Model Predictive Control
for Max-Plus-Linear and Piecewise Affine Systems

Ion Necoara

Model Predictive Control for Max-Plus-Linear and Piecewise Affine Systems

Proefschrift

ter verkrijging van de graad van doctor
aan de Technische Universiteit Delft,
op gezag van de Rector Magnificus prof.dr.ir. J.T. Fokkema,
voorzitter van het College voor Promoties,
in het openbaar te verdedigen op dinsdag 17 oktober 2006 om 10:00 uur
door

Ion NECOARA

Master of Science in Optimization and Statistics
University of Bucharest
geboren te Vrancea (Tulnici), Romania.

Dit proefschrift is goedgekeurd door de promotor:
Prof. dr. ir. J. Hellendoorn

Toegevoegd promotor:
Dr. ir. B. De Schutter

Samenstelling promotiecommissie:

Rector Magnificus	voorzitter
Prof. dr. ir. J. Hellendoorn	Technische Universiteit Delft, promotor
Dr. ir. B. De Schutter	Technische Universiteit Delft, toegevoegd promotor
Prof. dr. J. Maciejowski	University of Cambridge
Prof. dr. A. Rantzer	Lund University
Prof. dr. ir. C. Roos	Technische Universiteit Delft
Prof. dr. ir. P.P.J. van den Bosch	Technische Universiteit Eindhoven
Dr. ir. A.J.J. van den Boom	Technische Universiteit Delft
Prof. dr. R. Babuška, M.Sc.	Technische Universiteit Delft (reserve lid)

ISBN-10: 90-9020993-X
ISBN-13: 978-90-9020993-7

Keywords: model predictive control, robust control, Lyapunov stability, hybrid systems, piecewise affine systems, discrete event systems, max-plus-linear systems.

Printed in The Netherlands

to my parents Ion and Aurica

Contents

Acknowledgments

Tulnici, Romania, August 2006. The only thing that remained to complete my Ph.D. thesis is to write the acknowledgments and the mountain that I see through my window gives me some inspiration although it is not easy to write down what I think and feel.

This thesis is a reflection of the support, encouragement, and inspiration I have received from so many people over the last years.

First of all this thesis would not have been possible without my supervisor Bart De Schutter. Definitely, he is the person to whom I owe the most. Without his help, confidence and support I would not be now in the position to write these lines. His energy and enthusiasm have been a constant source of motivation for me. I thank him for the freedom he left me in doing my research, but also for the valuable discussions about my work. It has been a great honor to work with you and learn from you. *Hartelijk bedankt, Bart!*

I would like to thank also my promoter Hans Hellendoorn for his constant encouragement. I always felt his support during my staying here in Delft and I enjoyed his supervising style. It was a unique experience having you as a promoter.

I am deeply indebted to Ton van den Boom who had a very important impact on my thesis. He always made time to answer my questions, no matter how trivial. His wise advise and depth of knowledge in the field have been inspiring and eyeopening.

My wholehearted gratitude goes to Eric Kerrigan for the many illuminating discussions, his valuable ideas and also for making my life in Cambridge easier. He has taught me that results can always be presented better.

I would like to thank those professors who took the time to be members of my thesis committee: Prof. Jan Maciejowski, Prof. Anders Rantzer, Prof. Kees Roos and Prof. Paul van den Bosch. Their sharp comments and suggestion on how to improve the thesis helped me a lot in the last stage of my writing.

The research was made possible by the financial support of the Dutch Technology Foundation STW, Applied Science division of NWO and the Technology Programme of the Dutch Ministry of Economic Affairs through the project "Model Predictive Control for Hybrid Systems" (DMR.5675).

I am grateful for the friendship and support of the many former or present members of our group. Thanks goes to Hakan Koroglu for his help in the office and for the long discussions about religion. I also thank former group members Ricardo Lopezlena and Marcel Oosterom for their help during my first year in the group. I would especially like to acknowledge Andreas Hegyi for teaching me so much about Latex, Linux and Matlab. I do remember that he was always ready when I needed help. I thank Tudor Ionescu for allowing me to send sms to my family in Romania.

Also thanks for the help of the secretaries, Kitty Dukker and Ellen van den Berg, who helped me with countless questions, forms and financial matters.

The love and sweetness of Ameli made my live easier. Thank you, Ramiro, for our evening discussions which remembered me about home.

I am grateful to my family: my parents Ion and Aurica, my brother Ghita and my sister Marioara. Without your support and guidance through the years of my life, I would not be where I am today. It is to them that I dedicate this thesis.

Ion Necoara

Chapter 1

Introduction

1.1 Motivation

Many physical, biological and economical systems can be modeled by mathematical constructs such as differential or difference equations. These systems evolve in time or in any other independent variable according to some dynamical relations. We can enforce these systems to fulfill some requirements over a finite period of time (*horizon*) by the application of some external inputs or controls. However, these requirements must be met within the limitations and restrictions that those systems manifest such as equipment or safety constraints. If these requirements can be achieved, there may exist different controls for achieving them. If there exist different controls for achieving the same requirements, then there might exist one control that is achieving it in the "best" way. The measure of "best" or performance is called *cost function*, the limitations on inputs and outputs are called *constraints*, the best control is called *optimal control* and we refer to the associated design method as *finite-horizon optimal control problem*.

We can design an *infinite*-horizon optimal control by repeatedly solving a finite-horizon optimal control problem in a receding horizon fashion as it is explained next. At each step, a finite-horizon optimal control problem is solved for which an optimal control sequence is computed. Only the first control sample of the obtained optimal control sequence is applied to the system. At the next step, a new finite-horizon optimal control problem is solved. The resulting design method will be referred to as *model predictive control* (MPC), or *model-based predictive control* as it is sometimes known.

But, a real system is much too complicated to allow anything but approximations. Therefore, the mathematical description does not copy exactly the relevant physical phenomena taking place into the system. Such a mismatch is called *uncertainty*. Moreover, the system is often affected with *disturbances* from various sources. If the effect of the uncertainty and of the disturbances in the model is not taken into account, then the real and theoretical behavior of the system will differ and the requirements might not be met. Furthermore, the constraints might also be violated. Depending on the requirements we can have system failure, which in turn might lead to huge losses or even endanger human lives. That is why it is so important to design an optimal controller that can cope with these issues. Optimal control that is robust against disturbances and model uncertainty is referred to as *robust optimal control*. The receding horizon version of such control will be referred to as *robust model predictive control* (robust MPC).

However, the techniques available for nonlinear systems cannot be extended easily to hybrid systems and discrete event systems since many concepts have to be adapted adequately or new concepts have to be introduced. This constitutes the motivation for the work reported in this thesis. The present thesis concentrates on synthesizing (robust) optimal controllers and their (robust) MPC versions for some specific classes of nonlinear dynamical systems that in recent decades have become an integral part of our world: *hybrid systems* and *discrete event systems*.

Hybrid systems are mixtures of continuous dynamics and discrete events. Both dynamics interact and changes occur in response to discrete events but also in response to continuous dynamics (e.g. dynamics described by the difference or differential equations). Two basic hybrid system modeling approaches can be distinguished, an implicit and an explicit one [161]. The explicit approach is often represented by a hybrid automaton or a hybrid Petri net. The implicit approach is often represented by guarded equations to result in a collection of systems of difference or differential equations, one system for each mode, where the active mode changes via some guard conditions. In this thesis we concentrate on the second modeling approach. In particular, we consider hybrid systems described by piecewise affine (PWA) and max-min-plus-scaling (MMPS) difference equations. These systems arise naturally in many applications such as:

- *air traffic control* where the air traffic controller uses a set of maneuvers (speed change, short cut, etc) in order to obtain a conflict-free flight environment and the underlying aircrafts dynamics are or can be approximated as PWA difference or differential equations and various constraints must be obeyed.
- *automotive control* where the speed of a car engine is naturally modeled using several discrete modes corresponding to the position of the gear, while each mode is described by continuous dynamics (e.g. affine difference equations).
- *chemical process control* where to produce a substance, an instruction sequence is designed, in which each instruction could involve several continuous control elements.
- *electrical networks* by their very nature are hybrid (switching, diodes).
- *actuator saturation* in a linear system, etc.

We also consider a class of dynamic systems whose evolution equations change in time by the occurrence of events at possibly irregular time intervals nowadays often referred to as discrete event systems. Discrete event systems that model only synchronization aspects are called max-plus-linear (MPL) systems. However, we will introduce also the class of switching MPL systems that can model also choice by breaking synchronization and changing the order of events. This type of systems essentially consist of man-made systems that contain a finite number of resources (such as machines, communications channels, etc), that are shared by several users (such as jobs, information packets, etc) all of which contribute to the achievement of some common goal (the assembly of products, a parallel computation, etc.) [4] such as:

- *manufacturing systems* where the synchronization aspects are modeled using the $\max$ operator.
- *railway networks* where the connections between trains determine naturally (switching) max-plus-linear dynamics.
- *parallel computing* where several jobs processed on several computers involve the $\max$ operator.
- *queuing systems with finite capacity* where the service of a customer involves typically max-plus-linear dynamics.

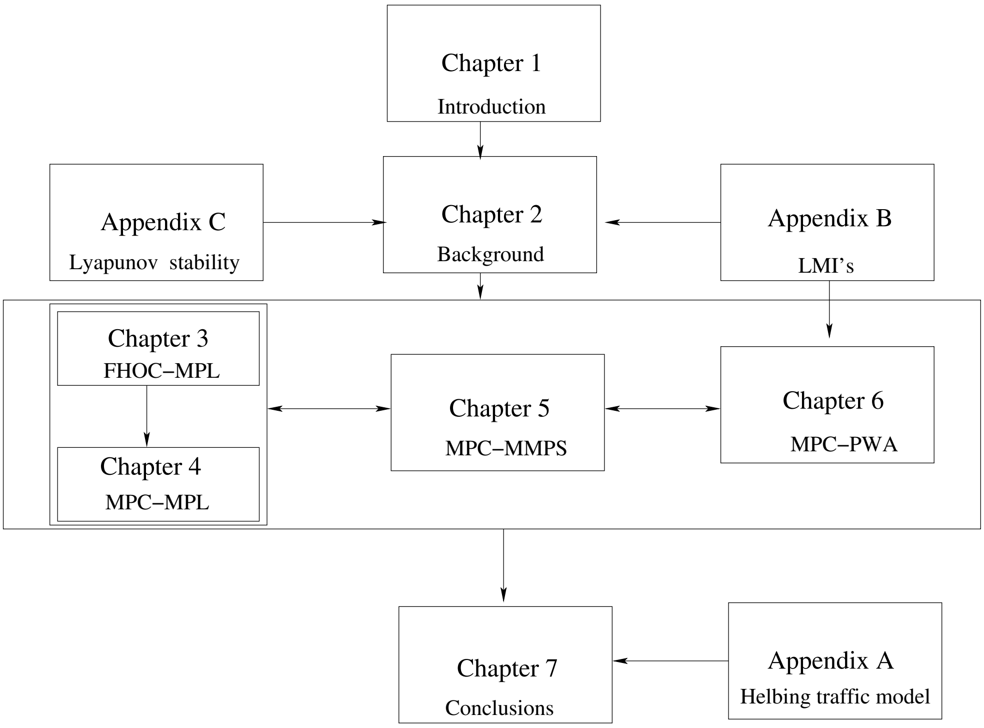

Figure 1.1: The interaction between the chapters.

- *planning* is one of the traditional fields where the $\max$ operator plays a crucial role (e.g. scheduling problems, the shortest path problem), etc.

This chapter proceeds now with a short summary of the Ph.D. thesis.

1.2 About this thesis

We have opted to make this Ph.D. thesis as self-contained as possible. Most of the notation used in this thesis is consistent with the literature, each new symbol is explained on the page where it is introduced, and in order to prevent confusion some notations and definitions are repeated in each chapter that makes use of them. The basic road map and the interaction between the chapters of this Ph.D. thesis are illustrated in Figure 1.1.

The background material is given in Chapter 2. Optimal control and MPC are the main control design techniques used in the thesis. In Chapter 2 we formulate the problem of finite-horizon optimal control and its receding horizon implementation (referred to as MPC) for general nonlinear systems. The focus in this chapter is on two major topics: techniques that are available to ensure stability of model predictive controllers for nonlinear systems and techniques by which the disturbances are handled in optimal control and MPC strategies.

Chapter 3 and Chapter 4 deal with optimal control and MPC for some special classes of discrete event systems: MPL systems and switching MPL systems. In the first part of Chapter 3 we study the finite-horizon optimal control problem for discrete event MPL systems. We derive sufficient conditions that ensure the constrained optimal control problem is solved via a linear program. Moreover, in the unconstrained case we demonstrate that for a proper choice of the cost function we obtain explicitly the state-space formula for the just-in-time controller. The robustification of the deterministic optimal control problem is considered in the second part of Chapter 3 where three types of min-max control problems are considered depending on the nature of the input over which we optimize: open-loop input sequences, disturbance feedback policies and state feedback policies. Despite the fact that the controlled system is nonlinear, we provide sufficient conditions that allow us to preserve convexity and consequently to recast the corresponding min-max problems as either linear programs or multi-parametric linear programs. In Chapter 4 we consider the receding horizon implementations of the optimal control problems discussed in Chapter 3. We mainly focus on stability of the closed-loop MPC. We introduce the notions of Lyapunov stability and positively invariant set for a normalized MPL system. Further, we design MPC strategies that guarantee a priori stability for the corresponding closed-loop system. The stability results are obtained either by deriving bounds on the tuning parameters or by using a terminal cost and a terminal set approach.

Chapter 5 represents the bridge that connects the previous chapters with Chapter 6. MMPS systems model on the one hand a large class of discrete event systems and on the other hand they are equivalent to some relevant subclasses of hybrid systems. In Chapters 5 and 6 we propose different MPC schemes for particular classes of discrete event systems and hybrid systems. In Chapter 5 we show that the open-loop min-max MPC optimization problem for uncertain MMPS systems can be recast as a finite sequence of linear programs if certain conditions on the stage cost and constraints are satisfied. We also introduce feedback by optimizing over disturbance feedback policies and an efficient algorithm is derived in order to solve this type of min-max control problems. In Chapter 6 different MPC strategies for deterministic and uncertain piecewise affine systems are presented that incorporate the property of nominal closed-loop stability and robust stability, respectively. Although the controlled system may be discontinuous we are able to show that the optimal value function in the deterministic case is continuous at the equilibrium point and thus it can serve as a Lyapunov function for the closed-loop system. In the disturbance case a new sufficient condition that preserves convexity of the predicted state set is introduced and based on this condition a robustly stable dual-mode MPC strategy is derived that considers only the extreme disturbance realizations.

Chapter 7 contains the conclusion of the thesis and suggestions for future research.

Appendix A presents some results that we obtained in the first year of the Ph.D. research. In this appendix we derive the main properties such as the formulas for the shock waves, rarefaction waves and the solution of the Riemann problem for a macroscopic gas-kinetic traffic flow model, called the Helbing traffic flow model.

Appendices B and C contain some additional background material on linear matrix inequalities and Lyapunov stability.

1.3 Contributions to the state of the art

The objective of the research presented in this thesis is to extend optimal control and MPC techniques to relevant classes of hybrid systems and discrete event systems. By focusing on specific classes of hybrid systems and discrete event systems, the goal is either to extend the framework by introducing new analysis techniques, to facilitate the design of the optimal controller, or to improve the performance by exploiting the particular structure of these model classes. We now explain how each chapter relates to this research objective and the main contributions of each of these chapters to the state of the art are also summarized.

Chapter 3 *"Finite-horizon optimal control for constrained max-plus-linear systems"* Most of the existing literature on optimal control of MPL systems uses a *residuation*-based approach[1] based on input-output models in order to design a controller. However, the residuation approach does not take into account uncertainties in a general framework, feedback is not incorporated, initial conditions are not considered, and the residuation approach does not handle general state and input constraints. In this chapter we derive novel methods to solve finite-horizon optimal control problems for nominal or uncertain MPL systems depending on the nature of the input over which we optimize that circumvent these issues. In the unconstrained case and for an appropriate cost function we obtain an analytic solution for the optimal control problem corresponding to the nominal case that leads to a just-in-time controller. Since MPL systems are nonlinear, non-convexity is clearly an issue if one seeks to develop efficient methods for solving optimal control problems for MPL systems. However, by employing recent results in polyhedral algebra and multi-parametric linear programming we prove that in the constrained case the optimal control problems for the nominal and uncertain MPL systems can be recast as linear programs or can be solved off-line via a set of multi-parametric linear programs. For the robust control problem we assume that the uncertainty lies in a polytope, we consider the min-max framework and feedback is incorporated using disturbance feedback policies or state feedback policies. These methods are useful for the design of MPC schemes in the next chapter.

Chapter 4 *"Model predictive control for max-plus-linear systems"* Recently, the MPC framework was extended to discrete event MPL systems. A major concern of MPC schemes is to prove closed-loop stability, since this property was not explicitly incorporated into the design procedure. In this chapter we introduce the notions of Lyapunov stability and positively invariant set for normalized MPL systems and their main features are derived. An MPC scheme for unconstrained MPL systems is proposed that, by a proper tuning of the design parameters, guarantees a priori closed-loop stability. In the constrained case we introduce a terminal inequality constraint (based on a positively invariant set) and an appropriate terminal cost that together guarantee closed-loop stability. Furthermore, in the disturbance case a robustly stable feedback min-max MPC scheme is developed that uses disturbance feedback policies or state feedback policies. This leads in general to a better performance and increased feasibility than with the existing open-loop MPC schemes. Using the results from Chapter 3 we obtain that the MPC optimization problems can be recast as linear programs or can be solved off-line via a set of multi-parametric linear programs. The equivalence between stability in terms of boundedness and asymptotic stability does not hold for MPL systems, as it holds for conventional linear systems, but under some additional assumptions we prove that both notions of stability hold for the closed-loop MPC.

Chapter 5 *"Model predictive control for uncertain max-min-plus-scaling systems"* MMPS systems have a dual interpretation: on the one hand they can be seen as an extension of discrete event MPL systems and on the other hand they are equivalent with some interesting classes of hybrid systems. In Chapter 5 we design a min-max MPC algorithm for uncertain MMPS systems based on open-loop input sequences. To the author's best knowledge the robust MPC problem for MMPS systems has not yet been addressed before by other authors. It is demonstrated that the resulting min-max problem can be solved efficiently by solving a finite sequence of linear programs. As an alternative to the open-loop MPC algorithm a feedback min-max MPC scheme is designed by optimizing over disturbance feedback

[1] *Residuation* is a general notion in lattice theory [20] which allows defining "pseudo-inverses" of some isotone maps (f is isotone if $x \leq y \Rightarrow f(x) \leq f(y)$).

policies, which leads to improved performance compared to the open-loop approach. This chapter can be regarded as the extension of some of the MPC algorithms of Chapter 4 towards a min-max MPC-MMPS algorithm.

Chapter 6 *"Model predictive control for piecewise affine systems"* A common way to guarantee nominal closed-loop stability for PWA systems using a model predictive controller is the introduction of an end point equality constraint. In order to guarantee feasibility of the corresponding MPC optimization problem we then need a long prediction horizon. However, the MPC framework often uses short prediction horizons in order to keep the MPC optimization problem tractable. In this chapter we develop a stabilizing MPC algorithm based on a terminal *in*equality constraint that allows us to choose short prediction horizons. In order to develop such an algorithm we make use of the special structure of our system. In particular, using the piecewise linear (PWL) dynamics of the PWA system we derive a stabilizing PWL local controller that enables us to construct a terminal set and a terminal cost. The PWL controller is obtained using the linear matrix inequality framework (via a relaxation procedure called the S-procedure) and the particular structure of the system. Despite the fact that the PWA system may be discontinuous we are able to show that the optimal value function of the corresponding MPC optimization problem is continuous at the equilibrium and can serve as a Lyapunov function for the closed-loop system. This is an interesting result since continuity is the classical and customary assumption on the optimal value function due apparently to the fact that MPC has often been applied to sufficiently smooth systems. The robustification of the MPC is considered in the second part of this chapter. We develop a robustly stable feedback MPC algorithm that considers only the extreme disturbance realizations by introducing a new sufficient condition that allows us to preserve convexity of the predicted state set.

Appendix A *"Structural properties of the Helbing traffic flow model"* In this appendix we study the main properties of the macroscopic, gas-kinetic, Helbing traffic flow model. For the first time it is demonstrated that this model does not give rise to negative flow and density. In addition, the main properties such as the formulas for the shock and rarefaction waves and the solution of the Riemann problem are derived.

1.4 Publications

Most of the material that is presented in this Ph.D. thesis has been published, or accepted for publication, in journals or conference proceedings. Some of the material has been submitted for publication recently. We detail below the links between these publications and each chapter of the thesis.

- Chapter 3 is based on the papers [122, 125, 126, 158].
- Chapter 4 is based on the papers [123, 124, 127, 128, 159].
- Chapter 5 is based on the papers [118, 119, 121].
- Chapter 6 is based on the papers [35, 114, 117, 120].
- Appendix A is based on the papers [115, 116].

Chapter 2

Background

In this chapter we describe briefly some special classes of hybrid systems (piecewise affine systems and max-min-plus-scaling systems) and discrete event systems (max-plus-linear systems and switching max-plus-linear systems) since the present thesis focuses on designing efficient optimal controllers for these particular classes of systems by exploiting their special properties. We also give an overview of optimal control and its receding horizon implementation which is referred to as model predictive control for general nonlinear systems. Besides their formulation, we concentrate on three important issues: feasibility, robustness and closed-loop stability.

2.1 Hybrid systems

The mathematical description of a system is, in general, associated with differential or difference equations that are typically derived from physical laws governing the dynamics of the system under consideration. Therefore, most of the literature on dynamic modeling and control is concerned with systems that are either completely continuous or completely discrete, and whose evolution is described by smooth linear or nonlinear state transition functions. However, most of the dynamical systems around us may be described in a *hybrid* framework: cars (gear shift), washing machines (on/off switches or valves), computers (if-then-else rules), etc.

The demands on modern technology have caused a considerable interest in the study of dynamical systems of a mixed continuous and discrete nature, called *hybrid systems*. The interest in hybrid systems has grown in both the academic community, due to the theoretical challenges, as well as in industry, due to their impact on applications. A more detailed and comprehensive review of the topic on hybrid systems can be found in [161].

There are many examples of hybrid systems [3, 161]. Air traffic control, automotive control, chemical processes, power electronics are hybrid by their very nature. As an illustrative example of a simple hybrid system, we mention a temperature control system in a room consisting of a heater and a thermostat. This is a system that can operate in two modes depending on whether the heater is on or off and the room temperature is described by differential equations. The variables of the system are the temperature (real-valued) and the operating mode of the heater (Boolean). Clearly, there must be a coupling between the continuous and discrete variables, so that for instance the operating mode will be switched to off when the temperature crosses a certain upper value (this example will be studied in more detail in Chapter 5).

Initially, the engineering solution to a hybrid control problem was based on a continuous or discrete formulation and dealt with the hybrid aspects in an *ad hoc* manner. Recently, because of their complexity, many analysis and synthesis techniques for such hybrid control problems have been proposed that rely on qualitative techniques, which have their roots in well-grounded classical methodologies. Among them, the class of optimal controllers is one of the most studied [11, 99, 139, 151]. Of course, the approaches differ in the hybrid model adopted, in the formulation of the optimal control problem and in the methods used to solve it.

In this section we briefly discuss a framework for modeling, analyzing and controlling hybrid systems. We will focus only on *discrete time* linear hybrid models. In this hybrid framework, the system is allowed to be discontinuous, both inputs and states are real-valued, an event occurs when the states reach a particular boundary and the states and inputs satisfy a given set of linear inequality constraints. We will focus on two special classes of hybrid systems:

- *piecewise affine* (PWA) systems
- *max-min-plus-scaling* (MMPS) systems

Each class has its own advantages over the others. For instance, stability criteria and control techniques were proposed for PWA systems in [22, 26, 74, 112, 139, 151] while control and verification techniques for MMPS systems are provided in [21, 42, 44, 46].

This section proceeds now with the introduction of the class of PWA systems and MMPS systems. In Section 2.2 we introduce two particular classes of discrete event systems: max-plus-linear systems and switching max-plus-linear systems. Finally, in Section 2.3 an overview of optimal control and model predictive control for nonlinear systems is given with particular interest in feasibility, closed-loop stability and robustness.

2.1.1 Piecewise Affine (PWA) systems

Among various classes of discrete time hybrid systems, PWA systems are the most studied since they represent the "simplest" extension of linear systems that can still model hybrid phenomena [9, 11, 22, 26, 49, 75, 151]. Discrete time PWA systems can model *exactly* a rich class of hybrid processes, such as actuator saturation in linear systems or switched systems where the dynamic behavior is described by a finite number of discrete time affine models, together with a set of logic rules for switching among these models. Moreover, they can *approximate* smooth nonlinear systems with arbitrary accuracy by sampling the continuous dynamics and linearizing at different operating points. Loosely speaking, PWA systems are defined by partitioning the state and/or input space of the system in a finite number of polyhedral regions and associating to each region a different affine dynamic. We will give in the sequel a precise definition for PWA systems.

We use some rather standard definitions:

Definition 2.1.1 *A polyhedron in the Euclidean space $\mathbb{R}^n$ is a set described as the intersection of a finite number of half spaces. Each half space can be either closed (i.e. $\{x \in \mathbb{R}^n : a^T x \leq b\}$, where $a \in \mathbb{R}^n,\ b \in \mathbb{R}$) or open (i.e. $\{x \in \mathbb{R}^n : a^T x < b\}$). A bounded polyhedron is called polytope.*

Definition 2.1.2 *Given a polyhedron $\mathcal{X}$, then a polyhedral partition of $\mathcal{X}$ is a finite collection of nonempty polyhedra $\{\mathcal{X}_i\}_{i\in\mathcal{I}}$ satisfying (i) $\cup_{i\in\mathcal{I}}\mathcal{X}_i = \mathcal{X}$, (ii) $\mathcal{X}_i \cap \mathcal{X}_j = \emptyset$ for all $i \neq j$.*

We define $\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$. For a given function $g : \mathbb{R}^n \to \bar{\mathbb{R}}$, its *effective domain* is defined as in [144] $\mathrm{dom}\ g := \{x \in \mathbb{R}^n : g(x) < \infty\}$. The function g is called *proper* if $g(x) < \infty$ for at least one $x \in \mathbb{R}^n$ and $g(x) > -\infty$ for all $x \in \mathbb{R}^n$. In other words the proper functions $g : \mathbb{R}^n \to \bar{\mathbb{R}}$ are thus the ones obtained by taking a set $\mathcal{X} \subseteq \mathbb{R}^n,\ \mathcal{X} \neq \emptyset$ and a function $g : \mathcal{X} \to \mathbb{R}$ and putting $g(x) = \infty$ for all $x \in \mathbb{R}^n \setminus \mathcal{X}$.

Definition 2.1.3 *The epigraph of a function* $g : \mathcal{X} \rightarrow \mathbb{R}$ *is defined as*

$$\operatorname{epi} g := \{(x,t) \in \mathcal{X} \times \mathbb{R} : \; g(x) \leq t\}. \tag{2.1}$$

Definition 2.1.4 *A function* $g : \mathcal{X} \rightarrow \mathbb{R}^k$*, where* $\mathcal{X} \subseteq \mathbb{R}^n$*, is PWA if there exists a polyhedral partition* $\{\mathcal{X}_i\}_{i\in\mathcal{I}}$ *of* $\mathcal{X}$ *and* g *is affine on each* $\mathcal{X}_i$*, i.e.* $g(x) := H_i x + k_i$ *for all* $x \in \mathcal{X}_i$ *and* $i \in \mathcal{I}$*. If in addition* g *is continuous, then* g *is called a continuous PWA function.*

Piecewise quadratic functions are defined analogously.

In [144] an alternative definition is given for a PWA function. Namely, $g : \mathcal{X} \rightarrow \mathbb{R}^k$, $g = [g_1 \; g_2 \cdots g_k]^T$ is a PWA function if the epigraph of each function g_i is a finite union of polyhedra. We now can define a PWA system.

Definition 2.1.5 *A PWA system is a dynamical system whose evolution is defined as follows:* $x(k+1) = f_{\mathrm{PWA}}(x(k), u(k)),\; y(k) = h_{\mathrm{PWA}}(x(k))$*, where* $f_{\mathrm{PWA}}(\cdot, u), h_{\mathrm{PWA}}$ *are PWA functions for each fixed* u *and* $f_{\mathrm{PWA}}(x, \cdot)$ *is an affine function for each fixed* x*. Here,* k *is a discrete time step,* $x \in \mathbb{R}^n$ *are the states,* $u \in \mathbb{R}^m$ *are the inputs and* $y \in \mathbb{R}^p$ *are the outputs. We say that a PWA system is continuous if the functions* f_{PWA} *and* h_{PWA} *are continuous.*

From Definition 2.1.5 we have the following explicit description of a PWA system:

$$\begin{array}{rcl} x(k+1) & = & A_i x(k) + B_i u(k) + a_i \\ y(k) & = & C_i x(k) + c_i \end{array} \quad \text{if } \; x(k) \in \mathcal{C}_i, \tag{2.2}$$

where $A_i \in \mathbb{R}^{n\times n},\; B_i \in \mathbb{R}^{n\times m},\; C_i \in \mathbb{R}^{p\times n}$ and $a_i \in \mathbb{R}^n,\; c_i \in \mathbb{R}^p$. Here, $\{\mathcal{C}_i\}_{i\in\mathcal{I}}$ is a polyhedral partition of the state space $\mathbb{R}^n$ and $\mathcal{I}$ is a finite index set. We may assume, without loss of generality, that the origin is an equilibrium point for the PWA system (2.2). We denote with $\mathcal{I}_0 \subseteq \mathcal{I}$ the set of indexes for the polyhedral sets $\mathcal{C}_i$ that contain the origin in their closure. It follows that the cardinality of $\mathcal{I}_0$ satisfies $|\mathcal{I}_0| \geq 1$. If $\mathcal{I}_0 = \mathcal{I}$ then the PWA system (2.2) is called *piecewise linear* (PWL). It follows that $a_i = 0,\; c_i = 0$ for all $i \in \mathcal{I}_0$.

Note that Definition 2.1.5 is the discrete time version of the definition of a PWA system in the system state x given in [74, 139] for the continuous time case. In order to avoid some issues in connection with the existence of a PWL controller (that will be derived in Section 6.1.2) we use this particular case of a more general definition of a PWA system:

Definition 2.1.6 *A general PWA system is a dynamical system whose evolution is defined as follows:* $x(k+1) = f_{\mathrm{PWA}}(x(k), u(k)),\; y(k) = h_{\mathrm{PWA}}(x(k))$*, where* $f_{\mathrm{PWA}}, h_{\mathrm{PWA}}$ *are PWA functions. When* f_{PWA} *and* h_{PWA} *are continuous PWA functions, we call such a system a general continuous PWA system.*

However, the most common situation is when the system equations are PWA in the system state x only. As we mentioned previously such model can, for example, arise from the linearization of nonlinear systems around different operating points, or from interconnections of linear systems and static PWL components, or even in practical applications (automotive control, air traffic control, chemical process control, etc.). Therefore, in this thesis the term "PWA" refers to PWA in the state space *only* (i.e. $x \in \mathcal{C}_i \subseteq \mathbb{R}^n$) while the term "general PWA" refers to PWA in the state *and* input space (i.e. $[x^T \; u^T]^T \in \mathcal{C}_i \subseteq \mathbb{R}^{n+m}$).

In [9, 64] the equivalence of discrete time general PWA systems and other classes of discrete time hybrid systems such as mixed logical dynamical (MLD) systems [11], max-min-plus-scaling (MMPS) systems [21, 44] (see also Section 2.1.2), linear complementarity (LC) systems [65, 160], extended linear complementarity (ELC) systems [43] is proved, possible under some additional assumptions related to boundedness of input, state, output or auxiliary variables. In particular, the following result can be deduced from [64]:

Lemma 2.1.7 *Every general PWA system can be written equivalently as an MMPS system provided that the set of feasible states and inputs is bounded.*

The equivalence of discrete time MLD, MMPS, LC, ELC and general PWA systems allows to easily transfer certain theoretical properties and tools developed for some class to other. It depends on the application, which of these classes is the best suited. It has been shown in [46] that in some particular applications the use of the MMPS framework has some advantages in the computations of optimal controllers.

2.1.2 Max-Min-Plus-Scaling (MMPS) systems

Another class of hybrid systems is the class of MMPS systems, the evolution equations of which can be described using the operations maximization, minimization, addition and scalar multiplication. In [61] the general properties of these class of systems are studied and conditions for stability are derived while in [44] optimal control, especially model predictive control, is proposed for the deterministic case (i.e. in the absence of disturbances). We introduce the following definitions:

Definition 2.1.8 *A scalar-valued MMPS function $g : \mathbb{R}^n \to \mathbb{R}$ is defined by the recursive relation:*

$$g(x) = x_i | \alpha | \max\{g_j(x), g_l(x)\} | \min\{g_j(x), g_l(x)\} | g_j(x) + g_l(x) | \beta g_j(x), \tag{2.3}$$

where $i \in \mathbb{N}_{[1,n]}$, $\alpha, \beta \in \mathbb{R}$ and $g_j, g_l : \mathbb{R}^n \to \mathbb{R}$ are again MMPS functions, and the symbol $|$ stands for "or". For vector-valued MMPS functions the above statements hold component-wise.

An MMPS function is e.g. $g : \mathbb{R}^2 \to \mathbb{R}$, $g(x) = -2x_1 + 3x_2 + \max\big\{\min\{5x_1 - 1, -x_2\}, -x_1 + x_2 - 1\big\}$. The following lemma taken from [131] states the equivalence between continuous PWA and MMPS functions:

Lemma 2.1.9 *Any continuous PWA function having domain $\mathbb{R}^n$ can be written as an MMPS function and vice versa.*

We now introduce the class of discrete time MMPS systems:

Definition 2.1.10 *An MMPS system is a dynamical system whose evolution can be expressed as follows:*

$$\begin{aligned} x(k+1) &= f_{\mathrm{MMPS}}(x(k), u(k)) \\ y(k) &= h_{\mathrm{MMPS}}(x(k)), \end{aligned} \tag{2.4}$$

where f_{MMPS} and h_{MMPS} are vector-valued MMPS functions.

From Lemma 2.1.7 it follows that under boundedness assumption on the states and the inputs, the class of general PWA systems is equivalent with the class of MMPS systems. From Lemma 2.1.9 we see that we can remove the boundedness assumption in the case of general continuous PWA systems:

Lemma 2.1.11 *Every general continuous PWA system can be written equivalently as an MMPS system and vice versa.*

2.2 Discrete event systems (DES)

In the previous section we saw that the state of a hybrid system generally changes as time changes. At every tick of the clock the state is expected to change and we refer to such systems as *time-driven* systems. In this case, the time variable k is a natural independent variable and the state transitions are synchronized by the clock, i.e. at every tick of the clock an event occurs that determines a change in the state. However, in the day-to-day life we encounter many systems that evolve in time by the occurrence of events at possible irregular time intervals, i.e. not necessarily coinciding with clock ticks. In this case state transitions are the result of asynchronous events and then time may no longer be an appropriate independent variable. We refer to such systems as *event-driven* systems. In this case, the variable k is an event counter. In [28] discrete event systems (DES) are defined as discrete-state, event-driven systems. In this thesis as in [4, 37] we refer to DES also to those event-driven dynamical systems for which the state space is a continuum.

Many practical systems, particularly technological ones are DES: manufacturing systems, telecommunications networks, railway systems, logistic systems and so on. All these systems are man-made and consist of a finite number of resources (e.g. machines, communication channels, memories) shared by several users (e.g. manufactured objects, jobs) all of which contribute to the achievement of some common goal (e.g. the assembly of products, a parallel computation). The behavior of such systems is governed by events rather than by ticks of a clock. An event corresponds to the start or the end of an activity. If we consider a production system then possible events are: the completion of a part on a machine, a buffer becoming empty, etc. All these events occur at discrete time instants. Moreover, the intervals between events are not necessarily identical; they can be deterministic or can vary stochastically.

In general the dynamics of DES can be described using the two paradigms of *synchronization* and *concurrency*. Synchronization requires the availability of several resources at the same time (e.g. before we assembly a product on a machine , the machine has to be idle and the various parts have to be available), whereas concurrency appears when some user must choose among several resources (e.g. in a manufacturing system a job may be executed on one of several machines that can handle that job and that are idle at that time) [4].

Although in general DES lead to a nonlinear description in conventional algebra, there exists a subclass of DES that contains only the paradigm of synchronization. For these DES the system equations become “linear” when we formulate it in the max-plus algebra. Such systems will be called *max-plus-linear* systems. We also consider dynamical systems in which we can switch between different modes of operation, each mode being described by a max-plus-linear model. We refer to such a system as *switching max-plus-linear* system. We can note that MMPS systems represent a general framework for modeling hybrid systems but also DES that includes max-plus-linear systems and switching max-plus-linear systems (provided that the switching function depends only on (x, u) and it is linear) as particular subclasses, as we will see in Section 5.1.2. Then, the index k might also have different interpretations: a time counter or an event counter, respectively. Therefore, depending on the meaning of k the constraints, the cost function and the implementation of the controller have to be adjusted adequately.

There are many modeling and analysis techniques for DES, such as queuing theory, max-plus algebra, perturbation analysis, computer simulation, etc. All these modeling and analysis techniques have particular advantages and disadvantages and it really depends on the systems we want to model and on the goals we want to achieve which one of the above procedures best suits our needs. Although the class of DES that can be described using max-plus algebra is somewhat limited, its analysis gives many insights in theoretical problems. Therefore, in the rest of this

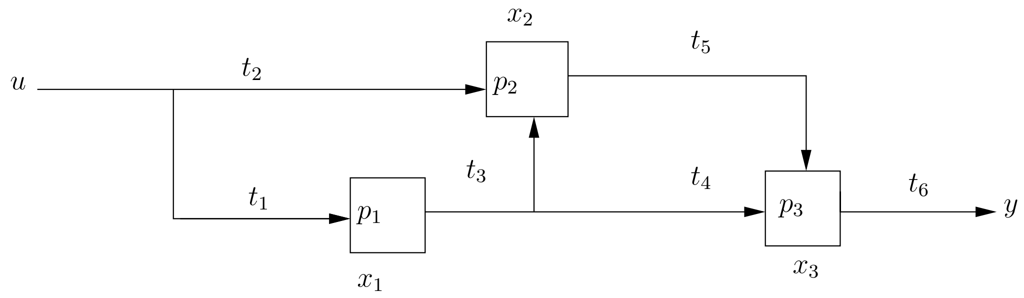

Figure 2.1: A production system.

section we concentrate on DES modeled using max-plus algebra.

2.2.1 Max-Plus-Linear (MPL) systems

The main goal of the next two sections is to introduce to the reader two classes of DES that can be modeled using the max-plus addition $\oplus$ and max-plus multiplication $\otimes$ operations.

First we give a very short introduction to the basic concepts of the max-plus-algebra that will be used in this section. A more detailed discussion on this topic will be given in Section 3.1.1. We consider the set of real numbers and $\varepsilon := -\infty$ denoted with $\mathbb{R}_\varepsilon := \mathbb{R} \cup \{\varepsilon\}$. For elements $x, y \in \mathbb{R}_\varepsilon$ we define two basic operations $\oplus$ and $\otimes$ by

$$x \oplus y := \max\{x, y\} \quad \text{and} \quad x \otimes y := x + y.$$

The set $\mathbb{R}_\varepsilon$ together with the operations $\oplus$ and $\otimes$ is called *max-plus algebra*.

For matrices $A, B \in \mathbb{R}_\varepsilon^{m\times n}$ and $C \in \mathbb{R}_\varepsilon^{n\times p}$ one can extend the max-plus operations in the usual way:

$$[A \oplus B]_{ij} := A_{ij} \oplus B_{ij} = \max\{A_{ij}, B_{ij}\} \quad \forall i \in \mathbb{N}_{[1,n]}, j \in \mathbb{N}_{[1,m]},$$
$$[A \otimes C]_{il} := \bigoplus_{k=1}^{n} A_{ik} \otimes C_{kl} = \max_{k\in\mathbb{N}_{[1,n]}} \{A_{ik} + C_{kl}\} \quad \forall i \in \mathbb{N}_{[1,n]}, l \in \mathbb{N}_{[1,p]}.$$

Note that in this thesis we use both max-plus and conventional algebra. Therefore, we will always write $\oplus$ and $\otimes$ explicitly in all equations. The operations '$+$' and '$\cdot$' denote the conventional summation and multiplication operators (only the conventional multiplication operator is omitted).

We now show by an example how certain classes of DES, characterized only by synchronization, can be modeled using max-plus algebra.

Example 2.2.1 A production system

Consider the production system of Figure 2.1. It consists of three processing units (machines). Raw material is fed to the first two units. The processing times for these three machines are $p_1 = 1,\ p_2 = 2$ and $p_3 = 2$ time units. We assume that it takes $t_1 = 1$ and $t_2 = 1$ time unit for the raw material to get from the input source to the first and second unit, respectively and $t_4 = 3$ time units for the finished products of the first machine to reach the third unit. The other transportation times (i.e. t_3, t_5 and t_6) are assumed to be negligible. At the input of the system and between the processing units there are buffers with a capacity that is large enough to ensure that no buffer overflow will occur. Each unit can only start working on a new product if it has finished processing the previous product. We assume that each processing unit starts working as soon as all parts are available. We denote with

- $u(k)$: time instant at which a batch of raw material is fed to the system for the $(k+1)^{th}$ cycle,
- $x_i(k)$: time instant at which unit i starts working for the k^{th} cycle,
- $y(k)$ time instant at which the k^{th} product leaves the system.

We explain in details the dynamical equation that describes the evolution of the first processing unit: unit 1 will start with job $k+1$ when

i) the previous job is finished, indicated by $x_1(k)+p_1$ (i.e. the start of the previous job $x_1(k)$ + the production time p_1) and

ii) the raw material has arrived at the machine at time $u(k)+t_1$ (i.e. the time the raw material is put into the system $u(k)$ + the transportation time t_1).

Since machine 1 starts working on a batch of raw material as soon as the raw material is available and the current batch has left the machine, this implies that we have $x_1(k+1) = \max\{x_1(k) + p_1,\ u(k)+t_1\}$. The same reasoning applies to the second and third machine. Therefore, the dynamical equations corresponding to this manufacturing system are:

$$\begin{aligned}
x_1(k+1) &= \max\{x_1(k)+1,\ u(k)+1\}\\
x_2(k+1) &= \max\{x_1(k)+2,\ x_2(k)+2,\ u(k)+2\}\\
x_3(k+1) &= \max\{x_1(k)+5,\ x_2(k)+4,\ x_3(k)+2,\ u(k)+5\}\\
y(k) &= x_3(k)+2.
\end{aligned}$$

If we write these dynamical equations in max-plus matrix notation, we obtain

$$\begin{aligned}
x(k+1) &= \begin{bmatrix} 1 & \varepsilon & \varepsilon \\ 2 & 2 & \varepsilon \\ 5 & 4 & 2 \end{bmatrix} \otimes x(k) \oplus \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \otimes u(k)\\
y(k) &= [\varepsilon\ \ \varepsilon\ \ 2] \otimes x(k),
\end{aligned} \tag{2.5}$$

where $x(k) = [x_1(k)\ \ x_2(k)\ \ x_3(k)]^T$ is the state vector.

Clearly, in Example 2.2.1 we only had synchronization and no concurrency. Synchronization requires the availability of several resources at the same time and this leads to the appearance of the max operator in the description of the dynamics of the system considered above. This example can be generalized. If we consider DES in which the sequence of the events are fixed (such as repetitive production systems, queuing systems with finite capacity, railway networks, logistic systems, etc.), then the behavior of such a process can be described by equations of the form

$$\begin{aligned}
x(k+1) &= A \otimes x(k) \oplus B \otimes u(k)\\
y(k) &= C \otimes x(k).
\end{aligned} \tag{2.6}$$

Here, $A \in \mathbb{R}_\varepsilon^{n\times n}$, $B \in \mathbb{R}_\varepsilon^{n\times m}$, $C \in \mathbb{R}_\varepsilon^{p\times n}$ and x represents the state, u the input and y the output vector. We refer to (2.6) as *max-plus-linear* (MPL) system.

Note that the description (2.6) closely resembles the state space description of classical discrete-time linear systems

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k). \end{aligned}$$

The term “max-plus-linear” comes from the fact that the input-output representation of (2.6) is linear in the max-plus algebra, i.e. if the input sequences u_1 and u_2 yield the output sequences y_1 and y_2 respectively, then the input sequence $\alpha \otimes u_1 \oplus \beta \otimes u_2$ yields the output sequence $\alpha \otimes y_1 \oplus \beta \otimes y_2$ [4,41].

2.2.2 Switching MPL systems

We have seen in the previous section that DES in which only synchronization occurs (i.e. availability of several resources at the same time) can be modeled as MPL systems. When at a certain time a user has to choose among several resources we have concurrency. This aspect cannot be described directly by a max operation but as we will see in this section, in some cases (see also [156, 157, 159]), we can still model such phenomenon via switching among different max expressions. The switching allows us to change the structure of the system, to break synchronization and to change the order of events. We illustrate this aspect through an example.

Example 2.2.2 A production system with concurrency

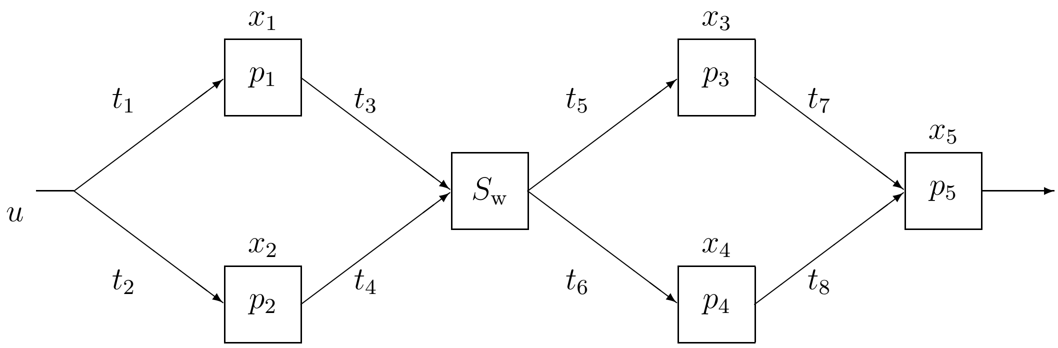

Figure 2.2: A production system with concurrency.

Consider the production system of Figure 2.2. This system consists of five processing units. The raw material is fed to the first two units, where preprocessing is done. Both intermediate products now have to be finished in either unit 3 or 4, which basically perform the same task, but the processing time of the third unit is longer than of the fourth unit. Therefore, the products coming from machine 1 and 2 are directed to a switching device S_w, that feeds the first product in the k^{th} cycle to the slower machine 3 and the second product to the faster machine 4. Finally, the products are assembled instantaneously (i.e. with a negligible processing time) in the fifth machine and become available. We assume that each machine starts working as soon as possible on each batch, i.e., as soon as the raw material or the required intermediate products are available, and as soon as the machine is idle. Similarly as in Example 2.2.1 we define

- $u(k)$: time instant at which raw material is fed to the system for the $(k+1)^{th}$ cycle,

- $x_i(k)$: time instant at which machine i starts working for the k^{th} cycle,
- $y(k)$ time instant at which the k^{th} product leaves the system.

The variables t_j are the transportation times and they take the following values: $t_1 = 4, t_2 = t_8 = 1$ time units and the other transportation times are assumed to be negligible. The processing times are $p_1 = 1, p_2 = 3, p_3 = 6, p_4 = 4$ and $p_5 = 0$ time units.

Using similar arguments as in Example 2.2.1, we can write easily the system equations for x_1 and x_2:

$$x_1(k+1) = \max\{x_1(k) + 1,\ u(k) + 4\} \tag{2.7}$$
$$x_2(k+1) = \max\{x_2(k) + 3,\ u(k) + 1\}. \tag{2.8}$$

It is clear that we have two subsystems (modes):

First mode: $x_1(k+1) + p_1 \leq x_2(k+1) + p_2$, i.e. if machine 1 finishes first and machine 2 finishes later in the $(k+1)^{th}$ cycle, the product of machine 1 will be directed to machine 3 and the product of machine 2 will be directed to machine 4. Machine 3 will start when

i) the previous job is finished at time instant $x_3(k) + p_3$ (i.e. the start of the previous job $x_3(k)$ + the production time p_3), and

ii) the intermediate product has arrived from machine 1 at time instant $x_1(k+1)+p_1$ (i.e. the start of the job $x_1(k+1)$ + the production time p_1 + transportation time t_3).

So,

$$x_3(k+1) = \max\{x_1(k+1) + 1,\ x_3(k) + 6\}.$$

By substitution of (2.7) we obtain:

$$x_3(k) = \max\{x_1(k) + 2,\ x_3(k) + 6,\ u(k) + 5\}.$$

In a similar way we derive

$$x_4(k) = \max\{x_2(k) + 2,\ x_4(k) + 4,\ u(k) + 4\},$$
$$x_5(k) = \max\{x_1(k) + 8,\ x_2(k) + 11,\ x_3(k) + 12,\ x_4(k) + 9, u(k) + 11\}.$$

For this first mode (i.e. $x_1(k+1)+p_1 \leq x_2(k+1)+p_2$) we obtain the max-plus-linear subsystem $x(k+1) = A_1 \otimes x(k) \oplus B_1 \otimes u(k)$ given explicitly by

$$x(k+1) \;=\; \begin{bmatrix} 1 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 3 & \varepsilon & \varepsilon & \varepsilon \\ 2 & \varepsilon & 6 & \varepsilon & \varepsilon \\ \varepsilon & 6 & \varepsilon & 4 & \varepsilon \\ 8 & 11 & 12 & 9 & \varepsilon \end{bmatrix} \otimes x(k) \oplus \begin{bmatrix} 4 \\ 1 \\ 5 \\ 4 \\ 11 \end{bmatrix} \otimes u(k).$$

Second mode: $x_1(k+1) + p_1 > x_2(k+1) + p_2$, i.e. machine 2 finishes first, machine 1 finishes later, we obtain the max-plus-linear subsystem $x(k+1) = A_2 \otimes x(k) \oplus B_2 \otimes u(k)$ given explicitly by

$$x(k+1) \;=\; \begin{bmatrix} 1 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 3 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 6 & 6 & \varepsilon & \varepsilon \\ 2 & \varepsilon & \varepsilon & 4 & \varepsilon \\ 7 & 12 & 12 & 9 & \varepsilon \end{bmatrix} \otimes x(k) \oplus \begin{bmatrix} 4 \\ 1 \\ 4 \\ 5 \\ 10 \end{bmatrix} \otimes u(k).$$

In both modes the output is given by: $y(k) = [\varepsilon\ \varepsilon\ \varepsilon\ \varepsilon\ 0] \otimes x(k)$. To decide the switching mechanism, we define the switching variable $z = [z_1\ z_2]^T$ as:

$$\begin{bmatrix} z_1(k+1) \\ z_2(k+1) \end{bmatrix} = \begin{bmatrix} x_1(k+1) + p_1 \\ x_2(k+1) + p_2 \end{bmatrix} = \begin{bmatrix} \max\{x_1(k) + 2,\ u(k) + 5\} \\ \max\{x_2(k) + 6,\ u(k) + 4\} \end{bmatrix},$$

and the sets

$$\mathcal{C}_1 = \{z \in \mathbb{R}_\varepsilon^2 | z_1 \leq z_2\},$$
$$\mathcal{C}_2 = \{z \in \mathbb{R}_\varepsilon^2 | z_1 > z_2\}.$$

Note that $z_1(k+1)$ and $z_2(k+1)$ are the time instants at which machines 1 and 2, respectively, finish their product in the cycle $k+1$. With that in mind, it is clear that mode 1 corresponds to "machine 1 finishes first, machine 2 finishes later" ($z_1 \leq z_2$) and mode 2 corresponds to "machine 2 finishes first, machine 1 finishes later" ($z_1 > z_2$).

In conclusion, if we consider DES that can switch between different modes of operation, where each mode corresponds to a set of required synchronizations and an event order schedule, then such systems can be modeled as follows:

$$\begin{array}{rcl} x(k+1) & = & A_i \otimes x(k) \oplus B_i \otimes u(k) \\ y(k) & = & C_i \otimes x(k) \end{array} \quad \text{if } \psi(x(k), z(k), u(k), \nu(k)) \in \mathcal{C}_i, \tag{2.9}$$

where the switching is determined by a function ψ which may depend on the previous state $x(k)$, the previous switching variable $z(k) \in \mathbb{R}_\varepsilon^{n_z}$, the input variable $u(k)$, and an (additional) control variable $\nu(k) \in \mathbb{R}^{n_\nu}$, and is denoted with:

$$z(k+1) = \psi(x(k), z(k), u(k), \nu(k)). \tag{2.10}$$

Here, $i \in \mathcal{I}$ is a finite index set and the moments of switching occur when the switching variable z reaches the boundary of a certain set $\mathcal{C}_i$, where $\{\mathcal{C}_i\}_{i \in \mathcal{I}}$ is a polyhedral partition of $\mathbb{R}_\varepsilon^{n_z}$. Moreover, the system matrices are $A_i \in \mathbb{R}_\varepsilon^{n \times n}$, $B_i \in \mathbb{R}_\varepsilon^{n \times m}$, $C_i \in \mathbb{R}_\varepsilon^{p \times n}$ for all $i \in \mathcal{I}$.

Besides production systems with concurrency, we can model using the difference equations (2.9) also railway networks in which we can change the order of trains, scheduling problems, etc. In some of these systems the switching mechanism will completely depend on the state x and input u, in other examples (e.g. railway networks) z will be equal to ν and so we can control the switching by choosing an appropriate ν. Usually ν takes finite discrete values (e.g. ν is binary, where $\nu = 0$ corresponds to the nominal case while $\nu = 1$ corresponds to the perturbed case, i.e. a synchronization is broken or the order of two events is switched). Note that the basic idea of switching MPL systems in the event-driven domain is parallel to that of PWA systems in the time-driven domain. In the analysis of PWA systems, the properties of the linear subsystems are often employed to derive properties for the PWA system [74, 139, 151]. Analogously, we will be able to use the properties of the MPL subsystems (i.e. the max-plus eigenvalues, the eigenvectors, the paths, etc.) for the analysis of the switching MPL system.

2.3 Optimal control and model predictive control

In engineering and mathematics, control theory deals with the behavior of dynamical systems over time. When the output variables of a system (process) need to show a certain behavior over time, a controller manipulates the inputs of the process to obtain the desired performance specifications on the output of the process. The performance specifications may include safety constraints, a certain level of performance, suppression of unknown disturbances, etc. Different design methods to achieve the performance specifications are proposed in the literature. Among them the class of optimal controllers is the most studied. In optimal control problems the control signal optimizes a certain *cost function* (cost criterion). Therefore, the performance specifications are imposed on the controls (inputs) while the sensors measure the outputs. The objective of the control system must be accomplished taking into account the dynamics of the process, the effects of the disturbances and the constraints.

Control systems are often based on exploiting the phenomenon of *feedback*. The basic principle of feedback is to measure through sensors the actual output of the process and then the controller processes the measurements and changes the inputs in an appropriate fashion [87, 100]. One of the most used optimal control method in process industry that makes use of feedback is *model predictive control*. The essence of model predictive control is to determine a control profile that optimizes a cost criterion over a prediction window (horizon) and then to apply this control profile until new process measurements become available at which time the whole procedure is repeated. Feedback is incorporated by using these measurements to update the optimization problem for the next step.

This section proceeds now with the problem formulation of the optimal control for a given nonlinear dynamical system.

2.3.1 Optimal control: problem formulation

In the optimal control literature the plant to be controlled is usually described in terms of difference equations of the form:

$$x(k+1) = f(x(k), u(k)) \tag{2.11}$$
$$y(k) = h(x(k)), \tag{2.12}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input and $y \in \mathbb{R}^p$ is the output. Note that we only consider discrete "time" systems (here k may denote time or any other independent variable, e.g. an event counter in the context of discrete event systems). We assume that $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous at the origin[1], $f(0,0) = 0$ and $h : \mathbb{R}^n \to \mathbb{R}^p$, with $h(0) = 0$. The control and state sequence must satisfy

$$x(k) \in X \quad \text{and} \quad u(k) \in U \quad \forall k \geq 0, \tag{2.13}$$

where usually X is a closed, convex subset of $\mathbb{R}^n$ and U is a compact, convex subset of $\mathbb{R}^m$, each set containing the origin in its interior. We employ $\mathbf{u}$ to denote a control sequence and $\phi(k; x, \mathbf{u})$ to denote the state solution of (2.11) at step k when the initial state is x at step 0 and the control sequence $\mathbf{u}$ is applied. By definition $\phi(0; x, \mathbf{u}) := x$.

The control objective is to steer the state of the system in a finite number of steps N to a "safe" region X_f, that for instance might be the origin or any other set point, in a "best" way. Performance is expressed via a performance measure (cost function) and the "best" way means that the plant has to be controlled so that the cost function in minimized. Other objectives like tracking (i.e. the plant should follow a predefined reference trajectory) can be reinterpreted to the objective of steering the system to a safe set by an appropriate extension of the model and a suitable change of coordinates or a certain choice of the cost function. We assume that $X_\mathrm{f} \subset X$

[1] Note that we can replace the origin with an invariant set. The presentation remains the same.

is also a closed, convex set, containing the origin. The cost criterion is written as a sum of *stage costs* $\ell(x,u)$ satisfying $\ell(0,0)=0$. Performance can also be expressed with respect to the safe region X_f, where we may have a cost criterion (*terminal cost*) V_f. Combining these two measures of performance we obtain a cost function (sometimes also referred to as a performance index) :

$$V_N(x,\mathbf{u}) = \sum_{i=0}^{N-1} \ell(x_i,u_i) + V_\mathrm{f}(x_N), \tag{2.14}$$

where $\mathbf{u} := [u_0^T \; u_1^T \cdots u_{N-1}^T]^T$ and $x_i := \phi(i;x,\mathbf{u})$ (and thus $x_0 = x$). For a given initial condition x, the set of feasible input sequences is defined by:

$$\Pi_N(x) := \{\mathbf{u} \in \mathbb{R}^{Nm} : x_i \in X,\; u_i \in U \;\; \forall i \in \mathbb{N}_{[0,N-1]}, x_N \in X_\mathrm{f}\}. \tag{2.15}$$

We denote with X_N the set of initial states for which a feasible input sequence exists, i.e.

$$X_N := \{x \in \mathbb{R}^n : \Pi_N(x) \neq \emptyset\}. \tag{2.16}$$

Then, the *finite-horizon optimal control problem* is formulated as follows:

$$\mathbb{P}_N(x): \qquad V_N^0(x) := \inf_{\mathbf{u}\in\Pi_N(x)} V_N(x,\mathbf{u}). \tag{2.17}$$

Corresponding to $\mathbb{P}_N(x)$, we introduce notation also for the set of input sequences **u** where the minimum of $V_N(x,\cdot)$ over $\Pi_N(x)$ is regarded as being attained [144]:

$$\arg\min_{\mathbf{u}\in\Pi_N(x)} V_N(x,\mathbf{u}) := \begin{cases} \{\mathbf{u}\in\Pi_N(x) : V_N(x,\mathbf{u}) = V_N^0(x)\} & \text{if } \; V_N^0(x) \neq \infty \\ \emptyset & \text{if } \; V_N^0(x) = \infty \end{cases} \tag{2.18}$$

The optimal control problem $\mathbb{P}_N(x)$ yields an optimal control sequence $\mathbf{u}_N^0(x) \in \arg\min_{\mathbf{u}\in\Pi_N(x)} V_N(x,\mathbf{u})$:

$$\mathbf{u}_N^0(x) = [(u_0^0(x))^T \; (u_1^0(x))^T \cdots (u_{N-1}^0(x))^T]^T. \tag{2.19}$$

For any initial condition $x \in X_N$ the optimal control sequence $\mathbf{u}_N^0(x)$ steers the plant to the safe region X_f in N steps without violating the constraints (2.13) in the best way. This control sequence is applied to the plant in an *open-loop* fashion.

The function $V_N^0 : X_N \to \bar{\mathbb{R}}$ associates to each state $x \in X_N$ the minimum value of the performance index. $V_N^0(x)$ and $\mathbf{u}_N^0(x)$ will be referred to as the *optimal value function* and the *optimizer* (minimizer), respectively. Note that the nonlinear program (2.17) depends on a parameter, the state x, appearing in the cost function but also in the constraints. Therefore, we can view (2.17) as a *multi-parametric program*:

$$J^0(x) = \inf_{u\in\mathcal{U}} \{J(x,u) : g(x,u) \leq 0\}, \tag{2.20}$$

where $u \in \mathcal{U} \subseteq \mathbb{R}^m$ is the optimization variable, $x \in \mathbb{R}^n$ is the parameter, $J : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is the cost function and $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n_g}$ are the constraints.

Important issues in multi-parametric programming are the behavior with respect to the parameter x of the optimal value and of the optimizer. A small perturbation of the parameter x in the nonlinear program (2.20) can cause different results. Depending on the properties of the functions J and g the optimizer $u^0(x) \in \arg\min_{u\in\mathcal{U}}\{J(x,u) : g(x,u) \leq 0\}$ may vary smoothly or change drastically as a function of the parameter x. Let us denote by $\mathcal{X}$ the set of feasible parameters, i.e.

$$\mathcal{X} = \{x \in \mathbb{R}^n : \exists u \in \mathcal{U} \text{ s. t. } g(x,u) \leq 0\}$$

and by $\Pi(x)$ the point-to-set-map that assigns to each parameter $x \in \mathcal{X}$ the set of feasible u, i.e.

$$\Pi(x) = \{u \in \mathcal{U} : g(x,u) \leq 0\}.$$

The following theorem, which can be found in [22, 51, 143], provides sufficient conditions under which the optimal value function J^0 and the optimizer u^0 are "well-behaved":

Theorem 2.3.1 *Suppose that $\mathcal{U}$ is a compact and convex set, J and g are continuous on $\mathbb{R}^n \times \mathcal{U}$, and each component of g is convex on $\mathbb{R}^n \times \mathcal{U}$. Then, $J^0 : \mathcal{X} \to \bar{\mathbb{R}}$ is a continuous function. If additionally J is strictly quasi-convex on $\mathcal{U}$ for each fixed $x \in \mathcal{X}$, then we can always select a continuous optimizer $u^0 : \mathcal{X} \to \bar{\mathbb{R}}^m$.* ♢

When J and g are linear, i.e. $J(x,u) = c^T u$ and $g(x,u) = \mathcal{H}x + \mathcal{G}u + \omega$, then the multi-parametric program (2.20) is called *multi-parametric linear program*:

$$J^0(x) = \min_{u \in \mathbb{R}^m} \{c^T u : \mathcal{H}x + \mathcal{G}u + \omega \leq 0\}. \tag{2.21}$$

Here, we consider that $\mathcal{U} = \mathbb{R}^m$ and the cost function $J(x,u) = c^T u$ instead of $J(x,u) = c^T [x^T \ u^T]^T$ since by adding an extra variable we can reduce in the optimization problem (2.21) the later cost function to first one. Moreover, for a linear program we use "min" instead of "inf" since the infimum is attained at a point in the feasible set. We define the set of feasible parameters $\mathcal{X}$: $\mathcal{X} = \{x \in \mathbb{R}^n : \exists u \text{ s. t. } \mathcal{H}x + \mathcal{G}u + \omega \leq 0\}$. Using duality for linear programming [52, 147] it can be easily shown that if there exists an $x_0 \in \mathcal{X}$ such that $J^0(x_0)$ is finite, then $J^0(x)$ is finite for all $x \in \mathcal{X}$. In the following we will summarize the main results of [52]:

Theorem 2.3.2 *Consider the multi-parametric linear program* (2.21) *such that there exists an $x_0 \in \mathcal{X}$ satisfying $J^0(x_0)$ is finite. Then, $\mathcal{X}$ is a closed polyhedral set, $J^0 : \mathcal{X} \to \mathbb{R}$ is a convex continuous PWA function, and we can always select a continuous PWA optimizer $u^0 : \mathcal{X} \to \mathbb{R}^m$.* ♢

The reader is referred to [23, 86] for geometric algorithms for computing the solution to a multi-parametric linear program. Methods of selecting a continuous optimizer can be found in [23, 76, 152].

2.3.2 Model Predictive Control (MPC)

"The only advanced control methodology which has made a significant impact on industrial control engineering is model predictive control. The main reasons for his success in applications are: (i) it handles multivariable control problems naturally; (ii) it can take account of actuator limitations; (iii) it allows operation closer to constraints, which frequently leads to more profitable operation."

J.M. Maciejowski [101]

In Section 2.3.1 we have presented the main ingredients of a constrained finite-horizon optimal control problem for a general nonlinear system (2.11)–(2.12). We can obtain an infinite-horizon controller by repeatedly solving the finite-horizon optimal control problem

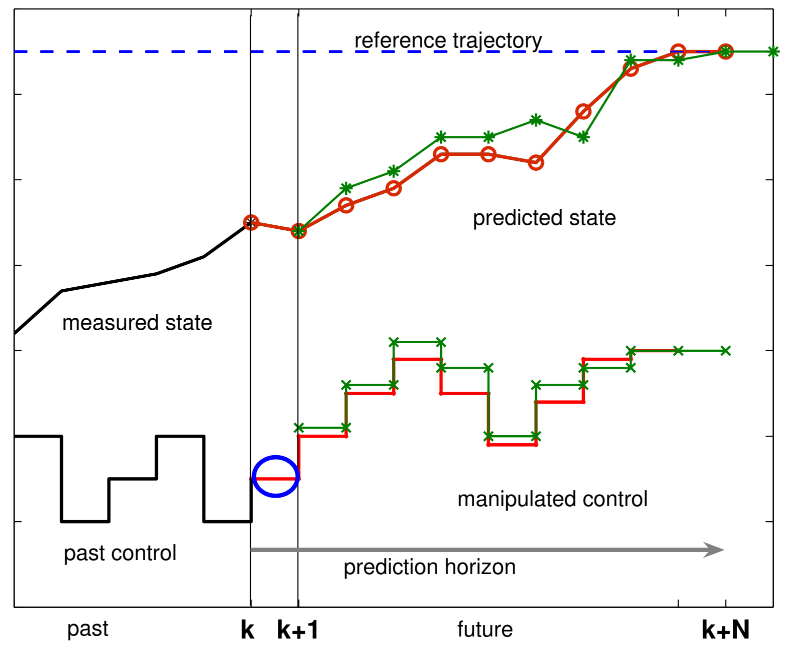

Figure 2.3: The model predictive control setup.

(2.17) where the current state of the plant is used as an initial state for the optimization. From the computed optimal control sequence only the first control sample is implemented and the whole procedure is repeated at the next step when new measurements of the state are available. This is referred to as the *receding horizon* implementation of the controller and the resulting design method is called *model predictive control* (MPC).

