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DOI
10.1088/1361-6404/adc357

Publication date
2025

Document Version
Final published version

Published in
European Journal of Physics

Citation (APA)
Idema, T. (2025). Stokes flow with Oseen tensors: Stokes' law, force fields, and flow near a wall. *European Journal of Physics*, *46*(3), Article 035802. https://doi.org/10.1088/1361-6404/adc357

European Journal of
Physics

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To cite this article: Timon Idema 2025 *Eur. J. Phys.* **46** 035802

OPEN ACCESS
IOP Publishing
European Journal of Physics
Eur. J. Phys. 46 (2025) 035802 (11pp)
https://doi.org/10.1088/1361-6404/adc357

Stokes flow with Oseen tensors: Stokes' law, force fields, and flow near a wall

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Received 15 January 2025, revised 10 March 2025
Accepted for publication 20 March 2025
Published 14 April 2025

Abstract

The quintessential example of a fluid flow problem with a known solution is the drag force required to move a sphere through a stationary fluid. While the equation describing this Stokes drag is simple, deriving it from the Stokes equations requires several pages of mathematics. In this paper, I present an alternative, more intuitive approach, based on the Oseen tensor, which gives the fluid flow due to an applied point force. For completeness, we first derive the expression for the Oseen tensor, then use it to re-derive Stokes law. As an additional application, we also study fluid flow near a wall, using a mirror images technique similar to the one used in electrostatics problems.

Keywords: Stokes' law, drag force, low Reynolds number flow, Oseen tensor

1. Introduction

Stokes' law relates the drag force on a sphere to the sphere's radius and velocity as the sphere is moving through a viscous fluid, in the limit where the drag force is much larger than the inertial forces, i.e., when the Reynolds number is small [1]. As the Reynolds number is proportional to the object's size and speed and the fluid density, and inversely proportional to the fluid's viscosity, there are various regimes in which Stokes' law applies: the slow motion of macroscopic objects through air and water, motion through highly viscous fluids, and any motion of microscopic objects like swimming bacteria or crawling cells [2, 3]. In this regime, the nonlinear Navier–Stokes equations simplify to the linear Stokes equations. From the Stokes equations, the drag force on a sphere can be calculated directly; an explicit calculation can be found in the textbook by Landau and Lifshitz [4]. This calculation however relies heavily on extended mathematical manipulations and the use of symmetry arguments, taking up several pages. For a sphere, one can also reduce the problem to an effective two-dimensional one, which is easier to solve, using a Stokes stream function [5–8], but does not generalize to non-axisymmetric shapes, and even the application to a cylinder is nontrivial [9]. Alternatively, one can motivate the functional form of Stokes law through dimensional analysis, once one has assumed that the drag force depends on the radius and velocity of the sphere, and the viscosity of the fluid medium. With this approach, however, it is impossible to determine the prefactor relating the drag force to the given quantities. In this paper, I present a third option: using the elementary solution to the Stokes equations, the Oseen tensor, which relates a point force acting in a fluid to the resulting fluid flow. As the derivation of the Oseen tensor itself is also difficult to find, I will present a derivation here as well. A benefit of this route is that it allows us to solve for much more than just the Stokes drag on a sphere; we will illustrate the possibilities by solving for flow near a wall as well. The derivation I present here is suitable for a graduate course in fluid mechanics, continuum physics, or a related topic.

The problem at hand is to find the flow of an incompressible fluid at zero Reynolds number. The equations describing this flow are the continuity equation, which follows from conservation of mass, and the Stokes equation, the zero Reynolds simplification of the Navier–Stokes equations that follow from conservation of momentum. For an incompressible flow, our system of equations is then given by

$$0 = \boldsymbol{\nabla} \cdot \boldsymbol{v} \tag{1}$$

$$0 = -\boldsymbol{\nabla} p + \eta \nabla^2 \boldsymbol{v} + \boldsymbol{f}^{\text{ext}}, \tag{2}$$

where η is the fluid's viscosity, $\boldsymbol{v}$ the velocity field, and p the pressure field. The flow is driven by the application of an external force field $\boldsymbol{f}^{\text{ext}}$. Because of the linearity of the equations, we can decompose the external force field into a collection of point forces, and find the total flow from the superposition of the flows due to those point forces. We will therefore first solve for an external force that takes the simple shape of a point force

$$\boldsymbol{f}^{\text{ext}} = \boldsymbol{F}\delta(\boldsymbol{x}), \tag{3}$$

under the condition that the fluid fills all of space, and both the fluid flow and the pressure are zero at infinity. In the theory of differential equations, the function that gives the solution in terms of such a point force is known as the system's *Green's function*; for the specific case of Stokes' equations it is also known as the *Stokeslet*. Note that $\boldsymbol{F}$ is an ordinary force, not a force per unit volume, as the Dirac delta function in equation (3) has dimension of $1/\text{volume}$. As we will derive below, the fluid flow field and the pressure resulting from a point force at the origin are given by:

$$\boldsymbol{v}(\boldsymbol{r}) = \frac{1}{8\pi\eta x}\left(\boldsymbol{F} + \frac{\boldsymbol{F} \cdot \boldsymbol{x}}{x^2}\boldsymbol{x}\right), \tag{4}$$

$$p(\boldsymbol{x}) = \frac{\boldsymbol{F} \cdot \boldsymbol{x}}{4\pi x^3}. \tag{5}$$

Here $\boldsymbol{x}$ is the position vector, and $x = \sqrt{\boldsymbol{x} \cdot \boldsymbol{x}}$ its length. We can write equation (4) in component form as well, separating out the magnitude of the force:

$$v_i = \frac{1}{8\pi\eta x}\left(\delta_{ij} + \frac{x_i x_j}{x^2}\right)F_j = J_{ij}(\boldsymbol{x})F_j. \tag{6}$$

The rank-2 tensor J_{ij} in equation (6) is known as the *Oseen tensor*. As explained above, to find the flow fields for multiple forces we now simply add those of individual forces. To find

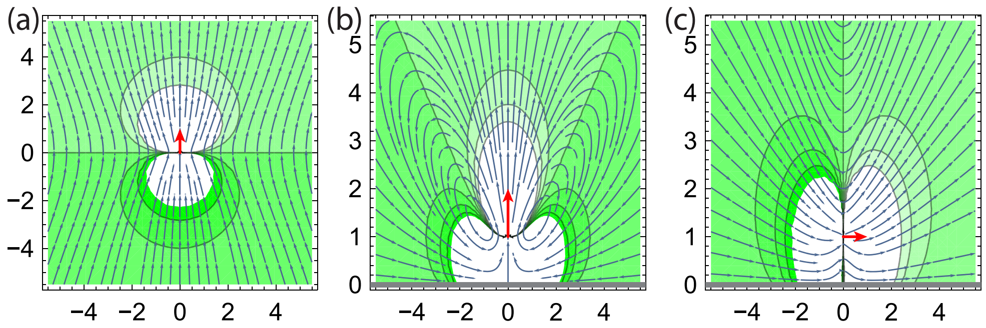

Figure 1. Flow and pressure fields due to a Stokeslet (point force). In all three figures, the red arrow indicates the point force, the blue arrows the resulting flow field, and the green contours the lines of constant pressure. (a) Free space. (b) Stokeslet above a wall, pointing away from the wall. (c) Stokeslet above a wall, pointing along the wall.

the flow and pressure fields of a continuous force distribution $\boldsymbol{f}(\boldsymbol{x})$, the sum becomes an integral, and we have:

$$\boldsymbol{v}(\boldsymbol{x}) = \int \boldsymbol{J}(\boldsymbol{x} - \boldsymbol{y}) \cdot \boldsymbol{f}(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y}, \tag{7}$$

$$p(\boldsymbol{x}) = \int \frac{\boldsymbol{f}(\boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y})}{4\pi|\boldsymbol{x} - \boldsymbol{y}|^3}\, \mathrm{d}\boldsymbol{y}, \tag{8}$$

where $\boldsymbol{J}$ is the matrix that has J_{ij} as its ijth component. Figure 1(a) shows the flow andpressure fields associated with a Stokeslet in free space.

2. The Oseen tensor

There are many ways to solve the Stokes equation, in combination with the incompressibility condition, including series expansions and Laplace and Fourier transformations. In this section we will use the latter transformation, which translates our partial differential equations into algebraic equations in Fourier space. These algebraic equations are easily solved, and give us the Oseen tensor (the Green's 'function' relating a point force to the resulting fluid flow) in Fourier space, as well as the pressure in Fourier space; the hardest part is actually the back-transform to real space. As there are a number of different conventions for the Fourier transformations, let us be explicit in stating which one we use (forward and back transforms):

$$\tilde{f}(\boldsymbol{q}) = \int \mathrm{d}\boldsymbol{x}\, f(\boldsymbol{x})\, \mathrm{e}^{\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{q}}, \tag{9}$$

$$f(\boldsymbol{x}) = \int \frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^n} \tilde{f}(\boldsymbol{q})\, \mathrm{e}^{-\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{q}}, \tag{10}$$

where n is the number of dimensions. Fourier transforms are so useful because they 'translate' derivatives to multiplications with wave vectors:

$$\int \mathrm{d}\boldsymbol{x}\, \frac{\partial f(\boldsymbol{x})}{\partial x_i}\, \mathrm{e}^{\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{q}} = -\int \mathrm{d}\boldsymbol{x}\, f(\boldsymbol{x})\, \frac{\partial \mathrm{e}^{\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{q}}}{\partial x_i} = -\mathrm{i}\, q_i \tilde{f}(\boldsymbol{q}). \tag{11}$$

