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CONFORMAL MAPPING OF EXTERIOR REGIONS BY THE METHOD OF TRIGONOMETRIC INTERPOLATION.

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Conformal Mapping of Exterior Regions by the Method of Trigonometric Interpolation.

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In this paper there is presented a numerical method which ensures any pre-assigned accuracy for constructing a mapping function for a simply-connected exterior region bounded by a simple closed curve. The boundary of the region may be given analytically, graphically or by a discrete series of points, i.e. in tabular form. Templates are constructed to simplify the calculation of the nodal points, which constitutes the principal difficulty in solving a given problem.

§ 1. Let us consider the mapping of the exterior of the unit circle $|\zeta| \geq 1$ onto the exterior of a given simply connected region of $z = x + iy$. We normalize the mapping function with the conditions $z = f(\zeta)_{\zeta=\infty} = \infty$, $z = f(\zeta)_{\zeta=1} = z_o$ and seek it in the form of a truncated series, i.e. in the form of the approximating polynomial:

$$z = \sum_{n=-1}^{m-2} C_n \zeta^{-n} = \sum_{n=-1}^{m-2} (A_n + i B_n) r^{-n} (\cos n\varphi - i \sin n\varphi) . \qquad (1)$$

To determine the coefficients C_n we shall divide the unit circle $r = 1$ into $2m$ equal parts $\Delta\varphi = \frac{\pi}{m}$ and we shall choose two systems of points: the even system $\varphi_{2k} = \frac{2k}{m}$ and the odd $\varphi_{2k-1} = \frac{(2k-1)}{m}$, $k = 1, 2, \ldots, m$. The images of these points $z_k = x_k + i y_k$ in the z-plane are called nodal points. According to (1) we have for them:

$$x_k = \sum_{n=-1}^{m-2} A_n \cos n\varphi_k + B_n \sin n\varphi_k ; \quad k = 1, 2, \ldots, m;$$

$$y_k = \sum_{n=-1}^{m-2} - A_n \sin n\varphi_k + B_n \cos n\varphi_k . \qquad (2)$$

The system of equations (2) is characterized by an interesting property, it is easily inverted with respect to the coefficients A_n and B_n. This is a consequence of the property of orthogonality, which trigonometric functions of discrete argument possess in the case of equally spaced data (Reference 1),

$$\sum_{k=1}^{m} \sin j\varphi_k \sin n\varphi_k = \sum_{k=1}^{m} \cos j\varphi_k \cos n\varphi_k = \begin{cases} 0, & j \neq n \\ \frac{m}{2}, & j = n \end{cases}$$

$$\sum_{k=1}^{m} \sin j\varphi_k \cos n\varphi_k = 0 \qquad (3)$$

Let us invert the system (2). For this we multiply the first equation by $\cos j\varphi_k$ and the second by $\sin j\varphi_k$ and sum with respect to k:

$$\sum_{k=1}^{m} x_k \cos j\varphi_k - y_x \sin j\varphi_k = \sum_{n=-1}^{m-2} A_n \Big(\sum_{k=1}^{m} \cos j\varphi_k \cos n\varphi_k +$$

$$+ \sum_{k=1}^{m} \sin j\varphi_k \sin n\varphi_k\Big) + \sum_{n=-1}^{m-2} B_n \Big(\sum_{k=1}^{m} \cos j\varphi_k \sin n\varphi_k - \sum_{k=1}^{m} \sin j\varphi_k \cos n\varphi_n\Big) = mA_j.$$

In an analogous manner, multiplying the first equation of system (2) by $\sin j\varphi_k$, the second by $\cos j\varphi_k$ and summing with respect to k, we determine B_j. Thus:

$$A_j = \frac{1}{m}\sum_{k=1}^{m} x_k \cos j\varphi_k - y_k \sin j\varphi_k, \quad j = -1, 0, \ldots, m-2$$

$$B_j = \frac{1}{m}\sum_{k=1}^{m} x_k \sin j\varphi_k + y_k \cos j\varphi_k \qquad (4)$$

Formula (4) permits the coefficients A_j, B_j to be easily calculated for known nodal points. However, the location of the nodal points is unknown at first and it is still necessary to construct an iteration process which would permit the nodal points to be determined with any pre-assigned degree of accuracy.

For this purpose we shall establish a relation between the nodal points of the even and odd systems, denoting the coefficients constructed on the basis of the m even points by $A_j^{(+m)}$, $B_j^{(+m)}$ and the coefficients constructed on the basis of the m odd points by $A_j^{(-m)}$, $B_j^{(-m)}$. Then in accordance with (4), we obtain:

$$A_j^{(+m)} = \frac{1}{m} \sum_{k=1}^{m} x_{2k} \cos j\varphi_{2k} - y_{2k} \sin j\varphi_{2k};$$

$$B_j^{(+m)} = \frac{1}{m} \sum_{k=1}^{m} x_{2k} \sin j\varphi_{2k} + y_{2k} \cos j\varphi_{2k}; \tag{5}$$

$$A_j^{(-m)} = \frac{1}{m} \sum_{k=1}^{m} x_{2k-1} \cos j\varphi_{2k-1} - y_{2k-1} \sin j\varphi_{2k-1};$$

$$B_j^{(-m)} = \frac{1}{m} \sum_{k=1}^{m} x_{2k-1} \sin j\varphi_{2k-1} + y_{2k-1} \cos j\varphi_{2k-1}. \tag{6}$$

In order to establish a relation between the even and odd nodal points, the coefficients $A_n^{(-m)}$, $B_n^{(-m)}$ must be eliminated from (2) by means of the formulae of (6). As a result we find:

$$x_{2\nu} = \sum_{n=-1}^{m-2} A_n^{(-m)} \cos n\varphi_{2\nu} + B_n^{(-m)} \sin n\varphi_{2\nu}$$

$$= \frac{1}{m}\Big[\sum_{k=1}^{m} x_{2k-1} \sum_{n=-1}^{m-2} \cos n(\varphi_{2k-1} - \varphi_{2\nu}) - y_{2k-1} \sum_{n=-1}^{m-2} \sin n(\varphi_{2k-1} - \varphi_{2\nu})\Big];$$

$$y_{2\nu} = \sum_{n=-1}^{m-2} -A_n^{(-m)} \sin n\varphi_{2\nu} + B_n^{(-m)} \cos n\varphi_{2\nu}$$

$$= \frac{1}{m}\Big[\sum_{k=1}^{m} x_{2k-1} \sum_{n=-1}^{m-2} \sin n(\varphi_{2k-1} - \varphi_{2\nu}) + y_{2k-1} \sum_{n=-1}^{m-z} \cos n(\varphi_{2k-1} - \varphi_{2\nu})\Big].$$

