Delft University of Technology
Faculty of Electrical Engineering, Mathematics and Computer Science
Delft Institute of Applied Mathematics

I. The Four Gap Theorem

and

II. Differences of random Cantor sets

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HENK DON
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“I. The Four Gap Theorem ”

and

“II. Differences of random Cantor sets ”

HENK DON

Delft University of Technology

Daily supervisor

Prof. dr. F.M. Dekking

Responsible professor

Prof. dr. F.M. Dekking

Other thesis committee members

Prof. dr. ir. K.I. Aardal

Dr. C. Kraaikamp

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Delft, the Netherlands

Summary

This thesis consists of two parts, which are separate with respect to content. The first part considers a subject in the field of number theory, while in the second part a subject from probability theory is studied.

The first part of this thesis deals with a variation on the Three Gap Theorem. The Three Gap Theorem states that the fractional parts of the first n multiples of an irrational number divide the interval $[0, 1]$ in subintervals of at most three different lengths. Instead of the fractional parts of these multiples, we considered the distances to the nearest integers, so the main question in this part of the thesis is:

What can we say about the distribution of the distances of multiples of an irrational number to the nearest integer?

We found a result similar to the Three Gap Theorem: these distances divide the interval $[0, 1/2]$ in subintervals of at most four different lengths. We give a proof of this result and also some additional properties are derived.[1]

The second part of this thesis is devoted to differences of random Cantor sets. The main question is the following:

Under which conditions does the algebraic difference between two random Cantor sets contain an interval?

There exist already some results in this direction. Dekking and Kuijvenhoven found some conditions under which the algebraic difference of two random Cantor sets contains an interval almost surely. In particular, they formulated the *joint survival condition* which they need to prove their main result. In the first chapter of Part II we try to find weaker conditions under which the result of Dekking and Kuijvenhoven can be proved. This results in the *triangle growth condition* and the *max-min growth condition*, of which the latter is the most promising condition. In the second chapter we consider a canonical class of random Cantor sets: *correlated fractal percolation*. Due to the elegant properties of this class, it is justified to pay special attention to it, but it also serves as a test case for the max-min growth condition. After derivation of some general results for correlated fractal percolation, we study some cases of *flimsy* correlated fractal percolation.

[1]This first part of the thesis is conditionally accepted by Acta Arithmetica.

Preface

This thesis contains the results of the research which I performed to complete my Master of Science Programme in Applied Mathematics at Delft University. It was me a great pleasure to do this job. Mathematics is a rich science, containing amazingly elegant structures. Working with and trying to solve problems in this discipline of science was both instructive and enjoyable. I hope that this thesis helps in convincing people of the beauty of mathematics.

Many people have contributed to the realization of this thesis. In the first place I want to thank my daily supervisor, Michel Dekking, who managed to find the perfect balance between letting me operate on my own and helping me with ideas on what direction to go or how to try to tackle the problems I encountered. He was almost always available to answer my questions or to have another interesting discussion on the topic of random Cantor sets.

I am also grateful to Cor Kraaikamp, who carefully read the first part of this thesis and gave me a lot of suggestions on how to improve the presentation of the proof of the Four Gap Theorem. Also Rob Tijdeman has to be mentioned, he came up with a suggestion to formulate the Four Gap Theorem in a more appealing way and he also gave many more useful comments on the first part of my work.

It is impossible to mention all those who supported me in doing this work, who made cups of coffee for me, who forgave me the moments that I was sunk into thoughts and who encouraged me to complete this thesis. I have appreciated all contributions, they relieved my task and motivated me to do the job.

Any comments or questions concerning the content of this thesis are welcome at henkdon@gmail.com.

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Part I

The Four Gap Theorem

Chapter 1

On the distribution of the distances of multiples of an irrational number to the nearest integer

1.1 Introduction

Take an arbitrary irrational number α and compute for the first n multiples the distance to the nearest integer. What can we say about the distribution of this sequence in the interval $[0, 1/2]$? In this paper we study the partition of the interval $[0, 1/2]$ induced by this sequence. The main result (Theorem 1.2) states that this sequence divides the interval in subintervals which can take at most four different lengths. This result is strongly related to the Three Gap Theorem, which states that for α irrational and $n \in \mathbb{N}$, the numbers

$$\{\alpha\}, \{2\alpha\}, \{3\alpha\}, \ldots, \{n\alpha\} \tag{1.1}$$

divide the interval $[0, 1]$ in subintervals of at most three different lengths. Here $\{x\} = x - \lfloor x \rfloor = x \bmod 1$ is the fractional part of x. The Three Gap Theorem was originally a conjecture of H. Steinhaus. Proofs were offered by various authors, for example by Sós [3], Świerckowski [5], Surányi [4], Slater [2] and van Ravenstein [1].

The Three Gap Theorem compares multiples of an irrational number with their floor. The floor of a number x is the nearest of the integers smaller than or equal to x. That inspired us to compare multiples of an irrational number with the nearest of *all* integers. We wrote a Matlab program, that takes a random number and computes the first n multiples of that number and their distances to the nearest integer, giving a sequence of n numbers in $[0,1/2]$. After that, the distances between consecutive numbers in this sequence were calculated. In doing this, we always found at most four different distances. We tried to prove that this was no coincidence, which resulted in the Four Gap Theorem. For the Matlab program, see Appendix A.

We start with Theorem 1.1, a variation on the Three Gap Theorem, which states that if we divide the interval $[0,1]$ in subintervals by the numbers

$$\{\alpha\},\{-\alpha\},\{2\alpha\},\{-2\alpha\},\ldots,\{n\alpha\},\{-n\alpha\} \tag{1.2}$$

then the subintervals again have at most three different lengths. We give an elementary proof for this theorem.

From Theorem 1.1 we extract the main result, Theorem 1.2. This 'Four Gap Theorem' gives an analogous statement about the distances to the nearest integers of the multiples of α: the numbers

$$||\alpha||,||2\alpha||,||3\alpha||,\ldots,||n\alpha|| \tag{1.3}$$

divide the interval $[0,\frac{1}{2}]$ in subintervals of at least two and at most four different lengths, where $||x||$ denotes the distance from x to the nearest integer. Here the number four is the best possible. We also derive some properties of the lengths of the subintervals in which $[0,\frac{1}{2}]$ is divided.

1.2 A variation on the three gap theorem

If we consider not only the fractional parts of the positive multiples of an irrational number α, but also of the negative multiples, we have the following result:

Theorem 1.1 *Let α be an irrational number between 0 and 1, and let $n\in\mathbb{N}, n\geq 1$. For the first n numbers in the sequence*

$$S_\alpha:\{\alpha\},\{-\alpha\},\{2\alpha\},\{-2\alpha\},\{3\alpha\},\{-3\alpha\},\ldots \tag{1.4}$$

the following assertions hold:

1. *They divide the interval $[0,1]$ in subintervals of either two or three different lengths, $l_1>l_2(>l_3)$. If we have three different lengths, $l_1>l_2>l_3$, then $l_1=l_2+l_3$.*
2. *By adding the $(n+1)$th element of the sequence S_α to the partition of $[0,1]$, one of the subintervals of length l_1 is divided in a subinterval of length l_2 and a subinterval of length l_1-l_2.*

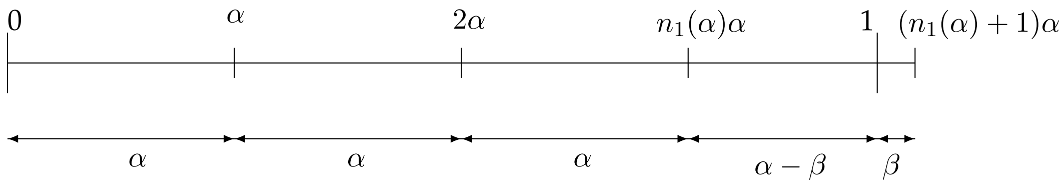

Figure 1.1: The first steps in the partition process.

Before proving the assertions we make some preparations by collecting observations that will be helpful in proving the assertions. Note that it makes no difference in Theorem 1.1 if we consider the open interval $(0, 1)$.

First note that for $x \in \mathbb{R} \setminus \mathbb{Z}$ we have {-x} = 1 - {x}, so the partition induced by the first $2n$ terms of the sequence S_α is symmetric with respect to $\frac{1}{2}$. This also means that without loss of generality we may assume that $\alpha < \frac{1}{2}$. Sometimes α will be called the *step size*.

It will prove useful to introduce some notation and definitions. For $n \geq 1$, $S_\alpha(n)$ denotes the nth term of S_α. For each $k \in \mathbb{N}, k \geq 1$ let $n_k(\alpha)$ be the unique integer for which:

$$n_k(\alpha)\alpha < k < (n_k(\alpha) + 1)\alpha. \tag{1.5}$$

Since α is irrational, k can never be a multiple of α. Define β by

$$\beta := (n_1(\alpha) + 1)\alpha - 1. \tag{1.6}$$

Note that $\beta = \{(n_1(\alpha) + 1)\alpha\}$. Figure 1.1 illustrates these definitions in case $n_1(\alpha) = 3$.

Definition 1.1 *For $k \in \mathbb{N}, k \geq 1$ the kth cycle of the sequence S_α consists of all those fractional parts $\{m\alpha\}, m \in \mathbb{Z}$ for which $k - 1 < |m\alpha| < k$, or equivalently $n_{k-1}(\alpha) < |m| \leq n_k(\alpha)$.*

Observe that a cycle consists of at least four partition points, because we assumed that $\alpha < \frac{1}{2}$. We are going to use this observation later. The next definition concerns intervals which are partitioned in the same way.

Definition 1.2 *For $0 \leq a, b < 1$, $y \leq \min\{1 - a, 1 - b\}$ and $n \in \mathbb{N}$ we write $(a, a + y)(n) \simeq (b, b + y)(n)$ if for all $x \in (0, y)$ the following equivalence holds:*

$$\exists k_1 \in \mathbb{Z}, |k_1| \leq n \text{ such that } a + x = \{k_1\alpha\}$$

$$\Longleftrightarrow$$

$$\exists k_2 \in \mathbb{Z}, |k_2| \leq n \text{ such that } b + x = \{k_2\alpha\}.$$

Note that $\simeq$ is an equivalence relation on the class of partitioned open subintervals of $(0,1)$. If we replace $b+x$ by $b+y-x$ in Definition 1.2, we get an equivalence for an interval and the mirror image of an other interval. If two intervals satisfy this adjusted definition, we will write $(a, a+y)(n) \overset{m}{\simeq} (b, b+y)(n)$.

Now let us investigate what happens in the interval $(0,\alpha)$. Note that S_α is a sequence in the *open* interval $(0,1)$. Therefore also here we investigate which values we get in the *open* interval $(0,\alpha)$. For $k \in \mathbb{N}, k \geq 1$ the interval $(k, k+\alpha)$ contains exactly one positive multiple of α and the interval $(-k, -k+\alpha)$ contains exactly one negative multiple of α. Hence in each cycle we get two values in $(0,\alpha)$, one of them being the fractional part of a positive multiple of α and the other being the fractional part of a negative multiple of α. The first cycle is an exception, since there is no positive multiple of α in $(0,\alpha)$.

The first positive multiple of α for which the fractional part is in $(0,\alpha)$ is $(n_1(\alpha)+1)\alpha = 1+\beta$, which gives β as a first hit in $(0,\alpha)$. Because $1+\beta$ is a positive multiple of α, the numbers $k+k\beta$ are also positive multiples of α, where $k \in \mathbb{N}$. The fractional parts of these numbers are (fractional parts of) multiples of β. As long as $k\beta < \alpha$ this gives hits in $(0,\alpha)$. As soon as $k\beta$ exceeds α, i.e. when $k = \lfloor \alpha/\beta \rfloor + 1$, we leave the interval $(0,\alpha)$, but in that case we already had hit the value $k\beta - \alpha$. This is exactly how it continues all the time: each next hit in $(0,\alpha)$ is shifted β in positive direction and as soon as we leave the interval, we come back modulo α. Hence the kth hit by the fractional part of a positive multiple of α in $(0,\alpha)$ is $k\beta \bmod \alpha$.

The first negative multiple of α for which the fractional part is in $(0,\alpha)$ is $-n_1(\alpha)\alpha$, giving the value $\{-n_1(\alpha)\alpha\} = 1-\{n_1(\alpha)\alpha\} = 1-n_1(\alpha)\alpha = \alpha-\beta$. Each next hit in $(0,\alpha)$ is shifted β to the left until $\alpha - k\beta$ dives under 0. In that case we leave $(0,\alpha)$, but we should note that the previous hit was $\alpha - k\beta + \alpha$, which is in $(0,\alpha)$. Hence the kth hit by the fractional part of a negative multiple of α in $(0,\alpha)$ is $\alpha - (k\beta \bmod \alpha)$.

By noting that the hits by fractional parts of positive and negative multiples of α are alternating we see that in $(0,\alpha)$ we get the following sequence of hits:

$$\begin{aligned} &\alpha - (\beta \bmod \alpha), \beta \bmod \alpha, \alpha - (2\beta \bmod \alpha), 2\beta \bmod \alpha, \\ &\qquad \alpha - (3\beta \bmod \alpha), 3\beta \bmod \alpha, \ldots \end{aligned} \tag{1.7}$$

By multiplying each term by $1/\alpha$ we get

$$1 - \Big(\frac{\beta}{\alpha} \bmod 1\Big), \frac{\beta}{\alpha} \bmod 1, 1 - \Big(\frac{2\beta}{\alpha} \bmod 1\Big), \frac{2\beta}{\alpha} \bmod 1,$$

$$1-\Big(\frac{3\beta}{\alpha} \bmod 1\Big), \frac{3\beta}{\alpha} \bmod 1, \ldots \tag{1.8}$$

By defining $\tilde{\alpha} := 1 - \dfrac{\beta}{\alpha}$, we can rewrite this as

$$\{\tilde{\alpha}\}, \{-\tilde{\alpha}\}, \{2\tilde{\alpha}\}, \{-2\tilde{\alpha}\}, \{3\tilde{\alpha}\}, \{-3\tilde{\alpha}\}, \ldots \tag{1.9}$$

Hence (1.7) is a scaled version of the sequence S_α (with a different irrational step size). That means that the partition of the subinterval $(0, \alpha)$ has exactly the same structure and properties as the partition of $(0, 1)$. The same self-similarity holds for the subintervals $(\alpha, 2\alpha), \ldots, ((n_{1/2}(\alpha) - 1)\alpha, n_{1/2}(\alpha)\alpha)$, where $n_{1/2}(\alpha)\alpha$ is the largest multiple of α smaller than $1/2$. In these subintervals we get the same sequence (1.7), but now shifted by a multiple of α to the corresponding positions in the subinterval. By using symmetry we also find the same structure of lengths for the intervals $(1 - n_{1/2}(\alpha)\alpha, 1 - (n_{1/2}(\alpha) - 1)\alpha), \ldots, (1 - \alpha, 1)$. These intervals are mirror images of the subintervals $(0, \alpha), \ldots, ((n_{1/2}(\alpha) - 1)\alpha, n_{1/2}(\alpha)\alpha)$.

