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The Ruijsenaars function transform

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"The Ruijsenaars function transform"

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Preface

This Master thesis is the final project for receiving the Master of Science degree in Applied Mathematics with a specialisation in Computational Science and Engineering. The research has been done within the Analysis Group of the Technical University of Delft. It has been carried out at the Technical University of Delft between March 2010 and April 2011.

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Thank you!

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Mick Kahmann

Chapter 1

Introduction

In this thesis we are going to study a function transform that involves a generalization of the hypergeometric function ${}_2F_1(a, b; c; z)$. This generalization is a part of a family of functions that we call hyperbolic hypergeometric functions. These functions will be discussed in chapter 2. The function transform involves Ruijsenaars' R-function that is defined in [7, (1.30)], which is why we will call it the Ruijsenaars Function Transform or RFT. The R-function is a particular case of the more general hyperbolic hypergeometric function that will be defined in section 2.2. $R_\lambda(x)$ is an eigenfunction of a second-order difference operator $\mathcal{L}_\gamma^{\omega_1,\omega_2}$ for every $\lambda \in \mathbb{R}$. The main goal of this thesis is to prove that the RFT is a unitary operator and we would like to find the inverse operator.

This introduction gives a short review of the hypergeometric series and hypergeometric functions and gives a few properties of the hypergeometric function ${}_2F_1(a, b; c; z)$. Section 1.2 gives an example of a particular type of hypergeometric functions $P_n^{(\alpha,\beta)}(x)$, called the Jacobi polynomials. They form an orthogonal basis for a certain L^2-space and are eigenfunctions of a second-order differential operator D. The corresponding integral transformation is a unitary operator.

1.1 Hypergeometric functions

Let us first recall the theory of hypergeometric functions. Define the shifted factorial $(a)_k$ with $a \in \mathbb{C}$ by

$$(a)_k = a(a+1)\cdots(a+k-1), \quad k \in \mathbb{Z}_{>0} \text{ and } (a)_0 = 1. \tag{1.1}$$

A hypergeometric function is the sum of a hypergeometric series, which is defined as follows: a series $\sum c_n$ is called hypergeometric if the ratio c_{n+1}/c_n is a rational function of n. By factorization this means that

$$\frac{c_{n+1}}{c_n} = \frac{(n+a_1)(n+a_2)\cdots(n+a_p)z}{(n+b_1)(n+b_2)\cdots(n+b_q)(n+1)}, \quad n \in \mathbb{Z}_{\geq 0}. \tag{1.2}$$

Iteration of (1.2) leads to

$$c_n = \frac{(a_1)_n(a_2)_n\cdots(a_p)_n z^n}{(b_1)_n(b_2)_n\cdots(b_q)_n n!}c_0, \quad n \in \mathbb{Z}_{\geq 0}.$$

Hence

$$\sum_{n=0}^{\infty} c_n = c_0 \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n(b_2)_n\cdots(b_q)_n}\frac{z^n}{n!}.$$

This leads to the definition of the hypergeometric function. The hypergeometric function ${}_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; z)$ is defined by means of a hypergeometric series as

$$
{}_pF_q\left(\begin{array}{c} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{array} ; z \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \frac{z^n}{n!}. \tag{1.3}
$$

The parameters must surely be such that the denominator factors in the terms of the series are never zero, so we set $b_1, b_2, \dots, b_q \in \mathbb{C} \setminus \mathbb{Z}_{<0}$. When at least one of the numerator parameters a_j equals a negative integer, the hypergeometric function is a polynomial in z. In all other cases, the radius of convergence ρ of the hypergeometric series is given by

$$
\rho = \begin{cases} \infty & \text{if} \quad p < q+1 \\ 1 & \text{if} \quad p = q+1 \\ 0 & \text{if} \quad p > q+1. \end{cases}
$$

This follows directly from d'Alembert's ratio test. In fact, we have

$$
\lim_{n \to \infty} \left| \frac{c_{n+1}}{c_n} \right| = \begin{cases} 0 & \text{if} \quad p < q+1 \\ |z| & \text{if} \quad p = q+1 \\ \infty & \text{if} \quad p > q+1. \end{cases}
$$

In case that $p = q+1$ the situation that $|z| = 1$ is of special interest. The hypergeometric series ${}_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; z)$ with $|z| = 1$ converges absolutely if $\text{Re}\left(\sum_{i=1}^{q} b_i - \sum_{j=1}^{p} a_j\right) > 0$. The series converges conditionally if $|z| = 1$ with $z \neq 1$ and $-1 < \text{Re}\left(\sum_{i=1}^{q} b_i - \sum_{j=1}^{p} a_j\right) \leq 0$ and the series diverges if $\text{Re}\left(\sum_{i=1}^{q} b_i - \sum_{j=1}^{p} a_j\right) \leq -1$.

Often the most general hypergeometric function ${}_pF_q$ is called a generalized hypergeometric function. By the words "hypergeometric function", we refer to the special case

$$
{}_2F_1\left(\begin{array}{c} a, b \\ c \end{array} ; z \right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}. \tag{1.4}
$$

Many elementary functions have representations as hypergeometric series. An example is

$$
\ln(1+z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} z^{n+1} = \sum_{n=0}^{\infty} \frac{(1)_n (1)_n}{(2)_n} \frac{(-1)^n z^{n+1}}{n!} = z\,{}_2F_1\left(\begin{array}{c} 1, 1 \\ 2 \end{array} ; -z \right),
$$

since $(1)_n = n!$ and $(2)_n = (n+1)!$. Another example is

$$
{}_0F_0\left(\begin{array}{c} - \\ - \end{array} ; z \right) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z, \quad z \in \mathbb{C}. \tag{1.5}
$$

An important role in the theory of hypergeometric functions is played by the gamma function $\Gamma(z)$. The gamma function is defined by

$$
\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt, \quad \text{Re}(z) > 0. \tag{1.6}
$$

An important property of the gamma function is the functional relation

$$
\frac{\Gamma(z+1)}{\Gamma(z)} = z, \quad \text{Re}(z) > 0, \tag{1.7}
$$

which by iteration implies the following relation between the shifted factorial and the gamma function

$$\frac{\Gamma(z+n)}{\Gamma(z)} = (z)_n, \quad \mathrm{Re}(z) > 0, \; n \in \mathbb{Z}_{\geq 0}. \tag{1.8}$$

Noting that we have

$$\Gamma(1) = \int_0^\infty e^{-t} dt = -e^{-t}\big|_0^\infty = 1,$$

this leads to

$$\Gamma(n+1) = n!, \quad n \in \mathbb{Z}_{\geq 0}. \tag{1.9}$$

The functional relation (1.7) can be used to find an analytic continuation for $\mathrm{Re}(z) \leq 0$ that is a meromorphic function with simple poles at $z \in \mathbb{Z}_{\leq 0}$.

We also introduce the beta function $B(u, v)$, that is also defined by means of an integral

$$B(u,v) = \int_0^1 t^{u-1}(1-t)^{v-1} dt, \quad \mathrm{Re}(u) > 0, \quad \mathrm{Re}(v) > 0. \tag{1.10}$$

This integral is often called the beta integral. From the definition we easily obtain the symmetry

$$B(u,v) = B(v,u), \tag{1.11}$$

since by using the substitution $t = 1 - s$, we have

$$B(u,v) = \int_0^1 t^{u-1}(1-t)^{v-1} dt = -\int_1^0 (1-s)^{u-1} s^{v-1} ds = \int_0^1 s^{v-1}(1-s)^{u-1} ds = B(v,u).$$

The connection between the beta function and the gamma function is given by the following theorem.

Theorem 1.1.

$$B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}, \quad Re(u) > 0, \quad Re(v) > 0. \tag{1.12}$$

Proof. By the definition (1.6) of the gamma function, we have

$$\Gamma(u)\Gamma(v) = \int_0^\infty e^{-t} t^{u-1} dt \int_0^\infty e^{-s} s^{v-1} ds = \int_0^\infty \int_0^\infty e^{-(t+s)} t^{u-1} s^{v-1} dt\, ds$$

Now we apply the change of variables $t = xy$ and $s = x(1-y)$ to this double integral. Note that $t + s = x$ and that $0 < t < \infty$ and $0 < s < \infty$ imply that $0 < x < \infty$ and $0 < y < 1$. The Jacobian of this transformation is

$$\frac{\partial(t,s)}{\partial(x,y)} = \begin{vmatrix} y & x \\ 1-y & -x \end{vmatrix} = -xy - x(1-y) = -x.$$

Since $x > 0$ we conclude that $dt\, ds = \left|\frac{\partial(t,s)}{\partial(x,y)}\right| dx\, dy = x\, dx\, dy$. Hence we have

$$\begin{aligned} \Gamma(u)\Gamma(v) &= \int_0^1 \int_0^\infty e^{-x} x^{u-1} y^{u-1} x^{v-1} (1-y)^{v-1} x\, dx\, dy \\ &= \int_0^\infty e^{-x} x^{u+v-1} dx \int_0^1 y^{u-1}(1-y)^{v-1} dy = \Gamma(u+v) B(u,v). \end{aligned}$$

This proves the theorem. □

Two examples of identities involving gamma functions are given in the theorem below. These identities are called the Euler- and Barnes integral representation for the hypergeometric function and they briefly explain why the gamma function is such an important building block in the field of hypergeometric functions. We are not going to prove the Barnes integral representation in detail, but we will give the main idea of this proof.

Theorem 1.2. *For Re(c) > Re(b) > 0 we have Euler's integral representation for the hypergeometric function*

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} ; z \right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt \tag{1.13} $$

for all z in the complex plane cut along the real axis from 1 to ∞.

Proof. First suppose that $|z| < 1$, then the binomial theorem implies that

$$ (1-zt)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n t^n. $$

This implies that

$$ \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n \int_0^1 t^{n+b-1}(1-t)^{c-b-1} dt. $$

The latter integral is a beta integral which by (1.12) equals

$$ \int_0^1 t^{n+b-1}(1-t)^{c-b-1} dt = B(n+b, c-b) = \frac{\Gamma(n+b)\Gamma(c-b)}{\Gamma(n+c)}. $$

Now we use the fact that

$$ \frac{\Gamma(n+b)}{\Gamma(b)} = b(b+1)(b+2)\cdots(b+n-1) = (b)_n, \quad n \in \mathbb{Z}_{\geq 0} $$

to obtain

$$ \begin{aligned} \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt &= \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(n+b)}{\Gamma(n+c)} \frac{(a)_n}{n!} z^n \\ &= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} = {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} ; z \right), \end{aligned} $$

which proves the theorem for $|z| < 1$. Since the integral is analytic in the cut plane $\mathbb{C} \setminus (1, \infty)$, the theorem holds in that region as well. □

Theorem 1.3. *We also have Barnes' integral representation for the hypergeometric function*

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} ; z \right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)} (-z)^s ds, \quad |\arg(-z)| < \pi. \tag{1.14} $$

The path of integration is curved, if necessary, to separate the poles $s = -a - n$ and $s = -b - n$ from the poles $s = n$ with $n \in \mathbb{Z}_{\geq 0}$. Such a contour always exists if $a, b \notin \mathbb{Z}_{\leq 0}$.

Proof. We will only give an idea of the proof. Let $\mathcal{C}$ be the closed contour formed by a part of the curve used in the theorem from $-(N+\frac{1}{2})i$ to $(N+\frac{1}{2})i$ together with the semicircle of radius $N+\frac{1}{2}$ to the right of the imaginary axis with 0 as center. It can be proved that the integral is an analytic function for $|\arg(-z)| < \pi$ and converges to zero on the semicircle for $N \to \infty$. Using Cauchy's residue theorem implies that the integral tends to the limit of the sums of the residues at $s = n$ with $n \in \mathbb{Z}_{\geq 0}$. This infinite sum is equal to

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}\, {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right),$$

which proves the theorem. □

1.2 Jacobi polynomials

The Jacobi polynomials are a well-known kind of orthogonal polynomials. A representation of the Jacobi polynomials is

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n}\sum_{k=0}^{n}(-1)^k\binom{n+\alpha}{k}\binom{n+\beta}{n-k}(1-x)^{n-k}(1+x)^k, \quad n \in \mathbb{Z}_{\geq 0}. \tag{1.15}$$

So $P_n^{(\alpha,\beta)}(x)$ is a polynomial of degree n. Note that by this representation we have the symmetry

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x), \quad n \in \mathbb{Z}_{\geq 0}. \tag{1.16}$$

A hypergeometric representation of $P_n^{(\alpha,\beta)}(x)$ is

$$P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n}\, {}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix}; \frac{1-x}{2}\right), \quad n \in \mathbb{Z}_{\geq 0}. \tag{1.17}$$

In view of the symmetry (1.16) we also have another hypergeometric representation, which is

$$P_n^{(\alpha,\beta)}(x) = (-1)^n\binom{n+\beta}{n}\, {}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \beta+1 \end{matrix}; \frac{1+x}{2}\right), \quad n \in \mathbb{Z}_{\geq 0}. \tag{1.18}$$

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are orthogonal on the interval $(-1, 1)$ with respect to the beta distribution $w(x) = (1-x)^\alpha(1+x)^\beta$.

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are solutions of the second-order linear differential equation

$$(1-x^2)y''(x) + [\beta-\alpha-(\alpha+\beta+2)x]y'(x) + n(n+\alpha+\beta+1)y(x) = 0, \quad n \in \mathbb{Z}_{\geq 0}. \tag{1.19}$$

This means that we can define the differential operator D by

$$D := (1-x^2)\frac{d^2}{dx^2} + [\beta-\alpha-(\alpha+\beta+2)x]\frac{d}{dx} \tag{1.20}$$

This implies that $P_n^{(\alpha,\beta)}(x)$ satisfies the eigenvalue equation

$$DP_n^{(\alpha,\beta)}(x) = \lambda_n P_n^{(\alpha,\beta)}(x), \tag{1.21}$$

where $\lambda_n = -n(n+\alpha+\beta+1)$. We can define an inner product $\langle\cdot,\cdot\rangle$ by

$$\langle f, g\rangle := \int_{-1}^{1} f(x)\overline{g(x)}w(x)dx \tag{1.22}$$

and it can be shown that $\langle Df, P_n^{(\alpha,\beta)}\rangle = \langle f, DP_n^{(\alpha,\beta)}\rangle$. It is now easily shown that the Jacobi polynomials are orthogonal with respect to this inner product. We have the following

$$\lambda_n \langle P_n^{(\alpha,\beta)}, P_m^{(\alpha,\beta)}\rangle = \langle DP_n^{(\alpha,\beta)}, P_m^{(\alpha,\beta)}\rangle = \langle P_n^{(\alpha,\beta)}, DP_m^{(\alpha,\beta)}\rangle = \lambda_m \langle P_n^{(\alpha,\beta)}, P_m^{(\alpha,\beta)}\rangle. \tag{1.23}$$

We know that for $n \neq m$ we have that $\lambda_n \neq \lambda_m$. Thus, in case that $n \neq m$ we have that $\langle P_n^{(\alpha,\beta)}, P_m^{(\alpha,\beta)}\rangle = 0$, which means that the Jacobi polynomials are orthogonal with respect to the inner product. Even more specific, the Jacobi polynomials satisfy the orthogonality relation

$$\int_{-1}^{1} P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x) w(x) dx = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)n!}\delta_{mn} = A_n \delta_{mn} \tag{1.24}$$

for $\alpha > -1, \beta > -1,\ m, n \in \mathbb{Z}_{\geq 0}$ and the Kronecker delta function which is defined by

$$\delta_{mn} := \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}$$

for $m, n \in \mathbb{Z}_{\geq 0}$. Together with the orthogonality relation (1.24) it can be shown that $\left\{P_n^{(\alpha,\beta)}\right\}_{n=0}^{\infty}$ is an orthogonal basis of $L^2\left[(-1,1), w(x)\right]$. This means that we may write every $f \in L^2\left[(-1,1), w(x)\right]$ as

$$f = \sum_{n=0}^{\infty} \frac{\hat{f}_n}{\left\|P_n^{(\alpha,\beta)}\right\|^2} P_n^{(\alpha,\beta)} = \sum_{n=0}^{\infty} A_n^{-1} \hat{f}_n P_n^{(\alpha,\beta)}, \tag{1.25}$$

where

$$\hat{f}_n = \langle f, P_n^{(\alpha,\beta)}\rangle = \int_{-1}^{1} f(x) P_n^{(\alpha,\beta)}(x) w(x) dx. \tag{1.26}$$

We may view $\hat{f}_n$ as a function transform on $L^2\left[(-1,1), w(x)\right]$ to $l^2(\mathbb{Z}_{\geq 0})$ that is similar to the Ruijsenaars Function Transform (5.34). It can be shown that $\langle Df, P_n^{(\alpha,\beta)}\rangle = \langle f, DP_n^{(\alpha,\beta)}\rangle$, so

$$\begin{aligned} \widehat{Df}_n &= \int_{-1}^{1} (Df)(x) P_n^{(\alpha,\beta)}(x) w(x) dx \\ &= \int_{-1}^{1} f(x) \left(DP_n^{(\alpha,\beta)}\right)(x) w(x) dx \\ &= \lambda_n \int_{-1}^{1} f(x) P_n^{(\alpha,\beta)}(x) w(x) dx \\ &= \lambda_n \hat{f}_n. \end{aligned} \tag{1.27}$$

This means that the integral transformation (1.26) interchanges the differential operator D by a multiplication operator.

1.3 Motivation for this thesis

The unitarity of the RFT is already mentioned in another form in [9, Corr. 3.2], but is far from an objective in that article. This thesis aims to achieve the unitarity of the RFT in another way. This is inspired by the way unitarity of the Askey-Wilson function transform is achieved in the article by E. Koelink and J.V. Stokman [1] and the way unitarity of the Wilson function transform is achieved in the article by W.G.M. Groenevelt [4]. A fundamental difference in the development in these articles, as opposed to Ruijsenaars' methods, is the c-function expansion they use for the Askey-Wilson function [1, (4.5)] and the Wilson function [4, (4.20)]. These expansions give an easier way to find asymptotics of these functions then the way Ruijsenaars [8] finds the asymptotics of the $\mathcal{E}$-function (4.26) that is essentially Ruijsenaars' R-function.

There are not a lot articles written on hyperbolic hypergeometric functions and we think that this thesis will be a welcome addition to the work that is done on hyperbolic hypergeometric functions. The way it is written is also consistent with the articles of Koelink, Stokman and Groenevelt that were just mentioned.

1.4 Overview of this thesis

In this section we will give a short overview of the different chapters of this thesis and the aspects of the theory which are presented there.

In Chapter 2 we consider in a uniform way the hyperbolic hypergeometric functions as they play an important role in this thesis. This discusses the general hyperbolic hypergeometric function and deals with two important degenerations of this function that are called the hyperbolic Barnes- and Euler integral. The hyperbolic gamma function is an important building block for these functions and the Barnes- and Euler integral are important for constructing difference equations for Ruijsenaars' R-function.

Chapter 3 studies the construction of contiguous relations for the hyperbolic hypergeometric function and the hyperbolic Euler integral. The contiguous relation for the hyperbolic Euler integral is the result of a succesful degeneration of the contiguous relation for the general hyperbolic hypergeometric function.

Ruijsenaars R-function will be defined in Chapter 4. A few important symmetries in its parameters will be discussed afterwards. The contiguous relation for the hyperbolic Euler integral along with its relation (2.48) to the hyperbolic Barnes integral imply four second-order difference equations of Askey-Wilson type. Section 4.2 defines a related difference operator $\mathcal{L}_\gamma^{\omega_1,\omega_2}$ for which Ruijsenaars' R-function is an eigenfunction and also discusses the $\mathcal{E}$-function which is important in Ruijsenaars' articles [8, 9].

Chapter 5 defines a Hilbert space $\mathcal{H}_w$ with an appropriate weight function $w(\gamma; x)$. The eigenvalue equation that involves $\mathcal{L}_\gamma^{\omega_1,\omega_2}$ and R give rise to equalities for the inner product that will also be defined in this chapter. For functions f that meet certain conditions we will define the RFT $\mathcal{F}$ as follows

$$\left(\mathcal{F}f\right)(\lambda) = \int_{-\infty}^{\infty} f(x) R_\lambda(x) w(\gamma; x) dx.$$

The inverse transformation will also be defined and altogether it will be made clear that the RFT is a unitary operator on $\mathcal{H}_w$.

Finally, in Chapter 6 we will discuss further research for the RFT. When the weight function w has poles, then we can make a corresponding weight that has a discrete spectrum. We will see that in this case the operator is also unitary, but it takes some more calculations to get there.

Chapter 2

Introduction to hyperbolic hypergeometric functions

Throughout this thesis we fix $\omega_1, \omega_2 \in \mathbb{R}_{\geq 0}$ satisfying $\omega_1/\omega_2 \notin \mathbb{Q}$ and we write

$$\omega = \frac{\omega_1 + \omega_2}{2}. \tag{2.1}$$

We will use a few shorthand notations in this thesis for expressions that occur frequently. A lot of functions (say f) will depend on ω_1 and ω_2, so we will write $f(z) = f(\omega_1, \omega_2; z)$ or $f(z) = f(z; \omega_1, \omega_2)$ if ω_1 and ω_2 are at their 'usual' places. Since the product $f(a+b)f(a-b)$ frequently occurs in this thesis, we use for this product the shorthand notation $f(a \pm b)$.

2.1 The hyperbolic gamma function

First, consider the integral

$$g(\omega_1, \omega_2; z) = \int_0^\infty \left(\frac{\sin(2yz)}{2\sinh(\omega_1 y)\sinh(\omega_2 y)} - \frac{z}{\omega_1 \omega_2 y} \right) \frac{dy}{y}. \tag{2.2}$$

Because $|\sin(2yz)| = \mathcal{O}\left(e^{2y|\mathrm{Im}(z)|}\right)$ and $|2\sinh(\omega_1 y)\sinh(\omega_2 y)| = \mathcal{O}(e^{2\omega y})$, we have that

$$\left| \frac{\sin(2yz)}{2y\sinh(\omega_1 y)\sinh(\omega_2 y)} \right| = \mathcal{O}\left(y^{-1} e^{2y(|\mathrm{Im}(z)|-\omega)}\right), \tag{2.3}$$

$$\left| -\frac{z}{\omega_1 \omega_2 y^2} \right| = \mathcal{O}\left(y^{-2}\right). \tag{2.4}$$

Defining the strip

$$S = \{ z \in \mathbb{C} \mid |\mathrm{Im}(z)| < \omega \}, \tag{2.5}$$

it is now clear from (2.2),(2.3) and (2.4) that the integral converges absolutely and uniformly on compact subsets of S. This implies that $g(\omega_1, \omega_2; z)$ is analytic in S. Ruijsenaars' [7] hyperbolic gamma function is now defined by

$$G(\omega_1, \omega_2; z) = \exp(ig(\omega_1, \omega_2; z)). \tag{2.6}$$

It is obviously analytic and zero-free in S. There exists (it is not obvious, but true) a unique meromorphic extension of $G(\omega_1, \omega_2; z)$ to $z \in \mathbb{C}$ satisfying

$$
\begin{aligned}
G(\omega_1, \omega_2; -z) &= 1/G(\omega_1, \omega_2; z), && (2.7)\\
G(\omega_2, \omega_1; z) &= G(\omega_1, \omega_2; z), && (2.8)\\
\overline{G(\omega_1, \omega_2; z)} &= G(\omega_1, \omega_2; -z) && (2.9)\\
G(\lambda\omega_1, \lambda\omega_2; \lambda z) &= G(\omega_1, \omega_2; z), \quad \lambda \in (0, \infty). && (2.10)
\end{aligned}
$$

These properties are easily derived by substitution into the definition of G and elementary mathematics. The first identity is called the reflection equation. We sometimes us the shorthand notation $G(a_1, a_2, \ldots, a_n)$, that stands for the product $G(a_1)G(a_2) \cdot \ldots \cdot G(a_n)$. We proceed by giving a few properties that are important throughout this thesis.

