TOKYO J. MATH.
VOL. 31, NO. 2, 2008

Metric Properties of Denjoy's Canonical Continued Fraction Expansion

Marius IOSIFESCU and Cor KRAAIKAMP

Romanian Academy and Technische Universiteit Delft

(Communicated by A. Tani)

Abstract. Metric properties of Denjoy's canonical continued fraction expansion are studied, and the natural extension of the underlying ergodic system is given. This natural extension is used to give simple proofs of results on mediant convergents obtained by W. Bosma in 1990.

1. Introduction

It is quite well-known—see e.g. [9]—that every real number x can be written as a regular continued fraction (RCF) expansion

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}} = [a_0; a_1, a_2, \dots], \tag{1}$$

where $a_0 = \lfloor x \rfloor$, and the a_i, $i \in \mathbf{N}_+ := \{1, 2, \dots\}$, are positive integers. In case x is irrational, the expansion (1) is unique and infinite; in case x is rational, (1) is finite and two possible expansions exist.

Apart from the RCF expansion (1), very many other continued fraction expansions of x exist. One such expansion, Denjoy's *canonical continued fraction expansion* ([5]), has hardly attracted any attention. Let x be a real number with RCF expansion (1), and let $d_0 \in \mathbf{Z}$ be such, that $d_0 \le x$. Then it was shown in [10] that x has an expansion of the form

$$x = [d_0; (0,1)^{a_0 - d_0}, (1,0)^{a_1 - 1}, 1, (1,0)^{a_2 - 1}, 1, \dots],$$

where $(1,0)^k$ is an abbreviation for the string $1, 0, 1, 0, \dots, 1, 0$ consisting of k pairs $(0, 1)$, which is empty if $k = 0$. Such a continued fraction expansion is called a *canonical continued fraction* (CCF) *expansion* of x.

Received May 10, 2007; revised March 20, 2008
2000 *Mathematics Subject Classification*: 28D05, 11K55
The first author was supported by Bezoekersbeurs B 61-617 of the Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO) and from Contract 2-CEx-06-11-97/19.09.06 of the Romanian Authority for Research.

In [10], a map $T_d : [0,\infty) \to [0,\infty)$ is studied, which 'generates' a unique CCF expansion for every $x > 0$. This 'Denjoy-map' is given by $T_d(0) = 0$, and

$$T_d(x) = \frac{1}{x} - d(x)\,, \quad x > 0\,,$$

where

$$d(x) = \begin{cases} 1\,, & \text{if } x \in (0,1]\,, \\ 0\,, & \text{if } x \in (1,\infty)\,. \end{cases}$$

The digits (or partial quotients) $d_n = d_n(x)$ of $x > 0$ are now given for $n \in \mathbf{N}_+$ by

$$d_n(x) = d(T_d^{n-1}(x)) \quad \text{whenever } T_d^{n-1}(x) > 0\,.$$

Using T_d, we find (see equation (14) further on) that $x > 0$ given by (1) has as CCF expansion

$$x = [0; d_1(x), d_2(x), \ldots]\,. \tag{2}$$

In [10] it has been shown that $([0,\infty), \mathcal{B}, \mu, T_d)$ is an ergodic system, where μ is a σ-finite, infinite T_d-invariant measure with density

$$f(x) = \frac{1}{x} 1_{(0,1]}(x) + \frac{1}{1+x} 1_{(1,\infty)}(x)\,, \quad x \in \mathbf{R}_+ := (0,\infty)\,.$$

Furthermore, it has been shown in [10] that the CCF-convergents p_n/q_n of x, which are obtained by taking finite truncations in (2), consist of the RCF-convergents P_n/Q_n of x and the RCF-mediants of x. The latter are defined as

$$\frac{aP_n - 1 + P_{n-1}}{aQ_n - 1 + Q_{n-1}} \quad \text{for all integers } a \text{ satisfying } 1 \le a \le a_n - 1\,.$$

One 'shortcoming' of the CCF algorithm is that every RCF-convergent P_n/Q_n of x appears a_{n+1} times as a CCF-convergent of it. There are several algorithms yielding the RCF-convergents and mediants, see for instance [6] or [11], where such algorithms together with the underlying ergodic systems are described. In [11], the RCF-convergents and mediants of any $x \in [0,1)$ are 'generated' in the same order, without the duplication of the RCF-convergents as in the case of the CCF expansion of x. The underlying transformation $S : [0,1] \to [0,1]$ in [11] is defined as

$$S(x) = \begin{cases} \dfrac{x}{1-x}\,, & x \in \left[0, \dfrac{1}{2}\right)\,, \\[2ex] \dfrac{1-x}{x}\,, & x \in \left[\dfrac{1}{2}, 1\right]\,, \end{cases}$$

and there is a σ-finite, infinite S-invariant measure ν with density $1/x$, $x \in \mathbf{R}_+$. Moreover, Ito showed in [11] that the dynamical system $([0,1], \mathcal{B}, S, \nu)$ is ergodic.

It is easy to check that

$$S(x) = \begin{cases} T_d^2(x)\,, & x \in \left[0, \frac{1}{2}\right)\,, \\ T_d(x)\,, & x \in \left[\frac{1}{2}, 1\right]\,, \end{cases}$$

i.e., S can be seen as a *jump transformation* of T_d. Hence, the ergodic properties of S can easily be carried over to T_d. Note that T_d^2 is used to avoid duplication of RCF-convergents. Since, as noticed in [10],

$$T(x) \;=\; T_d^{2(k-1)+1}(x)\,, \quad x \in \left[\frac{1}{k+1}, \frac{1}{k}\right)\,, \quad k \in \mathbf{N}_+\,,$$

where T is the usual Gauss map underlying the RCF expansion, i.e., $T(x) = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$, $x \in (0, 1)$, ergodic properties of T_d can also be obtained from the ergodic properties of the RCF expansion.

