Perturbations and sums of operators

Hytönen, Tuomas; van Neerven, Jan; Veraar, Mark; Weis, Lutz

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16

Perturbations and sums of operators

In this chapter we address a couple of topics in the theory of H^∞-calculus centering around the question what can be said about an operator of the form $A+B$ when A and B have certain "good" properties such as being (R-)sectorial or admitting a bounded H^∞-calculus. The chapter is divided into two sections. The first considers the case where B is "smaller" than A in certain ways, and the second considers the case where A and B are essentially on equal footing. The results of this chapter play an important role in applications as well in the further development of the abstract theory and will be needed in our treatment, in the next to chapter, of the maximal regularity problem.

16.1 Sums of unbounded operators

In general it is a rather delicate problem to give a meaning to the operator sum $A+B$ when A and B are unbounded operators acting in a Banach space X. The simplest approach is to define

$$\begin{aligned} \mathsf{D}(A+B) &:= \mathsf{D}(A) \cap \mathsf{D}(B), \\ (A+B)x &:= Ax + Bx, \quad x \in \mathsf{D}(A+B), \end{aligned} \tag{16.1}$$

but in concrete cases this definition may be vacuous due to the possibility that $\mathsf{D}(A) \cap \mathsf{D}(B)$ could be unreasonably small or even trivial, i.e., equal to $\{0\}$. Various methods to deal with this problem have been developed, such as the method of forms. In the context of evolution equations, the two prime applications one has in mind are cases where either A is the linear operator governing the equation, e.g., a linear differential operator in the space variables, and B is the derivative with respect to time, or both A and B are differential operators in the space variable, typically with B being of lower order than A. In both of these cases, the resolvent operators $R(\lambda, A)$ and $R(\mu, B)$ commute and $\mathsf{D}(A) \cap \mathsf{D}(B)$ is "large", in that it contains all elements of the form $R(\lambda, A)R(\mu, B)x$ with $x \in X$. In fact we have the following result.

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Proposition 16.1.1. *If A and B are sectorial operators acting in X whose resolvents commute, then $\mathsf{D}(A) \cap \mathsf{D}(B)$ is dense in both $\mathsf{D}(A)$ and $\mathsf{D}(B)$*

Proof. As a consequence of the resolvent commutation and Proposition 10.1.7 we have

$$\lim_{\lambda \to -\infty} \lambda R(\lambda, A)R(\mu, B)x = R(\mu, B)x \text{ in } \mathsf{D}(B)$$

for all $\mu \in \varrho(B)$ and

$$\lim_{\mu \to -\infty} \mu R(\lambda, A)R(\mu, B)x = R(\lambda, A)x \text{ in } \mathsf{D}(A)$$

for all $\lambda \in \varrho(A)$ □

It is for this reason that we will stick to the somewhat naive approach embodied in (16.1); the operator sum $A + B$ will always be understood as given in this way.

Let us briefly clarify the meaning of the term 'resolvent commutation' used in the above proposition. If commutation identity

$$R(\lambda, A)R(\mu, B) = R(\mu, B)R(\lambda, A)$$

holds for some $\lambda \in \varrho(A)$ and $\mu \in \varrho(B)$, then it holds for all $\lambda' \in \varrho(A)$ and $\mu' \in \varrho(B)$ in the connected components of $\varrho(A)$ containing λ and μ, respectively. This is an easy consequence of the Taylor series identities

$$R(\lambda', A) = \sum_{n=0}^{\infty} (\lambda - \lambda')^n R(\lambda, A)^{n+1},$$
$$R(\mu', B) = \sum_{n=0}^{\infty} (\mu - \mu')^n R(\mu, B)^{n+1},$$

which follow from repeated application of the resolvent identity (see Section 10.1.b). The following definition then suggests itself naturally:

Definition 16.1.2 (Resolvent commutation). *The sectorial operators A and B are said to* resolvent commute *when*

$$R(\lambda, A)R(\mu, B) = R(\mu, B)R(\lambda, A)$$

holds for some (or equivalently, all) λ, μ in the connected set $\complement\overline{\Sigma_\sigma} \cap \complement\overline{\Sigma_\tau}$ for some (or equivalently, all) $\omega(A) < \sigma < \pi$ and $\omega(B) < \tau < \pi$.

16.2 Perturbation theorems

When it comes to checking the boundedness of the H^∞-calculus of concrete operators, in particular elliptic differential operators, perturbation theorems are often the method of choice. Perturbation arguments compare a "complicated" operator with a more "basic" operator such as the Laplace operator or an elliptic operator with constant coefficients. In order to cover a multitude of concrete situations we phrase these perturbation arguments in the framework of sectorial operators and their scale of fractional domain spaces. The case of lower-order perturbations of the form

$$L = A + B \quad \text{with} \quad B : \mathsf{D}(A^\alpha) \to X \quad \text{for some} \ 0 < \alpha < 1$$

(Theorem 16.2.7) is readily obtained from the corresponding theorem about relatively bounded perturbations of the form

$$L = A + B \text{ with } \ \|Bx\| \leqslant \delta\|Ax\| \ \text{ for small } \delta > 0$$

(Theorem 16.2.3). In contrast to sectoriality, boundedness of the H^∞-functional calculus is not preserved under small relatively bounded perturbations, unless additional relative boundedness assumptions are made with respect to the fractional domains (Example 16.2.10 and Theorem 16.2.8). Analogous perturbation theorems for R-sectorial operators are proved as well.

Because of their importance in applications, in particular for the study of non-linear evolution equations, the literature on perturbation theorems is extensive. We can present only a representative selection of such theorems and some model applications serving as illustrations. Variants and extensions of these results, in particular to elliptic operators and pseudo-differential operators, will be discussed in the Notes.

We next introduce some notation which will be used throughout this chapter and the next ones. Recalling from Definition 10.1.1 that an operator A is called *σ-sectorial* if the set $\{\lambda \neq 0, \, |\arg(\lambda)| > \sigma\}$ is contained in the resolvent set $\varrho(A)$ and

$$\sup_{\lambda\neq 0, \, |\arg(\lambda)|\geqslant\sigma} \|\lambda R(\lambda, A)\| < \infty,$$

we define

$$M_{\sigma,A} := \sup\{\|\lambda R(\lambda, A)\| : \, \lambda \neq 0, \, |\arg(\lambda)| > \sigma\},$$
$$\widetilde{M}_{\sigma,A} := \sup\{\|AR(\lambda, A)\| : \, \lambda \neq 0, \, |\arg(\lambda)| > \sigma\}.$$

When A is *σ-R-sectorial* (the definition being similar), for $p \in [1, \infty)$ we set

$$\widetilde{M}^{R_p}_{\sigma,A} := \mathscr{R}_p(\{\lambda R(\lambda, A) : \, \lambda \neq 0, \, |\arg(\lambda)| > \sigma\}),$$
$$\widetilde{M}^{R_p}_{\sigma,A} := \mathscr{R}_p(\{AR(\lambda, A) : \, \lambda \neq 0, \, |\arg(\lambda)| > \sigma\}),$$

where $\mathscr{R}_p(\mathscr{T})$ denote the R-bound with exponent p (see Remark 8.1.2).

16.2.a Perturbations of sectorial operators

To set the stage for the results to follow, we begin with an elementary perturbation result for sectorial operators.

Proposition 16.2.1. *If A is an σ-sectorial operator on X and $B \in \mathscr{L}(X)$ is bounded, then for all $\lambda_0 \geqslant M\|B\|$ the operator $\lambda_0 + A + B$ is σ-sectorial.*

Proof. Set $M := M_{\sigma,A}$ for brevity. Fix a non-zero $\lambda \in \mathbb{C}$ with $|\arg(\lambda)| > \sigma$. Then $\lambda \in \varrho(A)$ and $\|R(\lambda, A)\| \leqslant M/|\lambda|$. Because

$$(\lambda - (A + B)) = (I - BR(\lambda, A))(\lambda - A)$$

and $\|BR(\lambda, A)\| \leqslant M\|B\|/|\lambda|$, for $|\lambda| > M\|B\|$ the operator $I - BR(\lambda, A)$ is invertible. For such λ it follows that $\lambda \in \varrho(A + B)$ and

$$R(\lambda, A + B) = R(\lambda, A) \sum_{n=0}^{\infty} [BR(\lambda, A)]^n$$

by the Neumann series. This gives the bound

$$\|R(\lambda, A + B)\| \leqslant \frac{M}{|\lambda|} \frac{1}{1 - M\|B\|/|\lambda|} = \frac{M}{|\lambda| - M\|B\|},$$

valid for non-zero $\lambda \in \mathbb{C}$ satisfying $|\arg(\lambda)| > \sigma$ and $|\lambda| > M\|B\|$. Shifting $A + B$ over $\lambda_0 \geqslant M\|B\|$, the result follows from this. □

The following lemma describes a useful technique that will enable us to deal with lower-order and relatively bounded perturbations.

Lemma 16.2.2 (The method of continuity). *Let E and F be Banach spaces. Let $(L_t)_{t\in[0,1]}$ be a family of bounded linear operators from E into F such that $t \mapsto L_t$ is continuous from $[0, 1]$ into $\mathscr{L}(E, F)$. Suppose furthermore that there exists a constant $C > 0$ such that for all $t \in [0, 1]$ and all $x \in E$ we have*

$$\|x\| \leqslant C\|L_t x\|.$$

Then L_0 is surjective if and only if L_1 is surjective.

Proof. Since $[0, 1]$ is compact, $t \mapsto L_t$ is uniformly continuous. Therefore we can find $\delta > 0$ such that $|t - s| < \delta$ implies $\|L_t - L_s\| \leqslant \frac{\varepsilon}{2C}$.

The assumption of the lemma imply that the operators L_t are injective. Now suppose that L_s is invertible for a given $s \in [0, 1]$. We will show that L_t is invertible for all $t \in [0, 1]$ satisfying $|t - s| < \delta$. Clearly, this implies the required result by an iteration argument.

Fix $f \in F$ and let $T : E \to E$ be the mapping given by $T(x) = y$, where $y \in E$ is the unique solution to $L_s y = f + L_s x - L_t x$. We claim that T is a uniform contraction. Indeed, by the assumed *a priori* estimate,

$$\|T(x_1) - T(x_2)\| = \|y_1 - y_2\| \leqslant C\|L_s y_1 - L_s y_2\|$$

Since $L_s y_1 - L_s y_2 = (L_s - L_t)(x_1 - x_2)$ we obtain

$$\|T(x_1) - T(x_2)\| \leqslant C\|L_s - L_t\| \, \|x_1 - x_2\| \leqslant \frac{1}{2}\|x_1 - x_2\|.$$

This proves the claim. By the Banach fixed point theorem, T has a unique fixed point x. It follows that $L_s x = f + L_s x - L_t x$, and hence $L_t x = f$. □

As a first application of this lemma we prove the following result on relatively bounded perturbations of sectorial operators.

Theorem 16.2.3 (Relatively bounded perturbations of sectorial operators). *Let A be an σ-sectorial operator, and let $B : \mathsf{D}(A) \to X$ be a linear operator that satisfies*

$$\|Bx\| \leqslant \delta\|Ax\| + K\|x\|, \quad x \in \mathsf{D}(A), \tag{16.2}$$

where $K \geqslant 0$ and $\delta \in (0,1)$ satisfies $\delta \widetilde{M}_{\sigma,A} < 1$. Then the operator $A+B$ with domain $\mathsf{D}(A+B) := \mathsf{D}(A)$ is closed, and the following assertions hold:

(1) *For all $\lambda \in \mathbb{R}$ large enough, $\lambda + A + B$ is σ-sectorial.*
(2) *If (16.2) holds with $K = 0$, then $A + B$ is σ-sectorial.*

Proof. Observe that for all $x \in \mathsf{D}(A)$,

$$\|Ax\| \leqslant \|(A+B)x\| + \|Bx\| \leqslant \|(A+B)x\| + \delta\|Ax\| + K\|x\|. \tag{16.3}$$

Therefore, $(1-\delta)\|Ax\| \leqslant \|(A+B)x\| + K\|x\|$. By a routine argument, (16.2) and (16.3) imply that $A+B$ is closed.

We will prove both assertions at the same time by showing that $\lambda_0 + A + B$ is sectorial for any fixed $\lambda_0 \geqslant 0$ large enough, permitting $\lambda_0 = 0$ if (16.2) holds with $K = 0$.

Fix $\lambda \in \lambda_0 + \Sigma_\sigma$. We will apply Lemma 16.2.2 to $E = \mathsf{D}(A)$, $F = X$, and the operators $L_t : \mathsf{D}(A) \to X$ given by

$$L_t x := (\lambda + A + tB)x, \qquad t \in [0,1],$$

where $\mathsf{D}(A)$ will be equipped with the equivalent norm

$$|\!|\!| x |\!|\!| = \|(\lambda - \lambda_0)x\| + \|Ax\|.$$

We first prove the following *a priori* estimate: For all $\lambda_0 \geqslant 0$ large enough there exists a constant $C \geqslant 0$ such that

$$|\!|\!| x |\!|\!| \leqslant C\|L_t x\|, \qquad x \in \mathsf{D}(A),\ t \in [0,1]. \tag{16.4}$$

Let $x \in \mathsf{D}(A)$ and set $y := L_t x$. Then $(\lambda + A)x = y - tBx$. Multiplying this identity with $A(\lambda + A)^{-1}$ on both sides and using (16.2), we obtain

$$\|Ax\| \leqslant \widetilde{M}_{\sigma,A}\|y\| + \widetilde{M}_{\sigma,A}\|Bx\| \leqslant \widetilde{M}_{\sigma,A}\|y\| + \widetilde{M}_{\sigma,A}\delta\|Ax\| + \widetilde{M}_{\sigma,A}K\|x\|.$$

Since $\widetilde{M}_{\sigma,A}\delta < 1$, it follows that

$$\|Ax\| \leqslant C_0\|y\| + C_0 K\|x\|, \tag{16.5}$$

where $C_0 = \widetilde{M}_{\sigma,A}(1 - \widetilde{M}_{\sigma,A}\delta)^{-1}$. To estimate $\|x\|$, writing $\lambda x = y - tBx - Ax$ we find that

$$\begin{aligned}|\lambda|\|x\| &\leqslant \|y\| + \|Bx\| + \|Ax\| \\ &\leqslant \|y\| + (\delta+1)\|Ax\| + K\|x\| \leqslant C_1\|y\| + C_2K\|x\|\end{aligned}$$

where $C_1 := 1 + (\delta+1)C_0$ and $C_2 := (\delta+1)C_0 + 1$, so that

$$\|x\| \leqslant \frac{C_1}{|\lambda| - C_2K}\|y\| =: D\|y\|,$$

provided we take $\lambda_0 \geqslant C_2K$ sufficiently large (in order that $|\lambda| > C_2K$). Such choices of λ_0 imply that $|\lambda - \lambda_0| \leqslant C_\sigma|\lambda|$ and, together with (16.5),

$$\begin{aligned}\|\!|x|\!\| = \|Ax\| + \|(\lambda-\lambda_0)x\| &\leqslant C_0\|y\| + C_0K\|x\| + |\lambda-\lambda_0|\|x\| \\ &\leqslant C_0\|y\| + C_0K\|x\| + C_\sigma(C_1\|y\| + C_2K\|x\|) \\ &\leqslant C\|y\| = C\|L_tx\|\end{aligned}$$

where $C := (C_0 + C_\sigma C_1) + (C_0 + C_\sigma C_2)DK$, which is (16.4). Scrutinising the proof, we see that $\lambda_0 = 0$ can be allowed if (16.2) holds with $K = 0$.

Since $L_0 = \lambda + A$ is surjective, Lemma 16.2.2 gives that $L_1 = \lambda + A + B$ is surjective, and hence boundedly invertible by (16.4). Also by (16.4), for all $y \in X$ and $\lambda \in \lambda_0 + \Sigma_{\pi-\sigma}$ (where we may take $\lambda_0 = 0$ if $K = 0$),

$$\|(\lambda-\lambda_0)(\lambda + A + B)^{-1}y\| \leqslant \|\lambda(\lambda + A + B)^{-1}y\| \leqslant C\|y\|,$$

which proves $\lambda_0 + A + B$ is σ-sectorial. □

Theorem 16.2.4 (Relatively bounded perturbations of R-sectorial operators). *Let A be σ-R-sectorial, and suppose that $B : \mathsf{D}(A) \to X$ is a linear operator which satisfies*

$$\|Bx\| \leqslant \delta\|Ax\| + K\|x\|, \quad x \in \mathsf{D}(A), \tag{16.6}$$

where $K \geqslant 0$ and $\delta \in (0,1]$ satisfies $\delta\widetilde{M}^{R_p}_{\sigma,A} < 1$ for some $p \in [1,\infty)$. Then the operator $A + B$ with domain $\mathsf{D}(A+B) := \mathsf{D}(A)$ is closed, and the following assertions hold:

(1) *For all $\lambda \in \mathbb{R}$ large enough, $\lambda + A + B$ is σ-R-sectorial.*
(2) *If (16.6) holds with $K = 0$, then $A + B$ is σ-R-sectorial.*

Proof. The method of proof is similar to that of Theorem 16.2.3. Again we will prove both assertions at the same time. Let $(\Omega, \mathbb{P})$ be a probability space supporting a Rademacher sequence $(\varepsilon_n)_{n\geqslant 1}$. For notational convenience we write $\|\cdot\|_p = \|\cdot\|_{L^p(\Omega;X)}$. We will show that $\lambda_0 + A + B$ is R-sectorial for all $\lambda_0 \geqslant 0$ large enough, and that we may take $\lambda_0 = 0$ if (16.6) holds with $K = 0$.

The assumptions of the theorem imply those of Theorem 16.2.3, and therefore $A + B$ satisfies its conclusions. It remains to prove the R-boundedness of the set

$$\big\{(\lambda-\lambda_0)(\lambda + A + B)^{-1} : \lambda \neq 0, \ \lambda \in \lambda_0 + \Sigma_{\pi-\sigma}\big\}.$$

To this end let $n \geqslant 1$, non-zero $\lambda_1, \ldots, \lambda_n \in \lambda_0 + \Sigma_{\pi-\sigma}$, and $y_1, \ldots, y_n \in X$ be arbitrary and fixed. Let $x_j \in X$ be the unique solution to $(\lambda_j + A + B)x_j = y_j$ for each $j \in \{1, \ldots, n\}$. It suffices to show that there is a constant $C \geqslant 0$ such that

$$\Big\| \sum_{j=1}^n \varepsilon_j (\lambda_j - \lambda_0) x_j \Big\|_p \leqslant C \Big\| \sum_{j=1}^n \varepsilon_j y_j \Big\|_p.$$

Since $Ax_j = A(\lambda_j + A)^{-1}[y_j - Bx_j]$, the R-sectoriality of A gives

$$\begin{aligned}\Big\| \sum_{j=1}^n \varepsilon_j A x_j \Big\|_p &\leqslant M \Big\| \sum_{j=1}^n \varepsilon_j y_j \Big\|_p + M \Big\| \sum_{j=1}^n \varepsilon_j B x_j \Big\|_p \\ &\leqslant M \Big\| \sum_{j=1}^n \varepsilon_j y_j \Big\|_p + M\delta \Big\| \sum_{j=1}^n \varepsilon_j A x_j \Big\|_p + MK \Big\| \sum_{j=1}^n \varepsilon_j x_j \Big\|_p,\end{aligned}$$

where $M := \widetilde{M}^{R_p}_{\sigma,A}$ for brevity. Therefore,

$$\Big\| \sum_{j=1}^n \varepsilon_j A x_j \Big\|_p \leqslant C_0 \Big\| \sum_{j=1}^n \varepsilon_j y_j \Big\|_p + C_0 K \Big\| \sum_{j=1}^n \varepsilon_j x_j \Big\|_p,$$

where $C_0 = CM(1 - \delta M)^{-1}$. Since $\lambda_j x_j = y_j - Bx_j - Ax_j$, we also find

$$\begin{aligned}\Big\| \sum_{j=1}^n \varepsilon_j \lambda_j x_j \Big\|_p &\leqslant \Big\| \sum_{j=1}^n \varepsilon_j y_j \Big\|_p + \Big\| \sum_{j=1}^n \varepsilon_j B x_j \Big\|_p + \Big\| \sum_{j=1}^n \varepsilon_j A x_j \Big\|_p \\ &\leqslant \Big\| \sum_{j=1}^n \varepsilon_j y_j \Big\|_p + K \Big\| \sum_{j=1}^n \varepsilon_j x_j \Big\|_p + (1+\delta) \Big\| \sum_{j=1}^n \varepsilon_j A x_j \Big\|_p \\ &\leqslant C_1 \Big\| \sum_{j=1}^n \varepsilon_j y_j \Big\|_p + C_1 K \Big\| \sum_{j=1}^n \varepsilon_j x_j \Big\|_p,\end{aligned} \tag{16.7}$$

where $C_1 = 1 + (1+\delta)C_0$.

Next we claim that there exist $D \geqslant 0$ and $\lambda_0 \geqslant 0$ such that

$$|\lambda_j - \lambda_0| \leqslant D(|\lambda_j| - 2C_1 K), \tag{16.8}$$

Writing $\lambda_j = \lambda_0 + re^{i\phi}$ with $|\phi| < \pi - \sigma$, (16.8) can be equivalently written as

$$(r + 2DC_1K)^2 \leqslant D^2(\lambda_0^2 + r^2 + 2\lambda_0 r \cos\phi).$$

If $|\phi| \leqslant \frac{1}{2}\pi$, then $\cos\phi \geqslant 0$ and the estimate holds with $D = \sqrt{2}$ and $\lambda_0 = C_1 K$. If $\frac{1}{2}\pi < |\phi| < \pi$, set $\delta := 1 + \cos\phi$ and note that $\delta \in (0, 1)$. It then follows that

$$\begin{aligned}\lambda_0^2 + r^2 + 2\lambda_0 r \cos\phi &= \lambda_0^2 + r^2 - 2\lambda_0 r(1-\delta) \\ &= \delta(\lambda_0^2 + r) + (1-\delta)(\lambda_0 - r)^2 \geqslant \delta(\lambda_0^2 + r^2)\end{aligned}$$

and the estimate holds with $D = \sqrt{8/\delta}$ and $\lambda_0 = DC_1K$. This proves the claim.

