Noncommutative and Vector-valued Rosenthal Inequalities

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Introduction

This thesis is dedicated to the study of a class of probabilistic inequalities, called *Rosenthal inequalities*. These inequalities provide two-sided estimates for the p-th moments of the sum of a sequence of independent, mean zero random variables, in terms of a suitable norm on the sequence itself. Rosenthal inequalities are named after the mathematician H.P. Rosenthal, who first discovered them for scalar-valued random variables around 1970. The original version of his result ([121], Theorem 3) reads as follows. If $2 \leq p < \infty$ and (f_i) is a sequence of independent, mean zero random variables in $L^p(\Omega)$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, then

$$\begin{aligned}\Big(\mathbb{E}\Big|\sum_{i=1}^n f_i\Big|^p\Big)^{\frac{1}{p}} &\leq c_p \max\Big\{\Big(\sum_{i=1}^n \mathbb{E}|f_i|^p\Big)^{\frac{1}{p}}, \Big(\sum_{i=1}^n \mathbb{E}|f_i|^2\Big)^{\frac{1}{2}}\Big\}, \\ \Big(\mathbb{E}\Big|\sum_{i=1}^n f_i\Big|^p\Big)^{\frac{1}{p}} &\geq \frac{1}{2} \max\Big\{\Big(\sum_{i=1}^n \mathbb{E}|f_i|^p\Big)^{\frac{1}{p}}, \Big(\sum_{i=1}^n \mathbb{E}|f_i|^2\Big)^{\frac{1}{2}}\Big\}.\end{aligned} \tag{0.1}$$

Here c_p is a constant depending only on p. Rosenthal's motivation for deriving these inequalities was to gain insight into which Banach spaces are linearly isomorphic to a complemented subspace of an L^p-space. Using (0.1), he showed that the span of a sequence of independent, mean zero random variables in $L^p(\Omega)$ is isomorphic to either $l^2, L^p, l^2 \oplus l^p$, or a novel space X_p, which is nowadays called *Rosenthal's space*. The latter space has several interesting properties. For example, it is isomorphic to both a complemented and an uncomplemented subspace of L^p. The space X_p is also important from a historical perspective. It was the first example of an $\mathscr{L}_p$-space which cannot be obtained by taking direct sums of l^2, l^p, and L^p.

Rosenthal's inequalities soon became a standard tool for probabilists and were subjected to intensive study. Several authors have investigated the best constant c_p appearing in the first inequality of (0.1). The constant c_p obtained in the original proof grows exponentially in p as $p \to \infty$. It was first determined by Johnson, Schechtman and Zinn [66] that the optimal order of growth is given by $\frac{p}{\log p}$. Different proofs of this fact are given in [10, 92]. The exact value of the best constant was subsequently determined in the special case where the f_i are, in addition, symmetric. Let n_{s} be a symmetrized Poisson random variable with parameter 1 and let g be a standard Gaussian random variable. It was shown by Utev [137] that $c_p = (\mathbb{E}|n_{\mathrm{s}}|^p)^{\frac{1}{p}}$ if $p > 4$ and by Figiel et al. [54] that $c_p = (1 + \mathbb{E}|g|^p)^{\frac{1}{p}}$ if $2 < p \leq 4$. Of course, $c_2 = 1$.

Rosenthal's inequalities have been extended in various directions. Soon after their discovery, Burkholder obtained a generalization in which the sequence (f_i) is replaced by a martingale difference sequence [27, 29]. These inequalities are usually called the Burkholder-Rosenthal inequalities. In another direction, several authors have considered versions of (0.1) in which the L^p-norm on $\sum_{i=1}^n f_i$ is replaced by the norm of a rearrangement invariant Banach function space [9, 64, 65].

We shall be interested in extending Rosenthal's inequalities in two different directions. In the first part of this thesis, we consider the situation in which the random variables f_i are *vector-valued*, i.e., they take values in a Banach space X. The inequalities we develop in this setting are principally designed to prove a novel Itô isomorphism for vector-valued stochastic integrals with respect to a compensated Poisson random measure. This isomorphism provides a key tool for the analysis of stochastic partial differential equations.

The second part of this thesis deals with the situation where the random variables f_i are replaced by *noncommutative* random variables. More precisely, we suppose that the f_i are elements of a noncommutative symmetric space associated with a von Neumann algebra. The noncommutative Rosenthal inequalities we establish are utilized to prove Itô isomorphisms for noncommutative stochastic integrals with respect to certain Brownian motions. These isomorphisms provide a tool to understand noncommutative (or quantum) stochastic differential equations.

We now describe our main results in these two directions and their applications in detail.

Stochastic partial differential equations

Many phenomena in physics, biology and financial mathematics can be described mathematically in the form of stochastic partial differential equations (SPDEs), i.e., partial differential equations driven by a random noise process. One can think here of models describing the erratic behavior of particles immersed in a fluid, turbulence, environmental pollution and the dynamics of financial instruments deriving their value from interest rates. In the functional analytic approach to SPDEs, one reformulates an SPDE as a stochastic ordinary differential equation (SDE) in a suitable infinite-dimensional state space X. This approach was pioneered by the schools of G. da Prato and J. Zabczyk for SPDEs driven by Gaussian noise [38]. By considering SDEs in Hilbert spaces they obtained existence and uniqueness results for a large class of SPDEs. For the study of the regularity of solutions to SPDEs, i.e., the smoothness of their paths, one must go beyond the framework of Hilbert spaces. Typically, one would like to consider SDEs in an L^p-space, or in a derived space such as an extrapolation space of an L^p-space or a Sobolev space, so that the regularity of a solution can be determined using the Sobolev embedding theorems. For this purpose, J. van Neerven, M. Veraar and L. Weis investigated the existence of Itô-type isomorphisms for stochastic integrals taking values in a general Banach space X. In [106] they showed that if X is a UMD Banach space and W is a standard Brownian motion, then for any $1 < p < \infty$ there exist constants $c_{p,X}, C_{p,X} > 0$, depending only on p and X, such that

$$c_{p,X}(\mathbb{E}\|R_F\|^p_{\gamma(0,t;X)})^{\frac{1}{p}} \leq \Big(\mathbb{E}\Big\|\int_0^t F_s\ dW_s\Big\|^p_X\Big)^{\frac{1}{p}} \leq C_{p,X}(\mathbb{E}\|R_F\|^p_{\gamma(0,t;X)})^{\frac{1}{p}}. \tag{0.2}$$

Here $R_F : \Omega \to \gamma(0,t;X)$ is the random integral operator

$$(R_F g)(\omega) := \int_0^t F_s(\omega)g(s)\ ds \qquad (\omega \in \Omega, g \in L^2(0,t))$$

associated with an adapted X-valued process F and $\gamma(0,t;X)$ is the space of γ-*radonifying* operators from $L^2(0,t)$ into X. These estimates give an exact description of the class of integrands F for which the p-th moment of the stochastic integral $\int_0^t F_s\ dW_s$ is finite.

The right hand side estimate in (0.2) is used to perform a fixed point argument that establishes the existence and uniqueness of solutions to abstract stochastic equations in the space X. In combination with Sobolev embedding theorems, the estimate also provides regularity of solutions and optimal convergence rates for numerical schemes. The left hand side estimate shows that the right hand side estimate is the best possible. This optimality proved crucial in the recent solution of the maximal regularity problem for SPDEs driven by Brownian motions [105]. Maximal regularity results in turn provide a powerful tool to study nonlinear SPDEs.

In the recent years there has been increased interest in SPDEs driven by Lévy noise. We refer to [111] for an introduction to the subject. For an effective treatment of such equations one again needs an Itô-type isomorphism as in (0.2) for stochastic integrals with respect to Lévy noise. By the celebrated Lévy-Itô decomposition, every scalar-valued Lévy process L can be written as

$$L_t = ct + \sigma W_t + \int_{\{|x|<1\}} x\ d\tilde{N}(t,dx) + \int_{\{|x|\geq 1\}} x\ dN(t,dx), \tag{0.3}$$

where c, σ are scalars, W is a standard Brownian motion, N is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R} - \{0\}$ which is independent of W, and $\tilde{N}$ is the compensated Poisson random measure associated with N (see e.g. [5]). Roughly speaking, this means that any Lévy process is the sum of a deterministic drift, a Brownian motion, and two integrals describing the small and large jumps of the process, respectively. For the problem of determining Itô-type isomorphisms for Lévy stochastic integrals, the decomposition suggests that one should first consider stochastic integrals with respect to Brownian motion and (compensated) Poisson random measures. The case of Brownian motion being well-understood, we consider the following question. Suppose we are given a compensated Poisson random measure $\tilde{N}$ on $\mathbb{R}_+ \times J$, where J is a σ-finite measure space (the 'jump space'), and a simple, adapted X-valued process F. Can one find a suitable norm $\|\|\cdot\|\|_{p,X}$ on the integrand F such that

$$c_{p,X} \, |||F|||_{p,X} \leq \Big(\mathbb{E}\Big\| \int_{\mathbb{R}_+ \times J} F \, d\tilde{N} \Big\|_X^p \Big)^{\frac{1}{p}} \leq C_{p,X} \, |||F|||_{p,X} \,, \tag{0.4}$$

for constants $c_{p,X}, C_{p,X}$ depending only on p and X?

In contrast to the well-established theory for Brownian motion, moment estimates for stochastic integrals of vector-valued processes with respect to Lévy processes are poorly developed even in finite-dimensional state spaces (see [89] for the best known result in this case). Previous approaches to this problem have yielded only (non-optimal) one-sided estimates under additional assumptions on the martingale type of the Banach space X [25, 31, 58, 143]. Other approaches define only weak or Pettis-type stochastic integrals and provide no moment estimates at all [4, 120, 122].

In Chapter 2 it will be demonstrated that one can find an Itô-type isomorphism (0.4) when X is a Hilbert space, an L^q-space, or even a non-commutative L^q-space, with $1 < q < \infty$. If X is an L^q-space it turns out that the norm $|||\cdot|||_{p,L^q}$ can always be expressed in terms of the norms of the three spaces $L^p(\Omega; L^q(S; L^2(\mathbb{R}_+ \times J)))$, $L^p(\Omega; L^p(\mathbb{R}_+ \times J; L^q(S)))$, and $L^p(\Omega; L^q(\mathbb{R}_+ \times J; L^q(S)))$ and takes a different form depending on the relative position of the parameters p, q and 2. For example, if $2 \leq q \leq p < \infty$, then $|||\cdot|||_{p,L^q}$ is given by the maximum of these three norms. The complete statement of our results can be found in Corollary 2.18 and Theorem 2.31. These Itô-isomorphisms can be combined with (0.2) to obtain Itô-type isomorphisms for stochastic integrals with respect to a Lévy process L through the Lévy-Itô decomposition, imposing suitable assumptions on the Lévy measure of L. This is sketched in Chapter 2 and will be explained in detail in forthcoming work.

Loosely speaking, our approach to (0.4) consists of two main steps. First we 'decouple' the stochastic integral, i.e., we use that if $1 < p < \infty$ and X is a UMD space, then there exist constants $c_{p,X}, C_{p,X} > 0$, depending only on p and X, such that

$$\begin{aligned} c_{p,X} \Big(\mathbb{E}\Big\| \int_{\mathbb{R}_+ \times J} F \, d\tilde{N}_c \Big\|_X^p \Big)^{\frac{1}{p}} &\leq \Big(\mathbb{E}\Big\| \int_{\mathbb{R}_+ \times J} F \, d\tilde{N} \Big\|_X^p \Big)^{\frac{1}{p}} \\ &\leq C_{p,X} \Big(\mathbb{E}\Big\| \int_{\mathbb{R}_+ \times J} F \, d\tilde{N}_c \Big\|_X^p \Big)^{\frac{1}{p}}, \end{aligned}$$

where $\tilde{N}_c$ is a copy of $\tilde{N}$ which is independent of F. Decoupling inequalities are a key ingredient in the proof of (0.2) [106, 138] and can be traced back to the work of Garling [55] and McConnell [101] (see also [91]). For a simple, adapted process F the decoupled stochastic integral is a sum of independent, mean zero L^q-valued random variables. Thus, the main work in proving (0.4) for L^q-spaces is to find Rosenthal-type inequalities for random vectors in L^q-spaces which yield the desired Itô isomorphism.

Vector-valued Rosenthal inequalities

Some of the most general inequalities for a sum of vector-valued random variables known in the literature were discovered by J. Hoffmann-Jørgensen [59]. He showed that, if X is a Banach space and (ξ_i) is a finite sequence of independent, mean zero X-valued random variables, then there is a constant $c > 0$ and, for all $1 \leq p < \infty$, a constant $C_p > 0$ depending only on p such that

$$\begin{aligned} \Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_X^p\Big)^{\frac{1}{p}} &\leq C_p\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_X + \Big(\mathbb{E}\max_i \|\xi_i\|_X^p\Big)^{\frac{1}{p}}\Big), \\ \Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_X^p\Big)^{\frac{1}{p}} &\geq c\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_X + \Big(\mathbb{E}\max_i \|\xi_i\|_X^p\Big)^{\frac{1}{p}}\Big). \end{aligned} \tag{0.5}$$

The original proof yields a constant C_p which grows exponentially in p. Different proofs were found by M. Talagrand ([133], see also [95]) and S. Kwapień and J. Szulga ([90], see also [91]), which yield that the optimal order of C_p is $\frac{p}{\log p}$ as $p \to \infty$. The inequalities in (0.5) can be considered as a significant generalization of (0.1) to the vector-valued case. For practical purposes, however, Hoffmann-Jørgensen's result lacks the power of Rosenthal's original inequalities, as it provides no direct way to compute the p-th moment $(\mathbb{E}\|\sum_i \xi_i\|_X^p)^{\frac{1}{p}}$ in terms of the *individual* elements ξ_i. It merely reduces the problem to computing two different quantities, which may not be a simpler task.

Motivated by our application to Poisson stochastic integration, we wish to obtain a generalization which captures the original flavour of Rosenthal's inequalities. More specifically, we consider the following question: given $1 \leq p < \infty$, a Banach space X and a finite sequence (ξ_i) of independent, mean zero X-valued random variables, can we find constants $c_{p,X}, C_{p,X}$ depending only on p and X such that

$$c_{p,X}\, |||\,(\xi_i)\,|||_{p,X} \leq \Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_X^p\Big)^{\frac{1}{p}} \leq C_{p,X}\, |||\,(\xi_i)\,|||_{p,X}\,, \tag{0.6}$$

for a suitable norm $|||\cdot|||$ on the sequence (ξ_i) which can be computed in terms of the (moments of the) individual elements ξ_i? Our main results give the following positive answer to this question in the case where X is an L^q-space. To state our results, let us introduce the following notation. If A, B are quantities depending on a parameter α, then we write $A \lesssim_\alpha B$ if there is a constant $c_\alpha > 0$ depending only on α such that $A \leq c_\alpha B$. We write $A \simeq_\alpha B$ if both $A \lesssim_\alpha B$ and $B \lesssim_\alpha A$ hold.

Theorem 0.1. *Let $1 < p, q < \infty$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and (S, Σ, μ) be a σ-finite measure space. Set*

$$\begin{aligned} S_q &= L^q(S; l^2(L^2(\Omega))); \\ D_{p,q} &= l^p(L^p(\Omega; L^q(S))). \end{aligned}$$

If (ξ_i) is a finite sequence of independent, mean zero $L^q(S)$-valued random variables, then

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}} \simeq_{p,q} \|(\xi_i)\|_{s_{p,q}},$$

where $s_{p,q}$ is given by

$$\begin{aligned} S_q \cap D_{q,q} \cap D_{p,q} \quad &\text{if} \quad 2 \leq q \leq p < \infty; \\ S_q \cap (D_{q,q} + D_{p,q}) \quad &\text{if} \quad 2 \leq p \leq q < \infty; \\ (S_q \cap D_{q,q}) + D_{p,q} \quad &\text{if} \quad 1 < p < 2 \leq q < \infty; \\ (S_q + D_{q,q}) \cap D_{p,q} \quad &\text{if} \quad 1 < q < 2 \leq p < \infty; \\ S_q + (D_{q,q} \cap D_{p,q}) \quad &\text{if} \quad 1 < q \leq p \leq 2; \\ S_q + D_{q,q} + D_{p,q} \quad &\text{if} \quad 1 < p \leq q \leq 2. \end{aligned}$$

The main ingredients in the proof of Theorem 0.1 are randomization techniques, Khintchine's inequalities, type and cotype inequalities for $L^q(S)$, Rosenthal's original inequalities (0.1) and Hoffmann-Jørgensen's inequalities (0.5). For $1 < p, q < \infty$ the spaces $s_{p,q}$ satisfy the duality relation

$$(s_{p,q})^* = s_{p',q'}, \ \tfrac{1}{p} + \tfrac{1}{p'} = 1, \ \tfrac{1}{q} + \tfrac{1}{q'} = 1.$$

This duality plays a prominent role in the proof.

Theorem 0.1 can be further generalized to apply to random vectors in a noncommutative L^q-space associated with a semi-finite von Neumann algebra $\mathcal{M}$. In this case, the role of the space S_q is taken over by the spaces $S_{q,c}$ and $S_{q,r}$, which we now briefly describe. If $1 \leq q < \infty$ and (ξ_i) is a finite sequence of $\mathcal{M}$-valued random variables we set

$$\begin{aligned} \|(\xi_i)\|_{S_{q,c}} &= \Big\|\Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}; \\ \|(\xi_i)\|_{S_{q,r}} &= \Big\|\Big(\sum_i \mathbb{E}|\xi_i^*|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}. \end{aligned}$$

From the work of M. Junge on conditional sequence spaces [68] one can deduce that these expressions define two norms on the linear space of all finite sequences of $\mathcal{M}$-valued random variables. We let $S_{q,c}, S_{q,r}$ denote the completions in the respective norms.

Theorem 0.2. *Let $1 < p, q < \infty$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{M}$ be a semi-finite von Neumann algebra. Set*

$$D_{p,q} = l^p(L^p(\Omega; L^q(\mathcal{M}))).$$

If (ξ_i) is a finite sequence of independent, mean zero $L^q(\mathcal{M})$-valued random variables, then

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}} \simeq_{p,q} \|(\xi_i)\|_{s_{p,q}},$$

where $s_{p,q}$ is given by

$$\begin{array}{rl} S_{q,c} \cap S_{q,r} \cap D_{q,q} \cap D_{p,q} & \text{if} \quad 2 \leq q \leq p < \infty; \\ S_{q,c} \cap S_{q,r} \cap (D_{q,q} + D_{p,q}) & \text{if} \quad 2 \leq p \leq q < \infty; \\ (S_{q,c} \cap S_{q,r} \cap D_{q,q}) + D_{p,q} & \text{if} \quad 1 < p < 2 \leq q < \infty; \\ (S_{q,c} + S_{q,r} + D_{q,q}) \cap D_{p,q} & \text{if} \quad 1 < q < 2 \leq p < \infty; \\ S_{q,c} + S_{q,r} + (D_{q,q} \cap D_{p,q}) & \text{if} \quad 1 < q \leq p \leq 2; \\ S_{q,c} + S_{q,r} + D_{q,q} + D_{p,q} & \text{if} \quad 1 < p \leq q \leq 2. \end{array}$$

The spirit of the proof of Theorem 0.2 is the same as in the case of classical L^q-spaces, but different arguments and additional machinery, such as the noncommutative Khintchine inequalities due to F. Lust-Piquard and G. Pisier [98], are required in the noncommutative setting. As a result, the proof will be different from the one presented for Theorem 0.1 even for a commutative von Neumann algebra.

The result in Theorem 0.2 for $p = q$ can also be deduced from the noncommutative Rosenthal inequality proved by M. Junge and Q. Xu (see (0.10) below). However, in the applications we are interested in one typically needs a version in which p and q are different.

Although it is not part of this thesis, we wish to point out that it is possible to deduce Burkholder-Rosenthal inequalities for $L^q(\mathcal{M})$-valued martingales from Theorem 0.2. This result is worked out in detail in [41].

Application to random matrices

Moment estimates and tail bounds for the largest singular value of random matrices play an important role in applications of random matrix theory in numerical error analysis, convex geometry and statistics, see [134, 139] and the references therein. As an application of one of our Rosenthal inequalities for random vectors in $L^q(\mathcal{M})$, we deduce quantitative bounds for the moments of the largest singular value of a random matrix in terms of its entries. Recall that the largest singular value of an $n \times n$ matrix a is equal to its operator norm $\|a\|$, when considering a as an operator on l_n^2.

Theorem 0.3. *Let $2 \leq p < \infty$. Suppose x_{ij} are independent, mean zero random variables in $L^p(\Omega)$. If x is the $n \times n$ random matrix $(x_{ij})_{i,j=1}^n$, then*

$$2^{1+\frac{1}{p}}(\mathbb{E}\|x\|^p)^{\frac{1}{p}} \geq \max\Big\{ \max_{j=1,\ldots,n} \Big(\sum_{i=1}^n \mathbb{E}x_{ij}^2 \Big)^{\frac{1}{2}}, \max_{i=1,\ldots,n} \Big(\sum_{j=1}^n \mathbb{E}x_{ij}^2 \Big)^{\frac{1}{2}}, \Big(\mathbb{E} \max_{i,j=1,\ldots,n} |x_{ij}|^p\Big)^{\frac{1}{p}} \Big\},$$

and

$$(\mathbb{E}\|x\|^p)^{\frac{1}{p}} \leq e(1+\sqrt{2})\alpha_{p,n} \max\Big\{ \max_{j=1,\ldots,n} \Big(\sum_{i=1}^n \mathbb{E}x_{ij}^2 \Big)^{\frac{1}{2}}, \max_{i=1,\ldots,n} \Big(\sum_{j=1}^n \mathbb{E}x_{ij}^2 \Big)^{\frac{1}{2}}, e\alpha_{p,n}\Big(\mathbb{E} \max_{i,j=1,\ldots,n} |x_{ij}|^p\Big)^{\frac{1}{p}} \Big\}, \tag{0.7}$$

with $\alpha_{p,n} < \max\{2\sqrt{\log n}, 2\sqrt{2}\sqrt{p-1}\}$.

More generally, in Chapter 3 we provide bounds for the moments of the largest singular value of a sum of random matrices and of a random matrix with independent rows or columns.

Unfortunately, the upper bound in (0.7) is not of the right order in terms of the dimension of the matrix. Indeed, if the entries of the matrix x are identically distributed and have finite fourth moment, then it has been known for a long time [11] that the largest singular value is asymptotically of order $\sqrt{n}$. More recently, R. Latala obtained the bound

$$\mathbb{E}\|x\| \leq C\Big(\max_{i=1,\ldots,n} \Big(\sum_{j=1}^n \mathbb{E}x_{ij}^2 \Big)^{\frac{1}{2}} + \max_{j=1,\ldots,n} \Big(\sum_{i=1}^n \mathbb{E}x_{ij}^2 \Big)^{\frac{1}{2}} + \Big(\sum_{i,j=1}^n \mathbb{E}x_{ij}^4 \Big)^{\frac{1}{4}} \Big), \tag{0.8}$$

for a matrix with independent, mean zero entries having finite fourth moment [93]. For comparison, observe that (0.5) and (0.8) together imply that there is a universal constant $C > 0$ such that for all $1 \leq p < \infty$,

$$(\mathbb{E}\|x\|^p)^{\frac{1}{p}} \leq C\frac{p}{\log p}\Big(\max_{i=1,\ldots,n} \Big(\sum_{j=1}^n \mathbb{E}x_{ij}^2 \Big)^{\frac{1}{2}} + \max_{j=1,\ldots,n} \Big(\sum_{i=1}^n \mathbb{E}x_{ij}^2 \Big)^{\frac{1}{2}} + \Big(\sum_{i,j=1}^n \mathbb{E}x_{ij}^4 \Big)^{\frac{1}{4}} + \Big(\mathbb{E} \max_{i,j=1,\ldots n} |x_{ij}|^p\Big)^{\frac{1}{p}} \Big).$$

The upper bound in Theorem 0.3 exhibits different growth behaviour in p and does not contain the factor $(\sum_{i,j=1}^n \mathbb{E}x_{ij}^4)^{\frac{1}{4}}$. In particular, the bound (0.7) is applicable to random matrices having entries without a finite fourth moment. On the other hand, we note that the bound in (0.8) is of the right order $\sqrt{n}$ if the entries of the matrix are, in addition, identically distributed. In (0.7) an additional factor $\alpha_{p,n}$ of order $\sqrt{\log n}$ appears. As will be discussed in Chapter 3, this factor is an inevitable product of our method to prove Theorem 0.3.

Further investigation is needed to discover the 'right' bounds for the moments of the largest singular value of a random matrix.

Noncommutative probability theory

In Part II of this thesis we migrate from the setting of vector-valued random variables into the realm of noncommutative probability theory. Loosely speaking, noncommutative probability theory is a generalization of classical probability theory, in which random variables are not modeled as measurable functions on a probability space, but instead by closed, densely defined operators on a Hilbert space. This mathematical formalism was initially developed to give a probabilistic description of quantum mechanical experiments [108]. In these experiments physical observables occur whose statistics violate simple probabilistic inequalities such as the famous Bell inequalities and hence cannot be described in terms of classical probability theory. An accessible introduction for mathematicians to the basic ideas of the probabilistic model for quantum mechanics can be found in [87]. In view of the origins of the subject, noncommutative probability theory is often referred to as quantum probability theory.

Let us now describe the setting of noncommutative measure theory. In its barest form a noncommutative measure space can be defined as a pair $(\mathcal{A}, \phi)$, where $\mathcal{A}$ is a unital algebra of bounded linear operators on a complex Hilbert space and ϕ is a weight on the positive cone $\mathcal{A}_+$ of $\mathcal{A}$. The elements of $\mathcal{A}$ are interpreted as bounded, measurable functions and the functional ϕ plays the role of (integration with respect to) a measure. In order to develop a satisfactory analogue of measure theory in the noncommutative context it turns out that one needs to impose some additional assumptions on the noncommutative measure space. We will always consider a pair $(\mathcal{M}, \tau)$, where $\mathcal{M}$ is a von Neumann algebra and τ is a normal, semi-finite, faithful trace on $\mathcal{M}_+$. Any Maharam measure space (S, Σ, μ) can be viewed as a noncommutative measure space, by identifying it with the pair $(L^\infty(S, \Sigma, \mu), \tau)$, where $\tau(f) = \int_S f \, d\mu$. Other natural examples that we will encounter in the main text are the algebra of bounded linear operators on a Hilbert space (equipped with its standard trace), random matrices, von Neumann algebras associated with groups and von Neumann subalgebras of the bounded linear operators acting on a Fock space.

Given a noncommutative measure space $(\mathcal{M}, \tau)$ one can construct the topological $*$-algebra $S(\tau)$ of *τ-measurable operators*, which is the noncommutative analogue of the space of measurable functions, equipped with the topology of convergence in measure. The trace τ can be extended to $S(\tau)_+$ and a good part of classical measure and integration theory can be recovered. For example, versions of Fatou's lemma, the dominated convergence theorem and Egorov-type results exist in the noncommutative setting. Moreover, every symmetric (quasi-)Banach function space E on $(0, \infty)$ gives rise in a natural way to a noncommutative version $E(\mathcal{M})$, called the *noncommutative (quasi-) Banach function space* associated with E. This construction yields noncommutative versions of many spaces of interest in probability theory, harmonic analysis and interpolation theory, such as L^p-spaces, weak L^p-spaces, Lorentz spaces and Orlicz spaces.

If the trace τ is finite and satisfies $\tau(\mathbf{1}) = 1$, then we can think of the pair $(\mathcal{M}, \tau)$ as a noncommutative probability space. Many of the classical probabilistic concepts, such as convergence in probability, distribution functions, conditional expectations, and martingales, have a natural noncommutative analogue. Other concepts require reformulation, e.g., almost sure convergence, or allow for different generalizations. The most prominent example in the latter category is the notion of independence. In the 1980's, D. Voiculescu discovered a new notion of independence, called *free independence*. This notion is different from *tensor independence*, which generalizes the concept familiar from classical probability theory. Voiculescu's discovery led to the birth of free probability theory, the branch of noncommutative probability theory which takes free independence as its axiom for independence (see [109, 140] for an introduction to this beautiful theory). Under a certain set of intuitive requirements on the notion of independence, it has been shown that free and tensor independence are the only possible notions of independence in a noncommutative probability space (cf. [17]). On the other hand, there are many examples of noncommutative random variables which satisfy a weaker notion of independence, which we discuss below.

In most cases, additional difficulties have to be overcome when generalizing probabilistic results to the noncommutative context. Simple arguments may break down as one cannot evaluate a noncommutative random variable pointwise, or because the triangle inequality for the absolute value on $S(\tau)$ does not hold. Moreover, in the development of martingale theory severe difficulties are posed by the lack of effective stopping time arguments, even though there exist many different constructions of noncommutative stopping times, see [34] and the references therein. Despite these difficulties, there has been considerable progress in the field. The early literature, summarized in [37, 61, 62], has mainly focused on central limit theorems, ergodic theorems, and almost sure and L^2-convergence results for noncommutative martingales. In the recent years, many classical probabilistic inequalities have been generalized to the context of noncommutative L^p-spaces. We mention in particular the noncommutative versions of Khintchine's inequalities [98], the Burkholder-Gundy inequalities for noncommutative martingales [114, 116], Doob's maximal inequality [68] and the Burkholder-Rosenthal inequalities [70]. These inequalities have proved to be fundamental for the study of the geometry of noncommutative L^p-spaces, free probability theory and noncommutative harmonic analysis.

In the second part of this thesis, we study Khintchine, Burkholder-Gundy, Rosenthal, Burkholder-Rosenthal and dual Doob inequalities in the setting of noncommutative symmetric spaces. Our efforts culminate in a generalization of (0.1), which is applied to obtain Itô-type isomorphisms for stochastic integrals with respect to Boson and free Brownian motion. We first give an exposition of this application.

Noncommutative stochastic integration

Noncommutative stochastic integration theory is concerned with the construction and analysis of integrals of the form $\int_0^t f_s \, d\Phi_s$, where f and Φ are noncommutative stochastic processes. Noncommutative stochastic integrals give a way to describe noncommutative continuous-time dynamical systems in terms of noncommutative stochastic differential equations. In contrast with the well-developed classical stochastic integration theory, noncommutative stochastic integration has so far been developed only in particular settings, usually where f and Φ are linear operators on a q-deformed Fock space. The richest theory is the stochastic integration theory for the symmetric or Boson Fock space (i.e., the case $q = 1$), initiated by Hudson and Parthasarathy in [60] (see also [110]). Boson stochastic integration not only includes integration with respect to Boson Brownian motion, but also Poisson processes and other noncommutative semimartingales. This theory is actively used in quantum optics, quantum measurement theory and quantum filtering theory. We refer to [97, 110] and the references therein for an exposition of the theory, and to [12, 22] for surveys on applications in the various areas of quantum physics. Stochastic integration theory with respect to Brownian motion for the anti-symmetric, or Fermion, Fock space ($q = -1$), initiated by Barnett, Streater and Wilde in [13] (see also [30, 114]) is also applied in quantum physics. Stochastic integrals have moreover been defined for operators on q-Fock spaces for $-1 < q < 1$ [51, 126]. Especially the theory for the full Fock space ($q = 0$), see [20, 88], is well developed under the impetus of free probability theory and can be considered important from a pure mathematical viewpoint.

Most of the existing stochastic integration theories either make explicit use of the underlying Fock space structure or use an L^2-isometry to define stochastic integrals. Apart from the estimates for stochastic integrals with respect to Fermionic Brownian motion given in [114], no estimates for the p-th moments of stochastic integrals are known. To explore whether one can formulate a canonical theory of stochastic integration in von Neumann algebras, we investigate whether one can find Itô isomorphisms for stochastic integrals with respect to a Boson or free Brownian motion.

In Theorems 8.17 and 8.21 we prove the following results. Let E be a symmetric Banach function space on $(0,\infty)$ and let p_E and q_E denote its lower and upper Boyd index, respectively. If Φ is a Boson Brownian motion and either $1 < p_E \leq q_E < 2$ or $2 < p_E \leq q_E < \infty$, then

$$\Big\| \int_0^t f \, d\Phi \Big\|_{E(\mathcal{M})} \simeq_E \|f\|_{\mathcal{H}^E(0,t)} \simeq_E \Big\| \int_0^t (d\Phi \; f) \Big\|_{E(\mathcal{M})}. \tag{0.9}$$

If E satisfies $1 < p_E \leq q_E < \infty$ and Φ is a free Brownian motion, then we find

$$\Big\| \int_0^t f \, d\Phi \Big\|_{E(\mathcal{M})} \simeq_E \|f\|_{\mathcal{H}_r^E(0,t)},$$
$$\Big\| \int_0^t (d\Phi \; f) \Big\|_{E(\mathcal{M})} \simeq_E \|f\|_{\mathcal{H}_c^E(0,t)}.$$

The spaces $\mathcal{H}^E(0,t)$, $\mathcal{H}_c^E(0,t)$ and $\mathcal{H}_r^E(0,t)$ are closed subspaces of noncommutative $L^2(0,t)$-valued symmetric spaces. The latter are examples of Hilbert-space valued noncommutative symmetric spaces, which are introduced in this thesis. These spaces generalize the Hilbert-space valued noncommutative L^p-spaces which were constructed earlier in [113] and studied in detail in [69].

The general ideas used in our approach to vector-valued Poisson stochastic integration can be used in the proof of the noncommutative Itô isomorphisms. First we prove new decoupling inequalities, which are used to decouple the noncommutative stochastic integrals. Concretely, we show that the stochastic integral of an adapted step process f can be viewed (in terms of equivalence of norms) as a randomized sum of the form $\sum_{k=1}^n f_k \otimes \Phi_k$ (or $\sum_{k=1}^n f_k * \Phi_k$ in the free case) defined in a tensor (free) product probability space. Here the Φ_k are increments of the integrator process defined on a 'copy' of the original probability space and the f_k are the values of f. The unconditionality of noncommutative martingale difference sequences in noncommutative symmetric spaces with nontrivial Boyd indices plays a prominent role in the proof. As a second step, we use novel noncommutative Khintchine-type inequalities to obtain two-sided estimates for the stochastic integral in terms of the integrand. These Khintchine inequalities are derived from new Rosenthal inequalities for independent random variables in a noncommutative symmetric space, which we present below.

Noncommutative Rosenthal inequalities

The classical Rosenthal and Burkholder-Rosenthal inequalities have been extended by M. Junge and Q. Xu to sequences of noncommutative random variables given by elements of a noncommutative (Haagerup) L^p-space [70, 71]. As is the case for Rosenthal's original result, these inequalities have been developed to study the classification and geometry of noncommutative L^p-spaces

[71]. The noncommutative random variables in Junge and Xu's generalization of (0.1) are only required to satisfy the following, very weak, notion of independence. Suppose that $(\mathcal{N}_k)$ is a sequence of von Neumann subalgebras of $\mathcal{M}$ and that $\mathcal{N}$ is a common von Neumann subalgebra of the $\mathcal{N}_k$ such that $\tau|_{\mathcal{N}}$ is semi-finite. Let $\mathcal{E}_{\mathcal{N}}$ be the conditional expectation with respect to $\mathcal{N}$. We say that $(\mathcal{N}_k)$ is *independent with respect to* $\mathcal{N}$ if for every k we have

$$\mathcal{E}_{\mathcal{N}}(xy) = \mathcal{E}_{\mathcal{N}}(x)\mathcal{E}_{\mathcal{N}}(y), \text{ for all } x \in \mathcal{N}_k \text{ and } y \in W^*((\mathcal{N}_j)_{j\neq k}),$$

where $W^*((\mathcal{N}_j)_{j\neq k})$ denotes the von Neumann subalgebra generated by the $\mathcal{N}_j$ with $j \neq k$. In [71] it is shown that if $2 \leq p < \infty$ and (x_k) is a sequence such that $x_k \in L^p(\mathcal{N}_k)$ and $\mathcal{E}_{\mathcal{N}}(x_k) = 0$ for all k, then

$$\Big\| \sum_{k=1}^n x_k \Big\|_{L^p(\mathcal{M})} \simeq_p \max\Big\{ \Big(\sum_{k=1}^n \|x_k\|^p_{L^p(\mathcal{M})} \Big)^{\frac{1}{p}}, \Big\| \Big(\sum_{k=1}^n \mathcal{E}_{\mathcal{N}}|x_k|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathcal{M})}, \Big\| \Big(\sum_{k=1}^n \mathcal{E}_{\mathcal{N}}|x_k^*|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathcal{M})} \Big\}. \tag{0.10}$$

They also observed that one can deduce a version of (0.10) for $1 < p < 2$ by duality.

The main part of the proof of (0.10) (and in fact of Rosenthal's classical proof) is a '$p \Rightarrow 2p$ argument': one proves that if (0.10) holds for some p, then it must hold for $2p$ as well. This type of argument can be traced back to [35] (see also [96], Lemma 2.c.4) and was used earlier to prove Burkholder-Gundy inequalities for noncommutative martingale difference sequences in noncommutative L^p-spaces [114]. An alternative proof of (0.10) was given by N. Randrianantoanina [117]. He proves a weak-type (1,1) inequality for a martingale difference sequence in $L^2(\mathcal{M})$. Subsequently he obtains the version for $1 < p < 2$ by real interpolation and finally finds the result for $2 \leq p < \infty$ by duality. This approach yields the optimal order of the constants in (0.10) and its dual version. In addition, he proves Rosenthal-type inequalities for independent random variables in noncommutative martingale BMO spaces. These can serve as a surrogate for the case $p = \infty$, in which case (0.10) does not hold. The proof of Randrianantoanina can be adapted to extend the Burkholder-Rosenthal to noncommutative Lorentz $L^{p,q}$-spaces associated with a finite von Neumann subalgebra [63].

We wish to obtain a version of (0.10) for a larger class of noncommutative symmetric spaces. The techniques used in both [71] and [117] are specific to L^p-spaces at several points and therefore not suited to this general setting. Using a different argument we can prove the following result. We let $\mathrm{diag}(x_k)_{k=1}^n$ denote the $n \times n$ diagonal matrix with $x_1, \ldots, x_n$ on its diagonal.

Theorem 0.4. *(Noncommutative Rosenthal theorem) Let $\mathcal{M}$ be a semi-finite von Neumann algebra equipped with a normal, semi-finite, faithful trace τ. Suppose that E is a symmetric Banach function space on $(0, \infty)$ satisfying any of the following conditions:*

(i) $2 < p_E \leq q_E < \infty$;
(ii) *E is an interpolation space for the couple (L^2, L^p) for some $2 \leq p < \infty$ and E is q-concave for some $q < \infty$.*

Let $(\mathcal{N}_k)$ be a sequence of von Neumann subalgebras of $\mathcal{M}$ and $\mathcal{N}$ a common von Neumann subalgebra of the $(\mathcal{N}_k)$ such that $\tau|_{\mathcal{N}}$ is semi-finite. Suppose that $(\mathcal{N}_k)$ is independent with respect to $\mathcal{E} := \mathcal{E}_{\mathcal{N}}$. Let (x_k) be a sequence such that $x_k \in E(\mathcal{N}_k)$ and $\mathcal{E}(x_k) = 0$ for all k. Then, for any n,

$$\Big\|\sum_{k=1}^n x_k\Big\|_{E(\mathcal{M})} \simeq_E \max\Big\{\|\mathrm{diag}(x_k)_{k=1}^n\|_{E(M_n(\mathcal{M}))}, \Big\|\Big(\sum_{k=1}^n \mathcal{E}|x_k|^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})}, \Big\|\Big(\sum_{k=1}^n \mathcal{E}|x_k^*|^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})}\Big\}. \tag{0.11}$$

More generally, we will prove under condition (i) that the noncommutative Burkholder-Rosenthal inequalities hold for noncommutative martingale difference sequences in $E(\mathcal{M})$. This result, formulated in Theorem 7.6 below, extends the known results for noncommutative L^p-spaces and Lorentz spaces [63, 70, 117].

The conditions (i) and (ii) in Theorem 0.4 ensure, in two different senses, that the space E is 'between L^2 and L^q', for some $q < \infty$. The presence of these conditions is not surprising, as the Rosenthal inequalities in noncommutative L^p-spaces do not hold for $p = \infty$ and, moreover, take a different form if $1 < p < 2$.

The Burkholder-Rosenthal inequalities in Theorem 7.6 hold, in particular, for the noncommutative weak L^p-space $L^{p,\infty}(\mathcal{M})$, for any $2 < p < \infty$. Since, by a result of H. Kosaki [84], we can always embed a Haagerup L^p-space into a noncommutative weak L^p-space $L^{p,\infty}$ over a suitable semi-finite von Neumann algebra, we recover the Burkholder-Rosenthal inequalities for Haagerup L^p-spaces proved in [70].

The techniques used in establishing (0.11) can be adapted to prove the following Rosenthal-type inequalities for independent random vectors in a noncommutative symmetric space.

Theorem 0.5. *Suppose $2 \leq p < \infty$ and let E be a symmetric Banach function space on $(0, \infty)$ which is 2-convex and q-concave for some $q < \infty$. Let $\mathcal{M}$ be a semi-finite von Neumann algebra. If (ξ_k) is a sequence of independent, mean zero $E(\mathcal{M})$-valued random variables, then*

$$\Big(\mathbb{E}\Big\|\sum_{k=1}^n \xi_k\Big\|_{E(\mathcal{M})}^p\Big)^{\frac{1}{p}} \simeq_{p,E} \max\Big\{\Big(\mathbb{E}||\mathrm{diag}(\xi_k)_{k=1}^n||_{E(M_n(\mathcal{M}))}^p\Big)^{\frac{1}{p}}, \Big\|\Big(\sum_{k=1}^n \mathbb{E}|\xi_k|^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})}, \Big\|\Big(\sum_{k=1}^n \mathbb{E}|\xi_k^*|^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})}\Big\}. \tag{0.12}$$

In principle it is possible to derive noncommutative and vector-valued Rosenthal inequalities for symmetric spaces which are 'between L^1 and L^2' by duality from Theorems 0.4 and 0.5, respectively. However, in applications it is usually easier to apply these results first to the case of interest and to use a duality argument only after making some additional calculations. An example of this situation can be found in Section 7.3, where Khintchine-type inequalities are derived from Theorem 0.4.

In the remainder of this introduction we discuss two main tools, noncommutative Khintchine inequalities and the noncommutative Boyd interpolation theorem, which are used to prove Theorem 0.4. These results are interesting in their own right.

Noncommutative Khintchine inequalities

Corresponding to the two conditions appearing in our noncommutative Rosenthal theorem, we need two different types of Khintchine inequalities for its proof. In the statement of these inequalities we use the notation

$$\|(x_k)\|_{E(\mathcal{M};l_c^2)} = \Big\|\Big(\sum_k x_k^* x_k\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})};$$
$$\|(x_k)\|_{E(\mathcal{M};l_r^2)} = \Big\|\Big(\sum_k x_k x_k^*\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})}.$$

If E is a symmetric quasi-Banach function space on $(0,\infty)$, then these expressions define two (quasi-)norms, called the column and row (quasi-)norm, on the space of all finite sequences (x_k) in $E(\mathcal{M})$.

Under condition (i) in Theorem 0.4, a key tool needed in proving this result is the following Khintchine type inequality. Let $(r_k)_{k=1}^{\infty}$ be a Rademacher sequence defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ and let $\mathcal{M}$ be a semi-finite von Neumann algebra. Let $L^\infty(\Omega)\overline{\otimes}\mathcal{M}$ denote the tensor product von Neumann algebra. Suppose that E is a symmetric quasi-Banach function space on $(0,\infty)$ which is p-convex for some $0<p<\infty$. In Theorem 6.1 we show that if the upper Boyd index q_E of E is finite, then

$$\Big\|\sum_k r_k\otimes x_k\Big\|_{E(L^\infty(\Omega)\overline{\otimes}\mathcal{M})} \lesssim_E \max\Big\{\|(x_k)\|_{E(\mathcal{M};l_c^2)}, \|(x_k)\|_{E(\mathcal{M};l_r^2)}\Big\}, \quad (0.13)$$

for any finite sequence (x_k) in $E(\mathcal{M})$. Our result complements the main result in [94], which provides the following dual inequality. It is shown there that if E is a symmetric Banach function space which is either separable or the dual of a separable space and satisfies $q_E<\infty$, then

$$\inf\Big\{\|(y_k)\|_{E(\mathcal{M};l_c^2)} + \|(z_k)\|_{E(\mathcal{M};l_r^2)}\Big\} \lesssim_E \Big\|\sum_k r_k\otimes x_k\Big\|_{E(L^\infty(\Omega)\overline{\otimes}\mathcal{M})},$$

where the infimum is taken over all decompositions $x_k = y_k + z_k$ in $E(\mathcal{M})$.

As an ingredient for our proof of Theorem 0.4 under condition (ii), as well as for our proof of Theorem 0.5, we establish a different Khintchine type of inequality. Suppose that E is a symmetric quasi-Banach function space on $(0,\infty)$ which is p-convex and q-concave for some $0 < p, q < \infty$. In Theorem 6.7 we show that

$$\mathbb{E}\Big\|\sum_k r_k x_k\Big\|_{E(\mathcal{M})} \lesssim_E \max\Big\{\|(x_k)\|_{E(\mathcal{M};l_c^2)}, \|(x_k)\|_{E(\mathcal{M};l_r^2)}\Big\}, \qquad (0.14)$$

for any finite sequence (x_k) in $E(\mathcal{M})$. This result complements a result by F. Lust-Piquard and Q. Xu [99], who showed that the inequality

$$\inf\Big\{\|(y_k)\|_{E(\mathcal{M};l_c^2)} + \|(z_k)\|_{E(\mathcal{M};l_r^2)}\Big\} \lesssim_E \mathbb{E}\Big\|\sum_k r_k x_k\Big\|_{E(\mathcal{M})} \qquad (0.15)$$

holds if E is a 2-concave symmetric Banach function space which is either separable or has the Fatou property. They derive this result from an interesting characterization which says that if E is a 2-convex symmetric Banach function space which is either separable or has the Fatou property, then $E(\mathcal{M})$ satisfies a version of the little Grothendieck inequality if and only if $E^\times(\mathcal{M})$ satisfies the Khintchine inequality (0.15). Here $E^\times$ denotes the Köthe dual of E.

Our results are optimal in the following sense: (0.13) holds for any semi-finite von Neumann algebra if and only if $q_E < \infty$ and (0.14) holds for any semi-finite von Neumann algebra if and only if E is q-concave for some $q < \infty$.

We apply the inequalities (0.13) and (0.14) to obtain Burkholder-Gundy inequalities for noncommutative martingale difference sequences in noncommutative Banach function spaces. Our main result in this direction, formulated in Theorem 6.29, improves several results in the literature [14, 15, 16].

In the Khintchine inequality (0.13) one can replace the Rademacher sequence (r_k) by a sequence of operator coefficients (α_k) in a finite von Neumann algebra $\mathcal{N}$, provided that the sequence (α_k) satisfies

$$\Big\|\sum_k \alpha_k \otimes x_k\Big\|_{L^q(\mathcal{N}\overline{\otimes}\mathcal{M})} \lesssim_q \max\Big\{\|(x_k)\|_{L^q(\mathcal{M};l_c^2)}, \|(x_k)\|_{L^q(\mathcal{M};l_r^2)}\Big\}, \qquad (0.16)$$

for all $1 \le q < \infty$. The situation is even better if (α_k) satisfies (0.16) for $q = \infty$, for example if (α_k) is a sequence of free group unitaries [57]. In this case we show that

$$\Big\|\sum_k \alpha_k \otimes x_k\Big\|_{E(\mathcal{N}\overline{\otimes}\mathcal{M})} \lesssim_E \max\Big\{\|(x_k)\|_{E(\mathcal{M};l_c^2)}, \|(x_k)\|_{E(\mathcal{M};l_r^2)}\Big\}, \qquad (0.17)$$

for *any* symmetric quasi-Banach function space E which is p-convex for some $0 < p < \infty$. In fact, for symmetric Banach function spaces (0.17) holds with a universal constant. This inequality is the basis for an alternative proof of the following special case of (0.13). Even though this method does not cover the full result, it yields a significantly better estimate on the constant in (0.13).

Theorem 0.6. *Let $1 \leq q < \infty$. Suppose that E is a fully symmetric quasi-Banach function space on $(0,\infty)$. If E is an exact interpolation space for the couple (L^1, L^q), then*

$$\Big\| \sum_k r_k \otimes x_k \Big\|_{E(L^\infty \overline{\otimes} \mathcal{M})} \leq 16\sqrt{2q} \max \Big\{ \|(x_k)\|_{E(\mathcal{M};l_c^2)}, \|(x_k)\|_{E(\mathcal{M};l_r^2)} \Big\}$$

for any finite sequence (x_k) in $E(\mathcal{M})$.

Using (0.17) we also show, under suitable conditions on the space E, that the intersection $E(\mathcal{M}; l_c^2) \cap E(\mathcal{M}; l_r^2)$ is isomorphic to a complemented subspace of $E(L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M})$, where $L(\mathbb{F}_\infty)$ is the free group von Neumann algebra. As a consequence, we find a useful interpolation result for intersections of row and column spaces. We refer to Theorem 6.25 for a precise statement.

Noncommutative Boyd interpolation theorem

In our proof of Theorem 0.4 we make extensive use of interpolation theory. In particular, we use the noncommutative version of the Boyd interpolation theorem, named after D. Boyd [23]. Let us first recall the classical version of Boyd's result. Fix $1 \leq p, q < \infty$. Let E be a symmetric Banach function space on $(0,\infty)$ which is separable or has the Fatou property. D. Boyd demonstrated that if $p < p_E \leq q_E < q$, then E is an interpolation space for the couple (L^p, L^q). Together with the Calderón-Mitjagin theorem, which characterizes the symmetric Banach function spaces which are an interpolation space for the couple (L^1, L^∞), the Boyd interpolation theorem provides an invaluable tool in the analysis of symmetric spaces.

To obtain interpolation tools for noncommutative Banach function spaces, one often appeals to an abstract lifting theorem from [48] which says that if E, E_0, E_1 are fully symmetric spaces such that E is an interpolation space for the interpolation couple (E_0, E_1), then $E(\mathcal{M})$ is an interpolation space for the couple $(E_0(\mathcal{M}), E_1(\mathcal{M}))$. In particular, we can 'lift' the classical Boyd interpolation theorem: if E is a symmetric Banach function space which satisfies $p < p_E \leq q_E < q$ and is either separable or has the Fatou property, then $E(\mathcal{M})$ is an interpolation space for the couple $(L^p(\mathcal{M}), L^q(\mathcal{M}))$.

We present a new, direct proof of the noncommutative Boyd interpolation theorem, which avoids the use of the lifting theorem. In this way, we remove some restrictions imposed by the lifting theorem and obtain an extension of the known version of the noncommutative Boyd theorem in two directions. Firstly, we find that the result is true for any $0 < p < q \leq \infty$ and any symmetric quasi-Banach function space E on $(0,\infty)$ which is s-convex for some $0 < s < \infty$. Secondly, we can interpolate *(sub)convex* operators defined only on the positive cone of a couple of noncommutative L^p-spaces. A special case of our main result, Theorem 5.19, reads as follows.

Theorem 0.7. *Let E be a symmetric quasi-Banach function space on $(0,\infty)$ which is s-convex for some $0 < s < \infty$. Let $\mathcal{M},\mathcal{N}$ be von Neumann algebras equipped with normal, semi-finite, faithful traces τ and σ, respectively. Suppose that $0 < p < q \leq \infty$ and let $T : L^p(\mathcal{M})_+ + L^q(\mathcal{M})_+ \to S(\sigma)$ be a subconvex map such that for some constants $C_p, C_q > 0$ depending only on p and q, respectively,*

$$\|Tx\|_{L^{r,\infty}(\mathcal{N})} \leq C_r \|x\|_{L^r(\mathcal{M})} \qquad (x \in L^r(\mathcal{M})_+,\ r = p,q).$$

If $0 < p < p_E$ and either $q_E < q < \infty$ or $q = \infty$, then there is a constant $c_{p,q,E}$ depending only on p, q and E such that

$$\|Tx\|_{E(\mathcal{N})} \leq c_{p,q,E} \max\{C_p, C_q\}\ \|x\|_{E(\mathcal{M})} \qquad (x \in E(\mathcal{M})_+).$$

Apart from the use of some basic properties of distribution functions for operators, our proof of Theorem 0.7 is completely elementary.

By modifying the proof of Theorem 0.7 we can deduce a version of the 'dual Doob' inequality, which is dual to Doob's maximal inequality, for noncommutative symmetric spaces. The following result is presented in Theorem 5.24 below.

Theorem 0.8. *Let E be a symmetric Banach function space on $(0,\infty)$ and let $\mathcal{M}$ be a finite von Neumann algebra. Let $(\mathcal{E}_i)_{i\geq 1}$ be an increasing sequence of conditional expectations in $\mathcal{M}$. If $1 < p_E \leq q_E < \infty$, then for any sequence $(x_i)_{i\geq 1}$ in $E(\mathcal{M})_+$,*

$$\Big\| \sum_{i\geq 1} \mathcal{E}_i(x_i) \Big\|_{E(\mathcal{M})} \lesssim_E \Big\| \sum_{i\geq 1} x_i \Big\|_{E(\mathcal{M})}. \tag{0.18}$$

In [68], this inequality was established for noncommutative L^p-spaces and used to prove a noncommutative version of Doob's maximal inequality by a duality argument. It is to be expected that this argument can be adapted to work for an appropriate class of noncommutative symmetric spaces. This is a topic for future research.

Organization

Part I is entirely dedicated to the study of vector-valued Rosenthal inequalities and their applications. In Chapter 1 we discuss Rosenthal inequalities for random variables taking values in Banach spaces and, in particular, in Hilbert spaces and L^p-spaces. In Chapter 2 these results are applied to vector-valued stochastic integration with respect to compensated Poisson random measures. The main results of this chapter will appear as part of [45]. The first two chapters can be read without any knowledge of noncommutative analysis. In Chapter 3 we prove Rosenthal inequalities for random variables taking values in a noncommutative L^p-space and apply these results to random matrices. This chapter is based on [41].

Chapter 4 contains a detailed introduction to symmetric quasi-Banach function spaces and their interpolation theory. Chapter 5 introduces noncommutative quasi-Banach function spaces, interpolation results for these spaces, Hilbert-space valued noncommutative symmetric spaces and conditional sequence spaces. In particular, the chapter contains the noncommutative Boyd interpolation theorem with a new proof based on [42]. Chapter 6 is devoted to the study of Khintchine inequalities in noncommutative symmetric spaces. Parts of Section 6.4 have appeared in [46]. Chapter 7 contains the main results on noncommutative Rosenthal and Burkholder-Rosenthal inequalities. Several of the results of Chapters 6 and 7 have been published in [44]. Finally, Chapter 8 gives an exposition of our results on stochastic integration in noncommutative symmetric spaces. The results in this chapter extend the results presented in [43], which contains an exposition in the specialized setting of noncommutative L^p-spaces.

Part I

Vector-valued Rosenthal inequalities

1

Vector-valued Rosenthal inequalities in L^p-spaces

In this chapter we consider Rosenthal-type inequalities for random vectors in Banach spaces. The main question is the following: given $1 \leq p < \infty$, a Banach space X and a sequence (ξ_i) of independent, mean zero X-valued random variables, can we find two-sided estimates of the form

$$c_{p,X} \,|||\, (\xi_i) \,|||\, \leq \Big(\mathbb{E}\Big\| \sum_i \xi_i \Big\|_X^p\Big)^{\frac{1}{p}} \leq C_{p,X} \,|||\, (\xi_i) \,|||\,, \tag{1.1}$$

for a suitably chosen norm $|||\cdot|||$ on the sequence (ξ_i) and constants $c_{p,X}, C_{p,X}$ depending only on p and X? After discussing some preliminary results from Banach space geometry, we will consider this question in a general Banach space. We shall see that if X satisfies a *type* assumption, then we obtain an upper estimate as in (1.1) and if it satisfies a *cotype* assumption, then we obtain a lower estimate. Unfortunately, unless the Banach space under consideration is (isomorphic to) a Hilbert space, the norm $|||\cdot|||$ we find in the upper estimate is in general different from the norm found in the lower estimate.

The main results of this chapter improve the latter estimates in the case where X is an L^q-space. We obtain a significant extension of Rosenthal's theorem for scalar-valued random variables, by finding a norm $|||\cdot|||_{p,q}$ such that the inequalities in (1.1) hold simultaneously. It turns out that one needs to consider six different norms $|||\cdot|||_{p,q}$, depending on the relative position of p, q and 2. These results can moreover be extended to the case where X is a *noncommutative* L^q-space, see Chapter 3 ahead.

The results of this chapter will be applied in Chapter 2 to the problem of vector-valued Poisson stochastic integration.

1.1 Probabilistic notions from Banach space geometry

Throughout this thesis we reserve the symbol $(r_i)_{i\geq 1}$ to denote a *Rademacher sequence*, i.e., a sequence of independent, $\{-1, 1\}$-valued random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which satisfy

$$\mathbb{P}(r_i = 1) = \mathbb{P}(r_i = -1) = \tfrac{1}{2} \qquad (i \geq 1).$$

Let us recall the following classical inequalities due to A. Khintchine.

Theorem 1.1. *[83] (Khintchine's inequalities) For any $0 < p < \infty$ and any finite sequence (a_i) in $\mathbb{C}$ we have*

$$\Big(\mathbb{E}\Big|\sum_i r_i a_i\Big|^p\Big)^{\frac{1}{p}} \simeq_p \Big(\sum_i |a_i|^2\Big)^{\frac{1}{2}}.$$

Using the Hölder and Hölder-Minkowski inequalities one can deduce the following result as a corollary.

Theorem 1.2. *For any $0 < p, q < \infty$ and any finite sequence (f_i) in $L^q(S)$ we have*

$$\Big(\mathbb{E}\Big\|\sum_i r_i f_i\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}} \simeq_{p,q} \Big\|\Big(\sum_i |f_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(S)}. \tag{1.2}$$

One cannot straightforwardly generalize Theorem 1.2 to the setting of Banach spaces, as the 'square function' appearing on the right hand side of (1.2) has no meaning for elements f_i from a Banach space. However, a useful substitute is given by the following inequalities due to J.P. Kahane [74]. The extension for quasi-Banach spaces is due to N. Kalton ([76], Theorem 2.1).

Theorem 1.3. *(Kahane's inequalities) Let $0 < p, q < \infty$ and let X be a quasi-Banach space. Then there exist constants $K_{q,p}$ depending only on p and q such that for any sequence $(x_i)_{i\geq 1}$ in X we have*

$$\Big(\mathbb{E}\Big\|\sum_{i\geq 1} r_i x_i\Big\|_X^q\Big)^{\frac{1}{q}} \leq K_{q,p}\Big(\mathbb{E}\Big\|\sum_{i\geq 1} r_i x_i\Big\|_X^p\Big)^{\frac{1}{p}}. \tag{1.3}$$

We now proceed to define several probabilistic concepts which are related to the geometry of a Banach space.

Definition 1.4. *A Banach space X is said to have* type p *for some $1 \leq p \leq 2$ if there is a constant $C > 0$ such that for any finite sequence (x_i) in X we have*

$$\Big(\mathbb{E}\Big\|\sum_i r_i x_i\Big\|_X^2\Big)^{\frac{1}{2}} \leq C\Big(\sum_i \|x_i\|_X^p\Big)^{\frac{1}{p}}. \tag{1.4}$$

The least possible constant C for which (1.4) holds is called the type p constant *of X and is denoted by $T_p(X)$.*

A Banach space X is said to have cotype q *for some $2 \leq q \leq \infty$ if there is a constant $C > 0$ such that for any finite sequence (x_i) in X we have*

$$\Big(\sum_i \|x_i\|_X^q\Big)^{\frac{1}{q}} \le C\Big(\mathbb{E}\Big\|\sum_i r_i x_i\Big\|_X^2\Big)^{\frac12} \text{ if } q<\infty$$
$$\max_i \|x_i\|_X \le C\Big(\mathbb{E}\Big\|\sum_i r_i x_i\Big\|_X^2\Big)^{\frac12} \text{ if } q=\infty. \tag{1.5}$$

The least possible constant C for which (1.5) holds is called the cotype q constant *of X and is denoted by $C_q(X)$.*

By the triangle inequality every Banach space has type 1 and cotype ∞ with constant 1. Note that by Kahane's inequalities we can replace $(\mathbb{E}\|\sum_i r_i x_i\|_X^2)^{\frac12}$ in the definition of type (and cotype) by $(\mathbb{E}\|\sum_i r_i x_i\|_X^r)^{\frac1r}$ for any $1\le r<\infty$. By the contractive embedding $l^r \subset l^p$, one immediately sees that if X has type p, then it has type r for any $r<p$ and similarly, if it has cotype q, then it has cotype s for any $s>q$.

It follows from the parallelogram law that a Banach space X is isometrically isomorphic to a Hilbert space if and only if X has both type and cotype 2 and $T_2(X)=C_2(X)=1$. The following fundamental result, due to S. Kwapień, extends this characterization to spaces which are only isomorphic to a Hilbert space. For a proof we refer to [1], Theorem 7.4.1.

Theorem 1.5. *A Banach space has both type 2 and cotype 2 if and only if it is isomorphic to a Hilbert space.*

If (S,Σ,μ) is a measure space, then $L^p(S)$ has type $\min\{p,2\}$ and cotype $\max\{p,2\}$. This fact is not difficult to deduce from Khintchine's inequalities for L^p-spaces (Theorem 1.2), see [1], Theorem 6.2.14.

To discuss duality results between type and cotype we need the following notion. Let $1<p<\infty$ and X be a Banach space. The *n-th Rademacher projection* π_n in $L^p(\Omega;X)$ is defined by

$$\pi_n f = \sum_{k=1}^n r_k \mathbb{E}(r_k f) \qquad (f\in L^p(\Omega;X)).$$

The space X is called *K-convex* if

$$K_{p,X} = \sup_{n\ge1} \|\pi_n\|_{B(L^p(\Omega;X))} < \infty.$$

This property does not depend on p, i.e. if $1<p,q<\infty$, then we have $K_{p,X}<\infty$ if and only if $K_{q,X}<\infty$. Moreover, a space X is K-convex if and only if its dual X^* is K-convex and $K_{p,X}=K_{p',X^*}$ whenever $1<p,p'<\infty$ satisfy $\frac1p+\frac1{p'}=1$. By a deep result due to G. Pisier, a Banach space X is K-convex if and only if it has type $p>1$. Moreover, if this is the case, then X has finite cotype. The proofs of these statements, as well as the following duality theorem, may be found in Chapter 13 of [39].

Theorem 1.6. *Let X be a Banach space and assume that $1 \le p \le 2 \le q \le \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. If X has type p, then X^* has cotype q and $C_q(X^*) \le T_p(X)$. On the other hand, if X is K-convex and has cotype $q < \infty$, then X^* has type p and $T_p(X^*) \le K_{2,X} C_q(X)$.*

In our treatment of vector-valued Poisson stochastic integration in Chapter 2, we will restrict ourselves to the class of spaces in which martingale difference sequences are unconditional.

Definition 1.7. *Let $1 < p < \infty$. A Banach space X is called a* UMD_p-space *if there is a constant $C > 0$ such that for any finite martingale difference sequence (x_i) in $L^p(\Omega; X)$ and any choice of signs $(\varepsilon_i)_{i \ge 1}$ in $\{-1, 1\}^{\mathbb{N}}$ we have*

$$\Big(\mathbb{E}\Big\| \sum_i \varepsilon_i x_i \Big\|_X^p\Big)^{\frac{1}{p}} \le C\Big(\mathbb{E}\Big\| \sum_i x_i \Big\|_X^p\Big)^{\frac{1}{p}}. \tag{1.6}$$

It is well known that the UMD_p-property in (1.6) is p-independent, i.e. if $1 < p, q < \infty$, then X is a UMD_p-space if and only if it is UMD_q. Therefore, if X is UMD_p for some $1 < p < \infty$, it is customary to simply call it a *UMD-space*. In the following theorem we collect some known properties of UMD Banach spaces.

Theorem 1.8. *If X is a UMD Banach space, then X has the following properties:*

(a) X is reflexive;
(b) X^* is a UMD space
(c) X has type $p > 1$ and cotype $q < \infty$.
(d) X is K-convex.

Although UMD spaces are very special from a geometrical point of view, many of the concrete spaces we will consider below in fact possess this property. For example, Hilbert spaces, L^p-spaces and noncommutative L^p-spaces with $1 < p < \infty$ are all UMD. For proofs of the stated facts and much more on UMD spaces, we refer to [28] and the references therein.

1.2 Banach spaces with type or cotype restriction

In this section we study sums of independent, mean zero random vectors in a Banach space. We use the following terminology and notation. Throughout, we let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a probability space. If X is a Banach space, then an *X-valued random variable* is a measurable map $\xi : \Omega \to X$, where X is equipped with its Borel σ-algebra $\mathcal{B}(X)$. We denote by μ_ξ the *distribution* of X, the probability measure on X given by

$$\mu_\xi(B) = \mathbb{P}(\xi \in B) \qquad (B \in \mathcal{B}(X)).$$

We will always assume that an X-valued random variable is strongly measurable, i.e. there is a sequence of X-valued step functions (ξ_i) such that $\xi_i \to \xi$ a.s. By the Pettis measurability theorem, this is equivalent to assuming that ξ is a.s. separably valued and $\langle \xi, x^* \rangle$ is measurable for any $x^* \in X^*$. We say two random variables ξ, η are *identically distributed* if $\mu_\xi = \mu_\eta$. Since ξ, η are strongly measurable, this is equivalent to saying that the scalar valued random variables $\langle \xi, x^* \rangle$ and $\langle \eta, x^* \rangle$ are identically distributed for any $x^* \in X^*$. Recall that a random variable ξ is *Bochner integrable* if there is a sequence of X-valued step functions (ξ_i) such that $\xi_i \to \xi$ a.s. and

$$\lim_{i \to \infty} \mathbb{E}\|\xi_i - \xi\|_X = 0.$$

If ξ is Bochner integrable, then we say it is *mean zero* if its Bochner integral $\mathbb{E}\xi$ is zero.

Given a Bochner integrable random variable ξ and a sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}$ we let $\mathbb{E}(\xi|\mathcal{G})$ be the vector-valued conditional expectation with respect to $\mathcal{G}$. If η is an X-valued random variable on $(\Omega, \mathcal{F}, \mathbb{P})$, then we set $\mathbb{E}(\xi|\eta) := \mathbb{E}(\xi|\sigma(\eta))$, where $\sigma(\eta)$ is the σ-algebra generated by η. Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ be a copy of $(\Omega, \mathcal{F}, \mathbb{P})$. On the product probability space $\Omega \times \tilde{\Omega}$ we define

$$\xi(\omega, \tilde{\omega}) = \xi(\omega) 1_{\tilde{\Omega}}(\tilde{\omega}), \qquad \tilde{\xi}(\omega, \tilde{\omega}) = \xi(\tilde{\omega}) 1_{\Omega}(\omega).$$

Notice that ξ and $\tilde{\xi}$ are independent and identically distributed. We shall refer to $\tilde{\xi}$ as an *independent copy* of ξ.

We commence by making the following well-known observation, see e.g. [95], Lemma 6.3. We provide a proof for the reader's convenience.

Lemma 1.9. *Let $F : \mathbb{R} \to \mathbb{R}$ be convex and increasing, let X be a Banach space and ξ be an X-valued mean zero random variable such that $\mathbb{E}|F(2\|\xi\|)| < \infty$. If $\tilde{\xi}$ is an independent copy of ξ, then*

$$\mathbb{E}F(\|\xi\|) \le \mathbb{E}\tilde{\mathbb{E}}F(\|\xi - \tilde{\xi}\|) \le \mathbb{E}F(2\|\xi\|).$$

Proof. Since F is convex and increasing, the function $F(\|\cdot\|)$ is convex as well. By Jensen's inequality,

$$\begin{aligned}
\mathbb{E}F(\|\xi\|) &= \mathbb{E}F(\|\xi - \tilde{\mathbb{E}}(\tilde{\xi})\|) \\
&= \mathbb{E}\tilde{\mathbb{E}}F(\|\mathbb{E}(\xi - \tilde{\xi}|\xi)\|) \\
&\le \mathbb{E}\tilde{\mathbb{E}}F(\|\tfrac{1}{2}(2\xi) + \tfrac{1}{2}(-2\tilde{\xi})\|) \\
&\le \mathbb{E}\tilde{\mathbb{E}}(\tfrac{1}{2}F(\|2\xi\|) + \tfrac{1}{2}F(\| - 2\tilde{\xi}\|)) \\
&= \mathbb{E}F(2\|\xi\|).
\end{aligned}$$

□

As a consequence we have the following randomization principle.

Corollary 1.10. *(Randomization) Suppose that X is a Banach space and that $1 \le p < \infty$. If (ξ_i) is a finite sequence of independent, mean zero X-valued random variables and (r_i) is a Rademacher sequence on a probability space $(\Omega_r, \mathcal{F}_r, \mathbb{P}_r)$, then*

$$\tfrac{1}{2}\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_X^p\Big)^{\frac{1}{p}} \le \Big(\mathbb{E}_r\mathbb{E}\Big\|\sum_i r_i\xi_i\Big\|_X^p\Big)^{\frac{1}{p}} \le 2\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_X^p\Big)^{\frac{1}{p}}.$$

Proof. Let $\tilde{\xi}_i$ be an independent copy of ξ_i. By Lemma 1.9,

$$\begin{aligned}
\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_X^p\Big)^{\frac{1}{p}} &\le \Big(\mathbb{E}\tilde{\mathbb{E}}\Big\|\sum_i \xi_i - \tilde{\xi}_i\Big\|_X^p\Big)^{\frac{1}{p}} \\
&= \Big(\mathbb{E}\tilde{\mathbb{E}}\mathbb{E}_r\Big\|\sum_i r_i(\xi_i - \tilde{\xi}_i)\Big\|_X^p\Big)^{\frac{1}{p}} \\
&\le 2\Big(\mathbb{E}\mathbb{E}_r\Big\|\sum_i r_i\xi_i\Big\|_X^p\Big)^{\frac{1}{p}} \\
&\le 2\Big(\mathbb{E}\tilde{\mathbb{E}}\mathbb{E}_r\Big\|\sum_i r_i(\xi_i - \tilde{\xi}_i)\Big\|_X^p\Big)^{\frac{1}{p}} \\
&\le 4\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_X^p\Big)^{\frac{1}{p}}.
\end{aligned}$$

□

Using Kahane's inequalities, this leads to the following observation.

Lemma 1.11. *If X is a Banach space and $1 \le p \le 2$, then X has type p if and only if for every finite sequence (η_i) of independent, mean zero X-valued random variables in $L^p(\Omega; X)$ we have*

$$\Big(\mathbb{E}\Big\|\sum_i \eta_i\Big\|_X^p\Big)^{\frac{1}{p}} \lesssim_{p,X} \Big(\sum_i \mathbb{E}\|\eta_i\|_X^p\Big)^{\frac{1}{p}}.$$

On the other hand, if $2 \le q < \infty$, then X has cotype q if and only if for every finite sequence (η_i) of independent, mean zero X-valued random variables in $L^q(\Omega; X)$ we have

$$\Big(\sum_i \mathbb{E}\|\eta_i\|_X^q\Big)^{\frac{1}{q}} \lesssim_{q,X} \Big(\mathbb{E}\Big\|\sum_i \eta_i\Big\|_X^q\Big)^{\frac{1}{q}}.$$

Lemma 1.12. *Fix $1 \le p < \infty$. Let X be a Banach space and (ξ_i) be a finite sequence of independent, mean zero X-valued random variables. If X has type $1 \le s \le 2$, then*

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_X^p\Big)^{\frac{1}{p}} \lesssim_{p,s,X} \Big(\mathbb{E}\Big(\sum_i \|\xi_i\|_X^s\Big)^{\frac{p}{s}}\Big)^{\frac{1}{p}}.$$

On the other hand, if X has cotype $2 \le s < \infty$, then

$$\Big(\mathbb{E}\Big(\sum_i \|\xi_i\|_X^s\Big)^{\frac{p}{s}}\Big)^{\frac{1}{p}} \gtrsim_{p,s,X} \Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_X^p\Big)^{\frac{1}{p}}.$$

Proof. Suppose X has type s. By Corollary 1.10, Kahane's inequalities and the type s inequality we obtain

$$\begin{aligned}\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_X^p\Big)^{\frac{1}{p}} &\simeq \Big(\mathbb{E}\mathbb{E}_r\Big\|\sum_i r_i\xi_i\Big\|_X^p\Big)^{\frac{1}{p}} \\ &\simeq_p \Big(\mathbb{E}\Big(\mathbb{E}_r\Big\|\sum_i r_i\xi_i\Big\|_X^2\Big)^{\frac{p}{2}}\Big)^{\frac{1}{p}} \\ &\lesssim_{p,s,X} \Big(\mathbb{E}\Big(\sum_i \|\xi_i\|_X^s\Big)^{\frac{p}{s}}\Big)^{\frac{1}{p}}.\end{aligned}$$

The second assertion is proved similarly. □

For the proof of the next lemma we need the following inequality, due to H.P. Rosenthal.

Theorem 1.13. *[121] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $2 \le p < \infty$. If (ξ_i) is a finite sequence of independent, mean zero random variables in $L^p(\Omega)$, then*

$$\Big(\mathbb{E}\Big|\sum_i \xi_i\Big|^p\Big)^{\frac{1}{p}} \simeq_p \max\Big\{\Big(\sum_i \mathbb{E}|\xi_i|^p\Big)^{\frac{1}{p}}, \Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{\frac{1}{2}}\Big\}.$$

Lemma 1.14. *Let X be a Banach space and (ξ_i) be a finite sequence of independent X-valued random variables. If $0 < s \le p < \infty$, then*

$$\Big(\mathbb{E}\Big(\sum_i \|\xi_i\|_X^s\Big)^{\frac{p}{s}}\Big)^{\frac{1}{p}} \lesssim_{p,s,X} \max\Big\{\Big(\sum_i \mathbb{E}\|\xi_i\|_X^p\Big)^{\frac{1}{p}}, \Big(\sum_i \mathbb{E}\|\xi_i\|_X^s\Big)^{\frac{1}{s}}\Big\}.$$

On the other hand, if $1 \le p \le s$, then

$$\Big(\mathbb{E}\Big(\sum_i \|\xi_i\|_X^s\Big)^{\frac{p}{s}}\Big)^{\frac{1}{p}} \gtrsim_{p,s,X} \inf\Big\{\Big(\sum_i \mathbb{E}\|\eta_i\|_X^p\Big)^{\frac{1}{p}} + \Big(\sum_i \mathbb{E}\|\theta_i\|_X^s\Big)^{\frac{1}{s}}\Big\},$$

where the infimum is taken over all finite sequences $(\eta_i) \in l^p(L^p(\Omega; X))$ and $(\theta_i) \in l^s(L^s(\Omega; X))$ such that $\xi_i = \eta_i + \theta_i$.

Proof. Suppose first that $0 < s \le p < \infty$. We may assume that $\mathbb{E}\|\xi_i\|_X^s < \infty$ for all i. By the triangle inequality,

$$\Big(\mathbb{E}\Big(\sum_i \|\xi_i\|_X^s\Big)^{\frac{p}{s}}\Big)^{\frac{s}{p}} \le \Big(\mathbb{E}\Big|\sum_i \|\xi_i\|_X^s - \mathbb{E}\|\xi_i\|_X^s\Big|^{\frac{p}{s}}\Big)^{\frac{s}{p}} + \sum_i \mathbb{E}\|\xi_i\|_X^s.$$

Notice that the sequence $(\|\xi_i\|_X^s - \mathbb{E}\|\xi_i\|_X^s)_{i\geq 1}$ is independent and mean zero. Therefore, by applying Lemma 1.12 if $p \leq 2s$ and Theorem 1.13 if $p > 2s$, we obtain

$$\Big(\mathbb{E}\Big|\sum_i \|\xi_i\|_X^s - \mathbb{E}\|\xi_i\|_X^s\Big|^{\frac{p}{s}}\Big)^{\frac{s}{p}}$$
$$\lesssim_{p,s} \max\Big\{\Big(\sum_i \mathbb{E}\Big|\|\xi_i\|_X^s - \mathbb{E}\|\xi_i\|_X^s\Big|^{\frac{p}{s}}\Big)^{\frac{s}{p}},$$
$$\chi_{(2s,\infty)}(p)\Big(\sum_i \mathbb{E}\Big(\|\xi_i\|_X^s - \mathbb{E}\|\xi_i\|_X^s\Big)^2\Big)^{\frac{1}{2}}\Big\}.$$

We estimate the two terms on the right hand side separately. For the first term, we have by the triangle inequality and Jensen's inequality,

$$\begin{aligned}\Big(\sum_i \mathbb{E}\Big|\|\xi_i\|_X^s - \mathbb{E}\|\xi_i\|_X^s\Big|^{\frac{p}{s}}\Big)^{\frac{s}{p}} &\leq \Big(\sum_i \mathbb{E}\|\xi_i\|_X^p\Big)^{\frac{s}{p}} + \Big(\sum_i (\mathbb{E}\|\xi_i\|_X^s)^{\frac{p}{s}}\Big)^{\frac{s}{p}} \\ &\leq 2\Big(\sum_i \mathbb{E}\|\xi_i\|_X^p\Big)^{\frac{s}{p}}.\end{aligned}$$

For the second term, suppose that $p > 2s$. By the triangle inequality in $l^2(L^2(\Omega))$,

$$\begin{aligned}\Big(\sum_i \mathbb{E}\Big(\|\xi_i\|_X^s - \mathbb{E}\|\xi_i\|_X^s\Big)^2\Big)^{\frac{1}{2}} &\leq \Big(\sum_i \mathbb{E}\|\xi_i\|_X^{2s}\Big)^{\frac{1}{2}} + \Big(\sum_i (\mathbb{E}\|\xi_i\|_X^s)^2\Big)^{\frac{1}{2}} \\ &\leq \Big(\sum_i \mathbb{E}\|\xi_i\|_X^{2s}\Big)^{\frac{1}{2}} + \sum_i \mathbb{E}\|\xi_i\|_X^s \\ &\leq 2\max\Big\{\Big(\sum_i \mathbb{E}\|\xi_i\|^p\Big)^{\frac{1}{p}}, \Big(\sum_i \mathbb{E}\|\xi_i\|_X^s\Big)^{\frac{1}{s}}\Big\}^s,\end{aligned}$$

where in the final step we use the contractive embedding

$$l^s(L^s(\Omega;X)) \cap l^p(L^p(\Omega;X)) \subset l^{2s}(L^{2s}(\Omega;X)),$$

which follows from Hölder's inequality. Collecting our estimates, we conclude that

$$\Big(\mathbb{E}\Big(\sum_i \|\xi_i\|_X^s\Big)^{\frac{p}{s}}\Big)^{\frac{1}{p}} \lesssim_{p,s,X} \max\Big\{\Big(\sum_i \mathbb{E}\|\xi_i\|_X^p\Big)^{\frac{1}{p}}, \Big(\sum_i \mathbb{E}\|\xi_i\|_X^s\Big)^{\frac{1}{s}}\Big\}.$$

Suppose now that $1 \leq p \leq s$. In this case we have the contractive embeddings $L^s(\Omega) \subset L^p(\Omega)$ and $l^p \subset l^s$. Hence,

$$\Big(\mathbb{E}\Big(\sum_i \|\xi_i\|_X^s\Big)^{\frac{p}{s}}\Big)^{\frac{1}{p}} \leq \Big(\sum_i \mathbb{E}\|\xi_i\|_X^s\Big)^{\frac{1}{s}}$$

and

$$\Big(\mathbb{E}\Big(\sum_i \|\xi_i\|_X^s\Big)^{\frac{p}{s}}\Big)^{\frac{1}{p}} \leq \Big(\sum_i \mathbb{E}\|\xi_i\|_X^p\Big)^{\frac{1}{p}}.$$

The result now follows by the triangle inequality. □

By combining Lemmas 1.12 and 1.14 we obtain the following estimates.

Theorem 1.15. *Let X be a Banach space with type $1 \leq s \leq 2$ and (ξ_i) be a finite sequence of independent, mean zero X-valued random variables. If $s \leq p < \infty$, then*

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_X^p\Big)^{\frac{1}{p}} \lesssim_{p,s,X} \max\Big\{\Big(\sum_i \mathbb{E}\|\xi_i\|_X^p\Big)^{\frac{1}{p}}, \Big(\sum_i \mathbb{E}\|\xi_i\|_X^s\Big)^{\frac{1}{s}}\Big\}.$$

On the other hand, if $1 \leq p \leq s$, then

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_X^p\Big)^{\frac{1}{p}} \lesssim_{p,s,X} \inf\Big\{\Big(\sum_i \mathbb{E}\|\eta_i\|_X^p\Big)^{\frac{1}{p}} + \Big(\sum_i \mathbb{E}\|\theta_i\|_X^s\Big)^{\frac{1}{s}}\Big\},$$

where the infimum is taken over all sequences $(\eta_i) \in l^p(L^p(\Omega;X))$ and $(\theta_i) \in l^s(L^s(\Omega;X))$ such that $\xi_i = \eta_i + \theta_i$.

Remark 1.16. Note that if the ξ_i in Theorem 1.15 are in addition symmetric, then the conclusion also holds for any quasi-Banach space with type $0 < s \leq 2$ and any $0 < p < \infty$. Indeed, the only place in the proof where it is required that X is a normed space and $p \geq 1$ is in the application of Corollary 1.10. For symmetric random variables, however, the conclusion of Corollary 1.10 is trivial.

Specialized to L^q-spaces we obtain the following result.

Corollary 1.17. *Suppose S is a σ-finite measure space and let (ξ_i) be a finite sequence of independent, mean zero $L^q(S)$-valued random variables. If $2 \leq p < \infty$ and $2 \leq q < \infty$, then*

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}} \lesssim_{p,q} \max\Big\{\Big(\sum_i \mathbb{E}\|\xi_i\|_{L^q(S)}^p\Big)^{\frac{1}{p}}, \Big(\sum_i \mathbb{E}\|\xi_i\|_{L^q(S)}^2\Big)^{\frac{1}{2}}\Big\}.$$

If $1 < q < 2$ and $q \leq p < \infty$, then

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}} \lesssim_{p,q} \max\Big\{\Big(\sum_i \mathbb{E}\|\xi_i\|_{L^q(S)}^p\Big)^{\frac{1}{p}}, \Big(\sum_i \mathbb{E}\|\xi_i\|_{L^q(S)}^q\Big)^{\frac{1}{q}}\Big\}.$$

If $1 < p < q \leq 2$, then

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}} \lesssim_{p,q} \inf\Big\{\Big(\sum_i \mathbb{E}\|\eta_i\|_{L^q(S)}^p\Big)^{\frac{1}{p}} + \Big(\sum_i \mathbb{E}\|\theta_i\|_{L^q(S)}^q\Big)^{\frac{1}{q}}\Big\},$$

where the infimum is taken over all sequences $(\eta_i) \in l^p(L^p(\Omega; L^q(S)))$ *and* $(\theta_i) \in l^q(L^q(\Omega; L^q(S)))$ *such that* $\xi_i = \eta_i + \theta_i$.
Finally, if $1 < p < 2$ *and* $2 \leq q < \infty$, *then*

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(S)}\Big)^{\frac{1}{p}} \lesssim_{p,q} \inf\Big\{\Big(\sum_i \mathbb{E}\|\eta_i\|^p_{L^q(S)}\Big)^{\frac{1}{p}} + \Big(\sum_i \mathbb{E}\|\theta_i\|^2_{L^q(S)}\Big)^{\frac{1}{2}}\Big\},$$

where the infimum is taken over all sequences $(\eta_i) \in l^p(L^p(\Omega; L^q(S)))$ *and* $(\theta_i) \in l^2(L^2(\Omega; L^q(S)))$ *such that* $\xi_i = \eta_i + \theta_i$.

We will now deduce lower bounds for $(\mathbb{E}\|\sum_i \xi_i\|^p_X)^{\frac{1}{p}}$ by duality from Theorem 1.15. We first recall some preliminary facts on duality for intersections and sums of Banach spaces.

Let (X, Y) be a *compatible couple of Banach spaces* (or more briefly, *couple* of Banach spaces), i.e., X, Y are continuously embedded in some Hausdorff topological vector space. Then the intersection $X \cap Y$ and the sum $X + Y$ are Banach spaces under the norms

$$\|z\|_{X\cap Y} = \max\{\|z\|_X, \|z\|_Y\}$$

and

$$\|z\|_{X+Y} = \inf\{\|x\|_X + \|y\|_Y : z = x + y,\ x \in X,\ y \in Y\}.$$

Suppose that $X \cap Y$ is dense in both X and Y. Then we have

$$(X \cap Y)^* = X^* + Y^*, \qquad (X + Y)^* = X^* \cap Y^* \tag{1.7}$$

isometrically. The duality brackets under these identifications are given by

$$\langle x, x^*\rangle = \langle x, x^*|_{X\cap Y}\rangle \qquad (x^* \in X^* + Y^*)$$

and

$$\langle x, x^*\rangle = \langle y, x^*\rangle + \langle z, x^*\rangle \qquad (x^* \in X^* \cap Y^*,\ x = y + z \in X + Y), \tag{1.8}$$

respectively, see e.g. [85], Theorem I.3.1.

Recall that a subset F of X^* is called *norming* for X if

$$\|x\| = \sup\{|\langle x, x^*\rangle| \ : \ x^* \in F,\ \|x^*\| \leq 1\}.$$

Let (S, Σ, μ) be a σ-finite measure space and suppose $1 \leq p, p' \leq \infty$ satisfy $\frac{1}{p} + \frac{1}{p'} = 1$. Then every function $g \in L^{p'}(S; X^*)$ defines an element of $\phi_g \in L^p(S; X)^*$ through the duality bracket

$$\langle f, \phi_g\rangle = \int_S \langle f(s), g(s)\rangle \, d\mu(s) \qquad (f \in L^p(S; X)).$$

In fact, $\|\phi_g\|_{L^p(S;X)^*} = \|g\|_{L^{p'}(S;X^*)}$. It may be shown that the map $g \mapsto \phi_g$ defines an isometry onto a closed subspace of $L^p(S; X)^*$ which is norming for $L^p(S;X)$. In general, this map is not surjective onto $L^p(S;X)^*$. If $1 \le p < \infty$, then the latter holds if and only if X^* has the so-called *Radon-Nikodým property* with respect to (S,Σ,μ). In particular this is the case if X^* is separable or reflexive. We refer to [40], Chapter III for proofs of these facts and a thorough treatment of the Radon-Nikodým property.

In the proof of Theorem 1.20 below we shall use the following observation.

Lemma 1.18. *Let (S,Σ,μ) be a σ-finite measure space, X a Banach space and suppose $1 \le p, p' \le \infty$. If Σ_0 is a sub-σ-algebra of Σ, then, for any $\xi \in L^p(\Omega,\Sigma_0;X)$ and $\eta \in L^{p'}(\Omega,\Sigma;X^*)$,*

$$\mathbb{E}(\langle \xi, \eta\rangle | \Sigma_0) = \langle \xi, \mathbb{E}(\eta|\Sigma_0)\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality bracket for X, X^*.

Proof. Suppose first that η is a simple function, i.e. $\eta = \sum_j \chi_{B_j} x_j^*$. For any $\mathcal{A} \in \Sigma_0$ we have

$$\begin{aligned}
\int_{\mathcal{A}} \langle \xi, \eta\rangle d\mu &= \int_{\mathcal{A}} \sum_j \chi_{B_j} \langle \xi, x_j^*\rangle d\mu \\
&= \int_{\mathcal{A}} \sum_j \mathbb{E}(\chi_{B_j}\langle \xi, x_j^*\rangle | \Sigma_0) d\mu \\
&= \int_{\mathcal{A}} \sum_j \mathbb{E}(\chi_{B_j}|\Sigma_0)\langle \xi, x_j^*\rangle d\mu \\
&= \int_{\mathcal{A}} \Big\langle \xi, \sum_j \mathbb{E}(\chi_{B_j}|\Sigma_0) x_j^* \Big\rangle d\mu \\
&= \int_{\mathcal{A}} \langle \xi, \mathbb{E}(\eta|\Sigma_0)\rangle d\mu.
\end{aligned}$$

By approximation we conclude that

$$\int_{\mathcal{A}} \langle \xi, \eta\rangle d\mu = \int_{\mathcal{A}} \langle \xi, \mathbb{E}(\eta|\Sigma_0)\rangle d\mu,$$

for any $\eta \in L^{p'}(\Omega; X^*)$. □

The following lemma is probably known to experts. We provide a proof of this result as we have not been able to trace a reference.

