceil}(x_t)_i \geq (2b+1)b\right) \\
&\leq \inf_{b\in\mathbb{R}_+} \frac{1}{(2b+1)b}\mathbb{E}_{\mu_N}^{ABEP_N}\left[\frac{1}{N}\sum_{i=\lfloor bN\rfloor}^{\lceil bN\rceil}(x_{tN^2})_i\right] \\
&\leq \inf_{b\in\mathbb{R}_+} \frac{C}{b} \\
&\leq \epsilon.
\end{aligned}$$

2. **Modulus of Continuity:** For this part of the proof we also follow the same line of arguments as for BEP, but this requires us to define a new Dynkin Martingale to capture the deviation from expected under ABEP dynamics.

   Definition 7.1. *Let $\left(\hat{M}_t^N\right)_{t\geq 0}$ be the Dynkin martingale with respect to $\mathbb{P}_{\mu_N}^{\mathrm{ABEP}_N}$ and function $\langle\Lambda_N(\cdot),\phi\rangle$, i.e.*

$$\begin{aligned}
\hat{M}_t^N &= \left\langle\Lambda_N(x_t^{(N)}),\phi\right\rangle - \left\langle\Lambda_N(x^{(N)}),\phi\right\rangle \\
&\quad - \int_0^t \mathcal{L}^{\mathrm{ABEP}_N}\langle\Lambda_N(\cdot),\phi\rangle^2(x_s^{(N)}) - \left\langle\Lambda_N(x_s^{(N)},\phi\right\rangle\mathcal{L}^{\mathrm{ABEP}_N}\langle\Lambda_N(\cdot),\phi\rangle(x_s^{(N)})ds.
\end{aligned}$$

   Note that

$$\begin{aligned}
&\mathbb{E}_{\mu_N}^{ABEP_N}\left[\sup_{\substack{0\leq s,t\leq T\\|s-t|\leq\delta}}\left|\left\langle\Lambda_N(x_t^{(N)})-\Lambda_N(x_s^{(N)}),\phi^j\right\rangle\right|\right] \\
&\leq \mathbb{E}_{\mu_N}^{ABEP_N}\left[\sup_{\substack{0\leq s,t\leq T\\|s-t|\leq\delta}}\left|\hat{M}_t^N-\hat{M}_s^N\right|\right] + \mathbb{E}_{\mu_N}^{ABEP_N}\left[\sup_{\substack{0\leq s,t\leq T\\|s-t|\leq\delta}}\left|\int_s^t\mathcal{L}^{\mathrm{ABEP}_N}\left\langle\Lambda_N(\cdot),g^j\right\rangle(x_u)du\right|\right].
\end{aligned}$$

   We now have to show that these terms can be made arbitrarily small through our choice of δ. We find the following:

$$\begin{aligned}
[\hat{M}^N]_t &= \int_0^t \mathcal{L}^{\mathrm{ABEP}_N}\left\langle\Lambda_N(\cdot),\phi^j\right\rangle^2(x_s) - \left\langle\Lambda_N(x_s^{(N)}),\phi^j\right\rangle\mathcal{L}^{\mathrm{ABEP}_N}\left\langle\Lambda_N(\cdot),\phi^j\right\rangle(x_s)ds \\
&= \int_0^t\sum_{i=-\infty}^{\infty}\frac{1}{4\sigma^2}(1-e^{-2\sigma(x_s)_i})(e^{2\sigma(x_s)_{i+1}}-1)(\phi_i^2+\phi_{i+1}^2-2\phi_i\phi_{i+1})du \\
&= \int_0^t\frac{1}{N^2}\sum_{i=-\infty}^{\infty}\left((x_s)_i(x_s)_{i+1}\partial_{xx}\phi_i+O(N^{-2})\right)du.
\end{aligned}$$

Which converges to 0 in expectation and

$$\begin{aligned}
&\int_s^t \mathcal{L}^{\mathrm{ABEP}_N} \left\langle \Lambda_N(\cdot), \phi^j \right\rangle (x_u) du \\
&= \int_0^t N \sum_{i=-\infty}^{\infty} \frac{1}{2\sigma} \left[-(1-e^{-2\sigma x_i})(e^{2\sigma x_{i+1}}-1) + (1-e^{-2\sigma x_{i-1}})(e^{2\sigma x_i}-1) \right] \phi_i \\
&+ \frac{2k}{2\sigma}(e^{-2\sigma x_i} + e^{2\sigma x_{i+1}} - e^{-2\sigma x_{i-1}} - e^{2\sigma x_i})\phi_i ds \\
&= \int_s^t \sum_{i=-\infty}^{\infty} N[2\sigma x_i(x_{i-1}-x_{i+1})\phi_i + 2k\phi_i(x_{i+1}-2x_i+x_{i-1})+ \\
&+ 2k\sigma(-x_{i+1}^2 - x_{i-1}^2) + h_i]\phi_i du \\
&= \int_s^t \sum_{i=-\infty}^{\infty} \left(\frac{4\gamma}{N} x_i x_{i+1} \partial_x \phi_i - \frac{2k\gamma}{N} x_i^2 \partial_x \phi_i + \frac{2k}{N} x_i \partial_{xx}\phi_i + \hat{h}_i \phi_i \right) du,
\end{aligned}$$

where h_i and $\hat{h}_i$ contain the lower-order terms from the Taylor expansions we used to simplify $\mathcal{L}_N \left\langle \Lambda_N(\cdot), \phi^j \right\rangle (x_u)$ to its highest-order terms. Since $\hat{h}_i$ is $O(N^{-2})$ clearly the integral resulting from the last term converges to 0 as $N \to \infty$.
By defining $u_i^1 = x_i x_{i+1}, u_i^2 = x_i^2, u_i^3 = x_i$ and $\Psi_i^1 = 2\partial_x, \Psi_i^2 = 2k\gamma\partial_x, \Psi_i^3 = 2k\partial_{xx}$ we can use again the fact that $\mathbb{E}_{\mu_N}^{ABEP_N}\left[u_i^j\right] < C$ by assumption 7.1 and $\Psi^j \phi_i < 2k(\gamma+1)K$ by smoothness assumption of ϕ (where we say $K := \sup_{\alpha \in \mathbb{N}_0}\{||(\partial_x)^\alpha \phi||_\infty\}$) to find

$$\begin{aligned}
&\mathbb{E}_{\mu_N}^{ABEP_N}\left[\sup_{\substack{t,s\in[0,T] \\ |t-s|\le\delta}} \left| \int_s^t \mathcal{L}^{\mathrm{ABEP}_N} \left\langle \Lambda_N(\cdot), \phi^j \right\rangle (x_u) \right| \right] \\
&\le \mathbb{E}_{\mu_N}^{ABEP_N}\left[\sup_{\substack{t,s\in[0,T] \\ |t-s|\le\delta}} \left| \int_s^t \sum_{i=-\infty}^{\infty} \left(\frac{1}{N} \sum_{j=1}^{3} \left| u_i^j \Psi_i^j \phi_i \right| \right) + |h_i \phi_i| ds \right| \right] \\
&\le \sum_{j=1}^{3} \mathbb{E}_{\mu_N}^{ABEP_N}\left[\sup_{\substack{t,s\in[0,T] \\ |t-s|\le\delta}} \frac{1}{N} \sum_{i=\lfloor -MN \rfloor}^{\lceil MN \rceil} \left(\int_s^t u_i^j du \right)^{1/2} \left(\int_s^t \Psi_i^j du \right)^{1/2} \right] + o(1) \\
&\le 2k(\gamma+1)K\sqrt{\delta} \sum_{j=1}^{3} \mathbb{E}_{\mu_N}^{ABEP_N}\left[\int_0^T \frac{1}{N} \sum_{i=-\infty}^{\infty} |u_i^j| du \right] + o(1) \\
&\le 6kCT(\gamma+1)K\sqrt{\delta} + o(1).
\end{aligned}$$

By taking $N \to \infty$ and decreasing δ we can make this expectation arbitrarily small and conclude that $\left\{\mathbb{P}_{\mu_N}^{\mathrm{ABEP}_N}\right\}_{N\in\mathbb{N}}$ is tight via Markov's inequality,

$$\mathbb{P}_{\mu_N}^{\mathrm{ABEP}_N}\left(\sup_{\substack{|s-t|\le\delta\\ s,t\in[0,T]}} d_M(\Lambda_N(x_t^{(N)}),\Lambda_N(x_s^{(N)}))\ge\epsilon\right)\le\mathbb{E}_{\mu_N}^{\mathrm{ABEP}_N}\left[\sup_{\substack{|s-t|\le\delta\\ s,t\in[0,T]}} d_M(\Lambda_N(x_t^{(N)}),\Lambda_N(x_s^{(N)}))\right]\epsilon^{-1}.$$

□

Corollary 7.3. *The limit* $\lim_{N\to\infty}\left(\Lambda_N(x_t^{(N)})\right)_{t\ge0}$ *in* $D([0,T],M_+)$ *endowed with the Skorokhod topology exists and has associated density that we call* $(\chi_t)_{t\ge0}$ *with for every* $t:\ \chi_t:\mathcal{B}(\mathbb{R})\to\mathbb{R}_+$. *The relation between* $(\chi_t)_{t\ge0}$ *and* $(\rho_t)_{t\ge0}$ *is given for each* $t>0$ *through*

$$\rho_t(x)=\chi_t(x)e^{-\gamma\int_x^\infty\chi_t(y)dy}. \tag{64}$$

Proof. The existence of this limit follows from Lemma 7.4. See again pages 123-126 of [18] for details. The relation (64) follows from the fact that we can apply Lemma 7.2 to any configuration $x_t^{(N)}$, as from the first part of the corollary we know that $\Lambda_N(x_t^{(N)})$ converges weakly in probability to $\chi_t(x)dx$, so that the assumptions for Lemma 7.2 are satisfied. □

7.2.5 Derivation of the viscous Burgers' equation

We now have everything in place to prove Theorem 7.1. From Corollary 7.3 we know that a limit trajectory $(\chi_t)_{t\ge0}$ for the ABEP exists and is related to $(\rho_t)_{t\ge0}$ via (64), and from Corollary 7.2 we know that ρ_t solves the heat equation. In order to find that χ_t solves the viscous Burgers' equation, we note that (64) corresponds to the Cole-Hopf transformation. This is a useful tool, introduced by E. Hopf in [13], in order to solve the viscous Burgers' equation.

