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Measuring the riskiness of financial assets

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"Measuring the riskiness of financial assets"

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Abstract

Measuring and managing risk is one of the foundations of the financial industry. This report formulates the basic terminology and tools for understanding the process of evaluating risk. For this purpose, financial assets can be represented mathematically as discrete and finite random variables called gambles. In order to see their wealth grow, investors devise strategies to allocate their resources among gambles depending on valuations of risk. Risky strategies open the possibility of bankruptcy, while safe strategies guarantee no-bankruptcy in the long run. Dean Foster and Sergiu Hart have developed a safe strategy based on an objective measure of riskiness for gambles. This measure can be interpreted operationally as the minimum wealth an investor should possess in order to be able to purchase the measured asset safely, if this strategy is used consistently in the long run. Several properties of this critical wealth are presented, as well as a comprehensive proof of the Critical Wealth Theorem that determines this wealth for every gamble. The important concepts of conditional expectations and of martingales, as well as the Convergence Theorem for martingales, are explored in the process of providing the aforementioned proof.

Contents

1 Introduction

Since the global financial crisis started in the year 2008, the price of almost every type of asset in the developed economies has dropped. It started with the American real-estate market, followed by most stock indexes worldwide. Currently, sovereign bond markets in the European Union are experiencing a decline in value. By bankrupting numerous banks, pension funds, hedge funds and private investors, the effects of this financial crisis have in turn propagated to the real economy through the depressing effect of limited credit and widespread deleveraging.

The business of institutional investors is to forecast extreme and risky developments, or at the very least build portfolios that can weather them and maintain their value in every scenario. But the development of the current crisis shows widespread misjudgement of the riskiness and value of financial assets. While the explanation of why so many institutions made such bad risk misjudgements lies beyond the scope of this report, these developments illustrate the importance of measuring and managing risk in financial markets, given the important effects it can have on the economy and on society.

This report will give the fundamental tools needed to understand the riskiness of assets and the decision-making that is based on it. The first part will model the assets as random variables and present the terminology that will further be used to describe the riskiness of assets and the strategies followed by investors to build portfolios. In the second part, we will present the Operational Measure of Riskiness developed by Foster and Hart and give some of their accompanying results. The last part will be dedicated to proving the results that Foster and Hart have derived, by exploring the concept of martingales.

2 The concept of risky assets

2.1 Gamble model

An asset can be described as an object that generates cash flows for its owner at given periods of time. Real estate will for instance generate monthly rents or capital gains upon resale, while shares in a firm can provide yearly dividend payments. Most assets generate their cash flows with a degree of uncertainty, as the real estate's rents might have to be pre-empted for unexpected repairs and the firm could run into trouble and decide to reduce or eliminate the dividend paid.

We will start by looking at the most basic assets, which are purchased for a given price at a given time and generate a stream of cashflow at a single time in the future. To reflect the uncertainty of such an asset's cashflow, we will model the asset as a discrete and finite random variable X characterized respectively by a set of possible outcomes and a corresponding set of probabilities

$$\{X = x_1, X = x_2, \ldots, X = x_n\} \text{ or } \{x_1, x_2, \ldots, x_n\}$$

$$\{P(X = x_1), P(X = x_2), \ldots, P(X = x_n)\} \text{ or } \{p_1, p_2, \ldots, p_n\}$$

together forming the random variable's probability distribution. We will consider the probability distribution to be fully known in advance.

In a further attempt to simplify reality, both the price paid to purchase the asset and the cashflow generated at a later time are consolidated into a single outcome. Similarly, despite the continuous nature of assets, we have approximated these with discrete valued variables, as the concepts involved will not change fundamentally, while the amount of calculations involved will.

Based on these considerations, we can now give a proper definition of the random variable mathematically representing the asset and which we will call a gamble.

Definition 2.1.1. *A gamble g is a discrete random variable characterized by its finite probability distribution. We call $\mathcal{G}$ the collection of all possible gambles and $\mathcal{G}_0 \in \mathcal{G}$ a finite subset from which the gambles under study are chosen.*

Definition 2.1.2. *The maximum possible loss of a gamble g is the number $L(g) = -\min x_i$. The maximum possible gain is the number $M(g) = \max x_i$. When the gamble to which they refer is made obvious from the context, the simpler notations L and M, may also be used.*

For each gamble, the expectation $\mathbb{E}[g]$ can be interpreted as the average outcome of g if the gamble were taken repeatedly. It is computed using the definition $\mathbb{E}[g] = \sum_{i=1}^{n} x_i \; P(X = x_i)$.

To further illustrate the way we model assets, we will now describe a few concrete examples.

Example 2.1.3. *Consider a one-year corporate bond with a face value of \$220, a coupon of 0% and a purchase price of \$100. These terms mean that, if all goes normally, after a year the bond issuer promises to pay its holder the face value of \$220 augmented by the coupon percentage* $220\$ \times 0\% = 0\$$. *The randomness of this seemingly deterministic asset is introduced by the risk of default. If the firm that issues the bond cannot make the payment after a year and defaults on it, the holder in our example receives nothing. The rating agencies consider that there exists a 50% chance that the firm will default on its payment and to keep our example simple, we will assume that their evaluation is correct. Figure 1 illustrates the cashflows that occur in each of these scenarios.*

Figure 1: Cashflows scenarios for one-year bond

According to our model, the bond is a gamble $g \in \mathcal{G}_0$ *characterized by the following probability distribution: set of outcomes* $\{x_1 = -100, x_2 = 120\}$ *and set of probabilities* $\{p_1 = 0,5, p_2 = 0,5\}$, *where* x_1 *equals the consolidated sum of the default scenario and* x_2 *equals the consolidated sum of the repayment scenario. The gamble's maximum loss* $L(g) = -x_1 = 100$ *occurs in case of default, the investor losing exactly the price he paid for the asset. The gamble's maximum gain equals* $M(g) = 120$ *and its expectation* $\mathbb{E}[g] = 10$.

Example 2.1.4. *Consider a stock priced \$100 whose owner could sell after a year in order to make a profit. Financial analysts consider that there exists a 80% chance that this stock will be priced \$115 after a year and a 20% chance that it will only be worth \$90. Furthermore, this firm is known not to distribute dividends to its shareholders.*

In the same way as we did for the bond in the previous example, we can model the stock as a gamble $g \in \mathcal{G}_0$ *characterized by the following probability distribution: set of outcomes* $\{x_1 = -10, x_2 = 15\}$ *and set of probabilities* $\{p_1 = 0,2, p_2 = 0,8\}$. *The gamble's maximum loss* $L = -x_1 = 10$ *occurs when the stock's price drops to \$90 towards the time that the investor sells it, after one year. The maximum gain and expectation are respectively* $M = 15$ *and* $\mathbb{E}[g] = 10$.

Example 2.1.5. *Consider an apartment priced \$100,000 which can be rented out for twelve months before being sold again after a year. A quick market research reveals that there exists an equal chance of any of the four following events to unfold:*

1. *the apartment is sold for \$105,000 and the cumulative rent for the twelve months equals \$6,000*
2. *the apartment is sold for \$105,000 but no tenants are found*
3. *the apartment is sold for \$80,000 and the cumulative rent equals \$6,000*
4. *the apartment is sold for \$80,000 and no tenants are found*

We assume that no other costs will have to be borne in either of the situations. We can model the real estate as a gamble $g \in \mathcal{G}_0$ *characterized by the following probability distribution: set of outcomes* $\{x_1 = -20,000, x_2 = -14,000, x_3 = 5,000, x_4 = 11,000\}$ *and set of probabilities* $\{p_1 = 0,25, p_2 = 0,25, p_3 = 0,25, p_4 = 0,25\}$. *The gamble's maximum loss* $L = -x_1 = 20,000$ *is suffered if event 4 occurs. The maximum gain and expectation are respectively* $M = 11,000$ *and* $\mathbb{E}[g] = -4,500$.

Example 2.1.6. *Consider an ounce of gold priced \$1,000 which can be resold after one year, for either \$1,000 or for \$1,030, each situation being considered equally likely to happen. We now have a gamble* $g \in \mathcal{G}_0$ *characterized by the following probability distribution: set of outcomes* $\{x_1 = 0, x_2 = 30\}$ *and set of probabilities* $\{p_1 = 0,5, p_2 = 0,5\}$. *The gamble's maximum loss, maximum gain and expectation are respectively* $L = 0$, $M = 30$ *and* $\mathbb{E}[g] = 15$.

2.2 Wealth levels

Investors allocate their capital between various assets in order to try to realize profits, or at least not suffer losses. This reflects the fact that there are usually more investment opportunities than capital available to invest and that, as such, the players are faced with a choice for each asset they encounter. They can decide either to buy it or not to buy it, in other words, they may either accept the gamble or reject the gamble.

Definition 2.2.1. *To each gamble g we associate a variable d such that it equals 0 if gamble g is rejected by the gambler and equals 1 if the gamble is accepted.*

As these decisions represent the only way the gambler can exert influence on the effects produced by the gamble, our research will focus on finding when and why a gamble is or should be accepted.

The next step in constructing our mathematical model is to acknowledge the fact that most investors hold their assets for some period of time, instead of systematically selling after a year, as our above examples implied. For practical purposes, we will divide the total time that an asset is held into several discrete periods of time $t = 1, 2, \dots$ and will model a new gamble for each period.

Definition 2.2.2. *At the start of each period $t = 1, 2, \dots$, the gambler is offered a gamble $g_t \in \mathcal{G}_0$. We call $(g_t)_{t=1,2,\dots}$ the resulting sequence of gambles. There are no restrictions on the stochastic dependence between consecutive gambles, or on the choice and order of gambles that are drawn from $\mathcal{G}_0$ to constitute the sequence.*

An investor could, for example, buy the same one-year bond from example 2.1.3 every year after having liquidated the previous one, so that the sequence this represents consists of the same gamble repeating itself. He could buy the stock from example 2.1.4 in year 1 and hold it several years before reselling it. According to our model, this would have the same effect as a resell after one year and an immediate repurchase of the differently priced stock which is then modelled by the same initial gamble but rescaled to denote the new purchasing price. Here the gambles in the sequence are similar in nature but not equal. Finally, whether actually selling and repurchasing or holding the initial asset, the more realistic sequence of gambles follows no pattern and is therefore totally unpredictable, past the current gamble on offer.

Assets hold value in the eyes of the investor insofar as the cashflow which they might generate can influence their wealth, where the concept of wealth is defined as follows as the cumulative result of all previously accepted gambles added to the initial wealth.

Definition 2.2.3. *The gambler's wealth at the beginning of period t is given by the stochastic process W_t where $t = 1, 2, \ldots$. The process is characterized by an initial wealth $W_1 > 1$ and the recurrence formula*

$$W_{t+1} = W_t + d_t g_t \tag{1}$$

for all $t = 2, 3, \ldots$.

The graph shown in figure 2 describes one possible evolution (unlucky in this case!) of the wealth of a gambler who starts with \$400 and is repeatedly offered and accepts the bond from example 2.1.3 for ten years.