We now apply the Fourier transform to our set of differential equations (1), (2), written out in coordinate form:

$$0 = \partial_i v_i, \tag{12}$$

$$0 = -\partial_i p + \eta \sum_k \partial_k \partial_k v_i + F_i\, \delta(\boldsymbol{x}), \tag{13}$$

which gives (note that the F_i are just numbers, so they remain the same under the Fourier transform; also, using the convention given in equation (9), the Fourier transform of the Dirac delta function is simply 1):

$$0 = -\mathrm{i} q_i \tilde{v}_i, \tag{14}$$

$$0 = \mathrm{i} q_i \tilde{p} - \eta \sum_k q_k q_k \tilde{v}_i + F_i. \tag{15}$$

Note that $\sum_k q_k q_k = \boldsymbol{q} \cdot \boldsymbol{q} = q^2$ with q the length of $\boldsymbol{q}$. From equations (14) and (15), we can easily solve for $\tilde{p}$, by contracting (i.e., taking the dot product) equation (15) with q_i, and applying equation (14):

$$0 = \mathrm{i} q^2 \tilde{p} + \sum_i q_i F_i,$$

$$\tilde{p} = \frac{i}{q^2} \sum_j q_j F_j. \tag{16}$$

Note that in equation (16) we replaced the summation (aka 'dummy') index i with j, to avoid confusion in the next step: we substitute equation (16) back into equation (15) and solve for $\tilde{v}_i$:

$$0 = -\sum_j \frac{q_i q_j}{q^2} F_j - \eta q^2 \tilde{v}_i + F_i$$

$$\tilde{v}_i = \frac{1}{\eta q^2} \sum_j \left(\delta_{ij} - \frac{q_i q_j}{q^2} \right) F_j = \sum_j \tilde{J}_{ij}(\boldsymbol{q}) F_i, \tag{17}$$

where we wrote $F_i = \sum_j \delta_{ij} F_j$ to be able to write the velocity as a tensor contracted with the vector $\boldsymbol{F}$, and introduced the (Fourier transform of) the Oseen tensor $\tilde{J}_{ij}(\boldsymbol{q})$. Equations (16) and (17) are the solution to our set of equations in Fourier space. To transform them back to real space, we note that $\tilde{p}$ is given by a dot product: that of $\boldsymbol{q}$ and $\boldsymbol{F}$, while $\boldsymbol{v}$ is the contraction of a symmetric two-tensor with the vector $\boldsymbol{F}$. We expect the same result in real space. For the pressure, the only candidate vector is the position vector, so we expect the pressure to take the form

$$p(\boldsymbol{x}) = a(x) \sum_i \frac{x_i}{x} F_i = \frac{a(x)}{x} \boldsymbol{x} \cdot \boldsymbol{F}, \tag{18}$$

where x is the length of the vector $\boldsymbol{x}$, and the function $a(x)$ can depend on x but not on its individual components. Requiring that $p(\boldsymbol{x})$ has the form equation (18), we find that

$$a(x) \frac{x_i}{x} = \int \frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^3} \frac{q_i}{q^2} \mathrm{e}^{-\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{q}}. \tag{19}$$

To find $a(x)$, we need to evaluate the integral on the right hand side of equation (19); moreover, we need to find the component of x_i/x. Fortunately, the latter is easy: we simply contract both sides of equation (18) with x_i/x. The left hand side becomes $\sum_i a(x)\, x_i\, x_i/x^2 = a(x)$, while on the right hand side we get a term of the form $\sum_i x_i\, q_i$, or the dot product of $\boldsymbol{x}$ and $\boldsymbol{q}$, the same as we have in the exponent that comes with the Fourier transform. To evaluate the integral, we choose coordinates of $\boldsymbol{q}$-space such that the $\boldsymbol{x}$ vector lies along the 3-axis. We then go to spherical coordinates, where the polar angle θ is the same as the angle between $\boldsymbol{x}$ and $\boldsymbol{q}$, so we can write $\boldsymbol{x} \cdot \boldsymbol{q} = \sum_i x_i q_i = xq\cos\theta$. We then find for $a(x)$:

$$a(x) = \mathrm{i}\int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} \int_0^{\pi} \frac{\sin\theta\, \mathrm{d}\theta}{2\pi} \int_0^{\infty} \frac{q^2\, \mathrm{d}q}{2\pi}\, q\cos\theta\, \mathrm{e}^{-\mathrm{i}qx\cos\theta}. \tag{20}$$

