Superconducting diode effect from Josephson junctions

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Conventional semiconductor diodes dissipate energy in the form of heat when current passes through them. This is unwanted in, for example, cryogenic environments. Using a superconducting diode could mitigate this problem. These have been made by using special materials or combining multiple different circuit elements. We provide a systematic method of designing a tunable superconducting diode using a circuit of solely Josephson tunnel junctions. We show that even for a small number of Josephson junctions a strong diode effect can be achieved and that this method is stable under manufacturing tolerances. This method involves solving computationally inexpensive linear least squares problems to tune the Josephson energies of the junctions used.

I. INTRODUCTION

A diode is an electronic component that conducts current in one direction with low resistance and high resistance in the opposite direction; asymmetric conductance if you will. There are multiple types of diodes, but the most commonly used type in present times is the semiconductor diode. The semiconductor diode is a crystalline piece of semiconductor material with a p-n junction.[1]

Semiconductor diodes have strongly asymmetric resistance, but one downside they all share is that resistance in the forward direction can never be truly zero, meaning that they will always heat up while conducting electricity. An alternative to this is to make a superconducting diode, which allows for dissipationless conduction of electricity.

A superconductor is a material in which electrical resistance vanishes and magnetic fields are expelled from the material. Different materials have different critical temperatures, the temperature under which a material is superconducting. Unlike with non-superconducting materials, the drop-off in resistance is sudden under this temperature threshold. For typical superconductors, the critical temperature ranges between $20K$ and less than $1K$. Aluminium metal, for example, becomes superconducting at $1.2K$. The basic idea behind superconductors is that below the critical temperature electrons will pair together to form Cooper pairs. This is due to phonon interactions with the material. These electron pairs have an integer total spin, making them behave more like bosons than fermions.[2] Because of this bosonic behaviour, multiple Cooper pairs can occupy the same quantum state. These pairs have a slightly lower energy, because the pairing costs some energy, so an energy gap will be created above the Cooper pair energy. This is what stops any collision interactions between electrons that would normally result in resistivity, thus making said material have no resistance at low temperatures.[3]

Multiple examples of such superconducting diodes have been produced in other recent papers.[4–11] Those are all relatively complicated, use special materials, and are harder to control the parameters of. We would like to find a method for making a superconducting diode which uses standard components and allows for more control of the various parameters of the diode.

Josephson junctions are one example of standard devices across which a supercurrent can flow.[12] They follow a symmetric current-phase relation. There are multiple Josephson junction devices, devices which are both non-linear and non-dissipative. The Josephson tunnel junction is an example of one such device. It consists of two superconducting electrodes separated by an insulating layer(tunnel barrier). A common material to use for the electrodes is aluminium and the insulating material is usually a thin aluminium oxide layer. In this device, electrons tunnel through the insulating barrier between the electrodes causing a supercurrent. The resulting Josephson junction(JJ), has a sinusoidal current-phase relation, the amplitude of which is proportional to both the area of the superconducting electrodes and the transparency of the tunnel barrier.[13,14] The energy-phase and current-phase relation[15–17] are given by:

$$\begin{aligned} U(\phi) =& - E_J \cos(\phi), \\ I(\phi) =& \frac{1}{\Phi_0}\frac{\partial U}{\partial \phi}, \\ I(\phi) =& I_c \sin(\phi) = \frac{E_J 2e}{\hbar}\sin(\phi). \end{aligned} \tag{1}$$

Here I_c is the critical current of the JJ, which is the maximum current a device can carry with zero resistance at a specified temperature and in the absence of magnetic fields. ϕ is the phase and E_J is the characteristic Josephson energy of a JJ, which determines the amplitude of the energy or current as previously mentioned. It is defined as $E_J = \frac{I_c \hbar}{2e}$, which is the critical current times the reduced magnetic flux quantum Φ_0. For easier calculations and derivations, we choose $\Phi_0 = 1$. $I(\phi)$ is now given in $[J/\Phi_0]$.