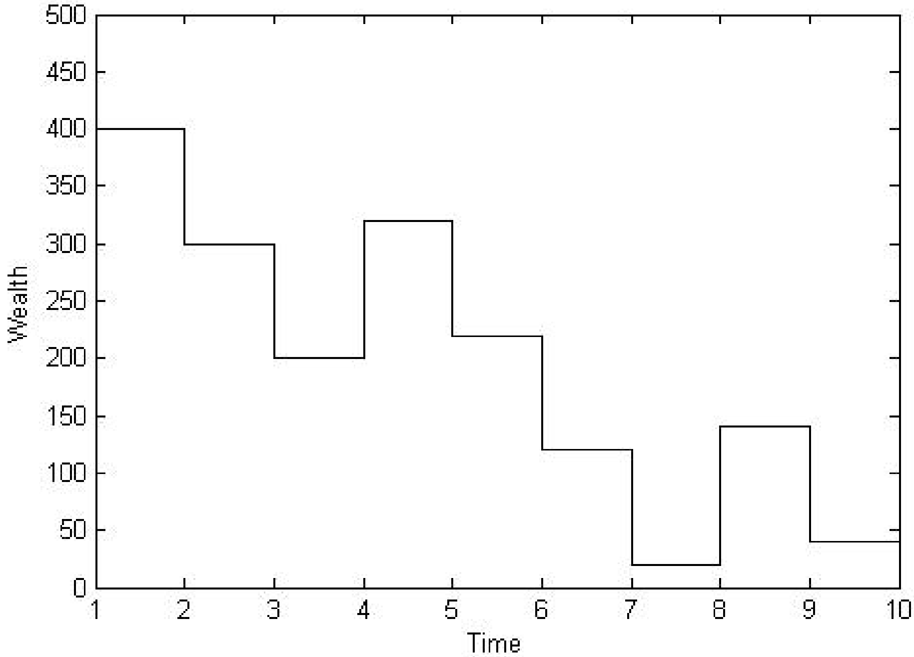

Figure 2: Graph of the development of wealth in time

Another way of looking at the wealth evolution is to measure the growth or decline of the wealth for every period of time. For this purpose, we will define a new random process that outputs the ratio of the change in wealth after the gamble is either accepted or rejected, by the wealth before the gamble was offered.

Definition 2.2.4. *We call growth factor the random variable Y_t defined by $Y_t = \log W_{t+1} - \log W_t$ This factor is well defined for all t for which both $W_t > 0$ and $W_{t+1} > 0$.*

A positive growth factor implies that $W_{t+1} \geq W_t$, whereas similarly the wealth decreases if and only if the growth factor is negative.

We conclude this section by noting that the overall money and asset supply being limited, the wealth W_t for every t is bounded by that total supply, i.e. an investor can't possess more than what is available in the world. The bounded character of the growth factors follows directly from that of the wealth and from the finite character of the gambles. In mathematical terms, this property translates to the following proposition.

Proposition 2.2.5. *There exists a finite K such that $|Y_t| < K$ for all t.*

2.3 Strategies

The investor will accept gambles if he feels that doing so might influence his wealth positively and will reject them if the perceived effect is negative. In taking these decisions, the gambler judges the degree to which the influence of the gamble on his wealth could be positive or negative and sets his strategy accordingly.

Definition 2.3.1. *We call strategy s any set of rules followed by the gambler to make decisions, in other words to determine the value of each d_t.*

It is natural to ask ourselves what criterion an investor can or should use to determine his strategy, whether there exist any objective measurements and conditions to be fulfilled for a gamble to have a positive influence on wealth. The worst negative impact on wealth is that which leads the wealth to vanish completely. As we will assume that no money can be borrowed, reaching the situation where $W_t \leq 0$ or where W_t is close to zero means that the investors not only has no wealth left but also doesn't have the means to invest in and accept any new gamble in order to recover his losses and try to accumulate new wealth. This situation of bankruptcy is a sinkhole from which there is no escape. We can define it formally as follows.

Definition 2.3.2. *A gambler is said to be bankrupt if $\lim_{t\to\infty} W_t = 0$ and in particular if at some period t, we have $W_t \leq 0$.*

This definition of bankruptcy in turn enables us to isolate strategies that can lead to bankruptcy under some scenarios.

Definition 2.3.3. *A strategy s is called risky if it leads to decisions that can cause bankruptcy, in other words if as a result of the decisions prescribed by the strategy we have $P(\lim_{t\to\infty} W_t = 0) \neq 0$ for at least one sequence of gambles $(g_t)_{1,2,\ldots}$. A strategy that is not risky is called a safe strategy and this kind of strategy guarantees non-bankruptcy for any sequence of gambles.*

A good strategist will therefore develop and use objective criterion to distinguish risky strategies from safe strategies and make the gambler either follow only the safe kind or follow the risky kind with full knowledge of the degree of riskiness of his behaviour.

The two simplest strategies would be either to accept or to reject all gambles. The latter case is of little interest for our research. The graph in figure 3 depicts the results of a simulation in which a gambler starting with $W_1 = 600$ is offered the bond from example 2.1.3 repeatedly for 20000 periods of time and follows the strategy of accepting all of them. The histogram in figure 4 records the final wealth when the previous simulation is run 100 times. The histogram shows that there exist runs for which the gambler becomes bankrupt, the graph illustrating one such instance. We can therefore say that the strategy that was followed is risky. We emphasize that the expectation of the gambles was positive and yet this characteristic did not prevent bankruptcy from occurring.

We have repeated the simulation, changing only the type of gamble on offer to the golden ounce from example 2.1.6. The distribution of final wealth is given by the histogram in figure 5. It appears that for this sequence of gambles, the strategy of accepting all of them does not lead to bankruptcy, in fact it doesn't even lead to any losses below the initial wealth.

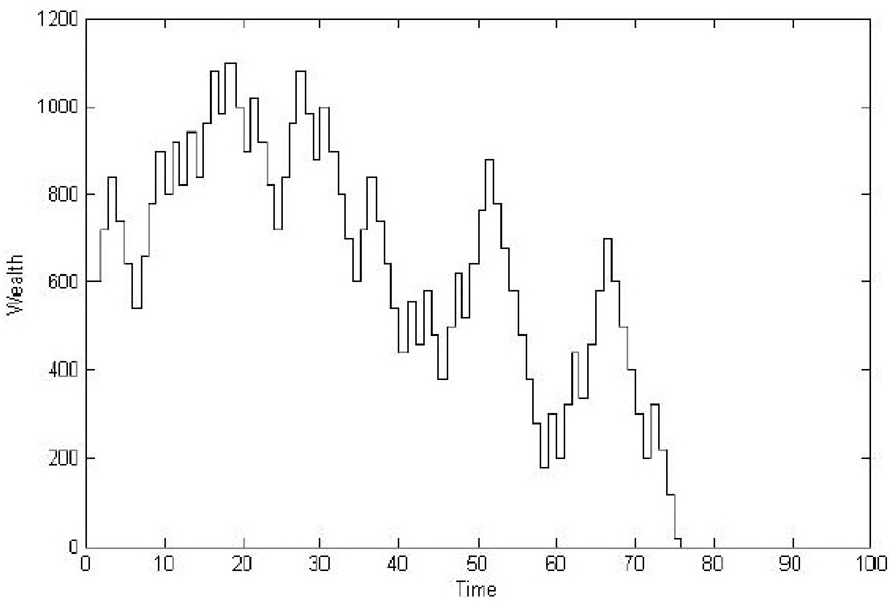

Figure 3: Graph of development of wealth in time, sequence of bonds from example 2.1.3

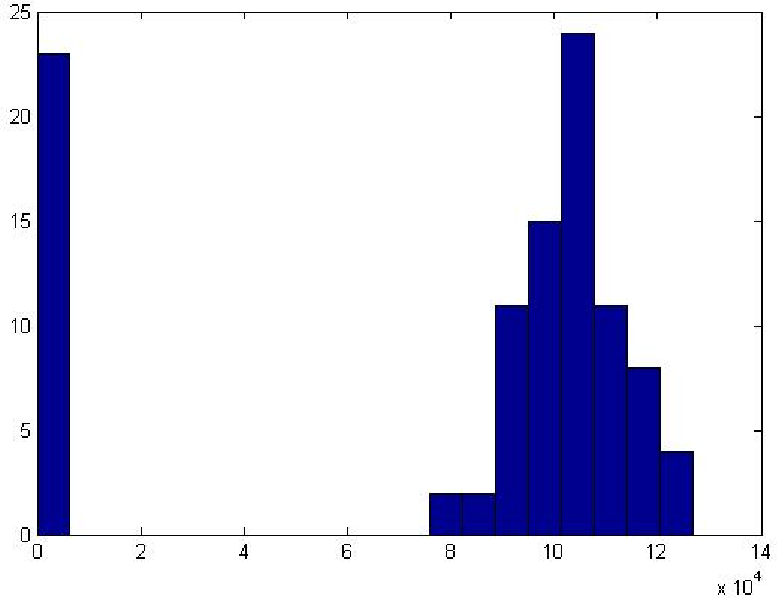

Figure 4: Histogram of final wealth, sequence of bonds from example 2.1.3

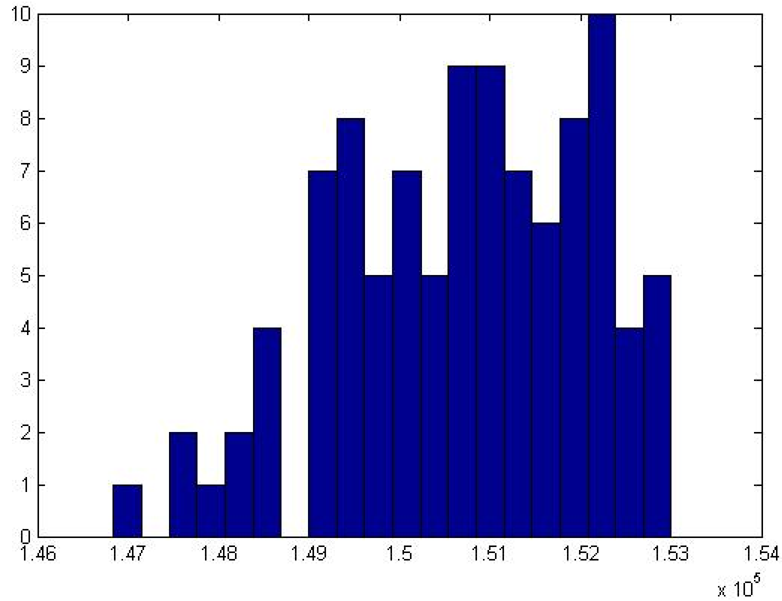

Figure 5: Histogram of final wealth, sequence of gold from example 2.1.6

This last example illustrates one of three basic criteria that help narrow the field of strategies that can be deemed safe. They are directly implied by the definition of bankruptcy and by equation (1). The first acknowledges the fact that if the gamble has no negative outcomes, the gambler's wealth can never decline, whatever the outcome. Since the initial wealth is always positive, there can in particular be no bankruptcy and therefore all safe strategies should lead to the gambler always accepting such gambles. The second recommendation derives from the fact that the gambler must at the very least "survive" the result of the gamble currently on offer, in other words, he must be certain not to go bankrupt immediately if the lowest outcome occurs. The third considers the gamble's expectation: consistently accepting gambles with a negative expectation means the wealth is more likely to drop than to grow and in particular in means that there exists a time T in the long run for which the gambler's wealth reaches or comes close to zero.