Lemma 1.19. *Let X be a Banach space and let (S,Σ,μ) be a σ-finite measure space. Suppose $1 < p, p', s, s' < \infty$ satisfy $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{s} + \frac{1}{s'} = 1$. Then the space $L^{p'}(S;X^*) \cap L^{s'}(S;X^*)$ is norming for $L^p(S;X) + L^s(S;X)$. The corresponding duality bracket is given by*

$$\langle f, g\rangle = \int_S \langle f(s), g(s)\rangle \; d\mu(s), \tag{1.9}$$

for any $f \in L^p(S;X) + L^s(S;X)$ and $g \in L^{p'}(S;X^*) \cap L^{s'}(S;X^*))$.

Proof. Note that the result holds if Σ is finite. Indeed, then

$$(L^p(S;X)+L^s(S;X))^* = L^p(S;X)^* \cap L^s(S;X)^* = L^{p'}(S;X^*) \cap L^{s'}(S;X^*).$$

For the general case we define a norm on $L^p(S;X)+L^q(S;X)$ by

$$\|f\|_* = \sup\{\langle f, x^*\rangle \ : \ x^* \in L^{p'}(S;X^*) \cap L^{s'}(S;X^*),\ \|x^*\| \le 1\}.$$

Since $L^{p'}(S;X^*)$ and $L^{s'}(S;X^*)$ isometrically embed into $L^p(S;X)^*$ and $L^s(S;X)^*$, respectively, we clearly have

$$\begin{aligned}\|f\|_* &\le \sup\{\langle f, x^*\rangle \ : \ x^* \in L^p(S;X)^* \cap L^s(S;X)^*,\ \|x^*\| \le 1\} \\ &= \|f\|_{L^p(S;X)+L^s(S;X)}.\end{aligned}$$

For the reverse inequality, suppose first that $f \in L^p(S;X)+L^s(S;X)$ is a simple function and let $g \in L^p(S;X)$ and $h \in L^s(S;X)$ be such that $f = g+h$. Then $f = \mathbb{E}(g|f) + \mathbb{E}(h|f)$ and moreover,

$$\|\mathbb{E}(g|f)\|_{L^p(S;X)} + \|\mathbb{E}(h|f)\|_{L^s(S;X)} \le \|g\|_{L^p(S;X)} + \|h\|_{L^s(S;X)}.$$

Thus, we can compute the norm of f using only $\sigma(f)$-measurable functions. Since $\sigma(f)$ is finitely generated, we have

$$\begin{aligned}&\|f\|_{L^p(S;X)+L^s(S;X)} \\ &\quad = \|f\|_{L^p(\sigma(f);X)+L^s(\sigma(f);X)} \\ &\quad = \sup\{\langle f, x^*\rangle \ : \ x^* \in L^{p'}(\sigma(f);X^*) \cap L^{s'}(\sigma(f);X^*),\ \|x^*\| \le 1\} \le \|f\|_*.\end{aligned}$$

If $f \in L^p(S;X)+L^s(S;X)$ then there is a sequence of simple functions f_n such that $f_n \to f$ in $L^p(S;X)+L^s(S;X)$. By the above,

$$\begin{aligned}\|f\|_* &\ge \|f_n\|_* - \|f-f_n\|_* \\ &\ge \|f_n\|_{L^p(S;X)+L^s(S;X)} - \|f-f_n\|_{L^p(S;X)+L^s(S;X)} \\ &\to \|f\|_{L^p(S;X)+L^s(S;X)},\end{aligned}$$

as $n \to \infty$.

Finally, recall that for $r = p, s$ the duality bracket between $L^r(S;X)$ and the norming subspace $L^{r'}(S;X^*)$ of $L^r(S;X)^*$ is given by

$$\langle f, g\rangle = \int_S \langle f(s), g(s)\rangle \, d\mu(s) \qquad (f \in L^r(S;X),\ g \in L^{r'}(S,X^*)).$$

Hence (1.9) follows immediately from (1.8). □

Theorem 1.20. *Let X be a Banach space with cotype $2 \le s < \infty$ and (ξ_i) be a finite sequence of independent, mean zero X-valued random variables. If $s \le p < \infty$, then*

$$\max\Big\{\Big(\sum_i \mathbb{E}\|\xi_i\|_X^p\Big)^{\frac{1}{p}}, \Big(\sum_i \mathbb{E}\|\xi_i\|_X^s\Big)^{\frac{1}{s}}\Big\} \lesssim_{p,s,X} \Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_X^p\Big)^{\frac{1}{p}}.$$

On the other hand, if $1 < p \le s$ *and* X *is K-convex, then*

$$\inf\Big\{\Big(\sum_i \mathbb{E}\|\eta_i\|_X^p\Big)^{\frac{1}{p}} + \Big(\sum_i \mathbb{E}\|\theta_i\|_X^s\Big)^{\frac{1}{s}}\Big\} \lesssim_{p,s,X} \Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_X^p\Big)^{\frac{1}{p}},$$

where the infimum is taken over all sequences $(\eta_i) \in l^p(L^p(\Omega;X))$ *and* $(\theta_i) \in l^s(L^s(\Omega;X))$ *such that* $\xi_i = \eta_i + \theta_i$.

Proof. Suppose first that $s \le p < \infty$. Since X has cotype s, it also has cotype p. Hence the first assertion follows immediately from Lemma 1.11.

We deduce the estimate in the case $1 < p \le s$ by duality from Theorem 1.15. Since X is K-convex and has cotype s, its dual X^* has type $s' \le p' < \infty$, where $\frac{1}{s} + \frac{1}{s'} = 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Let (η_i^*) be a finite sequence of X^*-valued random variables satisfying

$$\|(\eta_i)\|_* = \max\Big\{\Big(\sum_i \mathbb{E}\|\xi_i\|_{X^*}^{p'}\Big)^{\frac{1}{p'}}, \Big(\sum_i \mathbb{E}\|\xi_i\|_{X^*}^{s'}\Big)^{\frac{1}{s'}}\Big\} \le 1.$$

Then $(\mathbb{E}(\eta_i^*|\xi_i) - \mathbb{E}(\eta_i^*))$ is a finite sequence of independent, mean zero X^*-valued random variables. Therefore, by Lemma 1.18 and Theorem 1.15 we have

$$\begin{aligned}
\langle(\xi_i),(\eta_i^*)\rangle &= \sum_i \langle \xi_i, \eta_i^*\rangle \\
&= \sum_i \langle \xi_i, \mathbb{E}(\eta_i^*|\xi_i) - \mathbb{E}(\eta_i^*)\rangle \\
&= \sum_{i,j} \langle \xi_i, \mathbb{E}(\eta_j^*|\xi_j) - \mathbb{E}(\eta_j^*)\rangle \\
&= \Big\langle \sum_i \xi_i, \sum_j \mathbb{E}(\eta_j^*|\xi_j) - \mathbb{E}(\eta_j^*)\Big\rangle \\
&\le \Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_X^p\Big)^{\frac{1}{p}} \Big(\mathbb{E}\Big\|\sum_j \mathbb{E}(\eta_j^*|\xi_j) - \mathbb{E}(\eta_j^*)\Big\|_{X^*}^{p'}\Big)^{\frac{1}{p'}} \\
&\lesssim_{p',s',X^*} \Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_X^p\Big)^{\frac{1}{p}} \|(\mathbb{E}(\eta_j^*|\xi_j) - \mathbb{E}(\eta_j^*))\|_* \\
&\le 2\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_X^p\Big)^{\frac{1}{p}}.
\end{aligned}$$

Taking the supremum over all (η_i^*) as above we obtain by Lemma 1.19,

$$\|(\xi_i)\|_{l^p(L^p(\Omega;X))+l^s(L^s(\Omega;X))} \lesssim_{p,s,X} \Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_X^p\Big)^{\frac{1}{p}},$$

as asserted. □

Specialized to L^q-spaces Theorem 1.20 yields the following.

Corollary 1.21. *Suppose S is a σ-finite measure space and let (ξ_i) be a finite sequence of independent, mean zero $L^q(S)$-valued random variables. If $2 \le q \le p < \infty$, then*

$$\max\Big\{\Big(\sum_i \mathbb{E}\|\xi_i\|^p_{L^q(S)}\Big)^{\frac{1}{p}}, \Big(\sum_i \mathbb{E}\|\xi_i\|^q_{L^q(S)}\Big)^{\frac{1}{q}}\Big\} \lesssim_{p,q} \Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(S)}\Big)^{\frac{1}{p}}.$$

If $1 < q < 2$ and $2 \le p < \infty$, then

$$\max\Big\{\Big(\sum_i \mathbb{E}\|\xi_i\|^p_{L^q(S)}\Big)^{\frac{1}{p}}, \Big(\sum_i \mathbb{E}\|\xi_i\|^2_{L^q(S)}\Big)^{\frac{1}{2}}\Big\} \lesssim_{p,q} \Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(S)}\Big)^{\frac{1}{p}}.$$

If $1 < p, q \le 2$, then

$$\inf\Big\{\Big(\sum_i \mathbb{E}\|\eta_i\|^p_{L^q(S)}\Big)^{\frac{1}{p}} + \Big(\sum_i \mathbb{E}\|\theta_i\|^2_{L^q(S)}\Big)^{\frac{1}{2}}\Big\} \lesssim_{p,q} \Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(S)}\Big)^{\frac{1}{p}},$$

where the infimum is taken over all sequences $(\eta_i) \in l^p(L^p(\Omega; L^q(S)))$ and $(\theta_i) \in l^2(L^2(\Omega; L^q(S)))$ such that $\xi_i = \eta_i + \theta_i$.
Finally, if $1 < p < q$ and $2 \le q < \infty$, then

$$\inf\Big\{\Big(\sum_i \mathbb{E}\|\eta_i\|^p_{L^q(S)}\Big)^{\frac{1}{p}} + \Big(\sum_i \mathbb{E}\|\theta_i\|^q_{L^q(S)}\Big)^{\frac{1}{q}}\Big\} \lesssim_{p,q} \Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(S)}\Big)^{\frac{1}{p}},$$

where the infimum is taken over all sequences $(\eta_i) \in l^p(L^p(\Omega; L^q(S)))$ and $(\theta_i) \in l^q(L^q(\Omega; L^q(S)))$ such that $\xi_i = \eta_i + \theta_i$.

In the case where X has both type 2 and cotype 2 we obtain two-sided estimates for $(\mathbb{E}\|\sum_i \xi_i\|^p_X)^{\frac{1}{p}}$. Recall that by Theorem 1.5 such a space is isomorphic to a Hilbert space.

Corollary 1.22. *Let $2 \le p < \infty$ and H be a Hilbert space. If (ξ_i) is a finite sequence of independent, mean zero H-valued random variables, then*

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_H\Big)^{\frac{1}{p}} \simeq_p \max\Big\{\Big(\sum_i \mathbb{E}\|\xi_i\|^p_H\Big)^{\frac{1}{p}}, \Big(\sum_i \mathbb{E}\|\xi_i\|^2_H\Big)^{\frac{1}{2}}\Big\}. \tag{1.10}$$

On the other hand, if $1 < p < 2$, then

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_H\Big)^{\frac{1}{p}} \simeq_p \inf\Big\{\Big(\sum_i \mathbb{E}\|\eta_i\|^p_H\Big)^{\frac{1}{p}} + \Big(\sum_i \mathbb{E}\|\theta_i\|^2_H\Big)^{\frac{1}{2}}\Big\}, \tag{1.11}$$

where the infimum is taken over all sequences $(\eta_i) \in l^p(L^p(\Omega; H))$ and $(\theta_i) \in l^2(L^2(\Omega; H))$ such that $\xi_i = \eta_i + \theta_i$.

The inequality $\lesssim_p$ in (1.10) was also obtained in [112], Theorem 5.2.

1.3 Rosenthal inequalities for L^p-valued random variables

We now proceed to prove Rosenthal-type inequalities for random vectors in $L^q(S)$, where S is a σ-finite measure space. We start by making two elementary observations. First, note that by combining Theorem 1.2 and Corollary 1.10 we obtain the following result.

Lemma 1.23. *Let $1 \leq p < \infty$ and $1 \leq q < \infty$. Let (ξ_i) be a finite sequence of independent, mean zero $L^q(S)$-valued random variables. Then,*

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}} \simeq_{p,q} \Big(\mathbb{E}\Big\|\Big(\sum_i |\xi_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}}.$$

As a consequence, we find the following useful estimates.

Lemma 1.24. *Suppose $1 \leq p, q < 2$. Let (ξ_i) be a finite sequence of independent, mean zero $L^q(S)$-valued random variables. Then,*

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}} \lesssim_{p,q} \Big\|\Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(S)}.$$

On the other hand, if $2 \leq p, q < \infty$ then

$$\Big\|\Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(S)} \lesssim_{p,q} \Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}}.$$

Proof. Suppose $1 \leq p, q < 2$. By Lemma 1.23,

$$\begin{aligned}
\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}} &\simeq_{p,q} \Big(\mathbb{E}\Big\|\Big(\sum_i |\xi_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}} \\
&= \Big(\mathbb{E}\Big\|\sum_i |\xi_i|^2\Big\|_{L^{\frac{q}{2}}(S)}^{\frac{p}{2}}\Big)^{\frac{1}{p}} \\
&\leq \Big\|\sum_i \mathbb{E}|\xi_i|^2\Big\|_{L^{\frac{q}{2}}(S)}^{\frac{1}{2}} \\
&= \Big\|\Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(S)}.
\end{aligned}$$

Note that in the final inequality we apply Jensen's inequality, using that $\frac{p}{2}, \frac{q}{2} < 1$. If we assume $2 \leq p, q < \infty$ then this inequality is reversed. This completes the proof. □

The following Lemma is the key to the Rosenthal-type inequalities in the cases where $2 \leq p, q < \infty$.

Lemma 1.25. *Suppose* $0 < s \le 2$ *and* $s \le p, q < \infty$. *Let* (ξ_i) *be a finite sequence of independent, mean zero* $L^q(S)$*-valued random variables. Then,*

$$\begin{aligned}&\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(S)}\Big)^{\frac{1}{p}}\\ &\qquad \lesssim_{p,q,s} \max\Big\{\Big\|\Big(\sum_i \mathbb{E}|\xi_i|^s\Big)^{\frac{1}{s}}\Big\|_{L^q(S)}, \Big(\mathbb{E}\Big(\sum_i \|\xi_i\|^q_{L^q(S)}\Big)^{\frac{p}{q}}\Big)^{\frac{1}{p}}\Big\}. \qquad (1.12)\end{aligned}$$

Proof. By Lemma 1.23,

$$\begin{aligned}\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(S)}\Big)^{\frac{1}{p}} &\eqsim_{p,q} \Big(\mathbb{E}\Big\|\Big(\sum_i |\xi_i|^2\Big)^{\frac{1}{2}}\Big\|^p_{L^q(S)}\Big)^{\frac{1}{p}}\\ &\le \Big(\mathbb{E}\Big\|\Big(\sum_i |\xi_i|^s\Big)^{\frac{1}{s}}\Big\|^p_{L^q(S)}\Big)^{\frac{1}{p}}. \qquad (1.13)\end{aligned}$$

By the triangle inequality we obtain

$$\begin{aligned}&\Big(\mathbb{E}\Big\|\Big(\sum_i |\xi_i|^s\Big)^{\frac{1}{s}}\Big\|^p_{L^q(S)}\Big)^{\frac{1}{p}}\\ &\quad = \Big(\mathbb{E}\Big\|\sum_i |\xi_i|^s\Big\|^{\frac{p}{s}}_{L^{\frac{q}{s}}(S)}\Big)^{\frac{1}{p}}\\ &\quad \le \Big(\Big(\mathbb{E}\Big\|\sum_i |\xi_i|^s - \mathbb{E}|\xi_i|^s\Big\|^{\frac{p}{s}}_{L^{\frac{q}{s}}(S)}\Big)^{\frac{s}{p}} + \Big\|\sum_i \mathbb{E}|\xi_i|^s\Big\|_{L^{\frac{q}{s}}(S)}\Big)^{\frac{1}{s}}. \qquad (1.14)\end{aligned}$$

Suppose first that $q \le 2s$. Then, by Lemma 1.12,

$$\begin{aligned}&\Big(\mathbb{E}\Big\|\sum_i |\xi_i|^s - \mathbb{E}|\xi_i|^s\Big\|^{\frac{p}{s}}_{L^{\frac{q}{s}}(S)}\Big)^{\frac{s}{p}}\\ &\quad \lesssim_{p,q,s} \Big(\mathbb{E}\Big(\sum_i \|\,|\xi_i|^s - \mathbb{E}|\xi_i|^s\|^{\frac{q}{s}}_{L^{\frac{q}{s}}(S)}\Big)^{\frac{p}{q}}\Big)^{\frac{s}{p}}\\ &\quad \le \Big(\mathbb{E}\Big(\sum_i \|\xi_i\|^q_{L^q(S)}\Big)^{\frac{p}{q}}\Big)^{\frac{s}{p}} + \Big(\mathbb{E}\Big(\sum_i \|\mathbb{E}|\xi_i|^s\|^{\frac{q}{s}}_{L^{\frac{q}{s}}(S)}\Big)^{\frac{p}{q}}\Big)^{\frac{s}{p}}\\ &\quad \le 2\Big(\mathbb{E}\Big(\sum_i \|\xi_i\|^q_{L^q(S)}\Big)^{\frac{p}{q}}\Big)^{\frac{s}{p}},\end{aligned}$$

where in the final step we apply Jensen's inequality. By (1.13) and (1.14) this proves (1.12).

Suppose now that $q > 2s$. By applying Lemma 1.23 we find that

$$\Big(\mathbb{E}\Big\|\sum_i |\xi_i|^s - \mathbb{E}|\xi_i|^s\Big\|^{\frac{p}{s}}_{L^{\frac{q}{s}}(S)}\Big)^{\frac{s}{p}}$$

$$\begin{aligned}
&\simeq_{p,q,s} \Big(\mathbb{E}\Big\|\Big(\sum_i \big|\, |\xi_i|^s - \mathbb{E}|\xi_i|^s|^2\Big)^{\frac12}\Big\|^{\frac{p}{s}}_{L^{\frac{q}{s}}(S)}\Big)^{\frac{s}{p}} \\
&\le \Big(\mathbb{E}\Big\|\Big(\sum_i |\xi_i|^{2s}\Big)^{\frac12}\Big\|^{\frac{p}{s}}_{L^{\frac{q}{s}}(S)}\Big)^{\frac{s}{p}} + \Big\|\Big(\sum_i |\mathbb{E}|\xi_i|^s|^2\Big)^{\frac12}\Big\|_{L^{\frac{q}{s}}(S)} \\
&\le \Big(\mathbb{E}\Big\|\Big(\sum_i |\xi_i|^{2s}\Big)^{\frac{1}{2s}}\Big\|^{p}_{L^{q}(S)}\Big)^{\frac{s}{p}} + \Big\|\sum_i \mathbb{E}|\xi_i|^s\Big\|_{L^{\frac{q}{s}}(S)}. \qquad (1.15)
\end{aligned}$$

Since $q > 2s$ there is some $0 < \theta < \frac12$ such that $\frac{1}{2s} = \frac{\theta}{s} + \frac{1-\theta}{q}$. By applying Hölder's inequality three times we obtain

$$\begin{aligned}
&\Big(\mathbb{E}\Big\|\Big(\sum_i |\xi_i|^{2s}\Big)^{\frac{1}{2s}}\Big\|^{p}_{L^{q}(S)}\Big)^{\frac{s}{p}} \\
&\quad\le \Big(\mathbb{E}\Big\|\Big(\sum_i |\xi_i|^{s}\Big)^{\frac{\theta}{s}}\Big(\sum_i |\xi_i|^{q}\Big)^{\frac{1-\theta}{q}}\Big\|^{p}_{L^{q}(S)}\Big)^{\frac{s}{p}} \\
&\quad\le \Big(\mathbb{E}\Big(\Big\|\Big(\sum_i |\xi_i|^{s}\Big)^{\frac{\theta}{s}}\Big\|_{L^{\frac{q}{\theta}}(S)}\Big\|\Big(\sum_i |\xi_i|^{q}\Big)^{\frac{1-\theta}{q}}\Big\|_{L^{\frac{q}{1-\theta}}(S)}\Big)^{p}\Big)^{\frac{s}{p}} \\
&\quad\le \Big(\mathbb{E}\Big\|\Big(\sum_i |\xi_i|^{s}\Big)^{\frac{\theta}{s}}\Big\|^{\frac{p}{\theta}}_{L^{\frac{q}{\theta}}(S)}\Big)^{\frac{s\theta}{p}}\Big(\mathbb{E}\Big\|\Big(\sum_i |\xi_i|^{q}\Big)^{\frac{1-\theta}{q}}\Big\|^{\frac{p}{1-\theta}}_{L^{\frac{q}{1-\theta}}(S)}\Big)^{\frac{s(1-\theta)}{p}} \\
&\quad= \Big(\mathbb{E}\Big\|\Big(\sum_i |\xi_i|^{s}\Big)^{\frac{1}{s}}\Big\|^{p}_{L^{q}(S)}\Big)^{\frac{s\theta}{p}}\Big(\mathbb{E}\Big\|\Big(\sum_i |\xi_i|^{q}\Big)^{\frac{1}{q}}\Big\|^{p}_{L^{q}(S)}\Big)^{\frac{s(1-\theta)}{p}}. \qquad (1.16)
\end{aligned}$$

Combining (1.14), (1.15) and (1.16) we arrive at the inequality

$$a^2 \lesssim_{p,q,s} a^{2\theta} b^{2(1-\theta)} + c^2,$$

where $a = (\mathbb{E}\|(\sum_i |\xi_i|^s)^{\frac1s}\|^p_{L^q(S)})^{\frac{s}{2p}}$, $b = (\mathbb{E}(\sum_i \|\xi_i\|^q_{L^q(S)})^{\frac{p}{q}})^{\frac{s}{2p}}$ and $c = \|(\sum_i \mathbb{E}|\xi_i|^s)^{\frac1s}\|^{\frac{s}{2}}_{L^q(S)}$. Notice that if $a \le b$ then (1.12) immediately follows from (1.13). Hence, we may assume $a > b$. Since $0 < 2\theta < 1$ we then have

$$a^{2\theta} b^{2(1-\theta)} = b^2 (\tfrac{a}{b})^{2\theta} \le ab.$$

Thus we obtain the inequality

$$a^2 \lesssim_{p,q,s} ab + c^2.$$

Solving this quadratic equation we find that $a \lesssim_{p,q,s} \max\{b, c\}$ and hence $a^{\frac2s} \lesssim_{p,q,s} \max\{b^{\frac2s}, c^{\frac2s}\}$. That is,

$$\begin{aligned}
&\Big(\mathbb{E}\Big\|\Big(\sum_i |\xi_i|^{s}\Big)^{\frac{1}{s}}\Big\|^{p}_{L^{q}(S)}\Big)^{\frac{1}{p}} \\
&\qquad\lesssim_{p,q,s} \max\Big\{\Big(\mathbb{E}\Big(\sum_i \|\xi_i\|^q_{L^q(S)}\Big)^{\frac{p}{q}}\Big)^{\frac1p}, \Big\|\Big(\sum_i \mathbb{E}|\xi_i|^s\Big)^{\frac1s}\Big\|_{L^q(S)}\Big\}.
\end{aligned}$$

The result now follows from (1.13). This completes the proof. $\square$

We can now deduce estimates in the case where $p, q \geq 2$.

Theorem 1.26. *Suppose $2 \leq q \leq p < \infty$. Let (ξ_i) be a finite sequence of independent, mean zero $L^q(S)$-valued random variables. Then,*

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(S)}\Big)^{\frac{1}{p}} \simeq_{p,q} \max\Big\{\Big\|\Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(S)}, \Big(\sum_i \mathbb{E}\|\xi_i\|^p_{L^q(S)}\Big)^{\frac{1}{p}}, \Big(\sum_i \mathbb{E}\|\xi_i\|^q_{L^q(S)}\Big)^{\frac{1}{q}}\Big\}.$$

Proof. We have already proved that the maximum on the right hand side is dominated by $(\mathbb{E}\|\sum_i \xi_i\|^p_{L^q(S)})^{\frac{1}{p}}$. Indeed, by Corollary 1.21 we have

$$\max\Big\{\Big(\sum_i \mathbb{E}\|\xi_i\|^p_{L^q(S)}\Big)^{\frac{1}{p}}, \Big(\sum_i \mathbb{E}\|\xi_i\|^q_{L^q(S)}\Big)^{\frac{1}{q}}\Big\} \lesssim_{p,q} \Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(S)}\Big)^{\frac{1}{p}}$$

and in Lemma 1.24 we showed that

$$\Big\|\Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(S)} \lesssim_{p,q} \Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(S)}\Big)^{\frac{1}{p}}.$$

For the reverse we apply Lemma 1.25 with $s = 2$ and obtain

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(S)}\Big)^{\frac{1}{p}} \lesssim_{p,q} \max\Big\{\Big\|\Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(S)}, \Big(\mathbb{E}\Big(\sum_i \|\xi_i\|^q_{L^q(S)}\Big)^{\frac{p}{q}}\Big)^{\frac{1}{p}}\Big\}.$$

The result follows by applying Lemma 1.14 with $s = q$ and $X = L^q(S)$. $\square$

Remark 1.27. If $2 \leq q \leq p < \infty$ and $(\xi_i)_{i\geq 1}$ is an infinite sequence of independent, mean zero $L^q(S)$-valued random variables, then it follows from Theorem 1.26 that $\sum_{i\geq 1} \xi_i$ converges in $L^p(\Omega; L^q(S))$ if and only if

$$(\xi_i)_{i\geq 1} \in L^q(S; l^2(L^2(\Omega))) \cap l^p(L^p(\Omega; L^q(S))) \cap l^q(L^q(\Omega; L^q(S))).$$

Theorem 1.28. *Suppose $2 \leq p \leq q < \infty$. Let (ξ_i) be a finite sequence of independent, mean zero $L^q(S)$-valued random variables. Then,*

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(S)}\Big)^{\frac{1}{p}} \simeq_{p,q} \max\Big\{\Big\|\Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(S)}, \inf\Big\{\Big(\sum_i \mathbb{E}\|\eta_i\|^p_{L^q(S)}\Big)^{\frac{1}{p}} + \Big(\sum_i \mathbb{E}\|\theta_i\|^q_{L^q(S)}\Big)^{\frac{1}{q}}\Big\}\Big\},$$

where the infimum is taken over all sequences $(\eta_i) \in l^p(L^p(\Omega; L^q(S)))$ and $(\theta_i) \in l^q(L^q(\Omega; L^q(S)))$ such that $\xi_i = \eta_i + \theta_i$.

Proof. In Lemma 1.24 we proved

$$\Big\|\Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(S)} \lesssim_{p,q} \Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}},$$

and by Corollary 1.21,

$$\|(\xi_i)\|_{l^p(L^p(\Omega;L^q(S)))+l^q(L^q(\Omega;L^q(S)))} \lesssim_{p,q} \Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}}.$$

For the reverse inequality note that by applying Lemma 1.25 with $s = 2$ we obtain

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}} \lesssim_{p,q} \max\Big\{\Big\|\Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(S)}, \Big(\mathbb{E}\Big(\sum_i \|\xi_i\|_{L^q(S)}^q\Big)^{\frac{p}{q}}\Big)^{\frac{1}{p}}\Big\}.$$

The result follows by applying Lemma 1.14 with $s = q$. □

By duality we can deduce results for the cases $1 < p, q \le 2$.

Theorem 1.29. *Suppose $1 < p \le q \le 2$. Let (ξ_i) be a finite sequence of independent, mean zero $L^q(S)$-valued random variables. Then,*

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}} \simeq_{p,q} \inf\Big\{\Big\|\Big(\sum_i \mathbb{E}|\eta_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(S)} + \Big(\sum_i \mathbb{E}\|\theta_i\|_{L^q(S)}^p\Big)^{\frac{1}{p}} + \Big(\sum_i \mathbb{E}\|\kappa_i\|_{L^q(S)}^q\Big)^{\frac{1}{q}}\Big\}, \tag{1.17}$$

where the infimum is taken over all sequences $(\eta_i) \in L^q(S; l^2(L^2(\Omega)))$, $(\theta_i) \in l^p(L^p(\Omega; L^q(S)))$ and $(\kappa_i) \in l^q(L^q(\Omega; L^q(S)))$ such that $\xi_i = \eta_i + \theta_i + \kappa_i$.

Proof. Recall from Corollary 1.17 we have

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}} \lesssim_{p,q} \|(\xi_i)\|_{l^p(L^p(\Omega;L^q(S)))+l^q(L^q(\Omega;L^q(S)))},$$

and by Lemma 1.24,

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}} \lesssim_{p,q} \Big\|\Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(S)}.$$

Suppose we are given $(\eta_i) \in L^q(S; l^2(L^2(\Omega)))$, $(\theta_i) \in l^p(L^p(\Omega; L^q(S)))$ and $(\kappa_i) \in l^q(L^q(\Omega; L^q(S)))$ such that $\xi_i = \eta_i + \theta_i + \kappa_i$. Then

$$\xi_i = \mathbb{E}(\eta_i|\xi_i) - \mathbb{E}(\eta_i) + \mathbb{E}(\theta_i|\xi_i) - \mathbb{E}(\theta_i) + \mathbb{E}(\kappa_i|\xi_i) - \mathbb{E}(\kappa_i).$$

Moreover, $(\mathbb{E}(\eta_i|\xi_i) - \mathbb{E}(\eta_i))$, $(\mathbb{E}(\theta_i|\xi_i) - \mathbb{E}(\theta_i))$ and $(\mathbb{E}(\kappa_i|\xi_i) - \mathbb{E}(\kappa_i))$ are sequences of independent, mean zero random variables. Therefore, by the triangle inequality and the two estimates above,

$$\begin{aligned}
&\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}} \\
&\lesssim_{p,q} \Big\|\Big(\sum_i \mathbb{E}|\mathbb{E}(\eta_i|\xi_i) - \mathbb{E}(\eta_i)|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(S)} + \Big(\sum_i \mathbb{E}\|\mathbb{E}(\theta_i|\xi_i) - \mathbb{E}(\theta_i)\|_{L^q(S)}^p\Big)^{\frac{1}{p}} \\
&\qquad + \Big(\sum_i \mathbb{E}\|\mathbb{E}(\kappa_i|\xi_i) - \mathbb{E}(\kappa_i)\|_{L^q(S)}^q\Big)^{\frac{1}{q}} \\
&\leq \Big\|\Big(\sum_i \mathbb{E}|\eta_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(S)} + \Big(\sum_i \mathbb{E}\|\theta_i\|_{L^q(S)}^p\Big)^{\frac{1}{p}} + \Big(\sum_i \mathbb{E}\|\kappa_i\|_{L^q(S)}^q\Big)^{\frac{1}{q}},
\end{aligned}$$

where in the final estimate we apply Jensen's inequality and contractivity of conditional expectations. This concludes the proof that $(\mathbb{E}\|\sum_i \xi_i\|_{L^q(S)}^p)^{\frac{1}{p}}$ is dominated by the infimum on the right hand side of (1.17).

We deduce the reverse inequality by duality from Theorem 1.26. Let $2 \leq q' \leq p' < \infty$ be such that $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Let (η_i^*) be a sequence satisfying $\|(\eta_i^*)\|_* \leq 1$, where

$$\begin{aligned}
\|(\eta_i^*)\|_* := \max\Big\{&\Big\|\Big(\sum_i \mathbb{E}|\eta_i^*|^2\Big)^{\frac{1}{2}}\Big\|_{L^{q'}(S)}, \\
&\Big(\sum_i \mathbb{E}\|\eta_i^*\|_{L^{q'}(S)}^{p'}\Big)^{\frac{1}{p'}}, \Big(\sum_i \mathbb{E}\|\eta_i^*\|_{L^{q'}(S)}^{q'}\Big)^{\frac{1}{q'}}\Big\}.
\end{aligned}$$

Then, by Lemma 1.18 and Theorem 1.26 we have

$$\begin{aligned}
\langle(\xi_i), (\eta_i^*)\rangle &= \sum_i \langle \xi_i, \eta_i^*\rangle \\
&= \sum_i \langle \xi_i, \mathbb{E}(\eta_i^*|\xi_i) - \mathbb{E}(\eta_i^*)\rangle \\
&= \sum_{i,j} \langle \xi_i, \mathbb{E}(\eta_j^*|\xi_j) - \mathbb{E}(\eta_j^*)\rangle \\
&= \Big\langle \sum_i \xi_i, \sum_j \mathbb{E}(\eta_j^*|\xi_j) - \mathbb{E}(\eta_j^*)\Big\rangle \\
&\leq \Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}} \Big(\mathbb{E}\Big\|\sum_j \mathbb{E}(\eta_j^*|\xi_j) - \mathbb{E}(\eta_j^*)\Big\|_{L^{q'}(S)}^{p'}\Big)^{\frac{1}{p'}} \\
&\lesssim_{p',q'} \Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}} \|(\mathbb{E}(\eta_j^*|\xi_j) - \mathbb{E}(\eta_j^*))\|_*
\end{aligned}$$

$$\leq \quad 2\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(S)}\Big)^{\frac{1}{p}}.$$

If we now take the supremum over all (η_i^*) as above we obtain

$$\|(\xi_i)\|_{L^q(S;l^2(L^2(\Omega)))+l^p(L^p(\Omega;L^q(S)))+l^q(L^q(\Omega;L^q(S)))} \lesssim_{p,q} \Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(S)}\Big)^{\frac{1}{p}},$$

as required. □

Theorem 1.30. *Suppose $1 < q \leq p \leq 2$. Let (ξ_i) be a finite sequence of independent, mean zero $L^q(S)$-valued random variables. Then,*

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(S)}\Big)^{\frac{1}{p}} \simeq_{p,q} \inf\Big\{\Big\|\Big(\sum_i \mathbb{E}|\eta_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(S)} + \max\Big\{\Big(\sum_i \mathbb{E}\|\theta_i\|^p_{L^q(S)}\Big)^{\frac{1}{p}}, \Big(\sum_i \mathbb{E}\|\theta_i\|^q_{L^q(S)}\Big)^{\frac{1}{q}}\Big\}\Big\},$$

where the infimum is taken over all sequences $(\eta_i) \in L^q(S;l^2(L^2(\Omega)))$ and $(\theta_i) \in l^p(L^p(\Omega;L^q(S))) \cap l^q(L^q(\Omega;L^q(S)))$ such that $\xi_i = \eta_i + \theta_i$.

Proof. Let $\xi_i = \eta_i + \theta_i$, then $\xi_i = \mathbb{E}(\eta_i|\xi_i) - \mathbb{E}(\eta_i) + \mathbb{E}(\theta_i|\xi_i) - \mathbb{E}(\theta_i)$. Notice that $(\mathbb{E}(\eta_i|\xi_i) - \mathbb{E}(\eta_i))$ and $(\mathbb{E}(\theta_i|\xi_i) - \mathbb{E}(\theta_i))$ are sequences of independent, mean zero random variables. By the triangle inequality, Lemma 1.24 and Corollary 1.17 we have

$$\begin{aligned}
&\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(S)}\Big)^{\frac{1}{p}} \\
&\lesssim_{p,q} \Big\|\Big(\sum_i \mathbb{E}|\mathbb{E}(\eta_i|\xi_i) - \mathbb{E}(\eta_i)|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(S)} \\
&\qquad + \max\Big\{\Big(\sum_i \mathbb{E}\|\mathbb{E}(\theta_i|\xi_i) - \mathbb{E}(\theta_i)\|^p_{L^q(S)}\Big)^{\frac{1}{p}}, \\
&\qquad\qquad \Big(\sum_i \mathbb{E}\|\mathbb{E}(\theta_i|\xi_i) - \mathbb{E}(\theta_i)\|^q_{L^q(S)}\Big)^{\frac{1}{q}}\Big\} \\
&\leq \Big\|\Big(\sum_i \mathbb{E}|\eta_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(S)} + \max\Big\{\Big(\sum_i \mathbb{E}\|\theta_i\|^p_{L^q(S)}\Big)^{\frac{1}{p}}, \Big(\sum_i \mathbb{E}\|\theta_i\|^q_{L^q(S)}\Big)^{\frac{1}{q}}\Big\},
\end{aligned}$$

where in the final step we use Jensen's inequality and contractivity of vector-valued conditional expectations. This proves that $(\mathbb{E}\|\sum_i \xi_i\|^p_{L^q(S)})^{\frac{1}{p}}$ is dominated by the infimum on the right hand side.

The reverse inequality follows by duality from Theorem 1.28. This argument is analogous to the one in the proof of Theorem 1.29. We leave the details to the reader. □

For the case $1 < q < 2 \leq p < \infty$ we shall use the following result, due to J. Hoffmann-Jørgensen [59]. The inequality appearing below with constant of optimal order is due to M. Talagrand ([133], see also [95]). We refer to [90] (see also [91]) for a different proof based on hypercontractivity methods.

Theorem 1.31. *Let X be a Banach space and let (ξ_i) be a finite sequence of independent, mean zero X-valued random variables. Then there is a universal constant $C > 0$ such that for all $p \geq 1$,*

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_X^p\Big)^{\frac{1}{p}} \leq C\frac{p}{1+\log p}\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_X + \Big(\mathbb{E}\max_i \|\xi_i\|_X^p\Big)^{\frac{1}{p}}\Big).$$

Theorem 1.32. *Let $1 < q < 2 \leq p < \infty$. Let (ξ_i) be a finite sequence of independent, mean zero $L^q(S)$-valued random variables. Then,*

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}} \simeq_{p,q} \max\Big\{\Big(\sum_i \mathbb{E}\|\xi_i\|_{L^q(S)}^p\Big)^{\frac{1}{p}},$$
$$\inf\Big\{\Big\|\Big(\sum_i \mathbb{E}|\eta_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(S)} + \Big(\sum_i \mathbb{E}\|\theta_i\|_{L^q(S)}^q\Big)^{\frac{1}{q}}\Big\}\Big\},$$

where the infimum is taken over all sequences $(\eta_i) \in L^q(S; l^2(L^2(\Omega)))$ and $(\theta_i) \in l^q(L^q(\Omega; L^q(S)))$ such that $\xi_i = \eta_i + \theta_i$.

Proof. By Theorem 1.31 we have

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}} \lesssim_p \max\Big\{\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(S)}^q\Big)^{\frac{1}{q}}, \Big(\mathbb{E}\max_i \|\xi_i\|_{L^q(S)}^p\Big)^{\frac{1}{p}}\Big\}.$$

By Theorem 1.29 (with $p = q$) we have

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(S)}^q\Big)^{\frac{1}{q}} \simeq_{p,q} \|(\xi_i)\|_{L^q(S;l^2(L^2(\Omega)))+l^q(L^q(\Omega;L^q(S)))}$$

and obviously

$$\Big(\mathbb{E}\max_i \|\xi_i\|_{L^q(S)}^p\Big)^{\frac{1}{p}} \leq \Big(\sum_i \mathbb{E}\|\xi_i\|_{L^q(S)}^p\Big)^{\frac{1}{p}}.$$

For the reverse inequality, note that

$$\begin{aligned}\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}} &\geq \Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(S)}^q\Big)^{\frac{1}{q}} \\ &\simeq_{p,q} \|(\xi_i)\|_{L^q(S;l^2(L^2(\Omega)))+l^q(L^q(\Omega;L^q(S)))}.\end{aligned}$$

Moreover, by Corollary 1.21,

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}} \gtrsim_{p,q} \Big(\sum_i \mathbb{E}\|\xi_i\|_{L^q(S)}^p\Big)^{\frac{1}{p}}.$$

□

Theorem 1.33. *Let $1 < p < 2 \le q < \infty$. Let (ξ_i) be a finite sequence of independent, mean zero $L^q(S)$-valued random variables. Then,*

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(S)}\Big)^{\frac{1}{p}} \simeq_{p,q} \inf\Big\{\Big(\sum_i \mathbb{E}\|\theta_i\|^p_{L^q(S)}\Big)^{\frac{1}{p}} + \max\Big\{\Big\|\Big(\sum_i \mathbb{E}|\eta_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(S)}, \Big(\sum_i \mathbb{E}\|\eta_i\|^q_{L^q(S)}\Big)^{\frac{1}{q}}\Big\}\Big\},$$

where the infimum is taken over all sequences $(\eta_i) \in L^q(S; l^2(L^2(\Omega))) \cap l^q(L^q(\Omega; L^q(S)))$ and $(\theta_i) \in l^p(L^p(\Omega; L^q(S)))$ such that $\xi_i = \eta_i + \theta_i$.

Proof. By Theorem 1.28 (with $p = q$) we have

$$\begin{aligned}\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(S)}\Big)^{\frac{1}{p}} &\le \Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^q_{L^q(S)}\Big)^{\frac{1}{q}} \\ &\lesssim_{p,q} \max\Big\{\Big\|\Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(S)}, \Big(\sum_i \mathbb{E}\|\xi_i\|^q_{L^q(S)}\Big)^{\frac{1}{q}}\Big\}.\end{aligned}$$

On the other hand, by Corollary 1.17,

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(S)}\Big)^{\frac{1}{p}} \lesssim_{p,q} \Big(\sum_i \mathbb{E}\|\xi_i\|^p_{L^q(S)}\Big)^{\frac{1}{p}}.$$

Let $\xi_i = \eta_i + \theta_i$. Then, $\xi_i = \mathbb{E}(\eta_i|\xi_i) - \mathbb{E}(\eta_i) + \mathbb{E}(\theta_i|\xi_i) - \mathbb{E}(\theta_i)$. By applying the above to the sequences of independent, mean zero random variables $(\mathbb{E}(\eta_i|\xi_i) - \mathbb{E}(\eta_i))$ and $(\mathbb{E}(\theta_i|\xi_i) - \mathbb{E}(\theta_i))$ we obtain using Jensen's inequality

$$\begin{aligned}&\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(S)}\Big)^{\frac{1}{p}} \\ &\lesssim_{p,q} \Big\|\Big(\sum_i \mathbb{E}|\mathbb{E}(\eta_i|\xi_i) - \mathbb{E}(\eta_i)|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(S)} \\ &\quad + \Big(\sum_i \mathbb{E}\|\mathbb{E}(\eta_i|\xi_i) - \mathbb{E}(\eta_i)\|^q_{L^q(S)}\Big)^{\frac{1}{q}} + \Big(\sum_i \mathbb{E}\|\mathbb{E}(\theta_i|\xi_i) - \mathbb{E}(\theta_i)\|^p_{L^q(S)}\Big)^{\frac{1}{p}} \\ &\lesssim \max\Big\{\Big\|\Big(\sum_i \mathbb{E}|\eta_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(S)}, \Big(\sum_i \mathbb{E}\|\eta_i\|^q_{L^q(S)}\Big)^{\frac{1}{q}}\Big\} + \Big(\sum_i \mathbb{E}\|\theta_i\|^p_{L^q(S)}\Big)^{\frac{1}{p}}.\end{aligned}$$

The reverse inequality follows by duality from Theorem 1.32. We leave the details to the reader. □

2

Vector-valued Poisson stochastic integration

This chapter is devoted to the study of vector-valued stochastic integrals with respect to a compensated Poisson random measure. Motivated by applications of vector-valued stochastic integration to the study of stochastic partial differential equations, we aim to prove two-sided estimates for the p-th moments of a Poisson stochastic integral in terms of a suitable norm on the integrand. First we consider estimates for Poisson stochastic integrals of processes taking values in a Banach space. Depending on whether the Banach space satisfies a type or cotype restriction we obtain an upper or lower estimate, respectively, for the p-th moment of the stochastic integral. Only in the special case where the Banach space is a Hilbert space, our approach yields two-sided inequalities.

In the final section we consider Poisson stochastic integrals in L^p-spaces. We will restrict ourselves to classical L^p-spaces, as our main motivation is the application of moment estimates to stochastic partial differential equations. However, with some extra effort the results below can be deduced for processes taking values in a noncommutative L^p-space as well, using the results from Chapter 3. We leave this up to the interested reader.

Roughly speaking, the moment estimates for stochastic integrals are proved in three steps. First, by using decoupling inequalities (see Theorem 2.3 below) we reduce the problem of deriving moment estimates for stochastic integrals of simple processes to corresponding estimates for sums of independent, mean zero L^p-valued random variables, which were already obtained in Section 1.3. After applying the latter estimates to the decoupled stochastic integral, we finally use some elementary inequalities for Poisson random variables (Lemma 2.2 below) to obtain a suitable norm on the integrand.

2.1 Preliminaries

We start by giving a brief introduction to Poisson random measures.

Definition 2.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(E, \mathcal{E})$ be a measurable space. A* random measure *M is a collection $\{M(B) \ : \ B \in \mathcal{E}\}$ of random variables such that*

(i) $M(\emptyset) = 0$;
(ii) For any disjoint $A, B \in \mathcal{E}$ we have

$$M(A \cup B) = M(A) + M(B).$$

Let μ be a measure on $(E, \mathcal{E})$. We say a random measure N is a Poisson random measure *if the following conditions hold:*

(iii) For disjoint sets $A_1, \ldots, A_n$ in $\mathcal{E}$ the random variables $N(A_1), \ldots, N(A_n)$ are independent;
(iv) For any $A \in \mathcal{E}$ with $\mu(A) < \infty$ the random variable $N(A)$ is Poisson distributed with parameter $\mu(A)$.

The measure μ is called the intensity measure *of N.*

Let $\mathcal{E}_\mu = \{A \in \mathcal{E} \ : \ \mu(A) < \infty\}$. Then the random measure $\tilde{N}$ on $(E, \mathcal{E}_\mu, \mu)$ defined by

$$\tilde{N}(A) := N(A) - \mu(A) \qquad (A \in \mathcal{E}_\mu),$$

is called the compensated Poisson random measure *associated with N.*

Let N_1, N_2 be independent copies of the Poisson random measure N above. Then the random measure

$$N_s(A) := N_1(A) - N_2(A) \qquad (A \in \mathcal{E}) \tag{2.1}$$

is called the symmetrized Poisson random measure *associated with N.*

Given a σ-finite measure space $(E, \mathcal{E}, \mu)$ one can always construct a Poisson random measure with intensity measure μ, see e.g. [123], Proposition 19.4.

We now describe how (compensated) Poisson random measures arise naturally in the study of Lévy processes. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration satisfying the usual conditions, i.e. it is complete and right-continuous. Let $L = (L_t)_{t\geq 0}$ be a stochastic process adapted to this filtration. We will say L is a *Lévy process* if the following properties hold:

(i) $L_0 = 0$ a.s.;
(ii) L has a.s. càdlàg sample paths;
(iii) $L_t - L_s$ is independent of $\mathcal{F}_s$ for any $0 \leq s < t$;
(iv) $L_t - L_s$ and L_{t-s} are identically distributed for any $0 \leq s < t$;
(v) L is stochastically continuous, i.e. for $v > 0$ and $s \geq 0$,

$$\lim_{t \to s} \mathbb{P}(|L_t - L_s| > v) = 0.$$

In the literature Lévy-processes are usually defined under somewhat weaker assumptions (see e.g. [5], [123]).

Let L be a Lévy process and let Ω_0 be the set on which L has càdlàg paths. We can define the *jump process* ΔL of the Lévy process by

$$\Delta L_t(\omega) = \begin{cases} L_t(\omega) - L_{t-}(\omega), & \text{if } t \geq 0,\ \omega \in \Omega_0, \\ 0, & \text{if } t \geq 0,\ \omega \in \Omega_0^c, \end{cases}$$

where $L_{t-} = \lim_{s\uparrow t} L_s$. Then, for any $0 \leq t < \infty$, $A \in \mathcal{B}(\mathbb{R} - \{0\})$ and $\omega \in \Omega_0$ we define

$$N(t, A)(\omega) = \#\{0 \leq s \leq t \ : \ \Delta L_s(\omega) \in A\}$$

and we set $N(t, A)(\omega) = 0$ if $\omega \in \Omega_0^c$. That is, $N(t, A)(\omega)$ counts the number of jumps of a specified size in the interval $[0, t]$. It can be shown ([123], Theorem 19.2) that the map

$$N((s, t] \times A) = N(t, A) - N(s, A)$$

extends to a σ-additive Poisson random measure on $(\mathbb{R}_+ \times \mathbb{R} - \{0\}, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R} - \{0\}))$. Its intensity measure is given by $dt \times \nu$, where ν is the σ-finite measure

$$\nu(A) = \mathbb{E}(N(1, A)) \qquad (A \in \mathcal{B}(\mathbb{R} - \{0\})).$$

In fact, ν is precisely the Lévy measure of L. Moreover, one can show (see [123], Theorem 19.2) that there is a standard Brownian motion W and parameters $\sigma > 0$ and $b \in \mathbb{R}$ such that

$$L_t = bt + \sigma W_t + \int_{[0,t]\times\{|x|<1\}} x d\tilde{N} + \int_{[0,t]\times\{|x|\geq 1\}} x dN. \tag{2.2}$$

This is called the *Lévy-Itô decomposition* of L. Thus, together with Brownian motions, (compensated) Poisson random measures are the basic building blocks of Lévy processes.

We now discuss two results that we will be frequently used throughout this chapter. First we record the following elementary lemma.

Lemma 2.2. *Let N be a Poisson distributed random variable with parameter $0 \leq \lambda \leq 1$. Then for every $1 \leq p < \infty$ there exist constants $b_p, c_p > 0$ such that*

$$b_p \lambda \leq \|N - \lambda\|_p^p \leq c_p \lambda. \tag{2.3}$$

Proof. The inequalities are trivial if $\lambda = 0$, so we may assume $\lambda > 0$. We first prove the inequality on the left hand side of (2.3). Suppose first that $2 \leq p < \infty$. Then we have

$$\mathbb{E}|N - \lambda|^p = \sum_{k=0}^{\infty} |k - \lambda|^p \frac{\lambda^k e^{-\lambda}}{k!} \geq \sum_{k=2}^{\infty} |k - \lambda|^2 \frac{\lambda^k e^{-\lambda}}{k!} + |\lambda|^p e^{-\lambda} + |1 - \lambda|^p \lambda e^{-\lambda}.$$

Hence,

$$\mathbb{E}|N - \lambda|^p \geq \mathbb{E}|N - \lambda|^2 - |\lambda|^2 e^{-\lambda} - |1 - \lambda|^2 \lambda e^{-\lambda} + |\lambda|^p e^{-\lambda} + |1 - \lambda|^p \lambda e^{-\lambda}$$

$$
\begin{aligned}
&= \lambda + \lambda e^{-\lambda}(-\lambda - (1-\lambda)^2 + \lambda^{p-1} + (1-\lambda)^p) \\
&= \lambda(1 + e^{-\lambda} f_p(\lambda)),
\end{aligned}
$$

where

$$
f_p(\lambda) = \lambda^{p-1} - \lambda^2 + \lambda - 1 + (1-\lambda)^p. \tag{2.4}
$$

One easily sees that $\min_{0\le\lambda\le1}(1 + e^{-\lambda} f_p(\lambda)) = b_p > 0$. Indeed,

$$
1 + e^{-\lambda} f_p(\lambda) > 1 + e^{-\lambda}(-\lambda^2 + \lambda - 1) + e^{-\lambda}(1-\lambda)^p.
$$

Now,

$$
1 + e^{-\lambda}(-\lambda^2 + \lambda - 1) + e^{-\lambda}(1-\lambda)^p > 0
$$

if and only if

$$
(1-\lambda)^p > -e^{\lambda} + \lambda^2 - \lambda + 1 = -2\lambda + \frac{\lambda^2}{2} - \frac{\lambda^3}{6} - \frac{\lambda^4}{24} - \cdots .
$$

Clearly this holds if $0 \le \lambda \le 1$. This proves the left hand side inequality if $2 \le p < \infty$.

Suppose now that $1 \le p < 2$. Then, by the Cauchy-Schwartz inequality,

$$
\begin{aligned}
\lambda = \mathbb{E}|N-\lambda|^2 &= \mathbb{E}|N-\lambda|^{\frac{p}{2}}|N-\lambda|^{2-\frac{p}{2}} \\
&\le (\mathbb{E}|N-\lambda|^p)^{\frac{1}{2}}(\mathbb{E}|N-\lambda|^{4-p})^{\frac{1}{2}}.
\end{aligned}
$$

Since $4 - p \ge 2$ we find by the above that

$$
\lambda^2 \le \mathbb{E}|N-\lambda|^p \mathbb{E}|N-\lambda|^{4-p} \le \mathbb{E}|N-\lambda|^p c_{4-p}\lambda.
$$

We now consider the right hand side inequality of (2.3) for $2 \le p < \infty$. Clearly it suffices to prove this in the case where p is an even integer n. We first compute the moment generating function of $N - \lambda$.

$$
\mathbb{E}(e^{t(N-\lambda)}) = e^{-\lambda t}\mathbb{E}(e^{tN}) = e^{-\lambda t} e^{\lambda(e^t-1)} = e^{\lambda(e^t-1-t)} = \exp(\lambda \sum_{n=2}^{\infty} \frac{t^n}{n!}).
$$

It is now easy to see that the n-th moment of $N - \lambda$ can be written as $\lambda p_n(\lambda)$ for some polynomial p_n with positive coefficients. Since $\max_{0\le\lambda\le1}|p_n(\lambda)| \le c_n$ for some constant $c_n > 0$, our proof for the case $2 \le p < \infty$ is complete. If $1 \le p < 2$ then

$$
\begin{aligned}
\mathbb{E}|N-\lambda|^p &= \sum_{k=2}^{\infty} |k-\lambda|^p \frac{\lambda^k e^{-\lambda}}{k!} + |\lambda|^p e^{-\lambda} + |1-\lambda|^p \lambda e^{-\lambda} \\
&\le \sum_{k=2}^{\infty} |k-\lambda|^2 \frac{\lambda^k e^{-\lambda}}{k!} + |\lambda|^p e^{-\lambda} + |1-\lambda|^p \lambda e^{-\lambda}
\end{aligned}
$$

$$
\begin{aligned}
&= \mathbb{E}|N-\lambda|^2 - |\lambda|^2 e^{-\lambda} - |1-\lambda|^2 \lambda e^{-\lambda} + |\lambda|^p e^{-\lambda} + |1-\lambda|^p \lambda e^{-\lambda} \\
&= \lambda(1+e^{-\lambda} f_p(\lambda)) \leq \lambda \max_{0\leq\lambda\leq 1}(1+e^{-\lambda} f_p(\lambda)),
\end{aligned}
$$

where f_p is the continuous function defined in (2.4). □

We end this section by discussing a decoupling result that will be used intensively throughout the chapter. Its proof can be found in [138], Theorem 2.4.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(\mathcal{F}_i)_{i=0}^n$ be a filtration in $\mathcal{F}$. Fix $1 < p < \infty$ and let $(\xi_i)_{i=1}^n$ be a sequence of mean zero, $\mathcal{F}_i$-measurable random variables in $L^p(\Omega)$ such that ξ_i is independent of $\mathcal{F}_j$ for all $1 \leq j < i \leq n$. Let $\tilde{\Omega}$ be a copy of Ω. We define two independent copies of the sequence (ξ_i) on the product probability space $\Omega \times \tilde{\Omega}$ by setting

$$
\xi_i(\omega, \tilde{\omega}) = \xi_i(\omega) 1_{\tilde{\Omega}}(\tilde{\omega}), \qquad \tilde{\xi}_i(\omega, \tilde{\omega}) = \xi_i(\tilde{\omega}) 1_{\Omega}(\omega).
$$

Let X be a Banach space and let $(v_i)_{i=1}^n$ be a sequence of $\mathcal{F}_{i-1}$-measurable random variables in $L^\infty(\Omega; X)$. We identify v_i with its copy $v_i(\omega, \tilde{\omega}) = v_i(\omega) 1_{\tilde{\Omega}}(\tilde{\omega})$ in $L^\infty(\Omega \times \Omega; X)$.

Theorem 2.3. *(Decoupling) Fix $1 < p < \infty$ and let $(\xi_i)_{i=1}^n, (\tilde{\xi}_i)_{i=1}^n$ and $(v_i)_{i=1}^n$ be as above. If X is a UMD Banach space then*

$$
\Big(\mathbb{E}\Big\| \sum_{i=1}^n v_i \xi_i \Big\|_X^p\Big)^{\frac{1}{p}} \eqsim_{p,X} \Big(\mathbb{E}\tilde{\mathbb{E}}\Big\| \sum_{i=1}^n v_i \tilde{\xi}_i \Big\|_X^p\Big)^{\frac{1}{p}}
$$

Remark 2.4. In some applications one is only interested in the one-sided decoupling estimate

$$
\Big(\mathbb{E}\Big\| \sum_{i=1}^n v_i \xi_i \Big\|_X^p\Big)^{\frac{1}{p}} \lesssim_{p,X} \Big(\mathbb{E}\tilde{\mathbb{E}}\Big\| \sum_{i=1}^n v_i \tilde{\xi}_i \Big\|_X^p\Big)^{\frac{1}{p}}.
$$

This inequality can be proved under less restrictive assumptions on p and X, for example for $1 \leq p < \infty$ and $X = L^1(S)$, where S is a σ-finite measure space. We refer to [36] for results in this direction.

2.2 Poisson stochastic integration in Banach spaces

Let X be a Banach space and let $(J, \mathcal{J}, \nu)$ be a σ-finite measure space. Throughout this chapter we fix a Poisson random measure N on $(\mathbb{R}_+ \times J, \mathcal{B}(\mathbb{R}_+) \times \mathcal{J}, dt \times \nu)$. Moreover, we make the following assumption.

Assumption 2.5 *Throughout this chapter we fix a filtration $(\mathcal{F}_t)_{t\geq 0}$ such that for any $0 \leq s < t < \infty$ and any $A \in \mathcal{J}$ the random variable $\tilde{N}((s,t] \times A)$ is independent of $\mathcal{F}_s$.*

Definition 2.6. *Let $F : \Omega \times [0,\infty) \times J \to X$. We say F is a* simple, adapted X-valued process *if there is a finite partition $\pi = \{0 \le t_1 < \ldots < t_{l+1} < \infty\}$ of $[0,\infty)$, $F_{i,k} \in L^\infty(\mathcal{F}_{t_i})$, $x_{i,j,k} \in X$ and disjoint sets $A_1, \ldots A_m$ in $\mathcal{J}$ satisfying $\nu(A_j) < \infty$ for $i = 1, \ldots, l$, $j = 1, \ldots, m$ and $k = 1, \ldots, n$ such that*

$$F = \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} F_{i,k} \chi_{(t_i, t_{i+1}]} \chi_{A_j} x_{i,j,k}. \tag{2.5}$$

Let $t \ge 0$ and $B \in \mathcal{J}$. We define the (compensated) Poisson stochastic integral *of F on $[0,t] \times B$ with respect to $\tilde{N}$ by*

$$\int_{[0,t]\times B} F \, d\tilde{N} = \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} F_{i,k} \tilde{N}((t_i \wedge t, t_{i+1} \wedge t] \times (A_j \cap B)) x_{i,j,k}.$$

Remark 2.7. By refining the partition π in Definition 2.6 if necessary, we can and will always assume that $(t_{i+1} - t_i)\nu(A_j) \le 1$ for all $i = 1, \ldots, l$, $j = 1, \ldots, m$. This will allow us to apply Lemma 2.2 to the compensated Poisson random variables $\tilde{N}((t_i \wedge t, t_{i+1} \wedge t] \times (A_j \cap B))$.

From now on we will write $N_{i,j} := N((t_i, t_{i+1}] \times A_j)$ for brevity.

We will make use of the following inequality due to E.M. Stein (see [128], Chapter IV, the proof of Theorem 8).

Theorem 2.8. *(Stein's inequality) Let $1 < p < \infty$ and $1 \le s \le \infty$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(\mathcal{F}_i)_{i \ge 1}$ be a filtration in $\mathcal{F}$ with corresponding conditional expectations $(\mathbb{E}_i)_{i\ge1}$. Suppose $(f_i)_{i \ge 1}$ is a sequence of $\mathcal{F}$-measurable functions. Then,*

$$\Big\| \Big(\sum_i |\mathbb{E}_i(f_i)|^s \Big)^{\frac{1}{s}} \Big\|_{L^p(\Omega)} \lesssim_{p,s} \Big\| \Big(\sum_i |f_i|^s \Big)^{\frac{1}{s}} \Big\|_{L^p(\Omega)}.$$

Lemma 2.9. *Fix $1 \le p < \infty$ and let X be a Banach space. Let $0 \le t_0 < t_1 < \ldots < t_{l+1} < \infty$, $F_{i,k} \in L^\infty(\Omega)$, $x_{i,j,k} \in X$, and let $A_1, \ldots A_m$ be disjoint sets in $\mathcal{J}$ satisfying $\nu(A_j) < \infty$ for $i = 1, \ldots, l$, $j = 1, \ldots, m$ and $k = 1, \ldots, n$. Define*

$$F = \sum_{i,j,k} F_{i,k} \chi_{(t_i,t_{i+1}]} \chi_{A_j} x_{i,j,k}$$

and let $\tilde{F}$ be the associated simple adapted X-valued process given by

$$\tilde{F} = \sum_{i,j,k} \mathbb{E}(F_{i,k} | \mathcal{F}_{t_i}) \chi_{(t_i,t_{i+1}]} \chi_{A_j} x_{i,j,k}.$$

Then, if $1 < p < \infty$ and $1 \le s < \infty$,

$$\|\tilde{F}\|_{L^p(\Omega; L^s([0,t]\times J; X))} \lesssim_{p,s} \|F\|_{L^p(\Omega; L^s([0,t]\times J; X))}.$$

Proof. We may assume $t = t_{l+1}$. By Stein's inequality (c.f. Theorem 2.8) we have

$$\begin{aligned}
&\|\tilde{F}\|_{L^p(\Omega;L^s([0,t]\times J;X))} \\
&= \Big\| \Big(\sum_{i,j} (t_{i+1} - t_i)\nu(A_j) \Big\| \sum_k \mathbb{E}(F_{i,k}|\mathcal{F}_{t_i}) x_{i,j,k} \Big\|_X^s \Big)^{\frac{1}{s}} \Big\|_{L^p(\Omega)} \\
&= \Big\| \Big(\sum_{i,j} (t_{i+1} - t_i)\nu(A_j) \Big\| \mathbb{E}\Big(\sum_k F_{i,k} x_{i,j,k} \Big| \mathcal{F}_{t_i} \Big) \Big\|_X^s \Big)^{\frac{1}{s}} \Big\|_{L^p(\Omega)} \\
&\le \Big\| \Big(\sum_{i,j} (t_{i+1} - t_i)\nu(A_j) \Big(\mathbb{E}\Big(\Big\| \sum_k F_{i,k} x_{i,j,k} \Big\|_X \Big| \mathcal{F}_{t_i} \Big) \Big)^s \Big)^{\frac{1}{s}} \Big\|_{L^p(\Omega)} \\
&\lesssim_{p,s} \Big\| \Big(\sum_{i,j} (t_{i+1} - t_i)\nu(A_j) \Big\| \sum_k F_{i,k} x_{i,j,k} \Big\|_X^s \Big)^{\frac{1}{s}} \Big\|_{L^p(\Omega)} \\
&= \|F\|_{L^p(\Omega;L^s([0,t]\times J;X))}.
\end{aligned}$$

□

We will often use the following trivial observation.

Lemma 2.10. *Suppose (S, Σ, μ) is a σ-finite measure space and let X be a Banach space. Let $A_1, \dots, A_n$ be disjoint sets in Σ with $\mu(A_i) < \infty$ and let $\mathcal{A}$ be the σ-algebra generated by $A_1, \dots, A_n$. Then, for any $G \in L^1(S; X)$,*

$$\mathbb{E}(G|\mathcal{A}) = \sum_{i=1}^n \chi_{A_i} y_i,$$

for some $y_i \in X$.

Proof. Notice that since $A_1, \dots, A_n$ are disjoint, $\mathcal{A}$ is actually a finite algebra, consisting of $A_1, \dots, A_n$ and all their possible unions. Moreover, if $\mu(A_i) > 0$ then

$$\int_{A_i} G \, d\mu = (\mu(A_j))^{-1} \int_{A_i} \sum_{\{j \,:\, \mu(A_j) \neq 0\}}^{n} \Big(\int_{A_j} G \, d\mu \Big) \chi_{A_j} \, d\mu,$$

so

$$\mathbb{E}(G|\mathcal{A}) = \sum_{\{j \,:\, \mu(A_j) \neq 0\}} (\mu(A_j))^{-1} \chi_{A_j} \int_{A_j} G \, d\mu.$$

□

Theorem 2.11. *Let X be a UMD Banach space with type $s \in (1, 2]$ If $s \le p < \infty$ we have, for any simple, adapted X-valued process F, any $B \in \mathcal{J}$ and any $t \ge 0$,*

$$\Big\| \int_{[0,t]\times B} F \, d\tilde{N} \Big\|_{L^p(\Omega;X)} \lesssim_{p,s,X} \max\Big\{ \|F\|_{L^p(\Omega;L^s([0,t]\times B;X))}, \|F\|_{L^p(\Omega;L^p([0,t]\times B;X))} \Big\}.$$

On the other hand, if $1 < p < s$ *then*

$$\Big\| \int_{[0,t]\times B} F \, d\tilde{N} \Big\|_{L^p(\Omega;X)} \lesssim_{p,s,X} \|F\|_{L^p(\Omega;L^s([0,t]\times B;X)) + L^p(\Omega;L^p([0,t]\times B;X))}. \tag{2.6}$$

Proof. Let F be as in (2.5), taking Remark 2.7 into account. Without loss of generality, we may assume $t = t_{l+1}$ and $B = J$. Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ be a copy of $(\Omega, \mathcal{F}, \mathbb{P})$. By Theorem 2.3 we have

$$\Big\| \int_{[0,t]\times J} F \, d\tilde{N} \Big\|_{L^p(\Omega;X)} \simeq_p \Big\| \sum_{i,j,k} F_{i,k} \tilde{N}_{i,j} x_{i,j,k} \Big\|_{L^p(\Omega\times\tilde{\Omega};X)},$$

where we use the same letter $\tilde{N}_{i,j}$ to denote its copy on $\tilde{\Omega}$. Suppose first that $s \le p < \infty$. Fix $\omega \in \Omega$ and define

$$y_{i,j}(\omega) = \sum_k F_{i,k}(\omega) x_{i,j,k}.$$

It suffices to show that

$$\Big(\tilde{\mathbb{E}} \Big\| \sum_{i,j} \tilde{N}_{i,j} y_{i,j}(\omega) \Big\|_X^p \Big)^{\frac{1}{p}} \lesssim_{p,s,X} \max\Big\{ \|F(\omega)\|_{L^s([0,t]\times J;X)}, \|F(\omega)\|_{L^p([0,t]\times J;X)} \Big\}. \tag{2.7}$$

The result then follows by taking $L^p(\Omega)$-norms on both sides. Define $\xi_{i,j} = \tilde{N}_{i,j} y_{i,j}(\omega)$. Then $(\xi_{i,j})$ is a doubly indexed sequence of independent, mean zero X-valued random variables. By Theorem 1.15 we have

$$\Big(\tilde{\mathbb{E}} \Big\| \sum_{i,j} \xi_{i,j} \Big\|_X^p \Big)^{\frac{1}{p}} \lesssim_{p,s,X} \max\Big\{ \Big(\sum_{i,j} \tilde{\mathbb{E}} \|\xi_{i,j}\|_X^p \Big)^{\frac{1}{p}}, \Big(\sum_{i,j} \tilde{\mathbb{E}} \|\xi_{i,j}\|_X^s \Big)^{\frac{1}{s}} \Big\}.$$

If $r = s, p$ we have by Lemma 2.2,

$$\begin{aligned} \Big(\sum_{i,j} \tilde{\mathbb{E}} \|\xi_{i,j}\|_X^r \Big)^{\frac{1}{r}} &= \Big(\sum_{i,j} \tilde{\mathbb{E}} |\tilde{N}_{i,j}|^r \|y_{i,j}(\omega)\|_X^r \Big)^{\frac{1}{r}} \\ &\simeq_r \Big(\sum_{i,j} (t_{i+1} - t_i) \nu(A_j) \|y_{i,j}(\omega)\|_X^r \Big)^{\frac{1}{r}} \\ &= \|F(\omega)\|_{L^r([0,t]\times J;X)}. \end{aligned} \tag{2.8}$$

Thus (2.7) holds and our proof in the case $s \le p < \infty$ is complete.

Suppose now that $1 < p < s$. Again set $\xi_{i,j} = y_{i,j}(\omega)\tilde{N}_{i,j}$. If $r = p, s$ then by applying Theorem 1.15 to $(\xi_{i,j})$ and using (2.8) we obtain

$$\begin{aligned}\Big(\mathbb{E}\tilde{\mathbb{E}}\Big\|\sum_{i,j}\tilde{N}_{i,j}y_{i,j}\Big\|_X^p\Big)^{\frac{1}{p}} &\lesssim_{p,r,X} \Big(\mathbb{E}\Big(\sum_{i,j}\tilde{\mathbb{E}}|\tilde{N}_{i,j}|^r\|y_{i,j}\|_X^r\Big)^{\frac{p}{r}}\Big)^{\frac{1}{p}}\\ &\simeq_r \quad \|F\|_{L^p(\Omega;L^r([0,t]\times J;X))}.\end{aligned}$$

Since $L^\infty(\Omega)\otimes L^\infty([0,t])\otimes L^\infty(J)\otimes X$ is dense in both $L^p(\Omega;L^p([0,t]\times J;X))$ and $L^p(\Omega;L^s([0,t]\times J;X))$, we have, for F in $L^\infty(\Omega)\otimes L^\infty([0,t])\otimes L^\infty(J)\otimes X$,

$$\begin{aligned}&\|F\|_{L^p(\Omega;L^p([0,t]\times J;X)))+L^p(\Omega;L^s([0,t]\times J;X))}\\ &\qquad = \inf\Big\{\|F_1\|_{L^p(\Omega;L^p([0,t]\times J;X))} + \|F_2\|_{L^p(\Omega;L^s([0,t]\times J;X))}\Big\},\end{aligned}$$

where the infimum is taken over all decompositions $F = F_1 + F_2$ in $L^\infty(\Omega)\otimes L^\infty([0,t])\otimes L^\infty(J)\otimes X$. Fix such a decomposition. Let $\mathcal{A}$ be the sub-σ-algebra of $\mathcal{B}(\mathbb{R}_+)\times\mathcal{J}$ generated by the sets $(t_i, t_{i+1}]\times A_j$. Then $F = \mathbb{E}(F_1|\mathcal{A})+\mathbb{E}(F_2|\mathcal{A})$ and by Lemma 2.10 $\mathbb{E}(F_1|\mathcal{A}), \mathbb{E}(F_2|\mathcal{A})$ are of the form

$$F_\alpha = \sum_{i,j,k} F^\alpha_{i,k}\chi_{(t_i,t_{i+1}]}\chi_{A_j}x^\alpha_{i,j,k} \qquad (\alpha = 1,2).$$

Let $\tilde{F}_1, \tilde{F}_2$ be the associated simple adapted processes

$$\tilde{F}_\alpha = \sum_{i,j,k}\mathbb{E}(F^\alpha_{i,k}|\mathcal{F}_{t_i})\chi_{(t_i,t_{i+1}]}\chi_{A_j}x^\alpha_{i,j,k} \qquad (\alpha = 1,2),$$

then $F = \tilde{F}_1 + \tilde{F}_2$. By the above,

$$\begin{aligned}&\Big\|\int_{[0,t]\times J} F\, d\tilde{N}\Big\|_{L^p(\Omega;X)}\\ &\quad\le \Big\|\int_{[0,t]\times J}\tilde{F}_1\, d\tilde{N}\Big\|_{L^p(\Omega;X)} + \Big\|\int_{[0,t]\times J}\tilde{F}_2\, d\tilde{N}\Big\|_{L^p(\Omega;X)}\\ &\quad\lesssim_{p,s,X} \|\tilde{F}_1\|_{L^p(\Omega;L^p([0,t]\times J;X))} + \|\tilde{F}_2\|_{L^p(\Omega;L^s([0,t]\times J;X))}\\ &\quad\lesssim_{p,s} \|\mathbb{E}(F_1|\mathcal{A})\|_{L^p(\Omega;L^p([0,t]\times J;X))} + \|\mathbb{E}(F_2|\mathcal{A})\|_{L^p(\Omega;L^s([0,t]\times J;X))}\\ &\quad\le \|F_1\|_{L^p(\Omega;L^p([0,t]\times J;X))} + \|F_2\|_{L^p(\Omega;L^s([0,t]\times J;X))},\end{aligned}$$

where in the final two steps we use Lemma 2.9 and contractivity of vector-valued conditional expectations, respectively. Thus, taking the infimum over all decompositions $F = F_1 + F_2$ as above yields

$$\Big\|\int_{[0,t]\times J} F\, d\tilde{N}\Big\|_{L^p(\Omega;X)} \lesssim_{p,s,X} \|F\|_{L^p(\Omega;L^p([0,t]\times J;X))+L^p(\Omega;L^s([0,t]\times J;X))},$$

as asserted. $\square$

Remark 2.12. Note that for the second part of Theorem 2.11 to hold it is necessary that X has type s. Indeed, if the estimate (2.6) holds for some $1 < p < s$, then in particular,

$$\Big\| \int_{[0,t]\times J} F \, d\tilde{N} \Big\|_{L^p(\Omega;X)} \lesssim_{p,s,X} \Big\| \Big(\int_{[0,t]\times J} \|F\|_X^s dt \times d\nu \Big)^{\frac{1}{s}} \Big\|_{L^p(\Omega)},$$

for any simple, adapted X-valued process F and any $t \geq 0$. By taking $F = \sum_{i=0}^n \chi_{(i,i+1]} x_i$ in this inequality we obtain

$$\Big(\mathbb{E} \Big\| \sum_{i=1}^n \tilde{n}_i x_i \Big\|_X^p \Big)^{\frac{1}{p}} \lesssim_{p,s,X} \Big(\sum_{i=1}^n \|x_i\|_X^s \Big)^{\frac{1}{s}},$$

where $(\tilde{n}_i)$ is a sequence of independent compensated Poisson random variables with parameter 1. By [95], Proposition 9.15, we have

$$\|\tilde{n}_1\|_{1,\infty} \Big(\mathbb{E} \Big\| \sum_{i=1}^n r_i x_i \Big\|_X^p \Big)^{\frac{1}{p}} \lesssim \Big(\mathbb{E} \Big\| \sum_{i=1}^n \tilde{n}_i x_i \Big\|_X^p \Big)^{\frac{1}{p}},$$

and hence X has type s by Kahane's inequalities.

Remark 2.13. Using an entirely different approach, it is shown in [143] that Theorem 2.11 holds under the condition that X has martingale type s. This result is slightly more general, as every UMD space with type s automatically has martingale type s.

Specialized to L^q-spaces we obtain the following result.

Corollary 2.14. *Let S be a σ-finite measure space. If $2 \leq p, q < \infty$ we have for any simple adapted $L^q(S)$-valued process F, any $t \geq 0$ and $B \in \mathcal{J}$,*

$$\begin{aligned} &\Big\| \int_{[0,t]\times B} F \, d\tilde{N} \Big\|_{L^p(\Omega;L^q(S))} \\ &\qquad \lesssim_{p,q} \max\Big\{ \|F\|_{L^p(\Omega;L^2([0,t]\times B;L^q(S)))}, \|F\|_{L^p(\Omega;L^p([0,t]\times B;L^q(S)))} \Big\}. \end{aligned}$$

If $1 < q < 2$ and $q \leq p < \infty$ then

$$\begin{aligned} &\Big\| \int_{[0,t]\times B} F \, d\tilde{N} \Big\|_{L^p(\Omega;L^q(S))} \\ &\qquad \lesssim_{p,q} \max\Big\{ \|F\|_{L^p(\Omega;L^q([0,t]\times B;L^q(S)))}, \|F\|_{L^p(\Omega;L^p([0,t]\times B;L^q(S)))} \Big\}. \end{aligned}$$

If $1 < p < q \leq 2$ then

$$\Big\| \int_{[0,t]\times B} F \, d\tilde{N} \Big\|_{L^p(\Omega;L^q(S))}$$

$$\lesssim_{p,q} \|F\|_{L^p(\Omega;L^q([0,t]\times B;L^q(S)))+L^p(\Omega;L^p([0,t]\times B;L^q(S)))}.$$

Finally, if $1 < p < 2$ *and* $2 \leq q < \infty$ *then*

$$\Big\| \int_{[0,t]\times B} F \, d\tilde{N} \Big\|_{L^p(\Omega;L^q(S))} \lesssim_{p,q} \|F\|_{L^p(\Omega;L^2([0,t]\times B;L^q(S)))+L^p(\Omega;L^p([0,t]\times B;L^q(S)))}.$$

We now consider the 'dual' situation. From Theorem 1.8 it follows that the dual X^* of any UMD space X is reflexive and hence has the Radon-Nikodým property. By (1.7) we obtain

$$\Big(L^p(\Omega; L^p([0,t] \times J; X)) \cap L^p(\Omega; L^s([0,t] \times J; X))\Big)^* = L^{p'}(\Omega; L^{p'}([0,t] \times J; X^*)) + L^{p'}(\Omega; L^{s'}([0,t] \times J; X^*)), \quad (2.9)$$

where $1 \leq p, s < \infty$ and $1 < p', s' \leq \infty$ satisfy $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{s} + \frac{1}{s'} = 1$. Using this duality we obtain the following result.

Theorem 2.15. *Let* X *be a UMD Banach space with cotype* $2 \leq s < \infty$. *If* $s \leq p < \infty$ *we have, for any simple, adapted* X*-valued process* F,

$$\max\Big\{\|F\|_{L^p(\Omega;L^s([0,t]\times B;X))}, \|F\|_{L^p(\Omega;L^p([0,t]\times B;X))}\Big\} \lesssim_p \Big\| \int_{[0,t]\times B} F \, d\tilde{N} \Big\|_{L^p(\Omega;X)}. \quad (2.10)$$

On the other hand, if $1 < p < s$ *then*

$$\|F\|_{L^p(\Omega;L^s([0,t]\times B;X))+L^p(\Omega;L^p([0,t]\times B;X))} \lesssim_p \Big\| \int_{[0,t]\times B} F \, d\tilde{N} \Big\|_{L^p(\Omega;X)}.$$

Proof. Let F be the simple adapted process given in (2.5), taking Remark 2.7 into account. We may assume that $t = t_{l+1}$ and $B = J$. By Theorem 2.3,

$$\Big\| \int_{[0,t]\times J} F \, d\tilde{N} \Big\|_{L^p(\Omega;X)} \eqsim_p \Big\| \sum_{i,j,k} F_{i,k} \tilde{N}_{i,j} x_{i,j,k} \Big\|_{L^p(\Omega\times\tilde{\Omega};X)} = \Big(\mathbb{E}\tilde{\mathbb{E}}\Big\| \sum_{i,j} \tilde{N}_{i,j} y_{i,j} \Big\|_X^p\Big)^{\frac{1}{p}},$$

where $y_{i,j} = \sum_{k=1}^n F_{i,k} x_{i,j,k}$. Suppose that $s \leq p < \infty$. Set $\xi_{i,j}(\omega) = y_{i,j}(\omega)\tilde{N}_{i,j}$. By applying Theorem 1.20 for the sequence $(\xi_{i,j}(\omega))$ and using Lemma 2.2 we obtain for $r = p, s$,

$$\Big(\mathbb{E}\tilde{\mathbb{E}}\Big\| \sum_{i,j} \tilde{N}_{i,j} y_{i,j} \Big\|_X^p\Big)^{\frac{1}{p}} \gtrsim_{p,s,X} \Big(\mathbb{E}\Big| \sum_{i,j} \tilde{\mathbb{E}}|\tilde{N}_{i,j}|^r \|y_{i,j}\|_X^r \Big|^p\Big)^{\frac{1}{p}}$$

$$\simeq_r \quad \Big(\mathbb{E}\Big|\sum_{i,j}(t_{i+1}-t_i)\nu(A_j)\|y_{i,j}\|_X^r\Big|^p\Big)^{\frac{1}{p}}$$
$$= \quad \|F\|_{L^r(\Omega;L^r([0,t]\times J;X))}.$$

Taking the maximum over $r = p, s$ gives the result.

We deduce the inequality in the case $1 < p < s$ by duality from Theorem 2.11. Since X is a UMD space, it is K-convex (c.f. Theorem 1.8). Hence, by Theorem 1.6 we find that X^* has type $1 < s' \le 2$ and $s' \le p'$, where $\frac{1}{s} + \frac{1}{s'} = 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$.
Let G be an element of the algebraic tensor product $L^\infty(\Omega) \otimes L^\infty([0,t]) \otimes L^\infty(J) \otimes X^*$. Let $\mathcal{A}$ be the sub-σ-algebra of $\mathcal{B}(\mathbb{R}_+) \times \mathcal{J}$ generated by the sets $(t_i, t_{i+1}] \times A_j$. By Lemma 2.10 $\mathbb{E}(G|\mathcal{A})$ is of the form

$$\mathbb{E}(G|\mathcal{A}) = \sum_{i,j,k} G_{i,k}\chi_{(t_i,t_{i+1}]}\chi_{A_j} x^*_{i,j,k},$$

where $G_{i,k} \in L^\infty(\Omega)$. Let $\mathcal{E}_i$ be the conditional expectation with respect to $\mathcal{F}_{t_i}$ and let $\tilde{G}$ be the simple adapted process defined by

$$\tilde{G} = \sum_{i,j,k} \mathcal{E}_i(G_{i,k})\chi_{(t_i,t_{i+1}]}\chi_{A_j} x^*_{i,j,k}.$$

Let $\langle\cdot,\cdot\rangle$ denote the duality bracket for (2.9). By Lemma 1.18 and Theorem 2.11,

$$\begin{aligned}
&\langle F, G\rangle \\
&\quad = \langle F, \mathbb{E}(G|\mathcal{A})\rangle \\
&\quad = \sum_{i,j,k} \mathbb{E}(F_{i,k}G_{i,k})dt \times d\nu((t_i,t_{i+1}] \times A_j)\langle x_{i,j,k}, x^*_{i,j,k}\rangle \\
&\quad = \sum_{i,j,k} \mathbb{E}(F_{i,k}\mathcal{E}_i(G_{i,k}))dt \times d\nu((t_i,t_{i+1}] \times A_j)\langle x_{i,j,k}, x^*_{i,j,k}\rangle \\
&\quad = \sum_{i,j,k,l,m,n} \mathbb{E}(F_{i,k}\mathcal{E}_j(G_{l,n}))\tilde{\mathbb{E}}(\tilde{N}_{i,j}\tilde{N}_{l,m})\langle x_{i,j,k}, x^*_{l,m,n}\rangle \\
&\quad = \sum_{i,j,k,l,m,n} \mathbb{E}\tilde{\mathbb{E}}(F_{i,k}\tilde{N}_{i,j}\mathcal{E}_j(G_{l,n})\tilde{N}_{l,m}\langle x_{i,j,k}, x^*_{l,m,n}\rangle) \\
&\quad = \mathbb{E}\tilde{\mathbb{E}}\Big(\Big\langle \sum_{i,j,k} F_{i,k}\tilde{N}_{i,j}x_{i,j,k}, \sum_{l,m,n} \mathcal{E}_{l,n}(G_{l,n})\tilde{N}_{l,m}x^*_{l,m,n}\Big\rangle\Big) \\
&\quad \le \Big\|\sum_{i,j,k} F_{i,k}\tilde{N}_{i,j}x_{i,j,k}\Big\|_{L^p(\Omega\times\tilde{\Omega};X)}\Big\|\sum_{l,m,n} \mathcal{E}_{l,n}(G_{l,n})\tilde{N}_{l,m}x^*_{l,m,n}\Big\|_{L^{p'}(\Omega\times\tilde{\Omega};X^*)} \\
&\quad \lesssim_{p,s,X} \Big\|\sum_{i,j,k} F_{i,k}\tilde{N}_{i,j}x_{i,j,k}\Big\|_{L^p(\Omega\times\tilde{\Omega};X)} \\
&\qquad\quad \max\Big\{\|\tilde{G}\|_{L^{p'}(\Omega;L^{p'}([0,t]\times J;X^*))}, \|\tilde{G}\|_{L^{p'}(\Omega;L^{s'}([0,t]\times J;X^*))}\Big\}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim_{p,s} \Big\| \sum_{i,j,k} F_{i,k} \tilde{N}_{i,j} x_{i,j,k} \Big\|_{L^p(\Omega\times\tilde{\Omega};X)} \\
&\qquad \max\Big\{ \|\mathbb{E}(G|\mathcal{A})\|_{L^{p'}(\Omega;L^{p'}([0,t]\times J;X^*))}, \|\mathbb{E}(G|\mathcal{A})\|_{L^{p'}(\Omega;L^{s'}([0,t]\times J;X^*))} \Big\}, \\
&\lesssim_{p,s} \Big\| \sum_{i,j,k} F_{i,k} \tilde{N}_{i,j} x_{i,j,k} \Big\|_{L^p(\Omega\times\tilde{\Omega};X)} \\
&\qquad \max\Big\{ \|G\|_{L^{p'}(\Omega;L^{p'}([0,t]\times J;X^*))}, \|G\|_{L^{p'}(\Omega;L^{s'}([0,t]\times J;X^*))} \Big\},
\end{aligned}
$$

where the penultimate inequality follows by Lemma 2.9. Since $L^\infty(\Omega) \otimes L^\infty([0,t]) \otimes L^\infty(J) \otimes X^*$ is dense in $L^{p'}(\Omega; L^{p'}([0,t]\times J; X^*)) \cap L^{p'}(\Omega; L^{s'}([0,t]\times J; X^*))$, we conclude that

$$
\begin{aligned}
&\|F\|_{L^p(\Omega;L^p([0,t]\times J;L^p;X))+L^p(\Omega;L^s([0,t]\times J;X))} \\
&\qquad\qquad \lesssim_p \Big\| \sum_{i,j,k} F_{i,k} \tilde{N}_{i,j} x_{i,j,k} \Big\|_{L^p(\Omega\times\tilde{\Omega};X)}.
\end{aligned}
$$

□

Remark 2.16. Suppose X has finite cotype. Then, for the first part of Theorem 2.15 to hold it is necessary that X has cotype s. Indeed, suppose (2.10) holds for some $p \geq s$, then in particular

$$
\Big\| \Big(\int_{[0,t]\times J} \|F\|_X^s \, dt \times d\nu \Big)^{\frac{1}{s}} \Big\|_{L^p(\Omega)} \lesssim_{p,s,X} \Big\| \int_{[0,t]\times J} F \, d\tilde{N} \Big\|_{L^p(\Omega;X)}
$$

holds for any simple, adapted X-valued process F and any $t \geq 0$. By taking $F = \sum_{i=0}^n \chi_{(i,i+1]} x_i$ in this inequality we obtain

$$
\Big(\sum_{i=1}^n \|x_i\|_X^s \Big)^{\frac{1}{s}} \lesssim_{p,s,X} \Big(\mathbb{E} \Big\| \sum_{i=1}^n \tilde{n}_i x_i \Big\|_X^p \Big)^{\frac{1}{p}},
$$

where $(\tilde{n}_i)$ is a sequence of independent compensated Poisson random variables with parameter 1. Since X has finite cotype q, we find by [95], Proposition 9.14 that

$$
\Big(\mathbb{E} \Big\| \sum_{i=1}^n \tilde{n}_i x_i \Big\|_X^p \Big)^{\frac{1}{p}} \lesssim \|\tilde{n}_1\|_{r,1} \Big(\mathbb{E} \Big\| \sum_{i=1}^n r_i x_i \Big\|_X^p \Big)^{\frac{1}{p}},
$$

where $r = \max\{p, q\}$. By Kahane's inequalities we conclude that X has cotype s.

Specialized to L^q-spaces Theorem 2.15 yields the following.

