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ON WEAK* CONVERGENT SEQUENCES IN DUALS OF SYMMETRIC SPACES OF τ-MEASURABLE OPERATORS*

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ABSTRACT

It is shown that the pre-dual of a σ-finite von Neumann algebra has property (k) in the sense of Figiel, Johnson and Pelczyński [12]. This resolves in the affirmative an open question raised in [12]. It is shown further that a weakly sequentially complete symmetric space E of τ-measurable operators affiliated with a semifinite σ-finite von Neumann algebra has property (k).

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1. Introduction

The present paper finds its origins in the work of Figiel, Johnson and Pelczyński [12] which isolates a certain Banach space invariant which they call "property (k)" and which is a weakening of a stronger invariant, called "property (K)", introduced earlier and studied by Kalton and Pelczyński [16]. Precise definitions are given below.

It is shown in [12] that each separable subspace of the pre-dual of a von Neumann algebra has property (k) and that each weakly sequentially complete Banach lattice with a weak unit has property (k). A principal result of this paper (Theorem 4.4) is that the pre-dual of each σ-finite von Neumann algebra has property (K) and therefore has the weaker property (k). This resolves in the affirmative an open question raised in [12] Problem 6.6. Our approach here is to show that the Mackey topology on a σ-finite von Neumann algebra with respect to its pre-dual is metrizable on norm bounded sets. This, in turn, rests on combining the classical criterion of Akemann [26] that each weakly relatively compact subset of the pre-dual of a von Neumann algebra is of uniformly absolutely continuous norm (equi-integrable) with a characterisation of such sets in the pre-dual of a σ-finite von Neumann algebra, due to Raynaud and Xu [24].

We show further (Theorem 8.6) that if $(\mathcal{M}, \tau)$ is a semi-finite and σ-finite von Neumann algebra, then each weakly sequentially complete symmetric space of τ-measurable operators has property (k), and this is an exact non-commutative counterpart to the corresponding Banach lattice result in [12] referred to above.

Our approach to this latter result is based on the introduction and study, itself of independent interest, of natural variants of property (k) and property (K). Indeed, suppose that X is a Banach space and let $\mathfrak{S}$ be a collection of bounded subsets of X. The Banach space X will be said to have property $(K_{\mathfrak{S}})$ if every weak*-null sequence $\{x_n^*\}$ in X^* has a sequence of consecutive convex combinations which converges to 0 uniformly on each member of $\mathfrak{S}$. If $\mathfrak{S}$ is the class of all relatively weakly compact sets of X, then property $K_{\mathfrak{S}}$ coincides with property (K). On the other hand, if $\mathfrak{S}$ is the class of all subsets of X which are the continuous linear image of some relatively weakly compact subset of $L_1(0,1)$, then property $K_{\mathfrak{S}}$ coincides with property (k). The classes of sets which play a central role in establishing Theorem 8.6 are the class $\mathfrak{S}_{ob}$ of order-bounded subsets and the class $\mathfrak{S}_{an}$ (see Section 7) of bounded subsets of uniformly absolutely continuous norm in a symmetric space E of τ-measurable operators affiliated with a semifinite von Neumann algebra

A key step in the proof of Theorem 8.6 is that the topology of uniform convergence on the class $\mathfrak{S}_{an}$ is metrizable on norm bounded subsets of the Banach dual E^* in the case that E has order continuous norm and the underlying von Neumann algebra is σ-finite (Theorem 7.1). This result is based on a recent characterisation [7] of sets of uniformly absolutely continuous norm (or E-equi-integrable sets) in symmetric spaces E of τ-measurable operators, in terms of sets which are, in some sense, "almost" order-bounded in E. This characterisation goes back to a similar characterisation due to Raynaud and Xu [24] in the setting of the non-commutative Haagerup L_p-spaces

The final section of the paper studies property (K) in non-commutative symmetric spaces in the case that the trace is finite. In particular, it is shown that certain non-commutative Orlicz and Lorentz spaces have property (K) and Proposition 9.8 shows that property (K) can be "lifted" from a given symmetric function space E on the interval $[0, \tau(\mathbf{1}))$ with order continuous norm to the corresponding non-commutative symmetric space $E(\tau)$.

2. Preliminaries

In this section we recall some of the basic definitions from the theory of non-commutative integration and also introduce notation and terminology that will be used.

Let H be a Hilbert space. If $a : \mathfrak{D}(a) \to H$, where $\mathfrak{D}(a) \subseteq H$ is the domain of a, is a self-adjoint operator, the **spectral measure** of a is denoted by e^a. Suppose that $\mathcal{M}$ is a von Neumann algebra on H and that $\mathcal{M}$ is equipped with a fixed faithful normal semi-finite trace $\tau : \mathcal{M}^+ \to [0, \infty]$. The unit element of $\mathcal{M}$ will be denoted by $\mathbf{1}$ and $P(\mathcal{M})$ denotes the complete lattice of all self-adjoint projections in $\mathcal{M}$. The operator norm in $\mathcal{M}$ will be denoted by $\|\cdot\|_\infty$.

A linear operator $x : \mathfrak{D}(x) \to H$ is said to be **affiliated** with $\mathcal{M}$ if $xu = ux$ for all unitary $u \in \mathcal{M}'$, where $\mathcal{M}'$ is the commutant of $\mathcal{M}$. A self-adjoint operator a is affiliated with $\mathcal{M}$ if and only if $e^a(B) \in P(\mathcal{M})$ for all Borel sets $B \subseteq \mathbb{R}$.

A closed, densely defined linear operator x in H is called **τ-measurable** if x is affiliated with $\mathcal{M}$ and there exists $0 \le s \in \mathbb{R}$ such that $\tau(e^{|x|}(s, \infty)) < \infty$. The collection of all τ-measurable operators is denoted by $S(\tau)$, which is a $*$-algebra with respect to strong sum and product (and with respect to the measure topology, $S(\tau)$ is a complete metrizable topological algebra in which $\mathcal{M}$ is dense). For details we refer to, e.g., [20], [27].

For $x \in S(\tau)$, the **spectral distribution function** $d(|x|) : [0, \infty) \to [0, \infty]$ of $|x|$ is defined by setting

$$d(s; |x|) = \tau(e^{|x|}(s, \infty)), \quad s \geq 0,$$

which is right-continuous, decreasing and satisfies $\lim_{s\to\infty} d(s, |x|) = 0$. The **generalized singular value function** $\mu(x) : [0, \infty) \to [0, \infty]$ of $x \in S(\tau)$ is then defined to be the right-continuous inverse of $d(|x|)$, that is,

$$\mu(t; x) = \inf\{s \geq 0 : d(s; |x|) \leq t\}, \quad t \geq 0.$$

Note that $\mu(x)$ is decreasing, right-continuous and $\mu(t; x) < \infty$ for all $t > 0$. Furthermore, $\mu(0; x) < \infty$ if and only if $x \in \mathcal{M}$, in which case $\mu(0; x) = \|x\|_{B(H)}$, where $B(H)$ is the algebra of all bounded linear operators in the Hilbert space H.

If $x, y \in S(\tau)$ are such that

$$\int_0^t \mu(s; x) ds \leq \int_0^t \mu(s; y) ds, \quad t \geq 0,$$

then we say that x is **submajorized** by y, which is denoted by $x \prec\prec y$.

The following terminology will be used.

Definition 2.1: If $E \subseteq S(\tau)$ is a linear subspace equipped with a norm $\|\cdot\|_E$ such that $(E, \|\cdot\|_E)$ is a Banach space, then $(E, \|\cdot\|_E)$ is termed a:

(i) **Banach $\mathcal{M}$-bimodule** (of τ-measurable operators) if $vxw \in E$ and

$$\|vxw\|_E \leq \|v\|_\infty \|w\|_\infty \|x\|_E, \quad x \in E, \ v, w \in \mathcal{M};$$

(ii) **symmetric space** if it follows from $x \in S(\tau)$, $y \in E$ and $\mu(x) \leq \mu(y)$ that $x \in E$ and $\|x\|_E \leq \|y\|_E$;

(iii) **strongly symmetric space** if E is a symmetric space and its norm has the additional property that $\|x\|_E \leq \|y\|_E$ whenever $x, y \in E$ satisfy $x \prec\prec y$;

(iv) **fully symmetric space** if it follows from $x \in S(\tau)$, $y \in E$ and $x \prec\prec y$ that $x \in E$ and $\|x\|_E \leq \|y\|_E$.

Every symmetric space is a Banach $\mathcal{M}$-module and it is clear that every fully symmetric space is strongly symmetric. If $E \subseteq S(\tau)$ is a Banach $\mathcal{M}$-bimodule, then $x^* \in E$ and $\|x^*\|_E = \|x\|_E$ whenever $x \in E$. Furthermore, if $|x| \leq |y|$ in $S(\tau)$ and $y \in E$, then $x \in E$ and $\|x\|_E \leq \|y\|_E$.

If $x \in S(\tau)$, then the projection onto the closure of the range of $|x|$ is called the **support** of x and is denoted by $s(x)$.

The **carrier projection** $c_E \in P(\mathcal{M})$ of E is defined by

$$c_E = \bigvee \{s(x) : x \in E\},$$

which is a central projection. For further details, see [6], Section 3.1. Without loss of generality, we will always assume that $c_E = \mathbf{1}$.

Given a Banach $\mathcal{M}$-module $E \subseteq S(\tau)$, we define

$$E_h = \{a \in E : a^* = a\},$$

which is a real vector space. For $x \in E$, let

$$\operatorname{Re} x = \frac{1}{2}(x + x^*), \quad \operatorname{Im} x = \frac{1}{2i}(x - x^*),$$

and note that $x = \operatorname{Re} x + i \operatorname{Im} x$ with $\operatorname{Re} x, \operatorname{Im} x \in E_h$. Consequently, $E = E_h \oplus iE_h$. Furthermore, we define $E^+ = \{a \in E_h : a \geq 0\}$, a proper closed cone in E_h which is also generating (for any $a \in E_h$ we have $a = a^+ - a^-$). So E_h is an ordered Banach space.

Suppose now that $E \subseteq S(\tau)$ is a strongly symmetric space. By the assumption that $c_E = \mathbf{1}$, it follows that

$$L_1(\tau) \cap \mathcal{M} \subseteq E \subseteq L_1(\tau) + \mathcal{M}$$

with continuous embeddings. See [6], Lemma 25. Here

$$L_1(\tau) = \{x \in S(\tau) : \|x\|_{L_1(\tau)} < \infty\},$$

where

$$\|x\|_{L_1(\tau)} = \tau(|x|) = \int_{[0,\infty)} \mu(x) dm$$

and m denotes Lebesgue measure.

The **Köthe dual** $E^\times$ of E is defined by

$$E^\times = \{y \in S(\tau) : \sup(\tau(|xy|) : x \in E, \|x\|_E \leq 1) < \infty\}$$

and

$$\|y\|_{E^\times} = \sup(\tau(|xy|) : x \in E, \|x\|_E \leq 1), \quad y \in E^\times.$$

The space $(E^\times, \|\cdot\|_{E^\times})$ is a fully symmetric space with the **Fatou property** (that is, if $0 \leq y_\alpha \uparrow_\alpha$ in $E^\times$ and $\sup_\alpha \|y_\alpha\|_{E^\times} < \infty$, then there exists $0 \leq y \in E^\times$

such that $y_\alpha \uparrow_\alpha y$ and $\|y\|_{E^\times} = \sup_\alpha \|y_\alpha\|_{E^\times}$). For the details we refer the reader to [9]. If $y \in E^\times$ and we define

$$\langle x, \phi_y \rangle = \tau(xy), \quad x \in E,$$

then $\phi_y \in E^*$ (the Banach dual of E) and $\|\phi_y\|_{E^*} = \|y\|_{E^\times}$. In the sequel, we denote $\langle x, \phi_y \rangle$ by $\langle x, y \rangle$ and this is termed **trace duality**.

A functional $\phi \in E^*$ is called **normal** if $\langle x_\alpha, y \rangle \to 0$ whenever $x_\alpha \downarrow_\alpha 0$ in E, and ϕ is called **singular** if ϕ vanishes on some order dense order ideal in E (for the details we refer to [5]). The collections of normal and singular functionals on E are denoted by E_n^* and E_s^*, respectively, which are closed linear subspaces of E^*. Furthermore, $E^* = E_n^* \oplus E_s^*$, that is, every $\phi \in E^*$ has a unique decomposition $\phi = \phi_n + \phi_s$ with $\phi_n \in E_n^*$ and $\phi_s \in E_s^*$; this is called the **Yosida–Hewitt decomposition** of ϕ. The corresponding projection P_n in E^* onto E_n^* along E_s^* is called the **Yosida–Hewitt projection** and satisfies $\|P_n\| \leq 4$. Furthermore, the map $y \longmapsto \phi_y$, $y \in E^\times$, is a linear isometry from $E^\times$ onto E_n^*.

A strongly symmetric space $E \subseteq S(\tau)$ is said to have **order continuous norm** if $\|x_\alpha\|_E \downarrow_\alpha 0$ whenever $x_\alpha \downarrow_\alpha 0$ in E. It should be observed that E has order continuous norm if and only if $E^* = E_n^*$ and that any strongly symmetric space with order continuous norm is actually fully symmetric (see [9]). Furthermore, a strongly symmetric space $E \subseteq S(\tau)$ has the Fatou property if and only if $E^{\times\times} = E$ (isometrically).

A strongly symmetric space $E \subseteq S(\tau)$ is called a KB**-space** if every upwards directed system $(x_\alpha)_{n=1}^\infty \subseteq E^+$ satisfying $\sup_n \|x_\alpha\|_E < \infty$ is norm convergent in E.