Josephson junctions are a standard nonlinear building block for superconducting circuits, they are used in sensing devices such as SQUIDs,[18] SLUGs[19] and SNAILs.[20,21] We use them to make a tunable superconducting diode. The electrical symbol used in circuit schematics for a Josephson junction is a cross, as seen in the figure.

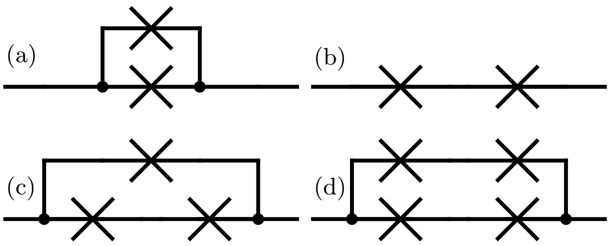

Figure 1: Schematic views of simple electrical circuits containing Josephson junctions. A Josephson junction is denoted by a cross. (a) and (b) do not work as a diode, (c) and (d) are the simplest working examples.

The general idea behind designing a superconducting diode is combining multiple Josephson junctions such that the resulting current-phase relation is asymmetric. This means that the critical current going one way is different from the critical current going the other way. It is useful to look at the simplest scenario of two JJs in parallel as depicted in Fig. 1(a). This is the simplest case; since one can add the currents of the two junctions together to find the total current. This is not sufficient to get an asymmetric current-phase relation. The current for one junction, $I_1(\phi) = E_{J_1}\sin(\phi)$, added to the current of the other junction, $I_2(\phi) = E_{J_2}\sin(\phi + \delta\phi)$, gives us another sinusoidal current-phase relation, which can't be asymmetric. Finding the current-phase relation for two JJs in series as depicted in Fig. 1(b) requires using the current for a single junction as seen in Eq.(1). We derive this relation by using the fact that because the JJs are in series; $I_1(\phi_1) = I_2(\phi_2)$. Here ϕ_1 and ϕ_2 are the phases through each of the junctions. We also use $\phi = \phi_1 + \phi_2$ because phases in series add. We first find $\phi_1(\phi)$ by using $\alpha = \frac{E_{J_1}}{E_{J_2}}$ and $\alpha\sin(\phi_1) = \sin(\phi - \phi_1)$:

$$\phi_1(\phi) = \arctan\left(\frac{\sin(\phi)}{\alpha + \cos(\phi)}\right). \tag{2}$$

We then substitute $\phi_1(\phi)$ into $I_1(\phi_1)$ following Eq.(1) to find $I(\phi)$, where we have substituted α back into the equation:

$$I(\phi) = \left(\frac{2E_{J_1}E_{J_2}}{E_{J_1} + E_{J_2}}\right)\frac{\sin(\frac{\phi}{2})\cos(\frac{\phi}{2})}{\sqrt{1 - \frac{4E_{J_1}E_{J_2}}{(E_{J_1}+E_{J_2})^2}\sin(\frac{\phi}{2})^2}}. \tag{3}$$

We simplify this equation by choosing A and τ as parameters we can finetune by changing E_{J_1} and E_{J_2}:

$$\begin{aligned} A &= E_{J_1} + E_{J_2}, \\ \tau &= \frac{4E_{J_1}E_{J_2}}{(E_{J_1} + E_{J_2})^2}. \end{aligned} \tag{4}$$

We recognise the resulting current-phase relationship as the derivative of the energy-phase relationship:[16,22]

$$I(\phi) = \frac{\partial}{\partial\phi}\left(A\sqrt{1 - \tau\sin(\frac{\phi}{2})^2}\right). \tag{5}$$

One will find that this current-phase relation is also symmetric, which can be seen in Fig. 3. So using two JJs in series will not yield a diode effect by itself.

The goal in this project is to find a way to combine just JJs to achieve a high diode efficiency, η:[23]

$$\eta = \frac{|I_c^+ - I_c^-|}{I_c^+ + I_c^-}. \tag{6}$$

Here, I_c^+ is the forward critical current and I_c^- is the reverse critical current.