Corollary 2.17. *Let S be a σ-finite measure space. If $2 \leq q \leq p < \infty$ we have for any simple adapted $L^q(S)$-valued process F, any $t \geq 0$ and any $B \in \mathcal{J}$,*

$$\max\Big\{\|F\|_{L^p(\Omega;L^q([0,t]\times B;L^q(S)))}, \|F\|_{L^p(\Omega;L^p([0,t]\times B;L^q(S)))}\Big\}$$
$$\gtrsim_{p,q} \Big\| \int_{[0,t]\times B} F \, d\tilde{N} \Big\|_{L^p(\Omega;L^q(S))}.$$

If $1 < q < 2$ and $2 \le p < \infty$ then

$$\max\Big\{\|F\|_{L^p(\Omega;L^2([0,t]\times B;L^q(S)))}, \|F\|_{L^p(\Omega;L^p([0,t]\times B;L^q(S)))}\Big\}$$
$$\gtrsim_{p,q} \Big\| \int_{[0,t]\times B} F \, d\tilde{N} \Big\|_{L^p(\Omega;L^q(S))}.$$

If $1 < p, q \le 2$ then

$$\|F\|_{L^p(\Omega;L^2([0,t]\times B;L^q(S)))+L^p(\Omega;L^p([0,t]\times B;L^q(S)))}$$
$$\gtrsim_{p,q} \Big\| \int_{[0,t]\times B} F \, d\tilde{N} \Big\|_{L^p(\Omega;L^q(S))}.$$

Finally, if $1 < p < q$ and $2 \le q < \infty$, then

$$\|F\|_{L^p(\Omega;L^q([0,t]\times B;L^q(S)))+L^p(\Omega;L^p([0,t]\times B;L^q(S)))}$$
$$\gtrsim_{p,q} \Big\| \int_{[0,t]\times B} F \, d\tilde{N} \Big\|_{L^p(\Omega;L^q(S))}.$$

In the case where X has both type 2 and cotype 2 we obtain two-sided estimates for the L^p-norm of the stochastic integral with respect to a compensated Poisson random measure. By Theorem 1.5, such a space is isomorphic to a Hilbert space.

Corollary 2.18. *Let $2 \le p < \infty$ and let H be a Hilbert space. Let N be a Poisson random measure. Then for any simple, adapted H-valued process F, any $B \in \mathcal{J}$ and any $t \ge 0$ we have*

$$\Big\| \int_{[0,t]\times B} F \, d\tilde{N} \Big\|_{L^p(\Omega;H)}$$
$$\simeq_p \max\Big\{\|F\|_{L^p(\Omega;L^2([0,t]\times B;H))}, \|F\|_{L^p(\Omega;L^p([0,t]\times B;H))}\Big\}. \quad (2.11)$$

On the other hand, if $1 < p < 2$ we have

$$\Big\| \int_{[0,t]\times B} F \, d\tilde{N} \Big\|_{L^p(\Omega;H)}$$
$$\simeq_p \inf\Big\{\|F_1\|_{L^p(\Omega;L^2([0,t]\times B;H))} + \|F_2\|_{L^p(\Omega;L^p([0,t]\times B;H))}\Big\}, \quad (2.12)$$

where the infimum is taken over all $F = F_1 + F_2$ with $F_1 \in L^p(\Omega; L^2([0,t] \times B; H))$ and $F_2 \in L^p(\Omega; L^p([0,t] \times B; H))$.

Remark 2.19. In the special case where $H = \mathbb{R}^n$, the upper estimate in (2.11) was obtained in [89], p.335, Corollary 2.12, by a completely different argument based on Itô's formula. Rather surprisingly, the other estimates in Corollary 2.18 seem to be unknown even in the scalar-valued case.

Motivated by Corollary 2.18, we introduce the notation

$$SI_{p,H} = \begin{cases} L^p(\Omega; L^2(\mathbb{R}_+ \times J; H)) \cap L^p(\Omega; L^p(\mathbb{R}_+ \times J; H)), & 2 \leq p < \infty, \\ L^p(\Omega; L^2(\mathbb{R}_+ \times J; H)) + L^p(\Omega; L^p(\mathbb{R}_+ \times J; H)), & 1 < p < 2. \end{cases}$$

We can now extend the class of stochastically integrable processes through the Itô-type isomorphism found in Corollary 2.18.

Definition 2.20. *Let $1 < p < \infty$, let H be a Hilbert space and let S be a σ-finite measure space. Let $(J, \mathcal{J}, \nu)$ be a σ-finite measure space and let N be a Poisson random measure on $\mathbb{R}_+ \times J$ with associated compensated Poisson random measure $\tilde{N}$. Let $t \geq 0$ and $B \in \mathcal{J}$. We say that an element $F \in SI_{p,H}$ is* L^p-stochastically integrable *on $[0,t] \times B$ if there exists a sequence of simple, adapted H-valued processes (F_n) such that $F_n \to F\chi_{[0,t]\times B}$ in $SI_{p,H}$ as $n \to \infty$. In this case we define the* L^p-stochastic integral *of F on $[0,t] \times B$ with respect to $\tilde{N}$ by*

$$\int_{[0,t]\times B} F \, d\tilde{N} = \lim_{n\to\infty} \int_{[0,t]\times B} F_n \, d\tilde{N},$$

where the limit is taken in $L^p(\Omega; H)$. We let $SI^{ad}_{p,H}$ denote the space of all L^p-stochastically integrable elements on $\mathbb{R}_+ \times J$.

Corollary 2.21. *If $1 < p < \infty$ and H is a Hilbert space, then $SI^{ad}_{p,H}$ is a Banach space. Moreover, if $F \in SI^{ad}_{p,H}$ and $2 \leq p < \infty$, then (2.11) holds. On the other hand, if $1 < p < 2$, then (2.12) holds.*

2.3 Itô isomorphisms for L^p-valued stochastic integrals

We now focus on Poisson stochastic integration in the special case where the integrand takes values in $L^q(S)$, where S is a σ-finite measure space. The main aim is to improve the results in Corollaries 2.14 and 2.17 and obtain two-sided estimates for the p-th moments of Poisson stochastic integrals. Throughout, we let N be a Poisson random measure on $(\mathbb{R}_+ \times J, \mathcal{B}(\mathbb{R}_+) \times \mathcal{J}, dt \times \nu)$ and let $\tilde{N}$ be the associated compensated random measure. We also fix a filtration $(\mathcal{F}_t)_{t\geq 0}$ in $(\Omega, \mathcal{F}, \mathbb{P})$ such that Assumption 2.5 holds.

Theorem 2.22. *Suppose $2 \leq q \leq p < \infty$. Then for any simple, adapted $L^q(S)$-valued process F, any $B \in \mathcal{J}$ and any $t \geq 0$ we have*

$$\Big\| \int_{[0,t]\times B} F \, d\tilde{N} \Big\|_{L^p(\Omega; L^q(S))}$$

$$\simeq_{p,q} \max\Big\{\|F\|_{L^p(\Omega;L^q(S;L^2([0,t]\times B)))}, \\ \|F\|_{L^p(\Omega;L^p([0,t]\times B;L^q(S)))}, \|F\|_{L^p(\Omega;L^q([0,t]\times B;L^q(S)))}\Big\}.$$

Proof. Let F be the simple adapted process given in (2.5), taking Remark 2.7 into account. We may assume that $t = t_{l+1}$ and $B = J$. By Theorem 2.3,

$$\Big\|\int_{[0,t]\times J} F\, d\tilde{N}\Big\|_{L^p(\Omega;L^q(S))} \simeq_p \Big\|\sum_{i,j,k} F_{i,k}\tilde{N}_{i,j}x_{i,j,k}\Big\|_{L^p(\Omega\times\tilde{\Omega};L^q(S))} \\ = \Big(\mathbb{E}\tilde{\mathbb{E}}\Big\|\sum_{i,j}\tilde{N}_{i,j}y_{i,j}\Big\|^p_{L^q(S)}\Big)^{\frac{1}{p}},$$

where $y_{i,j} = \sum_{k=1}^n F_{i,k}x_{i,j,k}$. Fix $\omega \in \Omega$ and set $\xi_{i,j}(\omega) = y_{i,j}(\omega)\tilde{N}_{i,j}$. Then $(\xi_{i,j}(\omega))$ is a sequence of independent, mean zero $L^q(S)$-valued random variables for every ω. Hence we may apply Theorem 1.26 pointwise and use Lemma 2.2 to obtain

$$\Big(\tilde{\mathbb{E}}\Big\|\sum_{i,j}\tilde{N}_{i,j}y_{i,j}(\omega)\Big\|^p_{L^q(S)}\Big)^{\frac{1}{p}} \\ \simeq_{p,q} \max\Big\{\Big(\sum_{i,j}\tilde{\mathbb{E}}|\tilde{N}_{i,j}|^p\|y_{i,j}(\omega)\|^p_{L^q(S)}\Big)^{\frac{1}{p}}, \\ \Big(\sum_{i,j}\tilde{\mathbb{E}}|\tilde{N}_{i,j}|^p\|y_{i,j}(\omega)\|^q_{L^q(S)}\Big)^{\frac{1}{q}}, \Big\|\Big(\sum_{i,j}\tilde{\mathbb{E}}|\tilde{N}_{i,j}|^2|y_{i,j}(\omega)|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(S)}\Big\} \\ \simeq_{p,q} \max\Big\{\Big(\sum_{i,j}(t_{i+1}-t_i)\nu(A_j)\|y_{i,j}(\omega)\|^p_{L^q(S)}\Big)^{\frac{1}{p}}, \\ \Big(\sum_{i,j}(t_{i+1}-t_i)\nu(A_j)\|y_{i,j}(\omega)\|^q_{L^q(S)}\Big)^{\frac{1}{q}}, \\ \Big\|\Big(\sum_{i,j}(t_{i+1}-t_i)\nu(A_j)|y_{i,j}(\omega)|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(S)}\Big\} \\ = \max\Big\{\|F(\omega)\|_{L^q(S;L^2([0,t]\times J))}, \\ \|F(\omega)\|_{L^p([0,t]\times J;L^q(S))}, \|F(\omega)\|_{L^q([0,t]\times J;L^q(S))}\Big\}.$$

The result now follows by taking the $L^p(\Omega)$-norm on both sides. □

We can deduce estimates in the case $1 < p \le q \le 2$ by duality from Theorem 2.22. First we need the following two lemmas.

Lemma 2.23. *Let $1 < p, q \le 2$. Then for any simple, adapted $L^q(S)$-valued process F, any $B \in \mathcal{J}$ and any $t \ge 0$ we have*

$$\Big\| \int_{[0,t]\times B} F \, d\tilde{N} \Big\|_{L^p(\Omega;L^q(S))} \lesssim_{p,q} \|F\|_{L^p(\Omega;L^q(S;L^2([0,t]\times B)))}.$$

On the other hand, if $2 \le p,q < \infty$,

$$\|F\|_{L^p(\Omega;L^q(S;L^2([0,t]\times B)))} \lesssim_{p,q} \Big\| \int_{[0,t]\times B} F \, d\tilde{N} \Big\|_{L^p(\Omega;L^q(S))}.$$

Proof. Suppose $1 < p,q \le 2$ and let F be as in (2.5). We may assume $t = t_{l+1}$ and $B = J$. Let $y_{i,j} = \sum_{k=1}^n F_{i,k} x_{i,j,k}$. By subsequently applying Theorem 2.3 and Lemma 1.24 we obtain

$$\begin{aligned}
\Big\| \int_{[0,t]\times J} F \, d\tilde{N} \Big\|_{L^p(\Omega;L^q(S))} &\simeq_{p,q} \Big(\mathbb{E}\tilde{\mathbb{E}} \Big\| \sum_{i,j} \tilde{N}_{i,j} y_{i,j} \Big\|_{L^q(S)}^p \Big)^{\frac{1}{p}} \\
&\lesssim_{p,q} \Big(\mathbb{E}\Big\| \Big(\sum_{i,j} \tilde{\mathbb{E}} |\tilde{N}_{i,j}|^2 |y_{i,j}|^2 \Big)^{\frac{1}{2}} \Big\|_{L^q(S)}^p \Big)^{\frac{1}{p}} \\
&= \Big(\mathbb{E}\Big\| \Big(\sum_{i,j} (t_{i+1} - t_i)\nu(A_j) |y_{i,j}|^2 \Big)^{\frac{1}{2}} \Big\|_{L^q(S)}^p \Big)^{\frac{1}{p}} \\
&= \|F\|_{L^p(\Omega;L^q(S;L^2([0,t]\times J)))}.
\end{aligned}$$

If $2 \le p,q < \infty$ then the inequality $\lesssim_{p,q}$ above is reversed. □

Lemma 2.24. *Fix* $1 \le p,q < \infty$. *Let* $0 \le t_0 < t_1 < \ldots < t_{l+1} < \infty$, $F_{i,k} \in L^\infty(\Omega)$, $x_{i,j,k} \in L^q(S)$, *and let* $A_1, \ldots A_m$ *be disjoint sets in* $\mathcal{J}$ *satisfying* $\nu(A_j) < \infty$ *for* $i = 1, \ldots, l$, $j = 1, \ldots, m$ *and* $k = 1, \ldots, n$. *Define*

$$F = \sum_{i,j,k} F_{i,k} \chi_{(t_i,t_{i+1}]} \chi_{A_j} x_{i,j,k}$$

and let $\tilde{F}$ *be the associated simple adapted* $L^q(S)$*-valued process given by*

$$\tilde{F} = \sum_{i,j,k} \mathbb{E}(F_{i,k}|\mathcal{F}_{t_i}) \chi_{(t_i,t_{i+1}]} \chi_{A_j} x_{i,j,k}.$$

Then,

$$\|\tilde{F}\|_{L^p(\Omega;L^q(S;L^2([0,t]\times J)))} \lesssim_{p,q} \|F\|_{L^p(\Omega;L^q(S;L^2([0,t]\times J)))}.$$

Proof. We may assume $t = t_{l+1}$. Let $(r_{i,j})$ be a doubly indexed Rademacher sequence. By Theorem 1.2 we have

$$\begin{aligned}
&\|\tilde{F}\|_{L^p(\Omega;L^q(S;L^2([0,t]\times J)))} \\
&\quad = \Big(\mathbb{E}\Big\| \Big(\sum_{i,j} (t_{i+1} - t_i)\nu(A_j) \Big| \sum_k \mathbb{E}(F_{i,k}|\mathcal{F}_{t_i}) x_{i,j,k} \Big|^2 \Big)^{\frac{1}{2}} \Big\|_{L^q(S)}^p \Big)^{\frac{1}{p}}
\end{aligned}$$

$$\begin{aligned}
&\simeq_{p,q} \Big(\mathbb{E}_r\mathbb{E}\Big\|\sum_{i,j} r_{i,j}(t_{i+1}-t_i)^{\frac12}\nu(A_j)^{\frac12}\sum_k \mathbb{E}(F_{i,k}|\mathcal{F}_{t_i})x_{i,j,k}\Big\|^p_{L^q(S)}\Big)^{\frac1p}\\
&\le \Big(\mathbb{E}_r\mathbb{E}\Big\|\sum_{i,j} r_{i,j}(t_{i+1}-t_i)^{\frac12}\nu(A_j)^{\frac12}\sum_k F_{i,k}x_{i,j,k}\Big\|^p_{L^q(S)}\Big)^{\frac1p}\\
&\simeq_{p,q} \|F\|_{L^p(\Omega;L^q(S;L^2([0,t]\times J)))},
\end{aligned}$$

where we use that $\mathbb{E}(\cdot|\mathcal{F}_{t_j})$ extends to a contraction on $L^p(\Omega;L^q(S))$. $\square$

In the proof of Theorem 2.25 below we will use the fact that the dual space of

$$\begin{aligned}
&L^p(\Omega;L^q(S;L^2([0,t]\times J)))\\
&\quad + L^p(\Omega;L^p([0,t]\times J;L^q(S))) + L^p(\Omega;L^q([0,t]\times J;L^q(S)))
\end{aligned} \tag{2.13}$$

is isometrically isomorphic to

$$\begin{aligned}
&L^{p'}(\Omega;L^{q'}(S;L^2([0,t]\times J)))\\
&\quad \cap L^{p'}(\Omega;L^{p'}([0,t]\times J;L^{q'}(S))) \cap L^{p'}(\Omega;L^{q'}([0,t]\times J;L^{q'}(S)))
\end{aligned}$$

whenever $1<p,p',q,q'<\infty$ satisfy $\frac1p+\frac1{p'}=1$ and $\frac1q+\frac1{q'}=1$. Indeed, the three spaces in (2.13) have a dense intersection (in fact, $L^\infty(\Omega)\otimes L^\infty([0,t])\otimes L^\infty(J)\otimes L^\infty(S)$ is dense in each of the spaces) and L^p, L^q and L^2 are all reflexive. Hence, the asserted duality follows from the general principles set out in Section 1.2. Moreover, it is clear from Fubini's theorem that for F,G in $L^\infty(\Omega)\otimes L^\infty([0,t])\otimes L^\infty(J)\otimes L^\infty(S)$, say, the corresponding duality bracket is given by

$$\langle F,G\rangle = \iiiint_{\Omega\times[0,t]\times J\times S} F(s)G(s)\; d\mathbb{P}dtd\nu d\mu. \tag{2.14}$$

Analogous remarks apply for similar spaces considered below, for example the dual of

$$\begin{aligned}
&L^p(\Omega;L^q(S;L^2([0,t]\times J)))\\
&\quad + (L^p(\Omega;L^p([0,t]\times J;L^q(S))) \cap L^p(\Omega;L^q([0,t]\times J;L^q(S))))
\end{aligned}$$

is isometrically isomorphic to

$$\begin{aligned}
&L^{p'}(\Omega;L^{q'}(S;L^2([0,t]\times J)))\\
&\quad \cap (L^{p'}(\Omega;L^{p'}([0,t]\times J;L^{q'}(S))) + L^{p'}(\Omega;L^{q'}([0,t]\times J;L^{q'}(S))))
\end{aligned}$$

and the corresponding duality bracket again satisfies (2.14) for F,G in $L^\infty(\Omega)\otimes L^\infty([0,t])\otimes L^\infty(J)\otimes L^\infty(S)$.

Theorem 2.25. *Suppose $1 < p \le q \le 2$. Then for any simple, adapted $L^q(S)$-valued process F, any $B \in \mathcal{J}$ and any $t \ge 0$ we have*

$$\Big\| \int_{[0,t]\times B} F \, d\tilde{N} \Big\|_{L^p(\Omega;L^q(S))}$$
$$\simeq_{p,q} \inf \Big\{ \|F_1\|_{L^p(\Omega;L^q(S;L^2([0,t]\times B)))}$$
$$+ \|F_2\|_{L^p(\Omega;L^p([0,t]\times B;L^q(S)))} + \|F_3\|_{L^p(\Omega;L^q([0,t]\times B;L^q(S)))} \Big\},$$

where the infimum is taken over all decompositions $F = F_1 + F_2 + F_3$ with $F_1 \in L^p(\Omega; L^q(S; L^2([0,t] \times B)))$, $F_2 \in L^p(\Omega; L^p([0,t] \times B; L^q(S)))$ and $F_3 \in L^p(\Omega; L^q([0,t] \times B; L^q(S)))$.

Proof. We first show that the infimum on the right hand side dominates the p-th moment of the stochastic integral. Let F be the simple adapted process given in (2.5), taking Remark 2.7 into account. We may assume that $t = t_{l+1}$ and $B = J$. By Theorem 2.3,

$$\Big\| \int_{[0,t]\times J} F \, d\tilde{N} \Big\|_{L^p(\Omega;L^q(S))} \simeq_p \Big\| \sum_{i,j,k} F_{i,k} \tilde{N}_{i,j} x_{i,j,k} \Big\|_{L^p(\Omega\times\tilde{\Omega};L^q(S))}$$
$$= \Big(\mathbb{E}\tilde{\mathbb{E}} \Big\| \sum_{i,j} \tilde{N}_{i,j} y_{i,j} \Big\|_{L^q(S)}^p \Big)^{\frac{1}{p}}, \tag{2.15}$$

where $y_{i,j} = \sum_{k=1}^n F_{i,k} x_{i,j,k}$. Set $\xi_{i,j}(\omega) = y_{i,j}(\omega)\tilde{N}_{i,j}$. By applying Theorem 1.29 for the sequence $(\xi_{i,j}(\omega))$ and using Lemma 2.2 we obtain for $r = p, q$,

$$\Big(\mathbb{E}\tilde{\mathbb{E}} \Big\| \sum_{i,j} \tilde{N}_{i,j} y_{i,j} \Big\|_{L^q(S)}^p \Big)^{\frac{1}{p}} \lesssim_{p,q} \Big(\mathbb{E} \Big| \sum_{i,j} \tilde{\mathbb{E}} |\tilde{N}_{i,j}|^r \|y_{i,j}\|_{L^q(S)}^r \Big|^p \Big)^{\frac{1}{p}}$$
$$\simeq_r \Big(\mathbb{E} \Big| \sum_{i,j} (t_{i+1} - t_i) \nu(A_j) \|y_{i,j}\|_{L^q(S)}^r \Big|^p \Big)^{\frac{1}{p}}$$
$$= \|F\|_{L^p(\Omega;L^r([0,t]\times J;L^q(S)))}.$$

Similarly, by Theorem 1.29 we obtain

$$\Big(\mathbb{E}\tilde{\mathbb{E}} \Big\| \sum_{i,j} \tilde{N}_{i,j} y_{i,j} \Big\|_{L^q(S)}^p \Big)^{\frac{1}{p}} \lesssim_{p,q} \|F\|_{L^p(\Omega;L^q(S;L^2([0,t]\times J)))}.$$

Since $L^\infty(\Omega) \otimes L^\infty([0,t]) \otimes L^\infty(J) \otimes L^\infty(S)$ is dense in $L^p(\Omega; L^q(S; L^2([0,t] \times J)))$, $L^p(\Omega; L^p([0,t] \times J; L^q(S)))$ and $L^p(\Omega; L^q([0,t] \times J; L^q(S)))$, we have, for F in $L^\infty(\Omega) \otimes L^\infty([0,t]) \otimes L^\infty(J) \otimes L^\infty(S)$,

$$\|F\|_{L^p(\Omega;L^q(S;L^2([0,t]\times J)))+L^p(\Omega;L^p([0,t]\times J;L^q(S)))+L^p(\Omega;L^q([0,t]\times J;L^q(S)))}$$
$$= \inf \Big\{ \|F_1\|_{L^p(\Omega;L^q(S;L^2([0,t]\times J)))}$$

$$+ \|F_2\|_{L^p(\Omega;L^p([0,t]\times J;L^q(S)))} + \|F_3\|_{L^p(\Omega;L^q([0,t]\times J;L^q(S)))}\Big\},$$

where the infimum is taken over all decompositions $F = F_1 + F_2 + F_3$ in $L^\infty(\Omega)\otimes L^\infty([0,t])\otimes L^\infty(J)\otimes L^\infty(S)$. Let $\mathcal{A}$ be the sub-σ-algebra of $\mathcal{B}(\mathbb{R}_+)\times\mathcal{J}$ generated by the sets $(t_i,t_{i+1}]\times A_j$. By Lemma 2.10 $\mathbb{E}(F_\alpha|\mathcal{A})$ is of the form

$$\mathbb{E}(F_\alpha|\mathcal{A}) = \sum_{i,j,k} F_{i,k,\alpha}\chi_{(t_i,t_{i+1}]}\chi_{A_j}x_{i,j,k,\alpha} \qquad (\alpha = 1,2,3).$$

For $\alpha = 1,2,3$ we let $\tilde{F}_\alpha$ be the associated simple, adapted process

$$\tilde{F}_\alpha = \sum_{i,j,k}\mathbb{E}(F_{i,k,\alpha}|\mathcal{F}_{t_i})\chi_{(t_i,t_{i+1}]}\chi_{A_j}x_{i,j,k,\alpha},$$

then $F = \tilde{F}_1 + \tilde{F}_2 + \tilde{F}_3$. By the triangle inequality and the above,

$$\begin{aligned}
&\Big\|\int_{[0,t]\times J} F\,d\tilde{N}\Big\|_{L^p(\Omega;L^q(S))}\\
&\quad\le \sum_{i=1}^3\Big\|\int_{[0,t]\times J}\tilde{F}_i\,d\tilde{N}\Big\|_{L^p(\Omega;L^q(S))}\\
&\quad\lesssim_{p,q} \|\tilde{F}_1\|_{L^p(\Omega;L^q(S;L^2([0,t]\times J)))} + \|\tilde{F}_2\|_{L^p([0,t]\times J;L^p(\Omega;L^q(S)))}\\
&\qquad\quad + \|\tilde{F}_3\|_{L^p(\Omega;L^q([0,t]\times J;L^q(S)))}\\
&\quad\lesssim_{p,q} \|\mathbb{E}(F_1|\mathcal{A})\|_{L^p(\Omega;L^q(S;L^2([0,t]\times J)))} + \|\mathbb{E}(F_2|\mathcal{A})\|_{L^p([0,t]\times J;L^p(\Omega;L^q(S)))}\\
&\qquad\quad + \|\mathbb{E}(F_3|\mathcal{A})\|_{L^p(\Omega;L^q([0,t]\times J;L^q(S)))}\\
&\quad\le \|F_1\|_{L^p(\Omega;L^q(S;L^2([0,t]\times J)))} + \|F_2\|_{L^p([0,t]\times J;L^p(\Omega;L^q(S)))}\\
&\qquad\quad + \|F_3\|_{L^p(\Omega;L^q([0,t]\times J;L^q(S)))},
\end{aligned}$$

where the penultimate inequality follows from Lemmas 2.9 and 2.24. By now taking the infimum over all decompositions $F = F_1 + F_2 + F_3$ as above we obtain

$$\begin{aligned}
&\Big\|\int_{[0,t]\times J} F\,d\tilde{N}\Big\|_{L^p(\Omega;L^q(S))}\\
&\quad\lesssim_{p,q} \|F\|_{L^p(\Omega;L^q(S;L^2([0,t]\times J)))+L^p(\Omega;L^p([0,t]\times J;L^q(S)))+L^p(\Omega;L^q([0,t]\times J;L^q(S)))},
\end{aligned}$$

as asserted.

We deduce the reverse inequality by duality from Theorem 2.22. Let p',q' be the Hölder conjugates of p and q, i.e. $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{q}+\frac{1}{q'}=1$. Let G be an element of the algebraic tensor product $L^\infty(\Omega)\otimes L^\infty([0,t])\otimes L^\infty(J)\otimes L^\infty(S)$. Then by Lemma 2.10 $\mathbb{E}(G|\mathcal{A})$ is of the form

$$\mathbb{E}(G|\mathcal{A}) = \sum_{i,j,k} G_{i,k}\chi_{(t_i,t_{i+1}]}\chi_{A_j}x^*_{i,j,k},$$

where $G_{i,k} \in L^\infty(\Omega)$. Let $\mathcal{E}_i$ be the conditional expectation with respect to $\mathcal{F}_{t_i}$ and let $\tilde{G}$ be the simple adapted process defined by

$$\tilde{G} = \sum_{i,j,k} \mathcal{E}_i(G_{i,k})\chi_{(t_i,t_{i+1}]}\chi_{A_j} x^*_{i,j,k}.$$

Let $\langle \cdot, \cdot \rangle$ be the duality bracket in (2.14). Then, by Lemma 1.18 and Theorem 2.22,

$$\begin{aligned}
&\langle F, G\rangle \\
&\quad = \langle F, \mathbb{E}(G|\mathcal{A})\rangle \\
&\quad = \sum_{i,j,k} \mathbb{E}(F_{i,k}G_{i,k}) dt \times d\nu((t_i,t_{i+1}] \times A_j)\langle x_{i,j,k}, x^*_{i,j,k}\rangle \\
&\quad = \sum_{i,j,k} \mathbb{E}(F_{i,k}\mathcal{E}_i(G_{i,k})) dt \times d\nu((t_i,t_{i+1}] \times A_j)\langle x_{i,j,k}, x^*_{i,j,k}\rangle \\
&\quad = \sum_{i,j,k,l,m,n} \mathbb{E}(F_{i,k}\mathcal{E}_j(G_{l,n}))\tilde{\mathbb{E}}(\tilde{N}_{i,j}\tilde{N}_{l,m})\langle x_{i,j,k}, x^*_{l,m,n}\rangle \\
&\quad = \sum_{i,j,k,l,m,n} \mathbb{E}\tilde{\mathbb{E}}(F_{i,k}\tilde{N}_{i,j}\mathcal{E}_j(G_{l,n})\tilde{N}_{l,m}\langle x_{i,j,k}, x^*_{l,m,n}\rangle) \\
&\quad = \mathbb{E}\tilde{\mathbb{E}}\Big(\Big\langle \sum_{i,j,k} F_{i,k}\tilde{N}_{i,j}x_{i,j,k}, \sum_{l,m,n} \mathcal{E}_{l,n}(G_{l,n})\tilde{N}_{l,m}x^*_{l,m,n}\Big\rangle\Big) \\
&\quad \le \Big\| \sum_{i,j,k} F_{i,k}\tilde{N}_{i,j}x_{i,j,k}\Big\|_{L^p(\Omega\times\tilde{\Omega};L^q(S))} \\
&\qquad\qquad \Big\| \sum_{l,m,n} \mathcal{E}_{l,n}(G_{l,n})\tilde{N}_{l,m}x^*_{l,m,n}\Big\|_{L^{p'}(\Omega\times\tilde{\Omega};L^{q'}(S))} \\
&\quad \lesssim_{p,q} \Big\| \sum_{i,j,k} F_{i,k}\tilde{N}_{i,j}x_{i,j,k}\Big\|_{L^p(\Omega\times\tilde{\Omega};L^q(S))} \max\Big\{\|\tilde{G}\|_{L^{p'}(\Omega;L^{q'}(S;L^2([0,t]\times J)))}, \\
&\qquad \|\tilde{G}\|_{L^{p'}(\Omega;L^{p'}([0,t]\times J;L^{q'}(S)))}, \|\tilde{G}\|_{L^{p'}(\Omega;L^{q'}([0,t]\times J;L^{q'}(S)))}\Big\} \\
&\quad \lesssim_{p,q} \Big\| \sum_{i,j,k} F_{i,k}\tilde{N}_{i,j}x_{i,j,k}\Big\|_{L^p(\Omega\times\tilde{\Omega};L^q(S))} \\
&\qquad \max\Big\{\|\mathbb{E}(G|\mathcal{A})\|_{L^{p'}(\Omega;L^{q'}(S;L^2([0,t]\times J)))}, \\
&\qquad\quad \|\mathbb{E}(G|\mathcal{A})\|_{L^{p'}(\Omega;L^{p'}([0,t]\times J;L^{q'}(S)))}, \|\mathbb{E}(G|\mathcal{A})\|_{L^{p'}(\Omega;L^{q'}([0,t]\times J;L^{q'}(S)))}\Big\} \\
&\quad \le \Big\| \sum_{i,j,k} F_{i,k}\tilde{N}_{i,j}x_{i,j,k}\Big\|_{L^p(\Omega\times\tilde{\Omega};L^q(S))} \max\Big\{\|G\|_{L^{p'}(\Omega;L^{q'}(S;L^2([0,t]\times J)))}, \\
&\qquad \|G\|_{L^{p'}(\Omega;L^{p'}([0,t]\times J;L^{q'}(S)))}, \|G\|_{L^{p'}(\Omega;L^{q'}([0,t]\times J;L^{q'}(S)))}\Big\},
\end{aligned}$$

where the penultimate inequality follows by Lemmas 2.9 and 2.24. Since $L^\infty(\Omega) \otimes L^\infty([0,t]) \otimes L^\infty(J) \otimes L^\infty(S)$ is dense in

$$L^{p'}(\Omega; L^{q'}(S; L^2([0,t] \times J))) \\ \cap L^{p'}(\Omega; L^{p'}([0,t] \times J; L^{q'}(S))) \cap L^{p'}(\Omega; L^{q'}([0,t] \times J; L^{q'}(S))),$$

we conclude that

$$\|F\|_{L^p(\Omega;L^q(S;L^2([0,t]\times J)))+L^p(\Omega;L^p([0,t]\times J;L^q(S)))+L^p(\Omega;L^q([0,t]\times J;L^q(S)))} \\ \lesssim_{p,q} \Big\| \sum_{i,j,k} F_{i,k} \tilde{N}_{i,j} x_{i,j,k} \Big\|_{L^p(\Omega\times\tilde{\Omega};L^q(S))}.$$

By (2.15) our proof is complete. □

Theorem 2.26. *Suppose $2 \le p \le q < \infty$. Then for any simple, adapted $L^q(S)$-valued process F, any $B \in \mathcal{J}$ and any $t \ge 0$ we have*

$$\Big\| \int_{[0,t]\times B} F \, d\tilde{N} \Big\|_{L^p(\Omega;L^q(S))} \\ \simeq_{p,q} \max\Big\{ \|F\|_{L^p(\Omega;L^q(S;L^2([0,t]\times B)))}, \\ \inf\Big\{ \|F_1\|_{L^p(\Omega;L^p([0,t]\times B;L^q(S)))} + \|F_2\|_{L^p(\Omega;L^q([0,t]\times B;L^q(S)))} \Big\}\Big\},$$

where the infimum is taken over all decompositions $F = F_1 + F_2$ with $F_1 \in L^p(\Omega; L^p([0,t] \times B; L^q(S)))$ and $F_2 \in L^p(\Omega; L^q([0,t] \times B; L^q(S)))$.

Proof. By Corollary 2.17 we have

$$\|F\|_{L^p(\Omega;L^q([0,t]\times B;L^q(S)))+L^p(\Omega;L^p([0,t]\times B;L^q(S)))} \\ \lesssim_{p,q} \Big\| \int_{[0,t]\times B} F \, d\tilde{N} \Big\|_{L^p(\Omega;L^q(S))}.$$

On the other hand, by Lemma 2.23,

$$\|F\|_{L^p(\Omega;L^q(S;L^2([0,t]\times B)))} \lesssim_{p,q} \Big\| \int_{[0,t]\times B} F \, d\tilde{N} \Big\|_{L^p(\Omega;L^q(S))}.$$

We now prove the reverse inequality. Let F be the simple adapted process defined in (2.5), taking Remark 2.7 into account. We may assume $t = t_{l+1}$ and $B = J$. Let $F = F_1 + F_2$ with F_1, F_2 in $L^\infty(\Omega) \otimes L^\infty([0,t]) \otimes L^\infty(J) \otimes L^\infty(S)$. Let $\mathcal{A}$ be the sub-σ-algebra of $\mathcal{B}(\mathbb{R}_+) \times \mathcal{J}$ generated by the sets $(t_i, t_{i+1}] \times A_j$. By Lemma 2.10 $\mathbb{E}(F_1|\mathcal{A}), \mathbb{E}(F_2|\mathcal{A})$ are of the form

$$\mathbb{E}(F_\alpha|\mathcal{A}) = \sum_{i,j,k} F_{i,k,\alpha} \chi_{(t_i,t_{i+1}]} \chi_{A_j} x_{i,j,k,\alpha} \qquad (\alpha = 1, 2).$$

For $\alpha = 1, 2$ set $y_{i,j,\alpha} = \sum_k F_{i,k,\alpha} x_{i,j,k,\alpha}$. By Theorem 2.3, Lemma 1.25 (with $s = 2$) and Lemma 2.2 we obtain

$$
\begin{aligned}
&\Big\| \int_{[0,t]\times J} F \, d\tilde{N} \Big\|_{L^p(\Omega;L^q(S))} \\
&\simeq_{p,q} \Big(\mathbb{E}\tilde{\mathbb{E}} \Big\| \sum_{i,j} \tilde{N}_{i,j} y_{i,j} \Big\|_{L^q(S)}^p \Big)^{\frac{1}{p}} \\
&\lesssim_{p,q} \max\Big\{ \Big(\mathbb{E} \Big\| \Big(\sum_{i,j} \tilde{\mathbb{E}} |\tilde{N}_{i,j}|^2 |y_{i,j}|^2 \Big)^{\frac{1}{2}} \Big\|_{L^q(S)}^p \Big)^{\frac{1}{p}}, \\
&\qquad\qquad \Big(\mathbb{E}\tilde{\mathbb{E}} \Big(\sum_{i,j} \| \tilde{N}_{i,j} y_{i,j} \|_{L^q(S)}^q \Big)^{\frac{p}{q}} \Big)^{\frac{1}{p}} \Big\} \\
&\leq \max\Big\{ \Big(\mathbb{E} \Big\| \Big(\sum_{i,j} \tilde{\mathbb{E}} |\tilde{N}_{i,j}|^2 |y_{i,j}|^2 \Big)^{\frac{1}{2}} \Big\|_{L^q(S)}^p \Big)^{\frac{1}{p}}, \\
&\quad \Big(\mathbb{E}\tilde{\mathbb{E}} \Big(\sum_{i,j} \| \tilde{N}_{i,j} y_{i,j,1} \|_{L^q(S)}^q \Big)^{\frac{p}{q}} \Big)^{\frac{1}{p}} + \Big(\mathbb{E}\tilde{\mathbb{E}} \Big(\sum_{i,j} \| \tilde{N}_{i,j} y_{i,j,2} \|_{L^q(S)}^q \Big)^{\frac{p}{q}} \Big)^{\frac{1}{p}} \Big\} \\
&\leq \max\Big\{ \Big(\mathbb{E} \Big\| \Big(\sum_{i,j} \tilde{\mathbb{E}} |\tilde{N}_{i,j}|^2 |y_{i,j}|^2 \Big)^{\frac{1}{2}} \Big\|_{L^q(S)}^p \Big)^{\frac{1}{p}}, \\
&\quad \Big(\mathbb{E} \sum_{i,j} \tilde{\mathbb{E}} |\tilde{N}_{i,j}|^p \| y_{i,j,1} \|_{L^q(S)}^p \Big)^{\frac{1}{p}} + \Big(\mathbb{E} \Big(\sum_{i,j} \tilde{\mathbb{E}} |\tilde{N}_{i,j}|^q \| y_{i,j,2} \|_{L^q(S)}^q \Big)^{\frac{p}{q}} \Big)^{\frac{1}{p}} \Big\} \\
&\simeq_{p,q} \max\Big\{ \Big(\mathbb{E} \Big\| \Big(\sum_{i,j} (t_{i+1}-t_i)\nu(A_j) |y_{i,j}|^2 \Big)^{\frac{1}{2}} \Big\|_{L^q(S)}^p \Big)^{\frac{1}{p}}, \\
&\quad \Big(\mathbb{E} \sum_{i,j} (t_{i+1}-t_i)\nu(A_j) \| y_{i,j,1} \|_{L^q(S)}^p \Big)^{\frac{1}{p}} \\
&\qquad + \Big(\mathbb{E} \Big(\sum_{i,j} (t_{i+1}-t_i)\nu(A_j) \| y_{i,j,2} \|_{L^q(S)}^q \Big)^{\frac{p}{q}} \Big)^{\frac{1}{p}} \Big\} \\
&= \max\Big\{ \|F\|_{L^p(\Omega;L^q(S;L^2([0,t]\times J)))}, \|\mathbb{E}(F_1|\mathcal{A})\|_{L^p(\Omega;L^p([0,t]\times J;L^q(S)))} + \\
&\qquad\qquad \|\mathbb{E}(F_2|\mathcal{A})\|_{L^p(\Omega;L^q([0,t]\times J;L^q(S)))} \Big\} \\
&\lesssim_{p,q} \max\Big\{ \|F\|_{L^p(\Omega;L^q(S;L^2([0,t]\times J)))}, \|F_1\|_{L^p(\Omega;L^p([0,t]\times J;L^q(S)))} \\
&\qquad\qquad + \|F_2\|_{L^p(\Omega;L^q([0,t]\times J;L^q(S)))} \Big\}.
\end{aligned}
$$

The result now follows by taking the infimum over all decompositions $F = F_1 + F_2$ as above. □

Theorem 2.27. *Suppose $1 < q \leq p \leq 2$. Then there exist constants depending only on p and q such that for any simple, adapted $L^q(S)$-valued process F, any $B \in \mathcal{J}$ and any $t \geq 0$ we have*

$$\Big\| \int_{[0,t]\times B} F \, d\tilde{N} \Big\|_{L^p(\Omega;L^q(S))}$$

$$\simeq_{p,q} \inf\Big\{\|F_1\|_{L^p(\Omega;L^q(S;L^2([0,t]\times B)))} + \max\Big\{\|F_2\|_{L^p(\Omega;L^p([0,t]\times B;L^q(S)))}, \|F_2\|_{L^p(\Omega;L^q([0,t]\times B;L^q(S)))}\Big\}\Big\}.$$

where the infimum is taken over all decompositions $F = F_1 + F_2$ *with* $F_1 \in L^p(\Omega;L^q(S;L^2([0,t]\times B)))$ *and* $F_2 \in L^p(\Omega;L^p([0,t]\times B;L^q(S))) \cap L^p(\Omega;L^q([0,t]\times B;L^q(S)))$.

Proof. We first show that the p-th moment of the stochastic integral is dominated by the right hand side. Let F be the simple adapted process defined in (2.5), taking Remark 2.7 into account. We may assume $t = t_{l+1}$ and $B = J$. Let $F = F_1 + F_2$ with F_1, F_2 in $L^\infty(\Omega)\otimes L^\infty([0,t])\otimes L^\infty(J)\otimes L^\infty(S)$. Let $\mathcal{A}$ be the sub-σ-algebra of $\mathcal{B}(\mathbb{R}_+)\times\mathcal{J}$ generated by the sets $(t_i,t_{i+1}]\times A_j$. By Lemma 2.10 $\mathbb{E}(F_1|\mathcal{A}), \mathbb{E}(F_2|\mathcal{A})$ are of the form

$$\mathbb{E}(F_\alpha|\mathcal{A}) = \sum_{i,j,k} F_{i,k,\alpha}\chi_{(t_i,t_{i+1}]}\chi_{A_j} x_{i,j,k,\alpha} \qquad (\alpha = 1,2).$$

For $\alpha = 1,2$ set $y_{i,j,\alpha} = \sum_k F_{i,k,\alpha}x_{i,j,k,\alpha}$. By Theorem 2.3, Corollary 2.14 and Lemma 2.23 we have

$$\begin{aligned}
&\Big\|\int_{[0,t]\times J} F\, d\tilde{N}\Big\|_{L^p(\Omega;L^q(S))} \\
&\quad\simeq_{p,q} \Big(\mathbb{E}\tilde{\mathbb{E}}\Big\|\sum_{i,j}\tilde{N}_{i,j}y_{i,j}\Big\|_X^p\Big)^{\frac{1}{p}} \\
&\quad\le \Big(\mathbb{E}\tilde{\mathbb{E}}\Big\|\sum_{i,j}\tilde{N}_{i,j}y_{i,j,1}\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}} + \Big(\mathbb{E}\tilde{\mathbb{E}}\Big\|\sum_{i,j}\tilde{N}_{i,j}y_{i,j,2}\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}} \\
&\quad\lesssim_{p,q} \|\mathbb{E}(F|\mathcal{A})\|_{L^p(\Omega;L^q(S;L^2([0,t]\times J)))} \\
&\qquad + \max\Big\{\|\mathbb{E}(F|\mathcal{A})\|_{L^p(\Omega;L^p([0,t]\times J;L^q(S)))}, \|\tilde{F}_2\|_{L^p(\Omega;L^q([0,t]\times J;L^q(S)))}\Big\} \\
&\quad\le \|F_1\|_{L^p(\Omega;L^q(S;L^2([0,t]\times J)))} \\
&\qquad + \max\Big\{\|F_2\|_{L^p(\Omega;L^p([0,t]\times J;L^q(S)))}, \|F_2\|_{L^p(\Omega;L^q([0,t]\times J;L^q(S)))}\Big\}.
\end{aligned}$$

The asserted inequality now follows by taking the infimum over all decompositions $F = F_1 + F_2$ as above.

The reverse inequality can be deduced by duality from Theorem 2.26. As the argument is very similar to the proof of Theorem 2.25 we leave the details to the reader. □

We now formulate the results for the remaining two cases.

Theorem 2.28. *Let* $1 < q < 2 \le p < \infty$. *Then for any simple, adapted* $L^q(S)$*-valued process* F, *any* $B \in \mathcal{J}$ *and any* $t \ge 0$ *we have*

$$\Big\|\int_{[0,t]\times B} F\, d\tilde{N}\Big\|_{L^p(\Omega;L^q(S))}$$

$$\simeq_{p,q} \max\Big\{ \inf\Big\{\|F_1\|_{L^p(\Omega;L^q(S;L^2([0,t]\times B)))} + \|F_2\|_{L^p(\Omega;L^q([0,t]\times B;L^q(S)))}\Big\}, \|F\|_{L^p(\Omega;L^p([0,t]\times B;L^q(S)))}\Big\},$$

where the infimum is taken over all decompositions $F = F_1 + F_2$ with $F_1 \in L^p(\Omega; L^q(S; L^2([0,t] \times B)))$ and $F_2 \in L^p(\Omega; L^q([0,t] \times B; L^q(S)))$.

Proof. We first show that the L^p-norm of the stochastic integral is dominated by the maximum on the right hand side. Let F be the simple adapted process defined in (2.5), taking Remark 2.7 into account. We may assume $t = t_{l+1}$ and $B = J$. Let $y_{i,j} = \sum_k F_{i,k}x_{i,j,k}$. By Theorem 2.3 and Theorem 1.31 we have

$$\begin{aligned}
&\Big\| \int_{[0,t]\times J} F\, d\tilde{N}\Big\|_{L^p(\Omega;L^q(S))} \\
&\quad\simeq_{p,q} \Big(\mathbb{E}\tilde{\mathbb{E}}\Big\|\sum_{i,j}\tilde{N}_{i,j}y_{i,j}\Big\|^p_{L^q(S)}\Big)^{\frac{1}{p}} \\
&\quad\lesssim_p \max\Big\{\Big(\mathbb{E}\Big(\tilde{\mathbb{E}}\Big\|\sum_{i,j}\tilde{N}_{i,j}y_{i,j}\Big\|^q_{L^q(S)}\Big)^{\frac{p}{q}}\Big)^{\frac{1}{p}}, \Big(\mathbb{E}\tilde{\mathbb{E}}\max_{i,j}\|\tilde{N}_{i,j}y_{i,j}\|^p_{L^q(S)}\Big)^{\frac{1}{p}}\Big\} \\
&\quad\leq \max\Big\{\Big(\mathbb{E}\Big(\tilde{\mathbb{E}}\Big\|\sum_{i,j}\tilde{N}_{i,j}y_{i,j}\Big\|^q_{L^q(S)}\Big)^{\frac{p}{q}}\Big)^{\frac{1}{p}}, \Big(\mathbb{E}\sum_{i,j}\tilde{\mathbb{E}}\|\tilde{N}_{i,j}y_{i,j}\|^p_{L^q(S)}\Big)^{\frac{1}{p}}\Big\}.
\end{aligned}$$

By Lemma 2.2 we have

$$\Big(\mathbb{E}\sum_{i,j}\tilde{\mathbb{E}}\|\tilde{N}_{i,j}y_{i,j}\|^p_{L^q(S)}\Big)^{\frac{1}{p}} \simeq_p \|F\|_{L^p(\Omega;L^p([0,t]\times J;L^q(S)))}.$$

Let $F = F_1 + F_2$ with F_1, F_2 in $L^\infty(\Omega) \otimes L^\infty([0,t]) \otimes L^\infty(J) \otimes L^\infty(S)$. Let $\mathcal{A}$ be the sub-σ-algebra of $\mathcal{B}(\mathbb{R}_+) \times \mathcal{J}$ generated by the sets $(t_i, t_{i+1}] \times A_j$. By Lemma 2.10 $\mathbb{E}(F_1|\mathcal{A}), \mathbb{E}(F_2|\mathcal{A})$ are of the form

$$\mathbb{E}(F_\alpha|\mathcal{A}) = \sum_{i,j,k} F_{i,k,\alpha}\chi_{(t_i,t_{i+1}]}\chi_{A_j}x_{i,j,k,\alpha} \qquad (\alpha = 1, 2).$$

For $\alpha = 1, 2$ set $y_{i,j,\alpha} = \sum_k F_{i,k,\alpha}x_{i,j,k,\alpha}$. By the triangle inequality and Theorem 1.29 we obtain

$$\begin{aligned}
&\Big(\mathbb{E}\Big(\tilde{\mathbb{E}}\Big\|\sum_{i,j}\tilde{N}_{i,j}y_{i,j}\Big\|^q_{L^q(S)}\Big)^{\frac{p}{q}}\Big)^{\frac{1}{p}} \\
&\quad\leq \Big(\mathbb{E}\Big(\tilde{\mathbb{E}}\Big\|\sum_{i,j}\tilde{N}_{i,j}y_{i,j,1}\Big\|^q_{L^q(S)}\Big)^{\frac{p}{q}}\Big)^{\frac{1}{p}} + \Big(\mathbb{E}\Big(\tilde{\mathbb{E}}\Big\|\sum_{i,j}\tilde{N}_{i,j}y_{i,j,2}\Big\|^q_{L^q(S)}\Big)^{\frac{p}{q}}\Big)^{\frac{1}{p}} \\
&\quad\lesssim_{p,q} \Big(\mathbb{E}\Big\|\Big(\sum_{i,j}\tilde{\mathbb{E}}|\tilde{N}_{i,j}y_{i,j,1}|^2\Big)^{\frac{1}{2}}\Big\|^p_{L^q(S)}\Big)^{\frac{1}{p}} + \Big(\mathbb{E}\Big(\sum_{i,j}\tilde{\mathbb{E}}\|\tilde{N}_{i,j}y_{i,j,2}\|^q_{L^q(S)}\Big)^{\frac{p}{q}}\Big)^{\frac{1}{p}}
\end{aligned}$$

$$\simeq_q \Big(\mathbb{E}\Big\|\Big(\sum_{i,j}(t_{i+1}-t_i)\nu(A_j)|y_{i,j,1}|^2\Big)^{\frac{1}{2}}\Big\|^p_{L^q(S)}\Big)^{\frac{1}{p}}$$
$$+\Big(\mathbb{E}\Big(\sum_{i,j}(t_{i+1}-t_i)\nu(A_j)\|y_{i,j,2}\|^q_{L^q(S)}\Big)^{\frac{p}{q}}\Big)^{\frac{1}{p}}$$
$$=\|\mathbb{E}(F_1|\mathcal{A})\|_{L^p(\Omega;L^q(S;L^2([0,t]\times J)))}+\|\mathbb{E}(F_2|\mathcal{A})\|_{L^p(\Omega;L^q([0,t]\times J;L^q(S)))}$$
$$\leq \|F_1\|_{L^p(\Omega;L^q(S;L^2([0,t]\times J)))}+\|F_2\|_{L^p(\Omega;L^q([0,t]\times J;L^q(S)))}$$

Taking the infimum over all F_1, F_2 as above yields the first inequality.

For the reverse inequality, note that

$$\Big\|\int_{[0,t]\times J} F\,d\tilde{N}\Big\|_{L^p(\Omega;L^q(S))}$$
$$\simeq_{p,q} \Big(\mathbb{E}\tilde{\mathbb{E}}\Big\|\sum_{i,j}\tilde{N}_{i,j}y_{i,j}\Big\|^p_{L^q(S)}\Big)^{\frac{1}{p}}$$
$$\geq \Big(\mathbb{E}\Big(\tilde{\mathbb{E}}\Big\|\sum_{i,j}\tilde{N}_{i,j}y_{i,j}\Big\|^q_{L^q(S)}\Big)^{\frac{p}{q}}\Big)^{\frac{1}{p}}$$
$$\gtrsim_{p,q} \|F\|_{L^p(\Omega;L^q(S;L^2([0,t]\times J)))+L^p(\Omega;L^q([0,t]\times J;L^q(S)))},$$

where the final inequality follows as in the proof of Theorem 2.25. Moreover, by Corollary 2.17,

$$\Big\|\int_{[0,t]\times J} F\,d\tilde{N}\Big\|_{L^p(\Omega;L^q(S))}\gtrsim_{p,q}\|F\|_{L^p(\Omega;L^p([0,t]\times J;L^q(S)))}.$$

□

Theorem 2.29. *Let $1<p<2\leq q<\infty$. Then for any simple, adapted $L^q(S)$-valued process F, any $B\in\mathcal{J}$ and any $t\geq 0$ we have*

$$\Big\|\int_{[0,t]\times B} F\,d\tilde{N}\Big\|_{L^p(\Omega;L^q(S))}$$
$$\simeq_{p,q}\inf\Big\{\max\Big\{\|F_1\|_{L^p(\Omega;L^q(S;L^2([0,t]\times B)))},\|F_1\|_{L^p(\Omega;L^q([0,t]\times B;L^q(S)))}\Big\}$$
$$+\|F_2\|_{L^p(\Omega;L^p([0,t]\times B;L^q(S)))}\Big\},$$

where the infimum is taken over all decompositions $F=F_1+F_2$ with $F_1\in L^p(\Omega;L^q(S;L^2([0,t]\times B)))\cap L^p(\Omega;L^q([0,t]\times B;L^q(S)))$ and $F_2\in L^p(\Omega;L^p([0,t]\times B;L^q(S)))$.

Proof. Let F be the simple adapted process defined in (2.5), taking Remark 2.7 into account. We may assume $t=t_{l+1}$ and $B=J$. Let $y_{i,j}=\sum_k F_{i,k}x_{i,j,k}$. We first show that the L^p-norm of the stochastic integral is dominated by the infimum on the right hand side. Let $F=F_1+F_2$ with F_1,F_2 in

$L^\infty(\Omega)\otimes L^\infty([0,t])\otimes L^\infty(J)\otimes L^\infty(S)$. Let $\mathcal{A}$ be the sub-σ-algebra of $\mathcal{B}(\mathbb{R}_+)\times\mathcal{J}$ generated by the sets $(t_i,t_{i+1}]\times A_j$. By Lemma 2.10 $\mathbb{E}(F_1|\mathcal{A}),\mathbb{E}(F_2|\mathcal{A})$ are of the form

$$\mathbb{E}(F_\alpha|\mathcal{A}) = \sum_{i,j,k} F_{i,k,\alpha}\chi_{(t_i,t_{i+1}]}\chi_{A_j}x_{i,j,k,\alpha} \qquad (\alpha=1,2).$$

For $\alpha=1,2$ set $y_{i,j,\alpha}=\sum_k F_{i,k,\alpha}x_{i,j,k,\alpha}$. By Theorem 2.3 we have

$$\begin{aligned}
&\Big\|\int_{[0,t]\times J} F\,d\tilde{N}\Big\|_{L^p(\Omega;L^q(S))}\\
&\quad\simeq_{p,q}\Big(\mathbb{E}\tilde{\mathbb{E}}\Big\|\sum_{i,j}\tilde{N}_{i,j}y_{i,j}\Big\|^p_{L^q(S)}\Big)^{\frac1p}\\
&\quad\le\Big(\mathbb{E}\tilde{\mathbb{E}}\Big\|\sum_{i,j}\tilde{N}_{i,j}y_{i,j,1}\Big\|^p_{L^q(S)}\Big)^{\frac1p}+\Big(\mathbb{E}\tilde{\mathbb{E}}\Big\|\sum_{i,j}\tilde{N}_{i,j}y_{i,j,2}\Big\|^p_{L^q(S)}\Big)^{\frac1p}\\
&\quad\le\Big(\mathbb{E}\Big(\tilde{\mathbb{E}}\Big\|\sum_{i,j}\tilde{N}_{i,j}y_{i,j,1}\Big\|^q_{L^q(S)}\Big)^{\frac pq}\Big)^{\frac1p}+\Big(\mathbb{E}\tilde{\mathbb{E}}\Big\|\sum_{i,j}\tilde{N}_{i,j}y_{i,j,2}\Big\|^p_{L^q(S)}\Big)^{\frac1p}.
\end{aligned}$$

By Theorem 1.28 and Lemma 2.2,

$$\begin{aligned}
&\Big(\mathbb{E}\Big(\tilde{\mathbb{E}}\Big\|\sum_{i,j}\tilde{N}_{i,j}y_{i,j,1}\Big\|^q_{L^q(S)}\Big)^{\frac pq}\Big)^{\frac1p}\\
&\quad\lesssim_{p,q}\max\Big\{\Big(\mathbb{E}\Big\|\Big(\sum_{i,j}\tilde{\mathbb{E}}|\tilde{N}_{i,j}y_{i,j,1}|^2\Big)^{\frac12}\Big\|^p_{L^q(S)}\Big)^{\frac1p},\\
&\qquad\qquad\Big(\mathbb{E}\Big(\sum_{i,j}\tilde{\mathbb{E}}\|\tilde{N}_{i,j}y_{i,j,1}\|^q_{L^q(S)}\Big)^{\frac pq}\Big)^{\frac1p}\Big\}\\
&\quad\simeq_q\max\Big\{\Big(\mathbb{E}\Big\|\Big(\sum_{i,j}(t_{i+1}-t_i)\nu(A_j)|y_{i,j,1}|^2\Big)^{\frac12}\Big\|^p_{L^q(S)}\Big)^{\frac1p},\\
&\qquad\qquad\Big(\mathbb{E}\Big(\sum_{i,j}(t_{i+1}-t_i)\nu(A_j)\|y_{i,j,1}\|^q_{L^q(S)}\Big)^{\frac pq}\Big)^{\frac1p}\Big\}\\
&\quad=\max\Big\{\|\mathbb{E}(F_1|\mathcal{A})\|_{L^p(\Omega;L^q(S;L^2([0,t]\times J)))},\\
&\qquad\qquad\|\mathbb{E}(F_1|\mathcal{A})\|_{L^p(\Omega;L^q([0,t]\times J;L^q(S)))}\Big\}\\
&\quad\le\max\Big\{\|F_1\|_{L^p(\Omega;L^q(S;L^2([0,t]\times J)))},\|F_1\|_{L^p(\Omega;L^q([0,t]\times J;L^q(S)))}\Big\}
\end{aligned}$$

On the other hand, by Corollary 1.17 and Lemma 2.2,

$$\Big(\mathbb{E}\tilde{\mathbb{E}}\Big\|\sum_{i,j}\tilde{N}_{i,j}y_{i,j,2}\Big\|^p_{L^q(S)}\Big)^{\frac1p}\lesssim_{p,q}\Big(\mathbb{E}\sum_{i,j}\tilde{\mathbb{E}}\|\tilde{N}_{i,j}y_{i,j,2}\|^p_{L^q(S)}\Big)^{\frac1p}$$

$$\begin{aligned}
&\simeq_p \Big(\mathbb{E}\sum_{i,j}(t_{i+1}-t_i)\nu(A_j)\|y_{i,j,2}\|^p_{L^q(S)}\Big)^{\frac{1}{p}}\\
&= \|\mathbb{E}(F_2|\mathcal{A})\|_{L^p(\Omega;L^p([0,t]\times J;L^q(S)))}\\
&\le \|F_2\|_{L^p(\Omega;L^p([0,t]\times J;L^q(S)))}.
\end{aligned}$$

Taking the infimum over all F_1, F_2 as above we obtain the result.

The reverse inequality follows by duality from Theorem 2.28. □

Remark 2.30. Using the one-sided decoupling inequality stated in Remark 2.4 one can show that the upper estimate given in Theorems 2.25, 2.27, 2.28 and 2.29 for the p-th moment $\|\int_{[0,t]\times B} F\, d\tilde{N}\|_{L^p(\Omega;L^q(S))}$ remains valid if either $p = 1$ or $q = 1$ (or both).

We now summarize the main results of this section.

Theorem 2.31. *Let $1 < p, q, r < \infty$. We set*

$$\begin{aligned}
S^p_q &= L^p(\Omega; L^q(S; L^2(\mathbb{R}_+ \times J)));\\
D^p_{r,q} &= L^p(\Omega; L^r(\mathbb{R}_+ \times J; L^q(S))).
\end{aligned}$$

Then for any $B \in \mathcal{J}$, any $t \ge 0$ and for any simple, adapted $L^q(S)$-valued process F,

$$\Big(\mathbb{E}\sup_{0\le s\le t}\Big\|\int_{[0,s]\times B} F\, d\tilde{N}\Big\|^p_{L^q(S)}\Big)^{\frac{1}{p}} \simeq_{p,q} \|F\chi_{[0,t]\times B}\|_{SI_{p,q}}, \tag{2.16}$$

where $SI_{p,q}$ is given by

$$\begin{aligned}
&S^p_q \cap D^p_{q,q} \cap D^p_{p,q} \text{ if } 2 \le q \le p < \infty;\\
&S^p_q \cap (D^p_{q,q} + D^p_{p,q}) \text{ if } 2 \le p \le q < \infty;\\
&(S^p_q \cap D^p_{q,q}) + D^p_{p,q} \text{ if } 1 < p < 2 \le q < \infty;\\
&(S^p_q + D^p_{q,q}) \cap D^p_{p,q} \text{ if } 1 < q < 2 \le p < \infty;\\
&S^p_q + (D^p_{q,q} \cap D^p_{p,q}) \text{ if } 1 < q \le p \le 2;\\
&S^p_q + D^p_{q,q} + D^p_{p,q} \text{ if } 1 < p \le q \le 2.
\end{aligned}$$

Proof. Observe that the map

$$s \mapsto \Big\|\int_{[0,s]\times B} F\, d\tilde{N}\Big\|_{L^q(S)}$$

defines a positive submartingale in $L^p(\Omega)$ and hence by Doob's inequality (see e.g. [119], Theorem 1.7) we have for any $p > 1$,

$$\Big(\mathbb{E}\sup_{0\le s\le t}\Big\|\int_{[0,s]\times B} F\, d\tilde{N}\Big\|^p_{L^q(S)}\Big)^{\frac{1}{p}} \le p'\Big(\mathbb{E}\Big\|\int_{[0,t]\times B} F\, d\tilde{N}\Big\|^p_{L^q(S)}\Big)^{\frac{1}{p}},$$

where $\frac{1}{p}+\frac{1}{p'} = 1$. The result now follows from Theorems 2.22, 2.25, 2.26, 2.27, 2.28 and 2.29. □

We can now extend the class of stochastically integrable processes through the Itô-type isomorphism stated in (2.16).

Definition 2.32. *Let $1 < p, q < \infty$ and let S be a σ-finite measure space. Let $(J, \mathcal{J}, \nu)$ be a σ-finite measure space and let N be a Poisson random measure on $\mathbb{R}_+ \times J$ with associated compensated Poisson random measure $\tilde{N}$. Let $t \geq 0$ and $B \in \mathcal{J}$. We say that an element $F \in SI_{p,q}$ is* L^p-stochastically integrable *on $[0,t] \times B$ if there exists a sequence of simple, adapted $L^q(S)$-valued processes (F_n) such that $F_n \to F\chi_{[0,t]\times B}$ in $SI_{p,q}$ as $n \to \infty$. In this case we define the* L^p-stochastic integral *of F on $[0,t] \times B$ with respect to $\tilde{N}$ by*

$$\int_{[0,t]\times B} F \, d\tilde{N} = \lim_{n\to\infty} \int_{[0,t]\times B} F_n \, d\tilde{N},$$

where the limit is taken in $L^p(\Omega; L^q(S))$. We let $SI_{p,q}^{ad}$ denote the space of L^p-stochastically integrable elements on $\mathbb{R}_+ \times J$.

Corollary 2.33. *If $1 < p, q < \infty$, then $SI_{p,q}^{ad}$ is a Banach space. Moreover, for any $F \in SI_{p,q}^{ad}$ the inequalities (2.16) hold.*

Remark 2.34. The results of this chapter are still valid if we replace the compensated Poisson random measure $\tilde{N}$ associated with N by the symmetrized Poisson random measure N_s defined in (2.1). Indeed, the following two properties of N_s, which it shares with $\tilde{N}$, are needed in the proofs.

(i) If $A_1, \ldots, A_n$ in $\mathcal{E}_\mu$ are disjoint then $N_s(A_1), \ldots, N_s(A_n)$ are independent, mean zero random variables;
(ii) For every $1 \leq p < \infty$ there exist constants $b_p, c_p > 0$ depending only on p such that whenever $A \in \mathcal{E}_\mu$ we have

$$b_p \mu(A) \leq \mathbb{E}|N_s(A)|^p \leq c_p \mu(A).$$

The proof of the second property is similar to the proof of Lemma 2.2.

Remark 2.35. Let us recall the following well-known result for vector-valued Gaussian stochastic integrals (see [106] for a more general statement). Suppose that $1 < p, q < \infty$ and let S be a σ-finite measure space. If W is a standard Brownian motion and F is a simple adapted $L^q(S)$-valued process, then for any $t \geq 0$,

$$\Big(\mathbb{E}\Big\| \int_{[0,t]} F \, dW \Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}} \simeq_{p,q} \Big(\mathbb{E}\Big\| \Big(\int_{[0,t]} |F|^2 dt\Big)^{\frac{1}{2}} \Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}}. \tag{2.17}$$

If L is a Lévy process without drift, then, under suitable assumptions on the Lévy measure of L, we can combine the Lévy-Itô decomposition (2.2) of L with the Itô isomorphisms induced by (2.16) and (2.17) to obtain Itô isomorphisms for $L^q(S)$-valued stochastic integrals with respect to L. The details of this procedure will be explained in future work.

3

Vector-valued Rosenthal inequalities in noncommutative L^p-spaces

We continue our investigation of Rosenthal-type inequalities for vector valued random variables. The main purpose of this chapter is to generalize our results of Section 1.3 to random variables taking values in a noncommutative L^p-space. Although the spirit of the proof will be the same as in the case of classical L^p-spaces, different arguments and additional machinery are required in the noncommutative setting. As a result, the proof will be different from the one presented in Section 1.3 even for a commutative von Neumann algebra.

The results presented in Section 3.2 can be applied to derive some estimates for the p-th moments of the operator norm, or equivalently, the largest singular value, of a sum of independent, mean zero random matrices in terms of the individual matrices. In particular, for a random matrix with independent, mean zero entries we find two-sided estimates for the p-th moments of the largest singular value of the matrix in terms of a suitable norm on its entries.

3.1 Noncommutative L^p-spaces

Throughout this thesis, we will use standard terminology and results from the theory of von Neumann algebras, which can for example be found in [131, 132] or [72, 73]. Let $\mathcal{M}$ be a von Neumann algebra acting on a complex Hilbert space H, which is equipped with a normal, semi-finite faithful trace τ. We say that a closed, densely defined linear operator x on H is *affiliated* with the von Neumann algebra $\mathcal{M}$ if $ux = xu$ for any unitary element u in the commutant $\mathcal{M}'$ of $\mathcal{M}$. For such an operator we define its *distribution function* by

$$d(v;x) = \tau(e^{|x|}(v,\infty)) \qquad (v \geq 0),$$

where $e^{|x|}$ is the spectral measure of $|x|$. The *decreasing rearrangement* or *generalized singular value function* of x is defined by

$$\mu_t(x) = \inf\{v > 0 \ : \ d(v;x) \leq t\} \qquad (t \geq 0).$$

We call x *τ-measurable* if $d(v;x) < \infty$ for some $v > 0$. We let $S(\tau)$ denote the linear space of all τ-measurable operators. One can show that $S(\tau)$ is a metrizable, complete topological $*$-algebra with respect to the measure topology. Moreover, the trace τ extends to a trace (again denoted by τ) on the set $S(\tau)_+$ of positive τ-measurable operators by setting

$$\tau(x) = \int_0^\infty \mu_t(x)\ dt \qquad (x \in S(\tau)_+). \tag{3.1}$$

For $0 < q < \infty$ we define

$$\|x\|_{L^q(\mathcal{M})} = (\tau(|x|^q))^{\frac{1}{q}} \qquad (x \in S(\tau)). \tag{3.2}$$

The linear space $L^q(\mathcal{M},\tau)$ of all $x \in S(\tau)$ satisfying $\|x\|_{L^q(\mathcal{M})} < \infty$ is called the *noncommutative L^q-space* associated with the pair $(\mathcal{M},\tau)$. We usually denote $L^q(\mathcal{M},\tau)$ by $L^q(\mathcal{M})$ for brevity. The map $\|\cdot\|_{L^q(\mathcal{M})}$ in (3.2) defines a norm (or q-norm if $0 < q < 1$) on the space $L^q(\mathcal{M})$ under which it becomes a Banach space (respectively, quasi-Banach space). It can alternatively be viewed as the completion of $\mathcal{M}$ in the (quasi-)norm $\|\cdot\|_{L^q(\mathcal{M})}$. We use the expression $L^\infty(\mathcal{M})$ to denote $\mathcal{M}$ equipped with its operator norm. By (3.1) and using that $\mu(|x|^q) = \mu(x)^q$, the noncommutative L^q-(quasi-)norm can alternatively be computed as

$$\|x\|_{L^q(\mathcal{M})} = \Big(\int_0^\infty \mu_t(x)^q\ dt\Big)^{\frac{1}{q}} \qquad (x \in L^q(\mathcal{M})). \tag{3.3}$$

We recall two familiar examples which are relevant to this chapter.

Example 3.1. (*Lebesgue spaces*) Let (S,Σ,μ) be a σ-finite measure space. Identify $f \in L^\infty(S)$ with the multiplication operator M_f on the Hilbert space $L^2(S)$ given by

$$M_f(h) := fh \qquad (h \in L^2(S)).$$

One can show that

$$\mathcal{M} := \{M_f \ :\ f \in L^\infty(S)\}$$

is a von Neumann subalgebra of $B(L^2(S))$. We will identify $L^\infty(S)$ with $\mathcal{M}$. The functional

$$\tau(f) := \int_S f\ d\mu \qquad (f \in L^\infty(S,\Sigma,\mu)_+)$$

defines a normal, semi-finite faithful trace and the associated noncommutative L^q-space coincides with the Lebesgue space $L^q(S)$, where the functions in $L^q(S)$ are identified with, in general unbounded, multiplication operators on $L^2(S)$.

Example 3.2. (*Schatten spaces*) Let H be a complex Hilbert space and let (e_α) be a maximal orthonormal system in H. The space $B(H)$ of bounded linear operators on H is a von Neumann algebra, which can be equipped with the normal, semi-finite faithful trace

$$\mathrm{Tr}(x) = \sum_{\alpha} \langle x e_\alpha, e_\alpha \rangle \qquad (x \in B(H)_+).$$

This is called the *standard trace* on $B(H)$. The associated noncommutative L^q-spaces, denoted by $S^q(H)$, are called the *Schatten spaces*.

We shall be interested in the space M_n of $n \times n$ matrices with complex coefficients, which can be identified with the von Neumann algebra $B(l_n^2)$. Under this identification Tr coincides with the usual trace on matrices, i.e. if $x = (x_{ij})_{i,j=1}^n$ then

$$\mathrm{Tr}(x) = \sum_{i=1}^n x_{ii}.$$

The associated noncommutative L^q-space is called the *n-th Schatten space* and denoted by S_n^q. Let x be an $n \times n$ matrix and let $\mu_1 \geq \ldots \geq \mu_n$ be its singular values, repeated according to multiplicity. Then its singular value function $\mu(x)$ is given by

$$\mu_t(x) = \sum_{i=1}^n \mu_i \chi_{[i-1,i)}(t) \qquad (t \geq 0).$$

According to (3.3) we have

$$\|x\|_{S_n^q} = \Big(\sum_{i=1}^n \mu_i^q \Big)^{\frac{1}{q}},$$

and moreover,

$$\|x\| = \max_{i=1,\ldots,n} \mu_i.$$

Let us note that if $n \geq 2$ and $r \geq \log n$ then for any sequence $(x_i)_{i=1}^n$ of complex numbers we have

$$\max_{i=1,\ldots,n} |x_i| \leq \Big(\sum_{i=1}^n |x_i|^r \Big)^{\frac{1}{r}} \leq e \max_{i=1,\ldots,n} |x_i|. \tag{3.4}$$

In particular, if $x \in S_n^q$ we obtain by taking $r = \log n$ and $x_i = \mu_i$,

$$\|x\|_{S_n^q} \leq e\|x\| \leq e\|x\|_{S_n^q} \qquad (\text{if } q = \log n). \tag{3.5}$$

This fact will be used frequently in Section 3.3 below.

Let us now state some properties that noncommutative L^q-spaces share with their classical counterparts. These facts can, for example, be found in [115]. First recall that Hölder's inequality holds: if $0 < q, r, s \leq \infty$ are such that $\frac{1}{q} = \frac{1}{r} + \frac{1}{s}$ and $x \in L^r(\mathcal{M})$, $y \in L^s(\mathcal{M})$, then $xy \in L^q(\mathcal{M})$ and

$$\|xy\|_{L^q(\mathcal{M})} \leq \|x\|_{L^r(\mathcal{M})}\|y\|_{L^s(\mathcal{M})}. \tag{3.6}$$

For $1 \leq q < \infty$ and $\frac{1}{q} + \frac{1}{q'} = 1$, the familiar duality $L^q(\mathcal{M})^* = L^{q'}(\mathcal{M})$ holds isometrically, with the duality bracket given by $\langle x, y\rangle = \tau(xy)$. In particular, $L^q(\mathcal{M})$ is reflexive if and only if $1 < q < \infty$ and $L^1(\mathcal{M}) = \mathcal{M}_*$ isometrically, where $\mathcal{M}_*$ is the predual of $\mathcal{M}$.

The noncommutative L^q-spaces form an interpolation scale with respect to both the real and complex interpolation method. Moreover, we recall that $L^q(\mathcal{M})$ is K-convex if and only if $1 < q < \infty$ and it is a UMD Banach space if and only if $1 < q < \infty$. Finally, it is known that noncommutative L^q-spaces have the same type and cotype as their commutative versions, but the proof is more involved than in the commutative case (see [52] or [115], Corollary 5.5). We state this result as a theorem for future reference.

Theorem 3.3. *If $\mathcal{M}$ is a semi-finite von Neumann algebra and $1 \leq q < \infty$, then $L^q(\mathcal{M})$ has type $\min\{q, 2\}$ and cotype $\max\{q, 2\}$.*

In other respects noncommutative L^q-spaces are radically different from their commutative counterparts. In particular, the isomorphism theory of these spaces is much more involved. We refer to the survey [115] and the references therein for a discussion of the differences and similarities between noncommutative and classical L^q-spaces.

We conclude this section by describing the column and row spaces and their conditional versions. Let $1 \leq q < \infty$. For a finite sequence (x_i) in $L^q(\mathcal{M})$ we define

$$\begin{aligned} \|(x_i)\|_{L^q(\mathcal{M};l^2_c)} &= \Big\|\Big(\sum_i x_i^* x_i\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}; \\ \|(x_i)\|_{L^q(\mathcal{M};l^2_r)} &= \Big\|\Big(\sum_i x_i x_i^*\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}. \end{aligned} \tag{3.7}$$

Given $x_1, \ldots, x_n$, we let $\mathrm{diag}(x_i)$, $\mathrm{row}(x_i)$ and $\mathrm{col}(x_i)$ denote the $n \times n$ matrix with the x_i on its diagonal, first row and first column, respectively, and zeroes elsewhere. Let $M_n(\mathcal{M})$ be the von Neumann algebra of $n \times n$ matrices with values in $\mathcal{M}$, equipped with the trace $\mathrm{Tr}\otimes\tau$, which it inherits when we identify $M_n(\mathcal{M})$ with the von Neumann tensor product $B(l^2_n)\overline{\otimes}\mathcal{M}$. By noting that

$$\begin{aligned} \Big\|\Big(\sum_{i=1}^n x_i^* x_i\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})} &= \|\mathrm{col}(x_i)\|_{L^q(M_n(\mathcal{M}))}; \\ \Big\|\Big(\sum_{i=1}^n x_i x_i^*\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})} &= \|\mathrm{row}(x_i)\|_{L^q(M_n(\mathcal{M}))}, \end{aligned}$$

one sees that the expressions in (3.7) define two norms on the linear space of all finitely nonzero sequences in $L^q(\mathcal{M})$. The completion of this space in these norms are called the *column* and *row space*, respectively.

We shall need a conditional version of these two spaces. Let us first state a well-known result on the existence of noncommutative conditional expectations (see e.g. [136]).

Proposition 3.4. *Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal, semi-finite, faithful trace τ and let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$ such that the restriction of τ to $\mathcal{N}$ is again semi-finite. Then there is a unique linear map $\mathcal{E} : L^1(\mathcal{M}) + L^\infty(\mathcal{M}) \to L^1(\mathcal{N}) + L^\infty(\mathcal{N})$ satisfying the following properties:*

(a) $\mathcal{E}(x^*) = \mathcal{E}(x)^*$;
(b) $\mathcal{E}(x) \geq 0$ if $x \geq 0$;
(c) if $x \geq 0$ and $\mathcal{E}(x) = 0$ then $x = 0$;
(d) $\mathcal{E}(x) = x$ for any $x \in L^1(\mathcal{N}) + L^\infty(\mathcal{N})$;
(e) $\mathcal{E}(x)^*\mathcal{E}(x) \leq \mathcal{E}(x^*x)$ for $x \in \mathcal{M}$;
(f) $\mathcal{E}$ is normal, i.e., $x_\alpha \uparrow x$ implies $\mathcal{E}(x_\alpha) \uparrow \mathcal{E}(x)$ for $(x_\alpha), x \in \mathcal{M}$;
(g) $\|\mathcal{E}(x)\|_1 \leq \|x\|_1$, for all $x \in L^1(\mathcal{M})$ and $\|\mathcal{E}(x)\|_\infty \leq \|x\|_\infty$, for all $x \in \mathcal{M}$;
(h) $\mathcal{E}(xy) = x\mathcal{E}(y)$ if $x \in L^1(\mathcal{N}), y \in L^\infty(\mathcal{M})$ and $\mathcal{E}(xy) = \mathcal{E}(x)y$ whenever $x \in L^1(\mathcal{M}), y \in L^\infty(\mathcal{N})$.

Moreover, for any $x \in L^1(\mathcal{M}) + L^\infty(\mathcal{M})$, $\mathcal{E}(x)$ is the unique element in $L^1(\mathcal{N}) + L^\infty(\mathcal{N})$ satisfying

$$\tau(xy) = \tau(\mathcal{E}(x)y), \tag{3.8}$$

for all $y \in L^1(\mathcal{N}) \cap L^\infty(\mathcal{N})$.

The inequality in (e) of Proposition 3.4 is called *Kadison's inequality*.

Let $1 \leq q < \infty$. Suppose that $\mathcal{N}$ is a von Neumann algebra equipped with a normal, semi-finite faithful trace σ and let $\mathcal{K}$ be a von Neumann subalgebra such that $\sigma|_{\mathcal{K}}$ is again semi-finite. Let $\mathcal{E} : \mathcal{N} \to \mathcal{K}$ be the conditional expectation with respect to $\mathcal{K}$. For a finite sequence (x_i) in $\mathcal{N}$ we define

$$\|(x_i)\|_{L^q(\mathcal{N};\mathcal{E},l^2_c)} = \Big\| \Big(\sum_i \mathcal{E}|x_i|^2 \Big)^{\frac{1}{2}} \Big\|_{L^q(\mathcal{N})};$$
$$\|(x_i)\|_{L^q(\mathcal{N};\mathcal{E},l^2_r)} = \Big\| \Big(\sum_i \mathcal{E}|x_i^*|^2 \Big)^{\frac{1}{2}} \Big\|_{L^q(\mathcal{N})}.$$

Using techniques from Hilbert C^*-modules it was shown by M. Junge [68] that

$$\{(x_i)_{i=1}^n \ : \ x_i \in \mathcal{N}, \ n \geq 1, \ \|(x_i)\|_{L^q(\mathcal{N};\mathcal{E},l^2_c)} < \infty\} \text{ and}$$
$$\{(x_i)_{i=1}^n \ : \ x_i \in \mathcal{N}, \ n \geq 1, \ \|(x_i)\|_{L^q(\mathcal{N};\mathcal{E},l^2_r)} < \infty\}$$

are normed linear spaces. By taking the completion of these spaces we obtain the *conditional column* and *row space*, respectively. Moreover, we have for $1 < q, q' < \infty$ with $\frac{1}{q} + \frac{1}{q'} = 1$

$$(L^q(\mathcal{N};\mathcal{E},l_c^2))^* = L^{q'}(\mathcal{N};\mathcal{E},l_r^2)$$

isometrically, with the duality bracket given by

$$\langle (x_i),(y_i)\rangle = \sum_i \tau(x_i y_i).$$

We shall use these results in the particular case where Ω is a probability space, $\mathcal{M}$ is a semi-finite von Neumann algebra and $\mathcal{N}$ is the tensor product von Neumann algebra $L^\infty \overline{\otimes} \mathcal{M}$, equipped with the tensor product trace $\mathbb{E}\otimes\tau$. Let us recall that, for any $1 \leq q < \infty$, the map defined on simple functions in the Bochner space $L^q(\Omega; L^q(\mathcal{M}))$ by

$$I_q\Big(\sum_i \chi_{A_i} x_i\Big) = \sum_i \chi_{A_i} \otimes x_i$$

extends to an isometric isomorphism

$$L^q(\Omega; L^q(\mathcal{M})) = L^q(L^\infty(\Omega)\overline{\otimes}\mathcal{M}). \tag{3.9}$$

Let $\mathcal{K}$ be the von Neumann subalgebra of $\mathcal{N}$ given by $\mathcal{K} = \mathbf{1}\otimes\mathcal{M}$ and let $\mathcal{E}$ be the associated conditional expectation. Under the identification (3.9), the element $\mathcal{E}(x)$ coincides with the Bochner integral $\mathbb{E}(x)$, whenever $x \in L^q(\mathcal{N})$. In particular, for any finite sequence (ξ_i) in $\mathcal{N}$,

$$\|(\xi_i)\|_{L^q(\mathcal{N};\mathcal{E},l_c^2)} = \Big\|\Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}.$$

With some abuse of notation we shall write $L^q(\mathcal{M};\mathbb{E},l_c^2)$ instead of $L^q(\mathcal{N};\mathcal{E},l_c^2)$.

3.2 Main result

We prove our main result, Theorem 3.15 below, in several steps. Throughout, we let $\mathcal{M}$ denote a semi-finite von Neumann algebra. Let us first recall the noncommutative version of Khintchine's inequalities [98].

Theorem 3.5. *(Noncommutative Khintchine inequalities) If $2 \leq q < \infty$, then, for any finite sequence (x_i) in $L^q(\mathcal{M})$,*

$$\Big(\mathbb{E}\Big\|\sum_i r_i x_i\Big\|_{L^q(\mathcal{M})}^q\Big)^{\frac{1}{q}} \leq B_q \max\Big\{\Big\|\Big(\sum_i |x_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}, \Big\|\Big(\sum_i |x_i^*|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}\Big\} \tag{3.10}$$

and

$$\Big(\mathbb{E}\Big\|\sum_i r_i x_i\Big\|^2_{L^q(\mathcal{M})}\Big)^{\frac{1}{2}} \geq \max\Big\{\Big\|\Big(\sum_i |x_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}, \Big\|\Big(\sum_i |x_i^*|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}\Big\},$$

where $B_q \leq C\sqrt{q}$, *with* $C \leq 2^{-\frac{1}{4}}\sqrt{\frac{\pi}{e}} < 1$.

On the other hand, if $1 \leq q \leq 2$, *then*

$$\Big(\mathbb{E}\Big\|\sum_i r_i x_i\Big\|^2_{L^q(\mathcal{M})}\Big)^{\frac{1}{2}} \leq \inf\Big\{\Big\|\Big(\sum_i |y_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})} + \Big\|\Big(\sum_i |z_i^*|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}\Big\}$$

and

$$\Big(\mathbb{E}\Big\|\sum_i r_i x_i\Big\|^q_{L^q(\mathcal{M})}\Big)^{\frac{1}{q}}$$
$$\geq c_q \inf\Big\{\Big\|\Big(\sum_i |y_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})} + \Big\|\Big(\sum_i |z_i^*|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}\Big\},$$

where the infimum is taken over all decompositions $x_i = y_i + z_i$ *in* $L^q(\mathcal{M})$.

Remark 3.6. It was first observed that the optimal constant in (3.10) is of order $\sqrt{q}$ in [113], p.106 and independently in [67]. It was proved by A. Buchholz ([26], Theorem 5 and the remark following it) that if (γ_i) is a sequence of independent standard Gaussian random variables, then

$$\Big(\mathbb{E}\Big\|\sum_i \gamma_i x_i\Big\|^{2n}_{L^{2n}(\mathcal{M})}\Big)^{\frac{1}{2n}}$$
$$\leq \Big(\frac{(2n)!}{2^n n!}\Big)^{\frac{1}{2n}} \max\Big\{\Big\|\Big(\sum_i |x_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^{2n}(\mathcal{M})}, \Big\|\Big(\sum_i |x_i^*|^2\Big)^{\frac{1}{2}}\Big\|_{L^{2n}(\mathcal{M})}\Big\}, \tag{3.11}$$

where the given constant is optimal. As remarked in [135], Proposition 10, one can deduce from (3.11) that the constant B_q in 3.10 is bounded by $2^{-\frac{1}{4}}\sqrt{\frac{\pi}{e}}\sqrt{q}$.

In the proof of the next result we will use that if $0 < q \leq 1$ and $\xi \in L^1(\Omega; L^q(\mathcal{M})_+)$ then

$$\mathbb{E}\|x\|_{L^q(\mathcal{M})} \leq \|\mathbb{E}x\|_{L^q(\mathcal{M})}. \tag{3.12}$$

This follows by approximation by step functions using the inequality

$$\|x+y\|_{L^q(\mathcal{M})} \geq \|x\|_{L^q(\mathcal{M})} + \|y\|_{L^q(\mathcal{M})} \qquad (x, y \in L^q(\mathcal{M})_+).$$

Lemma 3.7. *Let* (ξ_i) *be a finite sequence of independent, mean zero* $L^q(\mathcal{M})$*-valued random variables. If* $1 \leq p, q < 2$, *then*

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}}$$

$$\leq 4 \inf \Big\{ \Big\| \Big(\sum_i \mathbb{E}|\eta_i|^2 \Big)^{\frac{1}{2}} \Big\|_{L^q(\mathcal{M})} + \Big\| \Big(\sum_i \mathbb{E}|\theta_i^*|^2 \Big)^{\frac{1}{2}} \Big\|_{L^q(\mathcal{M})} \Big\},$$

where the infimum is taken over all sequences $(\eta_i) \in L^q(\mathcal{M}; \mathbb{E}, l^2_c)$ *and* $(\theta_i) \in L^q(\mathcal{M}; \mathbb{E}, l^2_r)$ *such that* $\xi_i = \eta_i + \theta_i$.