Proposition 2.3.4. *Any safe strategy should include or lead to the following prescriptions:*

$$\textit{accept all gambles } g \textit{ for which } L(g) \leq 0 \tag{2}$$

$$\textit{reject all gambles } g \textit{ for which } \mathbb{E}[g] < 0 \tag{3}$$

$$\textit{reject all gambles } g \textit{ for which } L(g) \geq W_t \tag{4}$$

According to these rules, every offer of the ounce of gold from example 2.1.6 must be accepted as the gamble has no negative outcomes. The histogram in figure 5 offers a clear illustration for this first rule. Similarly, every offer of the gamble representing the apartment from example 2.1.5, which has an expectation of $\mathbb{E}[g] = -4500$, must be rejected. Based on its expectation of $\mathbb{E}[g] = 10$, we cannot yet conclusively reject or accept the bond from example 2.1.3, unless the gambler's wealth is $W \leq 100 = L(g)$, in which case it should be rejected according to condition (4).

The three rules of proposition 2.3.4 are necessary for safe strategies but, as the inconclusive bond example illustrate, they need to be complemented by or generalised into a new set of rules that would be both necessay and sufficient.

Rule (2) says that all gambles without negative outcomes should be accepted, but this does not necessarily mean that a gambler should reject all gambles with at least one negative outcome. This should be made evident by looking at the example of a gamble with outcomes $\{-0.01, 1000\}$ and respective probabilities $\{0.001, 0.999\}$: it is hard to imagine going bankrupt given such a good deal, except if one starts with an initial wealth that is of the order of a few cents, and in particular if $W_1 \leq 0.01$, in which case rule (4) would be broken.

Rule (3) says that all gambles with negative expectation should be rejected. Again, the complementary rule that prescribes to accept all gambles with non-negative expectation does not follow naturally from it. The nature of random events is such that we can't reject the possibility of bad luck in the form of a long series of "bad" outcomes that would deplete the initial wealth of the gambler and make him bankrupt, even if the expectation is positive. Unless the gambler somehow possesses infinite wealth to start with.

Fortunately, rule (4) does provide us with fertile grounds to build upon, as it involves the gambler's wealth as well as characteristics from the gamble itself. Also we have seen in our short study of the other two rules that the wealth of the gambler is always indirectly involved: the only way the advantageous gamble from the first example could be ruinous was if the initial wealth was below a certain threshold, the complement of the third rule could only hold if in some way the gambler's wealth could be considered infinite. We recall that rule (4) requires a gambler to reject all gambles on offer when their current wealth does not exceed the maximum loss of that gamble. We can ask ourselves if a strategy that accepts gambles whenever the gambler's wealth exceeds the maximum loss of the gamble is a safe strategy.

In order to examine this question, it is useful to define a more general kind of strategy which we will call simple strategies.

Definition 2.3.5. *Let $Q(g)$ be a function that associates with each gamble g a number $Q(g)$ in $[0, \infty]$. We then call simple strategy with critical wealth function $Q(g)$ a strategy that rejects gamble g_t when $W_t < Q(g_t)$ and accepts it when $W_t \geq Q(g_t)$. Such a simple strategy is denoted s_Q.*

The number $Q(g)$ represents the lowest wealth which the gambler must possess in order to be able to accept gamble g. If $Q(g) = 0$, the gamble is always accepted and if it is infinite then the gamble is never accepted. We can now reformulate our previous question using the nomenclature of simple strategies: is the simple strategy s_Q with critical wealth $Q(g) = L(g)$ a safe strategy?

To answer this question, we have run a simulation of a gambler starting with $W_1 = 600$ and faced with a sequence of 20000 bonds from example 2.1.3 and using the simple strategy $Q(g) = L(g) = 100$. The simulation was then repeated 100 times and the results are shown in the histogram in figure 6.

Unfortunately, even though we notice that there are fewer bankruptcies in this example than in the example pictured in figure 3, where $Q(g) = 0$, the histogram indicates that there exist sequences of gambles that do lead a gambler to bankruptcy whenever he uses the aforementioned simple strategy.

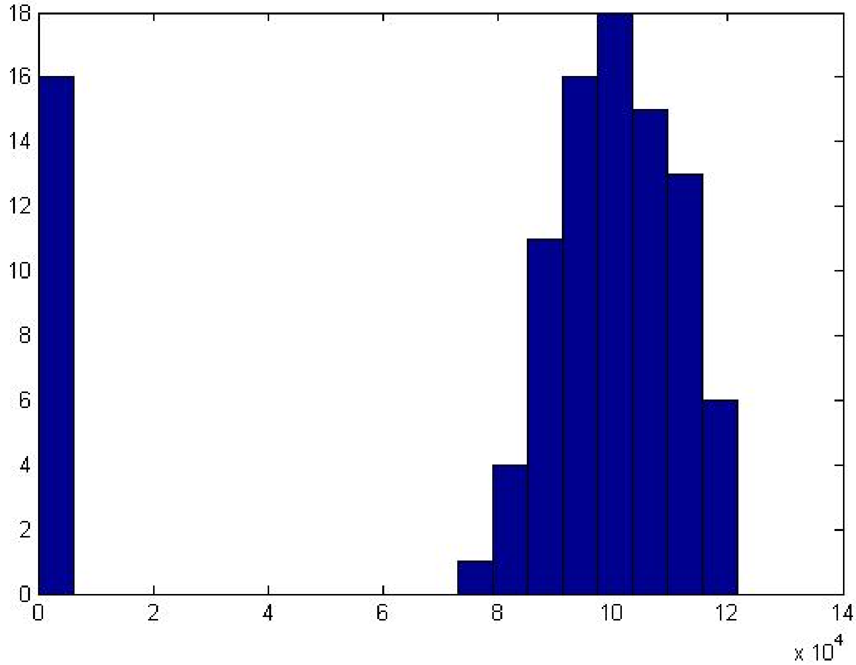

Figure 6: Histogram of final wealth, simple strategy $Q(g) = L(g)$

The simple strategy s_Q with $Q(g) = L(g)$ therefore is a risky strategy. We do note that rule (4) can be restated as follows: any simple strategy must at least satisfy $Q(g) > L(g)$ in order to be qualified as safe.

3 Foster and Hart's measure of riskiness

3.1 Critical Wealth Theorem

While a simple strategy using the maximum loss L as its critical wealth was shown to be risky, Foster and Hart have developed a critical wealth function $Q(g)$ that does guarantee no-bankruptcy. They interpret their function $R(g)$ as a measure of riskiness of gambles. In our terminology, the function represents the lowest critical wealth which ensures the safety of a simple strategy. Hence it gives us an objective rule that, we will see later, is both necessary and sufficient to design any safe strategy. Their main result is given in the following theorem.

Theorem 3.1.1 (Critical Wealth Theorem)**.** *There exists for every gamble $g \in \mathcal{G}$ a unique number $R(g) > 0$ such that, when faced with an infinite sequence $(g_t)_{t=1,2,...}$ of gambles, following the simple strategy s_Q with critical wealth function $Q(g)$ guarantees non-bankruptcy if and only if $Q(g) \geq R(g)$ for every gamble g in the sequence. This $R(g)$ is determined by the equation*

$$\mathbb{E}\Big[\log\Big(1 + \frac{g}{R(g)}\Big)\Big] = 0 \tag{5}$$

In accordance to our previous interpretation of the critical wealth involved in simple strategies, $R(g)$ represents the minimum wealth level at which g may be accepted. In can also be seen, intuitively, as a kind of buffer needed to withstand some degree of bad luck and still have enough wealth to accept new gambles and rebound. To give some insight into the measure introduced by Foster and Hart, we have computed the critical wealth for the bond from example 2.1.3.

Example 3.1.2. *We start by rewriting equation (5), where for simplicity we use R instead of $R(g)$:*

$$(5) \iff \sum_{i=1}^{n} p_i \log\Big(1 + \frac{x_i}{R}\Big) = 0 \tag{6}$$

$$\iff \log \prod_{i=1}^{n} \Big(1 + \frac{x_i}{R}\Big)^{p_i} = 0 \tag{7}$$

$$\iff \prod_{i=1}^{n} \Big(1 + \frac{x_i}{R}\Big)^{p_i} = 1 \tag{8}$$

We now subsitute the outcomes and probabilities of g and solve for R:

$$
\begin{aligned}
(8) &\iff \left(1 - \frac{100}{R}\right)\left(1 + \frac{120}{R}\right) = 1 \\
&\iff \frac{20}{R} - \frac{12000}{R^2} = 0 \\
&\iff 20R - 12000 = 0 \\
&\iff R = 600
\end{aligned}
$$

We will later give the proof for the Critical Wealth Theorem, but first we have chosen to explore the effect of the $R(g)$ boundary on the gambler's wealth. For this purpose, we have once more produced simulations of a gambler with $W_1 = 600$ faced with a sequence of 20000 gambles $g \in \mathcal{G}_0$ where $\mathcal{G}_0$ is the set of gambles that equal the bond from example 2.1.3 or multiples kg of it, with $k \in (0, 1)$ chosen randomly. The simulation is repeated 100 times for each of the following strategies. The histograms in figures 7 and 8 show the final wealth when the gambler follows a simple strategy with critical wealth respectively $Q(g) = R(g)$, $Q(g) = 9R(g)$ and $Q(g) = 0.9R(g)$.

These histograms give strong indications that a simple strategy based on critical wealth $R(g)$ guarantees no-bankruptcy, as none of the 100 runs