It has been demonstrated that using two junctions in parallel with an inductance has a slight diode effect.[24] It is another possible approach, but has the added complexity of introducing other types of circuit elements, which could have dissipation of their own or make it harder to control the parameters. In this article, the maximum diode efficiency achieved was relatively small compared to a normal, semiconductor diode,[1] with $\eta = 0.25$. This shows that a device this simple is not sufficient, but it does provide a clue on how to approach the problem.

It is not a given how to design an arbitrary superconducting diode, but we provide a way to systematically achieve exactly that. In this thesis, we describe a method using simple JJs in parallel and in series to achieve a strong diode effect. We do not consider using inductances like in the 2015 paper[24] for simplicity's sake. We provide an example with an $\eta = 0.70$ and show that this method is stable under manufacturing tolerances, meaning that this method could be viable for real-world applications.

II. MINIMAL DIODE CIRCUIT

The simplest circuit that produces a diode effect is one JJ in parallel to two JJs in series, as shown in Fig. 1(c). We need to use the two JJs in series because it has the simplest non-sinusoidal current-phase relation, which is necessary to achieve any kind of asymmetry. And indeed, when choosing a realistic value for τ for the two junctions in series, $\tau = 0.9$ in this case, and optimising for the ratio between the amplitudes of the parallel single junction and the two in series, we find $\eta = 0.43$. The current-phase relation of this simplest case is shown in Fig. 2, which clearly shows that the maximum positive current is larger than the minimal negative current. We do not use this simplest working case to build our diode with, because using multiple different building blocks for our diode would make the fabrication more complicated.

We use parallel arms that contain two Josephson Junctions in series each. By Kirchoff's current law, to get the full current-phase relation, we can simply add the currents of separate arms together. The current of one arm is given by Eq.(3), and it is useful to know the derivative of the current given by:

$$I'(\phi) = \frac{A\tau}{4}\frac{\sin\left(\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right)}{\left(1-\tau\sin\left(\frac{\phi}{2}\right)^2\right)^{\frac{3}{2}}} + \frac{A\tau}{4}\frac{\cos\left(\frac{\phi}{2}\right)-\sin\left(\frac{\phi}{2}\right)}{\sqrt{1-\tau\sin\left(\frac{\phi}{2}\right)^2}}. \quad (7)$$

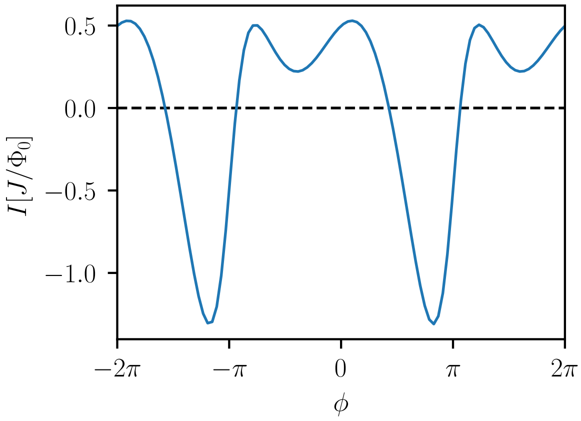

Figure 2: The current-phase relation of the circuit shown in Fig. 1(c) for $\tau = 0.9$. Optimisation here has given $E_J = 1.58$, $A = 5.79$ and the phase offset between the parallel branches is $\frac{11}{10}\pi$.

To give some better insight, in Fig. 3, Eq.(3) and Eq.(7) are plotted for $A = 1$ and $\tau = 0.9$. There it can clearly be seen that the current-phase relation is not sinusoidal.

Simply adding arms will not do to achieve a diode effect. Even if we change A or τ of one arm, the currents will simply add up to form the current-phase relationship of a single different arm. By applying a DC magnetic field however, we can induce a phase offset between multiple arms, thereby shifting the plot to the left or right as desired.