In this paper we first discuss some metric properties of the CCF expansion. Then we determine the natural extension of $([0, \infty), \mathcal{B}, \mu, T_d)$, and use it to study the metric properties of the so-called *approximation coefficients* $\theta_n = \theta_n(x)$ of x, defined by

$$\theta_n(x) = q_n^2 \left| x - \frac{p_n}{q_n} \right|\,, \quad n \in \mathbf{N} := \mathbf{N}_+ \cup \{0\}\,,$$

where p_n/q_n, $n \in \mathbf{N}$, are the CCF-convergents of x. In particular, the metric properties of the RCF-mediants are also studied. We show that the results from Bosma [2] can be obtained in a direct and elegant way.

2. Metric properties

2.1. The Perron-Frobenius operator. Let $L^1(\mu)$ and $L^\infty(\mu)$ denote the usual L^1- and L^∞- Banach spaces on $(\mathbf{R}_+, \mathcal{B}_+, \mu)$ with $\mathcal{B}_+ =$ Borel σ-algebra in $\mathbf{R}_+$.

According to the general theory (cf., e.g., [1], p. 33), the Perron-Frobenius operator U of T_d under μ takes $L^1(\mu)$ into itself and satisfies the equation

$$\int_{\mathbf{R}_+} (Uh) g \,\mathrm{d}\mu = \int_{\mathbf{R}_+} (g \circ T_d) h \,\mathrm{d}\mu \tag{3}$$

for any $h \in L^1(\mu)$ and $g \in L^\infty(\mu)$. Clearly, (3) implies that

$$\int_{\mathbf{R}_+} (U^n h) g \,\mathrm{d}\mu = \int_{\mathbf{R}_+} (g \circ T_d^n) h \,\mathrm{d}\mu \tag{4}$$

for any $n \in \mathbf{N}_+$. Let $\nu_h(A) = \int_A h \, \mathrm{d}\mu$ for $h \in L^1(\mu)$ and $A \in \mathcal{B}_+$. Then Uh can be expressed as a Radon-Nikodým derivative, namely, $\mathrm{d}(\nu_h \circ T_d^{-1})/\mathrm{d}\mu$. It is easy to check that we thus have

$$Uh(s) = \frac{1_{(0,1]}(s)}{s+1} h\left(\frac{1}{s}\right) + \left(1_{(1,\infty)}(s) + \frac{s 1_{(0,1]}(s)}{s+1}\right) h\left(\frac{1}{s+1}\right)$$

for μ-almost all $s \in \mathbf{R}_+$ and any $h \in L^1(\mu)$. Clearly, $U1 = 1$, even if the constant functions do not belong to $L^1(\mu)$. Actually, Uh thus defined makes sense for *any* function $h : \mathbf{R}_+ \to \mathbf{R}_+$ but, of course, without satisfying (3) for such an h.

The Gauss problem for the CCF transformation T_d can be approached in terms of the Perron-Frobenius operator U (cf. [9], Ch. 2, for the case of the RCF expansion.) This problem amounts to the asymptotic behavior of $m(T_d^{-n}(A))$ as $n \to \infty$ for probability measures $m \ll \lambda$ (Lebesgue measure on $\mathcal{B}_+$) and $A \in \mathcal{B}_+$. It is immediate from (4) by taking $g = 1_A$ that

$$\int_A U^n h \, \mathrm{d}\mu = \int_{T_d^{-n}(A)} h \, \mathrm{d}\mu \, .$$

Hence

$$m\left(T_d^{-n}(A)\right) = \int_A U^n \left(\frac{p}{f}\right) \mathrm{d}\mu \tag{5}$$

for any $n \in \mathbf{N}$ and $A \in \mathcal{B}_+$, where $p = \mathrm{d}m/\mathrm{d}\lambda$. Therefore, as in the case of the RCF expansion, the asymptotic behavior of $m\left(T_d^{-n}(A)\right)$ depends on the asymptotic behavior of U^n as $n \to \infty$. However, in our case here things are much simpler due to the properties of the T_d-invariant measure μ. Indeed, by Proposition 1.1.3 in [1], the transformation T_d is conservative since incompressible, that is, $A \in \mathcal{B}_+$ and $T_d^{-1}(A) \subset A$ imply $A = T_d^{-1}(A) \bmod \lambda$. [It is easy to check that $T_d^{-1}(A) \subset A$ only holds for either $A = \emptyset$ (the empty set) or $A = \mathbf{R}_+$.] Next, since T_d is conservative and ergodic, by [1], Exercise 2.2.1, p. 61, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} h \circ T_d^k = 0 \quad \mu\text{-a.e.}\,, \tag{6}$$

hence λ-a.e., for any $h \in L^1(\mu)$. In particular, for $h = 1_A$ with $A \in \mathcal{B}_+$ and $\mu(A) < \infty$, we deduce that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_A \circ T_d^k = 0 \quad \mu\text{-a.e.} \tag{7}$$

By dominated convergence, this clearly implies

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} m\left(T_d^{-k}(A)\right) = \int_{\mathbf{R}_+} \lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} 1_A \circ T_d^k}{n} p \, \mathrm{d}\lambda = 0 \tag{8}$$

for any $A \in \mathcal{B}_+$ such that $\mu(A) < \infty$.