The previous description of the MPC can be mathematically summarized as follows: given the event pair (k, x), i.e. $x(k) = x$, the optimization problem (2.17) is solved yielding the optimal control sequence $\mathbf{u}_N^0(x)$. Only the first control $u_0^0(x)$ is applied to the system at step k. At the next step $k+1$ a new optimization problem is solved over a shifted horizon (see also Figure 2.3). This defines an implicit MPC law

$$\kappa_N(x) = u_0^0(x). \tag{2.22}$$

Note that we can drop out the index k in the mathematical formula since f, ℓ and V_f are time invariant. An intrinsic feature of the MPC is that the optimization problem (2.17) is performed in open-loop. However, the MPC law (2.22) is a feedback law. Therefore, an open-loop control is used in the prediction although, the actual controller of the plant (i.e. the MPC) is in closed-loop form:

$$x(k+1) = f(x(k), \kappa_N(x(k))), \quad y(k) = h(x(k)). \tag{2.23}$$

There exists a vast literature dealing with MPC. Early industrial MPC algorithms like IDCOM (identification and command) [141] or DMC (dynamic matrix control) [38] used a quadratic cost function while the constraints were treated in an ad hoc fashion. One of the first MPC schemes that took explicitly into account the constraints in the optimization algorithm is QDMC (quadratic matrix control) [53]. Later an extensive number of publications appeared dealing with theoretical analysis of such algorithms , mainly concerning feasibility, robustness and closed-loop stability [1, 105, 111, 148, 149]. In fast applications (i.e. applications where the sampling time is small) the on-line computational burden of the nonlinear MPC algorithms may be too large and thus making these MPC schemes impracticable. Therefore, a large body of literature is devoted to the reduction of the on-line computations [7, 8, 13, 84, 145]. There are also several text books that discuss MPC, the most recent ones [27, 101] give an overview of the main MPC techniques.

We can find in the literature different formulations for MPC but all of them have common ingredients. One of these ingredients is the explicit constraints handling. Another important ingredient is the use of a model for the plant to be controlled. Using this model at each step, starting at the current state, we make a prediction of the "future" behavior of the plant over a finite-horizon. The actual input is computed on-line based on this future behavior. Despite the fact that we use future predictions in order to compute the actual input, the resulting controller remains causal. Although in some of the literature on predictive control there is a distinction between MPC and receding horizon control (see e.g. [22]), in this thesis we propose MPC as a generic title for that control method in which the current input action is computed by solving on-line a finite-horizon optimal control problem.

MPC is extensively used in industry, due to its ability to cope with hard constraints on inputs and states. Therefore, it has been widely applied in process industry where satisfaction of the constraints is particularly important because the most profitable operation is often obtained when a process is running at the constraints [101, 140]. There are also other attractive futures of the MPC: it is an easy-to-tune method, it is applicable to multivariable systems, it is capable of tracking pre-scheduled reference signals, etc. For an overview of applications of MPC schemes the reader is referred to [54, 137].

The main issues in MPC are *feasibility* of the on-line optimization and closed-loop *stability*. These two issues are connected to each other and we will discuss them in more details in the sequel. The treatment follows a similar reasoning as in [105]. Clearly, if the initial state $x(0) \notin X_N$, then the optimization problem (2.17) is infeasible from the beginning. Therefore, we are mainly concerned with providing *sufficient* conditions that guarantees feasibility of the optimization problem (2.17) at each step, once the initial state $x(0) \in X_N$. Before proceeding further a definition is given:

Definition 2.3.3 *For the autonomous system $x(k+1) = g(x(k))$ the set $\mathcal{X}$ is a positively invariant (PI) set if for all $x \in \mathcal{X}$, $g(x) \in \mathcal{X}$ (in other words if for any initial condition in $\mathcal{X}$ the subsequent trajectory remains in $\mathcal{X}$).*

Let us assume that inside X_f a local stabilizing controller $\kappa_\mathrm{f} : X_\mathrm{f} \to \mathbb{R}^m$ is available. The following conditions, if satisfied, ensure feasibility of the optimization problem (2.17) at each step, once the initial state is inside X_N:

$\mathcal{F}1$: $X_\mathrm{f} \subseteq X$ and $0 \in \mathrm{int}(X_\mathrm{f})$

$\mathcal{F}2$: $\kappa_\mathrm{f}(x) \in U$ for all $x \in X_\mathrm{f}$

$\mathcal{F}3$: $f(x, \kappa_\mathrm{f}(x)) \in X_\mathrm{f}$ for all $x \in X_\mathrm{f}$.

Here, $\mathrm{int}(X_\mathrm{f})$ denotes the interior of the set X_f. Note that the condition $\mathcal{F}3$ expresses the fact that X_f is a PI set for the closed-loop system $x(k+1) = f(x(k), \kappa_\mathrm{f}(x(k)))$.

Theorem 2.3.4 *Suppose that the conditions $\mathcal{F}1$–$\mathcal{F}3$ hold, then the set X_N is a PI set for the closed-loop system* (2.23).

Proof: Let $x \in X_N$ then, the optimization problem (2.17) has an optimal solution (minimizer) $\mathbf{u}_N^0(x) = [(u_0^0(x))^T \ (u_1^0(x))^T \cdots (u_{N-1}^0(x))^T]^T$. Let

$$\mathbf{x}^0 = [x^T \ (x_1^0)^T \cdots (x_N^0)^T]^T$$

denote the optimal state trajectory, i.e. $x_i^0 = \phi(i; x, \mathbf{u}_N^0(x))$ for all $i \in \mathbb{N}_{[1,N]}$. The MPC law $\kappa_N(x) = u_0^0(x)$ steers the plant from the state x to the successor state $x_1^0 = f(x, \kappa_N(x))$. Our goal is to show that the optimal control problem $\mathbb{P}_N(x_1^0)$ is also feasible. Since $x_N^0 \in X_{\mathrm{f}}$, then $\kappa_{\mathrm{f}}(x_N^0) \in U$ (according to $\mathcal{F}1$) and $f(x_N^0, \kappa_{\mathrm{f}}(x_N^0)) \in X_{\mathrm{f}} \subseteq X$ (according to $\mathcal{F}3$ and $\mathcal{F}1$). Furthermore, the control sequence $\mathbf{u}_N^0(x)$ is feasible for the optimization problem $\mathbb{P}_N(x)$ and thus the feasible control sequence $[(u_1^0(x))^T \cdots (u_{N-1}^0(x))^T]^T$ steers the plant from the state x_1^0 to $x_N^0 \in X_{\mathrm{f}}$. It follows that a feasible control sequence for $\mathbb{P}_N(x_1^0)$ is given by

$$\mathbf{u}^{\mathrm{f}} = [(u_1^0(x))^T \cdots (u_{N-1}^0(x))^T \ (\kappa_{\mathrm{f}}(x_N^0))^T]^T.$$

We conclude that $f(x, \kappa_N(x)) \in X_N$ and thus X_N is a PI set for (2.23). Moreover, $X_{\mathrm{f}} \subseteq X_N$. $\diamondsuit$

Once feasibility is guaranteed, the next step is to prove stability for the closed-loop system (2.23). Let us assume that the terminal cost V_{f} satisfies the following condition:

$\mathcal{S}1$: $V_{\mathrm{f}}(f(x, \kappa_{\mathrm{f}}(x))) - V_{\mathrm{f}}(x) + \ell(x, \kappa_{\mathrm{f}}(x)) \leq 0$ for all $x \in X_{\mathrm{f}}$.

Condition $\mathcal{S}1$ expresses the fact that V_{f} is a Lyapunov function for the system if additionally the conditions *(i)-(ii)* of Theorem C.1.2 are satisfied. For an elaborate discussion about Lyapunov stability see Appendix C.

In MPC, typically the stage cost satisfies $\ell(x, u) \geq \alpha(\|x\|)$ for all $x \in \mathbb{R}^n$, where α is a $\mathscr{K}$ function (see Appendix C for an appropriate definition) and $\|\cdot\|$ denotes some vector norm on $\mathbb{R}^n$. A typical example of such stage cost is the quadratic cost: $\ell(x, u) = x^T Q x + u^T R u$, where the weighting factors satisfy $Q \succ 0$ and $R \succ 0$ (i.e. Q, R are positive definite matrices as defined in Appendix B). If we define $\mathbb{R}_+ := [0, \ \infty)$, then from Theorem 2.3.4 it follows that $V_N^0 : X_N \to \mathbb{R}_+ \cup \{\infty\}$. Let us assume that V_N^0 is continuous at the origin. From Theorem 2.3.1 it follows that V_N^0 is continuous at the origin when the system is linear, i.e. $f(x, u) = Ax + Bu$, X and U are polytopes, the stage cost is quadratic, i.e. $\ell(x, u) = x^T Q x + u^T R u$, and terminal cost is a quadratic expression, i.e. $V_{\mathrm{f}}(x) = x^T P x$, where $P \succ 0$. We will show in Chapter 6 that V_N^0 is continuous at the origin also for a discontinuous PWA system subject to linear state and input constraints and quadratic stage cost.

The following theorem follows immediately:

Theorem 2.3.5 *Suppose that the conditions* $\mathcal{F}1$–$\mathcal{F}3$ *and* $\mathcal{S}1$ *hold. Suppose also that* V_N^0 *is continuous at the origin and* $\ell(x, u) \geq \alpha(\|x\|)$ *for all* $x \in \mathbb{R}^n$*, where* α *is a* $\mathscr{K}$ *function. Then, the closed-loop system* (2.23) *is asymptotically stable with a region of attraction* X_N.

Proof: We show that the conditions *(i)–(iii')* from Corollary C.1.4 given in Appendix C hold[2] for the function $V_N^0 : X_N \to \mathbb{R}_+ \cup \{\infty\}$. From Theorem 2.3.4 it follows that X_N is a PI set for (2.23), containing the origin in its interior (we recall that $0 \in \mathrm{int}(X_{\mathrm{f}})$ and $X_{\mathrm{f}} \subseteq X_N$). First, $V_N^0(0) = 0$, V_N^0 is continuous at the origin. Second, since the stage cost is bounded from below by a $\mathscr{K}$ function α, it follows that $V_N^0(x) \geq \alpha(\|x\|)$ for all $x \in X_N$. Third, the condition $\mathcal{S}1$ implies that the function V_N^0 satisfies *(iii')* on X_N, i.e.

$$V_N^0(f(x, \kappa_N(x))) - V_N^0(x) \leq V_N(f(x, \kappa_N(x)), \mathbf{u}^{\mathrm{f}}) - V_N^0(x) =$$
$$-\ell(x, \kappa_N(x)) + V_{\mathrm{f}}(f(x_N^0, \kappa_{\mathrm{f}}(x_N^0))) - V_{\mathrm{f}}(x_N^0) + \ell(x_N^0, \kappa_{\mathrm{f}}(x_N^0)) \leq -\alpha(\|x\|) \ \ \forall x \in X_N. \quad \diamondsuit$$

[2] Note that although in the MPC literature [80, 105] X_{f} is required in $\mathcal{F}1$ to be also closed, in order to prove stability we do not need this requirement (see Theorem C.1.2-*sufficiency*).

2.3.3 Robustness against uncertainty

The introduction of uncertainty in the mathematical description of the system raises the issue of *robustness*. A controlled system is robust when stability is maintained and the performance specifications are met for a certain range of model variations and a class of disturbances. Stability and performance robustness guarantee controlled systems' good behavior and safety. We can find different approaches in the literature to the study of robustness. In the present thesis we study robustness using a min-max game between the controller (acting as the minimizing player) and the plant model and the disturbance (acting as the maximizing player). In this case the robust optimal control approaches can be classified in two categories: *open-loop* min-max control and *feedback* min-max control. In an open-loop min-max control problem a single control sequence is used to minimize the worst-case cost while in a feedback min-max control problem the worst-case cost is minimized over a sequence of feedback control laws. We briefly address each approach below.

The robust control problem considered here is to steer an uncertain system subject to hard state and input constraints to a safe (target) set, while also minimizing a worst case performance function. This problem dates back to the late sixties and [16, 17, 48, 167] provide us first theoretical results on this topic. However, the main difficulty that had to be overcome was to find conditions to guarantee that the trajectory remains in the safe region once it had been reached. These conditions were provided in [18, 106] in terms of robust invariance: it was required that the safe region is a robustly invariant set. A more recent approach on this subject is based on set invariance theory and the reader is referred to [19, 79] for a survey.

Mathematically, this problem can be posed as follows. We assume that the plant is described in terms of difference equation of the form:

$$x(k+1) = f(x(k), u(k), w(k)) \tag{2.24}$$
$$y(k) = h(x(k)), \tag{2.25}$$

where the value of the uncertain parameters $w(k)$ is unknown, but is assumed to be time-varying and to take on values from a polytope $W = \{w \in \mathbb{R}^q : \Omega w \leq s\}$, where $\Omega \in \mathbb{R}^{n_\Omega \times q}$ and $s \in \mathbb{R}^{n_\Omega}$. Moreover, we assume that $0 \in W$, f is continuous in the origin and $f(0,0,0) = 0$.

Let $\mathbf{u} := [u_0^T\ u_1^T \ldots u_{N-1}^T]^T$ be an *open-loop* input sequence and $\mathbf{w} := [w_0^T\ w_1^T \cdots w_{N-1}^T]^T$ denote a realization of the disturbance over the prediction horizon N. Also, let $\phi(k; x, \mathbf{u}, \mathbf{w})$ denote the solution of (2.24) at step k when the initial state is x at step 0, the control is determined by $\mathbf{u}$ (i.e. $u(k) = u_k$) and the disturbance sequence is $\mathbf{w}$. By definition, $\phi(0; x, \mathbf{u}, \mathbf{w}) := x$. For a given initial state x, control sequence $\mathbf{u}$ and disturbance realization $\mathbf{w}$, the cost function $V_N(x, \mathbf{u}, \mathbf{w})$ is:

$$V_N(x, \mathbf{u}, \mathbf{w}) := \sum_{i=0}^{N-1} \ell(x_i, u_i) + V_{\mathrm{f}}(x_N), \tag{2.26}$$

where $x_i := \phi(i; x, \mathbf{u}, \mathbf{w})$ and thus $x_0 = x$.

For each initial condition x we define the set of feasible open-loop input sequences $\mathbf{u}$:

$$\Pi_N^{\mathrm{ol}}(x) := \{\mathbf{u} : x_i \in X, u_i \in U\ \ \forall i \in \mathbb{N}_{[0,N-1]}, x_N \in X_{\mathrm{f}},\ \forall \mathbf{w} \in \mathcal{W}\}, \tag{2.27}$$

where $\mathcal{W} := W^N$. Also, let X_N^{ol} denote the set of initial states for which a feasible input sequence exists:

$$X_N^{\mathrm{ol}} := \{x : \Pi_N^{\mathrm{ol}}(x) \neq \emptyset\}. \tag{2.28}$$

The *finite-horizon open-loop min-max* control problem is defined as:

$$\mathbb{P}_N^{\mathrm{ol}}(x): \qquad V_N^{0,\mathrm{ol}}(x) := \inf_{\mathbf{u}\in\Pi_N^{\mathrm{ol}}(x)} \max_{\mathbf{w}\in\mathcal{W}} V_N(x,\mathbf{u},\mathbf{w}). \tag{2.29}$$

Typically V_N is a continuous function and since W is a compact set, it follows that the maximum is attained in (2.29) and it is finite. Therefore, we use "$\inf\max$" instead of "$\inf\sup$".

When an open-loop min-max control is applied in a receding horizon fashion we refer to this design method as *open-loop min-max MPC*. In this case, the conditions from Theorem 2.3.4 do not guarantee feasibility and robust stability of the closed-loop system. We can find in the literature [5, 12, 105, 111] different modifications of the optimization problem $\mathbb{P}_N^{\mathrm{ol}}(x)$ and of the conditions $\mathcal{F}1$–$\mathcal{F}3$ and $\mathcal{S}1$ that guarantee robustness of the open-loop min-max MPC controller, i.e. robust feasibility and robust closed-loop stability. For instance in [111] the fixed receding horizon N is replaced by a variable receding horizon. In [5] an additional robust stability constraint is included in the optimal control problem that requires the control to reduce the cost associated to each possible realization of the system (assumed finite in number).

In general the open-loop formulation, although it is attractive from a computational point of view, is too conservative since the set of feasible trajectories may diverge severely from the origin [148]. It is known that effective control in the presence of disturbances requires one to optimize over *feedback policies* [16, 91, 103] rather than open-loop input sequences. A feedback control prevents the trajectory from diverging excessively and also the performance is improved compared to the open-loop case. This results from the increased number of degree of freedom in the optimal control problem.

We now present the feedback min-max optimal control formulation. In this case we define the decision variable in the optimal control problem, for a given initial condition x as a control *policy*

$$\pi := (\mu_0(\cdot), \mu_1(\cdot), \ldots, \mu_{N-1}(\cdot)),$$

where each $\mu_i(\cdot)$ is a feedback *law*. Also, let $x_k = \phi(k; x, \pi, \mathbf{w})$ denote the solution of (2.24) at step k when the initial state is x at step 0, the control is determined by the policy π and the disturbance sequence is $\mathbf{w}$.

For each initial condition x we define the set of feasible policies π:

$$\Pi_N^{\mathrm{fb}}(x) := \{\pi : \mu_i \in U,\ x_i \in X\ \forall i \in \mathbb{N}_{[0,N-1]},\ x_N \in X_{\mathrm{f}},\ \forall \mathbf{w} \in \mathcal{W}\} \tag{2.30}$$

Also, let X_N^{fb} denote the set of initial states for which a feasible policy exists, i.e.

$$X_N^{\mathrm{fb}} := \{x : \Pi_N^{\mathrm{fb}}(x) \neq \emptyset\}. \tag{2.31}$$

The *finite-horizon feedback min-max* control problem is defined as:

$$\mathbb{P}_N^{\mathrm{fb}}(x): \qquad V_N^{0,\mathrm{fb}}(x) := \inf_{\pi\in\Pi_N^{\mathrm{fb}}(x)} \max_{\mathbf{w}\in\mathcal{W}} V_N(x,\pi,\mathbf{w}). \tag{2.32}$$

The receding horizon implementation of a feedback min-max control is referred to as *feedback min-max MPC*. Let κ_N^{fb} denote the corresponding feedback MPC law, i.e. $\kappa_N^{\mathrm{fb}}(x) = \mu_0^0(x)$, where $\pi^0(x) = (\mu_0^0(x), \mu_1^0(\cdot), \cdots, \mu_{N-1}^0(\cdot))$ is a minimizer of (2.32). Before proceeding to study the behavior of the corresponding closed-loop system the notion of robust stability [80] is introduced. Because the disturbance is assumed to be bounded, the most that can be achieved with a controller is to steer the state to a neighborhood of the origin X_{f} and then with a local controller κ_{f} to maintain the state in X_{f} for any possible realizations of the disturbances. Therefore, the set

X_f is *robustly stable* if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $d(x, X_\mathrm{f}) \leq \delta$ implies $d(\phi(k; x, \kappa_N^{\mathrm{fb}}, \mathbf{w}), X_\mathrm{f}) \leq \epsilon$ for all $k \geq 0$ and for all admissible disturbance sequences **w**. If $\lim_{k\to\infty} d(\phi(k; x, \kappa_N^{\mathrm{fb}}, \mathbf{w}), X_\mathrm{f}) = 0$ for all admissible disturbance sequences **w** and for all $x \in X$, then the set X_f is *robustly asymptotically attractive* with a region of attraction X. When both conditions are satisfied we refer to X_f as *robustly asymptotically stable*. Here, $d(x, X_\mathrm{f})$ denotes the distance from a point x to the set X_f induced by some p-norm.

By some appropriate modifications of the conditions $\mathcal{F}1$–$\mathcal{F}3$ and $\mathcal{S}1$ (see e.g. [8,83,105,148]) robust feasibility and robust closed-loop stability can be recovered for the feedback min-max MPC controller. One method is proposed in [80,148] where the feedback min-max MPC problem for a linear system and a convex cost is solved by considering only the disturbance realizations that take on values at the vertexes of the disturbance polytope W. Robust stability is guaranteed under the following conditions on X_f, κ_f and V_f [80, 103, 105, 148]:

$\mathcal{F}1^w$: $X_\mathrm{f} \subseteq X$ and $0 \in \mathrm{int}(X_\mathrm{f})$

$\mathcal{F}2^w$: $\kappa_\mathrm{f}(x) \in U$ for all $x \in X_\mathrm{f}$

$\mathcal{F}3^w$: $f(x, \kappa_\mathrm{f}(x), w) \in X_\mathrm{f}$ for all $x \in X_\mathrm{f}$ and $w \in W$

$\mathcal{S}1^w$: $V_\mathrm{f} = 0$ and $\ell(x, u) = 0$ for all $x \in X_\mathrm{f}$, $\ell(x, u) \geq \alpha(d(x, X_\mathrm{f}))$ for all $x \in X \setminus X_\mathrm{f}$,

where α is a $\mathscr{K}$ function. Using these conditions one can follow a standard procedure of using the optimal value function as a candidate Lyapunov function [80,103,105] to show robust asymptotic stability.

As an alternative, in [8] it is proposed that a dynamic programming approach be used to obtain an explicit expression for the feedback MPC law. Provided the system is linear, the disturbance enters additively and the stage cost is piecewise affine (e.g. 1-norm or ∞-norm), a piecewise affine expression for the MPC law can be computed off-line using tools from multi-parametric linear programming.

2.4 Conclusions

In this chapter we have summarized some basic background on specific classes of hybrid systems (PWA systems and MMPS systems) and DES (MPL systems and switching MPL systems). The reader should note that we make distinction between PWA systems defined in the state space *only* and general PWA systems defined in the state *and* input space. It is also interesting to note that MMPS systems encompasses MPL systems and switching MPL systems, provided that some assumption on the switching function holds.

Furthermore, the main ingredients of finite-horizon optimal control and its receding horizon implementation, called MPC, for general nonlinear systems have been introduced and some general solutions to the main issues in MPC (e.g. feasibility, robustness and closed-loop stability) were presented, based on a terminal set and a terminal cost approach.

Chapter 3

Finite-horizon optimal control for constrained max-plus-linear systems

We provide in this chapter a solution to a class of finite-horizon optimal control problems for MPL systems. We first consider the deterministic case and then we extend the results to the disturbance case where we consider the min-max framework and where the uncertain parameters are assumed to lie in a given polytope. Despite the fact that the controlled system is nonlinear, we are able to provide sufficient conditions, which are often satisfied in practice, such that we can preserve convexity of the system and thus the optimal solution is computed by solving a linear program or multi-parametric linear programs. The key assumptions that allow us to guarantee convexity of the optimal value function and its domain are that the stage cost has a particular representation in which the coefficients corresponding to the state vector are nonnegative and that the matrix associated with the state constraints is also nonnegative.

3.1 Introduction

Conventional control theory uses systems which typically deal with quantities that are continuous variables, in the sense that they change as time passes and take on values in a continuum. However, in this technological era we encounter many quantities that are discrete and that evolve in time by the occurrence of events at possible irregular time intervals, i.e. not necessarily coinciding with clock ticks. We refer to these systems that contain such quantities as event-driven systems or *discrete event systems* (DES).

In general the dynamics of DES can be characterized by synchronization and concurrency (see Section 2.2). These two aspects make the dynamics of a general DES nonlinear in conventional algebra. However, there exists a class of DES that contains only the synchronization aspect for which the system equations become "linear" when we formulate it in the max-plus algebra. We refer to such a system as *max-plus-linear* (MPL) system.

In this chapter we focus on MPL systems and different versions of optimal control for such a class of systems will be presented. Before proceeding further, we give a short introduction to the basic concepts of the max-plus algebra. A more elaborate review of the topic can be found in [4, 37, 66]. In Section 3.2 we derive sufficient conditions under which the solution to a finite-horizon optimal control problem for constrained MPL systems can be computed by solving a linear program. Moreover, in the unconstrained case we derive the explicit state-space formula of the just-in-time controller. In Section 3.3 the robustification of the optimal control problem is discussed using the min-max paradigm. Since MPL systems are nonlinear, non-convexity is the main concern when we want to solve a min-max control problem for uncertain MPL systems. Using recent results in polyhedral algebra and multi-parametric linear programming it is demonstrated that the solutions to open-loop, disturbance feedback and state feedback min-max control problem can be computed by solving either a linear program or multi-parametric linear programs. This chapter is an extension of the work presented in [122, 125, 158].

3.1.1 Max-plus algebra

Define $\varepsilon := -\infty$ and denote $\mathbb{R}_\varepsilon := \mathbb{R} \cup \{\varepsilon\}$. For elements $x, y \in \mathbb{R}_\varepsilon$ we define the operations $\oplus$ (max-plus addition) and $\otimes$ (max-plus multiplication) by

$$x \oplus y := \max\{x, y\} \quad \text{and} \quad x \otimes y := x + y. \tag{3.1}$$

The set $\mathbb{R}_\varepsilon$ together with the operations $\oplus$ and $\otimes$ is called *max-plus algebra* and is denoted by $\mathcal{R}_\varepsilon = (\mathbb{R}_\varepsilon, \oplus, \otimes, \varepsilon, 0)$. It can be shown that the max-plus algebra $\mathcal{R}_\varepsilon$ is an algebraic structure called *semiring*: (i) $\oplus$ is associative and commutative with zero element ε; (ii) $\otimes$ is associative, distributes over $\oplus$ and has unit element 0; (iii) ε is absorbing for $\oplus$ (i.e. $x \otimes \varepsilon = \varepsilon \otimes x = x$ for all $x \in \mathbb{R}_\varepsilon$). Note that the semiring $\mathcal{R}_\varepsilon$ is also commutative (i.e. $x \otimes y = y \otimes x$) and idempotent (i.e. $x \oplus x = x$). The reason for using the symbols $\oplus$ and $\otimes$ for $\max$ and $+$, respectively, is that the remarkable analogy with the conventional algebra: many concepts and properties from conventional algebra (such as eigenvectors and eigenvalues, Cayley-Hamilton theorem, etc.) can be translated to max-plus algebra by replacing $+$ by $\oplus$ and $\cdot$ by $\otimes$, as we will see below.

For any $x \in \mathbb{R}_\varepsilon$ define

$$x^{\otimes^k} := \underbrace{x \otimes x \otimes \cdots \otimes x}_{k \text{ times}} \quad \forall\, k \in \mathbb{N} \setminus \{0\}, \quad x^{\otimes^0} := 0.$$

Observe that $x^{\otimes^k}$ corresponds to kx in conventional algebra.

The set of $m \times n$ matrices with entries in $\mathbb{R}_\varepsilon$ is denoted by $\mathbb{R}_\varepsilon^{m\times n}$. For matrices $A, B \in \mathbb{R}_\varepsilon^{m\times n}$ and $C \in \mathbb{R}_\varepsilon^{n\times p}$ one can extend the max-plus operations in the usual way:

$$[A \oplus B]_{ij} := A_{ij} \oplus B_{ij} = \max\{A_{ij}, B_{ij}\} \quad \forall i \in \mathbb{N}_{[1,n]}, j \in \mathbb{N}_{[1,m]},$$
$$[A \otimes C]_{il} := \bigoplus_{k=1}^{n} A_{ik} \otimes C_{kl} = \max_{k \in \mathbb{N}_{[1,n]}} \{A_{ik} + C_{kl}\} \quad \forall i \in \mathbb{N}_{[1,n]}, l \in \mathbb{N}_{[1,p]}.$$

Moreover, for any $A \in \mathbb{R}_\varepsilon^{m\times n}$ and $\lambda \in \mathbb{R}_\varepsilon$ we denote with $\lambda + A$, with some abuse of notation, the matrix from $\mathbb{R}_\varepsilon^{m\times n}$ defined as $[\lambda + A]_{ij} := \lambda + A_{ij}$ for all i, j. Similarly, in max-plus algebra we define $\lambda \otimes A$ as the matrix $\lambda + A$. The matrix $E \in \mathbb{R}_\varepsilon^{n\times n}$ is the identity matrix in max-plus algebra: $E_{ii} := 0$, for all $i \in \mathbb{N}_{[1,n]}$ and $E_{ij} := \varepsilon$, for all $i \neq j$ and the zero matrix is denoted with $\mathcal{E}$: $\mathcal{E}_{ij} := \varepsilon$, for all $i,\ j \in \mathbb{N}_{[1,n]}$. The dimensions of the matrices E and $\mathcal{E}$ are usually clear from the context.

For any matrix $A \in \mathbb{R}_\varepsilon^{n\times n}$, the k^{th} max-plus power of A is denoted with

$$A^{\otimes^k} := \underbrace{A \otimes A \otimes \cdots \otimes A}_{k \text{ times}} \quad \forall\, k \in \mathbb{N} \setminus \{0\}, \quad A^{\otimes^0} := E.$$

Moreover, we define A^*, whenever it exists[1], by

$$A^* := \lim_{k\to\infty} E \oplus A \oplus \cdots \oplus A^{\otimes^k}. \tag{3.2}$$

[1] See Lemma 3.1.1 (ii) below.

The k^{th} max-plus power of the matrix A has an interesting expression. For all $i, j \in \mathbb{N}_{[1,n]}$ let us define the set of *paths* between i and j of length k as

$$\mathrm{Path}(i,j;k) := \{(i_1, i_2, \cdots, i_{k+1}) \in \mathbb{N}_{[1,n]}^{k+1} : i_1 = i,\ i_{k+1} = j,\ A_{i_j i_{j+1}} \neq \varepsilon\, \forall j \in \mathbb{N}_{[1,k]}\} \quad (3.3)$$

When $i = j$ the path is called a *cycle*. Then, it follows that

$$[A^{\otimes^k}]_{ij} = \max\{A_{i_1 i_2} + A_{i_2 i_3} + \cdots + A_{i_k i_{k+1}} : (i_1, i_2, \cdots, i_{k+1}) \in \mathrm{Path}(i,j;k)\}. \quad (3.4)$$

The following consequence is immediate [66]:

Lemma 3.1.1 *Suppose that $A \in \mathbb{R}_\varepsilon^{n\times n}$ such that $A_{ij} < 0$ for all $i, j \in \mathbb{N}_{[1,n]}$.*
(i) The following relation holds: $\lim_{k\to\infty} A^{\otimes^k} = \mathcal{E}$.
(ii) A^* exists and is given by $A^* = E \oplus A \oplus \cdots \oplus A^{\otimes^{n-1}}$. ♢

A matrix $P \in \mathbb{R}_\varepsilon^{n\times n}$ is invertible in max-plus algebra if there exists a matrix $P^{\otimes^{-1}} \in \mathbb{R}_\varepsilon^{n\times n}$ such that $P^{\otimes^{-1}} \otimes P = P \otimes P^{\otimes^{-1}} = E$. It is well-known [4,66] that a matrix $P \in \mathbb{R}_\varepsilon^{n\times n}$ is invertible in max-plus algebra if and only if it can be factorized as $P = D \otimes T$, where $D \in \mathbb{R}_\varepsilon^{n\times n}$ is max-plus diagonal matrix with non-ε diagonal entries and $T \in \mathbb{R}_\varepsilon^{n\times n}$ is a max-plus permutation matrix[2]. Important notions in max-plus algebra are those of max-plus eigenvalue and eigenvector.

Definition 3.1.2 *Let $A \in \mathbb{R}_\varepsilon^{n\times n}$. Then, $\lambda \in \mathbb{R}_\varepsilon$ is a max-plus eigenvalue and $v \in \mathbb{R}_\varepsilon^n$ (where v has at least one finite entry) is a max-plus eigenvector if $A \otimes v = \lambda \otimes v$.* ♢

Note that Definition 3.1.2 allows an eigenvalue to be ε. Note further that a square matrix may have more than one max-plus eigenvalue. We denote with λ^* the largest max-plus eigenvalue of A. From [66] the following consequence can be deduced:

Lemma 3.1.3 *Suppose that the largest max-plus eigenvalue λ^* of the matrix $A \in \mathbb{R}_\varepsilon^{n\times n}$ is finite. Then, λ^* is given by*

$$\lambda^* = \max\left\{\frac{[A^{\otimes^k}]_{ii}}{k} : i, k \in \mathbb{N}_{[1,n]}\right\} \quad (3.5)$$

A matrix $A \in \mathbb{R}_\varepsilon^{n\times m}$ is *row-finite* if for any row $i \in \mathbb{N}_{[1,n]}$, $\max_{j\in\mathbb{N}_{[1,m]}} A_{ij} > \varepsilon$. Matrix A is *column-finite* if for any column $j \in \mathbb{N}_{[1,m]}$, $\max_{i\in\mathbb{N}_{[1,n]}} A_{ij} > \varepsilon$.

For any vector $x \in \mathbb{R}^n$ the ∞–norm is defined as $\|x\|_\infty := \max_{i\in\mathbb{N}_{[1,n]}}\{x_i, -x_i\}$. It is known [66] that for any row-finite matrix $A \in \mathbb{R}_\varepsilon^{n\times m}$, the map $\mathbb{R}^m \to \mathbb{R}^n,\ x \mapsto A \otimes x$ is *nonexpansive*, i.e. the following inequality holds:

$$\|(A \otimes x) - (A \otimes y)\|_\infty \leq \|x - y\|_\infty \ \ \forall\, x, y \in \mathbb{R}^m. \quad (3.6)$$

We also introduce the following notations:

$$x \oplus' y := \min\{x, y\} \quad \text{and} \quad x \otimes' y := x + y$$

for all $x, y \in \bar{\mathbb{R}}$. The operations $\otimes$ and $\otimes'$ differ only in that $(-\infty) \otimes (+\infty) := -\infty$, while $(-\infty)\otimes'(+\infty) := +\infty$. The matrix multiplication and addition for $(\oplus', \otimes')$ are defined similarly as to the case that we defined for $(\oplus, \otimes)$.

[2] A max-plus permutation matrix is obtained by permuting the rows or the columns of the max-plus identity matrix E.

For any matrix $A \in \mathbb{R}_\varepsilon^{m\times n}$ and any vectors $x, y \in \mathbb{R}^n$, the following inequalities hold:

$$x \leq y \Rightarrow \ A \otimes x \leq A \otimes y \quad \text{and} \quad A \otimes' x \leq A \otimes' y, \tag{3.7}$$

where "$\leq$" denotes the partial order defined by the *nonnegative orthant* $\mathbb{R}_+^n := \{x \in \mathbb{R}^n : x_i \geq 0 \ \forall i \in \mathbb{N}_{[1,n]}\}$ (i.e. $x \leq y$ if and only if $y - x \in \mathbb{R}_+^n$ or equivalently $x_i \leq y_i$ for all $i \in \mathbb{N}_{[1,n]}$). In the following theorem we summarize some basic results from max-plus algebra [4, 37, 66]:

Lemma 3.1.4 *(i) Suppose $A \in \mathbb{R}_\varepsilon^{m\times n}$ and $b \in \mathbb{R}_\varepsilon^m$. Then, the inequality*

$$A \otimes x \leq b$$

has the largest solution given by

$$x^0 = (-A^T) \otimes' b = -(A^T \otimes (-b)).$$

By the largest solution we mean that for all x satisfying $A \otimes x \leq b$ we have $x \leq x^0$.

(ii) Suppose $C \in \mathbb{R}_\varepsilon^{n\times n}$ and $b \in \mathbb{R}_\varepsilon^n$. Then, the equation

$$x = C \otimes x \oplus d$$

has a solution $x = C^* \otimes d$. If $C_{ij} < 0$ for all $i, j \in \mathbb{N}_{[1,n]}$, then the solution is unique. ♢

3.1.2 Constrained max-plus-linear systems

We now introduce a class of constrained DES that can be modeled using the max-plus addition $\oplus$ (corresponding to the order of events: e.g. a new job starts as soon as all preceding jobs were finished) and max-plus multiplication $\otimes$ (corresponding to durations: e.g. the finishing time of a job equals the starting time plus the duration). As we have seen in Section 2.2.1, DES with only synchronization and no concurrency, i.e. systems in which the sequence of events are fixed (for instance repetitive production systems like Example 2.2.1), can be described by equations of the form:

$$\begin{aligned} \bar{x}(k+1) &= \bar{A} \otimes \bar{x}(k) \oplus \bar{B} \otimes \bar{u}(k) \\ \bar{y}(k) &= \bar{C} \otimes \bar{x}(k). \end{aligned} \tag{3.8}$$

Here, $\bar{A} \in \mathbb{R}_\varepsilon^{n\times n}, \bar{B} \in \mathbb{R}_\varepsilon^{n\times m}, \bar{C} \in \mathbb{R}_\varepsilon^{p\times n}$ and $\bar{x}$ represents the state, $\bar{u}$ the input and $\bar{y}$ the output vector. We refer to (3.8) as a *max-plus-linear* (MPL) system. In the context of DES, k is an event counter while $\bar{u}, \bar{x}$ and $\bar{y}$ are times (feeding times, processing times and finishing times, respectively).

We consider a reference (due date) signal $\{r(k)\}_{k\geq 0} \subset \mathbb{R}^p$ which the output of the MPL system (3.8) may be required to "track" in the sense that, for instance, the tardiness $\max\{\bar{y}-r, 0\}$ is penalized.

Since time is not scalable, examples of typical *constraints* for an MPL system are:

$$\begin{aligned} \bar{y}(k) \leq r(k) + h^{\mathrm{y}}(k), \quad \bar{u}_i(k) - \bar{u}_j(k) \leq h_{ij}^{\mathrm{u}}(k),\ \bar{x}_i(k) - \bar{u}_j(k) \leq h_{ij}^{\mathrm{xu}}(k) \\ \bar{u}(k) - \bar{u}(k+1) \leq 0. \end{aligned} \tag{3.9}$$

The constraint $\bar{u}(k) - \bar{u}(k+1) \leq 0$ appears in the context of DES, where the input represents times, so it should be nondecreasing. The constraints in (3.9) can equivalently be written as:

$$\bar{H}_k \bar{x}(k) + \bar{G}_k \bar{u}(k) + \bar{F}_k r(k) \leq \bar{h}_k \tag{3.10}$$

$$\bar{u}(k) - \bar{u}(k+1) \leq 0, \tag{3.11}$$

for some matrices $\bar{H}_k, \bar{G}_k, \bar{F}_k$ and a vector $\bar{h}_k$ of appropriate dimensions.

Given a matrix $H \in \mathbb{R}^{n\times m}$, H is *nonnegative* if and only if $H_{ij} \geq 0$ for all $i \in \mathbb{N}_{[1,n]}, j \in \mathbb{N}_{[1,m]}$. Mathematically, we use the notation:

$$H \geq 0.$$

Similarly we define $H \leq 0$.

From the previous discussion it follows that $\bar{H}_k \geq 0$ for all $k \geq 0$. Note that the constraint (3.11) does not fit the form (3.10). However, we can include (3.11) into (3.10) as follows: we introduce a new state vector $x(k) = [\bar{x}^T(k) \;\; z^T(k)]^T$ with the dynamics

$$\begin{aligned} x(k+1) &= A \otimes x(k) \oplus B \otimes u(k) \\ y(k) &= C \otimes x(k) \end{aligned} \tag{3.12}$$

and the extra constraint[3]:

$$[0 \;\; I]x(k) \leq u(k), \tag{3.13}$$

where $u(k) = \bar{u}(k),\; y(k) = \bar{y}(k)$ and the system matrices are given by

$$A = \begin{bmatrix} \bar{A} & \bar{B} \\ \mathcal{E} & E \end{bmatrix}, \quad B = \begin{bmatrix} \bar{B} \\ E \end{bmatrix} \quad \text{and} \quad C = [\bar{C} \;\; \mathcal{E}].$$

Given any initial conditions $\bar{x}(0)$ and $u(-1)$ for the system (3.8) with the constraint (3.11) and the initial condition $x(0) = [\bar{x}(0)^T \;\; u(-1)^T]^T$ for the new system (3.12) with the constraint (3.13), then by applying the same input signal $\{u(k)\}_{k\geq 0}$ to both systems we obtain that the first n components of $x(k)$ coincide with $\bar{x}(k)$ and the last m components of $x(k)$ coincide with $u(k-1)$ for all $k \geq 0$. This can be proved by induction. For $k = 0$ the statement is obvious. Let us assume that the statement is true for k, i.e.

$$[I \;\; 0]x(k) = \bar{x}(k), \quad z(k) = u(k-1).$$

We prove that similar equalities hold for $k+1$. But, $z(k+1) = z(k) \oplus u(k)$ and from the inequality (3.13) it follows that $z(k+1) = u(k)$. Moreover, from our induction hypothesis we get that $[I \;\; 0]x(k+1) = \bar{A} \otimes \bar{x}(k) \oplus \bar{B} \otimes z(k+1) = \bar{x}(k+1)$. We also obtain that the output signals of these two systems coincide, i.e.

$$\bar{y}(k) = y(k) \quad \forall k \geq 0.$$

Moreover, the constraints (3.10)–(3.11) corresponding to the MPL system (3.8) can be written for the new system (3.12) as

$$[\bar{H}_k \;\; 0]x(k) + \bar{G}_k u(k) + \bar{F}_k r(k) \leq \bar{h}_k$$

and the extra constraint (3.13)

$$[0 \;\; I]x(k) - u(k) \leq 0,$$

i.e. combining them we obtain:

$$H_k x(k) + G_k u(k) + F_k r(k) \leq h_k, \tag{3.14}$$

where $H_k = \begin{bmatrix} \bar{H}_k & 0 \\ 0 & I \end{bmatrix}, G_k = \begin{bmatrix} \bar{G}_k \\ -I \end{bmatrix}, F_k = \begin{bmatrix} \bar{F}_k \\ 0 \end{bmatrix}$ and $h_k = \begin{bmatrix} \bar{h}_k \\ 0 \end{bmatrix}$.

Note that the condition on $\bar{H}_k$ ($\bar{H}_k \geq 0$) is preserved under the previous transformation, i.e. $H_k \geq 0$. Therefore, in the rest of this chapter we consider MPL systems of the form (3.12) subject to *hard state and input constraints* (3.14), where $\{r(k)\}_{k\geq 0}$ is a due date signal that the output should follow.

[3] Here, I denotes the identity matrix in conventional algebra of appropriate dimension.

3.2 Finite-horizon optimal control

In the last two decades there has been an increase in the amount of research on DES that can be modeled as MPL systems. Most of the earlier literature on this class of systems addresses performance analysis [4, 32, 34, 37, 55, 66] rather than control. Several authors have developed methods to compute optimal controllers for MPL systems [4, 33, 36, 85, 95, 109] using two main ingredients: residuation theory [20] and input-output models. In general, these methods use the residuation approach to design a just-in-time controller, i.e. so that the output of the controlled system is, on the one hand, less than the desired reference signal but as close as possible to the given reference and, on the other hand, the control is delayed as much as possible. However, the residuation approach does not cope with input and state constraints and thus the resulting control sequence is sometimes decreasing, i.e. it is not always physically feasible. Furthermore, the residuation approach cannot solve tracking problems corresponding to the case when the outputs do not occur before the chosen reference although these situations are often encountered in many practical applications, when e.g. the output of the process is already delayed with respect to the reference. Clearly, input-output models can easily be transformed into state-space models. The state-space approach, in addition, allows the time-invariant control design methods to easily be extended to the multi-input multi-output case and time-varying systems, and the initial state is included explicitly.

In this section we provide a solution to a class of finite-horizon optimal control problems for MPL systems where it is required that the closed-loop input and state sequence satisfy a given set of linear inequality constraints and the performance is measured via a cost function that may, in particular, be chosen to provide a trade-off between minimizing the due date error and a just-in-time control. We follow here a similar approach as in Section 2.3.1 on the finite-horizon optimal control of general nonlinear systems. Note that in [45] a finite-horizon optimal control problem, under similar settings as in this section, is solved in a receding horizon fashion. The main difference compared to [45] is that we determine sufficient conditions under which the corresponding optimization problem is a linear program or even has an analytic solution. Moreover, the analytic solution leads to a just-in-time controller. As an application of the finite-horizon optimal control we compute the solution to the "time" optimal control problem for MPL systems by solving a finite sequence of linear programs. The receding horizon implementation of the finite-horizon optimal control problems discussed in this chapter will be presented in Chapter 4.

3.2.1 Problem formulation

Before proceeding with the problem formulation we introduce a class of functions that will be used extensively in this chapter: $\mathcal{F}_{\text{mps}}$ denotes the set of *max-plus-scaling* functions, i.e. functions

$$g : \mathbb{R}^n \to \mathbb{R}, \quad x \mapsto \max_{j \in \mathbb{N}_{[1,l]}} \{\alpha_j^T x + \beta_j\},$$

where $\alpha_j \in \mathbb{R}^n$, $\beta_j \in \mathbb{R}$ and l is a finite positive integer. Furthermore, $\mathcal{F}_{\text{mps}}^+$ denotes the set of *max-plus-nonnegative-scaling* functions, i.e. functions defined by $g(x) = \max_{j \in \mathbb{N}_{[1,l]}} \{\alpha_j^T x + \beta_j\}$, where $\alpha_j \geq 0$ for all $j \in \mathbb{N}_{[1,l]}$. Note that the expression "max-plus-scaling function" is an equivalent terminology for a convex PWA function. Similarly, the expression "max-plus-nonnegative-scaling function" stands for a nondecreasing convex PWA function. We use these definitions since they closely resemble the definition of an MMPS function.

We consider the following MPL system:

$$x(k+1) = A \otimes x(k) \oplus B \otimes u(k) \tag{3.15}$$
$$y(k) = C \otimes x(k), \tag{3.16}$$

where $A \in \mathbb{R}_\varepsilon^{n\times n}, B \in \mathbb{R}_\varepsilon^{n\times m}$ and $C \in \mathbb{R}_\varepsilon^{p\times n}$. Since the states represent times we assume they can always be measured (see also Section 3.2.3 for a more elaborate discussion on this subject). Note that the function $(x, u) \mapsto A \otimes x \oplus B \otimes u$ belongs to $(\mathcal{F}^+_{\text{mps}})^n$.

We also consider a due date signal $\{r(k)\}_{k\geq 0} \subset \mathbb{R}^p$ which the output may be required to track. For the finite-horizon optimal control problem defined below, the system is subject to *hard* control and state constraints over a finite horizon N:

$$H_k x(k) + G_k u(k) + F_k r(k) \leq h_k \quad \forall k \in \mathbb{N}_{[0,N-1]}, \tag{3.17}$$

with the *terminal constraint*

$$H_N x(N) + F_N r(N) \leq h_N, \tag{3.18}$$

where $H_k \in \mathbb{R}^{n_k\times n}, G_k \in \mathbb{R}^{n_k\times m}, F_k \in \mathbb{R}^{n_k\times p}, h_k \in \mathbb{R}^{n_k}$. We can now formulate the problem of finite-horizon optimal control of a constrained MPL system. We will define the decision variable in the optimal control problem, for a given initial condition x and the due dates

$$\mathbf{r} := [r_0^T \ r_1^T \ldots r_N^T]^T,$$

as a control sequence

$$\mathbf{u} := [u_0^T \ u_1^T \ldots u_{N-1}^T]^T.$$

Let $\phi(i; x, \mathbf{u})$ denote the state solution of (3.15) at event step i when the initial state is x at event step 0 and the control is determined by **u** (i.e. $u(i) = u_i$). By definition, $\phi(0; x, \mathbf{u}) := x$. The cost function $V_N(x, \mathbf{r}, \mathbf{u})$, for the initial condition x, the due dates **r** and the control sequence **u**, is defined as:

$$V_N(x, \mathbf{r}, \mathbf{u}) := \sum_{i=0}^{N-1} \ell_i(x_i, u_i, r_i) + V_{\text{f}}(x_N, r_N), \tag{3.19}$$

where the *terminal cost* is given by

$$V_{\text{f}}(x_N, r_N) = \ell_N(x_N, r_N),$$

and where $x_i := \phi(i; x, \mathbf{u})$ (and thus $x_0 := x$). We usually denote with X_{f} the *terminal set*, i.e.

$$X_{\text{f}} := \{(x, r) : H_N x + F_N r \leq h_N\}.$$

For each initial condition x and due dates **r** we define the set of feasible control sequences **u**:

$$\Pi_N(x, \mathbf{r}) := \{\mathbf{u} : H_i x_i + G_i u_i + F_i r_i \leq h_i, \ (x_N, r_N) \in X_{\text{f}}, \forall i \in \mathbb{N}_{[0,N-1]}\}. \tag{3.20}$$

Also, let X_N denote the set of initial states and reference signals for which a feasible input sequence exists:

$$X_N := \{(x, \mathbf{r}) : \Pi_N(x, \mathbf{r}) \neq \emptyset\}. \tag{3.21}$$

The *finite-horizon optimal control problem* for the MPL system (3.15)–(3.16) is defined as:

$$\mathbb{P}_N(x,\mathbf{r}): \qquad V_N^0(x,\mathbf{r}) := \inf_{\mathbf{u}\in\Pi_N(x,\mathbf{r})} V_N(x,\mathbf{r},\mathbf{u}). \tag{3.22}$$

Let $\mathbf{u}_N^0(x,\mathbf{r}) =: [(u_0^0(x,\mathbf{r}))^T\ (u_1^0(x,\mathbf{r}))^T \cdots (u_{N-1}^0(x,\mathbf{r}))^T]^T$ denote a minimizer of the optimization problem $\mathbb{P}_N(x,\mathbf{r})$ (as defined in (2.18)), i.e.

$$\mathbf{u}_N^0(x,\mathbf{r}) \in \arg\min_{\mathbf{u}\in\Pi_N(x,\mathbf{r})} V_N(x,\mathbf{r},\mathbf{u}). \tag{3.23}$$

The following key assumptions will be used throughout this chapter:

A1: The matrices H_i in (3.17)–(3.18) are nonnegative (i.e. $H_i \geq 0$) for all $i \in \mathbb{N}_{[0,N]}$.

A2: The time-varying stage costs ℓ_i satisfies $\ell_i(\cdot,u,r) \in \mathcal{F}_{\text{mps}}^+$ for each fixed (u,r) and $\ell_i \in \mathcal{F}_{\text{mps}}$ for all $i \in \mathbb{N}_{[0,N]}$.

The conditions from assumptions **A1**-**A2** are not too restrictive and are often met in applications. Note that typical constraints for MPL systems satisfy assumption **A1** (see Section 3.1). A typical example of a stage cost that satisfies **A2** is the following:

$$\ell_i(x_i,u_i,r_i) = \sum_{j=1}^{p} \max\left\{\left[(C\otimes x_i) - r_i\right]_j,\ 0\right\} - \gamma\sum_{j=1}^{m}[u_i]_j \tag{3.24}$$

for all $i \in \mathbb{N}_{[0,N-1]}$. Here, $[v_i]_j$ denotes the j^{th} component of a vector v_i and $\gamma \geq 0$. In the context of manufacturing systems, this stage cost has the following interpretation: the first term penalizes the delay of the output with respect to the due dates, while the second term tries to maximize the feeding times, i.e. to feed the raw material as late as possible. The trade-off between these two terms is given by the size of γ. Clearly,

$$\ell_N(x_N,r_N) = \sum_{j=1}^{p} \max\left\{\left[(C\otimes x_N) - r_N\right]_j,\ 0\right\}. \tag{3.25}$$

For more examples of stage costs satisfying assumption **A2** see [45].

3.2.2 Linear programming solution

We now show that under the previous assumptions **A1**-**A2**, the optimization problem $\mathbb{P}_N(x,\mathbf{r})$ can be recast as a linear program. We denote with

$$\mathbf{x} := [x_0^T\ x_1^T \cdots x_N^T]^T.$$

Then, it follows that:

$$\mathbf{x} = \Theta\otimes x \oplus \Phi\otimes\mathbf{u}, \tag{3.26}$$

where

$$\Theta := \begin{bmatrix} E \\ A \\ \vdots \\ A^{\otimes^N} \end{bmatrix}, \Phi := \begin{bmatrix} \mathcal{E} & \mathcal{E} & \cdots & \mathcal{E} \\ B & \mathcal{E} & \cdots & \mathcal{E} \\ \vdots & \vdots & \ddots & \vdots \\ A^{\otimes^{N-1}}\otimes B & A^{\otimes^{N-2}}\otimes B & \cdots & B \end{bmatrix}. \tag{3.27}$$

The constraints (3.17)–(3.18) can be written more compactly as:

$$\mathbf{Hx} + \mathbf{Gu} + \mathbf{Fr} \leq \mathbf{h}$$

for some matrices $\mathbf{H}, \mathbf{G}, \mathbf{F}$ and a vector $\mathbf{h}$ of appropriate dimensions. Note that $\mathbf{H}$ has non-negative entries (according to assumption **A1**).

The next lemma states that some basic properties of max-plus-scaling functions are preserved under addition, composition and multiplication with a non-negative scalar.

Lemma 3.2.1 *Suppose the functions g_1, g_2 and $g_3 = [g_{31}, \ldots, g_{3n}]^T$ with g_1, g_2, g_{3j} of the form $g : Z \times W \to \mathbb{R}$, $(z, w) \mapsto g(z, w)$ have the property that for each $w \in W$, $g_i(\cdot, w)$, $g_{3j}(\cdot, w) \in \mathcal{F}^+_{\text{mps}}$ and for each $z \in Z$, $g_i(z, \cdot)$, $g_{3j}(z, \cdot) \in \mathcal{F}_{\text{mps}}$ for all i, j. Then, for any scalar $\lambda \geq 0$, $(\lambda g_1)(\cdot, w)$, $(g_1 + g_2)(\cdot, w)$, $g_1(g_3(\cdot, w), w) \in \mathcal{F}^+_{\text{mps}}$ for any fixed $w \in W$, and $(\lambda g_1)(z, \cdot)$, $(g_1 + g_2)(z, \cdot)$, $g_1(g_3(z, \cdot), \cdot) \in \mathcal{F}_{\text{mps}}$ for any fixed $z \in Z$.*

Proof : The proof is straightforward and uses some basic properties of the max operator:

$$\begin{aligned} \lambda \max\{a, b\} &= \max\{\lambda a, \lambda b\} \;\; \forall \lambda \geq 0, \\ \max\{a, b\} + \max\{c, d\} &= \max\{a + c, a + d, b + c, b + d\} \\ \max\{\max\{a, b\}, c\} &= \max\{a, b, c\}. \end{aligned}$$

$\diamondsuit$

Since $f \in (\mathcal{F}^+_{\text{mps}})^n$, from Lemma 3.2.1 and assumption **A2** it follows that the cost function V_N can be written as:

$$V_N(x, \mathbf{r}, \mathbf{u}) = \max_{j \in \mathcal{J}} \{\alpha_j^T \mathbf{x} + \beta_j^T \mathbf{u} + \delta_j(x, \mathbf{r})\},$$

where $\mathbf{x}$ is given by (3.26), α_j are non-negative vectors (i.e. $\alpha_j \geq 0$ for all $j \in \mathcal{J}$), $\delta_j(x, \mathbf{r})$ are affine expressions in $(x, \mathbf{r})$ and $\mathcal{J}$ is a finite index set. Define:

$$V(x, \mathbf{r}, \mathbf{u}, \mathbf{x}) := \max_{j \in \mathcal{J}} \{\alpha_j^T \mathbf{x} + \beta_j^T \mathbf{u} + \delta_j(x, \mathbf{r})\}.$$

We introduce the following relaxed set:

$$\Pi_{\text{rel}}(x, \mathbf{r}) = \{\mathbf{u} : \exists \mathbf{x} \text{ s. t. } \mathbf{Hx} + \mathbf{Gu} + \mathbf{Fr} \leq \mathbf{h},\; \mathbf{x} \geq \Theta \otimes x \oplus \Phi \otimes \mathbf{u}\}$$

and the following optimization problem

$$V^0_{\text{rel}}(x, \mathbf{r}) = \inf_{\mathbf{u} \in \Pi_{\text{rel}}(x, \mathbf{r})} V_N(x, \mathbf{r}, \mathbf{u}). \tag{3.28}$$

Let us define

$$\Pi(x, \mathbf{r}) = \{(\mathbf{u}, \mathbf{x}) : \mathbf{Hx} + \mathbf{Gu} + \mathbf{Fr} \leq \mathbf{h},\; \mathbf{x} \geq \Theta \otimes x \oplus \Phi \otimes \mathbf{u}\}.$$

Given a set $\mathcal{Z} \subseteq \mathbb{R}^n \times \mathbb{R}^m$, the operator Proj_n denotes the *projection* on $\mathbb{R}^n$, defined by

$$\text{Proj}_n \mathcal{Z} := \{x \in \mathbb{R}^n : \exists y \in \mathbb{R}^m \;\text{ s. t. }\; (x, y) \in \mathcal{Z}\}. \tag{3.29}$$

Clearly,

$$\Pi_{\text{rel}}(x, \mathbf{r}) = \text{Proj}_{Nm} \Pi(x, \mathbf{r}) \tag{3.30}$$

We consider the following optimization problem

$$V^0(x,\mathbf{r}) = \inf_{(\mathbf{u},\mathbf{x})\in\Pi(x,\mathbf{r})} V(x,\mathbf{r},\mathbf{u},\mathbf{x}) \tag{3.31}$$

and its minimizer is denoted with:

$$(\mathbf{u}^0(x,\mathbf{r}),\ \mathbf{x}^0(x,\mathbf{r})) \in \arg\min_{(\mathbf{u},\mathbf{x})\in\Pi(x,\mathbf{r})} V(x,\mathbf{r},\mathbf{u},\mathbf{x}).$$

The next theorem is a consequence of Theorem 2 in [45].

Theorem 3.2.2 *The optimization problem* (3.31) *is a linear program. Moreover,* $V_N^0(x,\mathbf{r}) = V_{\text{rel}}^0(x,\mathbf{r}) = V^0(x,\mathbf{r})$ *and* $\mathbf{u}^0(x,\mathbf{r}) \in \arg\min_{\mathbf{u}\in\Pi_N(x,\mathbf{r})} V_N(x,\mathbf{r},\mathbf{u})$ *for all* $(x,\mathbf{r}) \in X_N$.

Proof: Let $(x,\mathbf{r}) \in X_N$. The feasible set $\Pi(x,\mathbf{r})$ is a polyhedron and V is a convex PWA function. From basic results in convex optimization [25] it follows that the optimization problem (3.31) can be recast as a linear program.

Let $\mathbf{u} \in \Pi_{\text{rel}}(x,\mathbf{r})$. Since $\mathbf{H} \geq 0$, it follows that $\mathbf{u} \in \Pi_N(x,\mathbf{r})$ (we can take $\mathbf{x} = \Theta\otimes x\oplus\Phi\otimes\mathbf{u}$) and thus $\Pi_{\text{rel}}(x,\mathbf{r}) \subseteq \Pi_N(x,\mathbf{r})$. On the other hand let $\mathbf{u} \in \Pi_N(x,\mathbf{r})$, then clearly $\mathbf{u} \in \Pi_{\text{rel}}(x,\mathbf{r})$. Therefore, $\Pi_N(x,\mathbf{r}) \subseteq \Pi_{\text{rel}}(x,\mathbf{r})$. It follows that

$$\Pi_N(x,\mathbf{r}) = \Pi_{\text{rel}}(x,\mathbf{r})$$

Then, the optimization problem (3.22) is equivalent with the optimization problem (3.28), i.e. $V_N^0(x,\mathbf{r}) = V_{\text{rel}}^0(x,\mathbf{r})$. Moreover, the following inequalities are valid:

$$V_{\text{rel}}^0(x,\mathbf{r}) \geq V^0(x,\mathbf{r}) \geq V_N(x,\mathbf{r},\mathbf{u}^0(x,\mathbf{r})) \geq V_{\text{rel}}^0(x,\mathbf{r})$$

and thus $V_{\text{rel}}^0(x,\mathbf{r}) = V^0(x,\mathbf{r})$. The first and the last inequality follows from (3.30) (which in particular implies that $\mathbf{u}^0(x,\mathbf{r}) \in \Pi_{\text{rel}}(x,\mathbf{r})$) and the second inequality follows from the fact that $\alpha_j \geq 0$ for all $j \in \mathcal{J}$. This concludes our proof. $\diamondsuit$

The following corollary can also be proved

Corollary 3.2.3 *The sets* $\Pi_N(x,\mathbf{r})$ *for each* $(x,\mathbf{r}) \in X_N$ *and* X_N *are polyhedra.*

Proof: Note that $\Pi_N(x,\mathbf{r}) = \Pi_{\text{rel}}(x,\mathbf{r}) = \text{Proj}_{Nm}\Pi(x,\mathbf{r})$ and since $\Pi(x,\mathbf{r})$ is a polyhedron, it follows that $\Pi_N(x,\mathbf{r})$ is also a polyhedron. Moreover, the set

$$\tilde{X}_N = \{(x,\mathbf{r},\mathbf{u},\mathbf{x}) : \mathbf{H}\mathbf{x} + \mathbf{G}\mathbf{u} + \mathbf{F}\mathbf{r} \leq \mathbf{h},\ \mathbf{x} \geq \Theta\otimes x\oplus\Phi\otimes\mathbf{u}\}$$

is a polyhedron and $X_N = \text{Proj}_{n+Np}\tilde{X}_N$. Thus, X_N is also a polyhedron. $\diamondsuit$

3.2.3 Timing: deterministic case

Discrete event MPL systems are different from conventional time-driven systems in the sense that the event counter k is not directly related to a specific time, as we saw in Section 2.2. In this chapter and the next one we will use extensively the assumption that at event step k the state $x(k)$ is available. However, in general not all components of $x(k)$ are known at the same time instant (recall that $x(k)$ contains the time instants at which the internal activities or processes of the system start for the k^{th} cycle). Therefore, we will now present a method to address the availability issue of the state at a certain time t_0 of a deterministic MPL system.

We consider the case of full state information. This is possible due to the following fact. Let us note that in practical applications the entries of the system matrices are nonnegative or take the value ε. Since the components of x correspond to event times, they are in general easy to measure. Also note that measurements of occurrence times of events are in general not as susceptible to noise and measurement errors as measurements of continuous-time signals involving variables such as temperature, speed, pressure, etc. Let t_0 be the time when an optimal control problem is performed. We can define the initial cycle k_0 as follows:

$$k_0 := \arg\max\{k : x_i(k) \le t_0 \ \forall i \in \mathbb{N}_{[1,n]}\}.$$

Hence, the state $x(k_0)$ is completely known at time t_0 and thus $u(k_0 - 1)$ it is also available. Note that at time t_0 some components of the forthcoming states and of the forthcoming inputs might be known, i.e. $x_i(k_0 + \bar{l}) \le t_0$ and $u_j(k_0 + \tilde{l}) \le t_0$ for some positive integers $i, j, \bar{l}$ and $\tilde{l}$. Due to causality, the information about some components of the forthcoming states can be recast as linear equality and inequality constraints on some forthcoming inputs: $G_{\bar{l}} u(k_0 + \bar{l}) = h_{\bar{l}}$ and $G_{\tilde{l}} u(k_0 + \tilde{l}) \le h_{\tilde{l}}$ for some matrices $G_{\bar{l}}, G_{\tilde{l}}$ of appropriate dimensions and for some positive integers $\bar{l}, \tilde{l}$. Therefore, these linear equality and inequality constraints on the inputs must be taken into account by the optimal control problem that has to be solved at time t_0. Note that linear equality and inequality constraints on inputs fit in the formulation (3.14).

3.2.4 "Time" optimal control

As an application of the finite-horizon optimal control problem previously discussed, we consider the MPL counterpart of the conventional time optimal control problem: given a maximum horizon length $N_{\max}$ we consider the problem of ensuring that the completion times after N events (with $N \in \mathbb{N}_{[1,N_{\max}]}$) are less than or equal to a specified target time $\mathbf{T}$ (i.e. $y(N) \le \mathbf{T}$), using the latest controller that satisfies the input and state constraints (3.17)–(3.18). Note that such a problem, but without taking constraints into account, was considered also in [4] in terms of lattice theory. The time optimal control problem in our setting is different from the classical one (we want to *maximize*[4] N instead of minimizing it; so in fact a better term would be "throughput-optimal" control). Since we want the maximal N, the time optimal control problem can be posed in the framework of the finite-horizon optimal control problem considered in Section 3.2.1.

One proceeds by defining

$$N^0(x, \mathbf{T}) := \max_{(N,\mathbf{u})}\{N \in \mathbb{N}_{[1,N_{\max}]} : \mathbf{u} \in \Pi_N^{\mathbf{T}}(x, [0\ \ 0 \ldots 0\ \ \mathbf{T}^T]^T)\},$$

where $\Pi_N^{\mathbf{T}}(x, [0\ \ 0 \ldots 0\ \ \mathbf{T}^T]^T) = \Pi_N(x, [0\ \ 0 \ldots 0\ \ \mathbf{T}^T]^T)$ but with the substitutions $H_N \leftarrow [H_N^T\ \ I]^T \ge 0$, $F_N \leftarrow [F_N^T\ \ 0]^T$ and $h_N \leftarrow [h_N^T\ \ ((-C^T) \otimes' \mathbf{T})^T]^T$ (note that $\mathbf{r} = [0\, 0 \cdots 0\ \mathbf{T}^T]^T$ and thus $F_i r_i = 0$ for all $i \in \mathbb{N}_{[0,N-1]}$ and $r_N = \mathbf{T}$). It follows that

$$N^0(x, \mathbf{T}) = \max_N\{N \in \mathbb{N}_{[1,N_{\max}]} : (x, \mathbf{r}) \in X_N^{\mathbf{T}}\}, \tag{3.32}$$

where $X_N^{\mathbf{T}} = \{(x, \mathbf{r}) : \Pi_N^{\mathbf{T}}(x, \mathbf{r}) \ne \emptyset\}$. Since we want to feed the raw material as late as possible (see [4]), a suitable choice of stage cost is $\ell_i(x_i, u_i, r_i) := -\sum_{j=1}^m [u_i]_j$. Note that under these settings the assumptions **A1-A2** are still valid. The time optimal controller is implemented as follows:

[4] For a manufacturing system this requirement corresponds to producing as many products as possible by the target time.

1. For each $N \in \mathbb{N}_{[1,N_{\max}]}$, solve the optimization problem (3.22) or equivalently the linear program (3.31) where $\mathbf{r}$ is defined as $\mathbf{r} = [0\ 0 \cdots 0\ \mathbf{T}^T]^T \in \mathbb{R}^{Np}$.
2. Determine $N^0(x, \mathbf{T})$ according to (3.32).
3. Let $\mathbf{r}_N = [0\ 0 \cdots 0\ \mathbf{T}^T]^T \in \mathbb{R}^{Np}$, with $N = N^0(x, \mathbf{T})$.
4. Apply the control sequence $\mathbf{u}_N^0(x, \mathbf{r}_N)$.

The time optimal control problem involves solving $N_{\max}$ linear programs in step 1 above. The set $X_N^{\mathbf{T}}$ has the following interpretation: the boundary of the polyhedron $X_N^{\mathbf{T}}$ represents the latest starting times such that after N events the output is below the target time $\mathbf{T}$.

3.2.5 Just-in-time control for unconstrained MPL systems

We now study the finite-horizon optimal control problem for unconstrained MPL systems where the performance index (cost function) is designed to provide a trade-off between minimizing the due date error and a just-in-time production. By employing results in max-plus algebra, we provide sufficient conditions such that one can compute the analytic solution of the optimal control problem. For the MPL system (3.15)–(3.16) we consider the particular stage cost defined in (3.24)–(3.25). In this case, the finite-horizon optimal control problem (3.22) without state and input constraints can be written explicitly as:

$$V_N^0(x, \mathbf{r}) = \inf_{\mathbf{u} \in \bar{\mathbb{R}}^{Nm}} V_N(x, \mathbf{r}, \mathbf{u}), \tag{3.33}$$

where now the cost function has the particular form[5]

$$V_N(x, \mathbf{r}, \mathbf{u}) = \sum_{i=0}^{N-1} \left(\sum_{j=1}^{p} \max\{[y_i - r_i]_j, 0\} - \gamma \sum_{j=1}^{m} [u_i]_j \right) + \sum_{j=1}^{p} \max\{[y_N - r_N]_j, 0\}.$$

Here,

$$y_i := C \otimes x_i$$

denotes the output at event step i. It is obvious that the size of γ provides the trade-off between the delay of the finishing products with respect to the due dates and the feeding time. If $\gamma > 0$, then it follows that the optimizer of (3.33) satisfies $\mathbf{u}_N^0(x, \mathbf{r}) > \mathcal{E}$ (i.e. each component of the vector $\mathbf{u}_N^0(x, \mathbf{r})$ is larger than $\varepsilon = -\infty$).