The integrand in equation (20) does not depend on the azimuthal angle ϕ, so that integral is trivial. The remaining integral is of a more general form, which we will encounter again, so we give it its own name:

$$g_n(\boldsymbol{x}, f) = \int_0^{\pi} \frac{\sin\theta\, \mathrm{d}\theta}{2\pi} \int_0^{\infty} \frac{\mathrm{d}q}{2\pi}\, q^n f(\cos\theta)\, \mathrm{e}^{-\mathrm{i}qx\cos\theta}, \tag{21}$$

where n is an integer and $f(z)$ is any analytical function; in equation (20) we have $n = 1$ and $f(z) = z$ (or $f(\cos\theta) = \cos\theta$). Note that we canceled the q^2 from the transition to spherical coordinates with the $1/q^2$ term from the integrand. To evaluate the integral in equation (21), we make two coordinate transformations: $z = \cos\theta$, and $y = qx$, where we note that x, as the length of the position vector, is always positive:

$$g_n(\boldsymbol{x}, f) = \frac{1}{2\pi|\mathrm{x}|^{n+1}} \int_{-1}^{1} \mathrm{d}z\, f(z) \int_0^{\infty} \frac{\mathrm{d}y}{2\pi}\, y^n \mathrm{e}^{-\mathrm{i}yz}. \tag{22}$$

Since $f(z)$ does not depend on y, we have pulled it outside the y integral. That latter integral now strongly resembles an inverse Fourier transform of a one-dimensional variable, only over half the interval. We can evaluate it by taking successive z derivatives of the inverse Fourier transform of 1:

$$\int_0^{\infty} \frac{\mathrm{d}y}{2\pi}\, y^n\, \mathrm{e}^{-\mathrm{i}yz} = \mathrm{i}^{\mathrm{n}} \frac{\partial^{\mathrm{n}}}{\partial z^{\mathrm{n}}} \int_0^{\infty} \frac{\mathrm{d}y}{2\pi}\, \mathrm{e}^{-\mathrm{i}yz} = \frac{\mathrm{i}^{\mathrm{n}}}{2} \frac{\partial^{\mathrm{n}}}{\partial z^{\mathrm{n}}} \delta(z), \tag{23}$$

where $\delta(z)$ is the one-dimensional Dirac delta function of z. Substituting back into equation (22) and repeated integration by parts gives us $g_n(\boldsymbol{x}, f)$ for arbitrary functions $f(z)$:

$$g_n(\boldsymbol{x}, f) = \frac{\mathrm{i}^{\mathrm{n}}}{4\pi|\mathrm{x}|^{n+1}} \int_{-1}^{1} \mathrm{d}z\, f(z) \frac{\partial^n}{\partial z^n} \delta(z) = \frac{(-\mathrm{i})^n}{4\pi|\mathrm{x}|^{n+1}} \left(\frac{\partial^n f(z)}{\partial z^n} \right)_{z=0}. \tag{24}$$

For the pressure, we now substitute $n = 1$ and $f(z) = z$, to get:

$$a(x) = \mathrm{i}g_1(\boldsymbol{x}, z) = \mathrm{i}\frac{(-i)}{4\pi x^2}(1) = \frac{1}{4\pi x^2}, \tag{25}$$

$$p(\boldsymbol{x}) = a(x) \sum_i \frac{x_i}{x} F_i = \sum_i \frac{x_i}{4\pi x^3} F_i = \frac{\boldsymbol{x} \cdot \boldsymbol{F}}{4\pi x^3}. \tag{26}$$

To find the velocity profile, we use a similar tactic. We know that the Oseen tensor $J_{ij}(\boldsymbol{x})$ must be symmetric in i and j (because so is its Fourier transform), which limits us to two possible contributions: δ_{ij} and $x_i x_j / x^2$. In general, J_{ij} will be a linear combination of these two 'basis tensors'. However, they are not orthogonal: $\delta_{ij} \frac{x_i x_j}{x^2} = 1$, not 0, which makes determining their coefficients less efficient (we could do it for the case at hand, but in general, it is much easier to work with orthonormal bases). Instead, we therefore use $\frac{x_i x_j}{x^2}$ and $\frac{1}{\sqrt{2}}\left(\delta_{ij} - \frac{x_i x_j}{x^2}\right)$, which are orthonormal (the factor $1/\sqrt{2}$ comes from the fact that, in three dimensions, $\sum_i \delta_{ii} = 3$). We write for the Oseen tensor in real space:

$$J_{ij}(\boldsymbol{x}) = b(x)\frac{x_i x_j}{x^2} + \frac{c(x)}{\sqrt{2}}\left(\delta_{ij} - \frac{x_i x_j}{x^2}\right), \tag{27}$$

where as with the pressure, $b(x)$ and $c(x)$ are functions of the magnitude of x only. To find them, we contract the inverse Fourier transform of $\tilde{J}_{ij}(\boldsymbol{q})$ with the two basis tensors. Fortunately, all integrals are of the type $g_n(\boldsymbol{x}, f)$ which we already found; now we need $n = 0$ and two functions for $f(z)$: $f(z) = 1$ and $f(z) = z^2$:

$$\begin{aligned}
b(x) &= \sum_{i,j} \frac{x_i x_j}{x^2} J_{ij} \\
&= \sum_{i,j} \frac{x_i x_j}{x^2} \int_0^{\pi} \frac{\sin\theta \, \mathrm{d}\theta}{2\pi} \int_0^{\infty} \frac{q^2 \, \mathrm{d}q}{2\pi} \frac{1}{\eta q^2}\left(\delta_{ij} - \frac{q_i q_j}{q^2}\right) \mathrm{e}^{-\mathrm{i}\sum_k q_k x_k} \\
&= \frac{1}{\eta} \int_0^{\pi} \frac{\sin\theta \, \mathrm{d}\theta}{2\pi} \int_0^{\infty} \frac{\mathrm{d}q}{2\pi} (1 - \cos^2\theta) \mathrm{e}^{-\mathrm{i}qx\cos\theta} \\
&= \frac{1}{\eta}[g_0(\boldsymbol{x}, 1) - g_0(\boldsymbol{x}, z^2)] \\
&= \frac{1}{\eta}\frac{1}{4\pi x},
\end{aligned} \tag{28}$$

and

$$\begin{aligned}
c(x) &= \frac{1}{\sqrt{2}} \sum_{i,j} \left(\delta_{ij} - \frac{x_i x_j}{x^2}\right) J_{ij} \\
&= \frac{1}{\eta}\frac{1}{\sqrt{2}}[(3 - 1) g_0(\boldsymbol{x}, 1) - (g_0(\boldsymbol{x}, 1) - g_0(\boldsymbol{x}, z^2))] \\
&= \frac{1}{\eta}\frac{1}{\sqrt{2}}\frac{1}{4\pi x}.
\end{aligned} \tag{29}$$

The Oseen tensor is thus given by

$$J_{ij}(\boldsymbol{x}) = \frac{1}{\eta}\frac{1}{4\pi x}\left[\frac{x_i x_j}{x^2} + \frac{1}{2}\left(\delta_{ij} - \frac{x_i x_j}{x^2}\right)\right] = \frac{1}{\eta}\frac{1}{8\pi x}\left(\delta_{ij} + \frac{x_i x_j}{x^2}\right), \tag{30}$$

and for the velocity profile we get

$$\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{J} \cdot \boldsymbol{F} = \frac{1}{8\pi\eta r}\left(\boldsymbol{F} + \frac{\boldsymbol{F} \cdot \boldsymbol{x}}{x^2}\boldsymbol{x}\right). \tag{31}$$

Equations (26) and (31) together form the Stokeslet, the flow field in an incompressible flow at zero Reynolds number.

3. Stokes drag on a sphere

With the Oseen tensor, we can in principle find the flow field of any configuration of forces. If, instead of a single point force, we have a combination of multiple point forces, the velocity and pressure fields simply becomes the sum of the velocity and pressure fields of the individual forces; all we have to do is shift the position of each point force to the origin. Most problems in fluid mechanics, however, are not about point forces, but about forces on continuous surfaces. For those cases, we need to integrate over that surface; the velocity is then given by

$$\boldsymbol{v} = \int_{\partial V} \mathrm{d}S\, \boldsymbol{J}(\boldsymbol{x}) \cdot \frac{\boldsymbol{F}(\boldsymbol{x})}{A}, \tag{32}$$

where the integral is taken over the entire surface (written here as the boundary ∂V of the volume of the object immersed in the fluid) and $\boldsymbol{F}/A$ is the force per unit area.

The Oseen tensor gives the flow due to an applied force. In practice, we will often be interested in the force exerted by a flow on a stationary object, and the shape of the flow around the object. Luckily, a simple transformation allows us to translate this second problem back into the first. To find the force on an object at the origin, and the flow around that object, due to a flow which is given by the stationary field $\boldsymbol{v}$ far away from the object, all we need to do is calculate the force necessary to drag the object with velocity $-\boldsymbol{v}$ through a fluid that is stationary at infinity.