Theorem 7.5 (Cole-Hopf Transformation)**.** *Suppose we want to solve the viscous Burgers' equation*

$$u_t+uu_x=\mu u_{xx}\quad\text{with}\quad u(0,x)=g(x), \tag{65}$$

then the transformation

$$v=\exp\left\{-\frac{1}{2\mu}\int udx\right\}, \tag{66}$$

yields the following PDE with $C(t)$ *only depending on* t*,*

$$v_t=\mu v_{xx}+C(t)v\quad\text{with for } v=v(t,x)\quad v(0,x)=\exp\left\{-\frac{1}{2\mu}\int gdx\right\}.$$

Then through

$$w=v\exp\left\{-\int C(t)dt\right\}, \tag{67}$$

we end up with the linear heat equation

$$w_t=\mu w_{xx}\quad\textit{with for } w=w(t,x)\quad w(0,x)=\exp\left\{-\frac{1}{2\mu}\int gdx\right\} \tag{68}$$

Then by solving (68) *we can find the solution to* (65) *through tranformations* (66) *and* (67). *Under the restriction that for large* $|x|$ *we have*

$$\int_0^x u(0,y)dy = o(x^2),$$

this solution is given through

$$u(x,t) = \frac{\int\limits_{-\infty}^{\infty} \frac{x-y}{t} \exp\left\{-\frac{1}{2\mu}F(x,y,t)\right\} dy}{\int\limits_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\mu}F(x,y,t)\right\} dy},$$

with

$$F(x,y,t) = \frac{(x-y)^2}{2t} + \int_0^t u(0,z)dz.$$

Proof. See [13]. □

This theorem shows us the relation between the Cole-Hopf transformation given in (66) to the solution to the viscous Burgers' equation. Our approach differs from Theorem 7.5, as we go in opposite direction, starting out with a transformed variable ρ_t that solves the heat equation and looking for the PDE that is satisfied by un-transformed χ_t. However, the main idea is the same, namely to introduce a variable that solves the heat equation, and using it to prove that a related variable solves the viscous Burgers' equation.

Definition 7.2. *We define functions* $F_t : \mathbb{R} \to \mathbb{R}_+$ *and* $V_t : \mathbb{R} \to \mathbb{R}_+$ *as follows*

$$F_t(x) = \int_x^\infty \chi_t(x)dx \qquad V_t(x) := e^{-\gamma F_t}. \tag{69}$$

These functions will provide intermediate steps in deriving Burgers' equation for $\chi_t(x)$. $F_t(x)$ is the cumulative energy to the right of x at macro-scale, analogous to $E_i(x)$ for site i at micro-scale, and $V_t(x)$ is its Cole-Hopf transformation. We note that $\rho_t = \frac{1}{\gamma}\partial_x V_t$, and that from Corollary 7.2 we know that ρ_t solves the heat equation. The next step is then to prove that V_t also solves the heat equation. An important step before we can do this is showing that if we apply both the second-order space-derivative ∂_{xx} and integration to $\partial_x V_t(x)$, the order in which we do this does not matter.

Proposition 7.2. *For every* $x \in \mathbb{R}$ *we have*

$$\int_x^\infty \frac{\partial^3}{\partial y^3} V_t(y)dy = \frac{\partial^2}{\partial x^2} \int_x^\infty \frac{\partial}{\partial y} V_t(y)dy.$$

Proof. Straightforward calculation yields

$$\partial_t V_t = -\gamma \frac{\partial F_t(x)}{\partial t} e^{-\gamma F_t(x)},$$
$$\partial_x V_t = -\gamma \frac{\partial F_t(x)}{\partial x} e^{-\gamma F_t(x)},$$
$$\partial_{xx} V_t = \left(-\gamma \frac{\partial^2 F_t(x)}{\partial x^2} + \left(\gamma \frac{\partial F_t(x)}{\partial x} \right)^2 \right) e^{-\gamma F_t(x)},$$

with

$$\partial_t F_t(x) = \int_x^\infty \partial_t \chi_t(x) dy, \tag{70}$$
$$\partial_x F_t(x) = -\chi_t(x), \tag{71}$$
$$\partial_{xx} F_t(x) = -\partial_x \chi_t(x). \tag{72}$$

This means

$$\int_x^\infty \partial_{yy} \partial_y V_t(y) = (\partial_y \partial_y V_t(y))|_{y=x}^\infty \tag{73}$$
$$= (\gamma \partial_y \chi_t(y) + \gamma^2 \chi_t^2(y)) V_t(y)|_{y=x}^\infty$$
$$= -(\gamma \partial_x \chi_t(x) + \gamma^2 \chi_t^2(x)) V_t(x), \tag{74}$$

where in the last step we used the fact that $\chi_t(x)$ and its derivatives are 0 at $x = \infty$ and that V is bounded ($V(x) \leq V(\infty) = 1$). Similarly we find

$$\partial_{xx} \int_x^\infty \partial_y V_t(y) dy = \partial_{xx} V_t(y)|_{y=x}^\infty$$
$$= \partial_{xx}(1 - V_t(x))$$
$$= -(\gamma \partial_x \chi_t(x) + \gamma^2 \chi_t^2(x)) V_t(x),$$

which proves $\int_x^\infty \partial_{yy} \partial_y V_t(y) dy = \partial_{xx} \int_x^\infty \partial_y V_t(y) dy$. □

Lemma 7.6. *V_t solves the heat equation, i.e.*

$$\left(\frac{\partial}{\partial t} - 2k \frac{\partial^2}{\partial x^2} \right) V_t(x) = 0. \tag{75}$$

Proof. ρ_t solving the heat equation together with relation $\partial_x \rho_t(x) = V_t(x)$ means

$$\partial_x (\partial_t - 2k \partial_{xx}) V_t = 0.$$

This means that Proposition 7.2 yields

$$
\begin{aligned}
0 &= \int_x^\infty \partial_x(\partial_t - 2k\partial_{yy})V_t(y)dy \\
&= (\partial_t - 2k\partial_{xx}) \int_x^\infty \partial_y V_t(y)dy \\
&= (\partial_t - 2k\partial_{xx})V_t(y)|_{y=x}^\infty \\
&= (\partial_t - 2k\partial_{xx})(1 - V_t(x)).
\end{aligned}
$$

Since clearly $(\partial_t - 2k\partial_{xx})1 = 0$, we have $(\partial_t - 2k\partial_{xx})V_t(x) = 0$. □

This allows us to finish the proof of Theorem 7.1. Plugging in our expressions of $\partial_t V_t$ and $\partial_{xx} V_t$ from (73) and (74) we get

$$
\left(-\partial_t \gamma F_t(x) - 2k\partial_{xx}\gamma F_t(x) + 2k(\partial_x \gamma F_t(x))^2\right) e^{-\gamma F_t(x)} = 0.
$$

Since $F_t(x) < \infty \ \forall x \in \mathbb{R}$, we have $e^{-\gamma F_t(x)} > 0$, which means that we can divide it out of this differential equation, so that we find

$$
\begin{aligned}
0 &= -\gamma\partial_t F_t(x) - 2k\gamma\partial_{xx}F_t(x) + \gamma^2 2k(\partial_x F_t(x))^2 \qquad (76) \\
&= -\gamma \int_x^\infty \partial_t \chi_t(y)dy + 2k\gamma\partial_x\chi_t(x) + \gamma^2 2k\chi_t^2(x) \ \forall x \in \mathbb{R}.
\end{aligned}
$$

Thus taking the derivative with respect to x and dividing by $-\gamma$ yields Burgers' equation

$$
\partial_t \chi_t(x) - 2k\partial_{xx}\chi_t(x) - 2k\gamma\partial_x\chi_t^2(x) = 0. \qquad (77)
$$

□

7.3 Hydrodynamic limit of the Dynamic ABEP

7.3.1 Introduction and main result

Chapter 7.2 provides a nice framework for proving hydrodynamic limits of particle systems that can be reduced to the BEP via a (non-local) transformation. For this reason, it seems natural to continue along the same lines in order to find the hydrodynamic limit of the DABEP. In this section we will show that a large part of the proof works out in the same way for the DABEP as for the ABEP, but that in the final step the added complexity prevents us from finding a PDE for the hydrodynamic limit density of the DABEP, as we did for the ABEP in (77). We will first start by explaining the setting in which we take the limit, after which we will give the main result in Theorem 7.7.
We saw in (45) that a duality function between the DABEP and the SIP is given through

$$
D^d(x,\eta) = \prod_{i=-\infty}^{\infty} \frac{\Gamma(k)}{\Gamma(k+\eta_i)} \alpha^{\eta_i} \left(\frac{\cosh(\sigma\lambda + 2\sigma E_{i+1}(x)) - \cosh(\sigma\lambda + 2\sigma E_i(x))}{\sigma} \right)^{\eta_i},
$$

where the factor

$$\hat{g}_i(x) = \alpha \frac{\cosh(\sigma\lambda + 2\sigma E_{i+1}(x)) - \cosh(\sigma\lambda + 2\sigma E_i(x))}{\sigma},$$

forms a transformation from the DABEP to the BEP.
This means that, as was the case for the ABEP, it can be expected that this transformation allows us to express the hydrodynamic limit density of the DABEP through the known density of the BEP that we can create from it. In order to accommodate this, let again the asymmetry parameter be given through

$$\sigma = \sigma^{(N)} = \frac{\gamma}{2N}. \tag{78}$$

Furthermore, let

$$\lambda = \lambda^{(N)} = 2N\beta \tag{79}$$

so that the effect of the reservoir-like variable λ doesn't vanish as we increase N. Continuing with reusing the notation of the ABEP, we then define $\{x^{(N)}\}_{N\in\mathbb{N}}$ with for every $N \in \mathbb{N}$, $x^{(N)} \in \Omega_f$ a sequence of DABEP$(\sigma^{(N)}, k, \lambda^{(N)}, \alpha)$ configurations with distribution μ_N. We make the same assumptions about $\{x^{(N)}\}_{N\in\mathbb{N}}$ and $\{\mu_N\}_{N\in\mathbb{N}}$ as before.

