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Supplement to "Optimal shrinkage-based portfolio selection in high dimensions"¹

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1 Appendix: Proofs

Here the proofs of the theorems are given. Recall that the sample mean vector and the sample covariance matrix are given by

$$\bar{\mathbf{y}}_n = \frac{1}{n} \mathbf{Y}_n \mathbf{1}_n = \boldsymbol{\mu}_n + \boldsymbol{\Sigma}_n^{\frac{1}{2}} \bar{\mathbf{x}}_n \quad \text{with} \quad \bar{\mathbf{x}}_n = \frac{1}{n} \mathbf{X}_n \mathbf{1}_n \quad (1.1)$$

and

$$\mathbf{S}_n = \frac{1}{n} \mathbf{Y}_n (\mathbf{I} - \frac{1}{n} \mathbf{1} \mathbf{1}') \mathbf{Y}_n' = \boldsymbol{\Sigma}_n^{\frac{1}{2}} \mathbf{V}_n \boldsymbol{\Sigma}_n^{\frac{1}{2}} \quad \text{with} \quad \mathbf{V}_n = \frac{1}{n} \mathbf{X}_n (\mathbf{I} - \frac{1}{n} \mathbf{1} \mathbf{1}') \mathbf{X}_n', \quad (1.2)$$

respectively. Later on, we also make use of $\tilde{\mathbf{V}}_n$ defined by

$$\tilde{\mathbf{V}}_n = \frac{1}{n} \mathbf{X}_n \mathbf{X}_n' \quad (1.3)$$

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and the formula for the 1-rank update of usual inverse given by (c.f., Horn and Johnson (1985))

$$\mathbf{V}_n^{-1} = (\tilde{\mathbf{V}}_n - \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n')^{-1} = \tilde{\mathbf{V}}_n^{-1} + \frac{\tilde{\mathbf{V}}_n^{-1} \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^{-1}}{1 - \bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^{-1} \bar{\mathbf{x}}_n} \quad (1.4)$$

as well as the formula for the 1-rank update of Moore-Penrose inverse (see, Meyer (1973)) expressed as

$$\begin{aligned} \mathbf{V}_n^+ &= \left(\tilde{\mathbf{V}}_n' - \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' \right)^+ \\ &= \tilde{\mathbf{V}}_n^+ - \frac{\tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 + (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)}{\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n} + \frac{\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^3 \bar{\mathbf{x}}_n}{(\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n)^2} \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+. \end{aligned} \quad (1.5)$$

First, we present an important lemma which is a special case of Theorem 1 in Rubio and Mestre (2011).

Lemma 1.1. *Assume (A2). Let a nonrandom $p \times p$ -dimensional matrix Θ_p and a nonrandom $n \times n$ -dimensional matrix Θ_n possess a uniformly bounded trace norms (sum of singular values). Then it holds that*

$$\left| \text{tr} \left(\Theta_p (\tilde{\mathbf{V}}_n - z \mathbf{I}_p)^{-1} \right) - m(z) \text{tr}(\Theta_p) \right| \xrightarrow{a.s.} 0 \quad (1.6)$$

$$\left| \text{tr} \left(\Theta_n (1/n \mathbf{X}_n' \mathbf{X}_n - z \mathbf{I}_n)^{-1} \right) - \underline{m}(z) \text{tr}(\Theta_n) \right| \xrightarrow{a.s.} 0 \quad (1.7)$$

for $p/n \rightarrow c \in (0, +\infty)$ as $n \rightarrow \infty$, where

$$m(z) = (x(z) - z)^{-1} \quad \text{and} \quad \underline{m}(z) = -\frac{1-c}{z} + cm(z) \quad (1.8)$$

with

$$x(z) = \frac{1}{2} \left(1 - c + z + \sqrt{(1 - c + z)^2 - 4z} \right). \quad (1.9)$$

Proof of Lemma 1.1: The application of Theorem 1 in Rubio and Mestre (2011) leads to (1.6) where $x(z)$ is a unique solution in $\mathbb{C}^+$ of the following equation

$$\frac{1 - x(z)}{x(z)} = \frac{c}{x(z) - z}. \quad (1.10)$$

The two solutions of (1.10) are given by

$$x_{1,2}(z) = \frac{1}{2} \left(1 - c + z \pm \sqrt{(1 - c + z)^2 - 4z} \right). \quad (1.11)$$

In order to decide which of two solutions is feasible, we note that $x_{1,2}(z)$ is the Stieltjes transform with a positive imaginary part. Thus, without loss of generality, we can take $z = 1 + c + i2\sqrt{c}$ and get

$$\mathbf{Im}\{x_{1,2}(z)\} = \mathbf{Im} \left\{ \frac{1}{2} \left(2 + i2\sqrt{c} \pm i2\sqrt{2c} \right) \right\} = \mathbf{Im} \left\{ 1 + i\sqrt{c}(1 \pm \sqrt{2}) \right\} = \sqrt{c} (1 \pm \sqrt{2}), \quad (1.12)$$

which is positive only if the sign " + " is chosen. Hence, the solution is given by

$$x(z) = \frac{1}{2} \left(1 - c + z + \sqrt{(1 - c + z)^2 - 4z} \right). \quad (1.13)$$

The second assertion of the lemma follows directly from Bai and Silverstein (2010). $\square$

We note here that Lemma 1.1 is a special case of Theorem 1 in Rubio and Mestre (2011), where one has uniform convergence in the statement of the theorem. Although it is not precisely written in the statement of Theorem 1 in Rubio and Mestre (2011), this observation follows from its proof on page 600 where after showing pointwise convergence Rubio and Mestre additionally proved the uniform convergence by applying Montel's theorem. In short, they first show that the random sequence of analytic functions of interest forms a normal family and, thus, by Montel's theorem there exists a subsequence of it, which converges uniformly on each compact subset of $\mathbb{C} \setminus \mathbb{R}^+$ to an analytic function and this one vanishes almost surely on $\mathbb{C} \setminus \mathbb{R}^+$. And so, the entire sequence converges uniformly to zero on every compact subset of $\mathbb{C} \setminus \mathbb{R}^+$. Furthermore, it is mentioned on page 348 of Rubio et al. (2012) that the convergence in Theorem 1 of Rubio and Mestre (2011) is in fact uniform.

Moreover, the following result (see, e.g., Theorem 1 on page 176 in Ahlfors (1953)), known as the Weierstrass theorem on the uniform convergence, will be used in a sequel together with Lemma 1.1 in the proofs of the technical lemmas.

Theorem 1.1 (Weierstrass). *Suppose that $f_n(z)$ is analytic in the region Ω_n , and that the sequence $\{f_n(z)\}$ converges to a limit function $f(z)$ in a region Ω , uniformly on every compact subset of Ω . Then $f(z)$ is analytic in Ω . Moreover, $f'(z)$ converges uniformly to $f'(z)$ on every compact subset of Ω .*

Because the convergence in Lemma 1.1 is uniform over z on every compact subset of $\mathbb{C} \setminus \mathbb{R}^+$, the Weierstrass theorem allows us to interchange any derivative with respect to z and the limit $n \rightarrow \infty$. We will consider compact subsets, which are the small neighbourhoods of zero with $\Re(z) = 0$ (without loss of generality) because all of the times we will let $z \rightarrow 0$ in order to get specific limiting expressions of interest. For example, one may take Ω as a unit disk $|z| < 1$ and Ω_n as a disk $|z| < \varepsilon_n$ for some $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. The analyticity of the function $\text{tr} \left(\Theta_p (\tilde{\mathbf{V}}_n - z \mathbf{I}_p)^{-1} \right)$ follows immediately from the properties of the Stieltjes transform.

Lemma 1.2. *Assume (A2). Let $\boldsymbol{\theta}$ and $\boldsymbol{\xi}$ be universal nonrandom vectors with bounded Euclidean norms. Then it holds that*

$$\left| \boldsymbol{\xi}' \tilde{\mathbf{V}}_n^{-1} \boldsymbol{\theta} - (1 - c)^{-1} \boldsymbol{\xi}' \boldsymbol{\theta} \right| \xrightarrow{a.s.} 0, \quad (1.14)$$

$$\bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^{-1} \bar{\mathbf{x}}_n \xrightarrow{a.s.} c, \quad (1.15)$$

$$\bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^{-1} \boldsymbol{\theta} \xrightarrow{a.s.} 0, \quad (1.16)$$

$$\left| \boldsymbol{\xi}' \tilde{\mathbf{V}}_n^{-2} \boldsymbol{\theta} - (1 - c)^{-3} \boldsymbol{\xi}' \boldsymbol{\theta} \right| \xrightarrow{a.s.} 0, \quad (1.17)$$

$$\bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^{-2} \bar{\mathbf{x}}_n \xrightarrow{a.s.} \frac{c}{(1 - c)}, \quad (1.18)$$

$$\bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^{-2} \boldsymbol{\theta} \xrightarrow{a.s.} 0 \quad (1.19)$$

for $p/n \rightarrow c \in (0, 1)$ as $n \rightarrow \infty$.

Proof of Lemma 1.2: Since the trace norm of $\boldsymbol{\theta} \boldsymbol{\xi}'$ is uniformly bounded, i.e.

$$\|\boldsymbol{\theta} \boldsymbol{\xi}'\|_{tr} \leq \sqrt{\boldsymbol{\theta}' \boldsymbol{\theta}} \sqrt{\boldsymbol{\xi}' \boldsymbol{\xi}} < \infty,$$

we get from Lemma 1.1 that

$$|\text{tr}((\tilde{\mathbf{V}}_n - z \mathbf{I}_p)^{-1} \boldsymbol{\theta} \boldsymbol{\xi}') - m(z) \text{tr}(\boldsymbol{\theta} \boldsymbol{\xi}')| \xrightarrow{a.s.} 0 \quad \text{for } p/n \rightarrow c < 1 \text{ as } n \rightarrow \infty$$

Furthermore, the application of $m(z) \rightarrow (1 - c)^{-1}$ as $z \rightarrow 0$ leads to

$$|\boldsymbol{\xi}' \tilde{\mathbf{V}}_n^{-1} \boldsymbol{\theta} - (1 - c)^{-1} \boldsymbol{\xi}' \boldsymbol{\theta}| \xrightarrow{a.s.} 0 \text{ for } p/n \rightarrow c < 1 \text{ as } n \rightarrow \infty,$$

which proves (1.14).

For deriving (1.15) we consider

$$\begin{aligned} \bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^{-1} \bar{\mathbf{x}}_n &= \lim_{z \rightarrow 0} \text{tr} \left[\frac{1}{\sqrt{n}} \mathbf{X}_n' \left(\frac{1}{n} \mathbf{X}_n \mathbf{X}_n' - z \mathbf{I}_p \right)^{-1} \frac{1}{\sqrt{n}} \mathbf{X}_n \left(\frac{\mathbf{1}_n \mathbf{1}_n'}{n} \right) \right] \\ &= \lim_{z \rightarrow 0} \text{tr} \left[\left(\frac{\mathbf{1}_n \mathbf{1}_n'}{n} \right) \right] + z \text{tr} \left[\left(\frac{1}{n} \mathbf{X}_n' \mathbf{X}_n - z \mathbf{I}_n \right)^{-1} \left(\frac{\mathbf{1}_n \mathbf{1}_n'}{n} \right) \right], \end{aligned}$$

where the last equality follows from the Woodbury formula (e.g., Horn and Johnson (1985)).

The application of Lemma 1.1 and Theorem 1.1 lead to

$$\text{tr} \left[\left(\frac{\mathbf{1}_n \mathbf{1}_n'}{n} \right) \right] + z \text{tr} \left[\left(\frac{1}{n} \mathbf{X}_n' \mathbf{X}_n - z \mathbf{I}_n \right)^{-1} \left(\frac{\mathbf{1}_n \mathbf{1}_n'}{n} \right) \right] \xrightarrow{a.s.} [1 + (c - 1) + czm(z)] \text{tr} \left[\left(\frac{\mathbf{1}_n \mathbf{1}_n'}{n} \right) \right]$$

for $p/n \rightarrow c < 1$ as $n \rightarrow \infty$ where $m(z)$ is given by (1.8). Setting $z \rightarrow 0$ and taking into account $\lim_{z \rightarrow 0} m(z) = \frac{1}{1 - c}$ we get

$$\bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^{-1} \bar{\mathbf{x}}_n \xrightarrow{a.s.} 1 + c - 1 = c \text{ for } \frac{p}{n} \rightarrow c \in (0, 1) \text{ as } n \rightarrow \infty.$$

The result (1.16) was derived in Pan (2014) (see, p. 673 of this reference).

Next, we prove (1.17). It holds that

$$\boldsymbol{\xi}' \tilde{\mathbf{V}}_n^{-2} \boldsymbol{\theta} = \frac{\partial}{\partial z} \text{tr} \left[\left(\tilde{\mathbf{V}}_n - z \mathbf{I}_p \right)^{-1} \boldsymbol{\theta} \boldsymbol{\xi}' \right] \Big|_{z=0} = \frac{\partial}{\partial z} \zeta_n(z) \Big|_{z=0}$$

where $\zeta_n(z) = \text{tr} \left[\left(\tilde{\mathbf{V}}_n - z \mathbf{I} \right)^{-1} \boldsymbol{\theta} \boldsymbol{\xi}' \right]$. From Lemma 1.1 $\zeta_n(z)$ tends a.s. to $m(z) \boldsymbol{\xi}' \boldsymbol{\theta}$ as $n \rightarrow \infty$.

Furthermore,

$$\begin{aligned} \frac{\partial}{\partial z} m(z) \Big|_{z=0} &= \frac{\partial}{\partial z} \frac{1}{x(z) - z} \Big|_{z=0} = - \frac{x'(z) - 1}{(x(z) - z)^2} \Big|_{z=0} = - \frac{\frac{1}{2} \left(1 - \frac{1+c-z}{\sqrt{(1-c+z)^2 - 4z}} \right) - 1}{(x(z) - z)^2} \Big|_{z=0} = \frac{1}{(1 - c)^3}. \end{aligned} \tag{1.20}$$

Consequently, using Lemma 1.1 and Theorem 1.1 we conclude

$$|\boldsymbol{\xi}' \mathbf{S}_n^{-2} \boldsymbol{\theta} - (1-c)^{-3} \boldsymbol{\xi}' \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\theta}| \xrightarrow{a.s.} 0 \text{ for } p/n \rightarrow c < 1 \text{ as } n \rightarrow \infty.$$

Let $\eta_n(z) = \bar{\mathbf{x}}'_n (\tilde{\mathbf{V}}_n - z\mathbf{I})^{-1} \bar{\mathbf{x}}_n$ and $\boldsymbol{\Theta}_n = \left(\frac{\mathbf{1}_n \mathbf{1}'_n}{n} \right)$. Then

$$\bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^{-2} \bar{\mathbf{x}}_n = \left. \frac{\partial}{\partial z} \eta_n(z) \right|_{z=0},$$

where

$$\begin{aligned} \eta_n(z) &= \text{tr} \left[\frac{1}{\sqrt{n}} \mathbf{X}'_n \left(\frac{1}{n} \mathbf{X}_n \mathbf{X}'_n - z\mathbf{I}_p \right)^{-1} \frac{1}{\sqrt{n}} \mathbf{X}_n \boldsymbol{\Theta}_n \right] \\ &= \text{tr}(\boldsymbol{\Theta}_n) + z \text{tr} \left[(1/n \mathbf{X}'_n \mathbf{X}_n - z\mathbf{I}_n)^{-1} \boldsymbol{\Theta}_n \right] \xrightarrow{a.s.} 1 + z \underline{m}(z) = c + czm(z) \end{aligned}$$

for $\frac{p}{n} \rightarrow c \in (0, 1)$ as $n \rightarrow \infty$. Hence, application of Lemma 1.1 and Theorem 1.1 reveals

$$\bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^{-2} \bar{\mathbf{x}}_n \xrightarrow{a.s.} cm(0) + cz \left. \frac{\partial}{\partial z} m(z) \right|_{z=0} = \frac{c}{1-c}$$

for $\frac{p}{n} \rightarrow c \in (0, 1)$ as $n \rightarrow \infty$.

Finally, we get

$$\bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^{-2} \boldsymbol{\theta} = \left. \frac{\partial}{\partial z} \text{tr} \left[\bar{\mathbf{x}}'_n \left(\tilde{\mathbf{V}}_n - z\mathbf{I}_p \right)^{-1} \boldsymbol{\theta} \right] \right|_{z=0} \xrightarrow{a.s.} 0$$

for $\frac{p}{n} \rightarrow c \in (0, 1)$ as $n \rightarrow \infty$. □

Lemma 1.3. *Assume (A2). Let $\boldsymbol{\theta}$ and $\boldsymbol{\xi}$ be universal nonrandom vectors with bounded Eu-*

clidean norms. Then it holds that

$$|\xi' \mathbf{V}_n^{-1} \boldsymbol{\theta} - (1-c)^{-1} \xi' \boldsymbol{\theta}| \xrightarrow{a.s.} 0, \quad (1.21)$$

$$\bar{\mathbf{x}}'_n \mathbf{V}_n^{-1} \bar{\mathbf{x}}_n \xrightarrow{a.s.} \frac{c}{1-c}, \quad (1.22)$$

$$\bar{\mathbf{x}}'_n \mathbf{V}_n^{-1} \boldsymbol{\theta} \xrightarrow{a.s.} 0, \quad (1.23)$$

$$|\xi' \mathbf{V}_n^{-2} \boldsymbol{\theta} - (1-c)^{-3} \xi' \boldsymbol{\theta}| \xrightarrow{a.s.} 0, \quad (1.24)$$

$$\bar{\mathbf{x}}'_n \mathbf{V}_n^{-2} \bar{\mathbf{x}}_n \xrightarrow{a.s.} \frac{c}{(1-c)^3}, \quad (1.25)$$

$$\bar{\mathbf{x}}'_n \mathbf{V}_n^{-2} \boldsymbol{\theta} \xrightarrow{a.s.} 0 \quad (1.26)$$

for $p/n \rightarrow c \in (0, 1)$ as $n \rightarrow \infty$.