This shows that if the limit

$$\lim_{n\to\infty} m\left(T_d^{-n}(A)\right) \tag{9}$$

exists, then it should be 0; and if so then, from (5),

$$0 = \liminf_{n\to\infty} \int_A U^n\left(\frac{p}{f}\right) \mathrm{d}\mu \geq \int_A \liminf_{n\to\infty} U^n\left(\frac{p}{f}\right) \mathrm{d}\mu\,, \tag{10}$$

hence

$$\liminf_{n\to\infty} U^n\left(\frac{p}{f}\right) = 0 \quad \mu\text{-a.e.}$$

on any $A \in \mathcal{B}_+$ such that $\mu(A) < \infty$. This actually means that

$$\liminf_{n\to\infty} U^n\left(\frac{p}{f}\right) = 0 \quad \lambda\text{-a.e. in } \mathbf{R}_+\,.$$

In any case one can assert that we always have

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} U^k\left(\frac{p}{f}\right) = 0 \quad \lambda\text{-a.e. in } \mathbf{R}_+\,.$$

Recall that p above is any non-negative element of $L^1(\lambda)$.

Coming back to equation (7), let $A \in \mathcal{B}_+$ be such that $\mu(A) < \infty$. Let $A^c = \mathbf{R}_+ \setminus A$. Since $1_A + 1_{A^c} = 1$, we have

$$\frac{1}{n}\sum_{k=0}^{n-1} 1_A \circ T_d^k + \frac{1}{n}\sum_{k=0}^{n-1} 1_{A^c} \circ T_d^k = 1\,.$$

It follows from (7) that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} 1_{A^c} \circ T_d^k = 1 \quad \mu\text{-a.e.} \tag{11}$$

This means that μ-almost all orbits $(x, T_d(x), T_d^2(x), \ldots)$, $x \in \mathbf{R}_+$, hit A^c with relative frequency asymptotical to 1. Also, similarly to (8) we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} m\left(T_d^{-k}(A^c)\right) = 1\,. \tag{12}$$

Hence if the limit

$$\lim_{n\to\infty} m\left(T_d^{-n}(A^c)\right) \tag{13}$$

exists, then it should be 1.

We conjecture that both limits (9) and (13) do not exist, so that (8) and (12) appear to be the solution of Gauss' problem for T_d.

REMARK. The last conclusion holds, too, *mutatis mutandis*, for many other transformations with σ-finite, infinite invariant measure as, e.g., Ito's transformation S. □

2.2. A Markov chain and an infinite order chain associated with the CCF expansion. We shall consider in this subsection the CCF expansions of irrationals in the unit interval $I = [0, 1]$. With the usual definition (cf. Section 1) of $d(x)$ and $d_n = d_n(x) = d(T_d^{n-1}(x))$, $n \in \mathbf{N}_+$, we have

$$x = \cfrac{1}{d_1 + T_d(x)} = \cdots = \cfrac{1}{d_1 + \cfrac{1}{d_2 + \cfrac{1}{\ddots + \cfrac{1}{d_n + T_d^n(x)}}}} \tag{14}$$

for any $n \in \mathbf{N}_+$ and any $x \in \Omega$, where Ω is the set of irrationals in [0, 1]. Clearly, $d_1 = 1$. (When confusion cannot arise, we suppress the argument x in the notation.) Next, for the CCF-convergents p_n/q_n, $n \in \mathbf{N}_+$, the equations

$$p_n = d_n p_{n-1} + p_{n-2}\,, \quad q_n = d_n q_{n-1} + q_{n-2}\,, \tag{15}$$

do hold for any $n \in \mathbf{N}_+$ with $p_{-1} = 1$, $p_0 = 0$, $q_{-1} = 0$, and $q_0 = 1$. Hence *all* q_n, $n \in \mathbf{N}$, are positive for any $x \in \Omega$. It follows immediately from (14) and (15) that

$$x = \frac{(d_n + T_d^n(x))p_{n-1} + p_{n-2}}{(d_n + T_d^n(x))q_{n-1} + q_{n-2}} = \frac{p_n + p_{n-1}T_d^n(x)}{q_n + q_{n-1}T_d^n(x)} \tag{16}$$

for all $n \in \mathbf{N}_+$ and $x \in \Omega$. Also, it is easy to check that

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}\,, \quad n \in \mathbf{N}\,. \tag{17}$$

For any $n \in \mathbf{N}_+$ and any $i_k \in \{0, 1\}$, $1 \le k \le n$, consider the *admissible* n-tuples $i^{(n)} := (i_1, \ldots, i_n)$ with $i_1 = 1$, and where no two consecutive 0's appear in the finite sequence $i_1, \ldots, i_n$. Let $I(i^{(n)})$ denote the set of $x \in \Omega$ for which $d_k(x) = i_k$, $1 \le k \le n$. Such a set is called a *cylinder set of rank n*.

It is perhaps interesting to note that the number of admissible $i^{(n)}$ ending in 1 is F_{n-1}, $n \in \mathbf{N}_+$, while that of admissible $i^{(n)}$ ending in 0 is F_{n-2}, $n \ge 2$. Here the F_n are the Fibonacci numbers defined recursively by $F_0 = F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$, $n \in \mathbf{N}_+$.

PROPOSITION 1. *For any $n \in \mathbf{N}_+$ a cylinder set $I(i_1, \ldots, i_n)$ is either the set of irrationals in the interval with end points p_n/q_n and $(p_n + p_{n-1})/(q_n + q_{n-1})$ when $i_n = 0$ or the set of irrationals in the interval with end points p_{n-1}/q_{n-1} and p_n/q_n when $i_n = 1$.* [*Here, the p_n and q_n, $n \in \mathbf{N}$, are computed using the rules* (15) *with $p_{-1} = 1$, $p_0 = 0$, $q_{-1} = 0$, $q_0 = 1$, $d_k = i_k$, $1 \le k \le n$.*]

PROOF. The argument goes through as in the case of the RCF expansion (cf. [9], p. 18). We nevertheless give the details as there are some traps.

(a) If $i_n = 0$ then $i_{n+1} = 1$, so for $x \in I(i_1, \dots, i_n)$ we have

$$T_d^n(x) = \frac{1}{1 + \dots} < 1 \, .$$

Hence $T_d^n(x)$ varies through irrational values between 0 and 1, and by (16) we are done.