Each cycle of S_α gives two hits in each of those intervals. We conclude that for all $k \in \mathbb{N}, k \geq 1$

$$\begin{gathered}(0, \alpha)(n_k(\alpha)) \simeq \ldots \simeq ((n_{1/2}(\alpha) - 1)\alpha, n_{1/2}(\alpha)\alpha)(n_k(\alpha)) \overset{m}{\simeq} \\ (1 - n_{1/2}(\alpha)\alpha, 1 - (n_{1/2}(\alpha) - 1)\alpha)(n_k(\alpha)) \simeq \ldots \simeq (1 - \alpha, 1)(n_k(\alpha)).\end{gathered} \tag{1.10}$$

The only part which is not yet considered is the middle part of $(0, 1)$: the interval $(n_{1/2}(\alpha)\alpha, 1 - n_{1/2}(\alpha)\alpha)$, which will be denoted by I_m. First consider the positive multiples of α. Note that the length of the complement of I_m is a multiple of the step size α, which implies that the values we hit in I_m are of the form $n_{1/2}(\alpha)\alpha + (k\alpha \bmod L)$, for integer k, where L denotes the length of I_m. By symmetry we see that by adding the negative multiples of α too, we find the following sequence of hits in I_m:

$$\begin{gathered} n_{1/2}(\alpha)\alpha + (\alpha \bmod L), 1 - n_{1/2}(\alpha)\alpha - (\alpha \bmod L), \\ n_{1/2}(\alpha)\alpha + (2\alpha \bmod L), 1 - n_{1/2}(\alpha)\alpha - (2\alpha \bmod L), \\ n_{1/2}(\alpha)\alpha + (3\alpha \bmod L), 1 - n_{1/2}(\alpha)\alpha - (3\alpha \bmod L), \ldots, \end{gathered} \tag{1.11}$$

where the alternating order follows from the fact that the successor of $\{k\alpha\}$ in S_α is $\{-k\alpha\}$.

Subtract $n_{1/2}(\alpha)\alpha$ to get

$$\begin{gathered} \alpha \bmod L, L - (\alpha \bmod L), 2\alpha \bmod L, \\ L - (2\alpha \bmod L), 3\alpha \bmod L, L - (3\alpha \bmod L), \ldots \end{gathered} \tag{1.12}$$

Multiplying by $1/L$ yields

$$\begin{aligned}\frac{\alpha}{L} \bmod 1, 1 - \Big(\frac{\alpha}{L} \bmod 1\Big), \frac{2\alpha}{L} \bmod 1,\\ 1 - \Big(\frac{2\alpha}{L} \bmod 1\Big), \frac{3\alpha}{L} \bmod 1, 1 - \Big(\frac{3\alpha}{L} \bmod 1\Big), \ldots\end{aligned} \tag{1.13}$$

This is exactly S_α, with step size α/L. It follows that (1.11) is a scaled and translated version of the sequence S_α with a different step size.

The next step is to find the relation between the behavior of the partition process in I_m and its complement. The intervals $(0, L)$ and I_m have the same length (by definition of L) and the distance between their left endpoints is a multiple of α. From this we can conclude that in each cycle a value $x \in (0, L)$ is hit if and only if in the same cycle the point $x + n_{1/2}(\alpha)$ is hit in I_m. This reasoning is also valid when $(0, L)$ and I_m are not disjoint (which is possible when $L > \alpha$). By noting that $(0, L)(n_1(\alpha)) \simeq I_m(n_1(\alpha))$ and using induction on k it follows that $\forall k \in \mathbb{N}, k \geq 1$:

$$(0, L)(n_k(\alpha)) \simeq I_m(n_k(\alpha)). \tag{1.14}$$

In words: after each complete cycle the two intervals $(0, L)$ and I_m are partitioned in an equivalent way in the sense of Definition 1.2.

To prove Theorem 1.1, we use induction on the cycle number k. Note that if the theorem holds for n, then to go to $n + 1$ it suffices to check the second assertion of the theorem. We can see this as follows. If we had three lengths, then one of the longest subintervals is divided in two existing lengths, so we get nothing new. If we had two lengths, then we get one new length, being the difference of the two existing lengths. These remarks show that the 'at most three' part of the first assertion and the requirement $l_1 = l_2 + l_3$ in case of three lengths are not violated. The 'at least two' part of the first assertion of the theorem follows from the irrationality of α. If only one length is remaining, the interval $[0, 1]$ must be divided in equal parts. Hence in this case α would be a rational number.

Proof of Theorem 1.1

-Step 1- The first step in our induction argument is to show that during the first cycle (containing the first $2n_1(\alpha)$ terms of S_α) always one of the longest subintervals is divided in two intervals of which one has the second length occurring before the division. The first number in the sequence S_α is $\{\alpha\}$, so after adding this first number the interval $(0, 1)$ is divided in two subintervals, one of length α and one of length $1 - \alpha$, where the latter is the longest in view of our assumption that $\alpha < 1/2$. So now this longest subinterval should be divided in a part of length α (the second length) and

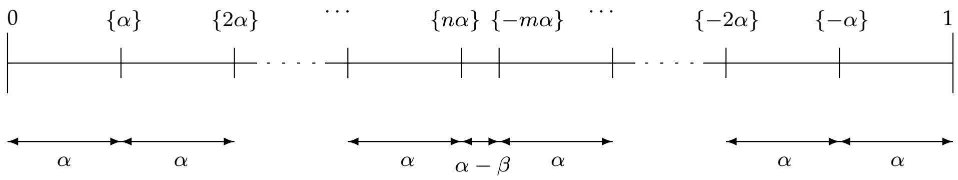

Figure 1.2: Halfway the first cycle: either $n = m$ or $n - 1 = m$.

a remaining part. Because the second hit is $\{-\alpha\} = 1 - \alpha$ this is indeed the case. The process continues in the same way, each time reducing the length of the middle subinterval by α, until the length of the middle subinterval becomes smaller than α. Now this middle subinterval has length $\alpha - \beta$, by definition of β.

At this point we have two different lengths: α and $\alpha - \beta$. The situation is illustrated by Figure 1.2. We now distinguish two cases.
If $n = m$, then the next hit will be $\{(n+1)\alpha\}$, dividing an interval of length α in a part of length $\alpha - \beta$ (which was the second length) and a part of length β (a new length). Now we have three different lengths and the sum of the two smallest equals the largest, as required. The next hit now is $\{-(n+1)\alpha\}$ and again this divides an interval of length α in a part of length $\alpha - \beta$ and a part of length β. The partition process continues in this way as long as we are in the first cycle.
If $n + 1 = m$, then the next hit will be $\{-(n+1)\alpha\}$ and also in this case all intervals of length α will successively be divided in a part of length $\alpha - \beta$ and a part of length β.
Hence we conclude that the theorem holds for $1 \leq n \leq 2n_1(\alpha)$.

-*Step 2*- The next step in the induction argument is to show that if the theorem holds in the first k cycles, then the theorem also holds in the next cycle. To prove this we use the observations made before, which state that the behavior of the partition process in each of the intervals

$$(0,\alpha),\ \ldots,\ ((n_{1/2}(\alpha)-1)\alpha, n_{1/2}(\alpha)\alpha),\ (n_{1/2}(\alpha)\alpha, 1 - n_{1/2}(\alpha)\alpha),$$
$$(1 - n_{1/2}(\alpha)\alpha, 1 - (n_{1/2}(\alpha)-1)\alpha),\ \ldots,\ (1-\alpha, 1) \qquad (1.15)$$

has after rescaling the same properties as the behavior in the entire interval $(0,1)$. From now on we will call these intervals *elementary intervals*.

A crucial remark is that all boundaries (except 0 and 1) of the elementary intervals belong to the first cycle of S_α. This implies that (at any point in one of the next cycles) the subintervals in which $(0,1)$ is divided can only intersect *one* of the elementary intervals. This guarantees that to find all lengths of subintervals in $(0, 1)$, it suffices to find all lengths in the elementary intervals.

For the elementary intervals we introduce the following abbreviations:

$$I^k := ((k-1)\alpha, k\alpha), \tag{1.16}$$

$$I^{-k} := (1 - k\alpha, 1 - (k-1)\alpha), \tag{1.17}$$

where $1 \leq k \leq n_{1/2}(\alpha)$. Recall that for the middle elementary interval we already introduced the symbol I_m. The sequence of hits in an elementary interval I will be denoted by S_α^I. For example, $S_\alpha^{I^1}$ is equal to the sequence (1.7). Because these sequences are scaled and translated versions of S_α (possibly with a different step size), we can also here introduce cycles. We are going to use this later, but we do not need to specify these cycles explicitly.

Induction Hypothesis: Assume that for all α the theorem holds as long as we are in one of the first k cycles of S_α, where $k \geq 1$.

Consider the partition of $(0, 1)$ in subintervals by the first n terms of S_α. Denote the lengths of the subintervals by $l_1 > l_2 (> l_3)$. To complete our proof of the theorem it suffices to show that the following three requirements are satisfied if $S_\alpha(n+1)$ is an element of the $(k+1)$th cycle of S_α:

Requirement 1: If $S_\alpha(n+1)$ is the very first hit in an elementary interval, then it should divide a subinterval of length l_1 in subintervals of length l_2 and $l_1 - l_2$.

Requirement 2: If $S_\alpha(n+1) \in I$, where I is one of the elementary intervals, then I should contain one of the subintervals of length l_1.

Requirement 3: If $S_\alpha(n+1)$ is not the very first hit in an elementary interval, denote the two largest lengths *in this elementary interval* by $\hat{l}_1 > \hat{l}_2$. $S_\alpha(n+1)$ should divide a subinterval of length $\hat{l}_1$ in subintervals of length $\hat{l}_2$ and $\hat{l}_1 - \hat{l}_2$.

We check each of the three requirements in the substeps below.

-Substep 2.1- The theorem only gives an assertion about the division in subintervals if we have already at least two lengths. Hence our induction hypothesis makes no statement about the very first hit in an elementary interval. Therefore we should start by checking that in each of the elementary intervals the partition process starts in the right way. If the very first hit in an elementary interval is an element of the first cycle of S_α, then we have no problems, because we already checked that the theorem holds for

$1 \leq n \leq 2n_1(\alpha)$.

The only elementary interval which possibly contains no element of the first cycle is I_m, if $L < \alpha$. Suppose the first value we hit in $(0, L)$ is x. Then in the same cycle we hit the value $n_{1/2}(\alpha)\alpha + x$. This is the first hit in I_m. This first hit divides I_m in exactly the same way as x has divided $(0, L)$. That means, two subintervals originate with lengths already occurring before the division (viz. l_2 and $l_3 = l_1 - l_2$).

-Substep 2.2- After each complete cycle of S_α, I_m is partitioned in a symmetric way. This means that the longest subinterval occurring in I_m is a subinterval of $(n_{1/2}(\alpha)\alpha, (n_{1/2}(\alpha) + 1)\alpha)$. From this observation, combined with (1.10) and (1.14) it follows that after each complete cycle all the intervals I^k,I^{-k}, where $1 \leq k \leq n_{1/2}(\alpha)$, contain a subinterval which has the maximal length. Now note that in each cycle the order in which the elementary intervals will get hits is as follows (writing $n_{1/2}(\alpha)$ as $n^\alpha_{1/2}$ for typographical reasons):

$$\underbrace{I^1, I^{-1}, I^2, I^{-2}, \ldots, I^{n^\alpha_{1/2}}, I^{-n^\alpha_{1/2}}}_{1st\ sequence}, \underbrace{I_m, \ldots, I_m}_{2nd\ sequence}, \underbrace{I^{-n^\alpha_{1/2}}, I^{n^\alpha_{1/2}}, \ldots, I^{-2}, I^2, I^{-1}, I^1}_{3rd\ sequence}, \tag{1.18}$$

where the second sequence can be empty. Because all elementary intervals in the first sequence (as indicated by (1.18)) contain a subinterval of the maximal length, the corresponding hits in these elementary intervals satisfy Requirement 2.

After the third sequence of hits a cycle is completed, so after this sequence again all the intervals I^k,I^{-k}, where $1 \leq k \leq n_{1/2}(\alpha)$, contain a subinterval which has the maximal length. Before the third sequence the maximal subinterval in each of these elementary intervals was certainly not smaller, which shows that the hits in the third sequence also satisfy Requirement 2.

The hits corresponding to the second sequence in (1.18) can only violate the Requirement 2 if the last of these hits does so. This last hit gives a value in $(n_{1/2}(\alpha)\alpha, (n_{1/2}(\alpha) + 1)\alpha)$. After the third sequence the $(k+1)$th cycle is complete and hence we have the equivalence $(0, \alpha)(n_{k+1}(\alpha)) \simeq (n_{1/2}(\alpha)\alpha, (n_{1/2}(\alpha) + 1)\alpha)(n_{k+1}(\alpha))$. The third sequence gives only one hit in $(0, \alpha)$. The distance between this hit and the last hit of the second sequence is $n_{1/2}(\alpha)\alpha$. It follows that the last hit of the second sequence gives exactly the same division in the middle elementary interval as the third sequence gives in $(0, \alpha)$. By equivalences and symmetry the same holds for the other elementary intervals. Hence the hits in the second sequence also satisfy Requirement 2.

-Substep 2.3- To check Requirement 3 we use our induction hypothesis. Suppose x is a hit in an elementary interval I. Suppose x is an element of one of the first k cycles of S_α^I. Then by our induction hypothesis it follows that x divides an interval of lenght $\hat{l}_1$ in a part of length $\hat{l}_2$ and a part of length $\hat{l}_1 - \hat{l}_2$, where $\hat{l}_1 > \hat{l}_2 (> \hat{l}_3)$ are the lengths of the subintervals in I.

If an elementary interval I has length not larger than α, then we get in one cycle of S_α at most 2 hits in I. After $k+1$ complete cycles of S_α, we have recorded at most $2(k+1)$ values in I. After k complete cycles of S_α^I, we have at least $4k$ hits in I. $k \geq 1$, so $4k \geq 2(k+1)$ which shows that Requirement 3 is satisfied for hits in I. The only elementary interval to which this argument possibly not applies is I_m, if $L > \alpha$.

Consider I_m, and suppose $L > \alpha$. Note that from the definitions it follows that $L = 2\alpha - \beta$. In the first cycle of S_α we get 2 hits in I_m. Each next cycle of S_α gives either 2 or 4 hits in I_m. After $k+1$ complete cycles of S_α, we have recorded at most $4k+2$ values in I_m. After k complete cycles of $S_\alpha^{I_m}$, we have at least $4k$ hits in I_m. It follows that if the $(k+1)$th cycle of S_α gives two hits in I_m, then Requirement 3 is satisfied. If the $(k+1)$th cycle of S_α gives four hits in I_m, then it suffices to check if the last two of these four hits satisfy Requirement 3.

Suppose the $(k+1)$th cycle of S_α gives four hits in I_m. Denote them by $x_1, \ldots, x_4$. Then x_3 and x_4 are in $((n_{1/2}(\alpha)+1)\alpha, 1-n_{1/2}(\alpha)\alpha)$ and $(n_{1/2}(\alpha)\alpha, 1-(n_{1/2}(\alpha)+1)\alpha)$ respectively. These intervals have both length $L - \alpha = \alpha - \beta$. The distance between x_4 and the next hit x in $(0, \alpha - \beta)$ is a multiple of α and by (1.14) we know that after adding both to the partition of $(0,1)$ we have $(n_{1/2}(\alpha)\alpha, 1-(n_{1/2}(\alpha)+1)\alpha)(n) \simeq (0, \alpha-\beta)(n)$, for some $n \in \mathbb{N}$. Hence x_4 divides $(n_{1/2}(\alpha)\alpha, 1-(n_{1/2}(\alpha)+1)\alpha)$ in exactly the same way as x divides $(0, \alpha - \beta)$. We already know that x satisfies Requirement 3 and therefore also x_4 satisfies Requirement 3. Using symmetry we see that an analogous argument applies to x_3, which completes the proof. □

1.3 A Four Gap Theorem

We are now in position to prove our main theorem, the 'Four Gap Theorem'.