The claim implies $|(\lambda_j - \lambda_0)/\lambda_j| \leqslant D$ and $2C_1DK \leqslant D|\lambda_j|$, and therefore the Kahane contraction principle (see Theorem 6.1.13) implies

$$\Big\| \sum_{j=1}^n \varepsilon_j (\lambda_j - \lambda_0) x_j \Big\|_p = \Big\| \sum_{j=1}^n \varepsilon_j \frac{\lambda_j - \lambda_0}{\lambda_j} \lambda_j x_j \Big\|_p \leqslant D \Big\| \sum_{j=1}^n \varepsilon_j \lambda_j x_j \Big\|_p$$

and

$$2C_1DK \Big\| \sum_{j=1}^n \varepsilon_j x_j \Big\|_p \leqslant D \Big\| \sum_{j=1}^n \varepsilon_j |\lambda_j| x_j \Big\|_p = D \Big\| \sum_{j=1}^n \varepsilon_j \lambda_j x_j \Big\|_p.$$

Taking the averages of the last two estimates we obtain

$$\begin{aligned}\frac{1}{2} \Big\| \sum_{j=1}^n \varepsilon_j (\lambda_j - \lambda_0) x_j \Big\|_p &\leqslant D \Big\| \sum_{j=1}^n \varepsilon_j \lambda_j x_j \Big\|_p - C_1DK \Big\| \sum_{j=1}^n \varepsilon_j x_j \Big\|_p \\ &\leqslant C_1 D \Big\| \sum_{j=1}^n \varepsilon_j y_j \Big\|_p,\end{aligned}$$

where in the last step we applied (16.7). □

As a simple corollary to the above results we show that the smallness conditions on the constants can be lifted in the case of lower order perturbations. The notation is as in Appendix C.

Corollary 16.2.5 (Lower order perturbations of (R-)sectorial operators). *Let A be sectorial (resp. R-sectorial) and let $\theta \in (0,1)$. If*

$$B : \mathsf{D}(A^\theta) \to X$$

is a bounded linear operator, then for all large enough $\lambda \in \mathbb{R}$ the operator $\lambda + A + B$ is sectorial (resp. R-sectorial) with $\omega(\lambda + A + B) \leqslant \omega(A)$ (resp. $\omega_R(\lambda + A + B) \leqslant \omega_R(A)$).

Proof. It suffices to check the conditions of Theorems 16.2.3 and 16.2.4. For $x \in \mathsf{D}(A)$, by the interpolation estimate of Theorem 15.2.8 we obtain

$$\|Bx\| \leqslant \|B\| \, \|x\|_{\mathsf{D}(A^\theta)} \leqslant \|B\| \, \|x\|^{1-\theta} \|x\|_{\mathsf{D}(A)}^\theta.$$

Using the inequality $a^{1-\theta} b^\theta \leqslant (1-\theta)a + \theta b$, for all $\varepsilon > 0$ we obtain

$$\|x\|^{1-\theta} \|x\|_{\mathsf{D}(A)}^\theta \leqslant (1-\theta)\varepsilon^{-\frac{1}{1-\theta}} \|x\| + \varepsilon^{\frac{1}{\theta}} \|x\|_{\mathsf{D}(A)}.$$

The result now follows by combining the estimates and choosing $\varepsilon > 0$ small enough. □

The same proof works if one assumes that $B : (X, \mathsf{D}(A))_{\theta,p} \to X$ is a bounded operator for some $\theta \in (0,1)$ and $p \in [1,\infty]$, or that $B : [X, \mathsf{D}(A)]_\theta \to X$ is a bounded operator for some $\theta \in (0,1)$. A similar remark applies to Theorem 16.2.8 below.

16.2.b Perturbations of the H^∞-calculus

Having studied perturbations of sectorial and R-sectorial operators, we now turn to perturbation of the H^∞-calculus. The first proposition addresses shifts by a positive multiple of the identity. In certain applications it enables one to improve "for sufficiently large $\nu > 0$" to "for all $\nu > 0$".

Proposition 16.2.6 (Perturbation by a multiple of the identity). *Let A be a sectorial operator on X.*

(1) *If A has a bounded $H^\infty(\Sigma_\sigma)$-calculus, then $A+\nu I$ has a bounded $H^\infty(\Sigma_\sigma)$-calculus for all $\nu > 0$, and $M^\infty_{\sigma,A+\nu} \leqslant M^\infty_{\sigma,A}$.*
(2) *If $A + \nu_0 I$ has a bounded $H^\infty(\Sigma_\sigma)$-calculus for some $\nu_0 > 0$, then $A + \nu I$ has a bounded $H^\infty(\Sigma_\sigma)$-calculus for all $\nu > 0$.*

Proof. Assertion (1) is obtained by applying the bounded H^∞-calculus of A to the function $f_\nu(z) = f(z + \nu)$, noting that $f_\nu(A) = f(A + \nu)$; since $\|f(\cdot+\nu)\|_{H^\infty(\Sigma_\sigma)} \leqslant \|f\|_{H^\infty(\Sigma_\sigma)}$, this also gives the bound for the boundedness constants of the H^∞-calculi.

For the proof of assertion (2) we fix $\nu > 0$. Writing $A+\nu = (A+\varepsilon)+(\nu-\varepsilon)$ we see that there is no loss of generality in assuming that A is invertible. We also may assume that $0 < \nu < \delta$, where $\delta > 0$ is to be specified later, for once we have the converse for such ν the general case follows by repeated application of the first part of the proposition.

For $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ consider

$$\begin{aligned}
&\frac{1}{2\pi i}\int_\Gamma f(\lambda)R(\lambda, A+\nu_0)\,\mathrm{d}\lambda - \frac{1}{2\pi i}\int_\Gamma f(\lambda)R(\lambda, A+\nu)\,\mathrm{d}\lambda \\
&\quad = (\nu_0-\nu)\frac{1}{2\pi i}\int_\Gamma f(\lambda)R(\lambda, A+\nu)R(\lambda, A+\nu_0)\,\mathrm{d}\lambda \\
&\quad = (\nu_0-\nu)\frac{1}{2\pi i}\int_\Gamma f(\lambda)R(\lambda, A+\nu)R(0, A+\nu_0)\,\mathrm{d}\lambda \\
&\qquad + (\nu_0-\nu)\frac{1}{2\pi i}\int_\Gamma f(\lambda)R(\lambda, A+\nu)[R(\lambda, A+\nu_0) - R(0, A+\nu_0)]\,\mathrm{d}\lambda \\
&\quad = (\nu_0-\nu)R(-\nu_0, A)\frac{1}{2\pi i}\int_\Gamma f(\lambda)R(\lambda, A+\nu)\,\mathrm{d}\lambda \\
&\qquad - (\nu_0-\nu)\frac{1}{2\pi i}\int_\Gamma \lambda f(\lambda)R(\lambda, A+\nu)R(\lambda-\nu_0, A)R(-\nu_0, A)\,\mathrm{d}\lambda.
\end{aligned}$$

If we call the last integral $I(f)$, we have

$$\frac{1}{2\pi i}\int_\Gamma f(\lambda)R(\lambda, A+\nu_0)\,\mathrm{d}\lambda$$
$$= [I + (\nu_0 - \nu)R(-\nu_0, A)]\frac{1}{2\pi i}\int_\Gamma f(\lambda)R(\lambda, A+\nu)\,\mathrm{d}\lambda - (\nu_0-\nu)I(f).$$

For invertible A the operator $-AR(-\nu_0, A) = I+\nu_0 R(-\nu_0, A)$ is invertible as well. Since the set of invertible operators is open in $\mathscr{L}(X)$, there exists an $r > 0$ so small that $I+(\nu_0-\nu)R(-\nu_0, A)$ is invertible if $\nu\|R(-\nu_0, A)\| < r$, i.e., if $\nu < \delta := r/\|R(-\nu_0, A)\|$. Under this assumption we have the representation

$$f(A+\nu) = [I + (\nu_0-\nu)R(-\nu_0, A)]^{-1}[f(A+\nu_0) + (\nu_0-\nu)I(f)].$$

Hence

$$\|f(A+\nu)\| \leqslant \big\|[I - (\nu_0-\nu)R(-\nu_0, A)]^{-1}\big\|\big(\|f(A+\nu_0)\| + (\nu_0-\nu)\|I(f)\|\big). \tag{16.9}$$

By the assumptions we have $\|f(A+\nu_0)\| \leqslant C\|f\|_{H^\infty(\Sigma_\sigma)}$. We estimate the integral $I(f)$ by splitting it into $\Gamma_1 = \Gamma \cap \{|\lambda| \leqslant 1\}$ and $\Gamma_2 = \Gamma \cap \{|\lambda| \geqslant 1\}$ and using

$$R(\lambda-\nu_0, A) = (\lambda-\nu_0)^{-1}[R(\lambda-\nu_0, A)A + I].$$

This gives

$$\begin{aligned}
I(f) = \frac{1}{2\pi i}&\int_{\Gamma_1} \lambda f(\lambda)R(\lambda, A+\nu)R(\lambda-\nu_0, A)\,\mathrm{d}\lambda\ [R(-\nu_0, A)]\\
&+ \frac{1}{2\pi i}\int_{\Gamma_2} \lambda f(\lambda)R(\lambda, A+\nu)(\lambda-\nu_0)^{-1}R(\lambda-\nu_0, A)\,\mathrm{d}\lambda\ [AR(-\nu_0, A)]\\
&+ \frac{1}{2\pi i}\int_{\Gamma_2} \lambda f(\lambda)R(\lambda, A+\nu)R(-\nu_0, A)(\lambda-\nu_0)^{-1}\,\mathrm{d}\lambda\\
&= (I) + (II) + (III).
\end{aligned}$$

The integrals (I) and (II) can be estimated by $C\|f\|_{H^\infty(\Sigma_\sigma)}$ with constant C only depending on A, ν_0, σ. The third can be rewritten with the help of the resolvent identity and Cauchy's formula:

$$\begin{aligned}
(III) &= \frac{1}{2\pi i}\int_{\Gamma_2}\frac{f(\lambda)}{\lambda-\nu_0}R(-\nu_0, A)\,\mathrm{d}\lambda - \frac{1}{2\pi i}\int_{\Gamma_2}\frac{f(\lambda)}{\lambda-\nu_0}R(\lambda, A+\nu)\,\mathrm{d}\lambda\\
&= f(\nu_0)R(-\nu, A) - \frac{1}{2\pi i}\int_{\Gamma_1}\frac{f(\lambda)}{\lambda-\nu_0}R(-\nu_0, A)\,\mathrm{d}\lambda\\
&\quad - \frac{1}{2\pi i}\int_{\Gamma_2}\frac{f(\lambda)}{\lambda-\nu_0}R(\lambda, A+\nu)\,\mathrm{d}\lambda.
\end{aligned}$$

The two remaining integrals can again be estimated by $C\|f\|_{H^\infty(\Sigma_\sigma)}$ with constant C only depending on A, ν_0, σ. With (16.9) we arrive at

$$\|f(A+\nu)\| \leqslant C'\|f\|_{H^\infty(\Sigma_\sigma)}, \quad f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma),$$

thus completing the proof. □

We continue with the following result for lower order perturbations.

Theorem 16.2.7 (Lower order perturbations of the H^∞-calculus). *Let A be a sectorial operator and suppose that B is linear operator in X satisfying*

$$\mathsf{D}(A^\alpha) \subseteq \mathsf{D}(B)$$

and

$$\|Bx\| \leqslant a\|A^\alpha x\| + b\|x\|, \quad x \in \mathsf{D}(A),$$

for suitable real numbers $a, b \geqslant 0$ and $\alpha \in (0,1)$. If A has a bounded $H^\infty(\Sigma_\sigma)$-calculus in X for some $\omega(A) < \sigma < \pi$, then $A+B+\nu$ has a bounded $H^\infty(\Sigma_\sigma)$-calculus in X for all sufficiently large $\nu > 0$.

Proof. By Proposition 16.2.3, for large enough $\nu > 0$ the operator $A + B + \nu$ is sectorial and $\omega(A+B+\nu) \leqslant \omega(A)$. By taking ν larger if necessary, we may assume that $0 \in \varrho(A + B + \nu)$.

We start from the identity

$$\begin{aligned} R(\lambda, A+B+\nu) &= R(\lambda, A+\nu) + R(\lambda, A+B+\nu)BR(\lambda, A+\nu) \\ &= R(\lambda, A+\nu) + M(\lambda), \end{aligned}$$

which may be verified by applying $\lambda - (A + B + \nu)$ on both sides, and where

$$M(\lambda) = R(\lambda, A+B+\nu)[B(A+\nu)^{-\alpha}](A+\nu)^\alpha R(\lambda, A+\nu).$$

For functions $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ this gives the Dunford integral

$$f(A+B) = f(A) + \frac{1}{2\pi i}\int_{\Gamma_\eta} f(\lambda)M(\lambda)\,\mathrm{d}\lambda,$$

where the contour $\Gamma_\eta = \partial\Sigma_\eta$ with $\omega(A) < \eta < \sigma$ is chosen as usual. Near the origin, the integrand is bounded since we assumed that $0 \in \varrho(A+B+\nu)$. For large values of $|\lambda|$ the integrand may be estimated pointwise by

$$\|f(\lambda)M(\lambda)\| \leqslant M_1 M_2 |\lambda|^{-\alpha} \|B(A+\nu)^{-\alpha}\| \|f\|_{H^\infty(\Sigma_\sigma)},$$

since

$$M_1 := \sup\{\|\lambda R(\lambda, A+B+\nu)\| : |\arg\lambda| = \eta\}$$

is finite by sectoriality of $A + B + \nu$ and

$$M_2 := \sup\{\|\lambda^{1-\alpha}(A+\nu)^\alpha R(\lambda, A+\nu)\| : |\arg\lambda| = \eta\}$$

is finite by sectoriality of $A + \nu$ and Corollary 15.2.14. It follows that the integral converges absolutely and its norm is bounded by a constant times $\|f\|_{H^\infty(\Sigma_\sigma)}$. This completes the proof. □

Our main perturbation theorem asserts that the H^∞-calculus of an R-sectorial operator is preserved under relatively bounded perturbations of the H^∞-calculus if we add an additional relative boundedness assumption in the fractional domains scale.

Theorem 16.2.8 (Relatively bounded perturbations of the H^∞-calculus). *Let A be a densely defined sectorial operator with a bounded $H^\infty(\Sigma_\sigma)$-calculus and let B be a densely defined τ-R-sectorial operator on X, with*

$$\mathsf{D}(A) \subseteq \mathsf{D}(B) \quad \textit{and} \quad 0 \in \varrho(A)$$

and satisfying the relative bound

(i) $\|Bx\| \leqslant C_0\|Ax\|$ *for all* $x \in \mathsf{D}(A)$.

Suppose that at least one of the following two additional relative bounds is also satisfied:

(ii) *there exists an* $\alpha \in (0,1)$ *such that* B *maps* $\mathsf{D}(A^{1+\alpha})$ *into* $\mathsf{D}(A^\alpha)$ *and*

$$\|A^\alpha Bx\| \leqslant C_1\|A^{1+\alpha}x\|, \quad x \in \mathsf{D}(A^{1+\alpha});$$

(iii) *there exists an* $\alpha \in (0,1)$ *such that*

$$\|A^{-\alpha} Bx\| \leqslant C_1\|A^{1-\alpha}x\|, \quad x \in \mathsf{D}(A^{1-\alpha}).$$

Then, given the constant C_1 in (ii) or (iii), there is a small enough constant $C \geqslant 0$ so that if (i) holds with $0 \leqslant C_0 \leqslant C$, then $A+B$ has a bounded $H^\infty(\Sigma_{\sigma\vee\tau})$-calculus.

If in (ii) or (iii) we have $C_1 < 1/\widetilde{M}_{\sigma\vee\tau,A}$, then the condition $0 \in \varrho(A)$ may be replaced by the weaker condition that A be injective and B maps $\mathsf{D}(A^{1-\alpha})$ into $\mathsf{D}(A^{-\alpha})$.

In the last line of the statement of the theorem, recall the notation $\widetilde{M}_{\theta,A} = \sup\{\|AR(\lambda,A)\| : \lambda \neq 0, \|\arg(\lambda)| \geqslant \theta\}$.

If X has the triangular contraction property, in particular if X is a UMD space, then by Theorem 10.3.4 we have $\omega_R(A) \leqslant \omega_{H^\infty}(A)$ and therefore the theorem applies.

At the end of the section, an example will be presented which shows that the additional assumptions (ii) and (iii) cannot be omitted.

We will reduce the theorem to the following technical lemma.

Lemma 16.2.9. *Let A be a densely defined sectorial operator with a bounded $H^\infty(\Sigma_\sigma)$-calculus and let B a densely defined R-sectorial operator on X. Let $\omega(A) < \sigma < \pi$ and $\omega_R(B) < \tau < \pi$, and set $\mu := \max\{\sigma,\tau\}$. Suppose there exists a holomorphic function $M : \{|\arg(\lambda)| > \mu\} \to \mathscr{L}(X)$ with R-bounded range and a $\beta \in (0,1)$ such that*

$$R(\lambda,B) = R(\lambda,A) + A^\beta R(\lambda,A)M(\lambda)A^{1-\beta}R(\lambda,A), \quad |\arg(\lambda)| \geqslant \mu. \quad (16.10)$$

Then B has a bounded $H^\infty(\Sigma_\mu)$-calculus.

Proof. Our aim is to prove that there exists a function $\phi \in H^1(\Sigma_\mu)$ and a constant $C \geqslant 0$ such that for all integers $N \geqslant 1$, all scalars $\epsilon_{-N}, \dots, \epsilon_N$ of modulus one, and all $t > 0$ we have

$$\Big\| \sum_{|n| \leqslant N} \epsilon_n \phi(t2^n B) \Big\| \leqslant C.$$

Once we have this, it follows from Proposition 10.4.11 (and tracking angles in its proof) that B has a bounded $H^\infty(\Sigma_\mu)$-calculus.

Let $\mu < \nu < \pi$ and consider the function $\psi_\nu \in H^1(\Sigma_\mu)$ given by

$$\psi_\nu(z) = \frac{z^{1/2}}{(e^{i\nu} - z)^{1/2}(2e^{i\nu} - z)^{1/2}}, \quad z \in \Sigma_\mu,$$

so that $\phi_\nu := \psi_\nu^2$ satisfies

$$\phi_\nu(z) = \frac{z}{(e^{i\nu} - z)(2e^{i\nu} - z)} = \frac{1}{e^{i\nu} - z} - \frac{2}{2e^{i\nu} - z}.$$

By (16.10),

$$\begin{aligned} R(\lambda, t2^n B) &= t^{-1}2^{-n} R(t^{-1}2^{-n}\lambda, B) \\ &= t^{-1}2^{-n} R(t^{-1}2^{-n}\lambda, A) \\ &\qquad + t^{-1}2^{-n} A^\beta R(t^{-1}2^{-n}\lambda, A) M(t^{-1}2^{-n}\lambda) A^{1-\beta} R(t^{-1}2^{-n}\lambda, A) \\ &= R(\lambda, t2^n A) \\ &\qquad + t2^n A^\beta R(\lambda, t2^n A) M(t^{-1}2^{-n}\lambda) A^{1-\beta} R(\lambda, t2^n A). \end{aligned}$$

By Corollary 15.2.14, the right-hand side has decay of order $|\lambda|^{-1}$ as $|\lambda| \to \infty$ in the complement of Σ_μ. Hence, by Cauchy's theorem and taking $\mu < \tau < \nu$,

$$\begin{aligned} \phi_\nu(t2^n B) &= \frac{1}{2\pi i} \int_{\partial \Sigma_\tau} \phi_\nu(\lambda) R(\lambda, t2^n B) \, \mathrm{d}\lambda \\ &= \phi_\nu(t2^n A) + t2^n A^\beta R(e^{i\nu}, t2^n A) M(t^{-1}2^{-n} e^{i\nu}) A^{1-\beta} R(e^{i\nu}, t2^n A) \\ &\quad - t2^{n+1} A^\beta R(2e^{i\nu}, t2^n A) M(t^{-1}2^{-n} 2e^{i\nu}) A^{1-\beta} R(2e^{i\nu}, t2^n A) \\ &= \phi_\nu(t2^n A) + t2^n A^\beta R(e^{i\nu}, t2^n A) M(t^{-1}2^{-n} e^{i\nu}) A^{1-\beta} R(e^{i\nu}, t2^n A) \\ &\quad - t2^{n-1} A^\beta R(e^{i\nu}, t2^{n-1} A) M(t^{-1}2^{-(n-1)} e^{i\nu}) A^{1-\beta} R(e^{i\nu}, t2^{n-1} A) \\ &= \phi_\nu(t2^n A) + \phi_{\beta,\nu}(t2^n A) M(t^{-1}2^{-n} e^{i\nu}) \phi_{1-\beta,\nu}(t2^n A) \\ &\quad - \phi_{\beta,\nu}(t2^{n-1} A) M(t^{-1}2^{-(n-1)} e^{i\nu}) \phi_{1-\beta,\nu}(t2^{n-1} A) \\ &= (I) + (II) + (III), \end{aligned} \tag{16.11}$$

where for $\alpha > 0$ we define $\phi_{\alpha,\nu} \in H^1(\Sigma_\mu)$ by

$$\phi_{\alpha,\nu}(z) := \frac{z^\alpha}{e^{i\nu} - z}.$$

In the penultimate identity of (16.11) we used the identity

$$\phi_{\alpha,\nu}(\tau A) = \tau^\alpha A^\alpha R(e^{i\nu}, \tau A),$$

which follows from Propositions 15.1.12 and 15.2.6.