(b) If $i_n = 1$ then either $i_{n+1} = 1$ or $i_{n+1} = 0$. If $i_{n+1} = 1$ then, as before, we get the irrationals in the interval with end points p_n/q_n and $(p_n + p_{n-1})/(q_n + q_{n-1})$ while if $i_{n+1} = 0$ we have

$$T_d^n(x) = \frac{1}{0 + \dots} > 1 \, .$$

That is, $T_d^n(x)$ varies through irrational values between 1 and ∞, and by (16) we get the interval with end points $(p_n + p_{n-1})/(q_n + q_{n-1})$ and p_{n-1}/q_{n-1}. It remains to note that $(p_n + p_{n-1})/(q_n + q_{n-1})$ always lies between p_{n-1}/q_{n-1} and p_n/q_n. □

Let us define $s_n = q_{n-1}/q_n$, $n \in \mathbf{N}$, so that, in particular, $s_0 = 0$ and $s_1 = 1/d_1 = 1$. It follows from the second equation in (15) that

$$s_n = \frac{1}{d_n + s_{n-1}} \, , \quad n \in \mathbf{N}_+ \, . \tag{18}$$

Note that in our sequence $(s_n)_{n \in \mathbf{N}}$ the values 0 and 1 appear no longer if $n \geq 3$. Indeed, s_2 takes on the values 1 and 1/2 both with probability 1/2 while the possible values of s_3 are 1/2, 2/3, and 2 with probabilities 1/2, 1/6, and 1/3, respectively. Then equation (18) shows that $s_n \neq 0$ and 1 for any $n \geq 3$. Hence the random events $\{s_n > 1\}$ and $\{d_n = 0\}$ are equivalent for $n \geq 3$.

Abusing notation, let λ denote the Lebesgue measure on I, too.

PROPOSITION 2. *For any $n \in \mathbf{N}_+$ we have*

$$\lambda(d_{n+1} = 1 \mid d_1, \dots, d_n) = 1_{\{0\}}(d_n) + 1_{\{1\}}(d_n) \frac{s_n}{s_n + 1}$$

and

$$\lambda(d_{n+1} = 0 \mid d_1, \dots, d_n) = 1 - \lambda(d_{n+1} = 1 \mid d_1, \dots, d_n) = 1_{\{1\}}(d_n) \frac{1}{s_n + 1} \, .$$

PROOF. The equations are simple consequences of Proposition 1 and equations (17). For example, if $d_n = 1$ and $n \geq 2$, then

$$\lambda(d_{n+1} = 1 \mid d_1, \dots, d_n) = \frac{\lambda(I(d_1, \dots, d_{n-1}, 1, 1))}{\lambda(I(d_1, \dots, d_{n-1}, 1))}$$
$$= \left| \frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}} \right| \Bigg/ \left| \frac{p_{n-1}}{q_{n-1}} - \frac{p_n}{q_n} \right| = \frac{q_{n-1}}{q_{n+1}}$$

$$= \frac{q_{n-1}}{q_n + q_{n-1}} = \frac{s_n}{s_n + 1}.$$

□

We also state

PROPOSITION 3. *For any $u \in \mathbf{R}_+$ and $n \in \mathbf{N}$ we have*

$$\lambda(T_d^n \geq u \mid d_1, \ldots, d_n) = \frac{1-u}{1+s_n u}\left(1_{\{0\}}(d_n) + \frac{1_{\{1\}}(d_n)s_n}{s_n+1}\right)1_{(0,1]}(u) + \frac{1_{\{1\}}(d_n)}{1+s_n u}1_{(1,\infty)}(u).$$

PROOF. The equation, that generalizes the result in Proposition 2, is again a consequence of Proposition 1 and equations (16) and (17), so that we skip the details. □

REMARK. Similar considerations can be made as for irrationals in $(1, \infty)$. Note that the invertible transformation $\tau : x \mapsto 1/x$, $x \in (0, 1)$, takes Ω into the set Ω' of irrationals in $(1, \infty)$ while Lebesgue measure on $(0, 1)$ is taken to the probability measure $\lambda\tau^{-1}$ on $(1, \infty)$ with density $1/y^2$, $y \in (1, \infty)$. If $x' \in \Omega'$ then we have $d_1(x') = 0$, so that $d_2(x') = 1$, and $d_n(x') = d_{n-1}(1/x')$ for any $n \geq 2$. □

Proposition 2 shows that the sequence $(d_n)_{n \in \mathbf{N}_+}$ is a $\{0, 1\}$-valued process which can be called an infinite order chain, cf. [8], Section 5.5. We will return to it a little bit later.

As already noticed, we have $s_n \neq 1$ for $n \geq 3$. Then, by (18) and Proposition 2, the sequence $(s_n)_{n \geq 3}$ is a $(\mathbf{Q}_+ \setminus \{0, 1\})$-valued Markov chain on $(\Omega, \mathcal{B}_{[0,1]}, \lambda)$ with the following transition mechanism. From state $s \in \mathbf{Q}_+ \setminus \{0, 1\}$ the only possible transitions are to states $1/s$ and $1/(1+s)$ with probabilities

$$\frac{1_{(0,1)}(s)}{s+1} \quad \text{and} \quad 1_{(1,\infty]}(s) + \frac{s1_{(0,1)}(s)}{s+1}.$$

More generally, we shall also consider an $\mathbf{R}_+$-valued Markov process with the same transition mechanism and states $s \in \mathbf{R}_+$. Obviously, the transition operator U of such a process can be expressed as

$$U\,h(s) = \frac{1_{(0,1]}(s)}{s+1}\,h\left(\frac{1}{s}\right) + \left(1_{(1,\infty)}(s) + \frac{s1_{(0,1]}(s)}{s+1}\right)h\left(\frac{1}{s+1}\right)$$

for any function $h \in L^\infty(\lambda)$ (or, equivalently, $h \in L^\infty(\mu)$ since $\lambda \equiv \mu$). It is not a coincidence that the transition operator and the Perron-Frobenius operator of T_d under μ have the same analytical expression (and both of them were denoted by the same letter). For a full explanation the reader is referred to [7], pp. 1–5, and Example (c), p. 6.