Proof of Lemma 1.3: From (1.4) we obtain

$$\xi' \mathbf{V}_n^{-1} \boldsymbol{\theta} = \xi' \tilde{\mathbf{V}}_n^{-1} \boldsymbol{\theta} + \frac{\xi' \tilde{\mathbf{V}}_n^{-1} \bar{\mathbf{x}}_n \bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^{-1} \boldsymbol{\theta}}{1 - \bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^{-1} \bar{\mathbf{x}}_n} \xrightarrow{a.s.} (1-c)^{-1} \xi' \boldsymbol{\theta}$$

for $p/n \rightarrow c \in (0, 1)$ as $n \rightarrow \infty$ following (1.14)-(1.16). Similarly, we get (1.22) and (1.23).

In case of (1.23), we get

$$\begin{aligned} \xi' \mathbf{V}_n^{-2} \boldsymbol{\theta} &= \xi' \tilde{\mathbf{V}}_n^{-2} \boldsymbol{\theta} + \frac{\xi' \tilde{\mathbf{V}}_n^{-2} \bar{\mathbf{x}}_n \bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^{-1} \boldsymbol{\theta}}{1 - \bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^{-1} \bar{\mathbf{x}}_n} + \frac{\xi' \tilde{\mathbf{V}}_n^{-1} \bar{\mathbf{x}}_n \bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^{-2} \boldsymbol{\theta}}{1 - \bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^{-1} \bar{\mathbf{x}}_n} \\ &+ \bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^{-2} \bar{\mathbf{x}}_n \frac{\xi' \tilde{\mathbf{V}}_n^{-1} \bar{\mathbf{x}}_n \bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^{-1} \boldsymbol{\theta}}{(1 - \bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^{-1} \bar{\mathbf{x}}_n)^2} \xrightarrow{a.s.} (1-c)^{-1} \xi' \boldsymbol{\theta} \end{aligned}$$

for $p/n \rightarrow c \in (0, 1)$ as $n \rightarrow \infty$. Similarly,

$$\begin{aligned} \bar{\mathbf{x}}'_n \mathbf{V}_n^{-2} \bar{\mathbf{x}}_n &= \bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^{-2} \bar{\mathbf{x}}_n + \frac{\bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^{-2} \bar{\mathbf{x}}_n \bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^{-1} \bar{\mathbf{x}}_n}{1 - \bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^{-1} \bar{\mathbf{x}}_n} + \frac{\bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^{-1} \bar{\mathbf{x}}_n \bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^{-2} \bar{\mathbf{x}}_n}{1 - \bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^{-1} \bar{\mathbf{x}}_n} \\ &+ \bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^{-2} \bar{\mathbf{x}}_n \frac{\bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^{-1} \bar{\mathbf{x}}_n \bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^{-1} \bar{\mathbf{x}}_n}{(1 - \bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^{-1} \bar{\mathbf{x}}_n)^2} = \frac{\bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^{-2} \bar{\mathbf{x}}_n}{(1 - \bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^{-1} \bar{\mathbf{x}}_n)^2} \xrightarrow{a.s.} \frac{c}{(1-c)^3} \end{aligned}$$

and

$$\begin{aligned} \bar{\mathbf{x}}'_n \mathbf{V}_n^{-2} \boldsymbol{\theta} &= \bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^{-2} \boldsymbol{\theta} + \frac{\bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^{-2} \bar{\mathbf{x}}_n \bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^{-1} \boldsymbol{\theta}}{1 - \bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^{-1} \bar{\mathbf{x}}_n} + \frac{\bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^{-1} \bar{\mathbf{x}}_n \bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^{-2} \boldsymbol{\theta}}{1 - \bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^{-1} \bar{\mathbf{x}}_n} \\ &+ \bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^{-2} \bar{\mathbf{x}}_n \frac{\bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^{-1} \bar{\mathbf{x}}_n \bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^{-1} \boldsymbol{\theta}}{(1 - \bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^{-1} \bar{\mathbf{x}}_n)^2} \xrightarrow{a.s.} 0 \end{aligned}$$

for $p/n \rightarrow c \in (0, 1)$ as $n \rightarrow \infty$. □

Lemma 1.4. Assume (A2). Let $\boldsymbol{\theta}$ and $\boldsymbol{\xi}$ be universal nonrandom vectors with bounded Euclidean norms and let $\mathbf{P}_n = \mathbf{V}_n^{-1} - \frac{\mathbf{V}_n^{-1}\boldsymbol{\eta}\boldsymbol{\eta}'\mathbf{V}_n^{-1}}{\boldsymbol{\eta}'\mathbf{V}_n^{-1}\boldsymbol{\eta}}$ where $\boldsymbol{\eta}$ is a universal nonrandom vectors with bounded Euclidean norm. Then it holds that

$$\boldsymbol{\xi}'\mathbf{P}_n\boldsymbol{\theta} \xrightarrow{a.s.} (1-c)^{-1} \left(\boldsymbol{\xi}'\boldsymbol{\theta} - \frac{\boldsymbol{\xi}'\boldsymbol{\eta}\boldsymbol{\eta}'\boldsymbol{\theta}}{\boldsymbol{\eta}'\boldsymbol{\eta}} \right), \quad (1.27)$$

$$\bar{\mathbf{x}}_n'\mathbf{P}_n\bar{\mathbf{x}}_n \xrightarrow{a.s.} \frac{c}{1-c}, \quad (1.28)$$

$$\bar{\mathbf{x}}_n'\mathbf{P}_n\boldsymbol{\theta} \xrightarrow{a.s.} 0, \quad (1.29)$$

$$\boldsymbol{\xi}'\mathbf{P}_n^2\boldsymbol{\theta} \xrightarrow{a.s.} (1-c)^{-3} \left(\boldsymbol{\xi}'\boldsymbol{\theta} - \frac{\boldsymbol{\xi}'\boldsymbol{\eta}\boldsymbol{\eta}'\boldsymbol{\theta}}{\boldsymbol{\eta}'\boldsymbol{\eta}} \right), \quad (1.30)$$

$$\bar{\mathbf{x}}_n'\mathbf{P}_n^2\bar{\mathbf{x}}_n \xrightarrow{a.s.} \frac{c}{(1-c)^3}, \quad (1.31)$$

$$\bar{\mathbf{x}}_n'\mathbf{P}_n^2\boldsymbol{\theta} \xrightarrow{a.s.} 0 \quad (1.32)$$

for $p/n \rightarrow c \in (0, 1)$ as $n \rightarrow \infty$.

Proof of Lemma 1.4: It holds that

$$\boldsymbol{\xi}'\mathbf{P}_n\boldsymbol{\theta} = \boldsymbol{\xi}'\mathbf{V}_n^{-1}\boldsymbol{\theta} - \frac{\boldsymbol{\xi}'\mathbf{V}_n^{-1}\boldsymbol{\eta}\boldsymbol{\eta}'\mathbf{V}_n^{-1}\boldsymbol{\theta}}{\boldsymbol{\eta}'\mathbf{V}_n^{-1}\boldsymbol{\eta}} \xrightarrow{a.s.} (1-c)^{-1} \left(\boldsymbol{\xi}'\boldsymbol{\theta} - \frac{\boldsymbol{\xi}'\boldsymbol{\eta}\boldsymbol{\eta}'\boldsymbol{\theta}}{\boldsymbol{\eta}'\boldsymbol{\eta}} \right)$$

for $p/n \rightarrow c \in (0, 1)$ as $n \rightarrow \infty$ following (1.21). Similarly, we get

$$\bar{\mathbf{x}}_n'\mathbf{P}_n\bar{\mathbf{x}}_n = \bar{\mathbf{x}}_n'\mathbf{V}_n^{-1}\bar{\mathbf{x}}_n - \frac{\bar{\mathbf{x}}_n'\mathbf{V}_n^{-1}\boldsymbol{\eta}\boldsymbol{\eta}'\mathbf{V}_n^{-1}\bar{\mathbf{x}}_n}{\boldsymbol{\eta}'\mathbf{V}_n^{-1}\boldsymbol{\eta}} \xrightarrow{a.s.} \frac{c}{1-c}$$

and

$$\bar{\mathbf{x}}_n'\mathbf{P}_n\boldsymbol{\theta} = \bar{\mathbf{x}}_n'\mathbf{V}_n^{-1}\boldsymbol{\theta} - \frac{\bar{\mathbf{x}}_n'\mathbf{V}_n^{-1}\boldsymbol{\eta}\boldsymbol{\eta}'\mathbf{V}_n^{-1}\boldsymbol{\theta}}{\boldsymbol{\eta}'\mathbf{V}_n^{-1}\boldsymbol{\eta}} \xrightarrow{a.s.} 0$$

for $p/n \rightarrow c \in (0, 1)$ as $n \rightarrow \infty$.

The rest of the proof follows from the equality

$$\mathbf{P}_n^2 = \mathbf{V}_n^{-2} - \frac{\mathbf{V}_n^{-2}\boldsymbol{\eta}\boldsymbol{\eta}'\mathbf{V}_n^{-1}}{\boldsymbol{\eta}'\mathbf{V}_n^{-1}\boldsymbol{\eta}} - \frac{\mathbf{V}_n^{-1}\boldsymbol{\eta}\boldsymbol{\eta}'\mathbf{V}_n^{-2}}{\boldsymbol{\eta}'\mathbf{V}_n^{-1}\boldsymbol{\eta}} + \boldsymbol{\eta}'\mathbf{V}_n^{-2}\boldsymbol{\eta} \frac{\mathbf{V}_n^{-1}\boldsymbol{\eta}\boldsymbol{\eta}'\mathbf{V}_n^{-1}}{(\boldsymbol{\eta}'\mathbf{V}_n^{-1}\boldsymbol{\eta})^2}$$

and Lemma 1.3. □

Proof of Theorem 2.1: Let $q_n = \max\{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n, \mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}\}$. We get that $q_n > 0$ uniformly in p , since $\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n \geq s$ and $s > 0$ uniformly in p by Assumption (A3).

The optimal shrinkage intensity can be rewritten in the following way

$$\begin{aligned}
\alpha_n^* &= \beta^{-1} \frac{\hat{\mathbf{w}}_S'(\boldsymbol{\mu}_n - \beta \boldsymbol{\Sigma}_n \mathbf{b}) - \mathbf{b}'(\boldsymbol{\mu}_n - \beta \boldsymbol{\Sigma}_n \mathbf{b})}{\hat{\mathbf{w}}_S' \boldsymbol{\Sigma}_n \hat{\mathbf{w}}_S - 2\mathbf{b}' \boldsymbol{\Sigma}_n \hat{\mathbf{w}}_S + \mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}} \\
&= \beta^{-1} \frac{\frac{\mathbf{1}' \mathbf{S}_n^{-1} (\boldsymbol{\mu}_n - \beta \boldsymbol{\Sigma}_n \mathbf{b})}{\mathbf{1}' \mathbf{S}_n^{-1} \mathbf{1}} + \gamma^{-1} \bar{\mathbf{y}}_n' \hat{\mathbf{Q}}_n (\boldsymbol{\mu}_n - \beta \boldsymbol{\Sigma}_n \mathbf{b}) - \mathbf{b}'(\boldsymbol{\mu}_n - \beta \boldsymbol{\Sigma}_n \mathbf{b})}{\frac{\mathbf{1}' \mathbf{S}_n^{-1} \boldsymbol{\Sigma}_n \mathbf{S}_n^{-1} \mathbf{1}}{(\mathbf{1}' \mathbf{S}_n^{-1} \mathbf{1})^2} + 2\gamma^{-1} \frac{\bar{\mathbf{y}}_n' \hat{\mathbf{Q}}_n \boldsymbol{\Sigma}_n \mathbf{S}_n^{-1} \mathbf{1}}{\mathbf{1}' \mathbf{S}_n^{-1} \mathbf{1}} + \gamma^{-2} \bar{\mathbf{y}}_n' \hat{\mathbf{Q}}_n \boldsymbol{\Sigma}_n \hat{\mathbf{Q}}_n \bar{\mathbf{y}}_n - 2 \frac{\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{S}_n^{-1} \mathbf{1}}{\mathbf{1}' \mathbf{S}_n^{-1} \mathbf{1}} - 2\gamma^{-1} \mathbf{b}' \boldsymbol{\Sigma}_n \hat{\mathbf{Q}}_n \bar{\mathbf{y}}_n + \mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}} \\
&= \beta^{-1} \frac{A_n^*}{B_n^*},
\end{aligned} \tag{1.33}$$

where

$$\begin{aligned}
A_n^* &= \frac{1}{q_n} \left(\frac{\mathbf{1}' \mathbf{S}_n^{-1} (\boldsymbol{\mu}_n - \beta \boldsymbol{\Sigma}_n \mathbf{b})}{\mathbf{1}' \mathbf{S}_n^{-1} \mathbf{1}} + \gamma^{-1} \bar{\mathbf{y}}_n' \hat{\mathbf{Q}}_n (\boldsymbol{\mu}_n - \beta \boldsymbol{\Sigma}_n \mathbf{b}) - \mathbf{b}'(\boldsymbol{\mu}_n - \beta \boldsymbol{\Sigma}_n \mathbf{b}) \right) \\
&= \frac{(1' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1/2} \sqrt{\boldsymbol{\mu}_n' \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n} (1' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1/2} (\boldsymbol{\mu}_n' \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n)^{-1/2} \mathbf{1}' \mathbf{S}_n^{-1} \boldsymbol{\mu}_n}{q_n (1' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1} \mathbf{1}' \mathbf{S}_n^{-1} \mathbf{1}} \\
&\quad - \beta \frac{(1' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1/2} \sqrt{\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}} (1' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1/2} (\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b})^{-1/2} \mathbf{1}' \mathbf{S}_n^{-1} \boldsymbol{\Sigma}_n \mathbf{b}}{q_n (1' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1} \mathbf{1}' \mathbf{S}_n^{-1} \mathbf{1}} \\
&\quad + \gamma^{-1} \frac{\boldsymbol{\mu}_n' \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n}{q_n} \frac{\bar{\mathbf{y}}_n' \hat{\mathbf{Q}}_n \boldsymbol{\mu}_n}{\boldsymbol{\mu}_n' \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n} - \beta \gamma^{-1} \frac{\sqrt{\boldsymbol{\mu}_n' \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n} \sqrt{\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}}}{q_n} \frac{\bar{\mathbf{y}}_n' \hat{\mathbf{Q}}_n \boldsymbol{\Sigma}_n \mathbf{b}}{\sqrt{\boldsymbol{\mu}_n' \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n} \sqrt{\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}}} \\
&\quad - \frac{\sqrt{\boldsymbol{\mu}_n' \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n} \sqrt{\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}}}{q_n} \frac{\mathbf{b}' \boldsymbol{\mu}_n}{\sqrt{\boldsymbol{\mu}_n' \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n} \sqrt{\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}}} + \beta \frac{\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}}{q_n}
\end{aligned}$$

and

$$\begin{aligned}
B_n^* &= \frac{1}{q_n} \left(\frac{\mathbf{1}' \mathbf{S}_n^{-1} \boldsymbol{\Sigma}_n \mathbf{S}_n^{-1} \mathbf{1}}{(\mathbf{1}' \mathbf{S}_n^{-1} \mathbf{1})^2} + 2\gamma^{-1} \frac{\bar{\mathbf{y}}_n' \hat{\mathbf{Q}}_n \boldsymbol{\Sigma}_n \mathbf{S}_n^{-1} \mathbf{1}}{\mathbf{1}' \mathbf{S}_n^{-1} \mathbf{1}} + \gamma^{-2} \bar{\mathbf{y}}_n' \hat{\mathbf{Q}}_n \boldsymbol{\Sigma}_n \hat{\mathbf{Q}}_n \bar{\mathbf{y}}_n \right. \\
&\quad \left. - 2 \frac{\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{S}_n^{-1} \mathbf{1}}{\mathbf{1}' \mathbf{S}_n^{-1} \mathbf{1}} - 2\gamma^{-1} \mathbf{b}' \boldsymbol{\Sigma}_n \hat{\mathbf{Q}}_n \bar{\mathbf{y}}_n + \mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b} \right) \\
&= \frac{(1' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1} (1' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1} \mathbf{1}' \mathbf{S}_n^{-1} \boldsymbol{\Sigma}_n \mathbf{S}_n^{-1} \mathbf{1}}{q_n (1' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-2} (\mathbf{1}' \mathbf{S}_n^{-1} \mathbf{1})^2} \\
&\quad + 2\gamma^{-1} \frac{(1' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1/2} \sqrt{\boldsymbol{\mu}_n' \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n} (1' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1/2} (\boldsymbol{\mu}_n' \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n)^{-1/2} \bar{\mathbf{y}}_n' \hat{\mathbf{Q}}_n \boldsymbol{\Sigma}_n \mathbf{S}_n^{-1} \mathbf{1}}{q_n (1' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1} \mathbf{1}' \mathbf{S}_n^{-1} \mathbf{1}} \\
&\quad + \gamma^{-2} \frac{\boldsymbol{\mu}_n' \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n}{q_n} \frac{\bar{\mathbf{y}}_n' \hat{\mathbf{Q}}_n \boldsymbol{\Sigma}_n \hat{\mathbf{Q}}_n \bar{\mathbf{y}}_n}{\boldsymbol{\mu}_n' \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n} \\
&\quad - 2 \frac{(1' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1/2} \sqrt{\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}} (1' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1/2} (\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b})^{-1/2} \mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{S}_n^{-1} \mathbf{1}}{q_n (1' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1} \mathbf{1}' \mathbf{S}_n^{-1} \mathbf{1}} \\
&\quad - 2\gamma^{-1} \frac{\sqrt{\boldsymbol{\mu}_n' \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n} \sqrt{\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}}}{q_n} \frac{\mathbf{b}' \boldsymbol{\Sigma}_n \hat{\mathbf{Q}}_n \bar{\mathbf{y}}_n}{\sqrt{\boldsymbol{\mu}_n' \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n} \sqrt{\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}}} + \frac{\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}}{q_n},
\end{aligned}$$

In the formulas for A_n^* and B_n^* the factors $q_n^{-1} \boldsymbol{\mu}_n' \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n$ and $q_n^{-1} \mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}$ are bounded by one.