On the other hand, if $2 \leq p, q < \infty$, *then*

$$2\Big(\mathbb{E}\Big\| \sum_i \xi_i \Big\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}}$$
$$\geq \max \Big\{ \Big\| \Big(\sum_i \mathbb{E}|\xi_i|^2 \Big)^{\frac{1}{2}} \Big\|_{L^q(\mathcal{M})}, \Big\| \Big(\sum_i \mathbb{E}|\xi_i^*|^2 \Big)^{\frac{1}{2}} \Big\|_{L^q(\mathcal{M})} \Big\}.$$

Proof. Suppose $1 \leq p, q < 2$. Let (α_i) be a finite sequence in $L^q(\mathcal{M}; \mathbb{E}, l^2_c)$ of independent, mean zero $L^q(\mathcal{M})$-valued random variables. By Corollary 1.10 and Theorem 3.5,

$$\begin{aligned}
\Big(\mathbb{E}\Big\| \sum_i \alpha_i \Big\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}} &\leq 2\Big(\mathbb{E}\mathbb{E}_r\Big\| \sum_i r_i\alpha_i \Big\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}} \\
&\leq 2\Big(\mathbb{E}\Big\| \Big(\sum_i |\alpha_i|^2 \Big)^{\frac{1}{2}} \Big\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}} \\
&= 2\Big(\mathbb{E}\Big\| \sum_i |\alpha_i|^2 \Big\|^{\frac{p}{2}}_{L^{\frac{q}{2}}(\mathcal{M})}\Big)^{\frac{1}{p}} \\
&\leq 2\Big(\mathbb{E}\Big\| \sum_i |\alpha_i|^2 \Big\|_{L^{\frac{q}{2}}(\mathcal{M})}\Big)^{\frac{1}{2}} \\
&\leq 2\Big\| \sum_i \mathbb{E}|\alpha_i|^2 \Big\|^{\frac{1}{2}}_{L^{\frac{q}{2}}(\mathcal{M})} \\
&= 2\Big\| \Big(\sum_i \mathbb{E}|\alpha_i|^2 \Big)^{\frac{1}{2}} \Big\|_{L^q(\mathcal{M})}.
\end{aligned}$$

Note that in the final two inequalities we apply Jensen's inequality and (3.12), respectively, using that $\frac{p}{2}, \frac{q}{2} < 1$. Similarly, if (α_i) is a finite sequence in $L^q(\mathcal{M}; \mathbb{E}, l^2_r)$ of independent, mean zero $L^q(\mathcal{M})$-valued random variables, then

$$\Big(\mathbb{E}\Big\| \sum_i \alpha_i \Big\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}} \leq \Big\| \Big(\sum_i \mathbb{E}|\alpha_i^*|^2 \Big)^{\frac{1}{2}} \Big\|_{L^q(\mathcal{M})}.$$

Let (η_i) and (θ_i) be finite sequences in $L^q(\mathcal{M}; \mathbb{E}, l^2_c)$ and $L^q(\mathcal{M}; \mathbb{E}, l^2_r)$, respectively, such that $\xi_i = \eta_i + \theta_i$, then $\xi_i = \mathbb{E}(\eta_i|\xi_i) - \mathbb{E}(\eta_i) + \mathbb{E}(\theta_i|\xi_i) - \mathbb{E}(\theta_i)$. Since $(\mathbb{E}(\eta_i|\xi_i) - \mathbb{E}(\eta_i))$ and $(\mathbb{E}(\theta_i|\xi_i) - \mathbb{E}(\theta_i))$ are sequences of independent, mean zero random variables, we obtain by the triangle inequality and the above,

$$\Big(\mathbb{E}\Big\| \sum_i \xi_i \Big\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}}$$

$$\leq 2\Big\|\Big(\sum_i \mathbb{E}|\mathbb{E}(\eta_i|\xi_i) - \mathbb{E}(\eta_i)|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}$$
$$+ 2\Big\|\Big(\sum_i \mathbb{E}|\mathbb{E}(\theta_i^*|\xi_i) - \mathbb{E}(\theta_i^*)|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}.$$

Therefore, by the triangle inequality in $L^q(\mathcal{M};\mathbb{E},l_c^2)$ and $L^q(\mathcal{M};\mathbb{E},l_r^2)$ we find

$$\begin{aligned}
&\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(\mathcal{M})}^p\Big)^{\frac{1}{p}} \\
&\leq 2\Big(\Big\|\Big(\sum_i \mathbb{E}|\mathbb{E}(\eta_i|\xi_i)|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})} + \Big\|\Big(\sum_i \mathbb{E}|\mathbb{E}(\eta_i)|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})} \\
&+ \Big\|\Big(\sum_i \mathbb{E}|\mathbb{E}(\theta_i^*|\xi_i)|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})} + \Big\|\Big(\sum_i \mathbb{E}|\mathbb{E}(\theta_i^*)|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}\Big) \\
&\leq 4\Big(\Big\|\Big(\sum_i \mathbb{E}|\eta_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})} + \Big\|\Big(\sum_i \mathbb{E}|\theta_i^*|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}\Big).
\end{aligned}$$

Note that the final step follows directly from Kadison's inequality for (non-commutative) conditional expectations if η_i, θ_i are, in addition, in $L^\infty\overline{\otimes}\mathcal{M}$. For general η_i and θ_i as above the asserted inequality then follows by a density argument. This proves the first statement.

Suppose now that $2 \leq p, q < \infty$. By Corollary 1.10 and Theorem 3.5,

$$\begin{aligned}
&2\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(\mathcal{M})}^p\Big)^{\frac{1}{p}} \\
&\geq \Big(\mathbb{E}\mathbb{E}_r\Big\|\sum_i r_i\xi_i\Big\|_{L^q(\mathcal{M})}^p\Big)^{\frac{1}{p}} \\
&\geq \max\Big\{\Big(\mathbb{E}\Big\|\Big(\sum_i |\xi_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}^p\Big)^{\frac{1}{p}}, \Big(\mathbb{E}\Big\|\Big(\sum_i |\xi_i^*|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}^p\Big)^{\frac{1}{p}}\Big\} \\
&= \max\Big\{\Big(\mathbb{E}\Big\|\sum_i |\xi_i|^2\Big\|_{L^{\frac{q}{2}}(\mathcal{M})}^{\frac{p}{2}}\Big)^{\frac{1}{p}}, \Big(\mathbb{E}\Big\|\sum_i |\xi_i^*|^2\Big\|_{L^{\frac{q}{2}}(\mathcal{M})}^{\frac{p}{2}}\Big)^{\frac{1}{p}}\Big\} \\
&\geq \max\Big\{\Big\|\sum_i \mathbb{E}|\xi_i|^2\Big\|_{L^{\frac{q}{2}}(\mathcal{M})}^{\frac{1}{2}}, \Big\|\sum_i \mathbb{E}|\xi_i^*|^2\Big\|_{L^{\frac{q}{2}}(\mathcal{M})}^{\frac{1}{2}}\Big\} \\
&= \max\Big\{\Big\|\Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}, \Big\|\Big(\sum_i \mathbb{E}|\xi_i^*|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}\Big\}.
\end{aligned}$$

This completes the proof. □

For our discussion in Section 3.3 below, we will keep track of the dependence of the constants on p and q in the inequalities (3.13) and (3.14) below.

Theorem 3.8. *Suppose $2 \le p, q < \infty$. If (ξ_i) is a finite sequence of independent, mean zero $L^q(\mathcal{M})$-valued random variables, then*

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}} \le C_{p,q}(1+\sqrt{2})\max\Big\{\Big\|\Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}, \Big\|\Big(\sum_i \mathbb{E}|\xi_i^*|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}, C_{p,q}\Big(\mathbb{E}\Big(\sum_i \|\xi_i\|^q_{L^q(\mathcal{M})}\Big)^{\frac{p}{q}}\Big)^{\frac{1}{p}}\Big\}, \tag{3.13}$$

where $C_{p,q} = 2B_qK_{p,q} < \max\{2\sqrt{2}\sqrt{p-1}, 2\sqrt{q}\}$ and B_q and $K_{p,q}$ are the constants in (3.10) and (1.3), respectively. Moreover,

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}} \ge \frac{1}{2}\max\Big\{(K_{q,p})^{-1}\Big(\mathbb{E}\Big(\sum_i \|\xi_i\|^q_{L^q(\mathcal{M})}\Big)^{\frac{p}{q}}\Big)^{\frac{1}{p}}, \Big\|\Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}, \Big\|\Big(\sum_i \mathbb{E}|\xi_i^*|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}\Big\}. \tag{3.14}$$

Proof. We first prove (3.14). By Lemma 3.7,

$$\max\Big\{\Big\|\Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}, \Big\|\Big(\sum_i \mathbb{E}|\xi_i^*|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}\Big\} \le 2\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}}.$$

Moreover, since $L^q(\mathcal{M})$ has cotype q (cf. Theorem 3.3), it follows that

$$\Big(\mathbb{E}\Big(\sum_i \|\xi_i\|^q_{L^q(\mathcal{M})}\Big)^{\frac{p}{q}}\Big)^{\frac{1}{p}} \le \Big(\mathbb{E}\Big(\mathbb{E}_r\Big\|\sum_i r_i\xi_i\Big\|^q_{L^q(\mathcal{M})}\Big)^{\frac{p}{q}}\Big)^{\frac{1}{p}}. \tag{3.15}$$

We refer to [52] for a proof that (3.15) holds with constant 1. By successively applying Kahane's inequalities and Corollary 1.10 we see that

$$\begin{aligned}\Big(\mathbb{E}\Big(\sum_i \|\xi_i\|^q_{L^q(\mathcal{M})}\Big)^{\frac{p}{q}}\Big)^{\frac{1}{p}} &\le K_{q,p}\Big(\mathbb{E}\mathbb{E}_r\Big\|\sum_i r_i\xi_i\Big\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}} \\ &\le 2K_{q,p}\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}}.\end{aligned}$$

We now prove (3.13). By Corollary 1.10 and Theorem 3.5 we have

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}} \le 2K_{p,q}B_q\max\Big\{\Big(\mathbb{E}\Big\|\Big(\sum_i |\xi_i|^2\Big)^{\frac{1}{2}}\Big\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}}, \Big(\mathbb{E}\Big\|\Big(\sum_i |\xi_i^*|^2\Big)^{\frac{1}{2}}\Big\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}}\Big\}. \tag{3.16}$$

By the triangle inequality in $L^{\frac{p}{2}}(\Omega; L^{\frac{q}{2}}(\mathcal{M}))$ it follows that

$$\begin{aligned}
&\Big(\mathbb{E}\Big\|\Big(\sum_i |\xi_i|^2\Big)^{\frac{1}{2}}\Big\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}} \\
&\quad = \Big(\mathbb{E}\Big\|\sum_i |\xi_i|^2\Big\|^{\frac{p}{2}}_{L^{\frac{q}{2}}(\mathcal{M})}\Big)^{\frac{1}{p}} \\
&\quad \leq \Big(\Big(\mathbb{E}\Big\|\sum_i |\xi_i|^2 - \mathbb{E}|\xi_i|^2\Big\|^{\frac{p}{2}}_{L^{\frac{q}{2}}(\mathcal{M})}\Big)^{\frac{2}{p}} + \Big\|\sum_i \mathbb{E}|\xi_i|^2\Big\|_{L^{\frac{q}{2}}(\mathcal{M})}\Big)^{\frac{1}{2}}. \qquad (3.17)
\end{aligned}$$

We now estimate the first term on the far right hand side. By applying Corollary 1.10 and Theorem 3.5 we obtain

$$\begin{aligned}
&\Big(\mathbb{E}\Big\|\sum_i |\xi_i|^2 - \mathbb{E}|\xi_i|^2\Big\|^{\frac{p}{2}}_{L^{\frac{q}{2}}(\mathcal{M})}\Big)^{\frac{2}{p}} \\
&\quad \leq 2K_{p,q}B_q\Big(\mathbb{E}\Big\|\Big(\sum_i \big|\, |\xi_i|^2 - \mathbb{E}|\xi_i|^2\big|^2\Big)^{\frac{1}{2}}\Big\|^{\frac{p}{2}}_{L^{\frac{q}{2}}(\mathcal{M})}\Big)^{\frac{2}{p}} \\
&\quad \leq C_{p,q}\Big(\Big(\mathbb{E}\Big\|\Big(\sum_i |\xi_i|^4\Big)^{\frac{1}{2}}\Big\|^{\frac{p}{2}}_{L^{\frac{q}{2}}(\mathcal{M})}\Big)^{\frac{2}{p}} + \Big\|\Big(\sum_i |\mathbb{E}|\xi_i|^2|^2\Big)^{\frac{1}{2}}\Big\|_{L^{\frac{q}{2}}(\mathcal{M})}\Big), \qquad (3.18)
\end{aligned}$$

where the final inequality is a consequence of the triangle inequality in $L^{\frac{p}{2}}(\Omega; L^{\frac{q}{2}}(\mathcal{M}; l^2_c))$. Notice that the second term on the right hand side is smaller than the first one. Indeed,

$$\begin{aligned}
\Big\|\Big(\sum_i |\mathbb{E}|\xi_i|^2|^2\Big)^{\frac{1}{2}}\Big\|_{L^{\frac{q}{2}}(\mathcal{M})} &= \|\mathrm{col}(\mathbb{E}|\xi_i|^2)\|_{L^{\frac{q}{2}}(M_n(\mathcal{M}))} \\
&= \|\mathbb{E}(\mathrm{col}(|\xi_i|^2))\|_{L^{\frac{q}{2}}(M_n(\mathcal{M}))} \\
&\leq \mathbb{E}\|\mathrm{col}(|\xi_i|^2)\|_{L^{\frac{q}{2}}(M_n(\mathcal{M}))} \\
&\leq (\mathbb{E}\|\mathrm{col}(|\xi_i|^2)\|^{\frac{p}{2}}_{L^{\frac{q}{2}}(M_n(\mathcal{M}))})^{\frac{2}{p}} \\
&= \Big(\mathbb{E}\Big\|\Big(\sum_i |\xi_i|^4\Big)^{\frac{1}{2}}\Big\|^{\frac{p}{2}}_{L^{\frac{q}{2}}(\mathcal{M})}\Big)^{\frac{2}{p}}. \qquad (3.19)
\end{aligned}$$

Write $x = \mathrm{col}(|\xi_i|)$ and $y = \mathrm{diag}(|\xi_i|)$. By the noncommutative Hölder inequality (3.6),

$$\begin{aligned}
&\Big(\mathbb{E}\Big\|\Big(\sum_i |\xi_i|^4\Big)^{\frac{1}{2}}\Big\|^{\frac{p}{2}}_{L^{\frac{q}{2}}(\mathcal{M})}\Big)^{\frac{2}{p}} \\
&\quad = (\mathbb{E}\|(x^*y^*yx)^{\frac{1}{2}}\|^{\frac{p}{2}}_{L^{\frac{q}{2}}(M_n(\mathcal{M}))})^{\frac{2}{p}} \\
&\quad = (\mathbb{E}\|yx\|^{\frac{p}{2}}_{L^{\frac{q}{2}}(M_n(\mathcal{M}))})^{\frac{2}{p}}
\end{aligned}$$

$$
\begin{aligned}
&\leq (\mathbb{E}|\ \|y\|_{L^q(M_n(\mathcal{M}))}\|x\|_{L^q(M_n(\mathcal{M}))}|^{\frac{p}{2}})^{\frac{2}{p}} \\
&\leq (\mathbb{E}\|y\|^p_{L^q(M_n(\mathcal{M}))})^{\frac{1}{p}}(\mathbb{E}\|x\|^p_{L^q(M_n(\mathcal{M}))})^{\frac{1}{p}} \\
&= \Big(\mathbb{E}\Big(\sum_i \|\xi_i\|^q_{L^q(\mathcal{M})}\Big)^{\frac{p}{q}}\Big)^{\frac{1}{p}}\Big(\mathbb{E}\Big\|\Big(\sum_i |\xi_i|^2\Big)^{\frac{1}{2}}\Big\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}}. \qquad (3.20)
\end{aligned}
$$

Collecting our estimates (3.17), (3.18), (3.19) and (3.20), we obtain the quadratic equation

$$a^2 \leq (2C_{p,q})ab + c^2,$$

where we set $a = (\mathbb{E}\|(\sum_i |\xi_i|^2)^{\frac{1}{2}}\|^p_{L^q(\mathcal{M})})^{\frac{1}{p}}$, $b = (\mathbb{E}(\sum_i \|\xi_i\|^q_{L^q(\mathcal{M})})^{\frac{p}{q}})^{\frac{1}{p}}$ and $c = \|(\sum_i \mathbb{E}|\xi_i|^2)^{\frac{1}{2}}\|_{L^q(\mathcal{M})}$. Solving this quadratic equation we obtain

$$a \leq \frac{1}{2}(2C_{p,q}b + ((2C_{p,q}b)^2 + 4c^2)^{\frac{1}{2}}) \leq \frac{1+\sqrt{2}}{2}\max\{2C_{p,q}b, 2c\},$$

that is,

$$
\begin{aligned}
&\Big(\mathbb{E}\Big\|\Big(\sum_i |\xi_i|^2\Big)^{\frac{1}{2}}\Big\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}} \\
&\quad \leq (1+\sqrt{2})\max\Big\{\Big\|\Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}, C_{p,q}\Big(\mathbb{E}\Big(\sum_i \|\xi_i\|^q_{L^q(\mathcal{M})}\Big)^{\frac{p}{q}}\Big)^{\frac{1}{p}}\Big\}.
\end{aligned}
$$

Applying this to the sequence (ξ_i^*) we obtain

$$
\begin{aligned}
&\Big(\mathbb{E}\Big\|\Big(\sum_i |\xi_i^*|^2\Big)^{\frac{1}{2}}\Big\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}} \\
&\quad \leq (1+\sqrt{2})\max\Big\{\Big\|\Big(\sum_i \mathbb{E}|\xi_i^*|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}, C_{p,q}\Big(\mathbb{E}\Big(\sum_i \|\xi_i\|^q_{L^q(\mathcal{M})}\Big)^{\frac{p}{q}}\Big)^{\frac{1}{p}}\Big\}.
\end{aligned}
$$

The inequality (3.13) now follows from (3.16).

Let us finally observe that $C_{p,q} = 2B_qK_{p,q} < \max\{2\sqrt{2}\sqrt{p-1}, 2\sqrt{q}\}$. Indeed, if $p \leq q$, then it is clear that $C_{p,q} = 2B_q < 2\sqrt{q}$. If $p > q \geq 2$ then the optimal constant in Kahane's inequality satisfies $K_{p,q} \leq \sqrt{\frac{p-1}{q-1}}$ (see e.g. the proof of [96], Theorem 1.e.13). Hence, in this case we can estimate $C_{p,q}$ by $2\sqrt{p-1}\sqrt{\frac{q}{q-1}} \leq 2\sqrt{2}\sqrt{p-1}$. $\square$

We are now ready to deduce the Rosenthal-type inequalities in the cases where $p, q \geq 2$.

Theorem 3.9. *Suppose $2 \leq q \leq p < \infty$. Let (ξ_i) be a finite sequence of independent, mean zero $L^q(\mathcal{M})$-valued random variables. Then,*

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}} \simeq_{p,q} \max\Big\{\Big(\sum_i \mathbb{E}\|\xi_i\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}}, \Big(\sum_i \mathbb{E}\|\xi_i\|^q_{L^q(\mathcal{M})}\Big)^{\frac{1}{q}},$$

$$\Big\|\Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}, \Big\|\Big(\sum_i \mathbb{E}|\xi_i^*|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}\Big\}.$$

Proof. By Lemma 1.14 and Theorem 3.8 we obtain

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}} \lesssim_{p,q} \max\Big\{\Big(\sum_i \mathbb{E}\|\xi_i\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}}, \Big(\sum_i \mathbb{E}\|\xi_i\|^q_{L^q(\mathcal{M})}\Big)^{\frac{1}{q}},$$
$$\Big\|\Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}, \Big\|\Big(\sum_i \mathbb{E}|\xi_i^*|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}\Big\}.$$

The reverse inequality follows from Lemma 3.7 and Theorem 1.20 (as $L^q(\mathcal{M})$ has cotype q). □

Theorem 3.10. *Suppose $2 \leq p \leq q < \infty$. Let (ξ_i) be a finite sequence of independent, mean zero $L^q(\mathcal{M})$-valued random variables. Then,*

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}}$$
$$\simeq_{p,q} \max\Big\{\inf\Big\{\Big(\sum_i \mathbb{E}\|\eta_i\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}} + \Big(\sum_i \mathbb{E}\|\theta_i\|^q_{L^q(\mathcal{M})}\Big)^{\frac{1}{q}}\Big\},$$
$$\Big\|\Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}, \Big\|\Big(\sum_i \mathbb{E}|\xi_i^*|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}\Big\},$$

where the infimum is taken over all sequences $(\eta_i) \in l^p(L^p(\Omega; L^q(\mathcal{M})))$ and $(\theta_i) \in l^q(L^q(\Omega; L^q(\mathcal{M})))$ such that $\xi_i = \eta_i + \theta_i$.

Proof. By Lemma 1.14 and Theorem 3.8 we obtain

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}}$$
$$\lesssim_{p,q} \max\Big\{\inf\Big\{\Big(\sum_i \mathbb{E}\|\eta_i\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}} + \Big(\sum_i \mathbb{E}\|\theta_i\|^q_{L^q(\mathcal{M})}\Big)^{\frac{1}{q}}\Big\},$$
$$\Big\|\Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}, \Big\|\Big(\sum_i \mathbb{E}|\xi_i^*|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}\Big\},$$

The reverse inequality follows from Lemma 3.7 and Theorem 1.20 (as $L^q(\mathcal{M})$ has cotype q). □

We deduce estimates for the cases $1 < p \leq q \leq 2$ and $1 < q \leq p \leq 2$ by duality from Theorems 3.9 and 3.10, respectively.

Theorem 3.11. *Suppose $1 < p \leq q \leq 2$. Let (ξ_i) be a finite sequence of independent, mean zero $L^q(\mathcal{M})$-valued random variables. Then,*

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}}$$

$$\simeq_{p,q} \inf\Big\{\Big\|\Big(\sum_i \mathbb{E}|\eta_{i,c}|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})} + \Big\|\Big(\sum_i \mathbb{E}|\eta_{i,r}^*|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}$$

$$+ \Big(\sum_i \mathbb{E}\|\theta_i\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}} + \Big(\sum_i \mathbb{E}\|\kappa_i\|^q_{L^q(\mathcal{M})}\Big)^{\frac{1}{q}}\Big\}, \qquad (3.21)$$

where the infimum is taken over all sequences $(\eta_{i,c}) \in L^q(\mathcal{M};\mathbb{E},l^2_c)$, $(\eta_{i,r}) \in L^q(\mathcal{M};\mathbb{E},l^2_r)$, $(\theta_i) \in l^p(L^p(\Omega;L^q(\mathcal{M})))$ *and* $(\kappa_i) \in l^q(L^q(\Omega;L^q(\mathcal{M})))$ *such that* $\xi_i = \eta_{i,c} + \eta_{i,r} + \theta_i + \kappa_i$.

Proof. Let $(\eta_{i,c}) \in L^q(\mathcal{M};\mathbb{E},l^2_c)$, $(\eta_{i,r}) \in L^q(\mathcal{M};\mathbb{E},l^2_r)$, $(\theta_i) \in l^p(L^p(\Omega;L^q(\mathcal{M})))$ and $(\kappa_i) \in l^q(L^q(\Omega;L^q(\mathcal{M})))$ be such that $\xi_i = \eta_{i,c} + \eta_{i,r} + \theta_i + \kappa_i$. Then,

$$\xi_i = \mathbb{E}(\eta_{i,c}|\xi_i) - \mathbb{E}(\eta_{i,c}) + \mathbb{E}(\eta_{i,r}|\xi_i) - \mathbb{E}(\eta_{i,r}) + \mathbb{E}(\theta_i|\xi_i) - \mathbb{E}(\theta_i) + \mathbb{E}(\kappa_i|\xi_i) - \mathbb{E}(\kappa_i).$$

By the proof of Lemma 3.7 we have

$$\Big(\mathbb{E}\Big\|\sum_i \mathbb{E}(\eta_{i,c}|\xi_i) - \mathbb{E}(\eta_{i,c})\Big\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}} \le 4\Big\|\Big(\sum_i \mathbb{E}|\eta_{i,c}|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}$$

and

$$\Big(\mathbb{E}\Big\|\sum_i \mathbb{E}(\eta_{i,r}|\xi_i) - \mathbb{E}(\eta_{i,r})\Big\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}} \le 4\Big\|\Big(\sum_i \mathbb{E}|\eta_{i,r}^*|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}.$$

Since $L^q(\mathcal{M})$ has type q we obtain by Theorem 1.15 and contractivity of vector-valued conditional expectations

$$\Big(\mathbb{E}\Big\|\sum_i \mathbb{E}(\theta_i|\xi_i) - \mathbb{E}(\theta_i)\Big\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}} \lesssim_{p,q} \Big(\sum_i \mathbb{E}\|\mathbb{E}(\theta_i|\xi_i) - \mathbb{E}(\theta_i)\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}}$$

$$\le 2\Big(\sum_i \mathbb{E}\|\theta_i\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}}.$$

Similarly,

$$\Big(\mathbb{E}\Big\|\sum_i \mathbb{E}(\kappa_i|\xi_i) - \mathbb{E}(\kappa_i)\Big\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}} \lesssim_{p,q} \Big(\sum_i \mathbb{E}\|\kappa_i\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}}.$$

The upper estimate in (3.21) now follows by the triangle inequality.

We deduce the reverse inequality by duality from Theorem 3.9. Let $2 \le q' \le p' < \infty$ be such that $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Let (η_i) be a finite sequence of $L^{q'}(\mathcal{M})$-valued random variables satisfying $\|(\eta_i)\|_* \le 1$, where

$$\|(\eta_i^*)\|_* = \max\Big\{\Big\|\Big(\sum_i \mathbb{E}|\eta_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^{q'}(\mathcal{M})}, \Big\|\Big(\sum_i \mathbb{E}|\eta_i^*|^2\Big)^{\frac{1}{2}}\Big\|_{L^{q'}(\mathcal{M})},$$

$$\Big(\sum_i \mathbb{E}\|\eta_i\|_{L^{q'}(\mathcal{M})}^{p'}\Big)^{\frac{1}{p'}}, \Big(\sum_i \mathbb{E}\|\eta_i\|_{L^{q'}(\mathcal{M})}^{q'}\Big)^{\frac{1}{q'}}\Big\}.$$

Then, by Theorem 3.9 we have

$$\begin{aligned}
\langle(\xi_i),(\eta_i)\rangle &= \sum_i \mathbb{E}\otimes\tau(\xi_i\eta_i) \\
&= \sum_i \mathbb{E}\otimes\tau(\xi_i(\mathbb{E}(\eta_i|\xi_i)-\mathbb{E}(\eta_i))) \\
&= \sum_{i,j} \mathbb{E}\otimes\tau(\xi_i(\mathbb{E}(\eta_j|\xi_j)-\mathbb{E}(\eta_j))) \\
&= \mathbb{E}\otimes\tau\Big(\Big(\sum_i \xi_i\Big)\Big(\sum_j \mathbb{E}(\eta_j|\xi_j)-\mathbb{E}(\eta_j)\Big)\Big) \\
&\leq \Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(\mathcal{M})}^p\Big)^{\frac{1}{p}}\Big(\mathbb{E}\Big\|\sum_j \mathbb{E}(\eta_j|\xi_j)-\mathbb{E}(\eta_j)\Big\|_{L^{q'}(\mathcal{M})}^{p'}\Big)^{\frac{1}{p'}} \\
&\lesssim_{p',q'} \Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}}\|(\mathbb{E}(\eta_j|\xi_j)-\mathbb{E}(\eta_j))\|_* \\
&\leq 2\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}}.
\end{aligned}$$

The asserted inequality now follows by taking the supremum over all (η_i) as above. □

Theorem 3.12. *Suppose $1 < q \leq p \leq 2$. Let (ξ_i) be a finite sequence of independent, mean zero $L^q(\mathcal{M})$-valued random variables. Then,*

$$\begin{aligned}
&\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{L^q(\mathcal{M})}^p\Big)^{\frac{1}{p}} \\
&\quad\simeq_{p,q} \inf\Big\{\Big\|\Big(\sum_i \mathbb{E}|\eta_{i,c}|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})} + \Big\|\Big(\sum_i \mathbb{E}|\eta_{i,r}^*|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})} \\
&\qquad\qquad + \max\Big\{\Big(\sum_i \mathbb{E}\|\theta_i\|_{L^q(\mathcal{M})}^p\Big)^{\frac{1}{p}}, \Big(\sum_i \mathbb{E}\|\theta_i\|_{L^q(\mathcal{M})}^q\Big)^{\frac{1}{q}}\Big\}\Big\},
\end{aligned}$$

where the infimum is taken over all sequences $(\eta_{i,c}) \in L^q(\mathcal{M};\mathbb{E},l_c^2)$, $(\eta_{i,r}) \in L^q(\mathcal{M};\mathbb{E},l_r^2)$ and $(\theta_i) \in l^p(L^p(\Omega;L^q(\mathcal{M}))) \cap l^q(L^q(\Omega;L^q(\mathcal{M})))$ such that $\xi_i = \eta_{i,c}+\eta_{i,r}+\theta_i$.

Proof. The proof is very similar to the one presented for Theorem 3.11. The upper estimate follows from Lemma 3.7 and the first part of Theorem 1.15. The lower estimate can be derived by duality from Theorem 3.10. We leave the details to the reader. □

Theorem 3.13. *Let $1 < q < 2 \leq p < \infty$. Let (ξ_i) be a finite sequence of independent, mean zero $L^q(\mathcal{M})$-valued random variables. Then,*

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}}$$
$$\simeq_{p,q} \max\Big\{\inf\Big\{\Big\|\Big(\sum_i \mathbb{E}|\eta_{i,c}|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})} + \Big\|\Big(\sum_i \mathbb{E}|\eta_{i,r}^*|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}$$
$$+\Big(\sum_i \mathbb{E}\|\theta_i\|^q_{L^q(\mathcal{M})}\Big)^{\frac{1}{q}}\Big\}, \Big(\sum_i \mathbb{E}\|\xi_i\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}}\Big\},$$

where the infimum is taken over all sequences $(\eta_{i,c}) \in L^q(\mathcal{M};\mathbb{E},l^2_c)$, $(\eta_{i,r}) \in L^q(\mathcal{M};\mathbb{E},l^2_r)$ and $(\theta_i) \in l^q(L^q(\Omega;L^q(\mathcal{M})))$ such that $\xi_i = \eta_{i,c} + \eta_{i,r} + \theta_i$.

Proof. By Theorem 1.31 we have

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}} \lesssim_p \max\Big\{\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^q_{L^q(\mathcal{M})}\Big)^{\frac{1}{q}}, \Big(\mathbb{E}\max_i \|\xi_i\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}}\Big\}.$$

By Theorem 3.11 (with $p = q$) we have

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^q_{L^q(\mathcal{M})}\Big)^{\frac{1}{q}} \simeq_{p,q} \|(\xi_i)\|_{L^q(\mathcal{M};\mathbb{E},l^2_c)+L^q(\mathcal{M};\mathbb{E},l^2_r)+l^q(L^q(\Omega;L^q(\mathcal{M})))}$$

and obviously

$$\Big(\mathbb{E}\max_i \|\xi_i\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}} \leq \Big(\sum_i \mathbb{E}\|\xi_i\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}}.$$

For the reverse inequality, note that

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}} \geq \Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^q_{L^q(\mathcal{M})}\Big)^{\frac{1}{q}}$$
$$\simeq_{p,q} \|(\xi_i)\|_{L^q(\mathcal{M};\mathbb{E},l^2_c)+L^q(\mathcal{M};\mathbb{E},l^2_r)+l^q(L^q(\Omega;L^q(\mathcal{M})))}.$$

Moreover, as $L^q(\mathcal{M})$ has cotype q we have by Theorem 1.20,

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}} \gtrsim_{p,q} \Big(\sum_i \mathbb{E}\|\xi_i\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}}.$$

□

Theorem 3.14. *Let $1 < p < 2 \leq q < \infty$. Let (ξ_i) be a finite sequence of independent, mean zero $L^q(\mathcal{M})$-valued random variables. Then,*

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}}$$

$$\simeq_{p,q} \inf\Big\{ \max\Big\{ \Big\|\Big(\sum_i \mathbb{E}|\eta_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}, \Big\|\Big(\sum_i \mathbb{E}|\eta_i^*|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}, \Big(\sum_i \mathbb{E}\|\eta_i\|^q_{L^q(\mathcal{M})}\Big)^{\frac{1}{q}}\Big\} + \Big(\sum_i \mathbb{E}\|\theta_i\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}}\Big\},$$

where the infimum is taken over all sequences (θ_i) *in* $l^p(L^p(\Omega; L^q(\mathcal{M})))$ *and* (η_i) *in* $L^q(\mathcal{M}; \mathbb{E}, l^2_c) \cap L^q(\mathcal{M}; \mathbb{E}, l^2_r) \cap l^q(L^q(\Omega; L^q(\mathcal{M})))$ *such that* $\xi_i = \eta_i + \theta_i$.

Proof. By Theorem 3.10 (with $p = q$) we have

$$\begin{aligned}\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}} &\leq \Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^q_{L^q(\mathcal{M})}\Big)^{\frac{1}{q}} \\ &\lesssim_{p,q} \max\Big\{\Big(\sum_i \mathbb{E}\|\xi_i\|^q_{L^q(\mathcal{M})}\Big)^{\frac{1}{q}}, \\ &\qquad \Big\|\Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}, \Big\|\Big(\sum_i \mathbb{E}|\xi_i^*|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}\Big\}.\end{aligned}$$

On the other hand, as $L^q(\mathcal{M})$ has type q we obtain by Theorem 1.15,

$$\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}} \lesssim_{p,q} \Big(\sum_i \mathbb{E}\|\xi_i\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}}.$$

Let $\xi_i = \eta_i + \theta_i$. Then, $\xi_i = \mathbb{E}(\eta_i|\xi_i) - \mathbb{E}(\eta_i) + \mathbb{E}(\theta_i|\xi_i) - \mathbb{E}(\theta_i)$. By applying the above to the sequences of independent, mean zero random variables $(\mathbb{E}(\eta_i|\xi_i) - \mathbb{E}(\eta_i))$ and $(\mathbb{E}(\theta_i|\xi_i) - \mathbb{E}(\theta_i))$ we obtain

$$\begin{aligned}&\Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}} \\ &\lesssim_{p,q} \max\Big\{\Big\|\Big(\sum_i \mathbb{E}|\mathbb{E}(\eta_i|\xi_i) - \mathbb{E}(\eta_i)|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}, \\ &\qquad \Big\|\Big(\sum_i \mathbb{E}|\mathbb{E}(\eta_i^*|\xi_i) - \mathbb{E}(\eta_i^*)|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}, \\ &\qquad \Big(\sum_i \mathbb{E}\|\mathbb{E}(\eta_i|\xi_i) - \mathbb{E}(\eta_i)\|^q_{L^q(\mathcal{M})}\Big)^{\frac{1}{q}}\Big\} \\ &\qquad + \Big(\sum_i \mathbb{E}\|\mathbb{E}(\theta_i|\xi_i) - \mathbb{E}(\theta_i)\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}} \\ &\leq \max\Big\{\Big\|\Big(\sum_i \mathbb{E}|\eta_i|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}, \Big\|\Big(\sum_i \mathbb{E}|\eta_i^*|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}, \\ &\qquad \Big(\sum_i \mathbb{E}\|\eta_i\|^q_{L^q(\mathcal{M})}\Big)^{\frac{1}{q}}\Big\} + \Big(\sum_i \mathbb{E}\|\theta_i\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}}.\end{aligned}$$

The reverse inequality follows by duality from Theorem 3.13. We leave the details to the reader. □

We now summarize the main results of this section.

Theorem 3.15. *Let $1 < p, q < \infty$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{M}$ be a semi-finite von Neumann algebra. Let $\mathcal{E}$ be the conditional expectation with respect to the von Neumann subalgebra $\mathbf{1} \otimes \mathcal{M}$ of $L^\infty \overline{\otimes} \mathcal{M}$. Set*

$$\begin{aligned} S_{q,c} &= L^q(L^\infty(\Omega)\overline{\otimes}\mathcal{M}; \mathcal{E}, l^2_c); \\ S_{q,r} &= L^q(L^\infty(\Omega)\overline{\otimes}\mathcal{M}; \mathcal{E}, l^2_r); \\ D_{p,q} &= l^p(L^p(\Omega; L^q(\mathcal{M}))). \end{aligned}$$

If (ξ_i) is a finite sequence of independent, mean zero $L^q(\mathcal{M})$-valued random variables, then

$$\Big(\mathbb{E}\Big\| \sum_i \xi_i \Big\|^p_{L^q(\mathcal{M})}\Big)^{\frac{1}{p}} \simeq_{p,q} \|(\xi_i)\|_{s_{p,q}},$$

where $s_{p,q}$ is given by

$$\begin{array}{rl} S_{q,c} \cap S_{q,r} \cap D_{q,q} \cap D_{p,q} & \text{if} \quad 2 \leq q \leq p < \infty; \\ S_{q,c} \cap S_{q,r} \cap (D_{q,q} + D_{p,q}) & \text{if} \quad 2 \leq p \leq q < \infty; \\ (S_{q,c} \cap S_{q,r} \cap D_{q,q}) + D_{p,q} & \text{if} \quad 1 < p < 2 \leq q < \infty; \\ (S_{q,c} + S_{q,r} + D_{q,q}) \cap D_{p,q} & \text{if} \quad 1 < q < 2 \leq p < \infty; \\ S_{q,c} + S_{q,r} + (D_{q,q} \cap D_{p,q}) & \text{if} \quad 1 < q \leq p \leq 2; \\ S_{q,c} + S_{q,r} + D_{q,q} + D_{p,q} & \text{if} \quad 1 < p \leq q \leq 2. \end{array}$$

If (S, Σ, μ) is a σ-finite measure space, then we obtain the results of Section 1.3 as a special case of Theorem 3.15 by viewing $L^q(S)$ as a noncommutative L^q-space (as explained in example 3.1). Indeed, in this case $S_{q,c}$ and $S_{q,r}$ coincide and are equal to $L^q(S; l^2(L^2(\Omega)))$.

3.3 Application to random matrices

The purpose of this section is to derive two-sided estimates for the p-th moments of the largest singular value of a random matrix with independent, mean zero columns or entries in terms of a suitable norm on its entries. The main results are Theorems 3.20, 3.21 and 3.24 below. The principal tool in our investigation is Theorem 3.8, used together with the estimate (3.5). Throughout, we let e_{ij} $(i, j = 1, \ldots, n)$ denote the standard matrix units of M_n, i.e. e_{ij} is the $n \times n$ matrix with (i, j)-th entry equal to 1 and zero entries elsewhere. We restrict our attention to random matrices with real-valued entries.

We shall use the following special case of [91], Proposition 1.1.1, which is a consequence of the Lévy-Octaviani inequalities.

Proposition 3.16. *Suppose $1 \le p < \infty$ and let X be a Banach space. If $\xi_1, \dots, \xi_n$ is a finite sequence of independent, symmetric X-valued random variables, then*

$$\Big(\mathbb{E} \max_{i=1,\dots,n} \|\xi_i\|_X^p\Big)^{\frac{1}{p}} \le 2^{\frac{1}{p}} \Big(\mathbb{E}\Big\| \sum_i \xi_i \Big\|_X^p\Big)^{\frac{1}{p}}.$$

By a randomization argument we obtain the following.

Lemma 3.17. *Suppose $1 \le p < \infty$ and let X be a Banach space. If (ξ_i) is a sequence of independent, mean zero X-valued random variables, then*

$$\Big(\mathbb{E} \max_{i=1,\dots,n} \|\xi_i\|_X^p\Big)^{\frac{1}{p}} \le 2^{1+\frac{1}{p}} \Big(\mathbb{E}\Big\| \sum_i \xi_i \Big\|_X^p\Big)^{\frac{1}{p}}.$$

Proof. Let (r_i) be a Rademacher sequence on a probability space $(\Omega_r, \mathcal{F}_r, \mathbb{P}_r)$. Then, by Corollary 1.10 and Proposition 3.16 it follows that

$$\begin{aligned}\Big(\mathbb{E} \max_{i=1,\dots,n} \|\xi_i\|_X^p\Big)^{\frac{1}{p}} &= \Big(\mathbb{E}\mathbb{E}_r \max_{i=1,\dots,n} \|r_i\xi_i\|_X^p\Big)^{\frac{1}{p}} \\ &\le 2^{\frac{1}{p}} \Big(\mathbb{E}\mathbb{E}_r\Big\| \sum_i r_i\xi_i \Big\|_X^p\Big)^{\frac{1}{p}} \le 2^{1+\frac{1}{p}} \Big(\mathbb{E}\Big\| \sum_i \xi_i \Big\|_X^p\Big)^{\frac{1}{p}}.\end{aligned}$$

□

Lemma 3.18. *Fix $n \in \mathbb{N}$ and $2 \le p < \infty$. If (x_i) is a finite sequence of independent, mean zero $n \times n$ random matrices, then*

$$\max\Big\{\Big\|\Big(\sum_i \mathbb{E}|x_i|^2\Big)^{\frac{1}{2}}\Big\|, \Big\|\Big(\sum_i \mathbb{E}|x_i^*|^2\Big)^{\frac{1}{2}}\Big\|\Big\} \le 2\Big(\mathbb{E}\Big\|\sum_i x_i\Big\|^p\Big)^{\frac{1}{p}}.$$

Proof. Observe that

$$\begin{aligned}\Big\|\Big(\sum_i \mathbb{E}|x_i|^2\Big)^{\frac{1}{2}}\Big\| &= \Big\|\mathbb{E}\mathbb{E}_r \sum_{i,j} r_i r_j x_i^* x_j\Big\|^{\frac{1}{2}} \\ &\le \Big(\mathbb{E}\mathbb{E}_r\Big\|\sum_{i,j} r_i r_j x_i^* x_j\Big\|\Big)^{\frac{1}{2}} \\ &= \Big(\mathbb{E}\mathbb{E}_r\Big\|\sum_i r_i x_i\Big\|^2\Big)^{\frac{1}{2}} \\ &\le 2\Big(\mathbb{E}\Big\|\sum_i x_i\Big\|^p\Big)^{\frac{1}{p}},\end{aligned}$$

where in the final step we apply Corollary 1.10. The result follows by applying this to the sequence (x_i^*). □

Lemma 3.19. *Let $2 \le p < \infty$. If $(x_i)_{i=1}^m$ is a finite sequence of independent, mean zero $n \times n$ random matrices, then*

$$2^{1+\frac{1}{p}}\Big(\mathbb{E}\Big\|\sum_{i=1}^m x_i\Big\|^p\Big)^{\frac{1}{p}} \ge \max\Big\{\Big\|\Big(\sum_{i=1}^m \mathbb{E}|x_i|^2\Big)^{\frac{1}{2}}\Big\|, \Big\|\Big(\sum_{i=1}^m \mathbb{E}|x_i^*|^2\Big)^{\frac{1}{2}}\Big\|, \Big(\mathbb{E}\max_{i=1,\dots m}\|x_i\|^p\Big)^{\frac{1}{p}}\Big\}, \tag{3.22}$$

and

$$\Big(\mathbb{E}\Big\|\sum_{i=1}^m x_i\Big\|^p\Big)^{\frac{1}{p}} \le e(1+\sqrt{2})\alpha_{p,n}\max\Big\{\Big\|\Big(\sum_{i=1}^m \mathbb{E}|x_i|^2\Big)^{\frac{1}{2}}\Big\|, \Big\|\Big(\sum_{i=1}^m \mathbb{E}|x_i^*|^2\Big)^{\frac{1}{2}}\Big\|, e^{\frac{\log m}{\log n}-1}\alpha_{p,n}\Big(\mathbb{E}\max_{i=1,\dots m}\|x_i\|^p\Big)^{\frac{1}{p}}\Big\}, \tag{3.23}$$

with $\alpha_{p,n} = C_{p,\max\{\log n,2\}} < \max\{2\sqrt{\log n}, 2\sqrt{2}\sqrt{p-1}\}$, where $C_{p,q}$ is the constant in Theorem 3.8.

Proof. Observe that (3.22) immediately follows from Lemmas 3.17 and 3.18.

By applying Theorem 3.8 with $q = \max\{\log n, 2\}$ we obtain

$$\begin{aligned}\Big(\mathbb{E}\Big\|\sum_{i=1}^m x_i\Big\|^p\Big)^{\frac{1}{p}} &\le (1+\sqrt{2})C_{p,q}\max\Big\{C_{p,q}\Big(\mathbb{E}\Big(\sum_{i=1}^m \|x_i\|_{S_n^q}^q\Big)^{\frac{p}{q}}\Big)^{\frac{1}{p}}, \\ &\qquad \Big\|\Big(\sum_{i=1}^m \mathbb{E}|x_i|^2\Big)^{\frac{1}{2}}\Big\|_{S_n^q}, \Big\|\Big(\sum_{i=1}^m \mathbb{E}|x_i^*|^2\Big)^{\frac{1}{2}}\Big\|_{S_n^q}\Big\} \qquad (3.24)\\ &\le (1+\sqrt{2})\alpha_{p,n}\max\Big\{\alpha_{p,n}\Big(\mathbb{E}\Big(\sum_{i=1}^m \|x_i\|^{\log n}\Big)^{\frac{p}{\log n}}\Big)^{\frac{1}{p}}, \\ &\qquad e\Big\|\Big(\sum_{i=1}^m \mathbb{E}|x_i|^2\Big)^{\frac{1}{2}}\Big\|, e\Big\|\Big(\sum_{i=1}^m \mathbb{E}|x_i^*|^2\Big)^{\frac{1}{2}}\Big\|\Big\},\end{aligned}$$

where the final inequality is a consequence of (3.5). The inequality (3.23) now follows by observing that

$$\begin{aligned}\Big(\mathbb{E}\Big(\sum_{i=1}^m \|x_i\|^{\log n}\Big)^{\frac{p}{\log n}}\Big)^{\frac{1}{p}} &\le m^{\frac{1}{\log n}}\Big(\mathbb{E}\max_{i=1,\dots m}\|x_i\|^p\Big)^{\frac{1}{p}} \\ &= e^{\frac{\log m}{\log n}}\Big(\mathbb{E}\max_{i=1,\dots m}\|x_i\|^p\Big)^{\frac{1}{p}}.\end{aligned}$$

□

Theorem 3.20. *Let $2 \le p < \infty$. Suppose $x = (x_{ij})_{i,j=1}^n$ is a random matrix with independent, mean zero columns and let $y = (y_{ik})_{i,k=1}^n$ be the matrix given by $y_{ik} = \mathbb{E}(\sum_{j=1}^n x_{ij}x_{kj})$. Then,*

$$2^{1+\frac{1}{p}}(\mathbb{E}\|x\|^p)^{\frac{1}{p}} \geq \max\Big\{ \max_{j=1,\dots,n} \Big(\sum_{i=1}^n \mathbb{E}x_{ij}^2\Big)^{\frac{1}{2}}, \|y\|^{\frac{1}{2}},$$
$$\Big(\mathbb{E} \max_{j=1,\dots n} \Big(\sum_{i=1}^n x_{ij}^2\Big)^{\frac{p}{2}}\Big)^{\frac{1}{p}}\Big\},$$

and

$$(\mathbb{E}\|x\|^p)^{\frac{1}{p}} \leq e(1+\sqrt{2})\alpha_{p,n} \max\Big\{ \max_{j=1,\dots,n} \Big(\sum_{i=1}^n \mathbb{E}x_{ij}^2\Big)^{\frac{1}{2}}, \|y\|^{\frac{1}{2}},$$
$$\alpha_{p,n}\Big(\mathbb{E} \max_{j=1,\dots n} \Big(\sum_{i=1}^n x_{ij}^2\Big)^{\frac{p}{2}}\Big)^{\frac{1}{p}}\Big\},$$

with $\alpha_{p,n} < \max\{2\sqrt{\log n}, 2\sqrt{2}\sqrt{p-1}\}$ *as in Lemma 3.19.*

Proof. Let $x_j = \sum_i x_{ij} \otimes e_{ij}$, i.e. the $n\times n$ matrix with j-th column equal to the j-th column of x and zeros elsewhere. Then (x_j) is a sequence of independent, mean zero random matrices and $x = \sum_j x_j$. Notice that

$$x_j^* x_j = \sum_{i,k} x_{ij}x_{kj} \otimes e_{ji}e_{kj} = \sum_i x_{ij}^2 \otimes e_{jj},$$

so

$$\Big\|\Big(\sum_j \mathbb{E}|x_j|^2\Big)^{\frac{1}{2}}\Big\| = \Big\|\sum_j \Big(\sum_i \mathbb{E}x_{ij}^2\Big)^{\frac{1}{2}} \otimes e_{jj}\Big\| = \max_{j=1,\dots n} \Big(\sum_i \mathbb{E}x_{ij}^2\Big)^{\frac{1}{2}}.$$

Moreover,

$$x_j x_j^* = \sum_{i,k} x_{ij}x_{kj} \otimes e_{ij}e_{jk} = \sum_{i,k} x_{ij}x_{kj} \otimes e_{ik}$$

and therefore

$$\Big\|\Big(\sum_j \mathbb{E}|x_j^*|^2\Big)^{\frac{1}{2}}\Big\| = \Big\|\Big(\sum_{i,k} \mathbb{E}\Big(\sum_{j=1}^n x_{ij}x_{kj}\Big) \otimes e_{ik}\Big)^{\frac{1}{2}}\Big\| = \|y\|^{\frac{1}{2}}.$$

Finally,

$$\Big(\mathbb{E} \max_{j=1,\dots,n} \|x_j\|^p\Big)^{\frac{1}{p}} = \Big(\mathbb{E} \max_{j=1,\dots,n} \|\ |x_j|\ \|^p\Big)^{\frac{1}{p}} = \Big(\mathbb{E} \max_{j=1,\dots n} \Big(\sum_{i=1}^n x_{ij}^2\Big)^{\frac{p}{2}}\Big)^{\frac{1}{p}}.$$

The result now follows from Lemma 3.19 (with $m = n$). □

Theorem 3.21. *Let* $2 \leq p < \infty$. *Suppose* x_{ij} *are independent, mean zero random variables in* $L^p(\Omega)$. *If* x *is the* $n \times n$ *random matrix* $(x_{ij})_{i,j=1}^n$, *then*

$$2^{1+\frac{1}{p}}(\mathbb{E}\|x\|^p)^{\frac{1}{p}} \geq \max\Big\{ \max_{j=1,\dots,n}\Big(\sum_{i=1}^n \mathbb{E}x_{ij}^2\Big)^{\frac{1}{2}}, \max_{i=1,\dots,n}\Big(\sum_{j=1}^n \mathbb{E}x_{ij}^2\Big)^{\frac{1}{2}}, \Big(\mathbb{E}\max_{i,j=1,\dots,n}|x_{ij}|^p\Big)^{\frac{1}{p}}\Big\},$$

and

$$(\mathbb{E}\|x\|^p)^{\frac{1}{p}} \leq e(1+\sqrt{2})\alpha_{p,n}\max\Big\{ \max_{j=1,\dots,n}\Big(\sum_{i=1}^n \mathbb{E}x_{ij}^2\Big)^{\frac{1}{2}}, \max_{i=1,\dots,n}\Big(\sum_{j=1}^n \mathbb{E}x_{ij}^2\Big)^{\frac{1}{2}}, e\alpha_{p,n}\Big(\mathbb{E}\max_{i,j=1,\dots,n}|x_{ij}|^p\Big)^{\frac{1}{p}}\Big\}, \tag{3.25}$$

with $\alpha_{p,n} < \max\{2\sqrt{\log n}, 2\sqrt{2}\sqrt{p-1}\}$ *as in Lemma 3.19.*

Proof. Set $y_{ij} = x_{ij} \otimes e_{ij}$, then $(y_{ij})_{i,j=1}^n$ is a doubly indexed sequence of independent, mean zero random matrices and $x = \sum_{i,j=1}^n y_{ij}$. Notice that

$$y_{ij}^* y_{ij} = x_{ij}^2 \otimes e_{ji}e_{ij} = x_{ij}^2 \otimes e_{jj},$$

so

$$\Big\|\Big(\sum_{i,j=1}^n \mathbb{E}|y_{ij}|^2\Big)^{\frac{1}{2}}\Big\| = \Big\|\sum_{j=1}^n\Big(\sum_{i=1}^n \mathbb{E}x_{ij}^2\Big)^{\frac{1}{2}} \otimes e_{jj}\Big\| = \max_{j=1,\dots,n}\Big(\sum_{i=1}^n \mathbb{E}x_{ij}^2\Big)^{\frac{1}{2}}. \tag{3.26}$$

Moreover,

$$y_{ij}y_{ij}^* = x_{ij}^2 \otimes e_{ij}e_{ji} = x_{ij}^2 \otimes e_{ii}$$

and therefore

$$\Big\|\Big(\sum_{i,j=1}^n \mathbb{E}|y_{ij}^*|^2\Big)^{\frac{1}{2}}\Big\| = \Big\|\sum_{i=1}^n\Big(\sum_{j=1}^n \mathbb{E}x_{ij}^2\Big)^{\frac{1}{2}} \otimes e_{ii}\Big\| = \max_{i=1,\dots,n}\Big(\sum_{j=1}^n \mathbb{E}x_{ij}^2\Big)^{\frac{1}{2}}. \tag{3.27}$$

Finally, it is clear that

$$\Big(\mathbb{E}\max_{i,j=1,\dots,n}\|y_{ij}\|^p\Big)^{\frac{1}{p}} = \Big(\mathbb{E}\max_{i,j=1,\dots,n}|x_{ij}|^p\Big)^{\frac{1}{p}}. \tag{3.28}$$

The result now follows from Lemma 3.19, applied with $m = n^2$. □

The constant $\alpha_{p,n}$ in (3.25) is of order $\sqrt{\log n}$ as $n \to \infty$. We shall now derive a bound similar to (3.25), in which the constant $\alpha_{p,n}$ is replaced by a constant of order $(\log n)^{\frac{1}{4}}$. We use the following result due to Y. Seginer (see [125], Theorems 3.1 and 3.2).

Theorem 3.22. *[125] If $(a_{ij})_{i,j=1}^n$ is a fixed matrix and (r_{ij}) is a doubly indexed Rademacher sequence, then there is a universal constant $C > 0$ such that*

$$(\mathbb{E}\|(r_{ij}a_{ij})\|^p)^{\frac{1}{p}} \leq C(\log n)^{\frac{1}{4}}\Big(\max_{i=1,\dots n}\Big(\sum_{j=1}^n x_{ij}^2\Big)^{\frac{1}{2}} + \max_{j=1,\dots n}\Big(\sum_{i=1}^n x_{ij}^2\Big)^{\frac{1}{2}}\Big),$$

provided $p \leq \log n^2$. *Moreover, the order of growth* $O((\log n)^{\frac{1}{4}})$ *is optimal.*

Let us recognize that the result above is actually a special case of the non-commutative Khintchine inequality (3.10), with a constant of improved order.

Corollary 3.23. *Let* $n \in \mathbb{N}$ *and set* $q = \log n$. *If* $(a_{ij})_{i,j=1}^n$ *is an* $n \times n$ *matrix and* (r_{ij}) *is a doubly indexed Rademacher sequence, then there is a universal constant* $C > 0$ *such that*

$$\Big(\mathbb{E}\Big\|\sum_{i,j=1}^n r_{ij}a_{ij}\otimes e_{ij}\Big\|_{S_n^q}^p\Big)^{\frac{1}{p}} \leq C(\log n)^{\frac{1}{4}}\max\Big\{\Big\|\Big(\sum_{i,j=1}^n |a_{ij}\otimes e_{ij}|^2\Big)^{\frac{1}{2}}\Big\|_{S_n^q}, \Big\|\Big(\sum_{i,j=1}^n |(a_{ij}\otimes e_{ij})^*|^2\Big)^{\frac{1}{2}}\Big\|_{S_n^q}\Big\}, \quad (3.29)$$

provided $p \leq \log n^2$. *Moreover, the order of growth* $O((\log n)^{\frac{1}{4}})$ *is optimal.*

Proof. By (3.5),

$$\begin{aligned}\max_{i=1,\dots n}\Big(\sum_{j=1}^n a_{ij}^2\Big)^{\frac{1}{2}} &= \Big\|\sum_{i=1}^n\Big(\sum_{j=1}^n a_{ij}^2\Big)^{\frac{1}{2}}\otimes e_{ii}\Big\| \\ &\leq \Big\|\Big(\sum_{i,j=1}^n a_{ij}^2\otimes e_{ii}\Big)^{\frac{1}{2}}\Big\|_{S_n^q} \\ &= \Big\|\Big(\sum_{i,j=1}^n |a_{ij}\otimes e_{ji}|^2\Big)^{\frac{1}{2}}\Big\|_{S_n^q} \\ &= \Big\|\Big(\sum_{i,j=1}^n |(a_{ij}\otimes e_{ij})^*|^2\Big)^{\frac{1}{2}}\Big\|_{S_n^q}.\end{aligned}$$

Similarly,

$$\max_{j=1,\dots n}\Big(\sum_{i=1}^n a_{ij}^2\Big)^{\frac{1}{2}} \leq \Big\|\Big(\sum_{i,j=1}^n |a_{ij}\otimes e_{ij}|^2\Big)^{\frac{1}{2}}\Big\|_{S_n^q}.$$

□

We can now improve the order of the constant in Theorem 3.21.

Theorem 3.24. *Let* $2 \leq p < \infty$. *Suppose* x_{ij} *are independent, mean zero random variables in* $L^p(\Omega)$. *If* x *is the* $n \times n$ *random matrix* $(x_{ij})_{i,j=1}^n$, *then*

$$(\mathbb{E}\|x\|^p)^{\frac{1}{p}} \leq C_p(\log n)^{\frac{1}{4}}\max\Big\{\max_{j=1,\dots,n}\Big(\sum_{i=1}^n \mathbb{E}x_{ij}^2\Big)^{\frac{1}{2}}, \max_{i=1,\dots,n}\Big(\sum_{j=1}^n \mathbb{E}x_{ij}^2\Big)^{\frac{1}{2}},$$

$$(\log n)^{\frac{1}{4}}\Big(\mathbb{E}\max_{i,j=1,\ldots,n}|x_{ij}|^p\Big)^{\frac{1}{p}}\Big\}, \tag{3.30}$$

where C_p is of order $\frac{p}{\log p}$ as $p\to\infty$.

Proof. Set $y_{ij}=x_{ij}\otimes e_{ij}$, then $(y_{ij})_{i,j=1}^n$ is a doubly indexed sequence of independent, mean zero S_n^q-valued random variables, with $q=\max\{\log n,2\}$ and $x=\sum_{i,j=1}^n y_{ij}$. To prove (3.30) we first apply Theorem 1.31 to obtain

$$(\mathbb{E}\|x\|^p)^{\frac{1}{p}}\lesssim\frac{p}{\log p}\max\{(\mathbb{E}\|x\|^2)^{\frac{1}{2}},(\mathbb{E}\max_{i,j=1,\ldots,n}\|y_{ij}\|^p)^{\frac{1}{p}}\}.$$

By replacing the use of the noncommutative Khintchine inequality (3.10) by (3.29) in the proof of Theorem 3.8, we see that

$$(\mathbb{E}\|x\|^2)^{\frac{1}{2}}\lesssim(\log n)^{\frac{1}{4}}\max\Big\{\Big\|\Big(\sum_{i,j=1}^n\mathbb{E}|y_{ij}|^2\Big)^{\frac{1}{2}}\Big\|_{S_n^q},\Big\|\Big(\sum_{i,j=1}^n\mathbb{E}|y_{ij}^*|^2\Big)^{\frac{1}{2}}\Big\|_{S_n^q},$$
$$(\log n)^{\frac{1}{4}}\Big(\mathbb{E}\Big(\sum_{i,j=1}^n\|y_{ij}\|_{S_n^q}^q\Big)^{\frac{2}{q}}\Big)^{\frac{1}{2}}\Big\}.$$

By (3.5) this implies that

$$(\mathbb{E}\|x\|^2)^{\frac{1}{2}}\lesssim(\log n)^{\frac{1}{4}}\max\Big\{\Big\|\Big(\sum_{i,j=1}^n\mathbb{E}|y_{ij}|^2\Big)^{\frac{1}{2}}\Big\|,\Big\|\Big(\sum_{i,j=1}^n\mathbb{E}|y_{ij}^*|^2\Big)^{\frac{1}{2}}\Big\|,$$
$$(\log n)^{\frac{1}{4}}\Big(\mathbb{E}\Big(\max_{i,j=1,\ldots,n}\|y_{ij}\|^2\Big)\Big)^{\frac{1}{2}}\Big\}$$
$$\leq(\log n)^{\frac{1}{4}}\max\Big\{\max_{j=1,\ldots,n}\Big(\sum_{i=1}^n\mathbb{E}x_{ij}^2\Big)^{\frac{1}{2}},\max_{i=1,\ldots,n}\Big(\sum_{j=1}^n\mathbb{E}x_{ij}^2\Big)^{\frac{1}{2}},$$
$$(\log n)^{\frac{1}{4}}\Big(\mathbb{E}\max_{i,j=1,\ldots,n}|x_{ij}|^p\Big)^{\frac{1}{p}}\Big\},$$

where the final step follows from (3.26),(3.27) and (3.28). □

Remark 3.25. If the entries x_{ij} in Theorem 3.24 are, in addition, identically distributed, then we can further improve the bound in (3.30). Indeed, by [125], Corollary 2.2 there is a constant $C>0$ such that for any $p\leq\log n^2$,

$$\mathbb{E}\|x\|^p\leq C\Big(\mathbb{E}\max_{i=1,\ldots n}\Big(\sum_{j=1}^n x_{ij}^2\Big)^{\frac{p}{2}}+\mathbb{E}\max_{j=1,\ldots n}\Big(\sum_{i=1}^n x_{ij}^2\Big)^{\frac{p}{2}}\Big).$$

If we use this inequality to replace (3.16) in the proof of Theorem 3.8 and follow the proof of Theorem 3.24, then we obtain

$$(\mathbb{E}\|x\|^p)^{\frac{1}{p}}\leq C_p\max\Big\{\max_{j=1,\ldots,n}\Big(\sum_{i=1}^n\mathbb{E}x_{ij}^2\Big)^{\frac{1}{2}},\max_{i=1,\ldots,n}\Big(\sum_{j=1}^n\mathbb{E}x_{ij}^2\Big)^{\frac{1}{2}},$$

$$(\log n)^{\frac{1}{4}}\Big(\mathbb{E}\max_{i,j=1,\dots,n}|x_{ij}|^p\Big)^{\frac{1}{p}}\Big\},$$

with C_p of order $\frac{p}{\log p}$ as $p\to\infty$. We leave the verification of the details to the interested reader.

The results in Theorems 3.21 and 3.24 give upper bounds for the L^p-norm of the largest singular value of an $n\times n$ random matrix with independent, mean zero entries, for any $2\le p<\infty$. However, these bounds are not of optimal order in terms of the dimensions of the matrix. In fact, using concentration inequalities for Gaussian random measures R. Latala proved in [93] that there is a universal constant $C>0$ such that

$$\mathbb{E}\|x\|\le C\Big(\max_{i=1,\dots,n}\Big(\sum_{j=1}^n\mathbb{E}x_{ij}^2\Big)^{\frac{1}{2}}+\max_{j=1,\dots,n}\Big(\sum_{i=1}^n\mathbb{E}x_{ij}^2\Big)^{\frac{1}{2}}+\Big(\sum_{i,j=1}^n\mathbb{E}x_{ij}^4\Big)^{\frac{1}{4}}\Big), \tag{3.31}$$

for any random matrix $x=(x_{ij})_{i,j=1}^n$ with independent, mean zero entries in $L^4(\Omega)$. For comparison, observe that (3.31) implies together with Theorem 1.31 that there is a universal constant $C>0$ such that for all $1\le p<\infty$,

$$\begin{aligned}(\mathbb{E}\|x\|^p)^{\frac{1}{p}}\le C\frac{p}{\log p}\Big(&\max_{i=1,\dots,n}\Big(\sum_{j=1}^n\mathbb{E}x_{ij}^2\Big)^{\frac{1}{2}}+\max_{j=1,\dots,n}\Big(\sum_{i=1}^n\mathbb{E}x_{ij}^2\Big)^{\frac{1}{2}}\\&+\Big(\sum_{i,j=1}^n\mathbb{E}x_{ij}^4\Big)^{\frac{1}{4}}+\Big(\mathbb{E}\max_{i,j=1,\dots n}|x_{ij}|^p\Big)^{\frac{1}{p}}\Big).\end{aligned}$$

The upper bound in Theorem 3.21 exhibits different growth behaviour in p and does not contain the factor $(\sum_{i,j=1}^n\mathbb{E}x_{ij}^4)^{\frac{1}{4}}$). In particular, the bound (3.25) is applicable to random matrices having entries with infinite fourth moment. On the other hand, we note that the bound in (3.31) is of order $\sqrt{n}$. Through our use of the noncommutative Khintchine inequality, we incur an extra factor of order $\sqrt{\log n}$ (or at least $(\log n)^{\frac{1}{4}}$). Unfortunately, as the order $(\log n)^{\frac{1}{4}}$ of the constant in (3.29) is optimal, this additional factor is an inevitable product of our method. Further investigation is needed to discover the 'right' bounds for the moments of the largest singular value of a random matrix.

Part II

Noncommutative Burkholder-Rosenthal inequalities

4

Quasi-Banach function spaces

In this preliminary chapter we introduce symmetric quasi-Banach function spaces and discuss their most important properties. These spaces play a pivotal role in many fields of mathematical analysis, especially probability theory, interpolation theory and harmonic analysis. Although our main interest is in symmetric Banach function spaces, we will find it necessary to work with the more general class of symmetric *quasi-Banach* function spaces (see in particular the proof of Theorem 7.4). The results presented in Sections 4.1, 4.2 and 4.3 below are all well known for Banach function spaces, but not easy to find for quasi-Banach function spaces.

Interpolation theory is a key tool in the analysis of symmetric spaces, which will be used intensively in the chapters to come. In Section 4.5 we review the basic definitions and collect several useful facts from this theory. As will become apparent in later chapters, an important problem is to determine if a particular symmetric quasi-Banach function space is an interpolation space for a couple of L^p-spaces. In the final section of this chapter we provide some sufficient conditions, formulated in terms of the convexity, concavity and Boyd indices of a symmetric space E, under which E has this property.

4.1 Basic definitions

Let us first recall some definitions and facts from the theory of quasi-Banach spaces. Let X be a vector space over $\mathbb{C}$ (or $\mathbb{R}$). A map $\|\cdot\| : X \to [0,\infty)$ is called a *quasi-norm* on X if it satisfies the following properties:

(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|\alpha x\| = |\alpha|\ \|x\|$, for all $x \in X$ and $\alpha \in \mathbb{C}$;
(iii) There is a constant $C \geq 1$ such that

$$\|x + y\| \leq C(\|x\| + \|y\|) \qquad (x, y \in X). \tag{4.1}$$

The inequality (4.1) is called the *quasi-triangle inequality.* If the map $\|\cdot\|$ satisfies

$$\|x+y\| \leq (\|x\|^p + \|y\|^p)^{\frac{1}{p}} \qquad (x, y \in X)$$

instead of (4.1), then $\|\cdot\|$ is called a *p-norm* on X. A basic, but most useful result is the following theorem due to T. Aoki and S. Rolewicz.

Theorem 4.1. *(Aoki-Rolewicz) Let X be a quasi-normed vector space. Then there is a $C > 0$ and $0 < p \leq 1$ such that for any $x_1, \ldots, x_n \in X$,*

$$\Big\|\sum_{i=1}^n x_i\Big\| \leq C\Big(\sum_{i=1}^n \|x_i\|^p\Big)^{\frac{1}{p}}.$$

By the Aoki-Rolewicz theorem, we can always equip a quasi-normed vector space with an equivalent p-norm, for a certain $0 < p \leq 1$.

If X is a quasi-normed vector space which is complete for the metric

$$d(x,y) = \|x-y\| \qquad (x, y \in X)$$

induced by the quasi-norm $\|\cdot\|$, then we call X a *quasi-Banach space.* The standard results from Banach space theory which depend only on the completeness of the space, such as the principle of uniform boundedness, the closed graph theorem and open mapping theorem, continue to be valid for quasi-Banach spaces. Results which rely on the convexity of the unit ball of a Banach space, such as the Hahn-Banach theorem, do not hold for quasi-Banach spaces. For example, if $0 < p < 1$, the Lebesgue space $L^p(0,1)$ is a quasi-Banach space. In sharp contrast to the L^p-spaces in the range $1 \leq p \leq \infty$, the space $L^p(0,1)$ has a *trivial* dual space if $0 < p < 1$. We refer to [78, 80] for a thorough treatment of the theory of quasi-Banach spaces. We shall be interested in quasi-Banach function spaces, which we now describe, and their noncommutative versions which will be introduced in Chapter 5 below.

Let $0 < \alpha \leq \infty$. For a measurable, a.e. finite function f on $(0, \alpha)$ we define its *distribution function* by

$$d(v; f) = \lambda(t \in (0,\alpha) \ : \ |f(t)| > v) \qquad (v > 0),$$

where λ denotes Lebesgue measure. Let $S(0, \alpha)$ denote the space of measurable, a.e. finite functions f on $(0, \alpha)$ such that $d(v; f) < \infty$ for some $v > 0$. For $f \in S(0, \alpha)$ we denote by $\mu(f)$ the *decreasing rearrangement* of f, defined by

$$\mu_t(f) = \inf\{v > 0 \ : \ d(v; f) \leq t\} \qquad (t \geq 0).$$

For $f, g \in S(0, \alpha)$ we say f is *submajorized* by g, and write $f \prec\prec g$, if

$$\int_0^t \mu_s(f)ds \leq \int_0^t \mu_s(g)ds, \qquad \text{for all } t > 0.$$

Recall the following terminology. A (quasi-)normed linear subspace E of $S(0, \alpha)$ is called a *(quasi-)Banach function space* on $(0, \alpha)$ if it is complete and if for $f \in S(0,\alpha)$ and $g \in E$ with $|f| \leq |g|$ we have $f \in E$ and $\|f\|_E \leq \|g\|_E$. A (quasi-)Banach function space E on $(0,\alpha)$ is called *symmetric* if for $f \in S(0,\alpha)$ and $g \in E$ with $\mu(f) \leq \mu(g)$ we have $f \in E$ and $\|f\|_E \leq \|g\|_E$. It is called *strongly symmetric* if, in addition, for $f, g \in E$ with $f \prec\prec g$ we have $\|f\|_E \leq \|g\|_E$. If, moreover, for $f \in S(0,\alpha)$ and $g \in E$ with $f \prec\prec g$ it follows that $f \in E$ and $\|f\|_E \leq \|g\|_E$, then E is called *fully symmetric*.

A symmetric (quasi-)Banach function space is said to have a *Fatou (quasi-)norm* if for every net (f_β) in E and $f \in E$ satisfying $0 \leq f_\beta \uparrow f$ we have $\|f_\beta\|_E \uparrow \|f\|_E$. The space E is said to have the *Fatou property* if for every net (f_β) in E and $f \in S(0,\alpha)$ satisfying $0 \leq f_\beta \uparrow$ and $\sup_\beta \|f_\beta\|_E < \infty$ the supremum $f = \sup_\beta f_\beta$ exists in E and $\|f_\beta\|_E \uparrow \|f\|_E$. We say that E has *order continuous* norm if for every net (f_β) in E such that $f_\beta \downarrow 0$ we have $\|f_\beta\|_E \downarrow 0$.

For further reference we record two elementary results on symmetric quasi-Banach function spaces. The first lemma is a consequence of Theorem 4.1 (see [103], Lemma 6).

Lemma 4.2. *Let E be a symmetric quasi-Banach function space. Then, for every $p > 0$ there exists a $c > 0$ and $0 < r \leq p$ such that for all $f_i \in E$,*

$$\Big\|\Big(\sum_{i=1}^{\infty} |f_i|^p\Big)^{\frac{1}{p}}\Big\|_E \leq c\Big(\sum_{i=1}^{\infty} \|f_i\|_E^r\Big)^{\frac{1}{r}}. \tag{4.2}$$

The following elementary Hölder-type inequality is well known for Banach function spaces ([96], Proposition 1.d.2 (i)).

Lemma 4.3. *Let E be a quasi-Banach function space and suppose $f, g \in E$. If $0 < \theta < 1$, then*

$$\|\, |f|^\theta |g|^{1-\theta}\|_E \leq C\|f\|_E^\theta \|g\|_E^{1-\theta},$$

where C is the constant in the quasi-triangle inequality.

Proof. We may assume $\|f\|_E > 0$, otherwise there is nothing to prove. Using that $x \mapsto \log x$ is concave on $(0,\infty)$, it follows that

$$s^\theta t^{1-\theta} \leq \theta s + (1-\theta)t, \quad s, t \in \mathbb{R}_+^2.$$

Applying this inequality pointwise it follows that for any $a > 0$ we have

$$\|\, |f|^\theta |g|^{1-\theta}\|_E = \|\, |a^{\frac{1}{\theta}} f|^\theta |a^{-\frac{1}{1-\theta}} g|^{1-\theta}\|_E \leq C(a^{\frac{1}{\theta}}\theta\|f\|_E + (1-\theta)a^{-\frac{1}{1-\theta}}\|g\|_E).$$

By setting $a = (\|g\|_E/\|f\|_E)^{\theta(1-\theta)}$ we obtain the desired inequality. $\square$

Let us finally discuss some results specific for symmetric Banach function spaces. The *Köthe dual* of a symmetric Banach function space E is the Banach function space $E^\times$ given by

$$E^\times = \Big\{ g \in S(0,\alpha) \ : \ \sup\Big\{ \int_0^\alpha |f(t)g(t)| \ dt \ : \ \|f\|_E \le 1 \Big\} < \infty \Big\};$$
$$\|g\|_{E^\times} = \sup\Big\{ \int_0^\alpha |f(t)g(t)| \ dt \ : \ \|f\|_E \le 1 \Big\}, \qquad g \in E^\times.$$

The space $E^\times$ is fully symmetric and has the Fatou property. It is isomorphic to a closed subspace of E^* via the map

$$g \mapsto L_g, \qquad L_g(f) = \int_0^\alpha f(t)g(t) \ dt \quad (f \in E).$$

A symmetric Banach function space on $(0,\alpha)$ has a Fatou norm if and only if E embeds isometrically into its second Köthe dual $E^{\times\times} = (E^\times)^\times$. It has the Fatou property if and only if $E = E^{\times\times}$ isometrically. It has order continuous norm if and only if it is separable, which is also equivalent to the statement $E^* = E^\times$. A symmetric Banach function space which has a Fatou norm is strongly symmetric. Moreover, a symmetric Banach function space which is separable or has the Fatou property is automatically fully symmetric. For proofs of these facts and more details we refer to [18, 85, 96, 142].

4.2 Boyd indices

We now discuss the *Boyd indices*, which were introduced by D.W. Boyd in [23]. Fix $0 < \alpha \le \infty$ and let E be a symmetric quasi-Banach function space on $(0,\alpha)$. For any $0 < a < \infty$ we define the dilation operator D_a on $S(0,\alpha)$ by

$$(D_a f)(s) = f(as)\chi_{(0,\alpha)}(as) \qquad (s \in (0,\alpha)).$$

The following lemma is well known for symmetric Banach function spaces (cf. [85]).

Lemma 4.4. *Let E be a symmetric quasi-Banach function space on $(0,\alpha)$. Then, for every $0 < a < \infty$, D_a defines a bounded linear operator on E. Moreover, $a \mapsto \|D_a\|$ is a decreasing, submultiplicative function on $(0,\infty)$.*

Proof. Since $\mu(f)$ is decreasing, we have for any $a \le b$,

$$D_b\mu(f)(s) = \mu_{bs}(f)\chi_{(0,\alpha)}(bs) \le \mu_{as}(f)\chi_{(0,\alpha)}(as) = D_a\mu(f)(s).$$

Hence, if D_a is bounded on E, then D_b is bounded on E as well and $\|D_b\| \le \|D_a\|$. In particular, $\|D_a\|$ is bounded on E if $a \ge 1$ and $\|D_a\| \le 1$. Moreover, it suffices to show that $D_{\frac{1}{n}}$ is bounded on E for every $n \in \mathbb{N}$.
Suppose first that $\alpha = \infty$. Fix $n \in \mathbb{N}$, let $f \in E_+$ and let f_i, $1 \le i \le n$, be mutually disjoint functions having the same distribution function as f. Then $D_{\frac{1}{n}} f$ and $\sum_{i=1}^n f_i$ have the same distribution function. Indeed,

$$\lambda(t \in (0,\infty) \ : \ (D_{\frac{1}{n}} f)(t) > v) = n\lambda(t \in (0,\infty) \ : \ f(t) > v)$$

$$= \sum_{i=1}^{n} \lambda(t \in (0,\infty) \ : \ f_i(t) > v)$$
$$= \lambda\Big(t \in (0,\infty) \ : \ \sum_{i=1}^{n} f_i(t) > v\Big). \tag{4.3}$$

Since E is symmetric it follows that $D_{\frac{1}{n}} f \in E$. Moreover, by Lemma 4.2, there exists some $c > 0$ and $0 < p \leq 1$ such that

$$\|D_{\frac{1}{n}} f\|_E = \Big\| \sum_{i=1}^{n} f_i \Big\|_E \leq c\Big(\sum_{i=1}^{n} \|f_i\|_E^p \Big)^{\frac{1}{p}} = cn^{\frac{1}{p}} \|f\|_E. \tag{4.4}$$

Suppose now that $0 < \alpha < \infty$. Observe that if $f \in E$ then $D_{\frac{1}{n}} f$ and $D_{\frac{1}{n}}(\chi_{(0,\frac{\alpha}{n})} f)$ have the same distribution function. Indeed,

$$\begin{aligned}\lambda(t \in (0,\alpha) \ : \ f(\frac{t}{n}) > v) &= \lambda(t \in (0,\alpha) \ : \ \chi_{(0,\frac{\alpha}{n})}(\frac{t}{n}) f(\frac{t}{n}) > v) \\ &= \lambda(t \in (0,\alpha) \ : \ D_{\frac{1}{n}}(\chi_{(0,\frac{\alpha}{n})} f)(t) > v). \end{aligned} \tag{4.5}$$

Therefore, by the argument given above,

$$\|D_{\frac{1}{n}} f\|_E = \|D_{\frac{1}{n}}(f\chi_{(0,\frac{\alpha}{n})})\|_E \leq cn^{\frac{1}{p}} \|f\chi_{(0,\frac{\alpha}{n})}\|_E \leq cn^{\frac{1}{p}} \|f\|_E.$$

We conclude that D_a is a bounded linear operator on E, for every $0 < a < \infty$.

From the above it is clear that $a \mapsto \|D_a\|$ is decreasing and, since $D_{ab} = D_a D_b$ if $a \leq b$, submultiplicative. □

Define the *lower Boyd index* p_E of E by

$$p_E = \sup\Big\{p > 0 : \ \exists c > 0 \ \forall 0 < a \leq 1 \ \|D_a f\|_E \leq ca^{-\frac{1}{p}} \|f\|_E\Big\}$$

and the *upper Boyd index* q_E of E by

$$q_E = \inf\Big\{q > 0 : \ \exists c > 0 \ \forall a \geq 1 \ \|D_a f\|_E \leq ca^{-\frac{1}{q}} \|f\|_E\Big\}.$$

It will be shown below that $0 < p_E \leq q_E \leq \infty$. Moreover, we will see in Lemma 4.9 that if E is a symmetric Banach function space then $1 \leq p_E \leq q_E \leq \infty$. In this case it is customary to say that E has *non-trivial Boyd indices* if $1 < p_E \leq q_E < \infty$. We will now deduce several different expressions for the Boyd indices.

The following lemma follows from a well-known property of submultiplicative functions, see [85], Theorem II.1.3.

Lemma 4.5. *If E is a symmetric quasi-Banach function space on $(0, \alpha)$, then*

$$\lim_{s\to\infty} \frac{\log s}{\log \|D_{\frac{1}{s}}\|} = \sup_{s>1} \frac{\log s}{\log \|D_{\frac{1}{s}}\|};$$
$$\lim_{s\downarrow 0} \frac{\log s}{\log \|D_{\frac{1}{s}}\|} = \inf_{0<s<1} \frac{\log s}{\log \|D_{\frac{1}{s}}\|}.$$

Proposition 4.6. *If E is a symmetric quasi-Banach function space on $(0,\alpha)$, then*

$$\begin{aligned} p_E &= \lim_{s\to\infty} \frac{\log s}{\log \|D_{\frac{1}{s}}\|} = \sup_{s>1} \frac{\log s}{\log \|D_{\frac{1}{s}}\|}; \\ q_E &= \lim_{s\downarrow 0} \frac{\log s}{\log \|D_{\frac{1}{s}}\|} = \inf_{0<s<1} \frac{\log s}{\log \|D_{\frac{1}{s}}\|}. \end{aligned} \tag{4.6}$$

Proof. Observe first that since $a \to \|D_a\|$ is decreasing, we have $\log \|D_a\| \geq 0$ if $0 < a \leq 1$ and $\log \|D_a\| \leq 0$ if $1 < a < \infty$. Set $\tilde{p}_E = \sup_{s>1} \frac{\log s}{\log \|D_{\frac{1}{s}}\|}$. Suppose first that $p_E, \tilde{p}_E < \infty$. For any $\varepsilon > 0$ there exists a $c_\varepsilon > 0$ such that for all $0 < a \leq 1$ we have $\|D_a\| \leq c_\varepsilon a^{-1/(p_E-\varepsilon)}$. Hence,

$$\log \|D_a\| \leq \frac{-1}{p_E - \varepsilon} \log a + c_\varepsilon$$

and so

$$p_E - \varepsilon \leq \frac{-\log a}{\log \|D_a\|} + c_\varepsilon \frac{-\log a}{\log \|D_a\|} \frac{1}{-\log a}(p_E - \varepsilon).$$

By taking the limit for $a \downarrow 0$ we obtain by Lemma 4.5 that $p_E - \varepsilon \leq \tilde{p}_E$. Since $\varepsilon > 0$ was arbitrary, we conclude that $p_E \leq \tilde{p}_E$.
On the other hand, for a given $\varepsilon > 0$ we can find $0 < a_* < 1$ such that for all $0 < a < a_*$ we have

$$\frac{\log a^{-1}}{\log \|D_a\|} > \tilde{p}_E - \varepsilon.$$

This implies

$$\log \|D_a\| < \log a^{-1/(\tilde{p}_E - \varepsilon)},$$

and so $\|D_a\| < a^{-1/(\tilde{p}_E-\varepsilon)}$ for any $0 < a < a_*$. Pick $m \in \mathbb{N}$ such that $2^{-m} < a_*$. Then for any $0 < a < 1$ we have

$$\begin{aligned} \|D_a f\|_E &= \|D_{2^m} D_{\frac{a}{2^m}} f\|_E \\ &\leq \|D_{2^m}\| (\frac{a}{2^m})^{-1/(\tilde{p}_E-\varepsilon)} \|f\|_E = \|D_{2^m}\| 2^{m/(\tilde{p}_E-\varepsilon)} a^{-1/(\tilde{p}_E-\varepsilon)} \|f\|_E. \end{aligned}$$

In other words, $\tilde{p}_E - \varepsilon \leq p_E$. Since this holds for any $\varepsilon > 0$ we get $\tilde{p}_E \leq p_E$. Similarly, one may show that $p_E = \infty$ if and only if $\tilde{p}_E = \infty$.

The proof of the equalities for q_E are similar and left to the interested reader. □

Corollary 4.7. *Let $0 < p \leq 1$. If E is a symmetric p-normed quasi-Banach function space on $(0,\alpha)$, then $p_E \geq p$. As a consequence, every symmetric quasi-Banach function space satisfies $p_E > 0$.*

Proof. It was observed in (4.4) that for every $n \geq 1$,

$$\|D_{\frac{1}{n}} f\|_E \leq n^{\frac{1}{p}} \|f\|_E$$

and therefore

$$\frac{\log \|D_{\frac{1}{n}}\|}{\log n} \leq \frac{1}{p}.$$

By taking the limit for $n \to \infty$ we see using Proposition 4.6 that $p_E \geq p$. Since every symmetric quasi-Banach function space can be equipped with an equivalent p-norm for some $0 < p \leq 1$ (cf. Theorem 4.1), the second assertion is an immediate consequence of the first statement. □

Remark 4.8. In many texts, the lower and upper Boyd indices of E are alternatively defined as the quantities

$$\underline{\alpha}_E = \sup_{0<s<1} \frac{\log \|D_{\frac{1}{s}}\|}{\log s}, \qquad \overline{\alpha}_E = \inf_{1<s<\infty} \frac{\log \|D_{\frac{1}{s}}\|}{\log s}.$$

It is clear that $\underline{\alpha}_E = \frac{1}{q_E}$ and $\overline{\alpha}_E = \frac{1}{p_E}$.

Finally, we recall the following duality for Boyd indices (see [85], Theorem II.4.11). If E is a symmetric Banach function space with Fatou norm, then

$$\frac{1}{p_E} + \frac{1}{q_{E^\times}} = 1, \qquad \frac{1}{p_{E^\times}} + \frac{1}{q_E} = 1. \tag{4.7}$$

4.3 Convexity and concavity

Let $0 < p, q \leq \infty$. A symmetric quasi-Banach function space E is said to be *p-convex* if there exists a constant $C > 0$ such that for any finite sequence $(f_i)_{i=1}^n$ in E we have

$$\Big\| \Big(\sum_{i=1}^n |f_i|^p \Big)^{\frac{1}{p}} \Big\|_E \leq C \Big(\sum_{i=1}^n \|f_i\|_E^p \Big)^{\frac{1}{p}} \qquad \text{(if } 0 < p < \infty\text{)},$$

or,

$$\Big\| \max_{1\leq i\leq n} |f_i| \Big\|_E \leq C \max_{1\leq i\leq n} \|f_i\|_E \qquad \text{(if } p = \infty\text{)}.$$

The least constant $M^{(p)}$ for which this inequality holds is called the *p-convexity constant* of E.
A symmetric quasi-Banach function space E is said to be *q-concave* if there exists a constant $C > 0$ such that for any finite sequence $(f_i)_{i=1}^n$ in E we have

$$\Big(\sum_{i=1}^n \|f_i\|_E^q \Big)^{\frac{1}{q}} \leq C \Big\| \Big(\sum_{i=1}^n |f_i|^q \Big)^{\frac{1}{q}} \Big\|_E \qquad \text{(if } 0 < q < \infty\text{)},$$

or,

$$\max_{1\leq i\leq n} \|f_i\|_E \leq C \Big\| \max_{1\leq i\leq n} |f_i| \Big\|_E \qquad \text{(if } q = \infty\text{)}.$$

The least constant $M_{(q)}$ for which this inequality holds is called the *q-concavity constant* of E. It is clear that every quasi-Banach function space is ∞-concave with $M_{(\infty)} = 1$ and any Banach function space is 1-convex with $M^{(1)} = 1$.

For $1 \leq r < \infty$, let the *r-concavification* and *r-convexification* of E be defined by

$$E_{(r)} := \{g \in S(0,\alpha) : \ |g|^{\frac{1}{r}} \in E\}, \ \|g\|_{E_{(r)}} = \| \ |g|^{\frac{1}{r}} \|_E^r,$$
$$E^{(r)} := \{g \in S(0,\alpha) : \ |g|^r \in E\}, \ \|g\|_{E^{(r)}} = \| \ |g|^r \|_E^{\frac{1}{r}},$$

respectively. As is shown in [96] (p. 53), if E is a Banach function space, then $E^{(r)}$ is a Banach function space. In general, $E_{(r)}$ is only a quasi-Banach function space. Using that $\mu(|f|^s) = \mu(f)^s$ for any $f \in S(0,\alpha)$ and $0 < s < \infty$, one sees that $E^{(r)}$ and $E_{(r)}$ are symmetric if E is symmetric. From the definitions one easily shows that if E is p-convex and q-concave for $0 < p \leq q \leq \infty$, then $E^{(r)}$ is pr-convex and qr-concave and $E_{(r)}$ is $\frac{p}{r}$-convex and $\frac{q}{r}$-concave. It is also follows from the definitions that

$$p_{E_{(r)}} = \frac{1}{r} p_E, \ q_{E_{(r)}} = \frac{1}{r} q_E, \ p_{E^{(r)}} = r p_E, \ q_{E^{(r)}} = r q_E.$$

The following lemma states a relationship between the convexity and concavity of a space and its Boyd indices. For symmetric Banach function spaces this result is classical (see e.g. [96]).

Lemma 4.9. *Let $0 \leq p, q \leq \infty$ and suppose E is a symmetric quasi-Banach function space on $(0,\alpha)$. If E is q-concave, then $q_E \leq q$. On the other hand, if E is p-convex, then $p_E \geq p$.*

Proof. Suppose first that E is a q-concave and $\alpha = \infty$. Fix $n \in \mathbb{N}$, let $f \in E$ and let f_i, $1 \leq i \leq n$, be mutually disjoint functions with the same distribution function as f. As observed in (4.3), $D_{\frac{1}{n}} f$ and $\sum_{i=1}^n f_i$ have the same distribution function. Since E is q-concave,

$$\begin{aligned}\|D_{\frac{1}{n}} f\|_E = \Big\| \sum_{i=1}^n |f_i| \Big\|_E &= \Big\| \Big(\sum_{i=1}^n |f_i|^q \Big)^{\frac{1}{q}} \Big\|_E \\ &\geq M_{(q)} \Big(\sum_{i=1}^n \|f_i\|_E^q \Big)^{\frac{1}{q}} = M_{(q)} n^{\frac{1}{q}} \|f\|_E.\end{aligned}$$

Letting $f = D_n g$ and using $D_{\frac{1}{n}} D_n g = g$, we obtain

$$\|D_n g\| \leq M_{(q)}^{-1} n^{-\frac{1}{q}} \|g\|_E.$$

Since this holds for all $g \in E$ we obtain $\log \|D_n\| \leq -\frac{1}{q} \log n - \log M_{(q)}$. Hence, for any $n \in \mathbb{N}$ we have

$$\frac{\log \|D_n\|}{\log n^{-1}} \geq \frac{1}{q} - \frac{\log M_{(q)}}{\log n^{-1}}.$$

Taking the limit for $n \to \infty$ we obtain by Proposition 4.6 that $\frac{1}{q_E} \geq \frac{1}{q}$ or $q_E \leq q$.
Suppose now that E is q-concave and $\alpha < \infty$. Notice that the norm of $D_{\frac{1}{n}}$ can be computed using only functions supported on $(0, \frac{\alpha}{n})$. Indeed, if $f \in E$ then by (4.5) $D_{\frac{1}{n}} f$ and $D_{\frac{1}{n}}(\chi_{(0,\frac{\alpha}{n})} f)$ have the same distribution function. Hence,

$$\frac{\|D_{\frac{1}{n}} f\|_E}{\|f\|_E} \leq \frac{\|D_{\frac{1}{n}} f\|_E}{\| \, |f|\chi_{(0,\frac{\alpha}{n})}\|_E} = \frac{\|D_{\frac{1}{n}}(|f|\chi_{(0,\frac{\alpha}{n})})\|_E}{\| \, |f|\chi_{(0,\frac{\alpha}{n})}\|_E}$$

and so

$$\|D_{\frac{1}{n}}\| = \sup_{\operatorname{supp}(f) \subset (0,\frac{\alpha}{n})} \frac{\|D_{\frac{1}{n}} f\|_E}{\|f\|_E}.$$

The argument above now yields $\|D_{\frac{1}{n}}\| \geq C n^{\frac{1}{q}}$, which completes the proof of the first assertion.