Concerning U as a transition operator of a Markov process we note the following.

(a) U is actually a typical operator dealt with in dependence-with-complete-connection theory. Cf. [8], especially Chapters 1 and 5. No known results can be used in our case to derive convergence of $U^n f$ as $n \to \infty$ for certain functions $f \in L^\infty(\lambda)$.

(b) The existence of a stationary measure different from μ for our Markov process is an open question. Such a measure would be a T_d-invariant one, leaving just the possibility of a λ-singular measure. Cf. [1], p. 45, on unicity of a λ-absolutely continuous invariant measure. We are unable to confirm or reject this possibility.

Coming back to the random sequence $(d_n)_{n\in\mathbf{N}_+}$, we remark that by Proposition 2 the probability $\lambda(d_{n+1}=1)$ that d_{n+1} takes on value 1 is equal to

$$\frac{1}{2}U^{n-3}h\left(\frac{1}{2}\right)+\frac{1}{6}U^{n-3}h\left(\frac{2}{3}\right)+\frac{1}{3}U^{n-3}h(2)\,,\quad n\geq 3\,,$$

where

$$h(s)=1_{(1,\infty)}(s)+\frac{s1_{(0,1]}(s)}{s+1}\,,\quad s\in\mathbf{R}_+\,.$$

Even if one cannot precise the asymptotic behavior of $U^n h$ as $n\to\infty$ (see (a) above), we can get instead some information about the asymptotic relative frequencies of 0 or 1 on the trajectories of $(d_n)_{n\in\mathbf{N}_+}$ not only for irrationals in $(0,1)$ but even for irrationals in $\mathbf{R}_+$.

Let $0<\varepsilon<1$ and $M>1$ be arbitrarily fixed. It follows from equation (11) – take $A=(\varepsilon,M)$ – that the sum of the relative frequencies of 0 and 1 when the values of T_d^k, $0\leq k\leq n-1$, are either less than ε or exceed M converges to 1 an $n\to\infty$ while the values between ε and M bring no asymptotic contribution. Cf. equation (7). It thus appears that the most common situation is that where the values $T_d^k(x)$, $x\in\mathbf{R}_+$, $k\in\mathbf{N}$, are clustering near 0 or ∞. We conjecture that there are no limiting relative frequencies of 0 or 1. See, however, [10], p. 237.

3. A natural extension, and its consequences

3.1. A natural extension. In the last two decades the concept of *natural extension* from ergodic theory has been very fruitful in the study of both arithmetic and metric properties of continued fractions; see, e.g., [14], [13], [3], [12], and [9]. A natural extension is a bijective system which "contains" the original system as a factor, and which is in some sense the "smallest' system with this property. One cannot speak of *the* natural extension; usually, various natural extension of a certain system exist, and these natural extensions are metrically isomorphic. In the natural extension for $([0,\infty),\mathcal{B},\mu,T_d)$, we want the second coordinate to act as "the past." In view of this, we define a map $\mathcal{T}_d: L\to L$ (where L still needs to be defined) by

$$\mathcal{T}_d(x,y)=\left(T_d(x),\frac{1}{d(x)+y}\right).$$

Setting

$$L=([0,1)\times[0,\infty))\cup([1,\infty)\times[0,1])\ ,$$

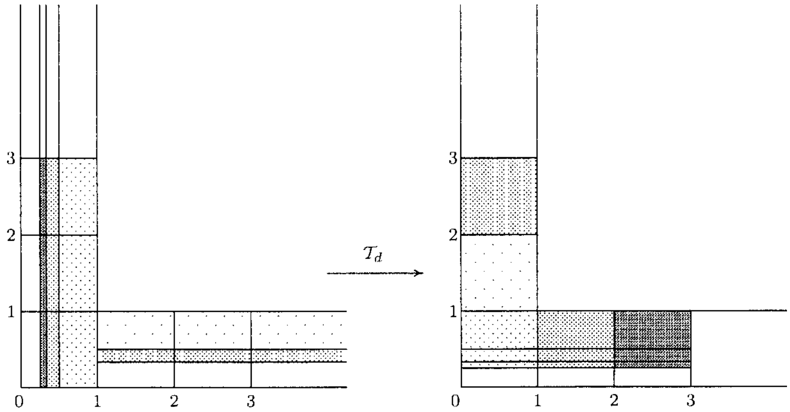

FIGURE 1. The map $\mathcal{T}_d$ acting on L.

one easily checks that a.e. $\mathcal{T}_d : L \to L$ is a bijective map, see also Figure 1.

Now, let $\bar{\mu}$ be the σ-finite, infinite measure on L with density

$$\frac{1}{(1+xy)^2}\,, \quad (x, y) \in L\,.$$

An easy Jacobian transformation shows that $\bar{\mu}$ is $\mathcal{T}_d$-invariant, i.e.,

$$\bar{\mu}\left(\mathcal{T}_d^{-1}(A)\right) = \bar{\mu}(A)\,, \quad A \in \mathcal{B}_L\,.$$

In [10] it was already noticed that if T is the Gauss-map, i.e., $T(x) = \frac{1}{x} - \lfloor\frac{1}{x}\rfloor$, $x \in (0, 1)$, then

$$T(x) = T_d^{2(k-1)+1}(x)\,, \quad x \in \left[\frac{1}{k+1}, \frac{1}{k}\right)\,.$$

