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SMEARED CRACKING, PLASTICITY, CREEP, AND THERMAL LOADING—A UNIFIED APPROACH

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A recently proposed smeared crack model which properly handles nonorthogonal cracks is further elaborated. It is proved that the model obeys the principle of material frame-indifference and some comments on possible stress-strain laws within a smeared crack are made. Algorithms are presented or indicated for the combination of plasticity and possibly multiple crack formation, for the combination of visco-elasticity and cracking, and for the combination of cracking and temperature-dependent material properties and phenomena like thermal dilatation and shrinkage.

1. Introduction

Two approaches may be distinguished for analyzing crack propagation in concrete structures, namely the discrete and the smeared crack approach. In the former method, nodes are disconnected if the tensile force in that particular node exceeds a threshold value [1, 2]. This concept has been refined in recent years as discrete crack models have been developed in which the cracks are no longer forced to align with the original interelement boundaries [3, 4].

A major disadvantage that adheres to the discrete crack approach is the fact that the topology of the finite element mesh is changed continuously. In practical applications such concepts are rather unwieldy and most researchers now apply smeared crack models [5, 6]. In this concept the crack is assumed to be distributed over the entire area belonging to an integration point. Indeed, we look upon the smeared crack concept as a genuine continuum approach in the sense that there is a representative domain for which we can define notions like "stress", "strain", and so on. We recognize that objections may be raised against such a conception, owing to the heterogeneity of concrete and the discontinuous nature of dominant cracks. However, current experiences indicate that concrete including phenomena like crack propagation can be described sufficiently accurately within the framework of continuum mechanics [7–17].

The enhanced performance of smeared crack models is largely caused by the introduction of a shear retention factor to model dowel action and aggregate interlock [6] and the application of strain softening models for concrete in tension [9, 10, 17]. But even with these improvements, a number of problems still adhere to most current smeared crack models. Prominent amongst these are the combination of cracking and nonlinear behavior of the concrete

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between the cracks and the rotation of principal stress axes after primary crack formation. This rotation of principal stress axes is caused by the development of shear tractions on the faces of the crack and results in secondary cracking if the current major principal stress exceeds the tensile strength. Yet, most analysts largely ignore this phenomenon as they allow secondary cracks to form only orthogonal to primary cracks.

Although we consider a solution to these problems of crucial importance for a further successful development of the smeared crack concept, these issues are rarely discussed in literature. Amongst the exceptions, we mention a recent paper by Bažant and Chern [18] who discuss the combination of cracking and creep of concrete, and some papers by Cope et al. (e.g. [11]), who have drawn attention to the rotation of the principal stress axes after primary crack formation. However, Cope's solution, namely to co-rotate the material axes when the stress rotation has exceeded a threshold angle, is theoretically not correct as it violates the principle of material frame-indifference [19, 20].

In this paper we shall further elaborate a smeared crack model which removes both above-mentioned deficiencies. We will present an extension and a generalization of previous work, in which attention was primarily directed to the development of nonorthogonal cracks in a smeared finite element model [13, 14]. In addition to proving that the model possesses the property of material frame-indifference, we shall now concentrate on the combination of crack formation and the nonlinear behavior of the concrete between the cracks. More specifically, we will present accurate algorithms for the combination of (possibly multiple) cracks and plasticity and for the combination of cracking and visco-elasticity, while an algorithm for the combination of cracking and temperature-dependent material properties will be indicated.

2. Nonorthogonal multiple cracking

The basic assumption in our treatment is that the total strain rate $\dot{\varepsilon}_{kl}$ is composed of a concrete strain rate $\dot{\varepsilon}^{\mathrm{co}}_{kl}$ and *several* crack strain rates which we will denote by $\dot{\varepsilon}^{\mathrm{I}}_{kl}$, $\dot{\varepsilon}^{\mathrm{II}}_{kl}$, etc., so that

$$\dot{\varepsilon}_{kl} = \dot{\varepsilon}^{\mathrm{co}}_{kl} + \dot{\varepsilon}^{\mathrm{I}}_{kl} + \dot{\varepsilon}^{\mathrm{II}}_{kl} + \cdots . \tag{1}$$

For the present, we will restrict ourselves to two active cracks, so that we have

$$\dot{\varepsilon}_{kl} = \dot{\varepsilon}^{\mathrm{co}}_{kl} + \dot{\varepsilon}^{\mathrm{I}}_{kl} + \dot{\varepsilon}^{\mathrm{II}}_{kl} . \tag{2}$$

This restriction is not essential and we will generalize to an arbitrary number of cracks in a subsequent section, but it serves the purpose of simplifying the algebraic expressions. The concrete strain rate $\dot{\varepsilon}^{\mathrm{co}}_{kl}$ is assumed to be related to some objective stress rate $\dot{\sigma}_{ij}$ (e.g. the Jaumann derivative of the Cauchy stress tensor) via

$$\dot{\sigma}_{ij} = D^{\mathrm{co}}_{ijkl}(\dot{\varepsilon}^{\mathrm{co}}_{kl} - \dot{\varepsilon}^{0}_{kl}) + \Sigma_{ij} , \tag{3}$$

the summation convention with respect to latin subscripts being implied. The fourth-order tensor D^{co}_{ijkl} contains the instantaneous moduli of the concrete and Σ_{ij} is a second-order tensor function which, e.g., represents relaxation effects. The autogeneous strain rate $\dot{\varepsilon}^0_{kl}$ is the part of the concrete strain rate which is independent of the stress (e.g., thermal dilatation or shrinkage).

The constitutive law (3) is quite general. It embraces most rate-independent plasticity theories including, for instance, Besseling's fraction model [21], but also rate-dependent models as visco-plasticity and visco-elasticity with degenerated kernels [22]. Please note that the concrete strain rate itself may also be conceived as a summation of several components, for instance of an elastic and a plastic part.

The relation between the stress rates in the crack $\dot{\sigma}'_{ij}$ and the crack strain rates $\dot{\varepsilon}'_{kl}$ of the primary crack is assumed to be given by

$$\dot{\sigma}'_{ij} = D'_{ijkl}\dot{\varepsilon}'_{kl} , \tag{4}$$

where the primes signify that the stress rate respectively the crack strain rate components of the primary crack are taken with respect to the coordinate system of this crack. The fourth-order tensor D'_{ijkl} represents the stress-strain relation within the primary crack and is also formulated with respect to the coordinate system of the crack. Analogously, we have for a secondary crack,

$$\dot{\sigma}''_{ij} = D''_{ijkl}\dot{\varepsilon}''_{kl} . \tag{5}$$

The double primes mean that the stress rate components, the crack strain rate components, and the crack stress-strain relation of the secondary crack are taken with respect to the coordinate system of the secondary crack.

The strain rate tensor is an objective second-order tensor. If α_{ik} are the direction cosines of the global coordinate system with respect to the coordinate system of the primary crack, and if β_{ik} are the direction cosines of the global coordinate system x, y, z with respect to that of the secondary crack, we have the identities

$$\dot{\varepsilon}^{\mathrm{I}}_{ij} = \alpha_{ik}\alpha_{jl}\dot{\varepsilon}'_{kl} , \tag{6}$$

$$\dot{\varepsilon}^{\mathrm{II}}_{ij} = \beta_{ik}\beta_{jl}\dot{\varepsilon}''_{kl} . \tag{7}$$

Moreover, as we restrict our considerations to Cartesian tensors we also have

$$\dot{\sigma}'_{ij} = \alpha_{ki}\alpha_{lj}\dot{\sigma}_{kl} , \tag{8}$$

$$\dot{\sigma}''_{ij} = \beta_{ki}\beta_{lj}\dot{\sigma}_{kl} . \tag{9}$$

To derive the final stress-strain law of the cracked concrete, we proceed as follows. First substitute the fundamental decomposition (1) in the constitutive law for the concrete and transform the crack strain rates in global coordinates $\dot{\varepsilon}^{\mathrm{I}}_{ij}$ and $\dot{\varepsilon}^{\mathrm{II}}_{ij}$ to local coordinates according to (6) and (7). This results in

$$\dot{\sigma}_{ij} = D^{\mathrm{co}}_{ijkl}(\dot{\varepsilon}_{kl} - \dot{\varepsilon}^0_{kl} - \alpha_{ko}\alpha_{lp}\dot{\varepsilon}'_{op} - \beta_{ko}\beta_{lp}\dot{\varepsilon}''_{op}) + \Sigma_{ij} . \tag{10}$$