The proof of the second assertion is similar. □

The following observation was originally made by J.L. Krivine for Banach lattices [86]. We shall use the following extension for quasi-Banach function spaces.

Proposition 4.10. *Let E be a quasi-Banach function space. If E is p-convex for some $0 < p \leq \infty$, then E is r-convex for any $0 < r \leq p$ and $M^{(r)} \leq M^{(p)}$. If E is q-concave for some $0 < q < \infty$, then E is s-concave for any $q \leq s \leq \infty$ and $M_{(s)} \leq M_{(q)}$.*

Proof. Notice that for any $0 < p < \infty$,

$$\begin{aligned} M^{(p)} &= \sup\Big\{\Big\|\Big(\sum_{i=1}^n |f_i|^p\Big)^{\frac{1}{p}}\Big\|_E;\ f_i \in E,\ \sum_{i=1}^n \|f_i\|^p = 1\Big\} \\ &= \sup\Big\{\Big\|\Big(\sum_{i=1}^n a_i|g_i|^p\Big)^{\frac{1}{p}}\Big\|_E;\ g_i \in E, \|g_i\|_E \leq 1,\ a_i \geq 0,\ \sum_{i=1}^n a_i = 1\Big\}. \end{aligned}$$

Fix $g_1, \ldots, g_n \in E$, $\|g_i\|_E \leq 1$ and $a_1, \ldots, a_n \geq 0$ with $\sum_{i=1}^n a_i = 1$. Let Ω be the probability space consisting of n atoms, indexed by $i = 1, \ldots, n$, with weights $a_1, \ldots, a_n$. Let $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ and let $h : \Omega \to \mathbb{C}$ be given by $h(i) = \alpha_i$. If $0 < r \leq p$, then by Hölder's inequality,

$$\Big(\sum_{i=1}^n a_i|\alpha_i|^r\Big)^{\frac{1}{r}} = \|h\|_{L^r(\Omega)} \leq \|h\|_{L^p(\Omega)} = \Big(\sum_{i=1}^n a_i|\alpha_i|^p\Big)^{\frac{1}{p}}.$$

Since $\|\cdot\|_E$ respects the order on E, we obtain

$$\Big\|\Big(\sum_{i=1}^n a_i|g_i|^r\Big)^{\frac{1}{r}}\Big\|_E \leq \Big\|\Big(\sum_{i=1}^n a_i|g_i|^p\Big)^{\frac{1}{p}}\Big\|_E \leq M^{(p)}.$$

Taking the supremum over all $a_1, \ldots, a_n$ and $g_1, \ldots, g_n$ yields the first statement. The modification of the proof for the case $p = \infty$ is obvious.

For the second statement we note that for any $0 < p < \infty$,

$$\frac{1}{M_{(p)}} = \inf\Big\{\Big\|\Big(\sum_{i=1}^n |f_i|^p\Big)^{\frac{1}{p}}\Big\|_E;\ f_i \in E,\ \sum_{i=1}^n \|f_i\|^p = 1\Big\}$$
$$= \inf\Big\{\Big\|\Big(\sum_{i=1}^n a_i|g_i|^p\Big)^{\frac{1}{p}}\Big\|_E;\ g_i \in E, \|g_i\|_E \le 1,\ a_i \ge 0,\ \sum_{i=1}^n a_i = 1\Big\}.$$

Let $0 < q \le s \le \infty$. By the above, we have for arbitrary $g_1, \ldots, g_n \in E$ with $\|g_i\|_E \le 1$ and $a_1, \ldots, a_n \ge 0$ satisfying $\sum_{i=1}^n a_i = 1$,

$$\frac{1}{M_{(q)}} \le \Big\|\Big(\sum_{i=1}^n a_i|g_i|^q\Big)^{\frac{1}{q}}\Big\|_E \le \Big\|\Big(\sum_{i=1}^n a_i|g_i|^s\Big)^{\frac{1}{s}}\Big\|_E.$$

By taking the infimum we obtain $M_{(s)} \le M_{(q)}$, as desired. □

Remark 4.11. By using the functional calculus for Banach lattices (see e.g. [124]) one easily deduces from the proof that Proposition 4.10 holds for any quasi-Banach lattice.

Lemma 4.12. *If E is a symmetric (quasi-)Banach function space which is q-concave for some $q < \infty$, then E has order continuous (quasi-)norm.*

Proof. By [2], Theorems 10.1 and 10.3, it suffices to show that if $(f_k)_{k=1}^\infty$ is a disjoint sequence in E with $0 \le f_k \le f$ for some $f \in E$ and all $k \ge 1$, then $\|f_k\|_E \to 0$. Since E is q-concave, we have for any $n \ge 1$,

$$\Big(\sum_{k=1}^n \|f_k\|_E^q\Big)^{\frac{1}{q}} \le M_{(q)}(E)\Big\|\Big(\sum_{k=1}^n |f_k|^q\Big)^{\frac{1}{q}}\Big\|_E = \Big\|\sum_{k=1}^n |f_k|\Big\|_E \le \|f\|_E.$$

Thus,

$$\Big(\sum_{k=1}^\infty \|f_k\|_E^q\Big)^{\frac{1}{q}} \le \|f\|_E < \infty$$

and the assertion follows. □

Finally we mention the following duality result due to J.L. Krivine ([86], théorème 6, see also [96], Proposition 1.d.4), which holds in fact for general Banach lattices.

Theorem 4.13. *Let E be a Banach function space and suppose $1 \le p, p', q, q' \le \infty$ are such that $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. If E is p-convex then E^* is p'-concave and $M_{(p')}(E^*) = M^{(p)}(E)$. On the other hand, if E is q-concave then E^* is q'-convex and $M^{(q')}(E^*) = M_{(q)}(E)$.*

4.4 Examples

Let us now look at some interesting classes of symmetric quasi-Banach function spaces.

Example 4.14. (Lorentz spaces $L^{p,q}$) Let $0 < p, q \leq \infty$. The *Lorentz space* $L^{p,q}(0,\alpha)$ is the subspace of all f in $S(0,\alpha)$ such that

$$\|f\|_{L^{p,q}} = \begin{cases} (\int_0^\alpha t^{\frac{q}{p}-1}\mu_t(f)^q\ dt)^{\frac{1}{q}} & (0 < q < \infty), \\ \sup_{0<t<\alpha} t^{\frac{1}{p}}\mu_t(f) & (q = \infty), \end{cases}$$

is finite. If $1 \leq q \leq p < \infty$ or $p = q = \infty$, then $L^{p,q}$ is a fully symmetric Banach function space. If $1 < p < \infty$ and $p \leq q$ then $L^{p,q}$ can be equivalently renormed to become a fully symmetric Banach function space ([18], Theorem 4.6). However, in general $L^{p,q}$ is only a symmetric quasi-Banach function space [77]. By the monotone convergence theorem, $L^{p,q}$ has the Fatou property. Its Boyd indices are determined by the first exponent, $p_{L^{p,q}} = q_{L^{p,q}} = p$.

The Lorentz space $L^{p,p}$ coincides with the Lebesgue space L^p. The spaces $L^{p,\infty}$ are referred to as *weak L^p-spaces*. Observe that the weak L^p-spaces are not q-concave for any $q < \infty$. Indeed, if we define $f_n(t) = t^{-\frac{1}{p}}\chi_{[0,\frac{1}{n})}(t)$, then $\|f_n\|_{L^{p,\infty}} = 1$ for all n, but $f_n \downarrow 0$. Thus the assertion follows from Lemma 4.12.

Example 4.15. (Lorentz spaces $\Lambda^{p,w}$) Let $0 < p < \infty$ and let w be a positive decreasing function on $(0,\alpha)$ such that $\int_0^\infty w(t)\ dt = \infty$ if $\alpha = \infty$. The *Lorentz space* $\Lambda^{p,w}$ is the subspace of all f in $S(0,\alpha)$ such that

$$\|f\|_{\Lambda^{p,w}} = \Big(\int_0^\alpha \mu_t(f)^p w(t)\ dt\Big)^{\frac{1}{p}}$$

is finite. Note that if $p \leq q$ and $w(t) = t^{\frac{p}{q}-1}$ we have $\Lambda^{p,w} = L^{q,p}$. If $1 \leq p < \infty$ then $L^{p,w}$ is a symmetric Banach function space with the Fatou property. It is p-convex with convexity constant equal to 1, but is not r-convex for any $r > p$. One can characterize the concavity of the space in terms of p and w (see [82], Theorem 7).

Example 4.16. (Orlicz spaces) Let $\Phi : [0,\infty) \to [0,\infty]$ be a Young's function, i.e., a convex, continuous and increasing function satisfying $\Phi(0) = 0$ and $\lim_{t\to\infty}\Phi(t) = \infty$. The *Orlicz space* $L_\Phi(0,\alpha)$ is the subspace of all f in $S(0,\alpha)$ such that for some $k > 0$,

$$\int_0^\infty \Phi\Big(\frac{|f(t)|}{k}\Big)dt < \infty.$$

If we equip L_Φ with the Luxemburg norm

$$\|f\|_{L_\Phi} = \inf\Big\{k > 0\ :\ \int_0^\infty \Phi\Big(\frac{|f(t)|}{k}\Big)dt \leq 1\Big\},$$

then L_Φ is a symmetric Banach function space with the Fatou property [18, 96]. If we weaken the convexity condition on Φ then we obtain examples of symmetric quasi-Banach function spaces [75]. The Boyd indices of L_Φ can be computed in terms of Φ, see e.g. [18], Theorem 8.18. A wealth of information on Orlicz spaces can be found in the monograph [118].

We refer to [18, 19, 85, 96] for many more concrete examples of quasi-Banach function spaces.

4.5 Fundamentals of interpolation theory

In this section we gather all the fundamental results we need from interpolation theory. Let (X_0, X_1) be a *compatible couple of (quasi-)Banach spaces* (or, stated more briefly, *couple* of quasi-Banach spaces), i.e. X_0, X_1 are continuously embedded in some Hausdorff topological vector space. Then the intersection $X_0 \cap X_1$ and the sum $X_0 + X_1$ are (quasi-)Banach spaces under the (quasi-)norms

$$\|f\|_{X_0 \cap X_1} = \max\{\|f\|_{X_0}, \|f\|_{X_1}\}$$

and

$$\|f\|_{X_0+X_1} = \inf\{\|f_0\|_{X_0} + \|f_1\|_{X_1} : f = f_0 + f_1,\ f_0 \in X_0,\ f_1 \in X_1\}.$$

A quasi-Banach space X is called an *intermediate space* for the couple (X_0, X_1) if

$$X_0 \cap X_1 \subset X \subset X_0 + X_1,$$

with continuous inclusions. Let $((X_0, X_1), (Y_0, Y_1))$ be an ordered pair consisting of two couples of quasi-Banach spaces. A linear map $T : X_0+X_1 \to Y_0+Y_1$ is called *admissible* for the pair $((X_0, X_1), (Y_0, Y_1))$ if $T|_{X_i}$ is a bounded linear operator from X_i into Y_i for $i = 0, 1$. An admissible map T is called a *contraction* for the pair $((X_0, X_1), (Y_0, Y_1))$ if $T|_{X_i}$ is a contraction for $i = 0, 1$. If X and Y are intermediate spaces for (X_0, X_1) and (Y_0, Y_1), respectively, then (X, Y) is called an *interpolation pair* for the pair $((X_0, X_1), (Y_0, Y_1))$ if every admissible operator T maps X into Y. It is then automatically bounded on X (c.f. the proof of [18], Proposition 1.11) and

$$\|T\|_{X \to Y} \leq C \max\{\|T\|_{X_0 \to X_1}, \|T\|_{Y_0 \to Y_1}\}, \tag{4.8}$$

for a certain constant $C \geq 1$, called the *interpolation constant*, which depends only on the spaces involved. The pair (X, Y) is called an *interpolation pair of exponent* $0 \leq \theta \leq 1$ if, moreover,

$$\|T\|_{X \to Y} \leq C \|T\|^{\theta}_{X_0 \to X_1} \|T\|^{1-\theta}_{Y_0 \to Y_1}. \tag{4.9}$$

If $C = 1$ in (4.8) (respectively, in (4.9)), then (X, Y) is called an *exact interpolation pair* (respectively, *exact interpolation pair of exponent* θ) for the pair $((X_0, X_1), (Y_0, Y_1))$. In the special case where $X = Y$, $X_0 = Y_0$ and $X_1 = Y_1$, we call X an *(exact) interpolation space* for the couple (X_0, X_1) if (4.8) holds (with $C = 1$) or an *(exact) interpolation space of exponent* θ if (4.9) holds (with $C = 1$).

The *K-functional* for a couple (X_0, X_1) is defined for each $t > 0$ and $f \in X_0 + X_1$ by

$$K(t, f; X_0, X_1) = \inf\{\|f_0\|_{X_0} + t\|f_1\|_{X_1} : f = f_0 + f_1,\ f_0 \in X_0,\ f_1 \in X_1\}.$$

We say that an interpolation space X for the couple (X_0, X_1) is *given by a K-method* if there exists a (quasi-)Banach function space Y on $(0, \infty)$ such that $f \in X$ if and only if $K(t, f; X_0, X_1) \in Y$ for all $t > 0$ and there exist constants $c, C > 0$ such that

$$c\|t \mapsto K(t, f; X_0, X_1)\|_Y \le \|f\|_X \le C\|t \mapsto K(t, f; X_0, X_1)\|_Y.$$

The *Gagliardo completion* $\overline{X_0}$ of X_0 is defined as the space of all $f \in X_0 + X_1$ for which $K(t, f; X_0, X_1)$ is bounded, which is a (quasi-)Banach space under the norm

$$\|f\|_{\overline{X_0}} = \sup_{0<t<\infty} K(t, f; X_0, X_1) = \lim_{t\to\infty} K(t, f; X_0, X_1).$$

The Gagliardo completion $\overline{X_1}$ is defined analogously as the space of all $f \in X_0 + X_1$ such that $K(t, f; X_1, X_0)$ is bounded, which is a (quasi-)Banach space under the norm

$$\|f\|_{\overline{X_1}} = \sup_{0<t<\infty} K(t, f; X_1, X_0) = \lim_{t\to\infty} K(t, f; X_1, X_0).$$

Using the identity

$$t^{-1}K(t, f; X_0, X_1) = K(t^{-1}, f; X_1, X_0) \qquad (t > 0)$$

we obtain the alternative expression

$$\|f\|_{\overline{X_1}} = \sup_{0<t<\infty} t^{-1}K(t, f; X_0, X_1) = \lim_{t\downarrow 0} t^{-1}K(t, f; X_0, X_1),$$

where the last equality follows as $t \mapsto t^{-1}K(t, f; X_0, X_1)$ is decreasing.

A couple (X_0, X_1) is called a *Gagliardo couple* (or *Gagliardo complete*) if $X_0 = \overline{X_0}$ and $X_1 = \overline{X_1}$. One can show ([18], Theorem V.1.4) that (X_0, X_1) is Gagliardo complete precisely when the unit balls of X_0 and X_1 are closed in the topology of $X_0 + X_1$.

An interpolation space X for a couple of quasi-Banach spaces (X_0, X_1) is called *K-monotone* if there exist a constant $C > 0$ such that if $f \in X_0 + X_1$ and $g \in X$ satisfy

$$K(t, f; X_0, X_1) \leq K(t, g; X_0, X_1) \quad (t > 0),$$

then $f \in X$ and $\|f\|_X \leq C\|g\|_X$. A couple (X_0, X_1) is called a *Calderón couple* if every interpolation space for (X_0, X_1) is K-monotone.

A couple (X_0, X_1) is said to be *divisible* if there is a constant $c > 0$ such that, whenever $f \in X_0 + X_1$ and ω_j, $j \geq 1$, are nonnegative concave functions on $(0, \infty)$ satisfying $\sum_{j\geq 1} \omega_j(1) < \infty$ and

$$K(t, f; X_0, X_1) \leq \sum_{j\geq 1} \omega_j(t) \qquad (t > 0),$$

there exist elements $f_j \in X_0 + X_1$ for which $f = \sum_{j\geq 1} f_j$ in $X_0 + X_1$ and

$$K(t, f_j; X_0, X_1) \leq c\omega_j(t) \qquad (j \geq 1,\ t > 0).$$

For a proof of the following theorem, due to J. Brudnyi and N. Krugljak we refer to [18], Theorem 5.3.6. One easily checks that this proof remains valid for a couple of quasi-Banach spaces.

Theorem 4.17. *(Brudnyi-Krugljak) Every Gagliardo couple is divisible.*

In the proof of Theorem 4.19 we use the following lemma.

Lemma 4.18. *Let $0 < p \leq 1$ and let E be a p-normed quasi-Banach function space on $(0, \alpha)$. Then the following are equivalent:*

(i) E is complete;
(ii) For any sequence (f_n) in E_+ satisfying $\sum_{n=1}^\infty \|f_n\|_E^p < \infty$ the element $\sum_{n=1}^\infty f_n$ exists in E and $\|\sum_{n=1}^\infty f_n\|_E \leq (\sum_{n=1}^\infty \|f_n\|_E^p)^{\frac{1}{p}}$.

Proof. $(i) \Rightarrow (ii)$: If $\sum_{n=1}^\infty \|f_n\|_E^p < \infty$ then the sequence $(\sum_{n=1}^m f_n)_{m\geq 1}$ is Cauchy in E and hence converges in norm to an element $f \in E$. Since $(\sum_{n=1}^m f_n)_{m\geq 1}$ is increasing, we must have $f = \sum_{n=1}^\infty f_n$.

$(ii) \Rightarrow (i)$: It suffices to show that if (f_n) is a sequence in E then $\sum_{n=1}^\infty f_n$ converges in E whenever $\sum_{n=1}^\infty \|f_n\|_E^p < \infty$. Clearly we may assume $f_n \geq 0$ for all n. By (ii), $g_m = \sum_{n=m}^\infty f_n$ exists in E for any $m \geq 1$. Moreover,

$$\Big\|g_1 - \sum_{n=1}^{m-1} f_n\Big\|_E = \|g_m\|_E \leq \Big(\sum_{n=m}^\infty \|f_n\|^p\Big)^{\frac{1}{p}},$$

so $\sum_{n=1}^\infty f_n$ converges in E. □

A proof of the following result, also due to J. Brudnyi and N. Krugljak, for Banach spaces may be found in [79], Theorem 6.3. For the convenience of the reader we provide a detailed proof for a couple of quasi-Banach spaces.

Theorem 4.19. *(Brudnyi-Krugljak) If (X_0, X_1) is a Gagliardo couple, then every K-monotone interpolation space X for (X_0, X_1) is given by a K-method.*

Proof. By Lemma 4.2, we may assume X is p-normed for some $0 < p \leq 1$ by passing to an equivalent norm. For $g \in S(0,\infty)$, define

$$\|g\|_Y = \inf\Big\{\Big(\sum_{i\geq 1}\|f_i\|_X^p\Big)^{\frac{1}{p}} : \ |g(t)| \leq \sum_{i\geq 1} K(t,f_i;X_0,X_1), \ t>0\Big\}$$

and set

$$Y = \{g \in S(0,\infty) : \ \|g\|_Y < \infty\}.$$

We will show that Y defines a p-normed quasi-Banach function space on $(0,\infty)$. Let $g_1, g_2 \in Y$. Then, for every $\varepsilon > 0$ there exist sequences $(f_i^1), (f_i^2)$ in X such that for $j = 1,2$,

$$|g_j(t)| \leq \sum_{i\geq 1} K(t,f_i^j;X_0,X_1) \qquad (t>0),$$

and $\|g_j\|_Y^p \leq \sum_{i\geq 1}\|f_i^j\|_X^p + \varepsilon$. Since

$$|g_1(t) + g_2(t)| \leq \sum_{j=1,2}\sum_{i\geq 1} K(t,f_i^j;X_0,X_1) \qquad (t>0)$$

we find that $g_1 + g_2 \in Y$ and

$$\|g_1+g_2\|_Y^p \leq \|g_1\|_Y^p + \|g_2\|_Y^p + 2\varepsilon,$$

for every $\varepsilon > 0$. Hence Y is p-normed. We will show that Y is complete by means of Lemma 4.18. Let (g_n) be a sequence in Y_+ such that $\sum_{n=1}^\infty \|g_n\|_Y^p < \infty$. Pick sequences (f_j^n) in X such that

$$\sum_{j=1}^\infty \|f_j^n\|_X^p < \|g_n\|_Y^p + \frac{\varepsilon}{2^n}.$$

Note that $g = \sum_{n=1}^\infty g_n$ exists in $S(0,\infty)$ and, moreover,

$$g(t) = \sum_{n=1}^\infty g_n(t) \leq \sum_{n=1}^\infty\sum_{j=1}^\infty K(t,f_j^n)$$

and also

$$\sum_{n=1}^\infty\sum_{j=1}^\infty \|f_j^n\|_X^p < \sum_{n=1}^\infty \|g_n\|_Y^p + \varepsilon.$$

This implies $g \in Y$ and, since $\varepsilon > 0$ was arbitrary,

$$\|g\|_Y^p \leq \sum_{n=1}^\infty \|g_n\|_Y^p.$$

By Lemma 4.18 we conclude that Y is complete.

We claim that there are constants $c, C > 0$ such that

$$c\|t \mapsto K(t, f; X_0, X_1)\|_Y \leq \|f\|_X \leq C\|t \mapsto K(t, f; X_0, X_1)\|_Y.$$

Clearly, if $f \in X$, then $K(\cdot, f) \in Y$ and $\|K(\cdot, f)\|_Y \leq \|f\|_X$. Suppose now that $f \in X_0 + X_1$ and $K(\cdot, f) \in Y$. Let $\varepsilon > 0$. Then there is a sequence (f_j) in X such that

$$K(t, f) \leq \sum_{j=1}^{\infty} K(t, f_j) \qquad (t > 0)$$

and

$$\sum_{j=1}^{\infty} \|f_j\|_X^p \leq (1 + \varepsilon)\|t \mapsto K(t, f)\|_Y^p.$$

Since (X_0, X_1) is divisible (c.f. Theorem 4.17), we can find a decomposition $f = \sum_{j=1}^{\infty} h_j$ in $X_0 + X_1$ such that $K(t, h_j) \leq CK(t, f_j)$, where C is a constant which does not depend on f. Since X is K-monotone, this implies that $h_j \in X$ for all j and $\|h_j\|_X \leq C_X\|f_j\|_X$, for some constant C_X depending only on X. By completeness this implies $f = \sum_{j=1}^{\infty} h_j$ is in X and

$$\|f\|_X^p \leq \sum_{j=1}^{\infty} \|h_j\|_X^p \leq C_X^p \sum_{j=1}^{\infty} \|f_j\|_X^p \leq C_X^p(1 + \varepsilon)\|t \mapsto K(t, f)\|_Y^p.$$

Hence $\|f\|_X \leq C_X\|K(\cdot, f)\|_Y$, as asserted. □

4.6 Interpolation spaces for couples of L^p-spaces

The purpose of this section is to give sufficient conditions for a symmetric quasi-Banach function space E to be an interpolation space for a couple of L^p-spaces. Throughout, we let $0 < \alpha \leq \infty$.

Let us first recall the celebrated Calderón-Mitjagin Theorem, which gives an exact description of the interpolation spaces for the couple (L^1, L^∞). For a proof see e.g. [85], Theorem II.4.3.

Theorem 4.20. *(Calderón-Mitjagin) Let $T : L^1 + L^\infty(0, \alpha) \to S(0, \alpha)$ be a linear operator. Then T is a contraction for the couple (L^1, L^∞) if and only if*

$$Tf \prec\prec f \qquad (f \in L^1 + L^\infty).$$

Moreover, if $f \in L^1 + L^\infty(0, \alpha)$ and $g \in S(0, \alpha)$ are such that $g \prec\prec f$, then there is a contraction T for the couple (L^1, L^∞) such that $Tf = g$.

Consequently, a symmetric quasi-Banach function space E on $(0, \alpha)$ is an exact interpolation space for the couple (L^1, L^∞) if and only if E is fully symmetric.

In the remainder of this section we provide four different sufficient conditions, formulated in terms of the convexity, concavity and Boyd indices of E, which ensure that E is an interpolation space for a couple of L^p-spaces. The main results are summarized in Theorem 4.31 below. This theorem was announced in, [79], Theorem 7.3, but a complete proof has not been published. We are grateful to S. Montgomery-Smith for providing a sketch of the proof [102].

As a first step, we will establish that if a (quasi-)Banach function space is an interpolation space for a couple of L^p-spaces, then all its concavifications and convexifications are interpolation spaces for a (different) couple of L^p-spaces. We shall use the following result, due to G. Sparr [127].

Theorem 4.21. *(Sparr) If (S, Σ, μ) is a σ-finite measure space, then the couple $(L^p(S), L^q(S))$ is a Calderón couple for any $0 < p, q \leq \infty$.*

In fact, the above theorem even holds for a couple of weighted L^p-spaces with different weights.

Theorems 4.19 and 4.21 together imply the following result.

Corollary 4.22. *Let (S, Σ, μ) be a σ-finite measure space and let $0 < p, q \leq \infty$. Then every interpolation space for the couple $(L^p(S), L^q(S))$ is given by a K-method.*

Proof. By Theorems 4.19 and 4.21, it remains to show that (L^p, L^q) is Gagliardo complete. Let (f_n) be a sequence in the unit ball of L^p and suppose that $f_n \to f$ in $L^p + L^q$. Then in particular, $f_n \to f$ in measure. Since L^p has the Fatou property, its unit ball is closed for the measure topology and therefore $f \in L^p$ and $\|f\|_{L^p} \leq 1$. By [18], Theorem 5.1.4, we conclude that (L^p, L^q) is Gagliardo complete. $\square$

We are now ready to prove the announced observation.

Proposition 4.23. *(S. Montgomery-Smith) Let $1 \leq p \leq q \leq \infty$. Suppose E is a symmetric quasi-Banach function space on $(0, \alpha)$ which is an interpolation space for the couple $(L^p(0, \alpha), L^q(0, \alpha))$. Then, for any $1 \leq s < \infty$, $E_{(s)}$ (respectively, $E^{(s)}$) is an interpolation space for the couple $(L^{\frac{p}{s}}(0, \alpha), L^{\frac{q}{s}}(0, \alpha))$ (respectively, $(L^{ps}(0, \alpha), L^{qs}(0, \alpha))$).*

Proof. By Corollary 4.22, there exists a quasi-Banach function space F such that

$$\|f\|_E \simeq_E \|t \mapsto K(t, f; L^p, L^q)\|_F.$$

Observe that

$$K(t, f; L^p, L^q) = \inf_{f_0 + f_1 \geq f} (\|f_0\|_{L^p} + t\|f_1\|_{L^q}). \tag{4.10}$$

Indeed, if $f_0 + f_1 \geq f$, then by the Riesz decomposition property (c.f. [3], Theorem 1.9) there exist g_0, g_1 such that $|g_0| \leq |f_0|$, $|g_1| \leq |f_1|$ and $f = g_0 + g_1$.

Using (4.10) and the Riesz decomposition property one also easily sees that $K(t,f;L^p,L^q) = K(t,|f|;L^p,L^q)$.

For any $a,b \geq 0$ we have $\alpha_s(a^s+b^s) \leq (a+b)^s \leq \beta_s(a^s+b^s)$, for some constants α_s, β_s depending only on s. Let $f \in E_{(s)}$ and pick $f_0 \in L^p, f_1 \in L^q$ such that $f_0 + f_1 \geq |f|^{\frac{1}{s}}$. Then $\beta_s(|f_0|^s + |f_1|^s) \geq |f|$ and so

$$\begin{aligned}
\|f_0\|_{L^p} + t\|f_1\|_{L^q} &= \beta_s^{-\frac{1}{s}}(\|\beta_s|f_0|^s\|_{L^{\frac{p}{s}}}^{\frac{1}{s}} + t\|\beta_s|f_1|^s\|_{L^{\frac{q}{s}}}^{\frac{1}{s}}) \\
&\geq \alpha_{\frac{1}{s}}^{-1}\beta_s^{-\frac{1}{s}}(\|\beta_s|f_0|^s\|_{L^{\frac{p}{s}}} + t^s\|\beta_s|f_1|^s\|_{L^{\frac{q}{s}}})^{\frac{1}{s}} \\
&\geq \alpha_{\frac{1}{s}}^{-1}\beta_s^{-\frac{1}{s}} \inf_{g_0+g_1\geq|f|}(\|g_0\|_{L^{\frac{p}{s}}} + t^s\|g_1\|_{L^{\frac{q}{s}}})^{\frac{1}{s}} \\
&= \alpha_{\frac{1}{s}}^{-1}\beta_s^{-\frac{1}{s}} K(t^s, f; L^{\frac{p}{s}}, L^{\frac{q}{s}})^{\frac{1}{s}}.
\end{aligned}$$

Hence,

$$K(t,|f|^{\frac{1}{s}};L^p,L^q) \geq \alpha_{\frac{1}{s}}^{-1}\beta_s^{-\frac{1}{s}} K(t^s,f;L^{\frac{p}{s}},L^{\frac{q}{s}})^{\frac{1}{s}}$$

and similarly we find that

$$K(t^s,f;L^{\frac{p}{s}},L^{\frac{q}{s}})^{\frac{1}{s}} \gtrsim_s K(t,|f|^{\frac{1}{s}};L^p,L^q).$$

Hence,

$$\begin{aligned}
\|f\|_{E_{(s)}} &= \|\,|f|^{\frac{1}{s}}\|_E^s \\
&\simeq_E \|t \mapsto K(t,|f|^{\frac{1}{s}};L^p,L^q)\|_F \simeq_s \|t \mapsto K(t^s,f;L^{\frac{p}{s}},L^{\frac{q}{s}})^{\frac{1}{s}}\|_F.
\end{aligned}$$

If T is an admissible linear operator for the couple $(L^{\frac{p}{s}}, L^{\frac{q}{s}})$, then

$$\begin{aligned}
K(t,Tf;L^{\frac{p}{s}},L^{\frac{q}{s}}) &= \inf_{f_0+f_1=Tf}(\|f_0\|_{L^{\frac{p}{s}}} + t\|f_1\|_{L^{\frac{q}{s}}}) \\
&\leq \inf_{g_0+g_1=f}(\|Tg_0\|_{L^{\frac{p}{s}}} + t\|Tg_1\|_{L^{\frac{q}{s}}}) \\
&\leq \max\{\|T\|_{\frac{p}{s}}, \|T\|_{\frac{q}{s}}\} \inf_{g_0+g_1=f}(\|g_0\|_{L^{\frac{p}{s}}} + t\|g_1\|_{L^{\frac{q}{s}}}) \\
&= \max\{\|T\|_{\frac{p}{s}}, \|T\|_{\frac{q}{s}}\} K(t,f;L^{\frac{p}{s}},L^{\frac{q}{s}}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|Tf\|_{E_{(s)}} &\simeq_{E,s} \|t \mapsto K(t^s,Tf;L^{\frac{p}{s}},L^{\frac{q}{s}})^{\frac{1}{s}}\|_F \\
&\lesssim \|t \mapsto K(t^s,f;L^{\frac{p}{s}},L^{\frac{q}{s}})^{\frac{1}{s}}\|_F \simeq_s \|f\|_{E_{(s)}}.
\end{aligned}$$

□

We now state the first sufficient condition for a space to be an interpolation space for a couple of L^p-spaces. This result was proved for rearrangement invariant symmetric spaces by D.W. Boyd [23] and later extended to symmetric quasi-Banach function spaces in [103], Theorem 3. We will present a different proof in Section 5.2 below, which also applies for noncommutative symmetric spaces.

Theorem 4.24. *(Boyd's theorem) Let* $0 < p < q \leq \infty$ *and let* E *be a symmetric quasi-Banach function space on* $(0,\alpha)$ *with* $p < p_E \leq q_E < q$*. Then* E *is an interpolation space for the couple* $(L^p(0,\alpha), L^q(0,\alpha))$.

In the proof of Theorem 4.26 we use the following notion.

Lemma 4.25. *Let* E *be a separable Banach function space on* $(0,\alpha)$ *and let* $T : E \to E$ *be a bounded linear operator. Then there exists a unique bounded linear operator* $T : E^\times \to E^\times$ *such that for all* $f \in E$, $g \in E^\times$,

$$\int_0^\alpha f(s)(T^\times g)(s)ds = \int_0^\alpha (Tf)(s)g(s)ds.$$

We call the operator $T^\times$ the *associated operator* to T. The existence and uniqueness of $T^\times$ follows from the existence and uniqueness of T^* and the isometric identification $E^* = E^\times$. Note that $\|T^\times\| = \|T^*\| = \|T\|$. The following result appears implicitly in the proof of Theorem 1 in [8].

Theorem 4.26. *Let* $1 \leq p < q \leq \infty$ *and* $\frac{1}{p'} + \frac{1}{p} = 1$, $\frac{1}{q'} + \frac{1}{q} = 1$*. Let* E *be a separable Banach function space on* $(0,\alpha)$*. If* $E^\times$ *is an interpolation space for the couple* $(L^{q'}(0,\alpha), L^{p'}(0,\alpha))$*, then* E *is an interpolation space for the couple* $(L^p(0,\alpha), L^q(0,\alpha))$.

Proof. Let T be an admissible operator for the couple (L^p, L^q). We claim that it suffices to show that $T|_E : E \to E^{\times\times}$ is bounded. Indeed, let $f \in E$ and let (f_n) be a sequence in $L^1 \cap L^\infty$ such that $f_n \to f$ in E. Then $Tf_n \in L^p \cap L^q \subset E$ and the claim implies that $Tf_n \to Tf$ in $E^{\times\times}$. Since E isometrically embeds into $E^{\times\times}$ and (Tf_n) is Cauchy in $E^{\times\times}$, we find that (Tf_n) is Cauchy in E as well. This implies that $Tf \in E$ and $Tf_n \to Tf$ in E.

We now prove our claim. Let T_p, T_q be the restrictions of T to L^p and L^q, respectively, and let $T_p^\times, T_q^\times$ be the corresponding associated operators. Suppose that $f \in L^p \cap L^q$ and $g \in L^{p'} \cap L^{q'}$. Then,

$$\int_0^\alpha (Tf)(s)g(s)ds = \int_0^\alpha f(s)(T_p^\times g)(s)ds = \int_0^\alpha f(s)(T_q^\times g)(s)ds.$$

By density of $L^p \cap L^q$ in both L^p and L^q, we see that $T_p^\times$ and $T_q^\times$ coincide on $L^{p'} \cap L^{q'}$. In fact, we can extend both operators by density to a bounded linear operator $T^\times : L^{p'} + L^{q'} \to L^{p'} + L^{q'}$. Since $T^\times$ is bounded on $L^{p'}$ and $L^{q'}$, we obtain by assumption that $T^\times$ is also bounded on $E^\times$.
By Lemma 4.25 there exist bounded linear operators $T_{p'}^{\times\times}, T_{q'}^{\times\times}$ associated to $T_p^\times, T_q^\times$. For $f \in L^p \cap L^q$ and $g \in L^{p'} \cap L^{q'}$ we have

$$\int_0^\alpha f(s)(T^\times g)(s)ds = \int_0^\alpha (T_{p'}^{\times\times} f)(s)g(s)ds = \int_0^\alpha (T_{q'}^{\times\times} f)(s)g(s)ds.$$

We find that $T_{p'}^{\times\times}, T_{q'}^{\times\times}$ coincide on $L^p \cap L^q$ and hence can be uniquely extended to a bounded linear operator $T^{\times\times}$ on $L^p + L^q$. In fact, if $f \in L^p \cap L^q$ and $g \in L^{p'} \cap L^{q'}$, then

$$\int_0^\alpha (T^{\times\times} f)(s)g(s)ds = \int_0^\alpha f(s)(T^\times g)(s)ds = \int_0^\alpha (Tf)(s)g(s)ds,$$

so in fact $T^{\times\times} = T$. If $h \in L^1 \cap L^\infty$, then

$$\begin{aligned}\|Th\|_{E^{\times\times}} &= \sup_{\|g\|_{E^\times} \le 1} \left| \int_0^\alpha (Th)(s)g(s)ds \right| \\ &= \sup_{\|g\|_{E^\times} \le 1} \left| \int_0^\alpha h(s)(T^\times g)(s)ds \right| \le \|T^\times\|_{E^\times \to E^\times} \|h\|_E,\end{aligned}$$

which proves our claim. □

Theorem 4.27. *Let $0 < p < q \le \infty$ and suppose that E is a symmetric quasi-Banach function space on $(0, \alpha)$ which is either separable or has the Fatou property. If E is p-convex with convexity constant equal to 1 and q-concave, then E is an interpolation space for the couple $(L^p(0, \alpha), L^q(0, \alpha))$.*

Proof. Suppose first that $p \ge 1, q = \infty$. In this case $E_{(p)}$ is a symmetric Banach function space which is either separable or has the Fatou property and is therefore fully symmetric. By Theorem 4.20 it follows that $E_{(p)}$ is an exact interpolation space for the couple (L^1, L^∞). By Proposition 4.23 we conclude that E is an interpolation space for the couple (L^p, L^∞).

Suppose now that $p = 1, q < \infty$. By Lemma 4.12 we see that E is a separable symmetric Banach function space and hence, by Theorem 4.13, $E^\times = E^*$ is q'-convex and ∞-concave. By renorming $E^\times$ if necessary, we may assume that the q'-convexity constant of $E^\times$ is equal to 1. By the above, $E^\times$ is an interpolation space for the couple $(L^{q'}, L^\infty)$. The result now follows by Theorem 4.26.

Next assume that $1 < p < q < \infty$. Then $E_{(p)}$ is a symmetric Banach function space which is $\frac{q}{p}$-concave and therefore separable by Lemma 4.12. Therefore, $E_{(p)}$ is an interpolation space for the couple $(L^1, L^{\frac{q}{p}})$. By now applying Proposition 4.23 we obtain the result.

Finally, if $0 < p < 1$ and $p < q \le \infty$, then $E^{(\frac{1}{p})}$ is a symmetric Banach function space which is either separable or has the Fatou property. Moreover, $E^{(\frac{1}{p})}$ is $\frac{q}{p}$-concave. By the above we conclude that $E^{(\frac{1}{p})}$ is an interpolation space for the couple $(L^1, L^{\frac{q}{p}})$. The result now follows by Proposition 4.23. □

Theorem 4.28. *Let $0 < p < q \le \infty$ and suppose that E is a symmetric quasi-Banach function space on $(0, \alpha)$ which is either separable or has the Fatou property. If E is p-convex with convexity constant equal to 1 and $q_E < q$, then E is an interpolation space for the couple $(L^p(0, \alpha), L^q(0, \alpha))$.*

Proof. The case $p = 1$ is proved in [7], Theorem 1. If $p > 1$, then $E_{(p)}$ is a symmetric Banach function space which is either separable or has the Fatou property. Moreover, $q_{E_{(p)}} < \frac{q}{p}$, and therefore E is an interpolation space for the couple $(L^1, L^{\frac{q}{p}})$. By Proposition 4.23, this implies that E is an interpolation space for the couple (L^p, L^q). Finally, if $0 < p < 1$, then $E^{(\frac{1}{p})}$ is a symmetric Banach function space which is either separable or has the Fatou property. Moreover, $q_{E^{(\frac{1}{p})}} < \frac{q}{p}$, and therefore E is an interpolation space for the couple $(L^1, L^{\frac{q}{p}})$. By Proposition 4.23, this implies that E is an interpolation space for the couple (L^p, L^q). □

Lemma 4.29. *Let $0 < p < \infty$. If E is a symmetric quasi-Banach function space on $(0, \alpha)$ with $p_E > p$, then E is an interpolation space for the couple $(L^p(0, \alpha), L^\infty(0, \alpha))$.*

Proof. Suppose first that $p_E > 1$. We claim that E is fully symmetric up to a constant, i.e. there is a constant $c_E > 0$ depending only on E, such that if $f \in S(0, \alpha)$, $g \in E$ and $f \prec\prec g$, then $f \in E$ and $\|f\|_E \le c_E \|g\|_E$.

Let $g^{**}(t) = \frac{1}{t} \int_0^t \mu_s(g) ds$ be the Hardy-Littlewood maximal function of g. By [103], Theorem 2i), the map $g \mapsto g^{**}$ is a bounded quasi-linear map on E and therefore $g^{**} \in E$ and $\|g^{**}\|_E \le c_E \|g\|_E$. By assumption $f^{**} \le g^{**}$, so $f^{**} \in E$ and $\|f^{**}\|_E \le \|g^{**}\|_E$, as E is symmetric. Finally, $\mu(f) \le f^{**}$, so $f \in E$ and $\|f\|_E = \|\mu(f)\|_E \le \|f^{**}\|_E$. This proves our claim.

Now let T be a contraction for the couple $(L^1(0, \alpha), L^\infty(0, \alpha))$. Then, by Theorem 4.20,

$$Tf \prec\prec f \qquad (f \in L^1 + L^\infty(0, \alpha)).$$

By our claim we obtain $\|Tf\|_E \lesssim_E \|f\|_E$ for all $f \in E$. This proves the case $p = 1$.

Suppose now that $p_E > p$. If $p > 1$ then $p_{E_{(p)}} > 1$ and by the above $E_{(p)}$ is an interpolation space for the couple (L^1, L^∞). On the other hand, if $p < 1$ then $E^{(\frac{1}{p})}$ is an interpolation space for the couple (L^1, L^∞). The result now follows by Proposition 4.23. □

Theorem 4.30. *Let $0 < p < q \le \infty$ and suppose that E is a symmetric quasi-Banach function space on $(0, \alpha)$ which is r-convex with convexity constant equal to 1, for some $0 < r < \infty$. If E is q-concave with concavity constant equal to 1 and $p_E > p$, then E is an interpolation space for the couple $(L^p(0, \alpha), L^q(0, \alpha))$.*

Proof. The case where $q = \infty$ is proved in Lemma 4.29. For the remaining cases we may assume, by Proposition 4.23, that $r = 1$. Under this assumption, E is a symmetric Banach function space and hence we can deduce the result by duality. Since $q < \infty$ it follows by Lemma 4.12 that E is separable. Moreover, by Theorem 4.13 the Köthe dual $E^\times = E^*$ is q'-convex with convexity constant equal to 1 and by (4.7) we have $q_{E^\times} < p'$, where $\frac{1}{p'} + \frac{1}{p} = 1$, $\frac{1}{q'} + \frac{1}{q} = 1$. By Theorem 4.28 we obtain that $E^\times$ is an interpolation space for the couple $(L^{q'}, L^{p'})$. The result now follows from Theorem 4.26. □

We now summarize the main results of this section.

Theorem 4.31. *[79] Let E be a symmetric quasi-Banach function space on $(0,\alpha)$ which is either separable or has the Fatou property and let $0 < p < q \le \infty$. Then E is an interpolation space for the couple $(L^p(0,\alpha), L^q(0,\alpha))$ whenever one of the following conditions holds:*

(i) $p < p_E \le q_E < q$;
(ii) *E is p-convex with convexity constant equal to 1 and $q_E < q$;*
(iii) *E is r-convex with convexity constant equal to 1, for some $0 < r < \infty$, E is q-concave with concavity constant equal to 1 and $p_E > p$;*
(iv) *E is p-convex with convexity constant equal to 1 and q-concave.*

Note that the conditions on the p-convexity and q-concavity constants in (ii)-(iv) are redundant if E is a symmetric Banach function space and $1 \le p, q < \infty$. Indeed, in this case it is well known that one can renorm the space to obtain a symmetric Banach function space with p-convexity and q-concavity constant equal to 1.

5

Noncommutative symmetric spaces

Every symmetric quasi-Banach function space on the positive real line satisfying a mild convexity condition induces a noncommutative function space of operators associated with a von Neumann algebra, called the associated *noncommutative quasi-Banach function space*. These noncommutative function spaces are the principal examples of *noncommutative symmetric spaces*. In the first section of this chapter we give a brief introduction to the basic properties and interpolation theory of these spaces. In the second section we present a new, direct proof of a noncommutative version of the celebrated Boyd interpolation theorem. In the third section we give a rigorous treatment of Hilbert-space valued noncommutative symmetric spaces. The special case where the Hilbert space is l^2 gives rise to the row and column spaces, which play a prominent role in the noncommutative version of Khintchine's inequalities. In the final chapter on noncommutative stochastic integration theory we will naturally encounter L^2-valued noncommutative symmetric spaces. The final section gives a brief treatment of conditional versions of the row and column spaces. These conditional sequence spaces will appear in our formulation of Rosenthal-type inequalities in noncommutative symmetric spaces.

5.1 Definition and basic properties

Let us first recall the terminology introduced in Section 3.1. Fix $0 < \alpha \leq \infty$. Let $\mathcal{M}$ be a semi-finite von Neumann algebra acting on a complex Hilbert space H, which is equipped with a normal, semi-finite, faithful trace τ satisfying $\tau(\mathbf{1}) = \alpha$. The *distribution function* of a closed, densely defined operator x on H, which is affiliated with $\mathcal{M}$, is given by

$$d(v; x) = \tau(e^{|x|}(v, \infty)) \qquad (v \geq 0),$$

where $e^{|x|}$ is the spectral measure of $|x|$. The *decreasing rearrangement* or *generalized singular value function* of x is defined by

$$\mu_t(x) = \inf\{v > 0 \ : \ d(v; x) \le t\} \qquad (t \ge 0).$$

We say that x is *τ-measurable* if $d(v;x) < \infty$ for some $v > 0$. We let $S(\tau)$ be the linear space of all τ-measurable operators, which is a metrizable, complete topological $*$-algebra with respect to the measure topology. We denote by $S_0(\tau)$ the linear subspace of all $x \in S(\tau)$ such that $d(v;x) < \infty$ for all $v > 0$. Moreover, let $\mathcal{F}(\tau)$ be the linear subspace of all $x \in \mathcal{M}$ with $\tau(s(x)) < \infty$, where $s(x)$ is the support projection of x. One can introduce a partial order on the linear subspace $S(\tau)_h$ of all self-adjoint operators in $S(\tau)$ by setting, for a self-adjoint operator x,

$$x \ge 0 \text{ if and only if } \langle x\xi, \xi\rangle_H \ge 0 \text{ for all } \xi \in D(x),$$

where $D(x)$ is the domain of x in H. We write $x \le y$ for $x, y \in S(\tau)_h$ if and only if $y - x \ge 0$. Under this partial ordering $S(\tau)_h$ is a partially ordered vector space. Let $S(\tau)_+$ denote the positive cone of all $x \in S(\tau)_h$ satisfying $x \ge 0$. It can be shown that $S(\tau)_+$ is closed with respect to the measure topology ([49], Proposition 1.4).

Throughout our exposition, we will tacitly use many properties of distribution functions and decreasing rearrangements. For the convenience of the reader we collect these facts in the following two propositions. The first result is essentially contained in the proof of [107], Theorem 1.

Proposition 5.1. *If $x, y \in S(\tau)$ and p is a projection in $\mathcal{M}$, then:*

(a) $d(v; x^*) = d(v; x)$ for all $v \ge 0$;
(b) $d(v; x) = d(v; \mu(x))$ for all $v \ge 0$;
(c) $d(v + w; x + y) \le d(v; x) + d(w; y)$ for all $v, w \ge 0$;
(d) $d(v; xp) \le \tau(p)$ for all $v \ge 0$;
(e) if $|x| \le |y|$ then $d(v; x) \le d(v; y)$ for all $v \ge 0$.

The following properties of decreasing rearrangements can be found in [53]. If p is a projection in $\mathcal{M}$, then we let $p^\perp := \mathbf{1} - p$ denote its orthogonal complement.

Proposition 5.2. *If $x, y \in S(\tau)$ and p is a projection in $\mathcal{M}$, then:*

(a) $\mu_t(\lambda x) = |\lambda|\mu_t(x)$ for all $\lambda \in \mathbb{C}$ and $t \ge 0$;
(b) $\mu_t(x^*) = \mu_t(x)$ for all $t \ge 0$;
(c) $\mu_{s+t}(x + y) \le \mu_s(x) + \mu_t(y)$ for all $s, t \ge 0$;
(d) $\mu_t(xp) = 0$ for all $t \ge \tau(p)$;
(e) if $|x| \le |y|$ then $\mu_t(x) \le \mu_t(y)$ for all $t \ge 0$;
(f) $\mu_t(uxv) \le \|u\| \ \mu_t(x) \ \|y\|$ for all $u, v \in \mathcal{M}$ and $t \ge 0$;
(g) $\mu_t(x^*x) = \mu_t(xx^*)$ for all $t \ge 0$.
(h) If $x_\alpha \uparrow x$ in $S(\tau)_+$, then $\mu_t(x_\alpha) \uparrow \mu_t(x)$ for all $t \ge 0$.

If $e = e^{|x|}(v, \infty)$, then

(i) $\mu_t(|x|e) = \mu_t(x)\chi_{[0,\tau(e))}(t)$ for all $t \ge 0$

(j) $\mu_t(|x|e^{\perp}) = \mu_{t+\tau(e)}(x)$ *for all* $t \geq 0$*, provided* $\tau(e) < \infty$.

Finally, suppose that $\phi : [0,\infty) \to [0,\infty)$ *is left-continuous on* $(0,\infty)$ *and satisfies* $\phi(0) = 0$*. If we define* $\phi(\infty) := \lim_{t\to\infty} \phi(t)$*, then*

(k) $\mu(\phi(|x|)) = \phi(\mu(x))$ *on* $[0,\infty)$.

For a symmetric (quasi-)Banach function space E on $(0,\alpha)$, we define

$$E(\mathcal{M},\tau) := \{x \in S(\tau) : \ \|\mu(x)\|_E < \infty\}.$$

We usually denote $E(\mathcal{M},\tau)$ by $E(\mathcal{M})$ for brevity. The following fundamental result is proved in [81], Theorem 8.11 (see also [49, 141] for earlier proofs of this result under additional assumptions).

Theorem 5.3. *If E is a symmetric (quasi-)Banach function space E on* $(0,\alpha)$ *which is p-convex for some* $0 < p < \infty$*, then* $E(\mathcal{M})$ *defines a p-convex (quasi-)Banach space under the (quasi-)norm* $\|x\|_{E(\mathcal{M})} := \|\mu(x)\|_E$*. The space* $E(\mathcal{M})$ *is continuously embedded in* $S(\tau)$ *with respect to the measure topology.*

We call $E(\mathcal{M})$ the *noncommutative (quasi-)Banach function space* associated with E and $\mathcal{M}$. Using the construction above, we obtain noncommutative versions of many important spaces in analysis, such as L^p-spaces, weak L^p-spaces, Lorentz spaces and Orlicz spaces. In particular, taking $E = L^p$ yields the noncommutative L^p-spaces introduced earlier in Chapter 3.

It is possible to define *noncommutative symmetric spaces* of measurable operators by analogy with the classical definition presented in Chapter 4. We refer to [49] for a detailed exposition of this approach. Although we shall restrict ourselves to the special class of noncommutative symmetric spaces which are induced by a classical symmetric space, we implicitly use some results from the general framework. Most importantly, we frequently use the following fact, which is established in [49], Theorem 5.6 and p. 745.

Theorem 5.4. *If E is a symmetric Banach function space E with order continuous norm, then* $E(\mathcal{M})^* = E^{\times}(\mathcal{M})$*. The associated duality bracket is given by*

$$\langle x, y\rangle = \tau(xy) \qquad (x \in E(\mathcal{M}),\ y \in E^{\times}(\mathcal{M})).$$

From this result it is possible to deduce that $\mathcal{F}(\tau)$ is norm dense in $E(\mathcal{M})$ if E has order continuous norm.

A natural question to ask is whether the interpolation results presented for symmetric quasi-Banach function spaces in Chapter 4 also hold for their noncommutative counterparts. The following theorem states that for *fully symmetric* Banach function spaces, one can always 'lift' interpolation results for commutative function spaces to their noncommutative versions. For the proof see [48], Theorem 3.4.

Theorem 5.5. *Let* $\mathcal{M},\mathcal{N}$ *be von Neumann algebras equipped with normal, semi-finite faithful traces* τ *and* σ*, respectively, satisfying* $\tau(\mathbf{1}) = \sigma(\mathbf{1}) = \alpha$.

Suppose that E_0, E_1, F_0, F_1 are fully symmetric Banach function spaces and that E, F are symmetric quasi-Banach function spaces on $(0, \alpha)$ which are p-convex for some $0 < p < \infty$. If (E, F) is an (exact) interpolation pair for the pair $((E_0, E_1), (F_0, F_1))$, then $(E(\mathcal{M}), F(\mathcal{N}))$ is an (exact) interpolation pair for the pair

$$((E_0(\mathcal{M}), E_1(\mathcal{M})), (F_0(\mathcal{N}), F_1(\mathcal{N}))).$$

Moreover, if (E, F) is an (exact) interpolation pair of exponent $0 \leq \theta \leq 1$, then $(E(\mathcal{M}), F(\mathcal{N}))$ is an (exact) interpolation pair of exponent θ.

Usually we will only need the following special case.

Theorem 5.6. *Fix $1 \leq p < q \leq \infty$. Suppose E is a symmetric quasi-Banach function space on $(0, \alpha)$ which is r-convex for some $0 < r < \infty$. Let $\mathcal{M}$ be a semi-finite von Neumann algebra equipped with a normal, semi-finite, faithful trace τ satisfying $\tau(\mathbf{1}) = \alpha$. If E is an interpolation space for the couple $(L^p(0, \alpha), L^q(0, \alpha))$, then $E(\mathcal{M})$ is an interpolation space for the couple $(L^p(\mathcal{M}), L^q(\mathcal{M}))$.*

For more details on measurable operators we refer to [50, 53, 107] and for the theory of noncommutative symmetric spaces to [32, 47, 48, 49, 50, 81, 129].

5.2 Noncommutative Boyd interpolation theorem

From the classical Boyd interpolation theorem and Theorem 5.6 one can deduce the following noncommutative version of the Boyd interpolation theorem.

Theorem 5.7. *Fix $1 \leq p < q \leq \infty$. Suppose E is a symmetric quasi-Banach function space on $(0, \alpha)$ which is r-convex for some $0 < r < \infty$. Assume that E has either the Fatou property or has order continuous quasi-norm. Let $\mathcal{M}$ be a semi-finite von Neumann algebra equipped with a normal, semi-finite, faithful trace τ satisfying $\tau(\mathbf{1}) = \alpha$. If $p < p_E \leq q_E < q$, then $E(\mathcal{M})$ is an interpolation space for the couple $(L^p(\mathcal{M}), L^q(\mathcal{M}))$.*

In this section we give an alternative, direct proof of the noncommutative Boyd interpolation theorem, which avoids the use of Theorem 5.6. In fact, apart from the use of the basic properties of distribution functions of operators given in Proposition 5.1, our proof is completely elementary. Our new proof yields three improvements of Theorem 5.7. Firstly, we show that the result holds for any $0 < p < q \leq \infty$ and any symmetric quasi-Banach function space E on $(0, \infty)$ which is r-convex for some $0 < r < \infty$. Secondly, we can interpolate (midpoint) convex and subconvex operators which are only defined on the positive cone of a couple of noncommutative L^p-spaces. Finally, in contrast to the classical version of Boyd's theorem, a noncommutative version of Lemma 4.29 is part of our result in a natural way. Our main results are stated in Theorems 5.19 and 5.21 below.

Our first observation is part of the argument in [96], Proposition 2.d.1. Since we deal with quasi-Banach function spaces, we give a full proof for the reader's convenience.

Lemma 5.8. *Fix $0 < \alpha \leq \infty$. Let E be a symmetric quasi-Banach function space on $(0,\alpha)$. For any $0 < q < \infty$ define $\phi_q : (0,1) \to (0,\infty)$ by $\phi_q(t) = t^{-\frac{1}{q}}$. If $q_E < q$, then there is a constant $c_{q,E} > 0$ such that*

$$\|f \otimes \phi_q\|_{E((0,\alpha)\times(0,1))} \leq c_{q,E}\|f\|_{E(0,\alpha)}, \tag{5.1}$$

for all $f \in E$.

Conversely, if (5.1) holds for every $f \in E$ then $q_E \leq q$.

Proof. Suppose that $q_E < q$ and let $f \in E_+$. Notice first that

$$\begin{aligned}\|f \otimes \phi_q\|_{E((0,\alpha)\times(0,1))} &= \|f(s)t^{-\frac{1}{q}}\|_{E((0,\alpha)\times(0,1))} \\ &\leq \Big\|f(s)\sum_{n=0}^{\infty} 2^{\frac{n+1}{q}}\chi_{(2^{-n-1},2^{-n}]}(t)\Big\|_{E((0,\alpha)\times(0,1))} \\ &\leq c\Big(\sum_{n=0}^{\infty} 2^{\frac{r(n+1)}{q}}\|f(s)\chi_{(2^{-n-1},2^{-n}]}(t)\|^r_{E((0,\alpha)\times(0,1))}\Big)^{\frac{1}{r}},\end{aligned}$$

where $c > 0$ and $0 < r \leq 1$ are as in (4.2).

Fix $q > q_0 > q_E$, then by definition of q_E there exists a constant $C_{q_0} > 0$ such that

$$\|D_u\| \leq C_{q_0}u^{-\frac{1}{q_0}},$$

for any $1 \leq u < \infty$. Observe that $f(s)\chi_{(2^{-n-1},2^{-n}]}(t)$ has the same distribution on $(0,\alpha)\times(0,1)$ as $D_{2^{n+1}}f$ on $(0,\alpha)$.
Hence, as E is symmetric, we finally obtain

$$\begin{aligned}\|f \otimes \phi_q\|_{E((0,\alpha)\times(0,1))} &\leq c\Big(\sum_{n=0}^{\infty} 2^{\frac{r(n+1)}{q}}\|D_{2^{n+1}}f(t)\|^r_{E(0,\alpha)}\Big)^{\frac{1}{r}} \\ &\leq cC_{q_0}\Big(\sum_{n=0}^{\infty} 2^{\frac{r(n+1)}{q}}2^{-\frac{r(n+1)}{q_0}}\Big)^{\frac{1}{r}}\|f\|_{E(0,\alpha)} \\ &\lesssim_{q,E} \|f\|_{E(0,\alpha)},\end{aligned}$$

as $q > q_0$.

To prove the second assertion, notice first that since $\mu(D_s(f)) \leq D_s\mu(f)$ for all $s \in (0,\infty)$ and $f \in E$, it suffices to show that there is a constant $c > 0$ such that for all $s > 1$ and $f \in E_+$ we have $\|D_sf\|_E \leq cs^{-\frac{1}{q}}\|f\|_E$. Fix $a \in (0,1]$ and observe that

$$\begin{aligned}\|f \otimes \phi_q\|_{E((0,\alpha)\times(0,1))} &= \|f(s)t^{-\frac{1}{q}}\|_{E((0,\alpha)\times(0,1))} \\ &\geq \|f(s)a^{-\frac{1}{q}}\chi_{(\frac{a}{2},a]}(t)\|_{E((0,\alpha)\times(0,1))} = a^{-\frac{1}{q}}\|D_{\frac{2}{a}}f\|_{E(0,\alpha)},\end{aligned}$$

where in the final step we use that $f(s)\chi_{(\frac{a}{2},a]}(t)$ has the same distribution on $(0,\alpha)\times(0,1)$ as $D_{\frac{2}{a}}f(t)$ on $(0,\alpha)$.
Hence,

$$\|D_{\frac{2}{a}}f\|_E \leq a^{\frac{1}{q}}\|f\otimes\phi_q\|_E \leq c_{q,E}(\tfrac{2}{a})^{-\frac{1}{q}}2^{\frac{1}{q}}\|f\|_E.$$

In other words, for any $s\geq 2$ we obtain

$$\|D_s f\|_E \leq c_{q,E}2^{\frac{1}{q}}s^{-\frac{1}{q}}\|f\|_E.$$

Clearly this implies that $q_E \leq q$. □

The corresponding result for the lower Boyd index reads as follows.

Lemma 5.9. *Let E be a symmetric quasi-Banach function space on* $(0,\infty)$. *For any* $0<p<\infty$ *define* $\psi_p:(0,\infty)\to(0,\infty)$ *by* $\psi_p(t)=t^{-\frac{1}{p}}\chi_{(1,\infty)}(t)$. *If* $p<p_E$, *then there is a constant* $c_{p,E}>0$ *such that*

$$\|f\otimes\psi_p\|_{E((0,\infty)^2)} \leq c_{p,E}\|f\|_{E(0,\infty)}, \tag{5.2}$$

for all $f\in E$.

Conversely, if (5.2) holds for every $f\in E$ *then* $p\leq p_E$.

Proof. Fix $p<p_0<p_E$. It clearly suffices to prove (5.2) for $f\in E_+$. Observe that $f\chi_{(2^n,2^{n+1}]}$ has the same distribution on $(0,\infty)^2$ as $D_{2^{-n}}f$ on $(0,\infty)$. Hence,

$$\begin{aligned}\|f(s)t^{-\frac{1}{p}}\|_{E((0,\infty)^2)} &\leq \Big\|f(s)\sum_{n=0}^{\infty}2^{-\frac{n}{p}}\chi_{(2^n,2^{n+1}]}(t)\Big\|_{E((0,\infty)^2)}\\ &\leq c\Big(\sum_{n=0}^{\infty}2^{-\frac{nr}{p}}\|f(s)\chi_{(2^n,2^{n+1}]}(t)\|^r_{E((0,\infty)^2)}\Big)^{\frac{1}{r}}\\ &= c\Big(\sum_{n=0}^{\infty}2^{-\frac{nr}{p}}\|D_{2^{-n}}f\|^r_{E(0,\infty)}\Big)^{\frac{1}{r}},\end{aligned}$$

where c and $0<r\leq 1$ are as in (4.2). By the definition of p_E, there is some constant $C_{p_0}>0$ such that

$$\|D_u\| \leq C_{p_0}u^{-\frac{1}{p_0}} \qquad (0<u\leq 1).$$

Hence,

$$\begin{aligned}\|f(s)t^{-\frac{1}{p}}\|_{E((0,\infty)^2)} &\leq cC_{p_0}\Big(\sum_{n=0}^{\infty}2^{-\frac{nr}{p}}2^{\frac{nr}{p_0}}\|f\|_{E(0,\infty)}\Big)^{\frac{1}{r}}\\ &\lesssim_{p,E} \|f\|_{E(0,\infty)},\end{aligned}$$

as $\frac{1}{p_0}-\frac{1}{p}<0$.

For the second assertion, notice first that $\mu(D_s(f)) = D_s\mu(f)$ for all $0 < s < \infty$ and $f \in E$. Therefore, it suffices to show that there is a constant $c > 0$ such that for all $0 < s \leq 1$ and $f \in E_+$ we have $\|D_s f\|_E \leq cs^{-\frac{1}{p}}\|f\|_E$. If $1 \leq a < \infty$, then

$$\begin{aligned}\|f(s)t^{-\frac{1}{p}}\|_{E((0,\infty)^2)} &\geq \|f(s)t^{-\frac{1}{p}}\chi_{(a,2a]}(t)\|_{E((0,\infty)^2)}\\ &\geq \|f(s)(2a)^{-\frac{1}{p}}\chi_{(a,2a]}(t)\|_{E((0,\infty)^2)}\\ &= 2^{-\frac{1}{p}}a^{-\frac{1}{p}}\|D_{a^{-1}}f\|_{E(0,\infty)},\end{aligned}$$

where we use that $f(s)\chi_{(a,2a]}(t)$ has the same distribution on $(0,\infty)^2$ as $D_{a^{-1}}f$ on $(0,\infty)$. By (5.2) we arrive at

$$\|D_{a^{-1}}f\|_E \leq 2^{\frac{1}{p}}a^{\frac{1}{p}}\|f \otimes \psi_p\|_E \lesssim_{p,E} a^{\frac{1}{p}}\|f\|_E.$$

Since this holds for any $1 \leq a < \infty$, we conclude that $p \leq p_E$. □

Combining Lemmas 5.8 and 5.9 yields the following.

Corollary 5.10. *Let $0 < p < q < \infty$ and let E be a symmetric quasi-Banach function space on $(0,\infty)$. Let $\theta_{p,q} : (0,\infty) \to (0,\infty)$ be defined by $\theta_{p,q} = \phi_q + \psi_p$. If $p < p_E \leq q_E < q$, then there is a constant $c_{p,q,E} > 0$ such that for any $f \in E$ we have*

$$\|f \otimes \theta_{p,q}\|_{E((0,\infty)^2)} \leq c_{p,q,E}\|f\|_{E(0,\infty)}. \tag{5.3}$$

Conversely, if (5.3) holds for all $f \in E$, then $p \leq p_E \leq q_E \leq q$.

We now compute the distribution function of $f \otimes \phi_q$, $f \otimes \psi_p$ and $f \otimes \theta_{p,q}$.

Lemma 5.11. *Let $0 < \alpha \leq \infty$. For $0 < q < \infty$ let $\phi_q : (0,1) \to (0,\infty)$ be given by $\phi_q(t) = t^{-\frac{1}{q}}$. If $f : (0,\alpha) \to [0,\infty]$ is measurable and a.e. finite, then for every $v > 0$,*

$$d(v; f \otimes \phi_q) = \int_{\{f \leq v\}} \Big(\frac{f(s)}{v}\Big)^q ds + d(v; f).$$

Proof. By a change of variable,

$$\begin{aligned}&\lambda\Big\{(s,t) \in (0,\alpha) \times (0,1) \ : \ f(s)\phi_q(t) > v\Big\}\\ &\quad= \int_0^1 \lambda\Big(s \in (0,\alpha) \ : \ f(s)t^{-\frac{1}{q}} > v\Big)dt\\ &\quad= \int_0^1 \lambda\Big(s \in (0,\alpha) \ : \ f(s) > vu\Big)qu^{q-1}du\\ &\quad= \int_0^\infty \lambda\Big(s \in (0,\alpha) \ : \ \min\Big(\frac{f(s)}{v}, 1\Big) > u\Big)qu^{q-1}du\end{aligned}$$

$$= \Big\| \min\Big(\frac{f}{v}, 1\Big) \Big\|_{L^q(0,\alpha)}^q$$
$$= \int_{\{f \le v\}} \Big(\frac{f(s)}{v}\Big)^q ds + \lambda(s \in (0,\alpha) \ : \ f(s) > v).$$

□

Lemma 5.12. *Let $\psi_p : (0,\infty) \to (0,\infty)$ be defined by $\psi_p(t) = t^{-\frac{1}{p}} \chi_{(1,\infty)}(t)$. If $f \in S_0(0,\infty)$, then for any $v > 0$,*

$$d(v; f \otimes \psi_p) = \int_{\{f > v\}} \Big(\frac{f(s)}{v}\Big)^p ds - d(v; f).$$

Proof. By assumption we have $d(v; f) < \infty$ for all $v > 0$. Using a change of variable,

$$\begin{aligned}
&\lambda\Big\{(s,t) \in (0,\infty)^2 \ : \ f(s)\psi_p(t) > v\Big\} \\
&\quad = \int_1^\infty \lambda\Big(s \in (0,\infty) \ : \ f(s) t^{-\frac{1}{p}} > v\Big) dt \\
&\quad = \int_1^\infty \lambda\Big(s \in (0,\infty) \ : \ f(s) > t^{\frac{1}{p}} v\Big) dt \\
&\quad = \int_1^\infty \lambda\Big(s \in (0,\infty) \ : \ f(s) > uv\Big) p u^{p-1} du \\
&\quad = \int_1^\infty \lambda\Big(s \in (0,\infty) \ : \ \frac{f(s)}{v} > u\Big) p u^{p-1} du \\
&\quad = \Big\|\frac{f}{v}\Big\|_{L^p(0,\infty)}^p - \int_0^1 \lambda\Big(s \in (0,\infty) \ : \ \frac{f(s)}{v} > u\Big) p u^{p-1} du \\
&\quad = \Big\|\frac{f}{v}\Big\|_{L^p(0,\infty)}^p - \int_{\{f \le v\}} \Big(\frac{f(s)}{v}\Big)^p ds - d(v; f) \\
&\quad = \int_{\{f > v\}} \Big(\frac{f(s)}{v}\Big)^p ds - d(v; f),
\end{aligned}$$

where in the penultimate step we apply Lemma 5.11. □

Corollary 5.13. *Let $\theta_{p,q} : (0,\infty) \to (0,\infty)$ be defined by*

$$\theta_{p,q}(t) = t^{-\frac{1}{q}} \chi_{(0,1]}(t) + t^{-\frac{1}{p}} \chi_{(1,\infty)}(t).$$

If $f \in S_0(0,\infty)$, then for any $v > 0$,

$$d(v; f \otimes \theta_{p,q}) = \int_{\{f > v\}} \Big(\frac{f(s)}{v}\Big)^p ds + \int_{\{f \le v\}} \Big(\frac{f(s)}{v}\Big)^q ds.$$

Proof. Since $f \in S_0(0,\infty)$ we have $d_f(v) < \infty$ for all $v > 0$. Since $d_{f\otimes\phi_q}$ and $d_{f\otimes\psi_p}$ have disjoint supports we have $d_{f\otimes\phi_q} + d_{f\otimes\psi_p} = d_{f\otimes\theta_{p,q}}$. The result now follows from Lemmas 5.11 and 5.12. □

Lemma 5.14. *Let $0 < p < q \leq \infty$ and let E be a symmetric quasi-Banach function space $(0,\infty)$ which is r-convex for some $0 < r < \infty$. If $E(\mathcal{M}) \subset L^p(\mathcal{M}) + L^q(\mathcal{M})$, then*

$$\|x\|_{E(\mathcal{M})} \lesssim_{p,q,E} \|x\|_{L^p(\mathcal{M})+L^q(\mathcal{M})}, \qquad \text{for all} x \in X.$$

Proof. By Theorem 4.1, there exists an equivalent s-norm on $E(\mathcal{M})$ for some $0 < s \leq 1$. Suppose the assertion is not true. Then there exist $x_n \in E(\mathcal{M})_+$ such that $\|x_n\|_{E(\mathcal{M})} \leq 1$, but $\|x_n\|_{L^p(\mathcal{M})+L^q(\mathcal{M})} > n^{2/s+1}$ for all $n \geq 1$. By completeness it follows that $\sum_{n\geq 1} n^{-2/s} x_n$ converges in $E(\mathcal{M})$ to some $x \in E(\mathcal{M})_+$ and since $E(\mathcal{M}) \subset L^p(\mathcal{M})+L^q(\mathcal{M})$ we have $x \in (L^p(\mathcal{M})+L^q(\mathcal{M}))+$. But $n^{-2/s} x_n \leq x$ and so $n < n^{-2/s}\|x_n\|_{L^p(\mathcal{M})+L^q(\mathcal{M})} \leq \|x\|_{L^p(\mathcal{M})+L^q(\mathcal{M})}$, a contradiction. □

Lemma 5.15. *Let $0 < p < q \leq \infty$ and let E be a symmetric quasi-Banach function space $(0,\infty)$ which is r-convex for some $0 < r < \infty$. If $0 < p < p_E$ and either $q_E < q < \infty$ or $q = \infty$, then for every semi-finite von Neumann algebra $\mathcal{M}$ we have*

$$L^p(\mathcal{M}) \cap L^q(\mathcal{M}) \subset E(\mathcal{M}) \subset L^p(\mathcal{M}) + L^q(\mathcal{M}),$$

with continuous inclusions.

Proof. Suppose first that $q_E < q < \infty$. Observe that $E(\mathcal{M}) \subset S_0(\tau)$. Indeed, otherwise we would have $\mathbf{1} \in E(\mathcal{M})$ and hence $q_E = \infty$. If $x \in E(\mathcal{M})$, then by Corollary 5.10 we have $\mu(x) \otimes \theta_{p,q} \in E$ and hence $d(v; \mu(x) \otimes \theta_{p,q}) < \infty$ for all $v > 0$. If $e_v = e^{|x|}[0,v]$, then by Corollary 5.13

$$\begin{aligned} v^{-q}\|xe_v\|_q^q + v^{-p}\|xe_v^{\perp}\|_p^p &= v^{-q}\int_{\{\mu(x)\leq v\}} \mu_t(x)^q dt + v^{-p}\int_{\{\mu(x)>v\}} \mu_t(x)^p dt \\ &= d(v; \mu(x) \otimes \theta_{p,q}) < \infty. \end{aligned}$$

Hence $x \in L^p(\mathcal{M}) + L^q(\mathcal{M})$. By Lemma 5.14 this implies that $E(\mathcal{M})$ embeds continuously into $L^p(\mathcal{M}) + L^q(\mathcal{M})$.

Suppose now that $q = \infty$. Pick $v > 0$ such that $d(v; \mu(x) \otimes \psi_p), d(v; \mu(x)) < \infty$. Then $xe_v \in \mathcal{M}$ and $xe_v^{\perp} \in L^p(\mathcal{M})$ since by Lemma 5.12,

$$v^{-p}\|xe_v^{\perp}\|_{L^p(\mathcal{N})}^p = v^{-p}\int_{\{\mu(x)>v\}} \mu_t(x)^p dt \leq d(v; \mu(x) \otimes \psi_p) + d(v; \mu(x)).$$

By Lemma 5.14 we conclude that $E(\mathcal{M})$ embeds continuously into $L^p(\mathcal{M}) + \mathcal{M}$.

For the first inclusion we observe that, by the proof of [96], Proposition 2.b.3., there exists a constant $c_{p,q,E}$ such that for any nonnegative simple function f in $L^p \cap L^q$ we have $\|f\|_E \leq c_{p,q,E}\|f\|_{L^p\cap L^q}$. Let f be a nonnegative function in $L^p \cap L^q$. Then there is a sequence of nonnegative simple functions $f_n \uparrow f$. Hence (f_n) is Cauchy in $L^p \cap L^q$ and therefore also in E. Hence $f_n \to g$

in E for some $g \in E$. Since this implies that $f_n \to g$ in measure, we must have $f = g$ and $\|f\|_E \le c_{p,q,E}\|f\|_{L^p \cap L^q}$. Hence $L^p(\mathcal{M}) \cap L^q(\mathcal{M}) = (L^p \cap L^q)(\mathcal{M}) \subset E(\mathcal{M})$ continuously. □

Remark 5.16. It is readily verified that $L^p(\mathcal{M}) + L^q(\mathcal{M}) \subset (L^p + L^q)(\mathcal{M})$ for any $0 < p, q \le \infty$. Therefore, it follows from Lemmas 5.14 and 5.15 that $L^p(\mathcal{M}) + L^q(\mathcal{M}) = (L^p + L^q)(\mathcal{M})$ isomorphically.

To formulate our main results the following definition is convenient.

Definition 5.17. *Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras equipped with normal, semi-finite, faithful traces τ and σ, respectively. Let D be a convex subset of $S(\tau)$. A map $T : D \to S(\sigma)_h$ is called* midpoint convex *if*

$$T(\tfrac{1}{2}x + \tfrac{1}{2}y) \le \tfrac{1}{2}T(x) + \tfrac{1}{2}T(y)$$

for all $x, y \in D$. A map $U : D \to S(\sigma)$ is called midpoint subconvex *if for every $x, y \in D$ there exist partial isometries $u, v \in \mathcal{N}$ such that*

$$|U(\tfrac{1}{2}x + \tfrac{1}{2}y)| \le \tfrac{1}{2}u^*|Ux|u + \tfrac{1}{2}v^*|Uy|v.$$

It is a well-known fact (see e.g. [53], Lemma 4.3) that for any $x, y \in S(\sigma)$ there are partial isometries $u, v \in \mathcal{N}$ such that

$$|x + y| \le u^*|x|u + v^*|y|v.$$

Therefore, any linear map is (midpoint) subconvex.

For further reference we state Chebyshev's inequality and include a short proof for the reader's convenience.

Lemma 5.18. *(Chebyshev's inequality) Let $0 < q < \infty$. If $x \in L^q(\mathcal{M})$, then for any $v > 0$,*

$$d(v; x) \le \frac{\|x\|^q_{L^q(\mathcal{M})}}{v^q}.$$

Proof. Let $v > 0$. Then,

$$\begin{aligned} v^q d(v; x) &= v^q \lambda(t \in (0, \infty) \ : \ \mu_t(x) > v) = \int_{\{\mu(x) > v\}} v^q \, dt \\ &\le \int_{\{\mu(x) > v\}} \mu_t(x)^q \, dt \le \int_0^\infty \mu_t(x)^q \, dt = \|x\|^q_{L^q(\mathcal{M})}. \end{aligned}$$

□

Observe that for any $0 < r < \infty$,

$$\|x\|_{L^{r,\infty}(\mathcal{M})} = \sup_{t>0} t^{\frac{1}{r}} \mu_t(x) = \sup_{v>0} v \ d(v; x)^{\frac{1}{r}}, \tag{5.4}$$

so Chebyshev's inequality implies that $L^r(\mathcal{M}) \subset L^{r,\infty}(\mathcal{M})$ contractively.

Theorem 5.19. *Let E be a symmetric quasi-Banach function space on $(0,\infty)$ which is s-convex for some $0<s<\infty$. Let $\mathcal{M},\mathcal{N}$ be von Neumann algebras equipped with normal, semi-finite, faithful traces τ and σ, respectively. Suppose that $0<p<q\leq\infty$ and let $T: L^p(\mathcal{M})_+ + L^q(\mathcal{M})_+ \to S(\sigma)$ be a midpoint subconvex map such that for some constants $C_p, C_q > 0$ depending only on p and q, respectively,*

$$\|Tx\|_{L^{r,\infty}(\mathcal{N})} \leq C_r \|x\|_{L^r(\mathcal{M})} \qquad (x\in L^r(\mathcal{M})_+,\ r=p,q). \tag{5.5}$$

If $0<p<p_E$ and either $q_E<q<\infty$ or $q=\infty$, then there is a constant $c_{p,q,E}$ depending only on p,q and E such that

$$\|Tx\|_{E(\mathcal{N})} \leq c_{p,q,E}\max\{C_p,C_q\}\ \|x\|_{E(\mathcal{M})} \qquad (x\in E(\mathcal{M})_+).$$

The same result holds if $T: L^p(\mathcal{M})_+ + L^q(\mathcal{M})_+ \to S(\sigma)_h$ is a midpoint convex map satisfying (5.5).

Proof. We may assume that $\max\{C_p,C_q\}\leq 1$. By Lemma 5.15 T is well-defined on $E(\mathcal{M})_+$. Let $x\in E(\mathcal{M})_+$ and let $e_v = e^x[0,v]$. By midpoint subconvexity, there exist partial isometries $u_1,u_2\in\mathcal{N}$ such that $|Tx|\leq \frac{1}{2}u_1^*|T(2xe_v)|u_1 + \frac{1}{2}u_2^*|T(2xe_v^\perp)|u_2$. It follows that

$$\begin{aligned} d(2v;Tx) &\leq d(v;\tfrac{1}{2}u_1^*|T(2xe_v)|u_1) + d(v;\tfrac{1}{2}u_2^*|T(2xe_v^\perp)|u_2) \\ &\leq d(2v;T(2xe_v)) + d(2v;T(2xe_v^\perp)). \end{aligned} \tag{5.6}$$

Suppose first that $q_E<q<\infty$. By (5.4) and (5.5) we have

$$d(v;Ty)\leq v^{-r}\|y\|_{L^r(\mathcal{M})}^r \qquad (v>0,\ y\in L^r(\mathcal{M})_+,\ r=p,q).$$

Therefore,

$$d(2v;Tx)\leq \max\{C_q^q,C_p^p\}\Big((2v)^{-q}\|2xe_v\|_{L^q(\mathcal{N})}^q + (2v)^{-p}\|2xe_v^\perp\|_{L^p(\mathcal{N})}^p\Big)$$

and from the Proposition 5.2 it follows that

$$\|xe_v\|_{L^q(\mathcal{N})}^q = \int_{\{\mu(x)\leq v\}}\mu_t(x)^q dt, \qquad \|xe_v^\perp\|_{L^p(\mathcal{N})}^p = \int_{\{\mu(x)>v\}}\mu_t(x)^p dt.$$

Therefore, by Corollary 5.13,

$$\begin{aligned} d(2v;Tx) &\leq v^{-q}\int_{\{\mu(x)\leq v\}}\mu_t(x)^q dt + v^{-p}\int_{\{\mu(x)>v\}}\mu_t(x)^p dt \\ &= d(v;\mu(x)\otimes\theta_{p,q}). \end{aligned}$$

Hence,

$$\mu_t(Tx)\leq 2\mu_t(\mu(x)\otimes\theta_{p,q}) \qquad (t\geq 0).$$

As E is symmetric, it follows that $Tx\in E(\mathcal{N})$ and moreover,

$$\|Tx\|_{E(\mathcal{N})} \le 2\|\mu(x)\otimes\theta_{p,q}\|_{E((0,\infty)^2)} \lesssim_{p,q,E} \|x\|_{E(\mathcal{M})},$$

where the final inequality follows from Corollary 5.10.

Suppose now that $q = \infty$. Then

$$\|\tfrac{1}{2}u_1^*T(2xe_v)u_1\|_{L^\infty(\mathcal{N})} \le C_\infty\|xe_v\|_{L^\infty(\mathcal{M})} \le v,$$

so $d(v; \frac{1}{2}u_1^*T(xe_v)u_1) = 0$. By (5.4) and (5.5) we have

$$d(v; Ty) \le v^{-p}\|y\|^p_{L^p(\mathcal{M})} \qquad (v > 0,\ y \in L^p(\mathcal{M})_+),$$

and therefore (5.6) implies that

$$\begin{aligned} d(2v; Tx) &\le C_p^p(2v)^{-p}\|2xe_v^\perp\|^p_{L^p(\mathcal{N})} \\ &\le v^{-p}\int_{\{\mu(x)>v\}} \mu_t(x)^p dt \end{aligned}$$

Observe that

$$v^{-p}\int_{\{\mu(x)>v\}} \mu_t(x)^p dt \le d(v; \mu(x)\otimes\psi_p) + d(v; \mu(x)).$$

Indeed, if $d(v;\mu(x)) < \infty$ then this holds (even with equality) by Lemma 5.12 and if $d(v;\mu(x)) = \infty$ then the inequality holds trivially. Since $\mu(x)$ and $\mu(x)\otimes\chi_{(0,1)}$ are identically distributed, we conclude using Proposition 5.1 that

$$d(2v; Tx) \le 2d(v; \mu(x)\otimes\psi_p + \mu(x)\otimes\chi_{(0,1)}).$$

Thus, $Tx \in E(\mathcal{N})$ and by Lemma 5.9,

$$\|Tx\|_{E(\mathcal{N})} \le 2\|D_{\frac{1}{2}}\mu(\mu(x)\otimes\psi_p + \mu(x)\otimes\chi_{(0,1)})\|_E \lesssim_{p,E} \|x\|_{E(\mathcal{M})},$$

as asserted. □

The same proof gives the following result for midpoint convex and subconvex maps defined on self-adjoint elements.

Corollary 5.20. *Let E be a symmetric quasi-Banach function space on $(0,\infty)$ which is s-convex for some $0 < s < \infty$. Let $\mathcal{M}, \mathcal{N}$ be von Neumann algebras equipped with normal, semi-finite, faithful traces τ and σ, respectively. Suppose that $0 < p < q \le \infty$ and let $T : L^p(\mathcal{M})_h + L^q(\mathcal{M})_h \to S(\sigma)$ be a midpoint subconvex map such that for some constants $C_p, C_q > 0$ depending only on p and q, respectively,*

$$\|Tx\|_{L^{r,\infty}(\mathcal{N})} \le C_r\|x\|_{L^r(\mathcal{M})} \qquad (x \in L^r(\mathcal{M})_h,\ r = p, q). \tag{5.7}$$

If $0 < p < p_E$ and either $q_E < q < \infty$ or $q = \infty$, then there is a constant $c_{p,q,E}$ depending only on p, q and E such that

$$\|Tx\|_{E(\mathcal{N})} \le c_{p,q,E} \max\{C_p, C_q\}\ \|x\|_{E(\mathcal{M})} \qquad (x \in E(\mathcal{M})_h).$$

The same result holds if $T : L^p(\mathcal{M})_h + L^q(\mathcal{M})_h \to S(\sigma)_h$ is a midpoint convex map satisfying (5.7).

Finally, we obtain the following noncommutative version of the Boyd interpolation theorem, which generalizes Theorem 5.7. Observe that Theorem 5.21 incorporates a noncommutative version of Lemma 4.29.

Theorem 5.21. *Let E be a symmetric quasi-Banach function space on $(0,\infty)$ which is s-convex for some $0 < s < \infty$. Suppose $0 < p < q \leq \infty$ and let $\mathcal{M},\mathcal{N}$ be semi-finite von Neumann algebras. If $0 < p < p_E$ and either $q_E < q < \infty$ or $q = \infty$, then $(E(\mathcal{M}),E(\mathcal{N}))$ is an interpolation pair for the pair $((L^p(\mathcal{M}),L^q(\mathcal{M})),(L^{p,\infty}(\mathcal{N}),L^{q,\infty}(\mathcal{N})))$, with interpolation constant depending only on p,q and E.*

Proof. Let T be an admissible linear operator for the pair of Banach couples $(L^p(\mathcal{M}),L^q(\mathcal{M})),(L^{p,\infty}(\mathcal{N}),L^{q,\infty}(\mathcal{N}))$. Fix $x \in E(\mathcal{M})$ and let $\mathrm{Re}(x),\mathrm{Im}(x) \in E(\mathcal{M})_h$ be its real and imaginary part. By Corollary 5.20,

$$\begin{aligned}\|Tx\|_{E(\mathcal{M})} &\lesssim_E \|T(\mathrm{Re}(x))\|_{E(\mathcal{M})} + \|T(\mathrm{Im}(x))\|_{E(\mathcal{M})}\\ &\lesssim_{p,q,E} \|\mathrm{Re}(x)\|_{E(\mathcal{M})} + \|\mathrm{Im}(x)\|_{E(\mathcal{M})} \leq 2\|x\|_{E(\mathcal{M})}.\end{aligned}$$

□

To illustrate the usefulness of the method used to prove the noncommutative Boyd interpolation theorem, we modify it to prove the dual version of Doob's maximal inequality in noncommutative symmetric spaces, see Theorem 5.24 below. Let us first recall the original result for noncommutative L^p-spaces, due to M. Junge.

Theorem 5.22. *[68] Let $\mathcal{M}$ be a finite von Neumann algebra and let $(\mathcal{E}_i)_{i\geq 1}$ be an increasing sequence of conditional expectations in $\mathcal{M}$. If $1 \leq p < \infty$, then for any sequence $(x_i)_{i\geq 1}$ in $L^p(\mathcal{M})_+$,*

$$\Big\|\sum_i \mathcal{E}_i(x_i)\Big\|_{L^p(\mathcal{M})} \lesssim_p \Big\|\sum_i x_i\Big\|_{L^p(\mathcal{M})}.$$

We shall need the following observation.

Lemma 5.23. *Let $x \in S(\tau)_+$. If e is a projection in $\mathcal{M}$, then*

$$x \leq 2(exe + e^{\perp}xe^{\perp}).$$

Proof. By writing

$$x = exe + e^{\perp}xe + exe^{\perp} + e^{\perp}xe^{\perp},$$

we see that the asserted inequality is equivalent to

$$exe - e^{\perp}xe - exe^{\perp} + e^{\perp}xe^{\perp} \geq 0.$$

But $x \geq 0$, so

$$exe - e^{\perp}xe - exe^{\perp} + e^{\perp}xe^{\perp} = (x^{\frac{1}{2}}e - x^{\frac{1}{2}}e^{\perp})^*(x^{\frac{1}{2}}e - x^{\frac{1}{2}}e^{\perp}) \geq 0$$

and the result follows. □

Theorem 5.24. *Let E be a symmetric quasi-Banach function space on $(0,\infty)$ which is s-convex for some $0<s<\infty$ and let $\mathcal{M}$ be a finite von Neumann algebra. Let $(\mathcal{E}_i)_{i\geq 1}$ be an increasing sequence of conditional expectations in $\mathcal{M}$. If $1<p_E\leq q_E<\infty$, then for any sequence $(x_i)_{i\geq 1}$ in $E(\mathcal{M})_+$,*

$$\Big\|\sum_{i\geq 1}\mathcal{E}_i(x_i)\Big\|_{E(\mathcal{M})}\lesssim_E\Big\|\sum_{i\geq 1}x_i\Big\|_{E(\mathcal{M})}, \tag{5.8}$$

where the sums converge in norm.