We estimate the terms (I)–(III) separately. We begin with (II). Fixing $x \in X$, by randomisation with a Rademacher sequence $(\varepsilon_n)_{n\in\mathbb{Z}}$,

$$\begin{aligned}
&\Big\| \sum_{|n|\leqslant N} \epsilon_n \phi_{\beta,\nu}(t2^n A) M(t^{-1}2^{-n}e^{i\nu})\phi_{1-\beta,\nu}(t2^n A)x\Big\| \\
&= \sup_{\|x^*\|\leqslant 1} \Big| \sum_{|n|\leqslant N} \epsilon_n \langle M(t^{-1}2^{-n}e^{i\nu})\phi_{1-\beta,\nu}(t2^n A)x, \phi_{\beta,\nu}(t2^n A)^* x^*\rangle \Big| \\
&= \sup_{\|x^*\|\leqslant 1} \Big| \mathbb{E}\Big\langle \sum_{|n|\leqslant N} \epsilon_n\varepsilon_n M(t^{-1}2^{-n}e^{i\nu})\phi_{1-\beta,\nu}(t2^n A)x, \sum_{|n|\leqslant N} \varepsilon_n \phi_{\beta,\nu}(t2^n A^*)x^* \Big\rangle \Big| \\
&\leqslant \mathbb{E}\Big\| \sum_{|n|\leqslant N} \epsilon_n\varepsilon_n M(t^{-1}2^{-n}e^{i\nu})\phi_{1-\beta,\nu}(t2^n A)x\Big\| \\
&\qquad\qquad \times \sup_{\|x^*\|\leqslant 1} \mathbb{E}\Big\| \sum_{|n|\leqslant N} \varepsilon_n \phi_{\beta,\nu}(t2^n A^*)x^*\Big\| \\
&\lesssim_M \mathbb{E}\Big\| \sum_{|n|\leqslant N} \varepsilon_n \phi_{1-\beta,\nu}(t2^n A)x\Big\| \sup_{\|x^*\|\leqslant 1} \mathbb{E}\Big\| \sum_{|n|\leqslant N} \varepsilon_n \phi_{\beta,\nu}(t2^n A^*)x^*\Big\|,
\end{aligned}$$

where the implicit constant in the last step is the R-boundedness constant of M. Similarly, shifting the index by one and using the contraction principle, we estimate (III) as follows:

$$\begin{aligned}
&\Big\| \sum_{|n|\leqslant N} \epsilon_n \phi_{\beta,\nu}(t2^n A) M(t^{-1}2^{-n}e^{i\nu})\phi_{1-\beta,\nu}(t2^n A)x\Big\| \\
&\lesssim_M \mathbb{E}\Big\| \sum_{|n|\leqslant N} \varepsilon_n \phi_{1-\beta,\nu}(t2^{n-1} A)x\Big\| \sup_{\|x^*\|\leqslant 1} \Big\| \sum_{|n|\leqslant N} \varepsilon_n \phi_{\beta,\nu}(t2^{n-1} A^*)x^*\Big\| \\
&\leqslant \mathbb{E}\Big\| \sum_{|n|\leqslant N+1} \varepsilon_n \phi_{1-\beta,\nu}(t2^n A)x\Big\| \sup_{\|x^*\|\leqslant 1} \mathbb{E}\Big\| \sum_{|n|\leqslant N+1} \varepsilon_n \phi_{\beta,\nu}(t2^n A^*)x^*\Big\|.
\end{aligned}$$

By the same argument, for (I) we obtain

$$\Big\| \sum_{|n|\leqslant N} \epsilon_n \phi_\nu(t2^n A)x\Big\| \leqslant \mathbb{E}\Big\| \sum_{|n|\leqslant N} \varepsilon_n \psi_\nu(t2^n A)x\Big\|.$$

Taking the supremum over $N \geqslant 1$ and $t > 0$, this proves the square function bound

$$\sup_{|n|\geqslant N}\sup_{t>0}\Big\|\sum_{|n|\leqslant N}\epsilon_n\phi_\nu(t2^nB)x\Big\| \leqslant C\|x\| + 2C'\sup_{t>0}\|x\|_{\phi_{1-\beta},A}\sup_{\|x^*\|\leqslant 1}\|x^*\|_{\phi_\beta,A^*}$$
$$\leqslant C''\|x\|,$$

where the estimate in the last step follows from the boundedness of the $H^\infty(\Sigma_\sigma)$-calculus of A through Theorem 10.4.4. □

Proof of Theorem 16.2.8. By the second part of Theorem 16.2.4, assumption (i) implies that $A + B$ is σ-R-sectorial operator provided the smallness condition on C_0 in (i) holds. Moreover, if we impose $C_0 < 1$, then for all $x \in \mathsf{D}(A + B) = \mathsf{D}(A)$ we have $\|Ax\| \leqslant \|(A + B)x\| + \|Bx\| \leqslant \|(A+B)x\| + C_0\|Ax\|$ and therefore $\|Ax\| \leqslant (1 - C_0)^{-1}\|(A + B)x\|$, while at the same time $\|(A+B)x\| \leqslant \|Ax\| + \|Bx\| \leqslant (1 + C_0)\|Ax$. We conclude that

$$\|Ax\| \eqsim_{C_0} \|(A + B)x\|, \quad x \in \mathsf{D}(A + B) = \mathsf{D}(A).$$

Furthermore, for $\lambda \in \complement\overline{\Sigma_{\sigma\vee\tau}}$ we have $\lambda \in \varrho(A+B)$ and the resolvent operator is represented by the perturbation formula of Proposition 16.2.1,

$$R(\lambda, A + B) = R(\lambda, A)\sum_{n=0}^{\infty}[BR(\lambda, A)]^n, \quad |\arg\lambda| > \sigma\vee\tau, \tag{16.12}$$

again provided C_0 is small enough, for then $\|BR(\lambda, A)\| \leqslant C_0\|AR(\lambda, A)\| \leqslant C_0\|\lambda R(\lambda, A) - I\| \leqslant C_0(1 + M_{\sigma\vee\tau,A}) < 1$ and the series converges absolutely.

First we assume that (i) and (iii) hold. For the time being, we do not assume that $0 \in \varrho(A)$ (in which case A^{-1} is bounded by Corollary 15.2.10), but only assume that A is invertible and B maps $\mathsf{D}(A^{1-\alpha})$ into $\mathsf{D}(A^{-\alpha})$. Then $U := A^{-\alpha}BA^{\alpha-1}$ is bounded on X of norm $\|U\| = C_1$ and we have

$$R(\lambda, A)BR(\lambda, A) = R(\lambda, A)A^\alpha UA^{1-\alpha}R(\lambda, A).$$

If $C_1 < \widetilde{M}_{\sigma\vee\tau,A}^{-1}$, then the sum

$$M(\lambda) := \sum_{k\geqslant 0}[UAR(\lambda, A)]^kU$$

converges in operator norm and defines a holomorphic function for $|\arg\lambda| > \sigma\vee\tau$. We then can rewrite (16.12) in the form

$$R(\lambda, A + B) = R(\lambda, A) + A^\alpha R(\lambda, A)\sum_{k\geqslant 0}[UAR(\lambda, A)]^kUA^{1-\alpha}R(\lambda, A)$$
$$= R(\lambda, A) + A^\alpha R(\lambda, A)M(\lambda)A^{1-\alpha}R(\lambda, A).$$

By the R-sectoriality of A and Proposition 8.1.24, the set $\{M(\lambda) : |\arg(\lambda)| > \sigma\}$ is R-bounded. Thus we derived the representation required in Lemma 16.2.9 and we can conclude that $A+B$ also has a bounded $H^\infty(\Sigma_{\sigma\vee\tau}$-calculus.

It remains to prove that the smallness assumption $C_1 < \widetilde{M}^{-1}_{\sigma\vee\tau,A}$ can be removed if $0 \in \varrho(A)$. Under this assumption, let (i) and (iii) hold, but not necessarily the smallness condition $C_1 < \widetilde{M}^{-1}_{\sigma\vee\tau,A}$.

Using the scale of homogeneous fractional domain spaces $X_\alpha := \mathsf{D}(A^\alpha)$ with norms $\|x\|_{X_\alpha} := \|A^\alpha x\|$ (recall that we are assuming $0 \in \varrho(A)$) we can restate our assumptions as stating that B extends to a bounded operator from X_1 to X and from $X_{1-\alpha}$ to $X_{-\alpha}$, with norm at most C_0 and C_1 respectively. By complex interpolation B acts as a bounded operator $[X_1, X_{1-\alpha}]_\theta$ to $[X, X_{-\alpha}]_\theta$ with norm $\leqslant C_0^{1-\theta} C_1^\theta$, $0 < \theta < 1$.

Since A has a bounded H^∞-calculus and therefore bounded imaginary powers, by Corollary 15.3.10 we have

$$[X_1, X_{1-\alpha}]_\theta = X_{1-\theta\alpha}, \quad [X, X_{-\alpha}]_\theta = X_{-\theta\alpha}$$

with equivalent norms, with equivalence constants which may be chosen independent of $\theta \in (0,1)$. Thus we obtain that B acts as a bounded operator from $X_{1-\theta\alpha}$ to $X_{-\theta\alpha}$ with norm $\lesssim C_0^{1-\theta} C_1^\theta$, $0 < \theta < 1$.

We can choose θ so small that B satisfies (iii) for $\alpha' = \theta\alpha$ with $C_1' < \widetilde{M}^{-1}_{\sigma\vee\tau,A}$ no matter how big C_1 was. This completes the proof of the case (iii).

Finally assume that (ii) holds for some $\alpha \in (0,1)$. By Proposition 15.1.12 we have $A^{\alpha-1} \subseteq A^\alpha A^{-1}$ and $A^{1+\alpha} \subseteq A^\alpha A$ (in fact we have equality in the second case by Theorem 15.2.5), and therefore

$$\|A^{\alpha-1} Bx\| = \|A^\alpha A^{-1} Bx\| \leqslant C_1 \|A^{1+\alpha} A^{-1} Bx\| = C_1 \|A^\alpha Bx\|$$

implies that (iii) holds for the exponent $1-\alpha \in (0,1)$ and $x \in \mathsf{D}(A^{1+\alpha})$. Since $\mathsf{D}(A^{1+\alpha})$ is dense in $\mathsf{D}(A^\alpha)$ by Proposition 15.1.13, (iii) holds for the exponent $1-\alpha \in (0,1)$ and $x \in \mathsf{D}(A^\alpha)$. □

We conclude this section with an example, due to McIntosh and Yagi, shows that boundedness of the H^∞-calculus is not preserved by small relatively bounded perturbations even when X is a Hilbert space. This shows that the additional assumptions (ii) or (iii) in Theorem 16.2.7 cannot be omitted, no matter how small the constant on (i) is chosen.

Example 16.2.10. We construct a bisectorial operator A on Hilbert space H admitting a bounded bisectorial H^∞-calculus with $\omega^{\mathrm{bi}}_{H^\infty}(A) = 0$, such that for any given $\varepsilon > 0$, an operator B_ε on H exists which is relatively bounded with respect to A, with $\|B_\varepsilon x\| \leqslant \varepsilon \|Ax\|$, and such that $A + B_\varepsilon$ fails to have a bounded bisectorial H^∞-calculus. This operator moreover satisfies $(A + B_\varepsilon)^2 = A^2 + C_\varepsilon$, where C is relatively bounded with respect to A^2, with $\|C_\varepsilon x\| \leqslant 2\varepsilon \|A^2 x\|$.

By the first part of Theorem 10.6.7, the operator A^2 has a bounded H^∞-calculus with $\omega_{H^\infty}(A) = 0$. If $A^2 + C_\varepsilon = (A + B_\varepsilon)^2$ had a bounded H^∞-calculus, then by the second part of Theorem 10.6.7 $A + B_\varepsilon$ would have a bounded bisectorial H^∞-calculus, and this is not the case. We conclude that $A^2 + C_\varepsilon$ does not have a bounded H^∞-calculus.

Let us proceed to the construction of the operators A and B_ε. Fix $\varepsilon > 0$. Omitting subscripts ε in what follows, for $n = 1, 2, \dots$ we will construct bounded operators A_n and B_n on a finite-dimensional H_n with the following properties for any $0 < \sigma < \frac{1}{2}\pi$:

- A_n and $A_n + B_n$ are σ-bisectorial with $\|B_n\| \leqslant \varepsilon\|A_n\|$;
- A_n^2 and $(A_n + B_n)^2 = A_n^2 + C_n$ with $\|C_n\| \leqslant 2\varepsilon\|A_n^2\|$;
- the spectra of A_n and $A_n + B_n$ is contained in $(-\infty, 1] \cup [1, \infty)$;
- the resolvents of A_n and $A_n + B_n$ satisfy
$$\|R(\lambda, A_n)\| \leqslant 1/\Im(\lambda), \quad \|R(\lambda, A_n + B_n)\| \leqslant (1+\varepsilon)/\Im(\lambda),$$
for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$;
- A_n and $A_n + B_n$ have contractive, respectively bounded, $H^\infty(\Sigma_\sigma^\pm)$-calculi;
- the spectral projections $\mathbf{1}_{\Sigma_\sigma^\pm}(A_n + B_n)$ have norm $\geqslant n$.

The counterexample with the stated properties is obtained by taking

$$H = \bigoplus_{n\geqslant 1} H_n, \quad A := \bigoplus_{n\geqslant 1} T_n, \quad B := \bigoplus_{n\geqslant 1} B_n, \quad C := \bigoplus_{n\geqslant 1} C_n.$$

The operator A has a contractive $H^\infty(\Sigma_\sigma^\pm)$-calculus. Furthermore, the inequalities $\|B_n\| \leqslant \varepsilon\|A_n\|$ imply that $\mathsf{D}(A) \subseteq \mathsf{D}(B)$ and B is relatively bounded with respect to A, with relative bound $\leqslant \varepsilon$. The operator $A + B$ with domain $\mathsf{D}(A + B) = \mathsf{D}(A)$ doesn't have a bounded $H^\infty(\Sigma_\sigma^{\rm bi})$-calculus: for if it had, then the associated spectral projections would be bounded; but if they were, then their restrictions to H_n would be uniformly bounded in n; but these restrictions have norm $\geqslant n$.

We now turn to the details of the construction. Choose $N_n \geqslant 1$ so large that $\frac{2\varepsilon}{3\pi}\log(N_n + 1)$. On $\mathbb{C}^{N_n+1}$ consider the matrices $T_n = (t_{jk}^{(n)})_{j,k=0}^{N_n}$ and $S_n = (s_{jk}^{(n)})_{j,k=0}^{N_n}$ given by

$$t_{jk}^{(n)} = 2^j\delta_{jk}, \qquad s_{jk}^{(n)} := \frac{\varepsilon}{\pi(k-j)}\delta_{j\neq k}.$$

Then T_n is self-adjoint and S_nT_n is skew-adjoint. The self-adjoint matrix iS_n is the $N_n \times N_n$ Toeplitz matrix with *generating function* $\varepsilon\theta/\pi$, $\theta \in (-\pi, \pi)$, that is, we have

$$s_{jk} = \widehat{f}_{j-k}, \quad j, k = 0, \dots, N_n.$$

Since the norm of a Toeplitz matrix with bounded real-valued generating function f is bounded by $\|f\|_{L^\infty(\mathbb{T})}$, we see that $\|S_n\| \leqslant \varepsilon$.

The matrix $Z_n = (z_{jk}^{(n)})_{j,k=0}^{N_n}$ given by

$$z_{jk}^{(n)} = \frac{2^k\varepsilon}{\pi(k-j)(2^j + 2^k)}\delta_{j\neq k}$$

has norm

$$\begin{aligned}\|Z_n\| \geqslant \|Ze_{N_n}\| &= \frac{2^{N_n}\varepsilon}{\pi}\sum_{j=0}^{N_n-1}\frac{1}{(N_n-j)(2^j+2^{N_n})}\\ &\geqslant \frac{\varepsilon}{\pi}\frac{2^{N_n}}{(2^{N_n-1}+2^{N_n})}\Big(\frac{1}{N_n}+\frac{1}{N_n-1}+\dots+1\Big)\\ &\geqslant \frac{2\varepsilon}{3\pi}\log(N_n+1)\geqslant n,\end{aligned} \tag{16.13}$$

where $(e_n)_{n=0}^{N_n}$ denote the standard unit vectors in $\mathbb{C}^{N_n+1}$, and it satisfies

$$T_nZ_n + Z_nT_n = S_nT_n. \tag{16.14}$$

On $H_n := C^{N_n+1}\times C^{N_n+1}$ define the operators

$$A_n := \begin{bmatrix} T_n & 0\\ 0 & -T_n\end{bmatrix}, \quad B_n := \begin{bmatrix} 0 & S_nT_n\\ 0 & 0\end{bmatrix}, \quad P_n^+ := \begin{bmatrix} I & Z_n\\ 0 & 0\end{bmatrix}, \quad P_n^- := \begin{bmatrix} 0 & -Z_n\\ 0 & I\end{bmatrix}.$$

One checks that

$$B_n = \begin{bmatrix} 0 & S_n\\ 0 & 0\end{bmatrix}\begin{bmatrix} T_n & 0\\ 0 & -T_n\end{bmatrix} = \begin{bmatrix} 0 & S_n\\ 0 & 0\end{bmatrix} A_n,$$

so

$$\|B_n\| \leqslant \|S_n\|\|A_n\| \leqslant \varepsilon\|A_n\|.$$

Also, using that $S_nT_n = -(T_nS_n)^*$, we have

$$(A_n+B_n)^2 = \begin{bmatrix} T_n^2 & T_nS_nT_n - S_nT_n^2\\ 0 & T_n^2\end{bmatrix} = A_n^2 + \begin{bmatrix} 0 & T_nS_nT_n - S_nT_n^2\\ 0 & 0\end{bmatrix} =: A_n^2 + C_n$$

with

$$\|C_n\| \leqslant \|T_nS_nT_n - S_nT_n^2\| \leqslant 2\|S_n\|\|T_n\|^2 \leqslant 2\varepsilon\|A_n^2\|,$$

where we used that $T_n^* = -T_n$, so T_n is normal and therefore $\|T_n\|^2 = \|T^2\|$.

Furthermore, one checks that $\sigma(A_n) = \sigma(A_n+B_n)$ and

$$R(\lambda, A_n+B_n) = \begin{bmatrix} R(\lambda,T_n) & R(\lambda,T_n)S_nT_nR(\lambda,-T_n)\\ 0 & R(\lambda,-T_n)\end{bmatrix} \tag{16.15}$$

for all $\lambda\in\varrho(A_n) = \varrho(A_n+B_n)$. In particular,

$$\begin{aligned}\sigma(A_n+B_n) &= \sigma(A_n)\\ &= \sigma(T_n)\cup\sigma(-T_n) = \{1,2,4,\dots,2^{N_n}\}\cup\{-1,-2,-4,\dots,-2^{N_n}\}\end{aligned}$$

By self-adjointness, for $\lambda\notin\mathbb{R}$ we have $\|A_n\| \leqslant |\Im(\lambda)|^{-1}$, so A_n is σ-bisectorial for all $0<\sigma<\frac{1}{2}\pi$. By (16.15), for $\lambda\notin\mathbb{R}$ we have $\lambda\in\varrho(A_n+B_n)$, and for $\lambda\notin\overline{\Sigma_\sigma}$

$$\|R(\lambda, A_n + B_n)\| \leqslant |\Im(\lambda)|^{-1} + \|R(\lambda, T_n)S_nT_nR(\lambda, T_n)\| \lesssim_\sigma (1+\varepsilon)|\Im(\lambda)|^{-1}.$$

It follows that $A_n + B_n$ is σ-bisectorial for all $0 < \sigma < \frac{1}{2}\pi$.

The operators P_n^+ and P_n^- are projections,

$$P_n^+ + P_n^- = I, \quad P_n^+ P_n^- = P_n^- P_n^+ = 0,$$

and by (16.13) their norms satisfy

$$\|P_n^+\| \geqslant \|Z_n\| \geqslant n, \quad \|P_n^-\| \geqslant \|Z_n\| \geqslant n.$$

To complete the construction we will show that

$$P_n^\pm = \mathbf{1}_{\Sigma_\sigma^\pm}(A_n + B_n).$$

Indeed, using (16.14) and (16.15), for $0 < \nu < \sigma$ we formally compute

$$\begin{aligned}
&\mathbf{1}_{\Sigma_\sigma^\pm}(A_n + B_n) \\
&\quad = \frac{1}{2\pi i}\int_{\partial\Sigma_\nu^\pm} R(z, A_n + B_n)\,\mathrm{d}z \\
&\quad = \frac{1}{2\pi i}\int_{\partial\Sigma_\nu^\pm} \begin{bmatrix} R(z,T_n) & R(z,T_n)S_nT_nR(z,-T_n) \\ 0 & R(z,-T_n)\end{bmatrix}\mathrm{d}z \\
&\quad = \frac{1}{2\pi i}\int_{\partial\Sigma_\nu^\pm} \begin{bmatrix} R(z,T_n) & R(z,T_n)(T_nZ_n + Z_nT_n)R(z,-T_n) \\ 0 & R(z,-T_n)\end{bmatrix}\mathrm{d}z \\
&\quad = \frac{1}{2\pi i}\int_{\partial\Sigma_\nu^\pm} \begin{bmatrix} R(z,T_n) & R(z,T_n)Z_n + Z_nR(z,-T_n) \\ 0 & R(z,-T_n)x\end{bmatrix}\mathrm{d}z \\
&\quad \overset{(*)}{=} \begin{bmatrix} I & Z_n \\ 0 & 0\end{bmatrix} = P_n^+,
\end{aligned}$$

where $(*)$ is a consequence of Cauchy's theorem, which gives

$$\frac{1}{2\pi i}\int_{\partial\Sigma_\nu^\pm} R(z,T_n)\,\mathrm{d}z = I, \quad \frac{1}{2\pi i}\int_{\partial\Sigma_\nu^\pm} R(z,-T_n)\,\mathrm{d}z = 0,$$

noting that $\sigma(T_n) = \{1, 2, 4, \dots, 2^{N_n}\}$ is contained in Σ_σ^+. To make the computation rigorous, one brings in additional terms $\zeta_k(T_n)$, where $\zeta_k(z) = \frac{k}{k+z} - \frac{1}{1+kz}$ as in Proposition 10.2.6, to be able to work with the Dunford calculus for functions in $H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ throughout; one passes to the limit $k \to \infty$ at the end. The proof that $\mathbf{1}_{\Sigma_\sigma^\pm}(A_n + B_n) = P_n^-$ is entirely similar.