Transforming this expression for the stress rate to local coordinates according to the identities (8) and (9), equating the resulting expressions to the right-hand sides of the crack stress-strain relations (4) and (5) and rearranging gives, respectively,

$$A_{ijop}\dot{\varepsilon}'_{op} + B_{ijop}\dot{\varepsilon}''_{op} = \alpha_{ki}\alpha_{lj}[D^{co}_{klmn}(\dot{\varepsilon}_{mn} - \dot{\varepsilon}^{0}_{mn}) + \Sigma_{kl}]\,, \tag{11}$$

$$C_{ijop}\dot{\varepsilon}'_{op} + E_{ijop}\dot{\varepsilon}''_{op} = \beta_{ki}\beta_{lj}[D^{co}_{klmn}(\dot{\varepsilon}_{mn} - \dot{\varepsilon}^{0}_{mn}) + \Sigma_{kl}]\,, \tag{12}$$

wherein we have put

$$A_{ijop} = D'_{ijop} + \alpha_{ki}\alpha_{lj}\alpha_{mo}\alpha_{np}D^{co}_{klmn}\,, \tag{13}$$

$$B_{ijop} = \alpha_{ki}\alpha_{lj}\beta_{mo}\beta_{np}D^{co}_{klmn}\,, \tag{14}$$

$$C_{ijop} = \beta_{ki}\beta_{lj}\alpha_{mo}\alpha_{np}D^{co}_{klmn}\,, \tag{15}$$

$$E_{ijop} = D''_{ijop} + \beta_{ki}\beta_{lj}\beta_{mo}\beta_{np}D^{co}_{klmn}\,. \tag{16}$$

Solving for $\dot{\varepsilon}'_{mn}$ and $\dot{\varepsilon}''_{mn}$ and substituting these expressions in (10) finally gives

$$\dot{\sigma}_{ij} = \Lambda_{ijkl}[D^{co}_{klmn}(\dot{\varepsilon}_{mn} - \dot{\varepsilon}^{0}_{mn}) + \Sigma_{kl}]\,, \tag{17}$$

with

$$\begin{aligned}
\Lambda_{ijkl} = {} & \delta_{ik}\delta_{jl} - D^{co}_{ijab}\alpha_{ac}\alpha_{bd}[A_{cdef} - B_{cduv}E^{-1}_{uvwz}C_{wzef}]^{-1}\alpha_{ke}\alpha_{lf} \\
& + D^{co}_{ijab}\alpha_{ac}\alpha_{bd}[A_{cdef} - B_{cduv}E^{-1}_{uvwz}C_{wzef}]^{-1}B_{efgh}E^{-1}_{ghmn}\beta_{km}\beta_{ln} \\
& + D^{co}_{ijab}\beta_{ac}\beta_{bd}E^{-1}_{cdef}C_{efgh}[A_{ghmn} - B_{ghuv}E^{-1}_{uvwz}C_{wzmn}]^{-1}\alpha_{km}\alpha_{ln} \\
& - D^{co}_{ijab}\beta_{ac}\beta_{bd}[E^{-1}_{cdqr} + E^{-1}_{cdef}C_{efgh}[A_{ghmn} \\
& \qquad - B_{ghuv}E^{-1}_{uvwz}C_{wzmn}]^{-1}B_{mnop}E^{-1}_{opqr}]\beta_{kq}\beta_{lr}\,.
\end{aligned} \tag{18}$$

Note that the major symmetry of the stress-strain law (17) with respect to the interchange of ij and mn is preserved when the constitutive tensors D^{co}_{ijkl}, D'_{ijkl} and D''_{ijkl} are symmetric with respect to the interchange of ij and kl.

The constitutive law (17) of the cracked concrete obeys the principle of material frame indifference (objectivity) if the constitutive law for the intact concrete (3) obeys this principle. This follows immediately from the objectivity of stress and strain rates, from the objectivity of the stress-strain relation for the cracks as expressed through (4)–(9), and from the assumed objectivity of (3). Hence, all quantities and constitutive assumptions which we use in deriving (17) are objective, and so (17) is objective since this equation is simply the result of algebraic manipulations with objective quantities.

More formally, material frame-indifference can be proved by considering the stress response to a given strain rate in a rotated reference frame. In such a system we have

$$\dot{\bar{\sigma}}_{ij} = \bar{\Lambda}_{ijkl}[\bar{D}^{co}_{klmn}(\dot{\bar{\varepsilon}}_{mn} - \dot{\bar{\varepsilon}}^{0}_{mn}) + \bar{\Sigma}_{kl}]\,, \tag{19}$$

where $\dot{\bar{\sigma}}_{ij}$ and $\dot{\sigma}_{kl}$ are interrelated through

$$\dot{\bar{\sigma}}_{ij} = Q_{ik}Q_{jl}\dot{\sigma}_{kl} \,. \tag{20}$$

Q_{ik} contains the direction cosines from the $\bar{x}$, $\bar{y}$, $\bar{z}$ reference frame with respect to the x, y, z reference frame. Similarly, we have

$$\dot{\bar{\varepsilon}}_{ij} = Q_{ik}Q_{jl}\dot{\varepsilon}_{kl} \,. \tag{21}$$

Since the constitutive law for the concrete has been assumed to be objective, we have from (3) and (21),

$$\bar{D}^{\mathrm{co}}_{mnqr}(\dot{\bar{\varepsilon}}_{qr} - \dot{\bar{\varepsilon}}^{0}_{qr}) + \bar{\Sigma}_{mn} = Q_{mo}Q_{np}[D^{\mathrm{co}}_{opqr}(\dot{\varepsilon}_{qr} - \dot{\varepsilon}^{0}_{qr}) + \Sigma_{op}] \,. \tag{22}$$

With (20) and (22), equation (19) can be rewritten as

$$\dot{\sigma}_{ij} = Q_{ki}Q_{lj}Q_{mo}Q_{np}\bar{\Lambda}_{ijkl}[D^{\mathrm{co}}_{opqr}(\dot{\varepsilon}_{qr} - \dot{\varepsilon}^{0}_{qr}) + \Sigma_{op}] \,. \tag{23}$$

Introducing the tensors $\bar{\alpha}_{ij}$ and $\bar{\beta}_{ij}$, which contain the direction cosines of the global coordinate system $\bar{x}$, $\bar{y}$, $\bar{z}$ with respect to the local coordinate system of the primary respectively the secondary crack,

$$\bar{\alpha}_{ki} = Q_{km}\alpha_{mi} \,, \tag{24}$$

$$\bar{\beta}_{ki} = Q_{km}\beta_{mi} \,, \tag{25}$$

we can derive that $\bar{A}_{ijop} = A_{ijop}$, $\bar{B}_{ijop} = B_{ijop}$, $\bar{C}_{ijop} = C_{ijop}$, and $\bar{E}_{ijop} = E_{ijop}$ when $\bar{\mathrm{A}}_{ijop}$, $\bar{B}_{ijop}$, $\bar{C}_{ijop}$, and $\bar{E}_{ijop}$ are defined similar to A_{ijop}, B_{ijop}, C_{ijop}, and E_{ijop}, but for the replacement of α_{ij} by $\bar{\alpha}_{ij}$ and β_{ij} by $\bar{\beta}_{ij}$. Using the identities $\bar{A}_{ijop} = A_{ijop}$, $\bar{B}_{ijop} = B_{ijop}$, $\bar{C}_{ijop} = C_{ijop}$, and $\bar{E}_{ijop} = E_{ijop}$ we can subsequently show that

$$Q_{ki}Q_{lj}Q_{mo}Q_{np}\bar{\Lambda}_{klmn} = \Lambda_{ijop} \,, \tag{26}$$

which completes the formal proof.

It is obvious that a generalization to an arbitrary number of cracks also results in an objective stress-strain relationship for the cracked concrete since an extra term in (1) does not affect the objectivity of this equation, and since the definition of the stress-strain law in subsequent cracks is essentially similar to the definitions (4)–(9) for the first two cracks.

Furthermore, the structure of (18) is quite similar to the structure of an elastoplastic stiffness tensor at a yield vertex when the tensor Σ_{kl} is omitted. Indeed, any constitutive law in which a decomposition in the sense of (1) is assumed will lead to an equation with a similar structure. This holds true for a yield vertex in which two yield surfaces are active, but for instance also for the intersection of a yield surface and a fracture surface, an issue to which we will return in a subsequent section.