Assumption 7.4.

$$\sup_{i\in\mathbb{Z},N\in\mathbb{N}} \mathbb{E}^{\mu_N}\left[\left(x_i^{(N)}\right)^2\right] \leq C \text{ for some } C > 0.$$

Assumption 7.5. $\Lambda_N(x^{(N)})$ *converges weakly in probability to a measure with density* $\chi : \mathbb{R} \to \mathbb{R}_+$*, i.e. for every* $\phi \in C_c^\infty(\mathbb{R})$ *and* $\epsilon > 0$ *we have*

$$\mu_N\left(\left|\left\langle \Lambda_N(x_i^{(N)}), \phi\right\rangle - \int \chi(x)\phi(x)dx\right| > \epsilon\right) \to 0.$$

Assumption 7.6. *For each* $N \in \mathbb{N}$ *the total amount of energy is bounded, i.e.*

$$\frac{1}{N}\sum_{i=-\infty}^{\infty} x_i^{(N)} \leq E < \infty.$$

Let $\mathbb{P}_{\mu_N}^{\mathrm{DABEP}_N}$ denote the probability measure of the accelerated DABEP$(\sigma^{(N)}, k, \lambda^{(N)}, \alpha)$. We now aim to find the hydrodynamic limit of the DABEP, i.e. a trajectory $(\chi_t(x)dx)_{t\geq 0}$ in $D([0,T], M_+)$ such that for each $\phi \in C_c^\infty(\mathbb{R})$ and $\epsilon > 0$ we have

$$\mathbb{P}_{\mu_N}^{\mathrm{DABEP}_N}\left(\sup_{t\in[0,T]}\left|\left\langle \Lambda_N(x^{(N)}), \phi\right\rangle - \int \chi_t(x)\phi(x)dx\right| > \epsilon\right) \to 0. \tag{80}$$

As we will find, altough we will be able to replicate much of the proof on the ABEP, our approach will not be able to derive a PDE for this $(\chi_t)_{t\geq 0}$. Instead we have the following PIDE.

Theorem 7.7. *Let* $(\chi_t)_{t\geq 0}$*, with for* $t > 0$*,* $\chi_t : \mathbb{R} \to \mathbb{R}_+$*, satisfy the following PIDE,*

$$\int_x^\infty \frac{\partial \chi_t(y)}{\partial t}dy + 2k\frac{\partial \chi_t(x)}{\partial x} - 2k\gamma\chi_t^2(x)\tanh\left(\gamma\beta + \gamma\int_x^\infty \chi_t(y)dy\right) = 0, \tag{81}$$

with $\chi_0 = \chi$*. Then* χ_t *is the density of the hydrodynamic limit of the DABEP, i.e.* (80) *holds.*

7.3.2 Relation BEP and DABEP

Under the parameterization of (78) and (79), we start the proof by creating the transformed process

$$z_i\left(x_t^{(N)}\right) = \alpha \frac{\cosh\left(2\sigma^{(N)}\beta N + 2\sigma^{(N)} E_{i+1}(x_t^{(N)})\right) - \cosh\left(2\sigma^{(N)}\beta N + 2\sigma^{(N)} E_i(x_t^{(N)})\right)}{\sigma^{(N)}}, \tag{82}$$

which is the BEP(k) (see Theorem 5.6). Leaving aside the question of convergence and assuming that both $\Lambda_N(x^{(N)})$ and $\left(\Lambda_N(x_t^{(N)})\right)_{t\geq 0}$ converge in same sense as for the ABEP and the BEP, we want to find the relation between the limiting densities of the DABEP and the BEP that we created with (82). In line with our approach for the ABEP, we do this for the initial state of the problem, and the then the relation will hold as we evolve the process.
Let $\rho : \mathbb{R} \to \mathbb{R}_+$ denote the density of the limiting measure towards which $\{\Lambda_N(z(x^{(N)})\}_{N\in\mathbb{N}}$ converges, i.e.

$$\lim_{N\to\infty} \mu_N \left(\left| \left\langle \Lambda_N(z_i(x^{(N)}), \phi \right\rangle - \int_{\mathbb{R}} \rho(x)\phi(x)dx \right| > \epsilon \right) = 0.$$

We then have the following relation between ρ and χ.

Lemma 7.8. *The relation between ρ and χ is given through*

$$\rho(x) = 2\alpha\chi(x)\cosh\left(\gamma\beta + \gamma\int_x^\infty \chi(y)dy\right).$$

Proof.

$$\begin{aligned} z_i(x) &= \frac{\alpha}{2\sigma}\left(e^{2\sigma\beta N + 2\sigma E_{i+1}(x)} + e^{-2\sigma\beta N - 2\sigma E_{i+1}(x)} - e^{2\sigma\beta N + 2\sigma E_{i+1}(x)} - e^{-2\sigma\beta N - 2\sigma E_{i+1}(x)}\right) \\ &= \frac{\alpha}{2\sigma}\sum_{k=0}^\infty \frac{1}{k!}\left[(\gamma\beta + 2\sigma E_{i+1}(x))^k - (\gamma\beta + 2\sigma E_i(x))^k + (-\gamma\beta - 2\sigma E_{i+1}(x))^k - (-\gamma\beta - 2\sigma E_i(x))^k\right]. \end{aligned}$$

We have

$$\begin{aligned} \frac{1}{k!}\left((\gamma\beta + 2\sigma E_{i+1}(x))^k - (\gamma\beta + 2\sigma E_i(x))^k\right) &= \frac{1}{k!}\left(\sum_{p=0}^k (\gamma\beta + 2\sigma E_i)^{k-p}(2\sigma x_j)^p \binom{k}{p}\right) - \frac{\gamma\beta + 2\sigma E_i}{k!} \\ &= \sum_{p=1}^k (\gamma\beta + 2\sigma E_i)^{k-p}(2\sigma x_j)^p \frac{1}{p!(k-p)!}. \end{aligned}$$

Since $\gamma\beta$ is constant in N and σE_i is bounded in N, terms become lower order as we increase p. Only keeping the highest order terms with $p = 1$ yields

$$\begin{aligned} &\frac{\alpha}{2\sigma}\sum_{k=0}^\infty \frac{1}{k!}\left((\gamma\beta + 2\sigma E_{i+1}(x))^k - (\gamma\beta + 2\sigma E_i(x))^k\right) \\ &= \frac{\alpha}{2\sigma}\sum_{k=1}^\infty (\gamma\beta + 2\sigma E_i)^{k-1} 2\sigma x_i \frac{1}{(k-1)!} + O(N^{-1}) \\ &= \alpha x_i e^{\gamma\beta + 2\sigma E_i} + O(N^{-1}). \end{aligned}$$

Similarly we find

$$\frac{\alpha}{2\sigma}\sum_{k=0}^{\infty}\frac{1}{k!}\Big((-\gamma\beta-2\sigma E_{i+1}(x))^k-(-\gamma\beta-2\sigma E_i(x))^k\Big)=\alpha x_i e^{-\gamma\beta-2\sigma E_i}+O(N^{-1}).$$

Thus

$$z_i(x)=2\alpha x_i\cosh\left(\gamma\beta+2\sigma E_i(x)\right)+O(N^{-1}).$$

From this we can use a similar approach to that of Prop. 7.3. in order to show that this relation between x and z on microscopic level leads to the relation between $\rho(x)$ and $\chi(x)$ from the Lemma,

$$\rho(x)=2\alpha\chi(x)\cosh\left(\gamma\beta+\gamma\int_x^{\infty}\chi(y)dy\right).$$

□

7.3.3 Derivation of PDE

Next we will try to use this result from Lemma 7.8 in order to derive a PDE for χ_t. Analogously to (69) we define functions F_t and V_t to facilitate the proof.

Definition 7.3.

$$F_t(x):=\beta+\int_x^{\infty}\chi_t(y)dy \qquad V_t(x):=\frac{2\alpha}{\gamma}\sinh\left(\gamma F_t(x)\right).$$

As before, $F_t(x)$ represents the cumulative energy, only this time with additional term β. Note that again we have

$$\partial_x V_t(x)=\rho_t(x).$$

We proceed with the analogue to Proposition 7.2.

Proposition 7.3.

$$\int_x^{\infty}\frac{\partial^3}{\partial y^3}V_t(y)dy=\frac{\partial^2}{\partial x^2}\int_x^{\infty}\frac{\partial}{\partial y}V_t(y)dy.$$

Proof. Explicit computation yields

$$\begin{aligned}
\partial_x V_t(x)&=\frac{2\alpha}{\gamma}(\partial_x F_t(x))\cosh(\gamma F_t(x)),\\
\partial_{xx} V_t(x)&=\frac{2\alpha}{\gamma}(\partial_x F_t(x))^2\sinh\left(\gamma F_t(x)\right)+\frac{2\alpha}{\gamma}(\partial_{xx} F_t(x))\cosh\left(\gamma F_t(x)\right), &(83)\\
\partial_t V_t(x)&=\frac{2\alpha}{\gamma}(\partial_t F_t(x))\cosh\left(\gamma F_t(x)\right), &(84)
\end{aligned}$$

where $F_t(x)$ is the same as before up to a constant term, so that equations (70), (71), (72) of its derivatives are still correct. We thus find

$$\begin{aligned}
\int_x^\infty \partial_{yy}\partial_y V_t(y)dy &= (\partial_{yy}V_t(y))|_{y=x}^{\infty} \\
&= 2\alpha\gamma\chi_t^2(y)\sinh\left(\sigma\beta+\gamma\int_y^\infty \chi_t(z)dz\right)+2\alpha(\partial_y\chi_t(y))\cosh\left(\sigma\beta+\gamma\int_y^\infty \chi_t(z)dz\right)\Bigg|_{y=x}^{\infty} \\
&= -2\alpha\gamma\chi_t^2(x)\sinh\left(\sigma\beta+\gamma\int_x^\infty \chi_t(y)dy\right)-2\alpha(\partial_x\chi_t(x))\cosh\left(\sigma\beta+\gamma\int_x^\infty \chi_t(y)dy\right) \\
&= -\partial_{xx}V_t(x),
\end{aligned}$$

and similarly

$$\begin{aligned}
\partial_{xx}\int_x^\infty \partial_y V_t(y)dy &= \partial_{xx}\left(V_t(y)|_{y=x}^{\infty}\right) \\
&= \partial_{xx}(V_t(\infty)-V_t(x)) \\
&= -\partial_{xx}V_t(x).
\end{aligned}$$

□

This finding allows us to interchange $\int_x^\infty \cdot dy$ and ∂_{xx}, which means in similar fashion to ABEP we find that $V_t(x)$ solves the heat equation

Corollary 7.4. *V_t solves the heat equation, i.e.*

$$\left(\frac{\partial}{\partial t}-2k\frac{\partial^2}{\partial x^2}\right)V_t(y)=0.$$

Proof. Works in the same way as the proof of Lemma 7.6, using Proposition 7.3. □

This finding allows us to derive a PDE for F_t.