In an analogous manner, eliminating the coefficients $A_n^{(+m)}$, $B_n^{(+m)}$ from (2) by means of the formulae of (5), we obtain:

$$x_{2\nu-1} = \frac{1}{m}\Big[\sum_{k=1}^{m} x_{2k} \sum_{n=-1}^{m-2} \cos n(\varphi_{2k} - \varphi_{2\nu-1}) - y_{2k} \sum_{n=-1}^{m-2} \sin n(\varphi_{2k} - \varphi_{2\nu-1})\Big];$$

$$y_{2\nu-1} = \frac{1}{m}\Big[\sum_{k=1}^{m} x_{2k} \sum_{n=-1}^{m-2} \sin n(\varphi_{2k} - \varphi_{2\nu-1}) + y_{2k} \sum_{n=-1}^{m-2} \cos n(\varphi_{2k} - \varphi_{2\nu-1})\Big].$$

Introducing now the designations:

$$\gamma^{I(m)}_{2k-1,2\nu} = \frac{1}{m}\sum_{n=-1}^{m-2} \sin n(\varphi_{2k-1} - \varphi_{2\nu}); \quad \gamma^{I(m)}_{2k,2\nu-1} = \frac{1}{m}\sum_{n=-1}^{m-2} \sin n(\varphi_{2k} - \varphi_{2\nu-1});$$

$$\gamma^{II(m)}_{2k-1,2\nu} = \frac{1}{m}\sum_{n=-1}^{m-2} \cos n(\varphi_{2k-1} - \varphi_{2\nu}); \quad \gamma^{II(m)}_{2k,2\nu-1} = \frac{1}{m}\sum_{n=-1}^{m-2} \cos n(\varphi_{2k} - \varphi_{2\nu-1}), \qquad (7)$$

we obtain the calculation formulae of the present iteration process:

$$x^{(n)}_{2\nu-1} = \sum_{k=1}^{m} x^{(n)}_{2k}\, \gamma^{II(m)}_{2k,2\nu-1} - y^{(n)}_{2k}\, \gamma^{I(m)}_{2k,2\nu-1};$$

$$y^{(n)}_{2\nu-1} = \sum_{k=1}^{m} x^{(n)}_{2k}\, \gamma^{I(m)}_{2k,2\nu-1} + y^{(n)}_{2k}\, \gamma^{II(m)}_{2k,2\nu-1}; \qquad (8)$$

$$x^{(n+1)}_{2\nu} = \sum_{k=1}^{m} x^{(n)}_{2k-1}\, \gamma^{II(m)}_{2k-1,2\nu} - y^{(n)}_{2k-1}\, \gamma^{I(m)}_{2k-1,2\nu};$$

$$y^{(n+1)}_{2\nu} = \sum_{k=1}^{m} x^{(n)}_{2k-1}\, \gamma^{I(m)}_{2k-1,2} + y^{(n)}_{2k-1}\, \gamma^{II(m)}_{2k-1,2\nu}. \qquad (9)$$

The process itself is carried out in the following manner. For some $m = 4, 8, 16, \ldots$, starting from graphical considerations, we select the zero-eth approximation to the values of the m even nodal points ($x^{(0)}_{2k}$, $y^{(0)}_{2k}$), $k = 1, 2, \ldots, m$ and by means of the formulae (8) we compute the approximate odd nodal points, which will not in general lie on the contour. We carry them to the contour along normals and obtain in zero-eth approximation the odd nodal points ($x^{(0)}_{2k-1}$, $y^{(0)}_{2k-1}$). We now compute by the formulae of (9) the approximate even nodal points, carry them to the contour and obtain as the first approximation of the even nodal points ($x^{(1)}_{2k}$, $y^{(1)}_{2k}$), etc.

We repeat the iteration process until a subsequent approximation coincides with prescribed accuracy with the previous approximation. To increase the accuracy one must pass from a smaller value of m to a larger value, for example, to 2m, taking the previous even and odd points as a system of new even points and repeating the iteration process for 2m. We thereby define more precisely the location of the nodal points, after which formulae (5), (6) give the opportunity of finding the coefficients of the mapping polynomial with the prescribed degree of accuracy.

We note that the quantities γ are not functions of the contour points and are computed only once. Moreover, to construct the matrices $\| \gamma^{I(m)} \|$ and $\| \gamma^{II(m)} \|$ it is sufficient to compute the first column of each matrix, respectively, and then with the elements of the first column to write the remaining elements of the first row of each matrix by means of the formula:

$$\gamma_{1,2m-(2k-2)} = \gamma_{2k-1,0}; \quad k = 2, \ldots, m$$

and to fill out the entire matrix writing identical elements along all downward diagonals.

§ 2. As simple as the formulae of (8) and (9) are, they require $4m^2$ multiplications to compute a single half-step of the iteration process. The calculation scheme for the iteration process can be significantly simplified by reducing similar sines to identical ones and by introducing the following grouping of the nodal points:

$$X_k^{\substack{+++\\-+-}} = x_k \pm x_{k+2} + x_{k+m} \pm x_{k+m+2}; \quad X_k^{\substack{--+\\+--}} = x_k \mp x_{k+2} - x_{k+m} \pm x_{k+m+2};$$

$$Y_k^{\substack{+++\\-+-}} = y_k \pm y_{k+2} + y_{k+m} \pm y_{k+m+2}; \quad Y_k^{\substack{--+\\+--}} = y_k \mp y_{k+2} - y_{k+m} \pm y_{k+m+2};$$

$$\mathfrak{X}^{\substack{--+\\+--}} = x_k \mp x_{k+6} - x_{k+m} \pm x_{k+m+6}; \quad \mathfrak{Y}^{\substack{--+\\+--}} = y_k \mp y_{k+6} - y_{k+m} \pm y_{k+m+6};$$