Moreover, since $(\mathbf{1}' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1} \leq \mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}$ for any $\mathbf{b}$ with $\mathbf{b}' \mathbf{1} = 1$, we also get $q_n^{-1} (\mathbf{1}' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1} \leq 1$.

The Euclidean norms of the following vectors

$$\frac{\Sigma_n^{-1/2}\mathbf{1}}{\sqrt{\mathbf{1}'\Sigma_n^{-1}\mathbf{1}}}, \quad \frac{\Sigma_n^{-1/2}\boldsymbol{\mu}_n}{\sqrt{\boldsymbol{\mu}_n'\Sigma_n^{-1}\boldsymbol{\mu}_n}} \quad \text{and} \quad \frac{\Sigma_n^{1/2}\mathbf{b}}{\sqrt{\mathbf{b}'\Sigma_n\mathbf{b}}}$$

are all equal to one. As a result, using $\bar{\mathbf{y}}_n = \boldsymbol{\mu}_n + \Sigma_n^{1/2}\bar{\mathbf{x}}_n$ and $\mathbf{S}_n = \Sigma_n^{1/2}\mathbf{V}_n\Sigma_n^{1/2}$ and applying Lemma 1.3 we get

$$\begin{aligned} \frac{\mathbf{1}'\Sigma_n^{-1}\mathbf{1}}{\mathbf{1}'\Sigma_n^{-1}\mathbf{1}} &= \frac{\mathbf{1}'\Sigma_n^{-1/2}\mathbf{V}_n^{-1}\Sigma_n^{-1/2}\mathbf{1}}{\mathbf{1}'\Sigma_n^{-1}\mathbf{1}} \xrightarrow{a.s.} (1-c)^{-1}, \\ \frac{\mathbf{1}'\Sigma_n^{-1}\boldsymbol{\mu}_n}{\sqrt{\mathbf{1}'\Sigma_n^{-1}\mathbf{1}}\sqrt{\boldsymbol{\mu}_n'\Sigma_n^{-1}\boldsymbol{\mu}_n}} &= \frac{\mathbf{1}'\Sigma_n^{-1/2}\mathbf{V}_n^{-1}\Sigma_n^{-1/2}\boldsymbol{\mu}_n}{\sqrt{\mathbf{1}'\Sigma_n^{-1}\mathbf{1}}\sqrt{\boldsymbol{\mu}_n'\Sigma_n^{-1}\boldsymbol{\mu}_n}} \xrightarrow{a.s.} (1-c)^{-1} \frac{\mathbf{1}'\Sigma_n^{-1}\boldsymbol{\mu}_n}{\sqrt{\mathbf{1}'\Sigma_n^{-1}\mathbf{1}}\sqrt{\boldsymbol{\mu}_n'\Sigma_n^{-1}\boldsymbol{\mu}_n}}, \\ \frac{\mathbf{1}'\Sigma_n^{-1}\Sigma_n\mathbf{b}}{\sqrt{\mathbf{1}'\Sigma_n^{-1}\mathbf{1}}\sqrt{\mathbf{b}'\Sigma_n\mathbf{b}}} &= \frac{\mathbf{1}'\Sigma_n^{-1/2}\mathbf{V}_n^{-1}\Sigma_n^{1/2}\mathbf{b}}{\sqrt{\mathbf{1}'\Sigma_n^{-1}\mathbf{1}}\sqrt{\mathbf{b}'\Sigma_n\mathbf{b}}} \xrightarrow{a.s.} (1-c)^{-1} \frac{1}{\sqrt{\mathbf{1}'\Sigma_n^{-1}\mathbf{1}}\sqrt{\mathbf{b}'\Sigma_n\mathbf{b}}}, \\ \frac{\mathbf{1}'\Sigma_n^{-1}\Sigma_n\mathbf{S}_n^{-1}\mathbf{1}}{\mathbf{1}'\Sigma_n^{-1}\mathbf{1}} &= \frac{\mathbf{1}'\Sigma_n^{-1/2}\mathbf{V}_n^{-2}\Sigma_n^{-1/2}\mathbf{1}}{\mathbf{1}'\Sigma_n^{-1}\mathbf{1}} \xrightarrow{a.s.} (1-c)^{-3} \end{aligned}$$

for $p/n \rightarrow c \in (0, 1)$ as $n \rightarrow \infty$.

Furthermore, from Lemma 1.3 and 1.4 using the equalities

$$\hat{\mathbf{Q}}_n = \Sigma_n^{-1/2}\mathbf{P}_n\Sigma_n^{-1/2} \quad \text{and} \quad \mathbf{P}_n\mathbf{V}_n^{-1} = \mathbf{V}_n^{-2} - \frac{\mathbf{V}_n^{-1}\boldsymbol{\eta}\boldsymbol{\eta}'\mathbf{V}_n^{-2}}{\boldsymbol{\eta}'\mathbf{V}_n^{-1}\boldsymbol{\eta}}$$

with $\boldsymbol{\eta} = \Sigma_n^{-1/2}\mathbf{1}/\sqrt{\mathbf{1}'\Sigma_n^{-1}\mathbf{1}}$ we obtain with $\boldsymbol{\mu}_n'\Sigma_n^{-1}\boldsymbol{\mu}_n > 0$

$$\begin{aligned} \frac{\bar{\mathbf{y}}_n'\hat{\mathbf{Q}}_n\boldsymbol{\mu}_n}{\boldsymbol{\mu}_n'\Sigma_n^{-1}\boldsymbol{\mu}_n} &= \frac{\boldsymbol{\mu}_n'\Sigma_n^{-1/2}\mathbf{P}_n\Sigma_n^{-1/2}\boldsymbol{\mu}_n}{\boldsymbol{\mu}_n'\Sigma_n^{-1}\boldsymbol{\mu}_n} + \frac{\bar{\mathbf{x}}_n'\mathbf{P}_n\Sigma_n^{-1/2}\boldsymbol{\mu}_n}{\boldsymbol{\mu}_n'\Sigma_n^{-1}\boldsymbol{\mu}_n} \\ &\xrightarrow{a.s.} (1-c)^{-1} \frac{\boldsymbol{\mu}_n'\mathbf{Q}_n\boldsymbol{\mu}_n}{\boldsymbol{\mu}_n'\Sigma_n^{-1}\boldsymbol{\mu}_n}, \\ \frac{\bar{\mathbf{y}}_n'\hat{\mathbf{Q}}_n\Sigma_n\mathbf{b}}{\sqrt{\boldsymbol{\mu}_n'\Sigma_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\Sigma_n\mathbf{b}}} &= \frac{\boldsymbol{\mu}_n'\Sigma_n^{-1/2}\mathbf{P}_n\Sigma_n^{1/2}\mathbf{b}}{\sqrt{\boldsymbol{\mu}_n'\Sigma_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\Sigma_n\mathbf{b}}} + \frac{\bar{\mathbf{x}}_n'\mathbf{P}_n\Sigma_n^{1/2}\mathbf{b}}{\sqrt{\boldsymbol{\mu}_n'\Sigma_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\Sigma_n\mathbf{b}}} \\ &\xrightarrow{a.s.} (1-c)^{-1} \frac{\boldsymbol{\mu}_n'\mathbf{Q}_n\Sigma_n\mathbf{b}}{\sqrt{\boldsymbol{\mu}_n'\Sigma_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\Sigma_n\mathbf{b}}}, \\ \frac{\bar{\mathbf{y}}_n'\hat{\mathbf{Q}}_n\Sigma_n\mathbf{S}_n^{-1}\mathbf{1}}{\sqrt{\boldsymbol{\mu}_n'\Sigma_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{1}'\Sigma_n^{-1}\mathbf{1}}} &= \frac{\boldsymbol{\mu}_n'\Sigma_n^{-1/2}\mathbf{P}_n\mathbf{V}_n^{-1}\Sigma_n^{-1/2}\mathbf{1}}{\sqrt{\boldsymbol{\mu}_n'\Sigma_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{1}'\Sigma_n^{-1}\mathbf{1}}} + \frac{\bar{\mathbf{x}}_n'\mathbf{P}_n\mathbf{V}_n^{-1}\Sigma_n^{-1/2}\mathbf{1}}{\sqrt{\boldsymbol{\mu}_n'\Sigma_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{1}'\Sigma_n^{-1}\mathbf{1}}} \\ &\xrightarrow{a.s.} (1-c)^{-3} \frac{\boldsymbol{\mu}_n'\mathbf{Q}_n\mathbf{1}}{\sqrt{\boldsymbol{\mu}_n'\Sigma_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{1}'\Sigma_n^{-1}\mathbf{1}}} = 0, \\ \frac{\bar{\mathbf{y}}_n'\hat{\mathbf{Q}}_n\Sigma_n\hat{\mathbf{Q}}_n\bar{\mathbf{y}}_n}{\boldsymbol{\mu}_n'\Sigma_n^{-1}\boldsymbol{\mu}_n} &= \frac{\bar{\mathbf{x}}_n'\mathbf{P}_n^2\bar{\mathbf{x}}_n}{\boldsymbol{\mu}_n'\Sigma_n^{-1}\boldsymbol{\mu}_n} + 2 \frac{\boldsymbol{\mu}_n'\Sigma_n^{-1/2}\mathbf{P}_n^2\bar{\mathbf{x}}_n}{\boldsymbol{\mu}_n'\Sigma_n^{-1}\boldsymbol{\mu}_n} + \frac{\boldsymbol{\mu}_n'\Sigma_n^{-1/2}\mathbf{P}_n^2\Sigma_n^{-1/2}\boldsymbol{\mu}_n}{\boldsymbol{\mu}_n'\Sigma_n^{-1}\boldsymbol{\mu}_n} \\ &\xrightarrow{a.s.} \frac{(1-c)^{-3}c + (1-c)^{-3}\boldsymbol{\mu}_n'\mathbf{Q}_n\boldsymbol{\mu}_n}{\boldsymbol{\mu}_n'\Sigma_n^{-1}\boldsymbol{\mu}_n} \end{aligned}$$

for $p/n \rightarrow c \in (0, 1)$ as $n \rightarrow \infty$.

Substituting the above results into the expressions of A_n^* and B_n^* , we get that

$$|A_n^* - A^*| \xrightarrow{a.s.} 0 \quad \text{and} \quad |B_n^* - B^*| \xrightarrow{a.s.} 0$$

with

$$\begin{aligned} A^* &= \frac{(1' \Sigma_n^{-1} \mathbf{1})^{-1/2} \sqrt{\mu_n' \Sigma_n^{-1} \mu_n} (1' \Sigma_n^{-1} \mathbf{1})^{-1/2} (\mu_n' \Sigma_n^{-1} \mu_n)^{-1/2} (1-c)^{-1} \mathbf{1}' \Sigma_n^{-1} \mu_n}{q_n (1-c)^{-1}} \\ &\quad - \beta \frac{(1' \Sigma_n^{-1} \mathbf{1})^{-1/2} \sqrt{\mathbf{b}' \Sigma_n \mathbf{b}} (1' \Sigma_n^{-1} \mathbf{1})^{-1/2} (\mathbf{b}' \Sigma_n \mathbf{b})^{-1/2} (1-c)^{-1}}{q_n (1-c)^{-1}} \\ &\quad + \gamma^{-1} \frac{\mu_n' \Sigma_n^{-1} \mu_n}{q_n} \frac{(1-c)^{-1} \mu_n' \mathbf{Q}_n \mu_n}{\mu_n' \Sigma_n^{-1} \mu_n} - \beta \gamma^{-1} \frac{\sqrt{\mu_n' \Sigma_n^{-1} \mu_n} \sqrt{\mathbf{b}' \Sigma_n \mathbf{b}}}{q_n} \frac{(1-c)^{-1} \mu_n' \mathbf{Q}_n \Sigma_n \mathbf{b}}{\sqrt{\mu_n' \Sigma_n^{-1} \mu_n} \sqrt{\mathbf{b}' \Sigma_n \mathbf{b}}} \\ &\quad - \frac{\sqrt{\mu_n' \Sigma_n^{-1} \mu_n} \sqrt{\mathbf{b}' \Sigma_n \mathbf{b}}}{q_n} \frac{\mathbf{b}' \mu_n}{\sqrt{\mu_n' \Sigma_n^{-1} \mu_n} \sqrt{\mathbf{b}' \Sigma_n \mathbf{b}}} + \beta \frac{\mathbf{b}' \Sigma_n \mathbf{b}}{q_n} \\ &= \frac{1}{q_n} \left(\frac{1' \Sigma_n^{-1} \mu_n}{1' \Sigma_n^{-1} \mathbf{1}} - \beta \frac{1}{1' \Sigma_n^{-1} \mathbf{1}} + \frac{\gamma^{-1}}{1-c} \mu_n' \mathbf{Q}_n \mu_n - \frac{\gamma^{-1} \beta}{1-c} \mu_n' \mathbf{Q}_n \Sigma_n \mathbf{b} - \mathbf{b}' \mu_n + \beta \mathbf{b}' \Sigma_n \mathbf{b} \right) \end{aligned}$$

and

$$\begin{aligned} B^* &= \frac{(1' \Sigma_n^{-1} \mathbf{1})^{-1} (1-c)^{-3}}{q_n (1-c)^{-2}} + 2\gamma^{-1} \frac{(1' \Sigma_n^{-1} \mathbf{1})^{-1/2} \sqrt{\mu_n' \Sigma_n^{-1} \mu_n}}{q_n} \frac{0}{(1-c)^{-1}} \\ &\quad + \gamma^{-2} \frac{\mu_n' \Sigma_n^{-1} \mu_n}{q_n} \frac{(1-c)^{-3} c + (1-c)^{-3} \mu_n' \mathbf{Q}_n \mu_n}{\mu_n' \Sigma_n^{-1} \mu_n} \\ &\quad - 2 \frac{(1' \Sigma_n^{-1} \mathbf{1})^{-1/2} \sqrt{\mathbf{b}' \Sigma_n \mathbf{b}} (1' \Sigma_n^{-1} \mathbf{1})^{-1/2} (\mathbf{b}' \Sigma_n \mathbf{b})^{-1/2} (1-c)^{-1}}{q_n (1-c)^{-1}} \\ &\quad - 2\gamma^{-1} \frac{\sqrt{\mu_n' \Sigma_n^{-1} \mu_n} \sqrt{\mathbf{b}' \Sigma_n \mathbf{b}}}{q_n} \frac{(1-c)^{-1} \mathbf{b}' \Sigma_n \mathbf{Q}_n \mu_n}{\sqrt{\mu_n' \Sigma_n^{-1} \mu_n} \sqrt{\mathbf{b}' \Sigma_n \mathbf{b}}} + \frac{\mathbf{b}' \Sigma_n \mathbf{b}}{q_n} \\ &= \frac{1}{q_n} \left(\frac{1}{1-c} \frac{1}{1' \Sigma_n^{-1} \mathbf{1}} + \gamma^{-2} \left(\frac{1}{(1-c)^3} \mu_n' \mathbf{Q}_n \mu_n + \frac{c}{(1-c)^3} \right) - 2 \frac{1}{1' \Sigma_n^{-1} \mathbf{1}} \right. \\ &\quad \left. - 2 \frac{\gamma^{-1}}{1-c} \left(\mathbf{b}' \mu_n - \frac{1' \Sigma_n^{-1} \mu_n}{1' \Sigma_n^{-1} \mathbf{1}} \right) + \mathbf{b}' \Sigma_n \mathbf{b} \right). \end{aligned}$$

Let $\alpha^* = A^*/B^*$. Then,

$$|\alpha_n^* - \alpha^*| \leq \left| \frac{1}{B_n^*} (A_n^* - A^*) \right| + \left| \frac{A^*}{B^* B_n^*} (B^* - B_n^*) \right| \xrightarrow{a.s.} 0$$

Using the notations $V_{GMV} = \frac{1}{1' \Sigma_n^{-1} \mathbf{1}}$, $R_{GMV} = \frac{1' \Sigma_n^{-1} \mu_n}{1' \Sigma_n^{-1} \mathbf{1}}$, $s = \mu_n' \mathbf{Q}_n \mu_n$, $R_b = \mathbf{b}' \mu_n$ and making some technical manipulations we get the statement of Theorem 2.1.

