Finite Formulation and Domain-Integrated Field Relations in Electromagnetics—A Synthesis

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***Abstract*—Complementary formulations of the integral type have established themselves as the most adequate approach to computational electromagnetics. This paper proposes a computational strategy that benefits from the advantages offered by the finite formulation of the electromagnetic (EM) field, employing integral field quantities and dual meshes, and by the domain-integrated field relations approach to EM field computation.**

***Index Terms*—Magnetic fields, numerical analysis.**

I. Introduction

COMPLEMENTARY formulations employing integral field equations were introduced in the realm of computational electromagnetics by means of the finite integration method (see [1] and the references therein). The advantages of the integral setup over a differential one were immediately recognized. Another essential feature of it was the use of dual grids as support for carrying out the integration.

The subsequent challenge was to implement a complementary formulation on a simplicial mesh, thus allowing an appropriate treatment of configurations containing curved surfaces. The computational methods that can be assigned to this class follow one of the following approaches:

- Employ "integral" field quantities; employ metric invariant equations making no reference to properties of matter; relate complementary field quantities by means of equivalent constitutive relations that are not defined pointwise.
- Employ computational field quantities expanded on the boundaries of elementary domains; employ integral field relations on those boundaries; employ in these relations field quantities that can be related without requiring any reference to properties of the matter; relate complementary field quantities by means of equivalent constitutive relations that hold for domains.

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The former approach is represented by the finite formulation of electromagnetic (EM) field (see [2], [3] and the references therein) and by a variety of formulations using differential forms [4]. The latter is represented by the domain-integrated field relations approach to electromagnetic (EM) field computation (see [5] and the references therein).

In the following, a comparative analysis of these two approaches is presented. This will suggest the manner to combine them into a computational method that is suitable for the analysis of strongly heterogeneous configurations.

II. Prerequisites

A. Local Field Quantities

Let the position in the three-dimensional (3-D) space be specified by the position vector $\boldsymbol{r}$ and let the time coordinate be t. The field quantities describing the EM field are: the electric field strength $\boldsymbol{E}$, the magnetic flux density $\boldsymbol{B}$, the electric flux density $\boldsymbol{D}$, and the magnetic field strength $\boldsymbol{H}$. The pairs $\{\boldsymbol{E}, \boldsymbol{B}\}$ and $\{\boldsymbol{D}, \boldsymbol{H}\}$ are composed of complementary field quantities. Following the standard method in linear, time-invariant systems theory, the source distributions that excite the configuration are incorporated into the constitutive relations. To this end, the impressed electric polarization $\boldsymbol{P}$ and the magnetization $\boldsymbol{M}$ are employed, these quantities being taken to be piecewise continuous in space and time. For allowing for the usual practice in electromagnetics of accounting for the presence/motion of electric charges, the electric charge density ρ and the electric conduction current density $\boldsymbol{J}$ are added to this list.

In subdomains of continuity of the material parameters, the field quantities are continuous functions of $\boldsymbol{r}$ and t, with finite limiting values assumed upon approaching the boundary of each subdomain. Across material interfaces, the quantities $\boldsymbol{\nu} \times \boldsymbol{E}$, $\boldsymbol{\nu} \cdot \boldsymbol{B}$, $\boldsymbol{\nu} \cdot \boldsymbol{D}$, and $\boldsymbol{\nu} \times \boldsymbol{H}$ (with $\boldsymbol{\nu}$ denoting the unit vector along the normal to the interface) are continuous, the remaining components showing a finite jump discontinuity. As far as the time behavior is concerned, the argumentation presented in [6] indicates that $\boldsymbol{B}$ and $\boldsymbol{D}$ are continuous across any jump discontinuity in the time behavior of the excitation of the configuration.

B. Integral EM Field Variables

In view of accounting for the geometrical properties of the (necessarily bounded) domain of computation, this domain is now interpreted as a finite collection of subdomains. This decomposition allows the definition of geometrical manifolds, namely: points $\mathbf{P}$, curves $\mathbf{C}$, surfaces $\mathbf{S}$ (with boundaries $\partial\mathbf{S}$), and "volumes" $\mathbf{V}$ (with boundaries $\partial\mathbf{V}$) that are attributed