We define the matrices

$$\bar{\Theta} := \begin{bmatrix} C \\ C \otimes A \\ \vdots \\ C \otimes A^{\otimes^N} \end{bmatrix}, \quad \bar{\Phi} := \begin{bmatrix} \mathcal{E} & \mathcal{E} & \cdots & \mathcal{E} \\ C \otimes B & \mathcal{E} & \cdots & \mathcal{E} \\ \vdots & \vdots & \ddots & \vdots \\ C \otimes A^{\otimes^{N-1}} \otimes B & C \otimes A^{\otimes^{N-2}} \otimes B & \cdots & C \otimes B \end{bmatrix}$$

and

$$\mathbf{y} := [y_0^T\ y_1^T \cdots y_N^T]^T.$$

[5] So, the assumption **A2** still holds in this section. However, we do not assume any constraints on inputs and states.

Clearly,

$$\mathbf{y} = \bar{\Theta} \otimes x \oplus \bar{\Phi} \otimes \mathbf{u}. \tag{3.34}$$

We introduce

$$\bar{\mathbf{y}} := \bar{\Theta} \otimes x \oplus \mathbf{r}$$

and the following linear program

$$\max_{\mathbf{u} \in \bar{\mathbb{R}}^{Nm}} \left\{ \sum_{i=0}^{N-1} \sum_{j=1}^{m} [u_i]_j : \bar{\Phi} \otimes \mathbf{u} \leq \bar{\mathbf{y}} \right\} \tag{3.35}$$

with the maximizer (as defined in (2.18))

$$\mathbf{u}^*(x, \mathbf{r}) \in \arg \max_{\mathbf{u} \in \bar{\mathbb{R}}^{Nm}} \left\{ \sum_{i=0}^{N-1} \sum_{j=1}^{m} [u_i]_j : \bar{\Phi} \otimes \mathbf{u} \leq \bar{\mathbf{y}} \right\}.$$

From Lemma 3.1.4 (i) it follows that $\mathbf{u}^*(x, \mathbf{r}) = (-\bar{\Phi}^T) \otimes' \bar{\mathbf{y}}$ or explicitly as a function of x and $\mathbf{r}$

$$\mathbf{u}^*(x, \mathbf{r}) = (-\bar{\Phi}^T) \otimes' (\bar{\Theta} \otimes x \oplus \mathbf{r}). \tag{3.36}$$

Since $\mathbf{r}$ is finite it follows that $\bar{\mathbf{y}}$ is also finite and thus $\mathbf{u}^*(x, \mathbf{r}) > \mathcal{E}$.

The following theorem, which is the extension to the multivariable case of a result in [158], provides sufficient conditions for which an analytic solution exists for the optimization problem (3.33).

Theorem 3.2.4 *Suppose* $0 < \gamma < \frac{1}{mN}$*, then* $\mathbf{u}^*(x, \mathbf{r})$ *is a minimizer of* (3.33).

Proof: Let $(x, \mathbf{r}) \in \mathbb{R}^n \times \mathbb{R}^{Np}$ be fixed. For simplicity, we drop out the dependence on $(x, \mathbf{r})$ of $\mathbf{u}^*(x, \mathbf{r})$. We will prove this theorem by contradiction. Define $\tilde{\mathbf{y}} := \bar{\Theta} \otimes x$, then $\bar{\mathbf{y}} := \tilde{\mathbf{y}} \oplus \mathbf{r}$.

First, let $\mathbf{u}^{\mathrm{f}} = [(u_0^{\mathrm{f}})^T \ (u_1^{\mathrm{f}})^T \cdots (u_{N-1}^{\mathrm{f}})^T]^T > \mathcal{E}$ be feasible for (3.35) such that $\mathbf{u}^{\mathrm{f}} \neq \mathbf{u}^*$. Then, from Lemma 3.1.4 (i) it follows:

$$\sum_{i=0}^{N-1} \sum_{j=1}^{m} [u_i^{\mathrm{f}}]_j < \sum_{i=0}^{N-1} \sum_{j=1}^{m} [u_i^*]_j.$$

Define $\mathbf{y}^{\mathrm{f}} = [(y_0^{\mathrm{f}})^T \ (y_1^{\mathrm{f}})^T \cdots (y_N^{\mathrm{f}})^T]^T$ as $\mathbf{y}^{\mathrm{f}} := \bar{\Theta} \otimes x \oplus \bar{\Phi} \otimes \mathbf{u}^{\mathrm{f}}$. Then, for each $i \in \mathbb{N}_{[0,N]}$ and $j \in \mathbb{N}_{[1,p]}$ it follows that

$$\max\{[y_i^{\mathrm{f}}]_j, [r_i]_j\} = \max\{[\tilde{y}_i]_j, \bar{\Phi}_{ip+j.} \otimes \mathbf{u}^{\mathrm{f}}, [r_i]_j\} = \max\{[\bar{y}_i]_j, \bar{\Phi}_{ip+j.} \otimes \mathbf{u}^{\mathrm{f}}\} = [\bar{y}_i]_j,$$

where we recall that $\bar{\Phi}_{ip+j.}$ denotes the $(ip + j)^{th}$ row of $\bar{\Phi}$. It follows that

$$V_N(x, \mathbf{r}, \mathbf{u}^{\mathrm{f}}) = \sum_{i=0}^{N} \sum_{j=1}^{p} \max\{[y_i^{\mathrm{f}} - r_i]_j, 0\} - \gamma \sum_{i=0}^{N-1} \sum_{j=1}^{m} [u_i^{\mathrm{f}}]_j =$$
$$\sum_{i=0}^{N} \sum_{j=1}^{p} ([\bar{y}_i]_j - [r_i]_j) - \gamma \sum_{i=0}^{N-1} \sum_{j=1}^{m} [u_i^{\mathrm{f}}]_j >$$
$$\sum_{i=0}^{N} \sum_{j=1}^{p} ([\bar{y}_i]_j - [r_i]_j) - \gamma \sum_{i=0}^{N-1} \sum_{j=1}^{m} [u_i^*]_j = V_N(x, \mathbf{r}, \mathbf{u}^*)$$

and thus $\mathbf{u}^{\mathrm{f}}$ cannot be the optimizer of (3.33).

Next, let us consider $\mathbf{u}^{\dagger} > \mathcal{E}$ that does not satisfy the inequality $\bar{\Phi} \otimes \mathbf{u} \leq \bar{\mathbf{y}}$ (i.e. $\mathbf{u}^{\dagger}$ is infeasible for the optimization problem (3.35)). Define

$$\delta := \max_{i \in \mathbb{N}_{[0,N]}, j \in \mathbb{N}_{[1,p]}} \{\bar{\Phi}_{ip+j\cdot} \otimes \mathbf{u}^{\dagger} - [\bar{y}_i]_j\} > 0.$$

Taking $\mathbf{y}^{\dagger} = \bar{\Theta} \otimes x \oplus \bar{\Phi} \otimes \mathbf{u}^{\dagger}$, then there exist i_0, j_0 such that $[y^{\dagger}_{i_0}]_{j_0} = \bar{\Phi}_{i_0 p + j_0 \cdot} \otimes \mathbf{u}^{\dagger} = [\bar{y}_{i_0}]_{j_0} + \delta$ and thus

$$\sum_{i=0}^{N} \sum_{j=1}^{p} \max\{[y^{\dagger}_i]_j - [r_i]_j, 0\} \geq \sum_{i=0}^{N} \sum_{j=1}^{p} \max\{[\bar{y}_i]_j - [r_i]_j, 0\} + \delta.$$

Note that $\mathbf{u}^{\ddagger} = \mathbf{u}^{\dagger} - \delta$ satisfies the constraint $\bar{\Phi} \otimes \mathbf{u} \leq \bar{\mathbf{y}}$ (i.e. $\mathbf{u}^{\ddagger}$ is feasible for (3.35)) and using the first part of the proof it follows that the corresponding cost satisfies:

$$V_N(x, \mathbf{r}, \mathbf{u}^{\ddagger}) \leq \sum_{i=0}^{N} \sum_{j=1}^{p} \max\{[y^{\dagger}_i]_j - [r_i]_j, 0\} - \delta - \gamma\Big(\sum_{i=0}^{N-1} \sum_{j=1}^{m} [u^{\dagger}_i]_j - Nm\delta\Big) =$$
$$V_N(x, \mathbf{r}, \mathbf{u}^{\dagger}) + (\gamma N m - 1)\delta < V_N(x, \mathbf{r}, \mathbf{u}^{\dagger})$$

and thus $\mathbf{u}^{\dagger}$ cannot be the optimizer of (3.33). This proves that $\mathbf{u}^*$ is also the optimizer of the original optimization problem (3.33). ♢

Remark 3.2.5 (i) From Theorem 3.2.4 it follows that the optimal control sequence $u^*(x, \mathbf{r})$ is a just-in-time control since given the desired output $\mathbf{r}$ we search for the latest input dates $\mathbf{u}$ such that the output dates occur at times as close as possible to the desired ones or at the latest before the desired ones (see (3.35)). Note that our state-space expression of the just-in-time controller (3.36) resembles the expression of the just-in-time controller in [109] obtained from an input-output model using residuation theory. The main differences consist in the fact that we take the initial state into account and we obtain a state-space formula compared to an input-output expression.

(ii) The reader might ask if the control sequence $u^*(x, \mathbf{r})$ is a nondecreasing input sequence. The original system is given by (3.8). Nevertheless, the just-in-time control sequence should satisfy the constraint (3.11). If we use the extended state approach from Section 3.1.2 we replace the constraint (3.11) with the constraint (3.13). Therefore, the optimization problem (3.33) should take into account the constraint (3.13). In this particular case however (i.e. when the constraints (3.10) are not present), a better approach is to use another extending state for the original system (3.8), as explained next: we introduce a new input vector $u(k)$ and a new state vector $x(k) = [\bar{x}^T(k) \;\; \bar{u}^T(k)]^T$ with the dynamics

$$\begin{aligned} x(k+1) &= A \otimes x(k) \oplus B \otimes u(k) \\ y(k) &= C \otimes x(k), \end{aligned} \tag{3.37}$$

where the system matrices are given by

$$A = \begin{bmatrix} \bar{A} & \bar{B} \\ \mathcal{E} & E \end{bmatrix}, \quad B = \begin{bmatrix} \bar{B} \\ E \end{bmatrix} \quad \text{and} \quad C = [\bar{C} \;\; \mathcal{E}].$$

Recall that $\bar{A}$, $\bar{B}$ and $\bar{C}$ denote the system matrices and $\bar{x}, \bar{u}$ denote the state and the input for the original system (3.8). It is clear that $\bar{u}(k+1) = \bar{u}(k) \oplus u(k)$ and thus $\bar{u}(k+1) \geq \bar{u}(k)$ (i.e.

the constraint (3.11) is satisfied for the original system). Therefore, we compute $u^*(x, \mathbf{r})$ for the extended system (3.37) as explained in this section, while the actual input applied to the original system (3.8) is given by

$$\bar{u}(k) = u_k^*(x, \mathbf{r}) \oplus \bar{u}(k-1) \tag{3.38}$$

for all $k \in \mathbb{N}_{[0,N-1]}$. Using similar arguments as in Section 3.1.2 we can easily verify via induction that by applying the input signal $u^*(x, \mathbf{r})$ to the extended system (3.37) and $\{\bar{u}(k)\}_{k \in \mathbb{N}_{[0,N-1]}}$ given by (3.38) to the original system (3.8) we obtain that the first n components of state corresponding to the extended system coincide with the state of the original system at each step k. Moreover, the output signals corresponding to these two systems coincide, i.e. $\bar{y}(k) = y(k)$ for all $k \geq 0$. $\diamondsuit$

3.3 Finite-horizon min-max control

As we have seen in the previous sections of this chapter considerable progress has been made in the synthesis of optimal controllers for *deterministic* MPL systems. However, progress has been slower for the more difficult problem of designing robust controllers for *uncertain* MPL systems. In robust control the goal is to expand the optimization problem to consider a class of models instead of a single, nominal model. Of course, the synthesis and analysis of robust controllers are more difficult than the corresponding deterministic counterpart since now we have to take into consideration an infinite number of realizations for the plant. The main approaches in designing robust controllers for MPL systems are based on either a min-max framework (e.g. open-loop min-max model predictive control [155]) or residuation theory [93, 102, 109]. Note that [93, 102, 109] do not take constraints into account. For instance in [102] closed-loop control based on residuation theory is derived that also guarantees nominal stability. However, the residuation approach used in [102] does not cope with input and state constraints and moreover the uncertainty is not taken into account. In [93] uncertainty is considered in terms of interval transfer functions, which is a particular case of our uncertainty description considered in this thesis. In [109] an adaptive control method is derived that takes into account possible mismatch between the system and its model. The open-loop min-max approach of [155] does not take mixed state and input constraints into account and does not incorporate feedback in the optimization problem. Moreover, the solution of the min-max control problem is obtained by resorting to computation of the vertexes of the uncertainty set. This makes in general difficult the computation of the optimal solution.

In this section we use a game-based or min-max approach. We analyze the solutions to three classes of finite-horizon min-max control problems for uncertain MPL systems, each class depending on the nature of the control sequence over which we optimize: open-loop control sequences, disturbance feedback policies, and state feedback policies. We assume that the uncertainty lies in a bounded polytope, and that the closed-loop input and state sequence should satisfy a given set of linear inequality constraints for all admissible disturbance realizations. Despite the fact that the controlled system is nonlinear, we provide sufficient conditions that allow to preserve convexity of the optimal value function and its domain. As a consequence, the min-max control problems can be either recast as a linear program or solved via N multi-parametric linear programs, where N is the prediction horizon. In some particular cases of the uncertainty description (e.g. interval matrices), by employing results from dynamic programming, we show that a min-max control problem can be recast as a deterministic optimal control problem. The main advantage of our approach compared to existing results on robust control of MPL systems [93, 102, 109, 155] is the fact that we also optimize over feedback policies, not only over open-loop input sequences, and that we incorporate state and input constraints directly into the problem formulation. In general, this results in increased feasibility and a better performance.

It is important to note that the assumptions **A1**-**A2** hold also in this section. Because MPL systems are nonlinear, non-convexity is clearly a problem if one seeks to develop "efficient" methods for solving min-max control problems for MPL systems. Based on the assumptions **A1**-**A2** and employing also recent results in polyhedral algebra and multi-parametric linear programming, we provide constructive proofs that allow one to compute robust optimal controllers for MPL systems in an efficient way. Robust performance and robust constraint fulfillment are considered with respect to all possible realizations of the disturbance in a worst-case framework (i.e. the opponent "nature" can pick a disturbance to maximally increase our cost). This section proceeds now by introducing the uncertainty description for an MPL system.

3.3.1 Uncertain MPL systems

Before giving the dynamical equations of an uncertain MPL system let us recall Example 2.2.1. In this example we have considered a production system with three units. The notation and the functioning rules are given in Section 2.2.1. The main difference is that now the processing times p_i and the transportation times t_j are not fixed but they are varying with each cycle. Let us write down explicitly the dynamical equations corresponding to this situation:

$$\begin{cases} x_1(k+1) & = \max\{x_1(k)+p_1(k-1),\ u(k)+t_1(k)\} \\ x_2(k+1) & = \max\big\{x_1(k)+p_1(k-1)+p_1(k)+t_3(k),\ x_2(k)+p_2(k-1), \\ & \qquad u(k)+\max\{t_2(k),\ t_1(k)+p_1(k)+t_3(k)\}\big\} \\ x_3(k+1) & = \max\big\{x_1(k)+\max\{p_1(k-1)+p_1(k)+t_4(k), \\ & \qquad p_1(k-1)+p_1(k)+t_3(k)+p_2(k)+t_5(k)\}, \\ & \qquad x_2(k)+p_2(k-1)+p_2(k)+t_5(k),\ x_3(k)+p_3(k-1), \\ & \qquad u(k)+\max\{p_1(k)+t_4(k)+t_1(k), t_2(k)+p_2(k)+t_5(k), \\ & \qquad t_1(k)+p_1(k)+t_3(k)+p_2(k)+t_5(k)\}\big\} \\ y(k) & = p_3(k-1)+t_6(k-1)+x_3(k). \end{cases} \tag{3.39}$$

As we have seen from this example, the entries of the system matrices of a given MPL system depend on the transportation times and the processing times. In practical applications, these parameters are uncertain since they can vary from one cycle to another, making the system matrices also event varying. In contrast to conventional linear systems, where the uncertainty and disturbances are usually modeled as an additive term, the uncertainty and disturbances in an MPL system enter max-plus multiplicative rather than max-plus additive. Indeed, from the mathematical description (3.39) of Example 2.2.1, we see that the uncertain parameters p_i and t_j appear in the entries of the system matrices (i.e. $f(x,u,w) = A(w)\otimes x \oplus B(w)\otimes u$) rather than as an max-plus additive term (i.e. $f(x,u,w) = A\otimes x \oplus B\otimes u \oplus E\otimes w$).

We gather in the vector w all the uncertainty caused by disturbances and errors in the estimation of the parameters p_i and t_i, i.e.

$$w(k) := [p_1(k)\dots p_l(k)\ t_1(k)\dots t_{\tilde{l}}(k)]^T,$$

such that $l + \tilde{l} = q$. Therefore, we consider the following *uncertain MPL system*:

$$x(k+1) = A(w(k-1), w(k)) \otimes x(k) \oplus B(w(k-1), w(k)) \otimes u(k) \quad (3.40)$$

$$y(k) = C(w(k-1)) \otimes x(k). \quad (3.41)$$

Note that $x(k)$ depends on $w(k-1)$ and thus $y(k) = h(x(k))$, i.e. (3.41) is in the form (2.25). Since the system matrices of a DES modeled as an MPL system usually consist of sums or maximization of internal process times and transportation times it follows that $A \in \mathcal{F}_{\text{mps}}^{n\times n}$, $B \in \mathcal{F}_{\text{mps}}^{n\times m}$ and $C \in \mathcal{F}_{\text{mps}}^{p\times n}$ (it is important to note that these matrix functions are nonlinear). We frequently use the short-hand notation

$$f_{\text{MPL}}(x, u, w_{\text{p}}, w_{\text{c}}) := A(w_{\text{p}}, w_{\text{c}}) \otimes x \oplus B(w_{\text{p}}, w_{\text{c}}) \otimes u, \quad (3.42)$$

and it is easy to verify that $f_{\text{MPL}} \in \mathcal{F}_{\text{mps}}^{n}$ and $f_{\text{MPL}}(\cdot, u, w_{\text{p}}, w_{\text{c}}) \in (\mathcal{F}_{\text{mps}}^{+})^n$ for each fixed $(u, w_{\text{p}}, w_{\text{c}})$. We also assume that the uncertain MPL system (3.40)–(3.41) is subject to hard control and state constraints (3.17)–(3.18) defined in Section 3.2.1.

As in previous sections, we assume that at event step k the state $x(k)$ is available, i.e. it can be measured or it can be computed (see Section 3.3.5 below for more details about timing issues in the disturbance case). However, the value of the disturbance $w(k)$ is unknown, but is assumed to be event-varying and to take on values from a polytope

$$W = \{w \in \mathbb{R}^q : \Omega w \leq s\},$$

where $\Omega \in \mathbb{R}^{n_\Omega \times q}$ and $s \in \mathbb{R}^{n_\Omega}$. We consider that $w(k-1)$ and $w(k)$ are independent. Moreover, at event step k we assume that the disturbance $w(k-1)$ can be also measured or computed. Note that since the past states $x(0), \cdots, x(k)$ are assumed to be known at event step k (we recall that they represent starting times of some activities), the past disturbances $w(0), \cdots, w(k-1)$ can also be measured or computed (we recall that w denotes processing and transportation times). This follows from the fact that if we are able to measure the state $x(k)$, then we are also able to measure the disturbance $w(k-1)$.

In the sequel, we will characterize the solutions to different min-max control problems and their main properties will be studied.

3.3.2 Open-loop min-max control

We start with the open-loop min-max control problem for an uncertain MPL system (3.40)–(3.41), i.e. when the optimization is performed over open-loop input sequences.

Let $\mathbf{u} := [u_0^T \; u_1^T \ldots u_{N-1}^T]^T$ be an open-loop input sequence and let

$$\mathbf{w} := [w_0^T \; w_1^T \ldots w_{N-1}^T]^T$$

denote a realization of the disturbance over the prediction horizon N. Also, let $\phi(i; x, w, \mathbf{u}, \mathbf{w})$ denote the solution of (3.40) at event step i when the initial state is x at event step 0, the initial value of the disturbance is w (i.e. $w(-1) = w$ or in other words $w_{-1} = w$), the control is determined by $\mathbf{u}$ (i.e. $u(i) = u_i$) and the disturbance sequence is $\mathbf{w}$. By definition, $\phi(0; x, w, \mathbf{u}, \mathbf{w}) := x$.

Given the initial state x, the initial disturbance w, the reference signal $\mathbf{r} := [r_0^T \; r_1^T \cdots r_N^T]^T$, the control sequence $\mathbf{u}$, and the disturbance realization $\mathbf{w}$, the cost function $V_N(x, w, \mathbf{r}, \mathbf{u}, \mathbf{w})$ is defined as:

$$V_N(x, w, \mathbf{r}, \mathbf{u}, \mathbf{w}) := \sum_{i=0}^{N-1} \ell_i(x_i, u_i, r_i) + V_{\text{f}}(x_N, r_N), \quad (3.43)$$

where $x_i := \phi(i; x, w, \mathbf{u}, \mathbf{w})$ (and thus $x_0 := x$).

Remark 3.3.1 The notations for the terminal cost and the terminal set correspond to those from Section 3.2.1. Recall that the assumptions **A1**-**A2** are assumed to hold also in this section. ♢

For each initial condition x, initial disturbance w and due dates $\mathbf{r}$ we define the set of feasible open-loop input sequences $\mathbf{u}$:

$$\Pi_N^{\text{ol}}(x, w, \mathbf{r}) := \{\mathbf{u} : H_i x_i + G_i u_i + F_i r_i \leq h_i \ \ \forall i \in \mathbb{N}_{[0,N-1]}, (x_N, r_N) \in X_{\text{f}}, \\ \forall \mathbf{w} \in \mathcal{W}\}, \tag{3.44}$$

where we recall that

$$\mathcal{W} := W^N.$$

Also, let X_N^{ol} denote the set of initial states and reference signals for which a feasible input sequence exists:

$$X_N^{\text{ol}} := \{(x, w, \mathbf{r}) : \Pi_N^{\text{ol}}(x, w, \mathbf{r}) \neq \emptyset\}. \tag{3.45}$$

The *finite-horizon open-loop min-max* control problem is defined as:

$$\mathbb{P}_N^{\text{ol}}(x, w, \mathbf{r}) : \qquad V_N^{0,\text{ol}}(x, w, \mathbf{r}) := \inf_{\mathbf{u} \in \Pi_N^{\text{ol}}(x,w,\mathbf{r})} \max_{\mathbf{w} \in \mathcal{W}} V_N(x, w, \mathbf{r}, \mathbf{u}, \mathbf{w}), \tag{3.46}$$

with the optimizer (whenever the infimum is attained)

$$\mathbf{u}_N^{0,\text{ol}}(x, w, \mathbf{r}) \in \arg \min_{\mathbf{u} \in \Pi_N^{\text{ol}}(x,w,\mathbf{r})} \max_{\mathbf{w} \in \mathcal{W}} V_N(x, w, \mathbf{r}, \mathbf{u}, \mathbf{w}). \tag{3.47}$$

Note that V_N is a continuous function and W is a compact set. Therefore, the maximum is attained in (3.46) and it is finite and thus we use "$\inf \max$" instead of "$\inf \sup$".

We define $\mathbf{x} := [x_0^T \ x_1^T \cdots x_N^T]^T$. It follows that:

$$\mathbf{x} = \begin{bmatrix} E \\ \Theta(1,1;w,\mathbf{w}) \\ \vdots \\ \Theta(N,1;w,\mathbf{w}) \end{bmatrix} \otimes x \oplus \begin{bmatrix} \mathcal{E} & \mathcal{E} & \cdots & \mathcal{E} \\ B(w, w_0) & \mathcal{E} & \cdots & \mathcal{E} \\ \vdots & \vdots & \ddots & \vdots \\ \Phi(N,1;w,\mathbf{w}) & \Phi(N,2;w,\mathbf{w}) & \cdots & B(w_{N-2}, w_{N-1}) \end{bmatrix} \otimes \mathbf{u},$$

where $\Theta(k, 1; w, \mathbf{w}) := A(w_{k-2}, w_{k-1}) \otimes \cdots \otimes A(w, w_0)$ and $\Phi(k, j; w, \mathbf{w}) := A(w_{k-2}, w_{k-1}) \otimes \cdots \otimes A(w_{j-1}, w_j) \otimes B(w_{j-2}, w_{j-1})$ (recall that $w_{-1} = w$). Therefore, $\mathbf{x}$ can be written more compactly as:

$$\mathbf{x} = \Theta(w, \mathbf{w}) \otimes x \oplus \Phi(w, \mathbf{w}) \otimes \mathbf{u}, \tag{3.48}$$

where $\Theta(w, \mathbf{w})$ and $\Phi(w, \mathbf{w})$ are appropriately defined. We recall to the reader (see also Section 3.2.2) that the inequalities (3.17)–(3.18) can be written as $\mathbf{Hx} + \mathbf{Gu} + \mathbf{Fr} \leq \mathbf{h}$ and that $\mathbf{H} \geq 0$ since assumption **A1** is assumed to hold also in this section. Now, the set of admissible open-loop input sequences $\Pi_N^{\text{ol}}(x, w, \mathbf{r})$ can be rewritten more compactly as:

$$\Pi_N^{\text{ol}}(x, w, \mathbf{r}) = \{\mathbf{u} : \mathbf{H}(\Theta(w, \mathbf{w}) \otimes x \oplus \Phi(w, \mathbf{w}) \otimes \mathbf{u}) + \mathbf{Gu} + \mathbf{Fr} \leq \mathbf{h}, \ \forall \mathbf{w} \in \mathcal{W}\}. \tag{3.49}$$

After some manipulations we obtain that the set of feasible $\mathbf{u}$ is given by:

$$\Pi_N^{\text{ol}}(x, w, \mathbf{r}) = \{\mathbf{u} : F\mathbf{u} + \Psi\mathbf{w} \leq c(x, w, \mathbf{r}), \ \forall \mathbf{w} \in \mathcal{W}\}, \tag{3.50}$$

where $F \in \mathbb{R}^{n_F \times Nm}$, $\Psi \in \mathbb{R}^{n_F \times Nq}$ and $c(x, w, \mathbf{r}) \in \mathbb{R}^{n_F}$ is an affine expression in $(x, w, \mathbf{r})$.

Proposition 3.3.2 *The sets X_N^{ol} and $\Pi_N^{\text{ol}}(x,w,\mathbf{r})$ are polyhedra.*

Proof: Note that $\Pi_N^{\text{ol}}(x,w,\mathbf{r}) = \{\mathbf{u} : F\mathbf{u} \leq c(x,w,\mathbf{r}) - \psi^0\}$, where the i^{th} component of the vector ψ^0 is given by $\psi_i^0 := \max_{\mathbf{w}\in\mathcal{W}} \Psi_{i.}\mathbf{w}$ (recall that $\Psi_{i.}$ denotes the i^{th} row of Ψ). Since W is a compact set it follows that ψ^0 is a finite vector. Therefore, $\Pi_N^{\text{ol}}(x,w,\mathbf{r})$ is a polyhedron.

Similarly $X_N^{\text{ol}} = \{(x,w,\mathbf{r}) : \exists\mathbf{u} \text{ s. t. } F\mathbf{u} \leq c(x,w,\mathbf{r}) - \psi^0\}$ and since $c(x,w,\mathbf{r})$ is an affine expression in $(x,w,\mathbf{r})$ it follows that X_N^{ol} is the projection of the polyhedron $\{(x,w,\mathbf{r},\mathbf{u}) : F\mathbf{u} - c(x,w,\mathbf{r}) \leq \psi^0\}$ onto a suitably-defined subspace. Therefore, X_N^{ol} is a polyhedron. $\diamondsuit$

Since $\ell_i(\cdot,u,r) \in \mathcal{F}_{\text{mps}}^+$ for all (u,r) (according to assumption **A2**), it follows that:

$$V_N(x,w,\mathbf{r},\mathbf{u},\mathbf{w}) = \max_{j\in\mathcal{J}}\{\alpha_j^T\mathbf{x} + \beta_j^T\mathbf{u} + \gamma_j^T\mathbf{w} + \delta_j(x,w,\mathbf{r})\}, \tag{3.51}$$

where $\mathcal{J}$ is a finite index set, $\alpha_j \geq 0$, and $\delta_j(x,w,\mathbf{r})$ are affine expressions in $(x,w,\mathbf{r})$, for all $j \in \mathcal{J}$.

Remark 3.3.3 Note that if the entries of matrix functions A, B and C are max-plus-nonnegative-scaling functions (i.e. A_{ij}, B_{il} and C_{kl} are in $\mathcal{F}_{\text{mps}}^+$ for all i,j,l and k), then the vectors γ_j are also nonnegative. We will make use of this property in Section 3.4.2. $\diamondsuit$

Equivalently, we can write $V_N(x,w,\mathbf{r},\mathbf{u},\mathbf{w})$ as:

$$V_N(x,w,\mathbf{r},\mathbf{u},\mathbf{w}) = \max_{i\in\mathcal{I}}\{p_i^T\mathbf{u} + q_i^T\mathbf{w} + s_i(x,w,\mathbf{r})\} \tag{3.52}$$

for some finite index set $\mathcal{I}$, some vectors p_i, q_i of appropriate dimensions and $s_i(x,w,\mathbf{r})$ are affine expressions in $(x,w,\mathbf{r})$ for all $i \in \mathcal{I}$. We define:

$$J_N(x,w,\mathbf{r},\mathbf{u}) := \max_{\mathbf{w}\in\mathcal{W}} V_N(x,w,\mathbf{r},\mathbf{u},\mathbf{w}). \tag{3.53}$$

Proposition 3.3.4 *The function $(x,w,\mathbf{r},\mathbf{u}) \mapsto J_N(x,w,\mathbf{r},\mathbf{u})$ is convex.*

Proof: From (3.52) we remark that $V_N(x,w,\mathbf{r},\mathbf{u},\mathbf{w})$ is a convex function in $(x,w,\mathbf{r},\mathbf{u})$ since $z \mapsto \max_i\{z_i\}$ is a convex map and convexity is preserved under composition of a convex function with affine maps. Since the point-wise supremum of an arbitrary, infinite set of convex functions is convex [144], it follows that $J_N(x,w,\mathbf{r},\mathbf{u})$ is a convex function. $\diamondsuit$

If we denote with $q_i^0 = \max_{\mathbf{w}\in\mathcal{W}} q_i^T\mathbf{w}$ (note that q_i^0 are finite since $\mathcal{W}$ is a compact set), then the open-loop min-max optimization problem (3.46) can be recast as a *linear program*:

$$\min_{(\mu,\mathbf{u})} \{\mu : \; F\mathbf{u} \leq c(x,w,\mathbf{r}) - \psi^0, \;\; p_i^T\mathbf{u} - \mu \leq -q_i^0 - s_i(x,w,\mathbf{r}) \;\; \forall i \in \mathcal{I}\}. \tag{3.54}$$

A finite-horizon open-loop min-max problem with only input constraints is also solved in [155] in a receding horizon fashion. The receding horizon implementation of the optimization problem $\mathbb{P}_N^{\text{ol}}(x,w,\mathbf{r})$ will be studied in the next chapter. Note that in [155] a solution is obtained by first computing the vertexes of $\mathcal{W}$. Let n_v be the number of vertexes of W. In the worst-case the number of vertexes of $\mathcal{W}$ may be exponential: $n_v^N \geq 2^{qN}$. Therefore, the computational complexity of our approach is better than the approach of [155], since in the corresponding linear program of [155] we have $|\mathcal{I}|(n_v^N - 1)$ more inequalities and also more variables than in our linear program (3.54).

3.3.3 Disturbance feedback min-max control

Effective control in the presence of disturbance requires one to optimize over *feedback policies* [16, 91, 103] rather than open-loop input sequences. A feedback controller prevents the trajectory from diverging excessively and also the performance is improved compared to the open-loop case. One way of including feedback is to consider semi-feedback control sequences, i.e. to search over the set of event-varying max-plus-scaling (or convex piecewise affine) state feedback control policies with memory of prior states [36, 102, 110]:

$$u_i = \bigoplus_{j=0}^{i} L_{i,j} \otimes x_j \oplus g_i \ \ \forall i \in \mathbb{N}_{[0,N-1]}, \tag{3.55}$$

where each $L_{i,j} \in \mathbb{R}_\varepsilon^{m\times n}$ and $g_i \in \mathbb{R}_\varepsilon^m$. We can also consider the affine approximation of (3.55), i.e. event-varying affine state feedback control policies with memory of prior states:

$$u_i = \sum_{j=0}^{i} \tilde{L}_{i,j} x_j + \tilde{g}_i \ \ \forall i \in \mathbb{N}_{[0,N-1]}, \tag{3.56}$$

where each $\tilde{L}_{i,j} \in \mathbb{R}^{m\times n}$ and $\tilde{g}_i \in \mathbb{R}^m$,

It is known [60, 97], even for linear systems, that given an initial state x and an initial disturbance w, the set of gains $\tilde{L}_{i,j}$ and $\tilde{g}_i$ such that the control sequence given by (3.56) satisfies the constraints (3.17)–(3.18) is a non-convex set (and thus a similar result holds for max-plus-scaling state feedback control policies (3.55)). Therefore, finding admissible $L_{i,j}$ and g_i ($\tilde{L}_{i,j}$ and $\tilde{g}_i$) given the current state x and current disturbance w is a very difficult problem. The state feedback policy (3.55) can be written more compactly as:

$$\mathbf{u} = \mathbf{L} \otimes \mathbf{x} \oplus \mathbf{g}, \tag{3.57}$$

where $\mathbf{L}$ and $\mathbf{g}$ have appropriate dimensions. Replacing the expression of $\mathbf{u}$ in (3.48) one gets that: $\mathbf{x} = \Phi(w,\mathbf{w}) \otimes \mathbf{L} \otimes \mathbf{x} \oplus \Theta(w,\mathbf{w}) \otimes x \oplus \Phi(w,\mathbf{w}) \otimes \mathbf{g}$. It follows from Lemma 3.1.4 (ii) that:

$$\mathbf{x} = \big(\Phi(w,\mathbf{w}) \otimes \mathbf{L}\big)^* \otimes \big(\Theta(w,\mathbf{w}) \otimes x \oplus \Phi(w,\mathbf{w}) \otimes \mathbf{g}\big).$$

Therefore, $\mathbf{u}$ can be rewritten as:

$$\begin{aligned}\mathbf{u} =& \mathbf{L} \otimes \big(\Phi(w,\mathbf{w}) \otimes \mathbf{L}\big)^* \otimes \big(\Theta(w,\mathbf{w}) \otimes x \oplus \Phi(w,\mathbf{w}) \otimes \mathbf{g}\big) \oplus \mathbf{g} = \\ & \big(\mathbf{L} \otimes \Phi(w,\mathbf{w})\big)^* \otimes \mathbf{L} \otimes \big(\Theta(w,\mathbf{w}) \otimes x \oplus \Phi(w,\mathbf{w}) \otimes \mathbf{g}\big) \oplus \mathbf{g} = \\ & \big(\mathbf{L}\otimes\Phi(w,\mathbf{w})\big)^* \otimes \big(\mathbf{L}\otimes\Phi(w,\mathbf{w})\otimes\mathbf{g}\big) \oplus \mathbf{g} \oplus \big(\mathbf{L}\otimes\Phi(w,\mathbf{w})\big)^* \otimes \mathbf{L} \otimes \Theta(w,\mathbf{w}) \otimes x = \\ & \big(\mathbf{L} \otimes \Phi(w,\mathbf{w})\big)^* \otimes \mathbf{g} \oplus \big(\mathbf{L} \otimes \Phi(w,\mathbf{w})\big)^* \otimes \mathbf{L} \otimes \Theta(w,\mathbf{w}) \otimes x = \\ & \big(\mathbf{L} \otimes \Phi(w,\mathbf{w})\big)^* \otimes \big(\mathbf{L} \otimes \Theta(w,\mathbf{w}) \otimes x \oplus \mathbf{g}\big),\end{aligned}$$

where in the second equality we used the following relation valid in max-plus algebra

$$A \otimes (X \otimes A)^* = (A \otimes X)^* \otimes A.$$

Define

$$\mathbf{u}(w,\mathbf{w}) := (\mathbf{L} \otimes \Phi(w,\mathbf{w}))^* \otimes (\mathbf{L} \otimes \Theta(w,\mathbf{w}) \otimes x \oplus \mathbf{g}),$$

then the function $(w,\mathbf{w}) \mapsto \mathbf{u}(w,\mathbf{w})$ is in $\mathcal{F}_{\text{mps}}^{Nm}$ (i.e. it is a convex piecewise affine function). Recall that we assume that at each step k the previous disturbances $w, w_0 \dots w_{k-1}$ are known (they can be computed or measured). Since $\mathbf{L}$ and $\Phi(w,\mathbf{w})$ are lower triangular matrices, it can be proved after some long but straightforward computations that $u_i(w,\mathbf{w})$ is a max-plus-scaling function depending only on the previous disturbances $w, w_0 \dots w_{i-1}$ for all $i \in \mathbb{N}_{[0,N-1]}$. It follows that the class of time-varying max-plus-scaling state feedback policies with memory of the prior states defined in (3.55) is included in the class of *max-plus-scaling disturbance feedback policies with memory of the prior disturbances*. Therefore, an alternative approach to state feedback policies (3.55) is to parameterize the control policy as a max-plus-scaling function of the previous disturbances. Unfortunately, this parametrization of the control will lead to non-convex inequalities as well. As an alternative, we propose to approximate the convex piecewise affine function $\mathbf{u}(w,\mathbf{w})$ with an affine one, i.e. to parameterize the controller as an affine function of the past disturbances:

$$u_i = \sum_{j=0}^{i-1} M_{i,j} w_j + v_i \quad \forall i \in \mathbb{N}_{[0,N-1]}, \tag{3.58}$$

where each $M_{i,j} \in \mathbb{R}^{m\times q}$ and $v_i \in \mathbb{R}^m$. A similar feedback policy was used in [14, 60, 97] in the context of robust control for linear systems. We will show in the sequel that contrary to state feedback policies (3.55) or (3.56), the set of gains $M_{i,j}$ and v_i such that the control sequence (3.58) satisfies the constraints (3.17)–(3.18) is a convex set.

Let us denote with

$$\mathbf{v} := [v_0^T \; v_1^T \cdots v_{N-1}^T]^T \tag{3.59}$$

and

$$\mathbf{M} := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ M_{1,0} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ M_{N-1,0} & M_{N-1,1} & \cdots & 0 \end{bmatrix} \tag{3.60}$$

so that the disturbance feedback policy becomes

$$\mathbf{u} = \mathbf{M}\mathbf{w} + \mathbf{v}.$$

Note that for $\mathbf{M} = 0$, (3.56) reduces to an open-loop control sequence. So, the extra degree of freedom given by the matrix M leads in general to a better performance and increased feasibility compared to the open-loop case.

For a given initial condition x, initial disturbance w and due dates $\mathbf{r}$ we define the set of feasible pairs $(\mathbf{M},\mathbf{v})$:

$$\Pi_N^{\mathrm{df}}(x,w,\mathbf{r}) = \{(\mathbf{M},\mathbf{v}) : \mathbf{M} \text{ as in (3.60)},\; u_i = \sum_{j=0}^{i-1} M_{i,j} w_j + v_i,\;\; H_i x_i + G_i u_i + F_i r_i \le h_i,\; \forall i \in \mathbb{N}_{[0,N-1]},\; (x_N, r_N) \in X_{\mathrm{f}},\;\; \forall \mathbf{w} \in \mathcal{W}\}.$$

Also, let X_N^{df} denote the set of initial states for which a solution to the optimization problem (3.61) exists, i.e.

$$X_N^{\mathrm{df}} = \{(x,w,\mathbf{r}) : \Pi_N^{\mathrm{df}}(x,w,\mathbf{r}) \neq \emptyset\}.$$

In this case, the *finite-horizon disturbance feedback min-max* control problem becomes:

$$\mathbb{P}_N^{\mathrm{df}}(x,w,\mathbf{r}): \qquad V_N^{0,\mathrm{df}}(x,w,\mathbf{r}) := \inf_{(\mathbf{M},\mathbf{v}) \in \Pi_N^{\mathrm{df}}(x,w,\mathbf{r})} \max_{\mathbf{w}\in\mathcal{W}} V_N(x,w,\mathbf{r},\mathbf{M}\mathbf{w}+\mathbf{v},\mathbf{w}) \tag{3.61}$$

and its optimizer is

$$\big(\mathbf{M}_N^{0,\mathrm{df}}(x,w,\mathbf{r}),\mathbf{v}_N^{0,\mathrm{df}}(x,w,\mathbf{r})\big)\in\arg\min_{(\mathbf{M},\mathbf{v})\in\Pi_N^{\mathrm{df}}(x,w,\mathbf{r})}\max_{\mathbf{w}\in\mathcal{W}}V_N(x,w,\mathbf{r},\mathbf{M}\mathbf{w}+\mathbf{v},\mathbf{w}). \tag{3.62}$$

We now show that the set $\Pi_N^{\mathrm{df}}(x,w,\mathbf{r})$ is polyhedral and moreover the optimization problem (3.61) is a linear program, for all $(x,w,\mathbf{r})\in X_N^{\mathrm{df}}$.

From (3.48) it follows that $\mathbf{x}$ can be written as :

$$\mathbf{x}=\Theta(w,\mathbf{w})\otimes x\oplus\Phi(w,\mathbf{w})\otimes(\mathbf{M}\mathbf{w}+\mathbf{v}).$$

The set of admissible affine disturbance feedback parameters $\Pi_N^{\mathrm{df}}(x,w,\mathbf{r})$ can be rewritten more compactly as follows:

$$\Pi_N^{\mathrm{df}}(x,w,\mathbf{r})=\{(\mathbf{M},\mathbf{v}):\mathbf{M}\text{ as in (3.60)},\ \ \mathbf{H}(\Theta(w,\mathbf{w})\otimes x\oplus\Phi(w,\mathbf{w})\otimes(\mathbf{M}\mathbf{w}+\mathbf{v}))+\\ \mathbf{G}(\mathbf{M}\mathbf{w}+\mathbf{v})+\mathbf{F}\mathbf{r}\le\mathbf{h},\forall\mathbf{w}\in\mathcal{W}\}.$$

Using (3.50) we obtain:

$$\Pi_N^{\mathrm{df}}(x,w,\mathbf{r})=\{(\mathbf{M},\mathbf{v}):\mathbf{M}\text{ as in (3.60)}, F\mathbf{v}+(F\mathbf{M}+\Psi)\mathbf{w}\le c(x,w,\mathbf{r}),\ \forall\mathbf{w}\in\mathcal{W}\}.$$

Proposition 3.3.5 *The sets X_N^{df} and $\Pi_N^{\mathrm{df}}(x,w,\mathbf{r})$ are polyhedra.*

Proof: We can write $\Pi_N^{\mathrm{df}}(x,w,\mathbf{r})$ equivalently as:

$$\Pi_N^{\mathrm{df}}(x,w,\mathbf{r})=\big\{(\mathbf{M},\mathbf{v}):\mathbf{M}\text{ as in (3.60)},\ F\mathbf{v}+\max_{\mathbf{w}\in\mathcal{W}}\{(F\mathbf{M}+\Psi)\mathbf{w}\}\le c(x,w,\mathbf{r})\big\},$$

where $\max_{\mathbf{w}\in\mathcal{W}}\{(F\mathbf{M}+\Psi)\mathbf{w}\}$ is the vector defined as follows

$$\max_{\mathbf{w}\in\mathcal{W}}\{(F\mathbf{M}+\Psi)\mathbf{w}\}:=\big[\max_{\mathbf{w}\in\mathcal{W}}\{[F\mathbf{M}+\Psi]_{1.}\mathbf{w}\}\cdots\max_{\mathbf{w}\in\mathcal{W}}\{[F\mathbf{M}+\Psi]_{n_F.}\mathbf{w}\}\big]^T,$$

and where $[F\mathbf{M}+\Psi]_{i.}$ denotes the i^{th} row of the matrix $F\mathbf{M}+\Psi$. Since $\mathcal{W}$ is a polytope, we can compute an admissible pair $(\mathbf{M},\mathbf{v})$ using dual optimization, by solving a single linear program (see also [14]). It is clear that

$$\mathcal{W}=\{\mathbf{w}\in\mathbb{R}^{Nq}:\boldsymbol{\Omega}\mathbf{w}\le\mathbf{s}\}, \tag{3.63}$$

where[6] $\boldsymbol{\Omega}=\mathrm{diag}(\Omega)$ and $\mathbf{s}=[s^T\cdots s^T]^T$. The dual problem [147] of the linear program

$$\max_{\mathbf{w}}\{[F\mathbf{M}+\Psi]_{i.}\mathbf{w}:\boldsymbol{\Omega}\mathbf{w}\le\mathbf{s}\}$$

is the following linear program

$$\min_{d_i}\{\mathbf{s}^Td_i:\boldsymbol{\Omega}^Td_i=[F\mathbf{M}+\Psi]_{i.}^T,\ d_i\ge 0\}.$$

In conclusion, we can write:

$$\Pi_N^{\mathrm{df}}(x,w,\mathbf{r})=\{(\mathbf{M},\mathbf{v}):\exists\mathbf{D}\ge 0,\mathbf{M}\text{ as in (3.60)},\ F\mathbf{v}+\mathbf{D}^T\mathbf{s}\le c(x,w,\mathbf{r}),\ F\mathbf{M}+\Psi=\mathbf{D}^T\boldsymbol{\Omega}\},$$

[6] $\mathrm{diag}(\Omega)$ denotes the block diagonal matrix having the entries on the diagonal equal to Ω and the rest equal to 0.

where $\mathbf{D} \in \mathbb{R}^{Nn_\Omega \times n_F}$ is defined as $\mathbf{D}_{.j} = d_j$ for all $j \in \mathbb{N}_{[1,n_F]}$ (recall that $\mathbf{D}_{.j}$ denotes the j^{th} column of the matrix $\mathbf{D}$).

It is clear that $\Pi_N^{\mathrm{df}}(x, w, \mathbf{r})$ is a polyhedron, since it is the projection of the polyhedron

$$\{(\mathbf{M}, \mathbf{v}, \mathbf{D}) : \mathbf{M} \text{ as in (3.60)},\ \mathbf{D} \geq 0,\ F\mathbf{v} + \mathbf{D}^T\mathbf{s} \leq c(x, w, \mathbf{r}),\ F\mathbf{M} + \Psi = \mathbf{D}^T\mathbf{\Omega}\}$$

onto a suitably defined subspace.

Similarly $X_N^{\mathrm{df}} = \{(x, w, \mathbf{r}) : \exists(\mathbf{M}, \mathbf{v}),\ \mathbf{M}$ as in (3.60), $\mathbf{D} \geq 0,\ F\mathbf{v} + \mathbf{D}^T\mathbf{s} \leq c(x, w, \mathbf{r}),\ F\mathbf{M} + \Psi = \mathbf{D}^T\mathbf{\Omega}\}$ and since $c(x, w, \mathbf{r})$ is an affine expression in $(x, w, \mathbf{r})$ it follows that X_N^{df} is also the projection of a polyhedron onto a suitably-defined subspace and thus X_N^{df} is a polyhedron. $\diamondsuit$

From (3.52) it follows that, as a function of $(\mathbf{M}, \mathbf{v})$, $V_N(x, w, \mathbf{r}, \mathbf{u}, \mathbf{w})$ can be expressed as:

$$V_N(x, w, \mathbf{r}, \mathbf{M}\mathbf{w} + \mathbf{v}, \mathbf{w}) = \max_{i \in \mathcal{I}} \{p_i^T\mathbf{v} + (p_i^T\mathbf{M} + q_i^T)\mathbf{w} + s_i(x, w, \mathbf{r})\}. \tag{3.64}$$

We define:

$$J_N(x, w, \mathbf{r}, \mathbf{M}, \mathbf{v}) := \max_{\mathbf{w} \in \mathcal{W}} V_N(x, w, \mathbf{r}, \mathbf{M}\mathbf{w} + \mathbf{v}, \mathbf{w}).$$

Proposition 3.3.6 *The function* $(x, w, \mathbf{r}, \mathbf{M}, \mathbf{v}) \mapsto J_N(x, w, \mathbf{r}, \mathbf{M}, \mathbf{v})$ *is convex.*

Proof: We use the same arguments as in the proof of Proposition 3.3.4. $\diamondsuit$

Theorem 3.3.7 *The robust optimal control problem* (3.61) *can be recast as a linear program.*

Proof: Note that $J_N(x, w, \mathbf{r}, \mathbf{M}, \mathbf{v}) = \max_{i \in \mathcal{I}} \left\{p_i^T\mathbf{v} + \max_{\mathbf{w} \in \mathcal{W}}\{(p_i^T\mathbf{M} + q_i^T)\mathbf{w}\} + s_j(x, w, \mathbf{r})\right\}$. Using again duality for linear programming it follows that

$$\max_{\mathbf{w} \in \mathcal{W}}\{(p_i^T\mathbf{M} + q_i^T)\mathbf{w}\} = \min_{z_i}\{\mathbf{s}^T z_i : \mathbf{\Omega}^T z_i = (p_i^T\mathbf{M} + q_i^T)^T,\ z_i \geq 0\}.$$

Therefore, the robust optimal control problem (3.61) can be recast as the linear program:

$$\begin{aligned} \min_{(\mu, \mathbf{M}, \mathbf{v}, \mathbf{D}, \mathbf{Z})} \big\{\mu : \mathbf{M} \text{ as in (3.60)},\ & F\mathbf{v} + \mathbf{D}^T\mathbf{s} \leq c(x, w, \mathbf{r}),\ F\mathbf{M} + \Psi = \mathbf{D}^T\mathbf{\Omega}, \\ & \mathbf{D} \geq 0, P^T\mathbf{M} + Q^T = \mathbf{Z}^T\mathbf{\Omega},\ P^T\mathbf{v} + \mathbf{Z}^T\mathbf{s} + S(x, w, \mathbf{r}) \leq \bar{\mu}, \\ & \mathbf{Z} \geq 0,\ \bar{\mu} = [\mu \ldots \mu]^T,\ \mathbf{D} \in \mathbb{R}^{Nn_\Omega \times n_F},\ \mathbf{Z} \in \mathbb{R}^{Nn_\Omega \times |\mathcal{I}|}\big\}, \end{aligned} \tag{3.65}$$

where $P_{.j} = p_j,\ Q_{.j} = q_j,\ S_j(x, w, \mathbf{r}) = s_j(x, w, \mathbf{r})$ and $Z_{.j} = z_j$ for all $j \in \mathcal{I}$. $\diamondsuit$

In the particular case when $\mathbf{M} = 0$ we obtain the open-loop control sequence $\mathbf{u}_N^{0,\mathrm{ol}}(x, w, \mathbf{r})$ derived in Section 3.3.2 and thus

$$X_N^{\mathrm{ol}} \subseteq X_N^{\mathrm{df}}, \quad V_N^{0,\mathrm{df}}(x, w, \mathbf{r}) \leq V_N^{0,\mathrm{ol}}(x, w, \mathbf{r})\ \forall (x, w, \mathbf{r}) \in X_N^{\mathrm{ol}}.$$

3.3.4 State feedback min-max control

In this section we consider full state feedback policies. Therefore, we will define the decision variable in the optimal control problem, for a given initial state x, initial disturbance w and the reference signal **r** as a control policy

$$\pi := (\mu_0, \mu_1, \dots, \mu_{N-1}),$$

where each $\mu_i : \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^{Np} \to \mathbb{R}^m$ is a state feedback control *law*. Also, let $x_i = \phi(i; x, w, \pi, \mathbf{w})$ denote the solution of (3.40) at event step i when the initial state is x at event step 0, the initial disturbance is w, the control is determined by the policy π, i.e. $u(i) = \mu_i(\phi(i-1; x, w, \pi, \mathbf{w}), w_{i-1}, \mathbf{r})$, and the disturbance sequence is **w** (where $w_{-1} := w$).

For each initial condition x, initial disturbance w and due dates **r** we define the set of feasible policies π:

$$\Pi_N^{\text{sf}}(x, w, \mathbf{r}) := \{\pi : \; H_i x_i + G_i \mu_i(x_{i-1}, w_{i-1}, \mathbf{r}) + F_i r_i \leq h_i \;\; \forall i \in \mathbb{N}_{[0,N-1]}, \\ (x_N, r_N) \in X_{\text{f}}, \; \forall \mathbf{w} \in \mathcal{W}\}. \quad (3.66)$$

Also, let X_N^{sf} denote the set of initial states *and* reference signals for which a feasible policy exists, i.e.

$$X_N^{\text{sf}} := \{(x, w, \mathbf{r}) : \Pi_N^{\text{sf}}(x, w, \mathbf{r}) \neq \emptyset\}. \quad (3.67)$$

The *finite-horizon state feedback min-max* control problem considered here is:

$$\mathbb{P}_N^{\text{sf}}(x, w, \mathbf{r}) : \qquad V_N^{0,\text{sf}}(x, w, \mathbf{r}) := \inf_{\pi \in \Pi_N^{\text{sf}}(x,w,\mathbf{r})} \max_{\mathbf{w} \in \mathcal{W}} V_N(x, w, \mathbf{r}, \pi, \mathbf{w}), \quad (3.68)$$

with the optimizer (as defined in (2.18)),

$$\pi_N^0(x, w, \mathbf{r}) \in \arg \min_{\pi \in \Pi_N^{\text{sf}}(x,w,\mathbf{r})} \max_{\mathbf{w} \in \mathcal{W}} V_N(x, w, \mathbf{r}, \pi, \mathbf{w}). \quad (3.69)$$

We will proceed to show how for the assumptions **A1–A2**, in conjunction with the convexity and monotonicity of the system dynamics (3.40)–(3.41), an explicit expression of the solution to the state feedback problem (3.68) can be computed using results from polyhedral algebra and multi-parametric linear programming. We use dynamic programming (DP) to derive the explicit solution.

DP [15, 103] is a well-known method for solving sequential, or multi-stage, decision problems. In DP the decision problem is broken into stages that are solved sequentially, where the solution of one stage is used to construct the solution of the subsequent stage. More specifically, one computes sequentially the partial return functions $\{V_i^0\}_{i=1}^N$, (as defined in (3.68) for $i = N$), the associated set-valued optimal control laws $\{\kappa_i\}_{i=1}^N$ (such that $\mu_{N-i}^0(x, w_{\text{p}}, \mathbf{r}) \in \kappa_i(x, w_{\text{p}}, \mathbf{r})$) and their domains $\{X_i\}_{i=1}^N$; here $i \in \mathbb{N}_{[1,N]}$ denotes "time-to-go". If we define

$$J_i(x, w_{\text{p}}, \mathbf{r}, u) := \max_{w_{\text{c}} \in W} \{\ell_{N-i}(f_{\text{MPL}}(x, u, w_{\text{p}}, w_{\text{c}}), u, r_{N-i}) + \\ V_{i-1}^0(f_{\text{MPL}}(x, u, w_{\text{p}}, w_{\text{c}}), w_{\text{c}}, \mathbf{r})\} \quad (3.70a)$$

for all $(x, w_{\text{p}}, \mathbf{r}, u) \in Z_i$, where

$$Z_i := \{(x, w_{\text{p}}, \mathbf{r}, u) : H_{N-i} f_{\text{MPL}}(x, u, w_{\text{p}}, w_{\text{c}}) + G_{N-i} u + F_{N-i} r_{N-i} \leq h_{N-i}, \\ w_{\text{p}} \in W, \; (f_{\text{MPL}}(x, u, w_{\text{p}}, w_{\text{c}}), w_{\text{c}}, \mathbf{r}) \in X_{i-1}, \; \forall w_{\text{c}} \in W\}, \quad (3.70b)$$

then we can compute $\{V_i^0, \kappa_i, X_i\}_{i=1}^N$ recursively as follows [16, 103]:

$$V_i^0(x, w_\mathrm{p}, \mathbf{r}) := \inf_u \{J_i(x, w_\mathrm{p}, \mathbf{r}, u) : (x, w_\mathrm{p}, \mathbf{r}, u) \in Z_i\},\ \forall (x, w_\mathrm{p}, \mathbf{r}) \in X_i, \tag{3.70c}$$

$$\kappa_i(x, w_\mathrm{p}, \mathbf{r}) := \arg\min_u \{J_i(x, w_\mathrm{p}, \mathbf{r}, u) : (x, w_\mathrm{p}, \mathbf{r}, u) \in Z_i\}, \tag{3.70d}$$

$$X_i := \{(x, w_\mathrm{p}, \mathbf{r}) : (x, \mathbf{r}) \in X_i^{(x,\mathbf{r})},\ w_\mathrm{p} \in W\},\ X_N := \mathrm{Proj}_{n+q+pN} Z_N, \tag{3.70e}$$

where for all $i \in \mathbb{N}_{[1,N-1]}$

$$X_i^{(x,\mathbf{r})} := \{(x, \mathbf{r}) : (x, w_\mathrm{p}, \mathbf{r}) \in \mathrm{Proj}_{n+q+pN} Z_i, \forall w_\mathrm{p} \in W\}, \tag{3.70f}$$

with the boundary conditions

$$X_0 := \{(x, w_\mathrm{p}, \mathbf{r}) :\ (x, r_N) \in X_\mathrm{f},\ w_\mathrm{p} \in W\}, \tag{3.70g}$$

$$V_0^0(x, w_\mathrm{p}, \mathbf{r}) := V_\mathrm{f}(x, r_N),\ \forall (x, w_\mathrm{p}, \mathbf{r}) \in X_0. \tag{3.70h}$$

From the principle of optimality of DP [16] it follows that

$$X_N^{\mathrm{sf}} = X_N,\ V_N^{0,\mathrm{sf}}(x, w, \mathbf{r}) = V_N^0(x, w, \mathbf{r})\ \ \forall (x, w, \mathbf{r}) \in X_N.$$

Moreover, the optimal solution is given by

$$\pi_N^0 = (\kappa_N,\ \kappa_{N-1}, \cdots, \kappa_1).$$

Note that at stage N of the DP recursion (i.e. $i = N$) one has $w_\mathrm{p} = w$, where w is the initial disturbance.

To simplify notation in the rest of this section, we define two prototype problems and we study their properties. The prototype maximization problem $\mathbb{P}_{\max}$ is defined as:

$$\mathbb{P}_{\max}:\ J(x, w_\mathrm{p}, \mathbf{r}, u) := \max_{w_\mathrm{c} \in W} \{\ell(f_{\mathrm{MPL}}(x, u, w_\mathrm{p}, w_\mathrm{c}), u, r) + V(f_{\mathrm{MPL}}(x, u, w_\mathrm{p}, w_\mathrm{c}), w_\mathrm{c}, \mathbf{r})\}, \tag{3.71}$$

for all $(x, w_\mathrm{p}, \mathbf{r}, u) \in Z$, where the domain of J is

$$Z := \{(x, w_\mathrm{p}, \mathbf{r}, u) : H f_{\mathrm{MPL}}(x, u, w_\mathrm{p}, w_\mathrm{c}) + Gu + Fr \le h,\ w_\mathrm{p} \in W, (f_{\mathrm{MPL}}(x, u, w_\mathrm{p}, w_\mathrm{c}), w_\mathrm{c}, \mathbf{r}) \in \Omega, \forall w_\mathrm{c} \in W\}, \tag{3.72a}$$

$$X := \{(x, w_\mathrm{p}, \mathbf{r}) : (x, \mathbf{r}) \in X^{(x,\mathbf{r})},\ w_\mathrm{p} \in W\} \quad \text{or} \quad X := \mathrm{Proj}_{n+q+pN} Z. \tag{3.72b}$$

with $X^{(x,\mathbf{r})} := \{(x, \mathbf{r}) : (x, w_\mathrm{p}, \mathbf{r}) \in \mathrm{Proj}_{n+q+pN} Z, \forall w_\mathrm{p} \in W\},\ \ell : \mathbb{R}^{n+m+p} \to \mathbb{R}, V : \Omega \to \mathbb{R}$, $\mathbf{r}$ has the form $\mathbf{r} = [\ldots r^T \ldots]^T$ (i.e., $\exists k : r_k = r$). The prototype minimization problem $\mathbb{P}_{\min}$ is defined as:

$$\mathbb{P}_{\min}:\ \ V^0(x, w_\mathrm{p}, \mathbf{r}) := \inf_u \{J(x, w_\mathrm{p}, \mathbf{r}, u) : (x, w_\mathrm{p}, \mathbf{r}, u) \in Z\}, \tag{3.73a}$$

$$\kappa(x, w_\mathrm{p}, \mathbf{r}) := \arg\min_u \{J(x, w_\mathrm{p}, \mathbf{r}, u) : (x, w_\mathrm{p}, \mathbf{r}, u) \in Z\}, \tag{3.73b}$$

for all $(x, w_\mathrm{p}, \mathbf{r}) \in X$.

In terms of these prototype problems, it is easy to identify the DP recursion (3.70) by setting $r \leftarrow r_{N-i}$, $\ell \leftarrow \ell_{N-i}$, $V \leftarrow V_{i-1}^0$, $V^0 \leftarrow V_i^0$, $X \leftarrow X_i$, $Z \leftarrow Z_i$ and $\Omega \leftarrow X_{i-1}$. Moreover, H, G, F, h are identified with $H_{N-i}, G_{N-i}, F_{N-i}, h_{N-i}$, respectively.

Clearly, we can now proceed to show, via induction, that a certain set of properties is possessed by each element in the sequence $\{V_i^0, \kappa_i, X_i\}_{i=1}^N$ by showing that if $\{V, \Omega\}$ has a given set of properties, then $\{V^0, X\}$ also has these properties, with the properties of κ being the same as those of each of the elements in the sequence $\{\kappa_i\}_{i=1}^N$. In the sequel, constructive proofs of the main results are presented, so that the reader can develop a prototype algorithm for computing the sequence $\{V_i^0, \kappa_i, X_i\}_{i=1}^N$.

Properties of X

The following lemma states that any set described by linear inequalities obtained from multiplication of f_{MPL} with a nonnegative matrix is a polyhedral set, a property that is crucial to the derivation of the rest of the results from this section, in particular Proposition 3.3.10.

Lemma 3.3.8 *The set* $\mathcal{Z} = \{(x, w_{\mathrm{p}}, \mathbf{r}, u) : \bar{H} f_{\mathrm{MPL}}(x, u, w_{\mathrm{p}}, w_{\mathrm{c}}) + \bar{G}u + \bar{F}w_{\mathrm{p}} + \bar{E}\mathbf{r} \leq \bar{h}, \forall w_{\mathrm{c}} \in W\}$ *with* $\bar{H} \geq 0$*, can be written equivalently as* $\mathcal{Z} = \{(x, w_{\mathrm{p}}, \mathbf{r}, u) : \tilde{H}x + \tilde{G}u + \tilde{F}w_{\mathrm{p}} + \tilde{E}\mathbf{r} \leq \tilde{h}\}$ *with* $\tilde{H} \geq 0$.

Proof: Recall that $f_{\mathrm{MPL}} \in \mathcal{F}_{\mathrm{mps}}$ and $f_{\mathrm{MPL}}(\cdot, u, w_{\mathrm{p}}, w_{\mathrm{c}}) \in (\mathcal{F}^{+}_{\mathrm{mps}})^{n}$ for each $(u, w_{\mathrm{p}}, w_{\mathrm{c}})$. From Lemma 3.2.1 it follows that the function $x \mapsto \bar{H} f_{\mathrm{MPL}}(x, u, w_{\mathrm{p}}, w_{\mathrm{c}})$ is in $(\mathcal{F}^{+}_{\mathrm{mps}})^{n_{\bar{H}}}$ for any $(u, w_{\mathrm{p}}, w_{\mathrm{c}})$. Moreover, given any finite set of scalar-valued functions $\{\varphi_j\}_{j \in \mathcal{I}}$ we have that

$$\{z : \max_{j \in \mathcal{I}}\{\varphi_j(z)\} \leq \alpha\} = \{z : \varphi_j(z) \leq \alpha, \forall j \in \mathcal{I}\}.$$

Hence, it is easy to verify that the set $\mathcal{Z}$ has the equivalent representation $\mathcal{Z} = \{(x, w_{\mathrm{p}}, \mathbf{r}, u) : \tilde{H}x + \tilde{G}u + \tilde{F}w_{\mathrm{p}} + \tilde{E}\mathbf{r} + \tilde{K}w_{\mathrm{c}} \leq h, \forall w_{\mathrm{c}} \in W\}$, where $\tilde{H} \geq 0$. If we define $\tilde{k}^{0}_{i} := \max_{w_{\mathrm{c}} \in W}\{\tilde{K}_{i.} w_{\mathrm{c}}\}$ (since W is a compact set it follows that $\tilde{k}^{0}_{i}$ is finite) then the result follows by letting $\tilde{h} := h - \tilde{k}^{0}$, where $\tilde{k}^{0} := [\tilde{k}^{0}_{1}\ \tilde{k}^{0}_{2} \ldots]^{T}$. ♢

Note that $\tilde{k}^{0}$ can be computed by solving a set of linear programs (recall that W is a polytope).

The next lemma shows that some useful properties of a class of polyhedra are inherited by its projection.

Lemma 3.3.9 *Let* $\mathcal{Z} = \{(x, r, t, u) \in \mathbb{R}^{n} \times \mathbb{R}^{p} \times \mathbb{R}^{q} \times \mathbb{R}^{m} : \bar{H}x + \bar{F}r + \bar{K}t + \bar{G}u \leq \bar{h}\}$ *be given, where* $\bar{H} \geq 0$ *and* $\bar{K} \leq 0$*. The set* $\mathcal{X} := \{(x, r, t) : \exists u \text{ s.t. } (x, r, t, u) \in \mathcal{Z}\}$ *is a polyhedral set of the form* $\mathcal{X} = \{(x, r, t) : \tilde{H}x + \tilde{F}r + \tilde{K}t \leq \tilde{h}\}$*, where* $\tilde{H} \geq 0$ *and* $\tilde{K} \leq 0$.

Proof: Since $\mathcal{X} = \mathrm{Proj}_{n+p+q}\mathcal{Z}$, it is clear that $\mathcal{X}$ is a polyhedron. We begin by considering the case $m = 1$. The proof for this case will lead to a solution for the case $m > 1$.

Note that for $m = 1$, $\bar{G}$ is a vector. Define the following index sets: $I_{+} := \{i \in \mathbb{N}_{[1, n_{\bar{H}}]} : \bar{G}_i > 0\}$, $I_{-} := \{i \in \mathbb{N}_{[1, n_{\bar{H}}]} : \bar{G}_i < 0\}$ and $I_{0} := \{i \in \mathbb{N}_{[1, n_{\bar{H}}]} : \bar{G}_i = 0\}$. We have the following cases:

1. $i \in I_0 \Rightarrow \bar{H}_{i.}x + \bar{F}_{i.}r + \bar{K}_{i.}t \leq h_i$ and $\bar{H}_{i.} \geq 0$, $\bar{K}_{i.} \leq 0$.
2. $j \in I_{+} \Rightarrow u \leq -\frac{1}{\bar{G}_j}\bar{H}_{j.}x - \frac{1}{\bar{G}_j}\bar{F}_{j.}r - \frac{1}{\bar{G}_j}\bar{K}_{j.}t + \frac{\bar{h}_j}{\bar{G}_j}$ and $\frac{1}{\bar{G}_j}\bar{H}_{j.} \geq 0$, $\frac{1}{\bar{G}_j}\bar{K}_{j.} \leq 0$.
3. $l \in I_{-} \Rightarrow u \geq -\frac{1}{\bar{G}_l}\bar{H}_{l.}x - \frac{1}{\bar{G}_l}\bar{F}_{l.}r - \frac{1}{\bar{G}_l}\bar{K}_{l.}t + \frac{\bar{h}_l}{\bar{G}_l}$ and $-\frac{1}{\bar{G}_l}\bar{H}_{l.} \geq 0$, $-\frac{1}{\bar{G}_l}\bar{K}_{l.} \leq 0$.