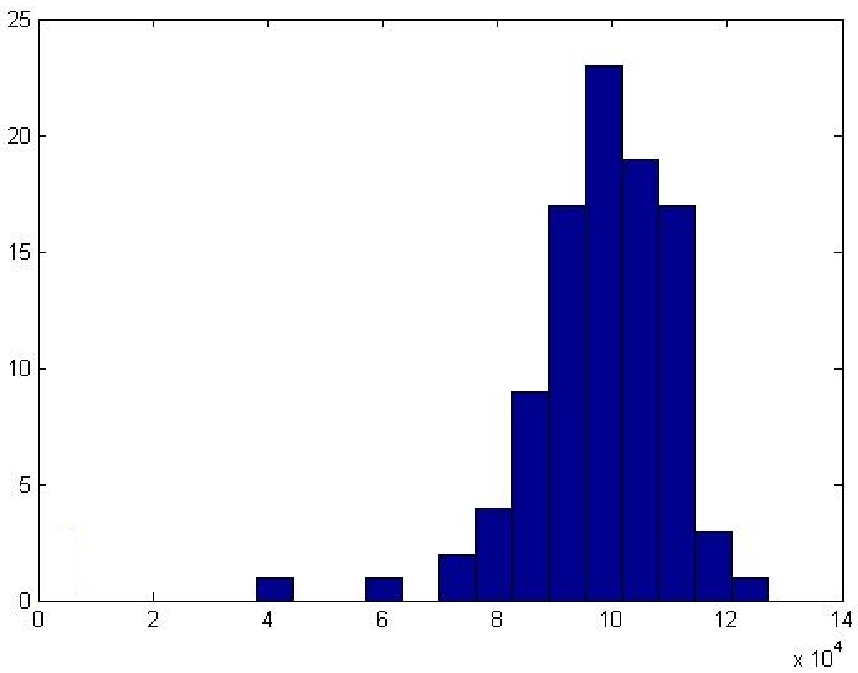

Figure 7: Histogram of final wealth, simple strategy $Q(g) = R(g)$

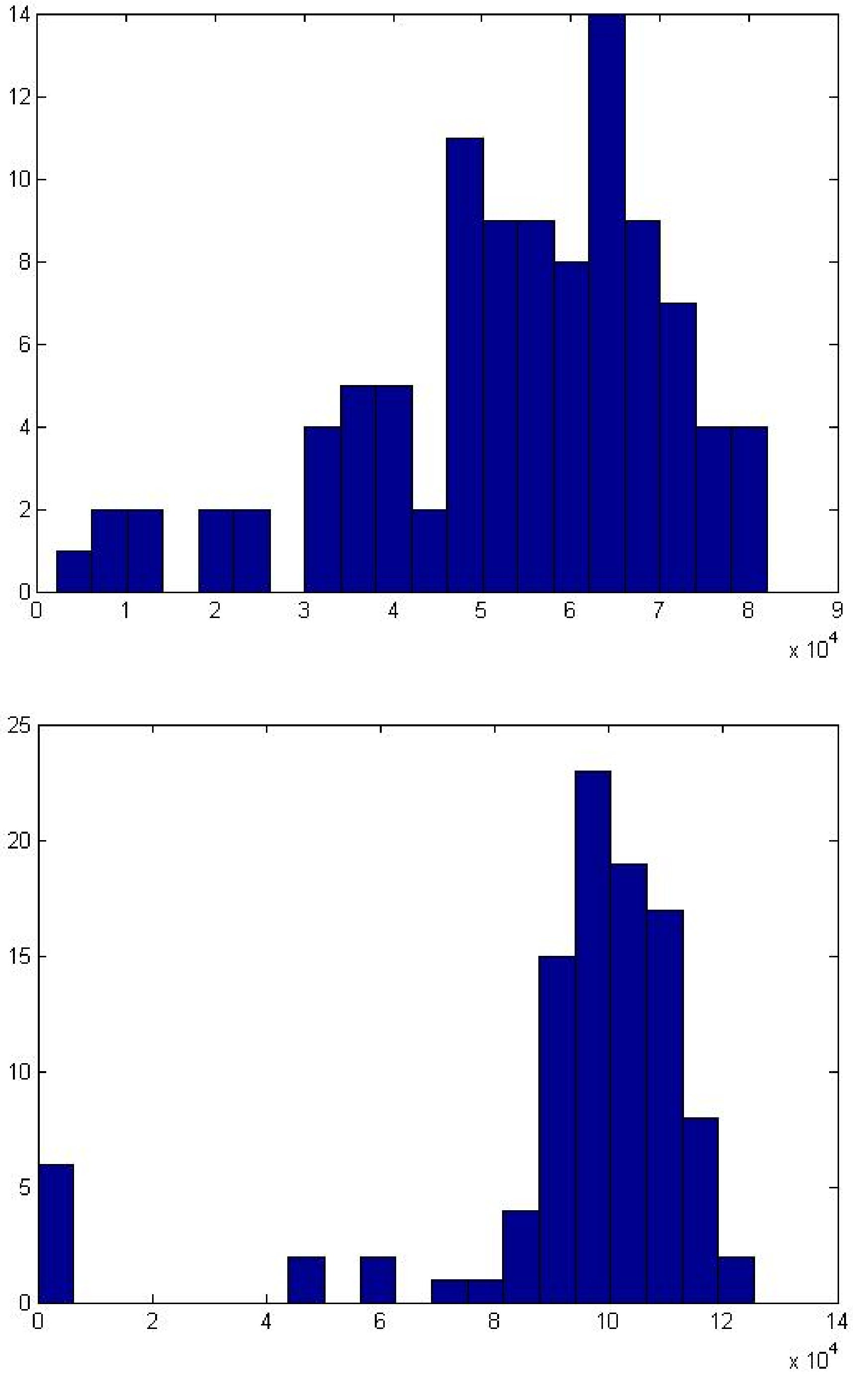

Figure 8: Top: histogram of final wealth, simple strategy $Q(g) = 9R(g)$. Bottom: histogram of final wealth, simple strategy $Q(g) = 0.9R(g)$

that led to figure 7 ended in bankruptcy. In fact, all of those runs led to an important increase in the gambler's wealth, all ending above 4000 and most between 8000 and 12000. The top histogram in figure 8 in turn indicates that a more conservative simple strategy is also safe, in accordance with the Critical Wealth Theorem. The average final wealth of the gambler does appear to suffer from such a strict strategy, as many gambles that would have been perfectly safe to accept according to the theory are rejected, forsaking an opportunity to add wealth. Finally, the third histogram reminds us that any simple strategy that is even slightly less conservative than the one based on critical wealth $R(g)$ contains the seeds of potential bankruptcy.

One last graph, in figure 9, showing the development of wealth for one of the previous runs with critical wealth $R(g)$, sheds more light on the way that such a simple strategy can avoid bankruptcy in the long run. After 20000 runs, we previously saw that the gambler never goes bankrupt. However, around the 700th run in the example pictured here, the gambler would have still found himself with less wealth than the wealth with which he had started, thus giving the appearance of near-bankruptcy. The simple strategy with critical wealth $R(g)$ therefore does not mean that bankruptcy may not seem close at times, only that it will always be far away enough to be able to

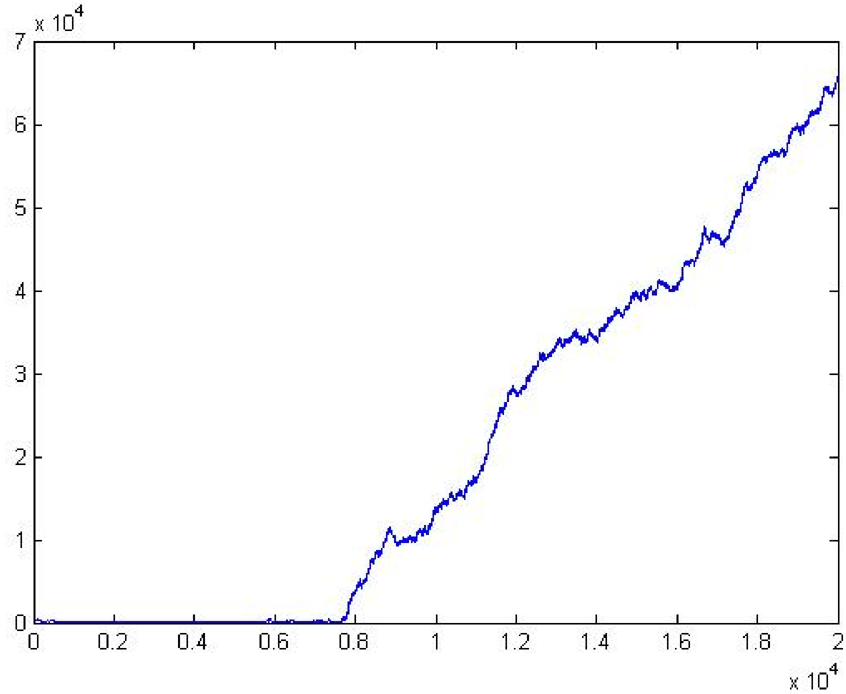

Figure 9: Graph of wealth in time, simple strategy $Q(g) = R(g)$

take advantage of a series of lucky outcomes to rebuild a safer situation and eventually escape the pull of bankruptcy completely. The graph indicates that, while non-bankruptcy is guaranteed in the long run, a long time horizon may indeed be needed in order for this effect to materialize with certainty.

3.2 Existence and uniqueness

The first step in proving the important result from the Critical Wealth Theorem will be to examine in turn the properties of the two related functions $\phi_g(\lambda) = \mathbb{E}[\log(1 + g\lambda)]$ and $\psi_g(r) = \mathbb{E}[\log(1 + \frac{g}{r})]$ and use those properties to show that for every gamble g, the critical wealth $R(g)$ is uniquely determined as the solution of the equation $\psi_g(r) = 0$. In the following, we will denote $R(g)$ simply as R whenever it is obvious to which gamble it refers.

Lemma 3.2.1. *For every gamble g, the function $\phi_g : \big[0, \frac{1}{L}\big) \longrightarrow \mathbb{R}$ with $\phi_g(\lambda) = \mathbb{E}[\log(1 + g\lambda)]$ is concave.*

Proof. Using the definition of the function ϕ_g, the concavity of the logarithm and the linearity of the expectation, we can say that for every $\lambda_1, \lambda_2 \in \big[0, \frac{1}{L}\big)$:

$$\begin{aligned}
\phi_g\Big(\frac{\lambda_1 + \lambda_2}{2}\Big) &= \mathbb{E}\Big[\log\Big(1 + \frac{g(\lambda_1 + \lambda_2)}{2}\Big)\Big] \\
&= \mathbb{E}\Big[\log\Big(\frac{1 + g\lambda_1}{2} + \frac{1 + g\lambda_2}{2}\Big)\Big] \\
&\geq \mathbb{E}\Big[\frac{1}{2}\log(1 + g\lambda_1) + \frac{1}{2}\log(1 + g\lambda_2)\Big] \\
&= \frac{1}{2}\Big(\mathbb{E}[\log(1 + g\lambda_1)] + \mathbb{E}[\log(1 + g\lambda_2)]\Big) \\
&= \frac{1}{2}\Big(\phi_g(\lambda_1) + \phi_g(\lambda_2)\Big)
\end{aligned}$$

which proves the concavity of the function ϕ_g. □

Lemma 3.2.2. *For every gamble g, the function $\phi_g : \big[0, \frac{1}{L}\big) \longrightarrow \mathbb{R}$ with $\phi_g(\lambda) = \mathbb{E}[\log(1 + g\lambda)]$ is smooth.*

Proof. We know that polynomials are smooth on $\mathbb{R}$, which includes $\big[0, \frac{1}{L}\big)$. We also know that the logarithm is a smooth function on $(0, \infty)$, which includes $(0, 1]$. From this we can say that each function $f(\lambda) = \log(1 + g\lambda)$ is also smooth on $\big[0, \frac{1}{L}\big)$. Moreover, any linear combination of smooth functions is itself smooth, hence taking the expectation preserves the smoothness, which proves that the function ϕ_g is smooth on its domain. □

Lemma 3.2.3. *For every gamble g, the function $\phi_g : \left[0, \frac{1}{L}\right) \longrightarrow \mathbb{R}$ with $\phi_g(\lambda) = \mathbb{E}[\log(1 + g\lambda)]$ has the following properties:*

1. $\phi_g(0) = 0$
2. $\phi_g'(0) = \mathbb{E}[g]$
3. $\lim_{\lambda \to \frac{1}{L}} \phi_g(\lambda) = -\infty$

Proof. Using the definition of the function ϕ_g, the fact that the logarithm vanishes at 1 and the fact that the expectation of a non-random variable equals that variable, it is obvious that $\phi_g(0) = \mathbb{E}[\log(1+g.0)] = \mathbb{E}[\log 1] = 0$, which proves the first property.