It is well-known that a strongly symmetric space E is a KB-space if and only if E has order continuous norm and the Fatou property. See, for example, [8, Proposition 3.2]. Furthermore, every strongly symmetric KB-space is actually fully symmetric. If $E \subseteq S(\tau)$ is a strongly symmetric KB-space, then E has order continuous norm and so the dual space E^* may be identified with its Köthe dual $E^\times$ via trace duality. Furthermore, since E has the Fatou property, we also have $E^{\times\times} = E$ (isometrically). It should also be noted that a strongly symmetric space E is weakly sequentially complete if and only if E is a KB-space ([4, Proposition 4.8], [8, Proposition 3.2]).

A very extensive class of strongly symmetric $\mathcal{M}$-bimodules $E(\tau)$ may be constructed from concrete symmetric spaces E on the positive semi-axis as may be found in [17].

If E is a symmetric space on the interval $[0, \tau(\mathbf{1}))$, set

$$E(\tau) = \{x \in S(\tau) : \mu(x) \in E\}, \quad \|x\|_{E(\tau)} = \|\mu(x)\|_E, \quad x \in E(\tau).$$

It is shown in [8], [9] (see also [2], [6], [18]) that if E is a strongly symmetric space, then $E(\tau)$ is a strongly symmetric space. It should be noted (see [Theorem 5.6][9]) that $E(\tau)^\times = E^\times(\tau)$.

Next we discuss some properties and terminology related to linear operators between Banach $\mathcal{M}$-bimodules, which will be used, in particular, in Section 8. Suppose that $\mathcal{M}$ and $\mathcal{N}$ are two semi-finite von Neumann algebras and that E and F are Banach $\mathcal{M}$- and $\mathcal{N}$-bimodules, respectively (note that this also includes the case $F = \mathbb{C}$). We denote by $\mathcal{L}(E, F)$ the Banach space of all bounded linear operators from E into F (equipped with the operator norm). For any $T \in \mathcal{L}(E, F)$ we define the operator $\bar{T} \in \mathcal{L}(E, F)$ by

$$\bar{T}x = (Tx^*)^*, \quad x \in E.$$

It is clear that $\|\bar{T}\| = \|T\|$. An operator $T \in \mathcal{L}(E, F)$ is called **hermitian** if $\bar{T} = T$ (the reader is warned that this notion of "hermitian" is not related to a similarly termed concept in the setting of general Banach space theory!).

Lemma 2.2: *An operator $T \in \mathcal{L}(E, F)$ is hermitian if and only if $Ta \in F_h$ for all $a \in E_h$.*

Proof. Suppose that $T \in \mathcal{L}(E, F)$ is hermitian and that $a \in E_h$. Then

$$(Ta)^* = (Ta^*)^* = \bar{T}a = Ta$$

and so $Ta \in F_h$. Assume now that $Ta \in F_h$ whenever $a \in E_h$. Given $x \in E$ we have that

$$\bar{T}x = (T(\operatorname{Re} x) - iT(\operatorname{Im} x))^* = T(\operatorname{Re} x) + iT(\operatorname{Im} x) = Tx$$

and so $\bar{T} = T$. ∎

The collection of all hermitian operators in $\mathcal{L}(E, F)$ is denoted by $\mathcal{L}_h(E, F)$, which is a real linear subspace of $\mathcal{L}(E, F)$. For $T \in \mathcal{L}(E, F)$ we define

$$\operatorname{Re} T = \frac{1}{2}(T + \bar{T}), \quad \operatorname{Im} T = \frac{1}{2i}(T - \bar{T}).$$

Observing that $(\operatorname{Re} T)(a) = \operatorname{Re}(Ta)$ and $(\operatorname{Im} T)(a) = \operatorname{Im}(Ta)$ for all $a \in E_h$, it is clear that $\operatorname{Re} T$ and $\operatorname{Im} T$ belong to $\mathcal{L}_h(E, F)$ and that $T = \operatorname{Re} T + i \operatorname{Im} T$.

Hence

$$\mathcal{L}(E,F)=\mathcal{L}_h(E,F)\oplus i\mathcal{L}_h(E,F).$$

If $T\in\mathcal{L}_h(E,F)$, then it follows from Lemma 2.2 that

$$T_h := T\,|_{E_h}\colon E_h\to F_h, \tag{1}$$

and so $T_h\in\mathcal{L}(E_h,F_h)$.

Lemma 2.3: *The map $T\longmapsto T_h$, $T\in\mathcal{L}_h(E,F)$, is an $\mathbb{R}$-isomorphism from $\mathcal{L}_h(E,F)$ onto $\mathcal{L}(E_h,F_h)$.*

Proof. Evidently, the map $T\longmapsto T_h$ is $\mathbb{R}$-linear and $\|T_h\|\le\|T\|$ for all $T\in\mathcal{L}_h(E,F)$. Furthermore, if $T\in\mathcal{L}_h(E,F)$ and $x\in E$, then

$$\begin{aligned}\|Tx\|_F =&\|T(\operatorname{Re}x)+iT(\operatorname{Im}x)\|_F\le\|T_h(\operatorname{Re}x)\|_F+\|T_h(\operatorname{Im}x)\|_F\\ \le&\|T_h\|\|\operatorname{Re}x\|_E+\|T_h\|\|\operatorname{Im}x\|_E\le 2\|T_h\|\|x\|_E,\end{aligned}$$

which shows that $\|T\|\le 2\|T_h\|$. .

Now, let $S\in\mathcal{L}(E_h,F_h)$ be given. Define the map $T:E\to F$ by setting

$$Tx=S(\operatorname{Re}x)+iS(\operatorname{Im}x),\quad x\in E.$$

It is easily verified that $T\in\mathcal{L}_h(E,F)$ and that $T_h=S$. The proof is complete. ∎

Remark 2.4: If in Lemma 2.3 we take $F=\mathbb{C}$, then it is easy to verify that the map $\phi\longmapsto\phi_h$, $\phi\in(E^*)_h$, is an isometric isomorphism from $(E^*)_h$ onto $(E_h)^*$ (and so there is no danger of confusion to use the notation E_h^*).

An operator $T\in\mathcal{L}(E,F)$ is called **positive** if $Tx\ge 0$ whenever $0\le x\in E$. The set of all positive operators in $\mathcal{L}(E,F)$ is denoted by $\mathcal{L}^+(E,F)$. Every $T\in\mathcal{L}^+(E,F)$ is hermitian. Indeed, if $a\in E_h$, then

$$Ta=T(a^+)-T(a^-)\in F^+-F^+=F_h.$$

Hence, by Lemma 2.2, $T\in\mathcal{L}_h(E,F)$.

Via the map $T\longmapsto T_h$ the cone $\mathcal{L}^+(E,F)$ may be identified (up to isomorphism) with the cone $\mathcal{L}^+(E_h,F_h)$.

Definition 2.5: An operator $T\in\mathcal{L}_h(E,F)$ is called **regular** if $T=T_1-T_2$ with $T_1,T_2\in\mathcal{L}^+(E,F)$. An operator $T\in\mathcal{L}(E,F)$ is called regular if $\operatorname{Re}T$ and $\operatorname{Im}T$ are both regular.

It should be observed that an operator $T \in \mathcal{L}_h(E,F)$ is regular if and only if the corresponding operator $T_h \in \mathcal{L}(E_h, F_h)$ is regular.

We will now conclude this section by making some additional remarks concerning regular operators.

Remark 2.6: The concept of regular operator, being a linear combination of positivity preserving operators, is a long established notion in the theory of partially ordered vector spaces and Banach lattices. For convenience, let E, F be (real Banach lattices) and recall that a linear operator $T : E \to F$ is said to be **order-bounded** if T maps order-bounded sets in E to order-bounded sets in F. Clearly, every regular linear operator $T : E \to F$ is order-bounded, and if F is Dedekind complete, then each order-bounded linear operator $T : E \to F$ is regular, by the classical Riesz–Kantorovich theorem. Consequently, if E is Dedekind complete, then the regular linear operators coincide with the order-bounded operators.

More recently, Pisier [22] introduced the following concept. A linear operator $T : E \to F$ is called **regular** (in the sense of Pisier) if there exists a constant $C > 0$ such that

$$\|\sup_{i\le n} |Tx_n|\|_F \le C \|\sup_{i\le n} |x_i|\|_E, \tag{2}$$

for all $x_1, x_2, \ldots, x_n \in E$ and all $n \in \mathbb{N}$. We shall refer to operators which satisfy (2) as **Pisier regular**. It is easy to see that if $T : E \to F$ is regular, then T is Pisier regular, and if F is an M-space, that is

$$\|u \vee v\|_F = \max\{\|u\|_F, \|v\|_F\}, \quad u, v \in F^+,$$

then every bounded linear operator $T : E \to F$ is Pisier regular. However, not every Pisier regular operator is regular. By way of example, let $E = L_1([0,\pi])$ and let $F = c_0$ equipped with the sup-norm $\|\cdot\|_\infty$. Evidently, c_0 is an M-space. For $f \in L_1([0,\pi])$ and $n \in \mathbb{N}$, define

$$a_n(f) = \frac{2}{\pi}\int_0^\pi f(t) cos(nt) dt.$$

By the Riemann–Lebesgue Lemma, $a_n(f) \to_n 0$. Now define the bounded linear operator $T : L_1([0,\pi]) \to c_0$ by setting

$$T(f) = \{a_n(f)\}_{n\ge 1}, \quad f \in L_1([0,\pi]).$$

From the above remarks, it follows that T is Pisier regular. However, T is not order-bounded, and so is not regular. Indeed, if $\varphi_n(t) = \cos(nt)$, $t \in [0, \pi]$, $n \in \mathbb{N}$, then the sequence $\{\varphi_n\}_{n\geq 1}$ is order-bounded in $L_1([0, \pi])$; however, T maps the sequence $\{\varphi_n\}_{n\geq 1}$ to the unit vector basis $\{e_n\}_{n\geq 1}$ in c_0 and the sequence $\{e_n\}_{n\geq 1}$ is not order-bounded in c_0.

To obtain a more positive result, recall first that a Banach lattice F is said to have the **weak Fatou property** if every norm-bounded increasing net in the positive cone of F has a least upper bound in F. It follows immediately that any Banach latttice with this property is necessarily Dedekind complete. It is a straightforward exercise now to show that, if the Banach lattice F has the weak Fatou property, then every Pisier regular operator is order-bounded. In particular, it follows that, if the Banach lattice F has the weak Fatou property, then the classes of regular, Pisier regular and order-bounded operators from the Banach lattice E into F coincide.

It should be observed that, in particular, for operators between L_p-spaces, the notions of regularity, Pisier regularity and order-boundedness all coincide.

3. Some general observations

We start this section with some general definitions and simple observations. First, we recall the following definition.

Definition 3.1: Let $(x_n)_{n=1}^{\infty}$ be a sequence in a (real or complex) vector space X. A sequence $(y_k)_{k=1}^{\infty}$ in X is called a **CCC sequence** of $(x_n)_{n=1}^{\infty}$ if there exists a sequence $1 = N_1 < N_2 < \cdots$ in $\mathbb{N}$ and a sequence $\{c_k\}_{k=1}^{\infty}$ in $\mathbb{R}^+$ such that

$$\sum_{j=N_k}^{N_{k+1}-1} c_j = 1$$

for all k and

$$y_k = \sum_{j=N_k}^{N_{k+1}-1} c_j x_j, \quad k = 1, 2, \ldots.$$

Here CCC stands for “concecutive convex combinations”.

Evidently, if (x_n) is a sequence in a locally convex space $(X, \mathcal{T})$ satisfying $x_n \xrightarrow{\mathcal{T}} 0$ and if (y_k) is a CCC sequence of (x_n), then $y_k \xrightarrow{\mathcal{T}} 0$. The following simple observation will be useful.

Lemma 3.2: *Suppose that* $(x_n)_{n=1}^{\infty}$ *is a sequence in a locally convex Hausdorff space* $(X,\mathcal{T})$ *and let* C_n *be the convex hull of the set* $\{x_k : k \geq n\}$, $n \in \mathbb{N}$. *Suppose, furthermore, that* $C \subseteq X$ *is such that* $C_n \subseteq C$ *for all* n *and that* $\mathcal{T}$ *is metrizable on* C. *If* $x \in C$ *and* $x \in \bigcap_{n=1}^{\infty} \overline{C_n}^{\mathcal{T}}$, *then there exists a CCC sequence* (y_k) *of* (x_n) *such that* $y_n \xrightarrow{\mathcal{T}} x$.

Proof. Denote by $\mathcal{T}_C$ the relative topology induced by $\mathcal{T}$ in C and let d be a metric on C which induces $\mathcal{T}_C$. Observe that

$$\overline{C_n}^{\mathcal{T}} \cap C = \overline{C_n}^{\mathcal{T}_C} \quad \text{for all } n.$$

Since $x \in \overline{C_1}^{\mathcal{T}} \cap C = \overline{C_1}^{\mathcal{T}_C}$, there exist $N_2 > 1$ and $0 \leq c_j \in \mathbb{R}$, $1 \leq j \leq N_2 - 1$ with $\sum_{j=1}^{N_2-1} c_j = 1$ such that the convex combination $y_1 = \sum_{j=1}^{N_2-1} c_j x_j$ satisfies $d(x, y_1) \leq 1$. Since $x \in \overline{C_{N_2}}^{\mathcal{T}} \cap C = \overline{C_{N_2}}^{\mathcal{T}_C}$, it follows that there exist $N_3 > N_2$ and a convex combination $y_2 = \sum_{j=N_2}^{N_3-1} c_j x_j$ such that $d(x, y_2) \leq 1/2$. Continuing this way, we obtain a CCC sequence (y_k) of (x_n) such that

$$d(x, y_k) \leq 1/k,$$

$k \in \mathbb{N}$, and the result follows. ∎

As a simple illustration of the above lemma, if X is a Banach space and if (x_n) is a sequence in X such that $x_n \xrightarrow{\sigma(X,X^*)} x \in X$, then there exists a CCC sequence (y_k) of (x_n) such that $y_k \to x$ in norm. Indeed, in view of Mazur's theorem (for convex sets in X the weak and norm closures coincide) we may apply the above lemma with $\mathcal{T}$ being the norm topology in X.