Theorem 1.2 *(The Four Gap Theorem) Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $n \in \mathbb{N}$. Let $||x||$ denote the distance from x to the nearest integer. The numbers*

$$||\alpha||, ||2\alpha||, ||3\alpha||, \ldots, ||n\alpha|| \tag{1.19}$$

divide the interval $[0, \frac{1}{2}]$ in subintervals of at least two and at most four different lengths. For these lengths the following assertions hold:

1. *The rightmost length, denoted by l_r, is unique.*
2. *There are two different lengths if and only if $n||\alpha|| < \frac{1}{2}$.*
3. *If we have three different lengths, denote the two lengths not equal to l_r by $l_1 > l_2$. Then exactly one of the following four equalities holds: $2l_r = l_1$, $2l_r = l_2$, $2l_r + l_2 = l_1$ or $l_1 + l_2 = 2l_r$.*
4. *If we have four different lengths, denote the three lengths not equal to l_r by $l_1 > l_2 > l_3$. Then $l_1 = l_2 + l_3$ and one of these lengths is equal to twice l_r.*

Proof. It is not possible to have only one length occurring, since α is irrational. Without loss of generality we assume that $\alpha \in [0, 1]$, since we may consider the fractional part of α.

Observe that $\min\{\{x\}, \{-x\}\} \in [0, \frac{1}{2}]$. So if we look at the sequence

$$\min\Big\{\{\alpha\}, \{-\alpha\}\Big\}, \min\Big\{\{2\alpha\}, \{-2\alpha\}\Big\}, \min\Big\{\{3\alpha\}, \{-3\alpha\}\Big\}, \dots \tag{1.20}$$

we get a subsequence of the sequence S_α. A term of the sequence S_α is also a term of the sequence (1.20) if and only if it is in $[0, \frac{1}{2}]$. Consequently, by Theorem 1.1, the first n terms of the sequence (1.20) divide the interval $[0, \frac{1}{2}]$ in subintervals of at least two and at most four different lengths. We already had three different lengths and possibly we get a fourth because we cut it off at $1/2$.

Now note that

$$\min\Big\{\{n\alpha\}, \{-n\alpha\}\Big\} = ||n\alpha||, \tag{1.21}$$

and it follows that the numbers in (1.19) divide $[0, \frac{1}{2}]$ in subintervals of at least two and at most four different lengths.

We now turn our attention to the four assertions about these lengths. If the rightmost length is not unique, then there exist integers $0 \leq k, l, m \leq n, l \neq m$ such that

$$\frac{1}{2} - ||k\alpha|| = ||l\alpha|| - ||m\alpha||, \tag{1.22}$$

which implies that $\frac{1}{2}$ is the sum of a multiple of α and an integer, contradicting the irrationality of α. Hence the rightmost length l_r is unique.

If $n||\alpha|| < \frac{1}{2}$, then the only lengths are $||\alpha||$ and l_r, so we have only two different lengths. Now assume that we have only two different lengths. The leftmost interval has length $\min_{1\leq k\leq n} ||k\alpha||$. It follows that the numbers $||\alpha||, \dots, ||n\alpha||$ are all multiples of $\min_{1\leq k\leq n} ||k\alpha||$. From the irrationality of α we conclude that $\min_{1\leq k\leq n} ||k\alpha|| = ||\alpha||$ and $||n\alpha|| = n||\alpha||$, which is only possible if $n||\alpha|| < \frac{1}{2}$.

Consider the partition of $[0, 1]$ by the numbers

$$\{\alpha\}\,,\{-\alpha\}\,,\{2\alpha\}\,,\{-2\alpha\}\,,\ldots,\{n\alpha\}\,,\{-n\alpha\}\,. \tag{1.23}$$

This partition is symmetric with respect to $\frac{1}{2}$. The subintervals in which $[0, 1]$ is divided by these numbers, have either two or three different lengths, according to Theorem 1.1. In the former case, if cutting it off at $\frac{1}{2}$ gives three lengths, either $2l_r = l_1$ or $2l_r = l_2$. In the latter case, if cutting it off at $\frac{1}{2}$ gives three lengths, either $2l_r + l_2 = l_1$ or $l_1 + l_2 = 2l_r$.

The last assertion of the Four Gap Theorem follows immediately from Theorem 1.1 and the observations made before. □

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Part II

Differences of random Cantor sets

Chapter 2

Introduction

We construct an $M-$adic random Cantor set F using the following mechanism: take the unit interval and divide it in M subintervals of equal length. Each of those subintervals corresponds to a number in the alphabet $\mathbb{A} = \{0, \dots, M-1\}$. It will be convenient to consider $\mathbb{A}$ as an Abelian group with addition. So for instance if $M = 6$ we have $5 + 3 = 2$. Now define a *joint survival distribution* μ on $2^{2^{\mathbb{A}}}$. According to this distribution we choose which subintervals are kept and which are discarded. Then in each next construction step, each of the surviving subintervals is again divided in M subintervals of equal length, of which again a subset survives according to the distribution μ.

The marginal probabilities p_i for $i \in \mathbb{A}$ are defined by

$$p_i := \sum_{X \subseteq \mathbb{A}: i \in X} \mu(X). \tag{2.1}$$

An important role is played by the cyclic correlation coefficients γ_k, $k \in \mathbb{A}$, defined by

$$\gamma_k := \sum_{i=0}^{M-1} p_i p_{i+k}. \tag{2.2}$$

Our main question is whether or not the algebraic difference $F_1 - F_2$ of two random Cantor sets F_1 and F_2 constructed in this way contains an interval. For both sets we take the same M, but not necessary the same joint survival distribution. We can distinguish between joint survival distributions selecting intervals independently and joint survival distributions not having this property. When we do not allow for dependence, the problem is somewhat less complicated, but still far from trivial. Intervals are selected and discarded independently if and only if the joint survival distribution satisfies the equality

$$\mu(X) = \prod_{i \in X} p_i \prod_{i \notin X} (1 - p_i) \tag{2.3}$$

for all $X \subseteq \mathbb{A}$. For the case where intervals are discarded independently, we already have the following result, due to Dekking and Simon in [1].

Theorem 2.1 *Consider two independent random Cantor sets F_1 and F_2 with survival probabilities $p_0, \ldots, p_{M-1}$.*

1. *If $\gamma_k > 1$ for all $k = 0, \ldots, M-1$, then $F_1 - F_2$ contains an interval a.s. on $\{F_1 - F_2 \neq \emptyset\}$.*
2. *If $\gamma_k, \gamma_{k+1} < 1$ for some k, then $F_1 - F_2$ contains no interval a.s.*

For general (by *general* we mean that dependent intervals are allowed) joint survival distributions the same result is proved in [2] (the result also holds for the asymmetric case, that is, the joint survival distributions of F_1 and F_2 need not to be the same), but an additional condition is needed in the proof: the joint survival distributions μ and λ of F_1 and F_2 should satisfy the following condition:

Condition 2.1 *A joint survival distribution $\mu : 2^{2^{\mathbb{A}}} \to [0,1]$ satisfies the* joint survival condition *if it assigns positive probability to its marginal support* $\mathrm{Supp}_m(\mu)$*, which is defined by*

$$\mathrm{Supp}_m(\mu) := \bigcup \{X \subseteq \mathbb{A} : \mu(X) > 0\} \qquad (2.4)$$

In Chapter 3 we will try to improve on this condition. The result of Dekking and Kuijvenhoven is the following:

Theorem 2.2 *Consider two independent random Cantor sets F_1 and F_2 whose joint survival distributions satisfy Condition 2.1, the joint survival condition.*

1. *If $\gamma_k > 1$ for all $k = 0, \ldots, M-1$, then $F_1 - F_2$ contains an interval a.s. on $\{F_1 - F_2 \neq \emptyset\}$.*
2. *If $\gamma_k, \gamma_{k+1} < 1$ for some $k \in \mathbb{A}$, then $F_1 - F_2$ contains no interval a.s.*

2.1 Notation and approach to the problem

The algebraic difference of F_1 and F_2 can be seen as a projection under $45°$ of the Cartesian product $F_1 \times F_2$. If $\phi : \mathbb{R}^2 \to \mathbb{R}$ is given by $\phi(x,y) = x - y$, then $F_1 - F_2 = \phi(F_1 \times F_2)$. From this point of view we will try to investigate the question if an interval will occur in the algebraic difference.

We denote the joint survival distributions of F_1 and F_2 by μ and λ, and the corresponding vectors of marginal probabilities by $\mathbf{p}$ and $\mathbf{q}$ respectively.

The n-th level approximations are denoted by F_1^n and F_2^n. We define the following subsets of the unit square $[0,1]^2$:

$$\Lambda^n := F_1^n \times F_2^n, \quad n \geq 0, \qquad \Lambda := F_1 \times F_2 = \bigcap_{n=0}^{\infty} \Lambda^n. \tag{2.5}$$

Strings over the alphabet $\mathbb{A}$ can be interpreted as M-ary expansions of numbers. Let $\mathcal{T}$ be the M-ary tree, the set of all strings over $\mathbb{A}$. The set of strings

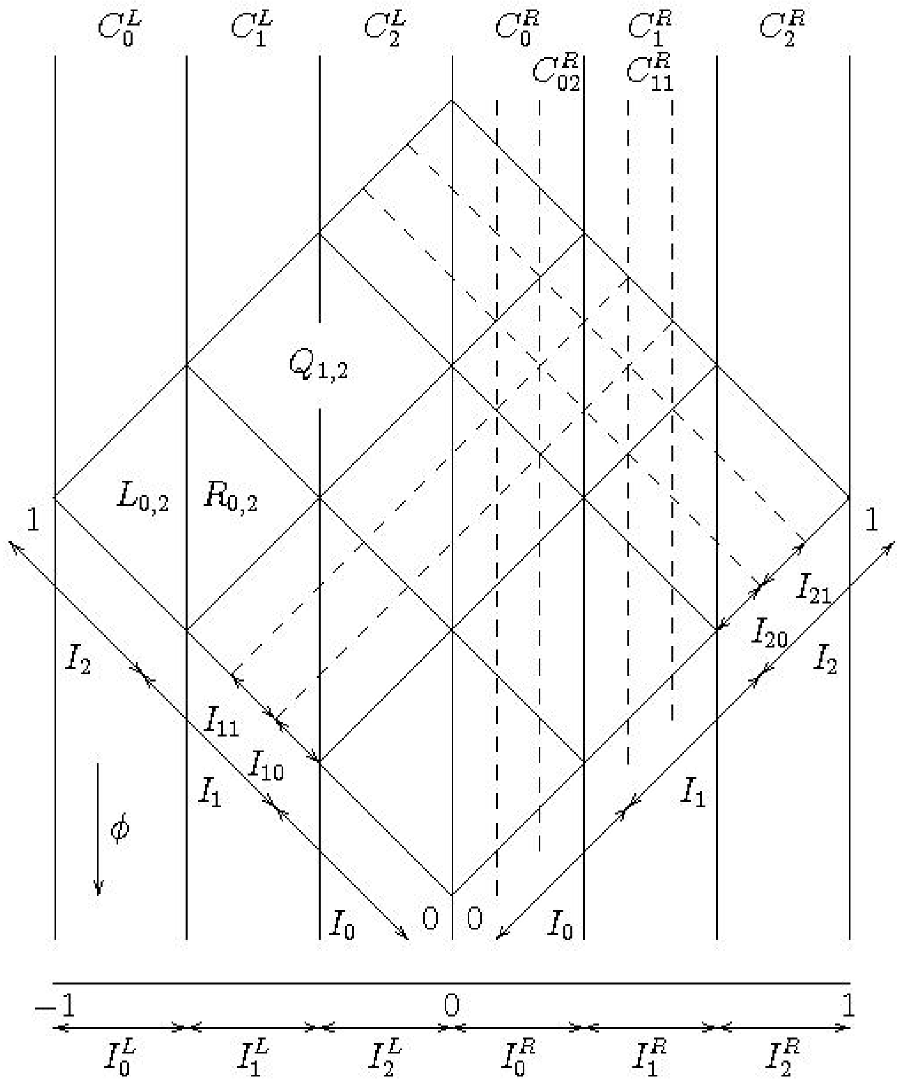

Figure 2.1: An illustration for $M = 3$ of the unit square $[0,1]^2$ rotated by $45°$, being projected by ϕ to a $\sqrt{2}$-scaled-down version of $[-1,1]$. The columns $C^U_{\underline{k}_n}$ split the n-th level squares $Q_{\underline{i}_n,\underline{j}_n} = I_{\underline{i}_n} \times I_{\underline{j}_n}$ into the 'left' and 'right' triangles $L_{\underline{i}_n,\underline{j}_n}$ and $R_{\underline{i}_n,\underline{j}_n}$.

of length n is denoted by $\mathcal{T}_n$ and corresponds to the set of all nodes at level n. For all $i_1 \dots i_n \in \mathcal{T}$ we let $[i_1 \dots i_n]$ denote the value of $i_1 \dots i_n$ as an M-ary number:

$$[i_1 \dots i_n]_M := \sum_{k=1}^{n} M^{n-k} i_k. \tag{2.6}$$

The n-th level M-adic subintervals of $[0,1]$ are defined by

$$I_{i_1 \dots i_n} := \frac{1}{M^n}[[i_1 \dots i_n]_M, [i_1 \dots i_n]_M + 1], \tag{2.7}$$

for all $i_1 \dots i_n \in \mathcal{T}_n$. The Λ^n are unions of M-adic squares

$$Q_{i_1 \dots i_n, j_1 \dots j_n} := I_{i_1 \dots i_n} \times I_{j_1 \dots j_n}. \tag{2.8}$$

The projection $\phi(Q_{i_1 \dots i_n, j_1 \dots j_n})$ of an M-adic square is equal to the union of two subsequent M-adic intervals in $[-1,1]$. We define left and right M-adic intervals and 'columns' in the plane $\mathbb{R}^2$ by

$$\begin{array}{ll} I^L_{i_1 \dots i_n} := I_{i_1 \dots i_n} - 1, & I^R_{i_1 \dots i_n} := I_{i_1 \dots i_n}, \\ C^L_{i_1 \dots i_n} := \phi^{-1}(I^L_{i_1 \dots i_n}), & C^R_{i_1 \dots i_n} := \phi^{-1}(I^R_{i_1 \dots i_n}). \end{array} \tag{2.9}$$

These M-adic columns split M-adic squares $Q_{i_1 \dots i_n, j_1 \dots j_n}$ in left (L-) and right (R-) triangles, denoted by $L_{i_1 \dots i_n, j_1 \dots j_n}$ and $R_{i_1 \dots i_n, j_1 \dots j_n}$. These definitions are illustrated by Figure 2.1. An M-adic interval $I^U_{\underline{k}_n}$ is absent in $\phi(\Lambda^n)$ if and only if there are no triangles in the corresponding column $C^U_{\underline{k}_n}$ in Λ^n. Therefore we count triangles: for $U, V \in \{L, R\}$ and $\underline{k}_n \in \mathcal{T}$ we let

$$Z^{UV}(\underline{k}_n) := \# \left\{ (\underline{i}_n, \underline{j}_n) : Q_{\underline{i}_n, \underline{j}_n} \subseteq \Lambda^n, V_{\underline{i}_n, \underline{j}_n} \subseteq C^U_{\underline{k}_n} \right\} \tag{2.10}$$

denote the number of level n V-triangles in $\Lambda^n \bigcap C^U_{\underline{k}_n}$. We denote the total number of V-triangles in columns $C^L_{\underline{k}_n}$ and $C^R_{\underline{k}_n}$ together by

$$Z^V(\underline{k}_n) := Z^{LV}(\underline{k}_n) + Z^{RV}(\underline{k}_n). \tag{2.11}$$

Squares $Q_{i_1 \dots i_n, j_1 \dots j_n}$ and $Q_{i'_1 \dots i'_n, j'_1 \dots j'_n}$ are called *aligned* if $i_1 \dots i_n = i'_1 \dots i'_n$ or $j_1 \dots j_n = j'_1 \dots j'_n$. The union of an unaligned left and right triangle surviving in the *same* column is called a Δ-pair. The following *expectation matrices* play an important role:

$$\mathcal{M}(\underline{k}_n) := \left[\begin{array}{cc} \mathbb{E}Z^{LL}(\underline{k}_n) & \mathbb{E}Z^{LR}(\underline{k}_n) \\ \mathbb{E}Z^{RL}(\underline{k}_n) & \mathbb{E}Z^{RR}(\underline{k}_n) \end{array} \right], \tag{2.12}$$

satisfying the relation $\mathcal{M}(k_1 \dots k_n) = \mathcal{M}(k_1) \dots \mathcal{M}(k_n)$. It is not difficult to prove that

$$[1 \quad 1]\mathcal{M}(k) = [\mathbb{E}Z^L(k) \quad \mathbb{E}Z^R(k)] = [\gamma_{k+1} \quad \gamma_k], \tag{2.13}$$

where the correlation coefficients for the asymmetric case are defined by

$$\gamma_k := \sum_{i=0}^{M-1} q_i p_{i+k}, \tag{2.14}$$

for $k \in \mathbb{A}$. The smallest correlation coefficient is denoted by

$$\gamma := \min_{k \in \mathbb{A}} \gamma_k. \tag{2.15}$$

An important tool in studying algebraic differences of random Cantor sets are *higher order random Cantor sets*. The idea is to collapse n (this is the order) steps of the construction into one single step. Let's give a simple example.