To prove the second, we first need to determine the derivative of ϕ_g. We do this by using in turn the definition of the derivative, the linearity of the expectation, the finiteness of the random variable g, which expresses itself in the finiteness of the sum involved in the expectation and enables us to interchange the limit and the expectation. Finally, we use the chain rule and recognize the derivative of the logarithm:

$$\begin{aligned}
\phi_g'(\lambda) &= \lim_{h \to 0} \frac{\mathbb{E}[\log(1 + g(\lambda + h))] - \mathbb{E}[\log(1 + g\lambda)]}{h} \\
&= \lim_{h \to 0} \mathbb{E}\Big[\frac{\log(1 + g(\lambda + h) - \log(1 + g\lambda)}{h}\Big] \\
&= \mathbb{E}\Big[\lim_{h \to 0} \frac{\log(1 + g(\lambda + h) - \log(1 + g\lambda)}{h}\Big] \\
&= \mathbb{E}\Big[\frac{g}{1 + g\lambda}\Big]
\end{aligned}$$

Now we can easily determine that

$$\phi_g'(0) = \mathbb{E}\big[\frac{g}{1 + g.0}\big] = \mathbb{E}[g]$$

Finally, we prove the limit by interchanging the order of the expectation and of the limit once more. We then split the sum implied by the expectation in two parts, one with limit to minus infinity, following the limit to minus infinity of the logarithm towards zero, and the other a sum of finite limits, before adding the limits back together:

$$
\begin{aligned}
\lim_{\lambda \to \frac{1}{L}} \phi_g(\lambda) &= \lim_{\lambda \to \frac{1}{L}} \mathbb{E}[\log(1+g\lambda)] \\
&= \mathbb{E}[\lim_{\lambda \to \frac{1}{L}} \log(1+g\lambda)] \\
&= \sum_{i=1}^{n} p_i \lim_{\lambda \to \frac{1}{L}} \log(1+x_i\lambda) \\
&= p_L \lim_{\lambda \to \frac{1}{L}} \log(1-L\lambda) + \sum_{\substack{i=1 \\ x_i > -L}}^{n} p_i \lim_{\lambda \to \frac{1}{L}} \log(1+x_i\lambda) \\
&= p_L \lim_{\lambda \to \frac{1}{L}} \log(1-L\lambda) + \sum_{\substack{i=1 \\ x_i > -L}}^{n} k_i \\
&= -\infty
\end{aligned}
$$

where $k_i \in \mathbb{R}$ for every i and p_L denotes the probability which corresponds to the maximum loss. □

Lemma 3.2.4. *For every gamble g for which $\mathbb{E}[g] > 0$, the function $\phi_g : \left[0, \frac{1}{L}\right) \longrightarrow \mathbb{R}$ with $\phi_g(\lambda) = \mathbb{E}[\log(1+g\lambda)]$ has exactly one zero λ_0 on $\left(0, \frac{1}{L}\right)$. For every gamble g for which $\mathbb{E}[g] \leq 0$, the function ϕ_g has no zeros on $\left(0, \frac{1}{L}\right)$.*

Proof. Where the expectation is positive, we know from lemma 3.2.3 that $\phi_g(0) = 0$ and $\phi_g'(0) > 0$. Combining these two pieces of information enables us to determine that there exists λ_1 such that $|\lambda_1 - 0| < \epsilon_1$ for any $\epsilon_1 > 0$ and that $\phi_g(\epsilon_1) > 0$. Similarly, lemma 3.2.3 tells us that ϕ_g has a negative limit towards $\frac{1}{L}$, so there exists a λ_2 such that $|\frac{1}{L} - \lambda_2| < \epsilon_2$ for any $\epsilon_2 > 0$ and that $\phi_g(\lambda_2) < 0$.

Thanks to the smoothness and therefore the continuity of ϕ_g, given by lemma 3.2.2, we can now apply the Intermediate Value Theorem to function ϕ_g on interval $[\epsilon_1, \epsilon_2]$ to show that there exists at least one zero for ϕ_g on the given interval. Taking the limits as $\epsilon_1 \to 0$ and $\epsilon_2 \to \frac{1}{L}$ yields the result that there exists at least one zero for ϕ_g on its domain. Finally, the concavity of ϕ_g, demonstrated in proposition 3.2.1, indicates that the function can have at the most one zero on the given interval. Combining these two results proves the existence and uniqueness of the zero, for gambles with positive expectation.

Where the expectation is non-positive, a similar reasoning proves the absence of zeros on the interval. □

Lemma 3.2.5. *For every gamble g for which $\mathbb{E}[g] > 0$, the function ψ_g : $(L, \infty) \longrightarrow \mathbb{R}$ with $\psi_g(r) = \mathbb{E}[\log(1 + \frac{g}{r})]$ has exactly one zero R on (L, ∞). For gambles g for which $\mathbb{E}[g] \leq 0$, the function ψ_g has no zeros on (L, ∞).*

Proof. From lemma 3.2.4 we know that the function ϕ_g has a unique zero λ_0 if the expectation of g is positive. If we now set $\lambda = \frac{1}{r}$, then $\phi_g(\lambda) = \mathbb{E}[\log(1+\frac{g}{r})] = \psi_g(r)$ has a zero in $\frac{1}{\lambda_0}$ which we define as R. A similar reasoning proves that there are no zeros when the expectation is non-positive. □

This concludes the proof of the part of theorem 3.1.1 which states that equation (5) uniquely determines the critical wealth R.

3.3 Properties of the critical wealth

We will now present useful properties of the critical wealth $R(g)$, starting with the shape of the ϕ_g and ψ_g functions on their domains, in cases where the expectation of g is positive.

Lemma 3.3.1. *For every gamble g for which $\mathbb{E}[g] > 0$, the function ϕ_g : $\left[0, \frac{1}{L}\right) \longrightarrow \mathbb{R}$ with $\phi_g(\lambda) = \mathbb{E}[\log(1 + g\lambda)]$ has the following properties:*

1. $\phi_g(\lambda) > 0$ *for* $0 < \lambda < \lambda_0$
2. $\phi_g(\lambda) < 0$ *for* $\lambda_0 < \lambda < \frac{1}{L}$

where λ_0 is the function's unique zero on interval $[0, 1/L)$.

Proof. From lemma 3.2.3 we know that $\phi_g(0) = 0$ and $\phi_g'(0) = \mathbb{E}[g] > 0$, so the function has positive values starting after $\lambda = 0$). From lemma 3.2.4 we also know that there is but one zero on the interval $(0, \frac{1}{L})$, so ϕ is positive on $(0, \lambda_0)$ and changes sign once at the most. Since lemma 3.2.3 also indicates a negative limit towards $\frac{1}{L}$, we can now conclude that the function indeed changes signs once and becomes negative on the interval $(\lambda_0, \frac{1}{L})$. □

This first result can be used directly to derive the following lemma for the function ψ_g.

Lemma 3.3.2. *For every gamble g for which $\mathbb{E}[g] > 0$, the function ψ_g : $(L, \infty) \longrightarrow \mathbb{R}$ with $\psi_g(r) = \mathbb{E}[\log(1 + \frac{g}{r})]$ has the following properties:*

1. $\psi_g(r) < 0$ *for* $L < r < R$
2. $\psi_g(r) > 0$ *for* $R < r$

where R is the function's unique zero on interval (L, ∞).

Proof. From lemma 3.3.1 we know that the function ϕ_g is positive on the interval $(0, \lambda_0)$ and negative on the interval $(\lambda_0, 1/L)$. If we now set $\lambda = \frac{1}{r}$ and $\lambda_0 = \frac{1}{R}$, then $\phi_g(\lambda) = \mathbb{E}[\log(1 + \frac{g}{r})] = \psi_g(r)$ has the properties that we wish to prove. □

Following the result from lemma 3.3.2, we are left to ask ourselves what shape the function ψ_g has when the expectation of the gamble is non-positive, knowing from lemma 3.2.5 that it has no zeros. More generally, we will research the effect on the critical wealth $R(g)$ of a variation in the expectation $\mathbb{E}[g]$ of the gamble, all else being equal.

We have plotted in figure 10 several graphs of the function ψ_g for the bond from example 2.1.3. These differ only by their expectation, obtained by varying the probability of the maximum loss occurring and adjusting the probability of the maximum gain accordingly. We observe that the higher the expectation, the closer critical wealth $R(g)$ becomes to the maximum loss $L(g) = 100$. On the other hand, the closer the critical wealth comes towards the maximum loss, the less beneficial effect a further gain in expectation has on lowering the critical wealth, as it obviously cannot drop below $L(g)$ according to lemma 3.2.5.

We further notice that the bottom graph, corresponding to a probability of 0.55 for the maximum loss and a negative expectation, does not have a zero. In this situation, by convention we will set the critical wealth $R(g) = \infty$, in accordance to the asymptotic behaviour of $\psi_g(r)$ when $r \longrightarrow \infty$.

We have also plotted graphs of the function ψ_g, for the bond from example 2.1.3, to isolate the effect on the critical wealth of varying the value of $L(g)$ while keeping the value of the maximum gain and of the expectation constant. To achieve this result, we adjust the probability p_1 along with the value of $x_1 = -L$. These graphs can be found in figure 11. The higher the maximum loss, the higher the critical wealth needed to manage it, which always includes a substantial buffer $R - L$.

In the particular situation where the maximum loss is not in fact a loss ($L = 0$), which is represented by the top graph, there is no zero. By convention, we will set the critical wealth $R(g) = 0$, in analogy to the behaviour of the graph for $L = 6$ and in general for graphs of values of L close to 0. These graphs appear to go up towards ∞ before turning abruptly towards $-\infty$, crossing the r-axis very close to $r = 0$.

The conventions of setting $R = \infty$ for gambles with non-positive expectation and $R = 0$ for gambles without negative outcomes allows us to integrate rules (2), (3) and (4) into the general framework of Foster and Hart's Critical Wealth Theorem. Consequently, a gamble with negative expectation will be associated with an infinite critical wealth, such that no gambler will possess