For the simplest example, we consider a sphere of radius R, centered at the origin, which we drag at constant velocity $\boldsymbol{v}$ by applying a force $\boldsymbol{F} = F\hat{z}$ on it. When the sphere is moving at constant velocity, it exerts a net force F on the surrounding fluid; by Newton's third law, the surrounding fluid exerts an equal but opposite net drag force on the sphere. These sphere-fluid forces act at the surface of the sphere, resulting in an effective force per unit area of the sphere of magnitude $F/4\pi R^2$. For the velocity we then find

$$\boldsymbol{v} = \oint_{\partial V} \mathrm{d}S\, \boldsymbol{J}(\boldsymbol{x}) \cdot \frac{F\hat{z}}{4\pi R^2}. \tag{33}$$

The integral in equation (33) is taken over the surface of the sphere. Substituting our expression (30) for the Oseen tensor into the integral, we get

$$\begin{aligned} \boldsymbol{v} &= \frac{1}{8\pi\eta R}\frac{F}{4\pi R^2}\oint_{\partial V} \mathrm{d}S \left[\hat{z} + \frac{\boldsymbol{x}\cdot\hat{z}}{R^2}\boldsymbol{x}\right], \\ &= \frac{1}{8\pi\eta R}\frac{F}{4\pi}\int_0^{\pi} \sin(\theta)\,\mathrm{d}\theta \\ &\qquad \cdot \int_0^{2\pi} \mathrm{d}\phi\, [\cos(\phi)\sin(\theta)\,\hat{\boldsymbol{x}} + \sin(\phi)\sin(\theta)\,\hat{\boldsymbol{y}} + (1+\cos^2(\theta))\hat{z}] \\ &= \frac{1}{8\pi\eta R}\frac{F\hat{z}}{2}\int_0^{\pi} \sin(\theta)\,\mathrm{d}\theta\, [1+\cos^2(\theta)] \\ &= \frac{F}{6\pi\eta R}\hat{z}. \end{aligned} \tag{34}$$

where we used that $|\boldsymbol{x}| = R$ at the surface of the sphere in the second line, expressed $\boldsymbol{x}$ and dS in spherical coordinates in the fourth, and used the fact that the ϕ integrals over the $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ components are both a full-period integral over a sine/cosine, and thus evaluate to zero, in the fifth line. Because (by Newton's third law) the drag force on the sphere is minus the force of the sphere on the fluid, the Stokes law follows:

$$\boldsymbol{F}^{\mathrm{drag}} = 6\pi\eta R\boldsymbol{v}. \tag{35}$$

4. Flow near a wall

The presence of a wall with no-slip boundary conditions severely impacts the shape of the fluid flow, impacting processes like sedimentation [10] and bacterial swimming [11, 12]. Fortunately, we can solve for the flow of a point force (and hence, by superposition, for any flow) near a wall by borrowing a technique from electrostatics: we can construct a flow that satisfies our boundary conditions using mirror images. In contrast to electrostatics, however, because the Stokeslet has a magnitude and a direction, just adding the mirror Stokeslet is not sufficient. We will illustrate with two examples. For both of them, we put the wall in the xy

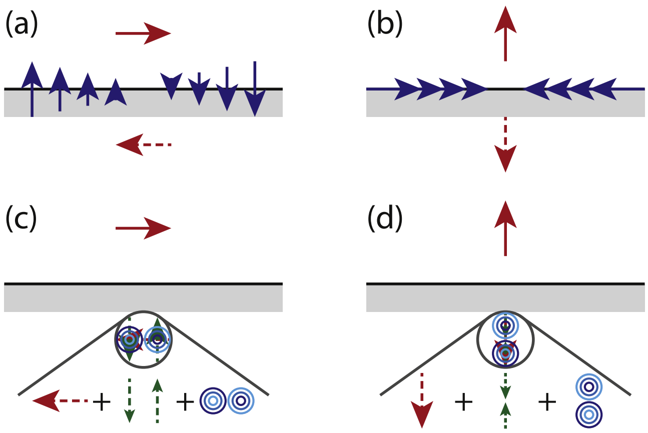

Figure 2. Mirror images to find the flow due to a point force near a wall. We consider the cases of a point force parallel ((a) and (c)) and perpendicular to ((b) and (d)) a wall. In both cases, simply adding the mirror image of the point force is not sufficient, as at the wall, the flow field then does not vanish ((a) and (b)). To compensate, we must add two more components to the 'mirror': a force dipole and a source dipole ((c) and (d)).