3. Crack stress-strain relation

A salient characteristic of crack formation concerns the fact that in the most general case of a three-dimensional solid only 3 out of 6 components of the crack strain rate vector are possibly nonzero, viz. the normal strain rate and two shear strain rates. We therefore assume that the stress-strain law for the crack has a structure such that the other strain rate components vanish. Moreover, we assume that the nonvanishing strain rate components are related to the components of the stress rate vector via

$$\begin{bmatrix} \dot{\sigma}'_{xx} \\ \dot{\sigma}'_{xy} \\ \dot{\sigma}'_{xz} \end{bmatrix} = \begin{bmatrix} D'_{11} & D'_{12} & D'_{13} \\ D'_{21} & D'_{22} & D'_{23} \\ D'_{31} & D'_{32} & D'_{33} \end{bmatrix} \begin{bmatrix} \dot{\varepsilon}'_{xx} \\ \dot{\varepsilon}'_{xy} \\ \dot{\varepsilon}'_{xz} \end{bmatrix}, \tag{27}$$

the meaning of the subscripts being explained in Fig. 1.

Equation (27) is a very general constitutive law for a crack as it allows for coupling effects in the sense that, for instance, the normal stress rate in the crack not only depends on the normal crack strain rate, but also on both shear crack strain rates. Similarly, any one of the shear stress rates may depend on all nonvanishing crack strain rate components. Such coupling effects occur, for instance, in crack-dilatancy theories [23]. Most applications are however restricted to small crack strains and then the off-diagonal terms are less important. Consequently, we have set the off-diagonal terms equal to zero, so that (27) reduces to:

$$\begin{bmatrix} \dot{\sigma}'_{xx} \\ \dot{\sigma}'_{xy} \\ \dot{\sigma}'_{xz} \end{bmatrix} = \begin{bmatrix} C & 0 & 0 \\ 0 & \beta^*\mu & 0 \\ 0 & 0 & \beta^*\mu \end{bmatrix} \begin{bmatrix} \dot{\varepsilon}'_{xx} \\ \dot{\varepsilon}'_{xy} \\ \dot{\varepsilon}'_{xz} \end{bmatrix}. \tag{28}$$

Herein, the tangent modulus C represents the relation between the normal crack strain rate and the normal stress rate (Fig. 2), μ is the elastic shear modulus, and β^* is a shear stiffness reduction factor.

In practice, the modulus C will be negative as we will normally have a descending relation between the stress rate normal to a crack and the normal crack strain rate. Here, the

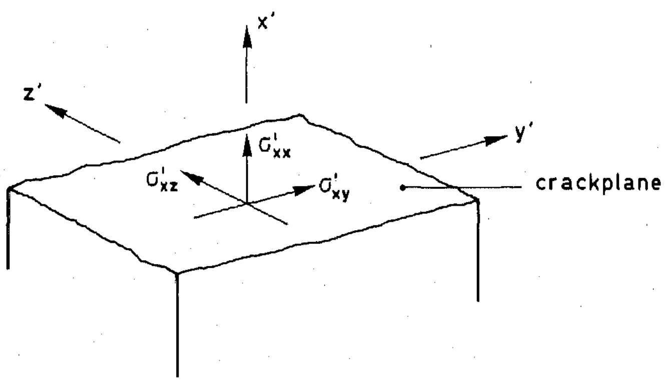

Fig. 1. Sign convention of stresses in the coordinate system of a primary crack.

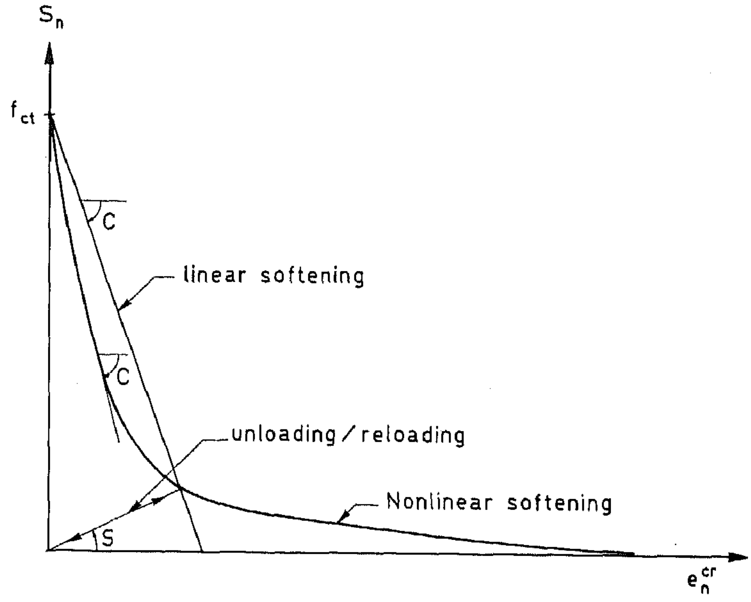

Fig. 2. Stress-strain relation normal to the crack.

evaluation of C from test data entails a complication, as recent research [9, 17] indicates that a straightforward translation from experimental data in a value for C leads to results which are not objective with regard to mesh refinement. To overcome this problem, it has been proposed to consider the fracture energy G_f, which is defined as the amount of energy needed to create one unit of area of a continuous crack [17, 24] as the fundamental parameter which governs crack propagation. This so-called "fictitious crack" or "tension-softening" model has also been adopted in this study, although it is beginning to emerge that the concept is not entirely free from deficiencies. This is particularly so when we allow for the possibility of multiple cracks. Suppose that a primary crack has been created with a softening modulus C determined from the fracture energy G_f. If upon formation of a secondary crack the same crack stress-strain relation is adopted for the second crack, the fracture energy will be consumed twice. If both cracks are orthogonal to each other, this is not unrealistic, but for any other inclination angle it seems incorrect. A solution to this problem seems only possible within a comprehensive stress-strain relation for a crack which incorporates at least an objective (with regard to mesh refinement) relation for shear softening and possibly also some theory for normal-shear coupling. Hence, the concept of a fracture energy as outlined above does not seem to suffice for multiple crack formation. Indeed, a solution in which the fracture energy is distributed over both cracks is not correct as the fracture energy G_f is not a scalar, but a vector, although this does not seem to have been recognized widely. In this respect, use of the term fracture energy for G_f is perhaps somewhat misleading.

Other problems with the application of fracture energy concepts in smeared crack analysis relate to axisymmetric configurations where the integration of the strain over the crack bandwidth entails complications owing to the $1/r$ term, and to the shape of the softening branch which may influence the results significantly [10].

The term $\beta^*\mu$ gives the relation between the shear stress rate in a crack and the shear crack strain rate and accounts for effects like aggregate interlock. The meaning of the reduction factor β^* differs from the classical shear retention factor β as introduced by Suidan and Schnobrich [6]. A relation between β and β^* can be derived from the consideration that the total shear strain rate is resolved in a crack strain rate and a concrete strain rate. Assuming that the concrete behaves in a linearly elastic manner, we may then derive that $\beta^* = \beta/(1-\beta)$ if we have only one crack and $\beta^* = 2\beta/(1-\beta)$ if we have two orthogonal cracks. For multiple nonorthogonal cracks or for inelastic concrete behavior, more complicated relations ensue. It should be emphasized that in the present approach the relation between the shear stress and the shear crack strain is considered as being fundamental. Hence, β^* is conceived as a material parameter and the relation with the more familiar β has only been derived to give the reader an idea of the range of values which can be used for β^*. In the sample problems which we will present, β^* has been assumed to be a constant both for loading and unloading, but a more realistic approach would be to make β^* a function at least of the crack strain [25]. Unfortunately, few experimental data exist to support a particular expression for β^*.

Especially when we allow the formation of nonorthogonal multiple cracks, we face the problem of crack arrest, unloading, and even closing of existing cracks. This nearly always occurs when a secondary crack arises in an integration point. It is thus very important that a crack closing option is included in a crack model. At present, we have adopted a secant approach for the unloading branch, so that for unloading the tangent softening modulus C is replaced by a secant modulus S for the normal stiffness D'_{11} (Fig. 2).