□

Proof of Corollary 2.1: (a) We first compute U_S/q_n with $q_n = \max\{\boldsymbol{\mu}'_n \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n, \mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}\}$ given by

$$\frac{1}{q_n} U_S = \frac{1}{q_n} \hat{\mathbf{w}}'_S \boldsymbol{\mu}_n - \frac{\gamma}{2} \frac{1}{q_n} \hat{\mathbf{w}}'_S \boldsymbol{\Sigma}_n \hat{\mathbf{w}}_S,$$

where

$$\begin{aligned} \frac{1}{q_n} \hat{\mathbf{w}}'_S \boldsymbol{\mu}_n &= \frac{1}{q_n} \frac{\mathbf{1}' \mathbf{S}_n^{-1} \boldsymbol{\mu}_n}{\mathbf{1}' \mathbf{S}_n^{-1} \mathbf{1}} + \gamma^{-1} \frac{1}{q_n} \bar{\mathbf{y}}'_n \hat{\mathbf{Q}}_n \boldsymbol{\mu}_n \\ &= \frac{(\mathbf{1}' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1/2} \sqrt{\boldsymbol{\mu}'_n \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n} (\mathbf{1}' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1/2} (\boldsymbol{\mu}'_n \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n)^{-1/2} \mathbf{1}' \mathbf{S}_n^{-1} \boldsymbol{\mu}_n}{q_n (\mathbf{1}' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1} \mathbf{1}' \mathbf{S}_n^{-1} \mathbf{1}} \\ &\quad + \gamma^{-1} \frac{\boldsymbol{\mu}'_n \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n}{q_n} \frac{\bar{\mathbf{y}}'_n \hat{\mathbf{Q}}_n \boldsymbol{\mu}_n}{\boldsymbol{\mu}'_n \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n} \\ &\xrightarrow{a.s.} \frac{(\mathbf{1}' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1/2} \sqrt{\boldsymbol{\mu}'_n \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n} (\mathbf{1}' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1/2} (\boldsymbol{\mu}'_n \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n)^{-1/2} (1-c)^{-1} \mathbf{1}' \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n}{q_n (1-c)^{-1}} \\ &\quad + \gamma^{-1} (1-c)^{-1} \frac{\boldsymbol{\mu}'_n \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n}{q_n} \frac{\boldsymbol{\mu}'_n \mathbf{Q}_n \boldsymbol{\mu}_n}{\boldsymbol{\mu}'_n \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n} = \frac{1}{q_n} (R_{GMV} + \gamma^{-1} (1-c)^{-1} s) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{q_n} \hat{\mathbf{w}}'_S \boldsymbol{\Sigma}_n \hat{\mathbf{w}}_S &= \frac{\mathbf{1}' \mathbf{S}_n^{-1} \boldsymbol{\Sigma}_n \mathbf{S}_n^{-1} \mathbf{1}}{(\mathbf{1}' \mathbf{S}_n^{-1} \mathbf{1})^2} + 2\gamma^{-1} \frac{\mathbf{1}' \mathbf{S}_n^{-1} \boldsymbol{\Sigma}_n \hat{\mathbf{Q}}_n \bar{\mathbf{y}}_n}{\mathbf{1}' \mathbf{S}_n^{-1} \mathbf{1}} + \gamma^{-2} \bar{\mathbf{y}}'_n \hat{\mathbf{Q}}_n \boldsymbol{\Sigma}_n \hat{\mathbf{Q}}_n \bar{\mathbf{y}}_n \\ &= \frac{(\mathbf{1}' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1} (\mathbf{1}' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1} \mathbf{1}' \mathbf{S}_n^{-1} \boldsymbol{\Sigma}_n \mathbf{S}_n^{-1} \mathbf{1}}{q_n (\mathbf{1}' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-2} (\mathbf{1}' \mathbf{S}_n^{-1} \mathbf{1})^2} + \gamma^{-2} \frac{\boldsymbol{\mu}'_n \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n}{q_n} \frac{\bar{\mathbf{y}}'_n \hat{\mathbf{Q}}_n \boldsymbol{\Sigma}_n \hat{\mathbf{Q}}_n \bar{\mathbf{y}}_n}{\boldsymbol{\mu}'_n \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n} \\ &\quad + 2\gamma^{-1} \frac{(\mathbf{1}' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1/2} \sqrt{\boldsymbol{\mu}'_n \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n} (\mathbf{1}' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1/2} (\boldsymbol{\mu}'_n \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n)^{-1/2} \mathbf{1}' \mathbf{S}_n^{-1} \boldsymbol{\Sigma}_n \hat{\mathbf{Q}}_n \bar{\mathbf{y}}_n}{q_n (\mathbf{1}' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1} \mathbf{1}' \mathbf{S}_n^{-1} \mathbf{1}} \\ &\xrightarrow{a.s.} \frac{(\mathbf{1}' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1} (1-c)^{-3}}{q_n (1-c)^{-2}} + \gamma^{-2} \frac{\boldsymbol{\mu}'_n \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n}{q_n} \frac{(1-c)^{-3} c + (1-c)^{-3} \boldsymbol{\mu}'_n \mathbf{Q}_n \boldsymbol{\mu}_n}{\boldsymbol{\mu}'_n \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n} \\ &\quad + 2\gamma^{-1} \frac{(\mathbf{1}' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1/2} \sqrt{\boldsymbol{\mu}'_n \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n}}{q_n} \frac{0}{(1-c)^{-1}} \\ &= \frac{1}{q_n} \left((1-c)^{-1} V_{GMV} + \gamma^{-2} \frac{c}{(1-c)^3} + \gamma^{-2} (1-c)^{-3} s \right) \end{aligned}$$

for $p/n \rightarrow c \in (0, 1)$ as $n \rightarrow \infty$.

Finally, the equality

$$\frac{1}{q_n} U_{EU} = \frac{1}{q_n} R_{GMV} + \frac{1}{2} \gamma^{-1} \frac{1}{q_n} s - \frac{\gamma}{2} \frac{1}{q_n} V_{GMV},$$

implies the statement of the first part of the corollary.

(b) It holds that

$$\begin{aligned} \frac{1}{q_n} U_{GSE} &= \alpha^* \frac{1}{q_n} \hat{\mathbf{w}}'_S \boldsymbol{\mu}_n + (1 - \alpha^*) \frac{1}{q_n} \mathbf{b}' \boldsymbol{\mu}_n \\ &- \frac{\gamma}{2} \left((\alpha^*)^2 \frac{1}{q_n} \hat{\mathbf{w}}'_S \boldsymbol{\Sigma}_n \hat{\mathbf{w}}_S + 2\alpha^*(1 - \alpha^*) \frac{1}{q_n} \mathbf{b}' \boldsymbol{\Sigma}_n \hat{\mathbf{w}}_S + (1 - \alpha^*)^2 \frac{1}{q_n} \mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b} \right), \end{aligned}$$

where the asymptotic values of $\hat{\mathbf{w}}'_S \boldsymbol{\mu}_n / q_n$ and $\hat{\mathbf{w}}'_S \boldsymbol{\Sigma}_n \hat{\mathbf{w}}_S / q_n$ are found in part (a) and

$$\begin{aligned} \frac{1}{q_n} \mathbf{b}' \boldsymbol{\Sigma}_n \hat{\mathbf{w}}_S &= \frac{1}{q_n} \frac{\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{S}_n^{-1} \mathbf{1}}{\mathbf{1}' \mathbf{S}_n^{-1} \mathbf{1}} + \gamma^{-1} \frac{1}{q_n} \mathbf{b}' \boldsymbol{\Sigma}_n \hat{\mathbf{Q}}_n \bar{\mathbf{y}}_n \\ &= \frac{(\mathbf{1}' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1/2} \sqrt{\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}} (\mathbf{1}' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1/2} (\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b})^{-1/2} \mathbf{1}' \mathbf{S}_n^{-1} \boldsymbol{\Sigma}_n \mathbf{b}}{q_n (\mathbf{1}' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1} \mathbf{1}' \mathbf{S}_n^{-1} \mathbf{1}} \\ &+ \gamma^{-1} \frac{\sqrt{\boldsymbol{\mu}'_n \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n} \sqrt{\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}}}{q_n} \frac{\mathbf{b}' \boldsymbol{\Sigma}_n \hat{\mathbf{Q}}_n \bar{\mathbf{y}}_n}{\sqrt{\boldsymbol{\mu}'_n \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n} \sqrt{\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}}} \\ &\xrightarrow{a.s.} \frac{(\mathbf{1}' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1/2} \sqrt{\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}} (\mathbf{1}' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1/2} (\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b})^{-1/2} (1 - c)^{-1}}{q_n (1 - c)^{-1}} \\ &+ \gamma^{-1} \frac{\sqrt{\boldsymbol{\mu}'_n \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n} \sqrt{\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}}}{q_n} \frac{(1 - c)^{-1} \left(\mathbf{b}' \boldsymbol{\mu}_n - \frac{\mathbf{1}' \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n}{\mathbf{1}' \boldsymbol{\Sigma}_n^{-1} \mathbf{1}} \right)}{\sqrt{\boldsymbol{\mu}'_n \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n} \sqrt{\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}}} \\ &= \frac{1}{q_n} (V_{GMV} + \gamma^{-1} (1 - c)^{-1} (R_b - R_{GMV})). \end{aligned}$$

for $p/n \rightarrow c \in (0, 1)$ as $n \rightarrow \infty$.

Hence,

$$\begin{aligned} \frac{1}{q_n} (U_{EU} - U_{GSE}) &= (\alpha^*)^2 \frac{1}{q_n} U_{EU} + (1 - \alpha^*)^2 \frac{1}{q_n} U_{EU} + 2\alpha^*(1 - \alpha^*) \frac{1}{q_n} U_{EU} \\ &- (\alpha^*)^2 \frac{1}{q_n} U_S - \alpha^*(1 - \alpha^*) \frac{1}{q_n} \hat{\mathbf{w}}'_S \boldsymbol{\mu}_n - (1 - \alpha^*)^2 \frac{1}{q_n} U_{\mathbf{b}} - \alpha^*(1 - \alpha^*) \frac{1}{q_n} \mathbf{b}' \boldsymbol{\mu}_n + \gamma \alpha^*(1 - \alpha^*) \frac{1}{q_n} \mathbf{b}' \boldsymbol{\Sigma}_n \hat{\mathbf{w}}_S \\ &\xrightarrow{a.s.} (\alpha^*)^2 \frac{1}{q_n} (U_{EU} - U_S) + (1 - \alpha^*)^2 \frac{1}{q_n} (U_{EU} - U_{\mathbf{b}}) \\ &+ \alpha^*(1 - \alpha^*) \frac{1}{q_n} \left(2R_{GMV} + \gamma^{-1} s - \gamma V_{GMV} - R_{GMV} - \frac{\gamma^{-1}}{1 - c} s - R_b + \gamma V_{GMV} + \frac{R_b - R_{GMV}}{1 - c} \right) \\ &= (\alpha^*)^2 \frac{1}{q_n} (U_{EU} - U_S) + (1 - \alpha^*)^2 \frac{1}{q_n} (U_{EU} - U_{\mathbf{b}}) + \alpha^*(1 - \alpha^*) \frac{c}{1 - c} \frac{1}{q_n} (R_b - R_{GMV} - \gamma^{-1} s). \end{aligned}$$

for $p/n \rightarrow c \in (0, 1)$ as $n \rightarrow \infty$. □

For the proof of Theorem 2.2 we need several results about the properties of Moore-Penrose inverse which are summarized in the following three lemmas. Similarly as the proof of Lemma 1.2, we will use Lemma 1.1 and Theorem 1.1 in a sequel every time a derivative must be interchanged with the limit $n \rightarrow \infty$.

Lemma 1.5. *Assume (A2). Let $\boldsymbol{\theta}$ and $\boldsymbol{\xi}$ be universal nonrandom vectors with bounded Eu-*

clidean norms. Then it holds that

$$\boldsymbol{\xi}' \tilde{\mathbf{V}}_n^+ \boldsymbol{\theta} \xrightarrow{a.s.} c^{-1}(c-1)^{-1} \boldsymbol{\xi}' \boldsymbol{\theta}, \quad (1.34)$$

$$\boldsymbol{\xi}' (\tilde{\mathbf{V}}_n^+)^2 \boldsymbol{\theta} \xrightarrow{a.s.} (c-1)^{-3} \boldsymbol{\xi}' \boldsymbol{\theta}, \quad (1.35)$$

$$\bar{\mathbf{x}}' \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n = 1, \quad (1.36)$$

$$\bar{\mathbf{x}}' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n \xrightarrow{a.s.} \frac{c}{c-1}, \quad (1.37)$$

$$\bar{\mathbf{x}}' (\tilde{\mathbf{V}}_n^+)^3 \bar{\mathbf{x}}_n \xrightarrow{a.s.} \frac{c^3}{(c-1)^3}, \quad (1.38)$$

$$\bar{\mathbf{x}}' (\tilde{\mathbf{V}}_n^+)^4 \bar{\mathbf{x}}_n \xrightarrow{a.s.} \frac{c^4 + c^5}{(c-1)^5}, \quad (1.39)$$

$$\bar{\mathbf{x}}' \tilde{\mathbf{V}}_n^+ \boldsymbol{\theta} \xrightarrow{a.s.} 0, \quad (1.40)$$

$$\bar{\mathbf{x}}' (\tilde{\mathbf{V}}_n^+)^2 \boldsymbol{\theta} \xrightarrow{a.s.} 0, \quad (1.41)$$

$$\bar{\mathbf{x}}' (\tilde{\mathbf{V}}_n^+)^3 \boldsymbol{\theta} \xrightarrow{a.s.} 0 \quad (1.42)$$

for $p/n \rightarrow c \in (1, +\infty)$ as $n \rightarrow \infty$.

Proof of Lemma 1.5: It holds that

$$\tilde{\mathbf{V}}^+ = \left(\frac{1}{n} \mathbf{X}_n \mathbf{X}_n' \right)^+ = \frac{1}{\sqrt{n}} \mathbf{X}_n \left(\frac{1}{n} \mathbf{X}_n' \mathbf{X}_n \right)^{-2} \frac{1}{\sqrt{n}} \mathbf{X}_n'$$

and, similarly,

$$(\tilde{\mathbf{V}}^+)^i = \frac{1}{\sqrt{n}} \mathbf{X}_n \left(\frac{1}{n} \mathbf{X}_n' \mathbf{X}_n \right)^{-(i+1)} \frac{1}{\sqrt{n}} \mathbf{X}_n' \quad \text{for } i = 2, 3, 4.$$

Let $\boldsymbol{\Theta} = \boldsymbol{\theta} \boldsymbol{\xi}'$. It holds that

$$\begin{aligned} \boldsymbol{\xi}' \tilde{\mathbf{V}}_n^+ \boldsymbol{\theta} &= \text{tr} \left[\frac{1}{\sqrt{n}} \mathbf{X}_n \left(\frac{1}{n} \mathbf{X}_n' \mathbf{X}_n \right)^{-2} \frac{1}{\sqrt{n}} \mathbf{X}_n' \boldsymbol{\Theta} \right] = \frac{\partial}{\partial z} \text{tr} \left[\frac{1}{\sqrt{n}} \mathbf{X}_n \left(\frac{1}{n} \mathbf{X}_n' \mathbf{X}_n - z \mathbf{I}_n \right)^{-1} \frac{1}{\sqrt{n}} \mathbf{X}_n' \boldsymbol{\Theta} \right] \Big|_{z=0}, \\ \boldsymbol{\xi}' (\tilde{\mathbf{V}}_n^+)^2 \boldsymbol{\theta} &= \text{tr} \left[\frac{1}{\sqrt{n}} \mathbf{X}_n \left(\frac{1}{n} \mathbf{X}_n' \mathbf{X}_n \right)^{-3} \frac{1}{\sqrt{n}} \mathbf{X}_n' \boldsymbol{\Theta} \right] = \frac{1}{2} \frac{\partial^2}{\partial z^2} \text{tr} \left[\frac{1}{\sqrt{n}} \mathbf{X}_n \left(\frac{1}{n} \mathbf{X}_n' \mathbf{X}_n - z \mathbf{I}_n \right)^{-1} \frac{1}{\sqrt{n}} \mathbf{X}_n' \boldsymbol{\Theta} \right] \Big|_{z=0} \end{aligned}$$

The application of Woodbury formula (matrix inversion lemma, see, e.g., Horn and Johnson (1985)),

$$\frac{1}{\sqrt{n}} \mathbf{X}_n \left(\frac{1}{n} \mathbf{X}_n' \mathbf{X}_n - z \mathbf{I}_n \right)^{-1} \frac{1}{\sqrt{n}} \mathbf{X}_n' = \mathbf{I}_p + z \left(\frac{1}{n} \mathbf{X}_n \mathbf{X}_n' - z \mathbf{I}_p \right)^{-1} \quad (1.43)$$

leads to

$$\begin{aligned}\xi' \tilde{\mathbf{V}}_n^+ \boldsymbol{\theta} &= \frac{\partial}{\partial z} z \text{tr} \left[\left(\frac{1}{n} \mathbf{X}_n \mathbf{X}_n' - z \mathbf{I}_p \right)^{-1} \boldsymbol{\Theta} \right] \Big|_{z=0}, \\ \xi' (\tilde{\mathbf{V}}_n^+)^2 \boldsymbol{\theta} &= \frac{1}{2} \frac{\partial^2}{\partial z^2} z \text{tr} \left[\left(\frac{1}{n} \mathbf{X}_n \mathbf{X}_n' - z \mathbf{I}_p \right)^{-1} \boldsymbol{\Theta} \right] \Big|_{z=0}.\end{aligned}$$

From the proof of Lemma 1.2 we know that the matrix $\boldsymbol{\Theta}$ possesses the bounded trace norm.