The hyperbolic gamma function $G(z)$ is a generalization of the 'regular' gamma function $\Gamma(z)$ in the following way

$$
\lim_{v \downarrow 0} G(\omega_1, \omega_2/v; i\omega_1(1/2 - z) + i\omega_2/2v) \left(\frac{2\pi v \omega_1}{\omega_2}\right)^{\frac{1}{2}-z} = \frac{\Gamma(z)}{\sqrt{2\pi}}. \tag{2.11}
$$

This limit is due to Ruijsenaars [6, Prop. III.6]. We will make a lot of use of the analytic difference equations in the next proposition.

Proposition 2.1. *The hyperbolic gamma function $G(\omega_1, \omega_2, z)$ satisfies the following first order Analytic Difference Equations (also called $A\Delta E$'s)*

$$
\frac{G(z + i\omega_1/2)}{G(z - i\omega_1/2)} = 2\cosh(\pi z/\omega_2) \qquad \frac{G(z + i\omega_2/2)}{G(z - i\omega_2/2)} = 2\cosh(\pi z/\omega_1), \tag{2.12}
$$

Proof. Define Γ_h for $0 < \mathrm{Im}(z) < \mathrm{Im}(\omega_1 + \omega_2)$ by

$$
\Gamma_h(z; \omega_1, \omega_2) = \exp\left(i \int_0^\infty \left(\frac{2z - \omega_1 - \omega_2}{2t\omega_1\omega_2} - \frac{\sin(t(2z - \omega_1 - \omega_2))}{2\sin(\omega_1 t)\sin(\omega_2 t)}\right) \frac{dt}{t}\right). \tag{2.13}
$$

This function is in fact another way of writing down the hyperbolic gamma function. A careful look at the integral expression for $\Gamma_h(z; \omega_1, \omega_2)$ lets us see that it can be related to our definition of the hyperbolic gamma function via

$$
\Gamma_h(z; \omega_1, \omega_2) = G(-i\omega_1, -i\omega_2; z - \omega_1/2 - \omega_2/2).
$$

Note that this implies

$$
G(\omega_1, \omega_2; z) = \Gamma_h(z + i\omega; i\omega_1, i\omega_2). \tag{2.14}
$$

We are first going to obtain a difference equation for Γ_h and afterwards use (2.14) to get to the difference equations (2.12) for G. We find

$$
\begin{aligned}
\frac{\Gamma_h(z + \omega_2)}{\Gamma_h(z)} &= \exp\left(i \int_0^\infty \frac{2z + \omega_2 - \omega_1}{2t^2\omega_1\omega_2} - \frac{\sin(t(2z + \omega_2 - \omega_1))}{2t\sin(\omega_1 t)\sin(\omega_2 t)} dt\right.\\
&\qquad \left. - i \int_0^\infty \frac{2z - \omega_1 - \omega_2}{2t^2\omega_1\omega_2} - \frac{\sin(t(2z - \omega_1 - \omega_2))}{2t\sin(\omega_1 t)\sin(\omega_2 t)} dt\right)\\
&= \exp\left(i \int_0^\infty \frac{1}{t^2\omega_1} - \frac{\sin(t(2z + \omega_2 - \omega_1)) - \sin(t(2z - \omega_1 - \omega_2))}{2t\sin(\omega_1 t)\sin(\omega_2 t)} dt\right)\\
&= \exp\left(i \int_0^\infty \frac{1}{t^2\omega_1} - \frac{\cos(t(2z - \omega_1))}{t\sin(\omega_1 t)} dt\right), && (2.15)
\end{aligned}
$$

using the trigonometric identity $\sin(a) - \sin(b) = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$ in the final equality. The integral between brackets is the sum of the integrals

$$\int_0^\infty \frac{1}{t^2\omega_1} - \frac{1}{t\sin(\omega_1 t)}dt = -i\ln(2),$$

which is a rescaled version of [5, (3.529), no. 2], and

$$\int_0^\infty \frac{1-\cos(t(2z-\omega_1))}{t\sin(\omega_1 t)}dt = -i\ln(\sin(\pi z/\omega_1)),$$

which also follows from [5, (3.529), no. 2]. Together this means that

$$\int_0^\infty \frac{1}{t^2\omega_1} - \frac{\cos(t(2z-\omega_1))}{t\sin(\omega_1 t)}dt = -i\ln(2\sin(\pi z/\omega_1)).$$

Continuing with (2.15), this implies

$$\begin{aligned}\Gamma_h(z+\omega_2;\omega_1,\omega_2) &= \exp(i\cdot -i\ln(2\sin(\pi z/\omega_1)))\Gamma_h(z;\omega_1,\omega_2)\\ &= 2\sin\left(\frac{\pi z}{\omega_1}\right)\Gamma_h(z). \qquad (2.16)\end{aligned}$$

Because we have that $\Gamma_h(z;\omega_1,\omega_2) = \Gamma_h(z;\omega_2,\omega_1)$, we can easily see that (2.16) implies that we also have

$$\Gamma_h(z+\omega_1;\omega_1,\omega_2) = 2\sin\left(\frac{\pi z}{\omega_2}\right)\Gamma_h(z;\omega_1,\omega_2). \qquad (2.17)$$

Taking $\omega_1 \to i\omega_1, \omega_2 \to i\omega_2$ and afterwards $z \to z + i\omega_2/2$ in (2.17) gives

$$\Gamma_h(z+i\omega+i\omega_1/2;i\omega_1,i\omega_2) = 2\sin\left(\frac{\pi(z+i\omega_2/2)}{i\omega_2}\right)\Gamma_h(z+i\omega_2/2;i\omega_1,i\omega_2) \qquad (2.18)$$

Using the identities $\sin(a/i) = \sin(-ia) = \sinh(a)$, $\sinh(b+\pi i/2) = \cosh(b)$ and (2.14), we have

$$G(\omega_1,\omega_2;z+i\omega_1/2) = 2\cosh(\pi z/\omega_2)G(\omega_1,\omega_2;z-i\omega_1/2). \qquad (2.19)$$

This is the first relation in (2.12) that we desired. The second one can again be obtained by using the $\omega_1 \leftrightarrow \omega_2$-symmetry of the hyperbolic gamma function. □

These AΔE's are the hyperbolic analogues of the difference equation (1.7) for $\Gamma(z)$. Note that these AΔE's imply the following expression, that we sometimes prove to be more useful

$$G(z+i\omega_1) = -2i\sinh(\pi(z+i\omega)/\omega_2)G(z). \qquad (2.20)$$

Furthermore, G satisfies

$$G(0) = 1, \text{ and } G(z) > 0, \ \mathrm{Im}(z) \in (-\omega,\omega), \ \mathrm{Re}(z) = 0. \qquad (2.21)$$

In Chapter 4, we are going to introduce Ruijsenaars' R-function. This function is defined as an integral which has an integrand that consists of a product of fifteen G-functions. To analyse the analyticity properties of R, we naturally need to know the analyticity properties of G.

Proposition 2.2. *The zeros and poles of $G(z)$ are given by*

$$z_{kl}^{+} \equiv i\omega + ik\omega_1 + il\omega_2, \quad k,l \in \mathbb{Z}_{\geq 0} \quad \textit{(zeros)}, \tag{2.22}$$
$$z_{kl}^{-} \equiv -z_{kl}^{+} \quad k,l \in \mathbb{Z}_{\geq 0} \quad \textit{(poles)}. \tag{2.23}$$

In particular, for $\omega_1/\omega_2 \notin \mathbb{Q}$ all poles and zeros are simple. The pole at z_{00}^{-} is simple and has residue

$$r_{00} = \frac{i\sqrt{\omega_1\omega_2}}{2\pi}. \tag{2.24}$$

More generally, if the quantity

$$t_{kl} \equiv \prod_{m=1}^{k} \sin(\pi m\omega_1/\omega_2) \prod_{n=1}^{l} \sin(\pi n\omega_2/\omega_1) \tag{2.25}$$

is non-zero, then the pole at z_{kl}^{-} is simple and has residue

$$r_{kl} = (-1)^{kl}(-1/2)^{k+l} r_{00}/t_{kl}. \tag{2.26}$$

Conversely, if z_{kl}^{-} is a simple pole, then one has $t_{kl} \neq 0$.

A short remark is that we already chose ω_1/ω_2 to be irrational, so throughout this thesis we can assume that the poles and zeros of the hyperbolic gamma functions are simple. We will also need to know the asymptotic behaviour of $G(z)$ as $\mathrm{Re}(z) \to \pm\infty$. For our purposes it is sufficient to know that for any $a, b \in \mathbb{C}$ we have

$$\lim_{\mathrm{Re}(z)\to\infty} \frac{G(z-a)}{G(z-b)} \exp\left(\frac{\pi i z}{\omega_1\omega_2}(b-a)\right) = \exp\left(\frac{\pi i}{2\omega_1\omega_2}(b^2-a^2)\right) \tag{2.27}$$

where the corresponding $o(\mathrm{Re}(z))$-tail as $\mathrm{Re}(z) \to \infty$ can be estimated uniformly. Furthermore, for periods ω_1 and ω_2 we have

$$|G(u+x)| \leq M \exp\left(\pi \mathrm{Im}\left(\frac{u}{\omega_1\omega_2}\right)|x|\right), \quad x \in \mathbb{R} \tag{2.28}$$

for some constant $M > 0$, provided that the line $u + \mathbb{R}$ does not hit a pole of G.

2.2 The hyperbolic hypergeometric function

The hypergeometric integrals that we will consider in this thesis depend meromorphically on a parameter $u \in \mathcal{G}_c$ with $\mathcal{G}_c \subset \mathbb{C}^8$ ($c \in \mathbb{C}$) the complex hyperplane

$$\mathcal{G}_c = \left\{ u = (u_1, u_2, \ldots, u_8) \in \mathbb{C}^8 \mid \sum_{j=1}^{8} u_j = 2c \right\}. \tag{2.29}$$

We now define the integrand of the hyperbolic hypergeometric function $I_h(u; z) = I_h(u; z; \omega_1, \omega_2)$ as

$$I_h(u; z; \omega_1, \omega_2) = \frac{G(i\omega \pm 2z)}{\prod_{j=1}^{8} G(u_j \pm z)} \tag{2.30}$$

for generic parameters $u \in \mathcal{G}_{2i\omega}$. The hyperbolic hypergeometric function $S_h(u) = S_h(u; \omega_1, \omega_2)$ is now defined as

$$S_h(u; \omega_1, \omega_2) = \int_{\mathcal{C}} I_h(u; z; \omega_1, \omega_2) dz. \tag{2.31}$$

The contour $\mathcal{C}$ is a deformation of the real line which separates the upward pole sequences of the integrand from the downward pole sequences. Note that the positively oriented real line can be chosen as an integration contour in the definition of $S_h(u)$ if $u \in \mathcal{G}_{2i\omega}$ satisfies $\mathrm{Im}(u_j - i\omega) < 0$ for $j = 1, \ldots, 8$. The asymptotic behaviour of $I_h(u; z)$ at $z = \pm\infty$ is $\mathcal{O}(\exp(-4\pi|z|\omega/\omega_1\omega_2))$, so the integral converges absolutely.

The hyperbolic integrals (like S_h) have an underlying symmetrygroup for the parameters $u \in \mathcal{G}_{2i\omega}$. These so-called Weyl groups are well treated in [3, Section 2]. We give now the explicit symmetry of $S_h(u)$.

Theorem 2.3. *Let v and w be two operations that act on $u \in \mathcal{G}_{2i\omega}$. These are*

$$wu = (u_1 + s, \ldots, u_4 + s, u_5 - s, \ldots, u_8 - s), \qquad vu = (i\omega - u_1, \ldots, i\omega - u_8) \tag{2.32}$$

with $s = i\omega - \frac{1}{2}(u_1 + u_2 + u_3 + u_4) = \frac{1}{2}(u_5 + u_6 + u_7 + u_8) - i\omega$. The hyperbolic hypergeometric function $S_h(u)$ $(u \in \mathcal{G}_{2i\omega})$ is invariant under permutations of $(u_1, u_2, \ldots, u_8)$ and it satisfies

$$\begin{aligned} S_h(u; \omega_1, \omega_2) &= S_h(wu; \omega_1, \omega_2) \prod_{1 \le j < k \le 4} G(i\omega - u_j - u_k; \omega_1, \omega_2) \\ &\quad \times \prod_{5 \le j < k \le 8} G(i\omega - u_j - u_k; \omega_1, \omega_2) \end{aligned} \tag{2.33}$$

as meromorphic functions in $u \in \mathcal{G}_{2i\omega}$. The hyperbolic hypergeometric function $S_h(u)$ also satisfies

$$S_h(u; \omega_1, \omega_2) = S_h(vu; \omega_1, \omega_2) \prod_{i \le j < k \le 8} G(i\omega - u_j - u_k; \omega_1, \omega_2) \tag{2.34}$$

as meromorphic functions in $u \in \mathcal{G}_{2i\omega}$.

2.3 The hyperbolic Barnes integral

In this section we degenerate the hyperbolic hypergeometric function $S_h(u)$, $u \in \mathcal{G}_{2i\omega}$ along $\beta = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_7 + \epsilon_8 - \epsilon_3 - \epsilon_4 - \epsilon_5 - \epsilon_6)$, where $\{\epsilon_j\}_{j=1}^{8}$ is the standard orthonormal basis of $\mathbb{R}^8$. This means that

$$u + r\beta = \left(u_1 + \frac{r}{2}, u_2 + \frac{r}{2}, u_3 - \frac{r}{2}, u_4 - \frac{r}{2}, u_5 - \frac{r}{2}, u_6 - \frac{r}{2}, u_7 + \frac{r}{2}, u_8 + \frac{r}{2}\right). \tag{2.35}$$

Concretely, for generic parameters $u \in \mathcal{G}_{2i\omega}$ we define $B_h(u) = B_h(u; \omega_1, \omega_2)$ by

$$B_h(u; \omega_1, \omega_2) = 2 \int_{\mathcal{C}} \frac{\prod_{j=3}^{6} G(z - u_j)}{\prod_{j=1,2,7,8} G(z + u_j)} dz. \tag{2.36}$$

This integral converges absolutely since the asymptotic behaviour of the integrand at $z = \pm\infty$ is $\exp(-4\pi\omega|z|/\omega_1\omega_2)$ due to (2.28). We may take the real line as integration contour if $u \in \mathcal{G}_{2i\omega}$ satisfies $\mathrm{Im}(u_j - i\omega) < 0$ for $j = 1, \ldots, 8$. Note that the hyperbolic Barnes integral is invariant under permutations of (u_1, u_2, u_7, u_8) and of (u_4, u_4, u_5, u_6). We call $B_h(u)$ the hyperbolic Barnes integral since it is essentially Ruijsenaars' [7] hyperbolic generalization of the Barnes integral representation of the Gauss hypergeometric function.

The degeneration of the hyperbolic hypergeometric function S_h to the hyperbolic Barnes integral B_h is made clear in the next proposition.

Proposition 2.4. *For $u \in \mathcal{G}_{2i\omega}$ satisfying $Im(u_j - i\omega) < 0$ for $j = 1, \ldots, 8$ we have*

$$\lim_{r\to\infty} S_h(u - r\beta) \exp\left(\frac{2\pi r\omega}{\omega_1\omega_2}\right) \exp\left(\frac{\pi i}{2\omega_1\omega_2}\left(\sum_{j=1,2,7,8} u_j^2 - \sum_{j=3}^{6} u_j^2\right)\right) = B_h(u). \tag{2.37}$$

Proof. The conditions on the parameters $u \in \mathcal{G}_{2i\omega}$ allow us to choose the real line as an integration contour in the integral expression of $S_h(u - r\beta)$, $r \in \mathbb{R}$, as well as in the integration expression of $B_h(u)$. Using that the integrand $I_h(u; z)$ of $S_h(u)$ is even in x, using the reflection equation for the hyperbolic gamma function (2.7) and by using the substitution $x = z + \frac{r}{2}$, we have

$$\begin{aligned} S_h(u - r\beta)e^{\frac{2\pi r\omega}{\omega_1\omega_2}} &= e^{\frac{2\pi r\omega}{\omega_1\omega_2}} \int_{-\infty}^{\infty} \frac{G(i\omega \pm 2x)}{\prod_{j=1,2,7,8} G(u_j - \frac{r}{2} \pm x)\prod_{j=3}^{6} G(u_j + \frac{r}{2} \pm x)} dx \\ &= 2e^{\frac{2\pi r\omega}{\omega_1\omega_2}} \int_{0}^{\infty} \frac{G(i\omega \pm 2x)}{\prod_{j=1,2,7,8} G(u_j - \frac{r}{2} \pm x)\prod_{j=3}^{6} G(u_j + \frac{r}{2} \pm x)} dx \\ &= 2e^{\frac{2\pi r\omega}{\omega_1\omega_2}} \int_{-\frac{r}{2}}^{\infty} \frac{\prod_{j=1,2,7,8} G(z + r - u_j)\prod_{j=3}^{6} G(z - u_j) G(i\omega \pm (2z + r))}{\prod_{j=1,2,7,8} G(u_j + z)\prod_{j=3}^{6} G(u_j + r + z)} dz \\ &= 2\int_{-\frac{r}{2}}^{\infty} k_1(2z + r)k_2(z + r)L(z)dz, \end{aligned}$$

where

$$\begin{aligned} L(z) &= \frac{\prod_{j=3}^{6} G(z - u_j)}{\prod_{j=1,2,7,8} G(z + u_j)}, \\ k_1(z) &= \frac{G(z + i\omega)}{G(z - i\omega)} e^{-\frac{2\pi\omega z}{\omega_1\omega_2}} = (1 - e^{-2\pi z/\omega_1})(1 - e^{-2\pi z/\omega_2}), \\ k_2(z) &= \frac{\prod_{j=1,2,7,8} G(z - u_j)}{\prod_{j=3}^{6} G(z + u_j)} e^{\frac{4\pi\omega z}{\omega_1\omega_2}}. \end{aligned}$$

The second expression of k_1 follows by applying both AΔE's for G in (2.12). The pointwise limits of k_1 and k_2 are

$$\lim_{z\to\infty} k_1(z) = 1, \quad \lim_{k\to\infty} k_2(z) = e^{\frac{\pi i}{2\omega_1\omega_2}\left(\sum_{j=3}^{6} u_j^2 - \sum_{j=1,2,7,8} u_j^2\right)}.$$

Moreover, observe that $k_1(z)$ is uniformly bounded by 4 for $z \in \mathbb{R}_{\geq 0}$, and that $k_2(z)$ is also uniformly bounded for $z \in \mathbb{R}_{\geq 0}$ because it is a continuous function on $\mathbb{R}_{\geq 0}$ which has a finite limit.

Denote by $\chi_{(-r/2,\infty)}(z)$ the indicator function on the interval $(-r/2, \infty)$. By Lebesgue's theorem of dominated convergence we now conclude that

$$\begin{aligned}
\lim_{r\to\infty} S_h(u-r\beta)e^{\frac{2\pi r\omega}{\omega_1\omega_2}} &= 2\lim_{r\to\infty}\int_{-\frac{r}{2}}^{\infty} k_1(2z+r)k_2(z+r)L(z)dz \\
&= 2\int_{-\infty}^{\infty}\lim_{r\to\infty}\chi_{(-r/2,\infty)}(z)k_1(2z+r)k_2(z+r)L(z)dz \\
&= 2e^{\frac{\pi i}{2\omega_1\omega_2}(\sum_{j=3}^{6}u_j^2-\sum_{j=1,2,7,8}u_j^2)}\int_{-\infty}^{\infty}L(z)dz \\
&= e^{\frac{\pi i}{2\omega_1\omega_2}(\sum_{j=3}^{6}u_j^2-\sum_{j=1,2,7,8}u_j^2)}B_h(u),
\end{aligned}$$

which is exactly the result we desired. □

In the following corollary we use Proposition 2.3 to degenerate the hyperbolic beta integral [10, (1.10)], which is

$$\int_{\mathcal{C}}\frac{G(i\omega\pm 2z)}{\prod_{j=1}^{6}G(u_j\pm z)}dz = 2\sqrt{\omega_1\omega_2}\prod_{1\le j<k\le 6}G(i\omega-u_j-u_k) \tag{2.38}$$

for generic $u_1,\dots,u_6\in\mathbb{C}$ satisfying the additive balancing condition $\sum_{j=1}^{6}u_j=4i\omega$.

Corollary 2.5. *For generic $u\in\mathbb{C}^6$ satisfying $Im(u_j-i\omega)<0$ for $j=1,\dots,8$ and $\sum_{j=1}^{6}u_j=4i\omega$ we have*

$$\int_{\mathcal{C}}\frac{G(z-u_4,z-u_5,z-u_6)}{G(z+u_1,z+u_2,z+u_3)}dz = \sqrt{\omega_1\omega_2}\prod_{j=1}^{3}\prod_{k=4}^{6}G(i\omega-u_j-u_k). \tag{2.39}$$

Proof. Substitute the parameters $u'=(u_1,u_2,u_4,u_5,u_6,0,u_3,0)$ in Proposition 2.3 with $u_j\in\mathbb{C}$ satisfying $\mathrm{Im}(u_j-i\omega)<0$ for $j=1,\dots,8$ and $\sum_{j=1}^{6}u_j=4i\omega$. Then $B_h(u')$ is equal to the lefthandside of (2.39), multiplied by 2. On the other hand, by proposition 2.3 and (2.38) we have

$$\begin{aligned}
B_h(u') &= \lim_{r\to\infty}S_h(u'-r\beta)\exp\left(\frac{2\pi r\omega}{\omega_1\omega_2}+\frac{\pi i}{2\omega_1\omega_2}\left(\sum_{j=1}^{3}u_j^2-\sum_{j=4}^{6}u_j^2\right)\right) \\
&= 2\sqrt{\omega_1\omega_2}\prod_{j=1}^{3}\prod_{k=4}^{6}G(i\omega-u_j-u_k) \\
&\quad\times\lim_{r\to\infty}\frac{\prod_{1\le j<k\le 3}G(i\omega-u_j-u_k+r)}{\prod_{4\le j<k\le 6}G(u_j+u_k-i\omega+r)}\exp\left(\frac{2\pi r\omega}{\omega_1\omega_2}+\frac{\pi i}{2\omega_1\omega_2}\left(\sum_{j=1}^{3}u_j^2-\sum_{j=4}^{6}u_j^2\right)\right) \\
&= 2\sqrt{\omega_1\omega_2}\prod_{j=1}^{3}\prod_{k=4}^{6}G(i\omega-u_j-u_k) \qquad (2.40)
\end{aligned}$$

where the last equality follows from a straightforward but tedious computation using the limit (2.27) for obtaining the last equality. □

2.4 The hyperbolic Euler integral

In this section we degenerate the hyperbolic hypergeometric function $S_h(u)$ ($u \in \mathcal{G}_{2i\omega}$) along $\alpha = \epsilon_7 - \epsilon_8$. This means that

$$u + r\eta = (u_1, \ldots, u_6, u_7 + r, u_8 - r). \tag{2.41}$$

The resulting degenerate integral $E_h(u) = E_h(u; \omega_1, \omega_2)$ is called the hyperbolic Euler integral and is defined by

$$E_h(u; \omega_1, \omega_2) = \int_{\mathcal{C}} \frac{G(i\omega \pm 2z)}{\prod_{j=1}^{6} G(u_j \pm z)} dz, \tag{2.42}$$

for generic parameters $u = (u_1, \ldots, u_6) \in \mathbb{C}^6$ satisfying

$$\mathrm{Im}\left(\frac{1}{\omega_1\omega_2}\sum_{j=1}^{6} u_j\right) > \frac{2\omega}{\omega_1\omega_2}. \tag{2.43}$$

It follows from the asymptotics (2.27) of the hyperbolic gamma function that the condition on the parameters ensures the absolute convergence of $E_h(u)$. Observe that $E_h(u)$ reduces to the hyperbolic beta integral (2.38) when the parameters $u \in \mathbb{C}^6$ satisfy the balancing condition $\sum_{j=1}^{6} u_j = 4i\omega$.