In the past, a number of other smeared crack models have been developed, some of which allow only one crack in an integration point [5, 6, 17], while other models permit the formation of two cracks in an integration point [26–28]. It can be shown that some of these models are obtained as a special case of the model considered here [14].

4. Matrix-vector formulation

For computational purposes, matrix-vector notation is usually preferable over tensor notation which was employed in the preceding sections. Further advantages of matrix-vector notation are that we can write (18) in a more compact form:

$$\begin{aligned} \boldsymbol{A} = \boldsymbol{I} - \boldsymbol{D}^{\mathrm{co}}\boldsymbol{N}_{\mathrm{I}}\boldsymbol{M}\boldsymbol{N}_{\mathrm{I}}^{\mathrm{t}} + \boldsymbol{D}^{\mathrm{co}}\boldsymbol{N}_{\mathrm{I}}\boldsymbol{M}\boldsymbol{B}\boldsymbol{E}^{-1}\boldsymbol{N}_{\mathrm{II}}^{\mathrm{t}} + \boldsymbol{D}^{\mathrm{co}}\boldsymbol{N}_{\mathrm{II}}\boldsymbol{E}^{-1}\boldsymbol{C}\boldsymbol{M}\boldsymbol{N}_{\mathrm{I}}^{\mathrm{t}} \\ -\boldsymbol{D}^{\mathrm{co}}\boldsymbol{N}_{\mathrm{II}}[\boldsymbol{E}^{-1} + \boldsymbol{E}^{-1}\boldsymbol{C}\boldsymbol{M}\boldsymbol{B}\boldsymbol{E}^{-1}]\boldsymbol{N}_{\mathrm{II}}^{\mathrm{t}} \,, \end{aligned} \tag{29}$$

the symbol t denoting a transpose, and that we can generalize to an infinite number of cracks. For convenience we will first define the matrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{E}$ and $\boldsymbol{M}$:

$$\boldsymbol{A} = \boldsymbol{D}' + \boldsymbol{N}_{\mathrm{I}}^{\mathrm{t}}\boldsymbol{D}^{\mathrm{co}}\boldsymbol{N}_{\mathrm{I}} \,, \tag{30}$$

$$\boldsymbol{B} = \boldsymbol{N}_{\mathrm{I}}^{\mathrm{t}}\boldsymbol{D}^{\mathrm{co}}\boldsymbol{N}_{\mathrm{II}} \,, \tag{31}$$

$$\boldsymbol{C} = \boldsymbol{N}_{\mathrm{II}}^{\mathrm{t}}\boldsymbol{D}^{\mathrm{co}}\boldsymbol{N}_{\mathrm{I}} \,, \tag{32}$$

$$E = D'' + N_{II}^{t} D^{co} N_{II} , \tag{33}$$

$$M = (A - BE^{-1}C)^{-1} , \tag{34}$$

and the transformation matrices N_I, N_{II} for the stress or strain vectors from the coordinate system of the first respectively the second crack to the global coordinate system:

$$N_I = \begin{bmatrix} l_x^2 & l_x m_x & l_x n_x \\ l_y^2 & l_y m_y & l_y n_y \\ l_z^2 & l_z m_z & l_z n_z \\ 2l_x l_y & l_x m_y + l_y m_x & l_x n_y + l_y n_x \\ 2l_y l_z & l_y m_z + l_z m_y & l_y n_z + l_z n_y \\ 2l_z l_x & l_z m_x + l_x m_z & l_z n_x + l_x n_z \end{bmatrix} , \tag{35}$$

the matrices N_{II}, N_{III}, etc. being defined similarly, l_x, l_y, and l_z are the cosines of the angle between the x-axis and the x'-axis, respectively the y'-axis and the z'-axis, and the other direction cosines are defined in accordance with this convention. As the stress rate vector and the strain rate vectors normally have six independent components, we would expect N_I etc. to be 6×6 matrices and not 6×3 matrices. However, it is recalled from the preceding section that the only nonvanishing crack strain components are the strain component normal to the crack and two shear crack strain components. If we also assume that the nonvanishing components of the crack strain rate are only related to the corresponding components (that is the normal and the two shear stress rates) of the stress rate vector in the coordinate system of the crack (see (27)), we may delete the appropriate columns from the transformation matrices N_I, N_{II}, etc., so that we end up with the 6×3 matrix (35).

As a first step to express Λ in a more compact form, we note that (29) is equivalent to

$$\Lambda = I - D^{co}[N_I \;\; N_{II}] \begin{bmatrix} A & B \\ C & E \end{bmatrix}^{-1} \begin{bmatrix} N_I^t \\ N_{II}^t \end{bmatrix} . \tag{36}$$

Next, define

$$D^{cr} = \begin{bmatrix} D' & 0 \\ 0 & D'' \end{bmatrix} , \tag{37}$$

so that the stress rate vector $\dot{s}$,

$$\dot{s} = \begin{bmatrix} \dot{\sigma}' \\ \dot{\sigma}'' \end{bmatrix} , \tag{38}$$

which assembles the stress rate components in the local coordinate systems of the cracks and the crack strain rate vector $\dot{e}^{cr}$,

$$\dot{e}^{cr} = \begin{bmatrix} \dot{\varepsilon}' \\ \dot{\varepsilon}'' \end{bmatrix} , \tag{39}$$

which contains the crack strain rate components in the local coordinate systems, are related by

$$\dot{s} = D^{cr}\dot{e}^{cr} . \tag{40}$$

With the additional definition

$$N = [N_I \; N_{II}] , \tag{41}$$

we can derive that

$$\Lambda = I - D^{co}N(D^{cr} + N^t D^{co} N)^{-1} N^t . \tag{42}$$

It is important to distinguish $\dot{e}^{cr}$, which assembles all individual crack strain rates with respect to their own coordinate system, from $\dot{\varepsilon}^{cr}$, which is the sum of all crack strain rates defined in the global xyz coordinate system. With the definition of the composite transformation matrix N, we observe that $\dot{\varepsilon}^{cr}$ and $\dot{e}^{cr}$ are related through (see (6) and (7)):

$$\dot{\varepsilon}^{cr} = N\dot{e}^{cr} . \tag{43}$$

Similarly, the vector $\dot{s}$, which assembles the stress rates in the individual cracks with respect to their own coordinate system, is related to the global stress rate $\dot{\sigma}$ by

$$\dot{s} = N^t\dot{\sigma} . \tag{44}$$

The generalization to more than two cracks in the same integration point is now straightforward, as we only have to expand the vectors $\dot{s}$:

$$\dot{s} = \begin{bmatrix} \dot{\sigma}' \\ \dot{\sigma}'' \\ \dot{\sigma}''' \\ \cdot \end{bmatrix} ; \tag{45}$$

and $\dot{e}^{cr}$:

$$\dot{e}^{cr} = \begin{bmatrix} \dot{\varepsilon}' \\ \dot{\varepsilon}'' \\ \dot{\varepsilon}''' \\ \cdot \end{bmatrix} ; \tag{46}$$

and the matrices D^{cr} and N to:

$$D^{cr} = \begin{bmatrix} D' & 0 & 0 & \cdot \\ 0 & D'' & 0 & \cdot \\ 0 & 0 & D''' & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} , \tag{47}$$

and

$$N = [N_I \quad N_{II} \quad N_{III} \quad \cdot] , \tag{48}$$

while (42) is unaffected.