plane at $z = 0$, and put our point force at $(0, 0, h)$; we will consider the case where the point force is perpendicular to the wall (positive z-direction) and where it is parallel to the wall (positive x-direction). For both cases, simply adding a 'mirror image' Stokeslet at $z = -h$ in the opposite direction gives the following flow field:

$$\begin{aligned} \boldsymbol{v}(\boldsymbol{x}) &= \boldsymbol{J}(\boldsymbol{x} - h\hat{z}) \cdot \boldsymbol{F} - \boldsymbol{J}(\boldsymbol{x} + h\hat{z}) \cdot \boldsymbol{F}, \\ v_i(\boldsymbol{x}) &= \frac{1}{8\pi\eta}\left[\frac{1}{|\boldsymbol{x} - h\hat{z}|}\left(\delta_{ij} + \frac{x_i x_j - x_i h\delta_{j3} - x_j h\delta_{i3} + h^2\delta_{i3}\delta_{j3}}{|\boldsymbol{x} - h\hat{z}|^2}\right)\right. \\ &\qquad \left. - \frac{1}{|\boldsymbol{x} + h\hat{z}|}\left(\delta_{ij} + \frac{x_i x_j + x_i h\delta_{j3} + x_j h\delta_{i3} + h^2\delta_{i3}\delta_{j3}}{|\boldsymbol{x} + h\hat{z}|^2}\right)\right] F_j. \end{aligned} \tag{36}$$

Simply substituting $x_3 = 0$ gives a nonzero flow field at the wall. If the force is parallel to the wall (positive x direction), the flow field is perpendicular to it, and vice versa, see figures 2(a) and (b).

It is easiest to see where the problem originates in the case that the point force is parallel to the wall (figure 2(a)). The point force and its mirror image together form a spatially separated force doublet, setting up an effective rotational flow. To compensate, we need additional terms in the 'mirror', which therefore no longer is simply the mirror of the actual Stokeslet. We have two options for generating a rotational field similar to the one we have: from a force doublet, or from a source doublet (i.e., a source and a sink placed next to each other). As we will see, we will need both to cancel the flow at the wall.

To get an expression for the flow field due to a force doublet in the k direction, we simply take the derivative of the Oseen tensor in the k direction. A helpful intermediate result is the derivative of $x = \sqrt{\boldsymbol{x} \cdot \boldsymbol{x}} = \sqrt{x_1^2 + x_2^2 + x_3^2}$:

$$\partial_k x = \frac{\partial x}{\partial x_k} = \frac{\partial\sqrt{x_1^2 + x_2^2 + x_3^2}}{\partial x_k} = \frac{1}{2\sqrt{x_1^2 + x_2^2 + x_3^2}} \cdot 2x_k = \frac{x_k}{x}. \tag{37}$$

For the derivative of the Oseen tensor we then find

$$\begin{aligned} D_{ijk} \equiv \partial_k J_{ij} &= \frac{1}{8\pi\eta}\left[-\frac{x_k}{x^3}\left(\delta_{ij} + \frac{x_i x_j}{x^2}\right) + \frac{1}{x}\left(\frac{\delta_{ik} x_j}{x^2} + \frac{\delta_{jk} x_i}{x^2} - \frac{2x_i x_j}{x^3}\frac{x_k}{x}\right)\right] \\ &= \frac{1}{8\pi\eta x^2}\left[\delta_{ik}\frac{x_j}{x} + \delta_{jk}\frac{x_i}{x} - \delta_{ij}\frac{x_k}{x} - 3\frac{x_i x_j x_k}{x^3}\right]. \end{aligned} \tag{38}$$

As the 'doublet tensor' D_{ijk} is a third-order tensor, we must contract it with a second-order tensor to get a vector field. This second-order tensor represents the strength of the doublet. We will use the symbol F_{jk} for this 'force doublet strength'; the velocity field is then given by $v_i = D_{ijk}F_{jk}$.

To get an expression for the flow field of a source doublet, we first need a simple source. If the source produces a fluid flux of magnitude Q, then at a distance R from the source, there will be a radial flow of magnitude $Q/4\pi R^2$. Expressing the unit vector in the ith direction as x_i/x, we can thus write for the flow field due to a source term

$$v_i = \frac{Q}{4\pi}\frac{x_i}{x^3}. \tag{39}$$

To get the flow field of a source doublet in the k direction, we take the derivative of this expression, which gives

$$\frac{Q}{4\pi}\frac{\partial}{\partial x_k}\frac{x_i}{x^3} = \frac{Q}{4\pi}\left(\frac{\delta_{ik}}{x^3} - 3\frac{x_i x_k}{x^5}\right). \tag{40}$$

To get a velocity field, we need to contract the tensor field for the source doublet with a vector, describing the source doublet strength. We will use S_k for the components of this vector, with a re-scaling such that the prefactor is identical to that of the Stokeslet and force doublet, writing for the source doublet flow

$$v_i = \frac{1}{8\pi\eta}\left(\frac{\delta_{ik}}{x^3} - 3\frac{x_i x_k}{x^5}\right)S_k \equiv Q_{ik}S_k. \tag{41}$$