16.3 Sum-of-operator theorems

The perturbations B studied in Section 16.2 have the property that $\mathsf{D}(B)$ is contained in $\mathsf{D}(A)$, so that the sum the sum $A + B$ may be defined unambiguously by the prescription $(A + B)x := Ax + Bx$. In all these cases, B is "small" in comparison with A. Under a resolvent commutation assumption, in the present section we treat A and B on a more equal footing.

We begin with a general result (Theorem 16.3.2) which says that the sum $A+B$ of two resolvent commuting sectorial operators A and B satisfying $\omega(A)+\omega(B)<\pi$ always has a sectorial extension, and that this extension is the closure of $A+B$ if both A and B are densely defined. In applications to maximal regularity of solution of evolution equations – the topic of the last two chapters of this book – more is needed, namely, that $A+B$ is closed and the following inequality holds:

$$\|Ax\|+\|Bx\|\leqslant C\|(A+B)x\|,\quad x\in\mathsf{D}(A)\cap\mathsf{D}(B) \tag{16.16}$$

with a constant C independent of $x\in\mathsf{D}(A)\cap\mathsf{D}(B)$. For later use we record the simple fact that this inequality in fact implies closedness:

Proposition 16.3.1. *If A and B are closed operators satisfying* (16.16)*, then the operator $A+B$ with its natural domain $\mathsf{D}(A+B)=\mathsf{D}(A)+\mathsf{D}(B)$ is closed.*

Proof. The proof is immediate: if $x_n\to x$ and $(A+B)x_n\to y$, then (16.16) implies that the sequences $(Ax_n)_{n\geqslant 1}$ and $(Bx_n)_{n\geqslant 1}$ are Cauchy. The closedness of A and B implies that $x\in\mathsf{D}(A)\cap\mathsf{D}(B)$ and $y=\lim_{n\to\infty}(A+B)x_n=Ax+Bx=\lim_{n\to\infty}Ax_n+\lim_{n\to\infty}Bx_n=Ax+Bx=(A+B)x$. □

As it turns out, the inequality (16.16) is rather delicate, and it only holds under additional assumptions on A, B, and X. We have already encountered one such situation: the Dore–Venni theorem (Theorem 15.4.11), which assumes that A and B resolvent commute and have bounded imaginary powers, with $\omega_{\rm BIP}(A)+\omega_{\rm BIP}(B)<\pi$, and the underlying Banach space X is a UMD space. In applications, however, one if often confronted with the situation where one of the operator is only (R-)sectorial, whilst the other operator has better properties such as a bounded H^∞-calculus. In the present section, for resolvent commuting sectorial operators A and B acting in a Banach X we will prove the following results:

- If A and B are densely defined, A has a bounded H^∞-calculus and B is R-sectorial, and if $\omega_{H^\infty}(A)+\omega_R(B)<\pi$, then $A+B$ is densely defined and sectorial, with

  $$\omega(A+B)\leqslant\max\{\omega_{H^\infty}(A),\omega_R(B)\}$$

  and the reverse triangle inequality (16.16) holds. If in addition X has the triangular contraction property, then $A+B$ is R-sectorial and

  $$\omega_R(A+B)\leqslant\max\{\omega_{H^\infty}(A),\omega_R(B)\}.$$

  (Theorem 16.3.6).

- If A and B are densely defined and have bounded H^∞-calculi with $\omega_{H^\infty}(A) + \omega_{H^\infty}(B) < \pi$ and X has Pisier's contraction property, then $A + B$ has a bounded H^∞-calculus with

$$\omega_{H^\infty}(A + B) \leqslant \max\{\omega_{H^\infty}(A), \omega_{H^\infty}(B)\}$$

and the reverse triangle inequality (16.16) holds (Theorem 16.3.10).
- If A has an absolute calculus with

$$\omega_{\mathrm{abs}}(A) + \omega(B) < \pi,$$

then the reverse triangle inequality (16.16) holds (Theorem 16.3.14). The same conclusion holds if X is a Hilbert space, A has bounded imaginary powers and B is densely defined, and $\omega_{\mathrm{BIP}}(A) + \omega(B) < \pi$ (Theorem 16.3.15).

To conclude this section we provide an example of the type of applications that will be studied in depth in the next two chapters and which indeed have motivated the development of the abstract approach to sums of operators presented here.

Suppose that $-A$ generates a C_0-semigroup on a Banach space X and consider the inhomogeneous abstract Cauchy problem

$$\begin{cases} u'(t) + Au(t) = f(t), & t \in [0, T], \\ u(0) = 0. \end{cases} \tag{ACP}$$

As we will explain in the next chapter, a thorough understanding of this problem is of paramount importance to the study of more general classes of nonlinear, possibly time-dependent, evolution equations. In order to connect (ACP) with operator sums we consider the weak derivative

$$Du := u'$$

viewed as a closed operator on $L^p(0, T; X)$ (with $1 \leqslant p \leqslant \infty$) with domain

$$\mathsf{D}(D) := {}_0W^{1,p}(0, T; X) = \big\{u \in W^{1,p}(0, T; X) :\ u(0) = 0\big\}$$

It will be checked in the next chapter (see Section 17.3.c) that this operator is sectorial of angle $\frac{1}{2}\pi$. Using this operator, we can rewrite (ACP) as the abstract operator equation

$$(D + \widetilde{A})u = f$$

in $L^p(0, T; X)$, where $\widetilde{A}$ is the natural extension of A to a closed operator acting in $\widetilde{X} := L^p(0, T; X)$, defined on $\mathsf{D}(\widetilde{A}) := L^p(0, T; \mathsf{D}(A))$ by

$$(\widetilde{A}f)(t) := A(f(t)), \quad t \in (0, T).$$

In the next chapter (see Propositions 17.3.14 and 17.3.15) we prove that the following assertions are equivalent:

(1) the inverse triangle inequality (16.16) holds, i.e., there is a constant $C \geqslant 0$ such that

$$\|\widetilde{A}u\|_p + \|Du\|_p \leqslant C\|(\widetilde{A} + D)u\|_p, \quad u \in \mathsf{D}(\widetilde{A}) \cap \mathsf{D}(D);$$

(2) $\widetilde{A} + D$ is closed;
(3) $\widetilde{A} + D$ boundedly invertible;
(4) A has maximal L^p-regularity on $(0, T)$.

For the problem (ACP), *maximal L^p-regularity* means that the unique mild solution of the problem, which is given in terms of the semigroup S generated by $-A$ as

$$u(t) = \int_0^t S(t-s)f(s)\,\mathrm{d}s$$

belongs to $L^p(0, T; \mathsf{D}(A)) \cap {}_0W^{1,p}(0, T; X) = \mathsf{D}(\widetilde{A}) \cap \mathsf{D}(D)$. As we will see in the next chapter, the bounded invertibility of $\widetilde{A} + D$ corresponds to the existence and uniqueness of mild solutions for (ACP). Maximal L^p-regularity will be studied in depth in the next chapter, where also a version of the above equivalences with $(0, T)$ replaced by $\mathbb{R}_+$ will be proved.

16.3.a The sum of two sectorial operators

We begin with a general result about sums of resolvent commuting operators. It is not quite as useful as the deeper sums-of-operator theorems proved in the next sections, but its virtue lies in the generality of its assumptions, namely, it is only required that A and B are sectorial with $\omega(A) + \omega(B) < \pi$. The price to be paid is that we do not obtain sectoriality, or even closedness, of $A + B$, but only the weaker result that $A + B$ has a sectorial extension. A second reason to present this result in fair detail is that some techniques that go into the proof will resurface in later proofs.

Theorem 16.3.2 (Sums of sectorial operators). *If A and B are resolvent commuting sectorial operators satisfying*

$$\omega(A) + \omega(B) < \pi$$

then the operator $A + B$ with its natural domain $\mathsf{D}(A + B) = \mathsf{D}(A) + \mathsf{D}(B)$ has a closed extension to a sectorial operator C which satisfies

$$\omega(C) \leqslant \max\{\omega(A), \omega(B)\}.$$

Furthermore,

(1) *If A or B is injective, then C is injective;*
(2) *If A and B are densely defined, then C is densely defined;*
(3) *If A and B are densely defined and A or B is standard sectorial, then C is standard sectorial.*

If (2) *holds (and hence if* (3) *holds), then* C *equals the closure of* $A+B$.

The proof of this theorem will be given shortly. We first pause a brief moment to explain why the condition

$$\omega(A) + \omega(B) < \pi$$

enters naturally in this theorem. Variants of this condition appear in all sum-of-operator theorems we are about to encounter. Arguing naively, one would like to realise the operator sum $A+B$ through a 'bivariate' extended Dunford calculus as $(z+w)(A+B)$, where $z+w$ is short-hand for the function $(z,w) \mapsto z+w$. With this notation, to prove sectoriality of $A+B$ one must estimate the norms of

$$\lambda R(\lambda, A+B) = \frac{\lambda}{\lambda - (z+w)}(A, B)$$

for all $\lambda \in \mathbb{C}$ in the complement of a sector Σ_ω containing all sums $z+w$ with $z \in \Sigma_\sigma$ and $w \in \Sigma_\tau$, where $\omega(A) < \sigma < \pi$ and $\omega(B) < \tau < \pi$ as usual. But the algebraic sum $\Sigma_\sigma + \Sigma_\tau$ is a *sector* only if $\sigma + \tau \leqslant \pi$! Under this condition, $\Sigma_\sigma + \Sigma_\tau = \Sigma_{\max\{\sigma,\tau\}}$. In contrast, when $\sigma+\tau > \pi$ the reader may check that $\Sigma_\sigma + \Sigma_\tau = \mathbb{C}$. Clearly, the condition $\sigma + \tau \leqslant \pi$ forces $\omega(A) + \omega(B) < \pi$, and in that case we may replace σ and τ by slightly smaller values to arrange that $\sigma + \tau < \pi$. Incidentally, this heuristic argument also shows that the inequality $\omega(A+B) \leqslant \max\{\omega(A), \omega(B)\}$ is natural to expect.

Let us now turn to the proof Theorem 16.3.2. Let A and B be resolvent commuting sectorial operators in X satisfying $\omega(A) + \omega(B) < \pi$, and let $\omega(A) < \sigma < \pi$ and $\omega(B) < \tau < \pi$ be such that $\sigma + \tau < \pi$. The construction of the sectorial operator C extending $A+B$ is based on the following observation, which makes use of the primary calculus involving the spaces $E(\Sigma)$ introduced in Section 15.1.a. For a holomorphic function $h \in E(\Sigma_\sigma) \otimes E(\Sigma_\tau)$ of the form

$$h(z,w) = \sum_{n=1}^{N} f_n(z) g_n(w)$$

with all $f_n \in E(\Sigma_\sigma)$ and $g_n \in E(\Sigma_\tau)$, we may define

$$h(A,B) := \sum_{n=1}^{N} f_n(A) g_n(B).$$

It is not difficult that the operator $h(A,B)$ is well defined, in the sense that it does not depend on the particular representation of h. We now observe that

$$h(z,w) := \frac{z+w}{(1+z)(1+w)} = \Big(1 - \frac{1}{1+z}\Big)\frac{1}{1+w} + \frac{1}{1+z}\Big(1 - \frac{1}{1+w}\Big).$$

This identifies the left-hand side as an element of $E(\Sigma_\sigma) \otimes E(\Sigma_\tau)$. Thinking of

$$\rho(z,w) := \frac{1}{(1+z)(1+w)}$$

as a regulariser for the function $(z,w) \mapsto z+w$, we define

$$C := (I+A)(I+B)h(A,B)$$

with domain

$$\mathsf{D}(C) := \big\{x \in X : \, h(A,B)x \in \mathsf{R}((I+B)^{-1}(I+A)^{-1}\big\}.$$

A bit of algebra reveals that

- $x \in \mathsf{D}(C) \iff (A+B)(I+B)^{-1}(I+A)^{-1}x \in \mathsf{R}((I+B)^{-1}(I+A)^{-1})$

  and, for $x \in \mathsf{D}(C)$,

  $$Cx = (I+A)(I+B)(A+B)(I+B)^{-1}(I+A)^{-1}x.$$

From this equivalence, by a standard argument one deduces that

- C is closed.

Proof of Theorem 16.3.2. We will prove that C defined by the above procedure has the required properties.

It is immediate from the definition that $\mathsf{D}(A) \cap \mathsf{D}(B)$ is contained in $\mathsf{D}(C)$; this is the same as saying that C is an extension of $A+B$. In fact, a moment's reflection shows that

$$\mathsf{D}(A) \cap \mathsf{D}(C) = \mathsf{D}(A) \cap \mathsf{D}(B) = \mathsf{D}(C) \cap \mathsf{D}(B). \tag{16.17}$$

Choose $\omega(A) < \sigma < \pi$ and $\omega(B) < \tau < \pi$ in such a way that $\sigma + \tau < \pi$. As was already observed above, the condition $\sigma + \tau < \pi$ implies that

$$\Sigma_\sigma + \Sigma_\tau := \{z + w : \, z \in \Sigma_\sigma, \, w \in \Sigma_\tau\} = \Sigma_{\max\{\sigma,\tau\}}.$$

For $z \in \Sigma_\sigma$, $w \in \Sigma_\tau$, and $\lambda \in \mathbb{C}$ with $\max\{\sigma,\tau\} < |\arg(\lambda)| < \pi$, we write

$$\frac{\lambda}{\lambda - (z+w)} = \frac{\lambda^2}{(\lambda - z)(\lambda - w)} + \frac{\lambda z w}{(\lambda - (z+w))(\lambda - z)(\lambda - w)}.$$

These functions are holomorphic on $\Sigma_\sigma \times \Sigma_\tau$ and one may check that

$$\lambda R(\lambda, C) = \lambda^2 R(\lambda, A)R(\lambda, B) + f_\lambda(A,B), \tag{16.18}$$

where $f_\lambda(A,B)$ can be defined in terms of the function

$$f_\lambda(z,w) := \frac{\lambda z w}{(\lambda - (z+w))(\lambda - z)(\lambda - w)}$$

as the absolutely convergent Dunford integral

$$f_\lambda(A,B) := \frac{1}{(2\pi i)^2}\int_{\partial\Sigma_\tau}\int_{\partial\Sigma_\sigma} f_\lambda(z,w)R(z,A)R(w,B)\,\mathrm{d}z\,\mathrm{d}w.$$

With $\mu := \lambda/|\lambda|$ we then have

$$\|f_\lambda(A,B)\| \lesssim_{\sigma,\tau,A,B} \int_{\partial\Sigma_\tau}\int_{\partial\Sigma_\sigma} \frac{1}{|\mu-(z+w)||\mu-z||\mu-w|}\,|\mathrm{d}z|\,|\mathrm{d}w| \quad (16.19)$$

and the sectoriality of C with angle $\omega(C) \leqslant \omega$ now easily follows from the fact that value of the integral on the right-hand side of (16.19) is uniformly bounded with respect to μ on the arc $\{|\mu|=1, \, |\arg(\mu)| \geqslant \max\{\sigma,\tau\}$.

Since the choices $\omega(A) < \sigma < \pi$ and $\omega(B) < \tau < \pi$ and $\max\{\sigma,\tau\} < \omega < \pi$ were arbitrary, it follows that $\omega(C) \leqslant \max\{\omega(A),\omega(B)\}$.

It remains to prove the assertions (1)–(3).

(1): Suppose that is injective and let $x \in \mathsf{D}(C)$ be such that $Cx = 0$. By the definition of C, this means that $h(A,B)x = (I+B)^{-1}(I+A)^{-1}y$ for some $y \in X$ and $Cx = (I+A)(I+B)h(A,B)x = y = 0$. Consider the function

$$g(z,w) = \frac{z}{(1+z)^2(z+w)}, \quad z,w \in \Sigma_\sigma.$$

By the primary calculus, for fixed $z \in \Sigma_\sigma$ we have

$$g(z,B) = z(1+z)^{-2}(z+B)^{-1}.$$

Borrowing some terminology from the next subsection, this function belongs to $H^1(\Sigma_\sigma;\mathscr{A})$, where $\mathscr{A}$ is the set of operators in $\mathscr{L}(X)$ commuting with the resolvent of A, and we may define a bounded operator $g(A,B)$ through the Dunford integral

$$g(A,B) := \frac{1}{2\pi i}\int_{\partial\Sigma_\nu} z(1+z)^{-2}(z+B)^{-1}(z-A)^{-1}\,\mathrm{d}z.$$

In view of

$$\begin{aligned}
&(z+B)^{-1}(z-A)^{-1}C \\
&\quad = (z+B)^{-1}(z-A)^{-1}[(I+A)(I+B)h(A,B)] \\
&\quad = [(1+z)(z-A)^{-1}-I][(1-z)(z+B)^{-1}+I]h(A,B) \\
&\quad = [(1+z)(z-A)^{-1}-I][(1-z)(z+B)^{-1}+I](A+B)(I+A)^{-1}(I+B)^{-1} \\
&\quad = [(1+z)(z-A)^{-1}-I][I-(I+A)^{-1}][(1-z)(z+B)^{-1}+I](I+B)^{-1} \\
&\qquad + [(1+z)(z-A)^{-1}-I][(1-z)(z+B)^{-1}+I][I-(I+B)^{-1}](I+A)^{-1} \\
&\quad = (z-A)^{-1}-(z+B)^{-1},
\end{aligned}$$

where the last line follows by the resolvent identity. By Cauchy's theorem,

$$\int_{\partial\Sigma_\nu} \frac{z}{(1+z)^2}\Big((z-A)^{-1}-(z+B)^{-1}\Big)\,\mathrm{d}z = \int_{\partial\Sigma_\nu} \frac{z}{(1+z)^2}(z-A)^{-1}\,\mathrm{d}z$$

$$= A(I+A)^{-2}x$$

It follows that

$$0 = g(A,B)Cx = A(I+A)^{-2}x.$$

Since A and $(I-A)^{-1}$ are injective, this forces $x = 0$.

If B is injective and $Cx = 0$, the same argument (with the roles of A and B reversed) again shows that $x = 0$.

(2): If A and B are densely defined, then so is C by (16.17). If $x \in \mathsf{D}(A)$, then the vectors $x_n := n^2(n+A)^{-1}(n+B)^{-1}x$ belong to $\mathsf{D}(A+B)$ and converge to x in X as $n \to \infty$. Similarly, the vectors $Cx_n = n^2(n+A)^{-1}(n+B)^{-1}Cx$ converge to Cx in X as $n \to \infty$. This shows that $\mathsf{D}(A+B)$ is dense in $\mathsf{D}(C)$ with respect to the graph norm.

(3): Suppose now that A and B are standard sectorial. Then $\mathsf{D}(A)$ and $\mathsf{D}(B)$ are dense, and therefore $\mathsf{D}(C)$ is dense by (2). Furthermore, arguing as in part (1) we see that for all $x \in X$ we have $g(A,B)x \in \mathsf{D}(C)$ and $Cg(A,B)x = A(I+A)^{-2}x$. Since $\mathsf{R}(A(I+A)^{-2}) = \mathsf{D}(A) \cap \mathsf{R}(A)$, it follows that $\mathsf{D}(A) \cap \mathsf{R}(A) \subseteq \mathsf{R}(C)$ and therefore $\mathsf{R}(C)$ is dense. By Proposition 10.1.8, this implies that $\mathsf{D}(C) \cap \mathsf{R}(C)$ is dense, i.e., C is standard sectorial. □

In the next proposition we assume that A and B are sectorial operators in X satisfying $\omega(A) + \omega(B) < \pi$, and choose $\omega(A) < \nu_A < \sigma_A < \pi$, $\omega(B) < \nu_B < \sigma_A < \pi$, and $\max\{\nu_A, \nu_B\} < \nu < \sigma < \pi$. The operator C is as in Theorem 16.3.2.

Proposition 16.3.3. *Every $\lambda \notin \overline{\Sigma_{\max\{\sigma,\tau\}}}$ belongs to $\varrho(C)$ and*

$$\varrho(A)R(\lambda, C)\varrho(B) = \frac{1}{(2\pi i)^2}\int_{\partial\Sigma_{\nu_A}}\int_{\partial\Sigma_{\nu_B}} \frac{\varrho(z)\varrho(w)}{\lambda - (z+w)} R(z,A)R(w,B)\,\mathrm{d}w\,\mathrm{d}z.$$

In its stated form, the proposition will be useful in the proof of Theorem 16.3.10. It is clear from the proof that the proposition could be stated with $\varrho(A)$, $\varrho(B)$, and $R(\lambda, C)$ replaced by more general operators $\phi(A)$, $\psi(B)$, and $f(A)$ under suitable conditions on the functions ϕ, ψ, and f. We leave the details to the interested reader.