Just like in the simplest working case of two JJs in series and one JJ in parallel to that, using four Josephson junctions as seen in Fig. 1(d), will also display a diode effect. If we optimise the Josephson energies of all junctions used and the phase offset between the arms, we can find a diode efficiency as high as 0.50. This comes with the caveat that $\tau = 1.0$ for one of the arms, which is very hard to achieve since it would require that $E_{J_1} = E_{J_2}$ precicely. This is not realistically achievable when taking tolerances into account. The optimisation also takes a long time to calculate for multiple arms. This is impractical at best and computationally impossible at worst.

A solution is to set τ to a constant value that is achievable to manufacture. Choosing τ reasonably close to 1 such as $\tau = 0.9$ is appropriate. The lower the τ, the less sharply peaked the derivative of the current in Fig. 3(b) is. This does lower the diode efficiency from $\eta = 0.5$ to $\eta = 0.27$ in the case with two arms. However, this result does give some insight into how to further approach the problem. Besides giving a hint, it also shows that, indeed, with two of these arms we already have a higher diode efficiency than was achieved when using two JJs and an inductance.[24]

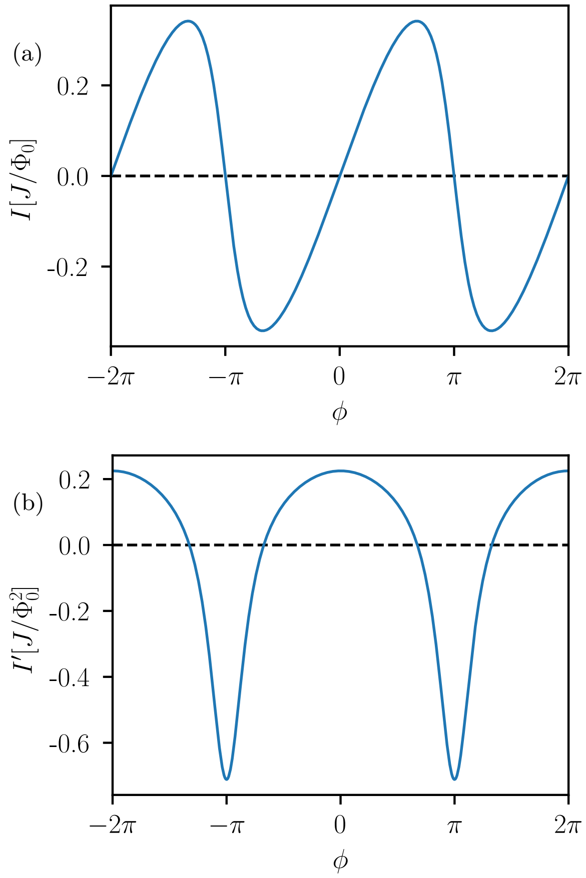

Figure 3: Plots of the current and derivative of the current of two Josephson junctions in series as given by Eq.(3) and Eq.(7) over two periods. Here $A = 1$ and $\tau = 0.9$.

The general shape we would ideally find for the second derivative is two symmetric peaks surrounded by a constant 0 area. This follows from the idea that a perfect diode would have a current-phase relation that looks similar to a square wave shifted up vertically: The minimum of the wave being near zero as seen in Fig. 4 while the integral over one period remains 0. The derivative of a square wave is known to have sharp peaks followed by a constant zero. The spacing of the peaks determines the duty cycle of the waveform. Now if one looks at Fig. 3(b), one can see that while not nearly perfect, this derivative somewhat resembles that of the square wave. There is a negative peak at $\phi = -\pi$ and a positive peak immediately next to it. By adding up multiple derivatives of this shape with different phase offsets and different amplitudes, we aim to get as close as possible to that ideal shape. Using multiple arms we cancel out the derivative of the current in such a way that we find a high diode efficiency.