It is then straightforward to combine the above and show that the set $\mathcal{X}$ is described by the following inequalities:

$$\bar{H}_{i.}x + \bar{F}_{i.}r + \bar{K}_{i.}t \leq \bar{h}_i \quad \forall i \in I_0,$$

$$(-\frac{1}{\bar{G}_l}\bar{H}_{l.} + \frac{1}{\bar{G}_j}\bar{H}_{j.})x + (-\frac{1}{\bar{G}_l}\bar{F}_{l.} + \frac{1}{\bar{G}_j}\bar{F}_{j.})r + (-\frac{1}{\bar{G}_l}\bar{K}_{l.} + \frac{1}{\bar{G}_j}\bar{K}_{j.})t \leq -\frac{\bar{h}_l}{\bar{G}_l} + \frac{\bar{h}_j}{\bar{G}_j}, \quad \forall j \in I_{+},\ l \in I_{-}$$

The result follows, since the rows of $\tilde{H}$ are composed of the vectors $\bar{H}_{i.} \geq 0$ and $-\frac{1}{\bar{G}_l}\bar{H}_{l.} + \frac{1}{\bar{G}_j}\bar{H}_{j.} \geq 0$ for all $i \in I_0$, $j \in I_+$, $l \in I_-$, while the rows of $\tilde{K}$ are composed of the vectors $\bar{K}_{i.} \leq 0$ and $-\frac{1}{\bar{G}_l}\bar{K}_{l.} + \frac{1}{\bar{G}_j}\bar{K}_{j.} \leq 0$ for all $i \in I_0, j \in I_+, l \in I_-$.

When $m > 1$, the previous reasoning implemented for the case $m = 1$ can be repeated m times, eliminating one component of the vector u at a time. ♢

It usually turns out that the matrix $[\tilde{H}\ \tilde{F}\ \tilde{K}\ \tilde{h}]$ obtained using the previous procedure contains many redundant rows that are not needed to represent the set $\mathcal{X}$. A row may be tested for redundancy by solving a linear programming problem in which the tested row determines the cost and the remaining rows form the constraints.

We are now in a position to show that X has the same structural properties as Ω.

Proposition 3.3.10 *Suppose Ω is a polyhedral set given by $\Omega = \{(x, w, \mathbf{r}) : \Gamma x + \Phi\mathbf{r} \leq \gamma,\ w \in W\}$ with $\Gamma \geq 0$, and assume that H in* (3.72a) *satisfies $H \geq 0$. Then, the set X defined in* (3.72b) *is a polyhedron given by either $X = \{(x, w_{\mathrm{p}}, \mathbf{r}) : \hat{H}x + \hat{E}\mathbf{r} \leq \hat{h},\ w_{\mathrm{p}} \in W\}$ or $X = \{(x, w_{\mathrm{p}}, \mathbf{r}) : \hat{H}x + \hat{F}w_{\mathrm{p}} + \hat{E}\mathbf{r} \leq \hat{h}\}$, where $\hat{H} \geq 0$.*

Proof: The set Z is described as follows:

$$Z = \{(x, w_{\mathrm{p}}, \mathbf{r}, u) : \bar{H} f_{\mathrm{MPL}}(x, u, w_{\mathrm{p}}, w_{\mathrm{c}}) + \bar{G}u + \bar{F}w_{\mathrm{p}} + \bar{E}\mathbf{r} \leq \bar{h},\ \forall w_{\mathrm{c}} \in W\}, \tag{3.74}$$

where $\bar{H} = [H^T\ \Gamma^T\ 0]^T \geq 0$, $\bar{G} = [G^T\ 0\ 0]^T$, $\bar{F} = [0\ 0\ \Omega^T]^T, \bar{E} = [(Fr)^T\ (\Phi\mathbf{r})^T\ 0]^T$ and $\bar{h} = [h^T\ \gamma^T\ s^T]^T$. From Lemma 3.3.8 it follows that Z can be written equivalently as $Z = \{(x, w_{\mathrm{p}}, \mathbf{r}, u) : \tilde{H}x + \tilde{G}u + \tilde{F}w_{\mathrm{p}} + \tilde{E}\mathbf{r} \leq \tilde{h}\}$ where $\tilde{H} \geq 0$. By applying a particular case of Lemma 3.3.9 it follows that

$$\mathrm{Proj}_{n+q+pN} Z = \{(x, w_{\mathrm{p}}, \mathbf{r}) : \hat{H}x + \hat{F}w_{\mathrm{p}} + \hat{E}\mathbf{r} \leq \hat{h}\},\quad \bar{H} \geq 0.$$

The rest follows immediately. ♢

Note that the set X_0 in (3.70g) is of the form given in Proposition 3.3.10 (since we assume that assumption **A1** holds). The reason for introducing assumption **A1** is now clear, since $H, \Gamma \geq 0$ are crucial in the proof of Proposition 3.3.10; it would not be possible to convert the expression for Z into a set of linear inequalities if some components of H or Γ were negative.

Properties of $\mathbb{P}_{\max}$

This section derives an invariance property of the prototype maximization problem $\mathbb{P}_{\max}$.

Proposition 3.3.11 *If ℓ, $V \in \mathcal{F}_{\mathrm{mps}}$ and $\ell(\cdot, u, r)$, $V(\cdot, w_{\mathrm{p}}, \mathbf{r}) \in \mathcal{F}^+_{\mathrm{mps}}$ for any fixed $(w_{\mathrm{p}}, \mathbf{r}, u)$, then J possesses the same properties, i.e. $J \in \mathcal{F}_{\mathrm{mps}}$ and $J(\cdot, w_{\mathrm{p}}, \mathbf{r}, u) \in \mathcal{F}^+_{\mathrm{mps}}$, for any fixed $(w_{\mathrm{p}}, \mathbf{r}, u)$.*

Proof: It follows from Lemma 3.2.1 that one can write $\ell(f_{\mathrm{MPL}}(x, u, w_{\mathrm{p}}, w_{\mathrm{c}}), u, r) + V(f_{\mathrm{MPL}}(x, u, w_{\mathrm{p}}, w_{\mathrm{c}}), w_{\mathrm{c}}, \mathbf{r}) = \max_{j \in \mathcal{J}}\{\alpha_j^T x + \beta_j^T w_{\mathrm{p}} + \mu_j^T w_{\mathrm{c}} + \gamma_j^T u + \delta_j^T \mathbf{r} + \tilde{\theta}_j\}$, where $\alpha_j \geq 0$ for all $j \in \mathcal{J}$, so that

$$\begin{aligned} J(x, w_{\mathrm{p}}, \mathbf{r}, u) = \max_{w_{\mathrm{c}} \in W} \big\{ \max_{j \in \mathcal{J}}\{\alpha_j^T x + \beta_j^T w_{\mathrm{p}} + \mu_j^T w_{\mathrm{c}} + \gamma_j^T u + \delta_j^T \mathbf{r} + \tilde{\theta}_j\}\big\} = \\ \max_{j \in \mathcal{J}} \big\{ \max_{w_{\mathrm{c}} \in W}\{\alpha_j^T x + \beta_j^T w_{\mathrm{p}} + \mu_j^T w_{\mathrm{c}} + \gamma_j^T u + \delta_j^T \mathbf{r} + \tilde{\theta}_j\}\big\} = \\ \max_{j \in \mathcal{J}}\{\alpha_j^T x + \beta_j^T w_{\mathrm{p}} + \gamma_j^T u + \delta_j^T \mathbf{r} + \theta_j\}, \end{aligned}$$

where $\theta_j := \tilde{\theta}_j + \max_{w_c \in W}\{\mu_j^T w_c\}$ for all $j \in \mathcal{J}$. Note that $\{\theta_j\}_{j\in\mathcal{J}}$ can be computed by solving a sequence of linear programs. Moreover, the coefficients of the variable x in J are the nonnegative vectors α_j. $\diamondsuit$

Recall that ℓ_i and V_0^0 given in (3.24) and (3.70h) satisfy the conditions of Proposition 3.3.11. If assumption **A2** did not hold, then one cannot guarantee that the cost function and value function of the maximization problem will be convex.

Properties of $\mathbb{P}_{\min}$

This section derives the main properties of V^0 and κ. Before proceeding, we show that if V^0 is proper, then V^0 is finite everywhere on X. Note that since we always have that $u(0)$ should be larger than the current time instant, i.e. the time instant at which we start performing the computations, $u(0)$ is bounded from below and V^0 will always be proper.

Lemma 3.3.12 *Suppose Ω is a polyhedral set given by $\Omega = \{(x, w, \mathbf{r}) : \Gamma x + \Phi\mathbf{r} \le \gamma,\ w \in W\}$ with $\Gamma \ge 0$, and assume that H in* (3.72a) *satisfies $H \ge 0$. Suppose also that $Z \ne \emptyset$ and $J \in \mathcal{F}_{\text{mps}}$. There exists an $(\bar{x}, \bar{w}_{\text{p}}, \bar{\mathbf{r}}) \in X$ such that $V^0(\bar{x}, \bar{w}_{\text{p}}, \bar{\mathbf{r}})$ is finite if and only if $V^0(x, w_{\text{p}}, \mathbf{r})$ is finite for all $(x, w_{\text{p}}, \mathbf{r}) \in \text{Proj}_{n+q+pN} Z$ and thus for all $(x, w_{\text{p}}, \mathbf{r}) \in X$.*

Proof: From the proof of Proposition 3.3.10 it follows that Z is a non-empty polyhedron: $Z = \{(x, w_{\text{p}}, \mathbf{r}, u) : \tilde{H}x + \tilde{G}u + \tilde{F}w_{\text{p}} + \tilde{E}\mathbf{r} \le \tilde{h}\}$, with $\tilde{H} \ge 0$. Since $J \in \mathcal{F}_{\text{mps}}$, we can write $J(x, w_{\text{p}}, \mathbf{r}, u) = \max_{j\in\mathcal{J}}\{\alpha_j^T x + \beta_j^T w_{\text{p}} + \gamma_j^T u + \delta_j^T \mathbf{r} + \theta_j\}$. The prototype minimization problem $\mathbb{P}_{\min}(x, \mathbf{r})$ becomes:

$$V^0(x,w_{\text{p}}, \mathbf{r}) = \inf_u \big\{ \max_{j\in\mathcal{J}}\{\alpha_j^T x + \beta_j^T w_{\text{p}} + \gamma_j^T u + \delta_j^T \mathbf{r} + \theta_j\} : (x, w_{\text{p}}, \mathbf{r}, u) \in Z\big\} =$$
$$\min_{(\mu,u)}\{\mu : \alpha_j^T x + \beta_j^T w_{\text{p}} + \gamma_j^T u + \delta_j^T \mathbf{r} + \theta_j \le \mu \ \ \forall j \in \mathcal{J},\ (x, w_{\text{p}}, \mathbf{r}, u) \in Z\}, \qquad (3.75)$$

i.e. we have obtained a feasible linear program for any $(x, w_{\text{p}}, \mathbf{r}) \in \text{Proj}_{n+q+pN} Z$.

Note that the feasible set of the dual of (3.75) does not depend on $(x, w_{\text{p}}, \mathbf{r})$. Assume that $V^0(\bar{x}, \bar{w}_{\text{p}}, \mathbf{r})$ is finite. From strong duality for linear programs [144, 147] it follows that the dual problem of (3.75) is feasible, independent of $(x, w_{\text{p}}, \mathbf{r})$. Using again strong duality for linear programs, we conclude that $V^0(x, w_{\text{p}}, \mathbf{r})$ is finite if $(x, w_{\text{p}}, \mathbf{r}) \in \text{Proj}_{n+q+pN} Z$ and $V^0(x, w_{\text{p}}, \mathbf{r}) = +\infty$ if $(x, w_{\text{p}}, \mathbf{r}) \notin \text{Proj}_{n+q+pN} Z$. From the definition of X in (3.72b) it follows that V^0 takes finite values on X as well. The reverse implication is obvious. $\diamondsuit$

The following proposition gives a characterization of the solution and of the optimal value of the prototype minimization problem $\mathbb{P}_{\min}$.

Proposition 3.3.13 *Suppose Ω is a polyhedral set given by $\Omega = \{(x, w, \mathbf{r}) : \Gamma x + \Phi\mathbf{r} \le \gamma,\ w \in W\}$ with $\Gamma \ge 0$, and assume that H in* (3.72a) *satisfies $H \ge 0$. Suppose also that $Z \ne \emptyset$, $J \in \mathcal{F}_{\text{mps}}$ and V^0 is proper. Then, the value function $V^0 \in \mathcal{F}_{\text{mps}}$ and has domain X, where X is a polyhedral set. The (set-valued) control law $\kappa(x, w_{\text{p}}, \mathbf{r})$ is a polyhedron for a given $(x, w_{\text{p}}, \mathbf{r}) \in X$. Moreover, it is always possible to select a continuous and PWA control law μ such that $\mu(x, w_{\text{p}}, \mathbf{r}) \in \kappa(x, w_{\text{p}}, \mathbf{r})$ for all $(x, w_{\text{p}}, \mathbf{r}) \in X$.*

Proof: It follows from the proof of Lemma 3.3.12 (i.e. equation (3.75)) that $\mathbb{P}_{\min}$ is a multi-parametric linear program as the one defined in Section 2.3.1. The properties stated above follow from Theorem 2.3.2. $\diamondsuit$

Now we can state the following key result, which, together with Propositions 3.3.10–3.3.13, allow one to deduce, via induction, some important properties of the sequence $\{V_i^0, \kappa_i, X_i\}_{i=1}^N$:

Theorem 3.3.14 *Suppose that the same assumptions as in Proposition 3.3.13 hold. If, in addition, $J(\cdot, w_\mathrm{p}, \mathbf{r}, u) \in \mathcal{F}_\mathrm{mps}^+$ for any fixed $(w_\mathrm{p}, \mathbf{r}, u)$, then the value function $V^0(\cdot, w_\mathrm{p}, \mathbf{r}) \in \mathcal{F}_\mathrm{mps}^+$ for any fixed $(w_\mathrm{p}, \mathbf{r})$.*

Proof: Using Proposition 3.3.10 it follows that $Z = \{(x, w_\mathrm{p}, \mathbf{r}, u) : \tilde{H}x + \tilde{G}u + \tilde{F}w_\mathrm{p} + \tilde{E}\mathbf{r} \leq \tilde{h}\}$, with $\tilde{H} \geq 0$. The function J can be written as: $J(x, w_\mathrm{p}, \mathbf{r}, u) = \max_{j \in \mathcal{J}}\{\alpha_j^T x + \beta_j^T w_\mathrm{p} + \gamma_j^T u + \delta_j^T \mathbf{r} + \theta_j\}$, where $\alpha_j \geq 0$ for all j. From Proposition 3.3.13 and the fact that V^0 is proper, it follows that $V^0 \in \mathcal{F}_\mathrm{mps}$ and its domain is $\mathrm{Proj}_{n+q+pN} Z$. The epigraph of V^0 is given by:

$$\begin{aligned}
\mathrm{epi}\, V^0 :=& \{(x, w_\mathrm{p}, \mathbf{r}, t) : V^0(x, w_\mathrm{p}, \mathbf{r}) \leq t,\ (x, w_\mathrm{p}, \mathbf{r}) \in \mathrm{Proj}_{n+q+pN} Z\} = \\
& \{(x, w_\mathrm{p}, \mathbf{r}, t) : \exists u \text{ s.t. } (x, w_\mathrm{p}, \mathbf{r}, u) \in Z,\ J(x, w_\mathrm{p}, \mathbf{r}, u) \leq t\} = \\
& \{(x, w_\mathrm{p}, \mathbf{r}, t) : \exists u \text{ s.t. } \tilde{H}x + \tilde{G}u + \tilde{F}w_\mathrm{p} + \tilde{E}\mathbf{r} \leq \tilde{h}, \\
& \qquad \alpha_j^T x + \beta_j^T w_\mathrm{p} + \gamma_j^T u + \delta_j^T \mathbf{r} + \theta_j \leq t\ \ \forall j \in \mathcal{J}\} = \\
& \{(x, w_\mathrm{p}, \mathbf{r}, t) : \exists u \text{ s.t. } \bar{H}x + \bar{G}u + \bar{F}w_\mathrm{p} + \bar{E}\mathbf{r} + \bar{K}t \leq \bar{h}\},
\end{aligned}$$

where $\bar{H} = [\tilde{H}^T\ \alpha_1^T\ \cdots\ \alpha_l^T]^T \geq 0$ and $\bar{K} = [0\ -1 \ldots -1]^T \leq 0$. From Lemma 3.3.9 we obtain that the epigraph of V^0 is a polyhedral set given by $\mathrm{epi}\ V^0 = \{(x, w_\mathrm{p}, \mathbf{r}, t) : \hat{H}x + \hat{F}w_\mathrm{p} + \hat{E}\mathbf{r} + \hat{K}t \leq \hat{h}\}$, where $\hat{H} \geq 0, \hat{K} \leq 0$. Let $l = n_{\hat{H}}$ be the number of inequalities describing $\mathrm{epi}\ V^0$. We arrange the indexes $i \in \mathbb{N}_{[1,l]}$ such that $\hat{K}_i < 0$ for $i \in \mathbb{N}_{[1,v]}$ but $\hat{K}_i = 0$ for $i \in \mathbb{N}_{[v+1,\, l]}$ (possibly $v = 0$, i.e. $\hat{K}_i = 0$ for all i). Taking $a_i = -\hat{H}_{i\cdot}/\hat{K}_i, b_i = -\hat{F}_{i\cdot}/\hat{K}_i, c_i = -\hat{E}_{i\cdot}/\hat{K}_i$ and $d_i = -\hat{h}_i/\hat{K}_i$ for all $i \in \mathbb{N}_{[1,v]}$, we get that the epigraph of V^0 is expressed as:

$$\begin{aligned}
\mathrm{epi}\, V^0 = \{(x, w_\mathrm{p}, \mathbf{r}, t) :& a_i x + b_i w_\mathrm{p} + c_i \mathbf{r} - d_i \leq t\ \ \forall i \in \mathbb{N}_{[1,v]}, \\
& \hat{H}_{i\cdot}x + \hat{F}_{i\cdot}w_\mathrm{p} + \hat{E}_{i\cdot}\mathbf{r} \leq \hat{h}_i\ \ \forall i \in \mathbb{N}_{[v+1,\, l]}\}.
\end{aligned} \tag{3.76}$$

But V^0 is proper and thus $v > 0$. Since $V^0 \in \mathcal{F}_\mathrm{mps}$, (3.76) gives us a representation of V^0 as $V^0(x, w_\mathrm{p}, \mathbf{r}) = \max_{i \in \mathbb{N}_{[1,v]}}\{a_i x + b_i w_\mathrm{p} + c_i \mathbf{r} - d_i\}$, where $a_i = -\hat{H}_{i\cdot}/\hat{K}_i \geq 0$, for all $i \in \mathbb{N}_{[1,v]}$, i.e. $V^0(\cdot, w_\mathrm{p}, \mathbf{r}) \in \mathcal{F}_\mathrm{mps}^+$ for any fixed $(w_\mathrm{p}, \mathbf{r})$. Moreover, we can derive also the domain where V^0 takes finite values: $\mathrm{Proj}_{n+q+pN} Z = \{(x, w_\mathrm{p}, \mathbf{r}) : \hat{H}_{i\cdot}x + \hat{F}_{i\cdot}w_\mathrm{p} + \hat{E}_{i\cdot}\mathbf{r} \leq \hat{h}_i\ \forall i \in \mathbb{N}_{[v+1,\, l]}\}$. $\diamondsuit$

Based on the invariance properties of the two prototype problems $\mathbb{P}_\mathrm{max}$ and $\mathbb{P}_\mathrm{min}$, we can now derive the properties of V_i^0, κ_i and X_i for all $i \in \mathbb{N}_{[1,N]}$. The following follows by applying Propositions 3.3.10–3.3.13 and Theorem 3.3.14 to the DP equations (3.70):

Theorem 3.3.15 *Suppose that **A1** and **A2** hold, Z_i is non-empty and V_i^0 is proper for all $i \in \mathbb{N}_{[1,N]}$. The following holds for each $i \in \mathbb{N}_{[1,N]}$:*

(i) X_i is a non-empty polyhedron,

(ii) V_i^0 is a convex, continuous PWA function with domain X_i,

(iii) $V_i^0(\cdot, w_\mathrm{p}, \mathbf{r}) \in \mathcal{F}_\mathrm{mps}^+$ for any fixed $(w_\mathrm{p}, \mathbf{r})$,

(iv) There exists a continuous PWA function μ_{N-i}^0 such that $\mu_{N-i}^0(x, w_\mathrm{p}, \mathbf{r}) \in \kappa_i(x, w_\mathrm{p}, \mathbf{r})$ for all $(x, w_\mathrm{p}, \mathbf{r}) \in X_i$.

Since the proofs of all the above results are constructive, it follows that the sequences $\{V_i^0, \kappa_i, X_i\}_{i=1}^N$ and $\{\mu_i^0\}_{i=1}^N$ can be computed iteratively, without gridding, by noting the following:

- Given X_{i-1}, one can compute X_i by first computing Z_i, as in the proof of Proposition 3.3.10, followed with a projection operation,

- Given V_{i-1}^0, a max-plus-scaling expressions of J_i can be computed by referring to the proof of Proposition 3.3.11,

- Given J_i and Z_i, one can compute V_i^0, κ_i and a μ_{N-i}^0 as in the proof of Proposition 3.3.13 or Theorem 3.3.14, either by using multi-parametric linear programming algorithms [23,152] or projection algorithms [76].

It follows that $X_N^{\mathrm{sf}} = X_N$ and $V_N^{0,\mathrm{sf}}(x, w_{\mathrm{p}}, \mathbf{r}) = V_N^0(x, w_{\mathrm{p}}, \mathbf{r})$ for all $(x, w_{\mathrm{p}}, \mathbf{r}) \in X_N$. Note that in order to get the state feedback solution we have to solve exactly N multi-parametric linear programs as in the linear case (see [7]). It is clear that:

$$\begin{array}{rl} & X_N^{\mathrm{ol}} \subseteq X_N^{\mathrm{df}} \subseteq X_N^{\mathrm{sf}} \\ V_N^{0,\mathrm{sf}}(x, w, \mathbf{r}) & \leq V_N^{0,\mathrm{df}}(x, w, \mathbf{r}) \leq \quad V_N^{0,\mathrm{ol}}(x, w, \mathbf{r}) \;\; \forall (x, w, \mathbf{r}) \in X_N^{\mathrm{ol}}. \end{array} \tag{3.77}$$

Note that if $\mathbf{r}$ is known in advance, then the only parameter in the multi-parametric linear programs is the initial state x.

3.3.5 Timing: disturbance case

In Section 3.2.3 we have discussed the timing issues for deterministic MPL systems. We now consider the deterministic counterpart in the presence of disturbances. We recall that in practical applications the entries of the system matrices are nonnegative or take the value ε. It follows that if $x(k)$ is completely available, then $u(k-1)$ and $w(k-1)$ are also available. The reader might ask how we determine the initial cycle k_0. Let t_0 be the time when one of the robust optimal control problems discussed in Sections 3.3.2–3.3.4 is solved. We can define the initial cycle k_0 as follows:

$$k_0 = \arg\max\{k : x_i(k) \leq t_0 \ \forall i \in \mathbb{N}_{[1,n]}\}.$$

This means that at time t_0 the state $x(k_0)$ is completely available and also $u(k_0-1), w(k_0-1)$ are completely known. However, at time t_0 also some components of the forthcoming inputs and states might be known. Due to causality, the information about the components of the forthcoming inputs and states can be recast as linear equality and inequality constraints on some inputs and disturbances, which thus fits in the framework presented in Sections 3.3.2–3.3.4.

In the open-loop case at time $u_i(k) \geq t_0$, where $k \geq k_0$ and $u_i(k) = [u_{k-k_0}^0(x(k_0), w(k_0 - 1), [r^T(k_0) \cdots r^T(k_0+N)]^T)]_i$, the i^{th} input is activated for the k^{th} cycle (e.g. for a manufacturing system at time $u_i(k)$ the raw material is fed to input i for the k^{th} cycle). In the disturbance or state feedback case $u_i(k) = [\mu_{k-k_0}^0(x(k), w(k_0-1), \cdots, w(k-1), [r^T(k_0) \cdots r^T(k_0+N)]^T)]_i$, where μ_k^0 is either the disturbance feedback policy computed in Section 3.3.3 or the state feedback policy computed in Section 3.3.4.

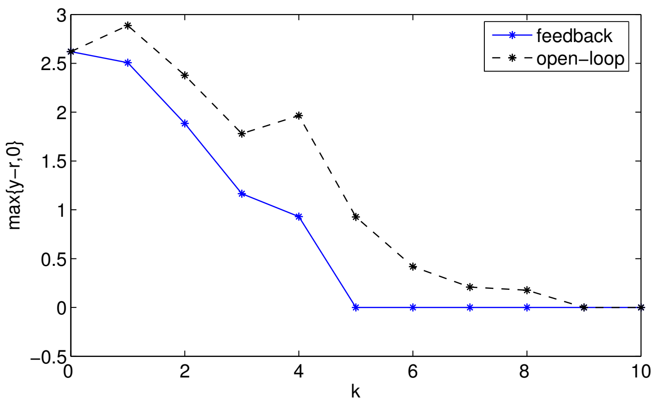

Figure 3.1: The tardiness $\max\{y-r,0\}$ *for the feedback controller (full) and the open-loop controller (dotted).*

N	1	2	3	4	5	6	7	8	9	10
n_R	2	7	7	10	13	15	19	23	25	25

Table 3.1: The number of regions in the multi-parametric linear programs as a function of N.

3.3.6 Example

We consider an example where we compare the three robust optimal control approaches presented in Sections 3.3.2– 3.3.4:

$$x(k+1) = \begin{bmatrix} -w_1(k)+w_2(k)+2 & \varepsilon \\ -w_1(k)-w_2(k)+5 & w_1(k)-2 \end{bmatrix} \otimes x(k) \oplus \begin{bmatrix} -w_1(k)+3 \\ -w_2(k)+2 \end{bmatrix} \otimes u(k)$$
$$y(k) = [0 \quad \varepsilon] \otimes x(k)$$

We assume a bounded disturbance set: $W = \{w \in \mathbb{R}^n : w_1 \in [2 \;\; 3],\; w_2 \in [1 \;\; 2], w_1 + w_2 \le 4\}$. We choose $N = 10$, the due date signal is $\mathbf{r} = [3.4\ 5\ 7\ 9.5\ 11.8\ 14\ 16.7\ 19.4\ 21.6\ 23.8\ 26]^T$ and the initial state is $x(0) = [6 \;\; 8]^T$. The system is subject to input-state constraints: $x_2(k) - u(k) \le 2, x_1(N) + x_2(N) \le 2r_N, -6 + r_k \le u(k) \le 6 + r_k$. We use the stage cost defined in (3.24)–(3.25) with $\gamma = 0.1$ and a random sequence of disturbances.

In this particular example we observe that the disturbance feedback controller from Section 3.3.3 coincides with the state feedback controller from Section 3.3.4. Moreover, the number of regions of the computed multi-parametric linear programs corresponding to the state feedback approach, as a function of the prediction horizon N, is given in Table 3.1.

Figure 3.1 shows the tardiness (i.e. the signal $\max\{y - r, 0\}$) for the open-loop controller derived in Section 3.3.2 and for the state feedback controller derived in Section 3.3.4. As we expected, the performance with respect to the feedback approach is better than the open-loop approach. The plot shows that the feedback controller gives a lower tardiness (i.e. better "tracking") than the open-loop controller. We conclude that the state and disturbance feedback approach outperform the open-loop approach. More simulation examples will be provided in the next chapter.

3.3.7 Robust "time" optimal control

As a direct application of the robust control problems discussed above, we consider the robust counterpart of the "time" optimal control problem presented in Section 3.2.4: given a maximum horizon length $N_{\max}$ we consider the problem of ensuring that the completion times after N

events (with $N \in \mathbb{N}_{[1,N_{\max}]}$) are less than or equal to a specified target time $\mathbf{T}$ (i.e. $y(N) \leq \mathbf{T}$), using the latest controller that satisfies the input and state constraints (3.17)–(3.18) regardless the values of the disturbance. The robust time optimal control problem can be posed in the framework of the finite-horizon min-max control problems considered in the previous section.

One proceeds by defining

$$N^0(x, w, \mathbf{T}) := \max_{N,\pi}\{N \in \mathbb{N}_{[1,N_{\max}]} : \pi \in \Pi_N^{\mathbf{T}}(x, w, [0 \ldots 0\ \mathbf{T}^T]^T)\}, \tag{3.78}$$

where $\Pi_N^{\mathbf{T}}(\cdot)$ is either $\Pi_N^{\mathrm{ol}}(\cdot)$ or $\Pi_N^{\mathrm{df}}(\cdot)$ or $\Pi_N^{\mathrm{sf}}(\cdot)$ depending whether π is an open-loop input sequence or a disturbance feedback policy or a state feedback policy, respectively, but with the substitutions $H_N \leftarrow [H_N^T\ I]^T \geq 0$, $F_N \leftarrow [F_N^T\ 0]^T$ and $h_N \leftarrow [h_N^T\ ((-C^T) \otimes' \mathbf{T})^T]^T$ (note that $F_i r_i = 0$, for all $i \in \mathbb{N}_{[0,N-1]}$ and $r_N = \mathbf{T}$). It follows that

$$N^0(x, w, \mathbf{T}) = \max_{N}\{N \in \mathbb{N}_{[1,N_{\max}]} : (x, w, \mathbf{r}) \in X_N^{\mathbf{T}}\}, \tag{3.79}$$

where $\mathbf{r} = [0\ 0 \cdots 0\ \mathbf{T}^T]^T$ and $X_N^{\mathbf{T}} = \{(x, w, \mathbf{r}) : \Pi_N^{\mathbf{T}}(x, w, \mathbf{r}) \neq \emptyset\}$. As a stage cost we consider $\ell_i(x_i, u_i, r_i) := -\sum_{j=1}^{m}(u_i)_j$. Clearly, the assumptions **A1–A2** also hold in this particular case. The robust time-optimal controller is implemented as follows:

1. For each $N \in \mathbb{N}_{[1,N_{\max}]}$, solve problem (3.46) or (3.61) or (3.68) where $\mathbf{r}$ is defined as $\mathbf{r} = [0\ 0 \cdots 0\ \mathbf{T}^T]^T \in \mathbb{R}^{pN}$. Determine $N^0(x, w, \mathbf{T})$ according to (3.79).
2. Let $\mathbf{r}_N = [0\ 0 \cdots 0\ \mathbf{T}^T]^T \in \mathbb{R}^{pN}$, with $N = N^0(x, w, \mathbf{T})$.
3. Let the optimal control policy be given by $\pi_N^0(x, w, \mathbf{T})$, with $N = N^0(x, w, \mathbf{T})$.
4. Apply the control policy $u(k) = \kappa_{N-k}^0(x(k-1), w_{\mathrm{p}}, \mathbf{r}_N)$ for $k = 1, 2, \ldots, N^0(x, w, \mathbf{T})$, where at step k, $w_{\mathrm{p}} = w(k-1)$.

The robust time optimal control problem involves solving either $N_{\max}$ linear programs or $\sum_{N=1}^{N_{\max}} N$ multi-parametric linear programs off-line in step 1 above. Steps 2–4 are implemented on-line.

3.4 Computational complexity

3.4.1 Worst-case computations

The main drawback of the min-max optimization problems described in Sections 3.3.2–3.3.4 is the computational complexity. Although the open-loop control problem (3.46) can be recast as a linear program (3.54) with $Nm+1$ variables, the number of inequalities that describe the feasible set in this linear program is $|\mathcal{I}| + n_F$, which, in the worst case, may be very large, i.e. in the worst case we have:

$$|\mathcal{I}| \geq |\mathcal{J}| \prod_{i=1}^{N}(n^i + n^{i-1}m + \ldots + m)^n \geq |\mathcal{J}|(n+m)^{Nn}$$

$$n_F \geq \sum_{i=1}^{N} n_i(n^i + n^{i-1}m + \ldots + m)^n \geq n^{nN}.$$

In the disturbance feedback approach (3.61), we still have to solve a linear program (3.65), as in the open-loop case, but the improvement in performance and feasibility compared to the open-loop case is obtained at the expense of introducing $N(N-1)mq/2 + Nn_F n_\Omega + |\mathcal{I}|Nn_\Omega$ extra variables and $n_F + |\mathcal{I}|$ extra inequalities.

For the state feedback approach (3.68) the solution is computed off-line, but the number of regions generated by the multi-parametric linear programming algorithm is also, in the worst case, exponential (see also Section 5.3 for more details about complexity of multi-parametric linear programming algorithms). In the next section we show that the computational complexity of the three min-max control problems considered in Sections 3.3.2–3.3.4 can be reduced significantly if the disturbance set has a certain description.

3.4.2 "Deterministic" min-max control

Now we assume a particular description of the uncertainty for an MPL system. From example (3.39) we have seen that the system matrices of an MPL system (3.40)–(3.41) depend on the consecutive disturbances $w(k-1)$ and $w(k)$. However, there are situations when the system matrices depend only on the disturbance $w(k)$. One possibility is described next. We could redefine the uncertainty as

$$w(k) := [p_1(k-1) \dots p_l(k-1)\ p_1(k) \dots p_l(k)\ t_1(k) \dots t_{\tilde{l}}(k)]^T, \tag{3.80}$$

but in this case we introduce some conservatism since we do not take into account that some components of $w(k-1)$ and $w(k)$ coincide.

A second possibility is the following. Note that in the context of MPL systems, the uncertainty comes from the parameters p_i and t_i. Moreover, only the parameters p_i depend on $k-1$. So. let us consider the situation where the parameters p_i are known and only the parameters t_i are uncertain. In this case the uncertainty vector becomes

$$w(k) := [t_1(k) \cdots t_q(k)]^T. \tag{3.81}$$

In these two situations it follows that the MPL system (3.40)–(3.41) can be rewritten as

$$\begin{aligned} x(k+1) &= A(w(k)) \otimes x(k) \oplus B(w(k)) \otimes u(k) \\ y(k) &= C(w(k)) \otimes x(k). \end{aligned} \tag{3.82}$$

Moreover, we assume that there exists a $\overline{w} \in W$ such that

$$A(w) \leq \overline{A},\ B(w) \leq \overline{B},\ C(w) \leq \overline{C}\ \ \forall w \in W, \tag{3.83}$$

where $\overline{A} := A(\overline{w}),\ \overline{B} := B(\overline{w}),\ \overline{C} := C(\overline{w})$.

Under the previous two situations described above, the inequalities in (3.83) typically hold since the parameters p_i and t_j denote processing times and transportation times and thus we can assume that each of them varies in some intervals: $p_i \in [\underline{p}_i\ \overline{p}_i]$ and $t_j \in [\underline{t}_j\ \overline{t}_j]$. Then, the uncertainty set W is given by a box in $\mathbb{R}^q$, $W := [\underline{w}\ \ \overline{w}]$, where

$$W = \left([\underline{p}_1\ \overline{p}_1] \times \cdots \times [\underline{p}_l\ \overline{p}_l]\right)^2 \times [\underline{t}_1\ \overline{t}_1] \times \dots \times [\underline{t}_{\tilde{l}}\ \overline{t}_{\tilde{l}}]$$

corresponds to the case (3.80), and where

$$W = [\underline{t}_1\ \overline{t}_1] \times \dots \times [\underline{t}_q\ \overline{t}_q]$$

corresponds to the case (3.81). Moreover, the entries of the system matrices corresponding to an MPL system are given by sums or maximization of processing times p_i and transportation times t_j and thus the entries of matrices A, B and C are max-plus-nonnegative-scaling functions:

$$A_{ij},\ B_{il},\ C_{kl} \in \mathcal{F}^+_{\text{mps}} \ \ \forall\, i, j, l, k, \tag{3.84}$$

i.e. each entry is a function defined as $w \mapsto \max_j\{\xi_j^T w + \eta_j\}$, where ξ_j are vectors with entries 0 and 1 (and thus $\xi_j \geq 0$) and $\eta_j \geq 0$. Since for any vector $\xi \geq 0$, it follows that $\xi^T \underline{w} \leq \xi^T w \leq \xi^T \overline{w}$ for all $w \in W$ $(= [\underline{w}\ \ \overline{w}])$, we can conclude that the inequalities (3.83) hold. Note that interval transfer functions for DES were also considered in [93] in an input-output framework.

We will show in the sequel a quite interesting result, namely that under the previous hypothesis (i.e. we assume that (3.82), (3.83) and (3.84) are valid) the finite-horizon min-max control problems discussed in Sections 3.3.2–3.3.4 reduce to an optimal control problem for a particular deterministic MPL system. It is straightforward to show that the following inequality holds in the max-plus algebra:

$$C_1 \leq D_1,\ C_2 \leq D_2 \ \ \Rightarrow \ \ C_1 \otimes C_2 \leq D_1 \otimes D_2, \tag{3.85}$$

for any matrices C_1, C_2, D_1 and D_2 of appropriate dimensions.

First let us consider the open-loop min-max case from Section 3.3.2. For an uncertain MPL system in the form (3.82), we do not have dependence on w anymore (e.g. $\Theta(w, \mathbf{w})$ becomes in this new settings $\Theta(\mathbf{w})$, etc). Let us define $\overline{\mathbf{w}} := [\overline{w}^T \ldots \overline{w}^T]^T \in \mathcal{W}$, $\overline{\Theta} := \Theta(\overline{\mathbf{w}})$ and[7] $\overline{\Phi} := \Phi(\overline{\mathbf{w}})$. From (3.85) it follows that

$$\Theta(\mathbf{w}) \otimes x \oplus \Phi(\mathbf{w}) \otimes \mathbf{u} \leq \Theta(\overline{\mathbf{w}}) \otimes x \oplus \Phi(\overline{\mathbf{w}}) \otimes \mathbf{u} = \overline{\Theta} \otimes x \oplus \overline{\Phi} \otimes \mathbf{u} \ \ \forall \mathbf{w} \in \mathcal{W}.$$

Since $\mathbf{H} \geq 0$, it follows from (3.49) that

$$\Pi_N^{\text{ol}}(x, \mathbf{r}) = \{\mathbf{u} :\ \mathbf{H}(\overline{\Theta} \otimes x \oplus \overline{\Phi} \otimes \mathbf{u}) + \mathbf{G}\mathbf{u} + \mathbf{F}\mathbf{r} \leq \mathbf{h}\},$$

which coincides with the set of feasible input sequences over the horizon N corresponding to the deterministic MPL system

$$\overline{x}(k+1) = \overline{A} \otimes \overline{x}(k) \oplus \overline{B} \otimes u(k), \quad \overline{y}(k) = \overline{C} \otimes \overline{x}(k). \tag{3.86}$$

Moreover, since in this section we assume that (3.84) hold, then according to Remark 3.3.3 it follows that $\alpha_j, \gamma_j \geq 0$ in (3.51) for all j. From (3.53) we conclude that

$$J_N(x, \mathbf{r}, \mathbf{u}) = V_N(x, \mathbf{r}, \mathbf{u}, \overline{\mathbf{w}}). \tag{3.87}$$

We now consider an optimal control problem for the deterministic system (3.86) over an horizon window of length N:

$$\mathbb{P}_N^{\text{upper}}(x, \mathbf{r}) : \qquad V_N^{0,\text{upper}}(x, \mathbf{r}) := \inf_{\mathbf{u} \in \Pi_N^{\text{ol}}(x, \mathbf{r})} V_N(x, \mathbf{u}, \mathbf{r}, \overline{\mathbf{w}}), \tag{3.88}$$

with a minimizer $\mathbf{u}_N^0(x, \mathbf{r})$ whenever the infimum is attained. From the previous discussion it follows that:

Lemma 3.4.1 *Suppose that* (3.82), (3.83) *and* (3.84) *hold. Then, the open-loop min-max control problem $\mathbb{P}_N^{\text{ol}}(x, \mathbf{r})$ is equivalent with the deterministic optimal control problem $\mathbb{P}_N^{\text{upper}}(x, \mathbf{r})$ for all $(x, \mathbf{r}) \in X_N^{\text{ol}}$.* ♢

[7] The reader should make distinction between $\overline{\Theta}$ defined above and $\bar{\Theta}$ defined in Section 3.2.5.

Let us now show that the state feedback min-max control problem $\mathbb{P}_N^{\text{sf}}(x, \mathbf{r})$ from Section 3.3.4 is equivalent with the same deterministic optimal control problem $\mathbb{P}_N^{\text{upper}}(x, \mathbf{r})$. Indeed, since $\ell_i(\cdot, u, r) \in \mathcal{F}_{\text{mps}}^+$ and using also Theorem 3.3.15 (iii), it follows that:

$$V_i^0(f_{\text{MPL}}(x, u, w), \mathbf{r}) \leq V_i^0(A(\overline{w}) \otimes x \oplus B(\overline{w}) \otimes u, \mathbf{r}) = V_i^0(\overline{A} \otimes x \oplus \overline{B} \otimes u, \mathbf{r}) \ \forall w \in W$$
$$\ell_i(f_{\text{MPL}}(x, u, w), u, r) \leq \ell_i(A(\overline{w}) \otimes x \oplus B(\overline{w}) \otimes u, u, r) = \ell_i(\overline{A} \otimes x \oplus \overline{B} \otimes u, u, r) \ \forall w \in W.$$

Therefore, $J_i(x, \mathbf{r}, u)$ as defined in (3.70a) is given by:

$$J_i(x, \mathbf{r}, u) = \ell_{N-i}(\overline{A} \otimes x \oplus \overline{B} \otimes u, u, r_{N-i}) + V_{i-1}^0(\overline{A} \otimes x \oplus \overline{B} \otimes u, \mathbf{r})$$

and the corresponding feasible set Z_i reduces to

$$Z_i = \{(x, \mathbf{r}, u) : H_{N-i}(\overline{A} \otimes x \oplus \overline{B} \otimes u) + G_{N-i}u + F_{N-i}r_{N-i} \leq h_{N-i},$$
$$\overline{A} \otimes x \oplus \overline{B} \otimes u \in X_{i-1}\}.$$

The next result follows:

Theorem 3.4.2 *Suppose that* (3.82), (3.83) *and* (3.84) *hold then* $X_N^{\text{ol}} = X_N^{\text{df}} = X_N^{\text{sf}}$ *and the robust control problems considered in Sections 3.3.2–3.3.4, i.e.* $\mathbb{P}_N^{\text{ol}}(x, \mathbf{r})$, $\mathbb{P}_N^{\text{df}}(x, \mathbf{r})$ *and* $\mathbb{P}_N^{\text{sf}}(x, \mathbf{r})$ *are reduced to the optimal control problem* $\mathbb{P}_N^{\text{upper}}(x, \mathbf{r})$ *corresponding to the deterministic system* (3.86) *for each* $(x, \mathbf{r}) \in X_N^{\text{sf}}$.

Proof : From the previous discussion (note that the optimal input sequence of the deterministic optimal control problem (3.88) can also be computed via dynamic programming approach) and using Bellman's principle of optimality for DP [15], it follows that the optimal problems $\mathbb{P}_N^{\text{sf}}(x, \mathbf{r})$ and $\mathbb{P}_N^{\text{upper}}(x, \mathbf{r})$ are equivalent. Therefore, from Lemma 3.4.1 and the inclusions (3.77) it follows that $X_N^{\text{ol}} = X_N^{\text{df}} = X_N^{\text{sf}}$ and robust control problems $\mathbb{P}_N^{\text{ol}}(x, \mathbf{r})$, $\mathbb{P}_N^{\text{sf}}(x, \mathbf{r})$ reduce to $\mathbb{P}_N^{\text{upper}}(x, \mathbf{r})$. Let $\mathbf{u}_N^0(x, \mathbf{r})$ be the optimal solution of these problems for an $(x, \mathbf{r}) \in X_N^{\text{sf}}$. Then, using now the inequalities from (3.77) it follows that the disturbance feedback control problem $\mathbb{P}_N^{\text{df}}(x, \mathbf{r})$ reduces to the same deterministic optimal control problem $\mathbb{P}_N^{\text{upper}}(x, \mathbf{r})$ (an optimal solution for the disturbance feedback approach is $\mathbf{M}_N^{0,\text{df}}(x, \mathbf{r}) = 0$ and $\mathbf{v}_N^{0,\text{df}}(x, \mathbf{r}) = \mathbf{u}_N^0(x, \mathbf{r})$). ♢

Using the relaxation from Theorem 3.2.2, the optimization problem (3.88) can be recast as a linear program with $N_v = Nn + Nm + 1$ variables and $N_c = |\mathcal{J}| + \sum_{i-1}^{N} n_i + n \sum_{i=1}^{N} (im + 1)$ constraints (recall that $|\mathcal{J}|$ is the number of affine terms in (3.51)). Thus there is a significant advantage in solving the linear program (3.88) that has fewer constraints and fewer number of variables than the linear program (3.54).

Remark 3.4.3 We can remark that the results presented in this chapter are in fact valid for a class of systems for which the MPL systems are a subclass, namely the class of max-plus-nonnegative-scaling systems, i.e. systems of the form

$$x(k+1) = f(x(k), u(k)) \quad y(k) = h(x(k)),$$

where $f(\cdot, u) \in (\mathcal{F}_{\text{mps}}^+)^n$ for any fixed u, $f \in \mathcal{F}_{\text{mps}}^n$ and $h \in (\mathcal{F}_{\text{mps}}^+)^p$. In the disturbance case, the results hold when $f(\cdot, u, w) \in (\mathcal{F}_{\text{mps}}^+)^n$ for any fixed (u, w), $f \in \mathcal{F}_{\text{mps}}^n$ and $h \in (\mathcal{F}_{\text{mps}}^+)^p$. ♢

3.5 Conclusions

In this chapter we have extended the finite-horizon optimal control framework to a class of nonlinear DES that models only the synchronization aspects, called MPL systems.

In Section 3.2 we have discussed finite-horizon optimal control for deterministic MPL systems subject to mixed state and input linear inequality constraints. We have provided sufficient conditions such that one can employ results from linear programming to compute an optimal control sequence over a finite horizon. In the unconstrained case and for a particular stage cost we have computed the analytic solution of the corresponding optimization problem, deriving bounds on the design parameters that lead to a just-in-time controller.

In Section 3.3 we have considered robust control for uncertain MPL systems. We have provided solutions to three types of finite-horizon min-max control problems, depending on the nature of the control input over which we optimize: open-loop input sequences, disturbances feedback policies, and state feedback policies. We have assumed that the uncertainty lies in a polytope and the state and input sequences should satisfy a given set of linear inequality constraints. Although the MPL system is nonlinear, we have shown that the open-loop and the disturbance feedback min-max problems can be recast as linear programs while the state feedback min-max problem can be solved exactly, without gridding, via N multi-parametric linear programs, where N is the prediction horizon. The main assumptions that allow us to preserve convexity in the min-max problems that we have considered, were that the stage cost be a max-plus-nonnegative-scaling expression in the state and the matrices associated with the state constraints have nonnegative entries. Finally, for a particular case of the uncertainty description we have proved that all three min-max problems are equivalent with a deterministic one. We have provided also an example illustrating that the performance is improved by including feedback in the min-max control problem.

Note that the finite-horizon optimal control problems presented in this chapter can be solved in a receding horizon fashion resulting in an infinite-horizon controller (also called MPC). The main properties of the MPC will be studied in the next chapter.

Chapter 4

Model predictive control for max-plus-linear systems

In this chapter we extend the conventional MPC framework to the class of discrete event MPL systems. We define the notion of stability (i.e. Lyapunov stability) and of positively invariant set for discrete event MPL systems, and their main features are derived. We provide here sufficient conditions that guarantee a priori stability of the closed-loop system obtained from applying an MPC law based on one of the finite-horizon optimal control problems derived in the previous chapter. We also provide a stabilizing MPC scheme for switching MPL systems.

4.1 Analysis of MPL systems

4.1.1 Introduction

In the previous chapter we have studied the solutions of different finite-horizon optimal control problems for MPL systems. The optimal control sequence, whenever it exists, steers the system towards the terminal set X_{f} in a finite number of steps while also satisfying the performance specifications. We can obtain an infinite-horizon feedback controller using the MPC framework. In MPC the current control action is obtained by solving on-line a finite-horizon optimal control problem. The current state of the plant is used as an initial state in the optimization problem. Only the first input of the optimal control sequence is applied to the system. The remaining optimal inputs are discarded, and the whole procedure is repeated at the next step. The prediction horizon denotes the length of the predictions of the future system behavior. This procedure, called also the *receding horizon* philosophy, defines an implicit (state feedback) MPC law. For more details on this topic the reader is referred to Section 2.3.2.

Relevant topics in the MPC literature are the issues of feasibility and stability of the closed-loop system. In general, for a finite prediction horizon closed-loop stability cannot be guaranteed *a priori* unless additional conditions are imposed. In [101] some tuning rules are given for the prediction horizon and for the weighting factors of the stage cost in order to achieve closed-loop stability. Common for many MPC schemes is the use of two "ingredients" in order to guarantee stability: a terminal set and a terminal cost. In this case the cost function serves as a Lyapunov function for the closed-loop system (according to Section 2.3.2). For an historic overview of different formulations of MPC and of different methods used to prove closed-loop stability a good reference is the survey paper [105].

In this chapter we focus on MPC for MPL systems obtained by repeatedly solving one of the finite-horizon optimal control problems defined in Chapter 3. We first define stability for DES and in particular for MPL systems: stability in terms of boundedness of the buffer levels and Lyapunov stability. Our main concern is to provide sufficient conditions such that we can guarantee a priori closed-loop stability in terms of Lyapunov and/or in terms of boundedness of the normalized state. In Section 4.2 we show that by a proper tuning of the design parameters the unconstrained model predictive controller is asymptotically stabilizing. In Section 4.3 the stabilizing properties of the model predictive controller in the constrained case are enforced through the introduction of an appropriate terminal inequality constraint derived from a positively invariant set and a terminal cost. In Section 4.4 we derive the main properties of a robust MPC scheme for constrained MPL systems with disturbances, in particular robust stability. Finally, in Section 4.5 we provide sufficient conditions for an MPC scheme to guarantee a priori closed-loop stability in terms of boundedness of a switching MPL system. Although stability in terms of boundedness and asymptotic stability are equivalent for linear systems, this equivalence does not hold anymore for MPL systems. In this chapter however, we show that under some additional assumptions both notions of stability hold for the closed-loop MPC. This chapter combines and extends the work of [123, 128, 159].

4.1.2 Stability for MPL systems

In this section we adopt the formulation developed in [4, 133] to the study of stability of MPL systems. We consider the following MPL system:

$$\begin{aligned}\bar{x}(k+1) &= \bar{A}\otimes\bar{x}(k)\oplus\bar{B}\otimes\bar{u}(k)\\ \bar{y}(k) &= \bar{C}\otimes\bar{x}(k).\end{aligned} \tag{4.1}$$

Let λ^* be the largest eigenvalue of $\bar{A}$ (see Section 3.1.1 for an appropriate definition). In classical linear system theory, the asymptotic behavior of the autonomous linear system $z(k+1)=Az(k)$ is characterized by the eigenvalues of the matrix A. A similar interpretation can be given to a max-plus eigenvalue. Let us consider the autonomous system defined over the max-plus algebra $z(k+1)=\bar{A}\otimes z(k)$. Initially, we assume that the matrix $\bar{A}\in\mathbb{R}_\varepsilon^{n\times n}$ is row finite and has the largest max-plus eigenvalue $\lambda^*>\varepsilon$ and the corresponding max-plus eigenvector $v\in\mathbb{R}^n$ (i.e. v is finite). For the initial condition $z(0)=v$ we see that $\phi(k;v)=(\lambda^*)^{\otimes^k}\otimes v=k\lambda^*+v$ (where $\phi(k;z)$ denotes the solution at step k associated to the system of difference equations $z(k+1)=\bar{A}\otimes z(k)$ with the initial state $z(0)=v$). Therefore, $\lim_{k\to\infty}\phi(k;v)/k=\lambda^*$. Moreover, from (3.6) it follows that for each initial state $z\in\mathbb{R}^n$ we have

$$\lim_{k\to\infty}\phi(k;z)/k=\lambda^*.$$

We conclude that λ^* gives the maximum *growth rate* of the system. When v is not finite it can be shown that only some components of the vector $\phi(k;z)/k$ converge towards λ^* (see Theorem 3.17 in [66] for more details).

In order to study stability for MPL systems we consider a particular expression for the reference (due date) signal $\{r(k)\}_{k\geq 0}\subset\mathbb{R}^p$ which the output may be required to "track":

$$r(k)=y_{\mathrm{t}}+\rho k, \tag{4.2}$$

where $y_{\mathrm{t}}\in\mathbb{R}^p$ and $\rho\in\mathbb{R}$. Note that we can consider a more general reference signal $\{r(k)\}_{k\geq 0}$ such that there exists a finite positive integer k_{r} for which $r(k)=y_{\mathrm{t}}+\rho k$ for all $k\geq k_{\mathrm{r}}$. The subsequent derivations remain the same. Since through the term $\bar{B}\otimes\bar{u}$ it is only possible to create delays in the starting times of activities, we should choose the growth rate ρ of the due dates such that it is larger than the maximum growth rate of the system, i.e. $\rho \geq \lambda^*$ (if $\rho < \lambda^*$, then stability for MPL systems, as it will be defined below, is not well-posed).

In conventional system theory *stability* is concerned with boundedness of the states. However, in MPL systems k is an event counter and $\bar{x}_i(k)$ refers to the occurrence time of event i. Therefore, the sequence $\{\bar{x}_i(k)\}_{k\geq 0}$ should always be nondecreasing and usually grows unbounded. We now show that by an appropriate change of coordinates stability for MPL systems can be posed in terms of boundedness of the states.

We assume that $\lambda^* > \varepsilon$ ($\lambda^* = \varepsilon$ does not make sense in practical applications). From Lemma 6.3.8 in [41], it follows that there exists an invertible matrix P in the max-plus algebra such that $[P^{\otimes^{-1}} \otimes \bar{A} \otimes P]_{ij} \leq \lambda^*$ for all $i, j \in \mathbb{N}_{[1,n]}$. Let us consider the following change of coordinates:

$$x(k) \leftarrow P^{\otimes^{-1}} \otimes \bar{x}(k) - \rho k, \;\; y(k) \leftarrow \bar{y}(k) - \rho k, \;\; u(k) \leftarrow \bar{u}(k) - \rho k.$$

Then, the new system matrices become:

$$A \leftarrow P^{\otimes^{-1}} \otimes \bar{A} \otimes P - \rho, \;\; B \leftarrow P^{\otimes^{-1}} \otimes \bar{B}, \;\; C \leftarrow \bar{C} \otimes P.$$

We refer to the new system as the *normalized* MPL system:

$$x(k+1) = A \otimes x(k) \oplus B \otimes u(k) \tag{4.3}$$
$$y(k) = C \otimes x(k). \tag{4.4}$$

Note that in the new coordinates the output should be regulated to the constant target y_{t}. We may assume without loss of generality that B is column-finite and C is row-finite since otherwise the corresponding inputs and outputs can be eliminated from the description model. Let us define the properties of controllability and observability for the MPL system (4.3)–(4.4).

Definition 4.1.1 *The MPL system* (4.3) *is controllable if and only if each state is connected to some input, i.e. the matrix*

$$\Gamma_n := [B \;\; A \otimes B \cdots A^{\otimes^{n-1}} \otimes B]$$

is row-finite. ♢

Note that the definition of controllability can be interpreted as follows: each component of the state can be made arbitrarily large by applying an appropriate controller to the system initially at rest.

Definition 4.1.2 *The system* (4.3)–(4.4) *is observable if and only if each state is connected to some output, i.e. the matrix*

$$\Upsilon_n := [C^T \;\; (C \otimes A)^T \cdots (C \otimes A^{\otimes^{n-1}})^T]^T$$

is column-finite. ♢

One can remark that these definitions are similar with the controllability and observability definitions given in [4, 56]. Moreover, controllability and observability are intrinsic properties of an MPL system and thus they do not depend on the choice of coordinates. Therefore, the normalized MPL system (4.3)–(4.4) is controllable and observable if and only if the original system (4.1) also possesses these properties.

The following assumption will be used throughout this chapter

A3: The MPL system (4.3)–(4.4) is controllable and observable, and $\rho > \lambda^* > \varepsilon$.

Note that in practical applications the assumption **A3** is almost always satisfied because we can choose ρ arbitrarily close to λ^*. Although $\lambda^* = \rho$ is an interesting case from a theoretical point of view, in this thesis we do not discuss it. However, from a practical point of view, due to the presence of disturbances in the plant, it is almost never possible to design a controller such that the controlled system has the growth rate λ^*.

Since $A = P^{\otimes^{-1}} \otimes \bar{A} \otimes P - \rho$ and $[P^{\otimes^{-1}} \otimes \bar{A} \otimes P]_{ij} \leq \lambda^*$ for all $i, j \in \mathbb{N}_{[1,n]}$, it follows from assumption **A3** that the matrix A satisfies:

$$A_{ij} < 0 \quad \forall i, j \in \mathbb{N}_{[1,n]}.$$

Because $A_{ij} < 0$ for all $i, j \in \mathbb{N}_{[1,n]}$, we have $A^* = E_n \oplus A \oplus \cdots \oplus A^{\otimes^{n-1}}$ (according to Lemma 3.1.1 (ii)). Note that for any finite vector u there exists a state equilibrium x corresponding to the normalized MPL system (4.3)–(4.4), i.e. $x = A \otimes x \oplus B \otimes u$, given by $x = A^* \otimes B \otimes u$. Note that x is unique, according to Lemma 3.1.4 (ii), and finite since Γ_n is row-finite. We associate to $y_{\rm t}$ the largest[1] equilibrium pair $(x_{\rm e}, u_{\rm e})$ satisfying $C \otimes x_{\rm e} \leq y_{\rm t}$. From the previous discussion and taking into account that Υ_n is also column-finite it follows that $(x_{\rm e}, u_{\rm e})$ is unique, finite, and given by:

$$u_{\rm e} = (-(C \otimes A^* \otimes B))^T \otimes' y_{\rm t}, \; x_{\rm e} = A^* \otimes B \otimes u_{\rm e}. \tag{4.5}$$

Now we consider a state feedback law $\kappa : \mathbb{R}^n \to \mathbb{R}^m$ (e.g. an MPC law) and the closed-loop system:

$$x(k+1) = A \otimes x(k) \oplus B \otimes \kappa(x(k)) \tag{4.6}$$

$$y(k) = C \otimes x(k), \tag{4.7}$$

i.e. the input at event step k is given by $u(k) = \kappa(x(k))$. Let $\{\phi(k; x, \kappa)\}_{k \geq 0}$ denote the closed-loop state trajectory, i.e. $\phi(k; x, \kappa)$ represents the state solution of (4.6) at event step k when the initial state is x and the feedback law κ is employed. Note that $\phi(0; x, \kappa) = x$. In [4, 133] stability for DES is defined in terms of boundedness of the buffer levels. Given a set $X \subseteq \mathbb{R}^n$ for the closed-loop system (4.6)–(4.7) this requirement can be translated mathematically as follows: for each $x \in X$

$$\|\phi(k; x, \kappa) - x_{\rm e}\|_\infty, \; \|y(k) - y_{\rm t}\|_\infty, \; \|u(k) - u_{\rm e}\|_\infty \tag{4.8}$$

should be bounded for all $k \geq 0$. Let us note that for a controllable and observable system the boundedness of $\{\phi(k; x, \kappa) - x_{\rm e}\}_{k \geq 0}$ is sufficient to guarantee also boundedness of $\{y(k) - y_{\rm t}\}_{k \geq 0}$ and $\{u(k) - u_{\rm e}\}_{k \geq 0}$. Therefore, under the controllability and observability assumption **A3**, the conditions (4.8) can be replaced with: for each $x \in X$

$$\|\phi(k; x, \kappa) - x_{\rm e}\|_\infty \tag{4.9}$$

should be bounded for all $k \geq 0$.

Remark 4.1.3 The stability condition (4.8) (or equivalently (4.9) for a controllable and observable MPL system) can be written for the original system (4.1) as: for each $x \in X$

$$\|\bar{x}(k) - \rho k\|_\infty, \; \|\bar{y}(k) - \rho k\|_\infty, \; \|\bar{u}(k) - \rho k\|_\infty \tag{4.10}$$

are bounded for all $k \geq 0$ (i.e. we have stability in terms of boundedness of the buffer levels as was defined in [4, 133]). ♢

[1] By the largest we mean that any other feasible equilibrium pair (x, u) satisfies $x \leq x_{\rm e}$ and $u \leq u_{\rm e}$.

Recall Definition 2.3.3 of a positively invariant set corresponding to the autonomous system (4.6):

Definition 4.1.4 *The set* $X_{\rm e}$ *is called positively invariant (PI) for the closed-loop system* (4.6) *if for all* $x \in X_{\rm e}$ *it follows that* $\phi(k; x, \kappa) \in X_{\rm e}$ *for all* $k \geq 0$. ♢

The distance from a point x to a set $\mathcal{X}$ induced by the ∞-norm is defined as:

$$d_\infty(x, \mathcal{X}) := \inf_{y \in \mathcal{X}} \|x - y\|_\infty .$$

We now introduce the notion of *Lyapunov stability* for the normalized MPL system (4.6)–(4.7).

Definition 4.1.5 *A PI set* $X_{\rm e}$ *is called stable with respect to the closed-loop system* (4.6)–(4.7) *if for any*[2] $\epsilon > 0$ *there exists a* $\delta > 0$ *such that for all* x *satisfying* $d_\infty(x, X_{\rm e}) \leq \delta$ *we have* $d_\infty(\phi(k; x, \kappa), X_{\rm e}) \leq \epsilon$ *for all* $k \geq 0$.

If $\lim_{k\to\infty} d_\infty(\phi(k; x, \kappa), X_{\rm e}) = 0$ *for all* $x \in X$*, then the set* $X_{\rm e}$ *is asymptotically attractive with respect to the closed-loop system* (4.6)–(4.7) *with a region of attraction* X.

When both conditions are satisfied we refer to $X_{\rm e}$ *as asymptotically stable with respect to the system* (4.6)–(4.7) *with a region of attraction* X*. When the convergence is attained in a finite number of steps we refer to* $X_{\rm e}$ *as finitely stable.* ♢

In the sequel we will study different MPC strategies for normalized MPL systems corresponding to one of the finite-horizon optimal control problems defined in Chapter 3. We will derive sufficient conditions that guarantee a priori closed-loop stability either in terms of Lyapunov and/or in terms of boundedness of the normalized state (as defined in (4.9)). It is well-known that stability in terms of boundedness and asymptotic stability are equivalent for linear systems. However, this equivalence does not hold anymore for MPL systems, but in this chapter we show that under some additional assumptions both notions of stability hold for the closed-loop MPC.

4.2 MPC for unconstrained MPL systems

First, we study the receding horizon implementation of the finite-horizon optimal control problem defined in Section 3.2.5. In this case we will prove that by appropriately tuning the design parameters the set $X_{\rm e} = \{x_{\rm e}\}$ is finitely stable with respect to the normalized MPL system (4.3)–(4.4) in closed-loop with the corresponding MPC law.

4.2.1 Problem formulation

We consider the following setting. Assumption **A3** holds and thus the normalized system

$$x(k+1) = A \otimes x(k) \oplus B \otimes u(k) \tag{4.11}$$

$$y(k) = C \otimes x(k) \tag{4.12}$$

satisfies $A_{ij} < 0$ for all $i, j \in \mathbb{N}_{[1,n]}$, the matrices Γ_n and Υ_n are row-finite and column-finite, respectively (i.e. the system is controllable and observable), and the finite equilibrium pair $(x_{\rm e}, u_{\rm e})$ is given by (4.5). The reference signal corresponding to the normalized system is constant, i.e. $y_{\rm t}$.

[2] The reader should make distinction between $\varepsilon = -\infty$ and ϵ used in stability contexts.

Although in general the MPC framework allows us to deal with state and input constraints, in this section we consider an unconstrained formulation of the MPC. The main advantage of the MPC scheme derived in this section compared to most of the results on optimal control and MPC for MPL systems [4,33,36,45,85,95,109] is the fact that we guarantee *a priori* closed-loop stability in terms of Lyapunov and in terms of boundedness of the buffer levels. We recall that in the absence of constraints by making the transformation given in Remark 3.2.5 the constraint (3.11), which expresses that the input should be nondecreasing, is satisfied automatically. Therefore, the MPC scheme derived below provides also a physical control sequence while most of the optimal control schemes based on residuation and input-output models found in the literature may provide a non-increasing control sequence.

We derive sufficient conditions such that one can employ results from max-plus algebra to compute a stabilizing model predictive controller for MPL systems. The usual approach for proving stability of the closed-loop MPC is to use a terminal cost and a terminal set such that the optimal cost is employed as a Lyapunov function (see Section 2.3.2). In this section however, we do not follow this classical proof for stability, but rather by taking advantage of the special properties, especially monotonicity, that MPL systems possess, we show that by a proper tuning of the MPC design parameters stability can still be guaranteed even in a finite number of event steps.

At event pair (k, x) (i.e. $x(k) = x$) we consider the optimization problem (3.33):

$$V_N^0(x) = \inf_{\mathbf{u}\in\bar{\mathbb{R}}^{Nm}} V_N(x, \mathbf{r}, \mathbf{u}), \tag{4.13}$$

where now the reference sequence has a particular expression, i.e. $\mathbf{r} = \mathbf{x}_\mathrm{e}$ with $\mathbf{x}_\mathrm{e}$ defined as

$$\mathbf{x}_\mathrm{e} := [x_\mathrm{e}^T \ x_\mathrm{e}^T \cdots x_\mathrm{e}^T]^T \in \mathbb{R}^{n(N+1)},$$

the cost function is given by

$$V_N(x, \mathbf{r}, \mathbf{u}) := \sum_{i=0}^{N-1}\left(\sum_{j=1}^{n}\max\{[x_i - x_\mathrm{e}]_j, 0\} - \gamma\sum_{j=1}^{m}[u_i]_j\right) + \sum_{j=1}^{n}\max\{[x_N - x_\mathrm{e}]_j, 0\},$$

and $x_i = \phi(i; x, \mathbf{u})$. Recall that $\phi(i; x, \mathbf{u})$ denotes the state solution of (4.11) at event step i when the initial condition is x and the control sequence $\mathbf{u} = [u_0^T \ u_1^T \cdots u_{N-1}^T]^T$ is applied. Clearly, $\phi(0; x, \mathbf{u}) = x$. Let

$$\mathbf{u}_N^0(x) = [(u_0^0(x))^T \ (u_1^0(x))^T \cdots (u_{N-1}^0(x))^T]^T$$

be an optimizer of (4.13) (as defined in (2.18)). If $\gamma > 0$, then clearly $\mathbf{u}_N^0(x) > \mathcal{E}$. The MPC law is given by

$$\kappa_N(x) := u_0^0(x) \tag{4.14}$$

and the closed-loop system becomes

$$x(k+1) = A \otimes x(k) \oplus B \otimes \kappa_N(x(k)), \tag{4.15}$$

$$y(k) = C \otimes x(k). \tag{4.16}$$

Since $\mathbf{r} = \mathbf{x}_\mathrm{e}$ at each event step k, we drop out the dependence on $\mathbf{r}$ of V_N^0, etc in order to simplify the notation in this section.