Proof. It has already been observed that every $\lambda \notin \overline{\Sigma_{\max\{\nu_A,\nu_B\}}}$ belongs to the resolvent set of C, and by (16.18) (using the notation introduced there) we have $R(\lambda, C) = \lambda R(\lambda, A)R(\lambda, B) + g_\lambda(A,B)$, where

$$g_\lambda(z,w) := \frac{zw}{(\lambda-(z+w))(\lambda-z)(\lambda-w)} = \frac{1}{(\lambda-(z+w))} - \frac{\lambda}{(\lambda-z)(\lambda-w)}. \tag{16.20}$$

Inserting this into the Dunford integral

$$\varrho(A)R(\lambda,C)\varrho(B) = \frac{1}{(2\pi i)^2}\int_{\partial\Sigma_{\nu_A}}\int_{\partial\Sigma_{\nu_B}} \varrho(z)\varrho(w)R(z,A)R(\lambda,C)R(w,B)\,\mathrm{d}w\,\mathrm{d}z,$$

we see that this results in the sum of three integrals, where (I) corresponds to the contribution $\lambda R(\lambda,A)R(\lambda,B)$, and (II) and (III) correspond to the splitting of g_λ by (16.20). By a simple computation involving Fubini's theorem and Cauchy's theorem, the integrals (I) and (III) cancel, and the integral (II) equals the one in the statement of the lemma. □

16.3.b Operator-valued H^∞-calculus and closed sums

In this section we extend the Dunford calculus of a sectorial operator A to an operator-valued Dunford calculus and study the question when this calculus is bounded with respect to the H^∞-norm. The idea is to obtain (16.16) from the boundedness of the operator $f(A,B)$ in terms of the function $f(\lambda,B) = B(\lambda+B)^{-1}$ in the operator-valued calculus. Loosely speaking, this gives a way to define an operator "$A(A+B)^{-1}$" even when $A+B$ fails to be bounded invertible. With the operator at hand, it is possible to run a rigorous version of the estimate

$$\|Ax\| = \|A(A+B)^{-1}(A+B)x\| \leqslant C\|(A+B)x\|$$

with $C = \|A(A+B)^{-1}\|$. From this one also obtains the estimate

$$\|Bx\| \leqslant \|(A+B)x\| + \|Ax\| \leqslant (1+C)\|(A+B)x\|,$$

and together these estimates give (16.16), with implied constant $1+2C$.

In what follows, A always denotes a sectorial operator on a Banach space X, and we fix $\omega(A) < \sigma < \pi$. Let $\mathscr{A}$ be a closed sub-algebra of $\mathscr{L}(X)$ resolvent commuting with A, i.e.,

$$TR(z,A) = R(z,A)T \quad \text{for all} \quad T\in\mathscr{A} \quad \text{and} \quad z\in\varrho(A).$$

We then denote by $H^1(\Sigma_\sigma;\mathscr{A})$ the space of all holomorphic functions $F : \Sigma_\sigma \to \mathscr{A}$ for which

$$\|F\|_{H^1(\Sigma_\sigma;\mathscr{A})} := \sup_{|\nu|<\sigma}\int_{\mathbb{R}_+} \|F(e^{i\nu}t)\|\,\frac{\mathrm{d}t}{t}$$

is finite. It is easily checked that, which respect to this norm, $H^1(\Sigma_\sigma;\mathscr{A})$ is a Banach space. For functions $F\in H^1(\Sigma_\sigma;\mathscr{A})$ we can define a bounded operator $F(A)\in\mathscr{A}$ by means of the operator-valued Dunford integral

$$F(A) = \frac{1}{2\pi i}\int_{\partial\Sigma_\nu} F(z)R(z,A)\,\mathrm{d}z,$$

where $\omega(A) < \nu < \sigma$. The resulting operator is independent of the particular choice of ν, and it satisfies

$$\|f(A)\| \leqslant \frac{M_{\sigma,A}}{\pi} \|F\|_{H^1(\Sigma_\sigma;\mathscr{A})},$$

where $M_{\sigma,A} = \sup_{\lambda\in \complement\overline{\Sigma_\sigma}} \|\lambda R(\lambda, A)\|$.

As in Proposition 10.2.2, this calculus is multiplicative and satisfies the following convergence property: if $F_n, F \in H^1(\Sigma_\sigma;\mathscr{A})$ are uniformly bounded and satisfy $F_n(z)x \to F(z)x$ for all $z \in \Sigma_\sigma$ and $x \in X$, then for all $g \in H^1(\Sigma_\sigma)$ we have

$$\lim_{n\to\infty} (f_n g)(A)x = (fg)(A)x, \quad x \in X.$$

Denote by $H^\infty(\Sigma_\sigma;\mathscr{A})$ the space of all holomorphic functions $F : \Sigma_\sigma \to \mathscr{A}$ for which the set $\{F(z) : z \in \Sigma_\sigma\}$ is uniformly bounded. Endowed with the norm

$$\|F\|_{H^\infty(\Sigma_\sigma;\mathscr{A})} := \sup\{\|F(z)\| : z \in \Sigma_\sigma\},$$

this space is easily seen to be Banach space. In the same way one defines $RH^\infty(\Sigma_\nu;\mathscr{A})$ as the space of all holomorphic functions $F : \Sigma_\nu \to \mathscr{A}$ for which the set $\{F(z) : z \in \Sigma_\nu\}$ is R-bounded. Endowed with the norm

$$\|F\|_{RH^\infty(\Sigma_\sigma;\mathscr{A})} := \mathscr{R}(\{F(z) : z \in \Sigma_\sigma\})$$

(the R-bound of $\{F(z) : z \in \Sigma_\sigma\}$), this space is a Banach space.

The main result of this section is the following theorem.

Theorem 16.3.4. *Let A be a sectorial operator on a Banach space X, let $\omega(A) < \sigma < \pi$, and suppose that A has a bounded $H^\infty(\Sigma_\sigma)$-calculus. Then there exists a unique bounded linear mapping $F \mapsto F(A)$ from $RH^\infty(\Sigma_\sigma;\mathscr{A})$ into $\mathscr{L}(\overline{\mathsf{D}(A)\cap\mathsf{R}(A)})$ with the following properties:*

(1) *For every function $F \in RH^\infty(\Sigma_\sigma;\mathscr{A}) \cap H^1(\Sigma_\sigma;\mathscr{A})$ the operator $F(A)$ coincides with the one defined by the Dunford integral;*
(2) *For all $F, G \in RH^\infty(\Sigma_\sigma;\mathscr{A})$ we have $FG \in RH^\infty(\Sigma_\sigma;\mathscr{A})$ and*

$$(FG)(A) = F(A)G(A) = G(A)F(A);$$

(3) *Whenever the functions $F_n, F \in RH^\infty(\Sigma_\sigma;\mathscr{A})$ are uniformly bounded and satisfy $F_n \to F$ pointwise on Σ_σ, then $\lim_{n\to\infty} F_n(A)x = F(A)x$ for all $x \in \overline{\mathsf{D}(A)\cap\mathsf{R}(A)}$.*

Furthermore, if X has Pisier's contraction property and $\mathscr{T}$ is an R-bounded subset of $\mathscr{A}$, then for all $0 < \sigma < \nu < \pi$ the family

$$\big\{F(A) : F \in RH^\infty(\Sigma_\nu;\mathscr{A}),\, F(z) \in \mathscr{T} \text{ for all } z \in \Sigma_\nu\big\}$$

is R-bounded.

Parts (2) and (3) are analogues of the corresponding results in Theorem 10.2.13 and the proofs are similar. The proof of (1), which is the non-trivial part of the theorem, is based on an extension of Lemma 10.3.13, which states that if A is a sectorial operator on a Banach space X and $F \in H^1(\Sigma_\sigma; \mathscr{A})$ is given, with $\omega(A) < \sigma < \pi$, then for all $\omega(A) < \nu < \sigma$ we have

$$F(A) = \frac{1}{2\pi i} \int_{\partial \Sigma_\nu} z^{1/2} F(z) \phi_z(A) \, \frac{\mathrm{d}z}{z}, \tag{16.21}$$

where $\phi_z(\lambda) := \lambda^{1/2}/(z - \lambda)$. The proof is identical to that of Lemma 10.3.13; all one needs to do is to replace $H^1(\Sigma_\sigma)$ by $H^1(\Sigma_\sigma; \mathscr{A})$ throughout, and so is the justification of the well-definedness of the operators $\phi_z(A)$ and the convergence of integral on the right-hand side of (16.21).

We also need the following strengthening of Lemma 10.3.8:

Lemma 16.3.5. *Let A be a sectorial operator on a Banach space X with a bounded H^∞-calculus, and let $\omega_{H^\infty}(A) < \nu < \sigma < \pi$. Suppose $\phi, \psi \in H^1(\Sigma_\sigma)$, and let $\mathscr{T} \subseteq \mathscr{A}$ be R-bounded. Then for all finite subsets $F \subseteq \mathbb{Z}$, all scalars $|a_j| \leqslant 1$ and operators $T_j \in \mathscr{T}$ $(j \in F)$, and all $x \in \overline{\mathsf{D}(A) \cap \mathsf{R}(A)}$,*

$$\sup_{t>0} \Big\| \sum_{j \in F} a_j T_j \phi(2^j t A) \psi(2^j t A) x \Big\| \leqslant C \|\phi\|_{H^1(\Sigma_\sigma)} \|\psi\|_{H^1(\Sigma_\sigma)} \|x\|,$$

where C is a constant depending only on ν, σ, and A.

Proof. Let A_0 denote the part of A in $X_0 := \overline{\mathsf{D}(A) \cap \mathsf{R}(A)}$. This operator is standard sectorial and has a bounded H^∞-calculus, with the same bounds, and the same holds for its adjoint A_0^*. Let $(\varepsilon_j)_{j \in \mathbb{Z}}$ be a Rademacher sequence. For norm one vectors $x \in X_0$ and $x^* \in X_0^*$, and for any fixed $t > 0$ and finite subset $F \subseteq \mathbb{Z}$ we may estimate

$$\begin{aligned}
&\Big| \Big\langle \sum_{j \in F} a_j T_j \phi(2^j t A) \psi(2^j t A) x, x^* \Big\rangle \Big| \\
&\quad = \Big| \sum_{j \in F} a_j \langle T_j \psi(2^j t A_0) x, \phi(2^j t A_0^*) x^* \rangle \Big| \\
&\quad = \Big| \mathbb{E} \Big\langle \sum_{j \in F} \varepsilon_j a_j T_j \psi(2^j t A_0) x, \sum_{k \in F} \varepsilon_k \phi(2^k t A_0^*) x^* \Big\rangle \Big| \\
&\quad \leqslant \Big(\mathbb{E} \Big\| \sum_{j \in F} \varepsilon_j a_j T_j \psi(2^j t A_0) x \Big\|^2 \Big)^{1/2} \Big(\mathbb{E} \Big\| \sum_{k \in F} \varepsilon_k \phi(2^k t A_0^*) x^* \Big\|^2 \Big)^{1/2} \\
&\quad \leqslant K_{\sigma - \nu} (M_{\nu, A}^\infty)^2 \mathscr{R}(\mathscr{T}) \|\phi\|_{H^1(\Sigma_\sigma)} \|\psi\|_{H^1(\Sigma_\sigma)} \|x\| \|x^*\|
\end{aligned}$$

using R-boundedness, the Kahane contraction principle, and Theorem 10.4.4 (and its notation) in the last step. The result now follows by taking the supremum over all $x^* \in X_0^*$ of norm at most 1. □

Proof of Theorem 16.3.4. Let $\omega(A) < \nu < \sigma < \pi$ and let F be as in (1). As in the proof of Theorem 10.3.4(3), for $x \in \mathsf{D}(A) \cap \mathsf{R}(A)$ and $F \in H^1(\Sigma_\sigma; \mathscr{A}) \cap RH^\infty(\Sigma_\sigma; \mathscr{A})$ we find

$$F(A)x = \sum_{j\in\mathbb{Z}} \sum_{\epsilon=\pm 1} \frac{1}{2\pi i} \epsilon e^{-\epsilon i\nu/2} \int_1^2 F(e^{-\epsilon i\nu} 2^j t) \phi_{e^{-\epsilon i\nu}}(t^{-1}2^{-j}A)x \, \frac{\mathrm{d}t}{t},$$

where $\phi_z(\lambda) = \lambda^{1/2}/(z-\lambda)$. Then, with $a_j(\epsilon) = \epsilon e^{-\epsilon i\nu/2}$,

$$\|F(A)x\| \leqslant \frac{1}{\pi} \sup_{\epsilon=\pm 1} \sup_{k\geqslant 1} \sup_{t>0} \Big\| \sum_{|j|\leqslant k} a_j(\epsilon) F(e^{-\epsilon i\nu} 2^j t) \phi_{e^{-\epsilon i\nu}}(t^{-1}2^{-j}A)x \Big\|.$$

Now we choose $\mathscr{T}$ to be the R-bounded range of F, and we let $\phi = \psi = (\phi_{e^{-i\nu}})^{1/2}$ if $\epsilon = 1$ and $\phi = \psi = (\phi_{e^{i\nu}})^{1/2}$ if $\epsilon = -1$. Applying the lemma twice, we obtain

$$\|F(A)x\| \leqslant \frac{2}{\pi} C \max_{\epsilon=\pm 1} \|\phi_{e^{-\epsilon i\nu}}\|_{H^1(\Sigma_\sigma)} \|x\|,$$

where C is the constant of the lemma.

The proofs of multiplicativity and the convergence property proceed as in Theorem 10.2.13.

Regarding the final assertion, we may adapt the proof of Theorem 10.3.4(3), replacing the scalar functions f_n and f by $\mathscr{A}$-valued functions F_n and F. □

As an application of the operator-valued calculus we prove a useful variant of the Dore–Venni theorem (Theorem 15.4.11). In that theorem, both A and B were assumed to have bounded imaginary powers and act in a UMD Banach space X. In the present theorem, we weaken the assumption on A and strengthen the assumption on B.

Theorem 16.3.6 (The sum of an R-sectorial operator and an operator with bounded H^∞-calculus). *Let A and B be resolvent commuting densely defined (respectively, standard) sectorial operators on a Banach space X. Assume that A has a bounded H^∞-calculus, B is R-sectorial, and*

$$\omega_{H^\infty}(A) + \omega_R(B) < \pi.$$

Then $A+B$ is a densely defined (respectively, standard) sectorial operator and

$$\omega(A+B) \leqslant \max\{\omega_{H^\infty}(A), \omega_R(B)\}.$$

Moreover, there exists a constant $C \geqslant 0$ such that

$$\|Ax\| + \|Bx\| \leqslant C\|(A+B)x\|, \quad x \in \mathsf{D}(A) \cap \mathsf{D}(B). \tag{16.22}$$

If X has the triangular contraction property, then $A+B$ is R-sectorial with

$$\omega_R(A+B) \leqslant \max\{\omega_R(A), \omega_{H^\infty}(B)\}.$$

Proof. The idea is to define $A(A+B)^{-1}$ and $B(A+B)^{-1}$ as bounded operators on $\overline{\mathsf{D}(A) \cap \mathsf{R}(A)}$ using the operator-valued functional calculus for A.

Step 1 – We first assume that A and B are standard sectorial. Let $\mathscr{A}$ denote the closed sub-algebra of $\mathscr{L}(X)$ comprised of all operators resolvent commuting with A. Choose $\omega_{H^\infty}(A) < \sigma < \pi$ and $\omega_R(B) < \tau < \pi$ and such that $\sigma + \tau < \pi$. We wish to apply the operator-valued calculus of A to the function

$$F(z) := B(z+B)^{-1} = I - zR(z,-B).$$

This function belongs to $RH^\infty(\Sigma_\sigma;\mathscr{A})$ since the spectrum of $-B$ is contained in the closure of $-\Sigma_\tau = \{z \in \mathbb{C} : |\arg(z)| > \pi - \tau\}$ and $\sigma < \pi - \tau$. Furthermore, the function

$$G(z) := \zeta_n(z)^2(z+B)\zeta_n(B)^2,$$

with $\zeta_n(z) = \frac{n}{n+z} - \frac{1}{1+nz}$ as in Proposition 10.2.6, is easily seen to belong to $H^1(\Sigma_\sigma;\mathscr{A}) \cap RH^\infty(\Sigma_\sigma;\mathscr{A})$ by R-sectoriality. We have

$$(FG)(z) = F(z)G(z) = \zeta_n(z)^2 B\zeta_n(B)^2,$$

and in the operator-valued Dunford calculus the operators $G(A)$ and $(FG)(A)$ are given by

$$\begin{aligned} G(A) &= B\zeta_n(B)^2\zeta_n(A)^2 + A\zeta_n(A)^2\zeta_n(B)^2, \\ (FG)(A) &= B\zeta_n(B)^2\zeta_n(A)^2, \end{aligned}$$

using resolvent commutation to do some rewriting. By the multiplicativity of the operator-valued H^∞-calculus of A we have

$$\begin{aligned} B\zeta_n(B)^2\zeta_n(A)^2 = (FG)(A) &= F(A)G(A) \\ &= F(A)\big(B\zeta_n(B)^2\zeta_n(A)^2 + A\zeta_n(A)^2\zeta_n(B)^2\big). \end{aligned}$$

The boundedness of the operator-valued H^∞-calculus of A then gives, for $x \in \mathsf{D}(A) \cap \mathsf{D}(B)$,

$$\|\zeta_n(A)^2\zeta_n(B)^2 Bx\| \lesssim_{\sigma,A} \big\|\big(\zeta_n(A)^2\zeta_n(B)^2 Bx + \zeta_n(B)^2\zeta_n(A)^2 Ax\big)\big\|$$

Letting $n \to \infty$ and using A and B are standard sectorial, we obtain the inequality

$$\|Bx\| \lesssim_{\sigma,A} \|(A+B)x\|.$$

From this we also obtain

$$\|Ax\| \leqslant \|Bx\| + \|Ax + Bx\| \lesssim_{\sigma,A} \|(A+B)x\|.$$

We have already observed in Proposition 16.3.1 that (16.22) implies the closedness of $A+B$. The standard sectoriality of $A+B$ now follows from Theorem 16.3.2.

Step 2 – We now assume that A and B are densely defined, but not necessarily standard sectorial. Then the operators $A_\varepsilon := A + \varepsilon$ and $B_\varepsilon : +B + \varepsilon$ are standard sectorial, and we may apply the above reasoning with $F_\varepsilon(z) := B_\varepsilon(z+B_\varepsilon)^{-1}$ and $G_\varepsilon(z) := \zeta_n(z)^2(z+B_\varepsilon)\zeta_n(B_\varepsilon)^2$. This results in the estimate

$$\|A_\varepsilon x\| + \|B_\varepsilon x\| \lesssim_{\sigma,A} \|(A_\varepsilon + B_\varepsilon)x\|$$

with an implied constant that is uniform in $\varepsilon > 0$ and independent of x. The estimate

$$\|Ax\| + \|Bx\| \lesssim_{\sigma,A} \|(A + B)x\|$$

follows from this by letting $\varepsilon \downarrow 0$.

Step 3 – Suppose finally that X has the triangular contraction property. From the proof of Theorem 16.3.2 (and keeping in mind that $A + B$ equals the operator C of that theorem by what we have already proved) we recall the identity

$$\begin{aligned}&\lambda R(\lambda, A + B)\\ &\quad = \lambda^2 R(\lambda, A)R(\lambda, B) + \frac{1}{(2\pi i)^2}\int_{\partial\Sigma_\tau}\int_{\partial\Sigma_\sigma} f_\lambda(z, w)R(z, A)R(w, B)\,\mathrm{d}z\,\mathrm{d}w.\end{aligned}$$

Outside the closure of $\Sigma_{\sigma+\tau}$ the operators $\lambda^2 R(\lambda, A)R(\lambda, B)$ are R-bounded, by the R-sectoriality of A (which follows from the second part of Theorem 10.3.4) and B (by assumption). The operators corresponding to the Dunford integral with f_λ are R-bounded by Theorem 8.5.2; the integrability properties required to apply theorem have already been observed in the proof of Theorem 16.3.2 (see (16.19)). □

Since standard sectorial operators with a bounded H^∞-calculus on a Banach space with the triangular contraction property are R-sectorial with $\omega_R(A) = \omega_{H^\infty}(A)$ (see Corollary 10.4.10), we have the following corollary.

Corollary 16.3.7. *Let A and B be resolvent commuting densely defined (respectively, standard) sectorial operators with bounded H^∞-calculi satisfying $\omega_{H^\infty}(A) + \omega_{H^\infty}(B) < \pi$ on a Banach space with the triangular contraction property. Then $A + B$ is a densely defined (respectively, standard) sectorial operator and* (16.22) *holds.*

16.3.c The joint H^∞-calculus

As a first application of the operator-valued functional calculus we construct the *joint functional calculus* of resolvent commuting standard sectorial operators.

Denote by $H^1(\Sigma_{\sigma_1} \times \cdots \times \Sigma_{\sigma_n})$ the space of holomorphic functions on $\Sigma_{\sigma_1} \times \cdots \times \Sigma_{\sigma_n}$ which obey the obvious integrability estimate extending the case $n = 1$. For functions $f \in H^1(\Sigma_{\sigma_1} \times \cdots \times \Sigma_{\sigma_n})$ we define the *joint Dunford calculus* by

$$f(A_1, \ldots, A_n) = \Big(\frac{1}{2\pi i}\Big)^n \int_{\partial \Sigma_{\nu_n}} \cdots \int_{\partial \Sigma_{\nu_1}} f(\lambda_1, \ldots, \lambda_n) \prod_{j=1}^{n} R(\lambda_j, A_j)\, \mathrm{d}\lambda_1 \ldots \mathrm{d}\lambda_n \tag{16.23}$$

where $\omega(A_i) < \nu_i < \sigma_i$ for $i = 1, \ldots, n$.