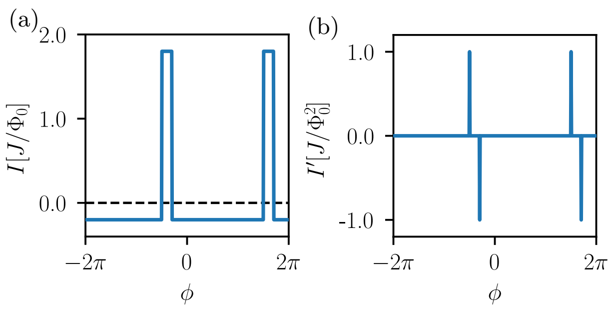

Figure 4: Current of an idealised diode and its derivative over two periods. For this specific current-phase relation; $\eta = 0.8$.

III. RECIPE FOR AN OPTIMAL DIODE

If multiple arms are used, the goal is to place the sharp peaks of the derivative such that adding all together, we get a large negative peak, a large positive peak and a flat area around 0. This can be done with manual tweaking and intuition if only two or three arms are used but this becomes impossible for more arms than that. Instead, we choose to have an unshifted reference arm with a larger amplitude. We then linearly space the offsets of the other arms such that the negative peaks are in the flat area. Then finally, we choose the amplitudes of all arms such that they cancel out the singular larger reference arm. The height of the peak of I', is given by Eq.(8) which can be derived from looking at Fig. 3(b) and Eq.(7). This then gives us the value of A to use to have a peak with a certain height at a chosen ϕ_{chosen}:

$$\begin{aligned} \min I'(\phi) = I'(\pi) &= \frac{A\tau}{4}\frac{-1}{\sqrt{1-\tau}}, \\ A &= \frac{4\sqrt{1-\tau}}{\tau} I'(\phi_{chosen}). \end{aligned} \tag{8}$$

From there we shift $I'(\phi)$ to the left or right by π to get the negative peak of the derivative at 0. We can then add a $\delta\phi$ to the phase to shift the peak to the final position as required. We show this approach working in Fig. 5, where all peaks are shifted to the area we want to be zero and the amplitudes are chosen according to Eq.(8). This gives us $\eta = 0.42$ for just three arms with $\tau = 0.9$.

Using this method and the appropriate number of arms for a given tau, a high diode efficiency can be achieved. The way to speed this up for more arms is by changing the problem into a least squares minimisation problem. Kirchoff's current law, as stated earlier, allows us to find the final current-phase relation by simply adding the separate elements together. We can also choose all amplitudes in such a way that we minimise the height of the flat area between the positive and negative peaks.

We first choose the offsets such that the positions of the peaks are linearly spaced in the area we want to be flattened, so we choose $\phi_0 < \phi_j < \phi_{max}$ where

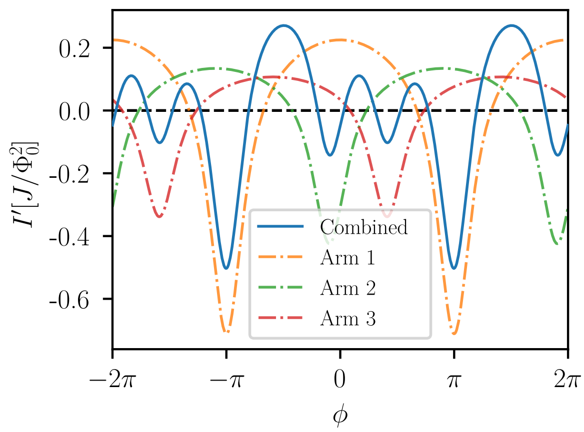

Figure 5: Derivative of current for 3 arms of two JJs in series each, $\tau = 0.9$, $\eta = 0.42$.

we can choose ϕ_0 and ϕ_{max}. These can be optimised for a given τ and the number of arms, but should generally be chosen such that the positive peak is as wide as the negative peak.