Lemma 7.9. *F_t is the unique solution to*

$$\frac{\partial F_t(x)}{\partial t}-2k\frac{\partial^2 F_t(x)}{\partial x^2}-2k\gamma\left(\frac{\partial F_t(x)}{\partial x}\right)^2\tanh(\gamma F_t(x))=0, \tag{85}$$

with initial condition

$$F_0(x)=\beta+\int_x^\infty \chi(y)dy.$$

Proof. Follows from Corollary 7.4. Filling in the derivatives of V_t given in (83) and (84) yields

$$\left(\partial_t \gamma F_t(x) - 2k(\partial_x \gamma F_t(x))^2\right)\cosh(\gamma F_t(x)) - 2k(\partial_{xx}\gamma F_t(x))\sinh(\gamma F_t(x)) = 0. \tag{86}$$

Since $\cosh(x) > 0$ for every $x \in \mathbb{R}$, we can divide (86) by $\gamma\cosh(\gamma F_t(x))$. Doing this and rearanging the terms yields (85). $\square$

This PDE for $F_t(x)$ is very similar to the one in (76) for the ABEP, with the main difference being the factor

$$\tanh(\gamma F_t(x)) = \tanh\left(\gamma\beta + \gamma \int_x^\infty \chi_t(y)dy\right),$$

in the quadratic term. Here we see the difference between the dynamics that the ABEP and the DABEP generate. Because this factor cannot be divided out or otherwise made easier (as far as we have found), we cannot derive a PDE for $\chi_t(x)$ such as (77) for the ABEP. Still this is a useful PDE for the cumulative energy function $F_t(x)$. Finally then inserting the derivatives of $F_t(x)$ as given in (70), (71) and (72) into (85) yields (81), which concludes the proof of Theorem 7.7.

$\square$

One final thing we can do is show that when we take the scaling limits that reduce the DABEP to the ABEP and the BEP, then (85) reduces to the PDEs of the limit density profiles of these processes.

Proposition 7.4.

1. *When we take the limit $\beta \to -\infty$,* (85) *reduces to Burgers' equation* (77)*.*
2. *When we take the limit $\gamma \downarrow 0$,* (85) *reduces to the heat equation* (50)*.*

Proof. For 1, we recognize that as $\lim_{x\to\infty} \tanh(x) = 1$, so that

$$\lim_{\beta\to\infty} 2k\gamma(\partial_x F_t(x))^2 \tanh\left(\gamma\beta + \gamma\int_x^\infty \chi_t(x)dy\right) = 2k\gamma(\partial_x F_t(x))^2,$$

which yields (76), which then gives us (77). Note that α has been divided out, so that we don't need to take $\alpha = \exp(\gamma\beta)$. Had we taken the limit $\beta \to -\infty$ at an earlier stage, this would have been required in order to prevent explosion of terms like $\exp(-\sigma\beta - 2\sigma E_j(x))$.
For 2, we note that the nonlinear term of (85) vanishes as $\gamma \downarrow 0$. $\square$

7.3.4 Conclusion

We have shown that the hydrodynamic limit of the DABEP is the solution to the PIDE given in (81). In the proof we made the assumption of existence of the limit of the trajectory, leaving aside a proof involving tightness, such as in Chapter 6.4.4 for the BEP and 7.2.2 for the ABEP. It seems that such a proof would work out very similarly to the proofs for the BEP and the ABEP, so decided not to do this in this thesis. In order to make the proof of 7.7 rigorous however, this would still need to be proven.
Through this proof of Theorem 7.7 we see that our proof of Theorem 7.1 for the hydrodynamic limit of the ABEP provides a relatively easy method for deriving hydrodynamic limits of similar processes. If future processes are derived which are related to the BEP in a similar way, this proof may provide a framework for proving their hydrodynamic limits.

8 Propagation of Chaos

8.1 Introduction

The hydrodynamic limits of Theorem 6.1 and Theorem 7.1 are interesting findings, and because of that the main topic of this thesis. In this section, however, we go in a slightly different direction, and prove the propagation of chaos for the BEP and ABEP.
Propagation of chaos was proven for the ASEP in the pivotal paper of Gärtner [10]. What 'propagation of chaos' means in the context of this paper is the following: Suppose we have a random configuration of the IPS of our choice (BEP/ABEP), distributed according to a "local equilibrium" measure. This is a measure with marginals similar to the stationary and reversible product measure, except that its parameterization is allowed to weakly depend on location. Then, under certain choices in the creation of this local equilibrium measure, we will find that as we evolve the IPS, the evolved measure of the distribution of its configuration will remain in local equilibrium form, i.e. like the reversible measure, but with marginals depending on location.
In fact, what we will see is that if we choose the reversible measures of the BEP and ABEP, given in Theorem 4.5 and 5.5 except with the scale parameter of the marginals determined by the value of $\rho : \mathbb{R} \to \mathbb{R}_+$ at the location corresponding to each marginal, then we can use duality of the BEP and ABEP with the n-SIP in order to prove that evolving the configuration under accelerated dynamics becomes the same as evolving ρ through the heat equation as we take the same scaling limit $N \to \infty$ as in Chapter 6 and 7.
Propagation of chaos is an interesting property, because while the hydrodynamic limit only tells us how the density profile evolves as it appears at macroscopic scale, propagation of chaos tells us that the local distribution of energy (at microscopic scale) evolves following the known evolution of a local equilibrium measure. For this reason, this property is also regularly referred to as "propagation of local equilibrium" in the literature.

8.2 BEP

8.2.1 Main result

We define $\nu^{2k,\infty}_{\rho,(N)}$ as the inhomogeneous infinite product of Gamma distributions with shape parameter 2k and scale parameters $\rho(\frac{i}{N})$ for $i \in \mathbb{Z}$ with $\rho : \mathbb{R} \to \mathbb{R}_+$, i.e.

$$\nu^{2k,\infty}_{\rho,(N)} = \otimes_{i\in\mathbb{Z}} \nu^{2k}_{\rho\left(\frac{i}{N}\right)}, \tag{87}$$

where $\nu^{2k}_{\rho\left(\frac{i}{N}\right)}$ is given in (39). We call $\nu^{2k,\infty}_{\rho,(N)}$ a "local equilibrium" measure as it is derived from (reversible) equilibrium measure $\nu^{2k,\infty}_{\theta}$ with constant parameter θ, where the slowly varying nature of $\rho\left(\frac{i}{N}\right)$ makes it so that $\nu^{2k,\infty}_{\rho,(N)}$ behaves like its counterpart with fixed parameter locally as we take $N \to \infty$. Similar to how the expected value of the BEP at each site is given by θ under $\nu^{2k,\infty}_{\theta}$, under $\nu^{2k,\infty}_{\rho,(N)}$ the profile is given through the function ρ, i.e. at site i the expected amount of energy is $\rho\left(\frac{i}{N}\right)$.
Roughly speaking, the goal of this section is to prove that as N increases, we get the following 'equality'.

$$\nu^{2k,\infty}_{\rho,(N)} S^{\mathrm{BEP}}_{N^2 t} \approx \nu^{2k,\infty}_{\rho_t,(N)}, \tag{88}$$

Where in the right-hand site the function from which the scale parameters are derived is ρ_t, the solution to the heat equation with initial condition ρ. Thus (88) claims that the evolved measure which combines the evolution as the BEP with product measure $\nu^{2k,\infty}_{\rho,(N)}$ is 'roughly equal' to a similar product measure $\nu^{2k,\infty}_{\rho_t,(N)}$, where the parameters are given through the same function evolved through the heat equation. The reason for this 'rough equality' is that the measures we are dealing with are infinite product measures which do not clearly converge when we integrate arbitrary functions. Therefore we mean by (88) that if we integrate a duality function $D^b(\xi,\cdot)$ with an arbitrary $\xi \in \Omega_n$ with respect to the measures on both sides of the equation, their difference converges to zero as $N \to \infty$.
This is formalized in the following theorem.

Theorem 8.1. *Suppose $\{\xi^{(N)}\}_{N\in\mathbb{N}}$ is a sequence of configurations with n particles (i.e. $\xi^{(N)} \in \Omega_n$) in such a way that under the usual rescaling of the space-dimension the locations of their n particles at macroscopic scale converge, i.e.*

$$\xi^{(N)} = \sum_{i=1}^{n} \delta_{X_i^{(N)}} \text{ with for each } i,\ \frac{X_i^{(N)}}{N} \to z_i. \tag{89}$$

Then we have

$$\lim_{N\to\infty} \int \mathbb{E}_y^{\mathrm{BEP}_N} D^b(\xi^{(N)}, y_t)\nu^{2k,\infty}_{\rho,(N)}(dy) - \int D^b(\xi^{(N)}, y)\nu^{2k,\infty}_{\rho_t,(N)}(dy) = 0, \tag{90}$$

where ρ_t is the solution to the heat equation with initial condition ρ.