Suppose that $(X, \|\cdot\|)$ is a Banach space with norm dual X^*. The following simple observation will be useful.

Lemma 3.3: *For a sequence* $(x_n^*)_{n=1}^{\infty}$ *in* X^* *satisfying* $x_n^* \to 0$ *with respect to* $\sigma(X^*, X)$, *the following two statements are equivalent.*

(i) *For every sequence* (x_n) *in* X *such that* $x_n \to 0$ *with respect to* $\sigma(X, X^*)$ *we have* $\langle x_n, x_n^* \rangle \to 0$ *as* $n \to \infty$.

(ii) *For every relatively* $\sigma(X, X^*)$*-compact set* $A \subseteq X$ *we have*

$$\sup_{x \in A} |\langle x, x_n^* \rangle| \to 0, \quad n \to \infty.$$

Proof. If $x_n \xrightarrow{\sigma(X,X^*)} 0$ in X, then the set $A = \{x_n : n \in \mathbb{N}\}$ is relatively $\sigma(X, X^*)$-compact and so the implication (ii)⇒(i) is evident.

(i)⇒(ii). It should be observed that if a sequence (x_n^*) has the stated property, then so has any subsequence of (x_n^*) this property. Let $A \subseteq X$ be a relatively $\sigma(X, X^*)$-compact set and suppose that $\sup_{x\in A} |\langle x, x_n^*\rangle| \nrightarrow 0$ as $n \to \infty$. By passing, if necessary, to a subsequence, we may assume that there exists $0 < \varepsilon \in \mathbb{R}$ such that $\sup_{x\in A} |\langle x, x_n^*\rangle| > \varepsilon$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, there exists $x_n \in A$ such that $|\langle x_n, x_n^*\rangle| > \varepsilon$. Since A is relatively $\sigma(X, X^*)$-compact, it follows from the Eberlein–Smulian theorem that (x_n) has a subsequence (x_{n_k}) such that $x_{n_k} \xrightarrow{\sigma(X,X^*)} x$ for some $x \in X$, that is,

$$x_{n_k} - x \xrightarrow{\sigma(X,X^*)} 0 \quad \text{as } k \to \infty.$$

By the assumption on the sequence (x_n^*), this implies that

$$\langle x_{n_k} - x, x_{n_k}^*\rangle \to 0, \quad k \to \infty.$$

Since

$$\varepsilon < |\langle x_{n_k}, x_{n_k}^*\rangle| \leq |\langle x_{n_k} - x, x_{n_k}^*\rangle| + |\langle x, x_{n_k}^*\rangle|$$

for all $k \in \mathbb{N}$ and $|\langle x, x_{n_k}^*\rangle| \to 0$ as $k \to \infty$ (as $x_n^* \xrightarrow{\sigma(X^*,X)} 0$ by hypothesis), this is a contradiction. Therefore, we may conclude that (ii) holds and the proof is complete. ∎

Let $(X, \|\cdot\|_X)$ be a Banach space. For any bounded subset $A \subseteq X$, the semi-norm $\rho_A : X^* \to [0, \infty)$ is defined by setting

$$\rho_A(x^*) = \sup\{|\langle x, x^*\rangle| : x \in A\}.$$

Given a non-empty collection $\mathfrak{S}$ of bounded subsets of X, the locally convex topology $\mathcal{T}_{\mathfrak{S}}$ in X^* generated by the semi-norms

$$\{\rho_A : A \in \mathfrak{S}\}$$

is called the **topology of uniform convergence on the sets of** $\mathfrak{S}$. It follows from the Hahn-Banach theorem that $\mathcal{T}_{\mathfrak{S}}$ is Hausdorff if and only if the linear span of $\bigcup_{A\in\mathfrak{S}} A$ is norm dense in X.

Recall that the **Mackey topology** $\tau(X^*, X)$ on X^* is defined as the topology of uniform convergence on all $\sigma(X, X^*)$-compact absolutely convex subsets of X. In the present setting, the Mackey–Arens theorem states: if $\mathcal{T}$ is a Hausdorff locally convex topology on X^*, then the dual space of $(X^*, \mathcal{T})$ equals X

if and only if $\mathcal{T} = \mathcal{T}_{\mathfrak{S}}$, where $\mathfrak{S}$ is some collection of $\sigma(X, X^*)$-compact absolutely convex subsets of X satisfying $\bigcup_{A\in\mathfrak{S}} A = X$ (see, e.g., [25, Chapter III, Section 7]). It follows, in particular, that any such topology $\mathcal{T}$ satisfies

$$\sigma(X^*, X) \subseteq \mathcal{T} \subseteq \tau(X^*, X).$$

Moreover, by [25, Proposition 8] it follows that for any convex set $C \subseteq X$ we have

$$\overline{C}^{\sigma(X^*,X)} = \overline{C}^{\tau(X^*,X)}$$

for any convex set $C \subseteq X^*$

It should be recalled furthermore that (by a theorem of M. Krein and V. Smulian; see, e.g., [11], V.6.4) the absolutely convex hull of any relatively $\sigma(X, X^*)$-compact subset of X is again relatively $\sigma(X, X^*)$-compact. Consequently, the Mackey topology $\tau(X^*, X)$ is also equal to the topology of uniform convergence on all relatively $\sigma(X, X^*)$-compact subsets of X. It should be observed that condition (ii) of Lemma 3.3 states that $x_n^* \to 0$ with respect to $\tau(X^*, X)$.

We recall the following definition (see [16], Section 2).

Definition 3.4: A Banach space X is said to have property (K) if every sequence (x_n^*) in X^* satisfying $x_n^* \to 0$ with respect to $\sigma(X^*, X)$ has a CCC sequence (y_k^*) such that $\langle x_k, y_k^*\rangle \to 0$ for every sequence (x_k) in X satisfying $x_k \to 0$ with respect to $\sigma(X, X^*)$.

Evidently, every reflexive space has property (K) (cf. the remarks following Lemma 3.2, applied to X^* instead of X). Recall now that a Banach space X is a **Grothendieck space** if every weak* null sequence in X^* is a weak null sequence. If X is a Grothendieck space, and if $\{x_n^*\} \subseteq X^*$ is a weak* null sequence, then it follows from the remarks following Lemma 3.2, that there exists a CCC sequence $\{y_n^*\}$ of $\{x_n^*\}$ such that $\{y_n^*\}$ converges to 0 in norm. In particular, it follows that X has property (K). Consequently, each space $L^\infty(\mu)$, and more generally, any von Neumann algebra, has property (K), since each of these spaces are Grothendieck spaces [21]. It is observed in [16] Proposition B that $L^1(\mu)$ has property (K) for any finite measure μ. It is noted also in [16] that c_0 fails to have property (K). A sharpened version of this result is given in [12, Proposition 4.9]. See the remark following Definition 4.5 below.

The remarks preceding Definition 3.4 yield immediately the following characterization of Banach spaces with property (K).

LEMMA 3.5: *If X is a Banach space, then the following two statements are equivalent:*

(i) *X has property (K);*

(ii) *every sequence (x_n^*) in X^* satisfying $x_n^* \to 0$ with respect to $\sigma(X^*, X)$ has a CCC sequence (y_k^*) such that $y_k^* \to 0$ with respect to the Mackey topology $\tau(X^*, X)$.*

Proof. Suppose that X has property (K) and let (x_n^*) be a sequence in X^* satisfying $x_n^* \xrightarrow{\sigma(X^*,X)} 0$. By hypothesis, there exists a CCC subsequence (y_k^*) of (x_n^*) such that $\langle x_k, y_k^* \rangle \to 0$ for every sequence (x_k) in X satisfying $x_k \to 0$ with respect to $\sigma(X, X^*)$. Since $y_k^* \to 0$ with respect to $\sigma(X^*, X)$, it follows from Lemma 3.3 that $y_k^* \to 0$ with respect to $\tau(X^*, X)$. This shows that (i) implies (ii).

Suppose now that (ii) holds and let (x_n^*) be a sequence in X^* satisfying $x_n^* \xrightarrow{\sigma(X^*,X)} 0$. Let (y_k^*) be a CCC sequence of (x_n^*) such that $y_k^* \xrightarrow{\tau(X^*,X)} 0$. If (x_k) is a sequence in X satisfying $x_k \xrightarrow{\sigma(X,X^*)} 0$, then the set $A = \{x_k : k \in \mathbb{N}\}$ is relatively $\sigma(X, X^*)$-compact and so $\sup_{x \in A} |\langle x, y_k^* \rangle| \to 0$ as $k \to \infty$, which implies, in particular, that $\langle x_k, y_k^* \rangle \to 0$ as $k \to \infty$. Hence X has property (K). ∎

In view of the above observations, the following definition seems to be natural.

Definition 3.6: Given a class of bounded subsets $\mathfrak{S}$ of a Banach space X, we will say that X has property $(K_{\mathfrak{S}})$ if every sequence (x_n^*) in X^* satisfying $x_n^* \to 0$ with respect to $\sigma(X^*, X)$ has a CCC sequence (y_k^*) such that $y_k^* \to 0$ as $k \to \infty$ with respect to $\mathcal{T}_{\mathfrak{S}}$, that is, $\sup_{x \in A} |\langle x, y_k^* \rangle| \to 0$ as $k \to \infty$ for all $A \in \mathfrak{S}$.

According to Lemma 3.5, property (K) corresponds to $(K_{\mathfrak{S}})$ where $\mathfrak{S}$ consists of all (absolutely convex) relatively $\sigma(X, X^*)$-compact sets. But we may also take for $\mathfrak{S}$ the collection of all order-bounded sets in the Banach lattice setting or the collection of all sets which are of uniformly absolutely continuous norm in a symmetric (non-commutative) space (for the definitions, see the next sections).

The following observation is useful.

PROPOSITION 3.7: *Let X be a Banach space and let $\mathfrak{S}$ be a collection of bounded subsets of X such that the linear span of $\bigcup_{A \in \mathfrak{S}} A$ is norm dense in X and $\mathcal{T}_{\mathfrak{S}} \subseteq \tau(X^*, X)$. If $\mathcal{T}_{\mathfrak{S}}$ is metrizable on norm bounded subsets of X^*, then X has property $(K_{\mathfrak{S}})$.*

Proof. Let (x_n^*) be a sequence in X^* such that $x_n^* \to 0$ with respect to $\sigma(X^*, X)$ and define the convex sets C_n by $C_n = \mathrm{co}\{x_k^* : k \geq n\}$ for $n \in \mathbb{N}$. By hypothesis, we have

$$0 \in \overline{C_n}^{\sigma(X^*,X)} = \overline{C_n}^{\tau(X^*,X)} \subseteq \overline{C_n}^{\mathcal{T}_{\mathfrak{S}}}, \quad n \in \mathbb{N},$$

and so $0 \in \bigcap_{n=1}^{\infty} \overline{C_n}^{\mathcal{T}_{\mathfrak{S}}}$. Denoting by K the absolutely convex hull of the sequence (x_n^*), it is clear that $0 \in K$ and that K is norm bounded in X^* (as the sequence (x_n^*) is norm bounded by the Uniform Boundedness Principle). By hypothesis, $\tau_{\mathfrak{S}}$ is metrizable on K and so the result now follows from Lemma 3.2. ∎

Via Lemma 3.5 we immediately obtain the following consequence.

Corollary 3.8: *If X is a Banach space such that the Mackey topology in X^* is metrizable on norm bounded subsets of X^*, then X has property (K).*

Remark 3.9: It may be of some interest to point out some ideas related to the discussion in the present section. As above, suppose that X is a Banach space and suppose that $\mathfrak{S} = \{A_n : n \in \mathbb{N}\}$ is a countable collection of relatively $\sigma(X, X^*)$-compact subsets of X such that the linear span of $\bigcup_n A_n$ is dense in X. Let $\mathcal{T}_{\mathfrak{S}}$ be the locally convex topology in X^* of uniform convergence on the sets A_n. Then it is clear that $\mathcal{T}_{\mathfrak{S}} \subseteq \tau(X^*, X)$ and that $\mathcal{T}_{\mathfrak{S}}$ is metrizable. Suppose that C is a convex subset of X^* and that $x_0^* \in \overline{C}^{\sigma(X^*,X)}$. It follows that $x_0^* \in \overline{C}^{\tau(X^*,X)}$ and so $x_0^* \in \overline{C}^{\mathcal{T}_{\mathfrak{S}}}$. Consequently, by Proposition 3.7, there exists a sequence $(x_k^*)_{k=1}^{\infty}$ in C such that $x_k^* \to x_0^*$ as $k \to \infty$ uniformly on all sets A_n, $n \in \mathbb{N}$.

This observation has the following interesting (but, sometimes overlooked) consequence, in the commutative setting due to A. Grothendieck [13] (with a similar proof). Let $\mathcal{M}$ be a von Neumann algebra equipped with a faithful normal trace τ satisfying $\tau(\mathbf{1}) = 1$. The corresponding non-commutative L_p-spaces ($1 \leq p \leq \infty$) satisfy $L_{p_1}(\tau) \subseteq L_{p_2}(\tau)$ and $\|\cdot\|_{p_2} \leq \|\cdot\|_{p_1}$ whenever $1 \leq p_2 \leq p_1 \leq \infty$. Recall furthermore that $L_p(\tau)^* = L_q(\tau)$ via trace duality, whenever $1 \leq p < \infty$ and $p^{-1} + q^{-1} = 1$. If B_q denotes the closed unit ball in $L_q = L_q(\tau)$, then B_q is $\sigma(L_q, L_p)$-compact $(p^{-1} + q^{-1} = 1)$ and hence (relatively) $\sigma(L_1, L_\infty)$-compact in L_1 whenever $1 < q \leq \infty$. Let $(q_n)_{n=1}^{\infty}$ be a sequence in $(1, \infty)$ satisfying $q_n \downarrow 1$ and put $A_n = B_{q_n}$, $n \in \mathbb{N}$. The collection $\mathfrak{S} = \{A_n : n \in \mathbb{N}\}$ of subsets of $X = L_1$ satisfies the conditions of the first part in this Remark. If $(y_k)_{k=1}^{\infty}$ is a sequence in L_∞ such that $y_k \to y \in L_\infty$ uniformly on the sets A_n, then $\|y - y_k\|_{p_n} \to 0$ as $k \to \infty$ for all n $(p_n^{-1} + q_n^{-1} = 1)$. Since

$p_n \uparrow \infty$, this implies that $\|y - y_k\|_p \to 0$ for all $1 \leq p < \infty$. Therefore, we may conclude that: if $C \subseteq L_\infty(\tau) = \mathcal{M}$ is a convex set and if $y \in \overline{C}^{\sigma(L_\infty, L_1)}$, then there exists a sequence (y_k) in C such that $\|y - y_k\|_p \to 0$ as $k \to \infty$ for all $1 \leq p < \infty$.