The next proposition shows exactly how the hyperbolic Euler integral is a degeneration of the hyperbolic hypergeometric function.

Proposition 2.6. *For $u \in \mathcal{G}_{2i\omega}$ satisfying $Im(u_j - \omega) < 0$ for $j = 1, \ldots, 8,$, the parameter condition* (2.43) *and $Im((u_7 + u_8)/\omega_1\omega_2) \geq 0$, we have*

$$\lim_{r\to\infty} S_h(u - r\eta_{\bar{78}}) \exp\left(-\frac{\pi i}{\omega_1\omega_2}(u_7 + u_8)(2r - u_7 + u_8)\right) = E_h(u_1, u_2, u_3, u_4, u_5, u_6). \tag{2.44}$$

Proof. The assumption $\mathrm{Im}(u_j - \omega) < 0$ for $j = 1, \ldots, 8$ ensures that the integration contours in S_h and E_h can be chosen as the positively oriented real line. We denote the integrand of the Euler integral by

$$J(z) = \frac{G(i\omega \pm 2z)}{\prod_{j=1}^{6} G(u_j \pm z)},$$

and we set

$$H(z) = \frac{G(z - u_7)}{G(z + u_8)} \exp\left(-\frac{\pi i z}{\omega_1\omega_2}(u_7 + u_8)\right).$$

This allows us to write

$$\begin{aligned} J(z)H(r+z)H(r-z) &= \frac{G(i\omega \pm 2z)}{\prod_{j=1}^{6} G(u_j \pm z)} \frac{G(r - u_7 \pm z)}{G(r + u_8 \pm z)} \exp\left(-\frac{2\pi i r}{\omega_1\omega_2}(u_7 + u_8)\right) \\ &= \frac{G(i\omega \pm 2z)}{G(u_7 - r \pm z)G(u_8 + r \pm z)\prod_{j=1}^{6} G(u_j \pm z)} \\ &\quad \times \exp\left(-\frac{2\pi i r}{\omega_1\omega_2}(u_7 + u_8)\right) \\ &= I_h(u - r\eta) \exp\left(-\frac{2\pi i r}{\omega_1\omega_2}(u_7 + u_8)\right), \end{aligned}$$

where $I_h(u)$ is the integrand (2.30) of the hyperbolic hypergeometric function $S_h(u)$. Observe that H is a continuous function on $\mathbb{R}$ satisfying

$$\begin{aligned}
\lim_{z\to\infty} H(z) &= \exp\left(\frac{\pi i}{2\omega_1\omega_2}(u_8^2-u_7^2)\right), && (2.45)\\
\lim_{z\to-\infty} H(z)\exp\left(\frac{2\pi i z}{\omega_1\omega_2}(u_7+u_8)\right) &= \lim_{z\to-\infty}\frac{G(z-u_7)}{G(z+u_8)}\exp\left(\frac{\pi i z}{\omega_1\omega_2}(u_7+u_8)\right)\\
&= \lim_{z\to\infty}\frac{G(z-u_8)}{G(z+u_7)}\exp\left(\frac{\pi i z}{\omega_1\omega_2}(-u_7-u_8)\right), \text{ by (2.7)}\\
&= \exp\left(\frac{\pi i}{2\omega_1\omega_2}(u_7^2-u_8^2)\right) && (2.46)
\end{aligned}$$

where we have used the limit (2.27) in both equations. Moreover, H is uniformly bounded on $\mathbb{R}$ in view of the parameter condition $\mathrm{Im}\left((u_7+u_8)/\omega_1\omega_2\right)\geq 0$ on the parameters, and we have

$$\begin{aligned}
&\lim_{r\to\infty} H(r+z)H(r-z)\\
=\ &\lim_{r\to\infty}\frac{G(r-u_7\pm z)}{G(r+u_8\pm z)}\exp\left(-\frac{2\pi i r}{\omega_1\omega_2}(u_7+u_8)\right)\\
=\ &\lim_{r\to\infty}\frac{G(r-u_7-z)}{G(r+u_8-z)}\exp\left(-\frac{\pi i r}{\omega_1\omega_2}(u_7+u_8)\right)\frac{G(r-u_7+z)}{G(r+u_8+z)}\exp\left(-\frac{\pi i r}{\omega_1\omega_2}(u_7+u_8)\right)\\
=\ &\exp\left(\frac{\pi i}{2\omega_1\omega_2}(u_8^2-u_7^2-2z(u_8+u_7))\right)\exp\left(\frac{\pi i}{2\omega_1\omega_2}(u_8^2-u_7^2+2z(u_8+u_7))\right)\\
=\ &\exp\left(\frac{\pi i}{\omega_1\omega_2}(u_8^2-u_7^2)\right)
\end{aligned}$$

for fixed $z\in\mathbb{R}$. By Lebesgue's theorem of dominated convergence we conclude that

$$\begin{aligned}
&\lim_{r\to\infty} S_h(u-r\eta)\exp\left(-\frac{\pi i}{\omega_1\omega_2}(u_7+u_8)(2r-u_7+u_8)\right)\\
=\ &\lim_{r\to\infty}\int_{\mathbb{R}} J(z)H(r+z)H(r-z)dz\\
=\ &\int_{\mathbb{R}} J(z)\lim_{r\to\infty} H(r+z)H(r-z)dz\\
=\ &E_h(u_1,\ldots,u_6)\exp\left(\frac{\pi i}{\omega_1\omega_2}(u_8^2-u_7^2)\right),
\end{aligned}$$

which implies the desired asymptotics. □

As a corollary of Proposition 2.5 we obtain the hyperbolic integral of Askey-Wilson type.

Corollary 2.7. *The hyperbolic Euler integral $E_h(u)$, $u\in\mathbb{C}^6$ is symmetric in $(u_1,\ldots,u_6)$ and for generic $u=(u_1,u_2,u_3,u_4)\in\mathbb{C}^4$ satisfying Im $\left(\frac{1}{\omega_1\omega_2}\sum_{j=1}^4 u_j\right)>\frac{2\omega}{\omega_1\omega_2}$ we have*

$$\begin{aligned}
\int_{\mathcal{C}}\frac{G(i\omega\pm 2z)}{\prod_{j=1}^4 G(u_j\pm z)}dz &= 2\sqrt{\omega_1\omega_2}\,G(u_1+u_2+u_3+u_4-3i\omega)\\
&\quad\times\prod_{1\leq j<k\leq 4} G(i\omega-u_j-u_k). && (2.47)
\end{aligned}$$

Proof. The permutation symmetry is trivial by looking at (2.42). For the second part of this corollary, apply Proposition 2.5 under the additional condition $u_5 = -u_6$ on the associated parameters $u \in \mathcal{G}_{2i\omega}$. Using the reflection equation for the hyperbolic gamma function (2.7), we see that the righthandside of (2.44) becomes the lefthandside of (2.47). On the other hand, $S_h(u - r\eta)$ can be evaluated by the hyperbolic beta integral (2.38), resulting in

$$\begin{aligned}
\int_{\mathcal{C}} \frac{G(i\omega \pm z)}{\prod_{j=1}^{4} G(u_j \pm z)} &= \int_{\mathcal{C}} \frac{G(i\omega \pm z)}{\prod_{j=1}^{6} G(u_j \pm z)}\Bigg|_{u_5=-u_6} \\
&= 2\sqrt{\omega_1\omega_2}G(i\omega - u_7 - u_8) \prod_{1\leq j<k\leq 4} G(i\omega - u_j - u_k) \\
&\quad \times \lim_{r\to\infty} \exp\left(-\frac{\pi i}{\omega_1\omega_2}(u_7 + u_8)(2r - u_7 + u_8)\right) \\
&\quad \times \prod_{j=1}^{4} \frac{G(i\omega - u_j - u_7 + r)}{G(-i\omega + u_j + u_8 + r)} \\
&= 2\sqrt{\omega_1\omega_2}G(u_1 + u_2 + u_3 + u_4 - 3i\omega) \prod_{1\leq j<k\leq 4} G(i\omega - u_j - u_k),
\end{aligned}$$

where we used the balancing condition $u_5 = -u_6$ in the first equality and the asymptotics (2.27) of the hyperbolic gamma function to obtain the last equality. □

Both the hyperbolic Euler integral E_h (2.42) and the hyperbolic Barnes integral B_h (2.36) are degenerations of the hyperbolic hypergeometric function S_h (2.31). We can use Propositions 2.3 and 2.5 to relate B_h to E_h. The next theorem states this relation that will prove to be useful in the following chapter.

Theorem 2.8. *We have*

$$\begin{aligned}
B_h(u) &= E_h(u_2 - s, u_7 - s, u_8 - s, u_3 + s, u_4 + s, u_6 + s) \\
&\quad \times \prod_{j=3}^{5} G(i\omega - u_1 - u_j) \prod_{j=2,7,8} G(i\omega - u_6 - u_j)
\end{aligned} \tag{2.48}$$

as meromorphic functions in $\{u \in \mathcal{G}_{2i\omega} \mid Im((u_1 + u_6)) < 2\omega,\ Im(u_j - \omega) < 0)$ *for* $j = 1, \ldots, 6\}$, *where*

$$s = \frac{1}{2}(u_2 + u_6 + u_7 + u_8) - i\omega = i\omega - \frac{1}{2}(u_1 + u_3 + u_4 + u_5).$$

Proof. We prove the theorem by analyzing the double integral

$$\frac{1}{\sqrt{\omega_1\omega_2}} \int_{\mathbb{R}^2} \frac{G(i\omega \pm 2z) \prod_{j=3}^{5} G(x - u_j)}{G(i\omega + s + x \pm z)G(x + u_1) \prod_{j=2,7,8} G(u_j - s \pm z)} dz dx$$

for $\omega_1, \omega_2 > 0, u \in \mathcal{G}_{2i\omega}$ and $s = \frac{1}{2}(u_2 + u_6 + u_7 + u_8) - i\omega$. We impose the additional parameter restraints

$$|\mathrm{Im}(s)| < \omega, \quad \mathrm{Im}(u_6 + s) < 0$$

to ensure the absolute convergence of the double integral, and

$$\mathrm{Im}(s) < 0, \quad \mathrm{Im}(i\omega - u_j) > 0\ (j = 1, 3, 4, 5), \quad \mathrm{Im}(i\omega - u_k + s) > 0\ (k = 2, 7, 8)$$

to ensure pole sequence separation by the integration contours. Note that these parameter restraints imply the parameter condition $\mathrm{Im}(u_1 + u_6) < 2\omega$ needed for the hyperbolic Euler integral in the righthandside of (2.48) to converge. Integrating the double integral first over x and using the integral evaluation formula (2.39) of Barnes type, we obtain an expression of the double integral as a multiple of $E_h(u_2 - s, u_3 + s, u_4 + s, u_5 + s, u_7 - s, u_8 - s)$. Integrating first over z and using the hyperbolic Askey-Wilson integral (2.47), we obtain an expression of the double integral as a multiple of $B_h(u)$. The resulting identity is (2.48) for a restricted parameter domain. Analytic continuation no completes the proof. □

Chapter 3

Contiguous relations

We are going to introduce some notational conventions such that we can write lengthy equations in a shorter way. We will come across a lot of hyperbolic functions, so we use the following notational conventions

$$s(a) = \sinh(\pi a), \quad c(a) = \cosh(\pi a). \tag{3.1}$$

In this section we write $\tau_{jk} = \tau_{jk}^{i\omega_1}$ $(1 \leq j \neq k \leq 8)$, which acts on $u \in \mathcal{G}_{2i\omega}$ by substracting $i\omega_1$ from u_j and adding $i\omega_1$ to u_k, i.e.

$$\tau_{jk}u = (u_1, \ldots, u_j - i\omega_1, \ldots, u_k + i\omega_1, \ldots, u_8). \tag{3.2}$$

We also write s_{jk} $(j \neq k)$ which acts on u by interchanging u_j and u_k, i.e.

$$\begin{aligned} s_{jk}u &= s_{jk}(u_1, \ldots, u_j, \ldots, u_k, \ldots, u_8) \\ &= (u_1, \ldots, u_k, \ldots, u_j, \ldots, u_8) \end{aligned} \tag{3.3}$$

We will encouter a lot of lengthy identities in the remainder of this thesis which contain terms that are equal up to a permutation of two parameters. For example, we write

$$f(u_1, u_2, u_3, \ldots, u_8) + f(u_2, u_1, u_3, \ldots, u_8) = f(u_1, \ldots, u_8) + (u_1 \leftrightarrow u_2) \tag{3.4}$$

or

$$g(x) + g(-x) = g(x) + (x \leftrightarrow -x), \tag{3.5}$$

where f and g are usually large expressions.

3.1 Constructing a contiguous relation for S_h

For a start, we are going to prove the following proposition.

Proposition 3.1. *For $x, y, z, v \in \mathbb{C}$, we have*

$$s(x \pm v)s(y \pm z) + s(x \pm y)s(z \pm v) + s(x \pm z)s(v \pm y) = 0. \tag{3.6}$$

Proof. First note,

$$\begin{aligned} s(a \pm b) &= \frac{1}{4}\left(e^{\pi(a+b)} - e^{-\pi(a+b)}\right)\left(e^{\pi(a-b)} - e^{-\pi(a-b)}\right) \\ &= \frac{1}{4}\left(e^{2\pi a} - e^{2\pi b} - e^{-2\pi b} + e^{-2\pi a}\right) \\ &= \frac{1}{2}c(2a) - \frac{1}{2}c(2b). \end{aligned}$$

This implies the following deduction

$$\begin{aligned} & s(x \pm v)s(y \pm z) + s(x \pm y)s(z \pm v) + s(x \pm z)s(v \pm y) \\ = \; & \frac{1}{4}\left(c(2x)c(2y) - c(2x)c(2z) - c(2v)c(2y) + c(2v)c(2z)\right) \\ + \; & \frac{1}{4}\left(c(2x)c(2z) - c(2x)c(2v) - c(2y)c(2z) + c(2y)c(2v)\right) \\ + \; & \frac{1}{4}\left(c(2x)c(2v) - c(2x)c(2y) - c(2z)c(2v) + c(2z)c(2y)\right) \\ = \; & 0, \end{aligned}$$

where we can just cancel out every term in the last step. □

If we substract $s(x \pm z)s(v \pm y)$, substitute $v = u_6 - i\omega,\ x = u_8 + i\omega,\ y = u_7 - i\omega$ and multiply by $-I_h(u)$ in equation (3.6), we obtain we obtain

$$\begin{aligned} & -s((u_8 + i\omega \pm (u_6 - i\omega))/\omega_2)s((u_7 - i\omega \pm z)/\omega_2)I_h(u) \\ + \; & s((u_8 + i\omega \pm (u_7 - i\omega))/\omega_2)s(u_6 - i\omega \pm z)/\omega_2)I_h(u) \\ = \; & s((u_8 + i\omega \pm z)/\omega_2)s(u_6 - i\omega \pm (u_7 - i\omega))/\omega_2)I_h(u), \end{aligned} \tag{3.7}$$

where we have also used that $s(a \pm b) = -s(b \pm a)$. Now, if we use $\sinh(a - \pi i) = -\sinh(a)$, it is easy to derive that $s(u_j - i\omega \pm z) = s(u_j - i\omega_1 + i\omega \pm z)$ for $j = 6, 7$. This turns (3.7) into

$$\begin{aligned} & s((u_8 + i\omega \pm (u_7 - i\omega))/\omega_2)s((u_6 - i\omega_1 + i\omega \pm z)/\omega_2)I_h(u) \\ - \; & s((u_8 + i\omega \pm (u_6 - i\omega))/\omega_2)s((u_7 - i\omega_1 + i\omega \pm z)/\omega_2)I_h(u) \\ = \; & s((u_8 + i\omega \pm z)/\omega_2)s(u_6 - i\omega \pm (u_7 - i\omega))/\omega_2)I_h(u). \end{aligned} \tag{3.8}$$

For constructing a contiguous relation, we need to make at least two differences in the argument of I_h. Using AΔE (2.20) for G, we compute the following

$$\begin{aligned} I_h(u) &= \frac{G(i\omega \pm 2z)}{G(u_6 \pm z)G(u_8 \pm z)\prod_{j=1,\ j\neq 6,8}^{8} G(u_j \pm z)} \\ &= \frac{s((u_8 + i\omega \pm z)/\omega_2)}{s((u_6 - i\omega_1 + i\omega \pm z)/\omega_2)} \cdot \frac{G(i\omega \pm 2z)}{G(u_6 - i\omega_1 \pm z)G(u_8 + i\omega_1 \pm z)\prod_{j=1,\ j\neq 6,8}^{8} G(u_j \pm z)} \\ &= \frac{s((u_8 + i\omega \pm z)/\omega_2)}{s((u_6 - i\omega_1 + i\omega \pm z)/\omega_2)} I_h(\tau_{68}u). \end{aligned} \tag{3.9}$$

We also have

$$\begin{aligned} I_h(u) &= \frac{G(i\omega \pm 2z)}{G(u_7 \pm z)G(u_8 \pm z)\prod_{j=1}^{6} G(u_j \pm z)} \\ &= \frac{s((u_8 + i\omega \pm z)/\omega_2)}{s((u_7 - i\omega_1 + i\omega \pm z)/\omega_2)} \cdot \frac{G(i\omega \pm 2z)}{G(u_7 - i\omega_1 \pm z)G(u_8 + i\omega_1 \pm z)\prod_{j=1}^{6} G(u_j \pm z)} \\ &= \frac{s((u_8 + i\omega \pm z)/\omega_2)}{s((u_7 - i\omega_1 + i\omega \pm z)/\omega_2)} I_h(\tau_{68}s_{67}u). \end{aligned} \tag{3.10}$$

If we now divide by $s((u_8 + i\omega \pm z)/\omega_2)s((u_6 - i\omega \pm (u_7 - i\omega))/\omega_2)$ in (3.8) and substitute (3.9) into the first line and (3.10) into the second line, we get the following difference equation when we integrate over the contour $\mathcal{C}$.

$$\frac{s((u_8 + i\omega \pm (u_7 - i\omega))/\omega_2)}{s((u_6 - i\omega \pm (u_7 - i\omega))/\omega_2)} S_h(\tau_{68}u) + \frac{s((u_8 + i\omega \pm (u_6 - i\omega))/\omega_2)}{s((u_7 - i\omega \pm (u_6 - i\omega))/\omega_2)} S_h(\tau_{78}u) = S_h(u), \tag{3.11}$$

where we have used that $S_h(\tau_{68}s_{67}u) = S_h(\tau_{78}u)$ due to the permutation symmetry of $S_h(u)$. We can obtain a different equation (which does not concern applying symmetry) by substituting the parameters vu in (3.11). The crux is that $\tau_{68}vu = v\tau_{86}u$, so we can subsequently use (2.34). For the first term this means

$$\begin{aligned}
&\left.\frac{s((u_8+i\omega\pm(u_7-i\omega))/\omega_2)}{s((u_6-i\omega\pm(u_7-i\omega))/\omega_2)}S_h(\tau_{68}u)\right|_{u\to vu}\\
=\ &\frac{s((-u_7-u_8+2i\omega)/\omega_2)s((u_7-u_8+2i\omega)/\omega_2)}{s((-u_6-u_7)/\omega_2)s((u_7-u_6)/\omega_2)}S_h(v\tau_{86}u)\\
=\ &\frac{s((u_7-u_8+2i\omega)/\omega_2)}{s((u_7-u_6)/\omega_2)}\prod_{m=1}^{5}\frac{s((u_m+u_6)/\omega_2)}{s((u_m+u_8-2i\omega)/\omega_2)}\frac{S_h(\tau_{86}u)}{\prod_{1\le j<k\le 8}G(i\omega-u_j-u_k)}
\end{aligned}\tag{3.12}$$

Because we also have $\tau_{68}s_{67}vu = v\tau_{86}s_{67}u$, we can obtain the same equality, but with u_6 and u_7 interchanged. The righthandside of (3.11) is equal to $\prod_{1\le j<k\le 8}G(i\omega-u_j-u_k)^{-1}S_h(u)$, so by multiplying by $\prod_{m=1}^{5}s((u_m+u_8-2i\omega)/\omega_2)\prod_{1\le j<k\le 8}G(i\omega-u_j-u_k)$ we obtain the following contiguous relation

$$\begin{aligned}
&\frac{s((u_7-u_8+2i\omega)/\omega_2)}{s((u_7-u_6)/\omega_2)}\prod_{j=1}^{5}s((u_j+u_6)/\omega_2)S_h(\tau_{86}u)+(u_6\leftrightarrow u_7)\\
=\ &\prod_{j=1}^{5}s((u_j+u_8-2i\omega)/\omega_2)S_h(u)
\end{aligned}\tag{3.13}$$

as meromorphic functions in $u\in\mathcal{G}_{2i\omega}$. Combining these contiguous relations and simplifying we obtain

$$A(u)S_h(\tau_{87}u)-(u_7\leftrightarrow u_8)=B(u)S_h(u),\quad u\in\mathcal{G}_{2i\omega}\tag{3.14}$$

where

$$\begin{aligned}
A(u)\ &=\ s((2i\omega-u_7+u_8)/\omega_2)\prod_{j=1}^{6}s((u_j+u_7)/\omega_2),\\
B(u)\ &=\ \frac{s((u_8\pm u_7)/\omega_2)s((2i\omega+u_8-u_7)/\omega_2)s((2i\omega-u_8+u_7)/\omega_2)}{s((2i\omega+u_8-u_6)/\omega_2)s((2i\omega+u_7-u_6)/\omega_2)}\\
&\quad\times\prod_{j=1}^{5}s((-2i\omega+u_j+u_6)/\omega_2)\\
&\quad-\frac{s((2i\omega-u_8+u_7)/\omega_2)s((u_7-u_6)/\omega_2)s((-2i\omega+u_6+u_7)/\omega_2)}{s((2i\omega+u_8-u_6)/\omega_2)}\\
&\quad\times\prod_{j=1}^{5}s((u_j+u_8)/\omega_2)\\
&\quad+\frac{s((2i\omega+u_8-u_7)/\omega_2)s((u_8-u_6)/\omega_2)s((-2i\omega+u_6+u_8)/\omega_2)}{s((2i\omega+u_7-u_6)/\omega_2)}\\
&\quad\times\prod_{j=1}^{5}s((u_j+u_7)/\omega_2).
\end{aligned}$$

This leads to the following theorem which is proven in [3, Thm. 4.3].