$$\alpha_x^+ = \sum_{j=0,2,\ldots}^{2m-2} x_j; \quad \alpha_y^+ = \sum_{j=0,2,\ldots}^{2m+2} y_j; \quad \alpha_x^- = \sum_{j=0,2,\ldots}^{2m-2} (-1)^{\frac{j}{2}} x_j; \quad \alpha_y^- = \sum_{j=0,2,\ldots}^{2m-2} (-1)^{\frac{j}{2}} y_j$$

for calculating the odd nodal points, and:

$$\alpha_x^+ = \sum_{j=1,3,\ldots}^{2m-1} x_j; \quad \alpha_y^+ = \sum_{j=1,3,\ldots}^{2m-1} y_j;$$

$$\alpha_x^- = \sum_{j=1,3,\ldots}^{2m-1} (-1)^{\frac{j-1}{2}} x_j; \quad \alpha_y^- = \sum_{j=1,3,\ldots}^{2m-1} (-1)^{\frac{j-1}{2}} y_j.$$

for calculating the even nodal points. As a result we obtain the computation tables 1, 2 for calculating the nodal points for $m = 4, 8, 16$. A table for computing the odd values of y_i from the even points for $m = 16$ is constructed analogous to table 2 in the following way: firstly it is necessary to replace x by y, y by x, X_i, $\mathfrak{X}_i$ by Y_i, $\mathfrak{Y}_i$ and Y_i, $\mathfrak{Y}_i$ by X_i, $\mathfrak{X}_i$; secondly the first column of α_y is constructed according to the rule $\alpha_y^+ - \alpha_x^-$, $\alpha_y^+ + \alpha_x^-$, etc. Similar tables can be constructed for computing the even points from the odd ones; it is necessary only to increase the index of x_i, y_i, X_i, Y_i, $\mathcal{X}_i$, and $\mathcal{Y}_i$ by unity, as is shown for the case $m = 4$.

All of these computations will be completely carried out by a single-type if advantage is taken of the previously mentioned templates which are equivalent to the tables 1, 2.

Computations with the given templates are carried out in the following way. Writing the given x_i, y_i in two columns for $m = 4, 16$ and for $m = 8$ writing y_i with an interval of three rows beneath x_i, we place the template on the column x_i, y_i so that the shaded line coincides with the first row. Then we carry out consecutively the indicated actions, not removing the readings of the indicator of the calculating machine and we compute respectively the quantities which are in the column headings of the templates I ($m = 4, 8, 16$). For example, to compute the odd points from the even points for $m = 4$ we have:

$$\alpha_x^+ = x_0 + x_2 + x_4 + x_6; \qquad \alpha_x^- = \alpha_x^+ - 2x_6 - 2x_2$$

$$\alpha_{1x_1} = \alpha_x^- + 2x_2 - 2x_4; \; \alpha_{1x_3} = \alpha_{1x_1} + 2x_4 - 2x_0$$

To use the templates II ($m = 8$) and III, IV ($m = 16$) it is necessary to determine the first value of α_i, as indicated above, write down the result, then remove it, shift the template down one row, carry out all the indicated actions, write down the result, remove it and shift the template again down one row, etc. We thereby compute consecutively all of the elements of the columns α_i (it is, in addition, necessary to watch the signs carefully). Template II ($m = 16$) is meant for the calculation of the columns α_2, α_6 from their first values computed according to template I ($m = 16$). We take the first value, then we apply template II to the first row of x_i, y_i and carrying out all of the indicated actions, we thereby determine α_2, α_6 for the points 3 or 4. Then, not removing the previous result, we shift the template down one row, we carry out all of the indicated actions and we determine α_2, α_6 for the points 5 or 6 etc. After determining all of the α, we find the corresponding p and q:

$$p_{x_{k,k+\frac{m}{2}}} = \frac{1}{16}\alpha_{x_k} + 0{,}08839\,\alpha_{4x_k} \pm 0{,}11548\,\alpha_{6x_k} \pm 0{,}04783\,\alpha_{2x_k};$$

$$q_x = 0{,}02439\,\alpha_{1x}(x) + 0{,}06945\,\alpha_{3x}(x) + 0{,}10393\,\alpha_{5x}(x) + 0{,}12260\,\alpha_{7x}(x) \quad (10)$$

$$+\, 0{,}29598\,\alpha_{7x}(y) + 0{,}04305\,\alpha_{5x}(y) - 0{,}02876\,\alpha_{3x}(y) - 0{,}05887\,\alpha_{1x}(y)$$

In an analogous manner we find p_y, q_y, after which we compute:

$$x_{k,\, m+k} = p_{kx} \pm q_{kx} \;;\quad y_{k,\, m+k} = p_{ky} \pm q_{ky} \qquad (10')$$

where the upper sign refers to the first index and the lower one to the second index.

Example. To find a function which maps the exterior of the circle $|\zeta| \geq 1$ onto the exterior of a turbine blade section $z = x + iy$, given in the tabular form:

x	-1.211	-1.200	-1.000	-0.700	-0.400	-0.000	+0.600	+0.900	+1.400	+1.925
y_u	-0.974	-0.930	-0.572	-0,128	+0.198	+0.429	+0.513	+0.502	+0.457	+0.400
y_l	-0.974	-0.980	-0.860	-0.656	-0.472	-0.255	-0.020	+0,129	+0.285	+0.400

By trial and error, we find the best approximation for the nodal points on the standard of $m = 4$, which we then make more accurate with $m = 8$. We select two systems of even points (crosses and circles) on the contour, which are located intentionally on different sides of the correct even nodal points. It is necessary to select the even crosses and circles for this so that the system of odd points computed from them lies on different sides of the contour. This can always be achieved if a sufficiently large bracket is taken. Joining the resulting off countour approximate odd crosses and circles with a straight line, we find their points of intersection with the contour (they are denoted on the drawing by V) and we take them as the zero-eth approximation of the odd nodal points, from which we compute the first approximation of the even nodal points. With this we complete the first step of the calculations and pass on to the second.

We carry the points so found along normals to the surface and select a second, significantly smaller, bracket placing the crosses and circles along the different sides of the deflection points at smaller distance than previously *. We compute the whole second step in a manner analogous to the first. As a result we obtain the second approximation of the even nodal points, we carry them to the contour and we determine the trend of the movement of the successive approximations to the even nodal points. After clarifying the trend of their movement, we select in succeeding steps crosses and circles significantly nearer to the deflection points on the basis of the results of the previous steps. In each step we constrict the bracket more and by the same token we localize more the intervals within which are found the correct nodal points. We terminate the process when all of the odd points, computed from the even points, lie on the contour with the prescribed accuracy.