4. Property (K) in pre-duals of von Neumann algebras

In the present section it will be shown that the pre-dual $\mathcal{M}_*$ of any σ-finite von Neumann algebra $\mathcal{M}$ has property (K), as defined in Definition 3.4. Recall first that a von Neumann algebra is said to be σ-finite if it admits at most countably many mutually orthogonal projections. See [26, Definition I.3.8].

For any $\varphi \in \mathcal{M}_*$ and $x \in \mathcal{M}$, the elements $x\varphi$ and φx of $\mathcal{M}_*$ are defined by setting

$$(x\varphi)(y) = \varphi(yx), \quad (\varphi x)(y) = \varphi(xy), \quad y \in \mathcal{M}.$$

A subset $A \subseteq \mathcal{M}_*$ is said to be of **uniformly absolutely continuous** norm if $p_\alpha \downarrow_\alpha 0$ in $P(\mathcal{M})$ implies that

$$\sup_{\varphi \in A} \|p_\alpha \varphi p_\alpha\|_{\mathcal{M}_*} \to_\alpha 0.$$

It is convenient now to recall the well-known theorem of C. A. Akemann (see [26], Theorem III.5.4) that a bounded set $A \subseteq \mathcal{M}_*$ is relatively $\sigma(\mathcal{M}_*, \mathcal{M})$ compact if and only if A is of uniformly absolutely continuous norm.

The unit ball in $\mathcal{M}$ will be denoted by $B_{\mathcal{M}}$. We shall need the following simple observation.

Lemma 4.1: *If $\varphi \in \mathcal{M}_*$, then each of the sets $B_{\mathcal{M}}\varphi$, $\varphi B_{\mathcal{M}}$ are of uniformly absolutely continuous norm (and hence are relatively $\sigma(\mathcal{M}_*, \mathcal{M})$ compact).*

Proof. It may clearly be assumed that $0 \leq \varphi$. It will suffice to show that $B_{\mathcal{M}}\varphi$ is of uniformly absolutely continuous norm. To this end, suppose that $p_\alpha \downarrow_\alpha 0$ holds in $P(\mathcal{M})$. Using the fact that $\varphi \geq 0$ and the Cauchy-Schwarz inequality ([26], Proposition I.9.5), it follows that

$$\begin{aligned} \sup_{x \in B_{\mathcal{M}}} \|p_\alpha x \varphi p_\alpha\|_{\mathcal{M}_*} &= \sup_{x, z \in B_{\mathcal{M}}} |\varphi(p_\alpha z p_\alpha x)| \\ &\leq \sup_{z \in B_{\mathcal{M}}} \varphi(p_\alpha z z^* p_\alpha)^{\frac{1}{2}} \sup_{x \in B_{\mathcal{M}}} \varphi(x^* p_\alpha x)^{\frac{1}{2}} \\ &\leq \varphi(p_\alpha)^{\frac{1}{2}} \varphi(\mathbf{1})^{\frac{1}{2}} \to_\alpha 0, \end{aligned}$$

where the final assertion follows from the normality of φ. ∎

The following result, due to Raynaud and Xu ([24], Proposition 4.13), will be crucial in what follows. It is a refinement in the special case of σ-finite von Neumann algebras of a characterisation of sets of uniformly absolutely continuous norm in arbitrary von Neumann algebra pre-duals. As details needed to derive this refinenent are not given in [24], they will be included here for the convenience of the reader.

Proposition 4.2: *Suppose that $\mathcal{M}$ is a σ-finite von Neumann algebra. There exists a $0 \leq \varphi_0 \in \mathcal{M}_*$ such that for every $A \subseteq \mathcal{M}_*$ of uniformly absolutely continuous norm and every $\varepsilon > 0$, there exists a constant $0 < C_\varepsilon \in \mathbb{R}$ such that*

$$A \subseteq C_\varepsilon(\varphi_0 B_{\mathcal{M}} + B_{\mathcal{M}}\varphi_0) + \varepsilon B_{\mathcal{M}_*}.$$

Proof. Let $\varepsilon > 0$ be given and suppose that $A \subseteq \mathcal{M}_*$ is a bounded set of of uniformly absolutely continuous norm. By [Proposition 4.13 (ii)][24], there exists $\varphi_\epsilon \in \mathcal{M}_*$ such that

$$A \subseteq \varphi_\epsilon B_{\mathcal{M}} + B_{\mathcal{M}}\varphi_\epsilon + \varepsilon B_{\mathcal{M}_*}.$$

Since $\mathcal{M}$ is σ-finite, it follows from [Proposition II.3.19][26] that there exists $0 \leq \varphi_0 \in \mathcal{M}_*$ with support $s(\varphi_0) = \mathbf{1}$. By [Lemma 4.7][24], $\mathcal{M} \cdot \varphi_0$ and $\varphi_0 \cdot \mathcal{M}$ are dense in $\mathcal{M}_*$. Consequentiy, there exist $x_\varepsilon, y_\varepsilon \in B_{\mathcal{M}}$ and positive constants $0 \leq A_\varepsilon, B_\varepsilon$, such that

$$\varphi_\varepsilon \in \varphi_0 B_\varepsilon y_\varepsilon + \varepsilon B_{\mathcal{M}_*}$$

and

$$\varphi_\varepsilon \in A_\varepsilon x_\varepsilon \varphi_0 + \varepsilon B_{\mathcal{M}_*}.$$

If now $C_\varepsilon = \max\{A_\varepsilon, B_\varepsilon\}$ then

$$\begin{aligned} A &\subseteq \varphi_0 B_\varepsilon y_\varepsilon B_{\mathcal{M}} + \varepsilon B_{\mathcal{M}_*} \cdot B_{\mathcal{M}} + A_\varepsilon B_{\mathcal{M}} x_\varepsilon \cdot \varphi_0 + \varepsilon B_{\mathcal{M}_*} \cdot B_{\mathcal{M}} + \varepsilon B_{\mathcal{M}_*} \\ &\subseteq C_\varepsilon(\varphi_0 \cdot B_{\mathcal{M}} + B_{\mathcal{M}} \cdot \varphi_0) + 3\varepsilon B_{\mathcal{M}_*}. \qquad \blacksquare \end{aligned}$$

Proposition 4.3: *If $\mathcal{M}$ is a σ-finite von Neumann algebra, then the Mackey topology $\tau(\mathcal{M}, \mathcal{M}_*)$ is metrizable on norm bounded subsets of $\mathcal{M}$.*

Proof. Let $0 \leq \varphi_0 \in \mathcal{M}_*$ be as in Proposition 4.2 and set $W = \varphi_0 B_{\mathcal{M}} + B_{\mathcal{M}}\varphi_0$. Define the semi-norm ρ on $\mathcal{M}$ by setting $\rho(x) = \sup_{\varphi \in W} |\varphi(x)|$, $x \in \mathcal{M}$, and let $\mathcal{T}_0$ denote the locally convex topology in $\mathcal{M}$ generated by ρ. By Lemma 4.1, W is $\sigma(\mathcal{M}_*, \mathcal{M})$-compact, so it is clear that $\mathcal{T}_0 \subseteq \tau(\mathcal{M}, \mathcal{M}_*)$.

We claim that $\mathcal{T}_0$ and $\tau(\mathcal{M}, \mathcal{M}_*)$ coincide on norm bounded subsets of $\mathcal{M}$. It suffices to show that the two topologies agree on the unit ball $B_{\mathcal{M}}$. For this purpose, suppose that (x_n) is a sequence in $B_{\mathcal{M}}$ and $x \in B_{\mathcal{M}}$ is such that $\rho(x - x_n) \to 0$ as $n \to \infty$. Let $A \subseteq \mathcal{M}_*$ be a relatively $\sigma(\mathcal{M}_*, \mathcal{M})$-compact set. By Akemann's theorem (cited above), A is of uniformly absolutely continuous norm. Consequently, given $\varepsilon > 0$, it follows from Proposition 4.2 that there exists a constant $0 < C_\varepsilon \in \mathbb{R}$ such that $A \subseteq C_\varepsilon W + \varepsilon B_{\mathcal{M}_*}$. If $\varphi \in A$ and we write $\varphi = \varphi_1 + \varphi_2$, with $\varphi_1 \in C_\varepsilon W$ and $\varphi_2 \in \varepsilon B_{\mathcal{M}_*}$, then we find that

$$\begin{aligned} |\varphi(x - x_n)| \leq & |\varphi_1(x - x_n)| + |\varphi_2(x - x_n)| \\ \leq & C_\varepsilon \rho(x - x_n) + 2\varepsilon \end{aligned}$$

and so

$$\sup_{\varphi \in A} |\varphi(x - x_n)| \leq C_\varepsilon \rho(x - x_n) + 2\varepsilon.$$

Since $\rho(x - x_n) \to 0$ as $n \to \infty$, this implies that

$$\limsup_{n \to \infty} \sup_{\varphi \in A} |\varphi(x - x_n)| \leq 2\varepsilon.$$

This holds for all $\varepsilon > 0$ and hence $\limsup_{n\to\infty} \sup_{\varphi \in A} |\varphi(x - x_n)| = 0$. This suffices to complete the proof of the proposition. ∎

It should be observed that this result may be considered a strengthening of the result that may be obtained by a combination of Theorem III.5.7 and Proposition II.2.7 in [26], as we do not assume that the underlying Hilbert space is separable.

The next result follows immediately from a combination of Proposition 4.3 and Corollary 3.8.

THEOREM 4.4: *If $\mathcal{M}$ is a σ-finite von Neumann algebra, then the pre-dual $\mathcal{M}_*$ has property (K).*

In their paper [12], T. Figiel, W. B. Johnson and A. Pelczyński introduced the following property in Banach spaces.

Definition 4.5: A Banach space X is said to have property (k) if for every $\sigma(X^*, X)$-null sequence $(x_n^*)_{n=1}^\infty$ in X^* there exists a CCC sequence $(y_k^*)_{k=1}^\infty$ of $(x_n^*)_{n=1}^\infty$ such that for every bounded linear operator $T : L_1(0,1) \to X$ and for every weakly null sequence $(f_k)_{k=1}^\infty$ in $L_1(0,1)$ satisfying $\sup_k \|f_k\|_\infty < \infty$, we have

$$\lim_{k \to \infty} \langle T f_k, y_k^* \rangle = 0.$$

Proposition 4.6: *If a Banach space X has property (K), then X has property (k).*

Proof. If $(f_k)_{k=1}^{\infty}$ is a weakly null sequence in $L_1(0,1)$ and

$$T : L_1(0,1) \to X$$

is a bounded linear operator, then (Tf_k) is a weak null sequence in X. Let $(x_n^*)_{n=1}^{\infty}$ be a weak* null sequence in X^*. Since X has property (K), there exists a CCC sequence (y_n^*) of (x_n^*) such that $y_n^* \to_n 0$ uniformly on each relatively weakly compact subset of X. In particular

$$\sup_k \langle Tf_k, y_n^* \rangle \to_n 0,$$

and this implies that X has property (k). ∎

It should be noted that it is shown in [12, Proposition 4.9] that, if a Banach space X contains a complemented subspace isomorphic to c_0, then X fails property (k) (and so also fails property (K)). In particular, c_0 does not have property (k).

Theorem 4.4 together with Proposition 4.6 now has the following immediate consequence, which answers affirmatively Problem 6.6 raised in [12] and is one of the main results of this paper.

Corollary 4.7: *The pre-dual $\mathcal{M}_*$ of every σ-finite von Neumann algebra $\mathcal{M}$ has property (k).*

The preceding Corollary 4.7 was established in [12, Proposition 4.7] in the special case that the von Neumann algebra $\mathcal{M}$ has separable pre-dual. It is worth noting, therefore, that a σ-finite von Neumann algebra $\mathcal{M}$ need not have separable pre-dual, even in the case that $\mathcal{M}$ is commutative. Indeed, if σ is the product measure on an uncountable number of copies of the unit interval $[0,1]$ equipped with Lebesgue measure, and if $\mathcal{M}$ is $L^\infty(\sigma)$ acting by multiplication on $L^2(\sigma)$, then $\mathcal{M}$ is a finite (and hence σ-finite) von Neumann algebra, but the pre-dual $\mathcal{M}_* = L^1(\sigma)$ is not separable. Details may be found in [28, Exercise 23.21].

It might be helpful for the reader's understanding to point out further that the σ-finiteness assumption in Theorem 4.4 and Corollary 4.7 is crucial. Indeed, this is the case even in the commutative setting as is shown by [12, Example 4.1] which exhibits an abstract L-space that fails property (k).

5. Property (K_{ob}) in Banach lattices

In the present section we will discuss property $(K_{\mathfrak{S}_{ob}})$ for Banach lattices E, where $\mathfrak{S}_{ob}$ is the class of order-bounded sets. Only real Banach lattices will be considered as the extension to the complex case is straightforward. For basic properties and terminology from the theory of Banach lattices, we refer to [19].