Note that we take a limit $N \to \infty$ that is very similar to the Chapters about the hydrodynamic limits, where function ρ corresponds to the appearance at macroscopic scale and the values y_i correspond to energy level at individual sites at microscopic sites. The fact the initial location of the particles in dual configurations $\xi^{(N)}$ are chosen so that they converge at macro-scale, is also very similar to the previous chapter. A key difference, as mentioned in the introduction, is that our focus here will remain on the microscopic level, where we investigate the distribution of energy levels at individual sites.

8.2.2 Proof

Our goal is to use duality between BEP and n-SIP to prove propagation of chaos of BEP as stated in (90). In [16] something very similar was done, where self-duality of the SIP was used to prove the propagation of chaos of the SIP. A lot of the work in that proof was on the behavior of the dual SIP, which means it is directly applicable here.
We start by elaborating a bit further on the relationship between the reversible measure $\nu_\theta^{2k,\infty}$ and the duality function D^b. As we have seen in Chapter 4 we can split duality function D^b into single-edge duality polynomials (see (30))

$$D^b(\xi, y) = \prod_{j=-\infty}^{\infty} \frac{y_j^{\xi_j}\Gamma(2k)}{\Gamma(2k+\xi_j)} = \prod_{j=-\infty}^{\infty} d^b(\xi_j, y_j).$$

Here ξ and y are arbitrary configurations of the n-SIP and ABEP respectively. Then the characterizing property of reversible measure $\nu_\theta^{2k,\infty}$ is the following (see the proof of Theorem 4.5)

$$\int d^b(n, y_i)\nu_\theta^{2k,\infty}(dy) = \theta^n.$$

We can similarly split the (non-reversible) product measure $\nu^{2k,\infty}_{\rho,(N)}$ into a product of marginal measures $\otimes_i \nu^{2k}_{\rho(\frac{i}{N})}$ so that we find the following:

$$\begin{aligned}
\int \mathbb{E}^{\mathrm{BEP}_N}_y \left[D^b(\xi^{(N)}, y_t)\right] \nu^{2k,\infty}_{\rho,(N)}(dy) &= \int \mathbb{E}^{\mathrm{BEP}}_y \left[D^b\left(\xi^{(N)}, y_{tN^2}\right)\right] \left(\otimes_i \nu^{2k}_{\rho\left(\frac{i}{N}\right)}\right)(dy) \\
&= \int \mathbb{E}^{\mathrm{SIP}}_{\xi^{(N)}} \left[D^b\left(\xi^{(N)}_{N^2t}, y\right)\right] \left(\otimes_i \nu^{2k}_{\rho(\frac{i}{N})}\right)(dy) \\
&= \mathbb{E}^{\mathrm{SIP}}_{\xi^{(N)}} \left[\prod_{i=-\infty}^{\infty} \int d^b((\xi^{(N)}_{N^2t})_i, y_i) \nu^{2k}_{\rho(\frac{i}{N})}(dy_i)\right] \\
&= \mathbb{E}^{\mathrm{SIP}}_{\xi^{(N)}} \left[\prod_{i=-\infty}^{\infty} \rho\left(\frac{i}{N}\right)^{\left(\xi^{(N)}_{N^2t}\right)_i}\right] \\
&= \mathbb{E}^{\mathrm{SIP}}_{X_1(0),\ldots,X_n(0)} \left[\prod_{i=1}^{n} \rho\left(\frac{X^{(N)}_i(tN^2)}{N}\right)\right]
\end{aligned}$$

where in the last step we rewrote

$$\xi^{(N)}_{N^2t} = \sum_{i=1}^{n} \delta_{X^{(N)}_i(tN^2)}.$$

If we then define $\mathcal{V}$ as the so-called "correlation function" of the BEP

$$\mathcal{V}(\nu, x_1, \ldots, x_n; t) := \int \mathbb{E}^{\mathrm{BEP}}_y D\left(\sum_{i=1}^{n} \delta_{x_i}, y_t\right) \nu(dy) - \prod_{i=1}^{n} \int \mathbb{E}^{\mathrm{BEP}}_y D(\delta_{x_i}, y_t)\nu(dy)$$

then our goal is to show convergence to 0 of

$$\begin{aligned}
&\mathcal{V}(\nu^{2k,\infty}_\rho, X^{(N)}_1(0), \ldots, X^{(N)}_n(0); N^2t) \\
&= \int \mathbb{E}^{\mathrm{BEP}}_y D^b(\xi^{(N)}, y_t)\nu^{2k,\infty}_{\rho(\frac{i}{N})}(dy) - \int D^b(\xi^{(N)}, y)\nu^{2k,\infty}_{\rho_t(\frac{i}{N})}(dy) \\
&= \mathbb{E}^{\mathrm{SIP}}_{X_1(0),\ldots,X_n(0)} \left[\prod_{i=1}^{n} \rho\left(\frac{X^{(N)}_i(N^2t)}{N}\right)\right] - \prod_{i=1}^{n} \rho_t\left(\frac{X^{(N)}_i(0)}{N}\right) \\
&= \mathbb{E}^{\mathrm{SIP}}_{X_1(0),\ldots,X_n(0)} \left[\prod_{i=1}^{n} \rho\left(\frac{X^{(N)}_i(N^2t)}{N}\right)\right] - \mathbb{E}^{\mathrm{IRW}}_{X_1(0),\ldots,X_n(0)} \left[\prod_{i=1}^{n} \rho\left(\frac{\hat{X}^{(N)}_i(N^2t)}{N}\right)\right] + o(1), \qquad (91)
\end{aligned}$$

where in the last step we used the fact that at macroscopic scale the accelerated evolution of a random walker is indistinguishable from Brownian motion, which has the Laplacian as generator. An interesting thing to note is that one may have thought that the reason for seeing the solution to the heat equation ρ_t in the right-hand side of (90) was because it is the hydrodynamic limit of the BEP, but this last step shows that it originates from the evolution of the input of ρ as IRW(2k) in the right term of (91).

Since ρ is continuous we thus need to prove that n SIP particles $X^{(N)}_1, \ldots, X^{(N)}_n$ behave similarly enough to n independent random walkers $\hat{X}^{(N)}_1, \ldots, \hat{X}^{(N)}_n$ that

$$\mathbb{E}\left[\frac{X^{(N)}_i(N^2t) - \hat{X}^{(N)}_i(N^2t)}{N}\right] \to 0.$$

This means we have to show $\mathbb{E}\left[X_i^{(N)}(t) - \hat{X}_i^{(N)}(t)\right]$ is $o(\sqrt{t})$.
This was proven in [16]. Later we will try to improve the bound that was found in that paper, but for now we will take this result and use it to conclude the proof of Theorem 8.1.

Lemma 8.2. *Let $(X_1^{(N)}, \dots, X_n^{(N)}, \hat{X}_1^{(N)}, \dots, \hat{X}_n^{(N)})$ be a coupling with respect to probability measure $\mathbb{P}$, where $X_1^{(N)}, \dots, X_n^{(N)}$ are jointly defined particles of the n-SIP(k) where for $X_i^{(N)} : \mathbb{R}_+ \to \mathbb{Z}$, $X_i^{(N)}(t)$ denotes the location of particle i at time t. Similarly $\hat{X}_1^{(N)}, \dots, \hat{X}_n^{(N)}$ are particles of the n-IRW(2k), in such a way that for every $i = 1, \dots, n$ we have $X_i^{(N)}(0) = \hat{X}_i^{(N)}(0)$.*
In this setting for each $i = 1, \dots, n$ and $t > 0$,

$$\mathbb{E}\left|\frac{X_i^{(N)}(N^2 t) - \hat{X}_i^{(N)}(N^2 t)}{N}\right| \to 0.$$

Proof. See [16] $\square$

We can now finish the proof of Theorem 8.1. Framing (91) via the coupling defined in Lemma 8.2 yields

$$\begin{aligned}
|\mathcal{V}| &= \left|\mathbb{E}^{\mathrm{SIP}}_{x_1,\dots,x_n}\left[\prod_{i=1}^n \rho\left(\frac{X_i^{(N)}(N^2t)}{N}\right)\right] - \mathbb{E}^{\mathrm{IRW}}_{x_1,\dots,x_n}\left[\prod_{i=1}^n \rho\left(\frac{\hat{X}_i^{(N)}(N^2t)}{N}\right)\right]\right| \\
&= \left|\mathbb{E}\left[\left(\prod_{i=1}^n \rho\left(\frac{X_i^{(N)}(N^2t)}{N}\right)\right) - \left(\prod_{i=1}^n \rho\left(\frac{\hat{X}_i^{(N)}(N^2t)}{N}\right)\right)\right]\right| \\
&\leq \mathbb{E}\left[\sum_{i=1}^n \left|\rho\left(\frac{X_i^{(N)}(N^2t)}{N}\right) - \rho\left(\frac{\hat{X}_i^{(N)}(N^2t)}{N}\right)\right| \prod_{\substack{j=1\\ j\neq i}}^n 2||\rho||_\infty\right] \\
&\leq \mathbb{E}\left[\sum_{i=1}^n \left|\frac{X_i^{(N)}(N^2t) - \hat{X}_i^{(N)}(N^2t)}{N}\right| ||\rho||_\infty \left(2||\rho||_\infty\right)^{n-1}\right] \\
&\leq n2^{n-1}||\rho||_\infty^n \mathbb{E}\left|\frac{X_i^{(N)}(N^2t) - \hat{X}_i^{(N)}(N^2t)}{N}\right|.
\end{aligned}$$

Thus by Lemma 8.2, $|\mathcal{V}|$ converges to 0, which concludes the proof of Theorem 8.1. $\square$