Note that we assume that at event step k the state $x(k)$ is available or can be measured. In Sections 3.2.3 and 3.3.5 we have provided some procedures to compute $x(k)$ at a certain time t_0 when the MPC optimization problem is performed. The reader might ask how to determine the next time instant when a new MPC optimization should be done. In principle, the appropriate input sequence **u** should be recomputed as soon as a new measurement of the state $x_i(k)$ becomes available or new information arrives. We can even refine this last statement in the following sense: if at a certain time a "lot" of new information (data) becomes available then we can stop the optimization problem and we restart a new one which takes into account all this new information available. The new data can be recast as linear inequalities on the input which thus fits the framework presented in this chapter. However, if new information is not available an optimization is superfluous and the already computed input sequence will be optimal.

We can consider also a time-driven approach, i.e. using a sampling time T we compute an optimal control problem at each $t_0 + jT$. If we have multiple inputs, (and so $u(k)$ is a vector), we will activate the i^{th} input at time $u_i(k) = [\kappa_N(x(k))]_i$. Let us assume that at time $t_0 + jT$ we solve an MPC optimization problem and we obtain $u_i(k) = t$ and let us consider an optimization of **u** for the next time, i.e. $t_0 + (j+1)T > t$. An event in the past cannot be changed any more, and so we will do the optimization of **u** subject to an additional equality constraint $[u_0]_i = t$.

We will show in the sequel that under certain conditions the set $X_\mathrm{e} = \{x_\mathrm{e}\}$ is finitely stable with respect to the closed-loop system (4.15)–(4.16).

4.2.2 Feedback controllers

For the normalized MPL system (4.11)–(4.12) we define two feedback controllers and we study their stabilizing properties: a feedback controller

$$\kappa_\mathrm{ub}(x) := (-B^T) \otimes' (A \otimes x \oplus x_\mathrm{e}) \tag{4.17}$$

and a "constant" controller:

$$\kappa_\mathrm{f}(x) := u_\mathrm{e}. \tag{4.18}$$

for all $x \in \mathbb{R}^n$. Note that for the original system the constant controller has the following expression: $\kappa_\mathrm{f}(x(k)) = u_\mathrm{e} + \rho k$. Later on, we will show that under some conditions the MPC law κ_N lies in between these two controllers. Here, $\{\phi(k; x, \kappa_\mathrm{ub})\}_{k\geq 0}$ denotes the closed-loop state trajectory corresponding to the feedback law κ_ub. Similarly, we define $\{\phi(k; x, \kappa_\mathrm{f})\}_{k\geq 0}$.

Lemma 4.2.1 *For all initial states $x \in \mathbb{R}^n$ the following inequalities hold:*

$$\phi(k; x, \kappa_\mathrm{f}) \leq \phi(k; x, \kappa_\mathrm{ub}), \quad \kappa_\mathrm{f}(\phi(k; x, \kappa_\mathrm{f})) \leq \kappa_\mathrm{ub}(\phi(k; x, \kappa_\mathrm{ub})) \quad \forall k \geq 0. \tag{4.19}$$

Proof : We prove the lemma by induction. For $k = 0$ we have that $\phi(k; x, \kappa_\mathrm{f}) = \phi(k; x, \kappa_\mathrm{ub}) = x$ and from the monotonicity property of the min operator (3.7) it follows that $\kappa_\mathrm{ub}(x) \geq (-B^T) \otimes' x_\mathrm{e} \geq u_\mathrm{e} = \kappa_\mathrm{f}(x)$ (the second inequality follows from the definition of u_e given in (4.5)). Let us assume that the inequalities of the lemma are valid for a given k. Now we prove that they also hold for $k+1$. We have $\kappa_\mathrm{ub}(\phi(k+1; x, \kappa_\mathrm{ub})) \geq (-B^T) \otimes' x_\mathrm{e} \geq u_\mathrm{e} = \kappa_\mathrm{f}(\phi(k+1; x, \kappa_\mathrm{f}))$. Moreover, using the induction hypothesis and the monotonicity property (3.7) of the max operator it follows that:

$$\begin{aligned}
\phi(k+1, x, \kappa_\mathrm{ub}) &= A \otimes \phi(k, x, \kappa_\mathrm{ub}) \oplus B \otimes \kappa_\mathrm{ub}(\phi(k; x, \kappa_\mathrm{ub})) \\
&\geq A \otimes \phi(k; x, \kappa_\mathrm{f}) \oplus B \otimes \kappa_\mathrm{ub}(\phi(k; x, \kappa_\mathrm{ub})) \\
&\geq A \otimes \phi(k; x, \kappa_\mathrm{f}) \oplus B \otimes \kappa_\mathrm{f}(\phi(k; x, \kappa_\mathrm{f})) = \phi(k+1, x, \kappa_\mathrm{f}).
\end{aligned}$$

This concludes our proof. ♢

The stabilizing properties of the two controllers defined above are summarized in the next theorem.

Theorem 4.2.2 *Suppose that assumption* **A3** *holds. Then, we have the following statements:*

(i) For any initial state $x \in \mathbb{R}^n$ *there exists a finite positive integer* $k_{\mathrm{f}}(x)$ *such that* $\phi(k; x, \kappa_{\mathrm{f}}) = x_{\mathrm{e}}$ *for all* $k \geq k_{\mathrm{f}}(x)$.

(ii) For any initial state $x \in \mathbb{R}^n$ *there exists a finite positive integer* $k_{\mathrm{ub}}(x)$ *such that* $\phi(k; x, \kappa_{\mathrm{ub}}) = x_{\mathrm{e}}$ *for all* $k \geq k_{\mathrm{ub}}(x)$.

(iii) The set $X_{\mathrm{e}} = \{x_{\mathrm{e}}\}$ *is finitely stable with respect to the closed-loop system* (4.6)–(4.7) *corresponding to the feedback laws* κ_{ub} *and* κ_{f}*, with a region of attraction* $\mathbb{R}^n$*. Moreover, the closed-loop state trajectory is bounded.*

Proof : (i) Note that for all $k \geq 0$ the following equality holds:

$$\phi(k; x, \kappa_{\mathrm{f}}) = A^{\otimes^k} \otimes x \oplus \big(\bigoplus_{t=1}^{k} A^{\otimes^{k-t}} \otimes B \otimes u_{\mathrm{e}} \big).$$

Recall that $A_{ij} < 0$ for all $i, j \in \mathbb{N}_{[1,n]}$ (according to assumption **A3**). Then, from Lemma 3.1.1 (i) it follows that for all $x \in \mathbb{R}^n$:

$$\lim_{k \to \infty} A^{\otimes^k} \otimes x = \mathcal{E}.$$

Since $x_{\mathrm{e}} = \bigoplus_{t=0}^{n} A^{\otimes^{n-t}} \otimes B \otimes u_{\mathrm{e}}$, it follows that there exists a finite $k_{\mathrm{f}}(x) \geq n$ (i.e. the positive integer k_{f} depends on the initial state x) such that $\phi(k; x, \kappa_{\mathrm{f}}) = x_{\mathrm{e}}$ for all $k \geq k_{\mathrm{f}}(x)$. In fact an upper bound on $k_{\mathrm{f}}(x)$ can be determined. Indeed, since $A_{ij} < 0$ for all i, j, then if $k_{\mathrm{f}}(x) \geq pn$ for some finite nonnegative integer p, it follows that $[A^{\otimes^{k_{\mathrm{f}}(x)}}]_{ij}$ is either ε or the largest path from i to j of length $k_{\mathrm{f}}(x)$ contains at least p cycles (see Section 3.1.1 for an appropriate definition of a path and of a cycle). Note that for any cycle $(i_1, i_2, \cdots, i_{k+1})$, where $i_1 = i_{k+1}$, we have $A_{i_1 i_2} + A_{i_2 i_3} + \cdots A_{i_k i_{k+1}} \leq \lambda^* - \rho < 0$. Since $[A^{\otimes^k} \otimes x]_i = \max_{j \in \mathbb{N}_{[1,n]}} \{[A^{\otimes^k}]_{ij} + x_j\}$, it follows that by choosing[3] $p = \lfloor \max_{i,j \in \mathbb{N}_{[1,n]}} \frac{[x_{\mathrm{e}}]_i - x_j}{\lambda^* - \rho} \rfloor$ we get $A^{\otimes^{pn}} \otimes x \leq x_{\mathrm{e}}$ and therefore pn is an upper bound on $k_{\mathrm{f}}(x)$.

(ii) First let us note that $\phi(k+1; x, \kappa_{\mathrm{ub}}) \leq A \otimes \phi(k; x, \kappa_{\mathrm{ub}}) \oplus x_{\mathrm{e}}$ for all $k \geq 0$. By induction it is straightforward to prove that:

$$\phi(k; x, \kappa_{\mathrm{ub}}) \leq A^{\otimes^k} \otimes x \oplus x_{\mathrm{e}} \;\; \forall k \geq 0.$$

Recall that for all $x \in \mathbb{R}^n$, $A^{\otimes^k} \otimes x \to \mathcal{E}$ as $k \to \infty$. Therefore, for any x there exists a finite integer $k'_{\mathrm{ub}}(x)$ such that $A^{\otimes^k} \otimes x \leq x_{\mathrm{e}}$ for any $k \geq k'_{\mathrm{ub}}(x)$. In conclusion, $\phi(k; x, \kappa_{\mathrm{ub}}) \leq x_{\mathrm{e}}$ for all $k \geq k'_{\mathrm{ub}}(x)$. Combining this inequality with first part of the theorem and with Lemma 4.2.1 it follows that there exists a finite $k_{\mathrm{ub}}(x) = \max\{k'_{\mathrm{ub}}(x), k_{\mathrm{f}}(x)\}$ such that $\phi(k; x, \kappa_{\mathrm{ub}}) = x_{\mathrm{e}}$ for all $k \geq k_{\mathrm{ub}}(x)$.

(iii) From (i) and (ii) we conclude that we have finite convergence of the closed-loop state trajectories towards the equilibrium state x_{e}. Let us now prove stability in the sense of Lyapunov. Let $\epsilon > 0$ and consider $\|x - x_{\mathrm{e}}\|_\infty \leq \epsilon$ (i.e. $\delta = \epsilon$).

Note that $x_{\mathrm{e}} = A^{\otimes^k} \otimes x_{\mathrm{e}} \oplus (\bigoplus_{t=1}^{k} A^{\otimes^{k-t}} \otimes B \otimes u_{\mathrm{e}})$, for all $k \geq 1$. The following inequality is an immediate consequence of (3.6): for all $x, y, u, v \in \mathbb{R}^n$

$$\|(A \otimes x \oplus B \otimes u) - (A \otimes y \oplus B \otimes v)\|_\infty \leq \|x - y\|_\infty \oplus \|u - v\|_\infty. \tag{4.20}$$

[3] $\lfloor x \rfloor$ is the largest integer less than or equal to x.

Using (4.20) it follows that

$$\|\phi(k;x,\kappa_{\rm f})-x_{\rm e}\|_\infty \le \|x-x_{\rm e}\|_\infty \le \epsilon \quad \forall k\ge 0. \tag{4.21}$$

Let us define $z_k = A^{\otimes^k}\otimes x$. From $\phi(k;x,\kappa_{\rm f})\le\phi(k;x,\kappa_{\rm ub})\le z_k\oplus x_{\rm e}$ it follows that:

$$\|\phi(k;x,\kappa_{\rm ub})-x_{\rm e}\|_\infty = \max_{i\in\mathbb{N}_{[1,n]}}\{[\phi(k;x,\kappa_{\rm ub})-x_{\rm e}]_i,[x_{\rm e}-\phi(k;x,\kappa_{\rm ub})]_i\}\le$$
$$\max_{i\in\mathbb{N}_{[1,n]}}\{[(z_k\oplus x_{\rm e})-x_{\rm e}]_i,[x_{\rm e}-\phi(k;x,\kappa_{\rm f})]_i\}\le\max_{i\in\mathbb{N}_{[1,n]}}\{[z_k-x_{\rm e}]_i,\epsilon\}\le$$
$$\max_{i\in\mathbb{N}_{[1,n]}}\{[A^{\otimes^k}\otimes x-x_{\rm e}]_i,\epsilon\}\le\max_{i\in\mathbb{N}_{[1,n]}}\{[A^{\otimes^k}\otimes x-A^{\otimes^k}\otimes x_{\rm e}]_i,\epsilon\}\le$$
$$\max_{i\in\mathbb{N}_{[1,n]}}\{[x-x_{\rm e}]_i,\epsilon\}=\epsilon \quad \forall k\ge 0,$$

where for the last inequality we have used the following formula[4]: $a^T\otimes x - a^T\otimes y \le \max_{i\in\mathbb{N}_{[1,n]}}\{x_i-y_i\}$, for any $a\in\mathbb{R}^n_\varepsilon$ and $x,y\in\mathbb{R}^n$.

It follows immediately that both closed-loop systems are also stable in terms of boundedness of the state (i.e. as defined in (4.9)). ♢

An immediate consequence of Theorem 4.2.2 is the following corollary:

Corollary 4.2.3 *Suppose that the state feedback law κ satisfies $\kappa_{\rm f}(\phi(k;x,\kappa_{\rm f})) \le \kappa(\phi(k;x,\kappa)) \le \kappa_{\rm ub}(\phi(k;x,\kappa_{\rm ub}))$ for all $k\ge 0$ and $x\in\mathbb{R}^n$. Then the corresponding closed-loop state trajectory satisfies $\phi(k;x,\kappa_{\rm f})\le\phi(k;x,\kappa)\le\phi(k;x,\kappa_{\rm ub})$ for all $k\ge0$ and $x\in\mathbb{R}^n$, i.e. the set $X_{\rm e}=\{x_{\rm e}\}$ is finitely stable with respect to the corresponding closed-loop system. Moreover, the closed-loop state trajectory is bounded.* ♢

4.2.3 Unconstrained MPC: closed-loop stability

Let us now consider the MPC law $\kappa_N(x) := u_0^0(x)$ defined in (4.14).

Lemma 4.2.4 *Suppose assumption **A3** holds. Then, we have the following inequalities:*

$$\kappa_{\rm f}(\phi(k;x,\kappa_{\rm f}))\le\kappa_N(\phi(k;x,\kappa_N)),\quad \phi(k;x,\kappa_{\rm f})\le\phi(k;x,\kappa_N)$$

for all $k\ge0$ and for all initial state $x\in\mathbb{R}^n$.

Proof: Define

$$\mathbf{u}_{\rm e} := [u_{\rm e}^T\ u_{\rm e}^T\cdots u_{\rm e}^T]^T\in\mathbb{R}^{Nm}.$$

First, let us show that $\mathbf{u}_N^0(x)\ge\mathbf{u}_{\rm e}$ for all $x\in\mathbb{R}^n$. The optimal state trajectory corresponding to $\mathbf{u}_N^0(x)$ is denoted with:

$$\mathbf{x}^0=[x^T\ (x_1^0)^T\cdots(x_N^0)^T]^T.$$

Note that $\mathbf{x}^0=\Theta\otimes x\oplus\Phi\otimes\mathbf{u}_N^0(x)$, where Θ and Φ are defined in (3.27). Let us assume that $\mathbf{u}_N^0(x)\not\ge\mathbf{u}_{\rm e}$. Define $\mathbf{u}^\dagger=\mathbf{u}_N^0(x)\oplus\mathbf{u}_{\rm e}$ and $\mathbf{x}^\dagger=\Theta\otimes x\oplus\Phi\otimes\mathbf{u}^\dagger$. Since $A^{\otimes^j}\otimes B\otimes u_{\rm e}\le x_{\rm e}$ for all $j\ge0$, we have $\mathbf{x}^\dagger=\mathbf{x}^0\oplus\Phi\otimes\mathbf{u}_{\rm e}\le\mathbf{x}^0\oplus\mathbf{x}_{\rm e}$. It follows that:

$$V_N(x,\mathbf{r},\mathbf{u}^\dagger)\le\sum_{j=0}^{N}\sum_{j=1}^{n}\max\{[x_i^0-x_{\rm e}]_j,0\}-\gamma\sum_{i=0}^{N-1}\sum_{j=1}^{m}[u_i^\dagger]_j$$
$$<\sum_{i=0}^{N}\sum_{j=1}^{n}\max\{[x_i^0-x_{\rm e}]_j,0\}-\gamma\sum_{i=0}^{N-1}\sum_{j=1}^{m}[u_i^0(x)]_j=V_N^0(x)$$

[4] Recall that by definition $\varepsilon-\varepsilon=\varepsilon$.

and thus we get contradiction with the optimality of $\mathbf{u}_N^0(x)$. Consequently, $\kappa_N(x) \geq u_{\mathrm{e}}$ for all $x \in \mathbb{R}^n$.

Now we go on with the proof of the lemma using induction. For $k = 0$ we have $\phi(k; x, \kappa_{\mathrm{f}}) = \phi(k; x, \kappa_N) = x$ and $\kappa_N(x) \geq u_{\mathrm{e}} = \kappa_{\mathrm{f}}(x)$. We assume that $\kappa_{\mathrm{f}}(\phi(k; x, \kappa_{\mathrm{f}})) \leq \kappa_N(\phi(k; x, \kappa_N))$ and $\phi(k; x, \kappa_{\mathrm{f}}) \leq \phi(k; x, \kappa_N)$ and we prove that these inequalities also hold for $k+1$. From the first part of the proof it follows that $\kappa(\phi(k+1; x, \kappa_N)) \geq u_{\mathrm{e}} = \kappa_{\mathrm{f}}(\phi(k+1; x, \kappa_{\mathrm{f}}))$. Moreover, from the monotonicity property of the max operator (3.7) and from the induction hypothesis it follows that $\phi(k+1; x, \kappa_{\mathrm{f}}) = A \otimes \phi(k; x, \kappa_{\mathrm{f}}) \oplus B \otimes \kappa_{\mathrm{f}}(\phi(k; x, \kappa_{\mathrm{f}})) \leq A \otimes \phi(k; x, \kappa_N) \oplus B \otimes \kappa_N(\phi(k; x, \kappa_N)) = \phi(k+1; x, \kappa_N)$. $\diamondsuit$

The following corollary is an immediate consequence of Theorem 3.2.4:

Corollary 4.2.5 *: Suppose* $0 < \gamma < \frac{1}{mN}$, *then* $\mathbf{u}_N^0(x) = (-\Phi^T) \otimes' (\Theta \otimes x \oplus \mathbf{x}_{\mathrm{e}})$.

From Corollary 4.2.5 we conclude that the MPC controller is a continuous piecewise affine (or equivalently a max-min-plus-scaling) function of the state: $\kappa_N(x) = \min_{i \in \mathcal{I}} \max_{j \in \mathcal{J}} \{\zeta_{ij} + x_j, \beta_{ij} + (x_{\mathrm{e}})_j\}$, where $\mathcal{I}$ and $\mathcal{J}$ are two finite index sets. It is important to note that although the controlled system is nonlinear, the continuity and piecewise affine properties of our MPC law are similar to the ones corresponding to the linear case [7].

The next theorem characterizes the stabilizing properties of the MPC. Contrary to the conventional MPC where stability is proved using the optimal value cost as a Lyapunov function [7, 101, 105], here the proof is based on the particular properties of the max-plus algebra, especially the monotonicity property (3.7).

Theorem 4.2.6 *Suppose* $0 < \gamma < \frac{1}{mN}$ *and assumption **A3** holds. Then,*
(i) The following inequalities hold:

$$\kappa_{\mathrm{f}}(\phi(k; x, \kappa_{\mathrm{f}})) \leq \kappa_N(\phi(k; x, \kappa_N)) \leq \kappa_{\mathrm{ub}}(\phi(k; x, \kappa_{\mathrm{ub}})) \tag{4.22}$$
$$\phi(k; x, \kappa_{\mathrm{f}}) \leq \phi(k; x, \kappa_N) \leq \phi(k; x, \kappa_{\mathrm{ub}}) \tag{4.23}$$

for all $k \geq 0$ *and* $x \in \mathbb{R}^n$. *Therefore,* $X_{\mathrm{e}} = \{x_{\mathrm{e}}\}$ *is finitely stable with respect to the closed-loop system* (4.15)–(4.16). *Moreover, the closed-loop state trajectory is bounded for each* $x \in \mathbb{R}^n$.

(ii) If $N = 1$, *then* $\kappa_1(x) = \kappa_{\mathrm{ub}}(x)$ *for all* $x \in \mathbb{R}^n$. *For two prediction horizons* $N_1 < N_2$ *the following inequalities hold:*

$$\kappa_{N_1}(\phi(k; x, \kappa_{N_1})) \geq \kappa_{N_2}(\phi(k; x, \kappa_{N_2})), \quad \phi(k; x, \kappa_{N_1}) \geq \phi(k; x, \kappa_{N_2}) \tag{4.24}$$

for all $k \geq 0$ *and for all* $x \in \mathbb{R}^n$.

Proof : (i) From Corollary 4.2.5 it follows that for all x

$$\kappa_N(x) \leq (-B^T) \otimes' (A \otimes x \oplus x_{\mathrm{e}}). \tag{4.25}$$

The left-hand side of inequalities (4.22)–(4.23) follows from Lemma 4.2.4. The right-hand side is proved using induction. For $k = 0$ we have that $\phi(k; x, \kappa_{\mathrm{ub}}) = \phi(k; x, \kappa_N) = x$ and $\kappa_N(x) \leq \kappa_{\mathrm{ub}}(x)$ (according to (4.25)). Let us assume that $\kappa_N(\phi(k-1; x, \kappa_N)) \leq \kappa_{\mathrm{ub}}(\phi(k-1; x, \kappa_{\mathrm{ub}}))$ and $\phi(k; x, \kappa_N) \leq \phi(k; x, \kappa_{\mathrm{ub}})$ are valid and we prove that they also hold for $k+1$. From (4.25) and our induction hypothesis we have:

$$B \otimes \kappa_N(\phi(k; x, \kappa_N)) \leq A \otimes \phi(k; x, \kappa_N) \oplus x_{\mathrm{e}} \leq A \otimes \phi(k; x, \kappa_{\mathrm{ub}}) \oplus x_{\mathrm{e}}.$$

On the other hand, $\kappa_{\rm ub}(\phi(k;x,\kappa_{\rm ub}))$ is the largest solution of the inequality

$$B \otimes u \leq A \otimes \phi(k;x,\kappa_{\rm ub}) \oplus x_{\rm e}.$$

From Lemma 3.1.4 (i) it follows that $\kappa_N(\phi(k;x,\kappa_N)) \leq \kappa_{\rm ub}(\phi(k;x,\kappa_{\rm ub}))$. Moreover, $\phi(k+1;x,\kappa_N) = A\otimes\phi(k;x,\kappa_N)\oplus B\otimes\kappa_N(\phi(k;x,\kappa_N)) \leq A\otimes\phi(k;x,\kappa_{\rm ub})\oplus B\otimes\kappa_{\rm ub}(\phi(k;x,\kappa_{\rm ub})) = \phi(k+1;x,\kappa_{\rm ub})$.

The rest follows from Corollary 4.2.3.

(ii) For $N = 1$ the result follows from Corollary 4.2.5. For two prediction horizons $N_1 < N_2$, we denote with $\Phi_{(N_1)}$, $\Theta_{(N_1)}$ the matrices Φ, Θ, respectively, from (3.27) corresponding to the prediction horizon $N = N_1$. Similarly, we define $\Phi_{(N_2)}, \Theta_{(N_2)}$. Note that $\Phi_{(N_2)} = \begin{bmatrix} \Phi_{(N_1)} & \mathcal{E} \\ * & * \end{bmatrix}$ and $\Theta_{(N_2)} = \begin{bmatrix} \Theta_{(N_1)} \\ * \end{bmatrix}$ (where $*$ stands for appropriate matrix blocks). Define $\mathbf{x}_{\mathrm{e}(N_1)} = [x_{\rm e}^T \cdots x_{\rm e}^T]^T \in \mathbb{R}^{n(N_1+1)}$ and the optimizer of (4.13) corresponding to the prediction horizon $N = N_1$ as $\mathbf{u}^0_{N_1}(x) = [(u^0_{0,(N_1)}(x))^T \cdots (u^0_{N_1-1,(N_1)}(x))^T]^T$. Similarly, we define $\mathbf{x}_{\mathrm{e}(N_2)}$ and $\mathbf{u}^0_{N_2}(x)$.

We prove the inequalities (4.24) by induction. For $k = 0$ we have $\phi(k;x,\kappa_{N_1}) = \phi(k;x,\kappa_{N_2}) = x$. From Corollary 4.2.5 it follows that:

$$\Phi_{(N_2)} \otimes \mathbf{u}^0_{(N_2)}(x) = \begin{bmatrix} \Phi_{(N_1)} & \mathcal{E} \\ * & * \end{bmatrix} \otimes \begin{bmatrix} [(u^0_{0,(N_2)}(x))^T \cdots (u^0_{N_1-1,(N_2)}(x))^T]^T \\ * \end{bmatrix} \leq$$

$$\Theta_{(N_2)} \otimes x \oplus \mathbf{x}_{\mathrm{e}(N_2)} \leq \begin{bmatrix} \Theta_{(N_1)} \otimes x \oplus \mathbf{x}_{\mathrm{e}(N_1)} \\ * \end{bmatrix}.$$

It follows that $\Phi_{(N_1)} \otimes [(u^0_{0,(N_2)}(x))^T \cdots (u^0_{N_1-1,(N_2)}(x))^T]^T \leq \Theta_{(N_1)} \otimes x \oplus \mathbf{x}_{\mathrm{e}(N_1)}$ and thus $[(u^0_{0,(N_2)}(x))^T \cdots (u^0_{N_1-1,(N_2)}(x))^T]^T \leq \mathbf{u}^0_{(N_1)}(x)$. Therefore, $u^0_{0,(N_2)}(x) = \kappa_{N_2}(x) \leq \kappa_{N_1}(x)$. Using exactly the same reasoning we can prove that the inequalities (4.24) hold for all $k \geq 0$. $\diamondsuit$

In this section we have considered MPC for state regulation. In the next section we extend these results to output regulation.

4.2.4 Output regulation

In the case of output regulation for normalized MPL systems, the goal is to bring the output as close as possible to the desired target $y_{\rm t}$. We discuss two options that allow us to achieve this goal. In the first option we can solve the optimal control problem (3.33) in a receding horizon fashion: at event pair (k, x) we solve

$$V_N^0(x) = \inf_{\mathbf{u}\in\bar{\mathbb{R}}^{Nm}} V_N(x,\mathbf{r},\mathbf{u}), \tag{4.26}$$

where now $\mathbf{r} = \mathbf{y}_{\rm t}$ and $\mathbf{y}_{\rm t} = [y_{\rm t}^T\ y_{\rm t}^T \cdots y_{\rm t}^T]^T$. This means that the cost function has the following form

$$V_N(x,\mathbf{r},\mathbf{u}) = \sum_{i=0}^{N-1}\left(\sum_{j=1}^{p}\max\{[y_i - y_{\rm t}]_j, 0\} - \gamma\sum_{j=1}^{m}[u_i]_j\right) + \sum_{j=1}^{p}\max\{[y_N - y_{\rm t}]_j, 0\},$$

where $y_i = C \otimes x_i$ (recall that $x_i = \phi(i;x,\mathbf{u})$). However, in this case stability cannot be guaranteed a priori (see Example 4.2.5 presented at the end of this section). Therefore, we will present in the sequel another approach to bring the system as close as possible to the desired target $y_{\rm t}$ that also guarantees a priori stability of the closed-loop system.

First let us prove that by regulating the output to the desired target we are also effectively regulating the state towards the corresponding equilibrium. Let $\mathbf{u}_N^0(x)$ be the optimizer of (4.26). Using similar arguments as in the proof of Lemma 4.2.4 we can prove that $\mathbf{u}_N^0(x) \geq \mathbf{u}_{\rm e}$ for all $x \in \mathbb{R}^n$. Note that we use the same notation as in the previous section (e.g. $\kappa_N(x) = u_0^0(x)$, etc.). It follows that for all $x \in \mathbb{R}^n$

$$\kappa_N(x) \geq u_{\rm e}. \tag{4.27}$$

Lemma 4.2.7 *Suppose that for any $x \in \mathbb{R}^n$ there exists a finite $k_0(x)$ such that $C \otimes \phi(k; x, \kappa_N) \leq y_{\rm t}$ for all $k \geq k_0(x)$. Then, there exists a finite $k_0'(x)$ such that $\phi(k; x, \kappa_N) = x_{\rm e}$ for all $k \geq k_0'(x)$.*

Proof: Denote $k_0(x) = k_0$. For all $j \geq 0$ we have:

$$\begin{aligned}
&C \otimes \phi(k_0; x, \kappa_N) \leq y_{\rm t} \\
&C \otimes A \otimes \phi(k_0; x, \kappa_N) \oplus C \otimes B \otimes \kappa_N(\phi(k_0; x, \kappa_N)) \leq y_{\rm t} \\
&C \otimes A^{\otimes^2} \otimes \phi(k_0; x, \kappa_N) \oplus C \otimes A \otimes B \otimes \kappa_N(\phi(k_0; x, \kappa_N)) \oplus \cdots \leq y_{\rm t} \\
&\cdots \\
&C \otimes A^{\otimes^j} \otimes \phi(k_0; x, \kappa_N) \oplus C \otimes A^{\otimes^{j-1}} \otimes B \otimes \kappa_N(\phi(k_0; x, \kappa_N)) \oplus \cdots \leq y_{\rm t}
\end{aligned}$$

So, $\bigoplus_{j \geq 0} C \otimes A^{\otimes^j} \otimes B \otimes \kappa_N(\phi(k_0; x, \kappa_N)) \leq y_{\rm t}$. From the definition of $u_{\rm e}$ given in (4.5) it follows that $\kappa_N(\phi(k_0; x, \kappa_N)) \leq u_{\rm e}$. From (4.27) it follows that $\kappa_N(\phi(k_0; x, \kappa_N)) \geq u_{\rm e}$ and thus $\kappa_N(\phi(k_0; x, \kappa_N)) = u_{\rm e}$. Using similar arguments it can be proved that $\kappa_N(\phi(k_0 + j; x, \kappa_N)) = u_{\rm e}$ for all $j \geq 0$. We can conclude that there exists a finite positive integer $k_0'(x)$ such that $\phi(k; x, \kappa_N) = x_{\rm e}$ for all $k \geq k_0'(x) := k_0(x) + k_{\rm f}(x)$ (where $k_{\rm f}(x)$ is defined as in Theorem 4.2.2 (i)). We remark that once the output is below the desired target $y_{\rm t}$, the state and the input will reach some steady-state (i.e. the equilibrium pair $(x_{\rm e}, u_{\rm e})$). ♢

Therefore, regulating the output towards the desired target is equivalent with regulating the state towards the corresponding equilibrium. As a consequence, we can apply the receding horizon controller from Section 4.2.1 in order to regulate the output to the desired target.

4.2.5 Example

The next example shows us that instability is really an issue when designing controllers for MPL systems. We consider the following example:

$$\bar{x}(k+1) = \begin{bmatrix} \varepsilon & 0 & \varepsilon & 9 \\ 4 & 3 & 4 & 5 \\ 8 & \varepsilon & 2 & 8 \\ 0 & 1 & \varepsilon & \varepsilon \end{bmatrix} \otimes \bar{x}(k) \oplus \begin{bmatrix} 0 \\ 5 \\ 2 \\ 8 \end{bmatrix} \otimes \bar{u}(k), \quad \bar{y}(k) = [6\ 5\ 8\ \varepsilon] \otimes \bar{x}(k). \tag{4.28}$$

Note that using the extended state discussed in Remark 3.2.5 the constraint $\bar{u}(k) - \bar{u}(k+1) \leq 0$ will be satisfied. The largest max-plus eigenvalue of the system matrix $\bar{A}$ of (4.28) is $\lambda^* = 5.25$. We choose the following reference signal $r(k) = 5 + 1.2\lambda^* k$ (i.e. $\rho = 1.2\lambda^*$). The initial

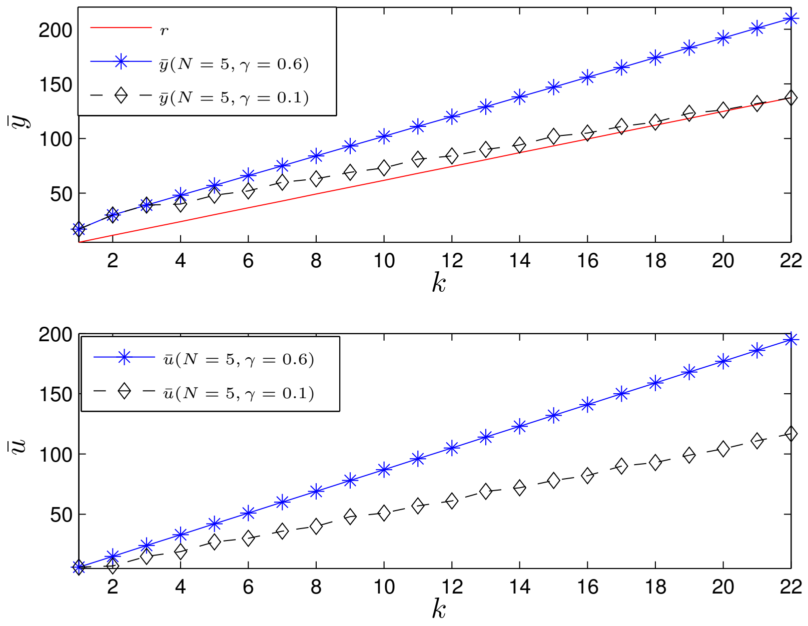

Figure 4.1: The closed-loop MPC simulations: unconstrained case.

conditions are given by $\bar{x}(0) = [6\ 12\ 9\ 14]^T$ and $\bar{u}(-1) = 6$. The closed-loop MPC simulations are given in Figure 4.1.

First, we choose the following values for the MPC design parameters: $N = 5$ and $\gamma = 0.6$. We solve the optimal control problem (4.13) in a receding horizon fashion. The corresponding MPC law makes the closed-loop system *unstable* as we see from the plots (the line marked by stars).

Next, we take $N = 5$ and $\gamma = 0.1$. The conditions from Theorem 4.2.6 are fulfilled. Therefore, solving the optimal control problem (4.13) in a receding horizon fashion, the corresponding MPC law makes the closed-loop system stable (see the line marked by diamonds).

4.3 MPC for constrained MPL systems

In this section we design a stabilizing MPC law for the normalized MPL system (4.11)–(4.12), where the input and state sequence must satisfy a given set of linear inequality constraints. We follow here a similar finite-horizon MPC approach as the one described in Section 2.3.2 for conventional, time-driven nonlinear systems and that uses a terminal set and a terminal cost as basic ingredients. However, the extension from classical time-driven systems to discrete-event MPL systems is not trivial since many concepts from system theory have to be adapted adequately. One of the key results of this section is to provide sufficient conditions based on a terminal set and a terminal cost framework such that one can compute an MPC law that guarantees a priori stability in terms of Lyapunov and in terms of boundedness, and constraint satisfaction for discrete-event MPL systems. The main difference between our approach and other methods that compute optimal controllers for MPL systems [4,33,36,45,85,95,102,109] is that in those papers the optimal controller does not fulfill both requirements: a priori stability of the closed-loop system and the closed-loop input and state sequence should satisfy a given set of inequality constraints.

4.3.1 Problem formulation

We consider the same settings as in Section 4.2. However, we now assume that the normalized MPL system (4.11)–(4.12) is subject to *hard* control and state constraints:

$$Hx(k) + Gu(k) \leq h. \tag{4.29}$$

In the rest of this chapter we also consider that the following assumption holds:

A4: The matrix H in (4.29) is nonnegative, i.e. $H \geq 0$.

Recall that typical constraints for MPL systems (3.9) can be rewritten, after normalization, as in (4.29), with $H \geq 0$. Note that assumption **A4** is a particular case of assumption **A1**. We frequently use the short-hand notation

$$f_{\mathrm{MPL}}(x, u) := A \otimes x \oplus B \otimes u.$$

We may assume that the equilibrium pair $(x_{\mathrm{e}}, u_{\mathrm{e}})$ defined in (4.5) belongs to the set $\{(x, u) : Hx + Gu \leq h\}$. Otherwise, $(x_{\mathrm{e}}, u_{\mathrm{e}})$ is determined as the optimal solution of the following linear programming problem:

$$\max_u \{\sum_{i=1}^m u_i : x = A^* \otimes B \otimes u, C \otimes x \leq y_{\mathrm{t}}, Hx + Gu \leq h\}. \tag{4.30}$$

We now give a lemma that will be used in the sequel:

Lemma 4.3.1 *(i) Let* $\mathcal{X} = \{x \in \mathbb{R}^n : Px \leq q\}$*, where* $P \geq 0$*. Then,*

$$d_\infty(x_0, \mathcal{X}) := \min_{x \in \mathcal{X}} \max_{i \in \mathbb{N}_{[1,n]}} \{[x_0 - x]_i,\ 0\}.$$

(ii) In particular if $\mathcal{X} := \{x \in \mathbb{R}^n : x \leq \zeta\}$*, then*

$$d_\infty(x_0, \mathcal{X}) = \max_{i \in \mathbb{N}_{[1,n]}} \{[x_0 - \ \zeta]_i, 0\}.$$

Proof: (i) It is straightforward to see that the statement is true when $x_0 \in \mathcal{X}$. Therefore, we consider the case when $x_0 \notin \mathcal{X}$, i.e. $d_\infty(x_0, \mathcal{X}) > 0$. We prove this case by contradiction. Let $x^* \in \mathcal{X}$ be the optimal solution, i.e. $0 < d_\infty(x_0, \mathcal{X}) = \|x_0 - x^*\|_\infty$. We define the set $\mathcal{I} \subseteq \mathbb{N}_{[1,n]}$ as follows: if $i \in \mathcal{I}$, then $\|x_0 - x^*\|_\infty = [x^* - x_0]_i > 0$ and for any $j \in \mathbb{N}_{[1,n]} \setminus \mathcal{I} : \|x_0 - x^*\|_\infty > [x_0 - x^*]_j$; otherwise, if such $\mathcal{I}$ does not exist, then define $\mathcal{I} = \emptyset$.

Assume that $\mathcal{I} \neq \emptyset$. Then, we define x_{feas} as: $[x_{\mathrm{feas}}]_i = [x_0]_i$, if $i \in \mathcal{I}$ and $[x_{\mathrm{feas}}]_i = x_i^*$, if $i \notin \mathcal{I}$. Since $P \geq 0$ and $x_{\mathrm{feas}} \leq x^*, x_{\mathrm{feas}} \neq x^*$, it follows that $x_{\mathrm{feas}} \in \mathcal{X}$. Moreover, $0 < d_\infty(x_0, \mathcal{X}) = \|x_0 - x^*\|_\infty = \max_{i \in \mathbb{N}_{[1,n]}} \{[x^* - x_0]_i, [x_0 - x^*]_i\} \leq \|x_{\mathrm{feas}} - x_0\|_\infty = \max_{i \notin \mathcal{I}} \{[x^* - x_0]_i, [x_0 - x^*]_i, 0\} < \max_{i \in \mathbb{N}_{[1,n]}} \{[x^* - x_0]_i, [x_0 - x^*]_i\} = \|x_0 - x^*\|_\infty$ i.e. we get a contradiction. Therefore, $\mathcal{I} = \emptyset$ and thus $\|x_0 - x^*\|_\infty = \max_{i \in \mathbb{N}_{[1,n]}} \{[x_0 - x^*]_i\}$.

(ii) If $x_0 \notin \mathcal{X}$ and $x \leq \zeta$, the following inequality is valid: $\max_{i \in \mathbb{N}_{[1,n]}} \{[x_0 - x]_i\} \geq \max_{i \in \mathbb{N}_{[1,n]}} \{[x_0 - \zeta]_i\}$. We conclude that $\min_{x \in \mathcal{X}} \max_{i \in \mathbb{N}_{[1,n]}} \{[x_0 - x]_i\} \geq \max_{i \in \mathbb{N}_{[1,n]}} \{[x_0 - \zeta]_i\}$. From (i) it follows that $d_\infty(x_0, \mathcal{X}) \geq \max_{i \in \mathbb{N}_{[1,n]}} \{[x_0 - \zeta]_i\} = \max_{i \in \mathbb{N}_{[1,n]}} \{[x_0 - \zeta]_i\}$ (according to the first part of this lemma). But $d_\infty(x_0, \mathcal{X}) \leq \max_{i \in \mathbb{N}_{[1,n]}} \{[x_0 - \zeta]_i\}$ since $\zeta \in \mathcal{X}$. It follows that $d_\infty(x_0, \mathcal{X}) = \max_{i \in \mathbb{N}_{[1,n]}} \{[x_0 - \zeta]_i\}$. ♢

Let X_f be an appropriate terminal set. Note that in the next section we provide a method to construct such a set. For a given set X_e such that $x_\mathrm{e} \in X_\mathrm{e} \subseteq X_\mathrm{f}$ we define a continuous stage cost $\ell(x,u,r)$, where now $r = x_\mathrm{e}$, with the following properties:

$\mathcal{P}1$: $\ell(x,u,r) = 0$ if and only if $x \in X_\mathrm{e}$ and $u = u_\mathrm{e}$.

$\mathcal{P}2$: $\ell(x,u,r) \geq \alpha(d_\infty(x, X_\mathrm{e}))$ for all x, where α is a $\mathscr{K}$ function.

Some examples of such stage costs are:

$$\ell(x,u,r) = \|x - x_\mathrm{e}\|_\infty + \gamma\|u - u_\mathrm{e}\|_\infty \tag{4.31}$$

$$\ell(x,u,r) = \max_{i\in\mathbb{N}_{[1,n]}} \{x_i - [x_\mathrm{e}]_i, 0\} + \gamma\|u - u_\mathrm{e}\|_\infty \tag{4.32}$$

$$\ell(x,u,r) = d_\infty(x, X_\mathrm{f}) + \gamma\|u - u_\mathrm{e}\|_\infty, \tag{4.33}$$

where $\gamma > 0$. For the stage cost (4.31) $X_\mathrm{e} = \{x_\mathrm{e}\}$, for (4.32) $X_\mathrm{e} = \{x : x \leq x_\mathrm{e}\}$ (according to Lemma 4.3.1 (ii)) and for (4.33) $X_\mathrm{e} = X_\mathrm{f}$. Note that the first term in the stage cost (4.31) penalizes the deviation from the state equilibrium x_e while in the stage costs (4.32)–(4.33) the first term penalizes the tardiness with respect to the boundary of the set X_e. The second term in these stage costs penalizes the deviation from the input equilibrium u_e.

We consider a prediction horizon $N \geq 1$. For event pair (k,x) (i.e. $x(k) = x$) the following optimal control problem is considered:

$$V_N^0(x) := \inf_{\mathbf{u}\in\Pi_N(x)} V_N(x, \mathbf{r}, \mathbf{u}), \tag{4.34}$$

where the reference sequence $\mathbf{r} = \mathbf{x}_\mathrm{e}$ and the set of feasible input sequences is defined as

$$\Pi_N(x) := \{\mathbf{u} : Hx_i + Gu_i \leq h \;\; \forall i \in \mathbb{N}_{[0,N-1]}, x_N \in X_\mathrm{f}\},$$

and where $\mathbf{u} = [u_0^T \; u_1^T \cdots u_{N-1}^T]^T$ and $x_i = \phi(i; x, \mathbf{u})$.

The cost criterion is defined as:

$$V_N(x, \mathbf{r}, \mathbf{u}) = \sum_{i=0}^{N-1} \ell(x_i, u_i, r) + V_\mathrm{f}(x_N, r).$$

The terminal cost is determined as follows:

$$V_\mathrm{f}(x, r) := \sum_{j=1}^{k_\mathrm{f}(x)} \ell(x_j, u_\mathrm{e}, r) \; \forall x \in X_\mathrm{f}$$

with $k_\mathrm{f}(x)$ defined as in the proof of Theorem 4.2.2 *(i)* and $x_j = \phi(j; x, \kappa_\mathrm{f})$. Typically $X_\mathrm{f} \subseteq \{x : x \leq a\}$ (see e.g. Remark 4.3.6) and then an upper bound on $k_\mathrm{f}(x)$ is $k_\mathrm{f}(a)$, where $k_\mathrm{f}(a)$ can be determined as in the proof of Theorem 4.2.2. Note that for the stage cost (4.33) we do not have a terminal cost, i.e. $V_\mathrm{f}(x,r) = 0$ for all $x \in X_\mathrm{f}$. For consistency with the previous chapter the reader should note that $\mathbf{r} = \mathbf{x}_\mathrm{e}$ at each event step k and we drop out the dependence on $\mathbf{x}_\mathrm{e}$ of V_N^0, etc for simplicity in notation.

The optimal control problem (4.34) yields an optimal control sequence $\mathbf{u}_N^0(x) = [(u_0^0(x))^T \; (u_1^0(x))^T \cdots (u_{N-1}^0(x))^T]^T$ and an optimal state trajectory $\mathbf{x}^0 = [x^T \; (x_1^0)^T \cdots (x_N^0)^T]^T$. The first control $u_0^0(x)$ is applied to the system (4.11)–(4.12) (at step k) according to the receding horizon principle. This defines an implicit MPC law $\kappa_N(x) := u_0^0(x)$. Let X_N denote the set of finite initial states for which a feasible input sequence exists, i.e.

$$X_N := \{x \in \mathbb{R}^n : \Pi_N(x) \neq \emptyset\}.$$

4.3.2 Positively invariant (PI) sets for MPL systems

In this section we introduce the notion of PI set for a discrete event MPL system and we derive sufficient conditions that allow us to determine efficiently the maximal PI set. We consider the following closed-loop MPL system:

$$x(k+1) = A \otimes x(k) \oplus B \otimes \kappa_{\mathrm{f}}(x(k)), \quad y(k) = C \otimes x(k). \tag{4.35}$$

We have seen in Theorem 4.2.2 that the set $X_{\mathrm{e}} = \{x_{\mathrm{e}}\}$ is finitely stable with respect to (4.35) with a region of attraction $\mathbb{R}^n$. Recall that $\kappa_{\mathrm{f}}(x) = u_{\mathrm{e}}$ for all $x \in \mathbb{R}^n$.

We define the input-state admissible set associated to the closed-loop system (4.35):

$$\mathcal{O}_0 := \{x \in \mathbb{R}^n : Hx + Gu_{\mathrm{e}} \leq h\}. \tag{4.36}$$

We want to compute the maximal PI set contained in the input-state admissible set $\mathcal{O}_0$. Therefore, we define recursively the sets :

$$\mathcal{O}_k := \{x \in \mathcal{O}_0 : \ f_{\mathrm{MPL}}(x, \kappa_{\mathrm{f}}(x)) \in \mathcal{O}_{k-1}\} \tag{4.37}$$

for all $k \geq 1$. It is trivial to see that $\mathcal{O}_k \subseteq \mathcal{O}_{k-1} \subseteq \cdots \subseteq \mathcal{O}_1 \subseteq O_0$. Then, the limit of $\mathcal{O}_k$ exists and we have

$$\mathcal{O}_\infty := \bigcap_{k \geq 0} \mathcal{O}_k = \lim_{k \to \infty} \mathcal{O}_k. \tag{4.38}$$

By induction we can prove that $x_{\mathrm{e}} \in \mathcal{O}_k$ for all $k \geq 0$ and thus $x_{\mathrm{e}} \in \mathcal{O}_\infty$, i.e. $\mathcal{O}_\infty$ is a non-empty set.

Lemma 4.3.2 *Suppose assumptions **A3** and **A4** hold. Then, the sets $\mathcal{O}_k$ are polyhedra described by*

$$\mathcal{O}_k = \{x \in \mathbb{R}^n : S_k x \leq \nu_k\} \tag{4.39}$$

for some matrix $S_k \geq 0$ and vector ν_k.

Proof: For $k = 0$ the statement holds according to assumption **A4** ($S_0 = H$ and $\nu_0 = h - Gu_{\mathrm{e}}$). Let us assume that $\mathcal{O}_{k-1} = \{x \in \mathbb{R}^n : S_{k-1}x \leq \nu_{k-1}\}$, with $S_{k-1} \geq 0$ and we prove that $\mathcal{O}_k$ has a similar form. Since $A \otimes x \oplus B \otimes u_{\mathrm{e}}$ is a max expression in x, it is straightforward to show that the inequality $S_{k-1} f_{\mathrm{MPL}}(x, \kappa_{\mathrm{f}}(x)) = S_{k-1}(A \otimes x \oplus B \otimes u_{\mathrm{e}}) \leq \nu_{k-1}$ can be rewritten in the form $\bar{S}_k x \leq \bar{\nu}_k$, with $\bar{S}_k \geq 0$. Then, $S_k = [S_{k-1}^T \ \bar{S}_k^T]^T \geq 0$ and $\nu_k = [\nu_{k-1}^T \ \bar{\nu}_k^T]^T$. ♢

From the previous lemma it is clear that the set $\mathcal{O}_\infty$ is convex (it is a countable intersection of polyhedral sets). We derive now conditions when $\mathcal{O}_\infty$ is a polyhedron.

Definition 4.3.3 *The set $\mathcal{O}_\infty$ is finitely determined if there exists a finite positive integer t^* such that $\mathcal{O}_\infty = \mathcal{O}_{t^*}$.* ♢

Proposition 4.3.4 *(i) If there exists a finite positive integer t^* such that $\mathcal{O}_{t^*} = \mathcal{O}_{t^*+1}$, then $\mathcal{O}_\infty$ is finitely determined and thus a polyhedral set.*

(ii) The set $\mathcal{O}_\infty$ is the maximal PI set for (4.35) *contained in $\mathcal{O}_0$.*

Proof : (i) Let us assume that there exists a t^* such that $\mathcal{O}_{t^*} = \mathcal{O}_{t^*+1}$. It is obvious that $\mathcal{O}_{t^*+2} \subseteq \mathcal{O}_{t^*+1}$. Moreover, for any $x \in \mathcal{O}_{t^*+1}$ it follows that $f_{\mathrm{MPL}}(x, \kappa_{\mathrm{f}}(x)) \in \mathcal{O}_{t^*} = \mathcal{O}_{t^*+1}$, i.e. $x \in \mathcal{O}_{t^*+2}$. In conclusion, $\mathcal{O}_{t^*+1} \subseteq \mathcal{O}_{t^*+2}$ and thus $\mathcal{O}_{t^*+2} = \mathcal{O}_{t^*+1} = \mathcal{O}_{t^*}$. Iterating this procedure and using (4.38) we conclude that $\mathcal{O}_\infty = \mathcal{O}_{t^*}$. Since $\mathcal{O}_{t^*}$ is a polyhedron it follows that $\mathcal{O}_\infty$ is also a polyhedron.

(ii) Let $\mathcal{O} \subseteq \mathcal{O}_0$ be a PI set for (4.35) and let $x \in \mathcal{O}$. Then from the definition of a PI set we have $S_0 f_{\mathrm{MPL}}(x, \kappa_{\mathrm{f}}(x)) \leq \nu_0$. This implies that $x \in \mathcal{O}_1$ (according to the recursion (4.37)). Therefore, $\mathcal{O} \subseteq \mathcal{O}_1$. By iterating this procedure we obtain that $\mathcal{O} \subseteq \mathcal{O}_k$ for all $k \geq 0$. In conclusion, for any PI set $\mathcal{O}$ it follows that $\mathcal{O} \subseteq \mathcal{O}_\infty$ and thus $\mathcal{O}_\infty$ is maximal. $\diamondsuit$

From Proposition 4.3.4 we have obtained that if $\mathcal{O}_\infty$ is finitely determined, then $\mathcal{O}_\infty$ is a polyhedron of the form $\mathcal{O}_\infty = \{x \in \mathbb{R}^n : S_\infty x \leq \nu_\infty\}$, where $S_\infty \geq 0$. Now, we give sufficient conditions under which the set $\mathcal{O}_\infty$ is finitely determined. Note that the recursive relation (4.37) can be written equivalently as:

$$\mathcal{O}_k = \{x \in \mathcal{O}_{k-1} : H\phi(k; x, \kappa_{\mathrm{f}}) + G u_{\mathrm{e}} \leq h\}, \tag{4.40}$$

where explicitly $\phi(k; x, \kappa_{\mathrm{f}}) = A^{\otimes^k} \otimes x \oplus A^{\otimes^{k-1}} \otimes B \otimes u_{\mathrm{e}} \oplus \cdots \oplus B \otimes u_{\mathrm{e}}$.

Theorem 4.3.5 *Suppose that there exists a finite positive integer t_0 and a vector $a \in \mathbb{R}^n$ such that $\mathcal{O}_{t_0} \subseteq \{x \in \mathbb{R}^n : x \leq a\}$, and assumptions **A3** and **A4** hold. Then, $\mathcal{O}_\infty$ is finitely determined.*

Proof : Since $A_{ij} < 0$ for all i, j, it follows that for all $x \in \mathbb{R}^n$: $A^{\otimes^k} \otimes x \to \varepsilon$ as $k \to \infty$. Moreover, for any $b \in \mathbb{R}^n$ we have: $b \oplus A \otimes b \oplus \cdots \oplus A^{\otimes^{k+n}} \otimes b = A^* \otimes b$ for all $k \geq 0$. Since $x_{\mathrm{e}} = A^* \otimes B \otimes u_{\mathrm{e}}$ is finite, there exists a finite positive integer $t^* \geq \max\{n, t_0\}$ such that $A^{\otimes^k} \otimes a \leq x_{\mathrm{e}}$ for all $k \geq t^*$. We now show that $\mathcal{O}_{t^*} = \mathcal{O}_{t^*+1}$. Since $\mathcal{O}_{t^*+1} \subseteq \mathcal{O}_{t^*}$, to complete the proof we now show that the other inclusion is also valid, i.e. $\mathcal{O}_{t^*} \subseteq \mathcal{O}_{t^*+1}$.

Let $x \in \mathcal{O}_{t^*} \subseteq \mathcal{O}_{t_0} \subseteq \{x \in \mathbb{R}^n : x \leq a\}$. Then, $A^{\otimes^{t^*+1}} \otimes x \leq A^{\otimes^{t^*+1}} \otimes a \leq x_{\mathrm{e}}$. It follows that: $H(A^{\otimes^{t^*+1}} \otimes x \oplus A^{\otimes^{t^*}} \otimes B \otimes u_{\mathrm{e}} \oplus \cdots \oplus B \otimes u_{\mathrm{e}}) = H(A^{\otimes^{t^*+1}} \otimes x \oplus A^* \otimes B \otimes u_{\mathrm{e}}) = H x_{\mathrm{e}} \leq h - G u_{\mathrm{e}}$, i.e. $x \in \mathcal{O}_{t^*+1}$. The rest follows from Proposition 4.3.4. $\diamondsuit$

Remark 4.3.6 It is often the case (see the constraints (3.9)) that the set $\mathcal{O}_0$ can be written as $\mathcal{O}_0 = \{x \in \mathbb{R}^n : x_i \leq a_i^0, \text{ for } i \in \mathbb{N}_{[1,n]}\}$, where a_i^0 is either a finite number or $+\infty$ (when there are no restrictions on x_i). Then, we can prove that all the sets $\mathcal{O}_k$ can be written in a similar form $\mathcal{O}_k = \{x \in \mathbb{R}^n : x_i \leq a_i^k \;\; \forall i \in \mathbb{N}_{[1,n]}\}$, where a_i^k is either a finite number or $+\infty$ (i.e. every $\mathcal{O}_k$ is described by at most n inequalities).

We prove this by induction. For $k = 0$ this statement is true. Let us assume that $\mathcal{O}_k = \{x \in \mathbb{R}^n : x_i \leq a_i^k \;\; \forall i \in \mathbb{N}_{[1,n]}\}$ and we prove that $\mathcal{O}_{k+1}$ has a similar form. We denote with $a^k = [a_1^k \cdots a_n^k]^T$. From the recursive relation (4.37) we have:

$$\begin{aligned}\mathcal{O}_{k+1} &= \{x \in \mathbb{R}^n : x \leq a^k, A \otimes x \leq a^k\} \\ &= \{x \in \mathbb{R}^n : x \leq a^k, x \leq (-A^T) \otimes' a^k\} = \{x \in \mathbb{R}^n : x \leq a^{k+1}\},\end{aligned}$$

where $a^{k+1} = \min\{a^k, (-A^T) \otimes' a^k\}$. We conclude that $\mathcal{O}_\infty$ is described by at most n inequalities and in fact $\mathcal{O}_\infty = \{x \in \mathbb{R}^n : x \leq a^\infty\}$, where a_i^∞ is either in $\mathbb{R}$ or equal to $+\infty$ for all $i \in \mathbb{N}_{[1,n]}$. $\diamondsuit$

Note that the results obtained in this section concerning the maximal PI set $\mathcal{O}_\infty$ for the MPL system (4.35) are similar to the ones obtained in [57] for conventional, time-driven linear systems.

4.3.3 Constrained MPC: closed-loop stability

The main advantage of MPC is that it can accommodate constraints on inputs and states. In this section it is assumed that the PI set $\mathcal{O}_\infty = \{x \in \mathbb{R}^n : S_\infty x \leq \nu_\infty\}$ is available, where $S_\infty \geq 0$. We also recall that assumptions **A3** and **A4** hold in the rest of this chapter.

The following theorem is a simple extension of Corollary C.1.4 given in Appendix C from an equilibrium point to a PI set X_e.

Theorem 4.3.7 *Let V be a function defined on a PI set X for the closed-loop system* (4.6)–(4.7) *satisfying the following properties:*

(i) $V(x) = 0$ for all $x \in X_\mathrm{e}$, where X_e is a PI set for (4.6)–(4.7) *satisfying $X_\mathrm{e} \subseteq \mathrm{int}(X)$ and V is continuous on a neighborhood of X_e.*

(ii) $V(x) \geq \alpha(d_\infty(x, X_\mathrm{e}))$ for all $x \in X$, where α is a $\mathscr{K}$ function.

(iii) $V(f_\mathrm{MPL}(x, \kappa(x)) - V(x) \leq -\beta(d_\infty(x, X_\mathrm{e}))$ for all $x \in X$, where β is a $\mathscr{K}$ function.

Then, X_e is asymptotically stable with respect to the closed-loop system (4.6)–(4.7) *with a region of attraction X.* ♢

Here, a neighborhood of a set X_e is defined as $\mathcal{N}(X_\mathrm{e}, \delta) := \{x : d_\infty(x, X_\mathrm{e}) < \delta\}$.

We define the terminal set

$$X_\mathrm{f} := \mathcal{O}_\infty.$$

The next theorem shows that the closed-loop system obtained from applying to the MPL system the feedback law derived in Section 4.3.1 enjoys some stabilizing properties.

Theorem 4.3.8 *Suppose that $X_\mathrm{e} \subseteq \mathrm{int}(X_N)$ and assumptions **A3** and **A4** hold.*

(i) The set X_e is asymptotically stable for the closed-loop system (4.15)–(4.16) *with a region of attraction X_N.*

(ii) If there exists an $a \in \mathbb{R}^n$ such that $X_\mathrm{e} \subseteq \{x \in \mathbb{R}^n : x \leq a\}$, then for each $x \in X_N$ the closed-loop state trajectory of the system (4.15)–(4.16) *is bounded.*

Proof: (i) We consider the function $V_N^0 : X_N \to \mathbb{R}$. We will show that V_N^0 satisfies the conditions from Theorem 4.3.7.

First note that the terminal set $X_\mathrm{f} = \mathcal{O}_\infty$ and the local controller $\kappa_\mathrm{f}(x) = u_\mathrm{e}$ for all $x \in X_\mathrm{f}$ satisfy the conditions $\mathcal{F}1$–$\mathcal{F}3$ from Section 2.3.2. From Theorem 2.3.4 it follows that the set X_N is PI for the closed-loop system (4.15). As a consequence we have that for any initial state $x \in X_N$ we can guarantee feasibility of the MPC-MPL optimization problem (4.34) at each step. Note that at the next step a feasible input sequence is given by $\mathbf{u}^\mathrm{f} = [(u_1^0(x))^T \cdots (u_{N-1}^0(x))^T \ \kappa_\mathrm{f}(x_N^0)]^T$.

From the properties of the stage cost $\mathcal{P}1$–$\mathcal{P}2$, convexity of the function f_MPL and linearity of the constraints we can easily see that the first two conditions from Theorem 4.3.7 are satisfied by V_N^0 (in particular continuity of V_N^0 for the stage cost (4.31) follows from (4.20) while for the stage costs (4.32) and (4.33) continuity of V_N^0 follows from multi-parametric linear programming arguments). It remains to prove the third condition:

$$\begin{aligned} V_N^0(f_\mathrm{MPL}(x, \kappa_N(x))) - V_N^0(x) \leq V_N(f_\mathrm{MPL}(x, \kappa_N(x)), \mathbf{r}, \mathbf{u}^\mathrm{f}) - V_N^0(x) = \\ - \ell(x, \kappa_N(x), r) \leq -\alpha(d_\infty(x, X_\mathrm{e})) \end{aligned} \quad (4.41)$$

according to the property $\mathcal{P}2$ of the stage cost. We obtain that the conditions from Theorem 4.3.7 are satisfied. Therefore, X_e is asymptotically stable for (4.15)–(4.16) with a region of attraction X_N.

(ii) For any finite initial state $x \in X_N$, from (4.41) it follows that the sequence $\{V_N^0(\phi(k;x,\kappa_N))\}_{k\geq 0}$ is non-increasing and bounded from below and thus convergent. Moreover, $\ell(\phi(k;x,\kappa_N),\kappa_N(\phi(k;x,\kappa_N)),r) \leq V_N^0(\phi(k;x,\kappa_N)) - V_N^0(\phi(k+1;x,\kappa_N))$. Therefore, $\lim_{k\to\infty} \ell(\phi(k;x,\kappa_N),\kappa_N(\phi(k;x,\kappa_N)),r) = 0$. Using continuity arguments and the property $\mathcal{P}1$ of the stage cost we conclude that

$$\lim_{k\to\infty} \kappa_N(\phi(k;x,\kappa_N)) = u_{\mathrm{e}} \tag{4.42}$$

$$\lim_{k\to\infty} d_\infty(\phi(k;x,\kappa_N),X_{\mathrm{e}}) = 0. \tag{4.43}$$

Since the system is controllable and observable (according to assumption **A3**), we cannot have finite escape, i.e. there does not exist a finite k_0 such that either $\phi(k_0;x,\kappa_N)$ or $\kappa_N(\phi(k_0;x,\kappa_N))$ or $y(k_0) = C \otimes \phi(k_0;x,\kappa_N)$ take infinite values. From (4.43) it follows that the sequence $\{d_\infty(\phi(k;x,\kappa_N),X_{\mathrm{e}})\}_{k\geq 0}$ is bounded for each finite initial state $x \in X_N$. If the set X_{e} is bounded (e.g. $X_{\mathrm{e}} = \{x_{\mathrm{e}}\}$ in (4.31)), then $\|\phi(k;x,\kappa_N) - x_{\mathrm{e}}\|_\infty$ is also bounded for all $k \geq 0$ (it follows from triangle inequality) and therefore the buffer levels remain bounded.

If X_{e} is not bounded, then from (4.42) we conclude that for any finite initial state $x \in X_N$ there exists a finite lower bound $\underline{u}(x)$ such that $\kappa_N(\phi(k;x,\kappa_N)) \geq \underline{u}(x)$ for all $k \geq 0$. From the monotonicity property of the max operator (3.7) it follows that there exists a finite lower bound[5] on the corresponding state trajectory $\phi(k;x,\kappa_N) \geq m(x)$ for all $k \geq 0$. Since $X_{\mathrm{e}} \subseteq \{x \in \mathbb{R}^n : x \leq a\}$, it follows that the set $X_{\mathrm{e}} \cap \{z : z \geq m(x)\}$ is bounded and then using the same arguments as before we conclude that $\|\phi(k;x,\kappa_N) - x_{\mathrm{e}}\|_\infty$ is also bounded for all $k \geq 0$ and so the buffer levels remain bounded. ◊

Remark 4.3.9 (i) For the constraints (4.29), $H \geq 0$, and for the terminal set X_{f}, $S_\infty \geq 0$, it follows that $\Pi_N(x)$ and X_N are polyhedra (according to Section 3.2.2). From Lemma 4.3.1 it follows that the stage costs (4.32) and (4.33) satisfy assumption **A2** and thus the optimization problem (4.34) can be recast as a *linear program* (according to Theorem 3.2.2). For the stage cost (4.31) the optimization problem (4.34) can be recast as a mixed-integer linear program.

(ii) If $X_{\mathrm{e}} \subset \mathrm{int}(X_{\mathrm{f}})$ then from (4.43) it follows that the trajectory enters the terminal set X_{f} in a finite number of steps. Inside X_{f}, we can use the feasible local controller $\kappa_{\mathrm{f}}(x) = u_{\mathrm{e}}$ (since X_{f} is a PI set for the system (4.35)) and so we can steer the trajectory towards the equilibrium x_{e} in finite number of steps as well (see Theorem 4.2.2). In conclusion, using such a dual-mode approach [105], we can guarantee that for any finite initial state $x \in X_N$ the trajectory reaches the steady-state in finite number of steps.

(iii) Note that by increasing the prediction horizon N, the region of attraction increases as well, i.e. for $N_1 < N_2$ it follows that $X_{N_1} \subseteq X_{N_2}$. Indeed, let $x \in X_{N_1}$, then there exists a feasible $\mathbf{u} = [u_0^T \cdots u_{N_1-1}^T]^T \in \Pi_{N_1}(x)$ and we can construct $\mathbf{u}^{\mathrm{f}} = [u_0^T \cdots u_{N_1-1}^T \; \underbrace{u_{\mathrm{e}}^T \cdots u_{\mathrm{e}}^T}_{N_2-N_1 \text{ times}}\,]^T \in \Pi_{N_2}(x)$ and thus $x \in X_{N_2}$. ◊

4.3.4 Output regulation

For a given set Y_{e} such that $y_{\mathrm{e}} := C \otimes x_{\mathrm{e}} \in Y_{\mathrm{e}}$, we define a stage cost $\ell(x,u,r)$, where now $r = y_{\mathrm{t}}$, with the following properties:

$\mathcal{P}1'$: $\ell(x,u,r) = 0$ if and only if $C \otimes x \in Y_{\mathrm{e}}$ and $u = u_{\mathrm{e}}$.

$\mathcal{P}2'$: $\ell(x,u,r) \geq \alpha(d_\infty(y,Y_{\mathrm{e}}))$ for all $y = C \otimes x$, where α is a $\mathscr{K}$ function.

[5] $m(x) := A^* \otimes x \oplus A^* \otimes B \otimes \underline{u}(x)$ which is a finite vector since A^* exists and Γ_n is row-finite.

Examples of such stage costs are (see [45] for more examples):

$$\ell(x,u,r) = \|y - y_\mathrm{e}\|_\infty + \gamma\|u - u_\mathrm{e}\|_\infty \tag{4.44}$$
$$\ell(x,u,r) = \max_{j\in\mathbb{N}_{[1,m]}} \{y_j - [y_\mathrm{t}]_j, 0\} + \gamma\|u - u_\mathrm{e}\|_\infty \tag{4.45}$$
$$\ell(x,u,r) = \sum_{j=1}^{m} \max\{y_j - [y_\mathrm{t}]_j, 0\} + \gamma\|u - u_\mathrm{e}\|_\infty, \tag{4.46}$$

where $\gamma > 0$ and $y = C \otimes x$. Note that for the stage cost (4.44) $Y_\mathrm{e} = \{y_\mathrm{e}\}$ and for (4.45) or (4.46) $Y_\mathrm{e} = \{y : y \leq y_\mathrm{t}\}$. In the stage cost (4.44) the first term penalizes the deviation from the due dates while in the stage costs (4.45)–(4.46) the first term penalizes the tardiness with respect to the due dates. The second term in these stage costs penalizes the deviation from the equilibrium input u_e. It is clear that now $\mathbf{r} = \mathbf{y}_\mathrm{t}$.