If $n = 2$, by Fubini's theorem we can formally rewrite (16.23) as

$$\begin{aligned} f(A_1, A_2) &= \frac{1}{2\pi i} \int_{\partial \Sigma_{\nu_1}} \Big(\frac{1}{2\pi i} \int_{\partial \Sigma_{\nu_2}} f(\lambda_1, \lambda_2) R(\lambda_2, A_2)\, \mathrm{d}\lambda_2 \Big) R(\lambda_1, A_1)\, \mathrm{d}\lambda_1 \\ &= \frac{1}{2\pi i} \int_{\partial \Sigma_{\nu_1}} f(\lambda_1, A_2) R(\lambda_1, A_1)\, \mathrm{d}\lambda_1 \\ &= \Phi_1(f(\cdot, A_2))(A_1), \end{aligned}$$

where $\Phi_1 : g \mapsto g(A_1)$ denotes the operator-valued calculus of A_1, provided of course that all terms are well defined. This indicates the way how to extend (16.23) to $H^\infty(\Sigma_{\sigma_1} \times \cdots \times \Sigma_{\sigma_n})$ using induction where each of the operator A_j has a bounded H^∞-calculus. Here, $H^\infty(\Sigma_{\sigma_1} \times \cdots \times \Sigma_{\sigma_n})$ denotes the space of bounded holomorphic functions on $\Sigma_{\sigma_1} \times \cdots \times \Sigma_{\sigma_n}$.

The following straightforward extension of Lemma 10.2.17 will be useful. As before, by $\mathscr{A}$ we denote the set of bounded operators commuting with the resolvent of A.

Lemma 16.3.8. *Let A have a bounded $H^\infty(\Sigma_\sigma)$-calculus on X. Suppose that $f : [a, b] \times \Sigma_\sigma \to \mathscr{A}$ is a measurable function with the following properties:*

(i) *$z \mapsto f(s, z)$ belongs to $RH^\infty(\Sigma_\sigma; \mathscr{A})$ for all $s \in [a, b]$;*

(ii) $\displaystyle\sup_{|\nu|<\sigma} \int_a^b \int_0^\infty \|f(s, e^{i\nu} t)\| \frac{\mathrm{d}t}{t}\, \mathrm{d}s < \infty.$

Then the function $g(z) = \int_a^b f(s, z)\, \mathrm{d}s$ belongs to $H^\infty(\Sigma_\sigma; \mathscr{A})$ and

$$g(A)x = \int_a^b f(s, A)x\, \mathrm{d}s, \qquad x \in X.$$

The straightforward proof is left to the reader.

We will now apply the operator-valued calculus to the sum-of-operators problem next.

Theorem 16.3.9. *Let $A_1, \ldots, A_n$ be densely defined resolvent commuting sectorial operators on a Banach space X with the Pisier contraction principle, and assume that A_j has a bounded $H^\infty(\Sigma_{\sigma_j})$-calculus, $j = 1, \ldots, n$. Then for $\sigma_j < \nu_j < \pi$, (16.23) extends to an algebra homomorphism $\Phi : H^\infty(\Sigma_{\nu_1} \times \cdots \times \Sigma_{\nu_n}) \to \mathscr{L}(\overline{\mathsf{D}(A) \cap \mathsf{R}(A)})$ with the following convergence property:*

If the functions f_m, f are uniformly bounded in $H^\infty(\Sigma_{\nu_1} \times \cdots \times \Sigma_{\nu_n})$ and $\lim_{m\to\infty} f_m = f$ pointwise on $\Sigma_{\nu_1} \times \cdots \times \Sigma_{\nu_n}$, then $\lim_{m\to\infty} \Phi(f_m)x = \Phi(f)x$ for all $x \in \overline{\mathsf{D}(A) \cap \mathsf{R}(A)}$.

Moreover, the set of operators

$$\big\{\Phi(f) : f \in H^\infty(\Sigma_{\sigma_1} \times \cdots \times \Sigma_{\sigma_n}), \|f\|_\infty \leqslant 1\big\}$$

is R-bounded.

Notation. In place of $\Phi(f)$ we shall write $f(A_1, \ldots, A_n)$.

Proof. By $\mathscr{A}_j$ we denote the sub-algebra of all operators in $\mathscr{L}(X)$ that commute with A_j and put $\mathscr{A} := \mathscr{A}_1 \cap \cdots \cap \mathscr{A}_n$. Note that $R(\lambda, A_j) \in \mathscr{A}$ for all $j = 1, \ldots, n$. The case $n = 1$ follows from the general properties of the H^∞-calculus. Assume now that $A_2, \ldots, A_n$ have a joint functional calculus $\Psi : H^\infty(\Sigma_{\sigma_2} \times \cdots \times \Sigma_{\sigma_n}) \to \mathscr{L}(X)$ with the required properties. Since X has Pisier's contraction property, the set

$$\mathscr{T} = \big\{g(A_2, \ldots, A_n) : g \in H^\infty(\Sigma_{\sigma_2} \times \cdots \times \Sigma_{\sigma_n}),\ \|g\|_{H^\infty} \leqslant 1\big\}$$

is an R-bounded subset of $\mathscr{A} \subseteq \mathscr{A}_1$ by Theorem 10.3.4(3). By Φ_1 we denote the operator-valued functional calculus of A_1 defined on $RH^\infty(\Sigma_{\sigma_1}; \mathscr{A}_1)$ as constructed in Theorem 16.3.4.

Given a function $f \in H^1(\Sigma_{\sigma_1} \times \cdots \times \Sigma_{\sigma_n}) \cap H^\infty(\Sigma_{\sigma_1} \times \cdots \times \Sigma_{\sigma_n})$ with $\|f\|_{H^\infty} \leqslant 1$, the set

$$\{f(\lambda_1, \cdot, \ldots, \cdot) : \lambda_1 \in \Sigma_{\sigma_1}\}$$

is uniformly bounded in $H^\infty(\Sigma_{\sigma_2} \times \cdots \times \Sigma_{\sigma_n})$. Hence

$$\Psi\big[f(\lambda_1, \cdot, \ldots, \cdot)\big] = f(\lambda_1, A_2, \ldots, A_n) \in \mathscr{T} \qquad \text{for all } \lambda_1 \in \Sigma_{\sigma_1}.$$

Furthermore, the function

$$\begin{aligned} &\lambda_1 \mapsto f(\lambda_1, A_2, \ldots, A_n) \\ &= \Big(\frac{1}{2\pi i}\Big)^{n-1} \int_{\partial \Sigma_{\nu_2}} \cdots \int_{\partial \Sigma_{\nu_n}} \prod_{j=2}^{n} f(\lambda_1, \ldots, \lambda_n) R(\lambda_j, A_j)\, \mathrm{d}\lambda_2 \ldots \mathrm{d}\lambda_n \end{aligned}$$

is holomorphic on Σ_{σ_1}. Again by Theorem 10.3.4(3), $f(\cdot, A_2, \ldots, A_n) \in RH^\infty(\Sigma_{\sigma_1}; \mathscr{A}_1)$. Consequently we can define

$$\Phi(f) = \Phi_1\big(f(\cdot, A_2, \ldots, A_n)\big)$$

using Theorem 16.3.13. We can extend this definition to arbitrary $f \in H^\infty(\Sigma_{\sigma_1} \times \cdots \times \Sigma_{\sigma_n})$. The required properties of Φ now follow from the corresponding properties of Ψ and Φ_1. For instance, Φ extends (16.23) by Fubini's theorem. To check the multiplicativity, choose $f, g \in H^\infty(\Sigma_{\sigma_1} \times \cdots \times \Sigma_{\sigma_n})$. Then

$$\begin{aligned}\Phi(f\cdot g) &= \Phi_1\big((f\cdot g)(\cdot,A_2,\dots,A_n)\big)\\ &= \Phi_1\big(f(\cdot,A_2,\dots,A_n)\cdot g(\cdot,A_2,\dots,A_n)\big)\\ &= \Phi_1\big(f(\cdot,A_2,\dots,A_n)\big)\Phi_1\big(g(\cdot,A_2,\dots,A_n)\big) = \Phi(f)\Phi(g).\end{aligned}$$

Also, if f_m, f are bounded in $H^\infty(\Sigma_{\sigma_1}\times\cdots\times\Sigma_{\sigma_n})$ and $f_m\to f$ pointwise as $m\to\infty$, then by the convergence property of the operator-valued calculus of A_1 we have

$$\lim_{m\to\infty} f_m(\lambda_1,A_2,\dots,A_n)x = f(\lambda_1,A_2,\dots,A_n)x$$

for every fixed $\lambda_1\in\Sigma_{\sigma_1}$ and all $x\in X$. Now apply the convergence property of Φ_1 to $F_m(\lambda) = f_m(\lambda,A_2,\dots,A_n)$ and $F(\lambda) = f(\lambda,A_2,\dots,A_n)$.

The final R-boundedness assertion follows directly from the final assertion of Theorem 16.3.4. □

As an application we have the following variant of Corollary 16.3.7. This is result is actually true for Banach space X with the triangular contraction property; we refer to the Notes for a discussion of this fact.

Theorem 16.3.10. *Let A and B be resolvent commuting standard sectorial operators with bounded H^∞-calculi satisfying $\omega_{H^\infty}(A)+\omega_{H^\infty}(B)<\pi$ on a Banach space X with Pisier's contraction property. Then $A+B$ admits a bounded H^∞-calculus with*

$$\omega_{H^\infty}(A,B)\leqslant \max\{\omega_{H^\infty}(A),\omega_{H^\infty}(B)\}.$$

For the proof we need a technical proposition. For the sake of its formulation, the joint Dunford calculus of two resolvent commuting sectorial operators A and B will be denoted by $\Phi_{A,B}: f\mapsto f(A,B)$, for functions $f\in H^1(\Sigma_\sigma)\times H^1(\Sigma_\tau)$. Likewise, the operator-valued Dunford calculus of A will be denoted by $\Phi_A: F\mapsto F(A)$, for operator-valued functions $F\in H^1(\Sigma_\sigma;\mathscr{A})$ where $\mathscr{A}$ is set of operators resolvent commuting with A.

Proposition 16.3.11. *Let A and B be resolvent commuting sectorial operators acting in a Banach space X satisfying $\omega(A)+\omega(B)<\pi$, and let* $\max\{\omega(A),\omega(B)\}<\sigma<\pi$. *Let C denote the operator sum of A and B as constructed above. Then for all $f\in H^1(\Sigma_\sigma)$ we have*

$$\begin{aligned}\varrho(A)f(C)\varrho(B) &= \Phi_{A,B}((z,w)\mapsto \varrho(z)f(z+w)\varrho(w))\\ &= \Phi_A(z\mapsto \varrho(z)f(z+B)\varrho(B)),\end{aligned}$$

where $\varrho(z)=z/(1+z)^2$.

Proof. By the definition of the joint Dunford calculus,

$$\Phi_{A,B}((z,w)\mapsto \varrho(z)(\lambda-(z+w))\varrho(w))$$

$$= \frac{1}{(2\pi i)^2} \int_{\partial\Sigma_{\nu_A}} \int_{\partial\Sigma_{\nu_B}} \frac{\varrho(z)\varrho(w)}{\lambda - (z+w)} R(z,A)R(w,B)\,\mathrm{d}w\,\mathrm{d}z.$$

On the other hand, by Proposition 16.3.3 and Cauchy's theorem,

$$\begin{aligned}
&\varrho(A)f(C)\varrho(B)\\
&\quad= \frac{1}{2\pi i}\int_{\partial\Sigma_\nu} f(\lambda)\varrho(A)R(\lambda,C)\varrho(B)\,\mathrm{d}\lambda\\
&\quad= \frac{1}{(2\pi i)^3}\int_{\partial\Sigma_\nu} f(\lambda)\int_{\partial\Sigma_{\nu_A}}\int_{\partial\Sigma_{\nu_B}} \frac{\varrho(z)\varrho(w)}{\lambda-(z+w)}R(z,A)R(w,B)\,\mathrm{d}w\,\mathrm{d}z\,\mathrm{d}\lambda\\
&\quad= \frac{1}{(2\pi i)^2}\int_{\partial\Sigma_{\nu_A}}\int_{\partial\Sigma_{\nu_B}} \varrho(z)\varrho(w)\\
&\qquad\qquad\times\Big(\frac{1}{2\pi i}\int_{\partial\Sigma_\nu}\frac{f(\lambda)}{\lambda-(z+w)}\,\mathrm{d}\lambda\Big)R(z,A)R(w,B)\,\mathrm{d}w\,\mathrm{d}z\\
&\quad= \frac{1}{(2\pi i)^2}\int_{\partial\Sigma_{\nu_A}}\int_{\partial\Sigma_{\nu_B}} \varrho(z)\varrho(w)f(z+w)R(z,A)R(w,B)\,\mathrm{d}w\,\mathrm{d}z
\end{aligned}$$

which equals

$$= \Phi_{A,B}((z,w)\mapsto \varrho(z)f(z+w)\varrho(w))$$

but also

$$\begin{aligned}
&= \frac{1}{2\pi i}\int_{\partial\Sigma_{\nu_A}} \varrho(z)\Big(\frac{1}{2\pi i}\int_{\partial\Sigma_{\nu_B}}\varrho(w)f(z+w)R(w,B)\,\mathrm{d}w\Big)R(z,A)\,\mathrm{d}z\\
&= \Phi_A(z\mapsto \varrho(z)f(z+B)\varrho(B)).
\end{aligned}$$

□

Proof of Theorem 16.3.10. Since X has Pisier's contraction property, A and B admit R-bounded operator-valued H^∞-calculi by Theorem 10.3.4. Choose $\omega(A) < \sigma_A < \pi$ and $\omega(B) < \sigma_B < \pi$ such that $\sigma_A + \sigma_B < \pi$, and let $\max\{\sigma_A, \sigma_B\} < \sigma < \pi$. For $f \in H^\infty(\Sigma_\sigma)$, the function

$$F(z,w) := f(z+w)$$

belongs to $H^\infty(\Sigma_{\sigma_A}) \times H^\infty(\Sigma_{\sigma_B})$. Since B has an R-bounded H^∞-calculus, the set $\{F(z,B) : z \in \Sigma_{\sigma_A}\}$ is an R-bounded subset of the set $\mathscr{A}$ of bounded operators resolvent commuting with A. Applying the operator-valued calculus of A we obtain a bounded operator $F(A,B)$ on $\overline{\mathsf{D}(A)\cap\mathsf{R}(A)}$. By Proposition 16.3.11,

$$\varrho(A)f(A+B)\varrho(B) = \varrho(A)F(A,B)\varrho(B),$$

where $\varrho(z) = z/(1+z)^2$. Since $\varrho(A)$ is injective and $\varrho(B)$ has dense range (we assumed that A and B are standard sectorial), we conclude that $f(A+B) = F(A,B)$ is a bounded operator on $\overline{\mathsf{D}(A)\cap\mathsf{R}(A)}$. The bound $\|f(A,B)\| \lesssim \|f\|_{H^\infty(\Sigma_\sigma)}$ follows by tracing the steps of the proof. □

Inspection of the proof shows that the 'standard' assumption on A may be weakened to 'densely defined and injective'. In reflexive spaces, however, these conditions imply standardness (see Proposition 10.1.9).

16.3.d The absolute calculus and closed sums

The main result of this section (Theorem 16.3.13) provides a version of Theorem 16.3.6 in which no assumption on the Banach space is needed, the assumption on a A is weakened to sectoriality, and the assumption on B is strengthened to having an absolute calculus.

Definition 16.3.12 (Absolute functional calculus). *Let A be a sectorial operator acting in a Banach space X, and let $\omega(A) < \sigma < \pi$. We say that A* admits an absolute calculus on Σ_σ *if there exist a constant $M \geqslant 0$ and $g, h \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$, with $\|h\|_{H^1(\Sigma_\sigma)} = 1$, such that for all $x, y \in \overline{\mathsf{D}(A) \cap \mathsf{R}(A)}$ the validity of the estimate*

$$\|h(tA)g(tA)x\| \leqslant \|g(tA)y\| \quad \textit{for all} \qquad t > 0$$

implies $\|x\| \leqslant M\|y\|$.

We denote

$$\omega_{\rm abs}(A) := \inf\big\{\sigma \in (0,\pi) : \, A \text{ admits an absolute calculus on } \Sigma_\sigma\big\}.$$

Examples of classes of operators with an absolute calculus will be given in the next subsection.

Theorem 16.3.13 (Absolute calculus implies operator-valued H^∞-calculus). *Let A be a sectorial operator in a Banach space X, let $\omega(A) < \sigma < \pi$, and suppose that A admits an absolute calculus on Σ_σ. Then A admits a bounded operator-valued $H^\infty(\Sigma_\sigma)$-calculus. In particular, A admits a bounded $H^\infty(\Sigma_\sigma)$-calculus.*

Proof. Let g, h be as in Definition 16.3.12, and let $F \in H^1(\Sigma_\sigma; \mathscr{A}) \cap H^\infty(\Sigma_\sigma; \mathscr{A})$. Choose $\omega(A) < \nu < \sigma$. For $z \in \mathsf{D}(A) \cap \mathsf{R}(A)$ we estimate

$$\begin{aligned} \|h(tA)F(A)z\| &\leqslant \frac{1}{2\pi} \int_{\partial\Sigma_\nu} |h(t\lambda)F(\lambda)| \|R(\lambda, A)z\| \, |\mathrm{d}\lambda| \\ &\leqslant \frac{M_{\nu,A}}{2\pi} \|F\|_\infty \int_{\partial\Sigma_\nu} |h(t\lambda)| \frac{|\mathrm{d}\lambda|}{|\lambda|} \|z\| \\ &\leqslant \frac{M_{\nu,A}}{\pi} \|F\|_\infty \|z\|, \end{aligned}$$

where $M_{\nu,A} = \sup_{\lambda \in \complement\overline{\Sigma_\nu}} \|\lambda R(\lambda, A)\|$ is finite by sectoriality. Now, given $y \in \mathsf{D}(A) \cap \mathsf{R}(A)$, we let $x := F(A)y$, note that $x \in \mathsf{D}(A) \cap \mathsf{R}(A)$, and apply the estimate with $z := g(tA)y$ to obtain

$$\|h(tA)g(tA)x\| \leqslant \frac{M_{\nu,A}}{\pi}\|F\|_\infty\|\|g(tA)y\|.$$

By the absolute calculus, this implies

$$\|F(A)y\| = \frac{M_{\nu,A}}{\pi}\|x\| \leqslant M\frac{M_{\nu,A}}{\pi}\|F\|_\infty\|y\|.$$

By the same argument as in the proof of Theorem 10.2.13, this proves the first assertion.

The second assertion follows by taking $F(\lambda) = f(\lambda)I_X$. □

Theorem 16.3.14 (Closedness from the absolute calculus). *Let A and B be resolvent commuting densely defined sectorial operators. If A has an absolute calculus on Σ_σ and $\sigma + \omega(B) < \pi$, then the operator $A + B$ with domain $\mathsf{D}(A + B) = \mathsf{D}(A) \cap \mathsf{D}(B)$ is closed and*

$$\|Ax\| + \|Bx\| \leqslant C(A + B)x\|, \quad x \in \mathsf{D}(A) \cap \mathsf{R}(A).$$

Proof. By Theorem 16.3.13, A admits a bounded operator-valued $H^\infty(\Sigma_\sigma)$-calculus. Since B is τ-sectorial, the family $f(z, B) = -zR(-z, B)$, $z \in \Sigma_{\pi-\tau}$, is uniformly bounded and commutes with the resolvent of A. Now Theorem 16.3.13 implies that $f(A, B)$ is well defined in the operator-valued calculus as a bounded operator on X. The reverse Hölder inequality

$$\|Ax\| + \|Bx\| \leqslant C\|(A + B)x\|, x \in \mathsf{D}(A) \cap \mathsf{D}(B),$$

is obtained by same argument as in Theorem 16.3.6. It was already observed that the closedness of $A + B$ follows from it. □

We will prove in the next section (see Theorem 16.3.18) that a standard sectorial operator on a Hilbert space has an absolute calculus if and only if it has a bounded imaginary powers. Taking this for granted for now, as a special case of Theorem 16.3.14 we recover the following classical result.

Theorem 16.3.15 (Da Prato–Grisvard). *Let A and B be resolvent commuting sectorial operators in a Hilbert space H. If A has bounded imaginary powers, B is densely defined, and $\omega_{\rm BIP}(A) + \omega(B) < \pi$, then the operator $A + B$ with domain $\mathsf{D}(A + B) = \mathsf{D}(A) \cap \mathsf{D}(B)$ is closed and we have*

$$\|Ax\| + \|Bx\| \leqslant C\|(A + B)x\|, \quad x \in \mathsf{D}(A) \cap \mathsf{D}(B),$$

with C a constant independent of x.

Comparing this result with the Dore–Venni theorem, where both A and B are assumed to have bounded imaginary powers, we observe that here, boundedness of the imaginary powers is imposed only on A.

16.3.e The absolute calculus and real interpolation

In this subsection and the next, we show connect the absolute calculus with the theory of real interpolation. The crucial observation is contained in the following theorem.

Theorem 16.3.16 (L^p-bounds imply absolute calculus). *Let A be a sectorial operator in a Banach space X and let $\omega(A) < \sigma < \pi$. Let $1 \leqslant p \leqslant \infty$, and suppose that there exist $\phi \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ such that*

$$\|x\|_{\phi,p} := \big\|t \mapsto \phi(tA)x\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)}, \quad x \in \mathsf{D}(A) \cap \mathsf{R}(A),$$

induces an equivalent norm on $\overline{\mathsf{D}(A) \cap \mathsf{R}(A)}$, the finiteness of the norms on the right-hand side being part of the assumptions. Then A has an absolute calculus on Σ_σ.

The proof depends on the following lemma.

Lemma 16.3.17. *Let A be a sectorial operator acting in X and let $\omega(A) < \sigma < \pi$. If for some $p \in [1, \infty]$ and some $\psi \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ one has $t \mapsto \psi(tA)x \in L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X)$ for all $x \in \overline{\mathsf{D}(A) \cap \mathsf{R}(A)}$, then for all $\phi \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ one has $t \mapsto \phi(tA)x \in L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X)$ for all $x \in \overline{\mathsf{D}(A) \cap \mathsf{R}(A)}$ and we have the equivalence of norms*

$$\big\|\phi(\cdot A)x\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)} \eqsim_{\phi,\psi,\sigma,A} \big\|\psi(\cdot A)x\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)}$$

with implied constants independent of x.