8.3 ABEP

8.3.1 Main result

As we saw in Theorem 5.5, we can find a reversible measure for the ABEP in finite volume via the pushforward measure $\mu_\theta^{2k,L} := (\nu_\theta^{2k,L}\det(\mathcal{J})) \circ g$, where g is the map from the ABEP to the BEP given in (33). To be more precise, we used map $\tilde{g}$, which is a version of map g defined for ABEP-configurations on $[-L, L]$. Since the lattice on which we define the ABEP in this section changes with N, we simply let g denote the appropriate version of the map. In this section we will use this finding in order to find an analogous result to Theorem 8.1 for the propagation of chaos of the ABEP, by pushing forward the slowly varying measure $\nu_{\rho,(N)}^{2k,\infty}$ from (31) through g^{-1}. As we

saw in Chapter 5.5, the pushforward measure of $\nu_\theta^{2k,\infty}$ by g^{-1} does not allow to be generalized to infinite volume, and we will see that the same holds for the pushforward of $\nu_{\rho,(N)}^{2k,\infty}$.
Because of this, the propagation of chaos for the ABEP will be proven for the ABEP($\sigma^{(N)}$,k) defined on a finite interval, corresponding to $[-L, L]$ at macro-scale. Let $\nu_\theta^{2k,L}$ denote the reversible measure in this finite setting, which is the product measure with marginals $\nu^{2k}_{\rho\left(\frac{i}{N}\right)}$ where i goes from $-NL$ to NL. One point we should pay attention to before we do this is the function ρ determining the scale parameter of $\nu_{\rho,(N)}^{2k,L}$. In finite volume, the evolution of independent random walkers still converges to the heat equation, but we now have boundaries. As in Chapter 5.5, we will take the simplest case of closed boundaries of the IPS, which corresponds at macroscopic scale to a PDE with Neumann boundary conditions. Thus we define ρ_t as the the solution to heat equation

$$\partial_t \rho_t = \partial_{xx} \rho_t,$$

with Neumann boundary conditions

$$\partial_x \rho_t(-L) = \partial_x \rho_t(L) = 0.$$

We continue by defining the pushforward measure of $\nu_{\rho,(N)}^{2k,L}$ through

$$\mu_{\rho,(N)}^{2k,L} := (\nu_{\rho,(N)}^{2k,L} \det(\mathcal{J})) \circ g.$$

Again this is not a reversible or even stationary measure anymore, but the slowly varying nature of $\rho(\frac{i}{N})$ allows for propagation of chaos. We find that $\mu_{\rho,(N)}^{2k,L}$ is given through

$$\mu_{\rho,(N)}^{2k,L}(dx) = \prod_{i=-NL}^{NL} \exp\left\{ \frac{e^{-2\sigma E_{i+1}(x)} - e^{-2\sigma E_i(x)}}{2\sigma\rho(\frac{i}{N})} \right\} \frac{(1 - e^{-2\sigma x_i})^{(2k-1)} e^{-(4\sigma k(i+L+1))x_i}}{(2\sigma\rho\left(\frac{i}{N}\right))^{2k}\Gamma(2k)}. \tag{92}$$

From (92) the nonlocal nature of $\mu^{2k}_{\rho\left(\frac{i}{N}\right)}$ becomes clear through the presence of the $E_i(x)$ and $E_{i+1}(x)$ terms. In this setting we make a similar claim to Theorem 8.1 about the propagation of chaos through the evolution of the ABEP under this measure.

Theorem 8.3. *Let $\{\xi^{(N)}\}_{N\in\mathbb{N}}$ be a sequence of configurations with n particles defined on increasingly large intervals $[-NL, NL]$ such that at macroscopic scale the locations of their particles converge, i.e. we have* (89) *with the additional constraint that $\left|\frac{X_i^{(N)}}{N}\right| \leq L$ for each i. Then*

$$\lim_{N\to\infty} \int \mathbb{E}_x^{\mathrm{ABEP}_N} D^a(x_t, \xi) \mu^{2k,L}_{\rho(\frac{i}{N})}(dx) - \int D^a(x, \xi) \mu^{2k,L}_{\rho_t(\frac{i}{N})}(dx) = 0. \tag{93}$$

8.3.2 Proof

Since

$$\int D^a(x, \xi) \mu^{2k,L}_{\rho(\frac{i}{N})}(dx) = \int (D^b(\cdot, \xi) \circ g)(x) \mu^{2k,L}_{\rho(\frac{i}{N})}(dx) = \int D^b(y, \xi) \nu^{2k,L}_{\rho\left(\frac{i}{N}\right)}(dy),$$

we have

$$\int \mathbb{E}_x^{\text{ABEP}} D^a(x_{N^2t}, \xi)\mu^{2k,L}_{\rho(\frac{i}{N})}(dx) = \int \mathbb{E}_\xi^{\text{SIP}} D^a(x, \xi_{N^2t})\mu^{2k,L}_{\rho(\frac{i}{N})}(dx)$$
$$= \mathbb{E}_\xi^{\text{SIP}} \int D^b(y, \xi_{N^2t})\nu^{2k,L}_{\rho(\frac{i}{N})}(dy)$$
$$= \mathbb{E}^{\text{SIP}}_{X_1,\dots,X_n}\left[\prod_{i=1}^n \rho\left(\frac{X_i(N^2t)}{N}\right)\right]. \tag{94}$$

Furthermore we have

$$\int D^a(x,\xi)d\mu^{2k,L}_{\rho_t(\frac{i}{N})} = \int D^b(y,\xi)d\nu^{2k,L}_{\rho_t(\frac{i}{N})}. \tag{95}$$

This brings us back to the setting from Theorem 8.1, which tells us that (94) and the right-hand side of (95) are equal. This concludes the proof of Theorem 8.3. □

8.4 Conjecture: An improved bound on the distance between coupled SIP and IRW particles

8.4.1 Introduction

As we saw, we could complete the proof of Theorem 8.1 and subsequently Theorem 8.3 via the proof of Lemma 8.2 from [16]. The approach in this paper was to create a coupling of n SIP-particles with n independent random walkers, and showing that the distance between a SIP-particle and a corresponding random walker is $o(\sqrt{t})$, i.e. of lower order than $\sqrt{t}$. In this thesis however, we propose a way to improve upon the results from that paper, by not only showing that that difference is $o(\sqrt{t})$, but that it is $O(\sqrt[4]{t})$, i.e. of the same order as $\sqrt[4]{t}$. We make the following conjecture.

Conjecture 8.1. *Let us be in the setting of 8.2, then for each $i = 1, \dots, n$ and $t > 0$,*

$$\lim_{N\to\infty} \mathbb{E}\left|\frac{X_i(N^4t) - \hat{X}_i(N^4t)}{N}\right| < C \text{ for some } C > 0. \tag{96}$$

The next paragraph outlines the first part of the approach of [16], which we will use in the same manner, after which we will deviate from this paper and improve upon their results.

8.4.2 Approach of previous work: 2 particles

The paper starts by proving the result in the special case where n = 2. As we will see later, the result for a general finite $n \in \mathbb{N}$ will then follow immediately. We start by defining a coupling $(X(t), Y(t), \hat{X}(t), \hat{Y}(t))$ where $X(t)$ and $Y(t)$ denote the location of two jointly defined SIP(k)-particles and and $\hat{X}, \hat{Y}$ are two IRW(2k)-particles. The generator of the coupling is given for continuous $f : \mathbb{Z}^4 \to \mathbb{R}$ through

$$\mathcal{L}f(\mathbf{x}) = \frac{1}{2}\sum_{\epsilon=\pm1} (f(\mathbf{x}+\epsilon e_{13}) - f(\mathbf{x})) + (f(\mathbf{x}+\epsilon e_{24}) - f(\mathbf{x}))$$
$$+ I(|x-y| = 1)\left(f(x,x,\hat{x},\hat{y}) + f(y,y,\hat{x},\hat{y}) - 2f(x,y,\hat{x},\hat{y})\right).$$

where $e_{13} = (1,0,1,0)$ and $e_{24} = (0,1,0,1)$. Here the first term containing e_{13} and e_{24} corresponds to simultaneous jumps of the pair x and $\hat{x}$ and pair y and $\hat{y}$, while the second term containing $I(|x-y|=1)$ corresponds to the additional attraction between 2-SIP particles x and y.
We are interested in $\phi(\mathbf{x}) = x - \hat{x}$. Defining $z = y - x$ we can express the evolution of $\phi(\mathbf{x}(t))$ in z via the Dynkin Martingale

$$M_t = \phi(\mathbf{x}(t)) - \phi(\mathbf{x}(0)) - \int_0^t I(|z(s)|=1)z(s)ds, \tag{97}$$

$$\langle M\rangle_t = \int_0^t (\mathcal{L}(\phi^2)(\mathbf{x}(s)) - 2\phi(\mathbf{x}(s))\mathcal{L}(\phi(\mathbf{x}(s))))ds = \int_0^t I(|z(s)|=1)ds. \tag{98}$$

Here z is a Markov Process with generator

$$Lf(z) = I(|z|=1)(f(0)-f(z)) + 2k(f(z+1)+f(z-1)-2f(z)).$$

We can see z as a random walker (at rate 2k) with an additional jumping rate from ±1 to 0 (at rate 1).
z spends as most as much time at ±1 as a regular symmetric random walk $\hat{z}$ which jumps to its neighbors at rate 2k and misses the additional pull to 0. Following the lines of the paper, such a random walker $\hat{z}(s)$ has the following bound on time spend at ±1 in expectation,

$$\int_0^t \mathbb{E}_{\hat{z}}(I(|\hat{z}(s)|=1)ds \leq C\sqrt{t} \text{ for some constant } C>0. \tag{99}$$

(99) must then also hold for z, which means that from (98) we get

$$\mathbb{E}\left[\frac{\langle M\rangle_t}{\sqrt{t}}\right] \leq C.$$

This brings us half the way of our prove, as we can conclude that

$$\mathbb{E}\left[\left(\frac{M_t}{\sqrt[4]{t}}\right)^2\right] \leq C.$$

Looking at (97), we see that what's left is to show that $H := \int_0^t I(|z(s)|=1)z(s)ds$ is $O(\sqrt[4]{t})$. In order to do this, we deviate from [16]. Our first step is breaking down time-interval [0,t] into distinct intervals between arrival times of $z(s)$ at 0. To warm us up to the full proof, we start out by assuming that we know exactly how many such arrivals there are.