Proof. By completeness it suffices to prove (5.8) for a finite sequence (x_i) in $E(\mathcal{M})_+$. Set $x=\sum_i x_i$. For any $v\geq 0$, let $e_v=e^x[0,v]$. By Lemma 5.23 and positivity of $\mathcal{E}_i$,

$$\sum_i\mathcal{E}_i(x_i)\leq 2\Big(\sum_i\mathcal{E}_i(e_vx_ie_v)+\sum_i\mathcal{E}_i(e_v^\perp x_ie_v^\perp)\Big).$$

Therefore,

$$d\Big(4v;\sum_i\mathcal{E}_i(x_i)\Big)\leq d\Big(v;\sum_i\mathcal{E}_i(e_vx_ie_v)\Big)+d\Big(v;\sum_i\mathcal{E}_i(e_v^\perp x_ie_v^\perp)\Big).$$

By Chebyshev's inequality and Theorem 5.22,

$$\begin{aligned}
&d\Big(4v;\sum_i\mathcal{E}_i(x_i)\Big)\\
&\quad\leq v^{-q}\Big\|\sum_i\mathcal{E}_i(e_vx_ie_v)\Big\|_{L^q(\mathcal{M})}^q+v^{-p}\Big\|\sum_i\mathcal{E}_i(e_v^\perp x_ie_v^\perp)\Big\|_{L^p(\mathcal{M})}^p\\
&\quad\lesssim_{p,q} v^{-q}\Big\|\sum_i e_vx_ie_v\Big\|_{L^q(\mathcal{M})}^q+v^{-p}\Big\|\sum_i e_v^\perp x_ie_v^\perp\Big\|_{L^p(\mathcal{M})}^p\\
&\quad= v^{-q}\int_{\{\mu(x)\leq v\}}\mu_t(x)^q dt+v^{-p}\int_{\{\mu(x)>v\}}\mu_t(x)^p dt\\
&\quad= d(v;\mu(x)\otimes\theta_{p,q}),
\end{aligned}$$

where the final equality follows from Corollary 5.13. Since E is symmetric, we conclude that $\sum_i\mathcal{E}_i(x_i)\in E(\mathcal{M})_+$ and by Corollary 5.10,

$$\Big\|\sum_i\mathcal{E}_i(x_i)\Big\|_{E(\mathcal{M})}\lesssim_{p,q}\|\mu(x)\otimes\theta_{p,q}\|_E\lesssim_{p,q,E}\Big\|\sum_i x_i\Big\|_{E(\mathcal{M})}.$$

□

5.3 Hilbert space-valued symmetric spaces

In this section we construct *noncommutative Hilbert space valued symmetric spaces* associated with a symmetric Banach function space E on $(0,\infty)$ and a semi-finite von Neumann algebra $\mathcal{M}$. These spaces were first defined for $E = L^p$ $(1 \le p \le \infty)$ in [69]. For our purposes in Chapter 8, the main point of this endeavour is to make rigorous sense of the spaces of all functions $f : [0,T] \to E(\mathcal{M})$ such that the 'square function norms'

$$\Big\|\Big(\int_0^T f(t)^* f(t)dt\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})} \text{ and } \Big\|\Big(\int_0^T f(t)f(t)^* dt\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})}$$

are finite. These spaces will later on be denoted by $E(\mathcal{M}; L^2(0,T)_c)$ and $E(\mathcal{M}; L^2(0,T)_r)$, respectively. Throughout, *we will assume that E is separable*. Recall that this implies that $\mathcal{F}(\tau)$ is dense in $E(\mathcal{M})$ and $E(\mathcal{M})^* = E^\times(\mathcal{M})$. We closely follow the exposition of [69].

Let H be a complex Hilbert space and let $\overline{H}$ denote its conjugate Hilbert space (i.e., its dual space). Let $\mathcal{M}\overline{\otimes}B(H)$ be the von Neumann tensor product equipped with the product trace $\tau \otimes tr$. For any vectors $\xi, \eta \in H$ we let $\xi \otimes \overline{\eta}$ denote the rank one projection $(\xi \otimes \overline{\eta})\zeta = \langle \zeta, \eta\rangle \xi$ in H. Similarly, we let $\overline{\xi} \otimes \eta$ denote the rank one projection $(\overline{\xi} \otimes \eta)\overline{\zeta} = \langle \overline{\zeta}, \overline{\eta}\rangle^- \overline{\xi}$ in $\overline{H}$, where $\langle \cdot, \cdot\rangle^-$ denotes the inner product in $\overline{H}$ (i.e. the conjugate inner product). Let e be a unit vector in H, let p_e be the rank one projection in H onto $span\{e\}$ (i.e. $p_e = e \otimes \overline{e}$) and let $p_{\overline{e}}$ be the rank one projection in $\overline{H}$ onto $span\{\overline{e}\}$ (i.e. $p_{\overline{e}} = \overline{e} \otimes e$. Then the *column* and *row* spaces associated with E and H are defined as

$$E(\mathcal{M}; H_c) = E(\mathcal{M}\overline{\otimes}B(H))\ (\mathbf{1}_{\mathcal{M}} \otimes p_e)$$

and

$$E(\mathcal{M}; H_r) = (\mathbf{1}_{\mathcal{M}} \otimes p_{\overline{e}})\ E(\mathcal{M}\overline{\otimes}B(\overline{H})).$$

Our first goal will be to show that these definitions are essentially independent of the choice of the unit vector e.

By identifying $x \in E(\mathcal{M})$ with $x \otimes p_e$ and $x \otimes p_{\overline{e}}$, we obtain an isometric embedding of $E(\mathcal{M})$ into $E(\mathcal{M}; H_c)$ and $E(\mathcal{M}, H_r)$, respectively. For the column space this is seen as follows (the row case is analogous). Notice first that for any $v > 0$, $e^{|x| \otimes p_e}(v, \infty) = e^{|x|}(v,\infty) \otimes p_e$. This implies that

$$\tau \otimes \mathrm{Tr}(e^{|x|\otimes p_e}(v,\infty)) = \tau \otimes \mathrm{Tr}(e^{|x|}(v,\infty) \otimes p_e) = \tau(e^{|x|}(v,\infty)) \qquad (v > 0).$$

Hence, $\mu(t; x \otimes p_e) = \mu(t; x)$ for every $t \ge 0$ and therefore $\|x \otimes p_e\|_{E(\mathcal{M}\overline{\otimes}B(H))} = \|x\|_{E(\mathcal{M})}$.

Lemma 5.25. *The algebraic tensor product $E(\mathcal{M}) \otimes \mathcal{F}(B(H))$ is norm dense in $E(\mathcal{M}\overline{\otimes}B(H))$.*

Proof. If (p_α) is the net of all finite rank projections in $B(H)$, then $\mathbf{1}_{\mathcal{M}} \otimes p_\alpha$ converges to $\mathbf{1}_{\mathcal{M}} \otimes \mathbf{1}_{B(H)}$ with respect to the ultra-strong operator topology. In particular, for any $x \in L^\infty \cap L^1(\mathcal{M}\overline{\otimes}B(H))$ we have $(\mathbf{1}_{\mathcal{M}} \otimes p_\alpha)x(\mathbf{1}_{\mathcal{M}} \otimes p_\alpha) \to x$ in the ultra-strong operator topology and hence in the ultra-weak operator topology. Hence, if $\phi \in E^*(\mathcal{M}\overline{\otimes}B(H)) = E^\times(\mathcal{M}\overline{\otimes}B(H))$ is zero on

$$(\mathbf{1}_{\mathcal{M}} \otimes p_\alpha) L^\infty \cap L^1(\mathcal{M}\overline{\otimes}B(H))(\mathbf{1}_{\mathcal{M}} \otimes p_\alpha),$$

for all α, then it is identically zero, as $L^\infty \cap L^1(\mathcal{M}\overline{\otimes}B(H))$ is norm dense in $E(\mathcal{M}\overline{\otimes}B(H))$. By the Hahn-Banach theorem,

$$\bigcup_\alpha (\mathbf{1}_{\mathcal{M}} \otimes p_\alpha) L^\infty \cap L^1(\mathcal{M}\overline{\otimes}B(H))(\mathbf{1}_{\mathcal{M}} \otimes p_\alpha)$$

is norm dense in $E(\mathcal{M}\overline{\otimes}B(H))$ (c.f. [33], Corollary III.6.14). Since, for all α,

$$(\mathbf{1}_{\mathcal{M}} \otimes p_\alpha) L^\infty \cap L^1(\mathcal{M}\overline{\otimes}B(H))(\mathbf{1}_{\mathcal{M}} \otimes p_\alpha) \subset E(\mathcal{M}) \otimes \mathcal{F}(B(H)),$$

the assertion follows. □

Let us observe that

$$E(\mathcal{M}) = (\mathbf{1}_{\mathcal{M}} \otimes p_e) E(\mathcal{M}\overline{\otimes}B(H))(\mathbf{1}_{\mathcal{M}} \otimes p_e), \tag{5.9}$$

where we identify $x \in E(\mathcal{M})$ with $x \otimes p_e$ as above. Indeed, it is clear that the inclusion '$\subset$' holds. On the other hand, if $a \in B(H)$, then $p_e a p_e = \langle ae, e\rangle p_e$. Thus, for finite sequences (x_i) and (a_i) in $E(\mathcal{M})$ and $E(B(H))$, respectively, $(\mathbf{1}_{\mathcal{M}} \otimes p_e)\sum_i x_i \otimes a_i(\mathbf{1}_{\mathcal{M}} \otimes p_e)$ is in $E(\mathcal{M})$. Since $x \mapsto (\mathbf{1}_{\mathcal{M}} \otimes p_e)x(\mathbf{1}_{\mathcal{M}} \otimes p_e)$ is a contractive projection on $E(\mathcal{M}\overline{\otimes}B(H))$, it follows by Lemma 5.25 that $(\mathbf{1}_{\mathcal{M}} \otimes p_e)E(\mathcal{M}\overline{\otimes}B(H))(\mathbf{1}_{\mathcal{M}} \otimes p_e) \subset E(\mathcal{M})$.

If $u \in E(\mathcal{M}; H_c)$, then by (5.9),

$$u^*u \in (\mathbf{1}_{\mathcal{M}} \otimes p_e) E_{(2)}(\mathcal{M}\overline{\otimes}B(H))(\mathbf{1}_{\mathcal{M}} \otimes p_e) = E_{(2)}(\mathcal{M})$$

and so $|u| \in E(\mathcal{M})$. Similarly we have $E(\mathcal{M}) = (\mathbf{1}_{\mathcal{M}} \otimes p_{\overline{e}})E(\mathcal{M}\overline{\otimes}B(H))(\mathbf{1}_{\mathcal{M}} \otimes p_{\overline{e}})$, where we identify $x \in E(\mathcal{M})$ with $x \otimes p_{\overline{e}}$, and so $|u^*| \in E(\mathcal{M})$ whenever $u \in E(\mathcal{M}; H_r)$.

For $u \in E(\mathcal{M}) \otimes H$ given by $u = \sum_i x_i \otimes \xi_i$, we define the element $\tilde{u}$ by

$$\tilde{u} = \sum_i x_i \otimes (\xi_i \otimes \overline{e}) = (\sum_i x_i \otimes (\xi_i \otimes \overline{e}))(\mathbf{1}_{\mathcal{M}} \otimes p_e).$$

Identifying u and $\tilde{u}$ gives a set inclusion $E(\mathcal{M}) \otimes H \subset E(\mathcal{M}; H_c)$. Similarly, we can identify u with

$$\hat{u} = \sum_i x_i \otimes (\overline{e} \otimes \xi_i) = (\mathbf{1}_{\mathcal{M}} \otimes p_{\overline{e}}) \sum_i x_i \otimes (\overline{e} \otimes \xi_i)$$

to obtain an inclusion of $E(\mathcal{M}) \otimes H$ into $E(\mathcal{M}; H_r)$. Under these identifications we have the following.

Lemma 5.26. *The algebraic tensor product $E(\mathcal{M}) \otimes H$ is norm dense in both $E(\mathcal{M}; H_c)$ and $E(\mathcal{M}; H_r)$.*

Proof. We only prove this for $E(\mathcal{M}; H_c)$. If (p_α) is the net of all finite rank projections in $B(H)$, then, by the proof of Lemma 5.25, we have for any $x \in E(\mathcal{M}\overline{\otimes}B(H))$,

$$\lim_\alpha \|(\mathbf{1}_\mathcal{M} \otimes p_\alpha)x(\mathbf{1}_\mathcal{M} \otimes p_e)(\mathbf{1}_\mathcal{M} \otimes p_\alpha) - x(\mathbf{1}_\mathcal{M} \otimes p_e)\|_{E(\mathcal{M}\overline{\otimes}B(H))} = 0.$$

Using that $p_e a p_e = \langle ae, e\rangle p_e$ for $a \in \mathcal{F}(B(H))$ it is straightforward to calculate that

$$(\mathbf{1}_\mathcal{M} \otimes p_\alpha)x(\mathbf{1}_\mathcal{M} \otimes p_e)(\mathbf{1}_\mathcal{M} \otimes p_\alpha) \in E(\mathcal{M}) \otimes H,$$

whenever $p_\alpha \geq p_e$. □

By calculating the norm of $|\tilde{u}|$ and $|\hat{u}^*|$ in $E(\mathcal{M}\overline{\otimes}B(H))$ we obtain

$$\Big\|\sum_{k=1}^n x_k \otimes \xi_k\Big\|_{E(\mathcal{M};H_c)} = \Big\|\Big(\sum_{i,j=1}^n \langle \xi_j, \xi_i\rangle\ x_i^* x_j\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})} \tag{5.10}$$

and

$$\Big\|\sum_{k=1}^n x_k \otimes \xi_k\Big\|_{E(\mathcal{M};H_r)} = \Big\|\Big(\sum_{i,j=1}^n \langle \xi_i, \xi_j\rangle\ x_i x_j^*\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})}, \tag{5.11}$$

respectively. By Lemma 5.26, we conclude that the definitions of $E(\mathcal{M}; H_c)$ and $E(\mathcal{M}; H_r)$ are essentially independent of the choice of the unit vector e, since we always obtain the completion of $E(\mathcal{M}) \otimes H$ in the respective norms given above. If $e_1, \ldots, e_n$ is an orthonormal system in H, then

$$\Big\|\sum_{k=1}^n x_k \otimes e_k\Big\|_{E(\mathcal{M};H_c)} = \Big\|\Big(\sum_{k=1}^n x_k^* x_k\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})};$$
$$\Big\|\sum_{k=1}^n x_k \otimes e_k\Big\|_{E(\mathcal{M};H_r)} = \Big\|\Big(\sum_{k=1}^n x_k x_k^*\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})}. \tag{5.12}$$

Observe that the column and row spaces are complemented subspaces of $E(\mathcal{M}\overline{\otimes}B(H))$ and $E(\mathcal{M}\overline{\otimes}B(\overline{H}))$, respectively. Therefore, by Theorem 5.6 we find that if E is an (exact) interpolation space for the couple (L^p, L^q), $1 \leq p \leq q \leq \infty$, then $E(\mathcal{M}; H_c)$ and $E(\mathcal{M}; H_r)$ are (exact) interpolation space for the couples $(L^p(\mathcal{M}; H_c), L^q(\mathcal{M}; H_c))$ and $(L^p(\mathcal{M}; H_r), L^q(\mathcal{M}; H_r))$, respectively.

We have the following useful duality for column and row spaces.

Lemma 5.27. *If E is a separable symmetric space on $(0, \infty)$, then*

$$E(\mathcal{M}; H_c)^* = E^\times(\mathcal{M}; \overline{H}_r)$$

isometrically.

Proof. Let $\phi \in E(\mathcal{M}; H_c)^*$. Then ϕ induces a continuous linear functional $\tilde{\phi}$ on $E(\mathcal{M}\overline{\otimes}B(H))$ given by

$$\tilde{\phi}(a) = \phi(a(\mathbf{1}_{\mathcal{M}} \otimes p_e)) \qquad (a \in E(\mathcal{M}\overline{\otimes}B(H))),$$

which, as $E(\mathcal{M}\overline{\otimes}B(H))^* = E^\times(\mathcal{M}\overline{\otimes}B(H))$ by Theorem 5.4, is in turn given by

$$\tilde{\phi}(a) = \tau \otimes \mathrm{Tr}(ab),$$

for some $b \in E^\times(\mathcal{M}\overline{\otimes}B(H))$. If $a \in E(\mathcal{M}; H_c)$, then

$$\phi(a) = \tilde{\phi}(a(\mathbf{1}_{\mathcal{M}} \otimes p_e)) = \tau \otimes \mathrm{Tr}(a(\mathbf{1}_{\mathcal{M}} \otimes p_e)b) = \tau \otimes \mathrm{Tr}(a\tilde{b}),$$

where $\tilde{b} := (\mathbf{1}_{\mathcal{M}} \otimes p_e)b \in E^\times(\mathcal{M}; \overline{H}_r)$. Now,

$$\begin{aligned}
\|\tilde{b}\|_{E^\times(\mathcal{M};\overline{H}_r)} &= \|(\mathbf{1}_{\mathcal{M}} \otimes p_e)b\|_{E(\mathcal{M}\overline{\otimes}B(H))^\times} \\
&= \sup_{\|a\|_{E(\mathcal{M}\overline{\otimes}B(H))} \le 1} |\tau \otimes \mathrm{Tr}(a(\mathbf{1}_{\mathcal{M}} \otimes p_e)b)| \\
&= \sup_{\|a\|_{E(\mathcal{M}\overline{\otimes}B(H))} \le 1} |\tilde{\phi}(a(\mathbf{1}_{\mathcal{M}} \otimes p_e))| \\
&= \sup_{a \in E(\mathcal{M}\overline{\otimes}B(H))} \frac{1}{\|a\|_{E(\mathcal{M}\overline{\otimes}B(H))}} |\tilde{\phi}(a(\mathbf{1}_{\mathcal{M}} \otimes p_e))| = \|\phi\|
\end{aligned}$$

This proves the desired isometric identity. □

Lemma 5.28. *If E is a separable symmetric space on $(0,\infty)$, then $E(\mathcal{M}; H_c)$ and $E(\mathcal{M}; H_r)$ can be contractively embedded in the injective tensor product $E(\mathcal{M}) \otimes_\lambda H$. In particular, $(E(\mathcal{M}; H_c), E(\mathcal{M}; H_r))$ is an interpolation couple of Banach spaces.*

Proof. Let $(x_k) \subset E(\mathcal{M})$ and $(\xi_k) \subset H$ be finite sequences. By (5.10),

$$\|x_1 \otimes \xi_1\|_{E(\mathcal{M};H_c)} = \|\xi_1\|_H \|x_1\|_{E(\mathcal{M})},$$

so the norm $\|\cdot\|_{E(\mathcal{M};H_c)}$ is a cross-norm on the algebraic tensor product $E(\mathcal{M}) \otimes H$. Similarly, by (5.11) we see that $\|\cdot\|_{E(\mathcal{M};H_r)}$ defines a cross-norm on $E(\mathcal{M}) \otimes H$. Recall that $E(\mathcal{M})^* = E^\times(\mathcal{M})$ and $H^* = \overline{H}$. Define the map $\|\cdot\|_*$ on $E(\mathcal{M})^* \otimes H^* = E^\times(\mathcal{M}) \otimes \overline{H}$ by

$$\Big\| \sum_k y_k \otimes \eta_k \Big\|_* = \sup \Big\{ \Big| \Big\langle u, \sum_k y_k \otimes \eta_k \Big\rangle \Big| : u \in E(\mathcal{M}) \otimes H, \|u\|_{E(\mathcal{M};H_c)} \le 1 \Big\}.$$

Since $E(\mathcal{M}) \otimes H$ is dense in $E(\mathcal{M}; H_c)$ and by the duality $E(\mathcal{M}; H_c)^* = E^\times(\mathcal{M}; \overline{H}_r)$ we obtain $\|\cdot\|_* = \|\cdot\|_{E^\times(\mathcal{M};\overline{H}_r)}$, which is a cross-norm on $E^\times(\mathcal{M}) \otimes \overline{H}$. In other words, $\|\cdot\|_{E(\mathcal{M};H_c)}$ is a reasonable cross-norm on $E(\mathcal{M}) \otimes H$ in the terminology of [131], Section IV.2. For $x = \sum_k x_k \otimes \xi_k$ let

$$\|x\|_\lambda = \sup\Big\{\Big|\sum_k \langle x_k, y\rangle\langle \xi_k, \eta\rangle\Big| : y \in E^\times(\mathcal{M})_1, \eta \in \overline{H}_1\Big\}$$

be the injective cross norm on $E(\mathcal{M}) \otimes H$. Then

$$\|x\|_\lambda \le \sup\{\|x\|_{E(\mathcal{M};H_c)}\|y \otimes \eta\|_* : y \in E^\times(\mathcal{M})_1, \eta \in \overline{H}_1\} \le \|x\|_{E(\mathcal{M};H_c)},$$

since $\|\cdot\|_*$ is a cross-norm. We conclude that the identity map on $E(\mathcal{M}) \otimes H$ extends to a contractive linear map $\iota : E(\mathcal{M}; H_c) \to E(\mathcal{M}) \otimes_\lambda H$. To see that this map is injective, let $\iota^* : (E(\mathcal{M}) \otimes_\lambda H)^* \to E^\times(\mathcal{M}; \overline{H}_r)$ denote the adjoint of ι. If $y \in E^\times(\mathcal{M})_1$ and $\eta \in \overline{H}_1$, then $y \otimes \eta$ satisfies

$$|\langle x, y \otimes \eta\rangle| \le \|x\|_\lambda \qquad (x \in E(\mathcal{M}) \otimes H)$$

and therefore uniquely extends to an element of $(E(\mathcal{M}) \otimes_\lambda H)^*_1$. Moreover, $y \otimes \eta \in E^\times(\mathcal{M}; \overline{H}_r)$ and it is easily seen that $\iota^* y \otimes \eta = y \otimes \eta$. Indeed, this holds on $E(\mathcal{M}) \otimes H$, which is dense in $E(\mathcal{M}; H_c)$. Suppose now that $\iota x = 0$ for some $x \in E(\mathcal{M}; H_c)$. Then for $y \in E^\times(\mathcal{M})_1$ and $\eta \in \overline{H}_1$

$$\langle x, y \otimes \eta\rangle = \langle x, \iota^* y \otimes \eta\rangle = \langle \iota x, y \otimes \eta\rangle = 0.$$

Thus, for any $y_1, \ldots, y_n \in E^\times(\mathcal{M})$ and $\eta_1, \ldots, \eta_n \in \overline{H}$ we have

$$\Big\langle x, \sum_k y_k \otimes \eta_k\Big\rangle = 0$$

and by density of $E^\times(\mathcal{M}) \otimes \overline{H}$ in $E^\times(\mathcal{M}; \overline{H}_r)$ we obtain $x = 0$. Hence ι is injective and our proof is complete. □

From now on, we identify $E(\mathcal{M}; H_c)$ and $E(\mathcal{M}; H_r)$ with their images in $E(\mathcal{M}) \otimes_\lambda H$.

By Lemma 5.26, the space $E(\mathcal{M}; H_c) \cap E(\mathcal{M}; H_r)$, with the intersection taken in $E(\mathcal{M}) \otimes_\lambda H$, is dense in $E(\mathcal{M}; H_c)$ and $E(\mathcal{M}; H_r)$ and therefore (see e.g. [85], Theorem I.3.1),

$$\begin{aligned}(E(\mathcal{M}; H_c) \cap E(\mathcal{M}; H_r))^* &= E(\mathcal{M}; H_c)^* + E(\mathcal{M}; H_r)^* \\ &= E^\times(\mathcal{M}; \overline{H}_c) + E^\times(\mathcal{M}; \overline{H}_r),\end{aligned}$$

and

$$\begin{aligned}(E(\mathcal{M}; H_c) + E(\mathcal{M}; H_r))^* &= E(\mathcal{M}; H_c)^* \cap E(\mathcal{M}; H_r)^* \\ &= E^\times(\mathcal{M}; \overline{H}_c) \cap E^\times(\mathcal{M}; \overline{H}_r), \qquad (5.13)\end{aligned}$$

if E is separable.

By taking the standard orthonormal basis for l^2 in (5.12) and using the density of $E(\mathcal{M}) \otimes l^2$ in both $E(\mathcal{M}; l^2_c)$ and $E(\mathcal{M}; l^2_r)$, we can alternatively describe these spaces as the completions of the linear space of all finite sequences $(x_k)_{k=1}^n$ in $E(\mathcal{M})$ in the norms

$$\begin{aligned}\|(x_k)_{k=1}^n\|_{E(\mathcal{M};l_c^2)} &= \Big\|\Big(\sum_{k=1}^n x_k^* x_k\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})} = \|\mathrm{col}(x_k)\|_{E(M_n(\mathcal{M}))};\\ \|(x_k)_{k=1}^n\|_{E(\mathcal{M};l_r^2)} &= \Big\|\Big(\sum_{k=1}^n x_k x_k^*\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})} = \|\mathrm{row}(x_k)\|_{E(M_n(\mathcal{M}))}.\end{aligned} \tag{5.14}$$

Observe that these expressions define two norms even if E is not separable. The spaces $E(\mathcal{M}; l_c^2)$ and $E(\mathcal{M}; l_r^2)$ will be main characters in Chapter 6.

5.4 Conditional sequence spaces

For our discussion in Chapter 7 we introduce conditional versions of the norms in (5.14). Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal, semi-finite, faithful trace τ and let $(\mathcal{M}_k)_{k\geq 1}$ be an increasing sequence of von Neumann subalgebras of $\mathcal{M}$ such that the restriction of τ to $\mathcal{M}_k$ is again semi-finite, for all $k \geq 1$. Let $\mathcal{E}_k$ denote the conditional expectation with respect to $\mathcal{M}_k$. In this section we give an elementary proof of the fact that the expressions

$$\|(x_k)\|_{E(\mathcal{M},(\mathcal{E}_k);l_c^2)} = \Big\|\Big(\sum_k \mathcal{E}_k|x_k|^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})} \tag{5.15}$$

and

$$\|(x_k)\|_{E(\mathcal{M},(\mathcal{E}_k);l_r^2)} = \Big\|\Big(\sum_k \mathcal{E}_k|x_k^*|^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})} \tag{5.16}$$

define norms on the space of all finite sequences in $E(\mathcal{M})$, whenever E is a symmetric Banach function space on $(0,\infty)$ which is 2-convex with convexity constant equal to 1 and $E_{(2)}$ is fully symmetric. For $E = L^p$ this result was obtained in [68] using a different method.

Proposition 5.29. *Let $\mathcal{M}$ be a semi-finite von Neumann algebra equipped with a normal, semi-finite, faithful trace τ and let $(\mathcal{M}_k)_{k\geq 1}$ be an increasing sequence of von Neumann subalgebras of $\mathcal{M}$ such that the restriction of τ to $\mathcal{M}_k$ is again semi-finite, for all $k \geq 1$. Let $\mathcal{E}_k$ denote the conditional expectation with respect to $\mathcal{M}_k$. Suppose E is a 2-convex symmetric Banach function space on $(0,\infty)$ with 2-convexity constant equal to 1 and suppose $E_{(2)}$ is fully symmetric. Then (5.15) and (5.16) define norms on the linear space of all finitely nonzero sequences in $E(\mathcal{M})$.*

Proof. It suffices to prove the assertion for (5.15). It is clear that $\|\cdot\|_{E(\mathcal{M},(\mathcal{E}_k);l_c^2)}$ is positive definite and homogeneous. It remains to show the triangle inequality. Let (x_k) and (y_k) be finite sequences in $E(\mathcal{M})$ and fix $\alpha > 0$. Using that $|\alpha x_k - \alpha^{-1} y_k|^2 \geq 0$, it follows that for all $k \geq 1$,

$$|x_k + y_k|^2 \leq (1+\alpha^2)|x_k|^2 + (1+\alpha^{-2})|y_k|^2.$$

As $\mathcal{E}_k$ is positive for $k \geq 1$, this implies that

$$\sum_k \mathcal{E}_k|x_k + y_k|^2 \leq (1+\alpha^2)\sum_k \mathcal{E}_k|x_k|^2 + (1+\alpha^{-2})\sum_k \mathcal{E}_k|y_k|^2.$$

Since E is 2-convex with 2-convexity constant equal to 1,

$$\Big\|\sum_k \mathcal{E}_k|x_k + y_k|^2\Big\|_{E_{(2)}(\mathcal{M})} \leq (1+\alpha^2)\Big\|\sum_k \mathcal{E}_k|x_k|^2\Big\|_{E_{(2)}(\mathcal{M})} + (1+\alpha^{-2})\Big\|\sum_k \mathcal{E}_k|y_k|^2\Big\|_{E_{(2)}(\mathcal{M})}.$$

Taking the infimum over all $\alpha > 0$ gives

$$\Big\|\sum_k \mathcal{E}_k|x_k + y_k|^2\Big\|_{E_{(2)}} \leq \Big(\Big\|\sum_k \mathcal{E}_k|x_k|^2\Big\|_{E_{(2)}}^{\frac{1}{2}} + \Big\|\sum_k \mathcal{E}_k|y_k|^2\Big\|_{E_{(2)}}^{\frac{1}{2}}\Big)^2,$$

which yields the result. □

The *conditional column space* $E(\mathcal{M},(\mathcal{E}_k);l_c^2)$ and the *conditional row space* $E(\mathcal{M},(\mathcal{E}_k);l_r^2)$ are defined as the completion in the norms given in (5.15) and (5.16), respectively, of the linear space of all finitely nonzero sequences in $E(\mathcal{M})$. If $\mathcal{M}_j = \mathcal{M}_k$ for all $j,k \geq 1$, then we set $\mathcal{E} := \mathcal{E}_k$ $(k \geq 1)$ and simply write $E(\mathcal{M},\mathcal{E};l_c^2)$ and $E(\mathcal{M},\mathcal{E};l_r^2)$ instead of $E(\mathcal{M},(\mathcal{E}_k);l_c^2)$ and $E(\mathcal{M},(\mathcal{E}_k);l_r^2)$, respectively.

6

Noncommutative Khintchine inequalities

In this chapter we study noncommutative Khintchine inequalities, which provide estimates for a randomized sum of elements of a noncommutative quasi-Banach function space in terms of a noncommutative 'square function norm' of the elements in question. In the first two sections of this chapter we prove two different types of noncommutative Khintchine inequalities for randomized sums involving Rademacher random variables. The main results, Theorems 6.1 and 6.7, state that these inequalities hold for symmetric spaces with finite upper Boyd index and finite concavity, respectively. As will be seen in the third section, the latter two conditions are necessary. In the fourth section we focus on Khintchine inequalities for randomized sums with operator coefficients and apply these inequalities to derive some new results in the interpolation theory for row and column spaces. In the final section we use Khintchine-type inequalities to derive new Burkholder-Gundy inequalities for noncommutative martingale differences sequences in a noncommutative Banach function space.

6.1 Spaces with finite upper Boyd index

Recall the notation

$$\|(x_i)\|_{E(\mathcal{M};l^2_c)} = \Big\|\Big(\sum_i x_i^* x_i\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})}; \ \|(x_i)\|_{E(\mathcal{M};l^2_r)} = \Big\|\Big(\sum_i x_i x_i^*\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})}.$$

At the end of Section 5.3 we observed that, if E is a symmetric (quasi-)Banach function space on $(0,\infty)$ and $\mathcal{M}$ is a semi-finite von Neumann algebra, then these expression define two (quasi-)norms on the linear space of all finite sequences in $E(\mathcal{M})$.

Throughout, we let (r_i) be a Rademacher sequence defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We first focus on the proof of the following noncommutative Khintchine inequality.

Theorem 6.1. *Let $0 < \alpha \leq \infty$ and let $\mathcal{M}$ be a von Neumann algebra equipped with a normal, semi-finite, faithful trace τ satisfying $\tau(\mathbf{1}) = \alpha$. Suppose E is a symmetric quasi-Banach function space on $(0, \alpha)$ which is p-convex for some $0 < p < \infty$ and satisfies $q_E < \infty$. Then*

$$\Big\| \sum_i r_i \otimes x_i \Big\|_{E(L^\infty \overline{\otimes} \mathcal{M})} \lesssim_E \max\Big\{ \|(x_i)\|_{E(\mathcal{M};l_c^2)}, \|(x_i)\|_{E(\mathcal{M};l_r^2)} \Big\}, \qquad (6.1)$$

for any finite sequence (x_i) in $E(\mathcal{M})$.

For a fully symmetric Banach function spaces with $1 < p_E \leq q_E < \infty$, which is separable or the dual of a separable space, this result was obtained in [94], Theorem 1.1. The case where $1 \leq p_E \leq q_E < \infty$ was left as an open question there. Our approach is completely different from the one in [94].

The main idea of our proof of Theorem 6.1 is to deduce (6.1) by a truncation argument from the case $E = L^q$, $1 \leq q < \infty$, which is stated in Theorem 3.5. We build on the work of ([96], Proposition 2.d.1), who used this strategy to prove Theorem 6.1 for (commutative) symmetric Banach function spaces.

Two key observations for our proof are Lemmas 5.8 and 5.11. Moreover, we use Chebyshev's inequality (Lemma 5.18) and the following result, which allows us to reduce the proof of Theorem 6.1 to the case of self-adjoint elements.

Lemma 6.2. *Let $\mathcal{M}$ be a semi-finite von Neumann algebra with a normal, semi-finite, faithful trace τ satisfying $\tau(\mathbf{1}) = \alpha$. Suppose that E is a symmetric quasi-Banach function space on $(0, \alpha)$ which is p-convex for some $0 < p < \infty$ and that for any finite sequence (x_k) of self-adjoint elements in $E(\mathcal{M})$ we have*

$$\Big\| \sum_k r_k \otimes x_k \Big\|_{E(L^\infty \overline{\otimes} \mathcal{M})} \lesssim_E \Big\| \Big(\sum_k x_k^2 \Big)^{\frac{1}{2}} \Big\|_{E(\mathcal{M})}.$$

Then, for any finite sequence (x_k) in $E(\mathcal{M})$,

$$\Big\| \sum_k r_k \otimes x_k \Big\|_{E(L^\infty \overline{\otimes} \mathcal{M})} \lesssim_E \max\Big\{ \|(x_k)\|_{E(\mathcal{M};l_c^2)}, \|(x_k)\|_{E(\mathcal{M};l_r^2)} \Big\}.$$

Furthermore, if

$$\mathbb{E}\Big\| \sum_k r_k x_k \Big\|_{E(\mathcal{M})} \lesssim_E \Big\| \Big(\sum_k x_k^2 \Big)^{\frac{1}{2}} \Big\|_{E(\mathcal{M})}$$

for any finite sequence (x_k) of self-adjoint elements in $E(\mathcal{M})$, then for any finite sequence (x_k) in $E(\mathcal{M})$,

$$\mathbb{E}\Big\| \sum_k r_k x_k \Big\|_{E(\mathcal{M})} \lesssim_E \max\Big\{ \|(x_k)\|_{E(\mathcal{M};l_c^2)}, \|(x_k)\|_{E(\mathcal{M};l_r^2)} \Big\}.$$

Proof. Given a finite sequence $(x_k)_{k=1}^n$ in $E(\mathcal{M})$, set

$$x_k = y_k + iz_k, \quad y_k^* = y_k, \quad z_k^* = z_k, \quad 1 \le k \le n,$$

and notice that

$$0 \le y_k^2, z_k^2 \le y_k^2 + z_k^2 = \tfrac{1}{2}(x_k^* x_k + x_k x_k^*), \quad 1 \le k \le n.$$

Hence, using that the square root is operator monotone,

$$\Big(\sum_k y_k^2\Big)^{\frac{1}{2}}, \Big(\sum_k z_k^2\Big)^{\frac{1}{2}} \le \Big(\sum_k \tfrac{1}{2}(|x_k|^2 + |x_k^*|^2)\Big)^{\frac{1}{2}} = \tfrac{1}{\sqrt{2}}\Big(\sum_k (|x_k|^2 + |x_k^*|^2)\Big)^{\frac{1}{2}}.$$

By our assumption,

$$\begin{aligned}
\Big\|\sum_k r_k \otimes x_k\Big\|_E &\lesssim_E \Big(\Big\|\sum_k r_k \otimes y_k\Big\|_E + \Big\|\sum_k r_k \otimes z_k\Big\|_E\Big) \\
&\lesssim_E \Big(\Big\|\Big(\sum_k y_k^2\Big)^{\frac{1}{2}}\Big\|_E + \Big\|\Big(\sum_k z_k^2\Big)^{\frac{1}{2}}\Big\|_E\Big) \\
&\lesssim \Big\|\Big(\sum_k |x_k|^2 + |x_k^*|^2\Big)^{\frac{1}{2}}\Big\|_E \\
&= \Big\|\sum_k |x_k|^2 + |x_k^*|^2\Big\|_{E_{(2)}}^{\frac{1}{2}} \\
&\lesssim \Big(\Big\|\sum_k |x_k|^2\Big\|_{E_{(2)}} + \Big\|\sum_k |x_k^*|^2\Big\|_{E_{(2)}}\Big)^{\frac{1}{2}} \\
&= \Big(\Big\|\Big(\sum_k |x_k|^2\Big)^{\frac{1}{2}}\Big\|_E^2 + \Big\|\Big(\sum_k |x_k^*|^2\Big)^{\frac{1}{2}}\Big\|_E^2\Big)^{\frac{1}{2}} \\
&\le 2\max\Big\{\Big\|\Big(\sum_k |x_k|^2\Big)^{\frac{1}{2}}\Big\|_E, \Big\|\Big(\sum_k |x_k^*|^2\Big)^{\frac{1}{2}}\Big\|_E\Big\}.
\end{aligned}$$

□

We are now ready to prove the first main result of this section.

Proof. (of Theorem 6.1) By Lemma 6.2, it suffices to consider the case where $x_1, \ldots, x_n$ are self-adjoint. Let B_q be the constant in (3.10). We begin by showing that for any $q \in [1, \infty)$ and $v > 0$

$$d\Big(B_q v; \sum_i r_i \otimes x_i\Big) \le 3d(v; f \otimes \phi_q), \tag{6.2}$$

where $f : (0, \alpha) \to [0, \infty]$ and $\phi_q : (0, 1) \to (0, \infty)$ are defined by $f(s) = \mu_s((\sum_i x_i^2)^{\frac{1}{2}})$ and $\phi_q(t) = t^{-\frac{1}{q}}$.

Fix $v > 0$. Define $\hat{e}_v = \mathbf{1} \otimes e_v$, where $e_v = e^{(\sum_i x_i^2)^{\frac{1}{2}}}[0, v]$, then $\hat{e}_v^\perp = \mathbf{1} \otimes e_v^\perp = \mathbf{1} \otimes e^{(\sum_i x_i^2)^{\frac{1}{2}}}(v, \infty)$. Since $d(v+w; a+b) \leq d(v; a) + d(w; b)$ for any $a, b \in S(\tau)$ and $v, w \geq 0$, we have

$$d\Big(B_q v; \sum_i r_i \otimes x_i\Big) \leq d\Big(B_q v; \hat{e}_v \sum_i r_i \otimes x_i \hat{e}_v\Big) + d\Big(0; \hat{e}_v \sum_i r_i \otimes x_i \hat{e}_v^\perp\Big)$$
$$+ d\Big(0; \hat{e}_v^\perp \sum_i r_i \otimes x_i \hat{e}_v\Big) + d\Big(0; \hat{e}_v^\perp \sum_i r_i \otimes x_i \hat{e}_v^\perp\Big).$$

Recall that if $y \in S(\tau)$ and e is a finite trace projection in $\mathcal{M}$, then $d(v; ye) = d(v; ey) \leq \tau(e)$. Hence,

$$d\Big(0; \hat{e}_v^\perp \sum_i r_i \otimes x_i \hat{e}_v\Big) \leq \mathbb{E} \otimes \tau(\hat{e}_v^\perp) = \tau(e_v^\perp) = d\Big(v; \Big(\sum_i |x_i|^2\Big)^{\frac{1}{2}}\Big) = d(v; f),$$

and analogously,

$$d\Big(0; \hat{e}_v \sum_i r_i \otimes x_i \hat{e}_v^\perp\Big), d\Big(0; \hat{e}_v^\perp \sum_i r_i \otimes x_i \hat{e}_v^\perp\Big) \leq d\Big(v; \Big(\sum_i |x_i|^2\Big)^{\frac{1}{2}}\Big) = d(v; f).$$

We estimate the remaining term using the noncommutative Khintchine inequality in $L^q(\mathcal{M})$ (Theorem 3.5) and Chebyshev's inequality (Lemma 5.18).

$$d\Big(B_q v; \hat{e}_v \Big(\sum r_i \otimes x_i\Big)\hat{e}_v\Big) \leq (B_q v)^{-q} \Big\| \sum_i r_i \otimes e_v x_i e_v \Big\|_{L^q(\mathcal{M})}^q$$
$$\leq v^{-q} \Big\| \Big(\sum_i |e_v x_i e_v|^2\Big)^{\frac{1}{2}} \Big\|_{L^q(\mathcal{M})}^q,$$

Observe that $E(\mathcal{M}) \subset S_0(\tau)$. Indeed, otherwise we would have $\mathbf{1} \in E(\mathcal{M})$ and hence $q_E = \infty$. Thus we have $\tau(e_v^\perp) < \infty$ for all $v > 0$ and so

$$\mu_t\Big(\Big(\sum_i x_i^2\Big)^{\frac{1}{2}} e_v\Big) = \mu_{t+\tau(e_v^\perp)}\Big(\Big(\sum_i x_i^2\Big)^{\frac{1}{2}}\Big) \qquad (t \geq 0).$$

Moreover, $\sum_i |e_v x_i e_v|^2 \leq |(\sum_i x_i^2)^{\frac{1}{2}} e_v|^2$, and hence we obtain

$$\Big\| \Big(\sum_i |e_v x_i e_v|^2\Big)^{\frac{1}{2}} \Big\|_{L^q(\mathcal{M})} \leq \Big\| \Big(\sum_i x_i^2\Big)^{\frac{1}{2}} e_v \Big\|_{L^q(\mathcal{M})}^q$$
$$= \int_{\mu((\sum_i x_i^2)^{\frac{1}{2}}) \leq v} \mu_s\Big(\Big(\sum_i x_i^2\Big)^{\frac{1}{2}}\Big)^q ds. \quad (6.3)$$

Collecting our estimates we obtain, using Lemma 5.11, for any $v > 0$ and $q \in [1, \infty)$

$$d\Big(B_q v; \sum_i r_i \otimes x_i\Big) \leq \Big(\int_{\{f \leq v\}} \Big(\frac{f(s)}{v}\Big)^q ds + 3d(v; f)\Big) \leq 3d(v; f \otimes \phi_q),$$

which proves (6.2). It follows that

$$\mu_t\Big(\sum_i r_i \otimes x_i\Big) \leq B_q \mu_{\frac{t}{3}}(f \otimes \phi_q) \qquad (t \geq 0).$$

Since E is symmetric and $D_{\frac{1}{3}}$ is bounded on E, we have

$$\begin{aligned}\Big\|\sum_i r_i \otimes x_i\Big\|_{E(L^\infty \overline{\otimes} \mathcal{M})} &= \Big\|\mu\Big(\sum_i r_i \otimes x_i\Big)\Big\|_{E(0,\alpha)} \\ &\leq B_q \|D_{\frac{1}{3}}\|_{E \to E} \|f \otimes \phi_q\|_{E((0,\alpha)\times(0,1))}.\end{aligned}$$

In particular this holds for any $q > q_E$ and hence, by Lemma 5.8, our proof is complete. □

By an argument similar to the one in Theorem 6.1 we obtain the following result for spaces with $q_E < 2$. We provide the full details for the reader's convenience.

Theorem 6.3. *Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal, faithful trace τ satisfying $\tau(\mathbf{1}) = \alpha$, E is a symmetric quasi-Banach function space on $(0, \alpha)$ which is p-convex for some $0 < p < \infty$ and suppose $q_E < 2$. Then for any finite sequence (x_i) in $E(\mathcal{M})$ we have,*

$$\Big\|\sum_i r_i \otimes x_i\Big\|_{E(L^\infty \overline{\otimes} \mathcal{M})} \leq c_E \inf\Big\{\|(y_i)\|_{E(\mathcal{M};l_c^2)} + \|(z_i)\|_{E(\mathcal{M};l_r^2)}\Big\}, \qquad (6.4)$$

where the infimum is taken over all decompositions $x_i = y_i + z_i$ in $E(\mathcal{M})$. If E is a symmetric Banach function space on $(0, \infty)$ which is separable or the dual of a separable space and satisfies $q_E < 2$ then

$$\Big\|\sum_i r_i \otimes x_i\Big\|_{E(L^\infty \overline{\otimes} \mathcal{M})} \simeq_E \inf\Big\{\|(y_i)\|_{E(\mathcal{M};l_c^2)} + \|(z_i)\|_{E(\mathcal{M};l_r^2)}\Big\}$$

Proof. Fix y_i, z_i in $E(\mathcal{M})$ such that $x_i = y_i + z_i$ for $1 \leq i \leq n$. Fix $v > 0$ and $q_E < q < 2$. Define $y = (\sum |y_i|^2)^{\frac{1}{2}}$, $z = (\sum |z_i^*|^2)^{\frac{1}{2}}$ and set $\hat{e}_v^y = 1 \otimes e_v^y$, $\hat{e}_v^z = 1 \otimes e_v^z$. Set $f_y(s) = \mu_s(y)$, $f_z(s) = \mu_s(z)$ and $f(s) = \mu_s(y+z)$. We first note that

$$\begin{aligned}&d\Big(v; \sum_i r_i \otimes x_i\Big) \\ &\leq d\Big(\frac{v}{16}; \hat{e}_v^y \hat{e}_v^z \Big(\sum_i r_i \otimes x_i\Big) \hat{e}_v^y \hat{e}_v^z\Big) + d\Big(\frac{v}{16}; \hat{e}_v^y \hat{e}_v^z \Big(\sum_i r_i \otimes x_i\Big) \hat{e}_v^y (\hat{e}_v^z)^\perp\Big) \\ &\quad + d\Big(\frac{v}{8}; \hat{e}_v^y \hat{e}_v^z \Big(\sum_i r_i \otimes x_i\Big)(\hat{e}_v^y)^\perp\Big) + d\Big(\frac{v}{4}; \hat{e}_v^y (\hat{e}_v^z)^\perp \Big(\sum_i r_i \otimes x_i\Big)\Big)\end{aligned}$$

$$+ d\Big(\frac{v}{2}; (\hat{e}^y_v)^\perp\Big(\sum_i r_i \otimes x_i\Big)\Big). \tag{6.5}$$

Reasoning as in the proof of Theorem 6.1 we obtain by Chebyshev's inequality and the noncommutative Khintchine inequality for $L^q(\mathcal{M})$,

$$d\Big(v; \hat{e}^y_v\hat{e}^z_v\Big(\sum_i r_i \otimes x_i\Big)\hat{e}^y_v\hat{e}^z_v\Big) \lesssim_q d(v; f_y \otimes \phi_q) + d(v; f_z \otimes \phi_q) \le 2d(v; f \otimes \phi_q).$$

Moreover,

$$\begin{aligned} &d\Big(\frac{v}{16}; \hat{e}^y_v\hat{e}^z_v\Big(\sum_i r_i \otimes x_i\Big)\hat{e}^y_v(\hat{e}^z_v)^\perp\Big) \\ &\qquad \le \mathbb{E}\otimes\tau((\hat{e}^z_v)^\perp) = d(v; z) \le d(v; f_z \otimes \phi_q) \le d(v; f \otimes \phi_q), \end{aligned}$$

and analogously it follows that that the remaining terms in (6.5) are bounded by $d(v; f \otimes \phi_q)$. We conclude that there is a constant C_q depending only on q such that for all $v > 0$,

$$d\Big(v; \sum_i r_i \otimes x_i\Big) \le C_q d(v; f \otimes \phi_q).$$

Since the dilation $D_{C_q^{-1}}$ is bounded on E, we obtain by Lemma 5.8

$$\begin{aligned} \Big\|\sum_i r_i \otimes x_i\Big\|_{E(L^\infty\overline{\otimes}\mathcal{M})} &\lesssim_{q,E} \|f \otimes \phi_q\|_{E((0,\alpha)\times(0,1))} \lesssim_{q,E} \|f\|_{E(0,\alpha)} \\ &\lesssim \Big\|\Big(\sum_i |y_i|^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})} + \Big\|\Big(\sum_i |z_i^*|^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})}. \end{aligned}$$

By taking the infimum over all possible decompositions $x_i = y_i + z_i$ in $E(\mathcal{M})$ we obtain (6.4).

The final statement follows from [94], Theorem 1.1 (1), which states that the reverse of the inequality in (6.4) holds if E is separable or the dual of a separable space and $q_E < \infty$. □

By a duality argument we obtain the following.

Corollary 6.4. *Let $\mathcal{M}$ be a semi-finite von Neumann algebra. If E is a separable symmetric Banach function space on $(0,\infty)$ with $p_E > 1$, then, for any finite sequence (x_i) in $E(\mathcal{M})$,*

$$\inf\Big\{\|(y_i)\|_{E(\mathcal{M};l^2_c)} + \|(z_i)\|_{E(\mathcal{M};l^2_r)}\Big\} \lesssim_E \Big\|\sum_i r_i \otimes x_i\Big\|_{E(L^\infty\overline{\otimes}\mathcal{M})}, \tag{6.6}$$

where the infimum is taken over all decompositions $x_i = y_i + z_i$ in $E(\mathcal{M})$. If $p_E > 2$, then

$$\max\Big\{\|(x_i)\|_{E(\mathcal{M};l^2_c)}, \|(x_i)\|_{E(\mathcal{M};l^2_r)}\Big\} \lesssim_E \Big\|\sum_i r_i \otimes x_i\Big\|_{E(L^\infty\overline{\otimes}\mathcal{M})}.$$

Proof. Since E is separable, we have by (5.13) the isometric identification

$$(E(\mathcal{M};l_c^2)+E(\mathcal{M};l_r^2))^* = E^{\times}(\mathcal{M};l_c^2)\cap E^{\times}(\mathcal{M};l_r^2).$$

If (y_i) is a finite sequence in $E^{\times}(\mathcal{M})$, then

$$\begin{aligned}
\sum_{i=1}^{n}\tau(x_iy_i) &= \sum_{i,j=1}^{n}\mathbb{E}(r_ir_j)\tau(x_iy_j)\\
&= \mathbb{E}\otimes\tau\Big(\Big(\sum_{i=1}^{n}r_i\otimes x_i\Big)\Big(\sum_{j=1}^{n}r_j\otimes y_j\Big)\Big)\\
&\leq \Big\|\sum_{i=1}^{n}r_i\otimes x_i\Big\|_{E(L^{\infty}\overline{\otimes}\mathcal{M})}\Big\|\sum_{j=1}^{n}r_j\otimes y_j\Big\|_{E^{\times}(L^{\infty}\overline{\otimes}\mathcal{M})}\\
&\lesssim_E \Big\|\sum_{i=1}^{n}r_i\otimes x_i\Big\|_{E(L^{\infty}\overline{\otimes}\mathcal{M})}\|(y_j)\|_{E^{\times}(\mathcal{M};l_c^2)\cap E^{\times}(\mathcal{M};l_r^2)},
\end{aligned}$$

where the final inequality follows by Theorem 6.1. The first statement now follows by taking the supremum over all finite sequences (y_i) in $E^{\times}(\mathcal{M})$.

The second statement follows similarly from Theorem 6.3. □

In the proof of Theorems 6.1 and 6.3 we can use the noncommutative Khintchine inequalities in [71], Remark 3.5, to obtain the following version where the Rademacher sequence is replaced by a sequence of independent noncommutative random variables (for the definition of independence see Definition 7.1 below).

Corollary 6.5. *Let $\mathcal{M},\mathcal{N}$ be von Neumann algebras equipped with normal, faithful, finite traces τ and σ, respectively, satisfying $\tau(\mathbf{1})=\alpha$ and $\sigma(\mathbf{1})=\beta$. Suppose E is a p-convex ($0<p<\infty$) symmetric quasi-Banach function space on $(0,\alpha\beta)$ with $q_E<\infty$. Let $q>\max\{2,q_E\}$ and $(\alpha_i)_{i\geq 1}$ be a sequence in $L^q(\mathcal{N})$ which is independent with respect to σ, satisfies $\sigma(\alpha_i)=0$ and is such that $d_q=\sup_{i\geq 1}\|\alpha_i\|_q<\infty$. Then*

$$\Big\|\sum_i\alpha_i\otimes x_i\Big\|_{E(\mathcal{N}\overline{\otimes}\mathcal{M})}\lesssim_{E,d_q}\max\Big\{\|(x_i)\|_{E(\mathcal{M};l_c^2)},\|(x_i)\|_{E(\mathcal{M};l_r^2)}\Big\},$$

for any finite sequence (x_i) in $E(\mathcal{M})$. If $q_E<2$, then

$$\Big\|\sum_i\alpha_i\otimes x_i\Big\|_{E(\mathcal{N}\overline{\otimes}\mathcal{M})}\lesssim_{E,d_q}\inf\Big\{\|(y_i)\|_{E(\mathcal{M};l_c^2)}+\|(z_i)\|_{E(\mathcal{M};l_r^2)}\Big\},$$

where the infimum is taken over all decompositions $x_i=y_i+z_i$ in $E(\mathcal{M})$.

The dual version of this result reads as follows.

Corollary 6.6. *Let $\mathcal{M},\mathcal{N}$ be von Neumann algebras equipped with a normal, faithful, finite trace. Suppose E is a separable symmetric Banach function space on $(0,\infty)$ with $p_E > 1$. Let $q > \max\{2, q_{E^\times}\}$ and $(\alpha_i)_{i\geq 1}$ be a sequence in $L^q(\mathcal{N})$ which is independent with respect to σ, satisfies $\sigma(\alpha_i) = 0$ and is such that $d_q = \sup_{i\geq 1}\|\alpha_i\|_q < \infty$. Then*

$$\inf\Big\{\|(y_i)\|_{E(\mathcal{M};l^2_c)}, \|(z_i)\|_{E(\mathcal{M};l^2_r)}\Big\} \lesssim_{E,d_q} \Big\|\sum_i \alpha_i\otimes x_i\Big\|_{E(\mathcal{N}\overline{\otimes}\mathcal{M})}, \tag{6.7}$$

where the infimum is taken over all decompositions $x_i = y_i + z_i$ in $E(\mathcal{M})$.

Proof. Since E is separable we have the isometric identification

$$(E(\mathcal{M};l^2_c) + E(\mathcal{M};l^2_r))^* = E^\times(\mathcal{M};l^2_c)\cap E^\times(\mathcal{M};l^2_r).$$

If (y_i) is a finite sequence in $E^\times(\mathcal{M})$, then

$$\begin{aligned}\sum_{i=1}^n \tau(x_iy_i) &= \sum_{i,j=1}^n \sigma(\alpha_i\alpha_j)\tau(x_iy_j)\\ &= \sigma\otimes\tau\Big(\Big(\sum_{i=1}^n \alpha_i\otimes x_i\Big)\Big(\sum_{j=1}^n \alpha_j\otimes y_j\Big)\Big)\\ &\leq \Big\|\sum_{i=1}^n \alpha_i\otimes x_i\Big\|_{E(\mathcal{N}\overline{\otimes}\mathcal{M})}\Big\|\sum_{j=1}^n \alpha_j\otimes y_j\Big\|_{E^\times(\mathcal{N}\overline{\otimes}\mathcal{M})}\\ &\lesssim_E \Big\|\sum_{i=1}^n \alpha_i\otimes x_i\Big\|_{E(\mathcal{N}\overline{\otimes}\mathcal{M})}\|(y_j)\|_{E^\times(\mathcal{M};l^2_c)\cap E^\times(\mathcal{M};l^2_r)},\end{aligned}$$

where the final inequality follows by Corollary 6.5. The result now follows by taking the supremum over all finite sequences (y_i) in $E^\times(\mathcal{M})$. □

6.2 Spaces with finite concavity

We now turn our attention to the following, different type of Khintchine inequality. Our proof proceeds along the same lines as in Theorem 6.1.

Theorem 6.7. *Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal, semi-finite, faithful trace τ satisfying $\tau(\mathbf{1}) = \alpha$. Suppose E is a symmetric quasi-Banach function space on $(0,\alpha)$ which is p-convex for some $0 < p < \infty$ and r-concave for some $r < \infty$. Then*

$$\mathbb{E}\Big\|\sum_i r_ix_i\Big\|_{E(\mathcal{M})} \lesssim_E \max\Big\{\|(x_i)\|_{E(\mathcal{M};l^2_c)}, \|(x_i)\|_{E(\mathcal{M};l^2_r)}\Big\}, \tag{6.8}$$

for any finite sequence (x_i) in $E(\mathcal{M})$.

Proof. By Lemma 6.2, it suffices to consider the case where $x_1, \dots, x_n$ are self-adjoint. Fix $q \geq 1$ such that $q > r$ and define $f : (0, \alpha) \to [0, \infty]$ by $f(s) = \mu_s((\sum_i x_i^2)^{\frac{1}{2}})$ and $\phi_q : (0,1) \to (0, \infty)$ by $\phi_q(t) = t^{-\frac{1}{q}}$. Since, by Proposition 4.10, E is q-concave for any $q \geq r$,

$$\mathbb{E}\Big\| \sum_i r_i x_i \Big\|_{E(\mathcal{M})} \leq \Big(\mathbb{E}\Big\| \mu\Big(\sum_i r_i x_i \Big) \Big\|^q_{E(\mathcal{M})} \Big)^{\frac{1}{q}}$$
$$\leq M_{(q)}(E) \Big\| \Big(\mathbb{E}\Big| \mu\Big(\sum_i r_i x_i \Big) \Big|^q \Big)^{\frac{1}{q}} \Big\|_E,$$

where $M_{(q)}(E)$ is the q-concavity constant of E. To see the last inequality, note that

$$\Big(\mathbb{E}\Big\| \mu\Big(\sum_i r_i x_i \Big) \Big\|^q_E \Big)^{\frac{1}{q}} = \Big(\sum_{(\varepsilon_i) \in \{-1,1\}^n} \frac{1}{2^n} \Big\| \mu\Big(\sum_i \varepsilon_i x_i \Big) \Big\|^q_E \Big)^{\frac{1}{q}}$$
$$\leq M_{(q)}(E) 2^{-\frac{n}{q}} \Big\| \Big(\sum_{(\varepsilon_i) \in \{-1,1\}^n} \Big| \mu\Big(\sum_i \varepsilon_i x_i \Big) \Big|^q \Big)^{\frac{1}{q}} \Big\|_E$$
$$= M_{(q)}(E) \Big\| \Big(\mathbb{E}\Big| \mu\Big(\sum_i r_i x_i \Big) \Big|^q \Big)^{\frac{1}{q}} \Big\|_E.$$

For any $v > 0$ we set $e_v = e^{(\sum_i |x_i|^2)^{\frac{1}{2}}}[0, v]$. Recall that $\mu_{s+t}(a + b) \leq \mu_s(a) + \mu_t(b)$ and $d(v+w; a+b) \leq d(v; a) + d(w; b)$ for all $a, b \in S(\tau)$ and $s, t, v, w \geq 0$. Let B_q be the constant in (3.10). By the triangle inequality in $L^q(\Omega)$, we have for any $v > 0$

$$d\Big(4B_q v; \Big(\mathbb{E}\Big| \mu\Big(\sum_i r_i x_i \Big) \Big|^q \Big)^{\frac{1}{q}} \Big) \tag{6.9}$$
$$\leq d\Big(4B_q v; \Big(\mathbb{E}\Big| D_{\frac{1}{4}} \mu\Big(e_v \sum_i r_i x_i e_v \Big) \Big|^q \Big)^{\frac{1}{q}} \Big)$$
$$+ d\Big(0; \Big(\mathbb{E}\Big| D_{\frac{1}{4}} \mu\Big(e_v^\perp \sum_i r_i x_i e_v \Big) \Big|^q \Big)^{\frac{1}{q}} \Big)$$
$$+ d\Big(0; \Big(\mathbb{E}\Big| D_{\frac{1}{4}} \mu\Big(e_v \sum_i r_i x_i e_v^\perp \Big) \Big|^q \Big)^{\frac{1}{q}} \Big)$$
$$+ d\Big(0; \Big(\mathbb{E}\Big| D_{\frac{1}{4}} \mu\Big(e_v^\perp \sum_i r_i x_i e_v^\perp \Big) \Big|^q \Big)^{\frac{1}{q}} \Big).$$

Recall that if e is a finite trace projection we have $\mu_t(ye) = \mu_t(ey) = 0$ for all $t \geq \tau(e)$. Therefore,

$$d\Big(0; \Big(\mathbb{E}\Big| D_{\frac{1}{4}} \mu\Big(e_v^\perp \sum_i r_i x_i e_v \Big) \Big|^q \Big)^{\frac{1}{q}} \Big)$$

$$\leq 4\tau(e_v^\perp) = 4d\Big(v; \Big(\sum_i |x_i|^2\Big)^{\frac{1}{2}}\Big) = 4d(v; f),$$

and analogously,

$$d\Big(0; \Big(\mathbb{E}\Big|D_{\frac{1}{4}}\mu\Big(e_v \sum_i r_i x_i e_v^\perp\Big)\Big|^q\Big)^{\frac{1}{q}}\Big), d\Big(0; \Big(\mathbb{E}\Big|D_{\frac{1}{4}}\mu\Big(e_v^\perp \sum_i r_i x_i e_v^\perp\Big)\Big|^q\Big)^{\frac{1}{q}}\Big).$$

are bounded by $d(v; f)$. We estimate the remaining term in (6.9) using Chebyshev's inequality (Lemma 5.18) and the noncommutative Khintchine inequality in $L^q(\mathcal{M})$ (Theorem 3.5). We obtain

$$\begin{aligned}
&d\Big(4B_q v; \Big(\mathbb{E}\Big(D_{\frac{1}{4}}\mu\Big(\sum_i r_i e_v x_i e_v\Big)^q\Big)\Big)^{\frac{1}{q}}\Big) \\
&\quad\leq (4B_q v)^{-q} \int_0^\infty \mathbb{E}\Big(\mu_{\frac{t}{4}}\Big(\sum_i r_i e_v x_i e_v\Big)^q\Big) dt \\
&\quad= (4B_q v)^{-q} \mathbb{E}\Big\|D_{\frac{1}{4}}\mu\Big(\sum_i r_i e_v x_i e_v\Big)\Big\|_{L^q(0,\infty)}^q \\
&\quad= (B_q v)^{-q} \mathbb{E}\Big\|\sum_i r_i e_v x_i e_v\Big\|_{L^q(\mathcal{M})}^q \\
&\quad\leq v^{-q} \Big\|\Big(\sum_i |e_v x_i e_v|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(\mathcal{M})}^q \\
&\quad\leq v^{-q} \int_{\{f \leq v\}} f(s)^q ds,
\end{aligned}$$

where the last inequality follows by (6.3).
By Lemma 5.11 we have

$$v^{-q} \int_{\{f \leq v\}} f(s)^q ds + d(v; f) = d(v; f \otimes \phi_q)$$

for all $v > 0$ and so,

$$\begin{aligned}
d\Big(4B_q v; \Big(\mathbb{E}\Big|\mu_t\Big(\sum_i r_i x_i\Big)\Big|^q\Big)^{\frac{1}{q}}\Big) &\leq v^{-q} \int_{\{f \leq v\}} f(s)^q ds + 12 d(v; f) \\
&\leq 12 d(v; f \otimes \phi_q).
\end{aligned}$$

Since the dilation operator $D_{\frac{1}{12}}$ is bounded on E we obtain

$$\Big\|\Big(\mathbb{E}\Big|\mu_t\Big(\sum_i r_i x_i\Big)\Big|^q\Big)^{\frac{1}{q}}\Big\|_E \lesssim_{q,E} \|f \otimes \phi_q\|_E.$$

By Lemma 4.9 the r-concavity of E implies that $q_E \leq r < q < \infty$ and hence the result follows from Lemma 5.8. □

By a duality argument we obtain the following.

Corollary 6.8. *Let $\mathcal{M}$ be a semi-finite von Neumann algebra. Suppose E is a separable symmetric Banach function space on $(0,\infty)$ which is p-convex for some $p>1$. Then, for any finite sequence (x_i) in $E(\mathcal{M})$,*

$$\inf\Big\{\|(y_i)\|_{E(\mathcal{M};l^2_c)}+\|(z_i)\|_{E(\mathcal{M};l^2_r)}\Big\}\lesssim_E \mathbb{E}\Big\|\sum_i r_i x_i\Big\|_{E(\mathcal{M})}, \tag{6.10}$$

where the infimum is taken over all decompositions $x_i=y_i+z_i$ in $E(\mathcal{M})$.

Proof. Since E is p-convex for $p>1$, its Köthe dual $E^\times=E^*$ is q-concave, where $\frac{1}{p}+\frac{1}{q}=1$ (cf. Theorem 4.13). Moreover, as E is separable we have the isometric identification

$$(E(\mathcal{M};l^2_c)+E(\mathcal{M};l^2_r))^*=E^\times(\mathcal{M};l^2_c)\cap E^\times(\mathcal{M};l^2_r).$$

If (y_i) is a finite sequence in $E^\times(\mathcal{M})$, then

$$\begin{aligned}
\sum_{i=1}^n \tau(x_iy_i) &= \sum_{i,j=1}^n \mathbb{E}(r_ir_j)\tau(x_iy_j)\\
&= \mathbb{E}\Big(\tau\Big(\Big(\sum_{i=1}^n r_ix_i\Big)\Big(\sum_{j=1}^n r_jy_j\Big)\Big)\Big)\\
&\leq \mathbb{E}\Big(\Big\|\sum_{i=1}^n r_ix_i\Big\|_{E(\mathcal{M})}\Big\|\sum_{j=1}^n r_jy_j\Big\|_{E^\times(\mathcal{M})}\Big)\\
&\leq \Big(\mathbb{E}\Big\|\sum_{i=1}^n r_ix_i\Big\|^2_{E(\mathcal{M})}\Big)^{\frac{1}{2}}\Big(\mathbb{E}\Big\|\sum_{j=1}^n r_jy_j\Big\|^2_{E^\times(\mathcal{M})}\Big)^{\frac{1}{2}}\\
&\lesssim_E \Big(\mathbb{E}\Big\|\sum_{i=1}^n r_ix_i\Big\|^2_{E(\mathcal{M})}\Big)^{\frac{1}{2}}\|(y_j)\|_{E^\times(\mathcal{M};l^2_c)\cap E^\times(\mathcal{M};l^2_r)}\\
&\lesssim \mathbb{E}\Big\|\sum_{i=1}^n r_ix_i\Big\|_{E(\mathcal{M})}\|(y_j)\|_{E^\times(\mathcal{M};l^2_c)\cap E^\times(\mathcal{M};l^2_r)},
\end{aligned}$$

where in the last two steps we used Theorem 6.7 and Kahane's inequalities. The result now follows by taking the supremum over all finite sequences (y_i) in $E^\times(\mathcal{M})$. □

Finally, we obtain a new proof of the following known result (see [99], Theorem 1.3(ii) for the first equivalence in (6.12) and [94], Corollary 4.3 for the second). In the proof we use the following facts on Rademacher subspaces. Let (r_i) be a Rademacher sequence defined on $(\Omega,\mathcal{F},\mathbb{P})$. Since $L^p(\mathcal{M})$ is K-convex for $1<p<\infty$ (see e.g. [39]), it follows from the isometric identification

$$L^p(\Omega;L^p(\mathcal{M}))=L^p(L^\infty(\Omega)\overline{\otimes}\mathcal{M})$$

that the n-th Rademacher projection

$$R_n(x) = \sum_{i=1}^{n} r_i \otimes \mathcal{E}_{\mathbb{C}\mathbf{1}\overline{\otimes}\mathcal{M}}((r_i \otimes \mathbf{1})x) \tag{6.11}$$

is bounded on $L^p(L^\infty(\Omega)\overline{\otimes}\mathcal{M})$ for any $1 < p < \infty$. Moreover, for all $n \geq 1$ we have $\|R_n\| \leq C_p$, for some constant C_p depending only on p. If E is a symmetric quasi-Banach function space on $(0, \alpha)$ with $1 < p_E \leq q_E < \infty$, we find by Theorem 5.21 that R_n defines a bounded projection in $E(L^\infty(\Omega)\overline{\otimes}\mathcal{M})$ and $\|R_n\| \leq C_E$ for all $n \geq 1$, where C_E is a constant depending only on E. We let $Rad_n(E)$ denote the image of R_n, i.e. the closed subspace of $E(L^\infty\overline{\otimes}\mathcal{M})$ spanned by the elements $\sum_{i=1}^n r_i \otimes x_i$, where $x_1, \ldots, x_n \in E(\mathcal{M})$.

Corollary 6.9. *Let E be a symmetric Banach function space on $(0, \alpha)$ and suppose E is 2-convex and q-concave for some $q < \infty$. Then, for any semi-finite von Neumann algebra equipped with a normal, semi-finite, faithful trace τ satisfying $\tau(\mathbf{1}) = \alpha$ and any finite sequence (x_i) in $E(\mathcal{M})$ we have*

$$\mathbb{E}\Big\|\sum_i r_i x_i\Big\|_{E(\mathcal{M})} \simeq_E \|(x_i)\|_{E(\mathcal{M};l_c^2)\cap E(\mathcal{M};l_r^2)} \simeq_E \Big\|\sum_i r_i \otimes x_i\Big\|_{E(L^\infty\overline{\otimes}\mathcal{M})}. \tag{6.12}$$

Proof. Since E is q-concave, it has order continuous norm and $q_E \leq q < \infty$ by Lemmas 4.12 and 4.9, respectively. Hence, by Theorems 6.1 and 6.7, it remains to show that

$$\|(x_i)\|_{E(\mathcal{M};l_c^2)\cap E(\mathcal{M};l_r^2)} \lesssim_E \mathbb{E}\Big\|\sum_i r_i x_i\Big\|_{E(\mathcal{M})}; \tag{6.13}$$

$$\|(x_i)\|_{E(\mathcal{M};l_c^2)\cap E(\mathcal{M};l_r^2)} \lesssim_E \Big\|\sum_i r_i \otimes x_i\Big\|_{E(L^\infty\overline{\otimes}\mathcal{M})}. \tag{6.14}$$

To prove (6.13), recall the fact that $E(\mathcal{M})$ is 2-convex whenever E has Fatou norm and E is 2-convex (see e.g. [50] for a proof of this fact). This implies that

$$\begin{aligned}\Big\|\Big(\sum_i |x_i|^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})} &= \Big\|\Big(\sum_i |r_i x_i|^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})} = \Big\|\Big(\mathbb{E}\Big|\sum_i r_i x_i\Big|^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})} \\ &\lesssim \Big(\mathbb{E}\Big\|\sum_i r_i x_i\Big\|_{E(\mathcal{M})}^2\Big)^{\frac{1}{2}} \lesssim \mathbb{E}\Big\|\sum_i r_i x_i\Big\|_{E(\mathcal{M})},\end{aligned}$$

where in the final inequality we apply Kahane's inequality. By applying this to (x_i^*) we see that (6.13) holds.

Since $L^p(\Omega; L^p(\mathcal{M})) = L^p(L^\infty(\Omega)\overline{\otimes}\mathcal{M})$ isometrically for $2 \leq p < \infty$, the above shows that for any finite sequence $(x_i)_{i=1}^n$ in $L^p(\mathcal{M})$,

$$\|(x_i)\|_{L^p(\mathcal{M};l_c^2)} \lesssim \Big\|\sum_i r_i \otimes x_i\Big\|_{L^p(L^\infty\overline{\otimes}\mathcal{M})}. \tag{6.15}$$

Since E is 2-convex and q-concave, E is an interpolation space for the couple (L^2, L^q) by Theorem 4.31. Hence $Rad_n(E)$ is a complemented subspace of $E(L^\infty\overline{\otimes}\mathcal{M})$ and by Theorem 5.5, we obtain

$$\|(x_i)\|_{E(\mathcal{M};l_c^2)} \lesssim_E \Big\|\sum_i r_i \otimes x_i\Big\|_{E(L^\infty\overline{\otimes}\mathcal{M})}$$

by interpolation from (6.15). By applying this to (x_i^*) we see that (6.14) holds. □

6.3 Optimality of the results

We shall now demonstrate that the results in Theorems 6.1 and 6.7 are, in a sense, the best possible.

The following result is shown in [96], Propositions 1.f.12 and 2.b.7, for symmetric Banach function spaces which are separable or have the Fatou property. The proof of these propositions goes through verbatim for symmetric quasi-Banach function spaces.

Lemma 6.10. *Let E be a symmetric quasi-Banach function space on $(0,\alpha)$. Then the following hold:*

(i) E is not q-concave for any $q < \infty$ if and only if for every $\varepsilon > 0$ and any $n \in \mathbb{N}$ there exists a sequence $(x_i)_{i=1}^\infty$ of mutually disjoint elements in E such that $\|x_i\| = 1$ for all $i \geq 1$ and

$$1 \leq \Big\|\sum_{i=1}^n x_i\Big\|_{E(0,\alpha)} < 1+\varepsilon.$$

(ii) $q_E = \infty$ if and only if for every $\varepsilon > 0$ and any $n \in \mathbb{N}$ there exists a sequence $(x_i)_{i=1}^\infty$ of mutually disjoint and identically distributed elements in E such that $\|x_i\| = 1$ for all $i \geq 1$ and

$$1 \leq \Big\|\sum_{i=1}^n x_i\Big\|_{E(0,\alpha)} < 1+\varepsilon.$$

The following observation is stated, without proof, in [100]. Let sign denote the sign function, with $\mathrm{sign}(0) := 0$.

Lemma 6.11. *Let $\varepsilon_1, \ldots, \varepsilon_n \in \{-1,1\}^n$. Then, for some $1 \leq k < 2^n$, we have*

$$\varepsilon_j = \varepsilon_n \mathrm{sign}\Big(\sin\Big(\frac{k\pi}{2^{n-j}}\Big)\Big),$$

for all $j = 1, \ldots, n$.

Proof. We prove the statement by induction. Note first that if $n = 1$ we can simply take $k = 1$. Suppose now that the statement holds for $n = m$. Let $\varepsilon_1, \ldots, \varepsilon_{m+1}$ be a sequence of signs. By applying the induction hypothesis to $\varepsilon_2, \ldots, \varepsilon_{m+1}$, we can find some $1 \leq l < 2^m$ such that

$$\varepsilon_{j+1} = \varepsilon_{m+1}\mathrm{sign}\Big(\sin \Big(\frac{l\pi}{2^{m-j}}\Big)\Big), \quad j = 1, \ldots, m+1.$$

Case I: $\varepsilon_1 = \varepsilon_{m+1}$. In this case we may take $k = l$. Indeed, then by our choice of l we clearly have

$$\varepsilon_j = \varepsilon_{m+1}\mathrm{sign}\Big(\sin \Big(\frac{k\pi}{2^{m+1-j}}\Big)\Big), \quad j = 2, \ldots, m+1,$$

and also,

$$\varepsilon_{m+1}\mathrm{sign}\Big(\sin \Big(\frac{k\pi}{2^{m}}\Big)\Big) = \varepsilon_{m+1} = \varepsilon_1.$$

Case I: $\varepsilon_1 = -\varepsilon_{m+1}$. In this case we may take $k = l + 2^m$. Indeed, then

$$\begin{aligned}
&\varepsilon_{m+1}\mathrm{sign}\Big(\sin \Big(\frac{k\pi}{2^{m+1-j}}\Big)\Big) \\
&\quad = \varepsilon_{m+1}\mathrm{sign}\Big(\sin \Big(\frac{l\pi}{2^{m+1-j}} + 2^{j-1}\pi\Big)\Big) \\
&\quad = \begin{cases} -\varepsilon_{m+1} = \varepsilon_1 & \text{if } j = 1, \\ \varepsilon_{m+1}\mathrm{sign}(\sin(\frac{l\pi}{2^{m+1-j}})) = \varepsilon_j & \text{if } j = 2, \ldots, m+1. \end{cases}
\end{aligned}$$

This completes the proof.

We obtain the following two implications. The proof of the first statement is due to B. Maurey ([100], Corollaire 1). We refer to [6], Theorem 7.1, for a different proof of the second statement if E has the Fatou property.

Proposition 6.12. *Let E be a symmetric quasi-Banach function space on $(0, \alpha)$. If E satisfies the Khintchine inequality*

$$\mathbb{E}\Big\| \sum_i r_i x_i \Big\|_{E(0,\alpha)} \lesssim_E \Big\| \Big(\sum_{i=1}^n |x_i|^2 \Big)^{\frac{1}{2}} \Big\|_{E(0,\alpha)}, \tag{6.16}$$

then E is q-concave for some $q < \infty$. On the other hand, if E satisfies the Khintchine inequality

$$\Big\| \sum_i r_i \otimes x_i \Big\|_{E((0,1)\times(0,\alpha))} \lesssim_E \Big\| \Big(\sum_{i=1}^n |x_i|^2 \Big)^{\frac{1}{2}} \Big\|_{E(0,\alpha)}, \tag{6.17}$$

then $q_E < \infty$.

Proof. To prove the first statement, suppose E is not q-concave for any $q < \infty$. We will show that (6.16) cannot hold. By Lemma 6.10 we can find, for any $n \geq 1$, disjoint elements $x_1, \dots, x_{2^n}$ in E with $\|x_i\| = 1$ and $\| \sum_{i=1}^{2^n} x_i \| \leq 2$. Define

$$\varepsilon_k^{(j)} = \operatorname{sign}\Big(\sin \Big(\frac{k\pi}{2^{n-j}} \Big) \Big) \quad (k = 1, \dots, 2^n,\ j = 1, \dots, n)$$

and set

$$y_j = \sum_{k=1}^{2^n} \varepsilon_k^{(j)} x_k \quad (j = 1, \dots, n).$$

As the x_i are disjoint, we have $|y_j| = | \sum_{i=1}^{2^n} x_i|$ and so

$$\Big\| \Big(\sum_{j=1}^n |y_j|^2 \Big)^{\frac{1}{2}} \Big\|_E = n^{\frac{1}{2}} \Big\| \sum_{i=1}^{2^n} x_i \Big\|_E \leq 2n^{\frac{1}{2}}.$$

Let $\varepsilon_1, \dots, \varepsilon_n$ be any sequence of signs. By Lemma 6.11, there is some $1 \leq k < 2^n$ such that

$$\varepsilon_j = \varepsilon_n \varepsilon_k^{(j)}, \quad j = 1, \dots, n.$$

Hence,

$$\begin{aligned} \Big| \sum_{j=1}^n \varepsilon_j y_j \Big| &= \Big| \sum_{j=1}^n \varepsilon_k^{(j)} y_j \Big| = \Big| \sum_{j=1}^n \varepsilon_k^{(j)} \sum_{i=1}^{2^n} \varepsilon_i^{(j)} x_i \Big| \\ &= \Big| \sum_{i=1}^{2^n} \sum_{j=1}^n \varepsilon_k^{(j)} \varepsilon_i^{(j)} x_i \Big| \\ &= |n x_k| + \Big| \sum_{i \neq k} \sum_{j=1}^n \varepsilon_k^{(j)} \varepsilon_i^{(j)} x_i \Big| \\ &\geq n|x_k|. \end{aligned}$$

It is now clear that for any Rademacher sequence $(r_j)_{j \geq 1}$,

$$\mathbb{E} \Big\| \sum_{j=1}^n r_j y_j \Big\|_E > n,$$

which shows that (6.16) cannot hold.

Suppose now that $q_E = \infty$. We will show that (6.17) cannot hold. By Lemma 6.10 we can find, for any $n \geq 1$, disjoint, identically distributed elements $x_1, \dots, x_{2^n}$ in E with $\|x_i\| = 1$ and $\| \sum_{i=1}^{2^n} x_i \| \leq 2$. Defining $y_1, \dots, y_n$ as above, we have $\|(\sum_{j=1}^n |y_j|^2)^{\frac{1}{2}} \|_E \leq 2n^{\frac{1}{2}}$ and, for any $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$, we can find a $1 \leq k < 2^n$ such that

$$\Big| \sum_{j=1}^n \varepsilon_j y_j \Big| \geq n|x_k|.$$

Since the x_i are identically distributed this shows that

$$d\Big(v;\sum_{j=1}^{n}\varepsilon_j y_j\Big)\geq d(v;n|x_1|)\qquad (v\geq 0).$$

Let $(r_i)_{j=1}^{\infty}$ be any Rademacher sequence. Then, for any $v\geq 0$,

$$d\Big(v;\sum_{j=1}^{n}r_j y_j\Big)=\mathbb{E}\Big(d\Big(v;\sum_{j=1}^{n}r_j y_j\Big)\Big)\geq \mathbb{E}(d(v;n|x_1|))=d(v;n|x_1|).$$

As E is symmetric, it follows that

$$\Big\|\sum_{j=1}^{n}r_j\otimes y_j\Big\|_{E((0,1)\times(0,\alpha))}\geq n\|x_1\|_{E(0,\alpha)}=n.$$

Since $n\geq 1$ was arbitrary, we conclude that (6.17) cannot hold. □

We have obtained the following two characterizations.

Theorem 6.13. *Suppose that E is a symmetric quasi-Banach function on $(0,\infty)$ which is p-convex for some $0<p<\infty$. Then the following are equivalent.*

(i) The inequality (6.1) holds for any semi-finite von Neumann algebra $\mathcal{M}$;
(ii) $q_E<\infty$.

Moreover, if this is the case and if E is either a separable symmetric Banach function space or the dual of a separable symmetric space, then

$$\|(x_k)\|_{E(\mathcal{M};l_c^2)+E(\mathcal{M};l_r^2)}\lesssim_E\Big\|\sum_k r_k\otimes x_k\Big\|_{E(L^\infty\overline{\otimes}\mathcal{M})}\lesssim_E\|(x_k)\|_{E(\mathcal{M};l_c^2)\cap E(\mathcal{M};l_r^2)}.$$

Note that the final assertion follows by [94], Theorem 1.1. (1).

Theorem 6.14. *Suppose that E is a symmetric quasi-Banach function space on $(0,\infty)$ which is p-convex for some $0<p<\infty$. Then the following are equivalent.*

(a) The inequality (6.8) holds for any semi-finite von Neumann algebra $\mathcal{M}$;
(b) E is q-concave for some $q<\infty$.

Moreover, if this is the case and if E is a separable symmetric Banach function space and $p>1$, then

$$\|(x_i)\|_{E(\mathcal{M};l_c^2)+E(\mathcal{M};l_r^2)}\lesssim_E\mathbb{E}\Big\|\sum_i r_i x_i\Big\|_{E(\mathcal{M})}\lesssim_E\|(x_i)\|_{E(\mathcal{M};l_c^2)\cap E(\mathcal{M};l_r^2)},$$

for any finite sequence (x_i) in $E(\mathcal{M})$.

6.4 Khintchine inequalities with operator coefficients

In this section we obtain some Khintchine-type inequalities for random sums with operator coefficients. These inequalities are utilized to give an alternative proof of a special case of Theorem 6.1. Even though this method does not cover the full result, it yields a better estimate on the constant in (6.1). At the end of the section we formulate a general interpolation result for intersections of row and column spaces. Throughout, we let $\mathcal{M}$ denote a von Neumann algebra equipped with a normal, semi-finite faithful trace τ.

Theorem 6.15. *Let $\mathcal{N}$ be a von Neumann algebra equipped with a normal, faithful trace σ such that $\sigma(\mathbf{1}) = 1$. Suppose (c_i) is a sequence in $\mathcal{N}$ and C a universal constant such that for any finite sequence (x_i) in $\mathcal{M}$,*

$$\Big\| \sum_i c_i \otimes x_i \Big\| \leq C \max\Big\{ \|(x_i)\|_{E(\mathcal{M};l_c^2)}, \|(x_i)\|_{E(\mathcal{M};l_r^2)} \Big\}. \tag{6.18}$$

If E is a symmetric quasi-Banach function space on $(0,\tau(\mathbf{1}))$ which is p-convex for some $0 < p < \infty$, then, for any finite sequence (x_i) in $E(\mathcal{M})$ we have

$$\Big\| \sum_i c_i \otimes x_i \Big\|_{E(\mathcal{N}\overline{\otimes}\mathcal{M})} \leq C_E \max\Big\{ \|(x_i)\|_{E(\mathcal{M};l_c^2)}, \|(x_i)\|_{E(\mathcal{M};l_r^2)} \Big\}, \tag{6.19}$$

where $C_E \leq C\|D_{\frac{1}{2}}\|_{E\to E}(2D_E)^{\frac{3}{2}}$, with D_E the constant in the quasi-triangle inequality for E. In particular, if E is a symmetric Banach function space, then $C_E \leq 4\sqrt{2}C$.

Proof. At the cost of a factor $(2D_E)^{\frac{3}{2}}$, we may assume that the x_i are self-adjoint (c.f. the proof of Lemma 6.2). Moreover, it suffices to prove that

$$d\Big(Cv; \sum_i c_i \otimes x_i\Big) \leq 2d\Big(v; \Big(\sum_i x_i^2\Big)^{\frac{1}{2}}\Big) \qquad (v > 0). \tag{6.20}$$

Indeed, then by taking the right continuous inverse on both sides we obtain

$$\mu_t\Big(\sum_i c_i \otimes x_i\Big) \leq C\mu_{\frac{t}{2}}\Big(\Big(\sum_i x_i^2\Big)^{\frac{1}{2}}\Big) \qquad (t > 0). \tag{6.21}$$

Since E is symmetric it follows that

$$\begin{aligned}\Big\| \sum_i c_i \otimes x_i \Big\|_{E(L^\infty(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M})} &\leq C\Big\| D_{\frac{1}{2}}\mu\Big(\Big(\sum_i x_i^2\Big)^{\frac{1}{2}}\Big)\Big\|_E \\ &\leq C\|D_{\frac{1}{2}}\|_{E\to E}\Big\|\Big(\sum_i x_i^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})}.\end{aligned}$$

To prove (6.20), fix $v > 0$ and let $e_v = e^{(\sum_i |x_i|^2)^{\frac{1}{2}}}[0,v]$. We make the decomposition

$$d\Big(Cv;\sum_i c_i\otimes x_i\Big)\le d\Big(0;\sum_i c_i\otimes(e_v^\perp x_i e_v^\perp+e_v x_i e_v^\perp)\Big)$$
$$+\,d\Big(0;\sum_i c_i\otimes e_v^\perp x_i e_v\Big)+d\Big(Cv;\sum_i c_i\otimes e_v x_i e_v\Big).$$

It is clear that

$$d\Big(0;\sum_i c_i\otimes(e_v^\perp x_i e_v^\perp+e_v x_i e_v^\perp)\Big),\ d\Big(0;\sum_i c_i\otimes e_v^\perp x_i e_v\Big)$$
$$\le \tau(e_v^\perp)=d\Big(v;\Big(\sum_i x_i^2\Big)^{\frac12}\Big).$$

Moreover, from (6.18) it follows that for any $t\ge 0$,

$$\mu_t\Big(\sum_i c_i\otimes e_v x_i e_v\Big)$$
$$\le\Big\|\sum_i c_i\otimes e_v x_i e_v\Big\|\le C\Big\|\Big(\sum_i|e_v x_i e_v|^2\Big)^{\frac12}\Big\|\le C\Big\|\Big(\sum_i x_i^2\Big)^{\frac12}\Big\|\le Cv.$$

Hence,

$$d\Big(Cv;\sum_i c_i\otimes e_v x_i e_v\Big)=\lambda\Big(t\in(0,\tau(\mathbf{1}))\ :\ \mu_t\Big(\sum_i c_i\otimes e_v x_i e_v\Big)>Cv\Big)=0.$$

This completes the proof. □

Example 6.16. (Free group unitaries) Let G be a discrete group and let $\mathbb{C}G$ be the associated group ring defined by

$$\mathbb{C}G=\Big\{\sum_{g\in G}a_g g\ :\ a_g\in\mathbb{C},\text{finitely many } a_g \text{ nonzero}\Big\}.$$

The multiplication on $\mathbb{C}G$ is defined by

$$\Big(\sum_g a_g g\Big)\Big(\sum_h b_h h\Big):=\sum_k\Big(\sum_{gh=k}a_g b_h\Big)k$$

We can define an involution on $\mathbb{C}G$ by setting

$$\Big(\sum_g a_g g\Big)^*:=\sum_g\overline{a_g}g^{-1}.$$

Consider the Hilbert space $l^2(G)$ with its canonical basis $\{\delta_g:\ g\in G\}$. The *left regular representation* of G is the linear $*$-homomorphism $\lambda:\mathbb{C}G\to B(l^2(G))$ which is defined by

$$\lambda(g)\delta_h:=\delta_{gh}\qquad(g,h\in G)$$

and extended by linearity. Its image, $\mathrm{im}(\lambda)$, is a $*$-subalgebra. The von Neumann algebra

$$L(G) = \mathrm{im}(\lambda)'' \subset B(l^2(G))$$

is called the *group von Neumann algebra* associated with G. Let e be the identity in G. If we define

$$\tau_G(x) := \langle x\delta_e, \delta_e \rangle \qquad (x \in L(G)),$$

then τ_G is a normal, faithful trace on $L(G)$ satisfying $\tau_G(\mathbf{1}) = 1$.