We obtain the following consequence:

Corollary 4.3.10 *Suppose assumptions* **A3** *and* **A4** *hold and there exists a* $b \in \mathbb{R}^p$ *such that* $Y_\mathrm{e} \subseteq \{y \in \mathbb{R}^p : y \leq b\}$. *Then, using as stage cost in the optimal control problem* (4.34) *one satisfying* $\mathcal{P}1' - \mathcal{P}2'$ *we obtain an MPC law* κ_N *for which the corresponding closed-loop buffers are bounded.*

Proof : With the same arguments as in the proof of Theorem 4.3.8 it follows that

$$\lim_{k\to\infty} \kappa_N(\phi(k;x,\kappa_N)) = u_\mathrm{e}$$
$$\lim_{k\to\infty} d_\infty(C \otimes \phi(k;x,\kappa_N), Y_\mathrm{e}) = 0$$

and that $\|\phi(k;x,\kappa_N) - x_\mathrm{e}\|_\infty$ is bounded for all $k \geq 0$ and thus the buffer levels remain bounded for any finite initial state $x \in X_N$. ♢

Note that the stage costs (4.45)–(4.46) satisfy assumption **A2** and thus the corresponding optimization problem (4.34) can be recast as a linear program (according to Theorem 3.2.2). For the stage cost (4.44) the optimization problem (4.34) can be recast as a mixed-integer linear program.

4.3.5 Example: production system

Consider the production system of Figure 2.1 with the dynamical equations (2.5):

$$\bar{x}(k+1) = \begin{bmatrix} 1 & \varepsilon & \varepsilon \\ 2 & 2 & \varepsilon \\ 5 & 4 & 2 \end{bmatrix} \otimes \bar{x}(k) \oplus \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \otimes \bar{u}(k), \quad \bar{y}(k) = [\varepsilon\ \varepsilon\ 2] \otimes \bar{x}(k).$$

For this example the largest max-plus eigenvalue of the system matrix $\bar{A}$ is $\lambda^* = 2$. We consider the reference signal for the output $r(k) = 5 + 1.5\lambda^* k$ (i.e. $\rho = 1.5\lambda^*$). The initial conditions are $\bar{x}(0) = [9\ \ 13\ \ 14]^T$ and $\bar{u}(-1) = 6$. We take the following constraints:

$$\bar{u}(k) - \bar{u}(k+1) \leq 0 \tag{4.47}$$
$$\bar{x}_2(k) - \bar{u}(k) \leq 2.5 \tag{4.48}$$

We now apply MPC. We choose the prediction horizon $N = 10$. We consider the stage cost (4.46) and we apply the MPC approach of Section 4.3.4. In this case the MPC optimization problem (4.34) can be recast as a linear program. The normalized system, obtained from extending the state as in Section 3.1.2 and normalization as in Section 4.1.2, becomes:

$$x(k+1) = \begin{bmatrix} -2 & \varepsilon & \varepsilon & -3 \\ -1 & -1 & \varepsilon & -2 \\ -1 & -2 & -1 & -2 \\ \varepsilon & \varepsilon & \varepsilon & -3 \end{bmatrix} \otimes x(k) \oplus \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \otimes u(k), \quad y(k) = [\varepsilon\ \varepsilon\ 6\ \varepsilon] \otimes x(k).$$

For the normalized system the PI set $\mathcal{O}_\infty$ is determined after 4 iterations:

$$\mathcal{O}_\infty = \mathcal{O}_4 = \{x \in \mathbb{R}^4_\varepsilon : I_4 x \leq [0.5 \ \ -0.5 \ \ 0 \ \ -2]^T\}.$$

We solve the linear program corresponding to (4.34) in a receding horizon fashion. For the original system the MPC sequence takes the following values:

$$\{\kappa_N + \rho k\}_{k=-1}^{14} = 6,\ 12.5,\ 14.5,\ 16.5,\ 18.5,\ 20.5,\ 22.5,\ 24.5,\ 26.5,\ 28.5,\ 30.5,\ 32.5,\ 34.5,\ 37,\ 40,\ 43.$$

The results of the closed-loop simulations are displayed in Figure 4.2. We observe from the first plot that although we start later than the initial due date the closed-loop output is able to track the due date signal after a finite transient behavior, i.e. we have closed-loop stability. The second plot displays the MPC input. We see that the MPC input reaches the steady-state behavior in a finite number of steps and that it is nondecreasing. The input-state constraints (4.48) are depicted in the third plot. Note that the MPC keeps the system behavior as close as possible to the constraints.

Let us now compare our MPC method with some other control design methods found in the literature. The max-plus control approaches proposed in [4, 36, 94, 102, 107] typically involve an open-loop optimal control problem over a simulation horizon and for a given due date signal r such that the output y of the system must satisfy $y \leq r$. The solution of this optimal control problem is computed using residuation, resulting in a just-in-time control input. The main disadvantage of these approaches is that it cannot cope with tracking problems where the outputs do not occur before the due dates and that the resulting control input sequence is sometimes decreasing, i.e. the constraint (4.47) might be violated. For instance, if we apply the method of [94] we get the following just-in-time control sequence $\{\bar{u}(k)\}_{k=-1}^{14} = 6,\ 1,\ 4,\ 7,\ 10,\ 13,\ 16,\ 19,\ 22,\ 25,\ 28,\ 31,\ 34,\ 37,\ 40,\ 43$. This sequence is not feasible since we have $\bar{u}(0) = 1 < \bar{u}(-1) = 6$, i.e. the constraint (4.47) is violated. This infeasibility is caused by the fact that the optimal input aims to fulfill the constraint $y \leq r$, which cannot be met using a nondecreasing input sequence. So, other control design methods that also include this constraint such as [4, 36, 102, 107] would also yield a non-increasing – and thus infeasible – input sequence. However, the MPC approach can cope with this constraint.

These issues are overcome in [109] by considering a residuation-based adaptive control that results in nondecreasing input sequences and allows violations of the due dates. However, the approach in [109] cannot cope with more complex state and input constraints, such as (4.48). For instance, using the adaptive control approach of [109] we obtain the following optimal input sequence $\{\bar{u}(k)\}_{k=-1}^{14} = 6,\ 6,\ 6,\ 7,\ 10,\ 13,\ 16,\ 19,\ 22,\ 25,\ 28,\ 31,\ 34,\ 37,\ 40,\ 43$. Note that applying this control the constraint (4.48) is violated (e.g. $\bar{x}(0) - \bar{u}(0) = 9 \not\leq 2.5$).

The MPC approach of [45] can cope with state–input constraints. However, this approach cannot guarantee a priori stability of the closed-loop system. Note that stability is really an issue when designing controllers for MPL systems, as we have seen in Example 4.2.5.

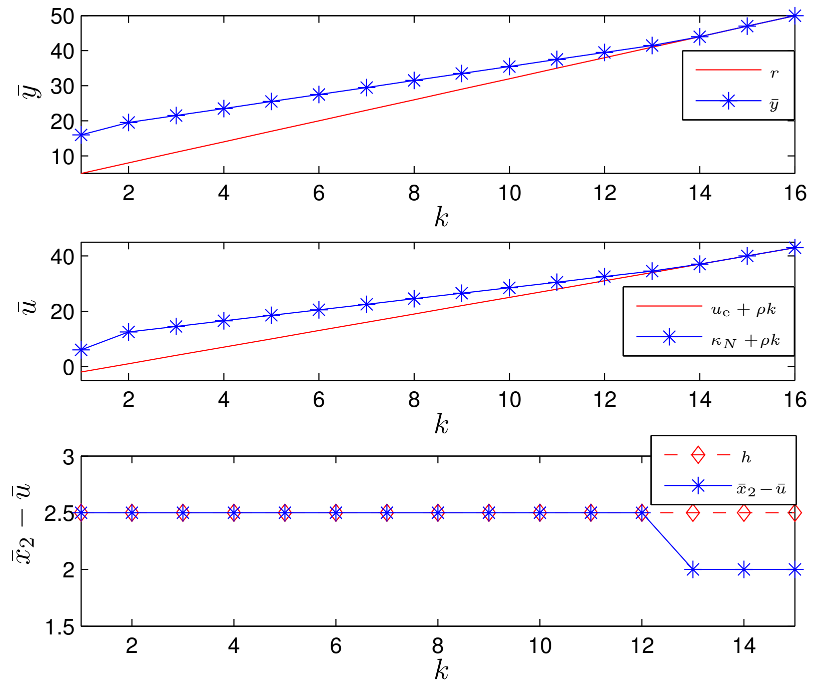

Figure 4.2: The closed-loop MPC simulations: constrained case.

4.4 Robust MPC for constrained MPL systems

In this section we propose a robustly stable MPC scheme for uncertain MPL systems based on solving one of the min-max control problems studied in Section 3.3. We consider the uncertain MPL system:

$$\begin{aligned}\bar{x}(k+1) &= \bar{A}(w(k-1), w(k)) \otimes \bar{x}(k) \oplus \bar{B}(w(k-1), w(k)) \otimes \bar{u}(k) \\ \bar{y}(k) &= \bar{C}(w(k-1)) \otimes \bar{x}(k).\end{aligned}$$

Recall that the largest eigenvalue λ^* of a matrix A gives also the maximum growth rate of the system $z(k+1) = A \otimes z(k)$. We introduce the notion of *worst-case growth rate* (according to (3.5)):

$$\lambda_W^* = \max_{j \in \mathbb{N}_{[1,n]}} \max_{\mathbf{w}_j \in W^{j+1}} \max_{((i_1 i_2)\cdots(i_j i_1))} (\bar{A}_{i_1 i_2}(w_0, w_1) + \cdots + \bar{A}_{i_j i_1}(w_{j-1}, w_j))/j,$$

where $\mathbf{w}_j = [w_0^T \; w_1^T \cdots w_j^T]^T$. Since $\bar{A}_{ij} \in \mathcal{F}_{\text{mps}}$, the maximum[6] is attained in a vertex of W^{j+1} for some $j \in \mathbb{N}_{[1,n]}$. We still assume that the slope of the reference signal (4.2) satisfies Assumption **A3**, i.e. $\rho > \lambda_W^*$.

We now consider a *normalized uncertain* MPL system obtained by subtracting in the conventional algebra from all entries of $\bar{x}, \bar{u}, \bar{y}$ and of $\bar{A}$ the values ρk and ρ, respectively, i.e. $x(k) \leftarrow \bar{x}(k) - \rho k,\; u(k) \leftarrow \bar{u}(k) - \rho k,\; y(k) \leftarrow \bar{y}(k) - \rho k,\; A(w_\text{p}, w_\text{c}) \leftarrow \bar{A}(w_\text{p}, w_\text{c}) - \rho$ and $B(w_\text{p}, w_\text{c}) \leftarrow \bar{B}(w_\text{p}, w_\text{c}), C(w_\text{p}) \leftarrow \bar{C}(w_\text{p})$:

$$\begin{aligned}x(k+1) &= A(w(k-1), w(k)) \otimes x(k) \oplus B(w(k-1), w(k)) \otimes u(k) \\ y(k) &= C(w(k-1)) \otimes x(k).\end{aligned} \tag{4.49}$$

[6] We used the fact that the maximum of a convex function over a polytope is attained in a vertex of the polytope [144].

We also consider a *nominal* value of the disturbance $w_{\rm n}$ and the corresponding nominal system is denoted with $x_{\rm n}(k+1) = A_{\rm n} \otimes x_{\rm n}(k) \oplus B_{\rm n} \otimes u_{\rm n}(k),\ y_{\rm n}(k) = C_{\rm n} \otimes x_{\rm n}(k)$ (where $A_{\rm n} := A(w_{\rm n}, w_{\rm n}),\ B_{\rm n} := B(w_{\rm n}, w_{\rm n})$ and $C_{\rm n} := C(w_{\rm n})$). Let $(x_{\rm e}, u_{\rm e})$ be a finite equilibrium pair corresponding to the nominal system and to the desired target $y_{\rm t}$, satisfying also the state-input constraints (4.29). It can be determined as the solution of the linear program (4.30).

Remark 4.4.1 Recall that $\kappa_{\rm f}(x) = u_{\rm e}$ for all $x \in \mathbb{R}^n$. From Theorem 4.2.2 (i) it follows that the set $\{x_{\rm e}\}$ is asymptotically stable with respect to the closed-loop system $x_{\rm n}(k+1) = A_{\rm n} \otimes x_{\rm n}(k) \oplus B_{\rm n} \otimes \kappa_{\rm f}(x_{\rm n}(k))$. Moreover, the closed-loop state trajectory is bounded for any finite initial condition. ♢

4.4.1 Robustly positively invariant (RPI) sets for uncertain MPL systems

We consider the normalized closed-loop system:

$$\begin{aligned} x(k+1) &= A(w(k-1), w(k)) \otimes x(k) \oplus B(w(k-1), w(k)) \otimes \kappa_{\rm f}(x(k)) \\ y(k) &= C(w(k-1)) \otimes x(k) \end{aligned} \tag{4.50}$$

subject to the state-input constraints

$$Hx(k) + Gu(k) \leq h, \tag{4.51}$$

where $H \geq 0$ and thus the constraints (4.51) satisfy assumption **A4**.

Definition 4.4.2 *[82] A set $Z \subseteq \{x : Hx + Gu_{\rm e} \leq h\}$ is a robustly positively invariant (RPI) set for the system* (4.50) *if for all initial states $x \in Z$ the subsequent state trajectories remain in Z for all possible disturbances. The maximal (minimal) RPI set for the system* (4.50) *is defined as the largest (smallest, non-empty) with respect to inclusion RPI set for* (4.50) *contained in* $\{x : Hx + Gu_{\rm e} \leq h\}$. ♢

Recall that $f_{\rm MPL}(x, u, w_{\rm p}, w_{\rm c}) = A(w_{\rm p}, w_{\rm c}) \otimes x \oplus B(w_{\rm p}, w_{\rm c}) \otimes u$. The maximal RPI set is computed iteratively as follows [82]: define

$$\mathcal{O}_0 = \{x : Hx + Gu_{\rm e} \leq h\}$$

and recursively

$$\mathcal{O}_k = \{x \in \mathcal{O}_0 : f_{\rm MPL}(x, u_{\rm e}, w_{\rm p}, w_{\rm c}) \in \mathcal{O}_{k-1},\ \forall w_{\rm p}, w_{\rm c} \in W\}.$$

It is trivial to see that $\mathcal{O}_k \subseteq \mathcal{O}_{k-1} \subseteq \cdots \subseteq \mathcal{O}_1 \subseteq O_0$ for all $k \geq 1$. Therefore, the limit of $\mathcal{O}_k$ exists and we have

$$\mathcal{O}_\infty = \bigcap_{k\geq 0} \mathcal{O}_k = \lim_{k\to\infty} \mathcal{O}_k.$$

Using similar arguments as in Lemma 4.3.2 (note that $H \geq 0$) it is easy to prove that $\mathcal{O}_k$ is a polyhedral set having the form $\mathcal{O}_k = \{x : T_k x \leq \tau_k\}$, where the matrix $T_k \geq 0$. Moreover, if there exists a t^* such that $\mathcal{O}_{t^*} = \mathcal{O}_{t^*+1}$ then $\mathcal{O}_\infty = \mathcal{O}_{t^*}$. In this case it follows that $\mathcal{O}_\infty = \{x \in \mathbb{R}^n : T_\infty x \leq \tau_\infty\}$, where $T_\infty \geq 0$.

4.4.2 Min-max MPC for MPL systems: closed-loop stability

We assume that the polyhedral RPI set $\mathcal{O}_\infty$ is available, where $T_\infty \geq 0$. Before proceeding to derive sufficient conditions for robust stability of an uncertain MPL system we recall some definitions (see Section 2.3.3). Let κ be a state feedback law and consider the closed-loop system

$$x(k+1) = A(w(k-1), w(k)) \otimes x(k) \oplus B(w(k-1), w(k)) \otimes \kappa(x(k)) \tag{4.52}$$

$$y(k) = C(w(k-1), w(k)) \otimes x(k). \tag{4.53}$$

Let $\phi(k; x, w, \kappa, \mathbf{w})$ denote the state solution of (4.52) at event step k when the initial state is x, the initial disturbance w, $\mathbf{w}$ is a realization of the disturbance signal and the feedback law κ is employed. We now introduce the notion of robust stability for the class of discrete event MPL systems with disturbances (see also Section 2.3.3).

Definition 4.4.3 *The RPI set X_f is robustly stable with respect to the closed-loop system* (4.52)–(4.53) *if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $d_\infty(x, X_\mathrm{f}) \leq \delta$ implies $d_\infty(\phi(k; x, w, \kappa, \mathbf{w}), X_\mathrm{f}) \leq \epsilon$ for all $k \geq 0$ and all admissible disturbance sequences $(w, \mathbf{w})$.*

If $d_\infty(\phi(k; x, \kappa, \mathbf{w}), X_\mathrm{f}) \to 0$ as $k \to \infty$ for all admissible disturbance sequences $(w, \mathbf{w})$ and for all $x \in X$, then the set X_f is robustly asymptotically attractive with a region of attraction X with respect to the system (4.52)–(4.53).

When both conditions are satisfied we refer to X_f as robustly asymptotically stable with respect to the system (4.52)–(4.53) *with a region of attraction X.* $\diamondsuit$

In this section we consider the following stage cost:

$$\ell(x, u, r) = d_\infty(x, X_\mathrm{f}) + \gamma \|u_\mathrm{e} - u\|_\infty, \tag{4.54}$$

where now $r = y_\mathrm{t}$, $X_\mathrm{f} := \mathcal{O}_\infty$, and $\gamma > 0$. Usually $X_\mathrm{f} = \{x : x \leq a_\infty\}$ and then from Lemma 4.3.1 it follows that in the context of DES the stage cost has the following significance: the first term expresses the tardiness with respect to a_∞ while the second term penalizes the deviations of the feeding times from the input equilibrium u_e. A similar stage cost was proposed in [80] in the context of min-max MPC for uncertain linear systems. From Lemma 4.3.1 it follows that this stage cost satisfies assumption **A2**. Note that in this case $\mathbf{r}$ is fixed for all $k \geq 0$, i.e. $\mathbf{r} = \mathbf{y}_\mathrm{t}$ at each step k, where we recall that $\mathbf{y}_\mathrm{t} = [y_\mathrm{t}^T \; y_\mathrm{t}^T \cdots y_\mathrm{t}^T]^T$. Therefore, we can omit $\mathbf{r}$.

The MPC formulation of the min-max optimal control problems discussed in Section 3.3 is as follows: at event triple (k, x, w) (i.e. for $x(k) = x,\; w(k-1) = w$) we consider

$$V_N^0(x, w) = \inf_{\pi \in \Pi_N(x,w)} \max_{\mathbf{w} \in \mathcal{W}} V_N(x, w, \pi, \mathbf{w}), \tag{4.55}$$

where $\Pi_N(x, w) = \{\pi : Hx_i + Gu_i \leq h \quad \forall i \in \mathbb{N}_{[0,N-1]},\; x_N \in X_\mathrm{f}\}$ and π is either the open-loop input sequence $\mathbf{u}$ (Section 3.3.2) or the disturbance feedback policy $(\mathbf{M}, \mathbf{v})$ (Section 3.3.3) or the state feedback policy $(\mu_0, \mu_1, \cdots, \mu_{N-1})$ (Section 3.3.4). Let $\pi^0(x, w) = (\mu_0^0(x, w), \mu_1^0(x, w), \cdots, \mu_{N-1}^0(x, w))$ be the optimal solution at step k (μ_i^0 are either vectors or control laws) and $\mathbf{x}^0(\mathbf{w}) = [x^T \; (x_1^0(\mathbf{w}))^T \cdots (x_N^0(\mathbf{w}))^T]^T$ be the optimal state trajectory for a certain realization of the disturbance $\mathbf{w}$. Then, only the first control in this sequence (i.e. $\kappa_N(x, w) := \mu_0^0(x, w)$) is applied to the uncertain MPL system at step k. Since assumptions **A1** and **A2** hold in this particular case ($H \geq 0$ according to **A4**, $T_\infty \geq 0$ and ℓ in (4.54) satisfies assumption **A2** according to Lemma 4.3.1), the solution to the min-max problem (4.55) can be obtained by solving either a linear program or N parametric linear programs. Note that with some abuse of terminology we use the term “MPC” even when we optimize over state feedback policies in (4.55) and thus the solution is computed off-line via multi-parametric linear programming.

The next theorem summarizes the main properties of this min-max MPC scheme.

Theorem 4.4.4 *The open-loop controller, the disturbance feedback controller and the state feedback controller obtained from* (4.55)*, applied to the normalized system in a receding horizon fashion, make the set X_{f} robustly asymptotically stable with respect to the closed-loop system* (4.52) *(where now $\kappa = \kappa_N$) and having a region of attraction X_N, where X_N is either X_N^{ol} or X_N^{df} or X_N^{sf}, respectively.*

Proof: *Feasibility*: Suppose that $(x, w) \in X_N$. The MPC input $u = \kappa_N(x, w)$ steers x to $x_1^0(\mathbf{w})$ and the disturbance takes a certain value w^+. Since X_{f} is an RPI set and since the final state $x_N^0(\mathbf{w}) \in X_{\mathrm{f}}$ for all $\mathbf{w} \in \mathcal{W}$, a feasible solution for (4.55) at the next step is given by $\pi^{\mathrm{f}} = (\mu_1^0(x, w), \cdots, \mu_{N-1}^0(x, w), \kappa_{\mathrm{f}}(x_N^0(\mathbf{w})))$.

Robustly asymptotically stable: From the first part of the proof it follows that $0 \leq V_N^0(x_1^0(\mathbf{w}), w^+) \leq V_N(x_1^0(\mathbf{w}), w^+, \pi^{\mathrm{f}}, \mathbf{w}) \leq V_N^0(x, w) - \ell(x, \kappa_N(x, w), r)$ for all $\mathbf{w} \in \mathcal{W}$. We conclude that $\ell(\phi(k; x, w, \kappa_N, \mathbf{w}), \kappa_N(\phi(k; x, w, \kappa_N, \mathbf{w}), w), r) \to 0$ as $k \to \infty$. From (4.54) it follows that

$$d_\infty(\phi(k; x, w, \kappa_N, \mathbf{w}), X_{\mathrm{f}}) \to 0, \;\; \kappa_N(\phi(k; x, w, \kappa_N, \mathbf{w}), w) \to u_{\mathrm{e}} \;\text{ as } k \to \infty, \tag{4.56}$$

i.e. X_{f} is robustly asymptotically attractive with domain of attraction X_N. Robust stability follows from the fact that the conditions $\mathcal{F}1^w - \mathcal{F}3^w$ and $\mathcal{S}1^w$ from Section 2.3.3 are verified in this case for $X_{\mathrm{f}} = \mathcal{O}_\infty$, $\kappa_{\mathrm{f}}(x) = u_{\mathrm{e}}$ and the stage cost ℓ defined in (4.54). Note that $\ell(x, u, r) \geq \alpha(d_\infty(x, X_{\mathrm{f}}))$ for all x, where α is a $\mathscr{K}$ function and $\ell(x, u, r) = 0$ for all $x \in X_{\mathrm{f}}, u = u_{\mathrm{e}}$. ♢

4.4.3 Example: production system with disturbances

Let us consider again the production system from Figure 2.1. In Example 2.2.1 or in Section 4.3.5 we have assumed that the processing and transportation times are fixed. We now assume that the parameters p_1, p_2, t_2, t_4 and t_6 are fixed at each cycle, taking the values $p_1 = 1, p_2 = 1, t_2 = 1, t_4 = 3$ and $t_6 = 0$, while the rest of the parameters are varying with each cycle: $p_3(k) \in [1.5 \;\; 2.5], t_1(k) \in [0 \;\; 2],\; t_3(k) \in [0 \;\; 1]$ and $t_5(k) \in [0 \;\; 1]$ for all $k \geq 0$. We define the uncertainty as $w(k) = [p_3(k-1) \;\; t_1(k) \;\; t_3(k) \;\; t_5(k)]$. Then, the uncertainty set is described by the following box $W = [1.5 \;\; 2.5] \times [0 \;\; 2] \times [0 \;\; 1] \times [0 \;\; 1]$. Moreover, the dynamical equations of the process (3.39) can be written in matrix form as in (3.82). There exists a feasible value of the uncertainty $\overline{w} = [2.5 \;\; 2 \;\; 1 \;\; 1] \in W$ for which the inequalities (3.83) hold. We assume the following constraints

$$\bar{u}(k) - \bar{u}(k+1) \leq 0 \tag{4.57}$$

$$\bar{x}_2(k) - \bar{u}(k) \leq 5. \tag{4.58}$$

We can easily remark that the conditions from Theorem 3.4.2 are fulfilled and that the corresponding deterministic system is given by

$$\overline{x}(k+1) = \begin{bmatrix} 1 & \varepsilon & \varepsilon \\ 3 & 1 & \varepsilon \\ 5 & 3 & 2.5 \end{bmatrix} \otimes \overline{x}(k) \oplus \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \otimes u(k), \;\; \overline{y}(k) = [\varepsilon \; \varepsilon \; 2.5] \otimes \overline{x}(k). \tag{4.59}$$

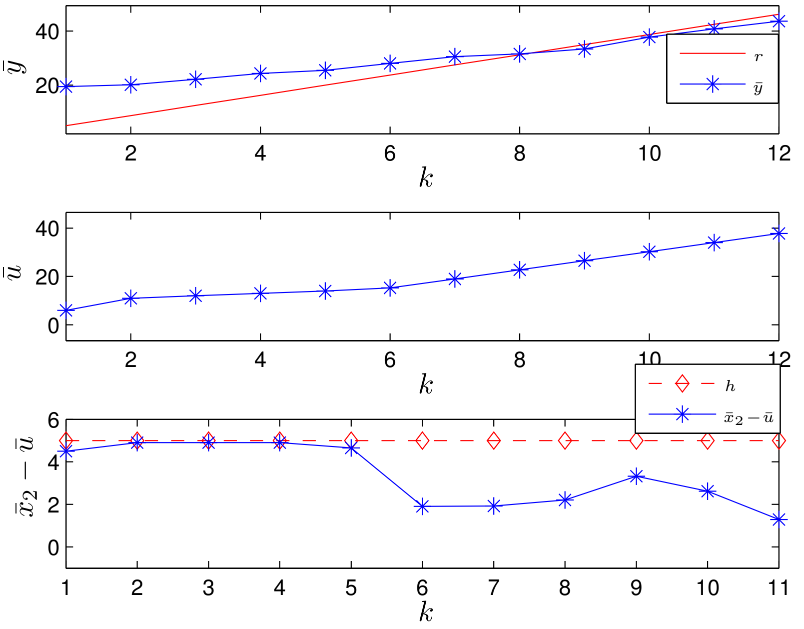

Figure 4.3: Robust MPC: closed-loop simulations.

We choose the following reference signal $r(k) = 5 + 1.5\lambda^* k$ (i.e. $\rho = 1.5\lambda^*$, where $\lambda^* = 2.5$), the prediction horizon $N = 8$ and the initial conditions are $\bar{x}(0) = [13\ 14.5\ 17]$ and $\bar{u}(-1) = 6$. Note that by extending the state as in Section 3.1.2 the constraint (4.57) is automatically satisfied. We obtain the following expression for X_f: $X_\mathrm{f} = \{x \in \mathbb{R}^4 : I_4 x \leq [1.75\ \ 0.5\ \ 5.5\ \ -3.5]^T\}$. Therefore, the optimization problem (4.55) corresponding to one of the optimal control problems $\mathbb{P}_N^{\mathrm{ol}}$ (Section 3.3.2) or $\mathbb{P}_N^{\mathrm{df}}$ (Section 3.3.3) or $\mathbb{P}_N^{\mathrm{sf}}$ (Section 3.3.4) are reduced to the deterministic optimal control problem $\mathbb{P}_N^{\mathrm{upper}}$ (i.e. (3.88) from Section 3.4.2) associated to the deterministic system (4.59).

Using the stage cost defined in (4.54) with $\gamma = 0.1$, the deterministic optimal control problem (3.88) applied in a receding horizon fashion yields the following optimal input sequence: $\{\kappa_N + \rho k\}_{k=-1}^{10} = 6,\ 11,\ 12,\ 13,\ 14,\ 15.2,\ 19,\ 22.7,\ 26.5,\ 30.2,\ 34,\ 37.7$. The results are displayed in Figure 4.3 using a feasible sequence of random disturbances.

We observe from the first plot that although we start later than the initial due date, the closed-loop output is able to track the due dates signal after a finite transient behavior. The second plot displays the MPC input. The input-state constraint (4.58) are depicted in the third plot. Note that sometimes the constraints are active.

Let us now compare our method with some other control design methods found in the literature. The adaptive control approach proposed in [109] has the most features in common with our approach in the sense that the approach of [109] allows violations of the due dates and tries to minimize this violations by updating the model at each step of the computation of the optimal control sequence. However, the approach in [109] cannot cope with state and input constraints. For instance, using the same disturbance realization as in our method and the adaptive control approach of [109] we obtain the following optimal input sequence $\{\bar{u}(k)\}_{k=-1}^{10} = 6,\ 6,\ 6.3,\ 8.8,\ 12.7,\ 17.3,\ 21.3,\ 25,\ 28.1,\ 31.6,\ 35.4,\ 39.2$. Note that $\bar{x}_2(0) - \bar{u}(0) = 9.5 \not\leq 5$. In [155] an open-loop min-max MPC scheme is derived using only input constraints and without guaranteeing a priori robust stability. However, the extension to mixed input and state constraints is straightforward according to Section 3.3.2. Moreover, from Section 3.3.2 we see that the optimal input sequence can be found without having to resort to computations of vertexes of W, as was done in [155] and stability is guaranteed a priori. Note that in this particular example, the open-loop approach discussed in Section 3.3.2 is equivalent with the state feedback approach derived in Section 3.3.4. However, from Example 3.3.6 we see that the state feedback approach outperforms the open-loop approach, in general.

4.5 MPC for switching MPL systems

As discussed previously, the class of MPL systems can only characterize synchronization and no concurrency or choice. However, the switching MPL framework allows us to break synchronization and to change the order of events. We recall from Section 2.2.2 or from the papers [156, 157, 159] that switching MPL systems are DES that can switch between different modes of operation. In each mode the switching MPL system is described by an MPL state equation with different system matrices for each mode:

$$\begin{array}{rcl} \bar{x}(k+1) & = & \bar{A}_i \otimes \bar{x}(k) \oplus \bar{B}_i \otimes \bar{u}(k) \\ \bar{y}(k) & = & \bar{C}_i \otimes \bar{x}(k) \end{array} \quad \text{if } \psi(\bar{x}(k), z(k), \bar{u}(k), \nu(k)) \in \mathcal{C}_i, \tag{4.60}$$

where the switching mechanism is determined by a variable $z \in \mathbb{R}_\varepsilon^{n_z}$ which is given by $z(k+1) = \psi(\bar{x}(k), z(k), \bar{u}(k), \nu(k))$ and $i \in \mathcal{I}$. We now provide sufficient conditions for the stability of switching MPL systems in terms of boundedness of the buffer levels (see also Section 4.1.2).

4.5.1 Sufficient conditions for stability of switching MPL systems

In this section we will concentrate on switching MPL systems with random mode switching, i.e. we do not take into account any knowledge of the mode switching function ψ. Note that if we are able to derive a stabilizing controller for a switching MPL system with random mode switching, the same controller will also be stabilizing if the mode switching is determined by some mode switching function ψ. A drawback however is that we will ignore the mode control signal ν for the purpose of control and concentrate fully on controlling the system with input signal u. We now give the definition for the maximum growth rate of a switching MPL system:

Definition 4.5.1 *For the switching MPL system* (4.60) *the maximum growth rate is defined as the smallest $\lambda^* > \varepsilon$ for which there exists an invertible matrix P in max-plus algebra such that the matrices $A_i = P^{\otimes^{-1}} \otimes \bar{A}_i \otimes P - \lambda^*$ satisfy $[A_i]_{lj} \leq 0$ for all $l, j \in \mathbb{N}_{[1,n]}$ and $i \in \mathcal{I}$.*

Note that for any switching MPL system the maximum grow rate λ^* exists and is finite. This fact is easily verified by noting that for $\lambda' = \max_{(i\in\mathcal{I};l,j\in\mathbb{N}_{[1,n]})}\{[\bar{A}_i]_{lj}\}$ and using the max-plus identity matrix $P = E$ the inequalities from the definition are fulfilled. Therefore, $\lambda^* \leq \lambda'$. We can easily see that λ^* can be determined by solving a sequence of linear programs. Indeed, from Section 3.1.1 we know that a matrix $P \in \mathbb{R}_\varepsilon^{n\times n}$ is invertible in max-plus algebra if and only if it can be factorized as $P = D \otimes T$, where $D = \text{diag}(d_1, d_2, \cdots, d_n)$ is a max-plus diagonal matrix with non-ε diagonal entries (i.e. $d_i \neq \varepsilon$ for all $i \in \mathbb{N}_{[1,n]}$) and $T \in \mathbb{R}_\varepsilon^{n\times n}$ is a max-plus permutation matrix. Note that once a permutation matrix T is fixed, $P_{ij} = d_i$ if $T_{ij} \neq \varepsilon$ and $P_{ij} = \varepsilon$ if $T_{ij} = \varepsilon$. Moreover, $P_{ij}^{\otimes^{-1}} = -d_j$ if $T_{ij} \neq \varepsilon$ and $P_{ij}^{\otimes^{-1}} = \varepsilon$ if $T_{ij} = \varepsilon$. In conclusion, once T is fixed, we have to solve the following linear program: $\min_{(\lambda,d)}\{\lambda : P^{\otimes^{-1}}_{qk} + [\bar{A}_i]_{kl} + P_{lj} \leq \lambda \;\; \forall q,k,i,l,j$ and $d = [d_1 \cdots d_n]^T\}$ and thus in order to determine λ^* we must solve $n!$ linear programs.

We consider the reference signal (4.2). Given a feedback controller $\bar{\kappa}$ for the switching MPL system (4.60), we define stability for the corresponding closed-loop system in terms of boundedness of the buffer levels (see Remark 4.1.3), i.e. $\|\bar{x}(k) - \rho k\|_\infty$, $\|\bar{y}(k) - \rho k\|_\infty$ and $\|\bar{u}(k) - \rho k\|_\infty$ are bounded for all $k \geq 0$.

We make the same change of coordinates as in Section 4.1.2:

$$x(k) \leftarrow P^{\otimes^{-1}} \otimes \bar{x}(k) - \rho k, \;\; y(k) \leftarrow \bar{y}(k) - \rho k, \;\; u(k) \leftarrow \bar{u}(k) - \rho k.$$

Then, the system matrices become:

$$A_i \leftarrow P^{\otimes^{-1}} \otimes \bar{A}_i \otimes P - \rho, \;\; B_i \leftarrow P^{\otimes^{-1}} \otimes \bar{B}_i, \;\; C_i \leftarrow \bar{C}_i \otimes P.$$

The *normalized free switching* MPL system is defined as:

$$\begin{aligned} x(k+1) &= A_{i(k)} \otimes x(k) \oplus B_{i(k)} \otimes u(k) \\ y(k) &= C_{i(k)} \otimes x(k) \\ i(k+1) &\in \mathcal{I}, \end{aligned} \tag{4.61}$$

where $i(\cdot)$ is a switching signal in $\mathcal{I}^{\mathbb{N}}$, i.e. any mode $i(k) \in \mathcal{I}$ can be active at event step k. We introduce now the definition of controllability for the normalized free switching system (4.61).

Definition 4.5.2 *The system* (4.61) *is controllable if there exists a finite positive integer $\tilde{n}$ such that the matrices*

$$\Gamma_{\left(i(1),i(2),\cdots,i(\tilde{n})\right)} = [B_{i(\tilde{n})} \;\; A_{i(\tilde{n})} \otimes B_{i(\tilde{n}-1)} \;\; \cdots \;\; A_{i(\tilde{n})} \otimes \cdots \otimes A_{i(2)} \otimes B_{i(1)}]$$

are row-finite for all $\left(i(1), i(2), \cdots, i(\tilde{n})\right) \in \mathcal{I}^{\tilde{n}}$.

Observability is defined in a similar fashion, i.e. there exists a finite positive integer $\tilde{n}$ such that the matrices

$$\Upsilon_{\left(i(1),i(2),\cdots,i(\tilde{n})\right)} = [(C_{i(1)})^T \;\; (C_{i(2)} \otimes A_{i(1)})^T \;\; \cdots \;\; (C_{i(\tilde{n})} \otimes A_{i(\tilde{n}-1)} \otimes \cdots \otimes A_{i(1)})^T]^T$$

are column-finite for all $\left(i(1), i(2), \cdots, i(\tilde{n})\right) \in \mathcal{I}^{\tilde{n}}$. Note that the controllability property means that each state is connected to some input, while the observability property means that each state is connected to some output (see also Definition 4.1.1 of controllability and Definition 4.1.2 of observability for MPL systems). We consider that the assumption **A3** also holds in this section, i.e. the slope of the reference signal (4.2) still satisfies $\rho > \lambda^* > \varepsilon$ and the system (4.61) is controllable and observable.

Since $\rho > \lambda^*$, it follows that $[A_i]_{lj} < 0$ for all $l, j \in \mathbb{N}_{[1,n]}$ and $i \in \mathcal{I}$ (according to the definitions of λ^* and A_i). Let us consider a feedback controller $\kappa : \mathbb{R}^n \to \mathbb{R}^m$ for the normalized free switching system (4.61). If we can show that the corresponding closed-loop state, output and input trajectories of the normalized free switching system are bounded, then the feedback controller $\kappa(\cdot) + \rho k$ stabilizes the original switching MPL system (4.60). The following lemma gives sufficient conditions for closed-loop stability of the normalized free switching system (4.61):

Lemma 4.5.3 *Suppose that the feedback controller κ is bounded, i.e. $\kappa_{\min} \leq \kappa(x) \leq \kappa_{\max}$ for all $x \in \mathbb{R}^n$, where $\kappa_{\min}$ and $\kappa_{\max}$ are given finite vectors. Then, the closed-loop normalized free switching system is stable in terms of boundedness of state, input and output trajectories.*

Proof: Let $\phi(k;x,\kappa)$ represent the state solution of (4.61) at event step k when the state at event step 0 is x and the feedback law κ is employed. Define $x_{\max}(k) = \max_{l\in\mathbb{N}_{[1,n]}}\{\phi_l(k;x,\kappa)\}$ and $b_{\max} = \max_{(i\in\mathcal{I};l,j\in\mathbb{N}_{[1,n]})}\{[B_i]_{lj} + (\kappa_{\max})_j\}$. Since $[A_i]_{lj} < 0$ for all $l,j \in \mathbb{N}_{[1,n]}$ and $i \in \mathcal{I}$ and using the monotonicity property of the max operator (3.7), it follows immediately that $\phi_l(k+1;x,\kappa) \leq \max\{x_{\max}(k), b_{\max}\}$. Therefore,

$$x_{\max}(k+1) \leq \max\{x_{\max}(k), b_{\max}\} \leq \max_{l\in\mathbb{N}_{[1,n]}}\{x_l, b_{\max}\}.$$

This means that the closed-loop state trajectory $\{\phi(k;x,\kappa)\}_{k\geq 0}$ is bounded from above. From the controllability assumption and using again the monotonicity property of the max operator (3.7) it follows that the closed-loop state trajectory $\{\phi(k;x,\kappa)\}_{k\geq 0}$ is also bounded from below. Using now the observability property of the system, it follows that the corresponding output trajectory is also bounded. This concludes our proof. $\diamondsuit$

4.5.2 MPC for switching MPL systems: closed-loop stability

We assume that the switching MPL system (4.60) is subject to state and input constraints:

$$H_k\bar{x}(k) + G_k\bar{u}(k) + \bar{G}_k\nu(k) + F_k r(k) \leq h_k,$$

where we recall that $r(k) = y_{\mathrm{t}} + \rho k$. We consider a stage cost $\ell(x,u,\nu,r)$ and we define a cost function over the prediction horizon N as

$$V_N(x,\mathbf{r},\mathbf{u},\boldsymbol{\nu}) = \sum_{i=0}^{N-1}\ell(x_i,u_i,\nu_i,r_i) + V_{\mathrm{f}}(x_N,r_N),$$

where x_i denotes the state solution of (4.60) at event step i when the initial condition of the state is x (and thus $x_0 = x$) and the control sequences $\mathbf{u} = [u_0^T\ u_1^T \cdots u_{N-1}^T]^T$ and $\boldsymbol{\nu} = [\nu_0^T\ \nu_1^T \cdots \nu_{N-1}^T]^T$ are employed, $\mathbf{r} = [r_0^T\ r_1^T \cdots r_N^T]^T$ denotes a reference sequence, and V_{f} is a terminal cost. The MPC optimization problem at event pair $(k,\bar{x})$ (i.e. $\bar{x}(k) = \bar{x}$) is defined as:

$$V_N^0(\bar{x}) = \inf_{(\mathbf{u},\boldsymbol{\nu})\in\Pi_N(k,\bar{x})} V_N(\bar{x},[r^T(k)\ \cdots r^T(k+N)]^T,\mathbf{u},\boldsymbol{\nu}), \tag{4.62}$$

where

$$\begin{aligned}\Pi_N(k,\bar{x}) = \{(\mathbf{u},\boldsymbol{\nu}) : H_{k+i}x_i + G_{k+i}u_i + \bar{G}_{k+i}\nu_i + F_{k+i}r(k+i) \leq h_{k+i},\\ \kappa_{\min} \leq u_i - \rho(k+i) \leq \kappa_{\max},\ \ \forall i \in \mathbb{N}_{[0,N-1]}\}\end{aligned}$$

and $\kappa_{\min}, \kappa_{\max}$ are chosen appropriately.

Let $(\mathbf{u}_N^0(\bar{x}), \boldsymbol{\nu}_N^0(\bar{x}))$ be an optimizer of (4.62). The MPC law is given by $\kappa_N(\bar{x}) = (u_0^0(\bar{x}), \nu_0^0(\bar{x}))$. Note that we imposed the stability constraint $\kappa_{\min} \leq u_i - \rho(k+i) \leq \kappa_{\max}$. Then, the next theorem is a straightforward consequence of Lemma 4.5.3:

Theorem 4.5.4 *Suppose that the MPC optimization problem* (4.62) *is feasible at each event step k. Then, the MPC law κ_N stabilizes the switching MPL system* (4.60) *in terms of boundedness of the closed-loop state, input and output trajectories.* $\diamondsuit$

In general, the stage cost ℓ satisfies $\ell \in \mathcal{F}_{\mathrm{mps}}$ (e.g. the stage cost (3.24) or if a 1-norm or ∞-norm is used). Moreover, we assume that the switching function ψ is linear. In this case, the MPC optimization problem (4.62) is a *mixed-integer linear program* or, as we will see in the next chapter, the optimal solution can be found by solving a finite set of linear programs.

4.5.3 Example: production system with concurrency

Consider the production system of Example 2.2.2. The input u is the control variable in this example (ν is not applicable here). We use the stage cost (3.24), i.e. $\ell(\bar{x}, \bar{u}, r) = \max\{\bar{y} - r, 0\} - \gamma u$, where $\bar{y} = [\varepsilon \;\; \varepsilon \;\; \varepsilon \;\; \varepsilon \;\; 0] \otimes \bar{x}$ and the terminal cost $V_\mathrm{f} = 0$. The prediction horizon is $N = 3$, $\gamma = 0.1$ and the initial state is $\bar{x}(0) = [4 \;\; 6 \;\; 3 \;\; 7 \;\; 8]^T$. Moreover, the maximum growth rate is $\lambda^* = 6$, we choose the reference signal $r(k) = 8 + 6.5k$ (i.e. $\rho = 6.5$) and $\kappa_{\min} = -2, \kappa_{\max} = 2$. We solve the MPC optimization problem (4.62) as explained above and we end up with a mixed-integer linear program that has to be solved on-line at each event step k. We observe in Figure 4.4 that the signal $\bar{y} - r$ is zero or negative, which means that our products

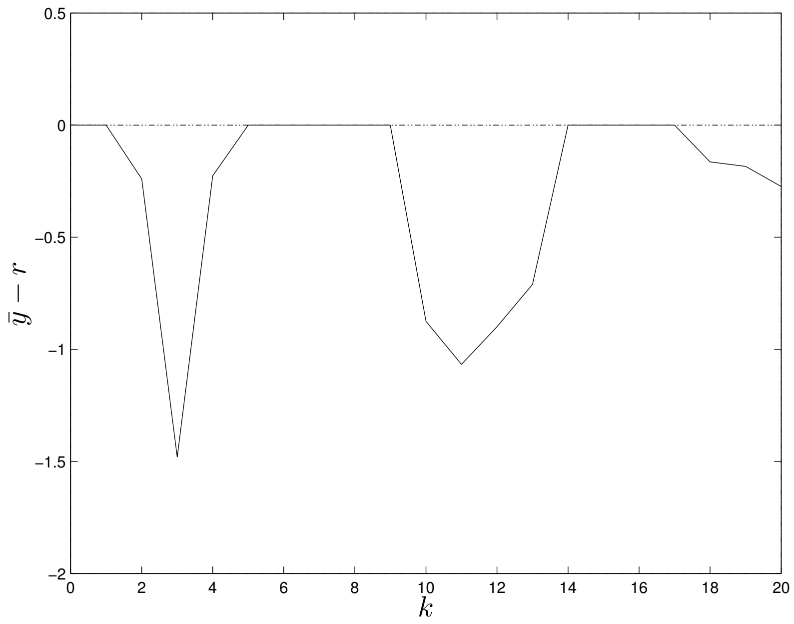

Figure 4.4: Due date error for a production system with concurrency

are delivered in time, and the MPC controller is stabilizing.

4.6 Conclusions

We have applied the MPC framework to some special classes of DES: MPL systems and switching MPL systems. The main goal of this chapter was to provide sufficient conditions that guarantee a priori that the MPC based on a finite-horizon optimal control problem discussed in the previous chapter stabilizes the closed-loop system. We have introduced the notions of stability in terms of Lyapunov and of PI set for MPL systems together with their main features.

In Section 4.2 we have considered unconstrained MPC for MPL systems and we have derived tuning rules for the MPC design parameters that guarantee stability of the equilibrium point with respect to the closed-loop system. The key assumptions that allow us to guarantee stability were that the growth rate of the due dates be larger than the growth rate of the system and that the cost function be designed to provide a just-in-time controller.

In Section 4.3 we have extended the MPC framework to the constrained MPL systems. We have derived an MPC law based on a terminal inequality constraint (obtained from a PI set) and a terminal cost approach that guarantee a priori stability for the closed-loop systems and the states and inputs sequences satisfy a given set of linear inequality constraints. We have extended the notion of PI set from classical time-driven systems to discrete-event MPL systems and the main properties have been demonstrated. Moreover, under additional assumptions we have proved that besides asymptotic stability, stability in terms of boundedness of the state also holds.

We have studied robust stability of the closed-loop MPC corresponding to uncertain MPL systems in Section 4.4. The main assumptions were that an RPI set is available and it is a polyhedron and that the stage cost has a certain representation.

Finally, sufficient conditions for guaranteeing stability of the closed-loop MPC corresponding to a switching MPL system have been derived in Section 4.5. We have shown that under the controllability and observability assumptions the boundedness of the MPC controller guarantees also boundedness of the closed-loop state trajectory. Each section is accompanied by an illustrative example.

Chapter 5

Model predictive control for uncertain max-min-plus-scaling systems

In the previous two chapters we have studied optimal control and in particular MPC for MPL systems. We have seen that MPL systems model discrete event phenomena with synchronization but no choice. The occurrence of choice could involve switching between different modes of operations, each mode being described by an MPL system, or could lead to the appearance of the minimum operator. This results in switching MPL systems or max-min-plus systems. A further extension is obtained by adding scalar multiplication which yields the class of MMPS systems. MMPS systems are also equivalent to certain classes of hybrid systems such as general continuous PWA systems. In this chapter we extend the classical min-max MPC framework to the class of uncertain MMPS systems. Provided that the stage cost is an MMPS expression (e.g. $1/\infty$-norm) and considering only linear input constraints we show that the open-loop min-max MPC problem for MMPS systems can be transformed into a finite sequence of linear programs, which can be solved efficiently. As an alternative to the open-loop MPC framework a feedback min-max MPC optimization problem over disturbance feedback policies is presented, which leads to improved performance compared to the open-loop approach.

5.1 MMPS systems

MMPS systems are dynamical systems whose evolution equations can be described using the operations maximization, minimization, addition, and scalar multiplication (see Section 2.1.2 for a formal definition). Typical examples of MMPS systems can be found in the area of both DES and hybrid systems, e.g. manufacturing plants, traffic networks, digital circuits, computer networks, etc.

In this thesis the class of MMPS systems constitutes the "bridge" that connects the last two chapters on control for some special classes of DES to the rest of the thesis. The present chapter unifies our previous results on optimal control for MPL systems (Chapter 3) and MPC for MPL systems and switching MPL systems (Chapter 4), since these classes of DES are in fact special subclasses of MMPS systems. However, the class of MMPS systems also encompasses bilinear max-plus systems, polynomial max-plus systems and min-max-plus systems. In addition, from Section 2.1.2 we have seen that the class of MMPS systems is equivalent, under some boundedness assumptions, to the class of general PWA systems and other important classes of hybrid systems. Therefore, MMPS systems form an interesting and relevant subclass of hybrid systems and thus this chapter anticipates the content of the rest of this thesis, namely optimal control (in particular MPC) for some special classes of hybrid systems.

This section proceeds now by introducing the canonical forms of MMPS systems. In Section 5.2 we present two min-max MPC algorithms for uncertain MMPS systems: in the first algorithm we optimize over open-loop input sequences while in the second one we introduce feedback by optimizing over disturbance feedback policies. Computational complexity of the two algorithms is discussed in Section 5.3 and in Section 5.4 we present a typical hybrid system (temperature control system in a room) where our approach is compared with other existing algorithms from the literature. This chapter is an extended version of [118, 119].

5.1.1 Canonical forms of MMPS systems

We recall from Definition 2.1.8 that an MMPS expression g of the variables $x_1, x_2, \cdots, x_n$ is defined recursively as:

$$g := x_i|\alpha|\max\{g_j, g_l\}|\min\{g_j, g_l\}|g_j + g_l|\beta g_j,$$

where g_j, g_l are again MMPS expressions. Now we provide some easily verifiable properties of the $\max$ and $\min$ operators that will be used extensively in this chapter:

$\mathcal{M}1$: the $\min$ operator is distributive with respect to the $\max$ operator, i.e.

$$\min\big\{\alpha, \max\{\beta,\gamma\}\big\} = \max\big\{\min\{\alpha,\beta\}, \min\{\alpha,\gamma\}\big\} \;\; \forall \alpha,\beta,\gamma \in \mathbb{R}^n.$$

Similarly, the $\max$ operator is distributive with respect to the $\min$ operator.

$\mathcal{M}2$: If $\theta \in \mathbb{R}_+$, then

$$\theta \max\{\alpha,\beta\} = \max\{\theta\alpha, \theta\beta\},$$
$$\theta \min\{\alpha,\beta\} = \min\{\theta\alpha, \theta\beta\}.$$

$\mathcal{M}3$: The $\min$ and $\max$ operator are related as follows:

$$\max\{\alpha,\beta\} = -\min\{-\alpha,-\beta\}.$$

$\mathcal{M}4$: Since addition is distributive with respect to the $\max$ and $\min$ operator, we have

$$\max\{\alpha,\beta\} + \max\{\gamma,\delta\} = \max\{\alpha+\gamma, \alpha+\delta, \beta+\gamma, \beta+\delta\},$$
$$\min\{\alpha,\beta\} + \min\{\gamma,\delta\} = \min\{\alpha+\gamma, \alpha+\delta, \beta+\gamma, \beta+\delta\}.$$

In the following we summarize the main result of [46, 131], which follows from the properties of the $\max$ and $\min$ operators mentioned above:

Lemma 5.1.1 *Any scalar-valued MMPS function $g : \mathbb{R}^n \to \mathbb{R}$ can be written into min-max canonical form*

$$g(x) = \min_{i\in\mathcal{I}} \max_{j\in\mathcal{J}_i} \{\alpha_{ij}^T x + \beta_{ij}\}, \tag{5.1}$$

or into max-min canonical form

$$g(x) = \max_{i\in\tilde{\mathcal{I}}} \min_{j\in\tilde{\mathcal{J}}_i} \{\tilde{\alpha}_{ij}^T x + \tilde{\beta}_{ij}\}, \tag{5.2}$$

where $\mathcal{I}, \mathcal{J}_i, \tilde{\mathcal{I}}$ and $\tilde{\mathcal{J}}_i$ are finite index sets and $\alpha_{ij}, \tilde{\alpha}_{ij} \in \mathbb{R}^n$, $\beta_{ij}, \tilde{\beta}_{ij} \in \mathbb{R}$ for all i, j. ♢

Using Lemma 5.1.1 it follows that any MMPS system can be written as (see Section 2.1.2)

$$\begin{aligned} x(k+1) &= f_{\mathrm{MMPS}}(x(k), u(k)) \\ y(k) &= h_{\mathrm{MMPS}}(x(k)), \end{aligned} \tag{5.3}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p$ and $f_{\mathrm{MMPS}}, h_{\mathrm{MMPS}}$ are vector-valued MMPS functions in min-max canonical form (5.1) or max-min canonical form (5.2).

From the results of [9, 64] we can conclude that MMPS systems are equivalent to other interesting classes of hybrid systems such as general PWA systems, MLD systems, LC systems, ELC systems under some mild assumptions related to boundedness of the states and of the inputs (see Lemma 2.1.7). This result is relevant since tools developed for a certain class can be used for the investigation of other classes of hybrid systems mentioned above. However, Lemma 2.1.11 allows us to conclude that a general continuous PWA system can be written equivalently as an MMPS system without imposing any additional assumptions concerning boundedness of the states and of the inputs, as was done in Lemma 2.1.7. Therefore, the MMPS models constitute an alternative modeling framework for hybrid systems. In that case k is a discrete time index.

On the other hand, many relevant subclasses of DES can be modeled using the MMPS framework, as it will be explained in the next section. In that case k becomes an event counter. Therefore, depending on the application, the state x, the input u, the output y and the index k can have different interpretations: they can represent either physical quantities or times of occurrence of some event (for x, u and y), and either a time step counter or an event step counter (for k).

5.1.2 MMPS systems and other classes of DES

In this section we will show that the MMPS model (5.3) can be considered as a generalized framework that encompasses several subclasses of DES such as MPL systems, max-plus-bilinear systems, max-plus-polynomial systems, switching MPL systems, max-min-plus systems.

MPL systems

From the definition of an MPL system (see e.g. Section 2.2.1) it follows that the MPL model

$$\begin{aligned} x(k+1) &= A \otimes x(k) \oplus B \otimes u(k) \\ y(k) &= C \otimes x(k) \end{aligned}$$

can be rewritten as

$$\begin{aligned} x_i(k+1) &= \max\big\{ \max_{j \in \mathbb{N}_{[1,n]}} \{a_{ij} + x_j(k)\}, \max_{j \in \mathbb{N}_{[1,m]}} \{b_{ij} + u_j(k)\}\big\} && \text{for } i \in \mathbb{N}_{[1,n]}, \\ y_i(k) &= \max_{j \in \mathbb{N}_{[1,n]}} \{c_{ij} + x_j(k)\}, && \text{for } i \in \mathbb{N}_{[1,p]}, \end{aligned}$$

which is clearly a special case of an MMPS system.

Max-plus-bilinear systems

Max-plus-bilinear systems are DES that can be described by a state space model of the following form:

$$\begin{aligned} x(k+1) &= A \otimes x(k) \oplus B \otimes u(k) \oplus \bigoplus_{i=1}^{m} L_i \otimes u_i(k) \otimes x(k) \\ y(k) &= C \otimes x(k) \end{aligned}$$

with $L_i \in \mathbb{R}_\varepsilon^{n \times n}$ for all $i \in \mathbb{N}_{[1,m]}$. This description is the max-plus algebraic equivalent of conventional bilinear discrete-time systems. Max-plus-bilinear systems could arise when some of the inputs of an MPL system are used as a switch to control the entries of the system matrix A, i.e. the constant system matrix A is replaced by the input-dependent system matrix $A \oplus L_1 \otimes u_1 \oplus \cdots \oplus L_m \otimes u_m$. Clearly, max-plus-bilinear systems are also a subclass of the MMPS systems.

Max-plus-polynomial systems

A max-plus-polynomial p of the variables $v_1, v_2, \cdots, v_n$ can be written as

$$p(v_1, v_2, \cdots, v_n) = \bigoplus_{i=1}^{q} c_i \otimes {v_1}^{\otimes^{r_{i,1}}} \otimes {v_2}^{\otimes^{r_{i,2}}} \otimes \cdots \otimes {v_n}^{\otimes^{r_{i,n}}}, \tag{5.4}$$

where c_i and $r_{i,j}$ are scalars.

Max-plus-polynomial systems are a further extension of MPL and max-plus-bilinear DES. They can be described by a state space model of the following form:

$$x(k+1) = f_{\text{pol}}(x(k), u(k))$$
$$y(k) = h_{\text{pol}}(x(k)),$$

where f_{pol} and h_{pol} are max-plus-polynomial functions. In [162] a subclass of max-plus-polynomial systems has been used in the design of traffic signal switching schemes.

Since (5.4) can be rewritten as

$$p(v_1, v_2, \ldots, v_n) = \max_{i=1,\ldots,q} (c_i + r_{i,1}v_1 + r_{i,2}v_2 + \cdots + r_{i,n}v_n),$$

which is an MMPS expression, it follows that a max-plus-polynomial system is also an MMPS system.

Switching MPL systems

It can be easily verified that switching MPL systems (see Section 2.2.2)

$$\begin{array}{rcl} x(k+1) & = & A_i \otimes x(k) \oplus B_i \otimes u(k) \\ y(k) & = & C_i \otimes x(k) \end{array} \quad \text{if } \ \psi(x(k), z(k), u(k), v(k)) \in \mathcal{C}_i, \tag{5.5}$$

where $\{\mathcal{C}_i\}_{i \in \mathcal{I}}$ is a polyhedral partition of $\mathbb{R}_\varepsilon^{n_z}$ and the switching function ψ is linear and depends only on (x, u) is a particular subclass of the MMPS systems. One simple proof is given next: under the previous assumptions on the switching function ψ and assuming that the set of feasible states and inputs is bounded it follows that (5.5) is a particular case of a general PWA system and thus can be written as an MMPS system according to Lemma 2.1.7.

Max-min-plus systems

Max-min-plus systems (or max-min systems as they are called in [129]) are described by the model

$$x(k+1) = f_{\text{mmp}}(x(k), u(k))$$
$$y(k) = h_{\text{mmp}}(x(k)),$$

where $f_{\text{mmp}}, h_{\text{mmp}}$ are max-min-plus expressions, i.e. expressions defined recursively by

$$f := x_i | f_k + \alpha | \max(f_k, f_l) | \min(f_k, f_l),$$

where α is a scalar, and f_k and f_l are again max-min-plus expressions. So max-min-plus expressions are special cases of MMPS expressions. This implies that max-min-plus systems are also a subclass of the MMPS systems.

5.1.3 Multi-parametric MMPS programming

In Section 2.3.1 we have briefly introduced multi-parametric programming (in particular, multi-parametric linear programming) and the main properties of the optimal value function and of the optimizer were presented. Multi-parametric MMPS programs are an extension of multi-parametric linear programs, as it will be proved next. Consider the multi-parametric linear program

$$J^0(x) = \max_{u \in \mathbb{R}^m} \{c^T u : \mathcal{H}x + \mathcal{G}u + \omega \leq 0\}, \tag{5.6}$$

where the vector of parameters x lies in $\mathbb{R}^n$. Since sup and inf are related as follows

$$\sup_{u \in \mathcal{U}} g(u) = -\inf_{u \in \mathcal{U}} -g(u),$$

from Theorem 2.3.2 we have that the set of parameters x such that the linear program (5.6) is feasible, i.e.

$$\mathcal{X} = \{x \in \mathbb{R}^n : \exists u \text{ s. t. } \mathcal{H}x + \mathcal{G}u + \omega \leq 0\}$$

is a closed polyhedral set, the optimal value function J^0 is a concave continuous PWA function and we can select a continuous PWA optimizer u^0, provided that there exists an $x_0 \in \mathcal{X}$ such that $J^0(x_0) \in \mathbb{R}$. It is well known [25] that any concave continuous PWA function can be written as:

$$J^0(x) = \min_{i \in \mathcal{I}} \{\alpha_i^T x + \beta_i\},$$

where $\mathcal{I}$ is a finite index set and $\alpha_i \in \mathbb{R}^n, \beta_i \in \mathbb{R}$. We conclude that $J^0 : \mathcal{X} \to \mathbb{R}$ is an MMPS function (in fact J^0 is a min-plus-scaling function[1] having domain $\mathcal{X}$). Moreover, from Lemma 2.1.11 we conclude that we can always select an MMPS optimizer u^0.

The following lemma deals with the special case of a multi-parametric program having as cost function an MMPS function. Note that related results were obtained in [81] for continuous PWA functions, but our proof is somewhat more intuitive and easier and moreover we obtain that the optimal value function J^0 is also continuous and an MMPS function (a property that is crucial in this chapter).

Lemma 5.1.2 *Let $J : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$ be an MMPS function and consider the following multi-parametric program:*

$$J^0(x) = \max_{u \in \mathcal{U}} \{J(x, u) : \mathcal{H}x + \mathcal{G}u + \omega \leq 0\}, \tag{5.7}$$

[1] A min-plus-scaling function is described by an expression defined recursively as

$$x_i | \alpha | \min\{g_j, g_l\} | g_j + g_l | \beta g_j,$$

where g_j, g_l are again min-plus-scaling expressions.

where $\mathcal{U}$ is a polytope. Then, the solution of the multi-parametric MMPS program (5.7) *can be obtained by solving a set of multi-parametric linear programs. Moreover, the optimal value function J^0 is an MMPS function and the optimizer u^0 is a PWA function having domain $\mathcal{X} = \{x \in \mathbb{R}^n : \exists u \in \mathcal{U}$ s. t. $\mathcal{H}x + \mathcal{G}u + \omega \leq 0\}$.*

Proof: Since J is an MMPS function and thus continuous and $\mathcal{U}$ is a polytope, from Theorem 2.3.1 it follows that the optimal value function J^0 is continuous on $\mathcal{X}$. Using the max-min canonical representation (5.2) of J, we have

$$J(x,u) = \max_{i \in \mathcal{I}} \min_{j \in \mathcal{J}_i} \{\alpha_{ij}^T x + \beta_{ij}^T u + \gamma_{ij}\}.$$

Then, the multi-parametric MMPS program (5.7) can be written as:

$$\max_{u \in \mathcal{U}} \big\{ \max_{i \in \mathcal{I}} \min_{j \in \mathcal{J}_i} \{\alpha_{ij}^T x + \beta_{ij}^T u + \gamma_{ij}\} : \mathcal{H}x + \mathcal{G}u + \omega \leq 0 \big\} =$$
$$\max_{i \in \mathcal{I}} \max_{u \in \mathcal{U}} \big\{ \min_{j \in \mathcal{J}_i} \{\alpha_{ij}^T x + \beta_{ij}^T u + \gamma_{ij}\} : \mathcal{H}x + \mathcal{G}u + \omega \leq 0 \big\}.$$

Therefore, for each $i \in \mathcal{I}$ we must solve the following multi-parametric program:

$$\max_{u \in \mathcal{U}} \big\{ \min_{j \in \mathcal{J}_i} \{\alpha_{ij}^T x + \beta_{ij}^T u + \gamma_{ij}\} : \mathcal{H}x + \mathcal{G}u + \omega \leq 0 \big\}.$$

The last multi-parametric program can be recast as a multi-parametric linear program:

$$J_i^0(x) = \max_{(u,\mu_i)} \{\mu_i : \alpha_{ij}^T x + \beta_{ij}^T u + \gamma_{ij} \geq \mu_i \ \forall j \in \mathcal{J}_i, \mathcal{H}x + \mathcal{G}u + \omega \leq 0, u \in \mathcal{U}\}.$$

It follows that J_i^0 is a min-plus-scaling expression in the variable the parameter x. In conclusion, we have to solve $|\mathcal{I}|$ multi-parametric linear programs and then $J^0(x) = \max_{i \in \mathcal{I}} J_i^0(x)$, i.e. J^0 is an MMPS function having domain $\mathcal{X}$. Similarly, we can prove that the optimizer u^0 is a PWA function on $\mathcal{X}$. However, in general nothing can be said about the continuity of the optimizer. $\diamondsuit$

5.2 Robust MPC for MMPS systems

Different control strategies (e.g. MPC) can be found in the literature for some specific subclasses of DES or hybrid systems [11,29], in particular for MPL systems [4,36,45,108] or PWA systems [46, 81, 138]. Using the work of [46] in which MPC for MMPS (and equivalently for general continuous PWA) systems for the deterministic case without disturbances is proposed, we further extend MPC for the cases with bounded disturbances. We consider uncertain MMPS systems, and thus also uncertain general continuous PWA systems. We model disturbances by including extra additive terms in the system equations of the MMPS system.

Note that there are some results in the literature on specific classes of uncertain DES and hybrid systems [81, 138] but to the authors' best knowledge this is the first time when the min-max optimal control approach is used for uncertain MMPS systems. The papers [81, 138] focus on worst-case approach. In [81, 138] dynamic programming was used to solve the min-max state feedback MPC problem for continuous PWA systems with bounded disturbances. The core difficulty with the dynamic programming approach is that optimizing over feedback policies with arbitrary nonlinear functions is in general a computationally hard problem. Moreover, in the dynamic programming approach it is difficult to take into consideration variable input constraints, (e.g. bounded rate variation $m \leq u(k+1) - u(k) \leq M$). In Chapter 4 we have shown that robust

MPC for MPL systems is a convex problem if the stage cost has a particular representation in which the coefficients corresponding to the state vector are nonnegative and the matrix associated with the state constraints is also nonnegative. The main difficulty in this case is represented by the - in the worst case - exponential number of constraints, that result from the transformation of max constraints in linear constraints. The approach proposed in this section addresses some of these issues. This section proceeds now with the problem formulation of the open-loop min-max MPC-MMPS.

5.2.1 Problem formulation

Let us define the class of uncertain MMPS systems. As in conventional linear systems, we model the uncertainty by including an extra term in the system equations for MMPS systems:

$$\begin{aligned} x(k+1) &= f_{\mathrm{MMPS}}(x(k), u(k), w(k)) \\ y(k) &= h_{\mathrm{MMPS}}(x(k)), \end{aligned} \tag{5.8}$$

where $f_{\mathrm{MMPS}},\ h_{\mathrm{MMPS}}$ are vector-valued MMPS functions. The uncertainty caused by disturbances in the estimation of the real system and measurements is gathered in the uncertainty vector w. As in previous chapters, we assume that this uncertainty is included in a polytope

$$W = \{w \in \mathbb{R}^q : \Omega w \leq s\}$$

and if consecutive disturbance samples $w(k)$,...,$w(k+j)$ are related (which is typically the case in the context of DES[2]), we assume that this relation is linear (e.g. a system of linear equalities or inequalities).