The results of the computations carried out on the standard of $m = 4$ are presented in table 3. Then we take the even and odd nodal points, found for $m = 4$, as a new system of even points and we carry out one cycle of the calculations for $m = 8$ to make the results more precise and then, in case it is necessary, for $m = 16$. All of these computations are carried out in tables 4, 5, where for $m = 8$ all of the computed odd points lay on the contour so that in the present example the computations for $m = 16$ were not necessary and were carried out only to provide an additional check. Knowing the nodal points, we find the coefficients of the mapping polynomial from formulas (5) or (6) for $m = 8$. For $m = 16$ we obtain the same result, and for $j \geq 3$ all $A_j = B_j = 0$.

In conclusion we note that the existance of a cusp on the contour does not create a complication and the entire calculation is carried out in a manner completely analogous to that considered above (Reference 2).

Bibliography.

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Table 1

$m=4$	$^1/_4$	$\frac{1}{2}S_1=0,353554$	$m=4$	$^1/_4$	$\frac{1}{2}S_1=0,353554$
unknowns	α	α_1	unknowns	α	α_1
$x_{1,5}$	$\alpha_x^+ + \alpha_y^-$	$\pm X_0^{+--}$	$x_{2,6}$	$\alpha_x^+ + \alpha_y^-$	$\pm X_1^{+--}$
$x_{3,7}$	$\alpha_x^+ - \alpha_y^-$	$\pm X_2^{+--}$	$x_{4;8}$	$\alpha_x^+ - \alpha_y^-$	$\pm X_3^{+--}$
$y_{1,5}$	$\alpha_y^+ - \alpha_x^-$	$\pm Y_0^{+--}$	$y_{2,6}$	$\alpha_y^+ - \alpha_x^-$	$\pm Y_1^{+--}$
$y_{3,7}$	$\alpha_y^+ + \alpha_x^-$	$\pm Y_2^{+--}$	$y_{4,8}$	$\alpha_y^+ + \alpha_x^-$	$\pm Y_3^{+--}$

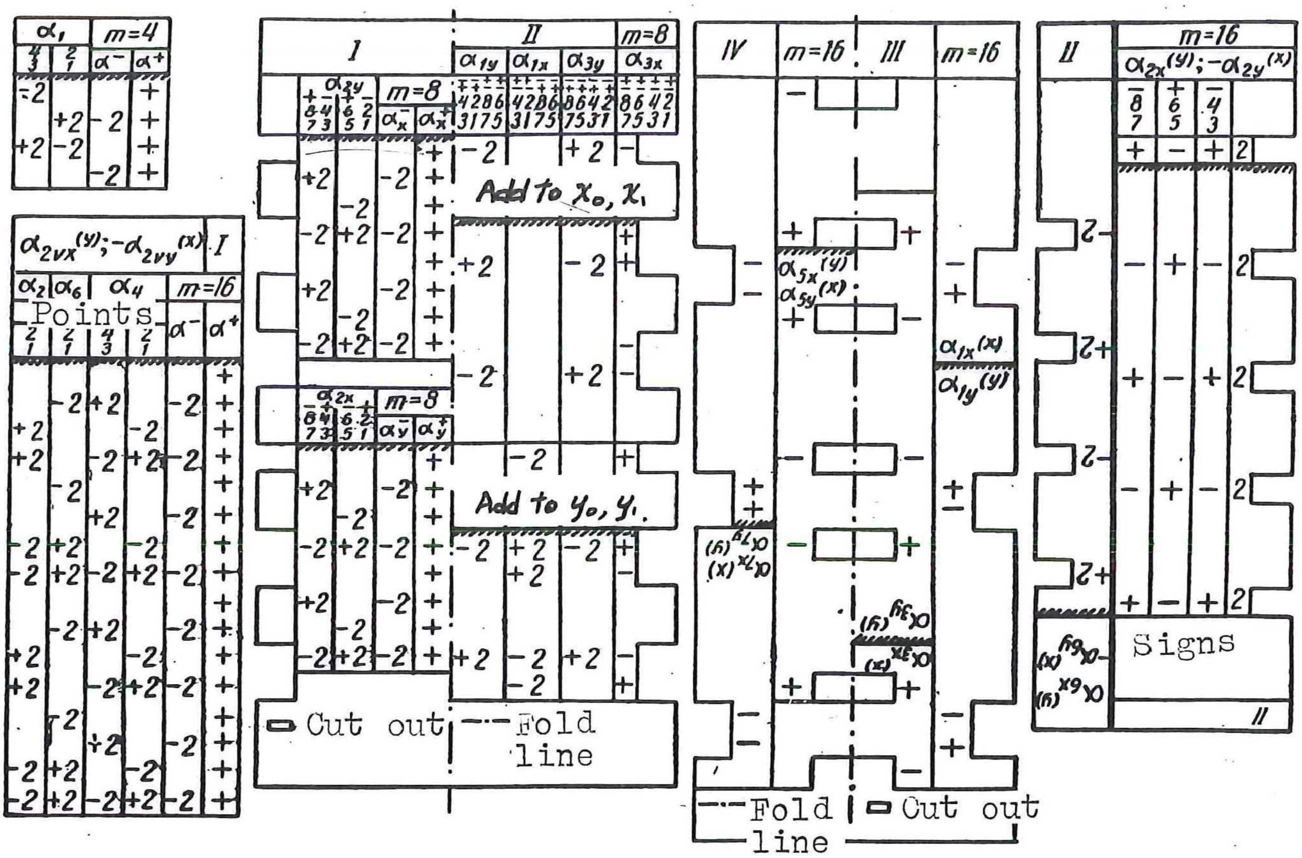
$m=8$	$^1/_8$	$\frac{1}{4}S_2=0,176777$	$\frac{1}{4}S_3=0,230970$	$\frac{1}{4}S_1=0,095671$
unknowns	α_x	α_{2x}	α_{3x}	α_{1x}
$x_{1,9}$	$\alpha_x^+ + \alpha_y^-$	$+(Y_0^{-+-} - Y_4^{-+-})$	$\pm(X_0^{+--} + Y_0^{--+})$	$\pm(X_4^{--+} + Y_4^{+--})$
$x_{3,11}$	$\alpha_x^+ - \alpha_y^-$	$+(Y_2^{-+-} - Y_6^{-+-})$	$\pm(X_2^{+--} + Y_2^{--+})$	$\pm(X_6^{--+} + Y_6^{+--})$
$x_{5,13}$	$\alpha_x^+ + \alpha_y^-$	$-(Y_0^{-+-} - Y_4^{-+-})$	$\pm(X_4^{+--} + Y_4^{--+})$	$\mp(X_0^{--+} + Y_0^{+--})$
$x_{7,15}$	$\alpha_x^+ - \alpha_y^-$	$-(Y_2^{-+-} - Y_6^{-+-})$	$\pm(X_6^{+--} + Y_6^{--+})$	$\mp(X_2^{--+} + Y_2^{+--})$
	α_y	α_{2y}	α_{3y}	α_{1y}
$y_{1,9}$	$\alpha_y^+ - \alpha_x^-$	$-(X_0^{-+-} - X_4^{-+-})$	$\mp(X_0^{--+} - Y_0^{+--})$	$\mp(X_4^{+--} - Y_4^{--+})$
$y_{3,11}$	$\alpha_y^+ + \alpha_x^-$	$-(X_2^{-+-} - X_6^{-+-})$	$\mp(X_2^{--+} - Y_2^{+--})$	$\mp(X_6^{+--} - Y_6^{--+})$
$y_{5,13}$	$\alpha_y^+ - \alpha_x^-$	$+(X_0^{-+-} - X_4^{-+-})$	$\mp(X_4^{--+} - Y_4^{+--})$	$\pm(X_0^{+--} - Y_0^{--+})$
$y_{7,15}$	$\alpha_y^+ + \alpha_x^-$	$+(X_2^{-+-} - X_6^{-+-})$	$\mp(X_6^{--+} - Y_6^{+--})$	$\pm(X_2^{+--} - Y_2^{--+})$