Let E be a Banach lattice and recall that a subset $A \subseteq E$ is called order-bounded if there exists $0 \leq w \in E$ such that $A \subseteq [-w, w]$, where

$$[-w, w] = \{x \in E : -w \leq x \leq w\}.$$

The collection of all order-bounded sets in E is denoted by $\mathfrak{S}_{ob}$. Let $\mathcal{T}_{ob}$ denote the locally convex topology in E^* of uniform convergence on order-bounded subsets of E, that is, $\mathcal{T}_{ob}$ is the locally convex topology in E generated by the collection $\{\rho_A : A \in \mathfrak{S}_{ob}\}$ of semi-norms on E^*, where

$$\rho_A(x^*) = \sup\{|\langle x, x^*\rangle| : x \in A\}, \quad A \in \mathfrak{S}_{ob}.$$

Since each $A \in \mathfrak{S}_{ob}$ is contained in $[-w, w]$ for some $0 \leq w \in E$, it is clear that $\mathcal{T}_{ob}$ is the locally convex topology generated by the lattice semi-norms $\{\rho_w : 0 \leq w \in E\}$, where

$$\rho_w(x^*) = \sup\{|\langle x, x^*\rangle| : x \in E, |x| \leq w\} = \langle w, |x^*|\rangle, \quad x^* \in E^* \tag{3}$$

where the last equality is given in [19, Theorem 1.3.2]. We will say that E has property (K_{ob}) if it has property $(K_{\mathfrak{S}_{ob}})$, as introduced in Definition 3.6. Note the following simple observation.

Lemma 5.1: *For a Banach lattice E, the following two conditions are equivalent:*

(i) *E has property (K_{ob});*
(ii) *every sequence (x_n^*) in E^* satisfying $x_n^* \to 0$ with respect to $\sigma(E^*, E)$ has a CCC sequence (y_k^*) such that $|y_k^*| \to 0$ with respect to $\sigma(E^*, E)$.*

Proof. It will be sufficient to show that a sequence (y_k^*) in E^* satisfies $\sup_{x\in A} |\langle x, y_k^*\rangle| \to 0$ as $k \to \infty$ for all order-bounded subsets $A \subseteq E$ if and only if $|y_k^*| \to 0$ with respect to $\sigma(E^*, E)$.

If the sequence (y_k^*) in E^* is such that $\sup_{x\in A} |\langle x, y_k^*\rangle| \to 0$ as $k \to \infty$ for all order-bounded subsets $A \subseteq E$, then

$$|y_k^*|(u) = \sup_{|x|\leq u} |\langle x, y_k^*\rangle| \to 0, \quad k \to \infty,$$

for all $0 \leq u \in E$, which implies that $|y_n^*| \to 0$ with respect to $\sigma(E^*, E)$.

Now suppose that (y_k^*) in E^* is such that $|y_k^*| \to 0$ with respect to $\sigma(E^*, E)$. If $A \subseteq E$ is order-bounded, then there exists $0 \leq u \in E$ such that $A \subseteq [-u, u]$. Consequently,

$$\sup_{x \in A} |\langle x, y_k^* \rangle| \leq \sup_{|x| \leq u} |\langle x, y_k^* \rangle| = |y_k^*|(u)$$

and so $\sup_{x \in A} |\langle x, y_k^* \rangle| \to 0$ as $k \to \infty$. ∎

We recall that an element $0 \leq e$ in the Banach lattice E is said to be a **weak order unit** for E if and only if $e \wedge x = 0, x \in E$ implies $x = 0$. The Banach lattice E is said to have **order continuous norm** if $\|x_\alpha\|_E \downarrow_\alpha 0$ whenever $x_\alpha \downarrow_\alpha 0$ in E.

Proposition 5.2: *If E is a Banach lattice with order continuous norm and weak order unit, then the topology $\mathcal{T}_{ob}$ is metrizable on norm bounded subsets of E^*.*

Proof. It is sufficient to show that $\mathcal{T}_{ob}$ is metrizable on the closed unit ball B_{E^*} in E^*. Let $0 \leq w \in E$ be a weak order unit. It will be shown that $\mathcal{T}_{ob}$ on B_{E^*} is the same as the topology induced by the semi-norm ρ_w (see (3)). The topology induced by ρ_w is evidently weaker than the one induced by $\mathcal{T}_{ob}$.

We claim that for every $0 \leq v \in E$ and $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that

$$[-v, v] \subseteq k[-w, w] + \varepsilon B_E. \tag{4}$$

Indeed, since w is a weak order unit, we have $v \wedge (kw) \uparrow_k v$ and so the order continuity of the norm implies that

$$\|v - v \wedge (kw)\| \to 0 \quad \text{as } k \to \infty.$$

Given $\varepsilon > 0$, let $k \in \mathbb{N}$ be such that $\|v - v \wedge (kw)\| \leq \varepsilon$, that is, $\|(v - kw)^+\| \leq \varepsilon$. If $x \in [-v, v]$, it follows that $|x| \wedge (kw) \in k[0, w]$ and $\|(|x| - kw)^+\| \leq \varepsilon$. Writing $x = x_1 + x_2$ with

$$x_1 = \{x^+ \wedge (kw) - x^- \wedge (kw)\}, \quad x_2 = \{(x^+ - kw)^+ - (x^- - kw)^+\}$$

we have

$$-(|x| \wedge (kw)) \leq x_1 \leq |x| \wedge (kw) \quad \text{and} \quad -(|x| - kw)^+ \leq x_2 \leq (|x| - kw)^+,$$

which implies that $x_1 \in k[-w, w]$ and $x_2 \in \varepsilon B_E$. This proves the claim.

Suppose that (x_n^*) is a sequence in B_{E^*} such that $\rho_w(x_0^* - x_n^*) \to 0$ for some $x_0^* \in B_{E^*}$. It will be sufficient to show that $\rho_v(x_0^* - x_n^*) \to 0$ for all $0 \leq v \in E$. Given $0 \leq v \in E$ and $\varepsilon > 0$, it follows from the first part of the present proof that there exists $k \in \mathbb{N}$ such that (4) holds. If $x \in [-v, v]$, then $x = x_1 + x_2$ with $x_1 \in k[-w, w]$ and $x_2 \in \varepsilon B_E$, so

$$\begin{aligned} |\langle x, x_0^* - x_n^* \rangle| \leq & |\langle x_1, x_0^* - x_n^* \rangle| + |\langle x_2, x_0^* - x_n^* \rangle| \\ \leq & k\rho_w(x_0^* - x_n^*) + 2\varepsilon, \end{aligned}$$

and hence $\rho_v(x_0^* - x_n^*) \leq k\rho_w(x_0^* - x_n^*) + 2\varepsilon$. This implies that

$$\limsup_{n\to\infty} \rho_v(x_0^* - x_n^*) \leq 2\varepsilon.$$

This holds for all $\varepsilon > 0$ and so we may conclude that $\rho_v(x_0^* - x_n^*) \to 0$ as $n \to \infty$. The proof is complete. ∎

The above proposition has the following consequence, which is the main result in the present section.

Theorem 5.3: *If E is a Banach lattice with order continuous norm and weak order unit, then E has property (K_{ob}).*

Proof. Since E has order continuous norm, each order interval in E is $\sigma(E, E^*)$-compact (see, e.g., [1], Theorem 12.9) and so $\sigma(E^*, E) \subseteq \mathcal{T}_{ob} \subseteq \tau(E^*, E)$. By Proposition 5.2, $\mathcal{T}_{ob}$ is metrizable on norm bounded subsets of E^* and hence, by Proposition 3.7, E has property (K_{ob}). ∎

Even for Banach lattices with a strong order unit, the converse of Theorem 5.3 is not valid. Indeed, it follows from the remarks following Definition 3.4 that each Grothendieck Banach lattice has property (K_{ob}). In particular, l^∞ has property (K_{ob}), but the norm on l^∞ is not order continuous.

In combination with Lemma 5.1, we obtain the following consequence of Theorem 5.3

Corollary 5.4: *Let E be a Banach lattice with order continuous norm and weak order unit. If (x_n^*) is a sequence in E^* satisfying $x_n^* \xrightarrow{\sigma(E^*,E)} 0$, then (x_n^*) has a CCC sequence (y_k^*) such that $|y_k^*| \to 0$ with respect to $\sigma(E^*, E)$.*

The above result is implicit in the proof of Proposition 4.5 in [12] and improves Sublemma 2.5 in [15].

6. Sets of uniformly absolutely continuous norm in non-commutative symmetric spaces

Let $\mathcal{M}$ be a semi-finite von Neumann algebra on a Hilbert space H, equipped with a fixed semi-finite, normal, faithful trace $\tau : \mathcal{M}^+ \to [0, \infty]$. We assume that $E \subseteq S(\tau)$ is a strongly symmetric space with carrier projection equal to $\mathbf{1}$, that is, $\bigvee\{s(x) : x \in E\} = \mathbf{1}$. In that case we have

$$(L_1 \cap L_\infty)(\tau) \subseteq E \subseteq (L_1 + L_\infty)(\tau),$$

with continuous embeddings. The hermitian part of E is denoted by E_h, that is,

$$E_h = \{a \in E : a = a^*\},$$

and the positive cone of E is denoted by E^+, that is,

$$E^+ = \{a \in E_h : a \geq 0\}.$$

For $w \in E^+$, the **order interval** $[-w, w] \subseteq E_h$ is defined by setting

$$[-w, w] = \{x \in E_h : -w \leq x \leq w\}.$$

It should be observed that any strongly symmetric space with order continuous norm is actually **fully symmetric**, that is, if $x \in S(\tau)$, $y \in E$ and $x \prec\prec y$, then $x \in E$ and $\|x\|_E \leq \|y\|_E$.

The following terminology and results will be used. See [7, Definition 3.3].

Definition 6.1: A subset $A \subseteq E$ is said to be of **uniformly absolutely continuous norm** if A is bounded and

$$\sup\{\|e_n x e_n\|_E : x \in A\} \to 0, \quad n \to \infty,$$

for all sequences $(e_n)_{n=1}^\infty$ in $P(\mathcal{M})$ satisfying $e_n \downarrow 0$.

Note that if the set $A \subseteq E$ is of uniformly absolutely continuous norm, then also the set $A^* = \{x^* : x \in A\}$ is of uniformly absolutely continuous norm and consequently, the sets $\operatorname{Re} A$ and $\operatorname{Im} A$ are both of uniformly absolutely continuous norm.

Examples of sets of uniformly absolutely continuous norm are provided in the following lemmas.

Lemma 6.2: *Suppose that E has order continuous norm. If $w \in E^+$, then the order interval $[-w, w]$ is of uniformly absolutely continuous norm.*

Proof. It should be observed that E^+ is a 2-normal cone, that is, it follows from $a \leq b \leq c$ in E_h that $\|b\|_E \leq 2\max(\|a\|_E, \|c\|_E)$ (see [5]). Indeed, it follows from $b \leq c$ that

$$0 \leq b^+ = e^b[0,\infty)be^b[0,\infty) \leq e^b[0,\infty)ce^b[0,\infty)$$

and so $\|b^+\|_E \leq \|e^b[0,\infty)ce^b[0,\infty)\|_E \leq \|c\|_E$. Similarly, $-b \leq -a$ implies that $\|b^-\|_E \leq \|a\|_E$, from which the claim follows.

Suppose now that $(e_n)_{n=1}^\infty$ in $P(\mathcal{M})$ is such that $e_n \downarrow 0$. If $x \in [-w, w]$, then $-e_nwe_n \leq e_nxe_n \leq e_nwe_n$ and so $\|e_nxe_n\|_E \leq 2\|e_nwe_n\|_E$ for all n. The order continuity of the norm implies that $\|e_nwe_n\|_E \to 0$ as $n \to \infty$ (see [7, Theorem 3.1]) and the result follows. ∎

Lemma 6.3: *If E has order continuous norm, then for every $y \in E$, the sets $yB_\mathcal{M}$ and $B_\mathcal{M}y$ are of uniformly absolutely continuous norm.*

Proof. Suppose that $(e_n)_{n=1}^\infty$ is a sequence in $P(\mathcal{M})$ satisfying $e_n \downarrow 0$. If $x \in B_\mathcal{M}$, then

$$\|e_nyxe_n\|_E \leq \|e_ny\|_E\|xe_n\|_\infty \leq \|e_ny\|_E$$

and so $\sup_{z\in yB_\mathcal{M}} \|e_nze_n\|_E \leq \|e_ny\|_E$. Since E has order continuous norm, it again follows from [7, Theorem 3.1] that $\|e_ny\|_E \to 0$ as $n \to \infty$ and we may conclude that $yB_\mathcal{M}$ is of uniformly absolutely continuous norm. The proof for $B_\mathcal{M}y$ is similar. ∎

The proposition which follows is a special case of [7, Theorem 3.12]. For the case of non-commutative L_p-spaces ($1 \leq p < \infty$), this result may be obtained from the paper [24] by Y. Raynaud and Q. Xu.

Proposition 6.4: *Suppose that $E \subseteq S(\tau)$ is a strongly symmetric space with order continuous norm and suppose that (p_n) is a sequence in $P(\mathcal{M})$ such that $p_n \uparrow_n \mathbf{1}$ and $\tau(p_n) < \infty$ for all n. If $A \subseteq E$ is of uniformly absolutely continuous norm, then for every $\varepsilon > 0$ there exists $n = n(\varepsilon) \in \mathbb{N}$ and $0 < C_\varepsilon \in \mathbb{R}$ such that*

$$A \subseteq C_\varepsilon(p_nB_\mathcal{M} + B_\mathcal{M}p_n) + \varepsilon B_E.$$

We shall need the following result, which is [7, Proposition 4.1].