TABLE I
INTEGRAL EM FIELD VARIABLES AND THEIR CORRESPONDING FIELD QUANTITIES

"Configuration" variables	"Source" variables
electric voltage impuls: $\mathcal{E} \leftrightarrow \int_{\mathbf{T}} \mathrm{d}t \int_{\mathbf{C}} \boldsymbol{E}\cdot\boldsymbol{\tau}\,\mathrm{d}L$	electric flux: $\Psi \leftrightarrow \int_{\tilde{\mathbf{S}}} \boldsymbol{D}\cdot\boldsymbol{n}\,\mathrm{d}A$
magnetic flux: $\Phi \leftrightarrow \int_{\mathbf{S}} \boldsymbol{B}\cdot\boldsymbol{n}\,\mathrm{d}A$	magnetic voltage impuls: $\mathcal{F} \leftrightarrow \int_{\tilde{\mathbf{T}}} \mathrm{d}t \int_{\tilde{\mathbf{C}}} \boldsymbol{H}\cdot\boldsymbol{\tau}\,\mathrm{d}L$
	total electric charge: $\mathcal{Q}^c \leftrightarrow \int_{\mathbf{V}} \rho\,\mathrm{d}V$
	electric charge flow: $\mathcal{Q}^f \leftrightarrow \int_{\mathbf{T}} \mathrm{d}t \int_{\tilde{\mathbf{S}}} \boldsymbol{J}\cdot\boldsymbol{n}\,\mathrm{d}A$

either an inner or an outer orientation [2], [3]. Similarly, time instants $\mathbf{I}$ and time intervals $\mathbf{T}$ (with boundaries $\partial\mathbf{T} = \{\mathbf{I}^+, \mathbf{I}^-\}$), that are also given an inner or an outer orientation, are defined. (For distinguishing between the two orientations, objects with an outer orientation will be marked with a tilde.)

With these choices, physical quantities are now associated with "space-time elements". The following set of integral electromagnetic field variables is introduced [2], [3]: the electric voltage impulse $\mathcal{E}$, the magnetic flux Φ, the electric flux Ψ, the magnetic voltage impulse $\mathcal{F}$, the total electric charge $\mathcal{Q}^c$, and the electric charge flow $\mathcal{Q}^f$. The correspondence between integral electromagnetic field variables and local field quantities is indicated in Table I. In this table, $\boldsymbol{n}$ denotes the unit vector along the normal to a surface and $\boldsymbol{\tau}$ the unit vector along the tangent to a curve, their orientation being chosen in accordance to the inner/outer orientations of the relevant geometrical manifolds. Note that the labeling of the field variables as "configuration" and "source" variables goes back to [7, (pp. 4–5)]. Finally, based on [8, (pp. 33)], it is inferred that the *density* of an integral quantity associated with a space element with an inner or an outer orientation should, likewise, be endowed with an inner or an outer orientation.

III. FINITE FORMULATION OF AN EM FIELD

In [2] and [3], a computational method that represents a consistent finite (as opposed to differential) model of the EM field was introduced. The philosophy of this method can be summarized as follows.

- Use global variables as main variables of physics; field quantities follow as densities and rates of global variables.
- Use configuration variables that are associated with space-time elements endowed with an inner orientation and source variables that are associated with space-time elements endowed with an outer orientation.
- Use field equations that relate global variables associated with a specified space-time element to global variables of the same kind, associated with its oriented boundary; these equations should be devoid of any notion of metric.
- Relate global variables of different kinds by means of constitutive relations, only.

Due to the particular manner in which configuration and source variables refer to space-time elements with inner and outer orientation, the application of this computational method calls in a natural manner for the use of dual meshes.

The following field equations are employed by the finite formulation of the EM field [2], [3]:

$$\Phi\big[\partial\mathbf{V}, \mathbf{I}\big] = 0 \tag{1}$$

$$\mathcal{E}\big[\partial\mathbf{S}, \mathbf{T}\big] = \Phi\big[\mathbf{S}, \mathbf{I}^-\big] - \Phi\big[\mathbf{S}, \mathbf{I}^+\big] \tag{2}$$

$$\Psi\big[\partial\tilde{\mathbf{V}}, \tilde{\mathbf{I}}\big] = \mathcal{Q}^c\big[\tilde{\mathbf{V}}, \tilde{\mathbf{I}}\big] \tag{3}$$

$$\mathcal{F}\big[\partial\tilde{\mathbf{S}}, \tilde{\mathbf{T}}\big] = \Psi\big[\tilde{\mathbf{S}}, \tilde{\mathbf{I}}^+\big] - \Psi\big[\tilde{\mathbf{S}}, \tilde{\mathbf{I}}^-\big] + \mathcal{Q}^f\big[\tilde{\mathbf{S}}, \tilde{\mathbf{T}}\big]. \tag{4}$$

For accounting for constitutive properties, the cell method was proposed for Voronoy–Delaunay dual meshes in [2] and for barycentric dual meshes in [3]. The cell method was extended to nonhomogeneous configurations in [9].