5. Cracking and plasticity

Neglecting stress-independent strain rates, a plasticity type constitutive law for the concrete between the cracks is obtained by reducing (3) to

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{D}^{\text{ep}}\dot{\boldsymbol{\varepsilon}}^{\text{co}} . \tag{49}$$

In (49) $\boldsymbol{D}^{\text{ep}}$ is the elastoplastic stress-strain matrix,

$$\boldsymbol{D}^{\text{ep}} = \boldsymbol{D}^{\text{e}} - \frac{\boldsymbol{D}^{\text{e}} \dfrac{\partial g}{\partial \boldsymbol{\sigma}} \dfrac{\partial f^{\text{t}}}{\partial \boldsymbol{\sigma}} \boldsymbol{D}^{\text{e}}}{h + \dfrac{\partial f^{\text{t}}}{\partial \boldsymbol{\sigma}} \boldsymbol{D}^{\text{e}} \dfrac{\partial g}{\partial \boldsymbol{\sigma}}} , \tag{50}$$

with $\boldsymbol{D}^{\text{e}}$ the elastic stress-strain matrix. h represents the rate of hardening of the material,

$$h = -\frac{\partial f}{\partial \kappa} \frac{\partial \kappa^{\text{t}}}{\partial \boldsymbol{\varepsilon}^{\text{p}}} \frac{\partial g}{\partial \boldsymbol{\sigma}} , \tag{51}$$

with $f = f(\boldsymbol{\sigma}, \kappa)$ and $g = g(\boldsymbol{\sigma}, \kappa)$ respectively the yield function and the plastic potential. κ is hardening parameter which is a functional of the plastic strain and the superscripts e and p denote elastic and plastic quantities, respectively. In such an inviscid plasticity formulation $\boldsymbol{\Sigma}$ vanishes and we obtain for the stress-strain law of the cracked, inelastic concrete

$$\dot{\boldsymbol{\sigma}} = [\boldsymbol{D}^{\text{ep}} - \boldsymbol{D}^{\text{ep}}\boldsymbol{N}[\boldsymbol{D}^{\text{cr}} + \boldsymbol{N}^{\text{t}}\boldsymbol{D}^{\text{ep}}\boldsymbol{N}]^{-1}\boldsymbol{N}^{\text{t}}\boldsymbol{D}^{\text{ep}}]\dot{\boldsymbol{\varepsilon}} . \tag{52}$$

Unfortunately, this rate law is not very suitable for numerical integration, because $\boldsymbol{D}^{\text{ep}}$, being a function of the stress tensor, varies during a loading step. To derive a rate law which is more suitable for integration to finite increments, we recall that the situation in which cracking and plasticity occur simultaneously can also be interpreted as a vertex in which a yield surface and a fracture surface intersect, or if we have multiple cracking, as a vertex in which a yield surface and several fracture surfaces intersect (Fig. 3). Consequently, we have the following

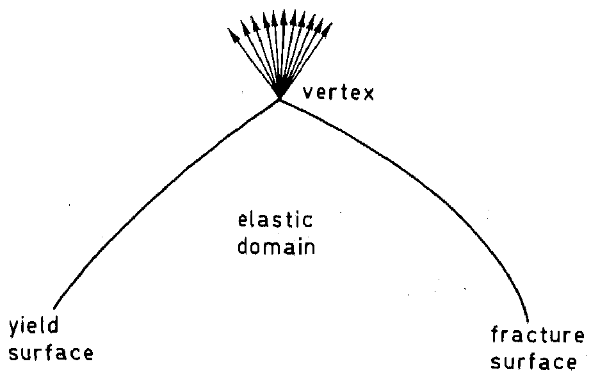

Fig. 3. Fan of possible inelastic strain increments at the intersection of a yield and a fracture surface.

decomposition of the total strain rate:

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^{\mathrm{e}} + \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} + \dot{\boldsymbol{\varepsilon}}^{\mathrm{cr}} , \tag{53}$$

when we assemble all crack strain rates in the vector $\dot{\boldsymbol{\varepsilon}}^{\mathrm{cr}}$.

In plasticity theory, the elastic strain rates are assumed to be related to the stress rates by

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{D}^{\mathrm{e}}\dot{\boldsymbol{\varepsilon}}^{\mathrm{e}} , \tag{54}$$

while the plastic strain rates are derivable from the plastic potential g,

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \lambda\, \partial g/\partial \boldsymbol{\sigma} , \tag{55}$$

with λ a nonnegative multiplier which can be determined from the condition that during loading, the stress must satisfy the consistency condition $\dot{f} = 0$. Using the definitions of f and h, $\dot{f} = 0$ can be elaborated as:

$$\frac{\partial f^{\mathrm{t}}}{\partial \boldsymbol{\sigma}}\, \dot{\boldsymbol{\sigma}} + h\lambda = 0 . \tag{56}$$

We now proceed in a manner which is essentially similar to the derivation in the preceding and we first substitute the decomposition (53) in (54). With the relations (43) and (55) for $\dot{\boldsymbol{\varepsilon}}^{\mathrm{cr}}$ and $\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}$ we obtain:

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{D}^{\mathrm{e}}\left[\dot{\boldsymbol{\varepsilon}} - \boldsymbol{N}\dot{\boldsymbol{e}}^{\mathrm{cr}} - \lambda \frac{\partial g}{\partial \boldsymbol{\sigma}}\right] . \tag{57}$$

Premultiplying this equation with $(\partial f/\partial \boldsymbol{\sigma})^{\mathrm{t}}$ and invoking (56) gives:

$$\left[\frac{\partial f^{\mathrm{t}}}{\partial \boldsymbol{\sigma}} \boldsymbol{D}^{\mathrm{e}} \frac{\partial g}{\partial \boldsymbol{\sigma}} + h\right]\lambda + \frac{\partial f^{\mathrm{t}}}{\partial \boldsymbol{\sigma}} \boldsymbol{D}^{\mathrm{e}}\boldsymbol{N}\dot{\boldsymbol{e}}^{\mathrm{cr}} = \frac{\partial f^{\mathrm{t}}}{\partial \boldsymbol{\sigma}} \boldsymbol{D}^{\mathrm{e}}\dot{\boldsymbol{\varepsilon}} . \tag{58}$$

Similarly, premultiplying (57) with $\boldsymbol{N}^{\mathrm{t}}$ and invoking (44) and (40), we get:

$$\boldsymbol{N}^{\mathrm{t}}\boldsymbol{D}^{\mathrm{e}} \frac{\partial g}{\partial \boldsymbol{\sigma}} \lambda + [\boldsymbol{D}^{\mathrm{cr}} + \boldsymbol{N}^{\mathrm{t}}\boldsymbol{D}^{\mathrm{e}}\boldsymbol{N}]\dot{\boldsymbol{e}}^{\mathrm{cr}} = \boldsymbol{N}^{\mathrm{t}}\boldsymbol{D}^{\mathrm{e}}\dot{\boldsymbol{\varepsilon}} . \tag{59}$$

Solving for λ and $\dot{\boldsymbol{e}}^{\mathrm{cr}}$ and substituting these expressions in (57) finally yields:

$$\dot{\boldsymbol{\sigma}} = \left\{\boldsymbol{D}^{\mathrm{ef}} - \frac{\boldsymbol{D}^{\mathrm{ef}} \dfrac{\partial g}{\partial \boldsymbol{\sigma}} \dfrac{\partial f^{\mathrm{t}}}{\partial \boldsymbol{\sigma}} \boldsymbol{D}^{\mathrm{ef}}}{h + \dfrac{\partial f^{\mathrm{t}}}{\partial \boldsymbol{\sigma}} \boldsymbol{D}^{\mathrm{ef}} \dfrac{\partial g}{\partial \boldsymbol{\sigma}}}\right\}\dot{\boldsymbol{\varepsilon}} , \tag{60}$$

with $\boldsymbol{D}^{\mathrm{ef}}$ the elastic-fracture matrix,

$$\boldsymbol{D}^{\mathrm{ef}} = \boldsymbol{D}^{\mathrm{e}} - \boldsymbol{D}^{\mathrm{e}}\boldsymbol{N}[\boldsymbol{D}^{\mathrm{cr}} + \boldsymbol{N}^{\mathrm{t}}\boldsymbol{D}^{\mathrm{e}}\boldsymbol{N}]^{-1}\boldsymbol{N}^{\mathrm{t}}\boldsymbol{D}^{\mathrm{e}} . \tag{61}$$

In principle, (60) and (61) are exactly identical with (50) and (52), but (60) is more feasible for integration to finite increments than (52). For a finite increment we have

$$\Delta\boldsymbol{\sigma} = \int_{t-\Delta t}^{t} \left\{ \boldsymbol{D}^{\mathrm{ef}} - \frac{\boldsymbol{D}^{\mathrm{ef}} \dfrac{\partial g}{\partial \boldsymbol{\sigma}} \dfrac{\partial f^{\mathrm{t}}}{\partial \boldsymbol{\sigma}} \boldsymbol{D}^{\mathrm{ef}}}{h + \dfrac{\partial f^{\mathrm{t}}}{\partial \boldsymbol{\sigma}} \boldsymbol{D}^{\mathrm{ef}} \dfrac{\partial g}{\partial \boldsymbol{\sigma}}} \right\} \dot{\boldsymbol{\varepsilon}} \, \mathrm{d}\tau \, . \tag{62}$$