Proposition 6.5: *Suppose that $E \subseteq S(\tau)$ has order continuous norm. If $A \subseteq E$ is of uniformly absolutely continuous norm, then E is relatively weakly compact.*

A particular consequence of the preceding proposition and Lemma 6.2 is that order intervals are relatively weakly compact in any strongly symmetric spaces $E \subseteq S(\tau)$ with order continuous norm.

7. Property $(K_{\mathfrak{S}})$ in symmetric spaces

As before, we assume that $E \subseteq S(\tau)$ is a strongly symmetric space, where $(\mathcal{M}, \tau)$ is a semi-finite von Neumann algebra. The collection of all subsets of E which are of uniformly absolutely continuous norm is denoted by $\mathfrak{S}_{an}$. The topology in $E^\times$ of uniform convergence on sets of $\mathfrak{S}_{an}$ is denoted by $\mathcal{T}_{an}$. Assuming that E has order continuous norm, it follows from Lemma 6.5 that

$$\sigma(E^*, E) = \sigma(E^\times, E) \subseteq \mathcal{T}_{an} \subseteq \tau(E^\times, E) = \tau(E^*, E),$$

where, as before, $\tau(E^\times, E)$ denotes the Mackey topology. The following result is one of the key ingredients in the present section. In contrast with some of the previous results, the hypothesis of being σ-finite is essential in the following theorem.

Theorem 7.1: *Suppose that $(\mathcal{M}, \tau)$ is a semi-finite and σ-finite von Neumann algebra. If $E \subseteq S(\tau)$ is a strongly symmetric space with order continuous norm, then the topology $\mathcal{T}_{an}$ is metrizable on norm bounded subsets of $E^\times$.*

Proof. Since $\mathcal{M}$ is semi-finite and σ-finite, there exists a sequence $(p_n)_{n=1}^\infty$ of projections in $P(\mathcal{M})$ such that $p_n \uparrow \mathbf{1}$ and $\tau(p_n) < \infty$ for all $n \in \mathbb{N}$. Define the set $W_n \subseteq E$ by setting

$$W_n = p_n B_{\mathcal{M}} + B_{\mathcal{M}} p_n, \quad n \in \mathbb{N},$$

and define the semi-norms $\rho_n : E^\times \to [0, \infty)$ by

$$\rho_n(y) = \sup_{x \in W_n} |\langle x, y \rangle|, \quad y \in E^\times.$$

Observe that the semi-norms ρ_n, $n = 1, 2, \ldots$, separate the points of $E^\times$. Indeed, if $y \in E^\times$ is such that $\rho_n(y) = 0$ for all n, then, in particular, $\tau(p_n x y) = 0$ for all $x \in B_{\mathcal{M}}$ and all n. Taking $x = v^*$, where $y = v|y|$ is the polar decomposition of y, it follows that $\tau(p_n|y|) = 0$ for all n. Since $p_n \uparrow \mathbf{1}$, the normality of the trace implies that $y = 0$.

Let $\mathcal{T}_0$ be the metrizable locally convex topology in $E^\times$ generated by $\{\rho_n : n \in \mathbb{N}\}$. By Lemma 6.3, each of the sets W_n is of uniformly absolutely continuous norm and so $\tau_0 \subseteq \tau_{an}$. We claim that $\mathcal{T}_{an}$ and $\mathcal{T}_0$ coincide on norm bounded subsets of $E^\times$. It suffices to show that $\mathcal{T}_{an}$ and $\mathcal{T}_0$ coincide on the closed unit ball $B_{E^\times}$. To this end, suppose that $(y_k)_{k=1}^\infty$ is a sequence in $B_{E^\times}$ and that $y \in B_{E^\times}$ such that $y_k \to y$ with respect to $\mathcal{T}_0$. Let $A \subseteq E$ be a set of uniformly absolutely continuous norm. Given $\varepsilon > 0$, it follows from Proposition 6.4 that there exist $n \in \mathbb{N}$ and $0 < C_\varepsilon \in \mathbb{R}$ such that

$$A \subseteq C_\varepsilon W_n + \varepsilon B_E.$$

If $x \in A$, then $x = x_1 + x_2$ with $x_1 \in C_\varepsilon W_n$ and $x_2 \in \varepsilon B_E$ and so

$$\begin{aligned} |\langle x, y - y_k\rangle| &\leq |\langle x_1, y - y_k\rangle| + |\langle x_2, y - y_k\rangle| \\ &\leq C_\varepsilon \rho_n(y - y_k) + 2\varepsilon, \end{aligned}$$

which shows that

$$\sup_{x\in A} |\langle x, y - y_k\rangle| \leq C_\varepsilon \rho_n(y - y_k) + 2\varepsilon$$

for all k. By hypothesis, $\rho_n(y - y_k) \to 0$ as $k \to \infty$ and so

$$\limsup_{k\to\infty} \sup_{x\in A} |\langle x, y - y_k\rangle| \leq 2\varepsilon.$$

This holds for all $\varepsilon > 0$, hence $\lim_{k\to\infty} \sup_{x\in A} |\langle x, y - y_k\rangle| = 0$. This holding for any set $A \subseteq E$ which is of uniformly absolutely continuous norm, we may conclude that $y_k \to y$ with respect to $\mathcal{T}_{an}$. This suffices to complete the proof. ∎

We will say that a strongly symmetric space $E \subseteq S(\tau)$ has property (K_{an}) if it has property $(K_{\mathfrak{S}_{an}})$. The following corollary is now an immediate consequence of Theorem 7.1 in combination with Proposition 3.7 and Proposition 6.5.

Corollary 7.2: *Suppose that $(\mathcal{M}, \tau)$ is a semi-finite and σ-finite von Neumann algebra. If $E \subseteq S(\tau)$ is a strongly symmetric space with order continuous norm, then E has property (K_{an}), that is, every sequence (z_n) in $E^\times$ satisfying $z_n \to 0$ with respect to $\sigma(E^\times, E)$ has a CCC sequence (y_k) such that $y_k \to 0$ uniformly on all subsets of E which are of uniformly absolutely continuous norm.*

Before formulating the next result, it is convenient to introduce the following terminology.

Definition 7.3: Let $E \subseteq S(\tau)$ be a strongly symmetric space. A subset $A \subseteq E$ is called **order-bounded** if there exist $w, v \in E^+$ such that

$$A \subseteq [-w, w] + i[-v, v].$$

Note that a subset $A \subseteq E$ is order-bounded if and only if there exists $w \in E^+$ such that

$$A \subseteq [-w, w] + i[-w, w]$$

(indeed, replace w and v in the definition by $w + v$). The collection of all order-bounded subsets of E will be denoted by $\mathfrak{S}_{ob}$. If E has order continuous norm, then it follows from Lemma 6.2 that every order-bounded subset of E is of uniformly absolutely continuous norm, that is, $\mathfrak{S}_{ob} \subseteq \mathfrak{S}_{an}$.

A strongly symmetric space $E \subseteq S(\tau)$ is said to have property (K_{ob}) if it has property $(K_{\mathfrak{S}_{ob}})$. In view of the above observations, the following result is now clear.

Corollary 7.4: *Suppose that $(\mathcal{M}, \tau)$ is a semi-finite and σ-finite von Neumann algebra. If $E \subseteq S(\tau)$ is a strongly symmetric space with order continuous norm, then E has property (K_{ob}).*

Proof. By Corollary 7.2, E has property (K_{an}). Since $\mathfrak{S}_{ob} \subseteq \mathfrak{S}_{an}$, it follows that E has property (K_{ob}). ∎

The following proposition is another consequence of Corollary 7.2 which may be noteworthy. It will be convenient to prove first the next lemma. For $x \in S(\tau)$ we denote

$$\Omega(x) = \{y \in S(\tau) : y \prec\prec x\}.$$

If $E \subseteq S(\tau)$ is a strongly symmetric space with order continuous norm, then $\Omega(x) \subseteq E$ for every $x \in E$, as E is fully symmetric.

Lemma 7.5: *If $\tau(\mathbf{1}) < \infty$ and $E \subseteq S(\tau)$ is a strongly symmetric space with order continuous norm, then for each $x \in E$, the set $\Omega(x)$ is of uniformly absolutely continuous norm.*

Proof. Let $x \in E$ be given. We claim that for every $\varepsilon > 0$ there exists $0 < C_\varepsilon \in \mathbb{R}$ such that

$$\Omega(x) \subseteq C_\varepsilon B_{\mathcal{M}} + \varepsilon B_E.$$

Indeed, since E has order continuous norm and $\tau(\mathbf{1}) < \infty$, it follows from [9, Proposition 1.8], that $\mathcal{M} = L_\infty(\tau)$ is norm dense in E. Therefore, there exists

$x_1 \in \mathcal{M}$ such that $\|x - x_1\|_E \leq \varepsilon$. If $y \in \Omega(x)$, then

$$y \prec\prec x = x_1 + (x - x_1)$$

and so, there exist $y_1, y_2 \in S(\tau)$ such that $y = y_1 + y_2$, $y_1 \prec\prec x_1$ and $y_2 \prec\prec x - x_1$ (see [9], Proposition 4.10). This implies that $\|y_1\|_\infty \leq \|x_1\|_\infty$ and $\|y_2\|_E \leq \|x - x_1\|_E$ and hence

$$y \in C_\varepsilon B_{\mathcal{M}} + \varepsilon B_E,$$

with $C_\varepsilon = \|x_1\|_\infty$. This proves the claim.

By Lemma 6.3, the set $B_{\mathcal{M}}$ is of uniformly absolutely continuous norm in E. It now follows easily that $\Omega(x)$ is also of uniformly absolutely continuous norm. ∎

Proposition 7.6: *Suppose that $\tau(\mathbf{1}) < \infty$ and that $E \subseteq S(\tau)$ is a strongly symmetric space with order continuous norm. If $(z_n)_{n=1}^\infty$ is a sequence in $E^\times$ such that $z_n \to 0$ with respect to $\sigma(E^\times, E)$, then there is a CCC sequence (y_k) of (z_n) such that*

$$\int_0^\infty \mu(t; x)\mu(t; y_k)dt \to 0, \quad k \to \infty$$

for every $x \in E$.

Proof. Suppose that $(z_n)_{n=1}^\infty$ is a sequence in $E^\times$ such that $z_n \to 0$ with respect to $\sigma(E^\times, E)$. By Corollary 7.2, the space E has property (K_{an}) and so there is a CCC sequence (y_k) of (z_n) such that $y_k \to 0$ uniformly on subsets of E which are of uniformly absolutely continuous norm. By Lemma 7.5, this implies that

$$\sup_{y \in \Omega(x)} |\langle y, y_k \rangle| \to 0$$

for all $x \in E$.

Recalling that

$$\int_0^\infty \mu(t; x)\mu(t; y_k)dt = \sup\{|\tau(yy_k)| : y \prec\prec x\} = \sup_{y \in \Omega(x)} |\langle y, y_k \rangle|$$

(see [8], Theorem 4.12), the result of the proposition follows. ∎

Remark 7.7: If $\tau(\mathbf{1}) = \infty$, then the result of Proposition 7.6 does not hold. By way of example, let $E = L_1(0, \infty)$ and define the sequence (x_n) in $E^\times = L_\infty(0, \infty)$ by setting $x_n = \chi_{(n,\infty)}$ for all $n \in \mathbb{N}$. If (y_k) is any CCC sequence of (x_n), then for every k there exists $N_k \in \mathbb{N}$ such that $y_k(t) = 1$ for all $t \geq N_k$ and so $\mu(y_k) = \mathbf{1}$

for all k. Consequently, if $x \in L_1(0,\infty)$, then $\int_0^\infty \mu(t;x)\mu(t;y_k)dt = \|x\|_1$ for all k.

8. Property (k) in symmetric spaces

The following characterization of property (k) will be convenient.

Lemma 8.1: *For a Banach space X the following statements are equivalent:*

(i) *X has property (k);*

(ii) *every weak* null sequence $(x_n^*)_{n=1}^\infty$ in X^* has a CCC sequence $(y_k^*)_{k=1}^\infty$ such that for every bounded linear operator $T : L_1(0,1) \to X$ and for every order-bounded set $A \subseteq L_1(0,1)$ we have*

$$\sup_{f\in A} |\langle Tf, y_k^*\rangle| \to 0, \quad k \to \infty;$$

(iii) *every weak* null sequence $(x_n^*)_{n=1}^\infty$ in X^* has a CCC sequence $(y_k^*)_{k=1}^\infty$ such that for every bounded linear operator $T : L_1(0,1) \to X$ and for every relatively weakly compact set $A \subseteq L_1(0,1)$ we have*

$$\sup_{f\in A} |\langle Tf, y_k^*\rangle| \to 0, \quad k \to \infty.$$

Proof. (i)⇒(ii). Let $(x_n^*)_{n=1}^\infty$ be a weak* null sequence in X^* and let (y_k^*) be a CCC sequence of (x_n^*) satisfying the condition of Definition 4.5. First, suppose that $A \subseteq L_1(0,1)$ satisfies $A \subseteq [-\mathbf{1},\mathbf{1}]$. If $\sup_{f\in A} |\langle Tf, y_k^*\rangle| \nrightarrow 0$ as $k \to 0$, then, by passing to a subsequence if necessary, we may assume that $|\langle Tf_k, y_k^*\rangle| \geq \delta > 0$ for all k and some sequence (f_k) in A. Since $[-\mathbf{1},\mathbf{1}]$ is weakly compact, we may also assume (by passing to a further subsequence) that $f_k \xrightarrow{\sigma(L_1,L_\infty)} f \in [-\mathbf{1},\mathbf{1}]$. Since $f_k - f \xrightarrow{\sigma(L_1,L_\infty)} 0$ and $\|f_k - f\|_\infty \leq 2$ for all k, it follows from the property of the sequence (y_k^*) that $\langle Tf_k - Tf, y_k^*\rangle \to 0$ as $k \to \infty$. Since (y_k^*) is a weak* null sequence (being a CCC sequence of the weak* null sequence (x_n^*)), it follows that $\langle Tf, y_k^*\rangle \to 0$ and hence, $\langle Tf_k, y_k^*\rangle \to 0$. This is a contradiction and so we may conclude that $\sup_{f\in A} |\langle Tf, y_k^*\rangle| \to 0$ as $k \to 0$.