8.4.3 Simpler problem: Deterministic number of arrival times

The holding times at $|z(s)|=1$ are exponentially distributed with rate $1+4k$, which makes holding times larger than $O(1)$ extremely unlikely. This means that if $|z(s)|$ spends $O(\sqrt{t})$ time at 1, then

there must be $O(\sqrt{t})$ instances of it arriving at 1.
This in turn means there are $O(\sqrt{t})$ instances of it arriving at 0 (as $t \to \infty$ the probability of at least half of the jumps from 1 going to 0 converges to one).
For now we assume for simplicity that the number of arrival times at 0 is exactly equal to $n = \lfloor m\sqrt{t} \rfloor$ for some $m > 0$.
We define $(\tau_j)_{j=1,..,n}$ as the arrival times of $z(s)$ at 0, i.e.

$$\tau_j := \inf\{s > \tau_{j-1} : z(s) = 0, z(s_-) \neq 0\} \wedge t \text{ with } \tau_0 = 0.$$

Then we define H_j as the signed time that $z(s)$ spends at ± 1 between two times of arrival at 0,

$$H_j := \int\limits_{\tau_{j-1}}^{\tau_j} (I(|z(s)| = 1)z(s)ds.$$

Note that these arrival times are integrable due to their boundedness by t, and that $\sum\limits_{j=1}^{n} H_j = H$.
For every j (except the last) we have that H_j is distributed as a random sum of independent holding times, where the amount of terms (i.e. amount of times $z(s) = \pm 1$ is visited before $z(s) = 0$) follows a geometric distribution and the terms are the holding times at $z(s) = \pm 1$, which are exponential.

$$H_j \overset{\mathrm{D}}{=} \sum_{i=1}^{V_j} h_{i,j} Z_j, \tag{100}$$

where

$$\begin{aligned}
&V_j \sim \text{Geo}\left(\frac{1+2k}{1+4k}\right) \text{ denotes the number of visits to } z(s) = \pm 1 \text{ before returning to } 0,\\
&h_{i,j} \sim \text{Exp}(1+4k) \text{ the holding times at } \pm 1,\\
&Z_j \sim Ber(1/2), \text{ where } Z_j = \begin{cases} 1 & \text{if } z(s) \geq 0 \text{ on } [\tau_{j-1}, \tau_j],\\ -1 & \text{if } z(s) \leq 0 \text{ on } [\tau_{j-1}, \tau_j]. \end{cases}
\end{aligned}$$

V_j, $h_{i,j}$ and Z_j are all nearly independent from each other, so for now we will treat them as such. Later will be argued why this is okay.
This means that $\mathbb{E}H_j = 0$ and $\sigma^2 := Var(H_j)$ is finite and can be explicitly calculated (under independence assumption) via

$$\begin{aligned}
Var(H_j) &= \mathbb{E}\left[Var\left(\sum_{i=1}^{V_j} h_{i,j} Z_j \middle| V_j\right)\right] + Var\left(\mathbb{E}\left[\sum_{i=1}^{V_j} h_{i,j} Z_j \middle| V_j\right]\right)\\
&= \mathbb{E}[V_j]Var(h_{i,j}) + 0\\
&= \frac{1+4k}{1+2k}\frac{1}{1+4k}\\
&= \frac{1}{1+2k}
\end{aligned}$$

by applying the total law of variation and noting that $\mathbb{E}[h_{i,j}Z_j|V_j] = 0 \; \forall V_j$, as Z_j is independent from Z_j and $h_{i,j}$.
We then apply the CLT to these H_j (or simply add their variances).
If $H_j \sim N(0, \sigma^2)$ for $j = 1, ..., n$ then since we have assumed that $n = \lfloor m\sqrt{t} \rfloor$, we have

$$\frac{\int\limits_0^t (I(|z(s)| = 1)z(s)ds}{\sqrt{\lfloor m\sqrt{t} \rfloor}} = \sqrt{n}\bar{H}_n \sim N(0, \sigma^2)$$

which shows that $\int\limits_0^t (I(|z(s)| = 1)z(s)ds$ is $O(\sqrt[4]{t})$, more specifically, its standard deviation is $\sigma\sqrt{\lfloor m\sqrt{t} \rfloor}$ and expectation is 0.

8.4.4 Random number of arrival times

The problem is that the number of arrival times at $z(s) = 0$ is not deterministically equal to some constant multiplied with $\sqrt{t}$. Let's call the number of arrivals at 0 on $[0, t]$ N_t^0. We then only know from point 3 that $\mathbb{E}[N_t^0] \leq C\sqrt{t}$ for some constant C. Intuitively, not knowing N_t^0 shouldn't be a problem. H is the sum of practically independent values of H_j, which means that the variance of H is approximately the sum of the variances of H_j. If this sum has a random number of terms that we assume to be independent from the values that are summed, we can use the Law of Total Variance. This is what we will do next.
Simplification: Treat N_t^0 and (H_j) as independent.
First let's investigate why this is a reasonable thing to do. The dependency between H_j and both H_k (for $k > j$) and N_t^0 comes from the following line of reasoning: Suppose H_j is large, i.e. $z(s)$ stayed at ± 1 for a long time on interval j. Then there is less time left until t for $z(s)$ to be at ± 1 in the future, making H_k smaller, and for $z(s)$ to revisit 0, making N_0^t smaller. There are 2 things to note in this reasoning:
One, the vast majority of time is spend at $|z(s)| \geq 2$, so even if H_j is relatively big, it will still be of order $o(\sqrt{t})$ and therefore not affect the total amount of time left, which is $O(t)$ for the vast majority of time.
Two, it's clear that H_j is actually negatively correlated both with H_k and N_t^0. The effect of this covariance on the variance of H will be negative, so not taking this covariance into account will lead to an overestimation of the variance. This is no problem, because we upper bound we create does not have to be tight, it just needs to prove H to be $O(\sqrt[4]{t})$.
Motivated by this line of reasoning we assume for the last step that for every $i, j \in \{1, \ldots, N_t^0\}$: $\mathrm{Cov}(H_i, H_j) < 0$ and similarly there is a negative covariance between H_j and N_t^0. Under this

assumption the Law of total variance tells us

$$
\begin{aligned}
Var(H) &= Var\left(\sum_{j=1}^{N_t^0} H_j\right) \qquad (101)\\
&= \mathbb{E}\left[Var\left(\sum_{j=1}^{N_t^0} H_j \middle| N_t^0\right)\right] + Var\left(E\left[\sum_{j=1}^{N_t^0} H_j \middle| N_t^0\right]\right)\\
&= \mathbb{E}\left[\mathbb{E}\left[\sum_{j=1}^{N_t^0} Var(H_j|N_t^0) \middle| N_t^0\right]\right] + \mathbb{E}\left[\mathbb{E}\left[\sum_{i=1}^{N_t^0}\sum_{j=1}^{N_t^0} Cov(H_i, H_j) \middle| N_t^0\right]\right] + 0\\
&\leq \mathbb{E}\left[\mathbb{E}\left[\sum_{j=1}^{N_t^0} Var(H_j) \middle| N_t^0\right]\right]\\
&\leq \mathbb{E}[N_t^0\sigma^2]\\
&\leq C\sigma^2\sqrt{t},
\end{aligned}
$$

where from line 3 to 4 our assumptions about negative covariance were used. This means

$$
Var\left(\frac{H}{t^{1/4}}\right) \leq C\sigma^2.
$$

Which concludes the proof. To summarize, we have

$$
\phi(\mathbf{x}(t)) := x(t) - \hat{x}(t) = M_t + \int_0^t I(|z(s)| = 1z(s)ds = M_t + H,
$$

with

$$
\frac{\langle M_t\rangle}{\sqrt{t}} \leq C \quad \text{and} \quad Var\left(\frac{H}{t^{1/4}}\right) \leq C\sigma^2 \quad \text{with} \quad \sigma^2 = \mathbb{E}\left[\text{Geo}\left(\frac{1+2k}{1+4k}\right)\right]\mathbb{E}\left[\text{Exp}(1+4k)\right],
$$

and of course the same for $y(t) - \hat{y}(t)$.
The simplification we made, that we can treat the values of N_t^0 and of H_j for $j \in \{1, \ldots, N_t^0\}$, and similarly H_j and H_k for $j \neq k$, as independent, is what prevents us from making the proof rigorous. We have argued that this simplification is not a problem, because as a result we overestimate the variance of H, which only means that the bound we find for $(\frac{H}{\sqrt[4]{t}})$ could have been tighter.
In order to make the proof rigorous, we would have to find an expression of $(\frac{H}{\sqrt[4]{t}})$ involving the covariances between N_t^0 and H_j from $\{H_j\}_{j\leq N_t^0}$, and prove that these covariances are nonpositive. Another approach may be to define $\hat{H}$, which is constructed via (100) and (101) but with independence of each of the random variables by construction. We may then argue that $|\hat{H}|$ stochastically dominates $|H|$, and thus boundedness of $\mathbb{E}\left|\frac{H}{\sqrt[4]{t}}\right|$ follows from boundedness of $\mathbb{E}\left|\frac{\hat{H}}{\sqrt[4]{t}}\right|$.

8.4.5 n-particle SIP/IRW coupling

Finally in order to finish this (informal) proof, we need to prove that this results does not just hold for a coupling between the 2-SIP and two independent random walkers, but holds for configurations with an arbitrary $n \in \mathbb{N}$ particles as well. This follows from the following two observations, which can again be taken directly from the [16].

1. The effect of binary collisions, i.e. interactions between exactly 2 neighboring particles, is the same in the setting with n-particle configuration as in 2-particle configurations. Such an effect can occur for every pair of particles, which means we have to sum this effect over each possible pair of particles. This means that we end up summing $\binom{n}{2}$ many $O(\sqrt[4]{t})$ terms, which remains $O(\sqrt[4]{t})$.