Theorem 3.2. *We have*

$$A(u)\left(S_h(\tau_{87}u) - S_h(u)\right) - (u_7 \leftrightarrow u_8) = B_2(u)S_h(u) \tag{3.15}$$

as meromorphic functions in $u \in \mathcal{G}_{2i\omega}$, where $A(u)$ is as above and with $B_2(u)$ defined by

$$\begin{aligned} B_2(u) \quad = \quad & \frac{s((u_7 \pm u_8)/\omega_2)s((u_7 - u_8 \pm 2i\omega)/\omega_2)}{4} \\ & \times \left(\sum_{j=7}^{8} s(2(i\omega + u_j)/\omega_2) - \sum_{j=1}^{6} s(2(i\omega - u_j)/\omega_2) \right). \end{aligned} \tag{3.16}$$

3.2 A contiguous relation for the hyperbolic Euler integral

What we would like to do now, is to take (3.15) and apply the achieved asymptotic (2.44). This yields a contiguous relation for the hyperbolic Euler integral, which we will use for later purposes. Because of the permutation symmetry of $S_h(u)$, we can write equality (3.15) in the following way

$$A(u)(S_h(\tau_{65}u) - S_h(u)) - (u_5 \leftrightarrow u_6) = B_2(s_{68}s_{57}u)S_h(u), \tag{3.17}$$

where u_7 and u_8 are interchanged with u_5 and u_6 respectively. Written out, this means

$$\begin{aligned} & s((2i\omega - u_5 + u_6)/\omega_2) \prod_{j=1,\ j\neq 5,6}^{8} s((u_j + u_5)/\omega_2)(S_h(\tau_{65}u) - S_h(u)) \\ & -s((2i\omega - u_6 + u_5)/\omega_2) \prod_{j=1,\ j\neq 5,6}^{8} s((u_j + u_6)/\omega_2)(S_h(\tau_{56}u) - S_h(u)) \\ = \quad & \frac{s((u_5 \pm u_6)/\omega_2)s((u_5 - u_6 \pm 2i\omega)/\omega_2)}{4} \\ & \times \left[\sum_{j=5}^{6} s(2(i\omega + u_j)/\omega_2) - \sum_{j=1,\ j\neq 5,6}^{8} s(2(i\omega - u_j)/\omega_2) \right] S_h(u), \end{aligned}$$

where $s_{56}u = (u_1, u_2, u_3, u_4, u_6, u_5, u_7, u_8)$. When we divide by $s((u_5 - u_6 \pm 2i\omega)/\omega_2)$ we obtain

$$\begin{aligned} & \frac{\prod_{j=1,\ j\neq 5,6}^{8} s((u_j + u_5)/\omega_2)}{s((u_5 - u_6 + 2i\omega)/\omega_2)}(S_h(\tau_{65}u) - S_h(u)) \\ & - \frac{\prod_{j=1,\ j\neq 5,6}^{8} s((u_j + u_6)/\omega_2)}{s((u_6 - u_5 + 2i\omega)/\omega_2)}(S_h(\tau_{56}u) - S_h(u)) \\ = \quad & \frac{1}{4}s((u_5 \pm u_6)/\omega_2) \left[\sum_{j=1,\ j\neq 5,6}^{8} s(2(i\omega - u_j)/\omega_2) - \sum_{j=5}^{6} s(2(i\omega + u_j)/\omega_2) \right] S_h(u), \end{aligned} \tag{3.18}$$

The trick to get the lefthandside correct, is to divide by $\prod_{j=7}^{8} s((u_j + u_5)/\omega_2)$, multiply by $\exp(-\frac{\pi i}{\omega_1\omega_2}(u_7 + u_8)(2r - u_7 + u_8))$, substitute $u - r\eta$ for u and afterwards take the limit $r \to \infty$. We will look at each term in the equation separately for obvious reasons. The first term on the lefthandside equals

$$\begin{aligned}
&\frac{\prod_{j=1}^4 s((u_j+u_5)/\omega_2)}{s((u_5-u_6+2i\omega)/\omega_2)} \lim_{r\to\infty} [S_h(\tau_{65}u - r\eta) - S_h(u-r\eta)] \\
&\times \exp\left(-\frac{\pi i}{\omega_1\omega_2}(u_7+u_8)(2r-u_7+u_8)\right) \\
= \quad &\frac{\prod_{j=1}^4 s((u_j+u_5)/\omega_2)}{s((u_5-u_6+2i\omega)/\omega_2)}(E_h(\tau_{65}u)-E_h(u)). \qquad (3.19)
\end{aligned}$$

Here we used the degeneration of the hyperbolic hypergeometric function (2.44). The second term on the lefthandside equals

$$\begin{aligned}
&\frac{\prod_{j=1}^4 s((u_j+u_6)/\omega_2)}{s((u_6-u_5+2i\omega)/\omega_2)} \lim_{r\to\infty} [S_h(\tau_{56}u - r\eta) - S_h(u-r\eta)] \\
&\times\ \exp\left(-\frac{\pi i}{\omega_1\omega_2}(u_7+u_8)(2r-u_7+u_8)\right) \frac{s((u_7+u_6-r)/\omega_2)s((u_8+u_6+r)/\omega_2)}{s((u_7+u_5-r)/\omega_2)s((u_8+u_5+r)/\omega_2)} \\
= \quad &\frac{\prod_{j=1}^4 s((u_j+u_6)/\omega_2)}{s((u_6-u_5+2i\omega)/\omega_2)}(E_h(\tau_{56}u)-E_h(u)) \lim_{r\to\infty} \frac{s((u_7+u_6-r)/\omega_2)}{s((u_7+u_5-r)/\omega_2)} \cdot \frac{s((u_8+u_6+r)/\omega_2)}{s((u_8+u_5+r)/\omega_2)} \\
= \quad &\frac{\prod_{j=1}^4 s((u_j+u_6)/\omega_2)}{s((u_6-u_5+2i\omega)/\omega_2)}(E_h(\tau_{56}u)-E_h(u)) \cdot e^{\frac{\pi}{\omega_2}(u_5-u_6)} \cdot e^{\frac{\pi}{\omega_2}(u_6-u_5)} \\
= \quad &\frac{\prod_{j=1}^4 s((u_j+u_6)/\omega_2)}{s((u_6-u_5+2i\omega)/\omega_2)}(E_h(\tau_{56}u)-E_h(u)) \qquad (3.20)
\end{aligned}$$

Now, (3.19) and (3.20) together imply that the lefthandside of (3.18) becomes

$$\frac{\prod_{j=1}^4 s((u_j+u_5)/\omega_2)}{s((u_5-u_6+2i\omega)/\omega_2)}(E_h(\tau_{65}u)-E_h(u)) - (u_5 \leftrightarrow u_6).$$

The righthandside of (3.18) now equals

$$\begin{aligned}
&\lim_{r\to\infty} \frac{s((u_5\pm u_6)/\omega_2)S_h(u-r\eta_{78}^{-})}{4s((u_7+u_5-r)/\omega_2)s((u_8+u_5+r)/\omega_2)} \exp\left(-\frac{\pi i}{\omega_1\omega_2}(u_7+u_8)(2r-u_7+u_8)\right) \\
\times \quad &\left[\sum_{j=1}^4 s(2(i\omega-u_j)/\omega_2) - \sum_{j=5}^6 s(2(i\omega+u_j)/\omega_2) + s(2(i\omega-u_7+r)/\omega_2) + s(2(i\omega-u_8-r)/\omega_2)\right] \\
= \quad &\frac{1}{4}s((u_5\pm u_6)/\omega_2)E_h(u) \times \lim_{r\to\infty} \frac{s(2(i\omega-u_7+r)/\omega_2)+s(2(i\omega-u_8-r)/\omega_2)}{s((u_7+u_5-r)/\omega_2)s((u_8+u_5+r)/\omega_2)}.
\end{aligned}$$

We have already assumed finite limits of the expressions above so that we could degenerate $S_h(u)$ apart from the limit that is still present in the equation above, i.e. we used that $\lim_{r\to\infty} f(r)g(r) = \lim_{r\to\infty} f(r) \lim_{r\to\infty} g(r)$ if $\lim_{r\to\infty} f(r)$ and $\lim_{r\to\infty} g(r)$ are finite. The six terms between large brackets which do not contain u_7 or u_8 converge to zero because

$$\lim_{r\to\infty} [s((u_7+u_5-r)/\omega_2)s((u_8+u_5+r)/\omega_2)]^{-1} = 0.$$

The only hurdle to take for obtaining the contiguous relation for the hyperbolic Euler integral, is to take the limit

$$\lim_{r\to\infty} \frac{1}{4} \frac{s(2(i\omega-u_7+r)/\omega_2)+s(2(i\omega-u_8-r)/\omega_2)}{s((u_7+u_5-r)/\omega_2)s((u_8+u_5+r)/\omega_2)} \qquad (3.21)$$

To do this, we are first going to look at the following general limit.

$$\begin{aligned}
\lim_{r\to\infty}\frac{s(2(a+r))}{s(b-r)s(c+r)} &= 2\lim_{r\to\infty}\frac{e^{2\pi(a+r)}-e^{-2\pi(a+r)}}{\left(e^{\pi(b-r)}-e^{-\pi(b-r)}\right)\left(e^{\pi(c+r)}-e^{-\pi(c+r)}\right)}\\
&= 2\lim_{r\to\infty}\frac{e^{2\pi(a+r)}-e^{-2\pi(a+r)}}{e^{\pi(b+c)}-e^{\pi(b-c-2r)}-e^{\pi(c-b+2r)}+e^{\pi(-b-c)}}\\
&= 2\lim_{r\to\infty}\frac{1-e^{-4\pi(a+r)}}{e^{\pi(b-2a+c-2r)}-e^{\pi(b-2a-c-4r)}-e^{\pi(c-2a-b)}+e^{\pi(-2a-b-c-2r)}}\\
&= \frac{2}{-e^{\pi(c-2a-b)}}=-2e^{-\pi(c-2a-b)} \qquad (3.22)
\end{aligned}$$

This also implies

$$\lim_{r\to\infty}\frac{s(2(a-r))}{s(b-r)s(c+r)}=-\lim_{r\to\infty}\frac{s(2((-a)+r))}{s(b-r)s(c+r)}=2e^{-\pi(c+2a-b)}. \qquad (3.23)$$

The degeneration of S_h to E_h needs u to be in $\mathcal{G}_{2i\omega}$. This restriction can be turned into the following equation

$$u\in\mathcal{G}_{2i\omega}\quad\Rightarrow\quad\sum_{j=1}^{8}u_j=4i\omega\quad\Rightarrow\quad 2i\omega-\sum_{j=1}^{6}u_j=u_7+u_8-2i\omega. \qquad (3.24)$$

By using (3.22) with $a=i\omega-u_7$ and (3.23) with $a=i\omega-u_8$ and letting $b=u_7+u_5,\ c=u_8+u_5$, we can now evaluate the limit (3.21)

$$\begin{aligned}
&\lim_{r\to\infty}\frac{1}{4}\frac{s(2(i\omega-u_7+r)/\omega_2)+s(2(i\omega-u_8-r)/\omega_2)}{s((u_7+u_5-r)/\omega_2)s((u_8+u_5+r)/\omega_2)}\\
=\ &\frac{1}{4}\left(-2e^{-\frac{\pi}{\omega_2}(u_7+u_8-2i\omega)}+2e^{\frac{\pi}{\omega_2}(u_7+u_8-2i\omega)}\right)\\
=\ &s((u_7+u_8-2i\omega)/\omega_2)\\
=\ &s\left((2i\omega-\sum_{j=1}^{6}u_j)/\omega_2\right). \qquad (3.25)
\end{aligned}$$

Altogether we have proved the following lemma

Lemma 3.3. *We have*

$$\begin{aligned}
&\frac{\prod_{j=1}^{4}s((u_j+u_5)/\omega_2)}{s((u_5-u_6+2i\omega)/\omega_2)}(E_h(\tau_{65}u)-E_h(u))-(u_5\leftrightarrow u_6)\\
=\ &s((u_5\pm u_6)/\omega_2)s\left((2i\omega-\sum_{j=1}^{6}u_j)/\omega_2\right)E_h(u) \qquad (3.26)
\end{aligned}$$

as meromorphic functions in $u\in\mathbb{C}^6$.

Chapter 4

Ruijsenaars' R-function

Motivated by the theory of quantum integrable, relativistic particle systems on the line, Ruijsenaars [7, 8, 9] introduced and studied a function R which is essentially the hyperbolic Barnes integral $B_h(u)$ with respect to a suitable reparametrization of the parameters $u \in \mathcal{G}_{2i\omega}$. The new parameters will be denoted by $(\gamma, x, \lambda) \in \mathbb{C}^6$ with $\gamma = (\gamma_0, \ldots, \gamma_3) \in \mathbb{C}^4$, where x is viewed as the geometric parameter and λ is viewed as the spectral parameter. We define the dual parameters $\hat{\gamma}$ by

$$\hat{\gamma} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\gamma. \tag{4.1}$$

It is an easy verification that $\gamma \mapsto \hat{\gamma}$ defines an involution on the parameters.

We have already established results for the hyperbolic Euler- and hyperbolic Barnes integral, which we are going to use to derive many of the properties of Ruijsenaars' R-function. The first goal of this section is to obtain an integral representation of R in terms of the hyperbolic Euler integral. Consequently, we are going to use (3.26) to establish difference equations for which R is a solution.

4.1 Constructing Askey-Wilson difference equations

Set

$$N(\gamma) = \prod_{j=1}^{3} G(i\gamma_0 + i\gamma_j + i\omega). \tag{4.2}$$

Ruijsenaars' [7] function $R(\gamma; x, \lambda; \omega_1, \omega_2) = R(\gamma; x, \lambda)$ is defined by

$$R(\gamma; x, \lambda) = \frac{1}{\sqrt{\omega_1\omega_2}} \int_{\mathcal{C}} \frac{G(z + i\gamma_0 \pm x)G(z + i\hat{\gamma}_0 \pm \lambda)}{G(z + i\omega)G(i\gamma_0 \pm x)G(i\hat{\gamma}_0 \pm \lambda)} \prod_{j=1}^{3} \frac{G(i\gamma_0 + i\gamma_j + i\omega)}{G(z + i\gamma_0 + i\gamma_j + i\omega)} dz. \tag{4.3}$$

A good look at this expression reveals that this is the same as

$$R(\gamma; x, \lambda) = \frac{1}{2\sqrt{\omega_1\omega_2}} \frac{N(\gamma)}{G(i\gamma_0 \pm x, i\hat{\gamma}_0 \pm \lambda)} B_h(u), \tag{4.4}$$

where $u \in \mathcal{G}_{2i\omega}/\mathbb{C}\beta_{1278}$ (note that for $\xi \in \mathbb{C}$ we have $B_h(u + \xi\beta_{1278}) = B_h(u)$ by (2.27) and Cauchy's theorem) with

$$\begin{aligned} u_1 &= i\omega, \quad u_2 = i\omega + i\gamma_0 + i\gamma_1, \quad u_3 = -i\gamma_0 + x, \quad u_4 = -i\gamma_0 - x, \\ u_5 &= -i\hat{\gamma}_0 + \lambda, \quad u_6 = -i\hat{\gamma}_0 - \lambda, \quad u_7 = i\omega + i\gamma_0 + i\gamma_2, \quad u_8 = i\omega + i\gamma_0 + i\gamma_3. \end{aligned} \tag{4.5}$$

By Proposition 2.2 we can deduce that the poles of R are located at

$$x = \pm\left(ik\omega_1 + il\omega_2 - \omega - \gamma_j\right), \quad \lambda = \pm\left(ik\omega_1 + il\omega_2 - \omega - \hat{\gamma}_j\right) \quad k, l \in \mathbb{Z}_{\geq 0}, \; j = 0, 1, 2, 3. \tag{4.6}$$

Note that $R(\gamma; x, \lambda; \omega_1, \omega_2)$ is invariant under permuting the role of the two periods ω_1 and ω_2. We will need the following auxiliary function, which we will call the c-function

$$c(\omega_1, \omega_2; \gamma; z) = \frac{1}{G(2z + i\omega)} \prod_{j=0}^{3} G(z - i\gamma_j). \tag{4.7}$$

Note that because of (2.9) we have

$$\overline{c(\gamma; z)} = c(\gamma; -z). \tag{4.8}$$

By Proposition 2.2 we can see that the zeros of $c(\gamma; z)$ are located at

$$z^1_{kl} = i\omega - ik\omega_1 - il\omega_2, \; k, l \in \mathbb{Z}_{\geq 0} \tag{4.9}$$

$$z^{2,j}_{kl} = i\gamma_j + i\omega + ik\omega_1 + il\omega_2, \; k, l \in \mathbb{Z}_{\geq 0}, \; j = 0, 1, 2, 3. \tag{4.10}$$

Note that c is invariant under interchangement of ω_1 and ω_2 and also under any permutation of $(\gamma_0, \gamma_1, \gamma_2, \gamma_3)$.

Proposition 4.1. *R is even in x and λ, i.e.*

$$R(\gamma; x, \lambda) = R(\gamma; -x, \lambda) = R(\gamma; x, -\lambda), \tag{4.11}$$

and is also self-dual, i.e.

$$R(\gamma; x, \lambda) = R(\hat{\gamma}; \lambda, x) \tag{4.12}$$

Furthermore, for an element $\sigma \in W(D_4)$, where $W(D_4)$ is the Weyl group of type D_4 acting on the parameters γ by permutations and even numbers of sign flips, we have

$$\frac{R(\gamma; x, \lambda)}{c(\gamma; x)c(\hat{\gamma}; \lambda)N(\gamma)} = \frac{R(\sigma\gamma; x, \lambda)}{c(\sigma\gamma; x)c(\widehat{\sigma\gamma}; \lambda)N(\sigma\gamma)} \tag{4.13}$$

The first symmetries are all direct consequences of the symmetries of the hyperbolic Barnes integral B_h and the second symmetries are proved in [8]. We can express R in terms of the hyperbolic Euler integral E_h by the following theorem.

Theorem 4.2. *We have*

$$R(\gamma; x, \lambda) = \frac{1}{2\sqrt{\omega_1\omega_2}} \frac{\prod_{j=1}^{3} G(i\gamma_0 + i\gamma_j + i\omega, \lambda - i\hat{\gamma}_j)}{G(\lambda + i\hat{\gamma}_0)} E_h(v), \tag{4.14}$$

where $v \in \mathbb{C}^6$ is given by

$$v_j = \frac{i\omega}{2} + i\gamma_{j-1} - \frac{i\hat{\gamma}_0}{2} + \frac{\lambda}{2}, \quad j = 1, \ldots, 4, \qquad v_{5,6} = \frac{i\omega}{2} \pm x - \frac{i\hat{\gamma}_0}{2} - \frac{\lambda}{2}. \tag{4.15}$$

Proof. Note first that by using the parameters u in (4.5), taking $\gamma \to (-\gamma_3, \gamma_1, \gamma_2, -\gamma_0)$ is the same as taking $u \to s_{18}u$. This can be seen in the following way. By first interchanging γ_0 and γ_3 and then sign-flipping them, we have

$$\begin{aligned} u_1' = i\omega, \quad u_2' = i\omega - i\gamma_3 + i\gamma_1, \qquad u_3' = -i\gamma_3 + x, \quad u_4' = -i\gamma_3 - x, \\ u_5' = i\hat{\gamma}_3 + \lambda, \quad u_6' = i\hat{\gamma}_3 - \lambda, \qquad u_7' = i\omega - i\gamma_3 + i\gamma_2, \quad u_8' = i\omega - i\gamma_0 - i\gamma_3. \end{aligned} \tag{4.16}$$

Then $u' + (i\gamma_0 + i\gamma_3)\beta_{1278}$ is the same as

$$\begin{aligned} u_1 = i\omega + i\gamma_0 + i\gamma_3, \quad u_2 = i\omega + i\gamma_0 + i\gamma_1, \qquad u_3 = -i\gamma_0 + x, \quad u_4 = -i\gamma_0 - x, \\ u_5 = -i\hat{\gamma}_0 + \lambda, \quad u_6 = -i\hat{\gamma}_0 - \lambda, \qquad u_7 = i\omega + i\gamma_0 + i\gamma_2, \quad u_8 = i\omega, \end{aligned} \tag{4.17}$$

which is the same as $s_{18}u$. Let $\sigma_0\gamma = (-\gamma_3, \gamma_1, \gamma_2, -\gamma_0)$, then $\widehat{\sigma_0\gamma} = \sigma_0\hat{\gamma}$. A careful calculation shows that

$$\frac{c(\gamma; x)c(\hat{\gamma}; \lambda)}{c(\sigma_0\gamma; x)c(\sigma_0\hat{\gamma}; \lambda)} = \prod_{j=0,3} \frac{G(x - i\gamma_j)G(\lambda - i\hat{\gamma}_j)}{G(x + i\gamma_j)G(\lambda + i\hat{\gamma}_j)}.$$

Using (4.13), we obtain

$$\begin{aligned} R(\gamma; x, \lambda) &= \frac{c(\gamma; x)c(\hat{\gamma}; \lambda)N(\gamma)}{c(\sigma_0\gamma; x)c(\sigma_0\hat{\gamma}; \lambda)N(\sigma_0\gamma)} R(\sigma_0\gamma; x, \lambda) \\ &= \frac{N(\gamma)}{N(\sigma_0\gamma)} \prod_{j=0,3} \frac{G(x - i\gamma_j)G(\lambda - i\hat{\gamma}_j)}{G(x + i\gamma_j)G(\lambda + i\hat{\gamma}_j)} \\ &\quad \times \frac{1}{2\sqrt{\omega_1\omega_2}} N(\sigma_0\gamma)G(i\gamma_3 \pm x, i\hat{\gamma}_3 \pm \lambda)B_h(s_{18}u) \\ &= \frac{1}{2\sqrt{\omega_1\omega_2}} \frac{G(x - i\gamma_0, \lambda - i\hat{\gamma}_0)}{G(x + i\gamma_0, \lambda + i\hat{\gamma}_0)} N(\gamma)B_h(u), \end{aligned} \tag{4.18}$$

where we have used the reflection equation (2.7) for the hyperbolic gamma function and the symmetry property for the hyperbolic Barnes integral. The last step will be to use equality (2.48), which gives the relation between $B_h(u)$ and $E_h(u)$. In the context of theorem 4 we calculate s

$$\begin{aligned} s &= i\omega - \frac{1}{2}(u_1 + u_3 + u_4 + u_5) \\ &= i\omega - \frac{1}{2}(i\omega - i\gamma_0 + x - i\gamma_0 - x - i\hat{\gamma}_0 + \lambda) \\ &= \frac{i\omega}{2} + i\gamma_0 + \frac{i\hat{\gamma}_0}{2} - \frac{\lambda}{2}, \end{aligned}$$

which leads to

$$\begin{aligned}
u_2 - s &= \frac{i\omega}{2} + i\gamma_1 - \frac{i\hat{\gamma}_0}{2} + \frac{\lambda}{2} = v_2,\\
u_7 - s &= \frac{i\omega}{2} + i\gamma_2 - \frac{i\hat{\gamma}_0}{2} + \frac{\lambda}{2} = v_3,\\
u_8 - s &= \frac{i\omega}{2} + i\gamma_3 - \frac{i\hat{\gamma}_0}{2} + \frac{\lambda}{2} = v_4,\\
u_3 + s &= \frac{i\omega}{2} + x + \frac{i\hat{\gamma}_0}{2} - \frac{\lambda}{2} = v_5,\\
u_4 + s &= \frac{i\omega}{2} - x + \frac{i\hat{\gamma}_0}{2} - \frac{\lambda}{2} = v_6,\\
u_5 + s &= \frac{i\omega}{2} + i\gamma_0 - \frac{i\hat{\gamma}_0}{2} + \frac{\lambda}{2} = v_1.
\end{aligned} \tag{4.19}$$

Furthermore, by again using the reflection equation (2.7), we calculate the following

$$\prod_{j=4}^{5} G(i\omega - u_1 - u_j) \prod_{j=2,7,8} G(i\omega - u_6 - u_j) = \frac{G(x+i\gamma_0)}{G(x-i\gamma_0, \lambda - i\hat{\gamma}_0)} \prod_{j=1}^{3} G(\lambda - i\hat{\gamma}_j).$$

Continuing with (4.18), this means that by using the symmetry properties for the hyperbolic Euler integral (see Corollary 2.6) we have

$$\begin{aligned}
R(\gamma; x, \lambda) &= \frac{1}{2\sqrt{\omega_1\omega_2}} \frac{G(x-i\gamma_0, \lambda - i\hat{\gamma}_0)}{G(x+i\gamma_0, \lambda + i\hat{\gamma}_0)} N(\gamma)\\
&\quad \times \frac{G(x+i\gamma_0)}{G(x-i\gamma_0, \lambda - i\hat{\gamma}_0)} \prod_{j=1}^{3} G(\lambda - i\hat{\gamma}_j) E_h(v_2, v_3, v_4, v_5, v_6, v_1)\\
&= \frac{1}{2\sqrt{\omega_1\omega_2}} \frac{\prod_{j=1}^{3} G(i\gamma_0 + i\gamma_j + i\omega, \lambda - i\hat{\gamma}_j)}{G(\lambda + i\hat{\gamma}_0)} E_h(v),
\end{aligned}$$

with v as we desired. □

We are now ready to make the final step in this section. We have obtained a contiguous relation for the hyperbolic Euler integral E_h in Lemma 3.3 and we have also obtained a representation of Ruijsenaars' R-function in terms of E_h in Theorem 4.2. We are going to combine the contiguous relation (3.26) and equality (4.14) to show that R satisfies the Askey-Wilson second-order difference equation in the next proposition.