Suppose now that A is an arbitrary order-bounded subset of $L_1(0,1)$, that is, $A \subseteq [-w,w]$ for some $0 \leq w \in L_1(0,1)$. Since $\|w - w \wedge n\mathbf{1}\|_1 \to 0$ as $n \to \infty$, it follows that for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $A \subseteq n[-\mathbf{1},\mathbf{1}] + \varepsilon B_{L_1}$. Since, by the first part of the present proof, $\sup_{f\in[-\mathbf{1},\mathbf{1}]} |\langle Tf, y_k^*\rangle| \to 0$, it now follows easily that $\sup_{f\in A} |\langle Tf, y_k^*\rangle| \to 0$ as $k \to 0$.

(ii)⇒(iii). If $A \subseteq L_1(0,1)$ is relatively weakly compact, then it follows from the well-known Dunford–Pettis theorem (see [19] Theorem 2.5.4) that for every

$\varepsilon > 0$ there exists $0 < w \in L_1(0,1)$ such that $A \subseteq [-w, w] + \varepsilon B_{L_1}$. The implication now readily follows.

(iii)⇒(i). This is evident, since for any weak null sequence (f_k) in $L_1(0,1)$ the set $A = \{f_k : k \in \mathbb{N}\}$ is relatively weakly compact. ∎

Remark 8.2: As was observed in [12], Remark 4.4, the condition $\sup_k \|f_k\|_\infty < \infty$ may be omitted in Definition 4.5. Note that this follows immediately from the equivalence (i)⇔(iii) in Lemma 8.1.

Let $\mathcal{M}$ be a semi-finite von Neumann algebra equipped with a fixed semi-finite normal faithful trace $\tau : \mathcal{M}^+ \to [0, \infty]$. The main purpose of the present section is to show that any strongly symmetric KB-space $E \subseteq S(\tau)$ has property (k) whenever $\mathcal{M}$ is σ-finite (Theorem 8.6). One of the main ingredients in the proof is Proposition 8.5. For the proof of this proposition we need some preparation.

Recall the following definitions and facts concerning ordered Banach spaces:

(1) Suppose that $(V, \|\cdot\|)$ is an ordered Banach space, that is, V is a real Banach space with a closed positive cone V^+ which induces a partial ordering in V. The positive cone is called **proper** if $V^+ \cap (-V^+) = \{0\}$ and it is called **generating** if $V = V^+ - V^+$.

(2) The positive cone V^+ of an ordered Banach space V is called α-**normal** (for some $0 < \alpha \in \mathbb{R}$) if it follows from $a \leq x \leq b$ in V that $\|x\| \leq \alpha \max(\|a\|, \|b\|)$. Note that any α-normal positive cone is proper. The cone V^+ is called α-**generating** if every $x \in V$ admits a decomposition $x = x_1 - x_2$, where $x_1, x_2 \in V^+$ and $\|x_1\| + \|x_2\| \leq \alpha \|x\|$.

(3) Let $(V, \|\cdot\|)$ be an ordered Banach space. The dual positive cone $(V^*)^+$ in V^* is defined by setting
$$(V^*)^+ = \{\phi \in V^* : \phi(x) \geq 0 \ \forall x \in V^+\}.$$

(4) According to a theorem of J. Grosberg and M. Krein (1939), if V is an ordered Banach space with α-normal cone V^+, then the dual cone $(V^*)^+$ is α-generating in V^* (see, e.g., [14], Theorem 3.6.7).

(5) A Banach lattice E is called **injective** if, whenever F is a Banach lattice, $G \subseteq F$ a Banach sublattice and $T : G \to E$ is a positive linear operator, there exists a positive linear operator $\hat{T} : F \to E$ such that $\hat{T}\mid_G = T$ and $\|\hat{T}\| = \|T\|$ (see [19], Definition 3.2.3).

(6) Suppose that F is a Banach lattice and that E is a Banach sublattice of F. If E is injective, then there exists a positive projection $P : F \to F$

such that $P(F) = E$ and $\|P\| = 1$. Indeed, the identity operator $I_E : E \to E$ has a positive extension $P : F \to E \subseteq F$.

(7) If K is an extremally disconnected compact Hausdorff space, then the Banach lattice $C(K; \mathbb{R})$ is injective (indeed, $C(K; \mathbb{R})$ is a Dedekind complete M-space with unit; see [19], Theorem 3.2.4).

The proposition which follows and its dual version Proposition 8.5 may be a kind of folklore, but we were not able to trace the relevant proofs in the literature.

Proposition 8.3: *If V is an ordered Banach space with an α-normal positive cone V^+, then every bounded linear map $T : V \to L_\infty^{\mathbb{R}}(0,1)$ is* **regular**, *that is, there exist positive bounded linear maps $T_j : V \to L_\infty^{\mathbb{R}}(0,1)$, $j = 1, 2$, such that*

$$T = T_1 - T_2$$

(and $\|T_j\| \le \alpha \|T\|$).

Proof. The space $L_\infty^{\mathbb{R}}(0,1)$ is Banach lattice isometrically isomorphic to a space $C(K; \mathbb{R})$, where K is an extremally disconnected compact Hausdorff space. Therefore, we may consider $T : V \to C(K; \mathbb{R})$. Let $\ell_\infty(K; \mathbb{R})$ be the Banach lattice of all real bounded functions on K (equipped with the sup-norm). Since $C(K; \mathbb{R})$ is a Banach sublattice of $\ell_\infty(K; \mathbb{R})$ and $C(K; \mathbb{R})$ is an injective Banach lattice, there exists a positive projection $P : \ell_\infty(K; \mathbb{R}) \to \ell_\infty(K; \mathbb{R})$ onto the subspace $C(K; \mathbb{R})$ with $\|P\| = 1$.

For $t \in K$ we define the positive linear functional δ_t on $\ell_\infty(K; \mathbb{R})$ by setting

$$\langle f, \delta_t \rangle = f(t), \quad f \in \ell_\infty(K; \mathbb{R}).$$

For each $t \in K$ define $\phi_t \in V^*$ by setting

$$\langle x, \phi_t \rangle = \langle Tx, \delta_t \rangle, \quad x \in V.$$

Note that $\|\phi_t\| \le \|T\|$ for all $t \in K$. Since the positive cone V^+ is assumed to be α-normal, the dual cone $(V^*)^+$ is α-generating and so for each $t \in K$ there exist $\phi_t^{(1)}, \phi_t^{(2)} \in (V^*)^+$ such that

$$\phi_t = \phi_t^{(1)} - \phi_t^{(2)}, \quad \|\phi_t^{(1)}\| + \|\phi_t^{(2)}\| \le \alpha \|\phi_t\|.$$

For $j = 1, 2$, define the linear operator $S_j : V \to \ell_\infty(K; \mathbb{R})$ by setting

$$(S_j x)(t) = \langle x, \phi_t^{(j)} \rangle, \quad t \in K, x \in V.$$

Note that

$$|(S_jx)(t)| = |\langle x, \phi_t^{(j)}\rangle| \leq \|\phi_t^{(j)}\|\|x\| \leq \alpha\|\phi_t\|\|x\| \leq \alpha\|T\|\|x\|, \quad t \in K,$$

so $S_jx \in \ell_\infty(K;\mathbb{R})$ and $\|S_jx\|_\infty \leq \alpha\|T\|\|x\|$, $x \in V$. Hence the operators S_j are bounded with $\|S_j\| \leq \alpha\|T\|$, $j = 1, 2$. Moreover, since the functionals $\phi_t^{(j)}$ are positive, it is also clear that the operators S_j are positive.

For every $x \in V$ we have

$$(Tx)(t) = \langle x, \phi_t\rangle = \langle x, \phi_t^{(1)}\rangle - \langle x, \phi_t^{(2)}\rangle = (S_1x)(t) - (S_2x)(t), \quad t \in K,$$

and so $T = S_1 - S_2$ (considering T as a map from V into $\ell_\infty(K;\mathbb{R})$).

Defining $T_j = PS_j$, $j = 1, 2$, it follows that $T_j : V \to C(K;\mathbb{R})$ are positive bounded linear operators such that $T = T_1 - T_2$ and $\|T_j\| \leq \alpha\|T\|$. ∎

Now we return to the situation where $(\mathcal{M}, \tau)$ is a semi-finite von Neumann algebra and $(E, \|\cdot\|_E)$ is a Banach $\mathcal{M}$-bimodule. It should be observed that the positive cone E^+ is 2-normal (see the proof of Lemma 6.2).

Corollary 8.4: *If $E \subseteq S(\tau)$ is a Banach $\mathcal{M}$-bimodule, then every bounded linear operator $T : E \to L_\infty(0, 1)$ is regular.*

Proof. Writing $T = \operatorname{Re} T + i \operatorname{Im} T$, it is sufficient to consider the case that T is hermitian. Let

$$S = T_h : E_h \to L_\infty^{\mathbb{R}}(0, 1)$$

be the corresponding operator $S \in \mathcal{L}(E_h, L_\infty^{\mathbb{R}}(0, 1))$ as defined in Section 2 via (1). As observed above, E_h is an ordered Banach space with 2-normal cone E^+. Hence it follows from Proposition 8.3 that there exist $S_j \in \mathcal{L}^+(E_h, L_\infty^{\mathbb{R}}(0, 1))$, $j = 1, 2$, such that $S = S_1 - S_2$. Now define $T_j \in \mathcal{L}^+(E, L_\infty^{\mathbb{R}}(0, 1))$, $j = 1, 2$, such that $S_j = (T_j)_h$ (see Lemma 2.3). It is now clear that $T = T_1 - T_2$ and so we may conclude that T is regular. ∎

The following result is well-known in the Banach lattice setting (see, e.g., [1], Theorem 15.3).

Proposition 8.5: *If $E \subseteq S(\tau)$ is a strongly symmetric KB-space, then every bounded linear operator $T : L_1(0, 1) \to E$ is regular.*

Proof. As in the proof of Corollary 8.4, it is sufficient to consider the case that T is hermitian. Consider now the adjoint operator $T^* : E^* = E^\times \to L_\infty(0, 1)$

and observe that T^* is hermitian as well. Indeed, if $y = y^* \in E^\times$, then for all $f \in L_1(0,1)$ we have

$$\begin{aligned}\langle f, \overline{T^*y}\rangle =& \overline{\langle \overline{f}, T^*y\rangle} = \overline{\langle T(\overline{f}), y\rangle} = \overline{\langle (Tf)^*, y\rangle}\\ =& \langle Tf, y^*\rangle = \langle Tf, y\rangle = \langle f, T^*y\rangle\end{aligned}$$

and , $\overline{T^*y} = T^*y$, that is, $T^*y \in L^{\mathbb{R}}_\infty(0,1)$.

It follows from Corollary 8.4 that there exist positive operators

$$S_j \in \mathcal{L}^+(E^\times, L_\infty(0,1)), \quad j = 1,2,$$

such that $T^* = S_1 - S_2$. This implies that $T^{**} = S_1^* - S_2^*$ and it is clear that S_1^* and S_2^* are positive operators. Let $P_n : (E^\times)^* \to (E^\times)^*$ be the Yosida–Hewitt projection (see Section 2). In particular, P_n is a positive projection onto $(E^\times)^*_n \cong E^{\times\times}$. Since E has the Fatou property, we also have $E^{\times\times} = E$. Hence P may be considered as a positive linear map $P : (E^\times)^* \to E$. Defining

$$T_j = PS_j^* \mid_{L_1(0,1)} : L_1(0,1) \to E, \quad j = 1,2,$$

it is now clear that $T_j \in \mathcal{L}^+(L_1(0,1), E)$ and $T = T_1 - T_2$. Therefore, we may conclude that T is regular. ∎

It was shown by Figiel, Johnson and Pelczyński (see [12], Proposition 4.5) that any weakly sequentially complete Banach lattice with a weak unit has property (k). Keeping in mind that a Banach lattice is weakly sequentially complete if and only if it is a (KB) space, we now present a non-commutative counterpart which is the principal result of this section.

Theorem 8.6: *Let $\mathcal{M}$ be a semi-finite and σ-finite von Neumann algebra. If $E \subseteq S(\tau)$ is a strongly symmetric KB-space, then E has property (k).*

Proof. We will show that E satisfies condition (ii) of Lemma 8.1. Let (x_n) be a sequence in $E^* = E^\times$ such that $x_n \xrightarrow{\sigma(E^\times, E)} 0$. By Corollary 7.4, E has property (K_{ob}) and so there is a CCC sequence (y_k) of (x_n) such that $y_k \to 0$ uniformly on all order-bounded subsets of E. Let $T : L_1(0,1) \to E$ be a bounded linear operator and $A \subseteq L_1(0,1)$ an order-bounded set. By Proposition 3.7, T is regular from which it follows that the set $T(A)$ is order-bounded in E. Consequently,

$$\sup_{f\in A} |\langle Tf, y_k^*\rangle| = \sup_{z\in T(A)} |\langle z, y_k^*\rangle| \to 0, \quad k \to \infty.$$

This suffices to complete the proof of the theorem. ∎

This theorem has the following immediate consequence.

COROLLARY 8.7: *Let $\mathcal{M}$ be a semi-finite and σ-finite von Neumann algebra. If E is a symmetric KB-space on $[0,\tau(\mathbf{1}))$, then the corresponding non-commutative symmetric space $E(\tau) \subseteq S(\tau)$ has property (k).*

Proof. Since E is a symmetric KB-space on $[0,\tau(\mathbf{1}))$, it follows that $E(\tau)$ is a strongly symmetric KB-space. Indeed this follows from [9], Proposition 3.6 and Proposition 5.2.