2. The probability of more than 2 particles interacting with each other at the same time (i.e. 3 or more particles being spread of two neighboring sites) converges to 0 very fast. According to [16], the time that at least 3 independent SIP particles are at nearest neighbor positions is dominated by $C + C_n ln(t)$, which means that this effect is negligible for our bound of order O($\sqrt[4]{t}$).

From these two observations we conclude that

$$\forall i = 1, \ldots, n, \forall t > 0 : \lim_{N \to \infty} \mathbb{E} \left| \frac{X_i(t) - \hat{X}_i(t)}{\sqrt[4]{t}} \right| < \tilde{C} \text{ for some finite } \tilde{C} > 0,$$

which yields (96).

9 Conclusion

In this thesis, we proved the hydrodynamic limit of the Brownian Energy Process in infinite volume. We showed the density field of this limit, representing the distribution of energy over the macroscopic space-dimension, evolves following dynamics described by the heat equation. This is in line with expectations brought forward in [17], where this finding was suggested, but not rigorously proven. Two useful tools were the Dynkin martingale, which allowed us to relate the evolution of the Brownian Energy Process to the Laplacian, and tightness, which allowed to prove that a hydrodynamic limit exists.

After this we proved the hydrodynamic limit of the Asymmetric Brownian Energy Process in infinite volume. We used the fact that this process can be transformed into its symmetric counterpart using the Cole-Hopf transformation, which we first proved, in order to express the density field of the Brownian Energy Process as a transformation of the density field of the Asymmetric Brownian Energy Process. This then allowed us to derive a PDE for the density field of the latter from the PDE of that of the former. Again via tightness the existence of such a limit density field was proven.

After this, we recreated this proof for the hydrodynamic limit of Dynamic Asymmetric Brownian Energy Process. We found that the method of proof relatively easily translated to this process, although for the resulting density field we found a PIDE that was not reducible to a PDE.

Lastly, propagation of chaos was proven for the Brownian Energy Process in infinite volume and for the Asymmetric Brownian Energy Process in finite volume. This result followed from the combination of findings in [16] and [3], where we proposed an informal proof of an improvement to a bound stated in [16].

This thesis leaves three questions unanswered that could be the focus of future studies. The first being whether conjecture 8.1 can be rigorously proven. This would require a more formal presentation of the arguments, and possibly an argument involving stochastic dominance, as suggested at the end of Chapter 8.4.4. Secondly, in our proof of the hydrodynamic limit of the DABEP from Theorem 7.7, we assumed the existence of a hydrodynamic limit, without proving this. We did this, because we expected that for this process the tightness arguments would work out in a very similar way to the BEP and the ABEP, however in order to make the proof rigorous, this should still be proven. Last, we may wonder whether it is possible to relax assumption 7.3 of finite total energy in the hydrodynamic limit of the Asymmetric Brownian Energy Process. This was left outside of the scope of this thesis, because relaxing this assumption makes the approach of the proof of hydrodynamic limit impossible, and introduces many other questions, among which the question whether the Asymmetric Brownian Energy Process in infinite volume with infinite energy even exists.

Apart from answering these three questions, the research of this thesis may be continued by proving the hydrodynamic limits of more general transport models with the property of attraction. The generalization from the ABEP to the DABEP is a good example of this. Another generalization for future work may be to focus on processes with location-dependent jumping rates. In recent years the trend in the study of interacting particles systems has been to move away from the fixed rate of $2k$ and instead have a vector $\alpha = \{\alpha_i,\ i \in V\}$, where V is the set of sites, determining the jumping rate from each site. It can be shown that duality holds for particle systems with corresponding arbitrary positive α in similar manner to those with a fixed parameter of $2k$ (see e.g. [4]). In future work one could investigate under what conditions one can find a hydrodynamic limit for such particle systems.

Appendix A. Overview of notation

	BEP	(D)ABEP	SIP	n-SIP	n-IRW
Reversible measure	$\nu_\theta^{2k,\infty}$	$\mu_\theta^{2k,L}$	$M_\theta^{2k,\infty}$	-	-
Initial distribution	ν_N	μ_N	-	-	-
(Accelerated) Generator	$\mathcal{L}^{\mathrm{BEP}_N}$	$\mathcal{L}^{\mathrm{ABEP}_N}$	$\mathcal{L}^{\mathrm{SIP}}$	$\mathcal{L}^{\mathrm{SIP}}$	-
(Accelerated) Path-space measure	$\mathbb{P}_{\nu_N}^{\mathrm{BEP}_N}$	$\mathbb{P}_{\mu_N}^{\mathrm{ABEP}_N}$	$\mathbb{P}_\eta^{\mathrm{SIP}}$	$\mathbb{P}_\xi^{\mathrm{SIP}}$	$\mathbb{P}_{x_1,\dots,x_n}^{\mathrm{IRW}}$
Hydrodynamic limit density	ρ_t	χ_t	-	-	-
Evolved configuration	y_t	x_t	η_t	ξ_t or $X_i(t)$	$\hat{X}_i(t)$
State space	$\mathbb{R}_+^{\mathbb{Z}}$	Ω_f	$\mathbb{N}_0^{\mathbb{Z}}$	Ω_n	Ω_n
Duality function with SIP	D^b	D^a	D^{SIP}	D^{SIP}	-
Algebraic operators	$\mathcal{K}_i^\alpha$	$\tilde{\mathcal{K}}_i^\alpha$	K_i^α	K_i^α	-

Appendix B. The Asymmetric Inclusion Process (ASIP)

A discrete IPS of interest in the asymmetric version of the SIP, the Asymmetric Inclusion Process (ASIP), introduced in [5]. How the asymmetry was achieved had to be carefully thought of, as adding asymmetry could mean the process generator is not longer associated to useful algebras, which means duality relations may be lost. Because of this, the ASIP(q,k) was constructed as the q-analog of the SIP(k). The following is the definition of the ASIP(q,k), where 2k is again the fixed jumping rate and q is a parameter representing the asymmetry:

Definition B.1 (The ASIP(q,k))**.** *For $q \in (0,1)$, let $[\cdot]_q$ denote a q-number, meaning for $n \geq 0$*

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

Then the ASIP(q,k) is the Markov process on $\mathbb{N}_0^{\mathbb{Z}}$ with generator defined on functions $f : \mathbb{N}_0^{\mathbb{Z}} \to \mathbb{R}$

$$\left(L^{\mathrm{ASIP}} f\right)(\eta) = \sum_{i=-\infty}^{\infty} \left(L_{i,i+1}^{\mathrm{ASIP}} f\right)(\eta), \text{ with}$$

$$\begin{aligned}\left(L_{i,i+1}^{\mathrm{ASIP}} f\right)(\eta) =& q_i^{\eta_i - \eta_{i+1} + (2k-1)} [\eta_i]_q [2k + \eta_{i+1}]_q (f(\eta^{i,i+1}) - f(\eta)) \\ &+ q^{\eta_i - \eta_{i+1} - (2k-1)} [2k + \eta_i]_q [\eta_{i+1}]_q (f(\eta^{i+1,i}) - f(\eta))\end{aligned}$$

This generator can be difficult to interpret, but one should think of $q \in (0,1]$ as a parameter quantifying the tendency for particles to preferentially jump to the right, where lower values of q correspond to a stronger asymmmetric. One can check that as we take $q \to 1$, the asymmetry dissapears and the SIP(k) is recovered.
The ASIP is an exception to the processes introduced in Chapter 4, in that its generator cannot be written via a representation of the su(1,1) Lie algebra. Instead we can write it a respresentation of the su$_q$(1,1) quantum Lie algebra, the q-analog of su(1,1). Discussing this algebra would require introducing quantum algebra's, which is not very relevant to this thesis, so instead we proceed by providing the self-duality function of the ASIP that can be found.

Proposition B.1. *The ASIP(q,k) is self-dual with self-duality function*

$$D^{\mathrm{ASIP}}(\eta,\xi)=\prod_{i=-\infty}^{\infty}\left(\frac{\binom{\eta_i}{\xi_i}_q}{\binom{\xi_i+2k-1}{\xi_i}_q}\cdot q^{(\eta_i-\xi_i)\left[2\sum_{m=1}^{i-1}\xi_m+\xi_i-4k\right]}\cdot 1_{\{\xi_i\le\eta_i\}}\right)$$

$$\text{with q-binomial }\binom{n}{m}_q=\frac{[n]_q!}{[m]_q![m-n]_q!}\text{ and q-factorial }[n]_q!=[n]_q[n-1]_q\ldots[1]_q.$$

Proof. See theorem 5.1 of [5]. □

Here we can use the fact that the ASIP(q,k) reduces to the SIP(k) as we let $q\to 1$. When we apply this limit to D^{ASIP}, we can see that D^{SIP} is recovered.
Another finding is that we can take a scaling limit of the ASIP in order to arrive at the ABEP, in similar fashion to how we showing that a scaling limit of the SIP produces the BEP in Theorem 4.3.

Proposition B.2. *Let $(\eta_t^N)_{N\in\mathbb{N}}$ denote a sequence of evolved ASIP($1-\frac{\sigma}{N}$, k)-configurations where $\lim_{N\to\infty}\frac{1}{N}\eta_0^N=x_0\in\Omega_f$. Then $x_t:=\lim_{N\to\infty}\frac{1}{N}\eta_t^N$ is the ABEP(σ, k) started form x_0, where convergence is weak in path space, i.e. for every local $f\in C_c^\infty(\Omega_f)$, we have*

$$\lim_{N\to\infty}\mathbb{P}_{\eta_0}^{\mathrm{ASIP}}\left(f\left(\frac{1}{N}\eta_t^N\right)\in B\right)=\mathbb{P}_{y_0}^{\mathrm{ABEP}}(f(y_t)\in B).$$

Proof. See [5]. □

Analogously to how one can derive the dynamic version of the ASEP (see [12]) and the ABEP (Definition 5.2), a dynamic version of the ASIP can be derived. Then in similar manner to Proposition B.2, it can be shown that the DABEP arises as a scaling limit of the dynamic ASIP. A publication including this finding is forthcoming.

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