Proposition 4.3. *Ruijsenaars' R-function satisfies the Askey-Wilson second-order difference equation*

$$A(\gamma; x; \omega_1, \omega_2)\left(R(\gamma; x+i\omega_1, \lambda) - R(\gamma; x, \lambda)\right) + (x \leftrightarrow -x) = B(\gamma; \lambda; \omega_1, \omega_2) R(\gamma; x, \lambda), \tag{4.20}$$

where

$$A(\gamma; x; \omega_1, \omega_2) = \frac{\prod_{j=0}^{3} s((i\omega + x + i\gamma_j)/\omega_2)}{s(2x/\omega_2)s(2(x+i\omega)/\omega_2)}, \tag{4.21}$$

$$B(\gamma; \lambda; \omega_1, \omega_2) = s((\lambda - i\omega - i\hat{\gamma}_0)/\omega_2)s((\lambda + i\omega + i\hat{\gamma}_0)/\omega_2). \tag{4.22}$$

Proof. We are going to use the contiguous relation (3.26) with the parameters (4.15) where $u_j = v_j$, $j = 1, \ldots, 6$. Careful calculations and using twice that $s(-z) = -s(z)$ give the following results

$$\begin{aligned}
\frac{\prod_{j=1}^{4} s((u_j+u_5)/\omega_2)}{s((u_5-u_6+2i\omega)/\omega_2)} &= \frac{\prod_{j=0}^{3} s((i\omega+x+i\gamma_j)/\omega_2)}{s(2(x+i\omega)/\omega_2)},\\
\frac{\prod_{j=1}^{4} s((u_j+u_6)/\omega_2)}{s((u_6-u_5+2i\omega)/\omega_2)} &= \frac{\prod_{j=0}^{3} s((i\omega-x+i\gamma_j)/\omega_2)}{s(2(-x+i\omega)/\omega_2)},\\
s((u_5\pm u_6)/\omega_2)s\left((2i\omega-\sum_{j=1}^{6}u_j)/\omega_2\right) &= s(2x/\omega_2)s((\lambda-i\omega-i\hat{\gamma}_0)/\omega_2)s((\lambda+i\omega+i\hat{\gamma}_0)/\omega_2).
\end{aligned}$$

Note that $E_h(\tau_{65}^{i\omega_1}u) = E_h(u')$, with $u'_j = u_j$ for $j = 1, \ldots, 4$ and $u'_{5,6} = \frac{i\omega}{2} \pm (x + i\omega_1) - \frac{i\hat{\gamma}_0}{2} - \frac{\lambda}{2}$ and also $E_h(\tau_{56}^{i\omega_1}u) = E_h(u'')$, with $u''_j = u_j$ for $j = 1, \ldots, 4$ and $u''_{5,6} = \frac{i\omega}{2} \pm (x - i\omega_1) - \frac{i\hat{\gamma}_0}{2} - \frac{\lambda}{2}$. If we multiply both sides of equation (3.26) by

$$\frac{1}{2\sqrt{\omega_1\omega_2}} \frac{\prod_{j=1}^{3} G(i\gamma_0+i\gamma_j+i\omega, \lambda-i\hat{\gamma}_j)}{G(\lambda+i\hat{\gamma}_0)},$$

which is independent of x, substitute the three expressions above into the equation and afterwards divide by $s(2x/\omega_2)$, we obtain

$$\begin{aligned}
&\frac{\prod_{j=0}^{3} s((i\omega+x+i\gamma_j)/\omega_2)}{s(2x/\omega_2)s(2(x+i\omega)/\omega_2)}\left(R(\gamma;x+i\omega_1,\lambda)-R(\gamma;x,\lambda)\right)\\
+\;&\frac{\prod_{j=0}^{3} s((i\omega-x+i\gamma_j)/\omega_2)}{s(-2x/\omega_2)s(2(-x+i\omega)/\omega_2)}\left(R(\gamma;x-i\omega_1,\lambda)-R(\gamma;-x,\lambda)\right)\\
=\;& s((\lambda-i\omega-i\hat{\gamma}_0)/\omega_2)s((\lambda+i\omega+i\hat{\gamma}_0)/\omega_2)R(\gamma;x,\lambda). \qquad (4.23)
\end{aligned}$$

Note that we have also used that $s(2x/\omega_2) = -s(-2x/\omega_2)$ in the second term of (4.23). Using that R is even in x, finally leaves us with

$$\begin{aligned}
&A(\gamma;x;\omega_1,\omega_2)\left(R(\gamma;x+i\omega_1,\lambda)-R(\gamma;x,\lambda)\right)\\
&+A(\gamma;-x;\omega_1,\omega_2)\left(R(\gamma;-x+i\omega_1,\lambda)-R(\gamma;-x,\lambda)\right) = B(\gamma;\lambda;\omega_1,\omega_2)R(\gamma;x,\lambda),
\end{aligned}$$

which is the desired result. □

4.2 An Askey-Wilson difference operator and the $\mathcal{E}$-function

We define the second-order Askey-Wilson (AW) difference operator $\mathcal{L}_\gamma^{\omega_1,\omega_2}$ by

$$\mathcal{L}_\gamma^{\omega_1,\omega_2} := A(\gamma;x;\omega_1,\omega_2)(T_{i\omega_1}^x - I) + A(\gamma;-x;\omega_1,\omega_2)(T_{-i\omega_1}^x - I), \qquad (4.24)$$

where I denotes the identity operator, T is the shift operator (i.e. $T_z^x f(x) = f(x+z)$) and $A(\gamma;x;\omega_1,\omega_2)$ as in (4.21). Looking at the previous section, we know that R satisfies the following eigenvalue equation

$$\left(\mathcal{L}_\gamma^{\omega_1,\omega_2} f\right)(x) = B(\gamma;\lambda;\omega_1,\omega_2)f(x), \qquad (4.25)$$

where $B(\gamma;\lambda;\omega_1,\omega_2)$ as in (4.22). Because R is symmetric in ω_1 and ω_2 and it is self-dual, i.e. $R(\gamma;x,\lambda) = R(\hat{\gamma},\lambda,x)$ (see (4.11)), we know that it is an eigenvalue solution to four different AW-difference operators. Those are two operators acting on the geometric variable x: $\mathcal{L}_\gamma^{\omega_1,\omega_2}, \mathcal{L}_\gamma^{\omega_2,\omega_1}$

(with eigenvalues $B(\gamma;\lambda;\omega_1,\omega_2)$ and $B(\gamma;\lambda;\omega_2,\omega_1)$ respectively) and two operators acting on the spectral parameter λ: $\mathcal{L}_{\hat{\gamma}}^{\omega_1,\omega_2},\mathcal{L}_{\hat{\gamma}}^{\omega_2,\omega_1}$ (with eigenvalues $B(\hat{\gamma};x;\omega_1,\omega_2)$ and $B(\hat{\gamma};x;\omega_2,\omega_1)$ respectively).

In [8] we can find the following definition for the $\mathcal{E}$-function, where $c(\gamma;x)$ is the function in (4.7),

$$\mathcal{E}(\omega_1,\omega_2,\gamma;x,\lambda)=\mathcal{E}(x,\lambda)\equiv\frac{K(\gamma;\lambda)}{c(\gamma;x)}R(\gamma;x,\lambda), \tag{4.26}$$

with $K(\gamma;\lambda)=\chi(\omega_1,\omega_2,\gamma)\left(N(\gamma)c(\hat{\gamma};\lambda)\right)^{-1}$ and where

$$\chi(\omega_1,\omega_2,p)\quad\equiv\quad\exp\left(\frac{2\pi i}{\omega_1\omega_2}\left[p\cdot p/4-(\omega_1^2+\omega_2^2+\omega_1\omega_2)/8\right]\right), \tag{4.27}$$

Because R is a solution to the eigenvalue equation (4.25) above and K is independent of x, the $\mathcal{E}$-function satisfies the equation

$$\left(M_{c(\gamma;x)}^{-1}\circ\mathcal{L}_{\gamma}^{\omega_1,\omega_2}\circ M_{c(\gamma;x)}\mathcal{E}\right)(x,\lambda)=\left(L_{\gamma}^{\omega_1,\omega_2}\mathcal{E}\right)(x,\lambda)=B(\gamma;\lambda)\mathcal{E}(x,\lambda).$$

Here, $M_{c(\gamma;x)}$ denotes multiplication by $c(\gamma;x)$.

We see that $\mathcal{E}(x,\lambda)$ is essentially Ruijsenaars' R-function up to multiplication with $c(\gamma;x)^{-1}$. Ruijsenaars has used this function in [8, 9]. The profit of using $\mathcal{E}$ is that it is an analytic function in $\mathbb{C}$ whereas Ruijsenaars R-function is not. It also has a D_4-invariance (see Proposition 4.1) on the parameters γ, i.e.

$$\mathcal{E}(\omega_1,\omega_2,\sigma\gamma;x,\lambda)=\mathcal{E}(\omega_1,\omega_2,\gamma;x,\lambda),\quad\text{for all }\sigma\in W(D_4), \tag{4.28}$$

where the D_4-symmetry on the parameters of R (4.13) are more complex.

The main reason why $\mathcal{E}$ is mentioned in this thesis, is that we are going to use the asymptotics of this function to say something about the asymptotics for Ruijsenaars' R-function. We introduce the Bachmann-Landau symbol $\sim$, which is a relation between two continuous functions. Let $f,g\in C(\mathbb{C})$, then

$$f\sim g\quad\Longleftrightarrow\quad\lim_{\mathrm{Re}(x)\to\infty}\frac{f(x)}{g(x)}=1. \tag{4.29}$$

Ruijsenaars also obtained a function $\mathcal{E}_{as}(\gamma;x,\lambda)$ (see [8, (1.31)]) for which we have $\mathcal{E}(\gamma;x,\lambda)\sim\mathcal{E}_{as}(\gamma;x,\lambda)$ with

$$\mathcal{E}_{as}(\omega_1,\omega_2,\gamma;x,\lambda)=e^{2\pi ix\lambda/\omega_1\omega_2}-u(\omega_1,\omega_2,\hat{\gamma};-\lambda)e^{-2\pi ix\lambda/\omega_1\omega_2}.$$

The u-function is defined as $u(\gamma;z)=-c(\gamma;z)/c(\gamma;-z)$.

Chapter 5

The Ruijsenaars function transform

In the previous chapter we have seen that Ruijsenaars' function $R(\gamma; x, \lambda)$ is a solution to the second-order difference equation of Askey-Wilson type. This equation implies a difference operator which yields a function transform that is similar to, say, the Fourier transform. We will refer to this function transform as the Ruijsenaars Function Transform or RFT, because we make the transformation by using Ruijsenaars' R-function. The main goal of this section will be to show that this transform is unitary. Furthermore, we are going to use a constant

$$\alpha = \frac{2\pi}{\omega_1\omega_2}. \tag{5.1}$$

5.1 The Hilbert space $\mathcal{H}_w$

This section defines a Hilbert space $\mathcal{H}_w$ with an appropriate weight function. We are briefly going to discuss the analiticity of this weight function. Afterwards we are going to prove Lemma 5.1, which gives an identity that involves the weight function and the function A that is defined in (4.21).

Let w be the weight function given by

$$w(\gamma; x) = \frac{1}{c(\gamma; x)c(\gamma; -x)}, \quad x \in \mathbb{R},\ \gamma \in \mathbb{R}^4, \tag{5.2}$$

where $c(\gamma; x)$ is given by (4.7). The weight w is positive because from (4.8) we obtain $w(\gamma; x) = |c(\gamma; x)|^{-2}$. It is also obviously an even function for $x \in \mathbb{R}$. We assume that the function w has only simple poles. These poles impose conditions on the parameters $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ that we will discuss later on.

Writing the w-function out in a product of hyperbolic gamma functions gives

$$w(\gamma; x) = \frac{G(i\omega \pm 2x)}{\prod_{j=0}^{3} G(-i\gamma_j \pm x)}.$$

Recall that the poles of $G(x)$ occur at $x_{kl} = -i\omega - ik\omega_1 - il\omega_2,\ k, l \in \mathbb{Z}_{\geq 0}$, and that it vanishes at $x_{kl} = i\omega + ik\omega_1 + il\omega_2,\ k, l \in \mathbb{Z}_{\geq 0}$. Note that $G(i\omega + 2x)$ and $G(i\omega - 2x)$ have the joint property that if one has a (simple) pole at $x = x_0$, the other has a zero at $x = x_0$. This means that there has to be another way to write this product, so we use the AΔE's (2.12) for the hyperbolic gamma function to obtain

$$\begin{aligned} G(-i\omega+2x) &= G(2x-i\omega_1/2-i\omega_2/2) \\ &= \frac{G(2x+i\omega_1/2-i\omega_2/2)}{2\cosh\left(\frac{\pi}{\omega_2}(2x-i\omega_2/2)\right)} \\ &= \frac{G(i\omega+2x)}{4\cosh\left(\frac{\pi}{\omega_2}(2x-i\omega_2/2)\right)\cosh\left(\frac{\pi}{\omega_1}(2x+i\omega_1/2)\right)}. \end{aligned}$$

So we have that

$$\begin{aligned} G(i\omega\pm 2x) &= \frac{G(i\omega+2x)}{G(-i\omega+2x)} \\ &= 4\cosh\left(\frac{\pi}{\omega_2}(2x-i\omega_2/2)\right)\cosh\left(\frac{\pi}{\omega_1}(2x+i\omega_1/2)\right) \\ &= \sinh(2\pi x/\omega_1)\sinh(2\pi x/\omega_2), \end{aligned}$$

which is analytic for $x\in\mathbb{C}$. From this we deduce that the poles of $w(\gamma;x)$ occur at

$$x_{kl}^{\delta}(\gamma_j)=\delta\left(i\gamma_j+i\omega+ik\omega_1+il\omega_2\right),\quad k,l\in\mathbb{Z}_{\geq 0},\quad j=0,1,2,3,\ \delta=+,-. \tag{5.3}$$

One could say that the $x_{kl}^{+}(\gamma_j)$ form an upward pole sequence and the $x_{kl}^{-}(\gamma_j)$ form a downward pole sequence. These poles are all simple in case

$$\gamma_0\neq\gamma_1\neq\gamma_2\neq\gamma_3 \text{ and } \omega_1/\omega_2\in\mathbb{R}\setminus\mathbb{Q}. \tag{5.4}$$

We define the measure $dw(\cdot)=dw(\gamma;\cdot)$ by

$$\int f(x)dw(x)=\frac{1}{4\pi}\int_{-\infty}^{\infty}f(x)w(\gamma;x)dx. \tag{5.5}$$

We assume from now on that $\gamma_0,\gamma_1,\gamma_2,\gamma_3,\omega_1$ and ω_2 satisfy the conditions (5.4) with the extra condition that $|\gamma_j|<\omega$ for all $j=0,1,2,3$, such that the measure $dw(\cdot)$ is a positive measure. We also define the measure $d\hat{w}(\cdot)=d\hat{w}(\hat{\gamma};\cdot)$ by

$$\int f(\lambda)d\hat{w}(\lambda)=\frac{1}{4\pi}\int_{-\infty}^{\infty}f(\lambda)\hat{w}(\gamma;\lambda)d\lambda. \tag{5.6}$$

For this measure to be positive we must have that $|\hat{\gamma}_j|<\omega$ for all $j=0,1,2,3$. Altogether this implies that if we want both measures dw and $d\hat{w}$ to be positive, we must have that $\gamma\in P$, where

$$P=\{p\in\mathbb{R}^4 \mid p_0\neq p_1\neq p_2\neq p_3,\ \max(|\hat{p}_0|,\ldots,|\hat{p}_3|)<\omega\}. \tag{5.7}$$

We can easily see that $|\gamma_j|<\omega$ for all $j=0,1,2,3$ if $\gamma\in P$. We surely see that if $|\gamma_j|<\frac{1}{2}\omega$ for all $j=0,1,2,3$, that we have

$$|\hat{\gamma}_j|<\frac{1}{2}(|\gamma_0|+|\gamma_1|+|\gamma_2|+|\gamma_3|)=\omega,\quad j=0,1,2,3.$$

We assume from now on that $\gamma\in P$

We define the Hilbert space $\mathcal{H}_w = \mathcal{H}_w(\gamma) = L^2_e(\mathbb{R}, w(\gamma;x)dx)$ to be the Hilbert space consisting of even functions that have finite norm with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}_w}$ defined by

$$\langle f,g\rangle_{\mathcal{H}_w} = \int f(x)\overline{g(x)}dw(x). \tag{5.8}$$

We also define the Hilbert space $\mathcal{H}_{\hat{w}} = \mathcal{H}_{\hat{w}}(\gamma) = L^2_e(\mathbb{R}, \hat{w}(\hat{\gamma};\lambda)d\lambda)$ to be the Hilbert space consisting of even functions that have finite norm with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}_{\hat{w}}}$ defined by

$$\langle f,g\rangle_{\mathcal{H}_{\hat{w}}} = \int f(\lambda)\overline{g(\lambda)}d\hat{w}(\lambda). \tag{5.9}$$

We conclude this small section with the following lemma which will prove to be useful in the proof of Proposition 5.2.

Lemma 5.1. *For $x \in \mathbb{R}$ we have the following identity*

$$A(\gamma;-x-i\omega_1)w(\gamma;x+i\omega_1) = A(\gamma;x)w(\gamma;x). \tag{5.10}$$

Proof. Let us first write them down

$$w(\gamma;x+i\omega_1) = \frac{G(i\omega+2x+2i\omega_1)G(i\omega-2x-2i\omega_1)}{\prod_{j=0}^3 G(-i\gamma_j+x+i\omega_1)G(-i\gamma_j-x-i\omega_1)} \tag{5.11}$$

$$A(\gamma;-x-i\omega_1) = \frac{\prod_{j=0}^3 s((i\omega-x-i\omega_1+i\gamma_j)/\omega_2)}{s(2(-x-i\omega_1)/\omega_2)s(2(-x-i\omega_1+i\omega)/\omega_2)} \tag{5.12}$$

We make use of the A∆E's (2.12) for the hyperbolic gamma function, but in a slightly different appearance.

$$\frac{G(x)}{G(x-i\omega_1)} = 2\cosh(\pi(x-i\omega_1/2)/\omega_2).$$

For example

$$\frac{1}{G(-i\gamma_j+x+i\omega_1)G(-i\gamma_j-x-i\omega_1)} = \frac{1}{G(-i\gamma_j\pm x)}\frac{\cosh\left(\frac{\pi}{\omega_2}(-i\gamma_j-x-i\omega_1/2)\right)}{\cosh\left(\frac{\pi}{\omega_2}(-i\gamma_j+x+i\omega_1/2)\right)},$$

together with

$$\begin{aligned}
\sinh\left(\frac{\pi}{\omega_2}(i\omega-x-i\omega_1+i\gamma_j)\right) &= \sinh\left(\frac{\pi}{\omega_2}(-i\omega_1/2+i\omega_2/2-x+i\gamma_j)\right)\\
&= \sinh\left(\frac{i\pi}{2}+\frac{\pi}{\omega_2}(-x+i\gamma_j-i\omega_1/2)\right)\\
&= i\cosh\left(\frac{\pi}{\omega_2}(i\gamma_j-x-i\omega_1/2)\right)\\
&= i\cosh\left(\frac{\pi}{\omega_2}(-i\gamma_j+x+i\omega_1/2)\right)
\end{aligned}$$

gives us

$$\begin{aligned}
\frac{s((i\omega-x-i\omega_1+i\gamma_j)/\omega_2)}{G(-i\gamma_j+x+i\omega_1)G(-i\gamma_j-x-i\omega_1)} &= \frac{i\cosh\left(\frac{\pi}{\omega_2}(-i\gamma_j-x-i\omega_1/2)\right)}{G(-i\gamma_j\pm x)}\\
&= \frac{s((i\omega+x+i\gamma_j)/\omega_2)}{G(-i\gamma_j\pm x)}, \quad j=0,1,2,3. \qquad (5.13)
\end{aligned}$$

With this we can rewrite a large part of the product $A(\gamma;-x-i\omega_1)w(\gamma;x+i\omega_1)$. Now we are left with rewriting the following fraction

$$\frac{G(i\omega+2x+2i\omega_1)G(i\omega-2x-2i\omega)}{s(2(-x-i\omega_1)/\omega_2)s(2(-x-i\omega_1+i\omega)/\omega_2)} = \frac{G(i\omega+2x+2i\omega_1)G(i\omega-2x-2i\omega)}{s(2(-x-i\omega_1)/\omega_2)s(2(x+i\omega)/\omega_2)},$$

where we have used standard rules for hyperbolic functions to obtain $s(2(-x-i\omega_1+i\omega)/\omega_2) = s(2(x+i\omega)/\omega_2)$. Using the AΔE's (2.12) for the hyperbolic gamma function again, we obtain

$$\begin{aligned} &G(i\omega+2x+2i\omega_1)G(i\omega-2x-2i\omega) \\ &= G(i\omega\pm 2x)\frac{\cosh\left(\frac{\pi}{\omega_2}(i\omega+2x+\frac{3}{2}i\omega_1)\right)\cosh\left(\frac{\pi}{\omega_2}(i\omega+2x+i\omega_1/2)\right)}{\cosh\left(\frac{\pi}{\omega_2}(i\omega-2x-\frac{3}{2}i\omega_1)\right)\cosh\left(\frac{\pi}{\omega_2}(i\omega-2x-i\omega_1/2)\right)}, \end{aligned}$$

which after multiple times of using standard rules for hyperbolic functions becomes

$$\begin{aligned} G(i\omega+2x+2i\omega_1)G(i\omega-2x-2i\omega) &= G(i\omega\pm 2x)\frac{s((-2x-2i\omega_1)/\omega_2)s((-2x-i\omega_1)/\omega_2)}{s((-2x-i\omega_1)/\omega_2)s(-2x/\omega_2)} \\ &= G(i\omega\pm 2x)\frac{s((2x+2i\omega_1)/\omega_2)}{s(2x/\omega_2)}. \end{aligned}$$

Altogether this means that

$$\frac{G(i\omega+2x+2i\omega_1)G(i\omega-2x-2i\omega)}{s(2(-x-i\omega_1)/\omega_2)s(2(-x-i\omega_1+i\omega)/\omega_2)} = \frac{G(i\omega\pm 2x)}{s(2x/\omega_2)s(2(x+i\omega)/\omega_2)}. \tag{5.14}$$

The product of the lefthandsides of (5.13) (for $j=1,2,3,4$) and (5.14) is equal to $A(\gamma;-x-i\omega_1)w(\gamma;x+i\omega_1)$, so

$$\begin{aligned} A(\gamma;-x-i\omega_1)w(\gamma;x+i\omega_1) &= \frac{\prod_{j=0}^{3} s((i\omega+x+i\gamma_j)/\omega_2)}{s(2x/\omega_2)s(2(x+i\omega)/\omega_2)}\cdot\frac{G(i\omega\pm 2x)}{\prod_{j=0}^{3} G(-i\gamma_j\pm x)} \\ &= A(\gamma;x)w(\gamma;x), \end{aligned}$$

which is the result we desired. □

5.2 The Wronskian

We are going to define a pairing $\langle\cdot,\cdot\rangle_N$, that for $N\to\infty$ gives the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$, and an appropriate wronskian. Proposition 5.5 lets us see how the pairing and the inner product are connected to eachother through an identity involving both. We will need to calculate some limits afterwards to finally obtain an expression for the inner product $\langle R_\lambda, R_{\lambda'}\rangle_{\mathcal{H}_w}$. This expression will be given in Proposition 5.7 and the proof involves using the wronskian. Obtaining this expression is the main goal of this section.