$m=16$	p_x			
	1/16	0,088388	0,115485	0,047835
unknowns	α_x	$\alpha_{4x}(y)$	$\alpha_{6x}(y)$	$\alpha_{2x}(y)$
$x_{1,17}$	$\alpha_x^+ + \alpha_y^-$	$+(Y_0^{-+-} + Y_8^{-+-} - Y_{12}^{-+-} - Y_4^{-+-})$	$+(Y_0^{-+-} - Y_8^{-+-} + Y_{12}^{+++} - Y_4^{+++})$	$+(Y_0^{-+-} - Y_8^{-+-} - Y_{12}^{+++} + Y_4^{+++})$
$x_{3,19}$	$\alpha_x^+ - \alpha_y^-$	$+(Y_2^{-+-} + Y_{10}^{-+-} - Y_{14}^{-+-} - Y_6^{-+-})$	$+(Y_2^{-+-} - Y_{10}^{-+-} + Y_{14}^{+++} - Y_6^{+++})$	$+(Y_2^{-+-} - Y_{10}^{-+-} - Y_{14}^{+++} + Y_6^{+++})$
$x_{5,21}$	$\alpha_x^+ + \alpha_y^-$	$+(Y_4^{-+-} + Y_{12}^{-+-} - Y_0^{-+-} - Y_8^{-+-})$	$+(Y_4^{-+-} - Y_{12}^{-+-} + Y_0^{+++} - Y_8^{+++})$	$+(Y_4^{-+-} - Y_{12}^{-+-} - Y_0^{+++} + Y_8^{+++})$
$x_{7,23}$	$\alpha_x^+ - \alpha_y^-$	$+(Y_6^{-+-} + Y_{14}^{-+-} - Y_2^{-+-} - Y_{10}^{-+-})$	$+(Y_6^{-+-} - Y_{14}^{-+-} + Y_2^{+++} - Y_{10}^{+++})$	$+(Y_6^{-+-} - Y_{14}^{-+-} - Y_2^{+++} + Y_{10}^{+++})$
$x_{9,25}$	$\alpha_x^+ + \alpha_y^-$	$+(Y_0^{-+-} + Y_8^{-+-} - Y_{12}^{-+-} - Y_4^{-+-})$	$-(Y_0^{-+-} - Y_8^{-+-} + Y_{12}^{+++} - Y_4^{+++})$	$-(Y_0^{-+-} - Y_8^{-+-} - Y_{12}^{+++} + Y_4^{+++})$
$x_{11,27}$	$\alpha_x^+ - \alpha_y^-$	$+(Y_2^{-+-} + Y_{10}^{-+-} - Y_{14}^{-+-} - Y_6^{-+-})$	$-(Y_2^{-+-} - Y_{10}^{-+-} + Y_{14}^{+++} - Y_6^{+++})$	$-(Y_2^{-+-} - Y_{10}^{-+-} - Y_{14}^{+++} + Y_6^{+++})$
$x_{13,29}$	$\alpha_x^+ + \alpha_y^-$	$+(Y_4^{-+-} + Y_{12}^{-+-} - Y_0^{-+-} - Y_8^{-+-})$	$-(Y_4^{-+-} - Y_{12}^{-+-} + Y_0^{+++} - Y_8^{+++})$	$-(Y_4^{-+-} - Y_{12}^{-+-} - Y_0^{+++} + Y_8^{+++})$
$x_{15,31}$	$\alpha_x^+ - \alpha_y^-$	$+(Y_6^{-+-} + Y_{14}^{-+-} - Y_2^{-+-} - Y_{10}^{-+-})$	$-(Y_6^{-+-} - Y_{14}^{-+-} + Y_2^{+++} - Y_{10}^{+++})$	$-(Y_6^{-+-} - Y_{14}^{-+-} - Y_2^{+++} + Y_{10}^{+++})$