We say that $\{I_1'(\phi + \delta\phi_1), I_2'(\phi + \delta\phi_2), \dots\}$ are the derivatives of the current following Eq.(7); all with the same τ and shifted by their respective $\delta\phi_i$. Then we can say that at any given ϕ, $I'(\phi) = A_0 I_0'(\phi + \delta\phi_0) + A_1 I_1'(\phi + \delta\phi_1) + \dots + A_n I_n'(\phi + \delta\phi_n)$, where n is however many arms you have. (We say that $A_0 = 1$, $\delta\phi_0 = 0$ and I_0' is the unshifted reference arm.) Then, we need to inspect this for multiple different ϕ_j to choose the A_i optimally. We solve the least squares minimalisation problem:

$$\min_{A_i} \sum_j I'(\phi_j)^2. \tag{9}$$

Solving this problem for a number of arms, we find a higher diode efficiency than for the three arms in Fig. 5: $\eta = 0.57$ for five arms in total and $\eta = 0.70$ for ten arms:

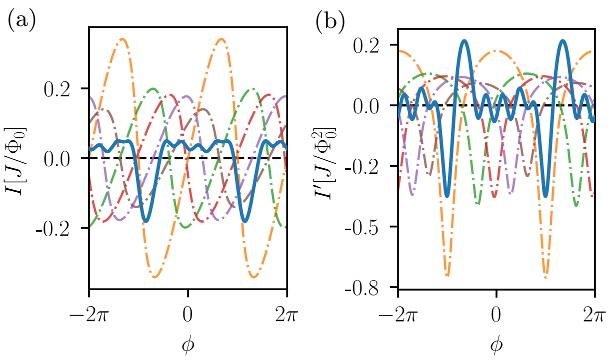

Figure 6: Two plots showing the current(a) and the derivative(b) of that current for 5 arms with $\tau = 0.9$ over two periods, $\eta = 0.57$. The thick line is the combined current or derivative and the dotted lines are for the individual arms.

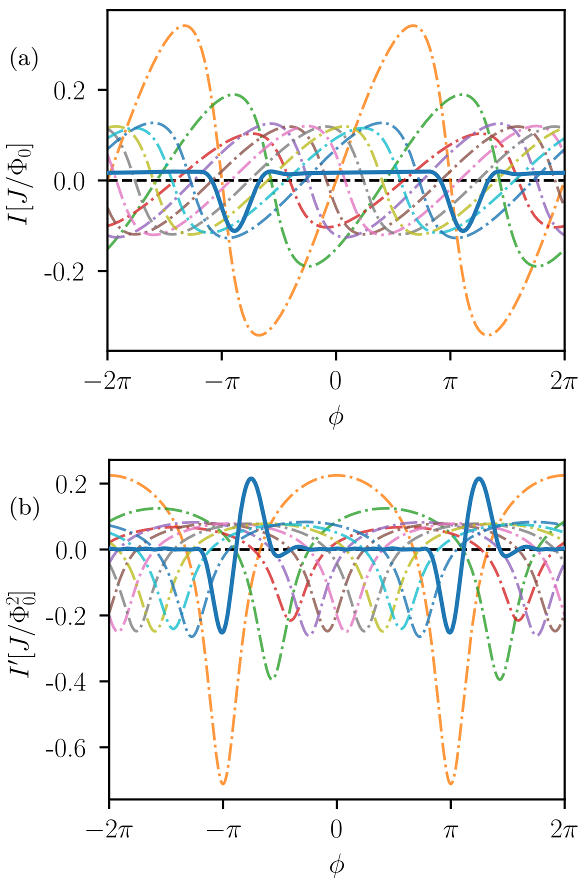

Figure 7: Two plots showing the current and the derivative of that current for 10 arms with $\tau = 0.9$ over two periods, $\eta = 0.70$. The thick line is the combined current or derivative and the dotted lines are for the individual arms.