During the calculation of the trial stress increment $\Delta\boldsymbol{\sigma}^*$,

$$\Delta\boldsymbol{\sigma}^* = \boldsymbol{D}^{\mathrm{ef}} \Delta\boldsymbol{\varepsilon} \, , \tag{63}$$

no plasticity is assumed to occur, but only the possibility of cracking is considered. This implies that during this predictor phase, we have the identities

$$\boldsymbol{D}^{\mathrm{ef}} \dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\sigma}}^* \, , \tag{64}$$

$$\frac{\partial f^{\mathrm{t}}}{\partial \boldsymbol{\sigma}} \dot{\boldsymbol{\sigma}}^* = \dot{f} \, , \tag{65}$$

so that we can rewrite (62) as

$$\Delta\boldsymbol{\sigma} = \int_{t-\Delta t}^{t} \left\{ \dot{\boldsymbol{\sigma}}^* - \frac{\dot{f} \boldsymbol{D}^{\mathrm{ef}} \dfrac{\partial g}{\partial \boldsymbol{\sigma}}}{h + \dfrac{\partial f^{\mathrm{t}}}{\partial \boldsymbol{\sigma}} \boldsymbol{D}^{\mathrm{ef}} \dfrac{\partial g}{\partial \boldsymbol{\sigma}}} \right\} \mathrm{d}\tau \, . \tag{66}$$

Introducing the notation

$$\boldsymbol{\sigma}^* = \boldsymbol{\sigma}^0 + \Delta\boldsymbol{\sigma}^* \, , \tag{67}$$

with $\boldsymbol{\sigma}^0$ either the contact stress at the intersection of the stress path and the yield surface or the stress at the beginning of the loading step, we get with a single-point numerical integration rule:

$$\Delta\boldsymbol{\sigma} = \Delta\boldsymbol{\sigma}^* - \frac{f(\boldsymbol{\sigma}^*, \kappa)}{h + \dfrac{\partial f^{\mathrm{t}}}{\partial \boldsymbol{\sigma}} \boldsymbol{D}^{\mathrm{ef}} \dfrac{\partial g}{\partial \boldsymbol{\sigma}}} \boldsymbol{D}^{\mathrm{ef}} \frac{\partial g}{\partial \boldsymbol{\sigma}} \, , \tag{68}$$

as by definition we have $f(\boldsymbol{\sigma}^0, \kappa) = 0$. Numerically, this condition need not be satisfied as the stresses resulting from the previous step may violate the yield criterion slightly. By putting $f(\boldsymbol{\sigma}^0, \kappa) = 0$ we strive to satisfy the yield criterion at any stage of the loading process, rather than to satisfy the consistency condition $\dot{f} = 0$, so that inaccuracies from previous loading steps are not carried along.

The approach becomes very simple when the gradients to the yield function f and the plastic potential g are evaluated for $\boldsymbol{\sigma} = \boldsymbol{\sigma}^*$. In this approach, there is no need to determine the intersection point of the stress path with the yield function if the response is partly elastic and partly plastic within the loading step, which may simplify the computer code significantly. In fact, this algorithm constitutes a generalization of the well-known radial return scheme used in metal plasticity [29, 30], for as we leave out cracking ($\boldsymbol{D}^{\mathrm{ef}} = \boldsymbol{D}^{\mathrm{e}}$) and adopt a von Mises yield criterion with an associated flow rule ($f = g$), the present procedure reduces to an elastic predictor-radial return scheme [15].

The algorithm for handling plasticity and fracture is not only relatively simple, but it is also quite accurate. Indeed, if we have linear hardening or softening for the yield function and for the fracture function, if we have a constant shear reduction factor in the crack β^*, and if we have no physical changes during the loading step (e.g., crack closing), we can prove that the algorithm guarantees a rigorous return to the fracture surface as well as to the yield surface for linear yield and fracture surfaces. Assume for this matter that some trial stress $\boldsymbol{\sigma}^*$ has been computed according to (63) and (67). If $\boldsymbol{\sigma}^*$ appears to lie outside the yield surface, a correction must be applied so that the final stress will be on the yield surface. The plastic part of the strain increment follows from

$$\Delta\boldsymbol{\varepsilon}^{\mathrm{p}} = \Delta\boldsymbol{\varepsilon} - \Delta\boldsymbol{\varepsilon}^{\mathrm{e}} - \Delta\boldsymbol{\varepsilon}^{\mathrm{cr}} , \tag{69}$$

but we must also require the incremental form of (55) to hold. By virtue of (55), (63), (67), and (69) and the identity

$$\Delta\boldsymbol{\varepsilon}^{\mathrm{e}} + \Delta\boldsymbol{\varepsilon}^{\mathrm{cr}} = [\boldsymbol{D}^{\mathrm{ef}}]^{-1}[\boldsymbol{\sigma}^1 - \boldsymbol{\sigma}^0] , \tag{70}$$

we obtain for the final stress state $\boldsymbol{\sigma}^1$:

$$\boldsymbol{\sigma}^1 = \boldsymbol{\sigma}^* - \lambda \boldsymbol{D}^{\mathrm{ef}} \frac{\partial g}{\partial \boldsymbol{\sigma}} . \tag{71}$$

The multiplier λ is determined implicitly by the condition that the final stress be on the yield surface:

$$f(\boldsymbol{\sigma}^1, \kappa^1) = 0 , \tag{72}$$

and must generally be determined by an iterative procedure. Alternatively, λ may be determined by expanding $f(\boldsymbol{\sigma}, \kappa)$ in a Taylor series around $\boldsymbol{\sigma} = \boldsymbol{\sigma}^*$, $\kappa = \kappa^0$. Omitting second- and higher-order terms, this yields:

$$f(\boldsymbol{\sigma}^1, \kappa^0) - \lambda\left[h + \frac{\partial f^{\mathrm{t}}}{\partial \boldsymbol{\sigma}} \boldsymbol{D}^{\mathrm{ef}} \frac{\partial g}{\partial \boldsymbol{\sigma}}\right] = 0 , \tag{73}$$

so that the following stress-strain relation is obtained:

$$\boldsymbol{\sigma}^1 = \boldsymbol{\sigma}^* - \frac{f(\boldsymbol{\sigma}^*, \kappa^0)}{h + \dfrac{\partial f^{\mathrm{t}}}{\partial \boldsymbol{\sigma}} \boldsymbol{D}^{\mathrm{ef}} \dfrac{\partial g}{\partial \boldsymbol{\sigma}}} \boldsymbol{D}^{\mathrm{ef}} \frac{\partial g}{\partial \boldsymbol{\sigma}} . \tag{74}$$

Comparing (68) and (74), we observe that this approach results in the same integration scheme as derived in (62)–(68). Hence, the present procedure offers a rigorous return to the yield surface for all yield functions which are linear in the principal stress space (such as the Mohr–Coulomb and Tresca criteria) if the aforementioned conditions are fulfilled. No drifting error is committed as may be the case with some other integration schemes. Especially when the stresses rotate strongly, these drifting errors may be considerable. Then, a correction procedure should be applied to bring the stresses back to the yield locus [31]. With the present approach, there is less need for such a correction procedure.

Several restrictions have been imposed in proving the rigorous return to the yield surface. When these restrictions are violated, the rigorous return is not obtained, although in these cases the algorithm is still competitive. The restriction which entails the most serious errors is the assumption that no physical changes may occur during the loading step. If the errors caused by this assumption cannot be tolerated, an inner iteration loop must be applied, or we must divide the strain path in several parts which are bounded by physical changes (e.g., crack formation).

Algorithms for the combination of cracking and plasticity in smeared models are not often described in literature, but an example thereof has been discussed by Owen et al. [32] for the combination of cracking and viscoplasticity. Their algorithm bears some resemblance to the treatment given here, as they employ a decomposition of the concrete strain rate into several components, but a rigorous decomposition of the total strain rate into several crack strain rate components and into several concrete strain rate components in the sense of (1) is not utilized. Such a decomposition would, however, have been implied in their equations if they had adopted a compliance formulation as given by Bažant and Oh for their elastic-fracture matrix instead of a stiffness formulation [14, 17]. Then, the explicit formulation of their algorithm would have been the only difference from the algorithm presented here, at least for only one active crack.