Table 2

$\pm q_x$							
0,024386	0,069446	0,103934	0,122598	0,295977	0,043050	−0,028765	−0,058873
$\alpha_{1x}(x)$	$\alpha_{3x}(x)$	$\alpha_{5x}(x)$	$\alpha_{7x}(x)$	$\alpha_{7x}(y)$	$\alpha_{5x}(y)$	$\alpha_{3x}(y)$	$\alpha_{1x}(y)$
$+X_8^{-}-+$	$+\mathfrak{X}_6^{-}-+$	$-\mathfrak{X}_{14}^{+}--$	$+X_0^{+}--$	$+Y_0^{-}-+$	$-\mathfrak{Y}_{14}^{-}-+$	$-\mathfrak{Y}_6^{+}--$	$-Y_8^{+}--$
$+X_{10}^{-}-+$	$+\mathfrak{X}_8^{-}-+$	$+\mathfrak{X}_0^{+}--$	$+X_2^{+}--$	$+Y_2^{-}-+$	$+\mathfrak{Y}_0^{-}-+$	$-\mathfrak{Y}_8^{+}--$	$-Y_{10}^{+}--$
$+X_{12}^{-}-+$	$+\mathfrak{X}_{10}^{-}-+$	$+\mathfrak{X}_2^{+}--$	$+X_4^{+}--$	$+Y_4^{-}-+$	$+\mathfrak{Y}_2^{-}-+$	$-\mathfrak{Y}_{10}^{+}--$	$-Y_{12}^{+}--$
$+X_{14}^{-}-+$	$+\mathfrak{X}_{12}^{-}-+$	$+\mathfrak{X}_4^{+}--$	$+X_6^{+}--$	$+Y_6^{-}-+$	$+\mathfrak{Y}_4^{-}-+$	$-\mathfrak{Y}_{12}^{+}--$	$-Y_{14}^{+}--$
$-X_0^{-}-+$	$+\mathfrak{X}_{14}^{-}-+$	$+\mathfrak{X}_6^{+}--$	$+X_8^{+}--$	$+Y_8^{-}-+$	$+\mathfrak{Y}_6^{-}-+$	$-\mathfrak{Y}_{14}^{+}--$	$+Y_0^{+}--$
$-X_2^{-}-+$	$-\mathfrak{X}_0^{-}-+$	$+\mathfrak{X}_8^{+}--$	$+X_{10}^{+}--$	$+Y_{10}^{-}-+$	$+\mathfrak{Y}_8^{-}-+$	$+\mathfrak{Y}_0^{+}--$	$+Y_2^{+}--$
$-X_4^{-}-+$	$-\mathfrak{X}_2^{-}-+$	$+\mathfrak{X}_{10}^{+}--$	$+X_{12}^{+}--$	$+Y_{12}^{-}-+$	$+\mathfrak{Y}_{10}^{-}-+$	$+\mathfrak{Y}_2^{+}--$	$+Y_4^{+}--$
$-X_6^{-}-+$	$-\mathfrak{X}_4^{-}-+$	$+\mathfrak{X}_{12}^{+}--$	$+X_{14}^{+}--$	$+Y_{14}^{-}-+$	$+\mathfrak{Y}_{12}^{-}-+$	$+\mathfrak{Y}_4^{+}--$	$+Y_6^{+}--$

Templates



m=4
m=8
m=16
Points
Add to x_0, x_1
Add to y_0, y_1
Cut out
Fold line
Signs

Table 3

m=4								$\frac{1}{2}S_1$=0,35355			
Step	k	$x_к$	$y_к$			k	$\frac{1}{4}\alpha_x$; $\frac{1}{4}\alpha_y$	$\pm\alpha_{1x}$; $\pm\alpha_{1y}$	k	$\widetilde{x}_k$	$\widetilde{y}_k$
OI	0	+1,925	+0,400	α_x^+	+2,479	1	+0,4905	±3,171	1	+1,612	+0,694
	2	+0,900	+0,500	$\pm\alpha_y^-$	∓0,517	3	+0,7490	∓2,479	3	−0,127	−0,098
	4	−0,900	−0,417	α_y^+	+0,483	1	+0,2280	±1,317	5	−0,631	−0,238
	6	+0,554	0,000	$\mp\alpha_x^-$	±0,429	3	+0,0135	∓0,317	7	+1,625	+0,125
XI	0	+1,925	+0,400	α_x^+	−0,485	1	−0,1330	±3,535	1	+1,117	+0,059
	2	−0,400	+0,200	$\pm\alpha_y^-$	∓0,047	3	−0,1095	∓2,735	3	−1,076	+0,048
	4	−1,210	−0,970	α_y^+	−1,093	1	−0,7520	±2,293	5	−1,383	−1,563
	6	−0,800	−0,723	$\mp\alpha_x^-$	∓1,915	3	+0,2055	∓0,447	7	+0,857	+0,363
VI	1	+1,430	+0,450	α_x^+	+1,094	2	+0,1210	±0,546	2	+0,314	+0,365
	3	−0,610	−0,020	$\pm\alpha_y^-$	∓0,610	4	+0,4260	∓4,254	4	−1,078	−0,634
	5	−0,970	−0,840	α_y^+	−0,170	2	+0,0010	±1,030	6	−0,072	−0,363
	7	+1,244	+0,240	$\mp\alpha_x^-$	±0,174	4	−0,0860	∓1,550	8	+1,930	+0,462
OII	0	+1,925	+0,400	α_x^+	+1,770	1	+0,3300	±3,280	1	+1,490	+0,575
	2	+0,600	+0,510	$\pm\alpha_y^-$	∓0,450	3	+0,5550	∓2,480	3	−0,322	+0,024
	4	−0,955	−0,500	α_y^+	+0,250	1	+0,0200	±1,570	5	−0,830	−0,535
	6	+0,200	−0,160	$\mp\alpha_x^-$	∓0,170	3	+0,1050	∓0,230	7	+1,432	+0,186
XII	0	+1,925	+0,400	α_x^+	+0,395	1	+0,0087	±3,455	1	+1,230	+0,334
	2	0,000	+0,430	$\pm\alpha_y^-$	∓0,360	3	+0,1887	∓2,655	3	−0,750	+0,083
	4	−1,130	−0,800	α_y^+	−0,440	1	−0,4087	±2,100	5	−1,213	−1,151
	6	−0,400	−0,470	$\mp\alpha_x^-$	∓1,195	3	+0,1887	∓0,300	7	+1,127	+0,295
VII	1	+1,365	+0,460	α_x^+	+1,020	2	+0,0700	±0,620	2	+0,289	+0,481
	3	−0,545	+0,055	$\pm\alpha_y^-$	∓0,740	4	+0,4400	∓4,230	4	−1,055	−0,685
	5	−1,060	−0,900	α_y^+	−0,140	2	+0,0675	±1,170	6	−0,149	−0,346
	7	+1,260	+0,245	$\mp\alpha_x^-$	±0,410	4	−0,1375	∓1,550	8	+1,935	+0,410
OIII	0	+1,925	+0,400	α_x^+	+1,305	1	+0,2255	±3,545	1	+1,479	+0,517
	2	+0,500	+0,510	$\pm\alpha_y^-$	∓0,403	3	+0,4270	∓2,345	3	−0,402	+0,062
	4	−1,020	−0,600	α_y^+	+0,003	1	−0,1255	±1,817	5	−1,028	−0,768
	6	−0,100	−0,307	$\mp\alpha_x^-$	∓0,505	3	+0,1270	∓0,183	7	+1,256	+0,192
XIII	0	+1,925	+0,400	α_x^+	+0,595	1	+0,0462	±3,655	1	+1,338	+0,414
	2	+0,200	+0,480	$\pm\alpha_y^-$	∓0,410	3	+0,2512	∓2,455	3	−0,617	+0,063
	4	−1,130	−0,800	α_y^+	−0,390	1	−0,3462	±2,150	5	−1,246	−1,106
	6	−0,400	−0,470	$\mp\alpha_x^-$	∓0,995	3	+0,1512	∓0,250	7	+1,119	+0,239