Example 2.1 *Take the standard triadic Cantor set (which in fact is not random but deterministic), then $M = 3$ and the joint survival distribution μ is defined by $\mu(\{0,2\}) = 1$, giving marginal probabilities $\boldsymbol{p} = (p_0, p_1, p_2) = (1, 0, 1)$. The same set can also be obtained by taking the corresponding second order set: let $M^{(2)} = 9$ and let $\mu^{(2)}$ be defined by $\mu^{(2)}(\{0, 2, 6, 8\}) = 1$, giving marginal probabilities $\boldsymbol{p}^{(2)} = (1, 0, 1, 0, 0, 0, 1, 0, 1)$.*

A more detailed and precise description of the construction of random Cantor sets and the approach to solve the problem can be found in [2].

A little lemma that we will need in Chapter 4 is the following:

Lemma 2.1 *Let X and Y be subsets of $\mathbb{A}$. Then*

$$\sum_{k \in \mathbb{A}} \sum_{j \in \mathbb{A}} \mathbf{1}_Y(j) \mathbf{1}_X(j+k) = \#X \#Y. \tag{2.16}$$

Proof.

$$\begin{aligned}
\sum_{k \in \mathbb{A}} \sum_{j \in \mathbb{A}} \mathbf{1}_Y(j) \mathbf{1}_X(j+k) &= \sum_{j \in \mathbb{A}} \sum_{k \in \mathbb{A}} \mathbf{1}_Y(j) \mathbf{1}_X(j+k) \\
&= \sum_{j \in \mathbb{A}} \left(\mathbf{1}_Y(j) \sum_{k \in \mathbb{A}} \mathbf{1}_X(j+k) \right) = \sum_{j \in \mathbb{A}} \left(\mathbf{1}_Y(j) \#X \right) \\
&= \#X \sum_{j \in \mathbb{A}} \mathbf{1}_Y(j) = \#X \#Y. \qquad (2.17)
\end{aligned}$$

□

Chapter 3

Improving on the joint survival condition

The joint survival condition is a rather strong and restrictive condition, and it is hard to believe that it is really an essential condition. Therefore, in this chapter we try to find alternative conditions, which of course should be weaker than the joint survival condition. In [2] the joint survival condition is needed to be able to prove that with positive probability the growth of left triangles and the growth of right triangles are both exponential in all subcolumns of a Δ-pair. In our search for better conditions we will focus on alternative ways to ensure positive probability of exponential growth.

The joint survival condition is only needed for the first part of Theorem 2.2. In the first section we give a rough sketch of the proof of the first part of this result for general joint survival conditions, as appearing in [2]. We mention the lemma's needed and develop a strategy to find conditions under which we can prove these lemma's. In the subsequent sections we discuss some alternatives for the joint survival condition.

3.1 Rough sketch of the proof

The idea of the proof that we almost surely can find an M-adic interval in the projection $\phi(\Lambda)$ is roughly as follows: suppose we have a Δ-pair in one of the columns with positive probability. If we can prove that there is a strictly positive probability that the number of L-triangles and R-triangles in *all subcolumns* of this column grows exponentially, then it can be shown that with positive probability the M-adic interval corresponding with this column is in the projection $\phi(\Lambda)$. Now we make use of the fact that conditioned on $\Lambda \neq \emptyset$ the Hausdorff dimension of Λ is almost surely larger than 1, which is implied by $\gamma > 1$. It can be shown (see [1]) that from this it follows that the number of unaligned squares grows to infinity. By self-similarity of the process each of the unaligned squares has positive probability to generate an interval in the projection, and hence with probability one there will be an interval in the projection.

To show that a Δ-pair occurs somewhere with positive probability it suffices that $\gamma > 1$. So the joint survival condition is only needed to ensure positive probability of exponential growth in all subcolumns of a Δ-pair. For any level m Δ-pair (L^m, R^m) that is contained in a level m column C, the distribution of the number of level $m+n$ V-triangles surviving in Λ^{m+n} in the $\underline{k}_n$-th subcolumn of (L^m, R^m), conditional on the survival of (L^m, R^m) in Λ^m, is independent of m, the particular choice of the column C and the Δ-pair in it. Therefore, we can unambigiously denote a random variable having this distribution by

$$\tilde{Z}^V(\underline{k}_n) \tag{3.1}$$

for all $V \in \{L, R\}$ and $\underline{k}_n \in \mathcal{T}$. In general $\tilde{Z}^V(\underline{k}_n)$ does not have the distribution of $Z^V(\underline{k}_n)$ because there is possible dependence between the offspring generation of two level 0 triangles, whereas there is no dependence between the offspring generation of the L-triangle and the R-triangle of a Δ-pair, because they are unaligned by definition of a Δ-pair. However, both do have the same expected value. Let

$$\tilde{N}(\underline{k}_n) := \min\left\{\tilde{Z}^L(\underline{k}_n), \tilde{Z}^R(\underline{k}_n)\right\} \tag{3.2}$$

be the minimum number of pairs of each triangle type that survive in the $\underline{k}_n$-th subcolumn of a level 1 Δ-pair. In [2] the following lemma on exponential growth of triangles is proved:

Lemma 3.1 *If $\gamma > 1$, and the joint survival distribution(s) satisfy the joint survival condition, then for all $n \geq 0$*

$$\mathbb{P}(\tilde{N}(\underline{k}_m) \geq \gamma^m \text{ for all } \underline{k}_m \in \mathcal{T}_m \text{ for all } 0 \leq m \leq n) > 0. \tag{3.3}$$

For the proof of the theorem it is not essential that the growth factor is equal to γ. The only requirement is that it is larger than 1. Our goal is to prove this lemma (possibly with a different growth factor) under weaker conditions than the joint survival condition.

3.2 Triangle growth condition

In this section we discuss the *triangle growth condition*, which is based on a geometric argument. Suppose that in the first construction step none of the level 1 squares is discarded. Observe that in this case for all $k \in \mathbb{A}$ the number of V-triangles in columns C_k^L and C_k^R together is given by $Z^V(k) =$

M. Note that these V-triangles are all pairwise unaligned. That means that if we take an arbitrary $i_0 \in \mathbb{A}$ and discard all squares $Q_{i_0,j}$, then in each column at most one of these triangles is lost. The same holds if we take an arbitrary $j_0 \in \mathbb{A}$ and discard all squares Q_{i,j_0}. By doing this $M-2$ times we get the following: let I and J be subsets of $\mathbb{A}$ which have together $M-2$ elements and discard all level 1 squares $Q_{i,j}$ for which $i \in I$ or $j \in J$. In doing this we lose at most $M-2$ triangles in each column and hence $Z^V(k) \geq M-(M-2) = 2$ for all $k \in \mathbb{A}$ and $V \in \{L, R\}$. That means that we have positive probability that the number of L- and R-triangles will double in each column in each level of the construction, up to a certain level n. The details of this reasoning are filled in in the proof of Lemma 3.2. Note that $\#(\mathbb{A} \setminus I) + \#(\mathbb{A} \setminus J) = M + 2$, a number which appears in the condition below. The idea of discarding described here leads to the following condition:

Condition 3.1 *A pair of joint survival distributions (μ, λ) satisfies the* triangle growth condition *if there exist sets $X, Y \subseteq \mathbb{A}$ for which $\mu(X) > 0$ and $\lambda(Y) > 0$ and $\#X + \#Y \geq M + 2$.*

With help of the triangle growth condition we can prove that we have positive probability to get the desired exponential growth:

Lemma 3.2 *If the pair of joint survival distributions (μ, λ) satisfies the triangle growth condition, then for all $n \geq 0$*

$$\mathbb{P}(\tilde{N}(\underline{k}_m) \geq 2^m \text{ for all } \underline{k}_m \in \mathcal{T}_m \text{ for all } 0 \leq m \leq n) > 0. \tag{3.4}$$

Proof. Let $X, Y \subseteq \mathbb{A}$ be such that $\mu(X) > 0$, $\lambda(Y) > 0$ and $\#X + \#Y \geq M + 2$. Define the joint survival distributions $\mu^\star$ and $\lambda^\star$ by $\mu^\star(X) = \lambda^\star(Y) = 1$. All entities corresponding to these survival distributions will be marked with a star superscript.

We take $k \in \mathbb{A}$ and investigate which squares possibly generate triangles in columns C_k^R and C_k^L. It is not so difficult to see that L-triangles in column C_k^L can be generated by squares $Q_{i,j}$ for which $j - i = M - (k+1)$, L-triangles in column C_k^R can be generated by squares $Q_{i,j}$ for which $i - j = k + 1$, R-triangles in column C_k^L can be generated by squares $Q_{i,j}$ for which $j - i = M - k$ and R-triangles in column C_k^R can be generated by squares $Q_{i,j}$ for which $i - j = k$.

Now let's focus on the L-triangles in columns C_k^L and C_k^R together. These triangles can occur in squares $Q_{i,j}$ for which either $i = k + 1 + j - M$ or $i = k + 1 + j$. For $i \in \mathbb{A}$ fixed, there is exactly one $j \in \mathbb{A}$ satisfying one of these two equalities. We have M choices for i, hence we find M unaligned squares $Q_{i,j}$ possibly generating a L-triangle in $C_k^L \cup C_k^R$. This gives

$$\begin{aligned} Z^{\star;L}(k) &\geq M - \#(\mathbb{A} \setminus X) - \#(\mathbb{A} \setminus Y) \\ &= M - (M - \#X) - (M - \#Y) \\ &= -M + \#X + \#Y \\ &\geq -M + M + 2 = 2. \end{aligned} \tag{3.5}$$

By an analogous argument it follows that $Z^{\star;R}(k) \geq 2$. This reasoning holds for all $k \in \mathbb{A}$. Consequently, the column sums of the matrices $\mathcal{M}^\star(k)$ are all at least equal to 2. This leads to the following componentwise (in)equalities:

$$\begin{aligned} [\tilde{Z}^{\star;L}(\underline{k}_m) \quad \tilde{Z}^{\star;R}(\underline{k}_m)] &= [\mathbb{E}\tilde{Z}^{\star;L}(\underline{k}_m) \quad \mathbb{E}\tilde{Z}^{\star;R}(\underline{k}_m)] \\ &= [\mathbb{E}Z^{\star;L}(\underline{k}_m) \quad \mathbb{E}Z^{\star;R}(\underline{k}_m)] \\ &= [1 \quad 1]\mathcal{M}^\star(\underline{k}_m) = [1 \quad 1]\mathcal{M}^\star(k_1)\ldots\mathcal{M}^\star(k_m) \\ &\geq [2^m \quad 2^m], \end{aligned} \tag{3.6}$$

for all $\underline{k}_m \in \mathcal{T}_m$. For $\underline{i}_n \in \mathcal{T}_n$ we write $\underline{i}_n \in X^n$ if $i_1, \ldots, i_n \in X$. Now consider for each $n \geq 0$ the event

$$J_n := \left\{ Q_{\underline{i}_n, \underline{j}_n} \subseteq \Lambda^n \text{ for all } \underline{i}_n \in X^n, \underline{j}_n \in Y^n \right\}. \tag{3.7}$$

Then $\mathbb{P}(J_n) = \mu(X)^{\sum_{j=1}^n (\#X)^j} \lambda(Y)^{\sum_{j=1}^n (\#Y)^j} > 0$ and $\mathbb{P}^\star(J_n) = 1$. By the self-similarity of the process and the requirement that the process runs independently in the triangles of a Δ-pair, the event that in the first $n \geq 0$ sublevels of the surviving Δ-pair all triangles (in the Δ-pair) $L_{\underline{i}_n, \underline{j}_n}$ and $R_{\underline{i}_n, \underline{j}_n}$ with $\underline{i}_n \in X^n, \underline{j}_n \in Y^n$ survive simultaneously, occurs with at least probability $(\mathbb{P}(J_n))^2 > 0$. Conditional on this latter event we have (using (3.6))

$$[\tilde{Z}^L(\underline{k}_m) \quad \tilde{Z}^R(\underline{k}_m)] = [\tilde{Z}^{\star;L}(\underline{k}_m) \quad \tilde{Z}^{\star;R}(\underline{k}_m)] \geq [2^m \quad 2^m] \tag{3.8}$$

elementwise for all $0 \leq m \leq n$ and $\underline{k}_m \in \mathcal{T}_m$. This implies the statement of the lemma. □

3.2.1 Discussion on the triangle growth condition

We now have found some alternative for the joint survival condition, the next question is how it relates to the joint survival condition. In some sense it is a somewhat more intuitive condition. Why the joint survival condition has some contra-intuitive properties is made clear in the following example.

Example 3.1 *Take $M = 6$ and $\mu = \lambda$ such that $\mu(\{0,1,2,3,4\}) = \mu(\{1,2,3,4,5\}) = 1/2$. The vector of marginal probabilities is given by $\boldsymbol{p} = (1/2, 1, 1, 1, 1, 1/2)$, yielding the following correlation coefficients:*

$$\gamma_0 = 4\frac{1}{2}, \quad \gamma_1 = \gamma_5 = 4\frac{1}{4}, \quad \gamma_2 = \gamma_3 = \gamma_4 = 4. \tag{3.9}$$

So all correlation coefficients are far above one. From the six subintervals always only one is discarded and hence intuitively one should somehow get an interval in the projection $\phi(\Lambda)$ (it will turn out that this is indeed the case). However, this example fails to satisfy the joint survival condition, because the marginal support is equal to

$$\mathrm{Supp}_m(\mu) = \{0,1,2,3,4\} \cup \{1,2,3,4,5\} = \{0,1,2,3,4,5\} \tag{3.10}$$

and $\mu(\{0,1,2,3,4,5\}) = 0$. So the joint survival condition is the breaking point in the proof. However, we still have not mentioned the most striking fact. If we adapt the joint survival distribution such that $\mu(\{1,2,3,4\}) = 1$, then the marginal probabilities become $\boldsymbol{p} = (0,1,1,1,1,0)$, giving correlation coefficients:

$$\gamma_0 = 4, \quad \gamma_1 = \gamma_5 = 3, \quad \gamma_2 = \gamma_3 = \gamma_4 = 2. \tag{3.11}$$

Moreover, it is trivial that the joint survival condition is satisfied now, and therefore we conclude that $\phi(\Lambda)$ almost surely contains an interval. Summarizing: we have an example, where we can not prove existence of an interval in $\phi(\Lambda)$, we adapt the joint survival distribution in such a way that all sets to which was assigned positive probability become smaller (!) and now suddenly we are able to prove the existence of an interval in $\phi(\Lambda)$.

In some cases where we couldn't solve the problem because the joint survival condition was not satisfied, we actually can solve the problem with help of the triangle growth condition. Let's illustrate this with an example.