Then the application of Lemma 1.1 leads to

$$\begin{aligned}\xi' \tilde{\mathbf{V}}_n^+ \boldsymbol{\theta} &\xrightarrow{a.s.} \frac{\partial}{\partial z} \frac{z}{x(z) - z} \Big|_{z=0} \xi' \boldsymbol{\theta}, \\ \xi' (\tilde{\mathbf{V}}_n^+)^2 \boldsymbol{\theta} &\xrightarrow{a.s.} \frac{1}{2} \frac{\partial^2}{\partial z^2} \frac{z}{x(z) - z} \Big|_{z=0} \xi' \boldsymbol{\theta}\end{aligned}$$

for $p/n \rightarrow c > 1$ as $n \rightarrow \infty$, where $x(z)$ is given in (1.9).

Let us make the following notations

$$\theta(z) = \frac{z}{x(z) - z} \quad \text{and} \quad \phi(z) = \frac{x(z) - zx'(z)}{z^2}.$$

Then the first and the second derivatives of $\theta(z)$ are given by

$$\theta'(z) = \theta^2(z)\phi(z) \quad \text{and} \quad \theta''(z) = 2\theta(z)\theta'(z)\phi(z) + \theta^2(z)\phi'(z). \quad (1.44)$$

Using L'Hopital's rule, we get

$$\theta(0) = \lim_{z \rightarrow 0} \theta(z) = \lim_{z \rightarrow 0} \frac{z}{x(z) - z} = \lim_{z \rightarrow 0} \frac{1}{(x'(z) - 1)} = \frac{1}{\frac{1}{2} \left(1 - \frac{1+c}{|1-c|} \right) - 1} = -\frac{c-1}{c}, \quad (1.45)$$

$$\phi(0) = \lim_{z \rightarrow 0} \phi(z) = \lim_{z \rightarrow 0} \frac{x(z) - zx'(z)}{z^2} = -\frac{1}{2} \lim_{z \rightarrow 0} x''(z) = -\frac{1}{2} \lim_{z \rightarrow 0} \frac{-2c}{((1-c+z)^2 - 4z)^{3/2}} = \frac{c}{(c-1)^3}, \quad (1.46)$$

and

$$\begin{aligned}\lim_{z \rightarrow 0} \phi'(z) &= -\lim_{z \rightarrow 0} \frac{2(x(z) - zx'(z)) + z^2 x''(z)}{z^2} \\ &= -\lim_{z \rightarrow 0} \frac{2\phi(z) + x''(z)}{z} = -\lim_{z \rightarrow 0} (2\phi'(z) + x'''(z)),\end{aligned} \quad (1.47)$$

which implies

$$\phi'(0) = \lim_{z \rightarrow 0} \phi'(z) = -\frac{1}{3} \lim_{z \rightarrow 0} x'''(z) = -\frac{1}{3} \lim_{z \rightarrow 0} \frac{6c(z-c-1)}{((1-c+z)^2 - 4z)^{5/2}} = \frac{2c(c+1)}{(c-1)^5}. \quad (1.48)$$

Combining (1.44), (1.45), (1.46), and (1.48), we get

$$\theta'(0) = \lim_{z \rightarrow 0} \theta'(z) = \theta^2(0)\phi(0) = \frac{1}{c(c-1)}$$

and

$$\theta''(0) = \lim_{z \rightarrow 0} \theta''(z) = 2\theta^3(0)\phi^2(0) + \theta^2(0)\phi'(0) = \frac{2}{(c-1)^3}.$$

Hence,

$$\begin{aligned} \boldsymbol{\xi}' \tilde{\mathbf{V}}_n^+ \boldsymbol{\theta} &\xrightarrow{a.s.} \frac{1}{c(c-1)} \boldsymbol{\xi}' \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\theta} \text{ for } p/n \rightarrow c > 1 \text{ as } n \rightarrow \infty, \\ \boldsymbol{\xi}' (\tilde{\mathbf{V}}_n^+)^2 \boldsymbol{\theta} &\xrightarrow{a.s.} \frac{1}{(c-1)^3} \boldsymbol{\xi}' \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\theta} \text{ for } p/n \rightarrow c > 1 \text{ as } n \rightarrow \infty. \end{aligned}$$

Taking into account that

$$\bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n = \frac{1}{n} \mathbf{1}_n' \mathbf{X}_n' \mathbf{X}_n (\mathbf{X}_n' \mathbf{X}_n)^{-2} \mathbf{X}_n' \mathbf{X}_n \mathbf{1}_n = \frac{1}{n} \mathbf{1}_n' \mathbf{1}_n = 1.$$

we get (1.36). Similarly, using

$$\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^i \bar{\mathbf{x}}_n = 1/n \mathbf{1}_n' (1/n \mathbf{X}_n' \mathbf{X}_n)^{-(i-1)} \mathbf{1}_n \text{ for } i = 2, 3, 4$$

we get

$$\begin{aligned} 1/n \mathbf{1}_n' (1/n \mathbf{X}_n' \mathbf{X}_n)^{-1} \mathbf{1}_n &= \lim_{z \rightarrow 0} \text{tr}[(1/n \mathbf{X}_n' \mathbf{X}_n - z\mathbf{I})^{-1} \boldsymbol{\Theta}_n] \xrightarrow{a.s.} \underline{m}(0), \\ 1/n \mathbf{1}_n' (1/n \mathbf{X}_n' \mathbf{X}_n)^{-2} \mathbf{1}_n &= \lim_{z \rightarrow 0} \frac{\partial}{\partial z} \text{tr}[(1/n \mathbf{X}_n' \mathbf{X}_n - z\mathbf{I})^{-1} \boldsymbol{\Theta}_n] \xrightarrow{a.s.} \underline{m}'(0), \\ 1/n \mathbf{1}_n' (1/n \mathbf{X}_n' \mathbf{X}_n)^{-3} \mathbf{1}_n &= \frac{1}{2} \lim_{z \rightarrow 0} \frac{\partial^2}{\partial z^2} \text{tr}[(1/n \mathbf{X}_n' \mathbf{X}_n - z\mathbf{I})^{-1} \boldsymbol{\Theta}_n] \xrightarrow{a.s.} \frac{1}{2} \underline{m}''(0) \end{aligned}$$

for $p/n \rightarrow c > 1$ as $n \rightarrow \infty$, where $\boldsymbol{\Theta}_n = 1/n \mathbf{1}_n \mathbf{1}_n'$.

Using that the elements of $\mathbf{X}_n$ are independent and identically distributed and the fact that

$n < p$ from Lemma 1.1 and Theorem 1.1 we get that

$$\underline{m}'(z) = \frac{1}{\underline{x}(z) - z} \quad \text{with} \quad \underline{x}(z) = \frac{1}{2} \left(1 - c^{-1} + z + \sqrt{(1 - c^{-1} + z)^2 - 4z} \right).$$

Thus,

$$\underline{m}(0) = \frac{1}{\underline{x}(0)} = \frac{1}{1 - c^{-1}} = \frac{c}{c - 1},$$

which proves (1.37). Furthermore, we get

$$\underline{m}'(z) = -\frac{\underline{x}'(z) - 1}{(\underline{x}(z) - z)^2}$$

with

$$\underline{x}'(z) = \frac{1}{2} \left(1 + \frac{1}{2} \frac{2(1 - c^{-1} + z) - 4}{\sqrt{(1 - c^{-1} + z)^2 - 4z}} \right) = \frac{1}{2} \left(1 + \frac{-1 - c^{-1} + z}{\sqrt{(1 - c^{-1} + z)^2 - 4z}} \right),$$

and, consequently,

$$\underline{m}'(0) = -\frac{\underline{x}'(0) - 1}{\underline{x}(0)^2} = \frac{1}{(1 - c^{-1})^3} = \frac{c^3}{(c - 1)^3}.$$

In order to prove (1.39), we compute

$$\underline{m}''(z) = -\frac{\underline{x}''(z)}{(\underline{x}(z) - z)^2} + 2\frac{(\underline{x}'(z) - 1)^2}{(\underline{x}(z) - z)^3}$$

where

$$\underline{x}''(z) = \frac{1}{2} \left(\frac{1}{\sqrt{(1 - c^{-1} + z)^2 - 4z}} - \frac{(-1 - c^{-1} + z)^2}{((1 - c^{-1} + z)^2 - 4z)^{3/2}} \right),$$

Hence,

$$\underline{m}''(0) = -\frac{\underline{x}''(0)}{\underline{x}(0)^2} + 2\frac{(\underline{x}'(0) - 1)^2}{\underline{x}(0)^3} = 2\frac{c^{-1} + 1}{(1 - c^{-1})^5} = 2\frac{c^4 + c^5}{(c - 1)^5}.$$

For (1.40) we consider

$$\bar{\mathbf{x}}' \tilde{\mathbf{V}}_n^+ \boldsymbol{\theta} = \text{tr} \left[\mathbf{I}_p + z \left(\frac{1}{n} \mathbf{X}_n \mathbf{X}_n' - z \mathbf{I}_p \right)^{-1} \boldsymbol{\theta} \bar{\mathbf{x}}' \right] = \bar{\mathbf{x}}' \boldsymbol{\theta} + z \bar{\mathbf{x}}' \left(\tilde{\mathbf{V}}_n - z \mathbf{I}_p \right)^{-1} \boldsymbol{\theta}.$$

Because of (1.16), it holds that $\bar{\mathbf{x}}' \left(\tilde{\mathbf{V}}_n - z \mathbf{I}_p \right)^{-1} \boldsymbol{\theta}$ is uniformly bounded as $z \rightarrow 0$. Moreover, $\bar{\mathbf{x}}' \boldsymbol{\theta} \xrightarrow{a.s.} 0$ as $p \rightarrow \infty$ following Kolmogorov's strong law of large numbers (c.f., Sen and Singer (1993, Theorem 2.3.10)), since $\boldsymbol{\theta}$ has a bounded Euclidean norm. Hence, $\bar{\mathbf{x}}' \tilde{\mathbf{V}}_n^+ \boldsymbol{\theta} \xrightarrow{a.s.} 0$ for $p/n \rightarrow c \in (1, +\infty)$ as $n \rightarrow \infty$.

Finally, in the case of (1.41) and (1.42), we get

$$\begin{aligned} \bar{\mathbf{x}}' (\tilde{\mathbf{V}}_n^+)^2 \boldsymbol{\theta} &= \frac{\partial}{\partial z} z \text{tr} \left[\left(\frac{1}{n} \mathbf{X}_n \mathbf{X}_n' - z \mathbf{I}_p \right)^{-1} \boldsymbol{\theta} \bar{\mathbf{x}}' \right] \Big|_{z=0} \xrightarrow{a.s.} 0, \\ \bar{\mathbf{x}}' (\tilde{\mathbf{V}}_n^+)^3 \boldsymbol{\theta} &= \frac{1}{2} \frac{\partial^2}{\partial z^2} z \text{tr} \left[\left(\frac{1}{n} \mathbf{X}_n \mathbf{X}_n' - z \mathbf{I}_p \right)^{-1} \boldsymbol{\theta} \bar{\mathbf{x}}' \right] \Big|_{z=0} \xrightarrow{a.s.} 0. \end{aligned}$$

for $p/n \rightarrow c \in (1, +\infty)$ as $n \rightarrow \infty$. □

Lemma 1.6. *Assume (A2). Let $\boldsymbol{\theta}$ and $\boldsymbol{\xi}$ be universal nonrandom vectors with bounded Euclidean norms. Then it holds that*

$$\boldsymbol{\xi}' \mathbf{V}_n^+ \boldsymbol{\theta} \xrightarrow{a.s.} c^{-1} (c-1)^{-1} \boldsymbol{\xi}' \boldsymbol{\theta}, \quad (1.49)$$

$$\bar{\mathbf{x}}_n' \mathbf{V}_n^+ \bar{\mathbf{x}}_n \xrightarrow{a.s.} \frac{1}{c-1}, \quad (1.50)$$

$$\bar{\mathbf{x}}_n' \mathbf{V}_n^+ \boldsymbol{\theta} \xrightarrow{a.s.} 0, \quad (1.51)$$

$$\boldsymbol{\xi}' (\mathbf{V}_n^+)^2 \boldsymbol{\theta} \xrightarrow{a.s.} (c-1)^{-3} \boldsymbol{\xi}' \boldsymbol{\theta}, \quad (1.52)$$

$$\bar{\mathbf{x}}_n' (\mathbf{V}_n^+)^2 \bar{\mathbf{x}}_n \xrightarrow{a.s.} \frac{c^2}{(c-1)^3}, \quad (1.53)$$

$$\bar{\mathbf{x}}_n' (\mathbf{V}_n^+)^2 \boldsymbol{\theta} \xrightarrow{a.s.} 0 \quad (1.54)$$

for $p/n \rightarrow c \in (1, +\infty)$ as $n \rightarrow \infty$.

Proof of Lemma 1.6: From (1.5) we get

$$\begin{aligned} \boldsymbol{\xi}' \mathbf{V}_n^+ \boldsymbol{\theta} &= \boldsymbol{\xi}' \tilde{\mathbf{V}}_n^+ \boldsymbol{\theta} - \frac{\boldsymbol{\xi}' \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \boldsymbol{\theta} + \boldsymbol{\xi}' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+ \boldsymbol{\theta}}{\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n} \\ &\quad + \frac{\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^3 \bar{\mathbf{x}}_n}{(\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n)^2} \boldsymbol{\xi}' \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+ \boldsymbol{\theta} \xrightarrow{a.s.} c^{-1} (c-1)^{-1} \boldsymbol{\xi}' \boldsymbol{\theta} \end{aligned}$$

for $p/n \rightarrow c \in (1, +\infty)$ as $n \rightarrow \infty$ following (1.34)-(1.36). Similarly, we get

$$\begin{aligned}\bar{\mathbf{x}}'_n \mathbf{V}_n^+ \bar{\mathbf{x}}_n &= \bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n - \frac{\bar{\mathbf{x}}'_n \mathbf{V}_n^+ \bar{\mathbf{x}}_n \bar{\mathbf{x}}'_n (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n + \bar{\mathbf{x}}'_n (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n \bar{\mathbf{x}}'_n \mathbf{V}_n^+ \bar{\mathbf{x}}_n}{\bar{\mathbf{x}}'_n (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n} \\ &+ \frac{\bar{\mathbf{x}}'_n (\tilde{\mathbf{V}}_n^+)^3 \bar{\mathbf{x}}_n}{(\bar{\mathbf{x}}'_n (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n)^2} \bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \xrightarrow{a.s.} \frac{1}{c-1}\end{aligned}$$

and

$$\begin{aligned} \bar{\mathbf{x}}'_n \mathbf{V}_n^+ \boldsymbol{\theta} &= \bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^+ \boldsymbol{\theta} - \frac{\bar{\mathbf{x}}'_n \mathbf{V}_n^+ \bar{\mathbf{x}}_n \bar{\mathbf{x}}'_n (\mathbf{V}_n^+)^2 \boldsymbol{\theta} + \bar{\mathbf{x}}'_n (\mathbf{V}_n^+)^2 \bar{\mathbf{x}}_n \bar{\mathbf{x}}'_n \mathbf{V}_n^+ \boldsymbol{\theta}}{\bar{\mathbf{x}}'_n (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n} \\ &+ \frac{\bar{\mathbf{x}}'_n (\tilde{\mathbf{V}}_n^+)^3 \bar{\mathbf{x}}_n}{(\bar{\mathbf{x}}'_n (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n)^2} \bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \bar{\mathbf{x}}'_n \tilde{\mathbf{V}}_n^+ \boldsymbol{\theta} \xrightarrow{a.s.} 0 \end{aligned}$$

for $p/n \rightarrow c \in (1, +\infty)$ as $n \rightarrow \infty$.