6. Cracking and visco-elasticity

Another constitutive relation which can be captured within (3) is visco-elasticity with degenerated kernels, i.e. visco-elastic models for which the kernel of the hereditary integral has been developed in a series so that the strain history can be described with a finite number of internal variables. An example of such a model is the Maxwell chain. This model arises if a relaxation-type hereditary integral is developed in a Dirichlet series [16, 33] and can mechanically be interpreted as a parallel arrangement of Maxwell chains. In the present conception, the Maxwell chain model is thought to apply only to the concrete and not to the cracks, so that the Maxwell elements all have the same concrete strain increment to which the crack strain increment must be added to obtain the total strain increment (Fig. 4). Here again we profit from the rigorous decomposition of the total strain increment into a concrete and into a crack strain increment, as we can independently specify the constitutive laws for the concrete and for the smeared-out cracks. With regard to the concrete, we have the following constitutive equation:

$$\dot{\sigma}_{ij} = \frac{E^0}{1+\nu}\left[\frac{\nu}{1-2\nu}\,\delta_{ij}\delta_{kl} + \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\right]\dot{\varepsilon}^{\,\mathrm{co}}_{kl} + \sum_{\alpha=1}^{N} \sigma^{\alpha}_{ij}\,. \tag{75}$$

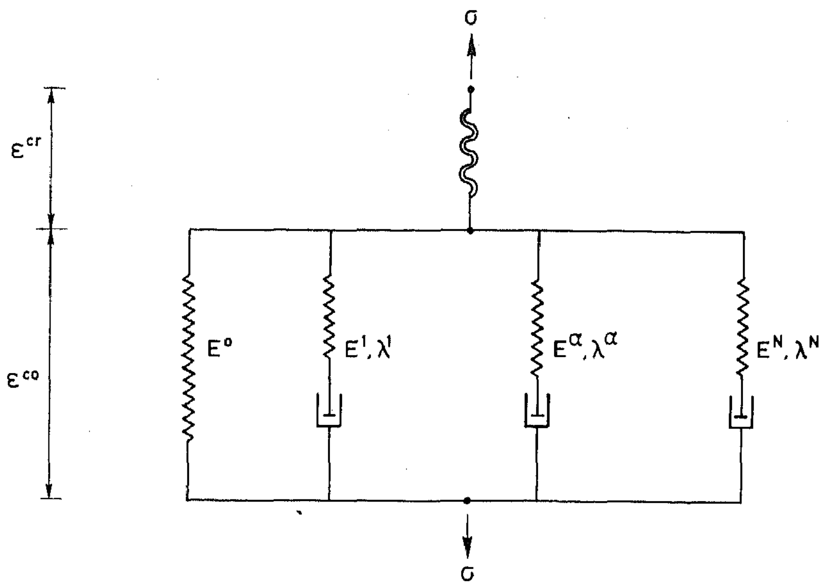

Fig. 4. Maxwell chain model for creep of concrete linked in series with a crack model.

E^0 is Young's modulus of the spring element in the chain (Fig. 4), ν is Poisson's ratio, and σ_{ij}^{α} are the internal stresses which follow from the evolution equation,

$$\dot{\sigma}_{ij}^{\alpha} = \frac{E^{\alpha}}{1+\nu}\left[\frac{\nu}{1-2\nu}\,\delta_{ij}\delta_{kl} + \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\right]\dot{\varepsilon}_{kl}^{\text{co}} - \frac{1}{\lambda^{\alpha}}\,\sigma_{ij}^{\alpha}\,, \tag{76}$$

with E^{α} and λ^{α} the Young's moduli and relaxation times of the Maxwell elements. For the smeared-out cracks, a tension-softening type model may be adopted.

Creep is a process which is highly stress-dependent and a single-point numerical integration scheme as adopted in the preceding section often leads to erroneous results. A possible solution is to employ a higher-order integration scheme [34], but such a solution is not attractive because the rate equations for fracture of concrete can be integrated accurately with a single-point integration rule. We therefore prefer the approach advocated by Taylor et al. [35] and by Bažant and Wu [33], who integrate (75) and (76) analytically over a time step, yielding

$$\begin{aligned}\Delta\sigma_{ij} = {} & \frac{1}{1+\nu}\left[E^0 + \sum_{\alpha=1}^{N}\left[1-\exp\left(-\frac{\Delta t}{\lambda^{\alpha}}\right)\right]\frac{E^{\alpha}}{\Delta t/\lambda^{\alpha}}\right]\left[\frac{\nu}{1-2\nu}\,\delta_{ij}\delta_{kl} + \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\right]\dot{\varepsilon}_{kl}^{\text{co}} \\ & - \sum_{\alpha=1}^{N}\left[1-\exp\left(-\frac{\Delta t}{\lambda^{\alpha}}\right)\right]\sigma_{ij}^{\alpha}(t-\Delta t)\,,\end{aligned} \tag{77}$$

We can now identify D_{ijkl}^{co} as

$$D_{ijkl}^{\text{co}} = \frac{1}{1+\nu}\left[E^0 + \sum_{\alpha=1}^{N}\left[1-\exp\left(-\frac{\Delta t}{\lambda^{\alpha}}\right)\right]\frac{E^{\alpha}}{\Delta t/\lambda^{\alpha}}\right]\left[\frac{\nu}{1-2\nu}\,\delta_{ij}\delta_{kl} + \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\right], \tag{78}$$

and Σ_{ij} as

$$\Sigma_{ij} = -\sum_{\alpha=1}^{N}\left[1-\exp\left(-\frac{\Delta t}{\lambda^{\alpha}}\right)\right]\sigma_{ij}^{\alpha}(t-\Delta t)\,. \tag{79}$$

With a single-point integration for the stress-strain law in the cracks, (17) can subsequently be integrated to yield

$$\Delta\boldsymbol{\sigma} = \Lambda[\boldsymbol{D}^{\mathrm{co}}\,\Delta\boldsymbol{\varepsilon} + \boldsymbol{\Sigma}]\,, \tag{80}$$

The accuracy of the above integration scheme hinges on the assumption that $\dot{\varepsilon}_{kl}^{\mathrm{co}}$ and E^{α} are nearly constant during the time step. Indeed, (77) constitutes an exact integration of (75) and (76) if $\dot{\varepsilon}_{kl}^{\mathrm{co}}$ and E^{α} are really constant during the time step. For a nonaging solid the assumption that E^{α} remains constant during a time step is satisfied rigorously, but even for an aging solid E^{α} usually varies so slowly with time that it mostly entails no serious integration errors. In the absence of cracks, the assumption that $\dot{\varepsilon}_{kl}^{\mathrm{co}}$ also remains constant, is usually also justified, since then $\dot{\varepsilon}_{kl}^{\mathrm{co}} = \dot{\varepsilon}_{kl}$ and $\dot{\varepsilon}_{kl}$ varies slowly during a time step. When we have crack formation or propagation, the assumption is more questionable. Since $\dot{\varepsilon}_{kl}^{\mathrm{cr}}$ may vary much more abruptly over a time step than $\dot{\varepsilon}_{kl}$ usually does, $\dot{\varepsilon}_{kl}^{\mathrm{co}}$ may also show more pronounced changes. The above observations imply that time steps must be chosen much smaller when cracks are present than when they are absent.

As an example, we will consider the unreinforced axisymmetric slab of Fig. 5. The slab is loaded uniformly with $q = 0.25\ \mathrm{N/mm^2}$, which is approximately 65% of the failure load. The creep behavior is modeled by 4 Maxwell elements with relaxation times and spring moduli as given in Fig. 5, whereby it is noted that although the algorithm described in the preceding section in principle allows for assigning the springs an age-dependent stiffness, this possibility has not been pursued in the present example. The results are given in the form of a time-deflection curve of the center of the slab (Fig. 6) and of crack patterns for $t = 0$ and $t = 1000$ days (Figs. 7–10). It is interesting to note that owing to the fact that a criterion for crack initiation has been employed which is a combination of principal stresses and strains [17], cracking gradually progresses in the course of time, both with regard to the radial and with regard to the circumferential cracks.