If $\mathbb{F}_\infty$ is the free group with countably many generators (g_i), then $L(\mathbb{F}_\infty)$ is called the *free group von Neumann algebra*. In this case we use τ_∞ to denote the trace on $L(\mathbb{F}_\infty)$. The elements $\lambda(g_i)$ are called *free group unitaries*. It is shown in [57], Proposition 1.1, that (6.18) holds with $C = 2$ if the c_i are equal to $\lambda(g_i)$ or $\lambda(g_i^{-1})$.

Remark 6.17. It is furthermore known that (6.18) holds if the c_i are elements of the semi-circular system, the circular system ([57], Proposition 4.8) or if the c_i are q-Gaussians, for some $-1 \leq q < 1$ ([24], Theorem 4.1).

From now on, we focus on the case where the c_i are the free group unitaries $\lambda(g_i)$. By a duality argument analogous to the one in the proof of Corollary 6.4 we obtain the following consequence of Theorem 6.15.

Corollary 6.18. *If E is a separable symmetric Banach function space on $(0,\infty)$, then for any finite sequence (x_i) in $E(\mathcal{M})$ we have*

$$\|(x_i)\|_{E(\mathcal{M};l^2_c)+E(\mathcal{M};l^2_r)} \leq C \Big\| \sum_i \lambda(g_i) \otimes x_i \Big\|_{E(L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M})},$$

with $C \leq 8\sqrt{2}$.

We will use the following inequality, which is the reverse of (6.18).

Lemma 6.19. *Suppose E is a 2-convex symmetric Banach function space on $(0, \tau(\mathbf{1}))$ such that $E_{(2)}$ is fully symmetric. Then,*

$$\max\Big\{ \|(x_i)\|_{E(\mathcal{M};l^2_c)}, \|(x_i)\|_{E(\mathcal{M};l^2_r)} \Big\} \leq \Big\| \sum_i \lambda(g_i) \otimes x_i \Big\|_{E(L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M})}.$$

Proof. Let $\mathcal{E}$ denote the conditional expectation onto the von Neumann subalgebra $\mathbb{C}\mathbf{1}\overline{\otimes}\mathcal{M}$. Then,

$$\begin{aligned} \Big\| \Big(\sum_i |x_i|^2 \Big)^{\frac{1}{2}} \Big\|_{E(\mathcal{M})} &= \Big\| \sum_i |x_i|^2 \Big\|^{\frac{1}{2}}_{E_{(2)}(\mathcal{M})} \\ &= \Big\| \sum_i \mathcal{E}|\lambda(g_i) \otimes x_i|^2 \Big\|^{\frac{1}{2}}_{E_{(2)}(L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M})} \end{aligned}$$

$$= \Big\| \mathcal{E} \Big| \sum_i \lambda(g_i) \otimes x_i \Big|^2 \Big\|^{\frac{1}{2}}_{E_{(2)}(L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M})}$$
$$\leq \Big\| \sum_i \lambda(g_i) \otimes x_i \Big\|_{E(L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M})}.$$

Similarly,

$$\Big\| \Big(\sum_i |x_i^*|^2 \Big)^{\frac{1}{2}} \Big\|_{E(\mathcal{M})} \leq \Big\| \sum_i \lambda(g_i) \otimes x_i \Big\|_{E(L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M})}.$$

□

By combining Theorem 6.15 and Lemma 6.19 we obtain the following generalization of the result for L^p-spaces given in [113], Theorem 8.4.10.

Theorem 6.20. *Suppose E is a 2-convex symmetric Banach function space on $(0, \tau(\mathbf{1}))$ such that $E_{(2)}$ is fully symmetric. Then, for any finite sequence (x_i) in $E(\mathcal{M})$,*

$$\Big\| \sum_i \lambda(g_i) \otimes x_i \Big\|_{E(L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M})} \simeq \max \Big\{ \|(x_i)\|_{E(\mathcal{M};l_c^2)}, \|(x_i)\|_{E(\mathcal{M};l_r^2)} \Big\}.$$

We shall use two results from [94], see Lemmas 6.21 and 6.22 below. To make our exposition self-contained, we reproduce the proofs of these results here.

Let $a \in L(\mathbb{F}_\infty)$ and let $\phi_a : L^1(L(\mathbb{F}_\infty)) \to \mathbb{C}$ be the linear functional given by

$$\phi_a(x) = \tau_\infty(xa) \qquad (x \in L^1(L(\mathbb{F}_\infty))).$$

Since $L^1(L(\mathbb{F}_\infty)) \otimes \mathcal{M}$ is dense in $L^1(L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M})$, the map $\phi_a \otimes \mathbf{1}_{L(\mathcal{M})}$ uniquely extends to a bounded linear operator $T_a^1 : L^1(L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M}) \to L^1(\mathcal{M})$ with $\|T\| \leq \|a\|$. On the other hand, $L(\mathbb{F}_\infty) \otimes \mathcal{M}$ is dense in $L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M}$ with respect to the ultra-weak operator topology. Since $\phi_a|_{\mathcal{M}} \otimes \mathbf{1}_{\mathcal{M}}$ is a normal map, it uniquely extends to a normal operator $T_a^\infty : L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M} \to \mathcal{M}$ with $\|T\| \leq \|a\|_{L^1(\mathbb{F}_\infty)}$. The maps T_a^1 and T_a^∞ coincide on the intersection of $L^1(L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M})$ and $L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M}$ and hence there is a bounded linear map

$$T_a : L^1(L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M}) + L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M} \to L^1(\mathcal{M}) + \mathcal{M}$$

which extends both these maps. For brevity, we write $\langle x, a \rangle = T_a(x)$.

For any $n \geq 1$ we define a projection

$$P_n : L^1(L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M}) + L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M} \to L^1(L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M}) + L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M}$$

by

$$P_n(x) = \sum_{i=1}^n \lambda(g_i) \otimes \langle x, \lambda(g_i)^* \rangle.$$

The following result is proved in [94], Lemma 5.4.

Lemma 6.21. *If E is a fully symmetric quasi-Banach function space on $(0, \tau(\mathbf{1}))$, then P_n is a bounded projection on $E(\mathbb{F}_\infty \overline{\otimes} \mathcal{M})$ with $\|P_n\| \leq 2$.*

Proof. Let P_n^∞ and P_n^1 denote the restriction of P_n to $L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M}$ and $L^1(L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M})$, respectively. Let $x_1 \in L(\mathbb{F}_\infty)$, $x_2 \in \mathcal{M}$, $y_1 \in L^1(L(\mathbb{F}_\infty))$ and $y_2 \in L^1(\mathcal{M})$ and set $x = x_1 \otimes x_2$, $y = y_1 \otimes y_2$. It is not difficult to calculate that

$$\tau_\infty \otimes \tau(P_n(x)y) = \tau_\infty \otimes \tau(x(P_n(y^*))^*).$$

By density it follows that P_n^∞ is the adjoint of the map $x \mapsto (P_n^1(x^*))^*$ and hence $\|P_n^\infty\| = \|P_n^1\|$. Let $S = \operatorname{span}\{\lambda(g) \ : \ g \in \mathbb{F}_\infty\}$. By [57], Proposition 1.3, the restriction of P_n to $S \otimes \mathcal{M}$ is bounded and has norm at most 2. Since $S \otimes \mathcal{M}$ is ultra-weakly dense in $L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M}$, it follows from Kaplansky's theorem that the unit ball of $S \otimes \mathcal{M}$ is ultra-weakly dense in the unit ball of $L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M}$. Since P_n^∞ is ultra-weakly continuous, we conclude that $\|P_n^\infty\| \leq 2$. Since $E(L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M})$ is an exact interpolation space for the couple $(L^1(L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M}), L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M})$, the result follows. □

The following result is essentially proved in [94], Lemma 5.5.

Lemma 6.22. *Let $1 \leq p_0 \leq p_1 < \infty$. Suppose E is a fully symmetric quasi-Banach function space on $(0, \tau(\mathbf{1}))$. If E is a κ-interpolation space for the couple (L^{p_0}, L^{p_1}), then for any finite sequence (x_i) in $E(\mathcal{M})$,*

$$\Big\| \sum_i r_i \otimes x_i \Big\|_{L^\infty \overline{\otimes} \mathcal{M}} \leq 2\kappa \max\{B_{p_0}, B_{p_1}\} \Big\| \sum_i \lambda(g_i) \otimes x_i \Big\|_{E(L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M})},$$

and if E is a κ-interpolation space of exponent θ, for some $0 \leq \theta \leq 1$, then

$$\Big\| \sum_i r_i \otimes x_i \Big\|_{L^\infty \overline{\otimes} \mathcal{M}} \leq 2\kappa B_{p_0}^\theta B_{p_1}^{1-\theta} \Big\| \sum_i \lambda(g_i) \otimes x_i \Big\|_{E(L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M})},$$

where B_q is the constant in (3.10).

Proof. We only prove the first statement, the proof of the second statement is similar. For $n \geq 1$ we define

$$Q_n : L^1(L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M}) + L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M} \to L^1(L^\infty(\Omega)\overline{\otimes}\mathcal{M}) + L^\infty(\Omega)\overline{\otimes}\mathcal{M}$$

by

$$Q_n(x) = \sum_{i=1}^n r_i \otimes \langle x, \lambda(g_i)^* \rangle$$

If $1 \leq p \leq 2$, then it follows from the noncommutative Khintchine inequalities and Corollary 6.18 that

$$\Big\| \sum_i r_i \otimes x_i \Big\|_{L^p(L^\infty \overline{\otimes} \mathcal{M})} \leq B_p \|(x_k)\|_{L^p(\mathcal{M}; l_c^2) + L^p(\mathcal{M}; l_r^2)}$$

$$\leq CB_p \Big\| \sum_i \lambda(g_i) \otimes x_i \Big\|_{L^p(L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M})},$$

and if $2 \leq p < \infty$ we obtain using Lemma 6.19,

$$\begin{aligned}\Big\| \sum_i r_i \otimes x_i \Big\|_{L^p(L^\infty\overline{\otimes}\mathcal{M})} &\leq B_p \|(x_k)\|_{L^p(\mathcal{M};l_c^2)\cap L^p(\mathcal{M};l_r^2)} \\ &\leq B_p \Big\| \sum_i \lambda(g_i) \otimes x_i \Big\|_{L^p(L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M})}.\end{aligned}$$

Together with Lemma 6.21 these estimates imply that

$$\|Q_n : L^p(L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M}) \to L^p(L^\infty(\Omega)\overline{\otimes}\mathcal{M})\| \leq 2B_p.$$

By interpolation we obtain

$$\|Q_n : E(L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M}) \to E(L^\infty(\Omega)\overline{\otimes}\mathcal{M})\| \leq 2\max\{B_{p_0}, B_{p_1}\}$$

and since

$$Q_n\Big(\sum_{i=1}^n \lambda(g_i) \otimes x_i\Big) = \sum_{i=1}^n r_i \otimes x_i$$

for any $x_1, \ldots, x_n \in E(\mathcal{M})$, the conclusion follows. □

We obtain the following extension of Theorem 6.13.

Corollary 6.23. *Let E be a rearrangement invariant Banach function space on $(0,\infty)$. Then the following are equivalent.*

(i) The inequality

$$\Big\| \sum_i r_i \otimes x_i \Big\|_{E(L^\infty\overline{\otimes}\mathcal{M})} \lesssim_E \Big\| \sum_i \lambda(g_i) \otimes x_i \Big\|_{E(L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M})}$$

holds for any semi-finite von Neumann algebra $\mathcal{M}$;

(ii) The inequality (6.1) holds for any semi-finite von Neumann algebra $\mathcal{M}$;

(iii) $q_E < \infty$.

Proof. The equivalence of (ii) and (iii) was observed in Theorem 6.13. If (i) holds then, by Theorem 6.15, we find that (6.1) holds for any semi-finite von Neumann algebra. On the other hand, if $q_E < \infty$, then by Theorem 4.31 we find that E is an interpolation space for the couple (L^1, L^q) for any $q > q_E$. By Lemma 6.22 we conclude that (i) holds. □

Lemma 6.22 and Theorem 6.15 together yield the following special case of Theorem 6.1. Recall that the constant B_q in (3.10) is (strictly) less than $\sqrt{q}$.

Theorem 6.24. *Let $1 \leq p_0 \leq p_1 < \infty$. Suppose E is a fully symmetric quasi-Banach function space on $(0, \alpha)$ and let C_E be the constant in (6.19). If E is a κ-interpolation space for the couple (L^{p_0}, L^{p_1}), then*

$$\Big\| \sum_i r_i \otimes x_i \Big\|_{E(L^\infty \overline{\otimes} \mathcal{M})} \leq 2C_E \kappa \sqrt{p_1} \max\Big\{ \|(x_i)\|_{E(\mathcal{M};l_c^2)}, \|(x_i)\|_{E(\mathcal{M};l_r^2)} \Big\},$$

and if E is a κ-interpolation space of exponent θ, for some $0 \leq \theta \leq 1$, then

$$\Big\| \sum_i r_i \otimes x_i \Big\|_{E(L^\infty \overline{\otimes} \mathcal{M})}$$
$$\leq 2C_E \kappa (p_0)^{\frac{\theta}{2}} (p_1)^{\frac{1-\theta}{2}} \max\Big\{ \|(x_i)\|_{E(\mathcal{M};l_c^2)}, \|(x_i)\|_{E(\mathcal{M};l_r^2)} \Big\},$$

for any finite sequence (x_i) in $E(\mathcal{M})$.

We conclude this section by proving the following interpolation result for intersections of column and row spaces.

Theorem 6.25. *Let $\mathcal{M}, \mathcal{N}$ be semi-finite von Neumann algebras and let E, E_0, E_1, F, F_0, F_1 be fully symmetric Banach function spaces on $(0, \infty)$. Suppose (E, F) is an (exact) interpolation pair for the pair $((E_0, E_1), (F_0, F_1))$. If E_0, E_1 are 2-convex and $(E_0)_{(2)}, (E_1)_{(2)}$ are fully symmetric, then*

$$(E(\mathcal{M}; l_c^2) \cap E(\mathcal{M}; l_r^2), F(\mathcal{N}))$$

is a (C-)interpolation pair for the pair

$$((E_0(\mathcal{M}; l_c^2) \cap E_0(\mathcal{M}; l_r^2), E_1(\mathcal{M}; l_c^2) \cap E_1(\mathcal{M}; l_r^2)), (F_0(\mathcal{N}), F_1(\mathcal{N}))),$$

where C is an absolute constant. Moreover, if (E, F) is an (exact) interpolation pair of exponent $0 \leq \theta \leq 1$, then $(E(\mathcal{M}; l_c^2) \cap E(\mathcal{M}; l_r^2), F(\mathcal{N}))$ is a (C-)interpolation pair of exponent θ.

Proof. Suppose (E, F) is an exact interpolation pair. For $n \geq 1$ let

$$I_n : (L^1 + L^\infty)(L(\mathbb{F}_\infty) \overline{\otimes} \mathcal{M}) \to (L^1 + L^\infty)(\mathcal{M}; l_c^2) \cap (L^1 + L^\infty)(\mathcal{M}; l_r^2)$$

be given by

$$I_n(x) = (\langle x, \lambda(g_i)^* \rangle)_{i=1}^n.$$

By Lemma 6.19,

$$\|I_n|_{\mathrm{Ran}(P_n)} : E_i(L(\mathbb{F}_\infty) \overline{\otimes} \mathcal{M}) \to E_i(\mathcal{M}; l_c^2) \cap E_i(\mathcal{M}; l_r^2)\| \leq 1, \qquad i = 0, 1.$$

Let T be a contraction for the pair

$$((E_0(\mathcal{M}; l_c^2) \cap E_0(\mathcal{M}; l_r^2), E_1(\mathcal{M}; l_c^2) \cap E_1(\mathcal{M}; l_r^2)), (F_0(\mathcal{N}), F_1(\mathcal{N}))).$$

Then, for $i = 0, 1$, the linear operator $T I_n P_n$ is a bounded linear operator from $E_i(L(\mathbb{F}_\infty) \overline{\otimes} \mathcal{M})$ into $F_i(\mathcal{N})$. By Theorem 5.5, we obtain

$$\begin{aligned}\|TI_nP_nx\|_{F(\mathcal{N})} &\leq \max_{i=0,1} \|TI_nP_n : E_i(L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M}) \to F_i(\mathcal{N})\| \, \|x\|_{E(L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M})} \\ &\leq 2\|x\|_{E(L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M})},\end{aligned}$$

where we use that $\|P_n : E_i(L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M}) \to E_i(L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M})\| \leq 2$ for $i = 0,1$ by Lemma 6.21. Taking $x = \sum_{i=1}^n \lambda(g_i) \otimes x_i$, with $x_1, \ldots, x_n \in E(\mathcal{M})$ yields

$$\begin{aligned}\|T((x_i)_{i=1}^n)\|_{F(\mathcal{N})} &\leq 2\Big\| \sum_{i=1}^n \lambda(g_i) \otimes x_i \Big\|_{E(L(\mathbb{F}_\infty)\overline{\otimes}\mathcal{M})} \\ &\leq 2C_E \max\{\|(x_i)_{i=1}^n\|_{E(\mathcal{M};l_c^2)}, \|(x_i)_{i=1}^n\|_{E(\mathcal{M};l_c^2)}\},\end{aligned}$$

where C_E is the constant in (6.19). Since the finite sequences $(x_i)_{i=1}^n$ in $E(\mathcal{M})$, $n \geq 1$, are dense in $E(\mathcal{M};l_c^2) \cap E(\mathcal{M};l_r^2)$, the result follows. □

6.5 Burkholder-Gundy inequalities

We apply the noncommutative Khintchine inequalities in Theorems 6.1 and 6.7 to prove Burkholder-Gundy inequalities for martingale difference sequences in certain noncommutative symmetric spaces. These inequalities will be utilized in the proof of the noncommutative Burkholder-Rosenthal theorem below (Theorem 7.6). The additional ingredient needed for the proof is the following randomization trick, explained in Lemma 6.27.

First recall the following definitions. Let E be a symmetric quasi-Banach function space on $(0,\alpha)$ which is p-convex for some $0 < p < \infty$ and let $\mathcal{M}$ be a von Neumann algebra with a normal, semi-finite, faithful trace τ satisfying $\tau(\mathbf{1}) = \alpha$. Suppose that $(\mathcal{M}_k)_{k=1}^\infty$ is a *(discrete-time) filtration*, i.e. an increasing sequence of von Neumann subalgebras such that $\tau|_{\mathcal{M}_k}$ is semi-finite, and let $\mathcal{E}_k$ be the conditional expectation with respect to $\mathcal{M}_k$. Then a sequence (x_k) in $E(\mathcal{M})$ is called a *martingale* with respect to $(\mathcal{M}_k)$ if $\mathcal{E}_k(x_{k+1}) = x_k$ for all $k \geq 1$. A sequence (y_k) in $E(\mathcal{M})$ is called a *martingale difference sequence* if $y_k = x_k - x_{k-1}$ for some martingale (x_k), with the convention $x_0 = 0$ and $\mathcal{M}_0 = \mathbb{C}\mathbf{1}$. It is called *finite* if there is some $N > 0$ such that $y_k = 0$ for all $k \geq N$.

The next proposition follows by interpolation, i.e. using Theorem 5.21, from the boundedness of martingale transforms in noncommutative L^p-spaces with $1 < p < \infty$ (c.f. [114], p. 668).

Proposition 6.26. *Let E be a symmetric quasi-Banach function space on $(0,\alpha)$ satisfying $1 < p_E \leq q_E < \infty$. For every $k \geq 1$, let $\xi_k \in \mathcal{M}_{k-1}$ and suppose that $\|\xi_k\| \leq 1$ and ξ_k commutes with $\mathcal{M}_k$. Then, for any martingale difference sequence $(y_k)_{k=1}^\infty$ with respect to $(\mathcal{M}_k)_{k=1}^\infty$ in $E(\mathcal{M})$ and any $n \geq 1$ we have*

$$\Big\| \sum_{k=1}^n \xi_k y_k \Big\|_{E(\mathcal{M})} \lesssim_E \Big\| \sum_{k=1}^n y_k \Big\|_{E(\mathcal{M})}.$$

In particular, taking $\xi_k \in \{-1,1\}$ yields the well known fact that noncommutative martingale difference sequences are unconditional in $E(\mathcal{M})$.

Lemma 6.27. *Let E be a symmetric p-convex ($0 < p < \infty$) quasi-Banach function space on $(0,\alpha)$ with $1 < p_E \leq q_E < \infty$ and suppose $\mathcal{M}$ is a von Neumann algebra equipped with a normal, semi-finite, faithful trace τ satisfying $\tau(\mathbf{1}) = \alpha$. Let $(\mathcal{M}_k)_{k=1}^{\infty}$ be an increasing sequence of von Neumann subalgebras such that $\tau|_{\mathcal{M}_k}$ is semi-finite. Then we have the equivalences*

$$\mathbb{E}\Big\|\sum_{k=1}^n r_k x_k\Big\|_{E(\mathcal{M})} \simeq_E \Big\|\sum_{k=1}^n x_k\Big\|_{E(\mathcal{M})} \simeq_E \Big\|\sum_{k=1}^n r_k \otimes x_k\Big\|_{E(L^\infty\overline{\otimes}\mathcal{M})}, \tag{6.22}$$

for any Rademacher sequence (r_k) and any martingale difference sequence $(x_k)_{k=1}^n$ in $E(\mathcal{M})$.

Proof. The first equivalence in (6.22) follows directly from the unconditionality of noncommutative martingale difference sequences in $E(\mathcal{M})$. For the second equivalence, observe that $(y_k) = (r_k \otimes x_k)$ is a martingale difference sequence with respect to the filtration $(L^\infty\overline{\otimes}\mathcal{M}_k)$. By applying Proposition 6.26 with $\xi_k = r_k \otimes \mathbf{1}$ we obtain

$$\Big\|\sum_{k=1}^n x_k\Big\|_{E(\mathcal{M})} = \Big\|\sum_{k=1}^n (r_k\otimes\mathbf{1})(r_k\otimes x_k)\Big\|_{E(L^\infty\overline{\otimes}\mathcal{M})} \lesssim_E \Big\|\sum_{k=1}^n r_k\otimes x_k\Big\|_{E(L^\infty\overline{\otimes}\mathcal{M})}.$$

The reverse inequality follows similarly from Proposition 6.26 with $(y_k) = (\mathbf{1}\otimes x_k)$ and $\xi_k = r_k \otimes \mathbf{1}$.

Let E be a symmetric Banach function space on $(0,\infty)$. For any finite martingale difference sequence (x_k) in $E(\mathcal{M})$ we set

$$\|(x_k)\|_{H_c^E} = \|(x_k)\|_{E(\mathcal{M};l_c^2)};\ \|(x_k)\|_{H_r^E} = \|(x_k)\|_{E(\mathcal{M};l_r^2)}.$$

These expressions define two norms on the linear space of all finite martingale difference sequences in $E(\mathcal{M})$.

For future reference we state the following version of Stein's inequality for noncommutative symmetric spaces. This result follows directly by interpolation from the Stein inequality for noncommutative L^p-spaces proved in [114], Theorem 2.3.

Lemma 6.28. *Let E be a symmetric quasi-Banach function space on $(0,\infty)$ which is s-convex for some $0 < s < \infty$ and let $\mathcal{M}$ be a semi-finite von Neumann algebra. Let $(\mathcal{E}_k)_{k\geq 1}$ be an increasing sequence of conditional expectations in $\mathcal{M}$. If $1 < p_E \leq q_E < \infty$, then for any finite sequence (x_k) in $E(\mathcal{M})$,*

$$\|(\mathcal{E}_k(x_k))\|_{E(\mathcal{M};l_c^2)} \lesssim_E \|(x_k)\|_{E(\mathcal{M};l_c^2)},\ \|(\mathcal{E}_k(x_k))\|_{E(\mathcal{M};l_r^2)} \lesssim_E \|(x_k)\|_{E(\mathcal{M};l_r^2)}.$$

Consequently, the map $(x_k) \mapsto (\mathcal{E}_k(x_k))$ extends to a bounded projection on both $E(\mathcal{M};l_c^2)$ and $E(\mathcal{M};l_r^2)$.

As was already noted in [14], if $1 < p_E \le q_E < \infty$ then it follows from the noncommutative Stein inequality that for any finite martingale difference sequence we have

$$\|(x_k)\|_{H_c^E + H_r^E} \simeq_E \|(x_k)\|_{E(\mathcal{M};l_c^2)+E(\mathcal{M};l_r^2)}.$$

By combining Lemma 6.27 with Theorems 6.1 and 6.3 and Corollary 6.4, we obtain the following result, which generalizes the Burkholder-Gundy inequalities for noncommutative L^p-spaces [114]. Part of this result was obtained in [14], Theorem 2.2.

Theorem 6.29. *Let E be a symmetric Banach function space on $(0,\infty)$ with $1 < p_E \le q_E < \infty$ and suppose that E is either separable or is the dual of a separable space. Suppose $\mathcal{M}$ is a von Neumann algebra equipped with a normal, semi-finite, faithful trace τ. Let $(\mathcal{M}_k)_{k=1}^\infty$ be an increasing sequence of von Neumann subalgebras such that $\tau|_{\mathcal{M}_k}$ is semi-finite. Then, for any finite martingale difference sequence (x_k) in $E(\mathcal{M})$ we have*

$$\|(x_k)\|_{H_c^E + H_r^E} \lesssim_E \Big\|\sum_k x_k\Big\|_{E(\mathcal{M})} \lesssim_E \|(x_k)\|_{H_c^E \cap H_r^E}.$$

Suppose that E is separable. If $p_E > 1$ and either $q_E < 2$ or E is 2-concave, then

$$\Big\|\sum_k x_k\Big\|_{E(\mathcal{M})} \simeq_E \|(x_k)\|_{H_c^E + H_r^E}.$$

On the other hand, if either E is 2-convex and $q_E < \infty$ or $2 < p_E \le q_E < \infty$ then

$$\Big\|\sum_k x_k\Big\|_{E(\mathcal{M})} \simeq_E \|(x_k)\|_{H_c^E \cap H_r^E}. \qquad (6.23)$$

7

Noncommutative Burkholder-Rosenthal inequalities

In this chapter we derive two different Rosenthal-type of inequalities. In the first part we derive new inequalities, stated in Theorem 7.4, for sums of independent noncommutative random variables which are elements of a noncommutative symmetric space. Under certain conditions we will, more generally, obtain Burkholder-Rosenthal inequalities for noncommutative martingale difference sequences. We present two applications of these results. Firstly, we give a new proof of the noncommutative Burkholder-Rosenthal inequalities in Haagerup L^p-spaces, which were established in [70]. Secondly, we derive new Khintchine-type inequalities with operator coefficients. The latter inequalities play an important role in Chapter 8.

In the final section of this chapter we return to the setting of Chapter 3 and consider sums of independent random vectors in a noncommutative symmetric space. Using similar techniques as in the first part, we can extend the result for random vectors in a noncommutative L^p-space established in Theorem 3.8.

7.1 Inequalities in noncommutative symmetric spaces

In the formulation of our noncommutative Rosenthal inequalities we use the following notion of conditional independence, which was introduced in [71]. Given a sequence $(\mathcal{N}_k)_{k\geq 1}$ of von Neumann subalgebras of a von Neumann algebra $\mathcal{M}$, we let $W^*((\mathcal{N}_k)_{k\geq 1})$ denote the von Neumann subalgebra generated by $\cup_{k\geq 1}\mathcal{N}_k$.

Definition 7.1. *Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal, semi-finite, faithful trace τ. Let $(\mathcal{N}_k)$ be a sequence of von Neumann subalgebras of $\mathcal{M}$ and $\mathcal{N}$ a common von Neumann subalgebra of the $\mathcal{N}_k$ such that $\tau|_{\mathcal{N}}$ is semi-finite. We call $(\mathcal{N}_k)$* independent *with respect to $\mathcal{E}_{\mathcal{N}}$ if for every k we have $\mathcal{E}_{\mathcal{N}}(xy) = \mathcal{E}_{\mathcal{N}}(x)\mathcal{E}_{\mathcal{N}}(y)$ for all $x \in \mathcal{N}_k$ and $y \in W^*((\mathcal{N}_j)_{j\neq k})$.*

Lemma 7.2. *Suppose that $(\mathcal{N}_k)$ is a sequence of von Neumann subalgebras of $\mathcal{M}$ which is independent with respect to $\mathcal{E}_{\mathcal{N}}$. Let $\mathcal{M}_k = W^*(\mathcal{N}_1,\ldots,\mathcal{N}_k)$.*

If $x_k \in (L^1 + L^\infty)(\mathcal{N}_k)$ satisfies $\mathcal{E}_{\mathcal{N}}(x_k) = 0$ for all $k \geq 1$, then (x_k) is a martingale difference sequence in $(L^1 + L^\infty)(\mathcal{M})$ with respect to the filtration $(\mathcal{M}_k)$.

Proof. Suppose first that $x_k \in \mathcal{N}_k$ and $\mathcal{E}_{\mathcal{N}}(x_k) = 0$ for all $k \geq 1$. Let $\mathcal{E}_k$ denote the conditional expectation with respect to $\mathcal{M}_k$. Fix $i \geq 1$ and let $y \in L^1 \cap L^\infty(\mathcal{M}_{i-1})$. By independence,

$$\tau(x_i y) = \tau(\mathcal{E}_{\mathcal{N}}(x_i y)) = \tau(\mathcal{E}_{\mathcal{N}}(x_i)\mathcal{E}_{\mathcal{N}}(y)) = 0.$$

By (3.8) in Proposition 3.4 this implies that $\mathcal{E}_{i-1}(x_i) = 0$.

Suppose now that $x_k \in (L^1 + L^\infty)(\mathcal{N}_k)$ for all $k \geq 1$. Since $\mathcal{N}_k$ is dense in $(L^1 + L^\infty)(\mathcal{N}_k)$, for every $i \geq 1$ there is a sequence $(x_{i,j})_{j\geq 1}$ in $\mathcal{N}_i$ converging to x_i in $(L^1 + L^\infty)(\mathcal{N}_i)$. Since $\mathcal{E}_{\mathcal{N}}$ is a contraction on $(L^1 + L^\infty)(\mathcal{M})$ and $\mathcal{E}_{\mathcal{N}}(x_i) = 0$, it follows that $x_{i,j} - \mathcal{E}_{\mathcal{N}}(x_{i,j})$ converges to x_i for $j \to \infty$. By the above, $\mathcal{E}_{i-1}(x_{i,j} - \mathcal{E}_{\mathcal{N}}(x_{i,j})) = 0$ for all $j \geq 1$ and by taking the limit for $j \to \infty$ we conclude that $\mathcal{E}_{i-1}(x_i) = 0$. Therefore, $(x_k)_{k\geq 1}$ is the martingale difference sequence for the martingale defined by $y_k = \sum_{i=1}^k x_i$. □

The following observation has its origins in [71] and [121].

Lemma 7.3. *Let $(\mathcal{M}, \tau)$ be a von Neumann algebra equipped with a normal, semi-finite, faithful trace τ satisfying $\tau(\mathbf{1}) = \alpha$ and let E be a p-convex ($0 < p < \infty$) quasi-Banach function space on $(0, \alpha)$ which is an interpolation space for the couple (L^1, L^∞). Let $(\mathcal{N}_k)$ be a sequence of von Neumann subalgebras of $\mathcal{M}$ and $\mathcal{N}$ a common von Neumann subalgebra of the $\mathcal{N}_k$ such that $\tau|_{\mathcal{N}}$ is semi-finite. Suppose that $(\mathcal{N}_k)$ is independent with respect to $\mathcal{E}_{\mathcal{N}}$. If $x_k \in E(\mathcal{N}_k)$ satisfy $\mathcal{E}_{\mathcal{N}}(x_k) = 0$, then*

$$\Big\|\sum_{k=1}^n x_k\Big\|_{E(\mathcal{M})} \simeq_E \mathbb{E}\Big\|\sum_{k=1}^n r_k x_k\Big\|_{E(\mathcal{M})}.$$

If E is moreover q-concave for some $q < \infty$, then

$$\Big\|\sum_{k=1}^n x_k\Big\|_{E(\mathcal{M})} \lesssim_E \max\Big\{\Big\|\Big(\sum_k |x_k|^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})}, \Big\|\Big(\sum_k |x_k^*|^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})}\Big\}.$$

Proof. It suffices to show that for any sequence of signs $(\varepsilon_k)_{k=1}^n \subset \{-1, 1\}^n$,

$$\Big\|\sum_{k=1}^n \varepsilon_k x_k\Big\|_{E(\mathcal{M})} \lesssim_E \Big\|\sum_{k=1}^n x_k\Big\|_{E(\mathcal{M})}.$$

Define $\mathcal{N}_+ = W^*(\{\mathcal{N}_k : \varepsilon_k = 1\})$ and $\mathcal{N}_- = W^*(\{\mathcal{N}_k : \varepsilon_k = -1\})$. Note that if $\varepsilon_i = -1$, then by independence and (3.8) it readily follows that $\mathcal{E}_{\mathcal{N}_+}(x_i) = \mathcal{E}_{\mathcal{N}}(x_i) = 0$. Hence,

$$\mathcal{E}_{\mathcal{N}_+}\Big(\sum_{k=1}^n x_k\Big) = \sum_{\varepsilon_k=1} x_k + \sum_{\varepsilon_k=-1} \mathcal{E}_{\mathcal{N}_+}(x_k) = \sum_{\varepsilon_k=1} x_k$$

and analogously, $\mathcal{E}_{\mathcal{N}_-}(\sum_{k=1}^n x_k) = \sum_{\varepsilon_k=-1} x_k$. Since E is an interpolation space for the couple (L^1, L^∞), it follows from Theorem 5.5 that $E(\mathcal{M})$ is an interpolation space for the couple $(L^1(\mathcal{M}), \mathcal{M})$. By Proposition 3.4 we find that conditional expectations are bounded on $E(\mathcal{M})$ by a constant c_E depending only on E. This implies that

$$\begin{aligned}\Big\|\sum_{k=1}^n \varepsilon_k x_k\Big\|_{E(\mathcal{M})} &= \Big\|\sum_{\varepsilon_k=1} x_k - \sum_{\varepsilon_k=-1} x_k\Big\|_{E(\mathcal{M})} \\ &= \Big\|(\mathcal{E}_{\mathcal{N}_+} - \mathcal{E}_{\mathcal{N}_-})\Big(\sum_{k=1}^n x_k\Big)\Big\|_{E(\mathcal{M})} \lesssim_E \Big\|\sum_{k=1}^n x_k\Big\|_{E(\mathcal{M})}.\end{aligned}$$

The final statement follows from Theorem 6.7. □

We are now ready to prove the main theorem of this section. For any $n \in \mathbb{N}$ we let $M_n(\mathcal{M})$ be the von Neumann algebra of all $n \times n$ matrices with entries in $\mathcal{M}$, equipped with its natural non-normalized trace. For a finite sequence $(x_k)_{k=1}^n$ we denote by $\mathrm{diag}(x_k), \mathrm{col}(x_k)$ and $\mathrm{row}(x_k)$ the $n \times n$ matrix with the x_k's on its diagonal, first column and first row, respectively (and zeros elsewhere).

Theorem 7.4. *(Noncommutative Rosenthal inequalities) Let $\mathcal{M}$ be a semi-finite von Neumann algebra equipped with a normal, semi-finite, faithful trace τ. Suppose that E is a symmetric Banach function space on $(0,\infty)$ satisfying any of the following conditions:*

(i) E is an interpolation space for the couple (L^2, L^p) for some $2 \leq p < \infty$ and E is q-concave for some $q < \infty$;
(ii) $2 < p_E \leq q_E < \infty$.

Let $(\mathcal{N}_k)$ be a sequence of von Neumann subalgebras of $\mathcal{M}$ and let $\mathcal{N}$ be a common von Neumann subalgebra of the $(\mathcal{N}_k)$ such that $\tau|_{\mathcal{N}}$ is semi-finite. Suppose that $(\mathcal{N}_k)$ is independent with respect to $\mathcal{E} := \mathcal{E}_{\mathcal{N}}$. Let (x_k) be a sequence such that $x_k \in E(\mathcal{N}_k)$ and $\mathcal{E}(x_k) = 0$ for all k. Then, for any $n \geq 1$,

$$\begin{aligned}\Big\|\sum_{k=1}^n x_k\Big\|_{E(\mathcal{M})} \simeq_E \max\Big\{&\|\mathrm{diag}(x_k)_{k=1}^n\|_{E(M_n(\mathcal{M}))}, \Big\|\Big(\sum_{k=1}^n \mathcal{E}|x_k|^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})}, \\ &\Big\|\Big(\sum_{k=1}^n \mathcal{E}|x_k^*|^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})}\Big\}. \qquad (7.1)\end{aligned}$$

Remark 7.5. Note that if $2 < p_E \leq q_E < \infty$, then E is an interpolation space for the couple (L^2, L^p), for any $p > q_E$. However, there are such spaces which are not q-concave for any $q < \infty$. Indeed, recall the Lorentz spaces $L^{p,q}$ on

$(0,\infty)$ introduced in example 4.14. The space $E = L^{3,\infty}$ has Boyd indices $p_E = q_E = 3$, but is not q-concave for any $q < \infty$. On the other hand, there are spaces which satisfy condition (i), but not condition (ii). For example, take $E = L^{2,r}$ for $2 \leq r < \infty$.

Theorem 7.4 generalizes the Rosenthal inequalities for commutative Banach function spaces ([64], Remark 7) and for noncommutative L^p-spaces ([71], Theorem 2.1). These two results can be recovered by taking $\mathcal{M} = L^\infty(\Omega)$, $\mathcal{N} = \mathbb{C}$ in the first case and by setting $E = L^p$ in the second.

We first prove Theorem 7.4 under condition (i).

Proof. (of Theorem 7.4, condition (i)) By Lemma 4.12 the space E has order continuous norm and therefore $L^1 \cap L^\infty(\mathcal{N}_k)$ is dense in $E(\mathcal{N}_k)$ for all $k \geq 1$. Therefore, by approximation it suffices to prove the result in the special case where the x_k are bounded.

By assumption, E is an interpolation space for the couple (L^2, L^p) for some $p < \infty$ and hence, by Proposition 4.23, the space $E_{(2)}$ is an interpolation space for the couple $(L^1, L^{\frac{p}{2}})$. By Theorem 5.5 and Proposition 3.4 we find that $\mathcal{E}$ is bounded on $E_{(2)}(\mathcal{M})$.

We first prove that the maximum on the right hand side is dominated by $\|\sum_k x_k\|_{E(\mathcal{M})}$. By our discussion preceding Corollary 6.9, the n-th Rademacher subspace $Rad_n(E)$ is C_E-complemented in $E(L^\infty\overline{\otimes}\mathcal{M})$, for some constant $C_E > 0$ independent of n. Recall from Theorem 3.3 that $L^q(\mathcal{M})$ has cotype q if $2 \leq q < \infty$, i.e.,

$$\|\mathrm{diag}(x_k)_{k=1}^n\|_{L^q(M_n(\mathcal{M}))} = \Big(\sum_{k=1}^n \|x_k\|_{L^q(\mathcal{M})}^q\Big)^{\frac{1}{q}} \leq \Big\|\sum_{k=1}^n r_k \otimes x_k\Big\|_{L^q(L^\infty\overline{\otimes}\mathcal{M})}.$$

By Theorem 5.5 we can interpolate this estimate for $q = 2$ and $q = p$ to obtain

$$\|\mathrm{diag}(x_k)_{k=1}^n\|_{E(M_n(\mathcal{M}))} \lesssim_E \Big\|\sum_{k=1}^n r_k \otimes x_k\Big\|_{E(L^\infty\overline{\otimes}\mathcal{M})}. \tag{7.2}$$

Moreover, by Lemma 6.27,

$$\Big\|\sum r_k \otimes x_k\Big\|_{E(L^\infty\overline{\otimes}\mathcal{M})} \simeq_E \Big\|\sum_k x_k\Big\|_{E(\mathcal{M})}. \tag{7.3}$$

Since the $(\mathcal{N}_k)$ are independent and $\mathcal{E}(x_k) = 0$ for all k, it follows that $\mathcal{E}(x_k^* x_j) = \mathcal{E}(x_k^*)\mathcal{E}(x_j) = 0$ if $j \neq k$. As $\mathcal{E}$ is bounded on $E_{(2)}(\mathcal{M})$, we find that

$$\begin{aligned}\Big\|\Big(\sum_k \mathcal{E}(x_k^* x_k)\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})} &= \Big\|\sum_k \mathcal{E}(x_k^* x_k)\Big\|_{E_{(2)}(\mathcal{M})}^{\frac{1}{2}} \\ &= \Big\|\mathcal{E}\Big(\Big(\sum_k x_k\Big)^*\Big(\sum_k x_k\Big)\Big)\Big\|_{E_{(2)}(\mathcal{M})}^{\frac{1}{2}}\end{aligned}$$

$$\lesssim_E \Big\| \sum_k x_k \Big\|_{E(\mathcal{M})},$$

and by applying this to the sequence (x_k^*) we get

$$\Big\| \Big(\sum_k \mathcal{E}(x_k x_k^*) \Big)^{\frac{1}{2}} \Big\|_{E(\mathcal{M})} \lesssim_E \Big\| \sum_k x_k \Big\|_{E(\mathcal{M})}.$$

We now prove the reverse inequality in (7.1). By Lemma 7.3,

$$\Big\| \sum_k x_k \Big\|_{E(\mathcal{M})} \lesssim_E \max \Big\{ \Big\| \Big(\sum_k x_k^* x_k \Big)^{\frac{1}{2}} \Big\|_{E(\mathcal{M})}, \Big\| \Big(\sum_k x_k x_k^* \Big)^{\frac{1}{2}} \Big\|_{E(\mathcal{M})} \Big\}. \quad (7.4)$$

By the quasi-triangle inequality in $E_{(2)}(\mathcal{M})$ we have

$$\begin{aligned} &\Big\| \Big(\sum_k x_k^* x_k \Big)^{\frac{1}{2}} \Big\|_{E(\mathcal{M})} \\ &\qquad \lesssim_E \Big(\Big\| \sum_k x_k^* x_k - \mathcal{E}(x_k^* x_k) \Big\|_{E_{(2)}(\mathcal{M})} + \Big\| \sum_k \mathcal{E}(x_k^* x_k) \Big\|_{E_{(2)}(\mathcal{M})} \Big)^{\frac{1}{2}}. \quad (7.5) \end{aligned}$$

Notice that $(|x_k|^2 - \mathcal{E}(|x_k|^2))_{k \geq 1}$ is independent with respect to $\mathcal{E}$, self-adjoint, and, moreover, $\mathcal{E}(|x_k|^2 - \mathcal{E}(|x_k|^2)) = 0$ for all k. By again applying Lemma 7.3 we find that

$$\begin{aligned} \Big\| \sum_k x_k^* x_k - \mathcal{E}(x_k^* x_k) \Big\|_{E_{(2)}(\mathcal{M})} &\lesssim_E \Big\| \Big(\sum_k (x_k^* x_k - \mathcal{E}(x_k^* x_k))^2 \Big)^{\frac{1}{2}} \Big\|_{E_{(2)}(\mathcal{M})} \\ &\lesssim_E \Big\| \Big(\sum_k |x_k|^4 \Big)^{\frac{1}{2}} \Big\|_{E_{(2)}(\mathcal{M})} + \Big\| \Big(\sum_k (\mathcal{E}(|x_k|^2))^2 \Big)^{\frac{1}{2}} \Big\|_{E_{(2)}(\mathcal{M})}, \end{aligned}$$

where in the final inequality we use the quasi-triangle inequality in $E_{(2)}(\mathcal{M}; l_c^2)$. Let $x = \mathrm{col}(|x_k|)$ and $y = \mathrm{diag}(|x_k|)$. Since $\mu(xy) \prec\prec \mu(x)\mu(y)$, it follows from Theorem 4.20 that there is a contraction T for the couple (L^1, L^∞) such that $\mu(xy) = T(\mu(x)\mu(y))$. Therefore,

$$\begin{aligned} \Big\| \Big(\sum_k |x_k|^4 \Big)^{\frac{1}{2}} \Big\|_{E_{(2)}(\mathcal{M})} &= \|(x^* y^* y x)^{\frac{1}{2}}\|_{E_{(2)}(M_n(\mathcal{M}))} \\ &= \|yx\|_{E_{(2)}(M_n(\mathcal{M}))} \lesssim_E \|\mu(x)\mu(y)\|_{E_{(2)}} \\ &= \|\mu(x)^{\frac{1}{2}} \mu(y)^{\frac{1}{2}}\|_E^2 \leq \|y\|_{E(M_n(\mathcal{M}))} \|x\|_{E(M_n(\mathcal{M}))} \\ &= \|\mathrm{diag}(x_k)\|_{E(M_n(\mathcal{M}))} \Big\| \Big(\sum_k |x_k|^2 \Big)^{\frac{1}{2}} \Big\|_{E(\mathcal{M})}, \quad (7.6) \end{aligned}$$

where in the final inequality we use the Hölder-type inequality in Lemma 4.3. Let $\mathcal{E}_n$ be the conditional expectation in $E(M_n(\mathcal{M}))$ with respect to the von

Neumann subalgebra $M_n(\mathcal{N})$, i.e. $\mathcal{E}_n = \mathcal{E} \otimes \mathbf{1}_{M_n(\mathbb{C})}$. Writing $z = \mathrm{col}(|x_k|^2)$, we have $\mathcal{E}_n(z) = \mathrm{col}(\mathcal{E}(|x_k|^2))$ and so by boundedness of $\mathcal{E}_n$ in $E_{(2)}(M_n(\mathcal{M}))$,

$$\begin{aligned}\Big\|\Big(\sum_k (\mathcal{E}(|x_k|^2))^2\Big)^{\frac{1}{2}}\Big\|_{E_{(2)}(\mathcal{M})} &= \|((\mathcal{E}_n(z))^*\mathcal{E}_n(z))^{\frac{1}{2}}\|_{E_{(2)}(M_n(\mathcal{M}))} \\ &= \|\mathcal{E}_n(z)\|_{E_{(2)}(M_n(\mathcal{M}))} \\ &\lesssim_E \|z\|_{E_{(2)}(M_n(\mathcal{M}))} = \Big\|\Big(\sum_k |x_k|^4\Big)^{\frac{1}{2}}\Big\|_{E_{(2)}(\mathcal{M})}.\end{aligned}$$

Putting our estimates together, starting from (7.5), we arrive at

$$\begin{aligned}&\Big\|\Big(\sum_k |x_k|^2\Big)^{\frac{1}{2}}\Big\|_E \\ &\quad\lesssim_E \Big(\|\mathrm{diag}(x_k)\|_{E(M_n(\mathcal{M}))}\Big\|\Big(\sum_k |x_k|^2\Big)^{\frac{1}{2}}\Big\|_E + \Big\|\Big(\sum_k \mathcal{E}(|x_k|^2)\Big)^{\frac{1}{2}}\Big\|_E^2\Big)^{\frac{1}{2}}.\end{aligned}$$

In other words, if we set $a = \|(\sum_k |x_k|^2)^{\frac{1}{2}}\|_{E(\mathcal{M})}$, $b = \|diag(x_k)\|_{E(M_n(\mathcal{M}))}$ and $c = \|(\sum_k \mathcal{E}(|x_k|^2))^{\frac{1}{2}}\|_{E(\mathcal{M})}$, we have $a^2 \lesssim_E ab + c^2$. Solving this quadratic equation we obtain $a \lesssim_E \max\{b, c\}$, or,

$$\begin{aligned}&\Big\|\Big(\sum_k |x_k|^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})} \\ &\qquad\lesssim_E \max\Big\{\|\mathrm{diag}(x_k)\|_{E(M_n(\mathcal{M}))}, \Big\|\Big(\sum_k \mathcal{E}(|x_k|^2)\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})}\Big\}.\end{aligned}$$

Applying this to the sequence (x_k^*) gives

$$\begin{aligned}&\Big\|\Big(\sum_k |x_k^*|^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})} \\ &\qquad\lesssim_E \max\Big\{\|\mathrm{diag}(x_k)\|_{E(M_n(\mathcal{M}))}, \Big\|\Big(\sum_k \mathcal{E}(|x_k^*|^2)\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})}\Big\}.\end{aligned}$$

The result now follows by (7.4). □

The result in Theorem 7.4 under condition (ii) follows directly from Lemma 7.2 and the following noncommutative version of the Burkholder-Rosenthal inequalities.

Theorem 7.6. *(Noncommutative Burkholder-Rosenthal inequalities) Let $\mathcal{M}$ be a semi-finite von Neumann algebra equipped with a normal, semi-finite, faithful trace τ. Suppose that E is a symmetric Banach function space on $(0, \infty)$ satisfying $2 < p_E \le q_E < \infty$. Let $(\mathcal{M}_k)$ be a filtration in $\mathcal{M}$ and, for every $k \ge 1$, let $\mathcal{E}_k$ denote the conditional expectation with respect to $\mathcal{M}_k$.*

Let (x_k) *be a martingale difference sequence in* $E(\mathcal{M})$ *with respect to* $(\mathcal{M}_k)$. *Then, for any* $n \geq 1$,

$$\Big\|\sum_{k=1}^{n} x_k\Big\|_{E(\mathcal{M})} \simeq_E \max\Big\{\|\mathrm{diag}(x_k)_{k=1}^n\|_{E(M_n(\mathcal{M}))}, \Big\|\Big(\sum_{k=1}^{n}\mathcal{E}_{k-1}|x_k|^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})}, \Big\|\Big(\sum_{k=1}^{n}\mathcal{E}_{k-1}|x_k^*|^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})}\Big\}. \tag{7.7}$$

Proof. We first prove that the maximum on the right hand side is dominated by $\|\sum_k x_k\|_{E(\mathcal{M})}$. By (7.2) and (7.3),

$$\|\mathrm{diag}(x_k)_{k=1}^n\|_{E(M_n(\mathcal{M}))} \lesssim_E \Big\|\sum_k x_k\Big\|_{E(\mathcal{M})}.$$

Since $1 < p_{E_{(2)}} \leq q_{E_{(2)}} < \infty$, we obtain by applying the noncommutative dual Doob inequality (Theorem 5.24) in $E_{(2)}(\mathcal{M})$,

$$\begin{aligned}\Big\|\Big(\sum_k \mathcal{E}_{k-1}(x_k^*x_k)\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})} &= \Big\|\sum_k \mathcal{E}_{k-1}(x_k^*x_k)\Big\|_{E_{(2)}(\mathcal{M})}^{\frac{1}{2}} \\ &\lesssim_E \Big\|\sum_k x_k^*x_k\Big\|_{E_{(2)}(\mathcal{M})}^{\frac{1}{2}} = \Big\|\Big(\sum_k x_k^*x_k\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})}.\end{aligned}$$

Therefore, by the Burkholder-Gundy inequality (6.23) in Theorem 6.29 we conclude that

$$\Big\|\Big(\sum_k \mathcal{E}_{k-1}(x_k^*x_k)\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})} \lesssim_E \Big\|\Big(\sum_k x_k^*x_k\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})} \lesssim_E \Big\|\sum_k x_k\Big\|_{E(\mathcal{M})}$$

and by applying this to the sequence (x_k^*) we get

$$\Big\|\Big(\sum_k \mathcal{E}_{k-1}(x_kx_k^*)\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})} \lesssim_E \Big\|\sum_k x_k\Big\|_{E(\mathcal{M})}.$$

We now prove the reverse inequality in (7.7). By Theorem 6.29,

$$\Big\|\sum_k x_k\Big\|_{E(\mathcal{M})} \lesssim_E \max\Big\{\Big\|\Big(\sum_k x_k^*x_k\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})}, \Big\|\Big(\sum_k x_kx_k^*\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})}\Big\}. \tag{7.8}$$

By the quasi-triangle inequality in $E_{(2)}(\mathcal{M})$ we have

$$\begin{aligned}&\Big\|\Big(\sum_k x_k^*x_k\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})} \\ &\quad\lesssim_E \Big(\Big\|\sum_k x_k^*x_k - \mathcal{E}_{k-1}(x_k^*x_k)\Big\|_{E_{(2)}(\mathcal{M})} + \Big\|\sum_k \mathcal{E}_{k-1}(x_k^*x_k)\Big\|_{E_{(2)}(\mathcal{M})}\Big)^{\frac{1}{2}}.\end{aligned} \tag{7.9}$$

Notice that $(|x_k|^2 - \mathcal{E}_{k-1}(|x_k|^2))_{k\geq 1}$ is a martingale difference sequence in $E_{(2)}(\mathcal{M})$. Since $1 < p_{E_{(2)}}, q_{E_{(2)}} < \infty$ we find by Theorem 6.29

$$\begin{aligned}\Big\|\sum_k x_k^* x_k - \mathcal{E}_{k-1}(x_k^* x_k)\Big\|_{E_{(2)}(\mathcal{M})} &\lesssim_E \Big\|\Big(\sum_k (x_k^* x_k - \mathcal{E}_{k-1}(x_k^* x_k))^2\Big)^{\frac{1}{2}}\Big\|_{E_{(2)}(\mathcal{M})} \\ &\lesssim_E \Big\|\Big(\sum_k |x_k|^4\Big)^{\frac{1}{2}}\Big\|_{E_{(2)}(\mathcal{M})} + \Big\|\Big(\sum_k (\mathcal{E}_{k-1}(|x_k|^2))^2\Big)^{\frac{1}{2}}\Big\|_{E_{(2)}(\mathcal{M})},\end{aligned}$$

where in the final inequality we use the quasi-triangle inequality in $E_{(2)}(\mathcal{M}; l_c^2)$. By applying the noncommutative Stein inequality (Lemma 6.28) to the second term on the right-hand side, we find that

$$\Big\|\sum_k x_k^* x_k - \mathcal{E}_{k-1}(x_k^* x_k)\Big\|_{E_{(2)}(\mathcal{M})} \lesssim_E \Big\|\Big(\sum_k |x_k|^4\Big)^{\frac{1}{2}}\Big\|_{E_{(2)}(\mathcal{M})}.$$

As observed in (7.6), we have

$$\Big\|\Big(\sum_k |x_k|^4\Big)^{\frac{1}{2}}\Big\|_{E_{(2)}(\mathcal{M})} \lesssim_E \|\mathrm{diag}(x_k)\|_{E(M_n(\mathcal{M}))}\Big\|\Big(\sum_k |x_k|^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})}.$$

Putting our estimates together, starting from (7.9), we arrive at

$$\begin{aligned}&\Big\|\Big(\sum_k |x_k|^2\Big)^{\frac{1}{2}}\Big\|_E \\ &\quad\lesssim_E \Big(\|\mathrm{diag}(x_k)\|_{E(M_n(\mathcal{M}))}\Big\|\Big(\sum_k |x_k|^2\Big)^{\frac{1}{2}}\Big\|_E + \Big\|\Big(\sum_k \mathcal{E}_{k-1}(|x_k|^2)\Big)^{\frac{1}{2}}\Big\|_E^2\Big)^{\frac{1}{2}}.\end{aligned}$$

In other words, if we set $a = \|(\sum_k |x_k|^2)^{\frac{1}{2}}\|_{E(\mathcal{M})}$, $b = \|diag(x_k)\|_{E(M_n(\mathcal{M}))}$ and $c = \|(\sum_k \mathcal{E}_{k-1}(|x_k|^2))^{\frac{1}{2}}\|_{E(\mathcal{M})}$, we have $a^2 \lesssim_E ab + c^2$. Solving this quadratic equation we obtain $a \lesssim_E \max\{b, c\}$, or,

$$\begin{aligned}&\Big\|\Big(\sum_k |x_k|^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})} \\ &\qquad\lesssim_E \max\Big\{\|\mathrm{diag}(x_k)\|_{E(M_n(\mathcal{M}))}, \Big\|\Big(\sum_k \mathcal{E}_{k-1}(|x_k|^2)\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})}\Big\}.\end{aligned}$$

Applying this to the sequence (x_k^*) gives

$$\begin{aligned}&\Big\|\Big(\sum_k |x_k^*|^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})} \\ &\qquad\lesssim_E \max\Big\{\|\mathrm{diag}(x_k)\|_{E(M_n(\mathcal{M}))}, \Big\|\Big(\sum_k \mathcal{E}_{k-1}(|x_k^*|^2)\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})}\Big\}.\end{aligned}$$

The result now follows by (7.8). $\square$

Remark 7.7. Notice that in the proof of Theorems 7.4 and 7.6 we cannot simply renorm $E_{(2)}$ to become a Banach space. Consider the Lorentz space $E = L^{p,q}(0,\infty)$, with $p > 2$ and $1 \leq q < 2$. Then E is a fully symmetric, separable Banach function space with $p_E = q_E = p > 2$. However, the space $E_{(2)} = L^{\frac{p}{2},\frac{q}{2}}$ contains a copy of $l^{\frac{q}{2}}$ and since $\frac{q}{2} < 1$, $L^{\frac{p}{2},\frac{q}{2}}$ cannot be isomorphically embedded into a Banach space (as $l^{\frac{q}{2}}$ is not locally convex).

The result in Theorem 7.6 generalizes the Burkholder-Rosenthal inequalities for noncommutative martingale difference sequences in noncommutative L^p-spaces and noncommutative Lorentz spaces found in [70], Theorem 5.1, and [63], Theorem 3.1, respectively. Note, however, that the result in [70] is also valid for Haagerup L^p-spaces (i.e., if τ is not a trace). We now sketch how to recover the noncommutative Burkholder-Rosenthal inequalities for Haagerup L^p-spaces from our result for noncommutative symmetric spaces.

7.2 Inequalities in Haagerup L^p-spaces

Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal, faithful state ϕ. Let $\sigma = \sigma^\phi$ denote the modular automorphism group of $\mathbb{R}$ on $\mathcal{M}$ associated with ϕ. For any von Neumann subalgebra $\mathcal{N}$ of $\mathcal{M}$ satisfying $\sigma(\mathcal{N}) \subset \mathcal{N}$, we let $\mathcal{R}(\mathcal{N}) = \mathcal{N} \rtimes_\sigma \mathbb{R}$ denote the von Neumann crossed product. It is known that $\mathcal{R}(\mathcal{M})$ is a semi-finite von Neumann algebra and that there exists a canonical normal semi-finite faithful trace τ on $\mathcal{R}(\mathcal{M})$ such that

$$\tau \circ \hat{\sigma}_t = e^{-t}\tau \qquad (t \in \mathbb{R}),$$

where $\hat{\sigma}$ is the dual action of $\mathbb{R}$ on $\mathcal{M}$ corresponding to σ. The Haagerup L^p-space $L^p(\mathcal{M},\phi)$ is defined as the space of all elements $x \in S(\tau)$ which satisfy $\hat{\sigma}_t(x) = e^{-\frac{t}{p}}x$. We let D denote the Radon-Nikodym derivative of the dual weight $\hat{\phi}$ with respect to τ, i.e.,

$$\hat{\phi}(x) = \tau(Dx) \qquad (x \in \mathcal{R}(\mathcal{M})).$$

The operator D is in $S(\tau)_+$ and, moreover, $D \in L^1(\mathcal{M},\phi)$. If $\mathcal{N}$ is any von Neumann subalgebra of $\mathcal{M}$, then the Radon-Nikodym derivative of $\widehat{\phi|_{\mathcal{N}}}$ with respect to τ is again equal to D. In particular, $D \in S(\mathcal{R}(\mathcal{N}), \tau|_{\mathcal{R}(\mathcal{N})})$ for any von Neumann subalgebra $\mathcal{N}$ of $\mathcal{M}$.

Let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$ and suppose that $\mathcal{N}$ is invariant under σ, i.e., $\sigma(\mathcal{N}) \subset \mathcal{N}$. Then there exists a unique normal, faithful conditional expectation $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ such that $\phi \circ \mathcal{E} = \phi$ (c.f. [130]). One can show that $\mathcal{E}$ extends to a normal faithful conditional expectation $\hat{\mathcal{E}} : \mathcal{R}(\mathcal{M}) \to \mathcal{R}(\mathcal{N})$ which satisfies $\tau \circ \hat{\mathcal{E}} = \tau$ (see e.g. [56], Theorem 4.1). One can then further extend $\hat{\mathcal{E}}$ as usual to a map on $L^1 + L^\infty(\mathcal{R}(\mathcal{M}))$ which satisfies the properties in Proposition 3.4. We refer to [56, 70] for more details on Haagerup L^p-spaces.

Recall that the Lorentz space $L^{p,\infty}(0,\infty)$ consists of all $f \in S(0,\infty)$ such that

$$\|f\|_{p,\infty} = \sup_{0<t<\infty} t^{\frac{1}{p}} \mu_t(f) < \infty.$$

If $1 < p \leq \infty$ then $L^{p,\infty}(0,\infty)$ can be equipped with an equivalent norm

$$\|f\|_{(p,\infty)} = \sup_{0<t<\infty} t^{\frac{1}{p}-1} \int_0^t \mu_s(f) ds.$$

Under this norm $L^{p,\infty}(0,\infty)$ is a fully symmetric Banach function space. Moreover, $L^{p,\infty}$ has the Fatou property and $p_{L^{p,\infty}} = q_{L^{p,\infty}} = p$. We wish to obtain the Burkholder-Rosenthal inequalities for Haagerup L^p-spaces by using the following embedding result due to H. Kosaki (see [84], Theorem 3.2).

Proposition 7.8. *If $1 < p < \infty$, then the Haagerup space $L^p(\mathcal{M},\phi)$ is a closed subspace of $L^{p,\infty}(\mathcal{R}(\mathcal{M}),\tau)$. Moreover, if $\frac{1}{p}+\frac{1}{p'}=1$, then*

$$\|x\|_{L^p(\mathcal{M},\phi)} = p'\|x\|_{L^{p,\infty}(\mathcal{R}(\mathcal{M}),\tau)} \qquad (x \in L^p(\mathcal{M},\phi)).$$

The following corollary yields an alternative proof of Theorem 5.1 from [70].

Corollary 7.9. *Fix $2 \leq p < \infty$. Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal, faithful state ϕ. Suppose that $(\mathcal{M}_k)$ is an increasing sequence of von Neumann subalgebras of $\mathcal{M}$. Suppose that, for every $k \geq 1$, $\mathcal{M}_k$ is invariant under σ and let $\mathcal{E}_k$ denote the associated conditional expectation. If (x_k) is a finite martingale difference sequence in $L^p(\mathcal{M},\phi)$ with respect to the filtration $(\mathcal{M}_k)$, then*

$$\Big\|\sum_k x_k\Big\|_{L^p(\mathcal{M},\phi)} \simeq_p \max\Big\{\|(x_k)\|_{l^p(L^p(\mathcal{M},\phi))}, \Big\|\Big(\sum_k \hat{\mathcal{E}}_{k-1}|x_k|^2\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathcal{M},\phi)}, \Big\|\Big(\sum_k \hat{\mathcal{E}}_{k-1}|x_k^*|^2\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathcal{M},\phi)}\Big\}.$$

Proof. The case where $p=2$ is trivial, so suppose that $2<p<\infty$. We can view (x_k) as a martingale difference sequence with respect to the increasing sequence of conditional expectations $(\hat{\mathcal{E}}_k)$. Therefore, Theorem 7.6 applied for in $L^{p,\infty}(\mathcal{R}(\mathcal{M}))$ yields

$$\begin{aligned}\Big\|\sum_{k=1}^n x_k\Big\|_{L^{p,\infty}(\mathcal{R}(\mathcal{M}))} &\\ \simeq_p \max\Big\{&\|\mathrm{diag}(x_k)\|_{L^{p,\infty}(M_n(\mathcal{R}(\mathcal{M})))}, \\ &\|(x_k)\|_{L^{p,\infty}(\mathcal{R}(\mathcal{M}),(\hat{\mathcal{E}}_{k-1});l^2_c)}, \|(x_k)\|_{L^{p,\infty}(\mathcal{R}(\mathcal{M}),(\hat{\mathcal{E}}_{k-1});l^2_r)}\Big\}. \quad (7.10)\end{aligned}$$

Consider the normal, semi-finite, faithful weight $\psi = \phi \otimes \mathrm{Tr}$ on $\mathcal{M}\overline{\otimes}B(l^2)$. As observed in [70], p. 995, the modular automorphism group σ^ψ associated with ϕ is given by $\sigma \otimes \mathbf{1}_{B(l^2)}$ and consequently,

$$(\mathcal{M}\overline{\otimes}B(l^2)) \rtimes_{\sigma^\psi} \mathbb{R} = (\mathcal{M} \rtimes_\sigma \mathbb{R})\overline{\otimes}B(l^2) = \mathcal{R}(\mathcal{M})\overline{\otimes}B(l^2).$$

Moreover, the canonical normal, semi-finite, faithful trace on $\mathcal{R}(\mathcal{M})\overline{\otimes}B(l^2)$ is given by $\tau \otimes \mathrm{Tr}$. By Proposition 7.8 we obtain

$$p'\|\mathrm{diag}(x_k)\|_{L^{p,\infty}(M_n(\mathcal{R}(\mathcal{M})))} = \|\mathrm{diag}(x_k)\|_{L^p(M_n(\mathcal{M}),\phi\otimes\mathrm{Tr})}.$$

Finally,

$$\|\mathrm{diag}(x_k)\|_{L^p(M_n(\mathcal{M}),\phi\otimes\mathrm{Tr})} = \|(x_k)\|_{l^p(L^p(\mathcal{M},\phi))}.$$

Therefore, we obtain (7.10) by applying Proposition 7.8 to every term in (7.10). □

By duality one can also deduce a version of Corollary 7.9 for $1 < p < 2$. We refer to [70], Theorem 6.1, for details.

7.3 Khintchine inequalities revisited

As an application of Theorem 7.4 we derive noncommutative Khintchine-type inequalities in which the Rademacher sequence is replaced by a sequence of independent noncommutative random variables. Similar inequalities in noncommutative L^p-spaces were considered in [71].

If (x_α) is a net in $S(\tau)$, then we say that x_α converges *locally in measure* to $x \in S(\tau)$ if $ex_\alpha e \to exe$ in $S(\tau)$ for every projection $e \in \mathcal{M}$ with $\tau(e) < \infty$.

Lemma 7.10. *Let $\mathcal{M}$ be a semi-finite von Neumann algebra and let p_α be a net of projections such that $p_\alpha \uparrow \mathbf{1}$. Then $p_\alpha x p_\alpha \to x$ locally in measure, for any $x \in S(\tau)$.*

Proof. Let e be a finite trace projection. If $y \in S_0(\tau)$ and x_α is a net in $S(\tau)_+$ such that $x_\alpha \downarrow 0$, then one can show (see e.g. [50]) that $x_\alpha y \to 0$ and $yx_\alpha \to 0$ in measure. In particular, $ep_\alpha \to e$ and $p_\alpha e \to e$. Since multiplication is bicontinuous with respect to the measure topology we obtain $ep_\alpha x p_\alpha e \to exe$, as asserted. □

For any $1 \le q \le \infty$ and any sequence (α_k) in $L^q(\mathcal{M})$ we use the notation

$$c_q := \inf_k \|\alpha_k\|_q, \ d_q := \sup_k \|\alpha_k\|_q.$$

Corollary 7.11. *Suppose that E is a symmetric Banach function space on $(0,\infty)$ which satisfies condition (i) or (ii) of Theorem 7.4 and let $2 \le p < \infty$ be such that E is an interpolation space for the couple (L^2, L^p). Suppose* *that E is either separable or has the Fatou property. Let $\mathcal{N}$ be a finite von Neumann algebra equipped with a normal, faithful, finite trace σ and let (α_k) be a sequence in $L^p(\mathcal{N})$ such that*

$$c_2 := \inf_k \|\alpha_k\|_2 > 0,\ d_p := \sup_k \|\alpha_k\|_p < \infty.$$

Assume that (α_k) is independent with respect to σ and that $\sigma(\alpha_k) = 0$ for all $k \geq 1$. Let $\mathcal{M}$ be a finite von Neumann algebra equipped with a normal, faithful, finite trace τ and (x_k) be a finite sequence in $E(\mathcal{M})$. Then,

$$\Big\|\sum_k \alpha_k \otimes x_k\Big\|_{E(\mathcal{N}\overline{\otimes}\mathcal{M})} \simeq_{E,c_2,d_p} \max\Big\{\|(x_k)\|_{E(\mathcal{M};l^2_c)}, \|(x_k)\|_{E(\mathcal{M};l^2_r)}\Big\}. \quad (7.11)$$

Proof. Note that for $(x_k)_{k=1}^n$ in $E(\mathcal{M})$ it is a priori not clear that $\sum \alpha_k \otimes x_k$ defines an element of $E(\mathcal{N}\overline{\otimes}\mathcal{M})$. We deduce this via an approximation argument. Suppose first that the α_k and x_k are bounded. Identify $\mathcal{M}$ with $\mathbb{C}\mathbf{1}_{\mathcal{N}}\overline{\otimes}\mathcal{M} \subset \mathcal{N}\overline{\otimes}\mathcal{M}$. It is easy to see that

$$\mathcal{E}_{\mathcal{M}}(\alpha \otimes x) = \sigma(\alpha)\mathbf{1}_{\mathcal{N}} \otimes x\ (\alpha \in L^p(\mathcal{N}),\ x \in \mathcal{M}).$$

Since (α_k) is independent with respect to σ, it follows that $(\alpha_k \otimes x_k)$ is independent with respect to $\mathcal{E}_{\mathcal{M}}$ and, moreover, $\mathcal{E}_{\mathcal{M}}(\alpha_k \otimes x_k) = 0$ for all k. By Theorem 7.4,

$$\begin{aligned}\Big\|\sum_k \alpha_k \otimes x_k\Big\|_E \simeq_E \max\Big\{&\|\mathrm{diag}(\alpha_k \otimes x_k)\|_{E(M_n(\mathcal{N}\overline{\otimes}\mathcal{M}))}, \\ &\|(\alpha_k \otimes x_k)\|_{E(\mathcal{N}\overline{\otimes}\mathcal{M},\mathcal{E}_{\mathcal{M}};l^2_c)}, \|(\alpha_k \otimes x_k)\|_{E(\mathcal{N}\overline{\otimes}\mathcal{M},\mathcal{E}_{\mathcal{M}};l^2_r)}\Big\}.\end{aligned}$$

Now,

$$\begin{aligned}\|(\alpha_k \otimes x_k)\|_{E(\mathcal{N}\overline{\otimes}\mathcal{M},\mathcal{E}_{\mathcal{M}};l^2_c)} &= \Big\|\Big(\sum_k \mathcal{E}_{\mathcal{M}}(|\alpha_k|^2 \otimes |x_k|^2)\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})} \\ &= \Big\|\Big(\sum_k x_k^* x_k \|\alpha_k\|_2^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})},\end{aligned}$$

so by assumption,

$$\|(\alpha_k \otimes x_k)\|_{E(\mathcal{N}\overline{\otimes}\mathcal{M},\mathcal{E}_{\mathcal{M}};l^2_c)} \simeq_{c_2,d_2} \|(x_k)\|_{E(\mathcal{M};l^2_c)}.$$

Applying this to $(\alpha_k^* \otimes x_k^*)$, we obtain

$$\|(\alpha_k \otimes x_k)\|_{E(\mathcal{N}\overline{\otimes}\mathcal{M},\mathcal{E}_{\mathcal{M}};l^2_r)} \simeq_{c_2,d_2} \|(x_k)\|_{E(\mathcal{M};l^2_r)}.$$

Notice that for any $2 \leq q < \infty$ we have $\|\alpha_k \otimes x_k\|_q = \|\alpha_k\|_q\|x_k\|_q$ and therefore,

$$\|(\alpha_k \otimes x_k)\|_{l^q(L^q(\mathcal{N}\overline{\otimes}\mathcal{M}))} \simeq_{c_q,d_q} \|(x_k)\|_{l^q(L^q(\mathcal{M}))}.$$

Also, $\|(x_k)\|_{l^q(L^q(\mathcal{M}))} \le \|(x_k)\|_{L^q(\mathcal{M};l^2_c)}$, which follows by interpolation of the cases $q = 2$ and $q = \infty$. Hence, for any $2 \le q < \infty$, the map $(x_k) \mapsto \mathrm{diag}(\alpha_k \otimes x_k)$ extends to a bounded map from $L^q(\mathcal{M}; l^2_c)$ into $L^q(M_n(\mathcal{N}\bar{\otimes}\mathcal{M}))$ with norm bounded by d_q. By interpolation, i.e., using Theorem 5.5,

$$\|\mathrm{diag}(\alpha_k \otimes x_k)\|_{E(M_n(\mathcal{N}\bar{\otimes}\mathcal{M}))} \lesssim_{E,d_p} \|(x_k)\|_{E(\mathcal{M};l^2_c)}.$$

By approximation (7.11) holds for $\alpha_k \in L^p(\mathcal{M})$ with $c_2 > 0$ and $d_p < \infty$.

Suppose now that E has the Fatou property and let $(x_k)_{k=1}^n$ be a finite sequence in $E(\mathcal{M})$. For every $n \ge 1$ set $e_n = e^{(\sum_k |x_k|^2)^{\frac{1}{2}}}[0, n]$. Then $e_n x_k e_n$ is bounded for all k and n, so by the above,

$$\begin{aligned}\Big\|\sum_k \alpha_k \otimes e_n x_k e_n\Big\|_{E(\mathcal{N}\bar{\otimes}\mathcal{M})} &\lesssim_{E,c_2,d_p} \|(e_n x_k e_n)\|_{E(\mathcal{M};l^2_c)\cap E(\mathcal{M};l^2_r)} \\ &\le \quad \|(x_k)\|_{E(\mathcal{M};l^2_c)\cap E(\mathcal{M};l^2_r)}.\end{aligned}$$

By Lemma 7.10 we find that $\sum_k \alpha_k \otimes e_n x_k e_n \to \sum_k \alpha_k \otimes x_k$ locally in measure. Since the closed unit ball of $E(\mathcal{M})$ is closed in $S(\tau)$ for convergence in the local measure topology if E has the Fatou property ([49], Proposition 5.14), we deduce that $\sum_k \alpha_k \otimes x_k \in E(\mathcal{M})$ and

$$\Big\|\sum_k \alpha_k \otimes x_k\Big\|_{E(\mathcal{N}\bar{\otimes}\mathcal{M})} \lesssim_{E,c_2,d_p} \max\Big\{\|(x_k)\|_{E(\mathcal{M};l^2_c)}, \|(x_k)\|_{E(\mathcal{M};l^2_r)}\Big\}.$$

The reverse estimate is proved similarly.

Suppose now that E is separable. Then there exists a sequence (x_k^m) in $E(\mathcal{M}; l^2_c) \cap E(\mathcal{M}; l^2_r)$ such that x_k^m is bounded and $(x_k^m) \to (x_k)$ in $E(\mathcal{M}; l^2_c) \cap E(\mathcal{M}; l^2_r)$. By the above, the sequence $(\sum \alpha_k \otimes x_k^m)$ is Cauchy in $E(\mathcal{N}\bar{\otimes}\mathcal{M})$ and hence converges to some $y \in E(\mathcal{N}\bar{\otimes}\mathcal{M})$ in norm and hence also with respect to the measure topology. On the other hand, it is clear that $x_k^m \to x_k$ in measure for all k and so $\sum \alpha_k \otimes x_k^m \to \sum \alpha_k \otimes x_k^m$ in measure. Therefore $y = \sum \alpha_k \otimes x_k$ and we conclude that $\sum \alpha_k \otimes x_k$ is in $E(\mathcal{N}\bar{\otimes}\mathcal{M})$ and that (7.11) holds. $\square$

By a duality argument we can now deduce the following result.

Corollary 7.12. *Let E be a separable symmetric space on $(0, \infty)$ which is either p-convex for some $p > 1$ and 2-concave or satisfies $1 < p_E \le q_E < 2$. Let $1 < r < p$ (respectively, $1 < r < p_E$) and let r' be such that $\frac{1}{r} + \frac{1}{r'} = 1$. Let $\mathcal{N}$ be a von Neumann algebra equipped with a normal, faithful, finite trace σ and let (α_k) be a sequence in $L^{r'}(\mathcal{N})$ such that*

$$c_2 := \inf_k \|\alpha_k\|_2 > 0, \;\; d_{r'} := \sup_k \|\alpha_k\|_{r'} < \infty.$$

Assume that (α_k) is independent with respect to σ and that $\sigma(\alpha_k) = 0$ for all $k \ge 1$. Let $\mathcal{M}$ be another von Neumann algebra equipped with a normal, faithful, finite trace τ. If (x_k) is a finite sequence in $E(\mathcal{M})$, then

$$\Big\|\sum_k \alpha_k \otimes x_k\Big\|_{E(\mathcal{N}\overline{\otimes}\mathcal{M})} \simeq_{E,c_2,d_{r'}} \inf\Big\{\|(y_k)\|_{E(\mathcal{M};l_c^2)} + \|(z_k)\|_{E(\mathcal{M};l_r^2)}\Big\},$$

where the infimum runs over all decompositions $x_k = y_k + z_k$ in $E(\mathcal{M})$.

Proof. Suppose first that $(\alpha_k)_{k=1}^n$ and $(x_k)_{k=1}^n$ are finite sequences in $\mathcal{N}$ and $\mathcal{M}$, respectively. Let $\mathcal{E}_{\mathcal{M}}$ denote the conditional expectation onto the von Neumann subalgebra $\mathbb{C}\mathbf{1}_{\mathcal{N}}\overline{\otimes}\mathcal{M}$. Let $1 \leq q < 2$, then $\frac{q}{2} < 1$ and it follows by [68], Theorem 7.1, that

$$\begin{aligned}\Big\|\sum_{k=1}^n \alpha_k \otimes x_k\Big\|_q^2 &= \Big\|\Big|\sum_{k=1}^n \alpha_k \otimes x_k\Big|^2\Big\|_{\frac{q}{2}} \leq \Big\|\mathcal{E}_{\mathcal{M}}\Big|\sum_{k=1}^n \alpha_k \otimes x_k\Big|^2\Big\|_{\frac{q}{2}} \\ &= \Big\|\sum_{k=1}^n x_k^* x_k \|\alpha_k\|_2^2\Big\|_{\frac{q}{2}} \leq \Big(\sup_{1\leq k\leq n} \|\alpha_k\|_2^2\Big)\|(x_k)\|_{L^q(\mathcal{M};l_c^2)}^2.\end{aligned}$$

By approximation, the estimate

$$\Big\|\sum_{k=1}^n \alpha_k \otimes x_k\Big\|_{L^q(\mathcal{N}\overline{\otimes}\mathcal{M})}^2 \leq \Big(\sup_{k\geq 1} \|\alpha_k\|_2^2\Big)\|(x_k)\|_{L^q(\mathcal{M};l_c^2)}^2$$

holds for any sequence $(\alpha_k)_{k\geq 1}$ in $L^2(\mathcal{M})$ such that $\sup_k \|\alpha_k\|_2 < \infty$. Hence, for any $1 \leq q \leq 2$, the map $(x_k) \mapsto \sum_k \alpha_k \otimes x_k$ extends to a bounded map from $L^q(\mathcal{M}; l_c^2)$ into $L^q(\mathcal{N}\overline{\otimes}\mathcal{M})$ with norm bounded by d_2. Notice that $E(\mathcal{M})$ is an interpolation space for the couple $(L^r(\mathcal{M}), L^2(\mathcal{M}))$ by Theorems 4.31 and 5.5. By interpolation we obtain,

$$\Big\|\sum_k \alpha_k \otimes x_k\Big\|_{E(\mathcal{N}\overline{\otimes}\mathcal{M})} \lesssim_{d_2} \|(x_k)\|_{E(\mathcal{M};l_c^2)}.$$

Applying this to $(\alpha_k^* \otimes x_k^*)$ yields

$$\Big\|\sum_k \alpha_k \otimes x_k\Big\|_{E(\mathcal{N}\overline{\otimes}\mathcal{M})} \lesssim_{d_2} \|(x_k)\|_{E(\mathcal{M};l_r^2)}.$$

By the triangle inequality, we arrive at

$$\Big\|\sum \alpha_k \otimes x_k\Big\|_{E(\mathcal{N}\overline{\otimes}\mathcal{M})} \lesssim_{d_2} \inf\Big\{\|(y_k)\|_{E(\mathcal{M};l_c^2)} + \|(z_k)\|_{E(\mathcal{M};l_r^2)}\Big\},$$

where the infimum runs over all decompositions $x_k = y_k + z_k$ in $E(\mathcal{M})$.

We deduce the opposite inequality by duality. Since $c_2 > 0$, we may assume that $\|\alpha_k\|_2 = 1$ for all k. Let $(x_k^{\times})$ be a finite sequence in $E^{\times}(\mathcal{M})$. Notice that $E^{\times} = E^*$ is either 2-convex and p'-concave for $p' < \infty$ (c.f. Theorem 4.13) or satisfies $2 < p_{E^{\times}} \leq q_{E^{\times}} < \infty$ by (4.7). Moreover, $E^{\times}$ has the Fatou property. Therefore we can apply Corollary 7.11 and obtain

$$\sum_k \tau(x_k x_k^{\times}) = \sigma \otimes \tau\Big(\Big(\sum_k \alpha_k \otimes x_k\Big)\Big(\sum_j \alpha_j \otimes x_j^{\times}\Big)\Big)$$

$$\leq \Big\| \sum_k \alpha_k \otimes x_k \Big\|_{E(\mathcal{N}\overline{\otimes}\mathcal{M})} \Big\| \sum_j \alpha_j \otimes x_j^\times \Big\|_{E^\times(\mathcal{N}\overline{\otimes}\mathcal{M})}$$
$$\lesssim_{d_{r'}} \Big\| \sum_k \alpha_k \otimes x_k \Big\|_{E(\mathcal{N}\overline{\otimes}\mathcal{M})} \max\Big\{ \|(x_j^\times)\|_{E^\times(\mathcal{M};l_c^2)}, \|(x_j^\times)\|_{E^\times(\mathcal{M};l_r^2)} \Big\}.$$

Taking the supremum over all finite sequences $(x_k^\times)$ in $E^\times(\mathcal{M})$ and using that $(E(\mathcal{M};l_c^2) + E(\mathcal{M};l_r^2))^* = E^\times(\mathcal{M};l_c^2) \cap E^\times(\mathcal{M};l_r^2)$ isometrically, we obtain the desired inequality. □

In Corollary 8.20 we will use Theorem 7.4 to derive Khintchine-type inequalities for free product von Neumann algebras.

7.4 Independent vectors in a noncommutative symmetric space

We return to the setting of Chapter 3 and consider sums of independent random vectors in a noncommutative Banach function spaces. The main result, Theorem 7.14, is an extension of the result for L^q-spaces stated in Theorem 3.8.