Nakada showed in [13] that the natural extension map $\mathcal{T}$ of the Gauss-map T is given by

$$\mathcal{T}(x, y) = \left(T(x), \frac{1}{a(x)+y}\right)\,, \quad (x, y) \in [0, 1) \times [0, 1]\,,$$

where $a(x) = \lfloor\frac{1}{x}\rfloor$ for $x \in (0, 1)$. He also showed that $([0, 1) \times [0, 1], \mathcal{B}, \bar{\gamma}, \mathcal{T})$ is a K-system, hence certainly ergodic. Here $\bar{\gamma}$ is a $\mathcal{T}$-invariant probability measure on $[0, 1) \times [0, 1]$ with density

$$\frac{1}{\log 2}\frac{1}{(1+xy)^2}\,, \quad (x, y) \in [0, 1) \times [0, 1]\,.$$

Nakada's system is 'sitting inside' the dynamical system $(L, \mathcal{B}_L, \bar{\mu}, \mathcal{T}_d)$ as an induced system. To see this, for $(x, y) \in [0, 1) \times [0, 1]$ define $r(x, y) = 0$ if $x = 0$, and

$$r(x, y) = \min\{k\,;\ k \geq 1, \mathcal{T}_d^k(x, y) \in [0, 1) \times [0, 1]\}$$

if $x \neq 0$. Define the induced transformation $\mathcal{K} : [0, 1) \times [0, 1] \to [0, 1) \times [0, 1]$ by

$$\mathcal{K}(x, y) = \mathcal{T}_d^{r(x,y)}(x, y)\,, \quad (x, y) \in [0, 1) \times [0, 1]\,.$$

We have the following result, which is stated here without proof.

PROPOSITION 4. *For $(x, y) \in (0, 1) \times [0, 1]$ let $k \in \mathbf{N}_+$ be such that $\frac{1}{k+1} \leq x < \frac{1}{k}$. Then*

$$r(x, y) = k \quad \textit{and} \quad \mathcal{K}(x, y) = \mathcal{T}(x, y)\,.$$

As an immediate consequence we find, see e.g. [15], that

$$(L, \mathcal{B}_L, \bar{\mu}, \mathcal{T}_d) \quad \text{is an ergodic system}\,.$$

Note that projecting on the first coordinate yields the ergodic system $([0, \infty), \mathcal{B}_{[0,\infty)}, \mu, T_d)$ considered in [10].

To obtain various results on the distribution of the RCF approximation coefficients Θ_n, $n \in \mathbf{N}$, Bosma *et al* derived from the ergodic system $\big([0, 1), \mathcal{B}_{[0,1)}, \mu, T\big)$ the important result below; see [3], or Chapter 4 in [9], for a proof of this result.

THEOREM 1 ([3]). *For almost all x the two-dimensional sequence*

$$(T_n, V_n) = \mathcal{T}^n(x, 0)\,, \quad n \in \mathbf{N}\,,$$

is distributed over $[0, 1) \times [0, 1]$ according to the density function

$$\frac{1}{\log 2} \frac{1}{(1 + xy)^2}\,, \quad (x, y) \in [0, 1) \times [0, 1]\,.$$

A similar result also holds in the present situation.

THEOREM 2. *For almost all x the two-dimensional sequence*

$$(t_k, v_k) = \mathcal{T}_d^k(x, 0)\,, \quad k \in \mathbf{N}\,,$$

is distributed over L according to the density function

$$\frac{1}{(1 + xy)^2}\,, \quad (x, y) \in L\,.$$

3.2. Approximation coefficients. Following [12], or Chapter 4 in [9], one can show that for any irrational $x > 0$ and for $k \in \mathbf{N}$ the approximation coefficients θ_k satisfy

$$\theta_{k-1} = \frac{v_k}{1 + t_k v_k}\,, \quad \text{and} \quad \theta_k = \frac{t_k}{1 + t_k v_k}\,, \tag{19}$$

where $(t_k, v_k) = \mathcal{T}_d^k(x, 0)$, $k \in \mathbf{N}$. Now, let x have RCF expansion (1), and suppose that $a_{n+1} \geq 2$ for some $n \in \mathbf{N}$. Setting $k(n) = a_0 + 2(a_1 - 1) + 1 + \cdots + 2(a_n - 1) + 1$, it was shown in [10] that

$$\frac{p_{k(n)}}{q_{k(n)}} = \frac{p_{k(n)+2j}}{q_{k(n)+2j}} = \frac{P_n}{Q_n} \quad \text{for } 1 \le j \le a_{n+1} - 1 \tag{20}$$

and

$$\frac{p_{k(n)+2j-1}}{q_{k(n)+2j-1}} = \frac{jP_n + P_{n-1}}{jQ_n + Q_{n-1}} \quad \text{for } 1 \le j \le a_{n+1} - 1\,. \tag{21}$$

In view of the "redundancy" (20), we will work with the ergodic system

$$(\bar{\Omega} = [0,1) \times [0,\infty), \mathcal{B}_{\bar{\Omega}}, \bar{\mu}, \mathcal{S})\,,$$

where $\mathcal{S} : \bar{\Omega} \to \bar{\Omega}$ is the induced transformation on $\bar{\Omega}$, i.e., for $(x, y) \in \bar{\Omega}$ we define

$$\mathcal{S}(x,y) = \begin{cases} \mathcal{T}_d(x,y)\,, & \text{if } \mathcal{T}_d(x,y) \in [0,1) \times [0,1]\,, \\ \mathcal{T}_d^2(x,y) & \text{if } \mathcal{T}_d(x,y) \in [1,\infty) \times [0,1]\,. \end{cases}$$

The next result follows at once from the dynamics of $\mathcal{T}_d$.