Now, we consider the equality

$$\begin{aligned}
\left[(\tilde{\mathbf{V}}_n - \bar{\mathbf{x}}_n \tilde{\mathbf{x}}_n')^+ \right]^2 &= \left(\tilde{\mathbf{V}}_n^+ - \frac{\tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \tilde{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 + (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n \tilde{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)}{\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n} + \frac{\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^3 \bar{\mathbf{x}}_n}{(\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n)^2} \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \tilde{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+ \right)^2 \\
&= (\tilde{\mathbf{V}}_n^+)^2 + \left[\frac{\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^3 \bar{\mathbf{x}}_n}{(\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n)^2} \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \tilde{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+ - \frac{(\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n \tilde{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+ + \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \tilde{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2}{\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n} \right]^2 \\
&\quad - \frac{2(\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n \tilde{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 + \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \tilde{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^3 + (\tilde{\mathbf{V}}_n^+)^3 \bar{\mathbf{x}}_n \tilde{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+}{\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n} \\
&\quad + \frac{\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^3 \bar{\mathbf{x}}_n}{(\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n)^2} (\tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \tilde{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 + (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n \tilde{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+) \\
&= (\tilde{\mathbf{V}}_n^+)^2 + \frac{\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^4 \bar{\mathbf{x}}_n}{(\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n)^2} \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \tilde{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+ - \frac{(\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^3 \bar{\mathbf{x}}_n)^2}{(\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n)^3} \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \tilde{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+ \\
&\quad + \frac{\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^3 \bar{\mathbf{x}}_n}{(\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n)^2} \left[(\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n \tilde{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+ + \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \tilde{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \right] \\
&\quad - \frac{(\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n \tilde{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 + \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \tilde{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^3 + (\tilde{\mathbf{V}}_n^+)^3 \bar{\mathbf{x}}_n \tilde{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+}{\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n}
\end{aligned}$$

Hence,

$$\begin{aligned} \xi'(\mathbf{V}_n^+)^2 \boldsymbol{\theta} &= \xi'(\tilde{\mathbf{V}}_n^+)^2 \boldsymbol{\theta} + \frac{\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^4 \bar{\mathbf{x}}_n}{(\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n)^2} \xi' \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+ \boldsymbol{\theta} - \frac{(\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^3 \bar{\mathbf{x}}_n)^2}{(\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n)^3} \xi' \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+ \boldsymbol{\theta} \\ &+ \frac{\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^3 \bar{\mathbf{x}}_n}{(\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n)^2} \left[\xi'(\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+ \boldsymbol{\theta} + \xi' \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \boldsymbol{\theta} \right] \\ &- \frac{\xi'(\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \boldsymbol{\theta} + \xi' \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^3 \boldsymbol{\theta} + \xi'(\tilde{\mathbf{V}}_n^+)^3 \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+ \boldsymbol{\theta}}{\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n} \xrightarrow{a.s.} (c-1)^{-3} \xi' \boldsymbol{\theta}, \end{aligned}$$

$$\begin{aligned}
\bar{\mathbf{x}}'_n(\mathbf{V}_n^+)^2\bar{\mathbf{x}}_n &= \bar{\mathbf{x}}'_n(\tilde{\mathbf{V}}_n^+)^2\bar{\mathbf{x}}_n + \frac{\bar{\mathbf{x}}'_n(\tilde{\mathbf{V}}_n^+)^4\bar{\mathbf{x}}_n}{(\bar{\mathbf{x}}'_n(\tilde{\mathbf{V}}_n^+)^2\bar{\mathbf{x}}_n)^2} - \frac{(\bar{\mathbf{x}}'_n(\tilde{\mathbf{V}}_n^+)^3\bar{\mathbf{x}}_n)^2}{(\bar{\mathbf{x}}'_n(\tilde{\mathbf{V}}_n^+)^2\bar{\mathbf{x}}_n)^3} + 2\frac{\bar{\mathbf{x}}'_n(\tilde{\mathbf{V}}_n^+)^3\bar{\mathbf{x}}_n}{\bar{\mathbf{x}}'_n(\tilde{\mathbf{V}}_n^+)^2\bar{\mathbf{x}}_n} \\
&- \bar{\mathbf{x}}'_n(\tilde{\mathbf{V}}_n^+)^2\bar{\mathbf{x}}_n - 2\frac{\bar{\mathbf{x}}'_n(\tilde{\mathbf{V}}_n^+)^3\bar{\mathbf{x}}_n}{\bar{\mathbf{x}}'_n(\tilde{\mathbf{V}}_n^+)^2\bar{\mathbf{x}}_n} \\
&= \frac{\bar{\mathbf{x}}'_n(\tilde{\mathbf{V}}_n^+)^4\bar{\mathbf{x}}_n}{(\bar{\mathbf{x}}'_n(\tilde{\mathbf{V}}_n^+)^2\bar{\mathbf{x}}_n)^2} - \frac{(\bar{\mathbf{x}}'_n(\tilde{\mathbf{V}}_n^+)^3\bar{\mathbf{x}}_n)^2}{(\bar{\mathbf{x}}'_n(\tilde{\mathbf{V}}_n^+)^2\bar{\mathbf{x}}_n)^3} \xrightarrow{a.s.} \frac{c^2}{(c-1)^3},
\end{aligned}$$

and

$$\begin{aligned}
\bar{\mathbf{x}}'_n(\mathbf{V}_n^+)^2\boldsymbol{\theta} &= \bar{\mathbf{x}}'_n(\tilde{\mathbf{V}}_n^+)^2\boldsymbol{\theta} + \frac{\bar{\mathbf{x}}'_n(\tilde{\mathbf{V}}_n^+)^4\bar{\mathbf{x}}_n}{(\bar{\mathbf{x}}'_n(\tilde{\mathbf{V}}_n^+)^2\bar{\mathbf{x}}_n)^2}\bar{\mathbf{x}}'_n\tilde{\mathbf{V}}_n^+\boldsymbol{\theta} - \frac{(\bar{\mathbf{x}}'_n(\tilde{\mathbf{V}}_n^+)^3\bar{\mathbf{x}}_n)^2}{(\bar{\mathbf{x}}'_n(\tilde{\mathbf{V}}_n^+)^2\bar{\mathbf{x}}_n)^3}\bar{\mathbf{x}}'_n\tilde{\mathbf{V}}_n^+\boldsymbol{\theta} \\
&+ \frac{\bar{\mathbf{x}}'_n(\tilde{\mathbf{V}}_n^+)^3\bar{\mathbf{x}}_n}{(\bar{\mathbf{x}}'_n(\tilde{\mathbf{V}}_n^+)^2\bar{\mathbf{x}}_n)^2}\left[\bar{\mathbf{x}}'_n(\tilde{\mathbf{V}}_n^+)^2\bar{\mathbf{x}}_n\bar{\mathbf{x}}'_n\tilde{\mathbf{V}}_n^+\boldsymbol{\theta} + \bar{\mathbf{x}}'_n(\tilde{\mathbf{V}}_n^+)^2\boldsymbol{\theta}\right] \\
&- \frac{\bar{\mathbf{x}}'_n(\tilde{\mathbf{V}}_n^+)^2\bar{\mathbf{x}}_n\bar{\mathbf{x}}'_n(\tilde{\mathbf{V}}_n^+)^2\boldsymbol{\theta} + \bar{\mathbf{x}}'_n(\tilde{\mathbf{V}}_n^+)^3\boldsymbol{\theta} + \bar{\mathbf{x}}'_n(\tilde{\mathbf{V}}_n^+)^3\bar{\mathbf{x}}_n\bar{\mathbf{x}}'_n\tilde{\mathbf{V}}_n^+\boldsymbol{\theta}}{\bar{\mathbf{x}}'_n(\tilde{\mathbf{V}}_n^+)^2\bar{\mathbf{x}}_n} \xrightarrow{a.s.} 0
\end{aligned}$$

for $p/n \rightarrow c \in (1, +\infty)$ as $n \rightarrow \infty$. □

Lemma 1.7. Assume (A2). Let $\boldsymbol{\theta}$ and $\boldsymbol{\xi}$ be universal nonrandom vectors with bounded Euclidean norms and let $\mathbf{P}_n^+ = \mathbf{V}_n^+ - \frac{\mathbf{V}_n^+\boldsymbol{\eta}\boldsymbol{\eta}'\mathbf{V}_n^+}{\boldsymbol{\eta}'\mathbf{V}_n^+\boldsymbol{\eta}}$ where $\boldsymbol{\eta}$ is a universal nonrandom vectors with bounded Euclidean norm. Then it holds that

$$\boldsymbol{\xi}'\mathbf{P}_n^+\boldsymbol{\theta} \xrightarrow{a.s.} c^{-1}(c-1)^{-1}\left(\boldsymbol{\xi}'\boldsymbol{\theta} - \frac{\boldsymbol{\xi}'\boldsymbol{\eta}\boldsymbol{\eta}'\boldsymbol{\theta}}{\boldsymbol{\eta}'\boldsymbol{\eta}}\right), \quad (1.55)$$

$$\bar{\mathbf{x}}'_n\mathbf{P}_n^+\bar{\mathbf{x}}_n \xrightarrow{a.s.} \frac{1}{c-1}, \quad (1.56)$$

$$\bar{\mathbf{x}}'_n\mathbf{P}_n^+\boldsymbol{\theta} \xrightarrow{a.s.} 0, \quad (1.57)$$

$$\boldsymbol{\xi}'(\mathbf{P}_n^+)^2\boldsymbol{\theta} \xrightarrow{a.s.} (c-1)^{-3}\left(\boldsymbol{\xi}'\boldsymbol{\theta} - \frac{\boldsymbol{\xi}'\boldsymbol{\eta}\boldsymbol{\eta}'\boldsymbol{\theta}}{\boldsymbol{\eta}'\boldsymbol{\eta}}\right), \quad (1.58)$$

$$\bar{\mathbf{x}}'_n(\mathbf{P}_n^+)^2\bar{\mathbf{x}}_n \xrightarrow{a.s.} \frac{c^2}{(c-1)^3}, \quad (1.59)$$

$$\bar{\mathbf{x}}'_n(\mathbf{P}_n^+)^2\boldsymbol{\theta} \xrightarrow{a.s.} 0 \quad (1.60)$$

for $p/n \rightarrow c \in (1, +\infty)$ as $n \rightarrow \infty$.

Proof of Lemma 1.7: It holds that

$$\boldsymbol{\xi}'\mathbf{P}_n^+\boldsymbol{\theta} = \boldsymbol{\xi}'\mathbf{V}_n^+\boldsymbol{\theta} - \frac{\boldsymbol{\xi}'\mathbf{V}_n^+\boldsymbol{\eta}\boldsymbol{\eta}'\mathbf{V}_n^+\boldsymbol{\theta}}{\boldsymbol{\eta}'\mathbf{V}_n^+\boldsymbol{\eta}} \xrightarrow{a.s.} c^{-1}(c-1)^{-1}\left(\boldsymbol{\xi}'\boldsymbol{\theta} - \frac{\boldsymbol{\xi}'\boldsymbol{\eta}\boldsymbol{\eta}'\boldsymbol{\theta}}{\boldsymbol{\eta}'\boldsymbol{\eta}}\right)$$

for $p/n \rightarrow c \in (1, +\infty)$ as $n \rightarrow \infty$ following (1.34). Similarly, we get

$$\bar{\mathbf{x}}'_n \mathbf{P}_n^+ \bar{\mathbf{x}}_n = \bar{\mathbf{x}}'_n \mathbf{V}_n^+ \bar{\mathbf{x}}_n - \frac{\bar{\mathbf{x}}'_n \mathbf{V}_n^+ \boldsymbol{\eta} \boldsymbol{\eta}' \mathbf{V}_n^+ \bar{\mathbf{x}}_n}{\boldsymbol{\eta}' \mathbf{V}_n^+ \boldsymbol{\eta}} \xrightarrow{a.s.} \frac{1}{c-1}$$

and

$$\bar{\mathbf{x}}'_n \mathbf{P}_n^+ \boldsymbol{\theta} = \bar{\mathbf{x}}'_n \mathbf{V}_n^+ \boldsymbol{\theta} - \frac{\bar{\mathbf{x}}'_n \mathbf{V}_n^+ \boldsymbol{\eta} \boldsymbol{\eta}' \mathbf{V}_n^+ \boldsymbol{\theta}}{\boldsymbol{\eta}' \mathbf{V}_n^+ \boldsymbol{\eta}} \xrightarrow{a.s.} 0$$

for $p/n \rightarrow c \in (1, +\infty)$ as $n \rightarrow \infty$.

The rest of the proof follows from the equality

$$(\mathbf{P}_n^+)^2 = (\mathbf{V}_n^+)^2 - \frac{(\mathbf{V}_n^+)^2 \boldsymbol{\eta} \boldsymbol{\eta}' \mathbf{V}_n^+}{\boldsymbol{\eta}' \mathbf{V}_n^+ \boldsymbol{\eta}} - \frac{\mathbf{V}_n^+ \boldsymbol{\eta} \boldsymbol{\eta}' (\mathbf{V}_n^+)^2}{\boldsymbol{\eta}' \mathbf{V}_n^+ \boldsymbol{\eta}} + \boldsymbol{\eta}' (\mathbf{V}_n^+)^2 \boldsymbol{\eta} \frac{\mathbf{V}_n^+ \boldsymbol{\eta} \boldsymbol{\eta}' \mathbf{V}_n^+}{(\boldsymbol{\eta}' \mathbf{V}_n^+ \boldsymbol{\eta})^2}$$

and Lemma 1.6. □

Proof of Theorem 2.2: Let $q_n = \max\{\boldsymbol{\mu}'_n \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n, \mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}\}$. From Assumption (A3) get that $q_n > 0$ uniformly in p , since $\boldsymbol{\mu}'_n \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n \geq s$ and $s > 0$ uniformly in p .

In case of $c > 1$, the optimal shrinkage intensity is given by

$$\begin{aligned} \alpha_n^+ &= \beta^{-1} \frac{\hat{\mathbf{w}}'_{S^*} (\boldsymbol{\mu}_n - \beta \boldsymbol{\Sigma}_n \mathbf{b}) - \mathbf{b}' (\boldsymbol{\mu}_n - \beta \boldsymbol{\Sigma}_n \mathbf{b})}{\hat{\mathbf{w}}'_{S^*} \boldsymbol{\Sigma}_n \hat{\mathbf{w}}_{S^*} - 2 \mathbf{b}' \boldsymbol{\Sigma}_n \hat{\mathbf{w}}_{S^*} + \mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}} \\ &= \beta^{-1} \frac{\frac{\mathbf{1}' \mathbf{S}_n^* (\boldsymbol{\mu}_n - \beta \boldsymbol{\Sigma}_n \mathbf{b})}{\mathbf{1}' \mathbf{S}_n^* \mathbf{1}} + \gamma^{-1} \bar{\mathbf{y}}'_n \hat{\mathbf{Q}}_n^* (\boldsymbol{\mu}_n - \beta \boldsymbol{\Sigma}_n \mathbf{b}) - \mathbf{b}' (\boldsymbol{\mu}_n - \beta \boldsymbol{\Sigma}_n \mathbf{b})}{\frac{\mathbf{1}' \mathbf{S}_n^* \boldsymbol{\Sigma}_n \mathbf{S}_n^* \mathbf{1}}{(\mathbf{1}' \mathbf{S}_n^* \mathbf{1})^2} + 2\gamma^{-1} \frac{\bar{\mathbf{y}}'_n \hat{\mathbf{Q}}_n^* \boldsymbol{\Sigma}_n \mathbf{S}_n^* \mathbf{1}}{\mathbf{1}' \mathbf{S}_n^* \mathbf{1}} + \gamma^{-2} \bar{\mathbf{y}}'_n \hat{\mathbf{Q}}_n^* \boldsymbol{\Sigma}_n \hat{\mathbf{Q}}_n^* \bar{\mathbf{y}}_n - 2 \frac{\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{S}_n^* \mathbf{1}}{\mathbf{1}' \mathbf{S}_n^* \mathbf{1}} - 2\gamma^{-1} \mathbf{b}' \boldsymbol{\Sigma}_n \hat{\mathbf{Q}}_n^* \bar{\mathbf{y}}_n + \mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}} \\ &= \beta^{-1} \frac{A_n^+}{B_n^+}, \end{aligned}$$

where

$$\begin{aligned} A_n^+ &= \frac{1}{q_n} \left(\frac{\mathbf{1}' \mathbf{S}_n^* (\boldsymbol{\mu}_n - \beta \boldsymbol{\Sigma}_n \mathbf{b})}{\mathbf{1}' \mathbf{S}_n^* \mathbf{1}} + \gamma^{-1} \bar{\mathbf{y}}'_n \hat{\mathbf{Q}}_n^* (\boldsymbol{\mu}_n - \beta \boldsymbol{\Sigma}_n \mathbf{b}) - \mathbf{b}' (\boldsymbol{\mu}_n - \beta \boldsymbol{\Sigma}_n \mathbf{b}) \right) \\ &= \frac{(\mathbf{1}' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1/2} \sqrt{\boldsymbol{\mu}'_n \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n} (\mathbf{1}' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1/2} (\boldsymbol{\mu}'_n \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n)^{-1/2} \mathbf{1}' \mathbf{S}_n^* \boldsymbol{\mu}_n}{q_n (\mathbf{1}' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1} \mathbf{1}' \mathbf{S}_n^* \mathbf{1}} \\ &\quad - \beta \frac{(\mathbf{1}' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1/2} \sqrt{\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}} (\mathbf{1}' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1/2} (\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b})^{-1/2} \mathbf{1}' \mathbf{S}_n^* \boldsymbol{\Sigma}_n \mathbf{b}}{q_n (\mathbf{1}' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1} \mathbf{1}' \mathbf{S}_n^* \mathbf{1}} \\ &\quad + \gamma^{-1} \frac{\boldsymbol{\mu}'_n \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n}{q_n} \frac{\bar{\mathbf{y}}'_n \hat{\mathbf{Q}}_n^* \boldsymbol{\mu}_n}{\boldsymbol{\mu}'_n \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n} - \beta \gamma^{-1} \frac{\sqrt{\boldsymbol{\mu}'_n \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n} \sqrt{\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}}}{q_n} \frac{\bar{\mathbf{y}}'_n \hat{\mathbf{Q}}_n^* \boldsymbol{\Sigma}_n \mathbf{b}}{\sqrt{\boldsymbol{\mu}'_n \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n} \sqrt{\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}}} \\ &\quad - \frac{\sqrt{\boldsymbol{\mu}'_n \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n} \sqrt{\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}}}{q_n} \frac{\mathbf{b}' \boldsymbol{\mu}_n}{\sqrt{\boldsymbol{\mu}'_n \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n} \sqrt{\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}}} + \beta \frac{\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}}{q_n} \end{aligned}$$