In the past, other analyses of concrete structures subjected to long-term loading have been presented, although most analysts employ Kelvin-type visco-elastic models [38–40]. It seems that neither of them has used a softening model for the cracks, but that all previous analyses

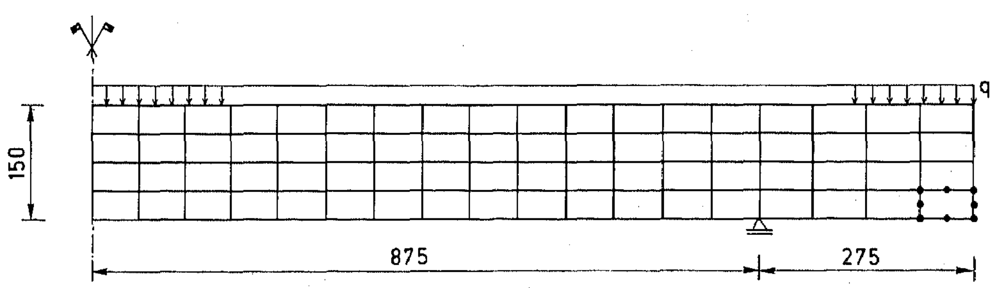

Fig. 5. Finite element mesh of axisymmetric slab used in creep analysis.

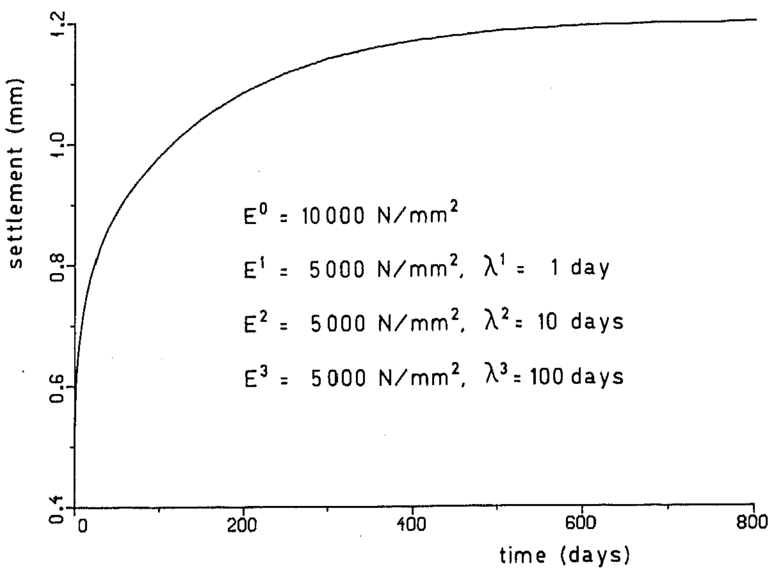

Fig. 6. Center deflection as a function of time.

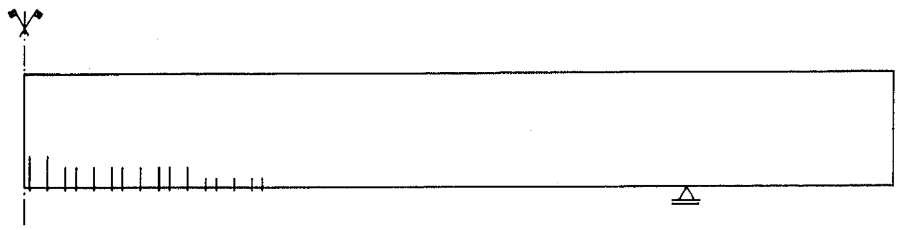

Fig. 7. Tangential crack pattern for $t = 0$.

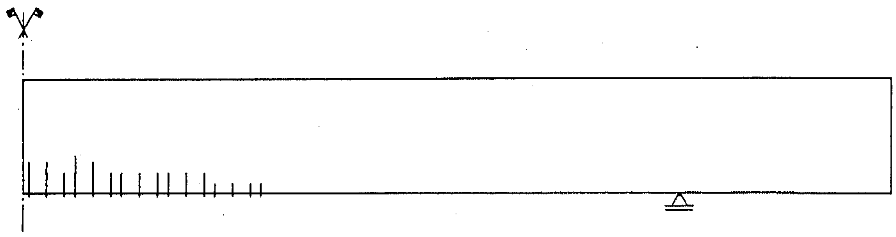

Fig. 8. Tangential crack pattern for $t = 1000$.

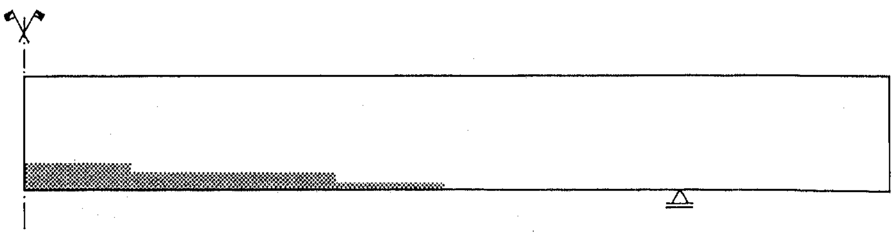

Fig. 9. Area in which radial cracks have developed for $t = 0$ (shaded).

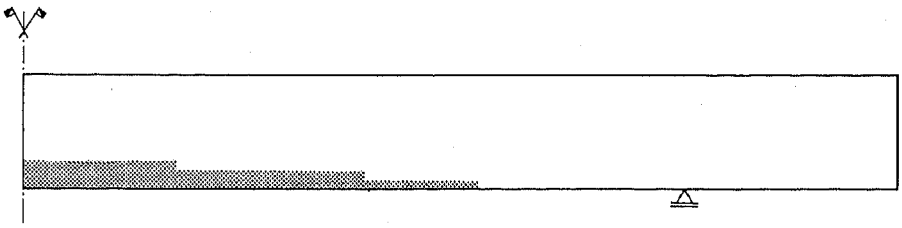

Fig. 10 Area in which radial cracks have developed for $t = 1000$ (shaded).

have been carried out under the assumption that all stiffness normal to a crack is lost upon crack formation. However, a comprehensive stress-strain relation for the smeared-out cracks including tension-softening and effects like aggregate interlock or crack dilatancy can only be incorporated if a decomposition of the strain increment is employed as starting point of the algorithm. If such a decomposition is not assumed, fracture and creep of the concrete between the cracks can not be distinguished properly. A similar algorithm has recently been proposed by Bažant and Chern [19], and indeed, if we restrict cracking to one crack in an integration point, their algorithm coincides with the present approach.

7. Temperature-dependent material properties

A third phenomenon which can be captured within the proposed concept is dependence of the material properties on the temperature. We will here only demonstrate that this phenomenon can be included in a natural fashion and we will therefore limit the treatment to the case that only the elastic properties depend on the temperature θ and that we have a thermal dilatation $\varepsilon^{\theta}_{kl}$. Then, the constitutive law for the (uncracked) concrete becomes:

$$\dot{\sigma}_{ij} = D^{e}_{ijkl}(\dot{\varepsilon}_{kl} - \dot{\varepsilon}^{\theta}_{kl}) + \dot{D}^{e}_{ijkl}(\varepsilon_{kl} - \varepsilon^{\theta}_{kl}) . \tag{81}$$

Identifying D^{co}_{ijkl} as the elasticity tensor D^{e}_{ijkl} at temperature θ and the autogeneous strain rate $\dot{\varepsilon}^{0}_{kl}$ as the thermal strain rate $\dot{\varepsilon}^{\theta}_{kl}$, we observe that a constitutive law in the sense of (3) is again obtained when

$$\Sigma_{ij} = \dot{D}^{e}_{ijkl}(\varepsilon_{kl} - \varepsilon^{\theta}_{kl}) . \tag{82}$$

The resulting equation for the cracked concrete can be integrated using a numerical integration scheme, for instance an Euler forward integration rule. Whether such a simple scheme is sufficiently accurate for this particular constitutive law will be reported in a future publication.

8. Conclusions

A smeared crack model which has the potential of properly describing nonorthogonal smeared cracks has been discussed. In this article, emphasis has been placed on the possibility to combine the proposed crack model with other nonlinear phenomena. In particular, algorithms have been described which are capable of simultaneously handling crack formation and plasticity, and crack formation and creep, while it has also been indicated how autogeneous strains due to shrinkage or thermal dilatation, and temperature-dependent material properties can be accommodated within the present concept. An example in the paper as well as several examples in other papers [9, 10, 14–16] have shown that the proposed algorithms can be implemented successfully in finite element codes.

It has furthermore been proved that the proposed crack model obeys the principle of material frame-indifference. This is by no means trivial as crack models have been proposed which do not meet this from a theoretical point of view desirable property [11].

Acknowledgment

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