Hence the result follows now from Theorem 8.6. ∎

As observed earlier (see the remarks following Corollary 4.7), the assumption that $\mathcal{M}$ be σ-finite is essential, even in the case that $\mathcal{M}$ is commutative.

9. Property (K) in non-commutative symmetric spaces

It might be observed that if $(\mathcal{M},\tau)$ is a semi-finite, σ-finite von Neumann algebra and if $E \subseteq S(\tau)$ is any strongly symmetric space with order continuous norm with the further property that each relatively weakly compact set is of uniformly absolutely continuous norm, then it follows from Corollary 7.2 that E has property (K). For the special case that $E = L^1(\tau)$, this is a special case of Theorem 7.3. However, there is a large class of spaces for which the notions of relative weak compactness and uniformly absolutely continuous norm coincide. For the case that $\tau(\mathbf{1}) < \infty$, such spaces may be characterised as those with the property that sequences which are null sequences for both the weak and measure topologies are null sequences for the norm topology. Spaces with this latter property are said to have Property (Wm). The following result is proved in [7], Proposition 6.10.

PROPOSITION 9.1: *Suppose $E \subseteq S(\tau)$ is strongly symmetric. If $\tau(\mathbf{1}) < \infty$ then the following statements are equivalent:*

(i) *E has property (Wm).*
(ii) *Each relatively weakly compact set in E is of uniformly absolutely continuous norm.*

It is worth noting that if $\tau(\mathbf{1}) < \infty$ and if $E \subseteq S(\tau)$ is a strongly symmetric space with property (Wm), then E is a KB-space. See [7], Lemma 6.9.

From the preceding proposition, and Corollary 7.2, we now obtain the following consequence.

Corollary 9.2: *Suppose $E \subseteq S(\tau)$ is strongly symmetric. If $\tau(\mathbf{1}) < \infty$ and if E has property (Wm), then E has property (K).*

By way of example, let ϕ denote an increasing concave function on $[0, \tau(\mathbf{1}))$ for which $\phi(0) = 0 = \phi(0+)$ and let Λ_ϕ be the corresponding Lorentz space on $[0, \tau(\mathbf{1}))$ with norm given by

$$\|x\|_{\Lambda_\phi} = \int_{[0,\tau(\mathbf{1}))} \mu(x) d\phi.$$

It is shown in [7, Corollary 6.14] that each Lorentz space $\Lambda_\phi(\tau)$ has property (Wm).

Corollary 9.3: *If $\tau(\mathbf{1}) < \infty$, then the Lorentz space $\Lambda_\phi(\tau)$ has property (K).*

It is, perhaps, not without interest to note that the spaces $\Lambda(\phi)$ are not Grothendieck spaces.

Suppose now that Φ is an increasing convex function on $[0, \infty)$ such that $\Phi(0) = 0$. Let L_Φ be the corresponding Orlicz space on $[0, \tau(\mathbf{1}))$ equipped with the norm

$$\|x\|_{L_\Phi} = \inf \left\{ \lambda : \lambda > 0, \int_{[0,\tau(\mathbf{1})} \Phi(|x(t)|/\lambda) dt \leq 1 \right\},$$

and let Ψ be the complementary function. The class of Orlicz spaces L_Φ for which the complementary function Ψ satisfies the condition

$$\lim_{t\to\infty} \Psi(Ct)/\Psi(t) = \infty$$

for some $C > 0$ will be denoted by Δ_3. It is shown in [7, Proposition 6.19] that if $\tau(\mathbf{1}) < \infty$ and if $L_\Phi \in \Delta_3$ then the corresponding non-commutative space $L_\Phi(\tau)$ has property (Wm). Consequently,

Corollary 9.4: *If $\tau(\mathbf{1}) < \infty$ and if $L_\Phi \in \Delta_3$, then the Orlicz space $L_\Phi(\tau)$ has property (K).*

We recall that a Banach lattice E is said to be a KB**-space** if every norm-bounded upwards directed system in E is convergent, and to have the **Fatou property** if $0 \leq x_\alpha \uparrow_\alpha \subseteq E$ and $\sup_\alpha \|x_\alpha\|_E < \infty$ implies that there exists $0 \leq x \in E$ such that $x = \sup x_\alpha$ holds in E and $\|x\|_E = \sup_\alpha \|x_\alpha\|_E$.

It is well known that the Banach lattice E is a KB-space if and only if E has order continuous norm and has the Fatou property.

PROPOSITION 9.5: *Let E be a Banach lattice with order continuous norm. If E has property (k) (in particular, if E has property (K)), then E has the Fatou property.*

Proof. If E does not have the Fatou property, then E is not a KB-space. It follows from [19, Theorem 2.4.12] that there exists a closed vector sublattice F of E which is a vector lattice and norm isomorphic to c_0. Since the norm on E is order continuous, it follows from [19] Corollary 2.4.3 that F is complemented in E. By [12, Proposition 4.9], it follows that F does not have property (k). It then follows that E does not have property (K) and this is a contradiction. ∎

Now suppose that $\tau(\mathbf{1}) = 1$ and that $\mathcal{N} \subseteq \mathcal{M}$ is a von Neumann subalgebra with trace σ given by the restriction $\tau|\mathcal{N}$ of τ to $\mathcal{N}$. The conditional expectation

$$\mathcal{E}_{\mathcal{N}} : L^1(\tau) \to L^1(\sigma)$$

is defined as in the commutative setting via the equality

$$\sigma(x\mathcal{E}_{\mathcal{N}}(y)) = \tau(xy), \quad x \in \mathcal{N}, \quad y \in L^1(\tau) \tag{5}$$

and an appeal to the fact that $L^1(\sigma), \mathcal{N}$ are dual in the sense of Köthe. See [9, Theorem 5.6]. The following observation will be needed. See [10, Lemma 5.1].

LEMMA 9.6: *If E is a fully symmetric space on $[0, \tau(\mathbf{1}))$, then $\mathcal{E}_{\mathcal{N}}(y) \in E(\sigma)$ for all $y \in E(\tau)$ and*

$$\sigma(x\mathcal{E}_{\mathcal{N}}(y)) = \tau(xy), \quad x \in E(\sigma)^{\times}, \quad y \in E(\tau). \tag{6}$$

Observe that $\mathcal{M}$ may be identified via the map $x \to x \otimes \mathbf{1}$ with the von Neumann subalgebra $\mathcal{M} \otimes \mathbb{C}\mathbf{1}$ of the von Neumann tensor product $\mathcal{M} \overline{\otimes} L^\infty[0,1)$ equipped with the tensor product trace $\tau \otimes dm$, with dm denoting Lebesgue measure. Let $\mathcal{E}$ denote the corresponding conditional expectation. Using Lemma 9.6, the result which follows is proved in [10, Theorem 5.2].

LEMMA 9.7: *If E is a fully symmetric space on $[0,1)$, then the mapping*

$$x \to x \otimes \mathbf{1} \in E(\tau \otimes dm)^{\times}, \quad x \in E(\tau)^{\times}$$

is $\sigma(E(\tau)^{\times}, E(\tau))$ to $\sigma(E(\tau \otimes dm)^{\times}, E(\tau \otimes dm))$ continuous.

PROPOSITION 9.8: *Suppose that E is a symmetric space on $[0,1)$ with order continuous norm. If E has property (K) then $E(\tau)$ has property (K).*

Proof. We assume first that $\mathcal{M}$ is non-atomic, that is, $\mathcal{M}$ does not have any minimal projections. Since E has order continuous norm, it follows from [9, Proposition 3.6] that $E(\tau)$ has order continuous norm and so $E(\tau)^* = E(\tau)^\times$. Suppose now that $(x_n)_{n=1}^\infty$ is a sequence in $E(\tau)^\times$ such that $x_n \to_n 0$ for the weak topology $\sigma(E(\tau)^\times, E(\tau))$. It follows from Proposition 7.6 that there exists a CCC sequence (y_n) of (x_n) such that

$$\int_{0,1} \mu(t;x)\mu(t;y_n)dt \to_n 0, \quad \forall x \in E(\tau).$$

This implies that $\mu(y_n) \to_n 0$ for the weak topology $\sigma(E^\times, E)$. Indeed, suppose that $f \in E$. Using the fact that $\mathcal{M}$ is non-atomic, let $\pi : E \to E(\tau)$ be an isometric *-isomorphism which preserves singular values. See, for example, [2], Proposition 2.2 and Remark 2.1. It follows that

$$\begin{aligned}\left|\int_{[0,1)} f(t)\mu(t;y_n)dt\right| &\le \int_{[0,1)} \mu(t,f)\mu(t,y_n)dt \\ &= \int_{[0,1)} \mu(t;\pi(f))\mu(t;y_n)dt \to_n 0.\end{aligned}$$

Since E has property (K), there exists a CCC sequence $(g_j)_{j=1}^\infty$ of the sequence $(\mu(y_k))_{k=1}^\infty$ such that

$$\sup_{f\in B}\int_{[0,1)} f(t)g_j(t)dt \to_j 0 \tag{7}$$

for each relatively $\sigma(E,E^*) = \sigma(E,E^\times)$ compact subset $B \subseteq E$. For each $j \ge 1$, set

$$g_j = \sum_{i=n_j}^{n_{j+1}-1} \alpha_{ij}\mu(y_i), \quad \sum_{i=n_j}^{n_{j+1}-1} \alpha_{ij} = 1, \quad \alpha_{ij} \ge 0, \quad n_1 < n_2 < n_3 \cdots.$$

Now set

$$z_j = \sum_{i=n_j}^{n_{j=1}-1} \alpha_{ij} y_i, \quad i \ge 1.$$

It follows that $(z_j)_{j=1}^\infty \subseteq E(\tau)^\times$ is a CCC sequence of the sequence $(y_k)_{k=1}^\infty$ and since the latter is a CCC sequence of the sequence $(x_n)_{n=1}^\infty$, it follows also that

$(z_j)_{j=1}^{\infty}$ is a CCC sequence of the sequence $(x_n)_{n=1}^{\infty}$. Note that

$$\mu(z_j) \prec\prec \sum_{i=n_j}^{n_{j+1}-1} \alpha_{ij}\mu(y_i) = g_j, \quad j \geq 1.$$

Suppose now that $A \subseteq E(\tau)$ is relatively $\sigma(E, E^{\times})$ compact and set

$$B = \{\mu(x) : x \in A\}.$$

By Proposition 9.5, E has the Fatou property. This implies that the natural embedding of E into $E^{\times\times}$ is a surjective isometry, so that E may be identified with $E^{\times\times}$. It now follows from [8, Proposition 2.10] (or [10, Theorem 5.4]) that B is relatively $\sigma(E, E^{\times}) = \sigma(E, E^*)$ compact. Now observe that

$$\begin{aligned}\sup_{x\in A} |\tau(xz_j)| &\leq \sup_{x\in A} \int_{[0,1)} \mu(t;x)\mu(t;z_j)dt \leq \sup_{x\in A} \int_{[0,1)} \mu(t;x)\mu(t;g_j)dt \\ &\leq \sup_{f\in B} \int_{[0,1)} f(t)\mu(t;g_j)dt \to_j 0,\end{aligned}$$

where the final assertion follows from (7).

To remove the assumption that $\mathcal{M}$ is non-atomic, we identify $\mathcal{M}$ via the map $z \to z \otimes \mathbf{1}$, $z \in \mathcal{M}$ with the von Neumann subalgebra $\mathcal{M} \otimes \mathbb{C}\mathbf{1}$ of the non-atomic von Neumann algebra tensor product $\mathcal{M}\overline{\otimes}L^{\infty}[0,1)$, equipped with the tensor product trace $\tau \otimes dm$. We denote by $\mathcal{E}$ the corresponding conditional expectation. Suppose then that $(x_n)_{n=1}^{\infty}$ is a sequence in $E(\tau)^{\times}$ such that $x_n \to_n 0$ for the weak topology $\sigma(E(\tau)^{\times}, E(\tau))$. It follows from Lemma 9.7 that $x_n \otimes \mathbf{1} \to_n 0$ for the weak topology $\sigma(E(\tau \otimes dm)^{\times}, E(\tau \otimes dm))$. Since $\mathcal{M}\overline{\otimes}L^{\infty}[0,1)$ is non-atomic, it now follows from the first part of the proof that there exists a CCC sequence $(y_n)_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that, if $B \subseteq E(\tau \otimes dm)$ is $\sigma(E(\tau \otimes dm), E(\tau \otimes dm)^{\times})$ relatively compact, then

$$\sup\{|\tau \otimes dm((y_n \otimes \mathbf{1})w)| : w \in B\} \to_n 0. \tag{8}$$

Now suppose that $A \subseteq E(\tau)$ is $\sigma(E(\tau), E(\tau)^{\times})$ relatively compact. Since $E^{\times}$ is fully symmetric and $E = E^{\times\times}$, the equality (6) may be applied to $E^{\times}$ rather than to E. Consequently, if $w \in E(\tau \otimes dm)^{\times}$, then there exists a unique $z \in E(\tau)^{\times}$ such that $\mathcal{E}(w) = z \otimes \mathbf{1}$. Consequently, for all $y \in E(\tau)$,

$$\tau \otimes dm((y \otimes \mathbf{1})w) = \tau \otimes dm((y \otimes \mathbf{1})\mathcal{E}(w)) = \tau(yz).$$

This implies that the map $y \to y \otimes \mathbf{1} \in E(\tau \otimes dm)$, $y \in E(\tau)$, is $\sigma(E(\tau), E(\tau)^{\times})$ to $\sigma(E(\tau \otimes dm), E(\tau \otimes dm)^{\times})$ continuous. Therefore, $B = A \otimes \mathbf{1}$ is relatively

$\sigma(E(\tau \otimes dm), E(\tau \otimes dm)^{\times})$ compact. It now follows from (8) that

$$\sup\{|\tau(y_n z)| : z \in A\} \to_n 0$$

and this suffices to complete the proof of the Proposition. ∎

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