Table 3 continued

Step	$m=4$							$\frac{1}{2}S_1=0{,}35355$			
	k	x_k	y_k			k	$\frac{1}{4}\alpha_x$; $\frac{1}{4}\alpha_y$	$\pm\alpha_{1x}$: $\pm\alpha_{1y}$	k	$\tilde{x}_k$	$\tilde{y}_k$
VIII	1	+1,398	+0,457	α_x^+	+0,896	2	+0,0285	±0,836	2	+0,324	+0,487
	3	−0,532	+0,063	$\pm\alpha_y^-$	∓0,782	4	+0,4195	∓4,250	4	−1,083	−0,706
	5	−1,145	−0,956	α_y^+	−0,216	2	+0,0435	±1,256	6	−0,267	−0,400
	7	+1,175	+0,220	$\mp\alpha_x^-$	±0,390	4	−0,1515	∓1,570	8	+1,922	+0,403
	0	+1,925	+0,400	α_x^+	+0,899	1	+0,1242	±3,601	1	+1,397	+0,459
	2	+0,325	+0,500	$\pm\alpha_y^-$	∓0,402	3	+0,3252	∓2,407	3	−0,526	+0,076
	4	−1,079	−0,702	α_y^+	−0,202	1	−0,2487	±2,002	5	−1,149	−0,956
	6	−0,272	−0,400	$\mp\alpha_x^-$	∓0,793	3	+0,1477	∓0,202	7	+1,176	+0,219

$m=8$				0,17678	0,23097	0,09567		
k	x_k	k	$^1/_8\alpha_x$	α_{2x}	α_{3x}	α_{1x}	k	x_k
0	+1,925	1	+0,2246	+0,390	±5,237	±3,056	1	+1,795
2	+1,397	3	+0,2246	−1,194	±3,658	±0,057	3	+0,864
4	+0,325	5	+0,2246	−0,390	∓0,062	∓2,975	5	−0,143
6	−0,526	7	+0,2246	+1,194	∓3,747	∓4,264	7	−0,838
8	−1,079		1/8	1	+0,2935	±1,5019	9	−1,208
10	−1,149	α_x^+	+1,797	3	+0,0135	±0,8503	11	−0,837
12	−0,272	$\pm\alpha_y^-$	0,000	5	+0,1556	∓0,2989	13	+0,454
14	+1,176			7	+0,4357	∓1,2734	15	+1,709

$m=8$				0,17678	0,23097	0,09567		
k	y_k	k	$^1/_8\alpha_y$	α_{2y}	α_{3y}	α_{1y}	k	y_k
0	+0,400	1	−0,0506	−1,195	±2,059	±2,148	1	+0,419
2	+0,459	3	−0,0503	−0,391	±0,366	±5,665	3	+0,507
4	+0,500	5	−0,0506	+1,195	∓1,542	±5,863	5	+0,365
6	+0,076	7	−0,0503	+0,391	∓2,547	±2,628	7	−0,318
8	−0,702		1/8	1	−0,2618	±0,6811	9	−0,943
10	−0,956	α_y^+	−0,404	3	−0,1194	±0,6265	11	−0,746
12	−0,400	$\mp\alpha_x^-$	∓0,001	5	+0,1606	±0,2047	13	−0,044
14	+0,219			7	+0,0188	∓0,3368	15	+0,356

Table 4

$m=16$		
k	x_k	y_k
0	+1,925	+0,400
2	+1,795	+0,419
4	+1,397	+0,459
6	+0,864	+0,507
8	+0,325	+0,500
10	−0,143	+0,365
12	−0,526	+0,076
14	−0,838	−0,318
16	−1,079	−0,702
18	−1,208	−0,943
20	−1,149	−0,956
22	−0,837	−0,746
24	−0,272	−0,400
26	+0,454	−0,044
28	+1,176	+0,219
30	+1,709	+0,356

	1/16
α_x^+	+3,593
$\pm\alpha_y^-$	0,000
$1/8s_4=0,08839$	
k	$\alpha_{4x}(y)$
1	+0,002
3	−0,002
5	−0,002
7	+0,002
α_y^+	−0,808
$\mp\alpha_x^-$	∓0,001
$1/8s_4=0,08839$	
k	$\alpha_{4y}(x)$
1	−0,001
3	−0,001
5	+0,001
7	+0,001