For $0<N<\infty$, we define a pairing $\langle\cdot,\cdot\rangle_N$ by

$$\langle f,g\rangle_N = \frac{1}{4\pi}\int_{-N}^{N} f(x)\overline{g(x)}w(\gamma;x)dx. \tag{5.15}$$

If f and g are real-valued functions in $\mathcal{H}_w$, the limit $N\to\infty$ gives the inner product $\langle f,g\rangle_{\mathcal{H}_w}$. For functions f,g that are analytic in the strip $S=\{z\in\mathbb{C}\mid |\mathrm{Im}(z)|<\omega_1\}$, we define the Wronskian $[f,g]$ by

$$[f,g](z) = \frac{1}{2\pi}\int_{z-i\omega_1}^{z}\left[f(x+i\omega_1)\overline{g(x)} - f(x)\overline{g(x+i\omega_1)}\right]A(\gamma;x)w(\gamma;x)dx. \tag{5.16}$$

Proposition 5.2. *Let f, g be analytic in $S = \{z \in \mathbb{C} \mid |Im(z)| < \omega_1\}$ and even, then $z \mapsto [f,g](z)$ is odd in z.*

Proof. Let I be the function given by

$$I(x) = \frac{1}{2\pi}[f(x+i\omega_1)\overline{g(x)} - f(x)\overline{g(x+i\omega_1)}]A(\gamma;x)w(\gamma;x),$$

then $[f,g](z) = \int_{z-i\omega_1}^{z} I(x)dx$. Since f, g, and w are even functions in x, and by (5.10) $A(\gamma;-x)w(\gamma;x) = A(\gamma;x-i\omega_1)w(\gamma;x-i\omega_1)$, we have $I(-x) = -I(x-i\omega_1)$. Therefore,

$$\int_{z-i\omega_1}^{z} I(x)dx = -\int_{-z+i\omega_1}^{-z} I(-x)dx = \int_{-z+i\omega_1}^{-z} I(x-i\omega_1)dx = -\int_{-z-i\omega_1}^{-z} I(x)dx,$$

which implies that $[f,g](z) = -[f,g](-z)$. We conclude that $z \mapsto [f,g](z)$ is odd in z. □

Proposition 5.3. *For $N \gg 0$ and for even analytic functions f and g in the strip $S = \{z \in \mathbb{C} \mid |Im(z)| < \omega_1\}$,*

$$\langle \mathcal{L}_\gamma^{\omega_1,\omega_2} f, g\rangle_N - \langle f, \mathcal{L}_\gamma^{\omega_1,\omega_2} g\rangle_N = [f,g](N). \tag{5.17}$$

Proof. For even functions f and g, we have

$$\langle f, g\rangle_N = \frac{1}{4\pi}\int_{-N}^{N} f(x)\overline{g(x)}w(\gamma;x)dx.$$

Now, we have by (4.24) and the fact that $\overline{\mathcal{L}_\gamma^{\omega_1,\omega_2} g} = \mathcal{L}_\gamma^{\omega_1,\omega_2}\overline{g}$ that

$$\begin{aligned}
&\langle \mathcal{L}_\gamma^{\omega_1,\omega_2} f, g\rangle_N - \langle f, \mathcal{L}_\gamma^{\omega_1,\omega_2} g\rangle_N \\
=\ & \frac{1}{4\pi}\int_{-N}^{N} (\mathcal{L}_\gamma^{\omega_1,\omega_2} f)(x)\overline{g(x)}w(\gamma;x)dx - \frac{1}{4\pi}\int_{-N}^{N} f(x)\overline{(\mathcal{L}_\gamma^{\omega_1,\omega_2} g)(x)}w(\gamma;x)dx \\
=\ & \frac{1}{4\pi}\int_{-N}^{N} \left[f(x-i\omega_1)\overline{g(x)} - f(x)\overline{g(x-i\omega_1)}\right] A(\gamma;-x)w(\gamma;x)dx \\
& + \frac{1}{4\pi}\int_{-N}^{N} \left[f(x+i\omega_1)\overline{g(x)} - f(x)\overline{g(x+i\omega_1)}\right] A(\gamma;x)w(\gamma;x)dx.
\end{aligned}$$

The first integral can be written as

$$-\frac{1}{4\pi}\int_{-N-i\omega_1}^{N-i\omega_1} [f(x+i\omega_1)\overline{g(x)} - f(x)\overline{g(x+i\omega_1)}]A(\gamma;-x-i\omega_1)w(\gamma;x+i\omega_1)dx,$$

Since $A(\gamma;-x-i\omega_1)w(\gamma;x+i\omega_1) = A(\gamma;x)w(\gamma;x)$ by Lemma 5.1, we have

$$\begin{aligned}
&\langle \mathcal{L}_\gamma^{\omega_1,\omega_2} f, g\rangle_N - \langle f, \mathcal{L}_\gamma^{\omega_1,\omega_2} g\rangle_N \\
=\ & \frac{1}{4\pi}\left(\int_{-N}^{N} - \int_{-N-i\omega_1}^{N-i\omega_1}\right)\left[f(x+i\omega_1)\overline{g(x)} - f(x)\overline{g(x+i\omega_1)}\right] A(\gamma;x)w(\gamma;x)dx.
\end{aligned}$$

Now we make a closed contour by connecting the straight line from $-N$ to N and the straight line from $-N-i\omega_1$ to $N-i\omega_1$ at the end points in a straight line, then the integrand has no poles inside the closed contour. So, by Cauchy's theorem,

$$\int_{-N}^{N} - \int_{-N-i\omega_1}^{N-i\omega_1} = \int_{N-i\omega_1}^{N} - \int_{-N-i\omega_1}^{-N},$$

and from this, we obtain

$$\begin{aligned}\langle \mathcal{L}_\gamma^{\omega_1,\omega_2} f, g\rangle_N - \langle f, \mathcal{L}_\gamma^{\omega_1,\omega_2} g\rangle_N &= \frac{1}{2}[f,g](N) - \frac{1}{2}[f,g](-N)\\ &= \frac{1}{2}[f,g](N) + \frac{1}{2}[f,g](N)\\ &= [f,g](N),\end{aligned}$$

where we have used Proposition 5.2 for the second equality. □

Corollary 5.4. *If f, g are analytic in the strip $S = \{z \in \mathbb{C} \mid |Im(z)| < \omega_1\}$, $f, g \in \mathcal{H}_w$ and $\mathcal{L}_\gamma^{\omega_1,\omega_2} f, \mathcal{L}_\gamma^{\omega_1,\omega_2} g \in \mathcal{H}_w$, we have*

$$\langle \mathcal{L}_\gamma^{\omega_1,\omega_2} f, g\rangle_{\mathcal{H}_w} = \langle f, \mathcal{L}_\gamma^{\omega_1,\omega_2} g\rangle_{\mathcal{H}_w}. \tag{5.18}$$

Proof. The idea of the proof is to look at identity (5.17) and to show that if $f, g \in \mathcal{H}_w$, we have $|[f,g](N)| \to 0$ as $N \to \infty$. Because $\lim_{N\to\infty}\langle f, g\rangle_N = \langle f, g\rangle_{\mathcal{H}_w}$ this would imply that we then have proved this corollary. Let $f, g \in \mathcal{H}_w$, then

$$\begin{aligned}[f,g](N) &= \frac{1}{2\pi}\int_{N-i\omega_1}^{N} \left\{ f(x+i\omega_1)\overline{g(x)} - f(x)\overline{g(x+i\omega_1)}\right\} A(\gamma;x)w(\gamma;x)dx\\ &= \frac{1}{2\pi}\int_{N-i\omega_1}^{N} f(x+i\omega_1)\overline{g(x)}A(\gamma;x)w(\gamma;x)dx\\ &\quad -\frac{1}{2\pi}\int_{N-i\omega_1}^{N} f(x)\overline{g(x+i\omega_1)}A(\gamma;x)w(\gamma;x)dx.\end{aligned}$$

Note that $A(\gamma;x)$ has its poles on the imaginary axis and we have the asymptotics (5.25), so it is a bounded function for all $x \in \mathbb{C}_R = \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$. Let $M = \sup_{x\in\mathbb{C}_R} |A(\gamma;x)|$, then

$$\begin{aligned}|[f,g](N)| &\leq \frac{M}{2\pi}\left|\int_{N-i\omega_1}^{N} f(x+i\omega_1)\overline{g(x)}w(\gamma;x)dx\right| + \frac{M}{2\pi}\left|\int_{N-i\omega_1}^{N} f(x)\overline{g(x+i\omega_1)}w(\gamma;x)dx\right|\\ &= \frac{M}{2\pi}\int_{-\omega_1}^{0} \left|f(N+i(t+\omega_1))\overline{g(N+it)}w(\gamma;N+it)\right| dt\\ &\quad +\frac{M}{2\pi}\int_{-\omega_1}^{0} \left|f(N+it)\overline{g(N+i(t+\omega_1))}w(\gamma;N+it)\right| dt\\ &\leq \frac{M\omega_1}{2\pi}\text{ess sup}_{x\in[N-i\omega_1,N+i\omega_1]}|f(x)g(x)w(\gamma;x)|\\ &\quad +\frac{M\omega_1}{2\pi}\text{ess sup}_{x\in[N-i\omega_1,N+i\omega_1]}|f(x)g(x)w(\gamma;x)|.\end{aligned}$$

All suprema will vanish as $N \to \infty$, because $f, g \in \mathcal{H}_w$. This implies that $\lim_{N\to\infty} |[f,g](N)| = 0$. Taking limits in (5.17) gives

$$\lim_{N\to\infty} \langle \mathcal{L}_\gamma^{\omega_1,\omega_2} f, g\rangle_N - \langle f, \mathcal{L}_\gamma^{\omega_1,\omega_2} g\rangle_N = \lim_{N\to\infty} [f,g](N) = 0,$$

which gives

$$\langle \mathcal{L}_\gamma^{\omega_1,\omega_2} f, g\rangle_{\mathcal{H}_w} = \langle f, \mathcal{L}_\gamma^{\omega_1,\omega_2} g\rangle_{\mathcal{H}_w}.$$

□

Since the R-functions are eigenfunctions of $\mathcal{L}_\gamma^{\omega_1,\omega_2}$ for eigenvalue $B(\gamma;\lambda)$, we obtain from Proposition 5.3 the following result.

Proposition 5.5. *For* $\lambda \neq \lambda'$ *and* $R(\gamma; x, \lambda) = R_\lambda(x)$,

$$\langle R_\lambda, R_{\lambda'} \rangle_N = \frac{[R_\lambda, R_{\lambda'}](N)}{B(\gamma;\lambda) - B(\gamma;\lambda')} = \frac{2[R_\lambda, R_{\lambda'}](N)}{c(2\lambda/\omega_2) - c(2\lambda'/\omega_2)}. \tag{5.19}$$

Remark. We used the definition (4.22) of B and the relation $s(\lambda-a)s(\lambda+a) = \frac{1}{2}[c(2\lambda)-c(2a)]$ to see that

$$\begin{aligned} B(\gamma;\lambda) - B(\gamma;\lambda') &= s((\lambda - i\omega - i\hat{\gamma}_0)/\omega_2)s((\lambda + i\omega + i\hat{\gamma}_0)/\omega_2) - (\lambda \leftrightarrow \lambda') \\ &= \frac{1}{2}[c(2\lambda/\omega_2) - c(2(i\omega + i\hat{\gamma}_0)/\omega_2)] - (\lambda \leftrightarrow \lambda') \\ &= \frac{1}{2}[c(2\lambda/\omega_2) - c(2\lambda'/\omega_2)] \end{aligned}$$

Next we want to let $N \to \infty$ in (5.19), so we need the asymptotic behaviour of the R-function and of $A(\gamma; x + i\omega_1 y)w(\gamma; x + i\omega_1 y)$ for $x \to \infty$ and $-1 \leq y \leq 0$.

The asymptotics for the weight w are calculated with the help of the asymptotics (2.27) for the hyperbolic gamma function.

$$\begin{aligned} w(\gamma; z) = \frac{G(i\omega \pm 2z)}{\prod_{j=0}^{3} G(-i\gamma_j \pm z)} &= \frac{G(2z + i\omega)}{G(2z - i\omega)} \prod_{j=0}^{3} \frac{G(z + i\gamma_j)}{G(z - i\gamma_j)} \\ &\sim \exp\left(\frac{4\pi\omega z}{\omega_1\omega_2}\right) \prod_{j=0}^{3} \exp\left(\frac{2\pi\gamma_j z}{\omega_1\omega_2}\right) \\ &= e^{\alpha z(2\hat{\gamma}_0 + 2\omega)} \end{aligned} \tag{5.20}$$

$A(\gamma; x)$ consists only of hyperbolic functions, so it should not be too hard to obtain the asymptotics for this function. Making use of the limits

$$\lim_{x\to\infty} \frac{s(a+x)}{s(b+x)} = \lim_{x\to\infty} \frac{s(a+x)}{c(b+x)} = e^{\pi(a-b)}, \quad a, b \in \mathbb{C}$$

and the identity $s(2a) = 2s(a)c(a)$ for $a \in \mathbb{C}$, we have

$$\begin{aligned} &\lim_{x\to\infty} A(\gamma; x + i\omega_1 y; \omega_1, \omega_2) \\ =\ &\lim_{x\to\infty} \frac{\prod_{j=0}^{3} s((i\omega + x + i\omega_1 y + i\gamma_j)/\omega_2)}{s(2(x + i\omega_1 y)/\omega_2)s(2(x + i\omega_1 y + i\omega)/\omega_2)} \\ =\ &\frac{1}{4}\lim_{x\to\infty} \frac{\prod_{j=0}^{3} s((i\omega + x + i\omega_1 y + i\gamma_j)/\omega_2)}{s((x + i\omega_1 y)/\omega_2)c((x + i\omega_1 y)/\omega_2)c((x + i\omega_1 y + i\omega)/\omega_2)s((x + i\omega_1 y + i\omega)/\omega_2)} \\ =\ &\frac{1}{4}e^{\frac{\pi}{\omega_2}(i\omega + i\gamma_0)}e^{\frac{\pi}{\omega_2}(i\omega + i\gamma_1)}e^{\frac{\pi i\gamma_2}{\omega_2}}e^{\frac{\pi i\gamma_3}{\omega_2}} \\ =\ &\frac{1}{4}e^{\frac{\pi}{\omega_2}(2i\omega + 2i\hat{\gamma}_0)}. \end{aligned}$$

Note that we have $R(\gamma; x, \lambda) = c(\gamma; x)\mathcal{E}(\gamma; x, \lambda)/K(\gamma; \lambda)$ by (4.26). Therefore we need to look for the asymptotics of the c-function (4.7). Ruijsenaars proved the following asymptotics for $g(\omega_1, \omega_2; z)$ in [6, Prop. III.4]: Fixing $\epsilon > 0$ and setting $\omega_m = \max(\omega_1, \omega_2)$, we have

$$g(\omega_1, \omega_2; z) = -\frac{\pi z^2}{2\omega_1\omega_2} - \frac{\pi(\omega_1^2 + \omega_2^2)}{24\omega_1\omega_2} + \mathcal{O}(\exp((\epsilon - 2\pi/\omega_m)z)), \quad \mathrm{Re}(z) \to \infty, \tag{5.21}$$

where g is the integral (2.2). Recall from (2.6) that $G(z) = \exp(ig(z))$, so the asymptotics for the hyperbolic gamma function are

$$G(\omega_1, \omega_2; z) \sim \exp\left(-\frac{2\pi i}{\omega_1\omega_2}[z^2/4 + (\omega_1^2 + \omega_2^2)/48]\right). \tag{5.22}$$

This means that

$$\begin{aligned}\prod_{j=0}^{3} G(z - i\gamma_j) &\sim \exp\left(-\frac{2\pi i}{\omega_1\omega_2}\left[\sum_{j=0}^{3}\frac{(z-i\gamma_j)^2}{4} + \frac{\omega_1^2+\omega_2^2}{12}\right]\right)\\ &= \exp\left(\frac{2\pi i}{\omega_1\omega_2}\left[-z^2 + iz\hat{\gamma}_0 + \sum_{j=0}^{3}\gamma_j^2/4 - (\omega_1^2+\omega_2^2)/12\right]\right)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{G(2z+i\omega)} &\sim \exp\left(\frac{2\pi i}{\omega_1\omega_2}[(2z+i\omega)^2/4 + (\omega_1^2+\omega_2^2)/48]\right)\\ &= \exp\left(\frac{2\pi i}{\omega_1\omega_2}[z^2 + iz\omega - \omega_2/4 + (\omega_1^2+\omega_2^2)/48]\right)\\ &= \exp\left(\frac{2\pi i}{\omega_1\omega_2}[z^2 + iz\omega - (\omega_1^2+\omega_2^2)/24 - \omega_1\omega_2/8]\right).\end{aligned}$$

For the asymptotics of the c-function, this means that

$$\begin{aligned}c(\gamma; z) &= \frac{1}{G(2z+i\omega)}\prod_{j=0}^{3} G(z - i\gamma_j)\\ &\sim \exp\left(\frac{2\pi i}{\omega_1\omega_2}[\gamma\cdot\gamma/4 + (\omega_1^2+\omega_2^2+\omega_1\omega_2)/8 + z(i\hat{\gamma}_0 + i\omega)]\right)\\ &= \chi(\gamma)e^{i\alpha z(i\hat{\gamma}_0+i\omega)},\end{aligned} \tag{5.23}$$

where $\chi(\gamma)$ is defined by (4.27). We are now ready to calculate the asymptotics of the R-function.

$$\begin{aligned}R_\lambda(\gamma; z) &= K(\gamma,\lambda)^{-1}\mathcal{E}(\gamma; z, \lambda)c(\gamma; z)\\ &\sim \frac{N(\gamma)c(\hat{\gamma};\lambda)}{\chi(\gamma)}\left(e^{i\alpha z\lambda} + \frac{c(\hat{\gamma};-\lambda)}{c(\hat{\gamma};\lambda)}e^{-i\alpha z\lambda}\right)\chi(\gamma)e^{i\alpha z(i\hat{\gamma}_0+i\omega)}\\ &= N(\gamma)e^{i\alpha z(i\hat{\gamma}_0+i\omega)}\left(c(\hat{\gamma};\lambda)e^{i\alpha z\lambda} + c(\hat{\gamma};-\lambda)e^{-i\alpha z\lambda}\right)\end{aligned} \tag{5.24}$$

Altogether we have proved the following proposition.

Proposition 5.6. *We have the following asymptotics*

$$A(\gamma; z) \sim \frac{1}{4}e^{i\alpha\omega_1(\hat{\gamma}_0+\omega)}, \tag{5.25}$$

$$w(\gamma; z) \sim e^{2\alpha z(\hat{\gamma}_0+\omega)}, \tag{5.26}$$

$$R_\lambda(\gamma; z) \sim N(\gamma)c(\hat{\gamma};\lambda)e^{i\alpha z(\lambda+i\hat{\gamma}_0+i\omega)} + N(\gamma)c(\hat{\gamma};-\lambda)e^{-i\alpha z(\lambda-i\hat{\gamma}_0-i\omega)}. \tag{5.27}$$

The asymptotic behaviour of the the pairing $\langle R_\lambda, R_{\lambda'}\rangle_N$ is crucial for the Ruijsenaars function transform that we are defining in the next section. The next proposition gives this result.