Example 3.2 *Take $M = 4$ and $\mu = \lambda$ such that $\mu(\{0,1,2\}) = \mu(\{1,2,3\}) = 1/2$. The vector of marginal probabilities is given by $\boldsymbol{p} = (1/2, 1, 1, 1/2)$, and the following correlation coefficients are given by:*

$$\gamma_0 = 2\frac{1}{2}, \quad \gamma_1 = \gamma_3 = 2\frac{1}{4}, \quad \gamma_2 = 2. \tag{3.12}$$

The marginal support is equal to

$$\mathrm{Supp}_m(\mu) = \{0,1,2\} \cup \{1,2,3\} = \{0,1,2,3\} \tag{3.13}$$

and $\mu(\{0,1,2,3\}) = 0$. So this example violates the joint support condition. Now let's check the triangle growth condition. Take

$$X = Y = \{0,1,2\}, \tag{3.14}$$

then we have $\#X + \#Y = 3 + 3 \geq M + 2 = 6$. Hence the triangle growth condition is satisfied and we can conclude that the projection $\phi(\Lambda)$ contains an interval almost surely.

The above example shows that in some cases the triangle growth condition holds, while the joint survival condition fails. Unfortunately, also the opposite case possibly occurs, and therefore an objective answer to the question which condition is better is hard to give. The example below shows that it is also possible to be able to prove occurrence of an interval in the projection almost surely with help of the joint survival condition, while the triangle growth condition is not satisfied.

Example 3.3 *Let $M = 7$ and define $\mu = \lambda$ by $\mu(\{0,1,2,4\}) = \frac{3}{4}$ and $\mu(\emptyset) = \frac{1}{4}$. Computing the marginal probabilities we find*

$$\boldsymbol{p} = (\tfrac{3}{4}, \tfrac{3}{4}, \tfrac{3}{4}, 0, \tfrac{3}{4}, 0, 0),$$

which means that the correlation coefficients are given by

$$\gamma_0 = \frac{9}{4}, \quad \gamma_i = \frac{9}{8} \text{ for } i \in \{1,2,3,4,5,6\}.$$

Hence all correlation coefficients are bigger than 1. Now let's check the triangle growth condition. The largest set having positive probability is $\{0,1,2,4\}$. Therefore we take $X = Y = \{0,1,2,4\}$. Then $\#X + \#Y = 4 + 4 = 8 \not\geq M + 2 = 7 + 2 = 9$. So using the triangle growth condition, we fail to prove existence of an interval in $\phi(\Lambda)$. Because the marginal support of μ (being equal to $\{0,1,2,4\}$) has positive probability, the joint survival condition is satisfied and here we succeed in proving occurrence of an interval in $\phi(\Lambda)$.

We have seen that neither of the joint survival condition and the triangle growth condition implies the other one. Although the triangle growth condition at first sight seems to be a more natural condition it has a big disadvantage: in contrast with the joint survival condition it does not propagate to higher order Cantor sets. For example, if $M = 9$ and the largest set to which is assigned positive probability by μ has 6 elements, then the triangle growth condition is satisfied because $2 \cdot 6 = 12 \geq 11 = M + 2$. However, for the corresponding second order Cantor set, $M^{(2)} = 81$ and the largest set having positive probability has 36 elements. Now $2 \cdot 36 = 72 \not\geq 83 = M^{(2)} + 2$. Thus, the corresponding second order Cantor set is not satisfying the triangle growth condition.

3.3 Max-min growth condition

If we have the algebraic difference of two deterministic Cantor sets, then we can describe this with two joint survival distributions assigning probability one to subsets X and Y of $\mathbb{A}$, i.e. $\mu(X) = \lambda(Y) = 1$. The marginal probabilities are then given by

$$p_i = \mathbf{1}_X(i) \quad q_i = \mathbf{1}_Y(i), \tag{3.15}$$

for $i \in \mathbb{A}$. Hence for the correlation coefficients (by (2.13) being the column sums of the expectation matrices $\mathcal{M}(k), k \in \mathbb{A}$) we find

$$\gamma_k = \sum_{i=0}^{M-1} q_i p_{i+k} = \sum_{i=0}^{M-1} \mathbf{1}_Y(i)\mathbf{1}_X(i+k). \tag{3.16}$$

To guarantee exponential growth of triangles, it suffices that the minimum of the column sums of the expectation matrices exceeds 1. This leads to the following criterion:

$$\min_{k\in\mathbb{A}} \sum_{i=0}^{M-1} \mathbf{1}_Y(i)\mathbf{1}_X(i+k) > 1. \tag{3.17}$$

We use this observation to formulate another condition under which we can prove to have exponential growth with positive probability. We only need *positive probability* of exponential growth and therefore we can manipulate with positive probabilities, as long as we keep them positive. What we mean by this is the following: suppose that the joint survival distributions μ_1 and λ_1 assign strictly positive probability to all (and only those) sets in $\mathcal{C}_\mu$ and $\mathcal{C}_\lambda$ respectively, where $\mathcal{C}_\mu$ and $\mathcal{C}_\lambda$ are collections of subsets of $\mathbb{A}$.

Now suppose that μ_2 and λ_2 assign positive probability to sets in a *smaller* collection of subsets of $\mathbb{A}$. So μ_2 and λ_2 assign positive probability only to sets in $\tilde{\mathcal{C}_\mu}$ and $\tilde{\mathcal{C}_\lambda}$ respectively, where $\tilde{\mathcal{C}_\mu} \subseteq \mathcal{C}_\mu$ and $\tilde{\mathcal{C}_\lambda} \subseteq \mathcal{C}_\lambda$. If the pair (μ_2, λ_2) gives positive probability to have exponential growth of triangles, then this implies that the pair (μ_1, λ_1) also satisfies this property, no matter what the exact probabilities assigned to the subsets of $\mathbb{A}$ are. Now take $\tilde{\mathcal{C}_\mu}$ and $\tilde{\mathcal{C}_\lambda}$ consisting of one single subset of $\mathbb{A}$, or to state it differently: take μ_2 and λ_2 deterministic, such that $\mu_2(X) = \lambda_2(Y) = 1$ for some $X, Y \in \mathbb{A}$. If it is possible to do this in such a way that X and Y satisfy the criterion in (3.17), then the pair (μ_1, λ_1) gives positive probability to get exponential growth of triangles.

The above reasoning motivates the formulation of the following condition.

Condition 3.2 *A pair of joint survival distributions $\mu, \lambda : 2^{2^{\mathbb{A}}} \to [0,1]$ satisfies the max-min growth condition (MMGC) if the following holds:*

$$\max_{X,Y\subseteq\mathbb{A}:\mu(X),\lambda(Y)>0} \min_{k\in\mathbb{A}} \sum_{j=0}^{M-1} \mathbf{1}_Y(j)\mathbf{1}_X(j+k) > 1. \tag{3.18}$$

Under this condition we can prove that we have positive probability to get the exponential growth we need. Basically, the proof is the same as the proof of Lemma 3.2.

Lemma 3.3 *If the joint survival distributions satisfy the MMGC, then for all* $n \geq 0$

$$\mathbb{P}(\tilde{N}(\underline{k}_m) \geq 2^m \textit{ for all } \underline{k}_m \in \mathcal{T}_m \textit{ for all } 0 \leq m \leq n) > 0. \tag{3.19}$$

Proof. Let $X, Y \subseteq \mathbb{A}$ be sets for which the maximum in the MMGC is attained. Now define the joint survival distributions $\mu^\star$ and $\lambda^\star$ by requiring that $\mu^\star(X) = \lambda^\star(Y) = 1$. Note that the MMGC now states that $\gamma^\star > 1$, and because this is a deterministic case, we even have $\gamma^\star \geq 2$. By noting that $[1 \quad 1]\mathcal{M}^\star(k_1)\ldots\mathcal{M}^\star(k_m) \geq [(\gamma^\star)^m \quad (\gamma^\star)^m]$, we can from here do exactly the same as in the proof of Lemma 3.2 starting from equation (3.6). □

3.3.1 Discussion on the max-min growth condition

In concrete situations, one frequently deals with the symmetric case $\mu = \lambda$. In that case it is a natural question if it makes sense to allow for asymmetry in the MMGC or that $X = Y$ always is a good choice. Obviously, allowing for asymmetry gives a theoretically weaker condition, but it also highly increases the required effort to check the condition. Possibly allowing for symmetry gives *in practice* an equivalent condition, although it is *theoretically* weaker. Do there exist concrete cases where $\mu = \lambda$ and the symmetric condition is not satisfied, while the asymmetric condition is satisfied? We searched for such examples and found one for $M = 7$, (which is the smallest M for which such an example exists), showing that allowing for asymmetry is useful.

Example 3.4 *Let $M = 7$ and define the sets $B := \{0, 2, 4, 6\}$ and $C := \{0, 1, 2, 3\}$. Consider the symmetric case and let the joint survival distribution μ be defined by $\mu(B) = \mu(C) = \frac{1}{2}$. The set B contains only even numbers, and therefore all terms in the sum below are zero, except the last one:*

$$\sum_{j=0}^{6} \mathbf{1}_B(j)\mathbf{1}_B(j+1) = \mathbf{1}_B(6)\mathbf{1}_B(0) = 1. \tag{3.20}$$

For the set C the following expression holds:

$$\sum_{j=0}^{6} \mathbf{1}_C(j)\mathbf{1}_C(j+3) = \mathbf{1}_C(0)\mathbf{1}_C(3) = 1. \tag{3.21}$$

Because B and C are the only sets to which μ assigns positive probability, we get the following result:

$$\max_{X \subseteq \mathbb{A}:\mu(X)>0} \min_{k \in \mathbb{A}} \sum_{j=0}^{6} \mathbf{1}_X(j)\mathbf{1}_X(j+k) \leq 1. \tag{3.22}$$

With some additional computations, it can be shown that

$$\sum_{j=0}^{6} \mathbf{1}_C(j)\mathbf{1}_B(j+k) \geq 2. \tag{3.23}$$

for all $k \in \mathbb{A}$. Consequently, the MMGC is satisfied, and according to (3.22) it is crucial that asymmetry in the MMGC is allowed.

One of the disadvantages of the triangle growth condition was that it failed to propagate to higher order Cantor sets. Fortunately, the MMGC scores better on this point, as the following lemma shows.

Lemma 3.4 *(Propagation of the MMGC to higher order Cantor sets) Suppose the pair of joint survival distributions (μ, λ) satisfies the MMGC, then for all $n \geq 1$, $(\mu^{(n)}, \lambda^{(n)})$ satisfies the MMGC.*

Proof. Let $X, Y \subseteq \mathbb{A}$ be sets for which the maximum in the MMGC is reached. Now define the joint survival distributions $\mu^\star$ and $\lambda^\star$ by requiring that $\mu^\star(X) = \lambda^\star(Y) = 1$. According to the MMGC we have $\gamma^\star > 1$. Let $n \geq 1$ and $k = [k_1 \dots k_n]_M \in \mathbb{A}^{(n)}$. Then the following holds componentwise:

$$\begin{aligned} [\gamma_{k+1}^{\star(n)} \quad \gamma_k^{\star(n)}] &= [1 \quad 1]\mathcal{M}^{\star(n)}(k) = [1 \quad 1]\mathcal{M}^\star(k_1)\dots\mathcal{M}^\star(k_n) \\ &\geq [(\gamma^\star)^n \quad (\gamma^\star)^n] > [1 \quad 1] \end{aligned} \tag{3.24}$$

Now define

$$X^{(n)} := \{[i_1 \dots i_n]_M : i_1, \dots, i_n \in X\}, \tag{3.25}$$

$$Y^{(n)} := \{[i_1 \dots i_n]_M : i_1, \dots, i_n \in Y\}. \tag{3.26}$$

Then $\mu^{\star(n)}(X^{(n)}) = \lambda^{\star(n)}(Y^{(n)}) = 1$ and hence for all $k \in \mathbb{A}^{(n)}$ we have

$$\gamma_k^{\star(n)} = \sum_{j=0}^{M^n-1} \mathbf{1}_{Y^{(n)}}(j)\mathbf{1}_{X^{(n)}}(j+k). \tag{3.27}$$

Therefore we can conclude that

$$\min_{k\in\mathbb{A}^{(n)}} \sum_{j=0}^{M^n-1} \mathbf{1}_{Y^{(n)}}(j)\mathbf{1}_{X^{(n)}}(j+k) > 1, \tag{3.28}$$

so it follows that $(\mu^{(n)}, \lambda^{(n)})$ satisfies the MMGC. □

An important question about the MMGC is how it relates to the joint survival condition. Did we make any progress in finding this alternative condition? Lemma 3.5 shows that the joint survival condition implies the MMGC, provided $\gamma > 1$. Recall that the cases which we are interested in, are those where $\gamma > 1$. Hence we conclude that the MMGC is a better condition than the joint survival condition from a theoretical point of view. The joint survival condition has the practical advantage that it is much more easy to check.

Lemma 3.5 *(The JSC implies the MMGC) Suppose that the joint survival distributions μ and λ satisfy the joint survival condition. If $\gamma > 1$, then the pair (μ, λ) satisfies the MMGC.*

Proof. Define the joint survival distributions $\mu^\star$ and $\lambda^\star$ by requiring that $\mu^\star(\text{Supp}_m(\mu)) = \lambda^\star(\text{Supp}_m(\lambda)) = 1$. Then for all $k \in \mathbb{A}$ we have

$$\begin{aligned} \gamma_k^\star &= \sum_{j=0}^{M-1} \mathbf{1}_{\text{Supp}_m(\lambda)}(j)\mathbf{1}_{\text{Supp}_m(\mu)}(j+k) \\ &\geq \sum_{j=0}^{M-1} q_j p_{j+k} = \gamma_k \geq \gamma > 1, \end{aligned} \tag{3.29}$$

since $q_j = 0$ if $j \notin \text{Supp}_m(\lambda)$, and similarly for p_i. Taking the minimum over k we find

$$\min_{k\in\mathbb{A}} \sum_{j=0}^{M-1} \mathbf{1}_{\text{Supp}_m(\lambda)}(j)\mathbf{1}_{\text{Supp}_m(\mu)}(j+k) > 1. \tag{3.30}$$

Now note that $\mu(\text{Supp}_m(\mu)) > 0$ and $\lambda(\text{Supp}_m(\lambda)) > 0$, because μ and λ satisfy the joint survival condition. It follows that the pair (μ, λ) satisfies the MMGC. □

Now let's compare the MMGC with the triangle growth condition. We already showed the existence of examples where the joint survival condition holds, while the triangle growth condition fails (Example 3.3). Using Lemma 3.5, it follows that it is also possible to find examples where the MMGC holds and the triangle growth condition fails. The opposite is not possible, because in the following lemma we show that the MMGC is a weaker condition than the triangle growth condition.

Lemma 3.6 *(The TGC implies the MMGC) Suppose that the pair of joint survival distributions (μ, λ) satisfies the triangle growth condition. Then the pair (μ, λ) satisfies the MMGC.*

Proof. Let $X \subseteq \mathbb{A}$ and $Y \subseteq \mathbb{A}$ be such that $\mu(X) > 0$, $\lambda(Y) > 0$ and $\#X + \#Y \geq M + 2$. Take $k \in \mathbb{A}$ and consider the sum $\sum_{j=0}^{M-1} \mathbf{1}_Y(j)\mathbf{1}_X(j+k)$. This sum has M terms and from the triangle growth condition it follows that at least two of these terms are equal to 1. Consequently

$$\min_{k \in \mathbb{A}} \sum_{j=0}^{M-1} \mathbf{1}_Y(j)\mathbf{1}_X(j+k) \geq 2, \tag{3.31}$$

and we conclude that the pair (μ, λ) satisfies the MMGC. □

3.4 Conclusion and research ideas

Of the three conditions we examined thus far, the MMGC is the best condition, in the sense that it is the weakest and hence covers most cases. However, both the joint survival condition and the triangle growth condition are much easier to check. Especially the joint survival condition can be checked at a glance. In checking the joint survival condition one should bear in mind that it can be useful to replace some sets having positive probability by subsets, as we did in Example 3.1.