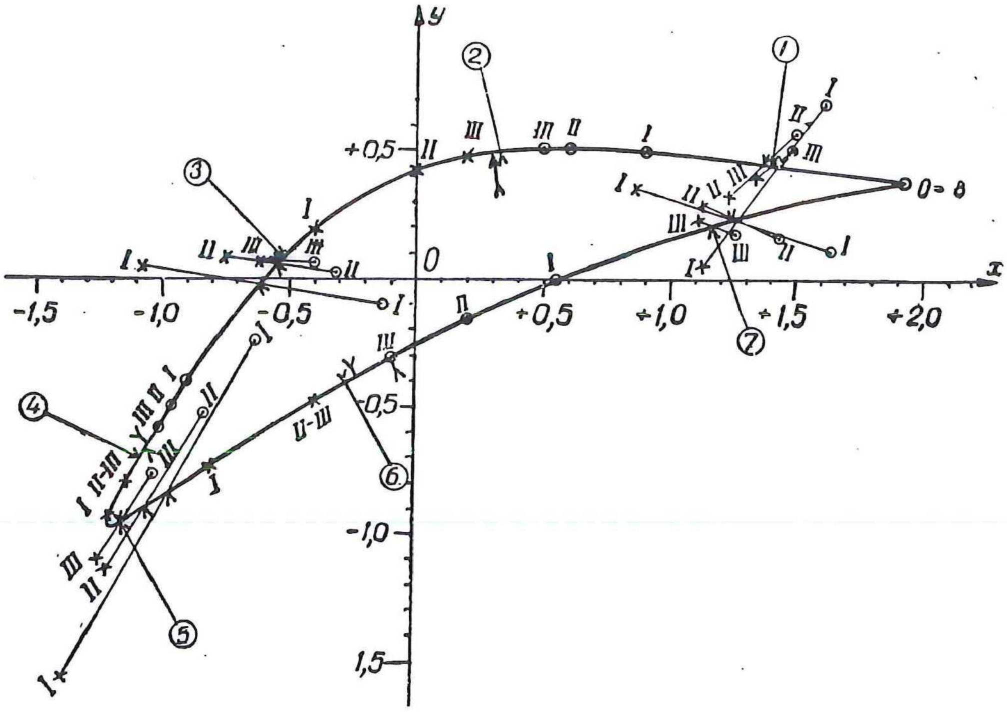
	p_x; p_y	
k	$^1/_{16}\alpha_x$	$\frac{1}{8}s_4\alpha_{4x}$
1	0,2246	+0,0018
3	0,2246	−0,0018
5	0,2246	−0,0018
7	0,2246	+0,0018
±	0,11548	0,04783
k	$\alpha_{6x}(y)$	$\alpha_{2x}(y)$
1	±1,512	∓0,626
3	∓0,178	±0,072
5	∓1,762	±0,732
7	∓2,316	±0,958
k	$1/16\alpha_y$	$1/8s_4\alpha_{4y}$
1	−0,0506	−0,0009
3	−0,0504	−0,0009
5	−0,0506	+0,0009
7	−0,0504	+0,0009
±	0,11548	0,04783
k	$\alpha_{6y}(x)$	$\alpha_{2y}(x)$
1	∓1,763	±0,729
3	∓2,315	±0,959
5	∓1,511	±0,627
7	±0,177	∓0,075

$x_{k,\,k+m}=p_x\pm q_x$ $y_{k,\,k+m}=p_y\pm q_y$	
p_x	$\pm q_x$
+0,3711	±1,5314
+0,2057	±1,4146
+0,0543	±1,0824
+0,0048	±0,5857
+0,0817	±0,0001
+0,2399	∓0,5862
+0,3913	∓1,0828
+0,4480	∓1,4148
p_y	$\pm q_y$
−0,2202	±0,6278
−0,2728	±0,7081
−0,1942	±0,6801
−0,0327	±0,5486
+0,1172	±0,3338
+0,1702	±0,0674
+0,0948	∓0,2084
−0,0663	∓0,4526

Table 5

m=16	±q_x, ±q_y			
	0,02439	0,06945	0,10393	0,12260
k	$\alpha_{1x}(x)$	$\alpha_{3x}(x)$	$\alpha_{5x}(x)$	$\alpha_{7x}(x)$
1	+1,194	+3,403	+5,093	+6,007
3	+1,105	+3,144	+4,705	+5,549
5	+0,845	+2,407	+3,600	+4,247
7	+0,457	+1,301	+1,949	+2,298
9	—0,001	—0,001	—0,001	0,000
11	—0,457	—1,303	—1,950	—2,299
13	—0,845	—2,406	—3,601	—4,249
15	—1,104	—3,143	—4,705	—5,551
k	$\alpha_{1y}(y)$	$\alpha_{3y}(y)$	$\alpha_{5y}(y)$	$\alpha_{7y}(y)$
1	+0,491	+1,396	+2,089	+2,464
3	+0,552	+1,574	+2,355	+2,777
5	+0,531	+1,511	+2,262	+2,668
7	+0,428	+1,219	+1,824	+2,153
9	+0,260	+0,741	+1,110	+1,309
11	+0,053	+0,151	+0,226	+0,266
13	—0,162	—0,462	—0,693	—0,817
15	—0,353	—1,006	—1,505	—1,776

±q_x, ±q_y						
0,29598	0,04305	—0,02876	—0,05887			
$\alpha_{7x}(y)$	$\alpha_{5x}(y)$	$\alpha_{3x}(y)$	$\alpha_{1x}(y)$	k	x_k	y_k
—0,260	—0,741	—1,110	—1,309	1	+1,902	+0,408
—0,053	—0,151	—0,226	—0,266	3	+1,620	+0,435
+0,162	+0,462	+0,693	+0,817	5	+1,137	+0,486
+0,353	+1,006	+1,505	+1,776	7	+0,590	+0,516
+0,491	+1,396	+2,089	+2,464	9	+0,082	+0,451
+0,552	+1,574	+2,355	+2,777	11	—0,346	+0,238
+0,531	+1,511	+2,262	+2,668	13	—0,691	—0,114
+0,428	+1,219	+1,824	+2,153	15	—0,967	—0,519
$\alpha_{7y}(x)$	$\alpha_{5y}(x)$	$\alpha_{3y}(x)$	$\alpha_{1y}(x)$	17	—1,160	—0,848
				19	—1,209	—0,981
				21	—0,028	—0,874
				23	—0,581	—0,581
				25	+0,082	—0,217
—0,001	—0,001	—0,001	0,000	27	+0,826	+0,103
—0,457	—1,303	—1,950	—2,299	29	+1,474	+0,303
—0,845	—2,406	—3,601	—4,249	31	+1,863	+0,386
—1,104	—3,143	—4,705	—5,551			
—1,194	—3,403	—5,093	—6,007			
—1,105	—3,144	—4,705	—5,549			
—0,845	—2,407	—3,600	—4,247			
—0,457	—1,301	—1,949	—2,298			



j	—1	0	1	2	3	4	5	6
A_j	0,976	0,225	0,526	0,198	0,000	0,000	0,000	0,000
B_j	0,126	—0,050	+0,425	—0,100	0,000	0,000	0,000	0,000