IV. DOMAIN-INTEGRATED FIELD RELATIONS METHOD

The method was originally developed in the context of stationary and static magnetic fields [10]. It was designed explicitly for investigating configurations where a discontinuous constitutive behavior persists down to the scale where actual measurements are still feasible, a scale referred to as the mesoscopic one. (In a computational scheme, this scale corresponds to the mesh size of the spatial discretization.) For such strongly heterogeneous configurations, the property of global differentiability throughout the domain of computation is lost. Consequently, a computational scheme stemming from the field equations in differential form does not make sense anymore, and field relations in integral form must be called upon. It is, however, noted that the existence and the uniqueness of the solution of field problems can be proven rigorously for the standard EM field equations in differential form, only. This is reflected in the domain-integrated field relations being constructed such that in any subdomain where the field quantities are continuously differentiable with respect to the space coordinates, they are equivalent to the field equations in differential form. As for the representation of field quantities, it was required that components that remain continuous upon crossing material interfaces are employed exclusively. To this end, an expansion technique using consistently linear edge and face expansion functions was chosen. (For compensating their reduced computational efficiency, an alternative, efficient expansion strategy was presented in [5].)

This computational approach was applied to the analysis of configurations with a very high degree of inhomogeneity, demonstrating its accuracy and robustness. Its success in the case of static and stationary magnetic fields justified the quest for a time-domain counterpart of it. A first proposal in this sense was formulated in [6]. There, the field relations and the pertaining compatibility relations were formulated on the boundaries of arbitrary, bounded domains, that were later identified with the simplices in a simplicial decomposition of the domain of computation. Nevertheless, from a physical (and computational) point of view, it is more adequate to express the relevant relations using a combination of contour and surface integrals.

Let $\mathcal{D}$ be a bounded domain with piecewise smooth boundary $\partial\mathcal{D}$. Let $\mathcal{S}$ be a simply connected subsurface of $\partial\mathcal{D}$ with piecewise smooth boundary $\partial\mathcal{S}$. Let $\boldsymbol{n}$ be the unit vector along the outward normal to $\partial\mathcal{D}$ and $\boldsymbol{\tau}$ be the unit vector along the tangent to $\partial\mathcal{S}$, the orientation on $\partial\mathcal{S}$ and that of $\boldsymbol{n}$ being related by

means of the screw rule. Let $\mathcal{T}$ be a bounded time interval with boundary $\partial\mathcal{T} = \{t_1, t_2\}$. Then, the space-time integrated field relations read

$$\int_{\partial\mathcal{S}\times\mathcal{T}} \boldsymbol{\tau}\cdot\boldsymbol{H}(\boldsymbol{r},t)\,dLdt - \int_{\mathcal{S}} \boldsymbol{n}\cdot\boldsymbol{D}(\boldsymbol{r},t),dA\Big|_{\partial\mathcal{T}} = 0 \quad (5)$$

$$\int_{\partial\mathcal{S}\times\mathcal{T}} \boldsymbol{\tau}\cdot\boldsymbol{E}(\boldsymbol{r},t)\,dLdt + \int_{\mathcal{S}} \boldsymbol{n}\cdot\boldsymbol{B}(\boldsymbol{r},t),dA\Big|_{\partial\mathcal{T}} = 0 \quad (6)$$

while the space-time integrated compatibility relations are

$$\int_{\partial\mathcal{D}} \boldsymbol{n}\cdot\boldsymbol{D}(\boldsymbol{r},t)dA\Big|_{\partial\mathcal{T}} = 0 \quad (7)$$

$$\int_{\partial\mathcal{D}} \boldsymbol{n}\cdot\boldsymbol{B}(\boldsymbol{r},t)dA\Big|_{\partial\mathcal{T}} = 0 \quad (8)$$

where the notation $\Big|_{\partial\mathcal{T}}$ stands for $f(t)\Big|_{\partial\mathcal{T}} = f(t_2) - f(t_1)$. Note that, unlike in the case of the formulation discussed in [6], (7) and (8) are true compatibility relations. For substantiating this, it is sufficient to observe that a summation of the contributions yielded by (5) and (6), when applied to any of the subsurfaces that compose the boundary $\partial\mathcal{D}$, yields automatically the conditions in (7) and (8).