The proof of Theorem 2.2 is based on two lemma's. In Lemma 1 in [2] it is shown that there is positive probability to get exponential growth of triangles in all subcolumns of a column containing a Δ-pair. Lemma 2 in [2] states that there is positive probability of existence of a level m Δ-pair in Λ^m. For both lemma's $\gamma > 1$ is needed, and for Lemma 1 also the joint survival condition is used. As we have seen, if we keep both lemma's the same, then we can replace the joint survival condition by the MMGC. An interesting idea is to adapt the two lemma's in the following way. Lemma 2 can be made stronger by requiring that there exists a certain number (say n, possibly dependent of M) of level m Δ-pairs in the same column of Λ^m. Lemma 1 can be weakened by only requiring that there is positive probability to get exponential growth of triangles in all subcolumns of a column containing n Δ-pairs. In doing this, one hopes that both lemma's can be proved under weaker conditions. We already made some investigations (without finding results worth mentioning) in this direction. The problem becomes rather complex due to dependencies between columns, but this certainly seems a promising direction to go.

Chapter 4

A canonical class of random Cantor sets

In this chapter we consider a class of random Cantor sets that deserves special attention due to its natural construction and interesting properties: correlated fractal percolation. Studying this class is also motivated since we want to test the MMGC. Does this condition help us to solve problems from the correlated fractal percolation class? Or possibly even more interesting: can we find Cantor sets in this class for which the sum of the Hausdorff dimensions is greater than one and for which the MMGC is *not* satisfied? This is an interesting question due to the Palis conjecture: if $\dim_H(F_1) + \dim_H(F_2) > 1$, then generically it should be true that $F_1 - F_2$ contains an interval. Although there exist examples where the Palis conjecture does not hold (see [2]), it would be rather surprising if the Palis conjecture fails for Cantor sets from the very symmetric correlated fractal percolation class. So if we can find examples of correlated fractal percolation not satisfying the MMGC, then this probably gives ideas to improve on the MMGC.

We will start with the definition and some general properties and results for correlated fractal percolation. Our goal is to answer the question whether or not an interval occurs in the algebraic difference of two random Cantor sets from the correlated fractal percolation class. Theorem 4.1 gives an answer to this question for most of the sets in this class. In the subsequent sections we look at some particular random Cantor sets for which the theorem gives no conclusion.

4.1 Construction and general properties

We start with the definition of the class of random Cantor sets which we will take into consideration.

Definition 4.1 *Consider the algebraic difference of two M-adic random Cantor sets. Consider the symmetric case $\mu = \lambda$ and suppose μ assigns the* same *positive probability to all subsets of $\mathbb{A}$ with m elements for some fixed integer $m \in \mathbb{A}$. Assume that μ assigns probability zero to all other non-empty subsets of $\mathbb{A}$. If $p := (1 - \mu\emptyset)\frac{m}{M}$ then we call this (m, M, p)-percolation.*

Now let's have a look at some properties of (m, M, p)-percolation. Due to symmetry, all marginal probabilities are the same, and we can compute them as follows. Let X be a subset of $\mathbb{A}$, chosen according to the survival distribution μ. The probability that X is non-empty is $1 - \mu(\emptyset)$. Given that X is non-empty, the probability that a fixed $k \in \mathbb{A}$ belongs to X equals $\frac{m}{M}$. It follows that for all $k \in \mathbb{A}$ the marginal probability p_k is given by

$$p_k = (1 - \mu(\emptyset))\frac{m}{M} = p, \tag{4.1}$$

which is exactly the reason why we defined (m, M, p)-percolation by requiring that $p = (1 - \mu\emptyset)\frac{m}{M}$. Because $0 \leq \mu(\emptyset) \leq 1$, (m, M, p)-percolation is only defined for $0 \leq p \leq \frac{m}{M}$. From now on we will assume that $p > 0$ and $m > 0$, since giving the empty set probability one does not yield the most exciting situation. As a consequence of the fact that all marginal probabilities are the same, also all correlation coefficients are the same. For all $k \in \mathbb{A}$ we have

$$\gamma_k = \sum_{j=0}^{M-1} p_j p_{j+k} = Mp^2 =: \gamma. \tag{4.2}$$

As we see, (m, M, p)-percolation is perfectly balanced and symmetric, and these features make this to be the most elegant and natural class of random Cantor sets.

Obviously for (m, M, p)-percolation the joint survival condition is not satisfied, unless we are in the case $m = M$, giving positive probability only to the full alphabet and the empty set (actually, this is *uncorrelated* fractal percolation, where intervals are discarded independently and the marginal probabilities p_k are all equal to p). This illustrates the importance of the MMGC, most of the research in this chapter involves attempting to find subsets of $\mathbb{A}$ satisfying the MMGC.

The following theorem gives an answer to the interval or not question for a large part of the (m, M, p)-percolation class.

Theorem 4.1 *Consider (m, M, p)-percolation. Then the following two assertions hold:*

1. *If* $m < \sqrt{M}$ *or* $p < \dfrac{1}{\sqrt{M}}$, *then* $F_1 - F_2$ *contains no interval a.s.*[1]

2. *If* $m \geq \sqrt{2M}$ *and* $p > \dfrac{1}{\sqrt{M}}$, *then* $F_1 - F_2$ *contains an interval a.s. on* $\{F_1 - F_2 \neq \emptyset\}$.

Proof. Suppose that $p < \frac{1}{\sqrt{M}}$, then for all $k \in \mathbb{A}$ we have

$$\gamma_k = Mp^2 < M\left(\frac{1}{\sqrt{M}}\right)^2 = 1, \tag{4.3}$$

and consequently $F_1 - F_2$ contains no interval a.s. by Theorem 2.2. If $m < \sqrt{M}$, then $p = (1-\mu\emptyset)\frac{m}{M} < \frac{1}{\sqrt{M}}$ and consequently the same argument is applicable, completing the proof of the first part of Theorem 4.1.

For the second part, suppose that $m \geq \sqrt{2M}$ and $p > \frac{1}{\sqrt{M}}$ and define the following subsets of $\mathbb{A}$:

$$X := \{0, 1, \ldots, m-1\}, \tag{4.4}$$

$$Y := \left\{\left\lfloor \frac{lM}{m} \right\rfloor ; l = 0, 1, \ldots, m-1\right\}. \tag{4.5}$$

Then for $k \in \mathbb{A}$

$$\sum_{j=0}^{M-1} \mathbf{1}_Y(j)\mathbf{1}_X(j+k) = \sum_{j=M-k}^{M-k+m-1} \mathbf{1}_Y(j) \tag{4.6}$$

by definition of X. Assume for some $k \in \mathbb{A}$ that the sum on the right hand side is smaller than or equal to 1. Since there are m terms in this sum, we can find m consecutive elements of $\mathbb{A}$ of which at most one is also an element of Y (where also $M-1$ and 0 are considered to be consecutive). Using the definition of Y it now follows that

$$\left\lfloor \frac{(l+2)M}{m} \right\rfloor - \left\lfloor \frac{lM}{m} \right\rfloor \geq m+1 \tag{4.7}$$

for some $l \in \{0, \ldots, m-1\}$.

Now observe that $M \leq \frac{1}{2}m^2$ and for a real number x, let $\{x\} = x - \lfloor x \rfloor$. Note that $\{x\} < 1$ for all $x \in \mathbb{R}$. Then we find

$$\left\lfloor \frac{(l+2)M}{m} \right\rfloor - \left\lfloor \frac{lM}{m} \right\rfloor = \frac{(l+2)M}{m} - \left\{\frac{(l+2)M}{m}\right\} - \frac{lM}{m} + \left\{\frac{lM}{m}\right\}$$

[1]Actually, $m < \sqrt{M}$ implies that $p < 1/\sqrt{M}$. Hence the statement *"If $p < 1/\sqrt{M}$, then $F_1 - F_2$ contains no interval a.s."* is equivalent to the first assertion of Theorem 4.1. We formulated the theorem in this way to emphasize what the bounds on m are.

$$
\begin{aligned}
&= \frac{2M}{m} + \left\{\frac{lM}{m}\right\} - \left\{\frac{(l+2)M}{m}\right\} \\
&\leq \frac{2(\frac{1}{2}m^2)}{m} + \left\{\frac{lM}{m}\right\} - \left\{\frac{(l+2)M}{m}\right\} \\
&< m+1, \qquad (4.8)
\end{aligned}
$$

contradicting (4.7). Hence the sum in (4.6) must be greater than one:

$$\sum_{j=0}^{M-1} \mathbf{1}_Y(j)\mathbf{1}_X(j+k) > 1 \qquad (4.9)$$

for all $k \in \mathbb{A}$. We conclude that X and Y satisfy the MMGC. On top of that, for γ we find

$$\gamma = Mp^2 > M\left(\frac{1}{\sqrt{M}}\right)^2 = 1, \qquad (4.10)$$

and therefore, by Theorem 2.2, $F_1 - F_2$ contains an interval a.s. on $\{F_1 - F_2 \neq \emptyset\}$.
□

The above result motivates the following definition.

Definition 4.2 *If we have (m, M, p)-percolation with $m < \sqrt{2M}$, then we call this* flimsy *percolation.*

In the proof of Theorem 4.1 it is shown that (m, M, p)-percolation satisfies the MMGC if $m \geq \sqrt{2M}$. This bound is sharp in the sense that flimsy (m, M, p)-percolation does not satisfy the MMGC, as is shown in the following property.

Property 4.1 *(m, M, p)-percolation satisfies the MMGC if and only if it is not flimsy.*

Proof. Consider (m, M, p)-percolation. If it is not flimsy, then the MMGC is satisfied, as is shown in the proof of Theorem 4.1. Therefore, it suffices to check that flimsy (m, M, p)-percolation fails to satisfy the MMGC.

Now consider flimsy (m, M, p)-percolation. Let X and Y be arbitrary nonempty sets to which μ assigns positive probability. By definition, X and Y both contain m elements. Using Lemma 2.1 it follows that for X and Y the following holds:

$$\sum_{k=0}^{M-1}\sum_{j=0}^{M-1} \mathbf{1}_Y(j)\mathbf{1}_X(j+k) = \#X\#Y = m^2 < 2M, \qquad (4.11)$$

where the last inequality follows from the fact that the percolation is flimsy. This means that in the sum $\sum_{k=0}^{M-1}\sum_{j=0}^{M-1} \mathbf{1}_Y(j)\mathbf{1}_X(j+k)$ at least one of the

M terms must be smaller than 2. By noting that each of these terms is an integer, it follows that at least one of the terms is smaller than or equal to 1. So we have

$$\min_{k\in\mathbb{A}} \sum_{j=0}^{M-1} \mathbf{1}_Y(j)\mathbf{1}_X(j+k) \leq 1. \tag{4.12}$$

Since X and Y were arbitrary sets, the MMGC does not hold. □

Due to the above lemma, studying higher order Cantor sets is the way to go when trying to solve the problem for flimsy (m, M, p)-percolation. An indication of the difficulty of the problem for a concrete case of (m, M, p)-percolation is given by the lowest order possibly satisfying the MMGC. The next lemma gives an expression of this difficulty measure in terms of m and M.

Lemma 4.1 *Consider (m, M, p)-percolation with $m > \sqrt{M}$. Let $\mu^{(n)}$ be the nth order joint survival distribution. If*

$$n < \frac{\log 2}{2\log m - \log M}, \tag{4.13}$$

then $\mu^{(n)}$ does not satisfy the MMGC.

Proof. Consider the nth order random Cantor sets corresponding to (m, M, p)-percolation. The largest subsets of $\mathbb{A}^{(n)}$ to which $\mu^{(n)}$ assigns positive probability consist of m^n elements. If $(m^n)^2 < 2M^n$, then by a reasoning very similar to the proof of Lemma 4.1, it can be shown that at least one of the terms in the sum

$$\sum_{k=0}^{M^n-1} \sum_{j=0}^{M^n-1} \mathbf{1}_{Y^{(n)}}(j)\mathbf{1}_{X^{(n)}}(j+k) \tag{4.14}$$

is smaller than or equal to 1. Here $X^{(n)}$ and $Y^{(n)}$ are arbitrary subsets of $\mathbb{A}^{(n)}$ for which $\mu^{(n)}(X^{(n)}) > 0$ and $\mu^{(n)}(Y^{(n)}) > 0$. Consequently, the MMGC is not satisfied. Solving the inequality $(m^n)^2 < 2M^n$ for n, we find the bound given in the lemma. □

The first order for which we can hope to find sets satisfying the MMGC is equal to the ceiling of the bound in (4.13). Evaluating the bound for the smallest m for which the percolation is not flimsy (that is $m = \sqrt{2M}$), we get

$$\frac{\log 2}{2\log\sqrt{2M} - \log M} = \frac{\log 2}{\log 2M - \log M} = \frac{\log 2}{\log 2 + \log M - \log M} = 1, \tag{4.15}$$

confirming that for non-flimsy percolation the first order possibly already satisfies the MMGC. The problem becomes more difficult when the denominator in (4.13) approaches zero. For example, if we take $(4, 15, p)$-percolation, then we have to go at least to the 11th order (bound ≈ 10.74), while for $(5, 15, p)$-percolation the 2nd order probably is sufficient (bound ≈ 1.36).

4.2 Some cases of flimsy (m, M, p)-percolation

In this section we are going to use the theory which we have developed so far. We will try to solve some cases of flimsy percolation for which $\sqrt{M} < m < \sqrt{2M}$.

4.2.1 $(2, 3, p)$-percolation

It is well known that $C - C = [-1, 1]$, where C is the standard deterministic triadic Cantor set. So a natural place to start our investigations is the case of percolation that most resembles this known case: $(2, 3, p)$-percolation. C is constructed by each time dividing each interval in three equal subintervals and discarding the middle of these three intervals. If we take $(2, 3, p)$-percolation with $p = \frac{2}{3}$, then again intervals are divided in three equal subintervals, the only difference with the construction of C is that the interval to be discarded can be chosen freely. For $M \leq 3$, $(2, 3, p)$-percolation is the only case of percolation which is flimsy. Figure 4.1 gives an illustration of $(2, 3, \frac{2}{3})$-percolation.

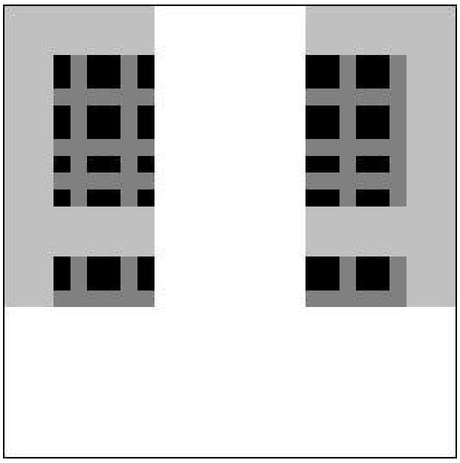

Figure 4.1: Illustration of the first construction steps of a realization of $(2, 3, \frac{2}{3})$-percolation. The black area is a 3rd level approximation of $F_1 \times F_2$, the union of black and dark grey is a 2nd level approximation and the union of black, dark grey and light grey is a 1st level approximation.

We now use Lemma 4.1 to compute the lowest order for which we can hope to succeed in finding sets satisfying the MMGC:

$$\frac{\log 2}{2\log 2 - \log 3} \approx 2.41, \tag{4.16}$$

so we can start our search at the 3rd order. We are looking for subsets $X^{(3)}$ and $Y^{(3)}$ of $\mathbb{A}^{(3)} = \{0, 1, \ldots, 26\}$ that should satisfy the following requirements:

1. $X^{(3)}$ should contain 8 elements. If we divide $\mathbb{A}^{(3)}$ in 3^k blocks of 3^{3-k} elements, then each of these blocks should contain either 0 or 2^{3-k} elements of $X^{(3)}$ for $k = 1, 2$. The same should hold for $Y^{(3)}$.