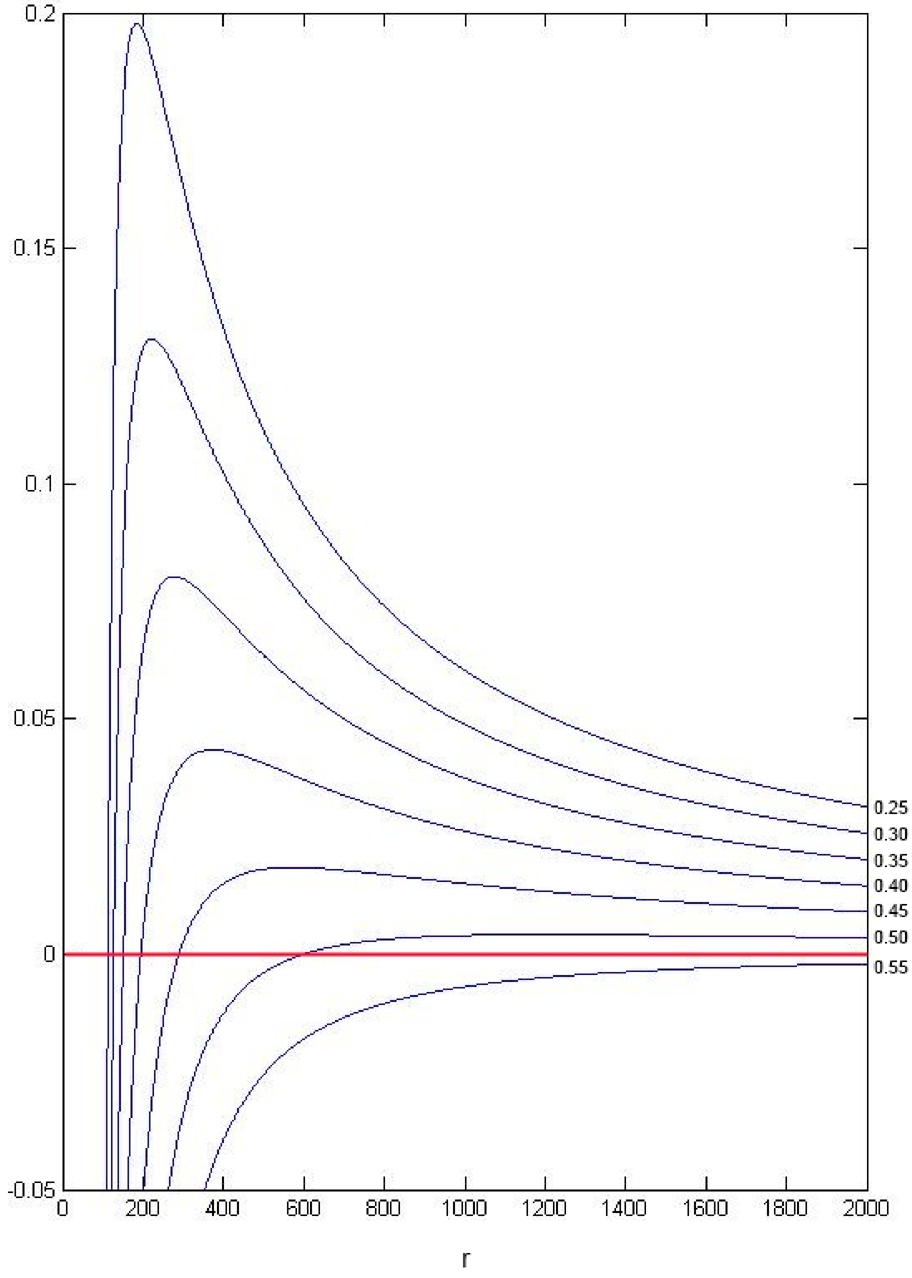

Figure 10: Graph of functions $\psi_g(r)$ according to values of p_1

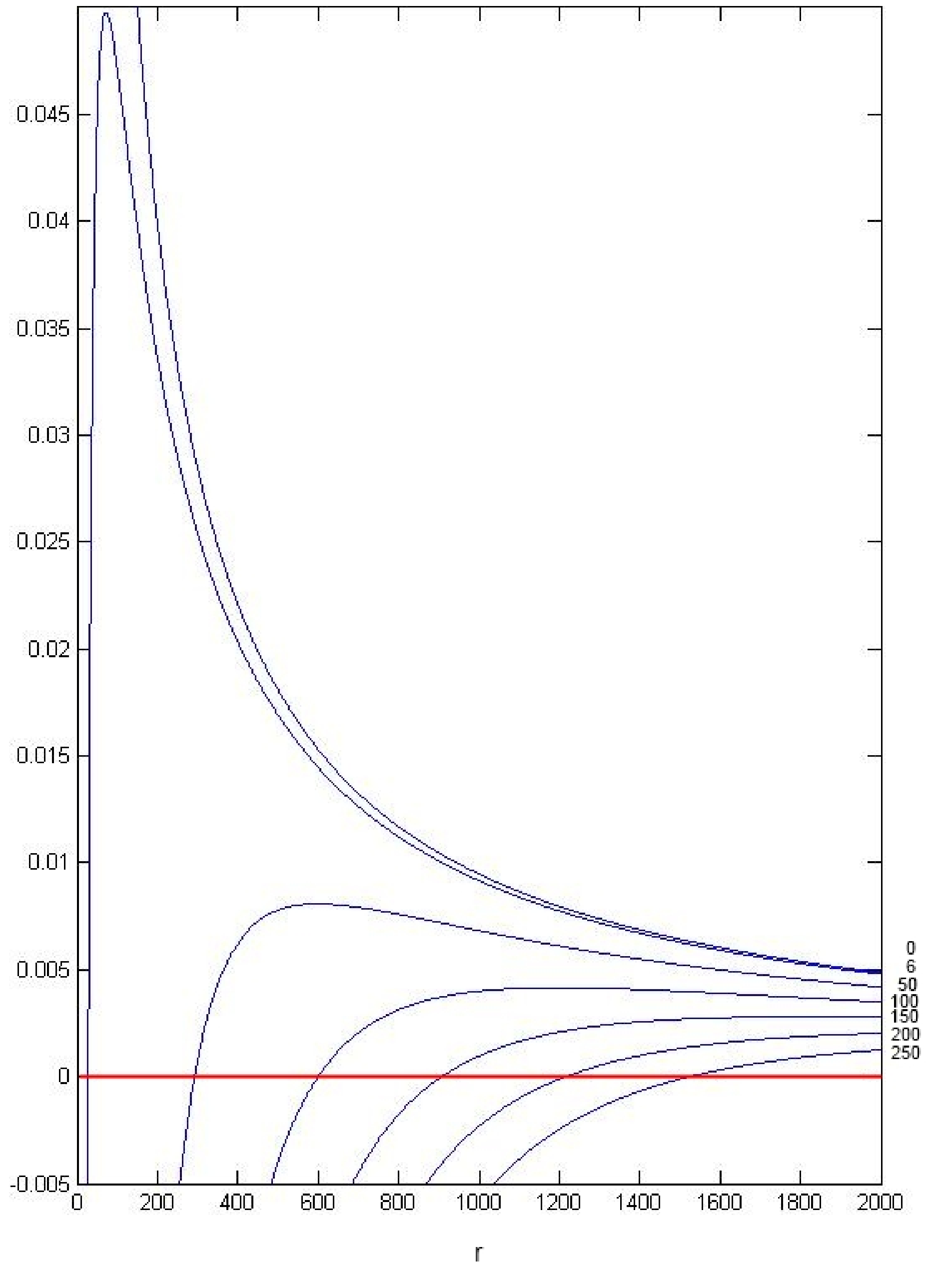

Figure 11: Graph of functions $\psi_g(r)$ according to values of L

enough wealth to accept the gamble, which will always be rejected, as was mandated by rule (3). Similarly, a gamble without negative outcomes has a critical wealth that equals zero, such that this gamble can always be accepted by all gamblers, even bankrupt ones, in accordance with rule (2). We have already established that the critical wealth of any gamble cannot equal or be lower than the maximum loss of the gamble, so that gambles can never be accepted by gamblers with a wealth equal to or lower than this maximum loss, as required by rule (4).

We conclude this section with the homogeneity property of the critical wealth.

Lemma 3.3.3. *For every gamble g, the critical wealth $R(g)$ is homogeneous, i.e. $R(kg) = kR(g)$.*

Proof. The critical wealth $R(g)$ of gamble g is uniquely determined by the equation $\mathbb{E}\Big[\log\Big(1+\frac{g}{R(g)}\Big)\Big] = 0$. We can therefore rewrite this equation as $\mathbb{E}\Big[\log\Big(1+\frac{kg}{kR(g)}\Big)\Big] = 0$. The latter expression can be seen as the equation determining the critical wealth of gamble kg to be $kR(g)$. □

The latter result will be used later in our proof for theorem 3.1.1, but it also has a direct financial interpretation. Homogeneity means that if the gamble is scaled then the critical wealth is scaled by the same amount. It feels appropriate that an asset should be as risky in dollars as it would be if first converted in euros, barring the exchange rate riskiness itself which really is an added gamble.

4 Proof of the Critical Wealth Theorem

4.1 Conditional expectations

Before we return to giving a proof of theorem 3.1.1, we will make a short aside to explore the concepts of conditional expectation and of martingales, which will be essential in understanding and developing the proof.

Let X and Y be two random variables with probability distributions given respectively by $\{x_1, \ldots, x_n\}$, $\{p(X = x_1), \ldots, p(X = x_n)\}$ and $\{y_1, \ldots, y_m\}$, $\{p(Y = y_1), \ldots, p(Y = y_m)\}$. For such variables, we recall that the conditional probability of an outcome x_i of X given that an outcome y_j of Y is known to have occurred is given by

$$p(X = x_i | Y = y_j) = \frac{p((X = x_i) \cap (Y = y_j))}{p(Y = y_j)}$$

where $p((X = x_i) \cap (Y = y_j))$ is the probability of both outcomes occurring simultaneously. We are now able to define conditional expectations as follows.

Definition 4.1.1. *The conditional expectation $\mathbb{E}[X|Y = y_j]$ of X knowing that the outcome $Y = y_j$ has occurred for Y is defined as the number*

$$\mathbb{E}[X|Y = y_j] = \sum_{i=1}^{n} x_i \; p(X = x_i | Y = y_j)$$

We note that in case X and Y are independent random variables, then $p((X = x_i) \cap (Y = y_j)) = p(X = x_i)p(Y = y_j)$ and so $p(X = x_i|Y = y_j) = p(X = x_i)$ and $\mathbb{E}[X|Y = y_j] = \mathbb{E}[X]$. We can also go a step further in defining conditional expectations:

Definition 4.1.2. *The conditional expectation $\mathbb{E}[X|Y]$ of X knowing the preceding random variable Y, is itself a random variable. It denotes a function $f(Y)$ of Y which attains the value $\mathbb{E}[X|Y = y_j]$ for every $Y = y_j$. In other words, its probability distribution is characterized by outcome set $\{\mathbb{E}[X|Y = y_1], \ldots, \mathbb{E}[X|Y = y_m]\}$ and probabilities $\{p(Y = y_1), \ldots, p(Y = y_m)\}$.*

The following example illustrates the concepts and calculations involved.