In a computational scheme, the domain $\mathcal{D}$ is identified with each of the simplicial cells $\mathcal{C}_m$ $(m = 1, \ldots, M)$ of the simplicial decomposition of the domain of computation, the surface $\mathcal{S}$ with any of the faces of the simplicial cell $\mathcal{C}_m$ and $\partial\mathcal{S}$ with the closed contour along the edges that enclose the relevant face. The interface boundary conditions then call for a spatial expansion of $\boldsymbol{E}$ and $\boldsymbol{H}$ in terms of consistently linear edge expansion functions and of $\boldsymbol{D}$ and $\boldsymbol{B}$ in terms of consistently linear face expansion functions, in conjunction with a linear interpolation on successive time intervals. As far as the source distributions are concerned, $\boldsymbol{P}$ and $\boldsymbol{M}$ are taken to be at most linearly varying in space and time in the interior of each cell.

For accounting for the constitutive properties, a procedure that is reminiscent of the method employed for stationary and static magnetic fields was proposed in [6]. This procedure minimizes the discrepancy in the stored electric and magnetic energies for the chosen (computational) field quantities. The main advantage of this approach is that electric and magnetic inhomogeneities that manifest themselves simultaneously can be accounted for. It also allows for employing the efficient normalization strategy, originally described in [5], that reduces the potential depreciation of the obtained accuracy due to round-off errors.

V. Comparative Analysis

The formulations described in Sections III and IV have some remarkable similarities.

1) The methods advocate the use of integral models of the electromagnetic field, the integration being carried out in both space and time.
2) Both methods employ a complete set of complementary field quantities.
3) Both methods compute quantities that are continuous across material interfaces, thus preventing possible detrimental round-off effects.

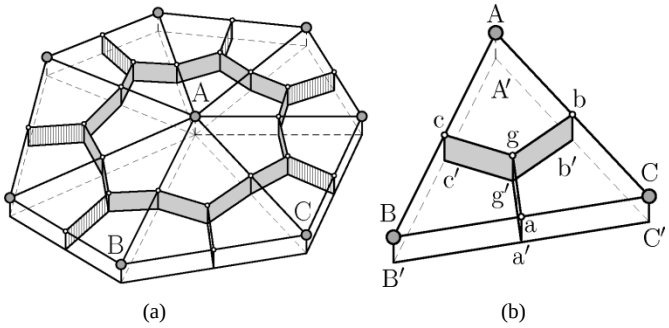
Notable differences between the two methods are as follows.

1) The main difference between the two methods concerns the approach to modeling the constitutive relations. These approaches aim at accommodating the choice for representing (integral) field quantities along lines and surfaces, only, with the fact that the constitutive properties of the matter always manifest themselves in a volume.

   In the finite formulation of electromagnetic field, the field is taken to be uniform in regions of space assimilated to cells [2], [3] or micro-cells [9]. The uniformity condition amounts to assuming that the densities of field variables (assimilated to the local field quantities) are constant over the relevant domains.

   In the case of the domain-integrated field relations method, following from the assumed mesoscopic structure of matter, the material parameters inside each cell may differ from the ones in any of the neighboring cells. Inside each cell, the material parameters are taken to be constant, while the field quantities themselves may vary. Since the field quantities were only defined on the boundaries of the simplicial cells, an extrapolation of these quantities into the interior of those cells is, thus, required. For the assumed simplicial decomposition, algebraic topology ensures the possibility to employ a consistently linear spatial expansion, based on the limiting values of the expanded quantities upon approaching nodes, edges and faces. The expansion coefficients of $\boldsymbol{D}$ are then related to those of $\boldsymbol{E}$ and $\boldsymbol{P}$, and the expansion coefficients of $\boldsymbol{B}$ are related to those of $\boldsymbol{H}$ and $\boldsymbol{M}$, by taking the value of the relevant expansions upon approaching the nodes of the mesh via the interior of each cell. This approach allows for taking into account electric and magnetic inhomogeneities that manifest themselves simultaneously.
2) The distinction made in the finite formulation of the EM field between "configuration" and "source" variables, corroborated with the choice for inner and outer orientations of the space-time elements, calls, naturally, upon the use of dual grids. The domain-integrated field relations method makes no distinction between field quantities, as long as integral relations are satisfied exactly and interface boundary conditions are enforced. For meeting these requirements, the use of a simplicial mesh suffices.
3) As far as the computational aspects are concerned, the application of the finite formulation results in a square system of linear, algebraic equations.

   For the kind of configuration considered above, the domain-integrated field relations method does so, as well. However, this method allows for more general variations of material parameters and source distributions in each cell. In such cases, a minimization of an energy type of norm yields an overdetermined system, the solution of which is computed in the least-squares sense. From a computational point of view, this fact will not influence too much the efficiency of the algorithm, the matrix-matrix multiplications involved by the least-squares solution being carried out at simplicial cell level. However, it can be argued that this approach to solving the system results



(a) (b)

Fig. 1. Primal mesh and its barycentric dual in the simplicial star of an (interior) node; the thickness of the 2-D slice is s: (a) the complete simplicial star and (b) one triangle in the simplicial star.

in the generation of an unphysical mixing of equations originating from field and constitutive relations.