2. $X^{(3)}$ and $Y^{(3)}$ should satisfy

$$\min_{k\in\mathbb{A}^{(3)}} \sum_{j=0}^{26} \mathbf{1}_{Y^{(3)}}(j)\mathbf{1}_{X^{(3)}}(j+k) > 1. \tag{4.17}$$

The first requirement is due to the special construction procedure and the second requirement is needed to let the MMGC be fulfilled. Note that the first of these two requirements only allows for subsets of $\mathbb{A}^{(3)}$ containing 8 elements, while there also exist smaller sets to which $\mu^{(3)}$ assigns positive probability if $p < \frac{2}{3}$. However, due to the second requirement it makes no sense to allow for smaller sets.

With some trial and error guesswork we failed to find appropriate subsets of $\mathbb{A}^{(3)}$. Therefore we switched to using a Matlab program. The implemented algorithm works as follows. Without loss of generality, we assume that in the first construction step the subsets $X, Y \subseteq \{0, 1, 2\}$ are given by:

$$X = Y = \{0, 1\}\,. \tag{4.18}$$

This does not violate the generality, since for the 3rd order we are interested in the family of sums

$$\sum_{j=0}^{26} \mathbf{1}_{Y^{(3)}}(j)\mathbf{1}_{X^{(3)}}(j+k), \quad k = 0, \ldots, 26, \tag{4.19}$$

which is invariant under the addition of a constant to all elements in X or Y. To see this, note that the two-element subsets of $\{0, 1, 2\}$ are $\{0, 1\}$, $\{1, 2\}$ and $\{0, 2\}$. The subset $\{1, 2\}$ can be obtained by adding the constant 1 to all elements of X and the subset $\{0, 2\}$ can be obtained by adding 2 to all elements of X. If we now for example take $X = \{0, 2\}$, this leads to replacement of k in the sum appearing in (4.19) by $k + 18$. But if k runs through the complete alphabet $\mathbb{A}^{(3)}$, then the same holds for $k + 18$. Consequently, we find the same family of sums as in (4.19).

A subset S of the nth order alphabet $\mathbb{A}^{(n)}$ can be represented as a vector of zeros and ones, where a 1 at the ith position means that $i - 1 \in S$. If we denote such a vector by $\vec{S}$, then $\vec{X}$ and $\vec{Y}$ are given by

$$\vec{X} = \vec{Y} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}. \tag{4.20}$$

Going to the next order means that each 1 in $\vec{X}$ and $\vec{Y}$ is replaced by a vector of length 3 of which two elements equal 1 and one element equals 0. Each 0 in $\vec{X}$ and $\vec{Y}$ is replaced by the length 3 zero vector.

Now the sum

$$\sum_{j=0}^{M^n-1} \mathbf{1}_{Y^{(n)}}(j)\mathbf{1}_{X^{(n)}}(j+k) \tag{4.21}$$

is the inner product of $\vec{X}^{(n)}$ and $\vec{Y}^{(n)}$, where the first k elements of $\vec{X}^{(n)}$ are moved to the end. For each pair $X^{(n)}, Y^{(n)}$ of subsets of $\mathbb{A}^{(n)}$, this gives M^n inner products. If the minimum of those inner products is at least 2, then we have succeeded and $X^{(n)}$ and $Y^{(n)}$ satisfy the MMGC. If the minimum of the inner products is 0, then we can stop searching in this direction, since the zero will never disappear if we go to higher orders.

Our algorithm now starts with the two vectors given in (4.20) and computes all pairs of second order sets $X^{(2)}$ and $Y^{(2)}$ for which the minimum of the inner products equals 1. Note that by Lemma 4.1 it is impossible to find second order sets with a minimum inner product of 2. For all those pairs of second order sets we compute the corresponding third order sets and check if they satisfy the MMGC. An implementation of this algorithm can be found in Appendix B.

We found the following six pairs of third order sets $X_j^{(3)}, Y_j^{(3)} \in \mathbb{A}^{(3)}$ satisfying the construction requirements and the MMGC:

$$\begin{cases} X_1^{(3)} & = & \{0,2,7,8,9,10,15,17\}, \\ Y_1^{(3)} & = & \{0,1,3,4,13,14,16,17\}, \end{cases} \tag{4.22}$$

$$\begin{cases} X_2^{(3)} & = & \{0,2,7,8,9,10,15,17\}, \\ Y_2^{(3)} & = & \{0,1,4,5,12,13,16,17\}, \end{cases} \tag{4.23}$$

$$\begin{cases} X_3^{(3)} & = & \{0,2,7,8,9,10,15,17\}, \\ Y_3^{(3)} & = & \{0,1,3,4,12,13,16,17\}, \end{cases} \tag{4.24}$$

$$\begin{cases} X_4^{(3)} & = & \{0,2,7,8,9,10,15,17\}, \\ Y_4^{(3)} & = & \{0,1,4,5,13,14,16,17\}, \end{cases} \tag{4.25}$$

$$\begin{cases} X_5^{(3)} & = & \{0,1,7,8,9,10,16,17\}, \\ Y_5^{(3)} & = & \{0,1,3,5,12,14,16,17\}, \end{cases} \tag{4.26}$$

$$\begin{cases} X_6^{(3)} & = & \{1,2,7,8,9,10,15,16\}, \\ Y_6^{(3)} & = & \{0,1,3,5,12,14,16,17\}, \end{cases} \tag{4.27}$$

and 6 other pairs of solution sets, where the roles of $X^{(3)}$ and $Y^{(3)}$ are reversed. Hence there are only 12 pairs of sets satisfying our requirements, while our algorithm searched in $3^{12} = 531441$ pairs of sets. So it is quite a hard problem to find appropriate sets.

The reason why we explicitly specified all pairs of sets satisfying the requirements is that we hope to see some structure in it. If we find some regularity in the solutions, then it possibly becomes easier to solve other cases of (m, M, p)-percolation. The first property of the solution sets that catches the eye is that most of them are symmetric with respect to $17/2$, only for $Y_3^{(3)}$ and $Y_4^{(3)}$ this does not hold. The second thing worth mentioning is that the range of the solution sets is maximal in all but one case. Due to our assumption (4.18) the maximal element possibly occurring in a third order solution set is equal to $2 * 3^2 - 1 = 17$. All solution sets, except $X_6^{(3)}$, contain both 0 and 17. A third characteristic is that for all j, $X_j^{(3)}$ contains the four-element cluster $\{7, 8, 9, 10\}$ (being in the middle between the extrema 0 and 17) and that for all j, $Y_j^{(3)}$ contains no element of the six-element cluster $\{6, 7, 8, 9, 10, 11\}$ (also being in the middle between the extrema).

Summarizing, we see that in the third order the MMGC is satisfied, and hence we conclude that for $(2, 3, p)$-percolation, $F_1 - F_2$ contains an interval a.s. on $\{F_1 - F_2 \neq \emptyset\}$, provided

$$\frac{1}{\sqrt{3}} < p \leq \frac{2}{3}, \tag{4.28}$$

which is needed to guarantee that $\gamma > 1$.

4.2.2 $(3, 5, p)$-percolation

For $M = 4$, there exists no integer m such that $\sqrt{M} < m < \sqrt{2M}$. Therefore, the next case which deserves attention is $(3, 5, p)$-percolation. An illustration of the first construction steps of a realization of $(3, 5, \frac{1}{2})$-percolation appears in Figure 4.2.

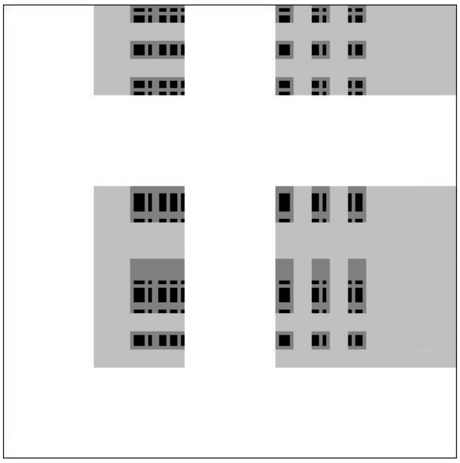

Figure 4.2: Illustration of the first construction steps of a realization of $(3, 5, \frac{1}{2})$-percolation. The colors have the same interpretation as in Figure 4.1. Note that in this case $p < \frac{m}{M} = \frac{3}{5}$, which means $\mu(\emptyset) > 0$. Consequently, possibly additional empty strips occur. This is seen in the rightmost vertical strip of the figure for the 2nd level approximation and somewhere in the middle we get an empty horizontal strip in the 3rd level.

According to Lemma 4.1, the lowest order for which it makes sense to search for appropriate sets is the 2nd order, since

$$\frac{\log 2}{2\log 3 - \log 5} \approx 1.18, \tag{4.29}$$

which is quite close to one, indicating that it might be not too difficult to find proper 2nd order sets. This time the requirements on the subsets $X^{(2)}$ and $X^{(2)}$ of $\mathbb{A}^{(2)}$ are:

1. $X^{(2)}$ should contain 9 elements. If we divide $\mathbb{A}^{(2)}$ in five blocks of five elements, then each of these blocks should contain either 0 or 3 elements of $X^{(2)}$. The same should hold for $Y^{(2)}$.

2. $X^{(2)}$ and $Y^{(2)}$ should satisfy

$$\min_{k\in\mathbb{A}^{(2)}} \sum_{j=0}^{24} \mathbf{1}_{Y^{(2)}}(j)\mathbf{1}_{X^{(2)}}(j+k) > 1. \tag{4.30}$$

An heuristic argument why this should be easier than the search for sets for $(2, 3, p)$-percolation in the previous subsection is that we now search for larger sets (9 elements instead of 8) while the alphabet is smaller (25 elements instead of 27). This is confirmed by the fact that we already found suitable sets in one of the first guesses. For example, as can be checked by hand, the following pair of sets satisfy both the construction requirement and the MMGC:

$$\left\{ \begin{array}{lcc} X^{(2)} & = & \{0,2,4,10,12,14,20,22,24\}, \\ Y^{(2)} & = & \{2,3,4,5,6,7,10,11,12\}, \end{array} \right. \tag{4.31}$$

Taking random pairs of subsets of $\mathbb{A}^{(2)}$ for which the construction requirement holds, we found 951 pairs satisfying the MMGC in 10000 realizations, which is much more than the 12 out of 531441 which we found for $(2,3,p)$-percolation.

Recapitulating, also for $(3,5,p)$-percolation $F_1 - F_2$ contains an interval a.s. on $\{F_1 - F_2 \neq \emptyset\}$. Here the condition on p is

$$\frac{1}{\sqrt{5}} < p \leq \frac{3}{5}. \tag{4.32}$$

4.2.3 $(3,6,p)$-percolation

The next and last percolation class we consider is $(3,6,p)$-percolation. Computing the bound of Lemma 4.1, we find

$$\frac{\log 2}{2\log 3 - \log 6} \approx 1.71, \tag{4.33}$$

so we are going to search for 2nd order sets $X^{(2)}, Y^{(2)} \subseteq \mathbb{A}^{(2)}$ satisfying

1. $\#X^{(2)} = \#Y^{(2)} = 9$, and dividing $\mathbb{A}^{(2)}$ in six blocks of six elements, each block should contain either 0 or 3 elements of $X^{(2)}$ and $Y^{(2)}$.

2. $X^{(2)}$ and $Y^{(2)}$ should satisfy

$$\min_{k \in \mathbb{A}^{(2)}} \sum_{j=0}^{35} \mathbf{1}_{Y^{(2)}}(j)\mathbf{1}_{X^{(2)}}(j+k) > 1. \tag{4.34}$$

Simple guessing, while keeping in mind the characteristics of the solutions for $(2,3,p)$-percolation, did not lead to success. The number of possible sets at the first level is

$$\begin{pmatrix} 6 \\ 3 \end{pmatrix} = 20, \tag{4.35}$$

which means that at the 2nd level we have 20^4 possible sets, and since we search for pairs of 2nd order sets, the number of possibilities is $20^8 = 25600000000$. Although we can correct this somewhat by accounting for symmetries, it costs far too much computing time to check all possibilities.

Inspired by the proof of Theorem 4.1, where the best choice of sets was given by (4.4) and (2.6), we restrict our search by the assumption that the first order approximations $X^{(1)}$ and $Y^{(1)}$ of $X^{(2)}$ and $Y^{(2)}$ are given by

$$X^{(1)} = \{0, 1, 2\}\,, \quad Y^{(1)} = \{0, 2, 4\}\,. \tag{4.36}$$

Still, the number of pairs to check is 64000000. Our approach now is to take a random pair of sets $X^{(2)}, Y^{(2)}$, for which the first order approximation is given by (4.37), and to check if the requirements are satisfied. We did this 600000 times, but unfortunately, we failed to find a solution pair.

4.3 Convolutions on cyclic groups

In this section we describe (m, M, p)-percolation problems from a different point of view. Using a convolution approach, an equivalent formal description of the problem will be derived. To explain the ideas, we take a realization of $(3, 5, \frac{3}{5})$-percolation as an example.

Suppose that in the first order the approximating sets are given by

$$X = \{0, 1, 3\}\,, \quad Y = \{0, 2, 4\}\,. \tag{4.37}$$

Then $\vec{X}$ and $\vec{Y}$, written as row vectors, are given by

$$\vec{X} = (1\;1\;0\;1\;0), \quad \vec{Y} = (1\;0\;1\;0\;1). \tag{4.38}$$

Taking the characteristic functions $\mathbf{1}_X(j)$ and $\mathbf{1}_Y(j)$, we get two mass distributions on $\mathbb{A}$:

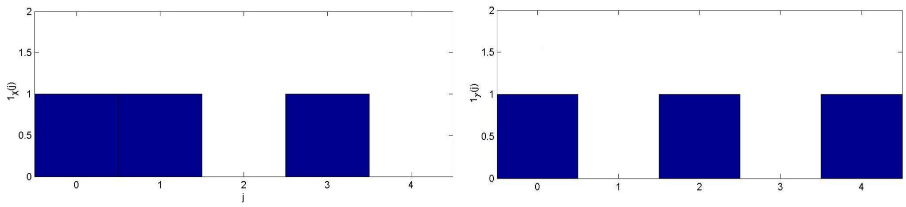

Figure 4.3: Mass distributions on $\mathbb{A}$ corresponding to X and Y.

Now let $X^{(n)}$ and $Y^{(n)}$ be nth order sets and consider the convolution $\mathbf{1}_{X^{(n)}} * \mathbf{1}_{Y^{(n)}}$ of the corresponding mass distributions. When k runs through $\mathbb{A}$, then we find the collection of sums

$$(\mathbf{1}_{X^{(n)}} * \mathbf{1}_{Y^{(n)}})(k) = \sum_{j=0}^{M^n-1} \mathbf{1}_{Y^{(n)}}(j)\mathbf{1}_{X^{(n)}}(k-j), \quad k = 0, \ldots, M^n - 1. \tag{4.39}$$

If we can find sets $X^{(n)}$ and $Y^{(n)}$ such that all those sums are greater than one, then by defining

$$\tilde{X}^{(n)} := \left\{ j \in \mathbb{A} : M - j \in X^{(n)} \right\}, \quad \tilde{Y}^{(n)} := Y^{(n)}, \tag{4.40}$$

we see that the collection of sums

$$\sum_{j=0}^{M^n-1} \mathbf{1}_{\tilde{Y}^{(n)}}(j)\mathbf{1}_{\tilde{X}^{(n)}}(j+k), \quad k = 0, \ldots, M^n - 1 \tag{4.41}$$

is exactly the same collection as (4.39). On top of that, if $X^{(n)}$ and $Y^{(n)}$ satisfy the construction requirements, then the same holds for $\tilde{X}^{(n)}$ and $\tilde{Y}^{(n)}$. So instead of searching for sets for which the sums in (4.41) are all greater than one, we can search for sets such that all sums in (4.39) exceed one.