Example 4.1.3. *Take two random variables X and Y, where Y precedes X. Their probability distributions are respectively given by $x_1 = 0, x_2 = 2$ and $p(X = x_1) = 0.6, p(X = x_2) = 0.4$ for X and by $y_1 = -5, y_2 = 0, y_3 = 5$ and $p(Y = y_1) = 0.2, p(Y = y_2) = 0.5, p(Y = y_3) = 0.3$ for Y. The detail of their combined probabilities is shown in the table from figure 12. Another way to presenting this information is by drawing the diagram from figure 13.*

$x_i \setminus y_j$	-5	0	5	p_i
0	0.1	0.4	0.1	0.6
2	0.1	0.1	0.2	0.4
p_j	0.2	0.5	0.3	1

Figure 12: Combined probability distribution of X and Y

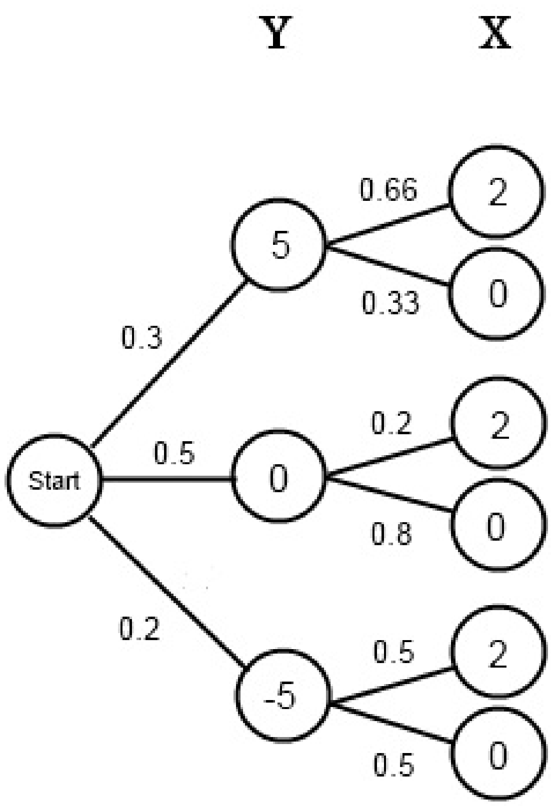

Figure 13: Probability tree for X and Y

We compute, for example, the conditional probability $p(X = 2|Y = 5) = 0.2/0.3 = 2/3 = 0.66$*, as reported directly in figure 13. Next we compute the following conditional expectations:*

$$\mathbb{E}(X|Y = -5) = 2 \cdot 0.5 + 0 \cdot 0.5 = 1$$
$$\mathbb{E}(X|Y = 0) = 2 \cdot 0.2 + 0 \cdot 0.8 = 0.4$$
$$\mathbb{E}(X|Y = 5) = 2 \cdot 0.66 + 0 \cdot 0.33 = 1.33$$

Using these latter results, we are able to construct the probability distribution for the random variable $\mathbb{E}[X|Y]$*, summarized in figure 14. Visually, we can construct this latter diagram by reproducing the part of figure 13 representing the random variable* Y*, on which the conditional variable is conditioned, and then replacing its outcomes with the corresponding conditional expectation of* X*.*

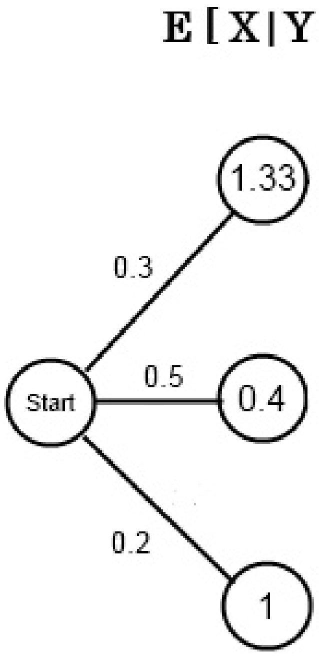

Figure 14: Probability tree for $\mathbb{E}[X|Y]$

Definition 4.1.2 of conditional expectation can also be applied to $\mathbb{E}[X|Y, Z]$ or even situations where the expectation of X is conditioned on more than two random variables. In such a case, we just have to remember that we can build a new random variable A which has outcomes $(Y = y_j, Z = z_k)$ with $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, l$. Then we can define $\mathbb{E}[X|Y, Z] = \mathbb{E}[X|A]$.

4.2 Martingales

The concept of conditional expectation will now be used to define a special type of random processes called martingales and give some of their most important properties.

Definition 4.2.1. *A martingale is a random process $(M_t)_{t=1,2,...}$ for which each M_t has conditional expectation $\mathbb{E}[M_{t+1}|M_1, M_2, \ldots, M_t] = M_t$.*

Such an object can be seen as a fair random process. Indeed, we recall the fact that $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$ and apply it to the defining property of martingales, restated as $\mathbb{E}[M_{t+1} - M_t|M_1, \ldots, M_t] = 0$. This leads to the following expression $\mathbb{E}[\mathbb{E}[M_{t+1} - M_t|M_1, \ldots, M_t]] = \mathbb{E}[M_{t+1} - M_t] = 0$. Another way to express this property of martingales is to say that $\mathbb{E}[M_{t+1}] = \mathbb{E}[M_t] = \cdots = \mathbb{E}[M_1] = M_1$.

This property enables seemingly complex problems to be solved simply if a martingale process can be recognized, as can be seen in the following example.

Example 4.2.2. *Consider a casino player starting with $M_1 = 100\$$. He participates in a series of gambles X_t which make the player either lose or add half of the money he currently possesses. We define the random process $(M_t)_{t=1,2,...}$, representing the amount of money owned by the gambler at time t, with initial wealth M_1 and recurrence formula $M_{t+1} = M_t \cdot X_t$, where X_t has outcomes 0.5 and 1.5, each with probability 0.5. We are now interested in knowing how much money the gambler is expected to end up with if he gambles 1000 times.*

One way to look at this problem is to draw a diagram recording all possible outcomes and their respective probabilities. When confronted with large numbers of gambles, it is clear that drawing such a graph is impractical. A faster and more elegant way to solve this problem is to notice that for all t, the gambles have expectation $\mathbb{E}[X_t] = 1$, so that $\mathbb{E}[M_{t+1}|M_1, M_2, \ldots, M_t] = M_t$ and therefore (M_t) is a martingale. This characteristic enables us to conclude that $\mathbb{E}[M_{1000}] = \mathbb{E}[M_1] = 100$. The player can expect to come out of the casino as rich as when he was entering it.

There is a specific type of martingale called the martingale with bounded increments.

Definition 4.2.3. *A random process for which there exists a finite K such that $|M_{t+1} - M_t| \leq K$ for all $t \geq 1$ is called a process with bounded increments.*

In practice, all martingales encountered in this report and in most financial problems are of the bounded increments type, so this requirement represents no big hurdle for what follows.

We can now present an important result for the convergence of martingales, as stated in the following theorem.

Theorem 4.2.4 (Convergence Theorem)**.** *If M_t is a martingale with bounded increments, then either of the following situations is true:*

1. $\lim_{t\to\infty} M_t$ *exists*

2. $\liminf_{t\to\infty} M_t = -\infty$ *and* $\limsup_{t\to\infty} M_t = \infty$

The formal proof of this theorem lies beyond the scope of this report, but the intuitive idea is illustrated by the drawing in figure 15. Let X_t be a martingale representing winnings per gamble and Y_t be the cumulated winnings. The strategy used here is to start accepting gambles whenever the outcome of X_t has been under a predetermined level a and to stop accepting them when it is above a predetermined b. Black circles in the drawing stand for the result of accepted gambles.

In this situation, the number $U_N[a,b]$ of upcrossings of interval $[a,b]$ made by X_t by time N is the number of times that the outcome of X_t has gone from under a to above b. In figure 15, there have been two such upcrossings. This number enables us to give an approximation of the cumulative winnings Y_N at time N, since Y_N is at least the product of the interval width by the number of upcrossings of that interval by definition, minus the maximum possible loss of the last running sequence of accepted gambles for which X_N has not yet reached b:

$$Y_N \geq (b-a)U_N[a,b] - [X_N - a]^- \tag{9}$$

where $[X_N - a]^-$ equals zero if its argument is non-negative and equals minus the argument if it is negative. Since X is a martingale, we can expect the cumulative winnings $\mathbb{E}[Y_N]$ to equal zero. Using this fact and taking the expectation of equation (9), we can establish the result of Doob's Upcrossing Lemma:

$$(b-a)\mathbb{E}[U_N[a,b]] \leq \mathbb{E}[[X_N - a]^-] \tag{10}$$

Assuming now that the increment X_N is bounded for every N then, as N goes to infinity, we can conclude that the expected number of upcrossings is smaller than a bounded number and thus is finite. In other words, the probability that the number of upcrossings becomes infinite as N tends to infinity is zero. This provides us with the result needed to prove the theorem as, if the number of upcrossings of X_t is finite, we know for sure that $\lim_{t\to\infty} X_t \neq \infty$, which leaves only the options of the limit for the martingale converging to a number or oscillating between ∞ and $-\infty$.

We will use this result in the next section to prove the guaranteed no-bankruptcy offered by the strategy presented in Critical Wealth Theorem.

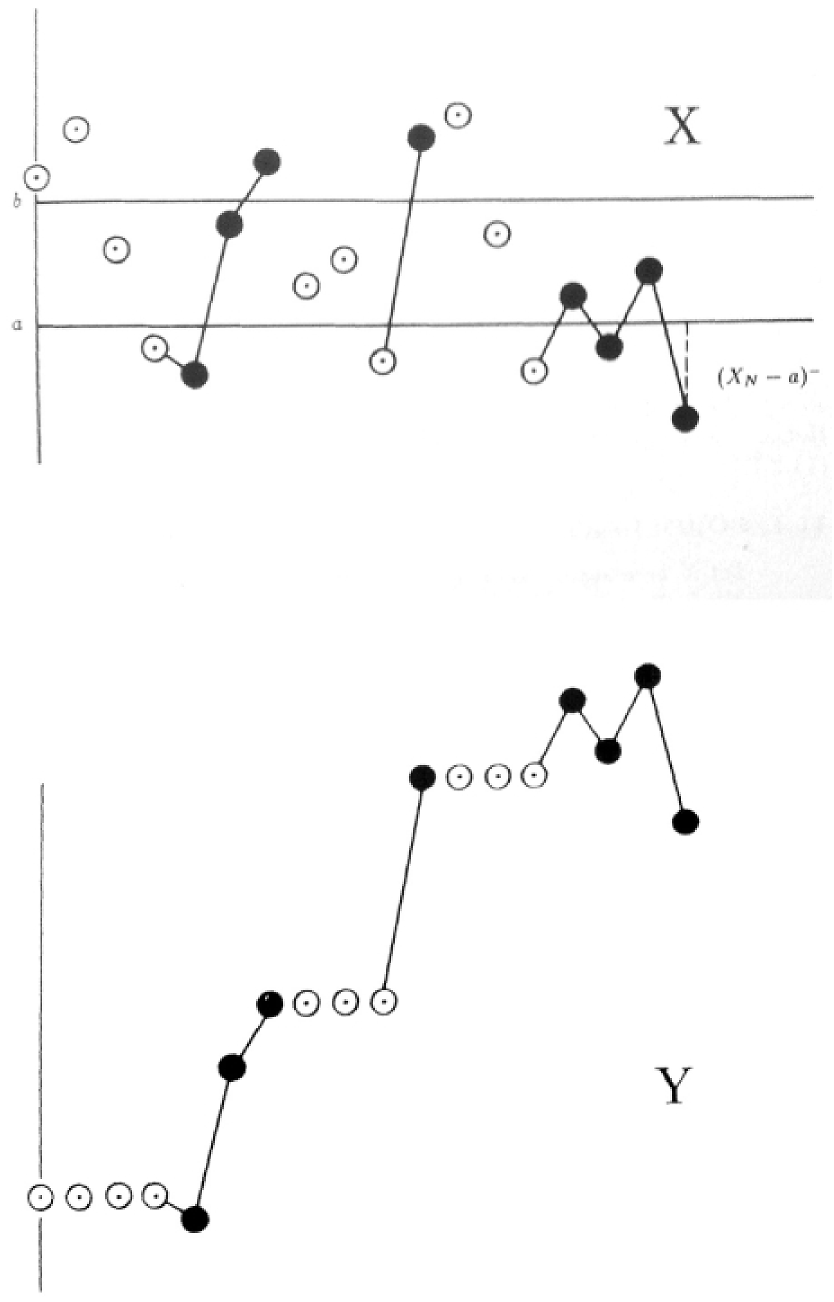

Figure 15: Illustration for upcrossing lemma

4.3 Guaranteed no-bankruptcy

The following lemmas combined form the proof for theorem 3.1.1. We prove first that the strategy described in the theorem does in fact guarantee no-bankruptcy and then that such a strategy is the simple strategy with the lowest critical wealth that guarantees no-bankruptcy.