Proof. Let $\psi \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ have the properties as stated, and let $\phi \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ be arbitrary and fixed. Choose an auxiliary function $g \in H^1(\Sigma_\sigma)$ such that

$$\int_0^\infty g(t)\psi(t) \frac{\mathrm{d}t}{t} = 1.$$

First we let $x \in \mathsf{D}(A) \cap \mathsf{R}(A)$. By the Calderón reproducing formula (Proposition 10.2.5) and the multiplicativity of the Dunford calculus,

$$\int_0^\infty g(tA)\psi(tA)x \frac{\mathrm{d}t}{t} = x \tag{16.24}$$

with improper convergence of the left-hand side integral. Fix $\omega(A) < \nu < \sigma$.

For all $s > 0$ and $0 < r < R < \infty$, by Fubini's theorem and multiplicativity we have

$$\begin{aligned}\int_r^R \phi(sA)g(tA)\psi(tA)x \frac{\mathrm{d}t}{t} &= \int_r^R \Big(\frac{1}{2\pi i}\int_{\partial\Sigma_\nu} \phi(s\lambda)g(t\lambda)R(\lambda,A)\,\mathrm{d}\lambda\Big)\psi(tA)x \frac{\mathrm{d}t}{t} \\ &= \frac{1}{2\pi i}\int_{\partial\Sigma_\nu} \phi(s\lambda)\Big(\int_r^R g(t\lambda)R(\lambda,A)\psi(tA)x \frac{\mathrm{d}t}{t}\Big)\,\mathrm{d}\lambda.\end{aligned} \tag{16.25}$$

By (16.24) (with x replaced by $\phi(sA)x$), upon passing to the limits $r \downarrow 0$ and $R \to \infty$ in (16.25) (using dominated convergence to deal with the right-hand side) we obtain

$$\begin{aligned}\phi(sA)x &= \int_0^\infty \phi(sA)g(tA)\psi(tA)x \,\frac{\mathrm{d}t}{t} \\ &= \int_0^\infty \Big(\frac{1}{2\pi i}\int_{\partial\Sigma_\nu} \phi(s\lambda)g(t\lambda)R(\lambda,A)\,\mathrm{d}\lambda\Big)\psi(tA)x\,\frac{\mathrm{d}t}{t} \\ &= \frac{1}{2\pi i}\int_{\partial\Sigma_\nu} \phi(s\lambda)G(\lambda)x\,\mathrm{d}\lambda\end{aligned}$$

with

$$G(\lambda) := \int_0^\infty g(t\lambda)R(\lambda,A)\psi(tA)\,\frac{\mathrm{d}t}{t}.$$

Applying Young's inequality for $L^1(\mathbb{R}_+, \frac{\mathrm{d}t}{t})$ twice (after parametrising $\partial\Sigma_\nu$ and substituting $s \mapsto s^{-1}$), we obtain that $\phi(\cdot A)x \in L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X)$ and

$$\big\|\phi(\cdot A)x\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)} \leqslant \frac{M_{\nu,A}}{\pi}\|\phi\|_{H^1(\Sigma_\sigma)}\|g\|_{H^1(\Sigma_\sigma)}\big\|\psi(\cdot A)x\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)}.$$

Now that we know that $t \mapsto \phi(tA)x$ belongs to $L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X)$, the opposite norm estimate is obtained by reversing the roles of ϕ and ψ. This proves the theorem for $x \in \mathsf{D}(A) \cap \mathsf{R}(A)$.

For $x \in \overline{\mathsf{D}(A) \cap \mathsf{R}(A)}$ the result follows by approximation, noting that $\psi(tA)x\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)} \leqslant C\|x$ by a closed graph argument. □

Proof of Theorem 16.3.16. Fix functions $g, h \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$, with $\|h\|_{H^1(\Sigma_\sigma)} = 1$ for some $\omega(A) < \nu < \sigma$, and assume that $x, y \in \mathsf{D}(A) \cap \mathsf{R}(A)$ satisfy

$$\|h(tA)g(tA)x\| \leqslant \|g(tA)y\|, \quad t > 0.$$

Let $1 \leqslant p \leqslant \infty$ and $f \in L^p(\mathbb{R}_+; \frac{\mathrm{d}t}{t}; X)$ be as in the assumptions in the theorem. Then, by Lemma 16.3.17, applied to $\phi = g$ and $\psi = g \cdot h$, the functions $t \mapsto g(tA)x$, $t \mapsto h(tA)g(tA)x = (hg)(tA)x$, and $t \mapsto g(tA)y$ belong to $L^p(\mathbb{R}, \frac{\mathrm{d}t}{t}; X)$ and

$$\begin{aligned}\|x\| &\eqsim \big\|f(\cdot A)x\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)} \eqsim \big\|h(\cdot A)g(\cdot A)x\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)} \\ &\leqslant \big\|g(\cdot A)y\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)} \eqsim \big\|f(\cdot A)y\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)} \eqsim \|y\|\end{aligned}$$

with implied constants independent of x and y. Hence, g and h satisfy the condition in the definition of the absolute calculus. □

An immediate application if the following characterisation of the absolute calculus in Hilbert spaces.

Theorem 16.3.18 (Hilbert space case). *For a standard sectorial operator A acting in a Hilbert space H, the following assertions are equivalent:*

(1) *A has a bounded H^∞-calculus;*
(2) *A has an absolute calculus;*
(3) *A has bounded imaginary powers.*

In this situation we have

$$\omega_{H^\infty}(A) = \omega_{\mathrm{abs}}(A) = \omega_{\mathrm{BIP}}(A).$$

Further equivalences are obtained in Theorems 10.4.21 (square function estimates) and 10.4.22 (generation of a contraction semigroup with respect to some equivalent Hilbertian norm).

Proof. The implication (2)⇒(1) has already been proved in Theorem 16.3.13. The implication (1)⇒(2) follows from the same theorem, because the boundedness of the H^∞-calculus of a sectorial operator on H implies the square function bounds

$$\|x\|_H \eqsim \|t \mapsto g(tA)x\|_{\gamma(L^2(\mathbb{R}_+,\frac{\mathrm{d}t}{t}),H)} = \Big(\int_0^\infty \|g(tA)x\|^2 \,\frac{\mathrm{d}t}{t}\Big)^{1/2}$$

by Theorem 10.4.16 and Proposition 9.2.9.

For standard sectorial operators A, the equivalence (1)⇔(3) is contained in Theorem 15.3.23. □

The main result of this section is Theorem 16.3.20 which asserts that invertible sectorial operators have a bounded H^∞-calculus on the real interpolation spaces $(X, \mathsf{D}(A))_{\theta,p}$. We begin with a general result which describes these interpolation spaces in terms of the Dunford calculus of A.

Theorem 16.3.19 (Real interpolation spaces between X and $\mathsf{D}(A)$). *Let $0 < \theta < 1$ and $p \in [1,\infty]$, and let A be a sectorial operator on X. Let $\omega(A) < \sigma < \pi$ and suppose that $0 \neq \phi \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ is such that the function $z \mapsto z^{-1}\phi(z)$ belongs to $H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ as well. Then*

$$(X, \mathsf{D}(A))_{\theta,p} = \Big\{x \in X : t \mapsto t^{-\theta}\phi(tA)x \in L^p(\mathbb{R}_+, \tfrac{\mathrm{d}t}{t}; X)\Big\}$$

with equivalence of norms

$$\|x\|_{(X,\mathsf{D}(A))_{\theta,p}} \eqsim \|x\| + \big\|t \mapsto t^{-\theta}\phi(tA)x\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)},$$

where the implied constants only depend on σ, A, and ϕ. If $0 \in \varrho(A)$, we also have equivalence of homogeneous norms

$$\|x\|_{(X,\mathsf{D}(A))_{\theta,p}} \eqsim \big\|t \mapsto t^{-\theta}\phi(tA)x\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)}.$$

The theorem should be compared with the first part of Proposition K.4.1, which asserts that If A is a sectorial operator in X, then

$$(X,\mathsf{D}(A))_{\theta,p} = \Big\{x\in X:\ \lambda\mapsto\lambda^\theta\|GR(\lambda,G)x\|\in L^p(\mathbb{R}_+,\tfrac{\mathrm{d}\lambda}{\lambda})\Big\}$$

with equivalence of norms

$$\|x\|_{(X,\mathsf{D}(G))_{\theta,p}} \eqsim \|x\| + \Big\|\lambda\mapsto\lambda^\theta\|G(\lambda+G)^{-1}x\|\Big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}\lambda}{\lambda})}.$$

In the $E(\Sigma_\sigma)$-calculus of A we have $G(\lambda+G)^{-1}=\phi(\lambda^{-1}G)$ with $\phi(z)=z/(z+1)$. The case treated in Theorem L.2.4 corresponds to the choice $\phi(z)=ze^{-z}$.

Proof. '$\subseteq$': For $t>0$ and $x=x_0+x_1$ with $x_0\in X$ and $x_1\in\mathsf{D}(A)$, write

$$\phi(tA)x=\phi(tA)x_0+\phi(tA)x_1$$

and note that $\phi(tA)x_1\in\mathsf{D}(A)$ with $A\phi(tA)x_1=\phi(tA)Ax_1$. Furthermore write $\phi_0(z):=\phi(z)$ and $\phi_1(z):=z^{-1}\phi(z)$. Then

$$\begin{aligned}\|\phi(tA)x\| &\leqslant \|\phi(tA)x\| + t\|(tA)^{-1}\phi(tA)Ax\|\\ &= \|\phi_0(tA)x\| + t\|\phi_1(tA)Ax\|\\ &\leqslant C_{\sigma,A}\big(\|\phi_0\|_{H^1(\Sigma_\sigma)}\|x\| + t\|\phi_1\|_{H^1(\Sigma_\sigma)}\|x\|_{\mathsf{D}(A)}\big),\end{aligned}$$

where $C_{\sigma,A}$ is a constant only depending on σ and A. Taking the infimum over all such decompositions, we obtain

$$\|\phi(tA)x\|\leqslant C_{\sigma,A}\max\{\|\phi_0\|_{H^1(\Sigma_\sigma)},\|\phi_1\|_{H^1(\Sigma_\sigma)}\}K(t,x;X,\mathsf{D}(A)).$$

It follows that if $x\in(X,\mathsf{D}(A))_{\theta,p}$, then $t\mapsto t^{-\theta}\phi(t,A)x\in L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)$ and

$$\begin{aligned}&\|t\mapsto t^{-\theta}\phi(t,A)x\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)}\\ &\qquad\leqslant C_{\sigma,A}\max\{\|\phi_0\|_{H^1(\Sigma_\sigma)},\|\phi_1\|_{H^1(\Sigma_\sigma)}\}\|x\|_{(X,\mathsf{D}(A))_{\theta,p}}.\end{aligned}$$

'$\supseteq$': Let $x\in X$ be such that $t\mapsto t^{-\theta}\phi(tA)x$ belongs to $L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)$. Choose $f\in H^1(\Sigma_\sigma)\cap H^\infty(\Sigma_\sigma)$ in such a way that $f_1(z):=zf(z)$ belongs to $H^1(\Sigma_\sigma)$ and the normalisation condition $\int_0^\infty f(s)\phi(s)\,\frac{\mathrm{d}s}{s}=1$ is satisfied. Noting that $f\phi\in H^1(\Sigma_\sigma)$, for $z\in\Sigma_\sigma$ we define

$$h(z):=\int_0^1 f(sz)\phi(sz)\,\frac{\mathrm{d}s}{s},\quad g(z):=\int_1^\infty f(sz)\phi(sz)\,\frac{\mathrm{d}s}{s}.$$

By substitution, $g(z)+h(z)=1$ for all $z\in\Sigma_\sigma$.

The assumption $t^{-\theta}\phi(tA)x\in L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)$ implies that $t^{-\theta}(f\phi)(tA)x\in L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)$ since $\|(f\phi)(tA)x\|=\|f(tA)\phi(tA)x\|\lesssim_{\sigma,A}\|f\|_{H^1(\Sigma_\sigma)}\|\phi(tA)x\|$ by multiplicativity. Hölder's inequality therefore implies that $(f\phi)(tA)x\in$

$L^p((0,1), \frac{\mathrm{d}t}{t}; X)$ for every $t > 0$. Hence, as in the proof of Proposition 10.2.5 we have

$$h(tA)x = \int_0^1 (f\phi)(stA) \frac{\mathrm{d}s}{s}. \tag{16.26}$$

Next, noting the identities

$$zg(z) = \int_1^\infty zf(sz)\phi(sz) \frac{\mathrm{d}s}{s} = \int_1^\infty s^{-1} f_1(sz)\phi(sz) \frac{\mathrm{d}s}{s},$$

reasoning similarly as for h we have $g(A)x \in \mathsf{D}(A)$ and, for $t > 0$,

$$Ag(tA)x = \int_1^\infty (st)^{-1}(f_1\phi)(stA)x \frac{\mathrm{d}s}{s} = \int_t^\infty s^{-1}(f_1\phi)(sA)x \frac{\mathrm{d}s}{s}.$$

Accordingly, for the decomposition $x = h(tA)x + g(tA)x \in X + \mathsf{D}(A)$ we obtain

$$\begin{aligned} K(t, x; X, \mathsf{D}(A)) &\leqslant \|h(tA)x\| + t\|g(tA)\|_{\mathsf{D}(A)} \\ &= \|h(tA)x\| + t\|g(tA)x\| + t\|Ag(tA)x\| \\ &\leqslant \|h(tA)x\| + t\|g(tA)x\| + t\Big\| \int_t^\infty s^{-1}(f_1\phi)(sA)x \frac{\mathrm{d}s}{s} \Big\|. \end{aligned} \tag{16.27}$$

We have $t\|g(tA)x\| \lesssim_{\sigma,A,\phi} t\|g\|_{H^1(\sigma_\sigma)}\|x\|$. Trivially, we also have

$$K(t, x; X, \mathsf{D}(A)) \leqslant \|x\|,$$

By taking the minimum of these estimates, it follows that

$$\begin{aligned} K(t, x; X, \mathsf{D}(A)) &\lesssim_{\sigma,A,\phi} \|h(tA)x\| + \min\{1,t\}\|x\| + t\Big\| \int_t^\infty s^{-1}(f_1\phi)(sA)x \frac{\mathrm{d}s}{s} \Big\| \\ &=: (I) + (II) + (III). \end{aligned}$$

We will estimate $(t^{-\theta}\times)$ (I), (II), (III) separately.

For term (II) it is immediately clear that $t^{-\theta}\min\{1,t\}\|x\|$ belongs to $L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t})$.

By (16.26) we may estimate term (I) by

$$\|h(tA)x\| = \Big\| \int_0^1 (f\phi)(stA) \frac{\mathrm{d}s}{s} \Big\| \leqslant \int_0^t \|(f\phi)(sA)x\| \frac{\mathrm{d}s}{s} =: \sigma_x(t).$$

We can now apply the first part of Hardy's inequality (Lemma L.3.2) to obtain that $t \mapsto t^{-\theta}\sigma_x(t) \in L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X)$ and

$$\|t \mapsto t^{-\theta}\sigma_x(t)\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X)} \lesssim_{\sigma,A} \frac{1}{\theta}\|t \mapsto t^{-\theta}(f\phi)(tA)x\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X)}$$

$$\lesssim_{\sigma,A} \frac{1}{\theta}\|f\|_{H^1(\Sigma_\sigma)}\|t \mapsto t^{-\theta}\phi(tA)x\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)}.$$

To estimate term (III), we note that

$$t\Big\|\int_t^\infty s^{-1}(f_1\phi)(sA)x\,\frac{\mathrm{d}s}{s}\Big\| \leqslant \|f_1\|_{H^1(\Sigma_\sigma)}\int_t^\infty \|\phi(sA)x\|\,\frac{\mathrm{d}s}{s}$$

by multiplicativity and since $s \geqslant t$ on the domain of integration, Therefore, by the second part of Lemma L.3.2,

$$\Big\|t \mapsto t^{-\theta}\int_t^\infty ts^{-1}(f_1\phi)(sA)x\,\frac{\mathrm{d}s}{s}\Big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)}$$
$$\lesssim_{\sigma,A} \frac{1}{\theta}\|f_1\|_{H^1(\Sigma_\sigma)}\|t \mapsto t^{-\theta}\phi(tA)x\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)}.$$

Combining things, we have shown that $t^{-\theta}K(t,x;X,\mathsf{D}(A))$ belongs to $L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t})$ and

$$\|t^{-\theta}K(t,x;X,\mathsf{D}(A))\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t})} \lesssim_{\sigma,A} \|x\| + \|t^{-\theta}\phi(tA)x\|_{L^1(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)}.$$

This is the same as saying that $x \in (X,\mathsf{D}(A))_{\theta,p}$ and

$$\|x\|_{(X,\mathsf{D}(A))_{\theta,p}} \lesssim_{\sigma,A} \|cx\| + \|t^{-\theta}\phi(tA)x\|_{L^1(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)}.$$

Finally, if $0 \in \varrho(A)$ we may endow $\mathsf{D}(A)$ with the equivalent norm $x \mapsto \|Ax\|$. In doing so, the term (II) disappears and the first equivalence of homogeneous norms is obtained.

Suppose next that A is invertible, in (16.27) we can estimate

$$\|g(tA)x\| \leqslant \|A^{-1}\|\|g(tA)Ax\|$$

and therefore the second term can be estimated in the same way at the third term appearing in the second line of (16.27). □

Theorem 16.3.20 (Absolute calculus on real interpolation spaces). *Let A be a densely defined sectorial operator with $0 \in \varrho(A)$. Then for all $1 \leqslant p \leqslant \infty$ and $0 < \theta < 1$, the part $A_{\theta,p}$ of A in the real interpolation space $(X,\mathsf{D}(A))_{\theta,p}$ has an absolute calculus.*

The proof of this theorem depends on the following lemma.

Lemma 16.3.21. *Under the assumptions of the theorem, for every $\alpha < 1-\theta$ the norm*

$$\|x\|_\alpha = \Big(\int_0^\infty \big(t^{-\theta}\zeta_\alpha(tA)x\|\big)^p\,\frac{\mathrm{d}t}{t}\Big)^{1/p} \quad \textit{with} \quad \zeta_\alpha(z) = \frac{z^\alpha}{(1+z)^{2\alpha}}, \qquad (16.28)$$

is an equivalent norm on $(X,\mathsf{D}(A))_{\theta,p}$.

Proof. This is an immediate consequence of Theorem 16.3.20, as the condition $\alpha > 1-\theta$ ensures that the conditions of the theorem hold for ζ_α. □

Proof of Theorem 16.3.20. We write $Y := (X, \mathsf{D}(A))_{\theta,p}$ for brevity.

Step 1 – We begin by preparing two helpful estimates.

First, for all $s, t > 0$ we have

$$\|\zeta_\alpha(sA)\zeta_\alpha(tA)\| \leqslant C_A\Big(\min\{\frac{s}{t}, \frac{t}{s}\}\Big)^\alpha. \tag{16.29}$$

Indeed, by multiplicativity of the $E(\Sigma_\sigma)$-functional calculus and the identity

$$\begin{aligned} T := &(tA)^\alpha(1+tA)^{-2\alpha}(sA)^\alpha(1+sA)^{-2\alpha} \\ &= \big(\frac{s}{t}\big)^\alpha\Big(A^{2\alpha}(t^{-1}+A)^{-2\alpha}\Big)\Big((s^{-1})^{2\alpha}(s^{-1}+A)^{-2\alpha}\Big) \end{aligned}$$

we obtain

$$\|T\| \leqslant \big(\frac{s}{t}\big)^\alpha \sup_{t\geqslant 0}\|A(t^{-1}+A)^{-1}\|^{2\alpha}\cdot\sup_{s\geqslant 0}\|s^{-1}(s^{-1}+A)^{-1}\|^{2\alpha} \leqslant C_A\big(\frac{s}{t}\big)^\alpha.$$

Since the same estimate holds with s and t interchanged, this gives (16.29).

Second, for all $s > 0$, $t \in [s, 2s]$, and $x \in X$ we have

$$\|\zeta_\alpha(sA)x\| = \|\zeta_\alpha((s-t/2)A)\zeta((t/2)A)x\| \leqslant C\|\zeta_\alpha((t/2)A)x\|, \tag{16.30}$$

where $C = \sup_{r>0}\|\zeta_\alpha(rA)\|$ is finite by (10.9).

Step 2 – Now we turn to the actual proof of the theorem.