Continuing with our example, consider the sum

$$\sum_{j=0}^{4} \mathbf{1}_Y(j)\mathbf{1}_X(k-j), \tag{4.42}$$

for X and Y as in (4.37). This is the convolution of the mass distributions on $\mathbb{A}$ corresponding to X and Y. Plotting this convolution on the integers we get:

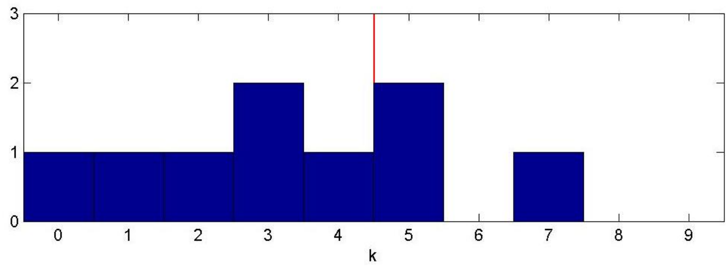

Figure 4.4: The sum $\sum_{j=0}^{4} \mathbf{1}_Y(j)\mathbf{1}_X(k-j)$ plotted on the integers. This is the convolution of the mass distributions plotted in Figure 4.3.

If we now adjust this plot for the group structure of $\mathbb{A}$, then we get a convolution of the mass distributions on the cyclic group $\mathbb{A}$, which is shown in Figure 4.5.

What we need is to find sets X and Y such that the corresponding plot representing the sum $\sum_{j=0}^{4} \mathbf{1}_Y(j)\mathbf{1}_X(k-j)$ as a function of k (as in Figure 4.5) has height at least two on the domain $\{0, \ldots, 4\}$. However, for the first order this is not possible, since the area under the plot is always 9, where

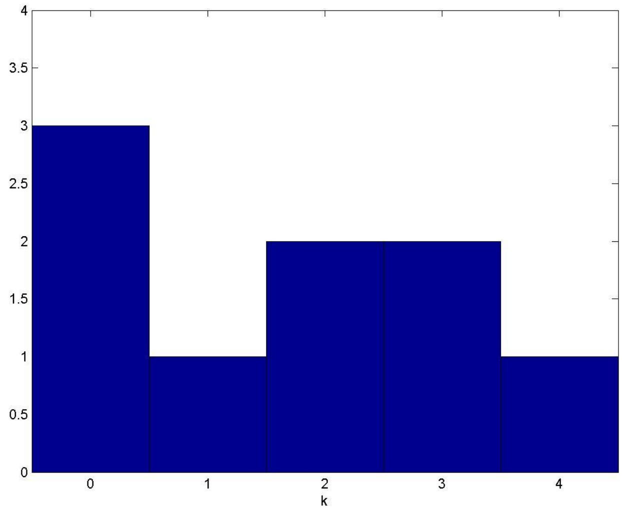

Figure 4.5: The same convolution plotted on the group $\mathbb{A}$.

10 is needed to make a minimum height of 2 possible.

The corresponding graph for 2nd order sets $X^{(2)}$ and $Y^{(2)}$ can be constructed by starting with their vector representation and following the same procedure as above. It seems more interesting to look at the graph for 2nd order sets in a different way. This graph can be obtained by moving all the blocks (by *blocks*, we mean the 1×1 squares together building the graph) in Figure (4.4) to the position which is 5 times the original position. After having done that, each of the blocks has to be replaced by a nine-block cluster. For example, the rightmost square in the plot of Figure (4.4), being at position 7, should be moved to position 35. This block was present in the graph because $3 \in X$ and $4 \in Y$. Since $3 \in X$, $X^{(2)}$ will contain three numbers from the set $\{15, 16, 17, 18, 19\}$ and $Y^{(2)}$ contains three numbers from $\{20, 21, 22, 23, 24\}$, because $4 \in Y$. Those two times three numbers form nine pairs, and the sum of each of these pairs causes the presence of a block at the corresponding position in the graph for $X^{(2)}$ and $Y^{(2)}$. In this way a nine-block cluster in constituted. The same should be done for all blocks in Figure 4.4. The last step is to subtract 25 from the position of all blocks out of the range of $\mathbb{A}^{(2)}$.

Note that the area under the 2nd order graph obtained now is 81, such that a minimal height of two is possible since the domain is $\{0, 1, \ldots, 24\}$. One of the ideas is that in choosing clusters of nine blocks and building up the 2nd order graph, one has a visualization of the problem, one can see where still gaps occur and what clusters fit best to build a graph with a minimal height of two. We tried out this approach for $(3, 5, p)$-percolation, and succeeded in the first try. Unfortunately, for $(3, 6, p)$-percolation we did not find appropriate sets. Complications arose due to dependencies between clusters: choosing a cluster in a certain position gives restrictions on clusters that are allowed in other positions. The situation became indistinct and it was not clear how to determine what good choices for the clusters would be.

The above convolution approach can be used to derive a formal description of the (m, M, p)-percolation problems by looking at the characteristic function of the mass distributions corresponding to X and Y. This yields for X and Y as in (4.37):

$$P_X(x) := 1 + x + x^3, \quad P_Y(x) := 1 + x^2 + x^4. \tag{4.43}$$

The characteristic function of the convolution of the two mass distributions is given by:

$$P_{XY}(x) = P_X(x)P_Y(x) = (1+x+x^3)(1+x^2+x^4) = 1+x+x^2+2x^3+x^4+2x^5+x^7. \tag{4.44}$$

If we define $\mathcal{P}$ to be the class of polynomials of degree four, with exactly 3 coefficients equal to 1 and 2 coefficients equal to 0, then the characteristic polynomials of the 2nd order sets $X^{(2)}$ and $Y(2)$ can be found in the following way. Compute $P_X(x^5)$ and $P_Y(x^5)$ and multiply each of the terms with a polynomial from $\mathcal{P}$, leading to

$$P_{X^{(2)}}(x) = P_1 + x^5P_2 + x^{15}P_3, \quad P_{Y^{(2)}}(x) = P_4 + x^{10}P_5 + x^{20}P_6, \tag{4.45}$$

for some $P_1, \ldots, P_6 \in \mathcal{P}$. Multiplying these two polynomials gives the characteristic polynomial of the 2nd order we are interested in:

$$P_{X^{(2)}Y^{(2)}}(x) = (P_1 + x^5P_2 + x^{15}P_3)(P_4 + x^{10}P_5 + x^{20}P_6). \tag{4.46}$$

Note that this product can also be obtained by evaluating $P_{XY}(x^5)$ and multiplying each of its terms (where $2x^3$ and $2x^5$ should be regarded as two terms each) by a product of two polynomials from $\mathcal{P}$. Observe that only 6 of these polynomials can be chosen freely, indicating that the polynomials are dependent, which are exactly the same dependencies as we found before for the clusters described above.

Accounting for the group structure of the alphabet, all powers in the polynomial (4.46) should be taken modulo 25. What is required now is that all coefficients of the resulting polynomial are at least equal to two. The coefficient for x^k can be found by evaluating the kth derivative for $x = 0$ and dividing by $k!$.

Summarizing, the formulation of the problem is now as follows:
Take polynomials $P_X, P_Y \in \mathcal{P}$. Compute $P_X(x^5)$ and $P_Y(x^5)$ and multiply each of the terms with a polynomial from $\mathcal{P}$. Find the product of the two resulting polynomials and call it $P_{X^{(2)}Y^{(2)}}$. Calculate for $k \in \mathbb{A}^{(2)}$

$$\frac{1}{k!}\frac{d^k}{dx^k}P_{X^{(2)}Y^{(2)}}(x) + \frac{1}{(k+25)!}\frac{d^{(k+25)}}{dx^{(k+25)}}P_{X^{(2)}Y^{(2)}}(x). \tag{4.47}$$

Is it possible that these numbers are at least 2 for all $k \in \mathbb{A}^{(2)}$?

We did not yet manage to solve this problem. An approach that might work is to consider the coefficients in the 8 polynomials $P_X, P_Y, P_1, \ldots, P_6$ as unknown variables that can take only the values 0 and 1. These variables should satisfy some requirements: the five coefficients of each of the polynomials should sum up to 3 and the numbers in (4.47) should be at least two. So this leads to an integer programming problem with 8 equality constraints and 25 inequality constraints.

Adapting this reformulation of the problem for other cases of (m, M, p)-percolation is straightforward.

4.4 Conclusion

In studying correlated fractal percolation, the MMGC proved to be an useful tool to answer the question whether or not an interval occurs in the algebraic difference of two random Cantor sets from the correlated fractal percolation class. For most cases of (m, M, p)-percolation ($m < \sqrt{M}$ and $m > \sqrt{2M}$) an answer to this question is found. For m somewhere in between these two bounds, the problem seems to be quite hard. Although it is possible to solve some particular cases (if m is close to $\sqrt{2M}$ the problem is relatively easy), we did not manage to prove general results for flimsy percolation.

One of the reasons to study (m, M, p)-percolation was to see if we could find random Cantor sets for which the MMGC is not satisfied, in spite of the sum of the Hausdorff dimensions being greater than one. As we have seen, finding sets satisfying the MMGC in some cases costs much (computing) time. Checking that such sets do not exist by simply going through all possibilities is a hopeless mission, since the number of possibilities extremely rapidly increases with m and M. Also here, more understanding of the general structures of (m, M, p)-percolation is needed to get some results.

In further investigations to tackle the problem of flimsy percolation, it is recommended to have a close look at the solutions for $(2, 3, p)$-percolation. If there exist some general structural properties, then studying these solutions is probably helpful in discovering them. Examining the formulation of the problem derived in Section 4.3 could also give new insights in the problem.

Bibliography

[1] F. M. Dekking and K. Simon. *On the size of the algebraic difference of two random Cantor sets.* Random Structures Algorithms **32** (2008), no. 2, 205-222.

[2] F. M. Dekking and B. Kuijvenhoven. *Differences of random Cantor sets and lower spectral radii.* ArXiv: 0811.0525.

Appendix A

Matlab program to check the Four Gap Theorem

The following Matlab program was used to come to our conjecture that the distances of the first n multiples of an irrational number α to the nearest integer partition the interval $[0, 1/2]$ in subintervals of at most four different lengths.

```
function lengths = fourgaps(alpha,n)

% This function computes the first n multiples of alpha
% and after that for each of those multiples, the dis-
% tance to the nearest integer is calculated. Those n
% distances are sorted and lenghts of subintervals in
% which [0,1/2] is divided by those numbers are calcula-
% ted. If alpha is not specified (input 0), we take a
% random number between zero and one.

% Print alpha:
if alpha == 0
    alpha = rand
else
    alpha
end
x = [];      % x will be used to store the distance of
             % multiples of alpha to the nearest integer.


for i = 1:n
    mult = i*alpha;     % mult is the multiple of alpha
                        % for which the distance to the
                        % nearest integer will be calcu-
```

```
                        % lated.
    x = [x;abs(i*alpha-round(i*alpha))];
end

% Now sort the hitted values (including 0 and 1/2):
x = sort([0;x;1/2]);

% We now are going to compute the lengths:
y = []; for i = 2:length(x)
    y = [y;x(i)-x(i-1)];
end

% Remove lengths that occur multiple times in y:
k = 1; while k < length(y)
    len = length(y);
    for i = 1:len-k
        index = len+1-i;
        if y(k) == y(index)
            y(index) = [];
        end
    end
    k = k+1;
end

% sort the lengths and multiply them by n (otherwise we
% get output like 0.000000000... if n is large):
lengths = n*sort(y)
```

Appendix B

Matlab programs for $(2, 3, p)$-percolation

The code below is an implementation of the algorithm described in Section 4.2.1. The first two functions, `autocor.m` and `nextlevel2uit3.m`, are functions needed for `TwoThreePerco.m`, the program performing the search for sets satisfying the MMGC.

B.1 autocor.m

```
function [autocor] = autocor(x,y);

% This function computes the correlation coefficients
% for two row vectors x and y.

% Check if x and y have the same lengths:
if(length(x) ~= length(y))
    'Error: lengths of x and y not the same.'
end M = length(x);

autocor = zeros(M,1); y = [y,y]; for i = 0:M-1
    autocor(i+1) = x*y(i+1:i+M)';
end
```

B.2 nextlevel2uit3.m

```
function [nextlevel2uit3] = nextlevel2uit3(A)

% Each row in A represents a pair of subsets of the
% alphabet. This function computes all possible pairs
```

```
% of sets on the next level, given the pairs of sets
% in the matrix A.

B = [];
% In the matrix B, we record all possible pairs of
% sets on the next level.

for j = 1:size(A,1)
    % j indicates the row of A which is in
    % consideration.
    C = [];
    % In the matrix C we collect next order pairs of
    % sets that correspond to the jth row of A.
    for k = 1:size(A,2)
        % k indicates the column of A being in
        % consideration.
        rijen = max(1,size(C,1));
        kolommen = size(C,2);
        if A(j,k) == 0
            % If the element in A equals zero:
            % Add three zeros in each row of C
            C(1,kolommen+3) = 0;
        end
        if A(j,k) == 1
            % If the element in A equals one:
            % Make two copies of C, such that each row
            % appears three times. Now add 110 to the
            % first, 101 to the second and 011 to the
            % third.
            C = [C;C;C];
            C([1:2*rijen],kolommen+1) = ones(2*rijen,1);
            C([1:rijen],kolommen+2) = ones(rijen,1);
            C([2*rijen+1:3*rijen],kolommen+2) =
                                            ones(rijen,1);
            C([rijen+1:3*rijen],kolommen+3) =
                                            ones(2*rijen,1);
        end
    end
    % Add the pairs of sets found for this row of A to
    % the pairs found for the previous rows:
    B = [B;C];
end A = B;

% And the result is:
```

```
nextlevel2uit3 = A;
```

B.3 TwoThreePerco.m

```
function [success, solutions] = TwoThreePerco(n)

% For (2,3,p)-percolation and a given n (the input
% argument), this function computes all nth order sets
% satisfying the MMGC and the construction requirements.

A = [1,1,0,1,1,0];

% Each row in the matrix A represents two sets (e.g. the
% above row represents the sets X = {0,1} and Y = {0,1}.
% For each row in A we compute the correlation coeffi-
% cients. If the minimum of these coefficients is 2,
% then we have succeeded. If the minimum is 0, then the
% row is deleted, since this will not lead to success.

k = size(A,1);
rijteller = 1;      % Indicates which row of A is in
                    % consideration.
success = 0;        % Will be set to 1 as soon as a
                    % solution is found.
solutions = [];     % This matrix specifies all
                    % solutions. Each row represents a
                    % pair of sets satisfying the MMGC.
deletions = 0;      % Counts the number of deleted rows.

for j = 1:n
    % Check all possibilities from A:
    for i = 1:k
        x = A(rijteller,1:3^j);
        y = A(rijteller,3^j+1:2*3^j);
        cor = autocor(x,y)';
        gamma = min(cor);
        if gamma == 0
            % This will never lead to success, so
            % delete this row:
            deletions = deletions+1
            A(rijteller,:) = [];
            rijteller = rijteller-1;
        end
```

```
        if gamma > 1
            % We have succeeded and found a solution.
            'hurrah'
            success = 1;
            solutions = [solutions;A(rijteller,:)];
        end
        rijteller = rijteller+1;
    end
    if j ~= n
        % We have finished checking all rows of A. Now
        % replace A by sets of the next level.
        A = nextlevel2uit3(A);
        k = size(A,1);
        rijteller = 1;
    end
end
```