To write the flow field due to the original Stokeslet at $z = h$, its mirror image at $z = -h$, and an additional force doublet and source doublet at the position of the mirror image, it is convenient to introduce coordinates centered at the position of the Stokeslet and the mirror. Following Blake [13], we define r as the distance to the Stokeslet, and R as the distance to its mirror image. We also have $r_1 = R_1 = x_1$, $r_2 = R_2 = x_2$, $r_3 = x_3 - h$, $R_3 = x_3 + h$, $r = \sqrt{x_1^2 + x_2^2 + (x_3 - h)^2}$ and $R = \sqrt{x_1^2 + x_2^2 + (x_3 + h)^2}$. For the complete vector field, we then get the following expression:

$$\begin{aligned} v_i &= J_{ij}(\boldsymbol{r})F_j - J_{ij}(\boldsymbol{R})F_j + D_{ijk}(\boldsymbol{R})F_{jk} + Q_{ik}(\boldsymbol{R})S_k \\ &= \frac{1}{8\pi\eta}\left(\frac{\delta_{ij}}{r} + \frac{r_i r_j}{r^3} - \frac{\delta_{ij}}{R} - \frac{R_i R_j}{R^3}\right)F_j \\ &\quad + \frac{1}{8\pi\eta}\left(\delta_{ik}\frac{R_j}{R^3} + \delta_{jk}\frac{R_i}{R^3} - \delta_{ij}\frac{R_k}{R^3} - 3\frac{R_i R_j R_k}{R^5}\right)F_{jk} \\ &\quad + \frac{1}{8\pi\eta}\left(\frac{\delta_{ik}}{R^3} - 3\frac{R_i R_k}{R^5}\right)S_k. \end{aligned} \tag{42}$$

We can now find the required strength of the force and source doublet simply by setting $\boldsymbol{v}(x_3 = 0) = 0$. If the force of magnitude F is in the positive x direction, parallel to the wall, we find that we need a force doublet in the z-direction of magnitude $-2hF$ and a source doublet in the x-direction of magnitude $2h^2F$, i.e., $F_{jk} = -2hF\delta_{j3}\delta_{k1}$ and $S_k = 2h^2F\delta_{k1}$. Likewise, if the force is in the positive z direction, perpendicular to the wall, we need both a force doublet of magnitude $2hF$ and a source doublet of magnitude $-2h^2F$, both perpendicular to the wall, i.e., $F_{ij} = 2hF\delta_{j3}\delta_{k3}$ and $S_k = -2h^2F\delta_{k3}$.

For a full solution, we also need to know the pressure. We already found the pressure of a Stokeslet (equation (26)). The pressure of a force doublet in the k direction is then the k derivative of this pressure, which gives

$$\frac{\partial}{\partial x_k}\frac{x_j}{4\pi x^3} = \frac{1}{4\pi}\left(\frac{\delta_{jk}}{x^3} - 3\frac{x_j x_k}{x^5}\right). \tag{43}$$

To get the pressure for a Stokes doublet, we contract this tensor with the tensor F_{jk} describing the doublet strength. The pressure field corresponding to a source term is zero, as can be checked easily by calculating the Laplacian of equation (39); consequently, the pressure field of a source doublet is also zero. For the total pressure, we thus get

$$p = \frac{1}{4\pi}\left(\frac{r_j}{r^3} - \frac{R_j}{R^3}\right)F_j + \frac{1}{4\pi}\left(\frac{\delta_{jk}}{R^3} - 3\frac{R_j R_k}{R^5}\right)F_{jk}. \tag{44}$$

The flows and pressure fields due to a Stokeslet near a wall are shown in figures 1(b) and (c). This problem was first solved by Blake [13], who solved the Stokes equation with the boundary conditions on the wall through a Fourier transformation, then decomposed the solution into the same mirror components we found here.

5. Conclusion

Fluid flows can be difficult to calculate for specific boundary conditions. In the case that the inertial term is negligible however (i.e., when the Reynolds number vanishes, and the Navier–Stokes equations simplify to the Stokes equations), a general solution can readily be written down using the Oseen tensor. In this paper, we have used this principle to re-derive the equation for Stokes drag on a sphere. Likewise, we have solved for the flow near a wall, using an extended version of the mirror image approach from electrostatics. This approach builds on physical intuition (combination of forces and the superposition principle) rather than on advanced mathematical techniques, allowing for an easier understanding of the resulting flow and pressure fields.

Data availability statement

No new data were created or analyzed in this study.

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