Lemma 4.3.1. *If for every gamble $g_t \in \mathcal{G}_0$ a strategy s rejects g_t when $W_t < R(g)$, then the growth factors Y_t is well defined for all $t = 1, 2, \ldots$ and $\mathbb{E}[Y_t|Y_1, \ldots, Y_{t-1}] \geq 0$ for all $t = 1, 2, \ldots$.*

Proof. We start by recalling that at $t = 1$ it is assumed that $W_1 > 0$, or else the gambler would be bankrupt to start with. If the gamble is rejected, then $W_{t+1} = W_t$. If the gamble is accepted, according to strategy s, it means that $W_t \geq R(g_t) > L$ and thus $W_{t+1} = W_t + g_t \geq R(g_t) + g_t \geq R(g_t) - L > 0$. By induction we can conclude that $W_t > 0$ for all t, whether the gamble is accepted or rejected, and therefore growth factor Y_t is well defined for all t.

If gamble g_t is rejected, then $Y_t = \log W_{t+1} - \log W_t = \log W_t - \log W_t = 0$ and so $\mathbb{E}[Y_t|Y_1, \ldots, Y_{t-1}] = 0$. If the gamble is accepted, we have $W_t \geq R(g_t)$ and according to lemma 3.2.5 we must have $\mathbb{E}\Big[\log\Big(1+\frac{g_t}{W_t}\Big)|Y_1, \ldots, Y_{t-1}\Big] \geq 0$. Moreover $Y_t = \log(W_t + g_t) - \log(W_t) = \log(1 + \frac{g_t}{W_t})$ and so combining these two expressions leads us to the conclusion that $\mathbb{E}[Y_t|Y_1, \ldots, Y_{t-1}] \geq 0$. □

Our next step is to construct a new variable X_T, related to the growth factors Y_t, in a way that makes it a martingale with bounded increments. This method is an example of a more general construction called Doob's decomposition.

Definition 4.3.2. *The random process $(X_T)_{T=1,2,\ldots}$ is defined by the expression $X_T = \sum_{t=1}^{T}(Y_t - \mathbb{E}[Y_t|Y_1, \ldots, Y_{t-1}])$.*

Lemma 4.3.3. *The random process $(X_T)_{T=1,2,\ldots}$ is a martingale with bounded increments.*

Proof. First we prove that the random process is a martingale by showing that $\mathbb{E}[X_{T+1}|X_1, \ldots, X_T] = X_T$.

$$\begin{aligned}
\mathbb{E}[X_{T+1}|X_1, \ldots, X_T] &= \mathbb{E}[X_T + Y_{T+1} - \mathbb{E}[Y_{T+1}|Y_1, \ldots, Y_T]|X_1, \ldots, X_T] \\
&= X_T + \mathbb{E}[Y_{T+1}|Y_1, \ldots, Y_T] - \mathbb{E}[Y_{T+1}|Y_1, \ldots, Y_T] \\
&= X_T
\end{aligned}$$

Next we decompose $|X_{T+1} - X_T|$ into its constituting elements and use the fact that $\mathbb{E}[Y_t|Y_1, \ldots, Y_{t-1}] \geq 0$ for all t and that $|Y_t| < K$ for every t, according to proposition 2.2.5, to prove that the sum of those elements is bounded by a finite K.

$$\begin{aligned}
|X_{T+1} - X_T| &= |\sum_{t=1}^{T+1}(Y_t - \mathbb{E}[Y_t|Y_1, \ldots, Y_{t-1}]) - \sum_{t=1}^{T}(Y_t - \mathbb{E}[Y_t|Y_1, \ldots, Y_{t-1}])| \\
&= |Y_{T+1} - \mathbb{E}[Y_{T+1}|Y_1, \ldots, Y_{T-1}]| \\
&\leq |Y_{T+1}| \\
&\leq K
\end{aligned}$$

□

We can now prove the part of theorem 3.1.1 that says that all strategies that reject gambles whenever the wealth lies under critical wealth $R(g)$ are strategies that guarantee no-bankruptcy.

Lemma 4.3.4. *If for every gamble $g_t \in \mathcal{G}_0$ a strategy s rejects g_t when $W_t < R(g)$, then the strategy guarantees no-bankruptcy.*

Proof. We know from lemma 4.3.1 that $\mathbb{E}[Y_t|Y_1, \ldots, Y_{t-1}] \geq 0$ for all t. Therefore, we can approximate martingale X_T as follows:

$$\begin{aligned}
X_T &= \sum_{t=1}^{T}(Y_t - \mathbb{E}[Y_t|Y_1, \ldots, Y_{t-1}]) \\
&\leq \sum_{t=1}^{T} Y_t = \sum_{t=1}^{T}(\log W_{t+1} - \log W_t) = \log W_{T+1} - \log W_1 \\
&\leq \log W_{T+1}
\end{aligned}$$

For bankruptcy to occur would require that $\lim_{T\to\infty} W_T = 0$ with positive probability, according to the definition, or $\lim_{T\to\infty} \log W_T = -\infty$. Therefore it would require that $\lim_{T\to\infty} X_T = -\infty$ with positive probability. But according to theorem 4.2.4, the martingale X_T must either have a finite limit or must oscillate between arbitrarily high and low values, none of these two cases allowing for the martingale to have the limit required for bankruptcy to occur. Consequently, bankruptcy occurs with zero probability. □

The reciprocal is also true, that only such strategies that reject gambles when the wealth lies under $R(g)$ guarantee no-bankruptcy, as any strategy that doesn't opens the possibility of bankruptcy.

Lemma 4.3.5. *If strategy s_Q is a simple strategy for which $Q(h) < R(h)$ for some gamble $h \in \mathcal{G}_0$, then there exists a sequence $(g_t)_{t=1,2,\ldots}$ such that* $\lim_{t\to\infty} W_t = 0$*. Moreover, all the gambles g_t that make up that sequence are multiples of gamble h.*

Proof. In case $Q(h) < L(h)$ it is easy to see that bankruptcy can occur by accepting h. Now we will further assume that $R(h) > Q(h) > L(h)$ and so by lemma 3.2.5 we have $\mathbb{E}[1 + h/Q(h)] < 0$. We can now construct a sequence of gambles $(g_t)_{t=1,2,\ldots}$ with the properties that we seek. Take a sequence of independent and identically distributed gambles with the same distribution as h and the added characteristic that each $g_t = \frac{W_t}{Q(h)}h$, to make each the biggest acceptable gamble possible given the wealth at that moment. If these conditions are met then, using the homogeneity of the critical wealth, we have $Q(g_t) = Q(\frac{W_t}{Q(h)}h) = \frac{W_t}{Q(h)}Q(h) = W_t$ which means that gamble g_t is accepted at wealth W_t and therefore that growth factor $Y_t = \log(1+g_t/W_t) = \log(1 + h/Q(h))$.

Since all g_t are i.i.d there follows that (Y_t) also is an i.i.d sequence on which the Strong Law of Large Numbers can be applied resulting in

$$\lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} Y_t = \mathbb{E}[Y_t] = \mathbb{E}\Big[\log\Big(1 + g_t/W_t\Big)\Big] = \mathbb{E}\Big[\log\Big(1 + h/Q(h)\Big)\Big] < 0$$

$$\begin{aligned}
&\iff \lim_{T\to\infty} \frac{1}{T}(\log W_{T+1} - \log W_1) < 0 \\
&\iff \lim_{T\to\infty} \log W_{T+1} = -\infty \\
&\iff \lim_{T\to\infty} W_{T+1} = 0 \\
&\iff \lim_{T\to\infty} W_T = 0
\end{aligned}$$

which means bankruptcy. □

5 Conclusion

We have seen that there exists an objective measure of the riskiness of a financial asset. This measure has a clear operational interpretation as the minimum wealth that an investor must hold prior to purchasing the asset if he wishes to be guaranteed non-bankruptcy in the long term while following this strategy. We have proved and illustrated that purchasing an asset with any wealth level under the asset's critical wealth may lead to bankruptcy. On the other hand, an investor whose wealth exceeds the critical wealth of the asset can safely buy it in the knowledge that he is guaranteed non-bankruptcy.

The model that has been developed helps us understand the mechanisms of risk but it remains a model that cannot be applied to the real world without refinements and careful interpretation. In the real world, inflation, time discounting or even the mortality of man mean that the time horizon is never in fact infinite. Also, the existence of debt and of corporate and bankruptcy legislation tends to lead to lowering the risk, for the investor, of assets below the riskiness defined by our model, as they remove the strict limits of wealth reaching zero.

Finally, one important limitation to our model is that the probability distribution of the assets is almost never known with certainty, especially the tail probabilities. Current practices in the financial industry are to make precise estimates of the probability distributions of non-tail outcomes and general estimates of the tail probability that some game-changing events occur. All non-tail risk is thereafter managed by portfolio diversification while the Basel II Accords require all banks to keep a buffer of unused capital to survive the unexpected tail hardships. The current financial crisis is proof that these practices are insufficient to prevent widespread bankruptcies. According to investor Nassim Taleb, the fault lies in the fact that humans have a psychological tendency to undervalue the risk of rare events. In any case, the crisis underlies the fact that mainstream economists have insufficient understanding of their subject, as the rare events come as complete surprises.

A Appendix: Simulation programs

For our simulations, we have developed Matlab programs. The first program generates gamble outcomes for a given gamble with two possible outcomes. It also outputs the minimum outcome and the value of $R(g)$:

```
function G = Gamble

% compute outcome of gamble

U = unifrnd(0,1);

if U <= 0.5
    g = -100;
else
    g = 120;
end

% compute extra characteristics

L = 100;
R = 600;

% put together output vector

G(1) = g;
G(2) = L;
G(3) = R;
```

Our second program was written to determine the development of the gambler's wealth in time, using the output from the Gamble function and following a simple strategy with critical wealth to be defined:

```
function [W_T]= Simulation(W_1,T)
% input initial wealth W_1
% input number of time periods T

% outputs final wealth W_T
% displays min W, average W
% displays graph W against t

% creating wealth matrix

W = zeros(1,T);
W(1) = W_1;

for i = 2:T;
    k = unifrnd(0,1); % random scaling factor
    Ga = k*Gamble;    % gamble outcome
    Q = Ga(3);        % critical wealth for simple strategy
                      %(Ga(2) = L, Ga(3) = R, 0 = accept all)

    if W(i-1) <= Q
        W(i) = W(i-1);
    else
        W(i ) = W(i-1) + Ga(1);
        if W(i) <= 0
            W(i) = 0;
            break
        end
    end
end

plot(W)
W_T = W(T)

return
```

The last program is used to run a simulation and requires the input of the initial wealth, the number of periods of time and the number of runs:

```
% requests input and feeds them to the relevant
functions to create a simulation run

clc
clear

% requests input

    W_1 = 600; % initial wealth
    T = 20000;   % number of time periods
    X = 100;   % number of runs

% initialize final wealth vector

W = zeros(1,X);

% calls Simulation function and feeds it the input

for i = 1:X;
    W(i) = Simulation(W_1, T);
end

% graphing the histogram of final wealths

if X ~= 1;
    hist(W, 20)
    min = min(W)
end
```

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