and

$$\begin{aligned}
B_n^+ &= \frac{1}{q_n} \left(\frac{\mathbf{1}' \mathbf{S}_n^* \Sigma_n \mathbf{S}_n^* \mathbf{1}}{(\mathbf{1}' \mathbf{S}_n^* \mathbf{1})^2} + 2\gamma^{-1} \frac{\bar{\mathbf{y}}_n' \hat{\mathbf{Q}}_n^* \Sigma_n \mathbf{S}_n^* \mathbf{1}}{\mathbf{1}' \mathbf{S}_n^* \mathbf{1}} + \gamma^{-2} \bar{\mathbf{y}}_n' \hat{\mathbf{Q}}_n^* \Sigma_n \hat{\mathbf{Q}}_n^* \bar{\mathbf{y}}_n \right. \\
&\quad \left. - 2 \frac{\mathbf{b}' \Sigma_n \mathbf{S}_n^* \mathbf{1}}{\mathbf{1}' \mathbf{S}_n^* \mathbf{1}} - 2\gamma^{-1} \mathbf{b}' \Sigma_n \hat{\mathbf{Q}}_n^* \bar{\mathbf{y}}_n + \mathbf{b}' \Sigma_n \mathbf{b} \right) \\
&= \frac{(\mathbf{1}' \Sigma_n^{-1} \mathbf{1})^{-1}}{q_n} \frac{(\mathbf{1}' \Sigma_n^{-1} \mathbf{1})^{-1} \mathbf{1}' \mathbf{S}_n^* \Sigma_n \mathbf{S}_n^* \mathbf{1}}{(\mathbf{1}' \Sigma_n^{-1} \mathbf{1})^{-2} (\mathbf{1}' \mathbf{S}_n^* \mathbf{1})^2} \\
&\quad + 2\gamma^{-1} \frac{(\mathbf{1}' \Sigma_n^{-1} \mathbf{1})^{-1/2} \sqrt{\boldsymbol{\mu}_n' \Sigma_n^{-1} \boldsymbol{\mu}_n}}{q_n} \frac{(\mathbf{1}' \Sigma_n^{-1} \mathbf{1})^{-1/2} (\boldsymbol{\mu}_n' \Sigma_n^{-1} \boldsymbol{\mu}_n)^{-1/2} \bar{\mathbf{y}}_n' \hat{\mathbf{Q}}_n^* \Sigma_n \mathbf{S}_n^* \mathbf{1}}{(\mathbf{1}' \Sigma_n^{-1} \mathbf{1})^{-1} \mathbf{1}' \mathbf{S}_n^* \mathbf{1}} \\
&\quad + \gamma^{-2} \frac{\boldsymbol{\mu}_n' \Sigma_n^{-1} \boldsymbol{\mu}_n}{q_n} \frac{\bar{\mathbf{y}}_n' \hat{\mathbf{Q}}_n^* \Sigma_n \hat{\mathbf{Q}}_n^* \bar{\mathbf{y}}_n}{\boldsymbol{\mu}_n' \Sigma_n^{-1} \boldsymbol{\mu}_n} \\
&\quad - 2 \frac{(\mathbf{1}' \Sigma_n^{-1} \mathbf{1})^{-1/2} \sqrt{\mathbf{b}' \Sigma_n \mathbf{b}}}{q_n} \frac{(\mathbf{1}' \Sigma_n^{-1} \mathbf{1})^{-1/2} (\mathbf{b}' \Sigma_n \mathbf{b})^{-1/2} \mathbf{b}' \Sigma_n \mathbf{S}_n^* \mathbf{1}}{(\mathbf{1}' \Sigma_n^{-1} \mathbf{1})^{-1} \mathbf{1}' \mathbf{S}_n^* \mathbf{1}} \\
&\quad - 2\gamma^{-1} \frac{\sqrt{\boldsymbol{\mu}_n' \Sigma_n^{-1} \boldsymbol{\mu}_n} \sqrt{\mathbf{b}' \Sigma_n \mathbf{b}}}{q_n} \frac{\mathbf{b}' \Sigma_n \hat{\mathbf{Q}}_n^* \bar{\mathbf{y}}_n}{\sqrt{\boldsymbol{\mu}_n' \Sigma_n^{-1} \boldsymbol{\mu}_n} \sqrt{\mathbf{b}' \Sigma_n \mathbf{b}}} + \frac{\mathbf{b}' \Sigma_n \mathbf{b}}{q_n}.
\end{aligned}$$

Using the equalities

$$\bar{\mathbf{y}}_n = \boldsymbol{\mu}_n + \Sigma_n^{1/2} \bar{\mathbf{x}}_n \text{ and } \mathbf{S}_n^* = \Sigma_n^{-1/2} \mathbf{V}_n^+ \Sigma_n^{-1/2},$$

the facts that $q_n^{-1} \boldsymbol{\mu}_n' \Sigma_n^{-1} \boldsymbol{\mu}_n \leq 1$, $q_n^{-1} \mathbf{b}' \Sigma_n \mathbf{b} \leq 1$, $q_n^{-1} (\mathbf{1}' \Sigma_n^{-1} \mathbf{1})^{-1} \leq 1$ and that the Euclidean norms of the following vectors

$$\frac{\Sigma_n^{-1/2} \mathbf{1}}{\sqrt{\mathbf{1}' \Sigma_n^{-1} \mathbf{1}}}, \quad \frac{\Sigma_n^{-1/2} \boldsymbol{\mu}_n}{\sqrt{\boldsymbol{\mu}_n' \Sigma_n^{-1} \boldsymbol{\mu}_n}} \quad \text{and} \quad \frac{\Sigma_n^{1/2} \mathbf{b}}{\sqrt{\mathbf{b}' \Sigma_n \mathbf{b}}}$$

are equal to one, the application of Lemma 1.6 yields

$$\begin{aligned}
\frac{\mathbf{1}' \mathbf{S}_n^* \mathbf{1}}{\mathbf{1}' \Sigma_n^{-1} \mathbf{1}} &= \frac{\mathbf{1}' \Sigma_n^{-1/2} \mathbf{V}_n^+ \Sigma_n^{-1/2} \mathbf{1}}{\mathbf{1}' \Sigma_n^{-1} \mathbf{1}} \xrightarrow{a.s.} c^{-1} (c-1)^{-1}, \\
\frac{\mathbf{1}' \mathbf{S}_n^* \boldsymbol{\mu}_n}{\sqrt{\mathbf{1}' \Sigma_n^{-1} \mathbf{1}} \sqrt{\boldsymbol{\mu}_n' \Sigma_n^{-1} \boldsymbol{\mu}_n}} &= \frac{\mathbf{1}' \Sigma_n^{-1/2} \mathbf{V}_n^+ \Sigma_n^{-1/2} \boldsymbol{\mu}_n}{\sqrt{\mathbf{1}' \Sigma_n^{-1} \mathbf{1}} \sqrt{\boldsymbol{\mu}_n' \Sigma_n^{-1} \boldsymbol{\mu}_n}} \xrightarrow{a.s.} c^{-1} (c-1)^{-1} \frac{\mathbf{1}' \Sigma_n^{-1} \boldsymbol{\mu}_n}{\sqrt{\mathbf{1}' \Sigma_n^{-1} \mathbf{1}} \sqrt{\boldsymbol{\mu}_n' \Sigma_n^{-1} \boldsymbol{\mu}_n}}, \\
\frac{\mathbf{1}' \mathbf{S}_n^* \Sigma_n \mathbf{b}}{\sqrt{\mathbf{1}' \Sigma_n^{-1} \mathbf{1}} \sqrt{\mathbf{b}' \Sigma_n \mathbf{b}}} &= \frac{\mathbf{1}' \Sigma_n^{-1/2} \mathbf{V}_n^+ \Sigma_n^{1/2} \mathbf{b}}{\sqrt{\mathbf{1}' \Sigma_n^{-1} \mathbf{1}} \sqrt{\mathbf{b}' \Sigma_n \mathbf{b}}} \xrightarrow{a.s.} c^{-1} (c-1)^{-1} \frac{1}{\sqrt{\mathbf{1}' \Sigma_n^{-1} \mathbf{1}} \sqrt{\mathbf{b}' \Sigma_n \mathbf{b}}}, \\
\frac{\mathbf{1}' \mathbf{S}_n^* \Sigma_n \mathbf{S}_n^* \mathbf{1}}{\mathbf{1}' \Sigma_n^{-1} \mathbf{1}} &= \frac{\mathbf{1}' \Sigma_n^{-1/2} (\mathbf{V}_n^+)^2 \Sigma_n^{-1/2} \mathbf{1}}{\mathbf{1}' \Sigma_n^{-1} \mathbf{1}} \xrightarrow{a.s.} (c-1)^{-3}
\end{aligned}$$

for $p/n \rightarrow c \in (1, +\infty)$ as $n \rightarrow \infty$.

Finally, from Lemma 1.6 and 1.7 as well as by using the equalities

$$\hat{\mathbf{Q}}_n^* = \Sigma_n^{-1/2} \mathbf{P}_n^+ \Sigma_n^{-1/2} \quad \text{and} \quad \mathbf{P}_n^+ \mathbf{V}_n^+ = (\mathbf{V}_n^+)^2 - \frac{\mathbf{V}_n^+ \boldsymbol{\eta} \boldsymbol{\eta}' (\mathbf{V}_n^+)^2}{\boldsymbol{\eta}' \mathbf{V}_n^+ \boldsymbol{\eta}}$$

with $\boldsymbol{\eta} = \Sigma_n^{-1/2} \mathbf{1} / \sqrt{\mathbf{1}' \Sigma_n^{-1/2} \mathbf{1}}$ we obtain

$$\begin{aligned} \frac{\bar{\mathbf{y}}_n' \hat{\mathbf{Q}}_n^* \boldsymbol{\mu}_n}{\boldsymbol{\mu}_n' \Sigma_n^{-1} \boldsymbol{\mu}_n} &= \frac{\boldsymbol{\mu}_n' \Sigma_n^{-1/2} \mathbf{P}_n^+ \Sigma_n^{-1/2} \boldsymbol{\mu}_n}{\boldsymbol{\mu}_n' \Sigma_n^{-1} \boldsymbol{\mu}_n} + \frac{\bar{\mathbf{x}}_n' \mathbf{P}_n^+ \Sigma_n^{-1/2} \boldsymbol{\mu}_n}{\boldsymbol{\mu}_n' \Sigma_n^{-1} \boldsymbol{\mu}_n} \\ &\xrightarrow{a.s.} c^{-1} (c-1)^{-1} \frac{\boldsymbol{\mu}_n' \mathbf{Q}_n \boldsymbol{\mu}_n}{\boldsymbol{\mu}_n' \Sigma_n^{-1} \boldsymbol{\mu}_n}, \\ \frac{\bar{\mathbf{y}}_n' \hat{\mathbf{Q}}_n^* \Sigma_n \mathbf{b}}{\sqrt{\boldsymbol{\mu}_n' \Sigma_n^{-1} \boldsymbol{\mu}_n} \sqrt{\mathbf{b}' \Sigma_n \mathbf{b}}} &= \frac{\boldsymbol{\mu}_n' \Sigma_n^{-1/2} \mathbf{P}_n^+ \Sigma_n^{1/2} \mathbf{b}}{\sqrt{\boldsymbol{\mu}_n' \Sigma_n^{-1} \boldsymbol{\mu}_n} \sqrt{\mathbf{b}' \Sigma_n \mathbf{b}}} + \frac{\bar{\mathbf{x}}_n' \mathbf{P}_n^+ \Sigma_n^{1/2} \mathbf{b}}{\sqrt{\boldsymbol{\mu}_n' \Sigma_n^{-1} \boldsymbol{\mu}_n} \sqrt{\mathbf{b}' \Sigma_n \mathbf{b}}} \\ &\xrightarrow{a.s.} c^{-1} (c-1)^{-1} \frac{\boldsymbol{\mu}_n' \mathbf{Q}_n \Sigma_n \mathbf{b}}{\sqrt{\boldsymbol{\mu}_n' \Sigma_n^{-1} \boldsymbol{\mu}_n} \sqrt{\mathbf{b}' \Sigma_n \mathbf{b}}}, \\ \frac{\bar{\mathbf{y}}_n' \hat{\mathbf{Q}}_n^* \Sigma_n \mathbf{S}_n^* \mathbf{1}}{\sqrt{\boldsymbol{\mu}_n' \Sigma_n^{-1} \boldsymbol{\mu}_n} \sqrt{\mathbf{1}' \Sigma_n^{-1} \mathbf{1}}} &= \frac{\boldsymbol{\mu}_n' \Sigma_n^{-1/2} \mathbf{P}_n^+ \mathbf{V}_n^+ \Sigma_n^{-1/2} \mathbf{1}}{\sqrt{\boldsymbol{\mu}_n' \Sigma_n^{-1} \boldsymbol{\mu}_n} \sqrt{\mathbf{1}' \Sigma_n^{-1} \mathbf{1}}} + \frac{\bar{\mathbf{x}}_n' \mathbf{P}_n^+ \mathbf{V}_n^+ \Sigma_n^{-1/2} \mathbf{1}}{\sqrt{\boldsymbol{\mu}_n' \Sigma_n^{-1} \boldsymbol{\mu}_n} \sqrt{\mathbf{1}' \Sigma_n^{-1} \mathbf{1}}} \\ &\xrightarrow{a.s.} (c-1)^{-3} \frac{\boldsymbol{\mu}_n' \mathbf{Q}_n \mathbf{1}}{\sqrt{\boldsymbol{\mu}_n' \Sigma_n^{-1} \boldsymbol{\mu}_n} \sqrt{\mathbf{1}' \Sigma_n^{-1} \mathbf{1}}} = 0, \\ \frac{\bar{\mathbf{y}}_n' \hat{\mathbf{Q}}_n^* \Sigma_n \hat{\mathbf{Q}}_n^* \bar{\mathbf{y}}_n}{\boldsymbol{\mu}_n' \Sigma_n^{-1} \boldsymbol{\mu}_n} &= \frac{\bar{\mathbf{x}}_n' (\mathbf{P}_n^+)^2 \bar{\mathbf{x}}_n}{\boldsymbol{\mu}_n' \Sigma_n^{-1} \boldsymbol{\mu}_n} + 2 \frac{\boldsymbol{\mu}_n' \Sigma_n^{-1/2} (\mathbf{P}_n^+)^2 \bar{\mathbf{x}}_n}{\boldsymbol{\mu}_n' \Sigma_n^{-1} \boldsymbol{\mu}_n} + \frac{\boldsymbol{\mu}_n' \Sigma_n^{-1/2} (\mathbf{P}_n^+)^2 \Sigma_n^{-1/2} \boldsymbol{\mu}_n}{\boldsymbol{\mu}_n' \Sigma_n^{-1} \boldsymbol{\mu}_n} \\ &\xrightarrow{a.s.} \frac{(c-1)^{-3} c^2 + (c-1)^{-3} \boldsymbol{\mu}_n' \mathbf{Q}_n \boldsymbol{\mu}_n}{\boldsymbol{\mu}_n' \Sigma_n^{-1} \boldsymbol{\mu}_n} \end{aligned}$$

for $p/n \rightarrow c \in (1, +\infty)$ as $n \rightarrow \infty$.

Substituting the above results into the expressions of A_n^* and B_n^* , we get that

$$|A_n^* - A^*| \xrightarrow{a.s.} 0 \quad \text{and} \quad |B_n^* - B^*| \xrightarrow{a.s.} 0$$

with

$$\begin{aligned} A^+ &= \frac{(\mathbf{1}' \Sigma_n^{-1} \mathbf{1})^{-1/2} \sqrt{\boldsymbol{\mu}_n' \Sigma_n^{-1} \boldsymbol{\mu}_n}}{q_n} \frac{(\mathbf{1}' \Sigma_n^{-1} \mathbf{1})^{-1/2} (\boldsymbol{\mu}_n' \Sigma_n^{-1} \boldsymbol{\mu}_n)^{-1/2} c^{-1} (c-1)^{-1} \mathbf{1}' \Sigma_n^{-1} \boldsymbol{\mu}_n}{c^{-1} (c-1)^{-1}} \\ &\quad - \beta \frac{(\mathbf{1}' \Sigma_n^{-1} \mathbf{1})^{-1/2} \sqrt{\mathbf{b}' \Sigma_n \mathbf{b}}}{q_n} \frac{(\mathbf{1}' \Sigma_n^{-1} \mathbf{1})^{-1/2} (\mathbf{b}' \Sigma_n \mathbf{b})^{-1/2} c^{-1} (c-1)^{-1}}{c^{-1} (c-1)^{-1}} \\ &\quad + \gamma^{-1} \frac{\boldsymbol{\mu}_n' \Sigma_n^{-1} \boldsymbol{\mu}_n}{q_n} \frac{c^{-1} (c-1)^{-1} \boldsymbol{\mu}_n' \mathbf{Q}_n \boldsymbol{\mu}_n}{\boldsymbol{\mu}_n' \Sigma_n^{-1} \boldsymbol{\mu}_n} - \beta \gamma^{-1} \frac{\sqrt{\boldsymbol{\mu}_n' \Sigma_n^{-1} \boldsymbol{\mu}_n} \sqrt{\mathbf{b}' \Sigma_n \mathbf{b}}}{q_n} \frac{c^{-1} (c-1)^{-1} \boldsymbol{\mu}_n' \mathbf{Q}_n \Sigma_n \mathbf{b}}{\sqrt{\boldsymbol{\mu}_n' \Sigma_n^{-1} \boldsymbol{\mu}_n} \sqrt{\mathbf{b}' \Sigma_n \mathbf{b}}} \\ &\quad - \frac{\sqrt{\boldsymbol{\mu}_n' \Sigma_n^{-1} \boldsymbol{\mu}_n} \sqrt{\mathbf{b}' \Sigma_n \mathbf{b}}}{q_n} \frac{\mathbf{b}' \boldsymbol{\mu}_n}{\sqrt{\boldsymbol{\mu}_n' \Sigma_n^{-1} \boldsymbol{\mu}_n} \sqrt{\mathbf{b}' \Sigma_n \mathbf{b}}} + \beta \frac{\mathbf{b}' \Sigma_n \mathbf{b}}{q_n} \\ &= \frac{1}{q_n} \left(\frac{\mathbf{1}' \Sigma_n^{-1} \boldsymbol{\mu}_n}{\mathbf{1}' \Sigma_n^{-1} \mathbf{1}} - \beta \frac{1}{\mathbf{1}' \Sigma_n^{-1} \mathbf{1}} + \frac{\gamma^{-1}}{c(c-1)} \boldsymbol{\mu}_n' \mathbf{Q}_n \boldsymbol{\mu}_n - \frac{\gamma^{-1} \beta}{c(c-1)} \boldsymbol{\mu}_n' \mathbf{Q}_n \Sigma_n \mathbf{b} - \mathbf{b}' \boldsymbol{\mu}_n + \beta \mathbf{b}' \Sigma_n \mathbf{b} \right) \end{aligned}$$