Proposition 5.7. *For $N \to \infty$, we have*

$$\langle R_\lambda, R_{\lambda'} \rangle_N \sim \sum_{\varepsilon,\xi\in\{-1,1\}} \frac{c(\hat{\gamma};\varepsilon\lambda)c(\hat{\gamma};\xi\lambda')}{\varepsilon\lambda+\xi\lambda'} e^{i\alpha N(\varepsilon\lambda+\xi\lambda')} \left(e^{-2\pi\varepsilon\lambda/\omega_2} - e^{-2\pi\xi\lambda'/\omega_2}\right)\left(1-e^{\alpha\omega_1(\varepsilon\lambda+\xi\lambda')}\right) \times \frac{N(\gamma)^2}{4\pi i\alpha[c(2\lambda/\omega_2)-c(2\lambda'/\omega_2)]} \tag{5.28}$$

Proof. Let $\Theta(x)$ be the function given by

$$\Theta(x) = R_\lambda(x+i\omega_1)R_{\lambda'}(x) - R_\lambda(x)R_{\lambda'}(x+i\omega_1).$$

From the asymptotic behaviour (5.27) of R_λ we find for $-1 \le y \le 0$ and $x \to \infty$,

$$\Theta(x+i\omega_1 y) = R_\lambda(x+i\omega_1(y+1))R_{\lambda'}(x+i\omega_1 y) - R_\lambda(x+i\omega_1 y)R_{\lambda'}(x+i\omega_1(y+1)),$$

where the second term can be expressed as

$$\begin{aligned} & R_\lambda(x+i\omega_1 y)R_{\lambda'}(x+i\omega_1(y+1)) \\ \sim\ & \left[N(\gamma)c(\hat{\gamma};\lambda)e^{i\alpha(x+i\omega_1 y)(\lambda+i\hat{\gamma}_0+i\omega)} + (\lambda \leftrightarrow -\lambda)\right] \\ & \times \left[N(\gamma)c(\hat{\gamma};\lambda')e^{i\alpha(x+i\omega_1(y+1))(\lambda'+i\hat{\gamma}_0+i\omega)} + (\lambda' \leftrightarrow -\lambda')\right] \\ =\ & N(\gamma)^2 e^{2i\alpha(x+i\omega_1(y+1/2))(i\hat{\gamma}_0+i\omega)} \sum_{\varepsilon,\xi\in\{-1,1\}} c(\hat{\gamma};\varepsilon\lambda)c(\hat{\gamma};\xi\lambda')e^{i\alpha(x+i\omega_1 y)(\varepsilon\lambda+\xi\lambda')-2\pi\xi\lambda'/\omega_2}. \end{aligned}$$

From this we can see that $\Theta(x+i\omega_1 y)$ for $x \to \infty$ is the following function

$$N(\gamma)^2 e^{-2\alpha(x+i\omega_1(y+1/2))(\hat{\gamma}_0+\omega)} \sum_{\varepsilon,\xi\in\{-1,1\}} c(\hat{\gamma};\varepsilon\lambda)c(\hat{\gamma};\xi\lambda')e^{i\alpha(x+i\omega_1 y)(\varepsilon\lambda+\xi\lambda')}\left(e^{-2\pi\varepsilon\lambda/\omega_2} - e^{-2\pi\xi\lambda'/\omega_2}\right)$$

Using the asymptotic behaviour of $A(\gamma; x+i\omega_1 y)w(\gamma; x+i\omega_1 y)$ for $x \to \infty$ gives

$$\begin{aligned} & \Theta(x+i\omega_1 y)A(\gamma;x+i\omega_1 y)w(\gamma;x+i\omega_1 y) \\ \sim\ & \frac{N(\gamma)^2}{4} e^{-2\alpha(x+i\omega_1(y+1/2))(\hat{\gamma}_0+\omega)} e^{i\alpha\omega_1(\hat{\gamma}_0+\omega)} e^{2\alpha(x+i\omega_1 y)(\hat{\gamma}_0+\omega)} \\ & \times \sum_{\varepsilon,\xi\in\{-1,1\}} c(\hat{\gamma};\varepsilon\lambda)c(\hat{\gamma};\xi\lambda')e^{i\alpha(x+i\omega_1 y)(\varepsilon\lambda+\xi\lambda')}\left(e^{-2\pi\varepsilon\lambda/\omega_2} - e^{-2\pi\xi\lambda'/\omega_2}\right) \\ =\ & \frac{N(\gamma)^2}{4} \sum_{\varepsilon,\xi\in\{-1,1\}} c(\hat{\gamma};\varepsilon\lambda)c(\hat{\gamma};\xi\lambda')e^{i\alpha(x+i\omega_1 y)(\varepsilon\lambda+\xi\lambda')}\left(e^{-2\pi\varepsilon\lambda/\omega_2} - e^{-2\pi\xi\lambda'/\omega_2}\right) \end{aligned}$$

Note that most of the factors in this expression are independent of y. Recalling the wronskian (5.16), we calculate by using the substitution $x = N + i\omega_1 y$

$$
\begin{aligned}
& [R_\lambda, R_{\lambda'}](N) \\
= \; & \frac{1}{2\pi}\int_{N-i\omega_1}^{N} \Theta(x)A(\gamma;x)w(\gamma;x)dx \\
= \; & \frac{i\omega_1}{2\pi}\int_{-1}^{0} \Theta(N+i\omega_1 y)A(\gamma;N+i\omega_1 y)w(\gamma;N+i\omega_1 y)dy \\
\sim \; & \frac{i\omega_1 N(\gamma)^2}{8\pi}\sum_{\varepsilon,\xi\in\{-1,1\}} c(\hat{\gamma};\varepsilon\lambda)c(\hat{\gamma};\xi\lambda')e^{i\alpha N(\varepsilon\lambda+\xi\lambda')}\left(e^{-2\pi\varepsilon\lambda/\omega_2}-e^{-2\pi\xi\lambda'/\omega_2}\right) \\
& \times\int_{-1}^{0} e^{-\alpha\omega_1 y(\varepsilon\lambda+\xi\lambda')}dy \\
= \; & \frac{N(\gamma)^2}{8\pi i\alpha}\sum_{\varepsilon,\xi\in\{-1,1\}} \frac{c(\hat{\gamma};\varepsilon\lambda)c(\hat{\gamma};\xi\lambda')}{\varepsilon\lambda+\xi\lambda'}e^{i\alpha N(\varepsilon\lambda+\xi\lambda')}\left(e^{-2\pi\varepsilon\lambda/\omega_2}-e^{-2\pi\xi\lambda'/\omega_2}\right)\left(1-e^{\alpha\omega_1(\varepsilon\lambda+\xi\lambda')}\right).
\end{aligned}
$$

Note that we have used dominated convergence to justify the interchangement of the limit and the integral. We obtain the desired result by using (5.19), which gives the link between the Wronskian $[R_\lambda, R_{\lambda'}](N)$ and the pairing $\langle R_\lambda, R_{\lambda'}\rangle_N$. □

5.3 The Ruijsenaars function transform

The last steps towards the main goal are taken in this section. The RFT $\mathcal{F}$ is defined and we will determine which conditions the functions f must satisfy in order to have a convergent integral transformation. We will also find an inverse integral transformation and finally show that the RFT is a unitary operator in Proposition 5.9. In this section we assume $\lambda, \lambda' \in \mathbb{R}$.

Proposition 5.8. *Let f be an even and continuous function, satisfying*

$$
f(\lambda) = \mathcal{O}\left(|\lambda|^{-1-\epsilon}e^{-\alpha|\lambda|(\gamma_0+\omega-\omega_1)}\right), \quad |\lambda|\to\infty,\ \epsilon>0. \tag{5.29}
$$

Then

$$
\frac{\alpha}{N(\gamma)^2}\lim_{N\to\infty}\int_{-\infty}^{\infty} f(\lambda)\langle R_\lambda, R_{\lambda'}\rangle_N d\lambda = \frac{f(\lambda')}{w(\hat{\gamma};\lambda')}. \tag{5.30}
$$

Proof. Proposition 5.7 gives us a representation of $\langle R_\lambda, R_{\lambda'}\rangle_N$. We multiply both sides with an arbitrary function $f(\lambda)$, and we integrate over λ from $-\infty$ to ∞. The function f must satisfy the condition (5.29) which will be proved at the end of this proof. When using Euler's formula

$$
e^{ix} = \cos(x) + i\sin(x), \quad x\in\mathbb{R}
$$

and letting $N\to\infty$ then gives

$$
\begin{aligned}
& \lim_{N\to\infty}\int_{-\infty}^{\infty} f(\lambda)\langle R_\lambda, R_{\lambda'}\rangle_N d\lambda \\
= \; & \frac{N(\gamma)^2}{4\pi i\alpha}\lim_{N\to\infty}\int_{-\infty}^{\infty} f(\lambda)\left\{\psi_1(\lambda)\cos(\alpha N(\lambda+\lambda'))+\psi_2(\lambda)\sin(\alpha N(\lambda+\lambda'))\right. \\
& \left. +\psi_3(\lambda)\cos(\alpha N(\lambda-\lambda'))+\psi_4(\lambda)\frac{\sin(\alpha N(\lambda-\lambda'))}{\lambda-\lambda'}\right\} d\lambda,
\end{aligned}
$$

where

$$
\begin{aligned}
\psi_1(\lambda) &= \sum_{\varepsilon\in\{-1,1\}} \varepsilon \frac{c(\hat{\gamma};\varepsilon\lambda)c(\hat{\gamma};\varepsilon\lambda')}{\lambda+\lambda'} \frac{\left(e^{-2\pi\varepsilon\lambda/\omega_2}-e^{-2\pi\varepsilon\lambda'/\omega_2}\right)}{c(2\lambda/\omega_2)-c(2\lambda'/\omega_2)} \left(1-e^{\alpha\omega_1\varepsilon(\lambda+\lambda')}\right) \\
\psi_2(\lambda) &= i\sum_{\varepsilon\in\{-1,1\}} \frac{c(\hat{\gamma};\varepsilon\lambda)c(\hat{\gamma};\varepsilon\lambda')}{\lambda+\lambda'} \frac{\left(e^{-2\pi\varepsilon\lambda/\omega_2}-e^{-2\pi\varepsilon\lambda'/\omega_2}\right)}{c(2\lambda/\omega_2)-c(2\lambda'/\omega_2)} \left(1-e^{\alpha\omega_1\varepsilon(\lambda+\lambda')}\right) \\
\psi_3(\lambda) &= \sum_{\varepsilon\in\{-1,1\}} \varepsilon \frac{c(\hat{\gamma};\varepsilon\lambda)c(\hat{\gamma};-\varepsilon\lambda')}{\lambda-\lambda'} \frac{\left(e^{-2\pi\varepsilon\lambda/\omega_2}-e^{2\pi\varepsilon\lambda'/\omega_2}\right)}{c(2\lambda/\omega_2)-c(2\lambda'/\omega_2)} \left(1-e^{\alpha\omega_1\varepsilon(\lambda-\lambda')}\right) \\
\psi_4(\lambda) &= i\sum_{\varepsilon\in\{-1,1\}} c(\hat{\gamma};\varepsilon\lambda)c(\hat{\gamma};-\varepsilon\lambda') \frac{\left(e^{-2\pi\varepsilon\lambda/\omega_2}-e^{2\pi\varepsilon\lambda'/\omega_2}\right)}{c(2\lambda/\omega_2)-c(2\lambda'/\omega_2)} \left(1-e^{\alpha\omega_1\varepsilon(\lambda-\lambda')}\right)
\end{aligned}
$$

From the Riemann-Lebesgue lemma, we find that the terms with $\psi_i,\ i=1,2,3$, vanish, provided that $f\psi_i \in L^1(-\infty,\infty)$. We recognize the term with ψ_4 as a Dirichlet integral. Using the well-known property (see [2, section 9.7]) for Dirichlet integrals

$$
\lim_{t\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty} g(x)\frac{\sin(t(x-y))}{x-y}dx = g(y), \tag{5.31}
$$

for a continuous function $g\in L^1(-\infty,\infty)$ we obtain

$$
\begin{aligned}
\lim_{N\to\infty}\int_{-\infty}^{\infty} f(\lambda)\langle R_\lambda, R_{\lambda'}\rangle_N d\lambda &= \frac{N(\gamma)^2}{4\pi i\alpha}\lim_{N\to\infty}\int_{-\infty}^{\infty} f(\lambda)\psi_4(\lambda)\frac{\sin(\alpha N(\lambda-\lambda'))}{\lambda-\lambda'}d\lambda \\
&= \frac{N(\gamma)^2}{4\pi i\alpha}\cdot \pi f(\lambda')\lim_{\lambda\to\lambda'}\psi_4(\lambda).
\end{aligned}
$$

We have to compute the last limit by using L'Hôpital's rule. If we compute the following

$$
\begin{aligned}
\lim_{\lambda\to\lambda'}\frac{1-e^{\alpha\omega_1\varepsilon(\lambda-\lambda')}}{c(2\lambda/\omega_2)-c(2\lambda'/\omega_2)} &= \lim_{\lambda\to\lambda'}\frac{-\alpha\omega_1\varepsilon e^{\alpha\omega_1\varepsilon(\lambda-\lambda')}}{\frac{2\pi}{\omega_2}\sinh(2\pi\lambda/\omega_2)} \\
&= \frac{-\frac{2\pi}{\omega_2}\varepsilon}{\frac{2\pi}{\omega_2}\sinh(2\pi\lambda'/\omega_2)} \\
&= \frac{-\varepsilon}{\sinh(2\pi\lambda'/\omega_2)},
\end{aligned}
$$

then

$$
\begin{aligned}
\lim_{\lambda\to\lambda'}\psi_4(\lambda) &= 2i\sum_{\varepsilon\in\{-1,1\}} c(\hat{\gamma};\varepsilon\lambda')c(\hat{\gamma};-\varepsilon\lambda')\varepsilon\sinh(-2\pi\lambda'/\omega_2)\cdot\lim_{\lambda\to\lambda'}\frac{1-e^{\alpha\omega_1\varepsilon(\lambda-\lambda')}}{c(2\lambda/\omega_2)-c(2\lambda'/\omega_2)} \\
&= 2i\sum_{\varepsilon\in\{-1,1\}} \varepsilon^2 c(\hat{\gamma};\varepsilon\lambda')c(\hat{\gamma};-\varepsilon\lambda') \\
&= 4ic(\hat{\gamma};\lambda')c(\hat{\gamma};-\lambda').
\end{aligned}
$$

Now, this implies that

$$
\lim_{N\to\infty}\int_{-\infty}^{\infty} f(\lambda)\langle R_\lambda, R_{\lambda'}\rangle_N d\lambda = \frac{N(\gamma)^2}{\alpha}f(\lambda')c(\hat{\gamma};\lambda')c(\hat{\gamma};-\lambda') = \frac{N(\gamma)^2}{\alpha}\frac{f(\lambda')}{w(\hat{\gamma};\lambda')}.
$$

The asymptotic behaviour of ψ_i, for $i = 1, 2, 3, 4$, can be obtained by using that we have

$$c(\hat{\gamma}; \lambda) = \mathcal{O}\left(e^{-\alpha|\lambda|(\gamma_0+\omega)}\right), \quad |\lambda| \to \infty$$

and

$$[c(2\lambda/\omega_2) - c(2\lambda'/\omega_2)]^{-1} = \mathcal{O}\left(e^{2\pi|\lambda|/\omega_2}\right) = \mathcal{O}\left(e^{\alpha\omega_1|\lambda|}\right), \quad |\lambda| \to \infty$$

where we have used the asymptotic behaviour of the c-function that we calculated in (5.23). Then we find, for $i = 1, 2, 3, 4$,

$$|\psi_i(\lambda)| = \mathcal{O}\left(e^{\alpha|\lambda|(\omega_1-\gamma_0-\omega)}\right), \quad |\lambda| \to \infty.$$

So, if f satisfies the conditions in the proposition, then $f\psi_i \in L^1(-\infty, \infty)$. $\square$

We are now ready to define the RFT, give its inverse and show that it is a unitary operator. Let us first define the space $\mathcal{H}_0$ which contains all functions f that are even and continuous, and satisfy

$$f(x) = \mathcal{O}\left(|x|^{-1-\epsilon}e^{-\alpha|x|(\hat{\gamma}_0+\omega+\omega_1)}\right), \quad |x| \to \infty, \ \epsilon > 0. \tag{5.32}$$

Let $\hat{\mathcal{H}}_0$ be the space that contains all functions g that are even and continuous, and satisfy

$$g(\lambda) = \mathcal{O}\left(|\lambda|^{-1-\epsilon}e^{-\alpha|\lambda|(\gamma_0+\omega+\omega_1)}\right), \quad |\lambda| \to \infty, \ \epsilon > 0. \tag{5.33}$$

Theorem 5.9. *For $f \in \mathcal{H}_0$ let the Ruijsenaars Function Transform be defined by*

$$(\mathcal{F}f)(\lambda) = \frac{1}{\sqrt{2\omega_1\omega_2}N(\gamma)}\int_{-\infty}^{\infty} f(x)R_\lambda(x)w(\gamma; x)dx \tag{5.34}$$

and for $f \in \hat{\mathcal{H}}_0$ let $\hat{\mathcal{F}}$ be given by

$$(\hat{\mathcal{F}}f)(x) = \frac{1}{\sqrt{2\omega_1\omega_2}N(\gamma)}\int_{-\infty}^{\infty} f(\lambda)R_\lambda(x)w(\hat{\gamma}; \lambda)d\lambda. \tag{5.35}$$

If $g \in \hat{\mathcal{H}}_0$, then $(\mathcal{F}(\hat{\mathcal{F}}g))(\lambda) = g(\lambda)$ and for $f \in \mathcal{H}_0$ we have $(\hat{\mathcal{F}}(\mathcal{F}f)(x) = f(x)$.

Proof. Let us first look at convergence of the RFT. By the asymptotics (5.26) and (5.27), we have

$$w(\gamma; x) = \mathcal{O}\left(e^{2\alpha|x|(\hat{\gamma}_0+\omega)}\right) \text{ and } R_\lambda(x) = \mathcal{O}\left(e^{-\alpha|x|(\hat{\gamma}_0+\omega)}\right), \quad |x| \to \infty.$$

This means that when $f \in \mathcal{H}_0$, that

$$f(x)R_\lambda(x)w(x) = \mathcal{O}\left(|x|^{-1-\epsilon}e^{-\alpha|x|\omega_1}\right) \implies f(x)R_\lambda(x)w(x) \in L^1(\mathbb{R}).$$

This means that $\mathcal{F}$ is well-defined on $\mathcal{H}_0$. In the same way we can show that $\hat{\mathcal{F}}$ is well-defined on $\hat{\mathcal{H}}_0$. Because $R_\lambda(x)$ is even in x and λ, we can directly see that $(\mathcal{F}f)(x)$ and $(\hat{\mathcal{F}}\hat{f})(\lambda)$ are even functions in x and λ respectively.

Let $g \in \hat{\mathcal{H}}_0$ and define $f(\lambda) = w(\hat{\gamma};\lambda)g(\lambda)$, then f satisfies the conditions given in Proposition 5.8. Then we have

$$\begin{aligned}
(\mathcal{F}(\hat{\mathcal{F}}g))(\lambda') &= \frac{1}{\sqrt{2\omega_1\omega_2}N(\gamma)}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\omega_1\omega_2}N(\gamma)}\int_{-\infty}^{\infty} g(\lambda)R_\lambda(x)w(\hat{\gamma};\lambda)d\lambda\right)R_{\lambda'}(x)w(\gamma;x)dx \\
&= \frac{2\pi}{\omega_1\omega_2 N(\gamma)^2}\lim_{N\to\infty}\int_{-\infty}^{\infty} f(\lambda)\left(\frac{1}{4\pi}\int_{-N}^{N} R_\lambda(x)R_{\lambda'}(x)w(\gamma;x)dx\right)d\lambda \\
&= \frac{\alpha}{N(\gamma)^2}\lim_{N\to\infty}\int_{-\infty}^{\infty} f(\lambda)\langle R_\lambda, R_{\lambda'}\rangle_N d\lambda \\
&= \frac{f(\lambda')}{w(\hat{\gamma};\lambda')} = g(\lambda').
\end{aligned}$$

By duality, we obtain $(\hat{\mathcal{F}}(\mathcal{F}f)(x) = f(x)$ for $f \in \mathcal{H}_0$. □

Theorem 5.10. *Let $f, g \in \mathcal{H}_0$, then*

$$\langle \mathcal{F}f, \mathcal{F}g\rangle_{\mathcal{H}_{\hat{w}}} = \langle f, g\rangle_{\mathcal{H}_w}. \tag{5.36}$$

Consequently, $\mathcal{F}$ extends uniquely to a unitary operator $\mathcal{F} : \mathcal{H}_w \to \mathcal{H}_{\hat{w}}$.

Proof. We are first going to show for $f, g \in \hat{\mathcal{H}}_0$ that $\langle \hat{\mathcal{F}}f, \hat{\mathcal{F}}g\rangle_{\mathcal{H}_w} = \langle f, g\rangle_{\mathcal{H}_{\hat{w}}}$. From Proposition 5.8, we obtain by using Fubini's theorem

$$\begin{aligned}
\langle \hat{\mathcal{F}}f, \hat{\mathcal{F}}g\rangle_{\mathcal{H}_w} &= \lim_{N\to\infty}\frac{1}{4\pi}\int_{-N}^{N}(\hat{\mathcal{F}}f)(x)\overline{(\hat{\mathcal{F}}g)(x)}w(\gamma;x)dx \\
&= \frac{\alpha}{4\pi N(\gamma)^2}\lim_{N\to\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\lambda)\overline{g(\lambda')}\langle R_\lambda, R_{\lambda'}\rangle_N w(\hat{\gamma};\lambda)w(\hat{\gamma};\lambda')d\lambda d\lambda' \\
&= \frac{1}{4\pi}\int_{-\infty}^{\infty} f(\lambda')\overline{g(\lambda')}w(\hat{\gamma};\lambda')d\lambda' = \langle f, g\rangle_{\mathcal{H}_{\hat{w}}} \qquad (5.37)
\end{aligned}$$

It is now clear from (5.37) that we have shown that

$$\langle \mathcal{F}f, \mathcal{F}g\rangle_{\mathcal{H}_{\hat{w}}} = \langle f, g\rangle_{\mathcal{H}_w}. \tag{5.38}$$

For the second part, we have $\mathcal{S}(\mathbb{R}) \subset \mathcal{H}_0 \subset \mathcal{H}_w \subset L^2(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ is the Schwartz space that consists of rapidly decreasing functions in the sense that

$$\sup_{x\in\mathbb{R}} |x|^k |f^{(l)}(x)| < \infty, \quad k, l \in \mathbb{Z}_{\geq 0}.$$

It is known that $\mathcal{S}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, which implies that $\mathcal{H}_0$ is dense in $\mathcal{H}_w$. So for every $f \in \mathcal{H}_w$ we can find a sequence of functions $(f_n)_{n=0}^{\infty} \subset \mathcal{H}_0$ such that $\|f_n - f\|_{\mathcal{H}_w} \to 0$ as $n \to \infty$. This means that $(f_n)_{n=0}^{\infty}$ is Cauchy:

$$\|f_n - f_m\|_{\mathcal{H}_w} \to 0, \text{ for } m, n \to \infty.$$

Note that $\mathcal{F}$ is a linear operator, so by (5.38)

$$\|\mathcal{F}f_n - \mathcal{F}f_m\|_{\mathcal{H}_{\hat{w}}} = \|\mathcal{F}(f_n - f_m)\|_{\mathcal{H}_{\hat{w}}} = \|f_n - f_m\|_{\mathcal{H}_w} \to 0. \tag{5.39}$$

So $(\mathcal{F}f_n)_{n=0}^{\infty}$ is Cauchy in $\mathcal{H}_{\hat{w}}$, hence it converges. We call the limit $\mathcal{F}f$. This defines uniquely $\mathcal{F}f$ for any $f \in \mathcal{H}_w$. □

A nice property that follows from the symmetry (5.18) is given in the next proposition. This proposition says that the RFT interchanges the AW-difference operator $\mathcal{L}_\gamma^{\omega_1,\omega_2}$ with a multiplication operator.

Proposition 5.11. *Let $f \in \mathcal{H}_w$ and $\mathcal{M}_{\gamma,\lambda}^{\omega_1,\omega_2}$ be the operator of multiplication by $B(\gamma;\lambda;\omega_1,\omega_2)$. Then we have*

$$\mathcal{F}\mathcal{L}_\gamma^{\omega_1,\omega_2} f = \mathcal{M}_{\gamma,\lambda}^{\omega_1,\omega_2}\mathcal{F}f. \tag{5.40}$$

Proof. The Ruijsenaars function transform $\mathcal{F}$ (5.34) can be written as $\mathcal{F}f = \langle f, R_\lambda\rangle_{\mathcal{H}_w}$ for every $f \in \mathcal{H}_0$. Let $(f_n)_{n=0}^\infty \subset \mathcal{H}_0$ such that $\|f_n - f\|_{\mathcal{H}_w} \to 0$ as $n \to \infty$. Thanks to the symmetry (5.18) and the eigenvalue equation (4.25), we can make the following deduction for every $f_n \in \mathcal{H}_0$

$$\begin{aligned}
\mathcal{F}\mathcal{L}_\gamma^{\omega_1,\omega_2} f_n &= \langle \mathcal{L}_\gamma^{\omega_1,\omega_2} f_n, R_\lambda\rangle_{\mathcal{H}_w} \\
&= \langle f_n, \mathcal{L}_\gamma^{\omega_1,\omega_2} R_\lambda\rangle_{\mathcal{H}_w} \\
&= \langle f_n, B(\gamma;\lambda) R_\lambda\rangle_{\mathcal{H}_w} \\
&= \mathcal{M}_{\gamma,\lambda}^{\omega_1,\omega_2}\mathcal{F}f_n.
\end{aligned} \tag{5.41}$$

Letting $n \to \infty$ gives the desired result. □

Remark. Note that this Proposition gives the same kind of result as identity (1.27) in the introduction. Here we have an integral transformation that interchanges a differential operator with a multiplication operator.