Using the link between MMPS and general continuous PWA systems, the uncertain MMPS system (5.8) can also be written as an *uncertain general continuous PWA* system:

$$\begin{aligned} x(k+1) &= f_{\mathrm{PWA}}(x(k), u(k), w(k)) \\ y(k) &= h_{\mathrm{PWA}}(x(k)), \end{aligned}$$

where $f_{\mathrm{PWA}},\ h_{\mathrm{PWA}}$ are continuous vector-valued PWA functions. Therefore, the algorithms derived in this section can also be applied to uncertain general continuous PWA systems. Note that in conventional uncertain PWA systems [74,81,138] the partition that generates the system is independent on the disturbance w. In our definition of an uncertain MMPS system and uncertain general continuous PWA system the partition will, in general, also depend on the disturbance (note that this is necessary to guarantee continuity of the system). Therefore, our modeling approach is more general than in [74, 81, 138].

As in previous chapters, we consider a reference signal $\{r(k)\}_{k\geq 0}$ which the output is required to track. Moreover, we assume that the stage cost $\ell(x,u,r)$ and the terminal cost $V_{\mathrm{f}}(x,r)$

[2] From Example 3.39 we see that we could define the disturbance vector as $w(k) = [p_1(k-1)\cdots p_l(k-1)\ p_1(k)\cdots p_l(k)\ t_1(k)\cdots t_{\tilde{l}}(k)]^T$ and then there exists a linear relation between $w(k)$ and $w(k+1)$ given by the equalities $w_i(k) = w_j(k+1)$ for all $i \in \mathbb{N}_{[1,l]}$ and $j \in \mathbb{N}_{[l+1,\ 2l]}$. In this case we can rewrite the uncertain MPL system (3.40)–(3.41) as follows:

$$\begin{aligned} x(k+1) &= A(w(k)) \otimes x(k) \oplus B(w(k)) \otimes u(k) \\ y(k) &= C(w(k-1)) \otimes x(k). \end{aligned}$$

are MMPS (or equivalently continuous PWA) functions: e.g. the stage cost (3.24)–(3.25) or if a 1-norm and ∞-norm is used

$$\begin{array}{rclrcl} \ell(x,u,r) & = & \|Q(y-r)\|_1 + \|Ru\|_1, & V_{\mathrm{f}}(x,r) & = & \|P(y-r)\|_1 \\ \ell(x,u,r) & = & \|Q(y-r)\|_\infty + \|Ru\|_\infty, & V_{\mathrm{f}}(x,r) & = & \|P(y-r)\|_\infty, \end{array}$$

where $y = h_{\mathrm{MMPS}}(x)$ denotes the output of the system and $Q \in \mathbb{R}^{n_Q \times p}$, $R \in \mathbb{R}^{n_R \times m}$, $P \in \mathbb{R}^{n_P \times p}$. For $z \in \mathbb{R}$ we have that its absolute value is given by $|z| = \max\{z, -z\}$ and for all $x \in \mathbb{R}^n$ we recall that 1-norm and ∞-norm are given by

$$\|x\|_1 := \sum_{i=1}^{n} |x_i|, \quad \|x\|_\infty := \max_{i \in \mathbb{N}_{[1,n]}} |x_i|,$$

i.e. these norms are defined by MMPS expressions.

Let $\mathbf{u} = [u_0^T \ u_1^T \ldots u_{N-1}^T]^T$ be a control sequence and $\mathbf{w} = [w_0^T \ w_1^T \ldots w_{N-1}^T]^T$ denote a realization of the disturbance over the prediction horizon N. Also, let $\phi(k; x, \mathbf{u}, \mathbf{w})$ denote the state solution of (5.8) at step k when the initial state is x at step 0, the control is determined by $\mathbf{u}$ and the disturbance sequence is $\mathbf{w}$. As in previous chapters $\phi(0; x, \mathbf{u}, \mathbf{w}) = x$. Using the properties $\mathcal{M}1 - \mathcal{M}4$ of the $\min$ and $\max$ operators it follows that $\phi(k; x, \mathbf{u}, \mathbf{w})$ is an MMPS expression of the variables $(x, \mathbf{u}, \mathbf{w})$ for all $k \geq 0$.

The cost function $V_N(x, \mathbf{r}, \mathbf{u}, \mathbf{w})$ is defined as:

$$V_N(x, \mathbf{r}, \mathbf{u}, \mathbf{w}) = \sum_{i=0}^{N-1} \ell(x_i, u_i, r_i) + V_{\mathrm{f}}(x_N, r_N),$$

where $x_i = \phi(i; x, \mathbf{u}, \mathbf{w})$ (and thus $x_0 = x$) and $\mathbf{r} = [r_0^T \ r_1^T \cdots r_N^T]^T$ denotes a reference sequence.

5.2.2 Open-loop min-max MPC for MMPS systems

Because the uncertainty vector w is in the polytope W and if $w_0, \cdots, w_j$ are related then this relation is linear, we conclude that $\mathbf{w}$ will also be in a bounded polyhedral set

$$\tilde{\mathcal{W}} = \{\mathbf{w} \in \mathbb{R}^{Nq} : \mathbf{\Omega}\mathbf{w} \leq \mathbf{s}\} \subseteq \mathcal{W},$$

where $\mathbf{\Omega} \in \mathbb{R}^{N_\Omega \times Nq}$, $\mathbf{s} \in \mathbb{R}^{N_\Omega}$ and we recall that $\mathcal{W} = W^N$.

We now assume that at each step k the state $x(k)$ is available. The *open-loop min-max* MPC-MMPS problem at event (k, x) (i.e. $x(k) = x$) is defined as follows:

$$V_N^{0,\mathrm{ol}}(x, \mathbf{r}) = \inf_{\mathbf{u} \in \Pi_N^{\mathrm{ol}}(k)} \max_{\mathbf{w} \in \tilde{\mathcal{W}}} V_N(x, \mathbf{r}, \mathbf{u}, \mathbf{w}), \tag{5.9}$$

and we assume that at each step k the uncertain MMPS system (5.8) is subject only to variable input constraints

$$\Pi_N^{\mathrm{ol}}(k) = \{\mathbf{u} \in \mathbb{R}^{Nm} : \mathbf{H}_k \mathbf{u} \leq \mathbf{h}_k\}$$

with $\mathbf{H}_k \in \mathbb{R}^{n_H \times Nm}$ and $\mathbf{h}_k \in \mathbb{R}^{n_H}$. As in (2.18) we define

$$\mathbf{u}_N^0(x, \mathbf{r}) \in \arg \min_{\mathbf{u} \in \Pi_N^{\mathrm{ol}}(k)} \max_{\mathbf{w} \in \tilde{\mathcal{W}}} V_N(x, \mathbf{r}, \mathbf{u}, \mathbf{w})$$

to be an optimizer (whenever the infimum is attained). Then, according to the receding horizon philosophy the actual control applied to the process at step k is the first sample in $\mathbf{u}_N^0(x,\mathbf{r})$, i.e. $u_0^0(x,\mathbf{r})$. The MPC law is given by

$$\kappa_N(x,\mathbf{r}) = u_0^0(x,\mathbf{r}).$$

Since the stage cost ℓ and the terminal cost V_f are MMPS expressions and since x_i are also MMPS expressions for all $i \in \mathbb{N}_{[0,N]}$ (recall that $\phi(k;x,\mathbf{u},\mathbf{w})$ is an MMPS expression of the variables $(x,\mathbf{u},\mathbf{w})$ for all $k \geq 0$), using again the properties $\mathcal{M}1 - \mathcal{M}4$ it follows that V_N is an MMPS expression. From Lemma 5.1.1 we conclude that V_N can be written in max-min canonical form:

$$V_N(x,\mathbf{r},\mathbf{u},\mathbf{w}) = \max_{i\in\mathcal{I}} \min_{j\in\mathcal{J}_i}\{\alpha_{ij}^T x + \beta_{ij}^T \mathbf{r} + \gamma_{ij}^T \mathbf{u} + \delta_{ij}\mathbf{w} + \theta_{ij}\}. \tag{5.10}$$

5.2.3 Solution based on multi-parametric MMPS programming

In this section we provide a solution to the open-loop min-max MPC-MMPS problem (5.9) based on multi-parametric MMPS programming. For a given $(x,\mathbf{r},\mathbf{u})$ we define the *inner* min-max MPC-MMPS problem

$$J_N(x,\mathbf{r},\mathbf{u}) = \max_{\mathbf{w}\in\tilde{\mathcal{W}}} V_N(x,\mathbf{r},\mathbf{u},\mathbf{w}) \tag{5.11}$$

and the optimizer is denoted with

$$\mathbf{w}^0(x,\mathbf{r},\mathbf{u}) \in \arg\max_{\mathbf{w}\in\tilde{\mathcal{W}}} V_N(x,\mathbf{r},\mathbf{u},\mathbf{w}). \tag{5.12}$$

The following lemma provides a method to compute the optimal solution to the inner min-max MPC-MMPS problem:

Lemma 5.2.1 *For a given $(x,\mathbf{r},\mathbf{u})$ the optimizer $\mathbf{w}^0(x,\mathbf{r},\mathbf{u})$ given by (5.12) can be computed by solving a set of linear programming problems.*

Proof: We determine for any fixed $(x,\mathbf{r},\mathbf{u})$ the optimizer $\mathbf{w}^0(x,\mathbf{r},\mathbf{u})$ using the max-min canonical form (5.10) of V_N, by solving the following optimization problem:

$$\max_{\mathbf{w}\in\tilde{\mathcal{W}}} \max_{i\in\mathcal{I}} \min_{j\in\mathcal{J}_i}\{\alpha_{ij}^T x + \beta_{ij}^T \mathbf{r} + \gamma_{ij}^T \mathbf{u} + \delta_{ij}^T\mathbf{w} + \theta_{ij}\}, \tag{5.13}$$

which is equivalent with:

$$\max_{i\in\mathcal{I}} \max_{\mathbf{w}} \big\{ \min_{j\in\mathcal{J}_j}\{\alpha_{ij}^T x + \beta_{ij}^T \mathbf{r} + \gamma_{ij}^T \mathbf{u} + \delta_{ij}^T\mathbf{w} + \theta_{ij}\} : \mathbf{\Omega w} \leq \mathbf{s}\big\}.$$

Now, for each $i \in \mathcal{I}$ we have to solve the following optimization problem:

$$\max_{\mathbf{w}} \big\{ \min_{j\in\mathcal{J}_j}\{\alpha_{ij}^T x + \beta_{ij}^T \mathbf{r} + \gamma_{ij}^T \mathbf{u} + \delta_{ij}^T\mathbf{w} + \theta_{ij}\} : \mathbf{\Omega w} \leq \mathbf{s}\big\},$$

which is equivalent with the following linear program:

$$\max_{(\mathbf{w},\mu_i)}\{\mu_i : \alpha_{ij}^T x + \beta_{ij}^T \mathbf{r} + \gamma_{ij}^T \mathbf{u} + \delta_{ij}^T\mathbf{w} + \theta_{ij} \geq \mu_i \ \ \forall j \in \mathcal{J}_i,\ \mathbf{\Omega w} \leq \mathbf{s}\}. \tag{5.14}$$

To obtain the optimizer corresponding to (5.13) we solve the linear program (5.14) for each $i \in \mathcal{I}$, with the optimizer $(\mathbf{w}_i^0(x,\mathbf{r},\mathbf{u}), \mu_i^0(x,\mathbf{r},\mathbf{u}))$ and then we select as $\mathbf{w}^0(x,\mathbf{r},\mathbf{u})$ the optimal solution $\mathbf{w}_{i^*}^0(x,\mathbf{r},\mathbf{u})$, where the index i^* is given by $\mu_{i^*}^0(x,\mathbf{r},\mathbf{u}) = \max_{i\in\mathcal{I}} \mu_i^0(x,\mathbf{r},\mathbf{u})$. ♢

Theorem 5.2.2 *The function* $(x, \mathbf{r}, \mathbf{u}) \mapsto J_N(x, \mathbf{r}, \mathbf{u})$ *is described by an MMPS expression.*

Proof: First, let us note that V_N is an MMPS function. Next, $\tilde{\mathcal{W}}$ is a polytope. Let us consider the multi-parametric MMPS program (5.11), where the vector of parameters is $(x, \mathbf{r}, \mathbf{u})$. The conditions from Lemma 5.1.2 are fulfilled and thus J_N is an MMPS function. $\diamondsuit$

Note that we can compute the expression of J_N explicitly as follows: for each $i \in \mathcal{I}$ we know from multi-parametric linear programming (see Section 5.1.3) that $\mu_i^0(x, \mathbf{r}, \mathbf{u})$ is a min-plus-scaling expression, i. e.

$$\mu_i^0(x, \mathbf{r}, \mathbf{u}) = \min_{j \in \tilde{\mathcal{J}}} \{\xi_{ij}^T x + \zeta_{ij}^T \mathbf{r} + \eta_{ij}^T \mathbf{u} + \nu_{ij}\}.$$

It follows that J_N is given by:

$$\begin{aligned} J_N(x, \mathbf{r}, \mathbf{u}) &= \max_{i \in \mathcal{I}} \mu_i^0(x, \mathbf{r}, \mathbf{u}) \\ &= \max_{i \in \mathcal{I}} \min_{j \in \tilde{\mathcal{J}}} \{\xi_{ij}^T x + \zeta_{ij}^T \mathbf{r} + \eta_{ij}^T \mathbf{u} + \nu_{ij}\}. \end{aligned} \tag{5.15}$$

We thus obtain directly the max-min canonical expression of J_N. Furthermore, $\mathbf{w}_i^0(x, \mathbf{r}, \mathbf{u})$ is a continuous PWA function and $\mathbf{w}^0(x, \mathbf{r}, \mathbf{u}) = \mathbf{w}_i^0(x, \mathbf{r}, \mathbf{u})$ if $\mu_i^0(x, \mathbf{r}, \mathbf{u}) \geq \mu_{i'}^0(x, \mathbf{r}, \mathbf{u})$ for all $i' \in \mathcal{I} \setminus \{i\}$. This implies that $\mathbf{w}^0(x, \mathbf{r}, \mathbf{u})$ is a PWA function. Note that $\mathbf{w}^0(x, \mathbf{r}, \mathbf{u})$ is not necessarily continuous.

The *outer* min-max MPC-MMPS problem is now defined as:

$$V_N^{0,\text{ol}}(x, \mathbf{r}) = \inf_{\mathbf{u} \in \Pi_N^{\text{ol}}(k)} J_N(x, \mathbf{r}, \mathbf{u}). \tag{5.16}$$

Note that $\mathbf{u}_N^0(x, \mathbf{r}) \in \arg\min_{\mathbf{u} \in \Pi_N^{\text{ol}}(k)} J_N(x, \mathbf{r}, \mathbf{u})$.

Theorem 5.2.3 *For given* $(x, \mathbf{r})$ *the outer min-max MPC-MMPS problem can be solved via a set of linear programming problems.*

Proof: From (5.15) we know that J_N is an MMPS function. Using Lemma 5.1.1 the max-min canonical form of J_N in (5.15) can be used to determine the equivalent min-max canonical form of J_N, i.e.

$$J_N(x, \mathbf{r}, \mathbf{w}) = \min_{i \in \bar{\mathcal{I}}} \max_{j \in \bar{\mathcal{J}}} \{\bar{\xi}_{ij}^T x + \bar{\zeta}_{ij}^T \mathbf{r} + \bar{\eta}_{ij}^T \mathbf{u} + \bar{\nu}_{ij}\}.$$

Then, the outer min-max MPC-MMPS problem (5.16) becomes

$$\begin{aligned} V_N^{0,\text{ol}}(x, \mathbf{r}) &= \inf_{\mathbf{u} \in \Pi_N^{\text{ol}}(k)} \min_{i \in \bar{\mathcal{I}}} \max_{j \in \bar{\mathcal{J}}} \{\bar{\xi}_{ij}^T x + \bar{\zeta}_{ij}^T \mathbf{r} + \bar{\eta}_{ij}^T \mathbf{u} + \bar{\nu}_{ij}\} \\ &= \min_{i \in \bar{\mathcal{I}}} \inf_{\mathbf{u} \in \Pi_N^{\text{ol}}(k)} \max_{j \in \bar{\mathcal{J}}} \{\bar{\xi}_{ij}^T x + \bar{\zeta}_{ij}^T \mathbf{r} + \bar{\eta}_{ij}^T \mathbf{u} + \bar{\nu}_{ij}\}. \end{aligned}$$

For each $i \in \bar{\mathcal{I}}$ we must solve the following linear programming problem:

$$\min_{(\mathbf{u}, \bar{\mu}_i)} \{\bar{\mu}_i : \bar{\xi}_{ij}^T x + \bar{\zeta}_{ij}^T \mathbf{r} + \bar{\eta}_{ij}^T \mathbf{u} + \bar{\nu}_{ij} \leq \bar{\mu}_i \ \forall j \in \bar{\mathcal{J}}, \ \mathbf{H}_k \mathbf{u} \leq \mathbf{h}_k\}. \tag{5.17}$$

In order to obtain the expression of the optimizer $\mathbf{u}_N^0(x, \mathbf{r})$ we solve the linear program (5.17) that yields the optimal solution $(\mathbf{u}^{0,i}(x, \mathbf{r}), \bar{\mu}_i^0(x, \mathbf{r}))$ for each $i \in \bar{\mathcal{I}}$ and then we select the optimal $\mathbf{u}_N^0(x, \mathbf{r})$ as the optimal solution $\mathbf{u}^{0,i^*}(x, \mathbf{r})$, where the index i^* is given by $\bar{\mu}_{i^*}^0(x, \mathbf{r}) = \min_{i \in \bar{\mathcal{I}}} \bar{\mu}_i^0(x, \mathbf{r})$. $\diamondsuit$

The following algorithm describes the main steps in computing the solution of the open-loop min-max MPC-MMPS problem (5.9):

Algorithm 5.2.4

(i) Compute the max-min expression of V_N. Solve *off-line* the inner min-max MPC-MMPS problem (5.11) using multi-parametric MMPS programming. According to Theorem 5.2.2 J_N is an MMPS function. Compute also off-line the min-max canonical form of J_N.

(ii) Compute *on-line* (at each step k) the solution[3] of the outer min-max MPC-MMPS problem (5.16) according to Theorem 5.2.3. ♢

Remark 5.2.5

(i) The second step of the Algorithm 5.2.4 consists in solving a set of linear programming problems of the form (5.17) according to Theorem 5.2.3.

(ii) It is clear from Theorem 5.2.3 that the outer min-max MPC-MMPS problem can also be solved off-line, using again multi-parametric MMPS programming. From Lemma 5.1.2 it follows that the MPC controller $\kappa_N(x, \mathbf{r})$ has a PWA expression. Then, the second step of Algorithm 5.2.4 consists in solving off-line the outer min-max MPC-MMPS problem and then on-line at each step k we need only to evaluate a PWA function corresponding to the MPC controller. ♢

5.2.4 Solution based on duality for linear programming

In Algorithm 5.2.4 we have to solve off-line the inner min-max MPC-MMPS problem using multi-parametric MMPS programming. In the case when the reference signal r is not constant we have to include $\mathbf{r}$ as additional vector of parameters in the multi-parametric MMPS program when we want to solve the inner min-max MPC-MMPS problem off-line, because the cost function depends also on $\mathbf{r}$. Of course, the computational complexity increases in that case because the dimension of the vector of parameters $(x, \mathbf{u}, \mathbf{r})$ is much larger than $(x, \mathbf{u})$, corresponding to the case $\mathbf{r}$ is constant at each step k. An alternative method is to use the duality theory of linear programming [132, 147]. For each $i \in \mathcal{I}$ the primal problem (5.14) can be written as:

$$\mathbb{P}: \qquad \max_{(\mathbf{w},\mu_i)} \{\mu_i : -\delta_{ij}^T \mathbf{w} + \mu_i \leq \alpha_{ij}^T x + \beta_{ij}^T \mathbf{r} + \gamma_{ij}^T \mathbf{u} + \theta_{ij} \ \ \forall j \in \mathcal{J}_i, \ \boldsymbol{\Omega} \mathbf{w} \leq \mathbf{s}\}.$$

We denote with $c_{ij}(x, \mathbf{r}, \mathbf{u}) = \alpha_{ij}^T x + \beta_{ij}^T \mathbf{r} + \gamma_{ij}^T \mathbf{u} + \theta_{ij}$, which is an affine expression in $(x, \mathbf{r}, \mathbf{u})$. In matrix notation the primal problem becomes:

$$\mathbb{P}: \qquad \max_{(\mathbf{w},\mu_i)} \left\{ \mu_i : \begin{bmatrix} -\delta_{i1}^T & 1 \\ \vdots & \vdots \\ -\delta_{i|\mathcal{J}_i|}^T & 1 \\ \boldsymbol{\Omega} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mu_i \end{bmatrix} \leq \begin{bmatrix} c_{i1}(x, \mathbf{r}, \mathbf{u}) \\ \vdots \\ c_{i|\mathcal{J}_i|}(x, \mathbf{r}, \mathbf{u}) \\ \mathbf{s} \end{bmatrix} \right\},$$

where we recall that $|\mathcal{J}_i|$ denotes the cardinality of the index set $\mathcal{J}_i$. The corresponding dual problem has the following form:

$$\mathbb{D}: \qquad \min_{y_i \geq 0} \left\{ c_i^T(x, \mathbf{r}, \mathbf{u}) y_i : \begin{bmatrix} -\delta_{i1} & \cdots & -\delta_{i|\mathcal{J}_i|} & \boldsymbol{\Omega}^T \\ 1 & \cdots & 1 & 0 \end{bmatrix} y_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\},$$

[3]We assume that at step k, the state $x = x(k)$ and $\mathbf{r} = [r^T(k)\ r^T(k+1) \cdots r^T(k+N)]^T$ are available.

where $c_i(x, \mathbf{r}, \mathbf{u}) = [c_{i1}(x, \mathbf{r}, \mathbf{u}) \cdots c_{i|\mathcal{J}_i|}(x, \mathbf{r}, \mathbf{u})\ \mathbf{s}_1 \cdots \mathbf{s}_{n_{\boldsymbol{\Omega}}}]^T$ and we recall that $n_{\boldsymbol{\Omega}}$ denotes the number of rows of the matrix $\boldsymbol{\Omega}$.

There are algorithms (e.g. the double description method of [113]) to compute a compact explicit description of the elements of the polyhedral set:

$$K_i = \left\{ y_i \geq 0 : \begin{bmatrix} -\delta_{i1} & \cdots & -\delta_{i|\mathcal{J}_i|} & \boldsymbol{\Omega}^T \\ 1 & \cdots & 1 & 0 \end{bmatrix} y_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

These elements can be expressed as follows (according to *the finite basis theorem* [147]):

$$y_i = \sum_{i=1}^{N_i} \alpha_{ij} y_i^j + \sum_{i=1}^{M_j} \beta_{ij} z_i^j$$

with $\sum_i \alpha_{ij} = 1, \alpha_{ij} \geq 0$ and $\beta_{ij} \geq 0$. The y_i^j are called vertexes and the z_i^j are called extremal rays (using the definitions of [132,147]). Note that the *lines* (i.e. linear subspaces) are not present in the description of the polyhedron K_i since $y_i \geq 0$. Assuming that the primal problem $\mathbb{P}$ has a finite optimum, we are interested only in the vertexes (as extremal rays give rise to infinite solutions):

$$\{y_i^1, ..., y_i^{N_i}\}.$$

Note that the vertexes $y_i^1, ..., y_i^{N_i}$ do not depend on the reference signal $\mathbf{r}$, since $\mathbf{r}$ appears in the expressions of c_{ij} but not in the expression of the polyhedral set K_i. According to strong duality theorem for linear programming we have:

$$\mu_i^0(x, \mathbf{r}, \mathbf{u}) = \min\{c_i^T(x, \mathbf{r}, \mathbf{u}) y_i^1, \cdots, c_i^T(x, \mathbf{r}, \mathbf{u}) y_i^{N_i}\}$$

Then,

$$J_N(x, \mathbf{r}, \mathbf{u}) = \max_{i \in \mathcal{I}} \mu_i^0(x, \mathbf{r}, \mathbf{u}) = \max_{i \in \mathcal{I}} \min\{c_i^T(x, \mathbf{r}, \mathbf{u}) y_i^1, \cdots, c_i^T(x, \mathbf{r}, \mathbf{u}) y_i^{N_i}\}.$$

Therefore, we obtain directly the max-min canonical form of J_N. Algorithm 5.2.4 of the previous section can also be applied for this case. Clearly, after we eliminate the redundant terms the max-min expression of J_N obtained applying duality coincides with the max-min expression of J_N obtained from solving a multi-parametric MMPS program. Note however that the computational complexity of the two approaches may differ (see Section 5.3).

5.2.5 Disturbance feedback min-max MPC for MMPS systems

We recall from the previous chapters that in the presence of disturbances a feedback controller performs better than an open-loop controller. Without imposing any structure on the feedback controller the state feedback solution to the min-max control problem (5.9) can be determined using tools from dynamic programming and multi-parametric MMPS programming as was done for continuous PWA systems with bounded disturbances in [81]. However, optimizing over feedback policies described by arbitrary nonlinear functions is computationally a hard problem. So, another approach to controlling an uncertain MMPS system different from the open-loop approach presented in Section 5.2.2 is to include feedback by searching over the set of affine functions of the past disturbances as it was done in Section 3.3.3 for uncertain MPL systems. Therefore, we consider *disturbance feedback policies* of the form (3.58):

$$u_i = \sum_{j=0}^{i-1} M_{i,j} w_j + v_i \quad \forall i \in \mathbb{N}_{[0,N-1]}, \tag{5.18}$$

where each $M_{i,j} \in \mathbb{R}^{m\times q}$ and $v_i \in \mathbb{R}^m$. Recall that a similar feedback policy was used in [14, 60, 97] for robust control of uncertain linear systems. Using the same notation as in Section 3.3.3 the disturbance feedback policy becomes

$$\mathbf{u} = \mathbf{M}\mathbf{w} + \mathbf{v}, \tag{5.19}$$

where $\mathbf{v}$ and $\mathbf{M}$ are defined as in (3.59) and (3.60).

Under this type of policy we define the *disturbance feedback min-max* MPC-MMPS problem at event (k, x) as:

$$V_N^{0,\mathrm{df}}(x, \mathbf{r}) = \inf_{(\mathbf{M},\mathbf{v})\in\Pi_N^{\mathrm{df}}(k)} \max_{\mathbf{w}\in\tilde{\mathcal{W}}} V_N(x, \mathbf{r}, \mathbf{M}\mathbf{w}+\mathbf{v}, \mathbf{w}), \tag{5.20}$$

where the disturbance feedback policy (5.19) satisfies the input constraints

$$\Pi_N^{\mathrm{df}}(k) = \{(\mathbf{M}, \mathbf{v})\colon \mathbf{M} \text{ as in (3.60)},\ \mathbf{H}_k(\mathbf{M}\mathbf{w} + \mathbf{v}) \le \mathbf{h}_k\ \forall \mathbf{w} \in \tilde{\mathcal{W}}\} \tag{5.21}$$

and its optimizer is

$$\big(\mathbf{M}_N^0(x, \mathbf{r}), \mathbf{v}_N^0(x, \mathbf{r})\big) \in \arg \min_{(\mathbf{M},\mathbf{v})\in\Pi_N^{\mathrm{df}}(k)} \max_{\mathbf{w}\in\tilde{\mathcal{W}}} V_N(x, \mathbf{r}, \mathbf{M}\mathbf{w} + \mathbf{v}, \mathbf{w}), \tag{5.22}$$

whenever the infimum is attained. Note that for $\mathbf{M} = 0$ the disturbance feedback min-max problem (5.20) reduces to the open-loop case (5.9). It follows that

$$V_N^{0,\mathrm{df}}(x, \mathbf{r}) \le V_N^{0,\mathrm{ol}}(x, \mathbf{r})\ \ \forall (x, \mathbf{r}) \in \mathbb{R}^n \times \mathbb{R}^{Np}, \tag{5.23}$$

i.e. the performance in the disturbance feedback approach will in general be better than the open-loop approach, since we have more degrees of freedom through the matrix $\mathbf{M}$.

We also split the optimization problem (5.20) into two subproblems, as it was done in Section 5.2.3. The inner min-max MPC-MMPS problem is formulated as:

$$J_N(x, \mathbf{r}, \mathbf{M}, \mathbf{v}) = \max_{\mathbf{w}\in\tilde{\mathcal{W}}} V_N(x, \mathbf{r}, \mathbf{M}\mathbf{w} + \mathbf{v}, \mathbf{w})$$

or equivalently using (5.10) as

$$J_N(x, \mathbf{r}, \mathbf{M}, \mathbf{v}) = \max_{\mathbf{w}\in\tilde{\mathcal{W}}} \max_{i\in\mathcal{I}} \min_{j\in\mathcal{J}_i} \{\alpha_{ij}^T x + \beta_{ij}^T \mathbf{r} + \gamma_{ij}^T \mathbf{v} + (\gamma_{ij}^T \mathbf{M} + \delta_{ij}^T)\mathbf{w} + \theta_{ij}\}.$$

Using similar arguments as in Lemma 5.2.1 we conclude that for a given $(x, \mathbf{r}, \mathbf{M}, \mathbf{v})$, $J_N(x, \mathbf{r}, \mathbf{M}, \mathbf{v})$ can be computed efficiently using a set of linear programming problems. Note that in this particular case we cannot obtain an explicit expression for $J_N(x, \mathbf{r}, \mathbf{M}, \mathbf{v})$ as in the open-loop case (see (5.15)) since the function $(\mathbf{M}, \mathbf{w}) \mapsto \gamma^T \mathbf{M}\mathbf{w}$, for some fixed γ, is neither convex nor concave.

The outer min-max MPC-MMPS problem becomes:

$$\inf_{(\mathbf{M},\mathbf{v})\in\Pi_N^{\mathrm{df}}(k)} J_N(x, \mathbf{r}, \mathbf{M}, \mathbf{v}). \tag{5.24}$$

It is clear that $\big(\mathbf{M}_N^0(x, \mathbf{r}), \mathbf{v}_N^0(x, \mathbf{r})\big) \in \arg\min_{(\mathbf{M},\mathbf{v})\in\Pi_N^{\mathrm{df}}(k)} J_N(x, \mathbf{r}, \mathbf{M}, \mathbf{v})$.

Note that the feasible set $\Pi_N^{\mathrm{df}}(k)$ defined in (5.21) is not described by linear inequalities in $(\mathbf{M}, \mathbf{v})$. However, in the sequel we show that the feasible set can be recast as a polyhedron. We can rewrite the input constraints as

$$\mathbf{H}_k \mathbf{M}\mathbf{w} \le -\mathbf{H}_k \mathbf{v} + \mathbf{h}_k\ \ \forall \mathbf{w} \in \tilde{\mathcal{W}}$$

or equivalently as

$$[\max_{\mathbf{w}\in\tilde{\mathcal{W}}}(\mathbf{H}_k\mathbf{M})_1\mathbf{w}\cdots\max_{\mathbf{w}\in\tilde{\mathcal{W}}}(\mathbf{H}_k\mathbf{M})_{n_{H_k}}\mathbf{w}]^T \leq -\mathbf{H}_k\mathbf{v}+\mathbf{h}_k,$$

where n_{H_k} denotes the number of rows of $\mathbf{H}_k$ and that $(\mathbf{H}_k\mathbf{M})_i$ denotes the i^{th} row of the matrix $\mathbf{H}_k\mathbf{M}$. Therefore, using duality for linear programming and the fact that $\tilde{\mathcal{W}} = \{\mathbf{w} : \mathbf{\Omega}\mathbf{w} \leq \mathbf{s}\}$ is a polytope, it follows that (recall that we dealt with a similar problem in Section 3.3.3)

$$\Pi_N^{\mathrm{df}}(k) = \{(\mathbf{M},\mathbf{v}) : \exists\mathbf{Z}\geq 0,\ \mathbf{M}\text{ as in (3.60)},\ \mathbf{H}_k\mathbf{M} = \mathbf{Z}^T\mathbf{\Omega},\ \mathbf{Z}^T\mathbf{s}+\mathbf{H}_k\mathbf{v}-\mathbf{h}_k\leq 0\},$$

where we recall that by $\mathbf{Z}\geq 0$ we mean $\mathbf{Z}_{ij}\geq 0$ for all i,j. It follows that the outer worst-case problem can be written as:

$$\inf_{(\mathbf{M},\mathbf{v},\mathbf{Z})}\big\{J_N(x,\mathbf{r},\mathbf{M},\mathbf{v}) : \exists\mathbf{Z}\geq 0,\ \mathbf{M}\text{ as in (3.60)},\ \mathbf{H}_k\mathbf{M} = \mathbf{Z}^T\mathbf{\Omega},\ \mathbf{Z}^T\mathbf{s}+\mathbf{H}_k\mathbf{v}-\mathbf{h}_k\leq 0\big\}.$$

Note that now the feasible set is described by linear inequalities in $(\mathbf{M},\mathbf{v},\mathbf{Z})$. The following algorithm provides a solution to the disturbance feedback min-max MPC-MMPS problem formulated in this section:

Algorithm 5.2.6

(i) Compute the max-min expression of $J_N(x,\mathbf{r},\mathbf{M},\mathbf{v})$

(ii) Solve (5.20) using a standard nonlinear optimization algorithm for nonlinear optimization problems with linear constraints (e.g., a gradient projection algorithm[4] [132]). $\diamondsuit$

Note that in each iteration step l of the algorithm for the outer problem the function values of J_N (and its gradient, which can be obtained using numerical approximation) have to be computed in the current iteration point $(\mathbf{M}^l,\mathbf{v}^l)$. This involves solving the inner problem for the given $(\mathbf{M}^l,\mathbf{v}^l)$. This can be done efficiently by solving a set of linear programming problems as was shown before.

5.3 Computational complexity

From a computational point of view, both approaches that we have derived before (the open-loop scheme and the disturbance feedback scheme) consist in two steps. In the first step we have to solve the maximization problem corresponding to the worst-case uncertainty. This can be done off-line solving a set of multi-parametric MMPS programs as in Section 5.2.3 (or alternatively by computing the vertexes of some polyhedral set as in Section 5.2.4). In the second step we have to solve on-line a set of linear programming problems or to apply an iterative procedure based on solving a set of linear programming problems in order to determine the MPC input. The main advantage of the second approach is that by introducing feedback the corresponding MPC controller will perform equal or better than the open-loop MPC controller. This improvement in performance is obtained at the expense of introducing $\frac{N(N-1)}{2}\,m\,q+n_{H_k}\,n_{\mathbf{\Omega}}$ extra variables and $n_{H_k}+n_{\mathbf{\Omega}}$ extra inequalities (recall that n_{H_k} and $n_{\mathbf{\Omega}}$ denote the number of rows of the matrices $\mathbf{H}_k$ and $\mathbf{\Omega}$, respectively). Note that the number of $\min$ terms in the max-min canonical form of

[4] Note that sequential quadratic programming is less suited due to the PWA nature of the objective function.

the cost is the same in both approaches. See also Table 5.1 for a comparison of computational times for different methods applied to an example.

From Table 5.1 we see that in the case of open-loop min-max MPC, the CPU time corresponding to the dual approach (Section 5.2.4) is less than the CPU time corresponding to the multi-parametric MMPS approach (Section 5.2.3). Theoretically, it is known [23] that the number of partitions n_R generated by a multi-parametric linear program (see (5.6)) is less than or equal to the number of vertexes n_v corresponding to the polyhedron generated by the associated dual (i.e. K_i). The complexity of algorithms [113, 132] for enumerating the vertexes of K_i with $n_0 = n_{\mathbf{\Omega}} + 1$ rows and n_1 columns is $\mathcal{O}(n_0^2 n_1 n_v)$. An upper bound on the number of vertexes is given by [59]:

$$N_{\mathrm{r}} \leq n_v \leq \binom{n_0 + n_1 - \lfloor n_1/2 \rfloor}{\lfloor n_1/2 \rfloor} + \binom{n_0 + n_1 - 1 - \lfloor (n_1 - 1)/2 \rfloor}{\lfloor (n_1 - 1)/2 \rfloor},$$

where $\lfloor x \rfloor$ is the largest integer less or equal to x and $\binom{m}{n} = \frac{m!}{n!(m-n)!}$. This means that in the worst-case the number of vertexes n_v can be of the order $\mathcal{O}((n_0 + n_1)^{\lfloor n_1/2 \rfloor})$ if $n_0 + n_1 \gg n_1$. Of course, we need extra computations in order to compute the optimal value in the case of multi-parametric MMPS programming approach. Since the execution time of a multi-parametric linear programming algorithm depends on many factors, it is difficult to give a net characterization of the computational complexity as a function of the number of variables, parameters and inequalities. But, after elimination of the redundant terms both approaches produce the same number of affine expressions, i.e. we get the same MMPS function for J_N. Moreover, when the reference signal r is not a constant vector, the dimension of the vector of parameters $(x, \mathbf{r}, \mathbf{u})$ is larger than the case when r is constant, which makes the computation of a multi-parametric MMPS programming solution more difficult.

The worst-case complexity of the approaches presented above is largely determined by the number of linear terms in the equivalent max-min canonical forms. In the worst case scenario this number increases rapidly as the prediction horizon, the number of states of the MMPS system, or the number of min-max nestings in the state equations or the objective function increases. However, although the number of terms in the full max-min canonical expression may be very large, this number can sometimes be reduced significantly (in [46] the authors provide an example where the full canonical form contains 216 max-terms, of which only 4 are necessary). Although to the authors' best knowledge there are currently not yet any efficient algorithms for the simplification and reduction to a minimal canonical form (i.e., the canonical form with the minimal number of terms), some ad-hoc methods can be used [46, 66] to reduce the number of $\min$ terms significantly. Furthermore, the complexity of the reduction process can also be reduced by already eliminating redundant terms during the intermediate steps of the transformations. In conclusion, although the reduction to canonical form is computationally intensive, it can be done off-line (for both the inner and the outer worst-case MPC-MMPS problems).

If we consider reference tracking (the reference signal $r \neq 0$) or if consecutive disturbances are related, using the dynamic programming approach [81] we must include $\mathbf{r}$ or $\mathbf{w}$ as parameters in the multi-parametric program (see also Chapter 3), which increases the computational complexity. Note that these issues can be easily handled with our approaches (open-loop or disturbance feedback MPC). From the above we can conclude however that also our algorithms (Algorithm 5.2.4 and 5.2.6) are not well suited for large problems with many states, inputs and inequalities. This is not surprising since the computation of optimal control laws for PWA systems reduces to mixed-integer linear/quadratic optimization problems, which are difficult to solve [11].

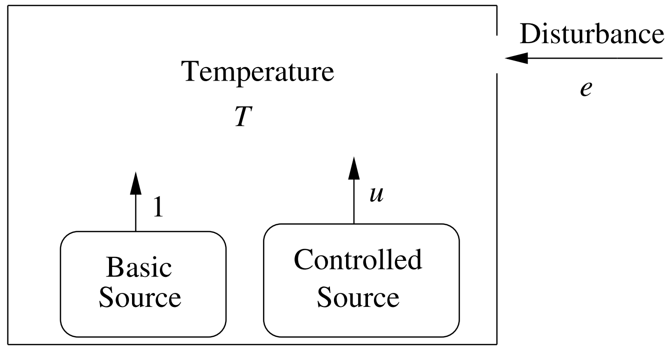

Figure 5.1: Temperature control system in a room.

5.4 Example: temperature control system in a room

In this section we present a typical example of a hybrid process for which we apply the methods described in this chapter. Consider a room with a basic heat source and an additional controlled heat source (see Figure 5.1). Let u be the contribution to the increase in room temperature per time unit caused by the controlled heat source (so $u \geq 0$). For the basic heat source, this value is assumed to be constant and equal to 1. The temperature in the room is assumed to be uniform and obeys the first-order differential equation

$$\dot{T}(t) = \alpha(T(t))T(t) + u(t) + 1 + e_1(t),$$

the disturbance being gathered in the scalar variable e_1. We assume that the temperature coefficient has a hybrid form, depending on a logic rule given by the following piecewise constant expression:

$$\alpha(T) = \begin{cases} -1/2 & \text{if} \quad T < 0 \\ -1 & \text{if} \quad T \geq 0. \end{cases}$$

We assume that the temperature is measured, but the measurement is noisy: $y(t) = T(t) + e_2(t)$. Using the Euler discretization scheme, with a sample time of 1 time unit and denoting the state $x(k) = T(k)$, we get the following continuous discrete-time PWA system:

$$\begin{aligned} x(k+1) &= \begin{cases} 1/2\, x(k) + u(k) + e_1(k) + 1 & \text{if } x(k) < 0 \\ u(k) + e_1(k) + 1 & \text{if } x(k) \geq 0 \end{cases} \\ y(k) &= x(k) + e_2(k). \end{aligned} \tag{5.25}$$

We define $w(k) = [e_1(k) \;\; e_2(k+1)]^T$ and assume that $-2 \leq w_1(k), w_2(k) \leq 2$, $w_1(k)+w_2(k) \leq 1$, i.e. the uncertainty set is given by the polytope

$$W = \big\{ w \in \mathbb{R}^2 : -2 \leq w_1, w_2 \leq 2,\ w_1 + w_2 \leq 1 \big\}.$$

The equivalent MMPS representation of (5.25) is given by:

$$\begin{aligned} x(k+1) &= \min\{1/2\, x(k) + u(k) + w_1(k) + 1, u(k) + w_1(k) + 1\} \\ y(k) &= x(k) + w_2(k-1). \end{aligned}$$

We take the prediction horizon $N = 3$. We consider the following constraints on the input[5]:

$$-4 \leq \Delta u(k) \leq 4, \qquad u(k) \geq 0 \ \ \forall\, k \geq 0, \tag{5.26}$$

[5] Because we have only heating, a physical constraint on input is $u(k) \geq 0$. Furthermore we assume that the rate of heating is bounded.

	off-line			on-line		
N	2	3	4	2	3	4
Nr. of param. LPs/LPs	7	12	18	4	8	16
Time param. LP saved (s)	10.2	45	130	0.12	0.35	0.79
Time dual (s)	0.35	0.9	2	0.06	0.08	0.1
Time dist. feedback (s)	0.65	2.3	7.5	0.09	0.3	0.95
Time param. LP ref. (s)	2	6.8	32	0.06	0.08	0.1

Table 5.1: The CPU time for $N \in \{2,\ 3,\ 4\}$ with different methods: param. LP saved=computes off-line and stores the controller for different values of r; dual=computes off-line the controller based on Section 5.2.4; dist. feedback=computes the controller based on Section 5.2.5; param. LP ref= the reference signal is considered as an extra parameter [81].

where $\Delta u(k) = u(k+1) - u(k)$. As stage cost we take

$$\ell(x, u, r) = \|y - r\|_\infty + \gamma \|u\|_1$$

and the terminal cost is $V_{\mathrm{f}}(x, r) = \|y - r\|_\infty$. The first term of the stage cost ℓ expresses the fact that we penalize the maximum difference between the reference and the output signal, while the second term penalizes the absolute value of the control effort. Because $u \geq 0$, we have $\|u\|_1 = u$ and therefore we get the following max-min canonical form for the cost function V_N:

$$V_N(x, \mathbf{r}, \mathbf{u}, \mathbf{w}) = \max\big\{ \min\{t_1, t_2\}, t_3, t_4, \min\{t_5, t_6, t_7\}, t_8, t_9, t_{10} \big\},$$

where t_j are appropriately defined affine functions of $(x, \mathbf{r}, \mathbf{u}, \mathbf{w})$.

We compute now the closed-loop MPC controller where $\gamma = 0.1$, initial state $x(0) = -6$, $u(-1) = 0$ and the reference signal $\{r(k)\}_{k=0}^{19} = -5, -5, -5, -5, -5, -3, -3, 1, 3, 3, 8, 8, 8, 8, 10,\ 10,\ 10,\ 7,\ 7,\ 7,\ 4,\ 3,\ 1,\ 1,\ 6,\ 7,\ 8,\ 9,\ 11,$ 11 using the methods given in Sections 5.2.3–5.2.5.

After we compute off-line the max-min canonical form of J_N and after elimination of the redundant terms, we obtain a min-max canonical form of J_N, which gives rise to only 4 linear programs that must be solved on-line at each sample step k in the open-loop approach.

In Table[6] 5.1 we provide the CPU time[7] for different steps of the algorithms and for different methods, where the values for the prediction horizon N are $2,\ 3$ and 4. Note that the number of multi-parametric MMPS programs or linear programs increases with N (see the third row). Note that in this example the computational time for the approach from Section 5.2.4 is less than the computational time for the multi-parametric MMPS programming approach from Section 5.2.3. Since the reference signal is not constant, we have to include $\mathbf{r}$ as an extra parameter when we apply the approach of [81], which results in a larger CPU time.

In Figure 5.2, the top plot represents the reference signal (dashed line) and the output of disturbance feedback approach (full line) and the open-loop approach (star line). We see that the MPC controller obtained using disturbance feedback policies performs the tracking better than the open-loop MPC controller. In the second plot we show the optimal input: we can see that always $u(k) \geq 0$. The third plot shows the absolute value of the tracking error. Note that the error from the open-loop approach is substantially above the error from disturbance feedback approach. Finally, we plot the input constraints (5.26). We can see that the constraint $|\Delta u(k)| \leq 4$ is also fulfilled, and that at some moments this constraint is indeed active.

[6] LP stands for linear program. Moreover, the off-line times do not include computation of the canonical forms and the reduction of the redundant terms since these operations were performed manually for this example.

[7] On a 1.5 GHz Pentium 4 PC with 512 MB RAM.

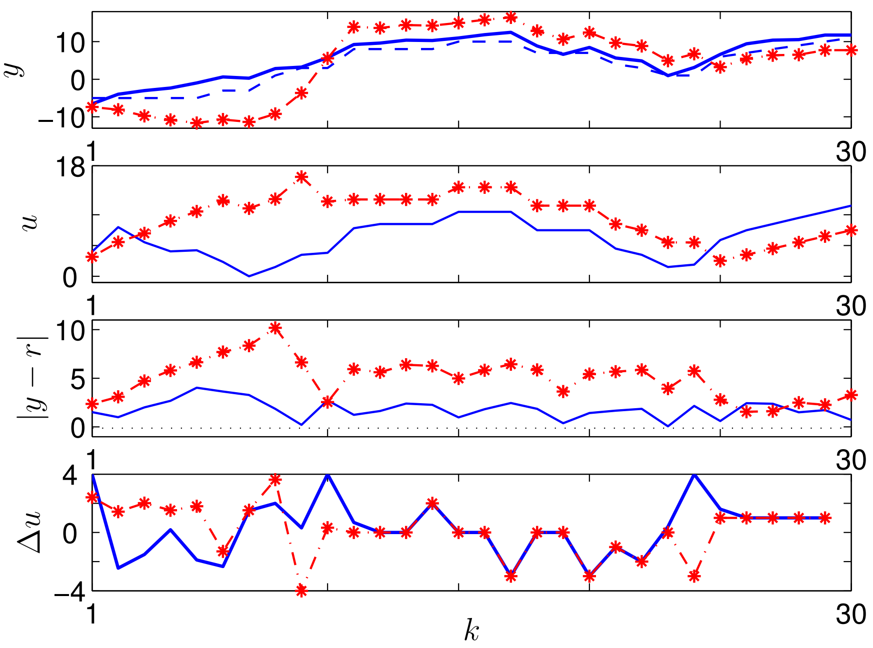

Figure 5.2: Illustration of the closed-loop MPC simulations for an uncertain MMPS system: full line-disturbance feedback approach, star line-open-loop approach, dashed line-reference signal r.

5.5 Conclusions

In this chapter we have extended the MPC framework for MMPS (or equivalently for general continuous PWA) systems to include also bounded disturbances. This allowed us to design a worst-case MPC-MMPS controller for such systems based on optimization over open-loop input sequences and disturbance feedback policies. We have shown that the resulting optimization problems can be computed efficiently using a two-step optimization approach that basically involves solving a set of linear programming problems. In the first step we have to solve off-line a multi-parametric MMPS program (or alternatively, we can compute the vertexes of some polyhedral set) and next we have to write the min-max expression of the worst-case performance criterion. In the second step we solve only a set of linear programming problems in both approaches. As we expected and was also illustrated in an example, the disturbance feedback based MPC controller performs better than the open-loop MPC controller, at the expense of introducing extra variables.

Chapter 6

Model predictive control for piecewise affine systems

Model predictive control (MPC) strategies for discrete time piecewise affine (PWA) systems are developed, which incorporate sufficient conditions to guarantee nominal and robust closed-loop stability. It is demonstrated how the structure of the PWA model can be exploited, both for designing an MPC strategy that is based on the combined use of a terminal set and a terminal cost, and for a robust MPC strategy that uses the dual-mode paradigm and the benefits of feedback. The MPC algorithms are illustrated with a couple of numerical examples.

6.1 Introduction

Hybrid systems model the interaction between continuous and logic components (see Section 2.1). Recently, hybrid systems have attracted the interest of both academia and industry [26, 49, 63, 151, 161], but tractable general analysis and control design methods for hybrid systems are not yet available. For this reason, several authors have studied special subclasses of hybrid systems for which analysis and control techniques are currently being developed: DES [29], (general) PWA systems [9–11, 75, 139, 151], MMPS systems [21, 44, 61], etc. A typical example of a hybrid system is the temperature control system in a room, discussed in Section 5.4.

Recently, research has been focused on developing stabilizing controllers for hybrid systems and in particular for PWA systems. PWA models are very popular, since they represent a powerful tool for approximating nonlinear systems with arbitrary accuracy and since a rich class of hybrid systems can be described by PWA systems. PWA systems are defined by partitioning *only* the state space of the system in a finite number of polyhedra and associating to each polyhedron a different affine dynamic. Several results about stability of PWA systems and MPC schemes for such systems can be found in the literature, see [11, 75, 89, 104, 112] and the references therein. For example, in [112] a piecewise linear (PWL) state feedback controller and also a quadratic Lyapunov function are derived, based on linear matrix inequalities (LMIs) that guarantees stability of the closed-loop PWA system. One of the first results in guaranteeing closed-loop stability of the MPC for PWA systems is obtained in [11], where a terminal equality constraint approach is employed. This type of constraint is rather restrictive, since in order to guarantee feasibility of the optimal problem we need a long prediction horizon, which leads to an optimization problem that is very demanding from a computational point of view. In [104] a terminal set and a terminal cost approach is presented to guarantee stability of the MPC scheme for continuous PWA systems in which the origin (i.e. the state equilibrium) lies in the interior of one of the polyhedra of the partition. In [89, 90] this approach is extended to PWA systems where the origin lies at the intersection of some polyhedra of the partition.

In this chapter we continue in the same line of research. In Section 6.1.2 we derive LMI conditions for stabilization of a PWL system using a PWL feedback controller and also a piecewise quadratic Lyapunov function. We also take into account the structure of the system, introducing less conservatism in the LMIs than in [112]. Moreover, from these LMIs we derive a static feedback controller that guarantees asymptotic stability of the closed-loop system on some region of attraction. We also derive LMI conditions that assure the static feedback controller satisfies given constraints on the inputs and outputs on that region. In general this region is small, therefore we will show in Section 6.2.1 that applying MPC we can also guarantee asymptotic stability, and we prove that this controller is better than the original static feedback controller, i.e. by this method the region of attraction increases such that for an infinite prediction horizon we obtain the maximal region of attraction. We derive a stable MPC strategy for PWA systems with an ellipsoidal terminal set and an upper bound on the infinite-horizon quadratic cost is used as a terminal cost. The terminal set and the terminal cost correspond to the PWL dynamics of the PWA system. It is important to note that although the PWA system may be discontinuous we will show that the optimal value function of the MPC optimization problem is continuous at the origin and can serve as a Lyapunov function for the closed-loop system. If the terminal set is small, we need a long prediction horizon. Therefore, we present in Section 6.2.3 an algorithm for enlarging this set based on *backward procedure* and then we show that a certain inner polytope approximation of this set can be used also as a polyhedral terminal set in Section 6.2.4. In this way this algorithm removes the drawback of the algorithms based on infinite recursive methods for constructing a terminal set. By enlarging the terminal set the prediction horizon can be chosen shorter and thus the computational complexity decreases.

Since disturbances are always present in the system, it is important that the designed controller be robust. In applications where safety and reliability are important requirements a robust controller designed on a worst-case scenario is in general preferable to other controllers. Some of the contributions in the literature on robust control for uncertain PWA systems include optimal control and min-max optimal control of continuous PWA systems with additive disturbances [50, 81, 138]. In [50] the H_∞ robust control problem for uncertain PWA systems is solved via LMIs. In [81, 138] robust control for the class of *continuous* PWA systems with additive disturbances is considered in the min-max framework, the optimal control problem being solved using dynamic programming (DP). In Chapter 5 of this thesis a min-max MPC strategy for the class of general continuous PWA systems with disturbances (i.e. systems whose dynamic equations are described by a PWA expression in the state *and* input space) is derived and the optimal problem is recast as a set of linear programs using the equivalent max-min canonical representation of a general continuous PWA system.

In Section 6.3 we consider the class of PWA systems with additive disturbances. We derive a stabilizing state feedback min-max MPC scheme based on a dual-mode approach and on the assumption that the mode is known. These allow us to preserve convexity of the state set evolution and thus to consider only the extreme disturbance realizations. We also provide an MPC strategy for PWL systems with additive disturbances in Section 6.4 where we remove this assumption on the mode. In this strategy we use the so-called *closed-loop paradigm* [145] by considering a semi-feedback policy that combines a local control law with an open-loop correction in order to guarantee satisfaction of the constraints. This scheme therefore renounces some degrees of freedom which in principle are available within a general state feedback policy formulation. On the other hand, it allows to well balance increased computational burden (that is met in the state feedback min-max framework) and reduction of conservativeness.

We recall from Section 2.1.1 the definition of a PWA system:

$$\begin{aligned} x(k+1) &= A_i x(k) + B_i u(k) + a_i \\ y(k) &= C_i x(k) + c_i \end{aligned} \quad \text{if } x(k) \in \mathcal{C}_i, \tag{6.1}$$

where $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $C_i \in \mathbb{R}^{p\times n}$ and $a_i \in \mathbb{R}^n$, $c_i \in \mathbb{R}^p$. Here, $\{\mathcal{C}_i\}_{i\in\mathcal{I}}$ is a polyhedral partition of the state space $\mathbb{R}^n$ and $\mathcal{I}$ is a finite index set. We may assume, without loss of generality that the origin is an equilibrium state for the PWA system (6.1). We denote with $\mathcal{I}_0 \subseteq \mathcal{I}$ the set of indexes for the polyhedral sets $\mathcal{C}_i$ that contain the origin in their closure. It follows that $|\mathcal{I}_0| \geq 1$ and that $a_i = 0,\ c_i = 0$ for all $i \in \mathcal{I}_0$. Each polyhedral cell is given by: $\mathcal{C}_i = \{x \in \mathbb{R}^n : E_i^1 x + e_i^1 \geq 0, E_i^2 x + e_i^2 > 0\}$, but the closure of $\mathcal{C}_i$ satisfies

$$\bar{\mathcal{C}}_i \subseteq \{x \in \mathbb{R}^n : E_i x + e_i \geq 0\}, \tag{6.2}$$

where $e_i = 0$ for all $i \in \mathcal{I}_0$.

The PWA system (6.1) is subject to hard input and output constraints:

$$X = \{x \in \mathbb{R}^n : |y_j| \leq y_{j,\max},\ \forall j \in \mathbb{N}_{[1,p]}\} \tag{6.3}$$

$$U = \{u \in \mathbb{R}^m : |u_j| \leq u_{j,\max},\ \forall\, j \in \mathbb{N}_{[1,m]}\}, \tag{6.4}$$

where $y_j = [C_i x + c_i]_j$ if $x \in \mathcal{C}_i$, and $y_{j,\max}, u_{j,\max} > 0$, i.e. X, U are compact sets, containing the origin in their interior.

This section proceeds now with the computation of lower and upper bounds on the infinite-horizon quadratic cost for the corresponding PWL dynamics of the system (6.1) based on a static PWL state feedback controller using an approach as in [75, 139] for continuous time PWA systems. It is important to note that in order to avoid the issues in connection with the existence of a stabilizing PWL state feedback controller we use the PWA formulation (6.1) (i.e. a system whose dynamic equations are described by a PWA expression *only* in the state space) instead of a general PWA system (i.e. a system whose dynamic equations are described by a PWA expression in the state *and* input space). The present chapter is an extension of [35, 114, 120].

6.1.1 Lower bound for the infinite-horizon quadratic cost

In this section we consider a generalization of the standard linear quadratic control for the discrete time PWL system obtained from the PWA system (6.1) corresponding to those modes $i \in \mathcal{I}_0$:

$$\begin{aligned} x(k+1) &= A_i x(k) + B_i u(k), \\ y(k) &= C_i x(k) \end{aligned} \quad \text{if } x(k) \in \mathcal{C}_i. \tag{6.5}$$

Throughout this chapter we assume that

$$0 \in \text{int}(\cup_{i\in\mathcal{I}_0} \mathcal{C}_i).$$

Let $\phi(k; x, \mathbf{u})$ denote the state solution of the PWL system (6.5) at step k when the initial state is x and the control sequence $\mathbf{u}$ is employed. We also use the short hand notation:

$$f_{\text{PWL}}(x, u) = A_i x + B_i u \quad \text{if } x \in \mathcal{C}_i.$$

The goal is to bring the system (6.5) to the origin from an arbitrary initial state x, satisfying the constraints on the inputs and outputs (6.3)–(6.4), limiting also the infinite-horizon quadratic cost:

$$V_\infty(x, \mathbf{u}) = \sum_{k=0}^{\infty} \ell(x_k, u_k), \tag{6.6}$$

where the stage cost is given by the quadratic expression

$$\ell(x,u) = x^T Q x + u^T R u, \tag{6.7}$$

such that $Q = Q^T \succ 0$, $R = R^T \succ 0$ (i.e. Q, R are positive definite matrices as defined in Appendix B), $\mathbf{u} = [u_0^T \; u_1^T \cdots]^T$ is an infinite control sequence and $x_k = \phi(k; x, \mathbf{u})$ (and thus $x_0 = x$). Note that the sets X and U can be written explicitly as:

$$\begin{array}{rcl} X \cap \mathcal{C}_i & = & \{x \in \mathbb{R}^n : H_i x + h_i \geq 0\} \\ U & = & \{u \in \mathbb{R}^m : Gu + g \geq 0\}. \end{array} \tag{6.8}$$

Using similar arguments as in [139] for continuous PWA systems the next theorem provides a lower bound on V_∞:

Theorem 6.1.1 *Suppose that there exists infinite control sequence* $\mathbf{u}$ *that asymptotically stabilizes the PWL system* (6.5).

(i) If there exist symmetric matrices $\overline{P}_i, \overline{U}_{ij}$ *such that* $\overline{U}_{ij} \geq 0$ *(i.e. all entries of* $\overline{U}_{ij}$ *are nonnegative) and that verify the following LMIs*[1]

$$\begin{bmatrix} R + B_i^T \overline{P}_j B_i & B_i^T \overline{P}_j A_i \\ * & A_i^T \overline{P}_j A_i - \overline{P}_i + \bar{Q} - E_i^T \overline{U}_{ij} E_i \end{bmatrix} \succcurlyeq 0 \tag{6.9}$$

for all $i, j \in \mathcal{I}_0$*, then the infinite-horizon quadratic cost verifies for every asymptotically stable trajectory with initial state* $x \in \mathcal{C}_{i_0}$*, where* $i_0 \in \mathcal{I}_0$*, the lower bound*

$$V_\infty(x, \mathbf{u}) \geq \sup_{(\overline{P}_i, \overline{U}_{ij})} \{x^T \overline{P}_{i_0} x : \overline{P}_i, \overline{U}_{ij} \text{ solution of } (6.9)\}. \tag{6.10}$$

(ii) If there exist symmetric matrices $\overline{P}_i, \overline{U}_{ij}$ *such that* $\overline{U}_{ij} \geq 0$ *and that verify the following LMIs*

$$\begin{bmatrix} R + B_i^T \overline{P}_j B_i & B_i^T \overline{P}_j A_i & 0 \\ * & A_i^T \overline{P}_j A_i - \overline{P}_i + Q & 0 \\ * & * & 0 \end{bmatrix} - \begin{bmatrix} \\ * \\ \end{bmatrix}^T \overline{U}_{ij} \begin{bmatrix} G & 0 & g \\ 0 & H_i & h_i \end{bmatrix} \succcurlyeq 0 \tag{6.11}$$

for all $i, j \in \mathcal{I}_0$*, then the infinite-horizon quadratic cost verifies for every asymptotically stable trajectory with initial state* $x \in \mathcal{C}_{i_0}$*, where* $i_0 \in \mathcal{I}_0$*, such that the input and the output constraints* (6.3)–(6.4) *are satisfied the lower bound* (6.10).

Proof: (i) We define the following piecewise quadratic function:

$$\overline{V}(x) = x^T \overline{P}_i x \;\text{ if } x \in \mathcal{C}_i,$$

where $\overline{P}_i$ are symmetric matrices for all $i \in \mathcal{I}_0$. Using a similar reasoning as in [75, 139] for continuous PWA systems it follows from (6.2) that

$$\mathcal{C}_i \subseteq \mathcal{E}_i := \{x \in \mathbb{R}^n : x^T E_i^T W E_i x \geq 0\}$$

for any symmetric matrix $W \geq 0$.

[1] In the sequel the symbol $*$ is used to induce a symmetric structure in an LMI.

We search for the matrices $\overline{P}_i$, where $i \in \mathcal{I}_0$, such that $\overline{V}$ satisfies the following inequality:

$$\overline{V}(x) - \overline{V}(f_{\text{PWL}}(x,u)) \leq \ell(x,u) \ \ \forall x \in \cup_{i\in\mathcal{I}_0}\mathcal{C}_i, u \in \mathbb{R}^m. \tag{6.12}$$

If we assume that $x \in \mathcal{C}_i$ and $f_{\text{PWL}}(x,u) \in \mathcal{C}_j$, then from (6.12) it follows that the matrices $\overline{P}_i$ must satisfy:

$$x^T\overline{P}_i x - (A_i x + B_i u)^T \overline{P}_j (A_i x + B_i u) \leq x^T Q x + u^T R u \ \ \forall x \in \mathcal{C}_i, u \in \mathbb{R}^m \tag{6.13}$$

for all $i, j \in \mathcal{I}_0$. Note that (6.13) can be replaced by the more conservative requirement: find the matrices $\overline{P}_i$ and the sets $\mathcal{E}_{ij} := \{x \in \mathbb{R}^n : x^T E_i^T \overline{U}_{ij} E_i x \geq 0\}$, where $\overline{U}_{ij} \geq 0$, satisfying

$$x^T\overline{P}_i x - (A_i x + B_i u)^T \overline{P}_j (A_i x + B_i u) \leq x^T Q x + u^T R u \ \ \forall x \in \mathcal{E}_{ij}, u \in \mathbb{R}^m \tag{6.14}$$

for all $i, j \in \mathcal{I}_0$. Applying the *S-procedure* to (6.14) (note that in this case the S-procedure is exact, provided that[2] int$(\mathcal{E}_{ij}) \neq \emptyset$, according to Section B.3 in Appendix B) we get: find $\overline{P}_i$ and $\overline{U}_{ij} \geq 0$ satisfying

$$x^T\overline{P}_i x - (A_i x + B_i u)^T \overline{P}_j (A_i x + B_i u) \leq x^T Q x + u^T R u - x^T E_i^T \overline{U}_{ij} E_i x \tag{6.15}$$

for all $x \in \mathbb{R}^n, u \in \mathbb{R}^m$ and for all $i, j \in \mathcal{I}_0$. The last inequalities lead us to the LMIs (6.9). Furthermore, since we assume that $x_\infty := \lim_{k\to\infty} x_k = 0$, from (6.12) we obtain

$$\big(\overline{V}(x_0) - \overline{V}(x_1)\big) + \big(\overline{V}(x_1) - \overline{V}(x_2)\big) + \cdots \leq \ell(x_0,u_0) + \ell(x_1,u_1) + \cdots$$

or equivalently (recall that $x_0 = x$)

$$\overline{V}(x) \leq V_\infty(x, \mathbf{u})$$

which leads us to the lower bound (6.10) on V_∞.
(ii) We now take into account the constraints (6.3)–(6.4). In this case (6.12) becomes:

$$\overline{V}(x) - \overline{V}(f_{\text{PWL}}(x,u)) \leq \ell(x,u) \quad \forall x \in \mathcal{C}_i \cap X, \ u \in U.$$

From (6.8) the constraints on the input can be written as $[G\ h_u][u^T\ 1]^T \geq 0$ and the constraints on the output as $[H_i\ h_i][x^T\ 1]^T \geq 0$ for all $x \in \mathcal{C}_i$. Then, by applying the S-procedure as in the first part of this proof we obtain: find $\overline{P}_i$ and $\overline{U}_{ij} \geq 0$ satisfying

$$\overline{V}(x) - \overline{V}(f_{\text{PWL}}(x,u)) \leq \ell(x,u) - \begin{bmatrix} u \\ x \\ 1 \end{bmatrix}^T \begin{bmatrix} * \end{bmatrix}^T \overline{U}_{ij} \begin{bmatrix} G & 0 & g \\ 0 & H_i & h_i \end{bmatrix} \begin{bmatrix} u \\ x \\ 1 \end{bmatrix}$$

for all $u \in \mathbb{R}^m, x \in \mathbb{R}^n$ and for all $i, j \in \mathcal{I}_0$. Using the equivalence (B.4)–(B.5) (see Appendix B) it follows immediately that we must replace the LMIs (6.9) with the LMIs (6.11). The rest follows from *(i)*. ♢

Remark 6.1.2

- Note that first, we can consider a quadratic function $\overline{V}(x) = x^T \bar{P} x$ (i.e. $P_1 = P_2 = \cdots = P$) and check if there exists a symmetric matrix P satisfying the LMIs (6.9) or (6.11). If such a matrix P does not exist, the next step is to consider a piecewise quadratic function as in the proof of Theorem 6.1.1.

[2] Recall that int$(\cdot)$ denotes the interior of a certain set.

- We can define a set $\Lambda = \{(i,j) \in \mathcal{I}_0^2 : \exists\, x \in \mathcal{C}_i, u \in U \text{ s.t. } f_{\mathrm{PWL}}(x,u) \in \mathcal{C}_j\}$ that gives us all possible transitions from one region to another and then to restrict $(i,j) \in \Lambda$ only. The set Λ can be determined via reachability analysis [9].