and

$$\begin{aligned}
B^+ &= \frac{(\mathbf{1}'\Sigma_n^{-1}\mathbf{1})^{-1}}{q_n} \frac{(c-1)^{-3}}{c^{-2}(c-1)^{-2}} + 2\gamma^{-1} \frac{(\mathbf{1}'\Sigma_n^{-1}\mathbf{1})^{-1/2} \sqrt{\mu_n' \Sigma_n^{-1} \mu_n}}{q_n} \frac{0}{c^{-1}(c-1)^{-1}} \\
&+ \gamma^{-2} \frac{\mu_n' \Sigma_n^{-1} \mu_n}{q_n} \frac{(c-1)^{-3} c^2 + (c-1)^{-3} \mu_n' \mathbf{Q}_n \mu_n}{\mu_n' \Sigma_n^{-1} \mu_n} \\
&- 2 \frac{(\mathbf{1}'\Sigma_n^{-1}\mathbf{1})^{-1/2} \sqrt{\mathbf{b}' \Sigma_n \mathbf{b}}}{q_n} \frac{(\mathbf{1}'\Sigma_n^{-1}\mathbf{1})^{-1/2} (\mathbf{b}' \Sigma_n \mathbf{b})^{-1/2} c^{-1} (c-1)^{-1}}{c^{-1}(c-1)^{-1}} \\
&- 2\gamma^{-1} \frac{\sqrt{\mu_n' \Sigma_n^{-1} \mu_n} \sqrt{\mathbf{b}' \Sigma_n \mathbf{b}}}{q_n} \frac{c^{-1}(c-1)^{-1} \mathbf{b}' \Sigma_n \mathbf{Q}_n \mu_n}{\sqrt{\mu_n' \Sigma_n^{-1} \mu_n} \sqrt{\mathbf{b}' \Sigma_n \mathbf{b}}} + \frac{\mathbf{b}' \Sigma_n \mathbf{b}}{q_n} \\
&= \frac{1}{q_n} \left(\frac{c^2}{c-1} \frac{1}{\mathbf{1}'\Sigma_n^{-1}\mathbf{1}} + \frac{\gamma^{-2}}{(c-1)^3} (c^2 + \mu_n' \mathbf{Q}_n \mu_n) - 2 \frac{1}{\mathbf{1}'\Sigma_n^{-1}\mathbf{1}} \right. \\
&\left. - 2 \frac{\gamma^{-1}}{c(c-1)} \left(\mathbf{b}' \mu_n - \frac{\mathbf{1}'\Sigma_n^{-1} \mu_n}{\mathbf{1}'\Sigma_n^{-1}\mathbf{1}} \right) + \mathbf{b}' \Sigma_n \mathbf{b} \right).
\end{aligned}$$

Let $\alpha^+ = A^+/B^+$. Hence,

$$|\alpha_n^+ - \alpha^+| \leq \left| \frac{1}{B_n^+} (A_n^+ - A^+) \right| + \left| \frac{A^+}{B^+ B_n^+} (B^+ - B_n^+) \right| \xrightarrow{a.s.} 0.$$

This completes the proof of Theorem 2.2. □

Proof of Corollary 2.2: (a) With $q_n = \max\{\mu_n' \Sigma_n^{-1} \mu_n, \mathbf{b}' \Sigma_n \mathbf{b}\}$ we get

$$\frac{1}{q_n} U_S = \frac{1}{q_n} \hat{\mathbf{w}}'_{S^*} \mu_n - \frac{\gamma}{2} \frac{1}{q_n} \hat{\mathbf{w}}'_{S^*} \Sigma_n \hat{\mathbf{w}}_{S^*},$$

where from the proof of Theorem 2.2 we have

$$\begin{aligned}
\frac{1}{q_n} \hat{\mathbf{w}}'_{S^*} \mu_n &= \frac{1}{q_n} \frac{\mathbf{1}' \mathbf{S}_n^* \mu_n}{\mathbf{1}' \mathbf{S}_n^* \mathbf{1}} + \gamma^{-1} \frac{1}{q_n} \bar{\mathbf{y}}_n' \hat{\mathbf{Q}}_n^* \mu_n \\
&= \frac{(\mathbf{1}'\Sigma_n^{-1}\mathbf{1})^{-1/2} \sqrt{\mu_n' \Sigma_n^{-1} \mu_n}}{q_n} \frac{(\mathbf{1}'\Sigma_n^{-1}\mathbf{1})^{-1/2} (\mu_n' \Sigma_n^{-1} \mu_n)^{-1/2} \mathbf{1}' \mathbf{S}_n^* \mu_n}{(\mathbf{1}'\Sigma_n^{-1}\mathbf{1})^{-1} \mathbf{1}' \mathbf{S}_n^* \mathbf{1}} \\
&+ \gamma^{-1} \frac{\mu_n' \Sigma_n^{-1} \mu_n}{q_n} \frac{\bar{\mathbf{y}}_n' \hat{\mathbf{Q}}_n^* \mu_n}{\mu_n' \Sigma_n^{-1} \mu_n} \\
&\xrightarrow{a.s.} \frac{(\mathbf{1}'\Sigma_n^{-1}\mathbf{1})^{-1/2} \sqrt{\mu_n' \Sigma_n^{-1} \mu_n}}{q_n} \frac{(\mathbf{1}'\Sigma_n^{-1}\mathbf{1})^{-1/2} (\mu_n' \Sigma_n^{-1} \mu_n)^{-1/2} c^{-1} (c-1)^{-1} \mathbf{1}' \Sigma_n^{-1} \mu_n}{c^{-1}(c-1)^{-1}} \\
&+ \gamma^{-1} c^{-1} (c-1)^{-1} \frac{\mu_n' \Sigma_n^{-1} \mu_n}{q_n} \frac{\mu_n' \mathbf{Q}_n \mu_n}{\mu_n' \Sigma_n^{-1} \mu_n} = \frac{1}{q_n} (R_{GMV} + \gamma^{-1} c^{-1} (c-1)^{-1} s)
\end{aligned}$$

and

$$\begin{aligned}
\hat{\mathbf{w}}'_{S^*} \Sigma_n \hat{\mathbf{w}}_{S^*} &= \frac{\mathbf{1}' \mathbf{S}_n^* \Sigma_n \mathbf{S}_n^* \mathbf{1}}{(\mathbf{1}' \mathbf{S}_n^* \mathbf{1})^2} + 2\gamma^{-1} \frac{\mathbf{1}' \mathbf{S}_n^* \Sigma_n \hat{\mathbf{Q}}_n^* \bar{\mathbf{y}}_n}{\mathbf{1}' \mathbf{S}_n^* \mathbf{1}} + \gamma^{-2} \bar{\mathbf{y}}_n' \hat{\mathbf{Q}}_n^* \Sigma_n \hat{\mathbf{Q}}_n^* \bar{\mathbf{y}}_n \\
&= \frac{(\mathbf{1}' \Sigma_n^{-1} \mathbf{1})^{-1}}{q_n} \frac{(\mathbf{1}' \Sigma_n^{-1} \mathbf{1})^{-1} \mathbf{1}' \mathbf{S}_n^* \Sigma_n \mathbf{S}_n^* \mathbf{1}}{(\mathbf{1}' \Sigma_n^{-1} \mathbf{1})^{-2} (\mathbf{1}' \mathbf{S}_n^* \mathbf{1})^2} + \gamma^{-2} \frac{\boldsymbol{\mu}'_n \Sigma_n^{-1} \boldsymbol{\mu}_n}{q_n} \frac{\bar{\mathbf{y}}_n' \hat{\mathbf{Q}}_n^* \Sigma_n \hat{\mathbf{Q}}_n^* \bar{\mathbf{y}}_n}{\boldsymbol{\mu}'_n \Sigma_n^{-1} \boldsymbol{\mu}_n} \\
&+ 2\gamma^{-1} \frac{(\mathbf{1}' \Sigma_n^{-1} \mathbf{1})^{-1/2} \sqrt{\boldsymbol{\mu}'_n \Sigma_n^{-1} \boldsymbol{\mu}_n}}{q_n} \frac{(\mathbf{1}' \Sigma_n^{-1} \mathbf{1})^{-1/2} (\boldsymbol{\mu}'_n \Sigma_n^{-1} \boldsymbol{\mu}_n)^{-1/2} \mathbf{1}' \mathbf{S}_n^* \Sigma_n \hat{\mathbf{Q}}_n^* \bar{\mathbf{y}}_n}{(\mathbf{1}' \Sigma_n^{-1} \mathbf{1})^{-1} \mathbf{1}' \mathbf{S}_n^* \mathbf{1}} \\
&\xrightarrow{a.s.} \frac{(\mathbf{1}' \Sigma_n^{-1} \mathbf{1})^{-1}}{q_n} \frac{(c-1)^{-3}}{c^{-2}(c-1)^{-2}} + \gamma^{-2} \frac{\boldsymbol{\mu}'_n \Sigma_n^{-1} \boldsymbol{\mu}_n}{q_n} \frac{(c-1)^{-3} c^2 + (c-1)^{-3} \boldsymbol{\mu}'_n \mathbf{Q}_n \boldsymbol{\mu}_n}{\boldsymbol{\mu}'_n \Sigma_n^{-1} \boldsymbol{\mu}_n} \\
&+ 2\gamma^{-1} \frac{(\mathbf{1}' \Sigma_n^{-1} \mathbf{1})^{-1/2} \sqrt{\boldsymbol{\mu}'_n \Sigma_n^{-1} \boldsymbol{\mu}_n}}{q_n} \frac{0}{c^{-1}(c-1)^{-1}} \\
&= \frac{1}{q_n} \left(c^2 (c-1)^{-1} V_{GMV} + \gamma^{-2} \frac{c^2}{(c-1)^3} + \gamma^{-2} (c-1)^{-3} s \right)
\end{aligned}$$

for $p/n \rightarrow c \in (1, +\infty)$ as $n \rightarrow \infty$.

Finally, using that

$$\frac{1}{q_n} U_{EU} = \frac{1}{q_n} R_{GMV} + \frac{1}{2} \gamma^{-1} \frac{1}{q_n} s - \frac{\gamma}{2} \frac{1}{q_n} V_{GMV},$$

we obtain the statement of the first part of the corollary.

(b) It holds that

$$\begin{aligned}
\frac{1}{q_n} U_{GSE} &= \alpha^+ \frac{1}{q_n} \hat{\mathbf{w}}'_{S^*} \boldsymbol{\mu}_n + (1 - \alpha^+) \frac{1}{q_n} \mathbf{b}' \boldsymbol{\mu}_n \\
&- \frac{\gamma}{2} \left((\alpha^+)^2 \frac{1}{q_n} \hat{\mathbf{w}}'_{S^*} \Sigma_n \hat{\mathbf{w}}_{S^*} + 2\alpha^+ (1 - \alpha^+) \frac{1}{q_n} \mathbf{b}' \Sigma_n \hat{\mathbf{w}}_{S^*} + (1 - \alpha^+)^2 \frac{1}{q_n} \mathbf{b}' \Sigma_n \mathbf{b} \right),
\end{aligned}$$

where

$$\begin{aligned}
\frac{1}{q_n} \mathbf{b}' \Sigma_n \hat{\mathbf{w}}_{S^*} &= \frac{1}{q_n} \frac{\mathbf{b}' \Sigma_n \mathbf{S}_n^* \mathbf{1}}{\mathbf{1}' \mathbf{S}_n^* \mathbf{1}} + \gamma^{-1} \frac{1}{q_n} \mathbf{b}' \Sigma_n \hat{\mathbf{Q}}_n^* \bar{\mathbf{y}}_n \\
&= \frac{(\mathbf{1}' \Sigma_n^{-1} \mathbf{1})^{-1/2} \sqrt{\mathbf{b}' \Sigma_n \mathbf{b}}}{q_n} \frac{(\mathbf{1}' \Sigma_n^{-1} \mathbf{1})^{-1/2} (\mathbf{b}' \Sigma_n \mathbf{b})^{-1/2} \mathbf{1}' \mathbf{S}_n^* \Sigma_n \mathbf{b}}{(\mathbf{1}' \Sigma_n^{-1} \mathbf{1})^{-1} \mathbf{1}' \mathbf{S}_n^* \mathbf{1}} \\
&+ \gamma^{-1} \frac{\sqrt{\boldsymbol{\mu}'_n \Sigma_n^{-1} \boldsymbol{\mu}_n} \sqrt{\mathbf{b}' \Sigma_n \mathbf{b}}}{q_n} \frac{\mathbf{b}' \Sigma_n \hat{\mathbf{Q}}_n^* \bar{\mathbf{y}}_n}{\sqrt{\boldsymbol{\mu}'_n \Sigma_n^{-1} \boldsymbol{\mu}_n} \sqrt{\mathbf{b}' \Sigma_n \mathbf{b}}} \\
&\xrightarrow{a.s.} \frac{(\mathbf{1}' \Sigma_n^{-1} \mathbf{1})^{-1/2} \sqrt{\mathbf{b}' \Sigma_n \mathbf{b}}}{q_n} \frac{(\mathbf{1}' \Sigma_n^{-1} \mathbf{1})^{-1/2} (\mathbf{b}' \Sigma_n \mathbf{b})^{-1/2} c^{-1} (c-1)^{-1}}{c^{-1} (c-1)^{-1}} \\
&+ \gamma^{-1} \frac{\sqrt{\boldsymbol{\mu}'_n \Sigma_n^{-1} \boldsymbol{\mu}_n} \sqrt{\mathbf{b}' \Sigma_n \mathbf{b}}}{q_n} \frac{c^{-1} (c-1)^{-1} \left(\mathbf{b}' \boldsymbol{\mu}_n - \frac{\mathbf{1}' \Sigma_n^{-1} \boldsymbol{\mu}_n}{\mathbf{1}' \Sigma_n^{-1} \mathbf{1}} \right)}{\sqrt{\boldsymbol{\mu}'_n \Sigma_n^{-1} \boldsymbol{\mu}_n} \sqrt{\mathbf{b}' \Sigma_n \mathbf{b}}} \\
&= \frac{1}{q_n} (V_{GMV} + \gamma^{-1} c^{-1} (c-1)^{-1} (R_b - R_{GMV}))
\end{aligned}$$

for $p/n \rightarrow c \in (1, +\infty)$ as $n \rightarrow \infty$.

Hence,

$$\begin{aligned}
& \frac{1}{q_n}(U_{EU} - U_{GSE}) = (\alpha^+)^2 \frac{1}{q_n} U_{EU} + (1 - \alpha^+)^2 \frac{1}{q_n} U_{EU} + 2\alpha^+(1 - \alpha^+) \frac{1}{q_n} U_{EU} \\
& - (\alpha^+)^2 \frac{1}{q_n} U_S - \alpha^+(1 - \alpha^+) \frac{1}{q_n} \hat{\mathbf{w}}'_{S^*} \boldsymbol{\mu}_n - (1 - \alpha^+)^2 \frac{1}{q_n} U_{\mathbf{b}} - \alpha^+(1 - \alpha^+) \frac{1}{q_n} \mathbf{b}' \boldsymbol{\mu}_n + \gamma \alpha^+(1 - \alpha^+) \frac{1}{q_n} \mathbf{b}' \boldsymbol{\Sigma}_n \hat{\mathbf{w}}_{S^*} \\
& \xrightarrow{a.s.} (\alpha^+)^2 \frac{1}{q_n} (U_{EU} - U_S) + (1 - \alpha^+)^2 \frac{1}{q_n} (U_{EU} - U_{\mathbf{b}}) \\
& + \alpha^+(1 - \alpha^+) \frac{1}{q_n} \left(2R_{GMV} + \gamma^{-1}s - \gamma V_{GMV} - R_{GMV} - \frac{\gamma^{-1}c^{-1}}{c-1} s - R_b + \gamma V_{GMV} + \frac{R_b - R_{GMV}}{c(c-1)} \right) \\
& = (\alpha^+)^2 \frac{1}{q_n} (U_{EU} - U_S) + (1 - \alpha^+)^2 \frac{1}{q_n} (U_{EU} - U_{\mathbf{b}}) + \alpha^+(1 - \alpha^+) \frac{1+c-c^2}{c(c-1)} \frac{1}{q_n} (R_b - R_{GMV} - \gamma^{-1}s).
\end{aligned}$$

for $p/n \rightarrow c \in (0, 1)$ as $n \rightarrow \infty$. □

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