Chapter 6

Further research: Discrete spectrum

In Chapter 5, we defined the RFT in case $|\gamma_j| < \omega$ for all $j = 0, 1, 2, 3$. In case we have $\gamma_j < -\omega$ for at least one $j = 0, 1, 2, 3$, we have the situation that the upward- and downward pole sequences (5.3) of $w(\gamma; x)$ will cross the real axis. In this case we need an integration contour that separates the upward pole sequence from the downward pole sequence. By deforming this contour to the real axis, we pick up residues of the weight w. So the measure dw has discrete mass points in this case. This chapter discusses how this can be done.

6.1 The Hilbert space $\mathcal{H}_w$ with discrete spectrum

Recall from (5.3) that the poles of the weight function $w(\gamma; x)$ occur at

$$x^{\delta}_{kl}(\gamma_j) = \delta\left(i\gamma_j + i\omega + ik\omega_1 + il\omega_2\right), \quad k, l \in \mathbb{Z}_{\geq 0}, \quad j = 0, 1, 2, 3, \ \delta = +, -. \tag{6.1}$$

For $j \in \{0, 1, 2, 3\}$, we define the set $\mathcal{D}_j$ by

$$\mathcal{D}_j = \left\{x = x^{+}_{kl}(\gamma_j) \mid k, l \in \mathbb{Z}_{\geq 0},\ \mathrm{Im}\left(x^{+}_{kl}(\gamma_j)\right) < 0\right\}, \tag{6.2}$$

and let $\mathcal{D} = \mathcal{D}_0 \cup \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$. Note that $\mathcal{D}$ is an empty set in case $\gamma_j > -\omega$ for all $j = 0, 1, 2, 3$ and that it is always a finite set. We also define $\hat{\mathcal{D}} = \hat{\mathcal{D}}_0 \cup \hat{\mathcal{D}}_1 \cup \hat{\mathcal{D}}_2 \cup \hat{\mathcal{D}}_3$, where

$$\hat{\mathcal{D}}_j = \left\{x = x^{+}_{kl}(\hat{\gamma}_j) \mid k, l \in \mathbb{Z}_{\geq 0},\ \mathrm{Im}\left(x^{+}_{kl}(\hat{\gamma}_j)\right) < 0\right\}. \tag{6.3}$$

We define the measure $dw(\cdot) = dw(\gamma; \cdot)$ by

$$\int_{-\infty}^{\infty} f(x) dw(x) = \frac{1}{4\pi}\int_{-\infty}^{\infty} f(x) w(\gamma; x) dx + \frac{i}{2}\sum_{x \in \mathcal{D}} f(x) \operatorname*{Res}_{z=x} w(\gamma; z). \tag{6.4}$$

In case $\gamma_j > -\omega$ for all $j = 0, 1, 2, 3$, the second term on the right-hand side of (6.4) will vanish beacause we will not have to deform the contour of integration. If $x \in \mathcal{D}_0$, we have explicitly

$$\begin{aligned}\operatorname*{Res}_{z=i\gamma_0+i\omega+ik\omega_1+il\omega_2} w(\gamma; z) \quad &= \quad \frac{G(2i\gamma_0 + 3i\omega + 2ik\omega_1 + 2il\omega_2)}{\prod_{j=1}^{3} G(-i\gamma_j \pm x^{+}_{kl}(\gamma_0))} \\ &\quad \times \frac{2\pi \prod_{m=1}^{k} \sin(\pi m\omega_1/\omega_2) \prod_{n=1}^{l} \sin(\pi n\omega_2/\omega_1)}{i\sqrt{\omega_1\omega_2}(-1)^{kl}(-1/2)^{k+l}},\end{aligned} \tag{6.5}$$

where we have used the residues for the hyperbolic gamma function in Proposition 2.2 (2.24)-(2.26). We want the residues of the measure $dw(\cdot)$ to be positive. Using the difference equations for the hyperbolic gamma function (2.12), we can always (for example) write $G(2i\gamma_0 + 3i\omega + 2ik\omega_1 + 2il\omega_2)$ as $S(\omega_1, \omega_2)G(u(\omega_1, \omega_2, \gamma_0))$. Here we have that $S(\omega_1, \omega_2)$ is a multiplication of hyperbolic functions and $u(\omega_1, \omega_2, \gamma_0) \in i(-\omega, \omega)$. In this case we know that $G(u(\omega_1, \omega_2, \gamma_0)) > 0$ because of (2.21). We can force all the arguments of the G-functions in (6.5) to be such that we have only positive G-functions, but we will be left with a lot of hyperbolic functions to be dealt with. And we certainly do not have a unique way of carrying out this procedure in case (say) ω_1 is very small compared to ω_2. We assume that $dw(\cdot)$ is a positive measure from now on.

We also define the measure $d\hat{w}(\cdot) = dw(\hat{\gamma}; \cdot)$ by

$$\int f(\lambda) d\hat{w}(\lambda) = \frac{1}{4\pi} \int_{-\infty}^{\infty} f(\lambda) w(\hat{\gamma}; \lambda) d\lambda + \frac{i}{2} \sum_{\lambda \in \hat{\mathcal{D}}} f(\lambda) \underset{z=\lambda}{\mathrm{Res}}\, w(\hat{\gamma}; z), \tag{6.6}$$

where we also assume that the residues in this expression are positive. We define again the Hilbert space $\mathcal{H}_w = \mathcal{H}_w(\gamma) = L_e^2(\mathbb{R}, w(\gamma; x)dx)$ to be the Hilbert space consisting of even functions that have finite norm with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}_w}$ defined by

$$\langle f, g \rangle_{\mathcal{H}_w} = \int f(x) \overline{g(x)} dw(x). \tag{6.7}$$

In the same way we define the Hilbert space $\mathcal{H}_{\hat{w}} = \mathcal{H}_{\hat{w}}(\hat{\gamma}) = L_e^2(\mathbb{R}, w(\hat{\gamma}; \lambda)d\lambda)$ with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}_{\hat{w}}}$ defined by

$$\langle f, g \rangle_{\mathcal{H}_{\hat{w}}} = \int f(\lambda) \overline{g(\lambda)} d\hat{w}(\lambda). \tag{6.8}$$

6.2 The Wronskian with discrete spectrum

We naturally have to change the definition of the pairing $\langle \cdot, \cdot \rangle_N$. For $0 < N < \infty$, we define a pairing $\langle \cdot, \cdot \rangle_N$ by

$$\langle f, g \rangle_N = \frac{1}{4\pi} \int_{-N}^{N} f(x) \overline{g(x)} w(\gamma; x) dx + \frac{i}{2} \sum_{x \in \mathcal{D}} f(x) \overline{g(x)} \underset{z=x}{\mathrm{Res}}\, w(\gamma; z). \tag{6.9}$$

If f and g are real-valued functions in $\mathcal{H}_w$, the limit $N \to \infty$ gives the inner product $\langle f, g \rangle_{\mathcal{H}_w}$. For functions f, g that are analytic in the strip $S = \{z \in \mathbb{C} \mid |\mathrm{Im}(z)| < \omega_1\}$, we define the Wronskian $[f, g]$ by

$$[f, g](z) = \frac{1}{2\pi} \int_{z - i\omega_1}^{z} \left[f(x + i\omega_1) \overline{g(x)} - f(x) \overline{g(x + i\omega_1)} \right] A(\gamma; x) w(\gamma; x) dx. \tag{6.10}$$

Note that the definition of the wronskian has not changed. The proofs of Proposition 5.2 till Proposition 5.7 remain unchanged in case we define the pairing as in (6.9), except for a slight change of the integration contour in the proof of Proposition 5.3. This means that all results in section 5.2 remain unchanged.

6.3 The Ruijsenaars Function Transform with discrete spectrum

In this section we define the RFT with the discrete spectrum. The discrete spectrum plays a role if the upward pole sequences $\mathcal{D}$ contain points under the real axis. Otherwise we can continue as in Section 5.3. The discrete spectrum brings some extra calculations with it, which are carried out from Section 6.3.2.

6.3.1 Continuous spectrum

We assume first that $\lambda, \lambda' \in \mathbb{R}$. Everything still remains the same until we reach Theorem 5.9. We will take it over from here and continue with the new measure.

Let the space $\mathcal{H}_0$ be defined as

$$\mathcal{H}_0 := \left\{ f \in \mathcal{H}_w, \text{ continuous}, f(x) = \mathcal{O}\left(|x|^{-1-\epsilon} e^{-\alpha|x|(\hat{\gamma}_0+\omega+\omega_1)} \right) \text{ for } |x| \to \infty \right\}. \tag{6.11}$$

Theorem 6.1. *For $f \in \mathcal{H}_0$ let the Ruijsenaars Function Transform be defined by*

$$(\mathcal{F}f)(\lambda) = \frac{4\pi}{\sqrt{2\omega_1\omega_2}N(\gamma)} \int f(x) R_\lambda(x) dw(x) \tag{6.12}$$

and for $f \in \hat{\mathcal{H}}_0$ let $\hat{\mathcal{F}}$ be given by

$$(\hat{\mathcal{F}}f)(x) = \frac{4\pi}{\sqrt{2\omega_1\omega_2}N(\gamma)} \int f(\lambda) R_\lambda(x) d\hat{w}(\lambda). \tag{6.13}$$

We denote the continuous part of the above integral by $\mathcal{F}_c f$, that is,

$$(\mathcal{F}_c f)(\lambda) = \frac{1}{\sqrt{2\omega_1\omega_2}N(\gamma)} \int_{-\infty}^{\infty} f(x) R_\lambda(x) w(\gamma; x) dx. \tag{6.14}$$

If $g \in \hat{\mathcal{H}}_0$, then $(\mathcal{F}(\hat{\mathcal{F}}_c g))(\lambda) = g(\lambda)$ and for $f \in \mathcal{H}_0$ we have $(\hat{\mathcal{F}}(\mathcal{F}f)(x) = f(x)$.

Proof. Let $g \in \hat{\mathcal{H}}_0$ and define $f(\lambda) = w(\hat{\gamma}; \lambda) g(\lambda)$, then f satisfies the conditions given in Proposition 5.8. Then we have

$$\begin{aligned}
(\mathcal{F}(\hat{\mathcal{F}}_c g))(\lambda') &= \frac{4\pi}{\sqrt{2\omega_1\omega_2}N(\gamma)} \int \left(\frac{1}{\sqrt{2\omega_1\omega_2}N(\gamma)} \int_{-\infty}^{\infty} g(\lambda) R_\lambda(x) w(\hat{\gamma}; \lambda) d\lambda \right) R_{\lambda'}(x) dw(x) \\
&= \frac{2\pi}{\omega_1\omega_2 N(\gamma)^2} \lim_{N\to\infty} \int_{-\infty}^{\infty} f(\lambda) \left(\frac{1}{4\pi} \int_{-N}^{N} R_\lambda(x) R_{\lambda'}(x) w(\gamma; x) dx \right. \\
&\quad \left. + \frac{i}{2} \sum_{x\in\mathcal{D}} R_\lambda(x) R_{\lambda'}(x) \operatorname*{Res}_{z=x} w(\gamma; z) \right) d\lambda \\
&= \frac{\alpha}{N(\gamma)^2} \lim_{N\to\infty} \int_{-\infty}^{\infty} f(\lambda) \langle R_\lambda, R_{\lambda'} \rangle_N d\lambda \\
&= \frac{f(\lambda')}{w(\hat{\gamma}; \lambda')} = g(\lambda').
\end{aligned}$$

By duality, we obtain $(\hat{\mathcal{F}}(\mathcal{F}f)(x) = f(x)$ for $f \in \mathcal{H}_0$. □

6.3.2 Discrete spectrum

In this subsection, we assume that $\lambda \in \hat{\mathcal{D}}$ and that the set $\hat{\mathcal{D}}$ (6.3) is not empty, so $\mathrm{Im}(\lambda) < 0$. First we show that R_λ is orthogonal to $R_{\lambda'}$ if $\lambda' \neq \lambda$.

Proposition 6.2. *For $\lambda \in \hat{\mathcal{D}}$, $\lambda' \in supp(d\hat{w})$, and $\lambda' \neq \lambda$,*

$$\langle R_\lambda, R_{\lambda'} \rangle_{\mathcal{H}_w} = 0. \tag{6.15}$$

Proof. Recall the definition of $\hat{\mathcal{D}}$ at (6.3). From Proposition 5.7 we know that

$$\begin{aligned}\langle R_\lambda, R_{\lambda'}\rangle_N \quad &\sim \quad \sum_{\varepsilon,\xi\in\{-1,1\}} \frac{c(\hat{\gamma};\varepsilon\lambda)c(\hat{\gamma};\xi\lambda')}{\varepsilon\lambda+\xi\lambda'} e^{i\alpha N(\varepsilon\lambda+\xi\lambda')}\left(e^{-2\pi\varepsilon\lambda/\omega_2}-e^{-2\pi\xi\lambda'/\omega_2}\right)\left(1-e^{\alpha\omega_1(\varepsilon\lambda+\xi\lambda')}\right)\\ &\times\frac{N(\gamma)^2}{4\pi i\alpha[c(2\lambda/\omega_2)-c(2\lambda'/\omega_2)]}.\end{aligned}$$

From the zeros of the c-function (4.10), we conclude that $c(\hat{\gamma};\lambda)=0$ when $\lambda\in\hat{\mathcal{D}}$. So in case $\lambda\in\hat{\mathcal{D}}$, we have

$$\begin{aligned}\langle R_\lambda, R_{\lambda'}\rangle_N \quad &\sim \quad \sum_{\xi\in\{-1,1\}} \frac{c(\hat{\gamma};-\lambda)c(\hat{\gamma};\xi\lambda')}{-\lambda+\xi\lambda'} e^{i\alpha N(-\lambda+\xi\lambda')}\left(e^{2\pi\lambda/\omega_2}-e^{-2\pi\xi\lambda'/\omega_2}\right)\left(1-e^{\alpha\omega_1(-\lambda+\xi\lambda')}\right)\\ &\times\frac{N(\gamma)^2}{4\pi i\alpha[c(2\lambda/\omega_2)-c(2\lambda'/\omega_2)]}.\end{aligned} \tag{6.16}$$

Recall that for $\lambda\in\hat{\mathcal{D}}$, we have $\lambda\in i\mathbb{R}_{<0}$. Then it is clear that for $\lambda'\in\mathbb{R}$, the right-hand side of (6.16) tends to zero for $N\to\infty$. In case $\lambda'\in\hat{\mathcal{D}}$, we have again by (4.10)

$$\begin{aligned}\langle R_\lambda, R_{\lambda'}\rangle_N \quad &\sim \quad \frac{c(\hat{\gamma};-\lambda)c(\hat{\gamma};-\lambda')}{\lambda+\lambda'} e^{-i\alpha N(\lambda+\lambda')}\left(e^{2\pi\lambda'/\omega_2}-e^{2\pi\lambda/\omega_2}\right)\left(1-e^{-\alpha\omega_1(\lambda+\lambda')}\right)\\ &\times\frac{N(\gamma)^2}{4\pi i\alpha[c(2\lambda/\omega_2)-c(2\lambda'/\omega_2)]},\end{aligned} \tag{6.17}$$

with $\mathrm{Im}(\lambda+\lambda')<0$. So in this case the right-hand side also tends to zero for $N\to\infty$. □

It remains to calculate the squared norm of R_λ in case $\lambda\in\hat{\mathcal{D}}$.

Proposition 6.3. *For $\lambda\in\hat{\mathcal{D}}$,*

$$\langle R_\lambda, R_\lambda\rangle_{\mathcal{H}_w} = \frac{N(\gamma)^2}{2\pi i\alpha}\left(\underset{\lambda'=\lambda}{Res}\ w(\hat{\gamma};\lambda')\right)^{-1} \tag{6.18}$$

Proof. We use expression (6.16), where we let $\lambda'\to\lambda$. Then for large N,

$$\begin{aligned}\lim_{\lambda'\to\lambda}\langle R_\lambda, R_{\lambda'}\rangle_N \quad &\sim \quad \frac{N(\gamma)^2c(\hat{\gamma};-\lambda)^2}{-8\pi i\alpha\lambda}e^{-2i\alpha\lambda N}\left(1-e^{-2\alpha\lambda\omega_1}\right)\lim_{\lambda'\to\lambda}\frac{e^{2\pi\lambda/\omega_2}-e^{2\pi\lambda'/\omega_2}}{c(2\lambda/\omega_2)-c(2\lambda'/\omega_2)}\\ &+\frac{N(\gamma)^2}{2\pi i\alpha}s(2\lambda/\omega_2)c(\hat{\gamma};-\lambda)\lim_{\lambda'\to\lambda}\frac{c(\hat{\gamma};\lambda')\left(1-e^{\alpha\omega_1(\lambda'-\lambda)}\right)}{(\lambda'-\lambda)[c(2\lambda/\omega_2)-c(2\lambda'/\omega_2)]}.\end{aligned} \tag{6.19}$$

By using L'Hôpital's rule, we are able to find the following limits

$$\lim_{\lambda'\to\lambda}\frac{e^{2\pi\lambda/\omega_2}-e^{2\pi\lambda'/\omega_2}}{c(2\lambda/\omega_2)-c(2\lambda'/\omega_2)} = \frac{e^{2\pi\lambda/\omega_2}}{s(2\lambda/\omega_2)} \tag{6.20}$$

$$\lim_{\lambda'\to\lambda}\frac{1-e^{\alpha\omega_1(\lambda'-\lambda)}}{c(2\lambda/\omega_2)-c(2\lambda'/\omega_2)} = \frac{1}{s(2\lambda/\omega_2)}. \tag{6.21}$$

Considering that (6.20) is independent of N as well as the second term, we can see from (6.19) that

$$\lim_{\lambda'\to\lambda}\langle R_\lambda, R_{\lambda'}\rangle_{\mathcal{H}_w} = \frac{N(\gamma)^2}{2\pi i\alpha}c(\hat{\gamma};-\lambda)\lim_{\lambda'\to\lambda}\frac{c(\hat{\gamma};\lambda')}{\lambda'-\lambda}.$$

Using that $c(\hat{\gamma};\lambda')/(\lambda'-\lambda)$ has only simple poles, we can calculate the last limit as follows

$$\lim_{\lambda'\to\lambda}\frac{c(\hat{\gamma};\lambda')}{\lambda'-\lambda} = \left(\lim_{\lambda'\to\lambda}\frac{\lambda'-\lambda}{c(\hat{\gamma};\lambda')}\right)^{-1} = \left(\operatorname*{Res}_{\lambda'=\lambda}\frac{1}{c(\hat{\gamma};\lambda')}\right)^{-1}. \tag{6.22}$$

This finally results in

$$\langle R_\lambda, R_\lambda\rangle_{\mathcal{H}_w} = \frac{N(\gamma)^2}{2\pi i\alpha}\left(\operatorname*{Res}_{\lambda'=\lambda} w(\hat{\gamma};\lambda')\right)^{-1}.$$

□

6.3.3 The Ruijsenaars Function Transform

Combining Propositions 5.10 and 5.11 with Theorem 5.9 gives the following theorem.

Theorem 6.4. *The Ruijsenaars Function Transform $\mathcal{F}$, defined by*

$$(\mathcal{F}f)(\lambda) = \frac{4\pi}{\sqrt{2\omega_1\omega_2}N(\gamma)}\int f(x)R_\lambda(x)dw(x) \tag{6.23}$$

extends to a unitary operator $\mathcal{F}:\mathcal{H}_w\to\mathcal{H}_{\hat{w}}$, and its inverse is given by $\mathcal{F}^{-1}=\hat{\mathcal{F}}$.

Proof. First we show that $\mathcal{F}\circ\hat{\mathcal{F}}$ is the identity operator on $\mathcal{H}_0$. The proof for the continuous part of $\hat{\mathcal{F}}$ is Theorem 5.9. Therefore we just write down the proof for the discrete part of $\hat{\mathcal{F}}$. We denote the discrete part of the RFT by $\mathcal{F}_d$, that is,

$$(\mathcal{F}_d f)(\lambda) = \frac{\pi i}{\sqrt{2\omega_1\omega_2}N(\gamma)}\sum_{x\in\mathcal{D}} f(x)R_\lambda(x)\operatorname*{Res}_{z=x} w(\gamma;z). \tag{6.24}$$

Let $g\in\hat{\mathcal{H}}_0$. Recall that $\hat{\mathcal{D}}$ is a finite set, then we obtain from Propositions 5.10 and 5.11,

$$\begin{aligned}
(\mathcal{F}(\hat{\mathcal{F}}_d g))(\lambda') &= \frac{4\pi}{\sqrt{2\omega_1\omega_2}N(\gamma)}\int\left(\frac{\pi i}{\sqrt{2\omega_1\omega_2}N(\gamma)}\sum_{\lambda\in\hat{\mathcal{D}}} g(\lambda)R_\lambda(x)\operatorname*{Res}_{z=x} w(\hat{\gamma};z)\right)R_{\lambda'}(x)dw(x)\\
&= \frac{2\pi i\alpha}{N(\gamma)^2}\sum_{\lambda\in\hat{\mathcal{D}}} g(\lambda)\operatorname*{Res}_{z=x} w(\hat{\gamma};z)\left(\int R_\lambda(x)R_{\lambda'}(x)dw(x)\right)\\
&= \frac{2\pi i\alpha}{N(\gamma)^2}\sum_{\lambda\in\hat{\mathcal{D}}} g(\lambda)\operatorname*{Res}_{z=x} w(\hat{\gamma};z)\langle R_\lambda, R_{\lambda'}\rangle_{\mathcal{H}_w}\\
&= \begin{cases} 0, & \lambda'\in\mathbb{R},\\ g(\lambda'), & \lambda'\in\hat{\mathcal{D}}.\end{cases}
\end{aligned} \tag{6.25}$$

The interchangement of the integral and the sum is justified by the fact that we take the sum over a finite set. Combining the above result with Theorem 5.9 and the fact that $(\mathcal{F}(\hat{\mathcal{F}}g))(\lambda') = (\mathcal{F}(\hat{\mathcal{F}}_c g+\hat{\mathcal{F}}_d g))(\lambda')$, we obtain the desired result. By duality, we obtain $(\hat{\mathcal{F}}(\mathcal{F}f))(x)=f(x)$ for $f\in\mathcal{H}_0$. The Plancherel identity can be proved as in Theorem 5.10 and $\mathcal{F}$ can be extended similar as in Theorem 5.10. □

The proof of Theorem 5.13 runs along the same lines in case dw and $d\hat{w}$ have discrete mass points. The proof of Proposition 5.14 remains the same. We see now that the results in this chapter are the same as in Chapter 5 if we take the discrete spectrum into account.

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