PROPOSITION 5. *Let $x > 0$ be a real number with RCF expansion* (1) *and CCF expansion* (2), *and suppose that $a_{n+1} > 1$. Furthermore, let*

$$k(n) = a_0 - d_0 + 2(a_1 - 1) + 1 + \cdots + 2(a_n - 1) + 1\,.$$

Then for $1 \le j \le a_{n+1} - 1$ we have

$$\mathcal{T}_d^{k(n)+2j-1}(x,0) = \left(t_{k(n)+2j-1}, v_{k(n)+2j-1}\right) \in (a_{n+1} - j - 1, a_{n+1} - j] \times [0,1]$$

and

$$\mathcal{T}_d^{k(n)+2j}(x,0) = \left(t_{k(n)+2j}, v_{k(n)+2j}\right) \in [0,1) \times [j, j+1] = I_j\,.$$

It follows from Proposition 5 that if $\mathcal{T}_d^k(x,0) = (t_k, v_k) \in I_B = [0,1) \times [B, B+1]$ for some $B \ge 1$, then p_k/q_k is equal to the RCF-convergent P_n/Q_n for some (unique) n, while p_{k-1}/q_{k-1} is equal to the mediant

$$\frac{BP_n + P_{n-1}}{BQ_n + Q_{n-1}} =: \frac{L_m^{(B)}}{M_m^{(B)}}$$

for some $m \in \mathbf{N}$, $m = m(n)$. Hence, for each $B \in \mathbf{N}$, the limiting distribution of the approximation coefficient $\Theta_n^{(B)}(x)$ defined as

$$\Theta_n^{(B)}(x) = \left(M_m^{(B)}\right)^2 \left| x - \frac{L_n^{(B)}}{M_m^{(B)}} \right|, \quad n \in \mathbf{N}\,,$$

can be derived for almost all x. We thus have the following result.

PROPOSITION 6 ([2]). *Let $B > 1$ be an integer.*

(i) *For any irrational number x with RCF expansion* (1) *and for any $n \in \mathbf{N}_+$ such that $0 < B < a_n$ we have*

$$\frac{B}{B+1} \leq \Theta_n^{(B)} \leq B+1 \, .$$

(ii) *For almost all x the sequence $\left(\Theta_n^{(B)}(x)\right)_{n \geq 1}$ is distributed according to the density function*

$$\frac{G^{(B)}(z)}{\log \frac{B+2}{B+1}} \, ,$$

where

$$G^{(B)}(z) = \begin{cases} G_0^{(B)}(z) = 0 \, , & \text{if } z \leq \dfrac{B}{B+1} \\ G_1^{(B)}(z) = -1 + \dfrac{B+1}{B} z - \log\left(\dfrac{B+1}{B} z\right) , & \text{if } \dfrac{B}{B+1} \leq z < \dfrac{B+1}{B+2} \\ G_2^{(B)}(z) = \dfrac{z}{B(B+1)} + \log\left(\dfrac{B(B+2)}{(B+1)^2}\right) , & \text{if } \dfrac{B+1}{B+2} \leq z < B \\ G_3^{(B)}(z) = 1 - \dfrac{z}{B+1} + \log\left(\dfrac{B+2}{(B+1)^2} z\right) , & \text{if } B \leq z < B+1 \\ G_4^{(B)}(z) = \log \dfrac{B+2}{B+1} \, , & \text{if } B+1 \leq z \, . \end{cases}$$

PROOF. It follows from (19) that

$$\Theta_n^{(B)}(x) = \frac{V}{1+TV} \, ,$$

where $(T, V) = \mathcal{T}_d^{k(n)+2B}(x, 0) \in I_B$. Since

$$\frac{\partial}{\partial t}\left(\frac{v}{1+tv}\right) = \frac{-v}{(1+tv)^2} < 0 \quad \text{and} \quad \frac{\partial}{\partial v}\left(\frac{v}{1+tv}\right) = \frac{1}{(1+tv)^2} > 0 \, ,$$

we find that $\Theta_n^{(B)}(x)$ attains its maximum and minimum on the boundary of I_B. Its maximum (which is $B+1$) is attained at $(0, B+1)$ while its minimum (which is $B/(B+1)$) is attained at $(1, B)$. This proves (i).

Now, $\Theta_n^{(B)}(x) \leq z$ if and only if $(t, v) \in I_B$ and

$$\frac{v}{1+tv} \leq z \, , \quad \text{i.e.,} \quad v \leq \frac{z}{1-zt} \, .$$

It follows from Theorem 2 that

$$\lim_{n \to \infty} \frac{1}{n} \#\left\{k;\ 1 \leq k \leq n,\ \Theta_n^{(B)}(x) \leq z\right\}$$

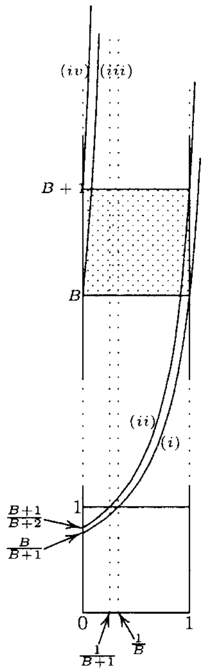

FIGURE 2. Five cases for $\Theta_n^{(B)}(x) \le z$. Depicted here are the graphs of $y = \frac{z}{1-zx}$ for (i): $z = \frac{B}{B+1}$, (ii): $z = \frac{B+1}{B+2}$, (iii): $z = B$ (iii), and (iv): $z = B + 1$.

exists for almost all x and is equal to

$$\frac{\bar{\mu}(D^{(B)}(z))}{\bar{\mu}(I_B)},$$

where

$$D^{(B)}(z) = \left\{(t, v) \in I_B;\ v \le \frac{z}{1 - zt}\right\}.$$

Therefore, since $\bar{\mu}(I_B) = \log \frac{B+2}{B+1}$, we are left with the computation of $\bar{\mu}(D^{(B)}(z))$. We distinguish 5 cases; see Figure 2. As an example, we deal with the case where $B \le z \le B+1$ (see Figure 3); the other cases can be treated in a similar way. Since

$$\bar{\mu}\big(D^{(B)}(z)\big) = \bar{\mu}(I_B) - \int_0^{\frac{B+1-z}{(B+1)z}} \left(\int_B^{B+1} \frac{\mathrm{d}y}{(1+xy)^2}\right)\mathrm{d}x$$