With the notation introduced above, we take $g := \zeta_{2\alpha}$ and $h := \zeta_\delta$, where $\alpha \in \mathbb{N}$ satisfies $\alpha > 1-\theta$ and $\delta > 0$. For $x, y \in Y$ assume that

$$\|h(tA)g(tA)x\|_Y \leqslant \|g(tA)y\|_Y, \quad t > 0. \tag{16.31}$$

Then for all $s > 0$,

$$\begin{aligned} \|\zeta_{3\alpha}(sA)x\| &\lesssim \Big(s^{\theta p}\int_s^{2s} t^{-\theta p}\,\frac{\mathrm{d}t}{t}\Big)\|\zeta_{3\alpha+\delta}(sA)x\|^p \\ &\lesssim s^{\theta p}\int_s^{2s} t^{-\theta p}\|\zeta_\alpha((t/2)A)\zeta_{2\alpha+\delta}(sA)x\|^p\,\frac{\mathrm{d}t}{t} && \text{(by (16.30))} \\ &= (2s)^{\theta p}\int_{2s}^{4s} t^{-\theta p}\|\zeta_\alpha(tA)\zeta_{2\alpha+\delta}(sA)x\|^p\,\frac{\mathrm{d}t}{t} \\ &\lesssim (2s)^{\theta p}\|\zeta_{2\alpha+\delta}(sA)x\|_Y^p && \text{(by (16.28))} \\ &= (2s)^{\theta p}\|h(sA)g(sA)x\|_Y^p \\ &\leqslant (2s)^{\theta p}\|g(sA)y\|_Y^p && \text{(by (16.31))} \\ &\lesssim (2s)^{\theta p}\int_0^\infty t^{-\theta p}\|\zeta_\alpha(tA)\zeta_\alpha(sA)\zeta_\alpha(sA)y\|^p\,\frac{\mathrm{d}t}{t} && \text{(by (16.28))} \end{aligned}$$

$$\begin{aligned}&\lesssim \|\zeta_\alpha(sA)y\|^p\Big((2s)^{\theta p}\int_0^s \Big(t^{-\theta p}\big(\frac{t}{s}\big)^{\alpha p}\,\frac{\mathrm{d}t}{t}\\ &\qquad\qquad + (2s)^{\theta p}\int_s^\infty t^{-\theta p}\big(\frac{s}{t}\big)^{\alpha p}\,\frac{\mathrm{d}t}{t}\Big) \quad \text{(by (16.29))}\\ &\lesssim_{\theta,p} \|\zeta_\alpha(sA)y\|^p\end{aligned}$$

with implied constants depending on A, σ, θ, p, and α. Integrating the left- and right-hand sides in this estimate with respect to $s^{-\theta p}\frac{\mathrm{d}s}{s}$ and using (16.28) twice, we see that $\|x\|_Y \lesssim \|y\|_Y$ with implied constant independent of x and y. This proves that $A_{\theta,p}$ has an absolute calculus on Y. □

Corollary 16.3.22 (Dore). *If A is a densely defined sectorial operator on X, with $0 \in \varrho(A)$, then for all $1 \leqslant p \leqslant \infty$ and $0 < \theta < 1$, the part $A_{\theta,p}$ of A in the real interpolation space $(X, \mathsf{D}(A))_{\theta,p}$ has a bounded H^∞-calculus.*

The invertibility assumption cannot be dropped in the corollary, and hence in the theorem. Indeed, let A be a bounded sectorial operator without a bounded H^∞-calculus (such operators exist, even on a separable Hilbert space, by Corollary 10.2.29). Then $\mathsf{D}(A) = X$ and therefore $(X, \mathsf{D}(A))_{\theta,p} = X$ for all $0 < \theta < 1$ and $p \in [1, \infty]$. By assumption, A doesn't have a bounded H^∞-calculus on this space.

16.4 Notes

Section 16.1

The problem of defining the sum of two unbounded operators A and B can be approached from various angles. Besides the direct approach of defining $A + B$ as (the closure of) the operator given on $\mathsf{D}(A) \cap \mathsf{D}(B)$ by $(A + B)x = Ax + Bx$, which works well if A and B have commuting resolvents, various other approaches can be taken. When A and B generate uniformly bounded C_0-semigroups S and T respectively, conditions can be formulated in order that the limit in the *Trotter product formula*

$$V(t)x := \lim_{n\to\infty} (S(t/n)T(t/n))^n x$$

exist for all $x \in X$, and defines a C_0-semigroup whose generator C is the closure of the operator $A+B$ initially defined on $\mathsf{D}(A)\cap\mathsf{D}(B)$ by $(A+B)x = Ax+Bx$ Engel and Nagel [2000]; resolvent commutation is not needed in these results. A different approach is the form method, suitable when A and B are defined on a Hilbert space with inner product $(\cdot|\cdot)$. This method in provides conditions under which the (closure of the) sum $\mathfrak{c} := \mathfrak{a} + \mathfrak{b}$ of the sesquilinear forms

$$\mathfrak{a}(x, y) := (Ax|y), \quad \mathfrak{b}(x, y) = (Bx|y)$$

is associated with a closed operator C satisfying

$$\mathfrak{c}(x, y) = (Cx|y).$$

Like always, subtle domain questions have to be taken care of. A detailed treatment is given in Kato [1995]; for a gentle introduction see, e.g., Van Neerven [2022].

Section 16.2

In this section some classical perturbation theorems for sectorial operators are extended to R-sectorial operators. Theorem 16.2.4 on relatively bounded perturbations of R-sectorial operators is basically from Kunstmann and Weis [2001], with some improvements of constants. More sophisticated perturbation results using real interpolation are contained in Haak, Haase, and Kunstmann [2006].

Proposition 16.2.6 on perturbations of the H^∞ by multiples of the identity and the main theorem of this section, Theorem 16.2.8 on relatively bounded perturbations of the H^∞-calculus, are from Kalton, Kunstmann, and Weis [2006]. This paper contains a number of variants of the relative boundedness conditions, some of them modelled after form perturbations. Part (iii) of Theorem 16.2.8 was proved independently by Denk, Dore, Hieber, Prüss, and Venni [2004]. The perturbation theorem 16.2.7 for lower order perturbations of the H^∞-calculus is due to Amann, Hieber, and Simonett [1994].

Example 16.2.10 is due to McIntosh and Yagi [1990]; a proof of the fact that the norm of a Toeplitz matrix with bounded real-valued generating function f is bounded by $\|f\|_{L^\infty(\mathbb{T})}$ can be found in Garoni and Serra-Capizzano [2017, Theorem 6.1]. A further example can be found in Kalton [2007]; see also the review paper Batty [2009].

The philosophy behind some of these perturbation theorems is that the boundedness of the H^∞-calculus is encoded in the fractional domain spaces of the operator in the following sense (see Kalton, Kunstmann, and Weis [2006]): If A and B are two standard sectorial operators on a reflexive Banach space X, and if for some $0 < \alpha_1 < \alpha_2 < \frac{3}{2}$ and $j = \{0, 1\}$ we have

$$\mathsf{D}(A^{\alpha_j}) = \mathsf{D}(B^{\alpha_j}) \quad \text{and} \quad \|A^{\alpha_j} x\| \eqsim \|B^{\alpha_j} x\| \tag{16.32}$$

for all x in this common domain, then if one of the operators has a bounded H^∞-calculus, then so does the other. Notice that there are no smallness assumptions here.

The basic idea of the proofs of Theorem 16.2.8, the comparison theorem just quoted, and their variants is to use the relative bounded or the equivalence of norms of (16.32) to show the equivalence of the discrete square function norms

$$x \mapsto \Big\| \sum_{j \in \mathbb{Z}} \varepsilon_j \phi(2^j A) x \Big\|$$

with the corresponding ones for B. Here, ϕ is usually of the form $\phi(z) = z^\alpha(1+z)^n$ with $\alpha < n$. The two conditions (ii) and (iii) of Theorem 16.2.8 and (16.32) correspond to the two sides of the square function estimate.

In Kalton, Kunstmann, and Weis [2006] it is also explained how to use these perturbation theorems to establish the boundedness of the H^∞-calculus for rather general classes of elliptic operators on $H^{s,p}(\mathbb{R}^d)$ or $H^{s,p}(D)$ for smooth domains $D \subseteq \mathbb{R}^d$ with Lopatinskii–Shapiro boundary conditions. The idea is to compare them to constant coefficients. A related approach is used in Denk, Hieber, and Prüss [2003]. For more recent results the reader is referred to the Notes of Chapter 17.

Let us mention two further topics related to the H^∞-calculus and its perturbations.

Extrapolation of the H^∞-calculus in the L^p-scale

We have seen in Chapter 11 that for singular integral it is often a successful strategy to first prove a Hilbert space result, then prove a weaker result on L^{p_0} or some endpoint of the L^p scale, and then extend the Hilbert space result to L^p-spaces by interpolation between 2 and p_0. This idea also proves fruitful for perturbation theorems; see Kunstmann and Weis [2017]. As in the case of classical singular integral operators, the Littlewood–Paley theory and so-called *off-diagonal estimates* play a crucial role. Here, the "Littlewood–Paley decomposition" of a standard sectorial operator B on a space $L^p(S)$ with a bounded H^∞-calculus is expressed as the equivalence of norms

$$\|x\|_{L^p(S)} \eqsim \Big\| \Big(\sum_{j\in\mathbb{Z}} |\phi(2^j B)x|^2 \Big)^{1/2} \Big\|_{L^p(S)}$$

for all $x \in L^p(S)$. Given a second standard sectorial operator A on $L^p(S)$, the following R-boundedness condition expresses that A is "close" to B in terms of the " Littlewood–Paley pieces" $\phi(2^j A)$ and $\phi(2^j B)$:

$$\mathscr{R}\Big(\phi(s2^{j+k}A)\psi(t2^k B) : j \in \mathbb{Z}\Big) \lesssim 2^{-\beta k} \tag{16.33}$$

for some $\beta > 0$ and all $k \in \mathbb{Z}$ and $s,t \in [1,2]$. Kunstmann and Weis [2017] contains the following theorem:

Theorem 16.4.1. *Let A and B be standard sectorial operators consistently defined on $L^2(S)$ and $L^{p_0}(S)$, with $p_0 \in (1,\infty)\setminus\{2\}$, and assume that A is R-sectorial on $L^{p_0}(S)$ and B has a bounded H^∞-calculus on both $L^2(S)$ and $L^{p_0}(A)$. If (16.33) holds for $p = 2$, then A has a bounded H^∞-calculus on both $L^2(S)$ and all spaces $L^p(S)$ with p between 2 and p_0.*

This theorem can be extended to the case where A is defined on a complemented subspace of L^p and A is a "retract" of B in a suitable sense. In this way one can, for example, derive the boundedness of the H^∞-calculus of the Stokes operator on the Helmholtz space $L^p_0(D)$ from the boundedness of the H^∞-calculus of the Laplace operator on $L^p(D)$, for bounded Lipschitz domains $D \subseteq \mathbb{R}^d$ with $d \geqslant 3$ and $|\frac{1}{2} - \frac{1}{p}| < \frac{1}{2d}$ (see Kunstmann and Weis [2017]).

Scales of fractional domain spaces and interpolation

Let A be a standard sectorial operator and denote by $\dot{X}_\alpha$ the completion of $\mathsf{D}(A^\alpha)$ with respect to the norm $\|x\|_\alpha := \|A^\alpha x\|$ for $\alpha \in \mathbb{R}$. The methods of Section 15.3.b show that a Hilbert space operator A has a bounded H^∞-calculus if and only if these fractional domain spaces can be identified with the complex interpolation spaces

$$[\dot{X}_\alpha, \dot{X}_\beta]_\theta = \dot{X}_\gamma$$

with $(1-\theta)\alpha + \theta\beta = \gamma$ for $\alpha \neq \beta$ and $\theta \in (0,1)$. This is not true anymore in Banach spaces, where complex interpolation is related to boundedness of the imaginary powers, rather than the boundedness of the H^∞-calculus. However, such an identification is possible with the help of the γ-interpolation method introduced in the Notes to Section 15.3. It is shown in Kalton, Kunstmann, and Weis [2006, Section 5.3] that a standard γ-sectorial operator A on a Banach space X with non-trivial type has a bounded H^∞-calculus if and only if

$$(\dot{X}_\alpha, \dot{X}_\beta)^\gamma_\theta = \dot{X}_\delta$$

with $(1-\theta)\alpha + \theta\beta = \delta$ for $\alpha \neq \beta$ and $\theta \in (0,1)$. Even when A does not have a bounded H^∞-calculus, the spaces $(\dot{X}_\alpha, \dot{X}_\beta)^\gamma_\theta$ can be identified with certain square function spaces $H^\gamma_{s,A}$ which are defined as the completion of $\mathsf{D}(A^m) \cap \mathsf{R}(A^m)$, $m > |s|+1$, with respect to the norm

$$\big\|t \mapsto t^{-s}\phi(tA)x\big\|_{\gamma(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X)}, \quad x \in \mathsf{D}(A^m) \cap \mathsf{R}(A^m)$$

for some $\phi \in H^1(\sigma_\sigma)$ such that $z \mapsto z^{-s}\phi(z)$ still belongs to $H^1(\sigma_\sigma)$. Complete proofs can be found in Kalton, Lorist, and Weis [2023].

Section 16.3

The operator-sum method as a purely functional analytic approach to evolution equations goes back to Da Prato and Grisvard [1975], where already Theorem 16.3.15 is proved. Our proof of Theorem 16.3.2 follows Haase [2006], where further properties of the operator C extending $A + B$ are discussed.

In the setting of Hilbert spaces, the operator-valued H^∞-calculus was introduced in Albrecht, Franks, and McIntosh [1998]. Theorem 16.3.4 is taken from Kalton and Weis [2001]. It is implicit in Lancien and Le Merdy [1998] (see also Lancien, Lancien, and Le Merdy [1998, Remark 6.5] and Albrecht, Franks, and McIntosh [1998]) that any sectorial operator on a Hilbert space with a bounded H^∞-calculus has a bounded operator-valued H^∞-calculus. In these papers the "right" method of proof for Theorem 16.3.4 was already found, but the crucial ingredient of R-boundedness was still missing.

Theorem 16.3.6 is due to Kalton and Weis [2001]. In the next chapter, the connections of Theorem 16.3.6 with maximal L^p-regularity will be discussed in detail. As we will see in Volume IV, the operator-valued functional calculus of Theorem 16.3.4 can be used to give a short proof for stochastic maximal L^p-regularity; see Van Neerven, Veraar, and Weis [2015b]. In Clément and Prüss [2001] it is shown that if A is an injective operator generating a bounded C_0-group on a UMD space X, and B is an invertible closed linear operator in X resolvent commuting with A such that $\pm iB$ is R-sectorial, then the operator $A+B$ with domain $\mathsf{D}(A)\cap\mathsf{D}(B)$ is closed and invertible. If B is also sectorial with angle $\omega(B) < \frac{1}{2}\pi$, then $A+B$ is sectorial as well, and $\omega(A+B) < \frac{1}{2}\pi$.

Theorem 16.3.9 is due to Lancien, Lancien, and Le Merdy [1998], Lancien and Le Merdy [1998], who extend an earlier result of Albrecht [1994] on L^p-spaces with $1 < p < \infty$.

That Theorem 16.3.10 holds more generally for Banach spaces with the triangular contraction property was shown by Le Merdy [2003].

The absolute functional calculus was introduced in Kalton and Kucherenko [2010], where Theorems 16.3.13, 16.3.14, and 16.3.20 were proved. The definition of the absolute calculus may be a little off-putting if one is accustomed to thinking in terms of spectral theory, but the benefits of this notion is considerable:

- It implies an operator-valued H^∞-calculus and sum-of-operators theorem without the complexities of R-boundedness, just as in Hilbert spaces (see Theorems 16.3.13 and 16.3.14).
- It leads to a simple sufficient condition for the abstract functional calculus of a sectorial operator A in terms of the equivalence

$$\|x\| \eqsim \Big(\int_0^\infty \|\phi(tA)x\|^p \,\frac{\mathrm{d}t}{t}\Big)^{1/p} \tag{16.34}$$

  (see Theorem 16.3.16, implicit in Kalton and Kucherenko [2010]). The criterion already suffices for the common applications and it shows that in L^1, L^2, and $C(K)$ spaces the absolute functional calculus is equivalent to the H^∞-calculus; see Kalton and Kucherenko [2010]. It also shows that the absolute calculus is mainly a tool for real interpolation spaces and non-UMD spaces. Indeed, for every operator A with a bounded H^∞-calculus on L^p with $1 < p < \infty$ we have (cf. Section 10.4)

$$\|x\|_p \eqsim \big\|t\mapsto \phi(tA)x\big\|_{\gamma(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)} \eqsim \Big\|\Big(\int_0^\infty |\phi(tA)x|^2\,\frac{\mathrm{d}t}{t}\Big)^{1/2}\Big\|_p,$$

which is a norm decidedly different from (16.34) when $p \neq 2$. However, in this setting it provides a unified approach to many results of Dore and Da Prato–Grisvard.

- The absolute functional calculus can be characterised in terms of generalised real interpolation spaces, where the role of $L^p(\mathbb{R}_+, \frac{dt}{t})$ is taken over by more general Banach function spaces E over $\mathbb{R}_+$. Essentially, a standard sectorial operator A, acting on an intermediate spaces X for a couple (X_0, X), where X_0 and X_1 are appropriate fractional domain spaces of X, has an absolute calculus if and only if $X = (X_0, X_1)_{\theta,E}$ for some $\theta \in (0, 1)$ and an appropriate choice of such a Banach function space E; see Kalton and Kucherenko [2010], where a precise statement of the result an be found. This characterisation of the absolute calculus compares nicely with the characterisation of the H^∞-calculus through the γ-interpolation method and the close relationship of bounded imaginary powers with the complex interpolation method described in the previous chapter. A necessary condition for the existence of a Banach function space E with $X = (X_0, X_1)_{\theta,E}$ is the monotonicity of the K-functional for the couple (X_0, X_1), in the sense that it has the property that $K(t, x; X_0, X_1) \leqslant K(t, y; X_0, X_1)$ for some $x \in X_0 + X_1$ and $y \in X$, and all $t \geqslant 0$, then $x \in X$ and $\|x\|_X \leqslant c\|y\|_X$. In fact, this is where the definition of the absolute calculus has its origin.

The proof of Theorem 16.3.19 follows Haase [2005], where some additional details have been written out. The theorem also admits a homogeneous version, which is presented in [Haase, 2006, Section 6.4]. A by-product of Theorem 16.3.19 is the equivalence of norms

$$\|x\| + \|t \mapsto t^{-\theta}\phi(tA)x\|_{L^p(\mathbb{R}_+, \frac{dt}{t}; X)} \eqsim \|x\| + \|t \mapsto t^{-\theta}\psi(tA)x\|_{L^p(\mathbb{R}_+, \frac{dt}{t}; X)}$$

for functions $\phi, \psi \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ satisfying the conditions of Theorem 16.3.19. Interestingly, this equivalence of norms remains true under somewhat weaker conditions on ϕ and ψ; see Haase [2006, Theorem 6.4.2]. The proof follows the lines of the equivalence of continuous square functions in Chapter 10, with simplifications due to the fact that various subtleties in the handling of γ-norms can now be avoided. This more general version of the equivalence of norms covers the function $\phi(z) = z/(z + 1)$ which is implicit in the first part of Proposition K.4.1.

Corollary 16.3.22 is a classical result due to Dore [1999]. This result was subsequently generalised to standard sectorial operators in Dore [2001], where it was shown that such operators have a bounded H^∞-calculus on $(X, \mathsf{D}(A) \cap \mathsf{R}(A))_{\theta,p}$; see also Kalton and Kucherenko [2010], who establish their absolute functional calculus.

Sums of non-commuting operators

In Section 16.3 we studied the closedness (and further properties) of sums of operator $A + B$ under the assumption that A and B are resolvent commuting. In this paragraph, we briefly comment on the closedness of sums of non-commuting operators, provided suitable condition bounds and, sometimes, domain compatibility assumptions, are imposed on the commutator $[A, B] = AB - BA$. The first such result was obtained by Da Prato and Grisvard [1975], who proved that the closure of $A + B$ is invertible and sectorial under commutator conditions. Closedness of $A + B$ itself was proved under further conditions in the case that X is a Hilbert space. Labbas and Terreni [1987] obtained similar results under a different type of commutator conditions, and Monniaux and Prüss [1997] proved a Dore–Veni type theorem for non-commuting operators under the commutator condition of Labbas–Terreni. In his PhD thesis, Štrkalj [2000] proved a version of Kalton and Weis [2001] for non-commuting operators under the same condition as Labbas–Terreni in the case that X is a B-convex space, and Prüss and Simonett [2007] proved a similar result under either one of the above commutator conditions without any restrictions on the space X. Moreover, under the condition that A and B has a bounded H^∞-calculus, with one of them R-bounded, it was shown in this paper that $A + B$ has a bounded H^∞-calculus. Similar results were proved under a different commutator conditions in Roidos [2018]. Products of non-commuting operators have been considered in Štrkalj [2000], Haller-Dintelmann and Hieber [2005].

Typical applications of these results include parabolic PDE on wedge or cone domains, where an elliptic operator C can be split into two space directions to obtain simpler operators A and B. This type of application was worked out in detail by Prüss and Simonett [2007], Prüss and Simonett [2006] and continued in Nau and Saal [2012], Maier and Saal [2014], Köhne, Saal, and Westermann [2021]. For another typical application to non-autonomous parabolic problems, in which case ∂_t and $A(t)$ are non-commuting on $L^p(0, T; X)$, the reader is referred to Di Giorgio, Lunardi, and Schnaubelt [2005a]. Applications to hyperbolic problems appear in Alouini and Goubet [2014].

An interesting class of operator sums for non-commuting operators arises in connection with (an abstract version of) the Weyl commutation relation for position and momentum operators. The general theory of such operators has an altogether different flavour due to its connections with the Heisenberg group; the reader is referred to Putnam [1967] for a general overview. In connection with the topics treated in this volume, the following result is worth mentioning. Suppose two d-tuples of operators $A = (A_1, \dots, A_d)$ and $B = (B_1, \dots, B_d)$ acting on a Banach space X are given such that $iA_1, \dots, iA_d$ and $iB_1, \dots, iB_d$ generate bounded C_0-groups satisfying the Weyl commutation relations

$$e^{isA_j} e^{itA_k} = e^{itA_k} e^{isA_j}, \quad e^{isB_j} e^{itB_k} = e^{itB_k} e^{isB_j}$$

$$e^{isA_j} e^{itB_k} = e^{-ist\delta_{jk}} e^{itB_k} e^{isA_j}.$$

Here, for clarity of exposition, we use exponential notation for the C_0-groups involved. Under this condition, the operator sum

$$\frac{1}{2}(A^2 + B^2) = \frac{1}{2}\sum_{j=1}^{d} A_j^2 + B_j^2$$

is the abstract counterpart of the quantum harmonic oscillator. Under the assumption that X is a UMD Banach lattice, it is shown in Van Neerven and Portal [2020] (under an additional boundedness assumption of the Weyl calculus associated with the pair (A, B)) and Van Neerven, Portal, and Sharma [2023] that the operator $\frac{1}{2}(A^2 + B^2) - \frac{1}{2}d$ is R-sectoriality and has a bounded of the H^∞-calculus.