VI. Proposed Computational Philosophy

Based on the arguments discussed above, it is tempting to try to combine the two methods. The combined method should benefit from the highly structured character of the equations pertaining to the finite formulation of electromagnetic field. Further, it should provide the possibility to compute the value of local field quantities, with acceptable accuracy, at any location inside the domain of computation (an aspect required by many practical applications). Finally, the intrinsic pointwise, up to material interfaces, character of the constitutive relations should be clearly manifest in this method. The following computational philosophy is, thus, proposed.

1) Employ *integral field relations* on the boundaries of space-time elements; formulate these relations in terms of field quantities that can be related without requiring any reference to properties of the matter.
2) Select as supports for carrying out integrations the elements of a *dual grid*, constituted from a primal simplicial mesh and its barycentric dual.
3) Expand local field quantities on the primal mesh, only; employ an expansion technique that is consistent with the interface boundary conditions referring to local field quantities; use these field expansions in integral field relations.
4) Relate complementary field quantities by means of constitutive relations derived from minimum *volume* energy arguments; use the constitutive relations for deriving one-to-one mappings of the expansion coefficients pertaining to the relevant complementary quantities.

Note that the combination of the principles 1)–3) is, by and large, equivalent to writing algebraic equations relating integral field quantities of the type employed by the finite formulation of the EM electromagnetic field.

This approach is illustrated by means of the two-dimensional (2-D) configuration in Fig. 1. Material parameters can, in principle, differ from one triangular prism to another. For simplicity, all media in the investigated configuration are taken to be isotropic. The EM field in the configuration is taken to be H-polarized. It then follows that $\boldsymbol{H}$ and, consequently, $\boldsymbol{B}$ are orthogonal to the ground plane, while $\boldsymbol{E}$ and $\boldsymbol{D}$ are parallel to that plane.

In view of complying to the interface boundary conditions, the following *spatial expansion* is employed: $\boldsymbol{E} \leftrightarrow$ consistently linear edge expansion, $\boldsymbol{D} \leftrightarrow$ consistently linear face expansion, $\boldsymbol{H} \leftrightarrow$ Cartesian expansion, with one degree of freedom (DoF) per each node and $\boldsymbol{B} \leftrightarrow$ local Cartesian expansion, with one DoF per each vertex of each triangle. As for the *time discretization* is concerned, linear interpolations based on the values of $\boldsymbol{E}$ and $\boldsymbol{B}$ "sampled" at instants $t = t_0 + n\Delta_t$ ($n = 0, \ldots, N$) and on the values of $\boldsymbol{D}$ and $\boldsymbol{H}$ "sampled" at the instants $\tilde{t} = t_0 + (n + 1/2)\Delta_t$ (with Δ_t denoting a constant time step) are employed.

Relations of type (6) are now invoked for closed contours along the primal mesh, enclosing the nodes where $\boldsymbol{E}$ is unknown [for example, the contour of the complete simplicial star in Fig. 1(a)]. Further, relations of type (5) are written for closed contours along the dual mesh, such as the contour $\{a, a', g', c', c, g, a\}$ in Fig. 1(b). Since expansions of $\boldsymbol{D}$ on the primal mesh are available, only, the relevant contour integrals are expressed for corresponding closed contours along the edges of this mesh (for the contour $\{a, a', g', c', c, g, a\}$ this being $\{a, a', B', c', c, B, a\}$). To this end, (7) is applied to quadrilateral prisms such as the one having the basis $\{B', a', g', c'\}$. Note that, by combining the relations of the type (5) relative to the three shaded contours in Fig. 1(b), the compatibility relation in (7), when applied to the triangular prism, is automatically satisfied. (In view of the configuration being 2-D, the compatibility relation in (8) is trivial.). These relations are supplemented by $\boldsymbol{D} \leftrightarrow \{\boldsymbol{E}, \boldsymbol{P}\}$ and $\boldsymbol{B} \leftrightarrow \{\boldsymbol{H}, \boldsymbol{M}\}$ mappings that follow from the constitutive relations. Finally, in order to reduce the number of DoFs to be computed, the ones associated with $\boldsymbol{E}$ and $\boldsymbol{H}$ are kept, while those associated with $\boldsymbol{B}$ and $\boldsymbol{D}$ are eliminated.

Application of this algorithm, by also taking into account the relevant boundary conditions, will yield a square system of linear algebraic equations that can be solved using suitably chosen standard (iterative) methods.

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