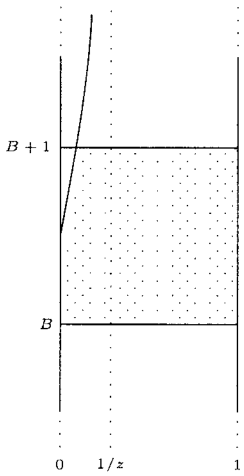

FIGURE 3. The case $B \le z \le B+1$.

and

$$\int_0^{\frac{B+1-z}{(B+1)z}} \left(\int_B^{B+1} \frac{\mathrm{d}y}{(1+xy)^2} \right) \mathrm{d}x = \int_0^{\frac{B+1-z}{(B+1)z}} \left(\frac{B+1}{1+(B+1)x} - z \right) \mathrm{d}x$$
$$= \log\left(\frac{B+1}{z}\right) - 1 + \frac{z}{B+1}\,,$$

we obtain

$$\bar{\mu}\,(D(B)(z)) = 1 - \frac{z}{B+1} + \log\left(\frac{B+2}{(B+1)^2} z\right) = G_3^{(B)}(z)\,.$$

□

REMARK. In [2], Bosma obtained Propositon 6 directly from the ergodic system $([0,1) \times [0,1], \mathcal{B}, \bar{\gamma}, \mathcal{T})$ underlying the RCF expansion, by cleverly combining two consecutive regular approximation coefficients. In this way, Bosma was able to express approximation coefficients of convergents and mediants in terms of the orbit $(\mathcal{T}^n(x,0))_{n\ge 0}$ in $[0,1) \times [0,1]$. To be precise, Bosma (see (1.8) in [2], p. 423) found that

$$\Theta_n^{(B)} = \frac{(1 - BT_{n-1}V_{n-1})}{1 + T_{n-1}V_{n-1}}\,.$$

Independently, Brown and Yin also obtained in [4] this relation. The outlook in this latter paper is quite different from that in [2]. □

References

[1] AARONSON, J. *An Introduction to Infinite Ergodic Theory*, Mathematical Surveys and Monographs, **50**. American Mathematical Society, Providence, RI, 1997.

[2] BOSMA, W., Approximation by mediants, Math. Comp. **54** (1990), no. 189, 421–434.

[3] BOSMA, W., JAGER, H. and WIEDIJK, F., Some metrical observations on the approximation by continued fractions, Nederl. Akad. Wetensch. Indag. Math. **45** (1983), no. 3, 281–299.

[4] BROWN, G. and YIN, Q., Metrical theory for Farey continued fractions, Osaka J. Math. **33** (1996), no. 4, 951–970.

[5] DENJOY, A., *Complément à la notice publiée en 1934 sur les travaux scientifiques de M. Arnaud Denjoy*, Hermann, Paris, 1942.

[6] DAJANI, K. and KRAAIKAMP, C., The mother of all continued fractions, Colloq. Math. **84/85** (2000), 109–123.

[7] FOGUEL, S. R., *The Ergodic Theory of Markov Processes*, Van Nostrand Mathematical Studies, No. 21. Van Nostrand Reinhold Co., New York-Toronto, Ont.-London, 1969.

[8] IOSIFESCU, MARIUS and GRIGORESCU, SERBAN, *Dependence with Complete Connections and its Applications*, Cambridge Tracts in Mathematics, **96**. Cambridge Univ. Press, Cambridge, 1990.

[9] IOSIFESCU, MARIUS and KRAAIKAMP, COR, *Metrical Theory of Continued Fractions*, Mathematics and its Applications, 547. Kluwer Academic Publishers, Dordrecht, 2002.

[10] IOSIFESCU, M. and KRAAIKAMP, C., On Denjoy's canonical continued fraction expansion, Osaka J. Math. **40** (2003), no. 1, 235–244.

[11] ITO, SH., Algorithms with mediant convergents and their metrical theory, Osaka J. Math. **26** (1989), no. 3, 557–578.

[12] KRAAIKAMP, C., A new class of continued fraction expansions, Acta Arith. **57** (1991), no. 1, 1–39.

[13] NAKADA, H., Metrical theory for a class of continued fraction transformations and their natural extensions, Tokyo J. Math. **4** (1981), no. 2, 399–426.

[14] NAKADA, HITOSHI, ITO, SHUNJI and TANAKA, SHIGERU, On the invariant measure for the transformations associated with some real continued-fractions, Keio Engrg. Rep. **30** (1977), no. 13, 159–175.

[15] WALTERS, P., *An Introduction to Ergodic Theory*, Springer-Verlag, New York, Heidelberg, Berlin, 1982.

Present Addresses:
MARIUS IOSIFESCU
ROMANIAN ACADEMY,
"GHEORGHE MIHOC–CAIUS IACOB" INSTITUTE OF MATHEMATICAL STATISTICS
AND APPLIED MATHEMATICS,
CASA ACADEMIEI ROMANE, CALEA 13 SEPTEMBRIE NR. 13, 050711 BUCHAREST 5, ROMANIA.
e-mail: miosifes@acad.ro

COR KRAAIKAMP
TECHNISCHE UNIVERSITEIT DELFT,
EWI (DIAM), THOMAS STIELTJES INSTITUTE FOR MATHEMATICS,
MEKELWEG 4, 2628 CD DELFT, THE NETHERLANDS.
e-mail: c.kraaikamp@tudelft.nl