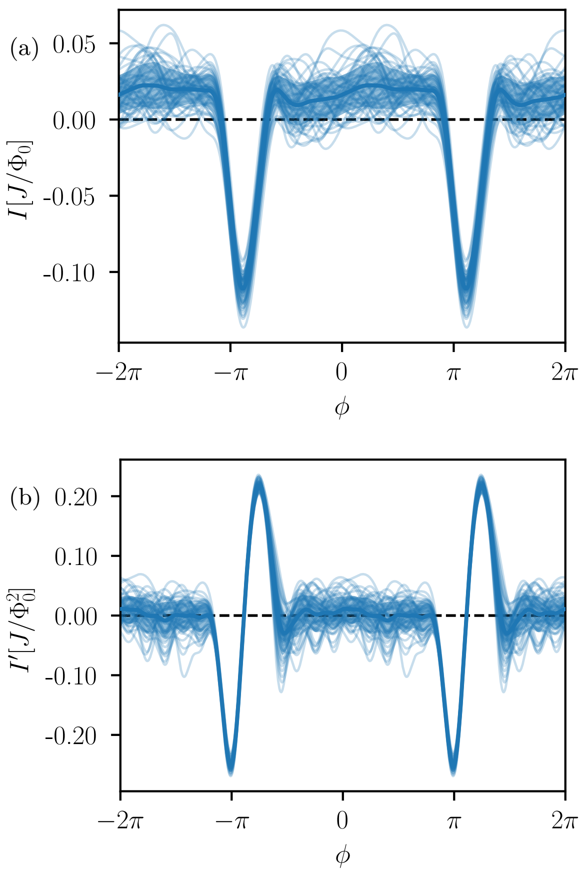

Figure 8: The current and derivative of the current plotted for 10 arms with a random 5% error applied to both τ and A over two periods. The thick line is the mean of the current or derivative and the shaded area is the standard deviation.

These figures clearly show that the current-phase relation converges and that the diode efficiency increases with the number of arms. The efficiency does not scale linearly, so while ten arms have a higher η, for the chosen $\tau = 0.9$, there may be a lower number of arms with high enough diode efficiency to be worth the easier manufacturing.

With our method, with even just 10 arms, the least squares problem converges and gives a high diode efficiency, but it is useful to examine how it is influenced if we introduce real-world tolerances of 5% for A and τ. Even if some errors such as these are made in the fabrication process, the resulting circuits are still diodes. In most cases, the positive and negative errors will cancel out and yield a good, though somewhat reduced, diode efficiency. In Fig. 8 multiple plots of the current and derivative of the current for 10 arms are plotted. Here all τ and all A have a random error as high as 5%. With a completely random error, the resulting diodes still have an average $\eta \geq 0.50$. So while decidedly not as good as without any errors, the diode effect is not lost completely and is still high enough that any ten arm 'failures' could be used as five arm substitutes, if the goal was to use a diode with $\eta = 0.50$. If the error is not random but instead multiplicative on all E_J, then the total amplitude of the current would be affected, but not the diode efficiency, as can be found by looking at Eq.(3) and Eq.(4). This is also how one would intentionally produce a diode that will handle a certain current.

IV. CONCLUSION

A good diode effect can be achieved using this method of determining the offsets and amplitudes. It is also computationally inexpensive and makes use of just Josephson tunnel junctions without the use of special materials.

We did not consider using other methods besides linearly spacing to decide the locations of the negative peaks in the derivative. In other words, we did not optimise the phase offsets of the arms. In this thesis, they are simply spaced linearly, but there may be a better way to place them. Regardless, even with the peaks spaced evenly, a diode efficiency of 0.70 can be achieved with just ten arms, if tolerances are tight enough.

This thesis can and should be expanded on by considering different current-phase relations. One could use a method that doesn't require knowing the original energy-phase relation. For this project, two Josephson junctions in series were used, and that composition has a known current-phase relation. Therefore, calculating the height of the peaks in the derivative of the current is relatively easy. In the future, other, sharper, or completely different current-phase relations may well be used, even if the exact relation is unknown, to achieve a diode effect by the method described in this paper. Combining multiple smaller devices in parallel to achieve an optimal current-phase relationship by matching the derivatives.

A superconducting diode made using this method will most likely be used in a circuit at some frequency. We did not consider frequency characteristics of the junctions or other inductances and capacitances this diode would be used in conjunction with. For any implementation of a circuit diode like described here, incorporating capacitances and calculating the frequency response, will be necessary.

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