Lemma 7.13. *Suppose that $\mathcal{M}$ is a semi-finite von Neumann algebra and let E be a 2-convex symmetric Banach function space on $(0,\infty)$ with Fatou norm. If (ξ_i) is a finite sequence of independent, mean zero $E(\mathcal{M})$-valued random variables, then*

$$\max\Big\{ \Big\| \Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{\frac12} \Big\|_{E(\mathcal{M})}, \Big\| \Big(\sum_i \mathbb{E}|\xi_i^*|^2\Big)^{\frac12} \Big\|_{E(\mathcal{M})} \Big\} \lesssim_E \Big(\mathbb{E}\Big\| \sum_i \xi_i \Big\|_{E(\mathcal{M})}^2\Big)^{\frac12}.$$

Proof. Since E is 2-convex and has Fatou norm, it follows that $E(\mathcal{M})$ is 2-convex as well (see e.g. [50]). By Corollary 1.10 and (6.13),

$$\Big(\mathbb{E}\Big\| \sum_i \xi_i \Big\|_{E(\mathcal{M})}^2\Big)^{\frac12}$$
$$\simeq \Big(\mathbb{E}\mathbb{E}_r\Big\| \sum_i r_i\xi_i \Big\|_{E(\mathcal{M})}^2\Big)^{\frac12}$$
$$\gtrsim_E \max\Big\{ \Big(\mathbb{E}\Big\| \Big(\sum_i |\xi_i|^2\Big)^{\frac12} \Big\|_{E(\mathcal{M})}^2\Big)^{\frac12}, \Big(\mathbb{E}\Big\| \Big(\sum_i |\xi_i^*|^2\Big)^{\frac12} \Big\|_{E(\mathcal{M})}^2\Big)^{\frac12} \Big\}$$
$$= \max\Big\{ \Big(\mathbb{E}\Big\| \sum_i |\xi_i|^2 \Big\|_{E_{(2)}(\mathcal{M})}\Big)^{\frac12}, \Big(\mathbb{E}\Big\| \sum_i |\xi_i^*|^2 \Big\|_{E_{(2)}(\mathcal{M})}\Big)^{\frac12} \Big\}$$
$$\gtrsim_E \max\Big\{ \Big\| \sum_i \mathbb{E}|\xi_i|^2 \Big\|_{E_{(2)}(\mathcal{M})}^{\frac12}, \Big\| \sum_i \mathbb{E}|\xi_i^*|^2 \Big\|_{E_{(2)}(\mathcal{M})}^{\frac12} \Big\}$$

$$= \max\Big\{\Big\|\Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})}, \Big\|\Big(\sum_i \mathbb{E}|\xi_i^*|^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})}\Big\},$$

where the final inequality follows by 2-convexity of $E(\mathcal{M})$. □

Theorem 7.14. *Suppose that $2 \leq p < \infty$ and let E be a symmetric Banach function space on $(0,\infty)$ which is 2-convex and q-concave for some $q < \infty$. Let $\mathcal{M}$ be a semi-finite von Neumann algebra. If (ξ_i) is a finite sequence of independent, mean zero $E(\mathcal{M})$-valued random variables, then*

$$\Big(\mathbb{E}\Big\|\sum_{i=1}^n \xi_i\Big\|_{E(\mathcal{M})}^p\Big)^{\frac{1}{p}} \simeq_{p,E} \max\Big\{\Big\|\Big(\sum_{i=1}^n \mathbb{E}|\xi_i|^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})}, \Big\|\Big(\sum_{i=1}^n \mathbb{E}|\xi_i^*|^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})}, \\ (\mathbb{E}\|\mathrm{diag}(\xi_i)_{i=1}^n\|_{E(M_n(\mathcal{M}))}^p)^{\frac{1}{p}}\Big\}. \tag{7.12}$$

Proof. By renorming we may assume that the 2-convexity constant of E is equal to 1. Note that E has Fatou norm by Lemma 4.12. We first show that the maximum on the right hand side dominates $(\mathbb{E}\|\sum_i \xi_i\|_{E(\mathcal{M})}^p)^{\frac{1}{p}}$. In Lemma 7.13 we observed that

$$\max\Big\{\Big\|\Big(\sum_i \mathbb{E}|\xi_i|^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})}, \Big\|\Big(\sum_i \mathbb{E}|\xi_i^*|^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})}\Big\} \lesssim_E \Big(\mathbb{E}\Big\|\sum_i \xi_i\Big\|_{E(\mathcal{M})}^p\Big)^{\frac{1}{p}}.$$

By Theorem 4.31, E is an interpolation space for the couple (L^2, L^q). Therefore, by our discussion preceding Corollary 6.9, the n-th Rademacher subspace $Rad_n(E)$ is C_E-complemented in $E(L^\infty\overline{\otimes}\mathcal{M})$, for some constant $C_E > 0$ independent of n. Recall from Theorem 3.3 that $L^r(\mathcal{M})$ has cotype r if $2 \leq r < \infty$, i.e.,

$$\|\mathrm{diag}(\xi_i)_{i=1}^n\|_{L^r(M_n(\mathcal{M}))} = \Big(\sum_{i=1}^n \|\xi_i\|_{L^r(\mathcal{M})}^r\Big)^{\frac{1}{r}} \leq \Big\|\sum_{i=1}^n r_i \otimes \xi_i\Big\|_{L^r(L^\infty\overline{\otimes}\mathcal{M})}.$$

By interpolation of this estimate for $r = 2$ and $r = q$ we obtain

$$\|\mathrm{diag}(\xi_i)_{i=1}^n\|_{E(M_n(\mathcal{M}))} \lesssim_E \Big\|\sum_{i=1}^n r_i \otimes \xi_i\Big\|_{E(L^\infty\overline{\otimes}\mathcal{M})}.$$

Moreover, by Corollary 6.9,

$$\Big\|\sum_{i=1}^n r_i \otimes \xi_i\Big\|_{E(L^\infty\overline{\otimes}\mathcal{M})} \simeq_E \mathbb{E}_r\Big\|\sum_{i=1}^n r_i\xi_i\Big\|_{E(\mathcal{M})} \leq \Big(\mathbb{E}_r\Big\|\sum_{i=1}^n r_i\xi_i\Big\|_{E(\mathcal{M})}^p\Big)^{\frac{1}{p}}.$$

By Corollary 1.10 we conclude that

$$(\mathbb{E}\|\mathrm{diag}(\xi_i)_{i=1}^n\|_{E(M_n(\mathcal{M}))}^p)^{\frac{1}{p}} \lesssim_E \Big(\mathbb{E}\mathbb{E}_r\Big\|\sum_{i=1}^n r_i\xi_i\Big\|_{E(\mathcal{M})}^p\Big)^{\frac{1}{p}}$$

$$\simeq \Big(\mathbb{E}\Big\|\sum_{i=1}^{n}\xi_i\Big\|_{E(\mathcal{M})}^{p}\Big)^{\frac{1}{p}}.$$

We now prove the reverse inequality. By randomization (c.f. Corollary 1.10) and Theorem 6.7 we have

$$\begin{aligned}&\Big(\mathbb{E}\Big\|\sum_i\xi_i\Big\|_{E(\mathcal{M})}^{p}\Big)^{\frac{1}{p}}\\&\quad\simeq_{p,E}\max\Big\{\Big(\mathbb{E}\Big\|\Big(\sum_i|\xi_i|^2\Big)^{\frac12}\Big\|_{E(\mathcal{M})}^{p}\Big)^{\frac1p},\Big(\mathbb{E}\Big\|\Big(\sum_i|\xi_i^*|^2\Big)^{\frac12}\Big\|_{E(\mathcal{M})}^{p}\Big)^{\frac1p}\Big\}.\end{aligned}\tag{7.13}$$

We estimate the first term on the right hand side. By the triangle inequality in $E_{(2)}(\mathcal{M})$ we obtain,

$$\begin{aligned}&\Big(\mathbb{E}\Big\|\Big(\sum_i|\xi_i|^2\Big)^{\frac12}\Big\|_{E(\mathcal{M})}^{p}\Big)^{\frac1p}\\&\quad\leq\Big(\Big(\mathbb{E}\Big\|\sum_i|\xi_i|^2-\mathbb{E}|\xi_i|^2\Big\|_{E_{(2)}(\mathcal{M})}^{\frac p2}\Big)^{\frac2p}+\Big\|\sum_i\mathbb{E}|\xi_i|^2\Big\|_{E_{(2)}(\mathcal{M})}\Big)^{\frac12}.\end{aligned}\tag{7.14}$$

We focus on the first term on the right hand side. By randomizing and applying Theorem 6.7 we find that

$$\begin{aligned}&\Big(\mathbb{E}\Big\|\sum_i|\xi_i|^2-\mathbb{E}|\xi_i|^2\Big\|_{E_{(2)}(\mathcal{M})}^{\frac p2}\Big)^{\frac2p}\\&\quad\simeq_{p,E}\Big(\mathbb{E}\Big\|\Big(\sum_i\big|\,|\xi_i|^2-\mathbb{E}|\xi_i|^2\big|^2\Big)^{\frac12}\Big\|_{E_{(2)}(\mathcal{M})}^{\frac p2}\Big)^{\frac2p}\\&\quad\leq\Big(\mathbb{E}\Big\|\Big(\sum_i|\xi_i|^4\Big)^{\frac12}\Big\|_{E_{(2)}(\mathcal{M})}^{\frac p2}\Big)^{\frac2p}+\Big\|\Big(\sum_i|\mathbb{E}|\xi_i|^2|^2\Big)^{\frac12}\Big\|_{E_{(2)}(\mathcal{M})},\end{aligned}$$

where we use the triangle inequality in $E_{(2)}(\mathcal{M};l_c^2)$. Notice that the second term on the right hand side is smaller than the first one. Indeed,

$$\begin{aligned}\Big\|\Big(\sum_i|\mathbb{E}|\xi_i|^2|^2\Big)^{\frac12}\Big\|_{E_{(2)}(\mathcal{M})}&=\|\mathrm{col}(\mathbb{E}|\xi_i|^2)\|_{E_{(2)}(M_n(\mathcal{M}))}\\&=\|\mathbb{E}(\mathrm{col}(|\xi_i|^2))\|_{E_{(2)}(M_n(\mathcal{M}))}\\&\leq(\mathbb{E}\|\mathrm{col}(|\xi_i|^2)\|_{E_{(2)}(M_n(\mathcal{M}))}^{\frac p2})^{\frac2p}\\&=\Big(\mathbb{E}\Big\|\Big(\sum_i|\xi_i|^4\Big)^{\frac12}\Big\|_{E_{(2)}(\mathcal{M})}^{\frac p2}\Big)^{\frac2p}.\end{aligned}$$

Write $x=\mathrm{col}(|\xi_i|)$ and $y=\mathrm{diag}(|\xi_i|)$. Since $E_{(2)}$ is a $\frac q2$-concave symmetric Banach function space, it is separable by Lemma 4.12 and therefore fully symmetric. Using that $\mu(yx)\prec\prec\mu(y)\mu(x)$, we obtain

$$\begin{aligned}\Big(\mathbb{E}\Big\|\Big(\sum_i|\xi_i|^4\Big)^{\frac{1}{2}}\Big\|_{E_{(2)}(\mathcal{M})}^{\frac{p}{2}}\Big)^{\frac{2}{p}} &= (\mathbb{E}\|(x^*y^*yx)^{\frac{1}{2}}\|_{E_{(2)}(M_n(\mathcal{M}))}^{\frac{p}{2}})^{\frac{2}{p}}\\ &= (\mathbb{E}\|yx\|_{E_{(2)}(M_n(\mathcal{M}))}^{\frac{p}{2}})^{\frac{2}{p}}\\ &\lesssim_E (\mathbb{E}\|\mu(y)\mu(x)\|_{E_{(2)}}^{\frac{p}{2}})^{\frac{2}{p}}\\ &= (\mathbb{E}\|\mu(y)^{\frac{1}{2}}\mu(x)^{\frac{1}{2}}\|_E^p)^{\frac{2}{p}}.\end{aligned}$$

By Lemma 4.3 this implies that

$$\begin{aligned}&\Big(\mathbb{E}\Big\|\Big(\sum_i|\xi_i|^4\Big)^{\frac{1}{2}}\Big\|_{E_{(2)}(\mathcal{M})}^{\frac{p}{2}}\Big)^{\frac{2}{p}}\\ &\le (\mathbb{E}|\ \|y\|_{E(M_n(\mathcal{M}))}\|x\|_{E(M_n(\mathcal{M}))}|^{\frac{p}{2}})^{\frac{2}{p}}\\ &\le (\mathbb{E}\|y\|_{E(M_n(\mathcal{M}))}^p)^{\frac{1}{p}}(\mathbb{E}\|x\|_{E(M_n(\mathcal{M}))}^p)^{\frac{1}{p}}\\ &= (\mathbb{E}\|\mathrm{diag}(\xi_i)\|_{E(M_n(\mathcal{M}))}^p)^{\frac{1}{p}}\Big(\mathbb{E}\Big\|\Big(\sum_i|\xi_i|^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})}^p\Big)^{\frac{1}{p}}.\end{aligned}$$

Collecting our estimates, starting from (7.14), we obtain the quadratic equation

$$a^2 \lesssim_{p,E} ab + c^2,$$

where we set $a = (\mathbb{E}\|(\sum_i|\xi_i|^2)^{\frac{1}{2}}\|_{E(\mathcal{M})}^p)^{\frac{1}{p}}$, $b = (\mathbb{E}\|\mathrm{diag}(\xi_i)\|_{E(M_n(\mathcal{M}))}^p)^{\frac{1}{p}}$ and $c = \|(\sum_i\mathbb{E}|\xi_i|^2)^{\frac{1}{2}}\|_{E(\mathcal{M})}$. Solving this quadratic equation we conclude that $a \lesssim_{p,E} \max\{b,c\}$, i.e.

$$\begin{aligned}&\Big(\mathbb{E}\Big\|\Big(\sum_i|\xi_i|^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})}^p\Big)^{\frac{1}{p}}\\ &\qquad\lesssim_{p,E}\max\Big\{\Big\|\Big(\sum_i\mathbb{E}|\xi_i|^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})},(\mathbb{E}\|\mathrm{diag}(\xi_i)\|_{E(M_n(\mathcal{M}))}^p)^{\frac{1}{p}}\Big\}.\end{aligned}$$

Applying this to the sequence (ξ_i^*) we obtain

$$\begin{aligned}&\Big(\mathbb{E}\Big\|\Big(\sum_i|\xi_i^*|^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})}^p\Big)^{\frac{1}{p}}\\ &\qquad\lesssim_{p,E}\max\Big\{\Big\|\Big(\sum_i\mathbb{E}|\xi_i^*|^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})},(\mathbb{E}\|\mathrm{diag}(\xi_i)\|_{E(M_n(\mathcal{M}))}^p)^{\frac{1}{p}}\Big\}.\end{aligned}$$

By (7.13) our proof is complete. □

8

Noncommutative stochastic integration

In this final chapter we apply the Khintchine-type inequalities obtained in Section 7.3 to prove Itô-type isomorphisms for stochastic integrals with respect to Boson and free Brownian motions. The key additional ingredients used to prove these results are two novel noncommutative decoupling inequalities, presented in Theorems 8.14 and 8.15.

8.1 Noncommutative decoupling inequalities

We start by discussing two preliminary notions from noncommutative probability theory. Firstly, we consider two 'strong' forms of independence and, secondly, we define probability distributions for normal operators affiliated to a finite von Neumann algebra. In the final part we prove two noncommutative decoupling inequalities.

8.1.1 Tensor and free independence

For the purpose of this section we introduce the following terminology.

Definition 8.1. *A pair $(\mathcal{M}, \tau)$ consisting of a finite von Neumann algebra and a normal, faithful trace τ on $\mathcal{M}$ satisfying $\tau(\mathbf{1}) = 1$ is called a* noncommutative probability space*.*

In this chapter we are concerned with the following two specific notions of independence: *tensor independence*, which is a straightforward generalization of the notion in classical probability theory and *free independence*, which was introduced by D.V. Voiculescu and led to the development of free probability theory which takes free independence as its axiom (see [140] and [109] for this beautiful theory). By axiomatizing the intuitive requirements a notion of independence should satisfy, one can in fact show that, in a sense, these are the only possible notions of independence in a noncommutative probability space (c.f. [17]).

Definition 8.2. *Let $(\mathcal{M}, \tau)$ be a noncommutative probability space. The von Neumann subalgebras $\mathcal{A}_1, \dots, \mathcal{A}_N$ of $\mathcal{M}$ are called* tensor independent *if we have the following factorization:*

$$\tau\Big(\prod_{j=1}^{n}\Big(\prod_{i=1}^{N} a_{ij}\Big)\Big) = \prod_{i=1}^{N}\tau\Big(\prod_{j=1}^{n} a_{ij}\Big),$$

whenever $a_{ij} \in \mathcal{A}_i$ $(j = 1, \dots, n;\ i = 1, \dots, N;\ n, N \in \mathbb{N})$.
The von Neumann subalgebras $\mathcal{A}_1, \dots, \mathcal{A}_N$ are called freely independent *if*

$$\tau\Big(\prod_{j=1}^{n} a_j\Big) = 0$$

whenever the following conditions hold:

(a) n is a positive integer;
(b) $a_j \in \mathcal{A}_{i(j)}$ for all $j = 1, \dots, n$;
(c) $\tau(a_j) = 0$ for all $j = 1, \dots, n$;
(d) neighboring elements are from different subalgebras, that is $i(1) \neq i(2)$, $i(2) \neq i(3), \dots, i(n-1) \neq i(n)$.

A collection $\{\mathcal{A}_i\}_{i\in I}$ of von Neumann subalgebras of $\mathcal{M}$ is called (tensor/freely) independent if every finite subcollection is (tensor/freely) independent.

One should observe that both tensor and free independence imply independence in the sense of Definition 7.1 with respect to the trivial von Neumann subalgebra $\mathbb{C}\mathbf{1}$.

Suppose that $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ are von Neumann subalgebras of $\mathcal{M}$. Then $\mathcal{A}_1$ is (tensor/freely) independent of $\mathcal{A}_2$ if and only if $\mathcal{A}_2$ is (tensor/freely) independent of $\mathcal{A}_1$. If this is the case, then any von Neumann subalgebra of $\mathcal{A}_1$ is (tensor/freely) independent of any von Neumann subalgebra of $\mathcal{A}_2$. Moreover, we note that $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ are (tensor/freely) independent if and only if $\mathcal{A}_1$, $\mathcal{A}_2$ are (tensor/freely) independent and $W^*(\mathcal{A}_1, \mathcal{A}_2)$ and $\mathcal{A}_3$ are (tensor/freely) independent if and only if $\mathcal{A}_2$, $\mathcal{A}_3$ are (tensor/freely) independent and $W^*(\mathcal{A}_2, \mathcal{A}_3)$ and $\mathcal{A}_1$ are (tensor/freely) independent. That is, tensor and free independence carry over to subalgebras, are commutative and associative.

Remark 8.3. For a τ-measurable operator a independence is always understood in terms of the von Neumann subalgebra $W^*(a)$ it generates. That is, we call a sequence $(a_k)_{k=1}^{\infty}$ in $S(\tau)$ (tensor/freely) independent if the sequence $(W^*(a_k))_{k=1}^{\infty}$ of von Neumann subalgebras in $\mathcal{M}$ is (tensor/freely) independent.

Roughly speaking, tensor and free independence correspond to two different ways of constructing products of noncommutative probability spaces. Suppose we are given some finite set $(\mathcal{M}_1, \tau_1), \dots, (\mathcal{M}_n, \tau_n)$ of noncommutative probability spaces. We wish to define a product probability space, that is, a noncommutative probability space $(\mathcal{M}, \tau)$ such that the $\mathcal{M}_i$ are contained in $\mathcal{M}$, τ equals τ_i on $\mathcal{M}_i$ $(i = 1, \ldots, n)$, $\mathcal{M}_1, \ldots, \mathcal{M}_n$ generate $\mathcal{M}$ as a von Neumann algebra and $\mathcal{M}_1, \ldots, \mathcal{M}_n$ are independent in a certain sense in $\mathcal{M}$ with respect to τ. We will consider two types of product probability spaces: the tensor product of noncommutative probability spaces, which corresponds to tensor independence and is a generalization of the product of classical probability spaces, and the free product of noncommutative probability spaces, in which case $\mathcal{M}_1, \ldots, \mathcal{M}_n$ are freely independent in the product space. The tensor product construction is classical, see e.g. [131].

Theorem 8.4. *(Tensor product probability space) Let $(\mathcal{M}_1, \tau_1), \ldots, (\mathcal{M}_n, \tau_n)$ be noncommutative probability spaces $(i = 1, \ldots, n)$. Set $\mathcal{M} = \mathcal{M}_1 \overline{\otimes} \cdots \overline{\otimes} \mathcal{M}_n$, the von Neumann algebra tensor product of $\mathcal{M}_1, \ldots, \mathcal{M}_n$, $\tau = \tau_1 \otimes \cdots \otimes \tau_n$ and define the maps $W^i : \mathcal{M}_i \to \mathcal{M}$ by $W_i(a) = 1 \otimes \cdots \otimes 1 \otimes a \otimes 1 \cdots \otimes 1$ (a on the i-th spot). Then $(\mathcal{M}, \tau)$ is a noncommutative probability space and the following properties are satisfied:*

- *The maps $W_i : \mathcal{M}_i \to \mathcal{M}$ are normal, injective, unital $*$-homomorphisms;*
- *The von Neumann algebras $W_i(\mathcal{M}_i)$ are tensor independent with respect to τ;*
- *$\cup_{i=1}^n W_i(\mathcal{M}_i)$ generates $\mathcal{M}$ as a von Neumann algebra;*
- *$\tau \circ W_i = \tau_i$ for $i = 1, \ldots, n$.*

The following lemma is a direct consequence of the associativity of tensor independence.

Lemma 8.5. *Let $(\mathcal{M}_1, \tau_1), \ldots, (\mathcal{M}_n, \tau_n)$ be noncommutative probability spaces and $(\mathcal{M}, \tau)$ be their tensor product probability space. For each i, let $\mathcal{A}_i$, $\mathcal{B}_i$ be von Neumann subalgebras of $\mathcal{M}_i$ such that $\mathcal{A}_i$ is tensor independent of $\mathcal{B}_i$ with respect to τ_i. Then $\mathcal{A}_1 \overline{\otimes} \cdots \overline{\otimes} \mathcal{A}_n$ is tensor independent of $\mathcal{B}_1 \overline{\otimes} \cdots \overline{\otimes} \mathcal{B}_n$ with respect to τ.*

The construction of a free product of C^*-probability spaces, i.e. C^*-algebras equipped with a state, is well-known (see e.g. Section 7 of [109]) and can be easily adapted to our present setting. We leave the details to the reader.

Theorem 8.6. *(Free product probability space) Let $(\mathcal{M}_1, \tau_1), \ldots, (\mathcal{M}_n, \tau_n)$ be noncommutative probability spaces $(i = 1, \ldots, n)$. Then there exists a noncommutative probability space $(\mathcal{M}, \tau)$ and a family of normal, injective, unital $*$-homomorphisms $W_i : \mathcal{M}_i \to \mathcal{M}$ such that*

- *The von Neumann algebras $W_i(\mathcal{M}_i)$ are freely independent with respect to τ;*
- *$\cup_{i=1}^n W_i(\mathcal{M}_i)$ generates $\mathcal{M}$ as a von Neumann algebra;*
- *$\tau \circ W_i = \tau_i$ for $i = 1, \ldots, n$.*

In what follows, we shall identify $\mathcal{M}_i$ with its image $W_i(\mathcal{M}_i)$ in $\mathcal{M}$. Note that this identification is trace preserving, since $\tau_i = \tau \circ W_i$.

The following lemma is the equivalent of Lemma 8.5 for free product probability spaces and a consequence of the associativity of free independence.

Lemma 8.7. *Let $(\mathcal{M}_1, \tau_1), \ldots, (\mathcal{M}_n, \tau_n)$ be noncommutative probability spaces and $(\mathcal{M}, \tau)$ be their free product probability space. For each i, let $\mathcal{A}_i, \mathcal{B}_i$ be von Neumann subalgebras of $\mathcal{M}_i$ such that $\mathcal{A}_i$ is freely independent of $\mathcal{B}_i$ with respect to τ_i. Then $\mathcal{A}_1 * \cdots * \mathcal{A}_n$ is freely independent of $\mathcal{B}_1 * \cdots * \mathcal{B}_n$ with respect to τ.*

For clarity, we shall write $a * b$ for the product of two elements a and b in a free product probability space.

8.1.2 Probability distributions

We now turn to the problem of defining probability distributions for random variables associated with a noncommutative probability space. We begin by recalling the following definition.

Definition 8.8. *Let $(\Omega, \mathcal{F})$ be a measurable space, $\mathscr{H}$ a complex Hilbert space and let $\mathcal{P}(\mathscr{H})$ denote the set of (orthogonal) projections in $\mathscr{H}$. Then a* spectral measure *e on $(\Omega, \mathcal{F})$ is a set map $e : \mathcal{F} \to \mathcal{P}(\mathscr{H})$ satisfying*

- $e(\Omega) = 1$;
- $e(A \cap B) = e(A)e(B) = e(B)e(A)$ *for any* $A, B \in \mathcal{F}$;
- *for any sequence $(A_n)_{n=1}^{\infty}$ of disjoint elements of $\mathcal{F}$ we have*

$$e\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} e(A_n),$$

  where the sum on the right hand side converges in the strong operator topology.

It is well-known that we can define a *spectral integral* with respect to a spectral measure e for any $\mathcal{F}$-measurable function $f : \Omega \to \mathbb{C}$, which we denote by

$$\int_{\Omega} f(\lambda)\ de(\lambda).$$

This defines a normal operator on $\mathscr{H}$, which is self-adjoint if f is real-valued and bounded if f is.
Conversely, suppose that a is a normal operator. By the spectral theorem, there exists a unique spectral measure e^a on the Borel subsets $\mathcal{B}(\mathbb{C})$ of $\mathbb{C}$ such that

$$a = \int_{\mathbb{C}} \lambda\ de^a(\lambda).$$

Moreover, for every Borel function $f : \mathbb{C} \to \mathbb{C}$,

$$f(a) := \int_{\mathbb{C}} f(\lambda)\ de^a(\lambda)$$

defines a normal operator.
If e is any spectral measure on a measurable space $(\Omega, \mathcal{F})$, $f : \Omega \to \mathbb{C}$ an $\mathcal{F}$-measurable function and $x = \int f\ de$, then the spectral measure of x is given by

$$e^x(B) = e(f^{-1}(B)) \qquad (B \in \mathcal{B}(\mathbb{C})).$$

If $\mathcal{M}$ is a von Neumann algebra, then a normal operator a is affiliated with $\mathcal{M}$ if and only if $e^a(B) \in \mathcal{M}$ for every $B \in \mathcal{B}(\mathbb{C})$. If this is the case,

$$W^*(a) = W^*(\{e^a(B) : B \in \mathcal{B}(\mathbb{C})\})$$

and moreover $f(a)$ is affiliated with $W^*(a)$, for any Borel function $f : \mathbb{C} \to \mathbb{C}$.
We can use the spectral measure e^a to define a probability distribution for a. Indeed, by the properties of e^a and τ it is not difficult to see that the map

$$(\tau e^a)(B) = \tau(e^a(B)) \qquad (B \in \mathcal{B}(\mathbb{C}))$$

defines a Borel probability measure on $\mathbb{C}$. Indeed, countable additivity follows by countable additivity of e^a and complete additivity of τ. We will call this the *probability distribution* of the normal operator a. The following property is well known and not difficult to prove using normality of τ and the monotone convergence theorem.

Lemma 8.9. *Let $(\mathcal{M}, \tau)$ be a noncommutative probability space and suppose a is a normal operator in $S(\tau)$. Then, for any Borel function $f : \mathbb{C} \to \mathbb{C}$, $f(a) \in L^1(\mathcal{M})$ if and only if $f \in L^1(\mathbb{C}, \tau e^a)$ and in this case*

$$\tau(f(a)) = \int_{\mathbb{C}} f(\lambda)\ d(\tau e^a)(\lambda).$$

We shall call two normal operators in $S(\tau)$ *identically distributed* if their probability distributions coincide.
For elements of $L^1(\mathcal{M})$ which are not normal, we cannot define a probability distribution as above. For an element $a \in \mathcal{M}$ we can still look at its $*$-*moments*, by which we mean the complex numbers $\tau(M(a, a^*))$, where $M(z, \overline{z})$ is any monomial in z and $\overline{z}$. For normal elements in $\mathcal{M}$, the probability distribution is completely determined by its $*$-moments.

Proposition 8.10. *Let $(\mathcal{M}, \tau)$ be a noncommutative probability space and let $a, b \in \mathcal{M}$ be normal elements. If a_1 and a_2 have identical $*$-moments and their spectra $\sigma(a_1)$ and $\sigma(a_2)$ coincide, then they are identically distributed.*

Proof. Set $\sigma = \sigma(a_1) = \sigma(a_2)$. By the Stone-Weierstrass theorem, we can approximate any $f \in C(\sigma)$ uniformly on σ by a sequence $(p_n)_{n=1}^{\infty}$ of polynomials in $z, \overline{z}$. In particular, the set $\{(p_n), f\}$ is uniformly bounded on σ. By the Borel functional calculus for normal operators, for $i = 1, 2$ we have $p_n(a_i, a_i^*) \to f(a_i)$ in the strong operator topology and since $\{(p_n(a_i, a_i^*), f(a_i)\}$ is norm bounded, this convergence actually holds in the

ultra-strong operator topology. Since a_1 and a_2 have identical $*$-moments, we have by linearity $\tau(p_n(a_1, a_1^*)) = \tau(p_n(a_2, a_2^*))$ for all n. By ultra-weak continuity of τ we obtain $\tau(f(a_1)) = \tau(f(a_2))$ for every $f \in C(\sigma)$.
Let any open subset $B \subset \sigma(a)$ be given and let $(f_n)_{n=1}^{\infty}$ be a sequence in $C(\sigma)$ approximating χ_B pointwise from below. Then $f_n(a_i) \uparrow e^{a_i}(B)$ $(i = 1, 2)$ and by normality of τ we obtain that a_1 and a_2 are identically distributed. $\square$

8.1.3 Decoupling

As a first step towards defining stochastic integrals, we now prove two decoupling results for stochastic integrals of simple adapted processes. The key results are Theorems 8.14 (for integrators with tensor independent increments) and 8.15 (for integrators with freely independent increments).

We need two preliminary observations. The first result generalizes the classical statement that conditioning on an independent σ-algebra is redundant. To facilitate computations with freely independent von Neumann algebras we use the notation $a^\circ := a - \tau(a)$ for $a \in L^1(\mathcal{M})$. Moreover, throughout this section we write $\tau(\cdot|\mathcal{A})$ to denote the conditional expectation with respect to a von Neumann subalgebra $\mathcal{A}$ of $\mathcal{M}$.

Lemma 8.11. *Let $(\mathcal{M}, \tau)$ be a noncommutative probability space. Fix $u \in L^1(\mathcal{M})$ and let $\mathcal{A}_1, \mathcal{A}_2$ be von Neumann subalgebras of $\mathcal{M}$ such that $W^*(u, \mathcal{A}_1)$ and $\mathcal{A}_2$ are either tensor independent or freely independent. Then,*

$$\tau(\tau(u|\mathcal{A}_1)v) = \tau(uv) \qquad (v \in L^\infty(W^*(\mathcal{A}_1, \mathcal{A}_2)). \tag{8.1}$$

In other words, $\tau(u|\mathcal{A}_1, \mathcal{A}_2) = \tau(u|\mathcal{A}_1)$. In particular, if $1 \le p \le \infty$ and $u \in L^p(\mathcal{M})$, then (8.1) holds for any $v \in L^{p'}(W^*(\mathcal{A}_1, \mathcal{A}_2))$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. Notice that to prove (8.1) it suffices to show for $u \in \mathcal{M}$ that

$$\begin{aligned} \tau(\tau(u|\mathcal{A}_1)v_{1,1}v_{2,1}\cdots v_{1,k}v_{2,k}) &= \tau(uv_{1,1}v_{2,1}\cdots v_{1,k}v_{2,k}); \\ \tau(\tau(u|\mathcal{A}_1)v_{2,1}v_{1,1}\cdots v_{2,k}v_{1,k}) &= \tau(uv_{2,1}v_{1,1}\cdots v_{2,k}v_{1,k}), \end{aligned} \tag{8.2}$$

for any $k \in \mathbb{N}$ and where $v_{i,j} \in \mathcal{A}_i$ for $i = 1, 2$, $j = 1, \dots, k$ (note that the $v_{i,j}$ are allowed to be equal to $\mathbf{1}$ as $\mathbf{1} \in \mathcal{A}_i$ for $i = 1, 2$). Indeed, suppose this is true. Then by linearity, we have

$$\tau(\tau(u|\mathcal{A}_1)g) = \tau(ug),$$

for any polynomial g in elements of $\mathcal{A}_1$ and $\mathcal{A}_2$. Since such polynomials (i.e. the algebra generated by $\mathcal{A}_1 \cup \mathcal{A}_2$) are ultra-weakly dense in $W^*(\mathcal{A}_1, \mathcal{A}_2)$, we obtain

$$\tau(\tau(u|\mathcal{A}_1)v) = \tau(uv),$$

for any $v \in W^*(\mathcal{A}_1, \mathcal{A}_2)$. Now (8.1) follows for $u \in L^1(\mathcal{M})$ by a density argument and L^1-contractivity of $\tau(\cdot|\mathcal{A}_1)$.
Suppose first that $W^*(u, \mathcal{A}_1)$ and $\mathcal{A}_2$ are tensor independent. Then,

$$\begin{aligned}\tau(uv_{1,1}v_{2,1}\cdots v_{1,k}v_{2,k}) &= \tau(uv_{1,1}\cdots v_{1,k})\tau(v_{2,1}\cdots v_{2,k})\\ &= \tau(\tau(u|\mathcal{A}_1)v_{1,1}\cdots v_{1,k})\tau(v_{2,1}\cdots v_{2,k})\\ &= \tau(\tau(u|\mathcal{A}_1)v_{1,1}v_{2,1}\cdots v_{1,k}v_{2,k}).\end{aligned}$$

This proves the first equation of (8.2) in the case of tensor independence, the proof of the second is analogous.
Suppose now that $W^*(u,\mathcal{A}_1)$ and $\mathcal{A}_2$ are freely independent. We will prove (8.2) by induction on k. Notice first that for $v \in \mathcal{A}_2$ we have $\tau(uv) = \tau(u)\tau(v) = \tau(\tau(u|\mathcal{A}_1))\tau(v) = \tau(\tau(u|\mathcal{A}_1)v)$. For $v \in \mathcal{A}_1$ we have $\tau(uv) = \tau(\tau(u|\mathcal{A}_1)v)$ by (3.8). Suppose that (8.2) holds for $k = n$. For the first equation of (8.2) we have

$$\begin{aligned}\tau(uv_{1,1}v_{2,1}\cdots v_{1,n}v_{2,n}v_{1,n+1}) &= \tau(uv_{1,1}v_{2,1}\cdots v_{1,n}v_{2,n}(v_{1,n+1})^\circ)\\ &\quad +\tau(uv_{1,1}v_{2,1}\cdots v_{1,n}v_{2,n})\tau(v_{1,n+1})\\ &= \tau(uv_{1,1}v_{2,1}\cdots v_{1,n}v_{2,n}(v_{1,n+1})^\circ)\\ &\quad +\tau(\tau(u|\mathcal{A}_1)v_{1,1}v_{2,1}\cdots v_{1,n}v_{2,n})\tau(v_{1,n+1}),\end{aligned}$$

where in the last step we use the induction hypothesis. We now proceed by writing $v_{2,n} = (v_{2,n})^\circ + \tau(v_{2,n})$ in the first term on the far right hand side to obtain

$$\begin{aligned}\tau(uv_{1,1}v_{2,1}\cdots v_{1,n}v_{2,n}(v_{1,n+1})^\circ) &= \tau(uv_{1,1}v_{2,1}\cdots v_{1,n}(v_{2,n})^\circ(v_{1,n+1})^\circ)\\ &\quad +\tau(uv_{1,1}v_{2,1}\cdots v_{1,n}(v_{1,n+1})^\circ)\tau(v_{2,n}).\end{aligned}$$

We can now write $v_{1,n} = (v_{1,n})^\circ + \tau(v_{1,n})$ in the first term on the right hand side and expand by linearity. Continuing in this fashion we arrive at

$$\begin{aligned}\tau(uv_{1,1}v_{2,1}\cdots v_{1,n}v_{2,n}v_{1,n+1}) &= \tau((uv_{1,1})^\circ(v_{2,1})^\circ\cdots(v_{1,n})^\circ(v_{2,n})^\circ(v_{1,n+1})^\circ)\\ &\quad +\text{lower order terms},\end{aligned}$$

where the lower order terms are products of elements of the form $\tau(v_{i,j})$ with $j \leq n+1$ and $\tau(uv_{1,1}w_{2,1}\cdots w_{1,l}w_{2,l})$ with $l \leq n$ and $w_{k,l} \in \mathcal{A}_k$ $(k = 1,2)$. To these terms we can apply the induction hypothesis, i.e. we can replace u by $\tau(u|\mathcal{A}_1)$. For the first term in the above equation we note that by free independence,

$$\begin{aligned}\tau((uv_{1,1})^\circ(v_{2,1})^\circ\cdots(v_{1,n})^\circ(v_{2,n})^\circ(v_{1,n+1})^\circ) &= 0\\ \tau((\tau(u|\mathcal{A}_1)v_{1,1})^\circ(v_{2,1})^\circ\cdots(v_{1,n})^\circ(v_{2,n})^\circ(v_{1,n+1})^\circ) &= 0.\end{aligned}$$

Now apply the above argument backwards with $\tau(u|\mathcal{A}_1)$ instead of u to obtain

$$\tau(\tau(u|\mathcal{A}_1)v_{1,1}v_{2,1}\cdots v_{1,n}v_{2,n}v_{1,n+1}) = \tau(uv_{1,1}v_{2,1}\cdots v_{1,n}v_{2,n}v_{1,n+1}).$$

For the second equation of (8.2) we use the same argument, with the only minor difference that we use the expansion

$$\tau(uv_{2,1}v_{1,1}\cdots v_{2,n}v_{1,n}v_{2,n+1}) = \tau(u^\circ(v_{2,1})^\circ(v_{1,1})^\circ\cdots(v_{2,n})^\circ(v_{1,n})^\circ(v_{2,n+1})^\circ) \\ +\text{lower order terms.}$$

By induction, we have proved (8.2). □

The following technical lemma is used to handle the case where the integrator has tensor independent, normal, but unbounded increments.

Lemma 8.12. *Let $(\mathcal{M}, \tau)$ be a noncommutative probability space. Suppose a, b are commuting normal elements in $L^1(\mathcal{M})$, which are identically distributed and tensor independent. Then,*

$$\tau(a - b|a + b) = 0.$$

Proof. Let e^a, e^b denote the spectral measures of a and b, respectively, then there is a unique product spectral measure $e = e^a \times e^b$ on $(\mathbb{C}^2, \mathcal{B}(\mathbb{C}^2))$ (c.f. [21]). Let e^{a+b} be the spectral measure of $a + b$. Clearly, by ultra-weak continuity of the map $c \mapsto \tau((a - b)c)$ on $\mathcal{M}$, it suffices to show that

$$\tau((a - b)e^{a+b}(B)) = 0, \tag{8.3}$$

for every $B \in \mathcal{B}(\mathbb{C})$.
Let $B \in \mathcal{B}(\mathbb{C})$ be arbitrary and set $A = \{(z_1, z_2) \in \mathbb{C}^2 : z_1 + z_2 \in B\}$. By the joint functional calculus of a and b,

$$(a - b)e^{a+b}(B) = \int_{\mathbb{C}^2} (z_1 - z_2)\chi_A(z_1, z_2)\ d(e)(z_1, z_2),$$

and by Lemma 8.9,

$$\tau((a - b)e^{a+b}(B)) = \int_{\mathbb{C}^2} (z_1 - z_2)\chi_A(z_1, z_2)\ d(\tau e)(z_1, z_2).$$

For any $C_1, C_2 \in \mathcal{B}(\mathbb{C})$,

$$\begin{aligned}\tau e(C_1 \times C_2) &= \tau(e^a \times e^b(C_1 \times C_2)) \\ &= \tau(e^a(C_1))\tau(e^b(C_2)) = \tau e^a \times \tau e^b(C_1 \times C_2),\end{aligned}$$

so τe is equal to the product probability measure $\tau e^a \times \tau e^b$. Since $\tau e^a = \tau e^b$ and $(z_1, z_2) \in A$ if and only if $(z_2, z_1) \in A$, we obtain

$$\begin{aligned}\int_{\mathbb{C}^2} (z_1 - z_2)\chi_A(z_1, z_2)\ d(\tau e) &= \int_{\mathbb{C}^2} (z_2 - z_1)\chi_A(z_2, z_1)\ d(\tau e^a \times \tau e^b) \\ &= -\int_{\mathbb{C}^2} (z_1 - z_2)\chi_A(z_1, z_2)\ d(\tau e).\end{aligned}$$

Hence, $\tau((a - b)e^{a+b}(B)) = 0$ and our proof is complete. □

Definition 8.13. *Let $(\mathcal{M},\tau)$ be a noncommutative probability space and let $(\mathcal{M}_n)_{n=0}^{\infty}$ be a discrete-time filtration in $\mathcal{M}$. A sequence $(x_n)_{n=1}^{\infty}$ in $\mathcal{M}$ is called* predictable *with respect to $(\mathcal{M}_n)_{n=0}^{\infty}$ if $x_n \in \mathcal{M}_{n-1}$ for all $n \geq 1$.*

The following two decoupling theorems are noncommutative versions of Theorem 2.3. The argument used to prove these results has its roots in [104], Theorem 6.1 (see also [106], Lemma 3.4).

Theorem 8.14. *Let E be a symmetric Banach function space on $(0,\infty)$ with $1 < p_E \leq q_E < \infty$. Suppose that $(\mathcal{M},\tau)$ and $(\widetilde{\mathcal{M}},\tilde{\tau})$ are noncommutative probability spaces. Let $(\mathcal{M}_n)_{n=0}^{N}$,$(\widetilde{\mathcal{M}}_n)_{n=0}^{N}$ be filtrations in $\mathcal{M}$ and $\widetilde{\mathcal{M}}$, and let $(v_n)_{n=1}^{N}, (w_n)_{n=1}^{N}$ be $(\mathcal{M}_n)_{n=0}^{N}$-predictable sequences in $\mathcal{M}$. Suppose that $\xi_1,\ldots,\xi_N \in E(\mathcal{M})$ and $\tilde{\xi}_1,\ldots,\tilde{\xi}_N \in E(\widetilde{\mathcal{M}})$ satisfy the following conditions:*

- *$\xi_n, \tilde{\xi}_n$ are centred, i.e. $\tau(\xi_n) = \tilde{\tau}(\tilde{\xi}_n) = 0$ $(n = 1,\ldots,N)$;*
- *$\xi_n \in E(\mathcal{M}_n)$, $\tilde{\xi}_n \in E(\widetilde{\mathcal{M}}_n)$ for $n = 1,\ldots,N$;*
- *ξ_n is tensor independent of $\mathcal{M}_{n-1}$ and $\tilde{\xi}_n$ is tensor independent of $\widetilde{\mathcal{M}}_{n-1}$ for $n = 1,\ldots,N$;*
- *If ξ_n and $\tilde{\xi}_n$ are bounded operators for every $n = 1,\ldots,N$, then we assume that ξ_n and $\tilde{\xi}_n$ have identical $*$-moments. Otherwise, we assume that for every $n = 1,\ldots,N$, ξ_n and $\tilde{\xi}_n$ are normal and identically distributed.*

Under these assumptions,

$$\Big\| \sum_{n=1}^{N} v_n \xi_n w_n \Big\|_{E(\mathcal{M})} \simeq_E \Big\| \sum_{n=1}^{N} v_n w_n \otimes \tilde{\xi}_n \Big\|_{E(\overline{\mathcal{M}})}, \tag{8.4}$$

where $(\overline{\mathcal{M}},\overline{\tau})$ denotes the tensor product probability space of $(\mathcal{M},\tau)$ and $(\widetilde{\mathcal{M}},\tilde{\tau})$.

Proof. For $n = 1,\ldots,N$ define

$$\begin{aligned} d_{2n-1} &:= \frac{1}{2} v_n \xi_n w_n \otimes 1 + \frac{1}{2} v_n w_n \otimes \tilde{\xi}_n, \\ d_{2n} &:= \frac{1}{2} v_n \xi_n w_n \otimes 1 - \frac{1}{2} v_n w_n \otimes \tilde{\xi}_n \end{aligned}$$

and

$$\begin{aligned} \mathcal{D}_{2n-1} &:= W^*(\mathcal{M}_{n-1} \overline{\otimes} \widetilde{\mathcal{M}}_{n-1}, \{\xi_n \otimes 1 + 1 \otimes \tilde{\xi}_n\}), \\ \mathcal{D}_{2n} &:= \mathcal{M}_n \overline{\otimes} \widetilde{\mathcal{M}}_n. \end{aligned}$$

Then $(\mathcal{D}_n)_{n=1}^{2N}$ is a filtration in $\overline{\mathcal{M}}$ and

$$\sum_{n=1}^{N} v_n \xi_n w_n \otimes 1 = \sum_{n=1}^{2N} d_n, \ \sum_{n=1}^{N} v_n w_n \otimes \tilde{\xi}_n = \sum_{n=1}^{2N} (-1)^{n+1} d_n.$$

Since noncommutative martingale difference sequences are unconditional in $E(\overline{\mathcal{M}})$ by Proposition 6.26, it suffices to show that $(d_n)_{n=1}^{2N}$ is a martingale difference sequence with respect to the filtration $(\mathcal{D}_n)_{n=1}^{2N}$.

Clearly, $(d_n)_{n=1}^{2N}$ is $(\mathcal{D}_n)_{n=1}^{2N}$-adapted. Due to Lemma 8.5 the von Neumann algebras $W^*(\xi_n)\overline{\otimes}W^*(\tilde{\xi_n})$ and $\mathcal{M}_{n-1}\overline{\otimes}\widetilde{\mathcal{M}}_{n-1}$ are tensor independent and hence, $W^*(\{\xi_n\otimes 1 - 1\otimes\tilde{\xi}_n, \xi_n\otimes 1 + 1\otimes\tilde{\xi}_n\})$ and $\mathcal{M}_{n-1}\overline{\otimes}\widetilde{\mathcal{M}}_{n-1}$ are tensor independent. By Lemma 8.11 we have

$$\begin{aligned}
&\overline{\tau}(d_{2n}|\mathcal{D}_{2n-1})\\
&\quad= \frac{1}{2}\tau\otimes\tilde{\tau}\Big(v_n\xi_n w_n\otimes 1 - v_n w_n\otimes\tilde{\xi}_n\Big|\mathcal{D}_{2n-1}\Big)\\
&\quad= \frac{1}{2}\tau\otimes\tilde{\tau}\Big(v_n\xi_n w_n\otimes 1 - v_n w_n\otimes\tilde{\xi}_n\Big|W^*(\mathcal{M}_{n-1}\overline{\otimes}\widetilde{\mathcal{M}}_{n-1},\{\xi_n\otimes 1+1\otimes\tilde{\xi}_n\})\Big)\\
&\quad= \frac{1}{2}\Big((v_n\otimes 1)\tau\otimes\tilde{\tau}\Big(\xi_n\otimes 1 - 1\otimes\tilde{\xi}_n\Big|\xi_n\otimes 1+1\otimes\tilde{\xi}_n\Big)(w_n\otimes 1)\Big).
\end{aligned}$$

We claim that the latter expression is zero. Suppose first that $\xi_n, \tilde{\xi}_n$ are bounded and have identical $*$-moments. Note that the map $c\mapsto\tau\otimes\tilde{\tau}((\xi_n\otimes 1 - 1\otimes\tilde{\xi}_n)c)$ is linear and ultra weakly-continuous and it is therefore sufficient to show that $\tau\otimes\tilde{\tau}((\xi_n\otimes 1 - 1\otimes\tilde{\xi}_n)P) = 0$ for any $*$-monomial P in $\xi_n\otimes 1 + 1\otimes\tilde{\xi}_n$. This follows by direct calculation, using the fact that ξ_n and $\tilde{\xi}_n$ have identical $*$-moments.

On the other hand, if $\xi_n, \tilde{\xi}_n$ are normal and identically distributed, we apply Lemma 8.12 with $a = \xi_n\otimes 1$ and $b = 1\otimes\tilde{\xi}_n$ to prove the claim.

Similarly we have,

$$\begin{aligned}
\overline{\tau}(d_{2n-1}|\mathcal{D}_{2n-2}) &= \frac{1}{2}\tau\otimes\tilde{\tau}\Big(v_n\xi_n w_n\otimes 1 + v_n w_n\otimes\tilde{\xi}_n\Big|\mathcal{M}_{n-1}\overline{\otimes}\widetilde{\mathcal{M}}_{n-1}\Big)\\
&= \frac{1}{2}\tau\otimes\tilde{\tau}\Big(\xi_n\otimes 1 + 1\otimes\tilde{\xi}_n\Big)v_n w_n\otimes 1 = 0,
\end{aligned}$$

where we use that ξ_n and $\tilde{\xi}_n$ are centred.
Thus $(d_n)_{n=1}^{2N}$ is the martingale difference sequence of the noncommutative martingale $(\sum_{k=1}^n d_k)_{n=1}^{2N}$ and our proof is complete. □

The free version of Theorem 8.14 reads as follows.

Theorem 8.15. *Let E be a symmetric Banach function space on $(0,\infty)$. Suppose that $(\mathcal{M},\tau)$ and $(\widetilde{\mathcal{M}},\tilde{\tau})$ are noncommutative probability spaces. Let $(\mathcal{M}_n)_{n=0}^N$, $(\widetilde{\mathcal{M}}_n)_{n=0}^N$ be filtrations in $\mathcal{M}$ and $\widetilde{\mathcal{M}}$, and let $(v_n)_{n=1}^N, (w_n)_{n=1}^N$ be $(\mathcal{M}_n)_{n=0}^N$-predictable sequences in $\mathcal{M}$. Suppose that the elements $\xi_1,\dots,\xi_N\in\mathcal{M}$ and $\tilde{\xi}_1,\dots,\tilde{\xi}_N\in\widetilde{\mathcal{M}}$ satisfy the following four conditions:*

- *$\xi_n, \tilde{\xi}_n$ are centred $(n = 1,\dots,N)$;*
- *$\xi_n\in\mathcal{M}_n$, $\tilde{\xi}_n\in\widetilde{\mathcal{M}}_n$ for $n = 1,\dots,N$;*
- *ξ_n is freely independent of $\mathcal{M}_{n-1}$ and $\tilde{\xi}_n$ is freely independent of $\widetilde{\mathcal{M}}_{n-1}$ for $n = 1,\dots,N$;*

- *For every $n = 1, \ldots, N$, ξ_n and $\tilde{\xi}_n$ have identical $*$-moments.*

Under these assumptions,

$$\Big\| \sum_{n=1}^{N} v_n \xi_n w_n \Big\|_{E(\mathcal{M})} \simeq_E \Big\| \sum_{n=1}^{N} v_n * \tilde{\xi}_n * w_n \Big\|_{E(\overline{\mathcal{M}})}, \tag{8.5}$$

where $(\overline{\mathcal{M}}, \overline{\tau})$ denotes the free product probability space of $(\mathcal{M}, \tau)$ and $(\widetilde{\mathcal{M}}, \tilde{\tau})$.

Proof. The proof is the same as the one for Theorem 8.14, once we replace all tensor product probability spaces by free product probability spaces and use Lemma 8.7 instead of Lemma 8.5. □

8.2 Itô isomorphisms

In this section we prove Itô-type isomorphisms for noncommutative stochastic integrals with respect to Boson and free Brownian motions. We extend the results obtained for L^p-spaces in [43].

Let $\mathcal{M}$ be a noncommutative probability space and let E be a separable symmetric Banach function space on $(0, \infty)$. Recall the following terminology. A *(continuous-time) filtration* is an increasing family of von Neumann subalgebras $(\mathcal{M}_t)_{t \geq 0}$ of $\mathcal{M}$, i.e., $\mathcal{M}_s \subset \mathcal{M}_t$ whenever $0 \leq s \leq t$, which generates $\mathcal{M}$, $\mathcal{M} = (\cup_{t \geq 0} \mathcal{M}_t)''$. An *$E(\mathcal{M})$-valued process adapted to the filtration* $(\mathcal{M}_t)_{t \geq 0}$ is a map $f : \mathbb{R}^+ \to E(\mathcal{M})$ such that $f(s) \in E(\mathcal{M}_s)$ for every $s \geq 0$. We call an $(\mathcal{M}_t)_{t \geq 0}$-adapted $E(\mathcal{M})$-valued process f *simple* if it is piecewise constant, i.e. if there exists a finite partition $\pi = \{0 = t_0 < t_1 < \ldots < t_{n+1} < \infty\}$ of $\mathbb{R}^+$ such that

$$f(t) = \sum_{k=0}^{n} f(t_k) \chi_{(t_k, t_{k+1}]}(t) \ (t \geq 0),$$

Notice that adaptedness of f means in this case that $f(t_k) \in E(\mathcal{M}_{t_k})$ for all $k \geq 0$.

Definition 8.16. *Let $\mathcal{M}$ be a noncommutative probability space and $(\mathcal{M}_t)_{t \geq 0}$ be a filtration in $\mathcal{M}$. A process $(\Phi_t)_{t \geq 0}$ in $L^1(\mathcal{M})$ (respectively, $\mathcal{M}$) is called a* Boson (free) Brownian motion *if*

- *$\Phi_t - \Phi_s$ is tensor (freely) independent of $\mathcal{M}_s$ for all $0 \leq s < t$,*
- *For any $0 \leq s < t$, $\Phi_t - \Phi_s$ has a normal (semicircular) distribution with mean 0 and variance $t - s$.*

Examples of Boson and free Brownian motions can be explicitly constructed using creation and annihilation operators on symmetric and full Fock spaces, see [43].

Let Φ be a Boson or free Brownian motion and let $(\mathcal{M}_t)_{t \geq 0}$ be the filtration generated by Φ. Following the classical approach of K. Itô, we define the left and right stochastic integral of a simple adapted $E(\mathcal{M})$-valued process f with respect to Φ by

$$\int_0^t f \; d\Phi := \sum_{k=0}^{n} f(t_k)(\Phi(t \wedge t_{k+1}) - \Phi(t \wedge t_k))$$
$$\int_0^t (d\Phi \; f) := \sum_{k=0}^{n} (\Phi(t \wedge t_{k+1}) - \Phi(t \wedge t_k)) f(t_k),$$

respectively, where $t \wedge t_k$ denotes the infimum of t and t_k. We let $\mathcal{S}_{ad}^E(0,T)$ be the linear space of simple adapted $E(\mathcal{M})$-valued processes supported on $[0,T]$. Let $\mathcal{H}_c^E(0,T)$ and $\mathcal{H}_r^E(0,T)$ be the closure of $\mathcal{S}_{ad}^E(0,T)$ in $E(\mathcal{M}; L^2(0,T)_c)$ and $E(\mathcal{M}; L^2(0,T)_r)$, respectively. We define

$$\mathcal{H}^E(0,T) = \begin{cases} \mathcal{H}_c^E(0,T) + \mathcal{H}_r^E(0,T), & \text{if } 1 < p_E \le q_E < 2, \\ \mathcal{H}_c^E(0,T) \cap \mathcal{H}_r^E(0,T), & \text{if } 2 < p_E \le q_E < \infty. \end{cases}$$

In the proof of Theorem 8.17 we shall use that $S_{ad}^{L^\infty}(0,T)$ is dense in $H^E(0,T)$ and that

$$\|f\|_{\mathcal{H}^E(0,T)} \simeq_E \inf\{\|g\|_{\mathcal{H}_c^E(0,T)} + \|h\|_{\mathcal{H}_r^E(0,T)}\}, \tag{8.6}$$

where the infimum is taken over all $g, h \in \mathcal{S}_{ad}^E(0,T)$ such that $f = g + h$. These facts were proved in [114], p. 687-688, for $E = L^p$ and can be obtained in the general case needed here by a straightforward modification of their argument.

Let $\mathcal{H}_{loc}^E(\mathbb{R}^+)$ denote the linear space of all processes $f : \mathbb{R}^+ \to E(\mathcal{M})$ such that, for every $T > 0$, the restriction of f to $[0,T]$ belongs to $\mathcal{H}^E(0,T)$. Analogously we define $\mathcal{H}_{c,loc}^E(\mathbb{R}^+)$ and $\mathcal{H}_{r,loc}^E(\mathbb{R}^+)$.

The following two theorems establish Itô-isomorphisms for $E(\mathcal{M})$-valued processes.

Theorem 8.17. *Let $\mathcal{M}$ be a noncommutative probability space and E be a separable symmetric Banach function space on $(0,\infty)$. Suppose that either $1 < p_E \le q_E < 2$ or $2 < p_E \le q_E < \infty$. If $f \in \mathcal{S}_{ad}^{L^\infty}(0,T)$ and Φ is a Boson Brownian motion, then*

$$\Big\| \int_0^T f \; d\Phi \Big\|_{E(\mathcal{M})} \simeq_E \|f\|_{\mathcal{H}^E(0,T)} \simeq_E \Big\| \int_0^T (d\Phi \; f) \Big\|_{E(\mathcal{M})}. \tag{8.7}$$

Hence, by density of $\mathcal{S}_{ad}^{L^\infty}(0,T)$ in $\mathcal{H}^E(0,T)$, for any $f \in \mathcal{H}^E(0,T)$ we can define the left and right stochastic integral $\int_0^T f \; d\Phi$ and $\int_0^T (d\Phi \; f)$ and (8.7) holds. Moreover, if $f \in \mathcal{H}_{loc}^E(\mathbb{R}^+)$, the processes $(\int_0^t f \; d\Phi)_{t\ge 0}, (\int_0^t (d\Phi \; f))_{t \ge 0}$ are continuous $E(\mathcal{M})$-valued martingales.

Proof. We only prove the first equivalence in (8.7), the second is proved analogously. Suppose $f \in S_{ad}^{L^\infty}(0,T)$ is given by

$$f = \sum_{k=0}^{n} f(t_k) \chi_{(t_k, t_{k+1}]},$$

where, refining the partition of $\mathbb{R}^+$ if necessary, we may assume that $T = t_{n+1}$. Then,

$$\int_0^T f\ d\Phi = \sum_{k=0}^n f(t_k)(\Phi(t_{k+1}) - \Phi(t_k)).$$

Let $(\widetilde{\mathcal{M}}, \tilde{\tau})$ be a copy of $(\mathcal{M}, \tau)$ and $\tilde{\Phi}$ a copy of Φ in $(\widetilde{\mathcal{M}}, \tilde{\tau})$. By decoupling (cf. Theorem 8.14) we obtain

$$\begin{aligned}
&\Big\| \int_0^T f\ d\Phi \Big\|_{E(\mathcal{M})} \\
&\quad \simeq_E \Big\| \sum_{k=0}^n f(t_k) \otimes (\tilde{\Phi}(t_{k+1}) - \tilde{\Phi}(t_k)) \Big\|_{E(\mathcal{M}\overline{\otimes}\widetilde{\mathcal{M}})} \\
&\quad = \Big\| \sum_{k=0}^n \sqrt{t_{k+1} - t_k} f(t_k) \otimes \frac{1}{\sqrt{t_{k+1} - t_k}} (\tilde{\Phi}(t_{k+1}) - \tilde{\Phi}(t_k)) \Big\|_{E(\mathcal{M}\overline{\otimes}\widetilde{\mathcal{M}})}.
\end{aligned} \tag{8.8}$$

Suppose first that $2 < p_E \le q_E < \infty$. By Corollary 7.11 we have

$$\begin{aligned}
&\Big\| \int_0^T f\ d\Phi \Big\|_E \\
&\quad \simeq_E \max \Big\{ \|(\sqrt{t_{k+1} - t_k} f(t_k))\|_{E(\mathcal{M}; l_c^2)}, \|(\sqrt{t_{k+1} - t_k} f(t_k))\|_{E(\mathcal{M}; l_r^2)} \Big\}.
\end{aligned}$$

It follows from (5.10) that

$$\begin{aligned}
&\|(\sqrt{t_{k+1} - t_k} f(t_k))\|_{E(\mathcal{M}; l_c^2)} \\
&\quad = \Big\| \sum_{k=0}^n \sqrt{t_{k+1} - t_k} f(t_k) \otimes \frac{1}{\sqrt{t_{k+1} - t_k}} \chi_{(t_k, t_{k+1}]} \Big\|_{E(\mathcal{M}, L^2(0,T)_c)} \\
&\quad = \|f\|_{\mathcal{H}_c^E(0,T)},
\end{aligned} \tag{8.9}$$

and by (5.11),

$$\|(\sqrt{t_{k+1} - t_k} f(t_k))\|_{E(\mathcal{M}; l_r^2)} = \|f\|_{\mathcal{H}_r^E(0,T)}. \tag{8.10}$$

Therefore,

$$\Big\| \int_0^T f\ d\Phi \Big\|_{E(\mathcal{M})} \simeq_E \|f\|_{\mathcal{H}^E(0,T)}.$$

Suppose now that E is $1 < p_E \le q_E < 2$. By (8.8) and Corollary 7.12 we have

$$\begin{aligned}
&\Big\| \int_0^T f\ d\Phi \Big\|_{E(\mathcal{M})} \\
&\quad \simeq_E \inf \Big\{ \|(\sqrt{t_{k+1} - t_k} a_k)\|_{E(\mathcal{M}; l_c^2)} + \|(\sqrt{t_{k+1} - t_k} b_k)\|_{E(\mathcal{M}; l_r^2)} \Big\},
\end{aligned} \tag{8.11}$$

where the infimum is taken over all decompositions $f(t_k) = a_k + b_k$ in $E(\mathcal{M})$, $0 \le k \le n$. Let $f(t_k) = c_k + d_k$, $0 \le k \le n$, be any such decomposition. Then, since $f(t_k) \in E(\mathcal{M}_{t_k})$, $f(t_k) = \tau(c_k|\mathcal{M}_{t_k}) + \tau(d_k|\mathcal{M}_{t_k})$ and by the noncommutative Stein inequality we have

$$\|(\tau(c_k|\mathcal{M}_{t_k}))\|_{E(\mathcal{M};l^2_c)} \lesssim_E \|(c_k)\|_{E(\mathcal{M};l^2_c)};$$
$$\|(\tau(d_k|\mathcal{M}_{t_k}))\|_{E(\mathcal{M};l^2_r)} \lesssim_E \|(d_k)\|_{E(\mathcal{M};l^2_r)}.$$

Hence, it is equivalent in (8.11) to take the infimum over all decompositions $f(t_k) = a_k + b_k$ in $E(\mathcal{M}_{t_k})$, $0 \le k \le n$. But any finite sequence $(c_k)_{k=0}^n$ with $c_k \in E(\mathcal{M}_{t_k})$ can be identified with an element of $g \in \mathcal{S}^E_{ad}(0,T)$ defined by

$$g = \sum_{k=1}^{n-1} c_k \chi_{(t_k,t_{k+1}]}.$$

Therefore, by (8.6), (8.9) and (8.10) we obtain

$$\Big\| \int_0^T f \, d\Phi \Big\|_{E(\mathcal{M})} \simeq_E \|f\|_{\mathcal{H}^E(0,T)}.$$

Suppose now that $f \in \mathcal{H}^E_{loc}(\mathbb{R}^+)$ and let $f_n \in \mathcal{S}^{L^\infty}_{ad}(0,T)$ converge to f in $\mathcal{H}^E(0,T)$. Then,

$$\Big\| \int_0^T f_n \, d\Phi - \int_0^T f_m \, d\Phi \Big\|_{E(\mathcal{M})} \simeq_E \|f_n - f_m\|_{\mathcal{H}^E(0,T)},$$

and by completeness of $\mathcal{H}^E(0,T)$ the sequence $(\int_0^T f_n \, d\Phi)$ converges to a limit $\int_0^T f \, d\Phi$ in $E(\mathcal{M}_T)$ which satisfies

$$\Big\| \int_0^T f \, d\Phi \Big\|_{E(\mathcal{M})} \simeq_p \|f\|_{\mathcal{H}^E(0,T)}.$$

Since $(\int_0^t f_n \, d\Phi)_{t\ge 0}$ is an $(\mathcal{M}_t)$-martingale for every n and conditional expectations are bounded on $E(\mathcal{M})$ by a constant depending only on E (cf. Proposition 3.4 and Theorem 5.21), we see that $(\int_0^t f \, d\Phi)_{t\ge 0}$ is an $E(\mathcal{M})$-valued martingale. To prove that the map $t \mapsto \int_0^t f \, d\Phi$ is continuous, suppose first that $f \in \mathcal{S}^\infty_{ad}(0,T)$ and let $u \in [0,T]$. By choosing $0 \le s < u$ close enough to u we may assume that $f\chi_{(s,u]} = a\chi_{(s,u]}$ for some $a \in \mathcal{M}_s$. We have

$$\begin{aligned}\Big\| \int_0^u f \, d\Phi - \int_0^s f \, d\Phi \Big\|_{E(\mathcal{M})} &= \Big\| \int_0^T f\chi_{(s,u]} \, d\Phi \Big\|_{E(\mathcal{M})} \\ &\simeq_E \|a\chi_{(s,u]}\|_{\mathcal{H}^E(0,T)} = \sqrt{u-s}\|a\|_{E(\mathcal{M})},\end{aligned}$$

so $t \mapsto \int_0^t f \, d\Phi$ is continuous at u. The continuity of $t \mapsto \int_0^t f \, d\Phi$ for $f \in \mathcal{H}^E(0,T)$ now follows by approximation. $\square$

Remark 8.18. From the proof of Theorem 8.17 it is not clear that for a simple adapted $E(\mathcal{M})$-valued process $f = \sum_{k=0}^{n} f(t_k)\chi_{(t_k,t_{k+1}]}$ the stochastic integrals $\int_0^T f\, d\Phi$ and $\int_0^T (d\Phi\, f)$ are given by (8.2). For the left stochastic integral this can be seen as follows (the other case is analogous). Since E is separable, $\mathcal{M}$ is dense in $E(\mathcal{M})$ and so we can find a simple, adapted $\mathcal{M}$-valued processes $f_i = \sum_{k=0}^{n} f_i(t_k)\chi_{(t_k,t_{k+1}]}$ such that $\sup_{0\le k\le n} \|f(t_k) - f_i(t_k)\|_{E(\mathcal{M})} < \frac{1}{i}$. Observe that $f_i \to f$ in $\mathcal{H}^E(0,T)$ and through the isomorphism proved above we obtain $\int_0^T f_i\, d\Phi \to \int_0^T f\, d\Phi$ in $E(\mathcal{M})$ and hence also with respect to the measure topology. On the other hand, for every $0 \le k \le n$ we have $f_i(t_k) \to f(t_k)$ in $E(\mathcal{M})$ and hence also in measure. Since addition and multiplication are continuous with respect to the measure topology, we obtain

$$\sum_{k=0}^{n-1} f_i(t_k)(\Phi(t_{k+1}) - \Phi(t_k)) \to \sum_{k=0}^{n-1} f(t_k)(\Phi(t_{k+1}) - \Phi(t_k))$$

in measure. Since the measure topology is Hausdorff, (8.2) holds.

To conclude this chapter we prove the free version of Theorem 8.17, see Theorem 8.21 below. We make use of the following Khintchine-type inequalities.

Proposition 8.19. *Fix $1 < p < \infty$. Let $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ be finite von Neumann algebras equipped with a normal, faithful, finite trace τ and $\tilde{\tau}$, respectively. Let $\overline{\mathcal{M}}$ be the free product von Neumann algebra of $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ and let $\overline{\tau}$ be the corresponding free product trace. Suppose that $(\xi_k)_{k=1}^{\infty}$ is a freely independent sequence in $L^\infty(\widetilde{\mathcal{M}})$ such that $\tau(\xi_k) = 0$ for all $k \ge 1$, $c_2 = \inf\{\|\xi_k\|_2\} > 0$ and $d_\infty = \sup\{\|\xi_k\|_\infty\} < \infty$. Then, for any $v_1, \dots, v_n \in L^p(\mathcal{M})$,*

$$\Big\| \sum_{k=1}^{n} v_k * \xi_k \Big\|_{L^p(\overline{\mathcal{M}})} \simeq_{p,c_2,d_\infty} \|(v_k)\|_{L^p(\mathcal{M};l_r^2)} \tag{8.12}$$

and

$$\Big\| \sum_{k=1}^{n} \xi_k * v_k \Big\|_{L^p(\overline{\mathcal{M}})} \simeq_{p,c_2,d_\infty} \|(v_k)\|_{L^p(\mathcal{M};l_c^2)}. \tag{8.13}$$

Proof. Throughout we use the notation $c_r = \inf\{\|\xi_k\|_r\}$ and $d_s = \sup\{\|\xi_k\|_s\}$. By assumption, $c_r > 0$ for $r \le 2 \le \infty$ and $d_s < \infty$ for every $1 \le s \le \infty$. Observe that (8.13) follows from (8.12) by taking adjoints.

To prove (8.12), we may assume without loss of generality that $\|\xi_k\|_2 = 1$ for all $k \ge 1$. Indeed, once we have proved the assertion under this additional assumption, the general case follows from the observations

$$\|(\|\xi_k\|_2 v_k)\|_{L^p(\mathcal{M};l_r^2)} \simeq_{c_2,d_2} \|(v_k)\|_{L^p(\mathcal{M};l_r^2)},$$

and

$$\sup\{\|\xi_k\|_2^{-1}\|\xi_k\|_\infty\} \le d_\infty c_2^{-1} < \infty.$$

Suppose first that $2 \leq p < \infty$. Recall the notation $\overline{\tau}(\cdot|\mathcal{M}) = \mathcal{E}_{\mathcal{M}}$. Observe that $\mathcal{E}_{\mathcal{M}}(v_j * \xi_j) = 0$ for every j. Moreover, the finite sequence $(v_k * \xi_k)$ is independent with respect to $\mathcal{E}_{\mathcal{M}}$. Indeed, for $j \neq k$, let $a \in W^*(\mathcal{M}, W^*(\xi_j))$, $b \in W^*(\mathcal{M}, W^*(\xi_k))$ and $c \in \mathcal{M}$ be arbitrary. Then,

$$\begin{aligned}\overline{\tau}(abc) &= \overline{\tau}(\overline{\tau}(a|\mathcal{M}, W^*(\xi_k))bc) \\ &= \overline{\tau}(\overline{\tau}(a|\mathcal{M})bc) = \overline{\tau}(\overline{\tau}(a|\mathcal{M})\overline{\tau}(b|\mathcal{M})c),\end{aligned}$$

where we use the free independence of $W^*(\mathcal{M}, W^*(\xi_j))$ and $W^*(\xi_k)$ and Lemma 8.11 in the second equality. Since $c \in \mathcal{M}$ was arbitrary, we conclude by Proposition 3.4 that $\mathcal{E}_{\mathcal{M}}(ab) = \mathcal{E}_{\mathcal{M}}(a)\mathcal{E}_{\mathcal{M}}(b)$.
By the noncommutative Rosenthal inequalities (Theorem 7.4) we have

$$\begin{aligned}\Big\| \sum_{k=1}^n v_k * \xi_k \Big\|_{L^p(\overline{\mathcal{M}})} \simeq_p \max\Big\{ &\|(v_k * \xi_k)\|_{l^p(L^p(\overline{\mathcal{M}}))}, \\ &\|(v_k * \xi_k)\|_{L^p(\overline{\mathcal{M}}, \mathcal{E}_{\mathcal{M}}; l^2_c)}, \|(v_k * \xi_k)\|_{L^p(\overline{\mathcal{M}}, \mathcal{E}_{\mathcal{M}}; l^2_r)} \Big\}.\end{aligned}$$

We shall estimate the three norms on the right hand side using the free independence of v_k and ξ_k. First observe that

$$\begin{aligned}\|(v_k * \xi_k)\|_{L^p(\overline{\mathcal{M}}, \mathcal{E}_{\mathcal{M}}; l^2_r)} &= \Big\| \Big(\sum_k \mathcal{E}_{\mathcal{M}}(v_k * \xi_k * \xi_k^* * v_k^*) \Big)^{\frac{1}{2}} \Big\|_p \\ &= \Big\| \Big(\sum_k v_k * \mathcal{E}_{\mathcal{M}}(\xi_k * \xi_k^*) * v_k^* \Big)^{\frac{1}{2}} \Big\|_p \\ &= \Big\| \Big(\sum_k v_k \tau(\xi_k \xi_k^*) v_k^* \Big)^{\frac{1}{2}} \Big\|_p = \|(v_k)\|_{L^p(\mathcal{M}; l^2_r)},\end{aligned}$$

as $\|\xi_k\|_2 = 1$ for all k. Using (3.8) one calculates that

$$\mathcal{E}_{\mathcal{M}}(\xi_k^* * v_k^* * v_k * \xi_k) = \tau(v_k^* v_k)\tilde{\tau}(\xi_k^* \xi_k) \qquad (k \geq 1). \tag{8.14}$$

Therefore,

$$\begin{aligned}\|(v_k * \xi_k)\|_{L^p(\overline{\mathcal{M}}, \mathcal{E}_{\mathcal{M}}; l^2_c)} &= \Big\| \Big(\sum_k \mathcal{E}_{\mathcal{M}}(\xi_k^* * v_k^* * v_k * \xi_k) \Big)^{\frac{1}{2}} \Big\|_p \\ &= \Big\| \Big(\sum_k \tau(v_k^* v_k)\tilde{\tau}(\xi_k^* \xi_k) \Big)^{\frac{1}{2}} \Big\|_p \\ &= \|(v_k)\|_{l^2(L^2(\mathcal{M}))} = \|(v_k)\|_{L^2(\mathcal{M}; l^2_r)}.\end{aligned}$$

It remains to estimate $\|(v_k * \xi_k)\|_{l^p(L^p(\overline{\mathcal{M}}))}$. As $\|v_k * \xi_k\|_p \leq \|v_k\|_p \|\xi_k\|_\infty$ it follows that

$$\|(v_k * \xi_k)\|_{l^p(L^p(\overline{\mathcal{M}}))} \lesssim_{d_\infty} \|(v_k)\|_{l^p(L^p(\overline{\mathcal{M}}))}.$$

Moreover,

$$\|(v_k)\|_{l^p(L^p(\mathcal{M}))} \leq \|(v_k)\|_{L^p(\mathcal{M};l_r^2)}.$$

For $p = 2$ this is clear. For $p = \infty$ we note that for any j we have $v_j v_j^* \leq \sum_k v_k v_k^*$ and therefore $\|v_j\|_\infty \leq \|(\sum_k v_k v_k^*)^{\frac{1}{2}}\|_\infty$. The remaining cases follow by interpolation. We have obtained

$$\begin{aligned}\Big\|\sum_{k=1}^n v_k * \xi_k\Big\|_p &\simeq_{p,d_\infty} \max\Big\{\|(v_k)\|_{l^p(L^p(\mathcal{M}))}, \|(v_k)\|_{L^2(\mathcal{M};l_r^2)}, \|(v_k)\|_{L^p(\mathcal{M};l_r^2)}\Big\} \\ &= \|(v_k)\|_{L^p(\mathcal{M};l_r^2)}.\end{aligned}$$

To obtain (8.12) in the case $1 < p < 2$ we use duality. Let v_k, ξ_k be as above and let p' denote the Hölder conjugate of p. Let (w_k) be a finite sequence in $L^{p'}(\mathcal{M})$. Using free independence one calculates that

$$\overline{\tau}(v_k \xi_k \xi_j^* w_j) = \begin{cases} \tau(v_k w_k)\|\xi_k\|_2^2, & \text{if } k = j, \\ 0, & \text{if } k \neq j. \end{cases} \tag{8.15}$$

By assumption $\|\xi_k\|_2 = 1$ for all $k \geq 1$, so

$$\begin{aligned}\Big|\sum_k \tau(v_k w_k)\Big| &= \Big|\overline{\tau}\Big(\Big(\sum_k v_k * \xi_k\Big)\Big(\sum_j \xi_j^* w_j\Big)\Big)\Big| \\ &\leq \Big\|\sum_k v_k * \xi_k\Big\|_{L^p(\overline{\mathcal{M}})}\Big\|\sum_j \xi_j^* w_j\Big\|_{L^{p'}(\overline{\mathcal{M}})} \\ &\lesssim_{p,d_\infty} \Big\|\sum_k v_k * \xi_k\Big\|_{L^p(\overline{\mathcal{M}})}\|(w_j)\|_{L^{p'}(\mathcal{M};l_c^2)}.\end{aligned}$$

By taking the supremum over all finite sequences (w_k) in $L^{p'}(\mathcal{M})$ satisfying $\|(w_k)\|_{L^{p'}(\mathcal{M};l_c^2)} \leq 1$ we obtain

$$\|(v_k)\|_{L^p(\mathcal{M};l_r^2)} \lesssim_{p,d_\infty} \Big\|\sum_k v_k * \xi_k\Big\|_{L^p(\overline{\mathcal{M}})}.$$

For the reverse inequality, first suppose that $v_1, \ldots, v_n \in L^\infty(\mathcal{M})$. Then,

$$\begin{aligned}\Big\|\sum_k v_k * \xi_k\Big\|_{L^p(\overline{\mathcal{M}})}^2 &= \Big\|\Big(\sum_k v_k * \xi_k\Big)\Big(\sum_j v_j * \xi_j\Big)^*\Big\|_{L^{\frac{p}{2}}(\overline{\mathcal{M}})} \\ &\leq \Big\|\mathcal{E}_{\mathcal{M}}\Big(\Big(\sum_k v_k * \xi_k\Big)\Big(\sum_j v_j * \xi_j\Big)^*\Big)\Big\|_{L^{\frac{p}{2}}(\overline{\mathcal{M}})} \\ &= \Big\|\sum_k \mathcal{E}_{\mathcal{M}}(v_k * \xi_k * \xi_k^* * v_k^*)\Big\|_{L^{\frac{p}{2}}(\mathcal{M})} \\ &= \Big\|\sum_k v_k v_k^* \tilde{\tau}(\xi_k \xi_k^*)\Big\|_{L^{\frac{p}{2}}(\mathcal{M})} = \|(v_k)\|_{L^p(\mathcal{M};l_r^2)}^2,\end{aligned}$$

where the inequality is a consequence of [70], Theorem 7.1, as $\frac{p}{2} < 1$, and the penultimate equality follows from (8.14). The asserted inequality for $v_1, \ldots, v_n \in L^p(\mathcal{M})$ now follows by approximation. This completes the proof. $\square$

By interpolation and duality we obtain the following result for noncommutative Banach function spaces.

Corollary 8.20. *(Free Khintchine inequalities) Let E be a separable symmetric Banach function space on $(0,\infty)$ with $1 < p_E \leq q_E < \infty$. Let $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ be finite von Neumann algebras equipped with a normal, faithful, finite trace τ and $\tilde{\tau}$, respectively. Let $\overline{\mathcal{M}}$ be the free product von Neumann algebra of $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ and let $\overline{\tau}$ be the corresponding free product trace. Suppose that $(\xi_k)_{k=1}^{\infty}$ is a freely independent sequence in $L^{\infty}(\widetilde{\mathcal{M}})$ such that $\tau(\xi_k) = 0$ for all $k \geq 1$, $c_2 = \inf\{\|\xi_k\|_2\} > 0$ and $d_{\infty} = \sup\{\|\xi_k\|_{\infty}\} < \infty$. Then, for any $v_1, \ldots, v_n \in E(\mathcal{M})$,*

$$\Big\| \sum_{k=1}^{n} v_k * \xi_k \Big\|_{E(\overline{\mathcal{M}})} \simeq_{E,c_2,d_\infty} \|(v_k)\|_{E(\mathcal{M};l_r^2)} \tag{8.16}$$

and

$$\Big\| \sum_{k=1}^{n} \xi_k * v_k \Big\|_{E(\overline{\mathcal{M}})} \simeq_{E,c_2,d_\infty} \|(v_k)\|_{E(\mathcal{M};l_c^2)}. \tag{8.17}$$

Proof. It suffices to prove (8.16), as (8.17) immediately follows by applying (8.16) for the sequence (v_k^*). Notice that by interpolation we immediately obtain

$$\Big\| \sum_{k=1}^{n} v_k * \xi_k \Big\|_{E(\overline{\mathcal{M}})} \lesssim_{E,c_2,d_\infty} \|(v_k)\|_{E(\mathcal{M};l_r^2)}$$

from (8.12). For the reverse inequality we use the isometric identification $(E(\mathcal{M};l_r^2))^* = E^{\times}(\mathcal{M};l_c^2)$ observed in Lemma 5.27. By (4.7) we have $1 < p_{E^{\times}} \leq q_{E^{\times}} < \infty$. Let (w_k) be a finite sequence in $E^{\times}(\mathcal{M})$. We may assume that $\|\xi_k\|_2 = 1$ for all k. By (8.15),

$$\begin{aligned}
\Big| \sum_k \tau(v_k w_k) \Big| &= \Big| \overline{\tau}\Big(\Big(\sum_k v_k * \xi_k \Big) \Big(\sum_j \xi_j^* w_j \Big) \Big) \Big| \\
&\leq \Big\| \sum_k v_k * \xi_k \Big\|_{E(\overline{\mathcal{M}})} \Big\| \sum_j \xi_j^* w_j \Big\|_{E^{\times}(\overline{\mathcal{M}})} \\
&\lesssim_{E,c_2,d_\infty} \Big\| \sum_k v_k * \xi_k \Big\|_{E(\overline{\mathcal{M}})} \|(w_j)\|_{E^{\times}(\mathcal{M};l_c^2)}.
\end{aligned}$$

By taking the supremum over all finite sequences (w_k) in $E^{\times}(\mathcal{M})$ satisfying $\|(w_k)\|_{E^{\times}(\mathcal{M};l_c^2)} \leq 1$ we obtain

$$\|(v_k)\|_{E(\mathcal{M};l_r^2)} \lesssim_{E,c_2,d_\infty} \Big\| \sum_k v_k * \xi_k \Big\|_{E(\overline{\mathcal{M}})}.$$

□

We are ready to prove the final result of this thesis.

Theorem 8.21. *Let $\mathcal{M}$ be a noncommutative probability space and E be a separable symmetric Banach function space on $(0,\infty)$. If $f \in \mathcal{S}_{ad}^{L^\infty}(0,T)$ and Φ is a free Brownian motion, then*

$$\begin{aligned} \Big\| \int_0^T f \; d\Phi \Big\|_{E(\mathcal{M})} &\simeq_E \|f\|_{\mathcal{H}_r^E(0,T)}, \\ \Big\| \int_0^T (d\Phi \; f) \Big\|_{E(\mathcal{M})} &\simeq_E \|f\|_{\mathcal{H}_c^E(0,T)}. \end{aligned} \tag{8.18}$$

Hence, by density of $\mathcal{S}_{ad}^{L^\infty}(0,T)$ in $\mathcal{H}_r^E(0,T)$ and $\mathcal{H}_c^E(0,T)$, for any f in $\mathcal{H}_r^E(0,T)$ (respectively, $\mathcal{H}_c^E(0,T)$) we can define the stochastic integral $\int_0^T f \, d\Phi$ (respectively, $\int_0^T (d\Phi \; f)$) and (8.18) holds. Moreover, if $f \in \mathcal{H}_{r,loc}^E(\mathbb{R}^+)$ (respectively, $f \in \mathcal{H}_{c,loc}^E(\mathbb{R}^+)$), then $(\int_0^t f \; d\Phi)_{t\geq 0}$ (respectively, $(\int_0^t (d\Phi \; f))_{t\geq 0}$) are continuous $E(\mathcal{M})$-valued martingales.

Proof. The proof is similar to, but simpler than, the one for Theorem 8.17. In this case we use the free version of the decoupling inequalities in Theorem 8.15 and the free Khintchine-type inequalities in Corollary 8.20 instead of their tensor counterparts in Theorem 8.14 and Corollaries 7.11 and 7.12, respectively. We leave the details to the interested reader. □

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List of symbols

General mathematics

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$
$\mathbb{R}$, real numbers
$\mathbb{C}$, complex numbers
p', Hölder conjugate of p
$A \lesssim_\alpha B$, p. 6
$A \simeq_\alpha B$, p. 6
$a \wedge b = \min\{a, b\}$

Functions and operators

χ_A, indicator of a set A
$(r_i)_{i \geq 1}$, Rademacher sequence, p. 23
$d(v; f)$, p. 106
$\mu_t(f)$, p. 106
$f \prec\prec g$, p. 106
$K(t, f; X_0, X_1)$, p. 117
$d(v; x)$, p. 77, 127
$\mu_t(x)$, p. 77, 128
$T^\times$, p. 123
$p^\perp$, p. 128
R_n, p. 160
$\lambda(g)$, p. 167

Probability

$(\Omega, \mathcal{F}, \mathbb{P})$, probability space
a.s., almost surely
$\mathcal{B}(X)$, Borel σ-algebra
$\mathbb{E}$, expectation
$\mathbb{E}(\cdot|\mathcal{G}), \mathbb{E}(\cdot|\xi)$, p. 27
$N, \tilde{N}, N_s$, p. 48

Noncommutative probability

$\mathcal{E}$, conditional expectation, p. 81
$W^*(\mathcal{N})$, p. 175
$W^*(a)$, p. 194
$\tau(\cdot|\mathcal{A})$, p. 198
a°, p. 198

Spaces

Miscellaneous

Index

Summary

Noncommutative and Vector-valued Rosenthal Inequalities

This thesis is dedicated to the study of a class of probabilistic inequalities, called *Rosenthal inequalities*. These inequalities provide two-sided estimates for the p-th moments of the sum of a sequence of independent, mean zero random variables, in terms of a suitable norm on the sequence itself. Rosenthal inequalities are named after the mathematician H.P. Rosenthal, who first discovered them for scalar-valued random variables around 1970. The main results of this thesis extend Rosenthal's inequalities in two different directions.

In Part I we consider sums of independent, mean zero random variables taking values in a Banach space. The main results give Rosenthal-type inequalities in the case where the Banach space is either a Hilbert space or an L^p-space. The inequalities we develop in this setting are principally designed to prove a novel Itô isomorphism for vector-valued stochastic integrals with respect to a compensated Poisson random measure. These kind of isomorphisms are a key tool for the analysis of stochastic partial differential equations.

The Rosenthal-type inequalities are further extended to apply to random variables taking values in a noncommutative L^p-space associated with a von Neumann algebra. By specializing this result to the von Neumann algebra of $n \times n$ matrices, we find quantitative bounds for the moments of the largest singular value of a random matrix in terms of its entries.

Part II of this thesis is dedicated to a generalization of Rosenthal's original inequalities to sequences of *noncommutative random variables*. The main result provides a generalization of Rosenthal's theorem for elements of a noncommutative symmetric space, which are independent in a noncommutative sense. For a suitable class of noncommutative symmetric spaces we moreover prove Burkholder-Rosenthal inequalities for noncommutative martingales.

As is the case in Part I, we apply the noncommutative Rosenthal inequalities to obtain norm estimates for stochastic integrals. We obtain Itô isomorphisms for stochastic integrals in a large class of noncommutative symmetric spaces, in the case where the integrator is either a Boson or a free Brownian motion.

For our proof of the noncommutative Rosenthal and Burkholder-Rosenthal inequalities we develop several new tools which are interesting in their own right. A good part of Part II is devoted to a new, direct proof of the 'upper' Khintchine inequalities for noncommutative symmetric spaces. Our results improve the best known results in the literature in this direction and they are shown to be optimal in a certain sense. The noncommutative Khintchine inequalities are utilized to prove Burkholder-Gundy inequalities for martingales in noncommutative symmetric spaces. Finally, we present several new results in the interpolation theory for noncommutative symmetric spaces. In particular, using a new method we find an extension of the noncommutative Boyd interpolation theorem. This method is adapted to yield a dual version of Doob's maximal inequality in noncommutative symmetric spaces.

Samenvatting

Niet-commutatieve en Vector-waardige Rosenthal Ongelijkheden

Dit proefschrift is gewijd aan de studie van een klasse van ongelijkheden in de kansrekening, genaamd *Rosenthal ongelijkheden.* Deze ongelijkheden geven tweezijdige afschattingen voor de p-de momenten van de som van een rij onafhankelijke, gecentreerde kansvariabelen, in termen van een gepaste norm op de rij zelf. Rosenthal ongelijkheden zijn vernoemd naar de wiskundige H.P. Rosenthal, die rond 1970 dergelijke ongelijkheden voor het eerst ondekte voor scalar-waardige kansvariabelen. De hoofdresultaten van dit proefschrift breiden Rosenthal's oorspronkelijke ongelijkheden uit in twee verschillende richtingen.

In Deel I beschouwen we sommen van onhankelijke, gecentreerde stochasten die waarden aannemen in een Banachruimte. De hoofdresultaten geven nieuwe Rosenthal-achtige ongelijkheden in het geval waar de Banachruimte een Hilbertruimte of een L^p-ruimte is. De ongelijkheden die wij ontwikkelen in deze setting zijn hoofdzakelijk ontworpen om nieuwe Itô isomorfismen te bewijzen voor vector-waardige stochastische integralen met betrekking tot een gecompenseerde Poisson kansmaat. Dergelijke isomorfismen spelen een belangrijke rol in de analyse van stochastische partiële differentiaalvergelijkingen.

De vector-waardige Rosenthal ongelijkheden worden verder uitgebreid voor stochasten met waarden in een niet-commutatieve L^p-ruimte behorend bij een von Neumann algebra. In het speciale geval waarin de von Neumann algebra gegeven wordt door de $n \times n$ matrices geeft dit resultaat kwantitatieve afschattingen voor de momenten van de grootste singuliere waarde van een stochastische matrix in termen van de individuele elementen van de matrix.

Deel II van dit proefschrift is gewijd aan een generalisatie van Rosenthal's originele ongelijkheden voor rijtjes bestaande uit *niet-commutatieve kansvariabelen.* Het hoofdresultaat geeft een generalisatie van Rosenthal's stelling voor

elementen van een niet-commutatieve symmetrische ruimte, die onafhankelijk zijn in een niet-commutatieve zin. Voor een geschikte klasse van niet-commutatieve symmetrische ruimten bewijzen we bovendien de Burkholder-Rosenthal ongelijkheden voor niet-commutatieve martingalen.

Net als in Deel I gebruiken we de niet-commutatieve Rosenthal ongelijkheden om norm afschattingen voor stochastische integralen te krijgen. We vinden Itô isomorfismen voor stochastische integralen in een grote klasse van niet-commutatieve symmetrische ruimten, in het geval waar de stochastische integrator gegeven wordt door een Boson of een vrije Brownse beweging.

Voor het bewijs van de niet-commutatieve Rosenthal en Burkholder-Rosenthal ongelijkheden ontwikkelen we verscheidene nieuwe instrumenten die op zichzelf staand interessant zijn. Een groot gedeelte van Deel II is gewijd aan een nieuw, direct bewijs van de 'bovenste' Khintchine ongelijkheden voor niet-commutatieve symmetrische ruimten. Onze resultaten in deze richting verbeteren de bekende resultaten uit de literatuur en we tonen aan dat deze, in zekere zin, optimaal zijn. De Khintchine ongelijkheden worden vervolgens gebruikt om Burkholder-Gundy ongelijkheden voor martingalen in niet-commutatieve symmetrische ruimten te bewijzen. Tot slot presenteren we verscheidene nieuwe resultaten in de interpolatietheorie voor niet-commutatieve symmetrische ruimten. In het bijzonder vinden we, via een nieuwe methode, een uitbreiding van de niet-commutatieve Boyd interpolatiestelling. Met behulp van deze methode wordt bovendien een duale versie van Doob's maximaalongelijkheid in niet-commutatieve symmetrische ruimten verkregen.

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Delft, September 2011 Sjoerd Dirksen

Curriculum Vitae

Sjoerd Dirksen was born on the 29th of April, 1983, in Rotterdam, The Netherlands. In 2001 he completed his secondary education at Het Stedelijk Gymnasium in Nijmegen. Between 2001 and 2005, he pursued the study of Econometrics and Operations Research at the Erasmus University Rotterdam and obtained his BSc. degree 'cum laude'. After an exchange semester at the University of Calgary, Canada, in the fall of 2004, he commenced his study in Mathematics at Utrecht University in January 2005. Between 2005 and 2007, he obtained his BSc. and MSc. degree 'cum laude'. In October 2007, he began his PhD research under supervision of prof. dr. J.M.A.M. van Neerven and prof. dr. B. de Pagter at Delft University of Technology. Part of this research was carried out during a five-month stay at the University of New South Wales in Sydney, Australia, and a two-month visit to the University of Illinois at Urbana-Champaign, USA.