- We can search for tighter outer approximations with ellipsoids $\bar{\mathcal{E}}_{ij}$ for each polytope $\mathcal{C}_i$ than $\mathcal{E}_{ij}$, i.e. $\mathcal{C}_i \subseteq \bar{\mathcal{E}}_{ij} \subseteq \mathcal{E}_{ij}$. In that case, in the inequalities (6.14) the conditions $x \in \mathcal{E}_{ij}$ will be replaced by the less conservative conditions $x \in \bar{\mathcal{E}}_{ij}$. ♢

6.1.2 Upper bound for the infinite-horizon quadratic cost

In order to obtain an upper bound for V_∞ we could consider a particular control law that asymptotically stabilizes[3] the PWL system (6.5). Our first impulse would be to take the ordinary linear quadratic control

$$\kappa_0(x) = F_i^0 x \;\text{ if }\; x \in \mathcal{C}_i, \tag{6.16}$$

where $F_i^0 = -(R + B_i^T \overline{P}_i B_i)^{-1} B_i \overline{P}_i A_i$. Note that the PWL control law (6.16) cannot guarantee stability for (6.5). However, if the closed loop-system $x(k+1) = f_{\mathrm{PWL}}(x(k), \kappa_0(x(k)))$, which explicitly can be written as

$$x(k+1) = (A_i + B_i F_i^0)x(k) \;\text{ if }\; x(k) \in \mathcal{C}_i, \tag{6.17}$$

where $i \in \mathcal{I}_0$, is asymptotically stable, then we can choose the controller κ_0 in order to obtain an upper bound for the infinite-horizon quadratic cost. We can check stability via LMI feasibility as we will see in the sequel. Indeed, similarly as in the proof of Theorem 6.1.1 we introduce the piecewise quadratic function:

$$V(x) = x^T P_i x \quad \text{if } \; x \in \mathcal{C}_i,$$

where P_i are symmetric matrices such that

$$x^T P_i x \succ 0 \;\; \forall x \in \mathcal{C}_i \setminus \{0\} \tag{6.18}$$

and

$$V(f_{\mathrm{PWL}}(x, \kappa_0(x))) - V(x) \le -\ell(x, \kappa_0(x)) \;\; \forall x \in \cup_{i \in \mathcal{I}_0} \mathcal{C}_i. \tag{6.19}$$

Lemma 6.1.3 *Suppose that $f_{\mathrm{PWL}}(\cup_{i\in\mathcal{I}_0}\mathcal{C}_i, \kappa_0) \subseteq \cup_{i\in\mathcal{I}_0}\mathcal{C}_i$ and the piecewise quadratic function V satisfies the inequalities* (6.18)–(6.19). *Then the origin is asymptotically stable with respect to the closed-loop system* (6.17) *with a region of attraction $\cup_{i\in\mathcal{I}_0}\mathcal{C}_i$.*

Proof: First, let us note that if the piecewise quadratic function V satisfies the inequalities (6.18)–(6.19), then V is a *Lyapunov function* (see Definition C.1.3) for the closed-loop system (6.17). Indeed, $V(0) = 0$ and V is continuous at the origin (recall that we have assumed $0 \in \mathrm{int}(\cup_{i\in\mathcal{I}_0})\mathcal{C}_i$.). Then, from (6.18) it follows that there exists $\zeta > 0$ sufficiently small such that $V(x) \ge \zeta \|x\|_2^2$ for all $x \in \cup_{i\in\mathcal{I}_0}\mathcal{C}_i$. Furthermore, $\ell(x,u) > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}, u \in \mathbb{R}^m \setminus \{0\}$ and thus V decreases along the trajectories of (6.17) starting in $\cup_{i\in\mathcal{I}_0}\mathcal{C}_i$. From Theorem C.1.2 it then follows that the origin is stable with respect to the system (6.17). In fact, we have that

$$V(f_{\mathrm{PWL}}(x, \kappa_0(x))) - V(x) \le -\lambda_{\min}(Q)\|x\|_2^2$$

for all $x \in \cup_{i\in\mathcal{I}_0}\mathcal{C}_i$, where $\lambda_{\min}(Q)$ is the smallest eigenvalue of the positive definite matrix Q (this means that $\lambda_{\min}(Q) > 0$). We conclude that the conditions from Corollary C.1.4 are also satisfied (here $\alpha(z) = \zeta z^2, \beta(z) = \lambda_{\min}(Q) z^2$ are the required $\mathscr{K}$ functions), which also proves attractiveness and thus asymptotic stability. ♢

[3]For a brief introduction to Lyapunov stability the reader is referred to Appendix C.

Remark 6.1.4 Note that V is not continuous on $\cup_{i\in\mathcal{I}_0}\mathcal{C}_i$, but only in the origin. So, in the discrete time case we do not have to impose continuity everywhere for the function V as in the continuous time case [75, 139]. ♢

If $x \in \mathcal{C}_i$ and $f_{\mathrm{PWL}}(x, \kappa_0(x)) \in \mathcal{C}_j$, then (6.19) can be written explicitly as:

$$x^T(A_i+B_iF_i^0)^T P_j(A_i+B_iF_i^0)x - x^T P_i x \le -x^T Q x - x^T F_i^{0,T} R F_i^0 x \quad \forall x \in \mathcal{C}_i \tag{6.20}$$

for all $i, j \in \mathcal{I}_0$. We can relax (6.18), (6.20) to the following inequalities: for all $i, j \in \mathcal{I}_0$ the following inequalities hold

$$\begin{aligned} x^T P_i x > 0 \qquad & \forall x \in \mathcal{E}_i \setminus \{0\} \\ x^T(A_i+B_iF_i^0)^T P_j(A_i+B_iF_i^0)x - x^T P_i x + x^T Q x + x^T F_i^{0,T} R F_i^0 x \le 0 \quad & \forall x \in \mathcal{E}_{ij}, \end{aligned}$$

where $\mathcal{E}_i := \{x \in \mathbb{R}^n : x^T E_i^T W_i E_i x \ge 0\}$, $\mathcal{E}_{ij} := \{x \in \mathbb{R}^n : x^T E_i^T U_{ij} E_i x \ge 0\}$, and $W_i, U_{ij} \ge 0$. Using a similar reasoning as in the proof of Theorem 6.1.1 we relax these inequalities to: find P_i, W_i, U_{ij} such that $W_i, U_{ij} \ge 0$ and that satisfy the following inequalities for all $i, j \in \mathcal{I}_0$

$$\begin{aligned} x^T P_i x &> x^T E_i^T W_i E_i x \\ x^T(A_i + B_iF_i^0)^T P_j(A_i + B_iF_i^0)x - x^T P_i x &\le -x^T Q x - x^T F_i^{0,T} R F_i^0 x - x^T E_i^T U_{ij} E_i x \end{aligned}$$

for all $x \in \mathbb{R}^n, x \ne 0$. We obtain the following LMIs in P_i, U_{ij}, W_i:

$$\begin{aligned} P_i &\succ E_i^T W_i E_i, \\ (A_i + B_iF_i^0)^T P_j(A_i + B_iF_i^0) - P_i + Q + F_i^{0,T} R F_i^0 + E_i^T U_{ij} E_i &\preccurlyeq 0, \end{aligned} \tag{6.21}$$

where $W_i, U_{ij} \ge 0$, for all $i, j \in \mathcal{I}_0$. As a consequence of the Lemma 6.1.3 we conclude that if the LMIs (6.21) are feasible, then the origin is asymptotically stable with respect to the closed-loop system (6.17). Furthermore, from (6.21) it follows that (6.19) is still valid and thus

$$\big(V(x_1) - V(x_0)\big) + \big(V(x_2) - V(x_1)\big) + \cdots \le -\ell(x_0, \kappa_0(x_0)) - \ell(x_1, \kappa_0(x_1)) - \cdots ,$$

where $x_k = \phi(k; x, \kappa_0)$ for all $k \ge 0$ and we recall that $\phi(k; x, \kappa_0)$ denotes the state solution of (6.17) at step k when the initial state is x (i.e. the linear quadratic control κ_0 is applied to the PWL system (6.5)). It follows that for all $x \in \mathcal{C}_{i_0}$ with $i_0 \in \mathcal{I}_0$

$$V_\infty(x, \kappa_0) \le \inf_{(P_i, U_{ij}, W_i)} \{x^T P_{i_0} x : P_i, U_{ij}, W_i \text{ solution of (6.21)}\}.$$

If the linear quadratic controller (6.16) is not stabilizing for (6.17), i.e. the LMIs (6.21) do not have a feasible solution, or if the controller κ_0 does not satisfy the constraints on output and input (6.3)–(6.4), we have to look for another state feedback controller. In the sequel we provide methods to find such a controller. Let us consider the PWL state feedback controller

$$\kappa_{\mathrm{f}}(x) = F_i x \ \text{ if } \ x \in \mathcal{C}_i \tag{6.22}$$

for the PWL system (6.5). The closed-loop system becomes

$$x(k+1) = f_{\mathrm{PWL}}(x(k), \kappa_{\mathrm{f}}(x(k))). \tag{6.23}$$

We want to determine the gains F_i such that the origin is asymptotically stable with respect to (6.23) and the piecewise quadratic function V defined as

$$V(x) = x^T P_i x \quad \text{if } \ x \in \mathcal{C}_i \tag{6.24}$$

satisfies the following inequalities

$$\left.\begin{aligned} V(x) &> 0 \\ V(f_{\mathrm{PWL}}(x, \kappa_{\mathrm{f}}(x))) - V(x) &\le -\ell(x, \kappa_{\mathrm{f}}(x)) \end{aligned}\right\} \quad \forall x \in \cup_{i\in\mathcal{I}_0}\mathcal{C}_i \setminus \{0\}. \tag{6.25}$$

Remark 6.1.5

- It is important to note that the inequalities (6.25) are sufficient to guarantee that V is a Lyapunov function and the origin is asymptotically stable with respect to the closed-loop system (6.23) (according to Lemma 6.1.3).
- First, we can search for a common linear feedback controller $\kappa_{\mathrm{f}}(x) = Fx$ and a common quadratic Lyapunov function candidate $V(x) = x^T Px$, where $P \succ 0$, such that the inequalities (6.25) are satisfied. If such matrices F and P do not exist, then we search for a PWL feedback controller κ_{f} and a piecewise quadratic function V. Piecewise quadratic Lyapunov function candidates were also used in [75, 139] for continuous time PWA systems. $\diamondsuit$

The inequalities (6.25) can be written explicitly as: for all $i, j \in \mathcal{I}_0$ (assuming that $x \in \mathcal{C}_i$ and $f_{\mathrm{PWL}}(x, \kappa_{\mathrm{f}}(x)) \in \mathcal{C}_j$)

$$\begin{array}{c} x^T P_i x > 0 \\ x^T(A_i+B_iF_i)^T P_j(A_i+B_iF_i)x - x^T P_i x + x^T Q x + x^T F_i^T R F_i x \leq 0 \end{array} \forall x \in \mathcal{C}_i \setminus \{0\} \tag{6.26}$$

The inequalities (6.26) are implied by the more conservative inequalities:

$$\begin{array}{ll} x^T P_i x > 0 & \forall x \in \mathcal{E}_i \setminus \{0\} \\ x^T(A_i + B_iF_i)^T P_j(A_i + B_iF_i)x - x^T P_i x + x^T Q x + x^T F_i^T R F_i x < 0 & \forall x \in \mathcal{E}_{ij} \setminus \{0\} \end{array}$$

for all $i, j \in \mathcal{I}_0$. Now, applying the S-procedure we obtain the following matrix inequalities in P_i, F_i, U_{ij}, W_i:

$$P_i \succ E_i^T W_i E_i \ \ \forall i \in \mathcal{I}_0 \tag{6.27}$$

$$(A_i + B_iF_i)^T P_j(A_i + B_iF_i) - P_i + Q + F_i^T R F_i + E_i^T U_{ij} E_i \prec 0 \ \ \forall i, j \in \mathcal{I}_0, \tag{6.28}$$

where $W_i, U_{ij} \geq 0$. The following theorem gives a solution to (6.27)-(6.28):

Theorem 6.1.6 *The matrix inequalities* (6.27)–(6.28) *have a solution if and only if the following matrix inequalities in P_i, F_i, U_{ij}, W_i and θ have a solution*

$$\begin{bmatrix} B_i^T P_j B_i + \theta R - I & B_i^T P_j A_i + F_i \\ * & A_i^T P_j A_i - P_i + E_i^T U_{ij} E_i + \theta Q - F_i^T F_i \end{bmatrix} \prec 0 \tag{6.29}$$

$$P_i > E_i^T W_i E_i$$

for all $i, j \in \mathcal{I}_0$, where $U_{ij}, W_i \geq 0$ and $\theta > 0$.

Proof: It is easy to see that (6.28) can be written as:

$$\begin{bmatrix} F_i \\ I \end{bmatrix}^T \begin{bmatrix} B_i^T P_j B_i + R & B_i^T P_j A_i \\ * & A_i^T P_j A_i - P_i + E_i^T U_{ij} E_i + \bar{Q} \end{bmatrix} \begin{bmatrix} F_i \\ I \end{bmatrix} \prec 0.$$

Here, $M^{\perp}$ denotes the orthogonal complement of the matrix M. Furthermore, $\ker(M)$ and $\mathrm{Im}(M)$ denote the kernel and the image of M, respectively (see Section B.4 for appropriate definitions). In our case we have that

$$\ker([-I \ \ F_i]) = \mathrm{Im}(\begin{bmatrix} F_i \\ I \end{bmatrix}).$$

It is known [72] that the orthogonal complement of the column space of M is the null space of M^T. Let us note that the following equality holds $\begin{bmatrix} -I \\ F_i^T \end{bmatrix}^{\perp} = \begin{bmatrix} F_i \\ I \end{bmatrix}^T$. Therefore, the previous matrix inequalities can be written as:

$$\begin{bmatrix} -I \\ F_i^T \end{bmatrix}^{\perp} Q_{ij} \begin{bmatrix} -I \\ F_i^T \end{bmatrix}^{\perp,T} \prec 0, \tag{6.30}$$

where $Q_{ij} = \begin{bmatrix} B_i^T P_j B_i + R & B_i^T P_j A_i \\ * & A_i^T P_j A_i - P_i + E_i^T U_{ij} E_i + Q \end{bmatrix}$. Using now the elimination lemma (see Section B.4) we obtain that (6.30) is equivalent to

$$Q_{ij} \prec \sigma_{ij} \begin{bmatrix} -I \\ F_i^T \end{bmatrix} \begin{bmatrix} -I & F_i \end{bmatrix} \tag{6.31}$$

with $\sigma_{ij} \in \mathbb{R}$. Of course (6.31) has a solution if and only if

$$Q_{ij} \prec \sigma \begin{bmatrix} -I \\ F_i^T \end{bmatrix} \begin{bmatrix} -I & F_i \end{bmatrix} \tag{6.32}$$

with $\sigma > 0$ has a solution (take $\sigma > \max_{i,j}\{0, \sigma_{ij}\}$ for the implication "(6.31) $\Rightarrow$ (6.32)"; the other implication is obvious). Now if we divide (6.32) by $\sigma > 0$ and denote with $P_i \leftarrow 1/\sigma P_i$, $U_{ij} \leftarrow 1/\sigma U_{ij}$, $W_i \leftarrow 1/\sigma W_i$ and $\theta \leftarrow 1/\sigma$, we obtain (6.29). $\diamondsuit$

The matrix inequalities (6.29) are not LMIs due to the terms $F_i^T F_i$. Therefore, we have to use standard algorithms for solving bilinear matrix inequalities (BMIs) (see Section B.5 for an appropriate definition). The algorithms for solving BMIs cover both local and global approaches. Local approaches are computationally less intensive, and they consist in searching a feasible solution: if it exists then we have solved the problem, otherwise one cannot tell whether there is a feasible solution or not. Global algorithms are able to find a solution if the problem is feasible. The branch-and-bound algorithm derived in [154] can be used to solve globally our problem, although in this case the computational time is increasing in comparison with the local approach.

Since the sets X, U contain the origin in their interiors, there exists a sufficiently small set containing the origin in its interior such that the PWL state feedback controller κ_{f} defined in (6.22) satisfies the output and input constraints. However, since the matrices P_i are not positive definite, this set is not convex in general and it is difficult to determine such a set. Thus, we now discuss some possible relaxations for (6.27)–(6.28) such that the output and input constraints are satisfied on a convex set and the obtained matrix inequalities are easier to solve than the matrix inequalities (6.29).

The first relaxation is to replace (6.27) with $P_i \succ 0$ for all $i \in \mathcal{I}_0$. In this case we can apply the Schur complement (Section B.2) to (6.28). Note that (6.28) is equivalent to

$$\begin{gathered}(A_i + B_i F_i)^T S_j^{-1}(A_i + B_i F_i) - P_i + Q + F_i^T R F_i + E_i^T U_{ij} E_i \prec 0 \\ 0 \prec P_i \preccurlyeq S_i^{-1},\end{gathered} \tag{6.33}$$

for all $i, j \in \mathcal{I}_0$. In this way we also take into account the case $S_i = P_i^{-1}$. We now give a sketch of the proof for this equivalence: it is clear that if (6.28) has a solution, then there exists an $\epsilon > 0$ such that $(A_i+B_iF_i)^T P_j (A_i+B_iF_i) - P_i + Q + F_i^T R F_i + E_i^T U_{ij} E_i \prec -\epsilon(A_i+B_iF_i)^T(A_i+B_iF_i)$. Then, we can take $S_i^{-1} = P_i + \epsilon I$ and thus we obtain (6.33). The other implication is obvious.

Now using the Schur complement, (6.33) is equivalent to

$$\begin{bmatrix} P_i - Q - E_i^T U_{ij} E_i & (A_i + B_i F_i)^T & F_i^T \\ * & S_j & 0 \\ * & * & R^{-1} \end{bmatrix} \succ 0 \tag{6.34}$$

$$0 \prec P_i \preccurlyeq S_i^{-1}. \tag{6.35}$$

We give an algorithm for finding a solution for (6.34)–(6.35) based on an idea from [73]:

Algorithm 6.1.7 We want to solve the feasibility problem: find P_i, S_i, F_i, where $i \in \mathcal{I}_0$, that satisfy the following matrix inequalities

$$\begin{array}{c} LMI(S_i, P_i, F_i) \prec 0 \\ 0 \prec P_i \preccurlyeq S_i^{-1} \end{array} \quad \forall i \in \mathcal{I}_0,$$

where $LMI(S_i, P_i, F_i) \prec 0$ are the LMIs (6.34). It is clear that $0 \prec P_i \preccurlyeq S_i^{-1}$ is equivalent to $0 \prec S_i \preccurlyeq P_i^{-1}$ or $\lambda_{\max}(PS) \leq 1$ ($\lambda_{\max}(PS)$ denotes[4] the largest eigenvalue of the matrix PS in the conventional algebra). We take $0 < \theta < 1$. The algorithm consists of three steps.

Step 1

Solve $LMI(S_i, P_i, F_i) \prec 0$ for all $i \in \mathcal{I}_0$. Therefore, we have available a solution P_i^0, S_i^0, F_i^0. If $P_i^0 \preccurlyeq (S_i^0)^{-1}$, then we stop because we have found a solution. Otherwise, choose $\beta_i^0 > \lambda_{\max}(P_i^0 S_i^0)$.

Step 2

For all $k \geq 0$ fix P_i^k. Solve the following LMIs:

$$\begin{array}{c} LMI(S_i, P_i^k, F_i) \prec 0 \\ 0 \prec S_i \prec \beta_i^k (P_i^k)^{-1} \end{array} \quad \forall i \in \mathcal{I}_0.$$

We obtain $\{S_i^{k+1}\}_{i \in \mathcal{I}_0}$ and we define $\alpha_i^k = (1-\theta)\lambda_{\max}(S_i^{k+1} P_i^k) + \theta \beta_i^k$.

Step 3

Fix S_i^{k+1}. Solve the following LMIs:

$$\begin{array}{c} LMI(S_i^{k+1}, P_i, F_i) \prec 0 \\ 0 \prec P_i \prec \alpha_i^k (S_i^{k+1})^{-1} \end{array} \quad \forall i \in \mathcal{I}_0.$$

We obtain P_i^{k+1}, F_i^{k+1} and we define $\beta_i^{k+1} = (1-\theta)\lambda_{\max}(P_i^{k+1} S_i^{k+1}) + \theta \alpha_i^k$. $\diamondsuit$

Properties of the algorithm:

1. If Step 1 is feasible then Step 2 and 3 are feasible for all $k \geq 0$.
2. If there exists a k such that $\alpha_i^k \leq 1$ in Step 2 or $\beta_i^k \leq 1$ in Step 3 for all $i \in \mathcal{I}_0$, then we stop the algorithm. We have found a solution.
3. $0 < \beta_i^{k+1} < \alpha_i^k < \beta_i^k$ for all $i \in \mathcal{I}_0$. Therefore, there exists $\beta_i^* = \lim_{k \to \infty} \beta_i^k$ for all $i \in \mathcal{I}_0$. If $\beta_i^* < 1$ for all $i \in \mathcal{I}_0$, then the algorithm yields a solution.

[4]Note that the eigenvalues of the matrix PS are the same as the eigenvalues of the symmetric matrix $S^{1/2} P S^{1/2}$ and thus they are real.

We may assume that $\cup_{i\in\mathcal{I}_0}\mathcal{C}_i$ is a polytope (otherwise we can take an inner approximation with a polytope of this set) given by:

$$\cup_{i\in\mathcal{I}_0}\mathcal{C}_i = \{x \in \mathbb{R}^n : Dx \leq 1\},$$

where $D \in \mathbb{R}^{n_D \times n}$ (since the origin is assumed to lie in the interior of $\cup_{i\in\mathcal{I}_0}\mathcal{C}_i$).

Theorem 6.1.8 *(i) If the following LMIs in P_i, S_i, F_i, U_{ij}: $P_i \succ 0, S_i \succ 0, U_{ij} \geq 0$,*

$$\begin{bmatrix} P_i - Q - E_i^T U_{ij} E_i & (A_i + B_i F_i)^T & F_i^T \\ * & S_j & 0 \\ * & * & R^{-1} \end{bmatrix} \succ 0 \tag{6.36}$$

and the following BMIs

$$S_i P_i + P_i S_i \leq 2I \tag{6.37}$$

have a solution for all $i, j \in \mathcal{I}_0$, then this solution is also a solution of the matrix inequalities (6.34)–(6.35).

(ii) If the following additional LMIs

$$\begin{bmatrix} 1/\rho & D_{l.} \\ * & P_i \end{bmatrix} \succcurlyeq 0 \tag{6.38}$$

are satisfied for all $l \in \mathbb{N}_{[1,n_D]}, i \in \mathcal{I}_0$, where $D_{l.}$ denotes the l^{th} row of D, then the set $X_\mathrm{f} = \{x \in \mathbb{R}^n : x^T P_i x \leq \rho,\ i \in \mathcal{I}_0\}$, where $\rho > 0$, is a positively invariant (PI) set for the closed-loop system (6.23)*, convex, compact, containing the origin in its interior.*

(iii) If the following additional LMIs

$$\begin{bmatrix} \Lambda - E_i^T W_i E_i & F_i \\ * & P_i \end{bmatrix} \succcurlyeq 0,\ \ \Lambda_{jj} \leq u_{j,\max}^2/\rho \ \ \forall j \in \mathbb{N}_{[1,m]} \tag{6.39}$$

are satisfied for all $i \in \mathcal{I}_0$, where the matrices W_i have all entries non-negative, then the state feedback controller κ_f satisfies the input constraints (6.4) *for all $x \in X_\mathrm{f}$.*

(iv) If the following additional LMIs

$$\begin{bmatrix} \Gamma - E_i^T \tilde{W}_i E_i & C_i(A_i + B_i F_i) \\ * & P_i \end{bmatrix} \succcurlyeq 0,\ \ \Gamma_{jj} \leq y_{j,\max}^2/\rho \ \ \forall j \in \mathbb{N}_{[1,p]}, \tag{6.40}$$

are satisfied for all $i \in \mathcal{I}_0$, where the matrices $\tilde{W}_i$ have all entries non-negative, then the output of (6.23) *satisfies the output constraints* (6.3) *for all initial states $x \in X_\mathrm{f}$. Taking $\gamma = 1/\rho$ all formulas* (6.36)–(6.40) *are LMIs except* (6.37).

Proof: (i) The BMIs (6.37) imply[5] (see e.g. [150]) that

$$0 \prec S_i \preccurlyeq P_i^{-1} \ \text{ or equivalently } \ 0 \prec P_i \preccurlyeq S_i^{-1}$$

Applying the Schur complement to (6.36) and using the last inequality we get:

$$\begin{aligned} 0 &\prec P_i - Q - E_i^T U_{ij} E_i - (A_i + B_i F_i)^T S_j^{-1} (*) - F_i^T R F_i \\ &\preccurlyeq P_i - Q - E_i^T U_{ij} E_i - (A_i + B_i F_i)^T P_j (*) - F_i^T R F_i, \end{aligned}$$

[5] However, there is no equivalence between (6.37) and (6.35) as it is shown in [150].

i.e. the LMIs (6.28) are valid with the requirement that U_{ij} has all entries non-negative.

(ii) An ellipsoid $\{x \in \mathbb{R}^n : x^T P x \leq \rho\}$ is contained in a half space $\{x \in \mathbb{R}^n : d^T x \leq 1\}$ if and only if [168]

$$\rho\, d^T P^{-1} d \leq 1.$$

Using this remark and the Schur complement formula we obtain the LMIs (6.38). Furthermore, if $x \in X_\mathrm{f} \cap \mathcal{C}_{i_0}$ for some $i_o \in \mathcal{I}_0$, then $x^T P_{i_0} x \leq \rho$ and $f_\mathrm{PWL}(x, \kappa_\mathrm{f}(x)) = (A_{i_0} + B_{i_0} F_{i_0})x$. Therefore, for any $j \in \mathcal{I}_0$ we have:

$$x^T (A_{i_0} + B_{i_0} F_{i_0})^T P_j (A_{i_0} + B_{i_0} F_{i_0}) x \leq x^T P_{i_0} x \leq \rho,$$

i.e. $f_\mathrm{PWL}(x, \kappa_\mathrm{f}(x)) \in X_\mathrm{f}$. It follows that X_f is a PI set for (6.23). The set X_f is convex[6], compact and contains the origin in the interior since it is the intersection of a finite number of ellipsoids (each ellipsoid $\{x \in \mathbb{R}^n : x^T P_i x \leq \rho\}$ is a convex, compact set, containing the origin in the interior because $P_i \succ 0$).

(iii) The constraint on the input (6.4) is equivalent to $u_j^2 \leq u_{j,\max}^2$ for all $j \in \mathbb{N}_{[1,m]}$. We have $X_\mathrm{f} \subseteq \{x \in \mathbb{R}^n : x^T P_i x \leq \rho\}$ and if $x \in \mathcal{C}_i$, then

$$[\kappa_\mathrm{f}(x)]_j^2 \leq \max_{x \in X_\mathrm{f}} [F_i x]_j^2 \leq \max_{x^T P_i x \leq \rho} [F_i x]_j^2 \leq \max_{x^T P_i / \rho x \leq 1} [F_i x]_j^2 \leq \| \sqrt{\rho} [F_i P_i^{-1/2}]_j \|_2^2 =$$
$$\rho [F_i P_i^{-1} F_i^T]_{jj} = \rho [F_i P_i^{-1} F_i^T]_{jj} \leq \rho \Lambda_{jj} \leq u_{j,\max}^2.$$

Taking W_i with all entries non-negative, and applying the S-procedure, the last inequality translates into:

$$\Lambda - F_i P_i^{-1} F_i^T - E_i^T W_i E_i \geq 0$$

and $\Lambda_{jj} \leq u_{j,\max}^2 / \rho$, i.e. the LMIs (6.39), once we define $\gamma := 1/\rho$.

(iv) The LMIs (6.40) are derived in the same way. ♢

We propose now a second relaxation that is based on solving only LMIs. If we do not apply the S-procedure, i.e. we replace in (6.26) the condition "$x \in \mathcal{C}_i$" with the more conservative condition "$x \in \mathbb{R}^n$", then we obtain the following matrix inequalities:

$$\begin{aligned} P_i &\succ 0 \\ (A_i + B_i F_i)^T P_j (A_i + B_i F_i) - P_i + Q + F_i^T R F_i &\preccurlyeq 0 \end{aligned} \tag{6.41}$$

for all $i, j \in \mathcal{I}_0$. We use the following linearization of (6.41):

$$P_i = S_i^{-1}, F_i = Y_i G^{-1}.$$

Using this change of variables we see that the determination of the control law does not depend explicitly on the Lyapunov matrices P_i. The extra degree of freedom introduced by the matrix G which is not considered symmetric, is incorporated in the control variable, removing the special structure of P_i to G. A similar linearizing method was used in [39] in the context of stabilizing linear parametric varying systems. Another method to linearize (6.41) can be found in [83], i.e. $S = P^{-1}, F_i = Y_i S$. Let us consider an inner approximation with an ellipsoid of the polytope $\cup_{i \in \mathcal{I}_0} \mathcal{C}_i$, i.e.

$$\mathcal{E}(L) = \{x \in \mathbb{R}^n : x^T L^{-1} x \leq 1\} \subseteq \cup_{i \in \mathcal{I}_0} \mathcal{C}_i, \tag{6.42}$$

where $L \succ 0$. The computation of a maximal volume ellipsoid included in a polytope can be done using convex optimization as we saw in the proof of Theorem 6.1.8 (ii).

[6] We observe that X_f is in particular a PI set for the free switching system with the modes $i \in \mathcal{I}_0$.

Theorem 6.1.9 *(i) If the following LMIs in G, Y_i, S_i: $S_i \succ 0$ and*

$$\begin{bmatrix} G+G^T-S_i & (A_iG+B_iY_i)^T & (Q^{1/2}G)^T & (R^{1/2}Y_i)^T \\ * & S_j & 0 & 0 \\ * & * & I & 0 \\ * & * & * & I \end{bmatrix} \succ 0 \tag{6.43}$$

have a solution for all $i,j \in \mathcal{I}_0$, then $F_i = Y_iG^{-1}, P_i = S_i^{-1}$ are solutions of (6.41).

(ii) If the following additional LMIs

$$\begin{bmatrix} \lambda L - S_i & 0 \\ * & -\lambda + 1/\rho \end{bmatrix} \succcurlyeq 0 \tag{6.44}$$

are satisfied for all $i \in \mathcal{I}_0$, where $\lambda > 0$, then the set $X_\mathrm{f} = \{x \in \mathbb{R}^n : x^T P_i x \le \rho,\ i \in \mathcal{I}_0\}$, where $\rho > 0$, is a PI set for the closed-loop system (6.23)*, convex, and compact, containing the origin in its interior.*

(iii) If the following additional LMIs

$$\begin{bmatrix} \Lambda & Y_i \\ * & G+G^T-S_i \end{bmatrix} \succcurlyeq 0, \quad \Lambda_{jj} \le u_{j,\max}^2/\rho \ \ \forall j \in \mathbb{N}_{[1,m]} \tag{6.45}$$

are satisfied for all $i \in \mathcal{I}_0$, then the state feedback controller κ_f satisfies the input constraints (6.4) *for all $x \in X_\mathrm{f}$.*

(iv) If the following additional LMIs

$$\begin{bmatrix} \Gamma & C_i(A_iG+B_iY_i) \\ * & G+G^T-S_i \end{bmatrix} \succcurlyeq 0, \quad \Gamma_{jj} \le y_{j,\max}^2/\rho \ \ \forall j \in \mathbb{N}_{[1,p]}. \tag{6.46}$$

are satisfied for all $i \in \mathcal{I}_0$, then the output of (6.23) *satisfies the output constraints* (6.3) *for all initial states $x \in X_\mathrm{f}$. Taking $\gamma = 1/\rho$ all previous matrix inequalities become LMIs.*

Proof: (i) From (6.43) using the Schur complement, we observe first that G is a nonsingular matrix because

$$G + G^T \succ S_i$$

and also

$$0 \prec S_i \Rightarrow (S_i - G)^T S_i^{-1}(S_i - G) \succcurlyeq 0.$$

Therefore we get the following relation:

$$G + G^T - S_i \preccurlyeq G^T S_i^{-1} G$$

and

$$\begin{aligned} 0 &\prec G + G^T - S_i - (A_iG+B_iY_i)^T S_j^{-1}(*) - G^TQG - Y_i^TRY_i \\ &\preccurlyeq G^TS_i^{-1}G - (A_iG+B_iY_i)^T S_j^{-1}(*) - G^TQG - Y_i^TRY_i \\ &= G^T\big(S_i^{-1} - (A_i + B_iY_iG^{-1})^T S_j^{-1}(*) - Q - G^{-T}Y_i^TRY_iG^{-1}\big)G. \end{aligned}$$

Since G is nonsingular, taking $F_i = Y_iG^{-1}, P_i = S_i^{-1}$, from the last inequality we obtain (6.41).

(ii) The LMIs (6.44) express the fact that

$$\{x : x^T S_i^{-1} x \le \rho\} \subseteq \mathcal{E}(L)$$

since $S_i^{-1} = P_i$ (see [164] for a detailed discussion about inclusion of ellipsoids) and the rest of the proof is similar to the proof of Theorem 6.1.8 (ii).

(iii) The constraint on the input (6.4) is equivalent to $u_j^2 \leq u_{j,\max}^2$. We have $X_{\rm f} \subseteq \{x : x^T P_i x \leq \rho\}$ and if $x \in \mathcal{C}_i$, then

$$[\kappa_{\rm f}(x)]_j^2 \leq \max_{x \in X_{\rm f}} [Y_i G^{-1} x]_j^2 \leq \max_{x^T P_i x \leq \rho} [Y_i G^{-1} x]_j^2 \leq \max_{x^T P_i/\rho x \leq 1} [Y_i G^{-1} x]_j^2$$
$$\leq \|\sqrt{\rho}[Y_i G^{-1} S_i^{1/2}]_j\|_2^2 = \rho [Y_i G^{-1} S_i G^{-T} Y_i^T]_{jj} \leq \rho \Lambda_{jj} \leq u_{j,\max}^2.$$

Making use again of the S-procedure we obtain (6.45). We recall that $P_i = S_i^{-1}$.

(iv) This proof is similar to (iii). ♢

Note that for simplicity of the presentation we have considered that the ellipsoids $\mathcal{E}^L$ and $\{x \in \mathbb{R}^n : x^T P_i x \leq \rho\}$ are centered at the origin although we can choose the centers of these ellipsoids any appropriate points. The results still hold.

Corollary 6.1.10 *Suppose that the matrices $\overline{P}_i$ from Theorem 6.1.1 are available. Suppose also that the matrices F_i, P_i are available. Then*

(i) The infinite-horizon quadratic cost is bounded:

$$\sup_{(\overline{P}_i, \overline{U}_{ij})} x_0^T \bar{P}_{i_0} x_0 \leq V_\infty(x_0, \kappa_{\rm f}) \leq \inf_{(P_i, F_i, U_{ij})} x_0^T P_{i_0} x_0 \quad \forall x_0 \in \mathcal{C}_{i_0},\ i \in \mathcal{I}_0. \tag{6.47}$$

(ii) The origin is asymptotically stable with respect to the closed-loop system (6.23) *with a region of attraction*

$$\mathcal{E} := \cup_{i \in \mathcal{I}_0} \big(\{x \in \mathbb{R}^n : x^T P_i x \leq \rho\} \cap \mathcal{C}_i\big)$$

and the closed-loop outputs and inputs satisfy the constraints (6.3)–(6.4) *on $\mathcal{E}$. Moreover, the infinite-horizon quadratic cost is bounded from above by:*

$$V_\infty(x_0, \kappa_{\rm f}) \leq \rho \quad \forall x_0 \in \mathcal{E}.$$

Proof: (i) Since F_i, P_i are available, it follows that the function

$$V(x) = x^T P_i x \ \text{ if } \ x \in \mathcal{C}_i$$

is a piecewise quadratic Lyapunov function for the closed-loop system (6.23). Contrary to the continuous time case the Lyapunov function can be discontinuous across cell boundaries for discrete time case.

(ii) We know that $X_{\rm f}$ is a convex PI set. However, a larger PI set is $\mathcal{E}$ (note that $X_{\rm f} \subseteq \mathcal{E}$), which is a union of convex sets. From the LMIs (6.39) or (6.45) it follows that the controller $\kappa_{\rm f}$ satisfies the input constraints on $\mathcal{E}$. The output constraints are satisfied on $\mathcal{E}$ due to the LMIs (6.40) or (6.46). Asymptotic stability is proved using the same Lyapunov function V. Moreover, $V_\infty(x_0, \kappa_{\rm f}) \leq V(x_0) \leq \rho$ for all $x_0 \in \mathcal{E}$. ♢

6.2 MPC for PWA systems

In the previous section we have provided methods to derive a PWL feedback controller

$$\kappa_{\rm f}(x) = F_i x \quad \text{if} \ \ x \in \mathcal{C}_i$$

that stabilizes only the PWL dynamics (6.5) corresponding to the PWA system (6.1). Moreover, the closed-loop output and input trajectories corresponding to the PWL dynamics satisfy the constraints (6.3)–(6.4) on the convex PI set

$$X_{\rm f} = \{x \in \mathbb{R}^n : x^T P_i x \le \rho,\ i \in \mathcal{I}_0\}.$$

In general, this set is small in comparison with X_∞ defined as the largest domain of attraction achievable by a control law that asymptotically stabilizes the PWA system (6.1) and that makes the closed-loop output and input trajectories to satisfy the constraints (6.3)–(6.4). In this section we prove that using an MPC law the corresponding domain of attraction approximates X_∞ arbitrarily closely. The MPC scheme derived in this section also uses the terminal set and terminal cost framework.

The following assumption is assumed to hold in the remainder of this chapter:

A5: The PWL feedback controller $\kappa_{\rm f}$ and the piecewise quadratic function V satisfy (6.25). Moreover, the convex PI set $X_{\rm f} \subseteq X$ and it contains the origin in its interior.

The matrices F_i and P_i that define $\kappa_{\rm f}$ and V are determined using one of the methods discussed in Section 6.1.2.

6.2.1 Problem formulation

We use similar notations as in the previous chapters: $\phi(k; x, \mathbf{u})$ denotes the state solution of the PWA system (6.1) at step k when the initial state is x and the control sequence $\mathbf{u}$ is applied. We consider a prediction horizon of length N. We choose the following cost function:

$$V_N(x, \mathbf{u}) = \sum_{i=0}^{N-1} \ell(x_i, u_i) + V_{\rm f}(x_N), \tag{6.48}$$

where ℓ is the quadratic stage cost defined in (6.7), $\mathbf{u} = [u_0^T\ u_1^T \cdots u_{N-1}^T]^T$ and $x_i = \phi(i; x, \mathbf{u})$. For the terminal cost $V_{\rm f}(x)$ ideally we should take the infinite-horizon value cost $V_\infty(x, \kappa_{\rm f})$ (in this way a stable MPC strategy for linear systems is constructed in [105]), but due to the nonlinearity of our PWA system, this cannot be computed explicitly as in the linear case. Therefore, we replace the infinite-horizon value cost with its upper bound that we have derived in Section 6.1.2, i.e. $V_{\rm f}(x) = V(x)$ or explicitly

$$V_{\rm f}(x) = x^T P_i x \quad \text{if}\ \ x \in \mathcal{C}_i.$$

From Corollary 6.1.10 it follows that $V_{\rm f}(x) \le \rho$ for all $x \in X_{\rm f}$. We assume that at each step k the state $x(k)$ is available (i.e. can be measured or estimated).

For each initial condition x we define the set of feasible control sequences $\mathbf{u}$:

$$\Pi_N(x) = \{\mathbf{u} : x_i \in X,\ u_i \in U\ \ \forall i \in \mathbb{N}_{[0,N-1]},\ \ x_N \in X_{\rm f}\}. \tag{6.49}$$

Note that additionally to the output and input constraints (6.3)–(6.4) we impose also a *terminal constraint* $x_N \in X_{\rm f}$. Also, let X_N denote the set of initial states for which a feasible input sequence exists:

$$X_N = \{x : \Pi_N(x) \ne \emptyset\}. \tag{6.50}$$

The MPC law is obtained as follows. At event (x, k) (i.e. the state of the PWA system (6.1) at step k is x) the following optimal control problem is solved:

$$V_N^0(x) = \inf_{\mathbf{u} \in \Pi_N(x)} V_N(x, \mathbf{u}). \tag{6.51}$$

Let $\mathbf{u}_N^0(x) = [(u_0^0(x))^T \ (u_1^0(x))^T \cdots (u_{N-1}^0(x))^T]^T$ denote a minimizer of the optimization problem (6.51) (as defined in (2.18)), i.e.

$$\mathbf{u}_N^0(x) \in \arg \min_{\mathbf{u} \in \Pi_N(x)} V_N(x, \mathbf{u}) \tag{6.52}$$

and let $\mathbf{x}^0 = [x^T \ (x_1^0)^T \cdots (x_N^0)^T]^T$ denote the optimal state trajectory (i.e. $x_i^0 = \phi(i; x, \mathbf{u}_N^0(x))$). We obtain an implicit MPC law:

$$\kappa_N(x) = u_0^0(x).$$

Similar *quasi-infinite* (we use this terminology since V_f approximates the infinite-horizon value cost) MPC strategies have been employed also in [30, 83, 98] in the context of MPC for linear parametric varying systems with polytopic uncertainty or nonlinear systems.

6.2.2 MPC for PWA systems: closed-loop stability

We now study the behavior of the PWA system (6.1) in closed-loop with the MPC law κ_N:

$$\begin{aligned} x(k+1) &= A_i x(k) + B_i \kappa_N(x(k)) + a_i \\ y(k) &= C_i x(k) + c_i \end{aligned} \quad \text{if } x(k) \in \mathcal{C}_i \tag{6.53}$$

Theorem 6.2.1 *Suppose that the assumption* **A5** *holds. Then, we have:*

(i) The set X_N is a PI set for the closed-loop system (6.53) *and $X_\mathrm{f} \subseteq X_N$ for all $N > 0$.*

(ii) The origin is asymptotically stable with respect to the closed-loop system (6.53) *with a region of attraction X_N.*

(iii) The following inclusions hold $X_N \subseteq X_{N+1}$ for all N and $\lim_{N\to\infty} X_N = X_\infty$. Moreover, if there exists an N^* such that $X_{N^*} = X_{N^*+1}$, then $X_\infty = X_{N^*}$.

Proof: (i) Since we assume that assumption **A5** holds, it follows that the PWL feedback controller κ_f and the terminal set X_f satisfy the conditions $\mathcal{F}1$–$\mathcal{F}3$ given in Section 2.3.2 for the system (6.53). From Theorem 2.3.4 it follows that X_N is a PI set for the closed-loop system (6.53). Moreover, for all $N > 0$ and for all $x \in X_\mathrm{f}$ a feasible input sequence $\mathbf{u}_\mathrm{t} \in \Pi_N(x)$ is given by

$$\mathbf{u}_\mathrm{t} = [(\kappa_\mathrm{f}(x))^T \ (\kappa_\mathrm{f}(\phi(1; x, \kappa_\mathrm{f})))^T \cdots (\kappa_\mathrm{f}(\phi(N-1; x, \kappa_\mathrm{f})))^T]^T.$$

Therefore, $\Pi_N(x) \neq \emptyset$ and thus $x \in X_N$. We conclude that $X_\mathrm{f} \subseteq X_N$.

(ii) We will show that V_N^0 satisfies the conditions from Theorem 2.3.5, i.e. V_N^0 is a Lyapunov function. First let us note that the terminal cost V_f defined above satisfies the condition $\mathcal{S}1$ given in Section 2.3.2, according to (6.25). Moreover, from the first part of this proof we see that the conditions $\mathcal{F}1$–$\mathcal{F}3$ are fulfilled. It remains to show that V_N^0 is continuous at the origin. It is clear that $V_N^0(x) \geq \lambda_{\min}(Q)\|x\|_2^2$ for all $x \in X_\mathrm{f}$, where $\lambda_{\min}(Q) \geq 0$ (we recall that $V_\mathrm{f}(x) \geq 0$ for all $x \in X_\mathrm{f}$). For all $x \in X_\mathrm{f}$ we have that

$$V_N^0(x) \leq V_N(x, \mathbf{u}_\mathrm{t}).$$

We denote with $A_{F_i} = A_i + B_i F_i$ for all $i \in \mathcal{I}_0$. Since for all $x \in X_\mathrm{f}$ only the PWL dynamics of the PWA system are active (recall that X_f is a PI set for the PWL dynamics), it follows that $\phi(k;x,\mathbf{u}_\mathrm{t}) = A_{F_{i(k)}}...A_{F_{i(1)}}x$ for all $k \in \mathbb{N}_{[1,N]}$, where $i(1),...,i(k)$ is a feasible switching sequence. It follows that the function $x \mapsto V_N(x,\mathbf{u}_\mathrm{t})$ is piecewise quadratic on X_f. Moreover, after some long but straightforward computations it can be proved that there exists a symmetric matrix $\tilde{P}$ satisfying $V_N(x,\mathbf{u}_\mathrm{t}) \leq x^T\tilde{P}x$ for all $x \in \mathbb{R}^n$ and $\tilde{P}$ has at least one positive eigenvalue (otherwise, $0 < V_N^0(x) \leq V_N(x,\mathbf{u}_\mathrm{t}) \leq x^T\tilde{P}x \leq 0$ for all $x \in X_\mathrm{f}, x \neq 0$, i.e. we obtain a contradiction). It follows that $V_N(x,\mathbf{u}_\mathrm{t}) \leq x^T\tilde{P}x \leq \lambda_{\max}(\tilde{P})\|x\|_2^2$ for all $x \in X_\mathrm{f}$, where $\lambda_{\max}(\tilde{P}) > 0$. We obtain that

$$\lambda_{\min}(Q)\|x\|_2^2 \leq V_N^0(x) \leq V_N(x,\mathbf{u}_\mathrm{t}) \leq \lambda_{\max}(\tilde{P})\|x\|_2^2 \quad \forall x \in X_\mathrm{f}.$$

Since X_f contains a ball centered at the origin (according to assumption **A5**), it follows that V_N^0 is continuous at the origin. Asymptotic stability follows now from Theorem 2.3.5. Note that we do not need V_N^0 be continuous on X_N but rather be continuous at the origin.

(iii) If $x \in X_N$, then there exists an $\mathbf{u}_N^0(x) \in \Pi_N(x)$. Therefore, $[(\mathbf{u}_N^0(x))^T \ (\kappa_\mathrm{f}(x_N^0))^T]^T \in \Pi_{N+1}(x)$. So, $x \in X_{N+1}$, i.e. $X_N \subseteq X_{N+1}$. Let us define $\bar{X}_\infty = \lim_{N\to\infty} X_N = \cup_{N\geq 0} X_N$. We prove that $\bar{X}_\infty = X_\infty$. It is clear that $\bar{X}_\infty \subseteq X_\infty$. It remains to prove that $X_\infty \subseteq \bar{X}_\infty$. Let $x \in X_\infty$. Then, from the definition of X_∞ there exists a feasible input sequence $\mathbf{u}_0^\infty = [u_0^T \ u_1^T \cdots]^T$ such that the state trajectory starting from x satisfies $\lim_{k\to\infty}\phi(k;x,\mathbf{u}_0^\infty) = 0$, and the input and the output constraints are also satisfied. Now, since the set X_f contains the origin in its interior and since $\lim_{k\to\infty}\phi(k;x,\mathbf{u}_0^\infty) = 0$, there exists a finite N such that $\phi(N;x,\mathbf{u}_0^\infty) \in X_\mathrm{f}$. Therefore, $x \in X_N \subseteq \bar{X}_\infty$, i.e. $X_\infty \subseteq \bar{X}_\infty$.

Furthermore, from the equality $X_{N^*} = X_{N^*+1}$ it follows that there does not exist a state $x \notin X_{N^*}$ such that with a feasible input $u \in U$ the next state $f_{\mathrm{PWA}}(x,u) \in X_{N^*}$. Therefore, $X_\infty = X_{N^*}$. ♢

Remark 6.2.2

- Note that although the PWA system may be discontinuous we have shown that the optimal value function V_N^0 of the MPC optimization problem is continuous at the equilibrium and can serve as a Lyapunov function for the closed-loop system (see the proof of Theorem 6.2.1 (ii)). This result is important since most of the literature on MPC for general nonlinear systems assumes that V_N^0 is continuous on X_N, which is a conservative requirement in the hybrid case.

- Point *(i)* is essential in order to ensure that it is worth replacing the auxiliary controller κ_f with the MPC controller. Point *(iii)* shows that at the cost of an increasing computational effort associated with the optimization problem (6.51), the domain of attraction can be enlarged towards the maximum achievable one. Therefore, N is a tuning parameter that realizes a trade-off between complexity and performance.

- When $N = 1$ we have to solve at each step k a convex optimization problem. If $N > 1$, the optimization problem (6.51) is a non-convex optimization problem: the objective function is convex subject to linear and convex inequality constraints and nonlinear equality constraints. In Section 6.2.4 we construct another PI set, different from X_f, that is given by linear inequalities. In that case we can solve (6.51) using mixed-integer quadratic programming, which is also a nonlinear optimization problem but using branch-and-bound methods it is more tractable. ♢

6.2.3 Enlargement of the terminal set

In the previous section we have derived a stable MPC strategy for PWA systems using a terminal cost and a terminal set approach. For the terminal cost we have chosen the piecewise quadratic Lyapunov function V corresponding to the PWL dynamics of the system, while the terminal set was given by a certain sub-level set of V. The resulting optimization problem that we have to solve on-line at each sample step k is non-convex, the computational time increasing with the prediction horizon N. If the terminal set is small, then we need a long prediction horizon in order to enlarge the domain of attraction X_N. Therefore, the optimization problem will be computationally intensive. In the sequel we develop a method to enlarge the terminal set based on *backward procedure*. This procedure can be performed off-line, and thus we can efficiently implement on-line the stable MPC scheme derived previously using a shorter prediction horizon N.

The backward procedure consists of three steps:

Algorithm 6.2.3

Step 1

Suppose that the LMIs from Theorem 6.1.9 are feasible. For simplicity we consider $\rho = 1$. Solve the following convex optimization problem:

$$\min_{(G,Y_i,S_i)} -\sum_{i\in\mathcal{I}_0} \log\det S_i$$

subject to:

$$\begin{bmatrix} G+G^T-S_i & (A_iG+B_iY_i)^T & (Q^{1/2}G)^T & (R^{1/2}Y_i)^T \\ * & S_j & 0 & 0 \\ * & * & I & 0 \\ * & * & * & I \end{bmatrix} \succ 0, \quad 0 \prec S_i \preccurlyeq \lambda L \tag{6.54}$$

$$\begin{bmatrix} \Lambda & Y_i \\ * & G+G^T-S_i \end{bmatrix} \succcurlyeq 0, \quad \begin{bmatrix} \Gamma & C_i(A_iG+B_iY_i) \\ * & G+G^T-S_i \end{bmatrix} \succcurlyeq 0, \tag{6.55}$$

with $\Lambda_{jj} \le u^2_{j,\max}, \Gamma_{jj} \le y^2_{j,\max},\ 0 < \lambda \le 1$ for all $i, j \in \mathcal{I}_0$ and define

$$F_{i,1} = Y_iG^{-1}, P_{i,1} = S_i^{-1}, \mathcal{E}^1 = \{x \in \mathbb{R}^n : x^TP_{i,1}x \le 1,\ i \in \mathcal{I}_0\}.$$

Recall that the matrix L represents the ellipsoid defined in (6.42). From Corollary 6.1.10 it follows that the origin is asymptotically stable with respect to the closed-loop system (6.23) and the closed-loop outputs and inputs satisfy the constraints (6.3)–(6.4) on $\mathcal{E}^1$.

Step 2

Using the previous terminal set $\mathcal{E}^{\text{prev}} = \{x \in \mathbb{R}^n : x^TP_{i,\text{prev}}x \le 1 \ \ \forall i \in \mathcal{I}_0\}$, we construct a new larger terminal set $\mathcal{E}^{\text{new}} = \{x \in \mathbb{R}^n : x^TP_{i,\text{new}}x \le 1 \ \ i \in \mathcal{I}_0\}$ based on a PWL feedback controller $\kappa(x) = F_{i,\text{new}}x$ if $x \in \mathcal{C}_i$, that steers the system from $\mathcal{E}^{\text{new}}$ but not within $\mathcal{E}^{\text{prev}}$ to the last terminal set $\mathcal{E}^{\text{prev}}$. The new set $\mathcal{E}^{\text{new}}$ is computed by solving the following convex optimization problem:

$$\min_{(G,Y_i,S_i)} -\sum_{i\in\mathcal{I}_0} \log\det S_i$$

subject to

$$\begin{bmatrix} G+G^T-S_i & (A_iG+B_iY_i)^T \\ * & P^{-1}_{j,\text{prev}} \end{bmatrix} \succ 0, \quad \lambda_1 P^{-1}_{i,\text{prev}} \preccurlyeq S_i \preccurlyeq \lambda_2 L \tag{6.56}$$

and the LMIs (6.55) for all $i, j \in \mathcal{I}_0$, where $\lambda_1 \geq 1, 0 < \lambda_2 \leq 1$.

We sketch the proof. We denote with $P_{i,\text{new}} = S_i^{-1}, F_{i,\text{new}} = Y_i G^{-1}$. The second LMI in (6.56) is equivalent with $\mathcal{E}^{\text{prev}} \subseteq \mathcal{E}^{\text{new}} \subseteq \mathcal{E}(L)$. The first LMI in (6.56), after applying the Schur formula, expresses the fact that:

$$P_{i,\text{new}} = S_i^{-1} \succcurlyeq (A_i + B_i F_{i,\text{new}})^T P_{j,\text{prev}} (A_i + B_i F_{i,\text{new}}),$$

i.e. if $x \in (\mathcal{E}^{\text{new}} \cap \mathcal{C}_i) - \mathcal{E}^{\text{prev}}$ and applying the input $u = F_{i,\text{new}} x$, then $f_{\text{PWL}}(x, u) = (A_i + B_i F_{i,\text{new}})x \in \mathcal{E}^{\text{prev}}$. The LMIs (6.55) guarantee that the input and output constraints are satisfied on $\mathcal{E}^{\text{new}}$.

Step 2 is an iterative procedure, i.e. we repeat it as long as we want, let us say M times (e.g. we stop when there is no more increase in the volume of the set $\mathcal{E}^{\text{new}}$ or when $\mathcal{E}^{\text{new}}$ is not contained in $\mathcal{E}(L)$) and we obtain the sets $\mathcal{E}^1 \subseteq \mathcal{E}^2 \subseteq \cdots \subseteq \mathcal{E}^M$.

Therefore, we have available a sequence of feedback controllers

$$\kappa(x) = F_{i,l} x \ \text{ if } \ x \in (\mathcal{E}^l \setminus \mathcal{E}^{l-1}) \cap \mathcal{C}_i$$

for all $i \in \mathcal{I}_0$ and $l \in \mathbb{N}_{[1,M]}$. By definition $\mathcal{E}^0$ is the empty set. We define the terminal set $X_\text{f} = \mathcal{E}^M$.

Step 3

At this stage we want to find a piecewise quadratic terminal cost

$$V_\text{f}(x) = x^T P_i x \ \text{ if } \ x \in \mathcal{C}_i$$

such that stability is guaranteed when we use the MPC strategy derived in the previous section with the terminal set $X_\text{f} = \mathcal{E}^M$. The matrices $P_i \succ 0$ are determined by solving the following LMIs:

$$\begin{gathered} P_i \succ 0 \\ (A_i + B_i F_{i,l})^T P_j (A_i + B_i F_{i,l}) - P_i + Q + F_{i,l}^T R F_{i,l} + E_i^T U_{ij} E_i \preccurlyeq 0 \end{gathered} \tag{6.57}$$

for all $i, j \in \mathcal{I}_0$, $l \in \mathbb{N}_{[1,M]}$, where $U_{ij} \geq 0$. The reader should note that the LMIs (6.57) guarantee that the terminal cost V_f defined above satisfies the condition $\mathcal{S}1$ given is Section 2.3.2 for the closed-loop system

$$x(k+1) = (A_i + B_i F_{i,l}) x(k) \ \text{ if } \ x(k) \in (\mathcal{E}^l - \mathcal{E}^{l-1}) \cap \mathcal{C}_i. \tag{6.58}$$

The following consequence follows:

Corollary 6.2.4 *(i) The feedback controller $\kappa(x) = F_{i,l} x$ if $x \in (\mathcal{E}^l \setminus \mathcal{E}^{l-1}) \cap \mathcal{C}_i$ asymptotically stabilizes the closed-loop system* (6.58).

(ii) The convex set $X_\text{f} = \mathcal{E}^M$ is a PI set for the closed-loop system (6.58).

(iii) Using $X_\text{f} = \mathcal{E}^M$ as a convex terminal set and the piecewise quadratic terminal cost $V_\text{f}(x) = x^T P_i x$ if $x \in \mathcal{C}_i$, with P_i solution of (6.57)*, Theorem 6.2.1 still holds.*

Proof: It is obvious that the origin is asymptotically stable with respect to the closed-loop system (6.58), because for any $x \in \mathcal{E}^M$ in at most M steps the state trajectory of (6.58) reaches $\mathcal{E}^1$ and then according to Corollary 6.1.10 it converges asymptotically towards zero. Moreover, the input and output constraints are satisfied on $\mathcal{E}^M$.

For the last part we observe that if $x \in (\mathcal{E}^l \cap \mathcal{C}_i) \subseteq \mathcal{E}^M$, then we have $(A_i + B_i F_{i,l})x \in \mathcal{E}^{l-1} \subseteq \mathcal{E}^M$. Therefore, $X_\text{f} = \mathcal{E}^M$ is a PI set for the closed-loop system (6.58). Furthermore, the LMIs (6.57) guarantee stability for the MPC scheme derived in the previous section with the new terminal set X_f and the new terminal cost V_f defined before. ♢

Remark 6.2.5 In the first two steps of the backward procedure defined in Algorithm 6.2.3 we could also use the matrix inequalities defined in Theorem 6.1.8. This means that in *Step 1* we should solve an optimization problem subject to the LMIs and BMIs from Theorem 6.1.8, i.e. a non-convex optimization problem. Similarly, we could proceed in *Step 2*, namely we should solve the optimization problem:

$$\min_{(F_i,P_i,U_{ij})} \sum_{i\in\mathcal{I}_0} \log\det P_i$$

$$\text{subject to: } \begin{bmatrix} P_i - E_i^T U_{ij} E_i & (A_i + B_i F_i) \\ * & P_{j,\text{prev}}^{-1} \end{bmatrix} \succ 0, \quad 0 \prec P_i \preccurlyeq \lambda P_{i,\text{prev}},$$

and the LMIs (6.39)–(6.40), where $U_{ij} \geq 0$ for all $i, j \in \mathcal{I}_0$ and $0 < \lambda \leq 1$. ♢

6.2.4 Polyhedral terminal set

In Section 6.2.3 we have presented an algorithm to enlarge an ellipsoidal terminal set. For this type of terminal sets the optimal control problem (6.51) is non-convex (except the case when $N = 1$) and thus difficult to solve on-line.

In this section we provide a method to construct a polyhedral terminal set. Note that if $X_\text{f} = \cup_{i\in\mathcal{I}_0}\mathcal{P}_i$, where each $\mathcal{P}_i$ is a polyhedral set, then the optimization problem (6.51) becomes a mixed-integer quadratic program [11] that can be solved using branch-and-bound algorithms. Our method is based on the backward procedure presented in Algorithm 6.2.3. Let us assume that we have available $\mathcal{E}^1 \subseteq \mathcal{E}^2 \subseteq \cdots \subseteq \mathcal{E}^M$. Note that

$$\mathcal{E}^{M-1} = \{x \in \mathbb{R}^n : x^T P_{i,M-1} x \leq 1 \ \forall i \in \mathcal{I}_0\}, \ \mathcal{E}^M = \{x \in \mathbb{R}^n : x^T P_{i,M} x \leq 1 \ \forall i \in \mathcal{I}_0\}$$

and

$$\{x \in \mathbb{R}^n : x^T P_{i,M-1} x \leq 1\} \subseteq \{x \in \mathbb{R}^n : x^T P_{i,M} x \leq 1\} \ \ \forall i \in \mathcal{I}_0. \tag{6.59}$$

We consider the case when the inclusions in (6.59) are strict. In this case we can derive a polyhedral terminal set as follows: we define the polyhedral sets

$$\mathcal{P}_i = \{x \in \mathbb{R}^n : H^i x \leq h^i\}$$

satisfying the following inclusions

$$\{x : x^T P_{i,M-1} x \leq 1\} \subseteq \mathcal{P}_i \subseteq \{x : x^T P_{i,M} x \leq 1\}.$$

Then, the polyhedral terminal set is defined as:

$$X_\text{f} = \cup_{i\in\mathcal{I}_0} (\mathcal{P}_i \cap \mathcal{C}_i). \tag{6.60}$$

Using similar arguments as in the proof of Corollary 6.2.4 we obtain that

- The union of polytopes X_f defined in (6.60) is a PI set for the closed-loop system (6.58).
- Using the union of polytopes X_f defined in (6.60) as a terminal set and the piecewise quadratic terminal cost $V_\text{f}(x) = x^T P_i x$ if $x \in \mathcal{C}_i$, with P_i the solution of (6.57), Theorem 6.2.1 still holds. Moreover, the optimization problem (6.51) becomes a mixed-integer quadratic program.

Note that in the case of a common matrix the determination of a polytope $\mathcal{P}$ is much simpler since it must satisfy $\{x : x^T P_{M-1} x \leq 1\} \subseteq \mathcal{P} \subseteq \{x : x^T P_M x \leq 1\}$.

Now, we provide a method to construct the polytopes $\mathcal{P}_i$:

(i): Choose s points $\{v_1, v_2, \cdots, v_s\}$ on the ellipsoid $\{x \in \mathbb{R}^n : x^T P_{i,M} x \leq 1\}$. Take $\mathcal{P}_i = \text{conv}\{v_1, v_2, \cdots, v_s\}$ (i.e. the convex hull of the set $\{v_1, v_2, \cdots, v_s\}$).

(ii): Check whether $\{x \in \mathbb{R}^n : x^T P_{i,M-1} x \leq 1\} \subseteq \mathcal{P}_i$ via the LMI condition (6.38). If the last inclusion does not hold increase the number of points s and repeat (i)–(ii).

6.2.5 Examples

Example 6.2.6 We consider the following PWL system taken from [6]:

$$x(k+1) = \begin{cases} A_1 x(k) + B_1 u(k) & \text{if } E_1 x(k) \geq 0 \\ A_2 x(k) + B_2 u(k) & \text{if } E_2 x(k) \geq 0 \end{cases}$$

The matrices of the system take the following values

$$A_1 = \begin{bmatrix} 0.35 & -0.6062 \\ 0.6062 & 0.35 \end{bmatrix}, A_2 = \begin{bmatrix} 0.35 & 0.6062 \\ -0.6062 & 0.35 \end{bmatrix}$$

$$B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ E_1 = [1 \ \ 0], \ E_2 = [-1 \ \ 0]$$

$$X = \{x \in \mathbb{R}^2 : |x_1| \leq 5, \ |x_2| \leq 5\}, \ U = \{u \in \mathbb{R} : |u| \leq 1\}.$$

Furthermore, we select $Q = I, \ R = 0.1$.

We now apply MPC. We construct a terminal set and a terminal cost using the backward procedure defined in Algorithm 6.2.3. For this system the LMIs from Step 1 have a solution for a common matrix P but with a PWL feedback controller:

$$P_{1,1} = P_{2,1} = \begin{bmatrix} 1.3593 & 0 \\ 0 & 1.967 \end{bmatrix},$$

$$F_{1,1} = [-0.4646 \ \ -0.1423], \ F_{2,1} = [0.4646 \ \ -0.1423].$$

Iterating Step 2 for $M = 3$ we obtain the following terminal set:

$$X_\text{f} = \mathcal{E}^3 = \{x \in \mathbb{R}^2 : x^T \begin{bmatrix} 0.0441 & 0 \\ 0 & 0.0627 \end{bmatrix} x \leq 1\}$$

and applying then Step 3 we obtain the following quadratic terminal cost:

$$V_\text{f}(x) = x^T \begin{bmatrix} 6.7534 & 0 \\ 0 & 9.2863 \end{bmatrix} x.$$

If we apply the MPC strategy derived in Section 6.2.1 for the terminal ellipsoidal set given by $P_{1,1}$ (i.e. $\{x \in \mathbb{R}^2 : x^T P_{1,1} x \leq 1\}$) and the terminal cost $x^T P_{1,1} x$ we need at least a prediction horizon $N = 4$ in order to have feasibility of the optimization problem (6.51) for all $x \in [-5 \ 5] \times [-5 \ 5]$. Therefore, we have to solve on-line a non-convex optimization problem, which is computationally intensive. However, using the terminal set X_f and the terminal cost V_f defined above, for $N = 1$ the optimization problem (6.51) is feasible for all $x \in [-5 \ 5] \times [-5 \ 5]$. So, at each step we have to solve a convex optimization problem. Figure 6.1 displays the ellipsoids found in Step 2 of the backward procedure and the closed-loop state trajectory obtained from using the MPC law κ_1. We observe that the state trajectory converges towards the origin, i.e. we have asymptotic stability.

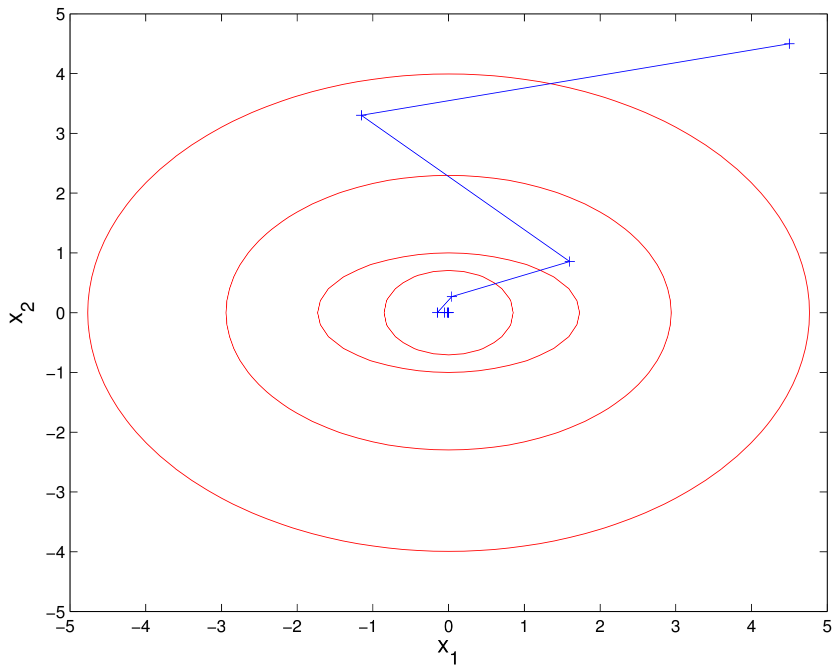

Figure 6.1: Enlargement of ellipsoidal terminal set and the closed-loop state trajectory corresponding to the MPC law κ_1 for the initial state $x = [4.5\ \ 4.5]^T$.

Example 6.2.7 We now give an example of a PWL system for which the LMIs from Theorem 6.1.9 are infeasible, while the LMIs from Theorem 6.1.8 are feasible:

$$x(k+1) = \begin{cases} A_1 x(k) + B_1 u(k) & \text{if } E_1 x(k) \geq 0 \\ A_2 x(k) + B_2 u(k) & \text{if } E_2 x(k) \geq 0 \\ A_3 x(k) + B_3 u(k) & \text{if } E_3 x(k) \geq 0 \\ A_4 x(k) + B_4 u(k) & \text{if } E_4 x(k) \geq 0 \end{cases}$$

The matrices of the system are given by

$$A_1 = \begin{bmatrix} 0.5 & 0.61 \\ 0.9 & 1.345 \end{bmatrix}, \ A_2 = \begin{bmatrix} -0.92 & 0.644 \\ 0.758 & -0.71 \end{bmatrix}$$
$$A_3 = A_1, \ A_4 = A_2, B_i = [1 \quad 0]^T \ \ \forall i \in \mathbb{N}_{[1,4]}.$$

The partitioning is given by:

$$E_1 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, \ E_2 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, E_3 = -E_1, \ E_4 = -E_2.$$

The tuning parameters Q and R are chosen as: $Q = 10^{-4} I, \ \ R = 10^{-3}$. We consider the following constraints: $X = \{x \in \mathbb{R}^2 : |x_1| \leq 6, \ |x_2| \leq 6\}, \ U = \{u \in \mathbb{R} : |u| \leq 2\}$.

For this example the LMIs from Theorem 6.1.9 are infeasible (using the Matlab LMI toolbox). We obtain conclusive results only if we are looking for a piecewise quadratic Lyapunov function and only if we apply the relaxations (i.e. the S-procedure) from Theorem 6.1.8. We

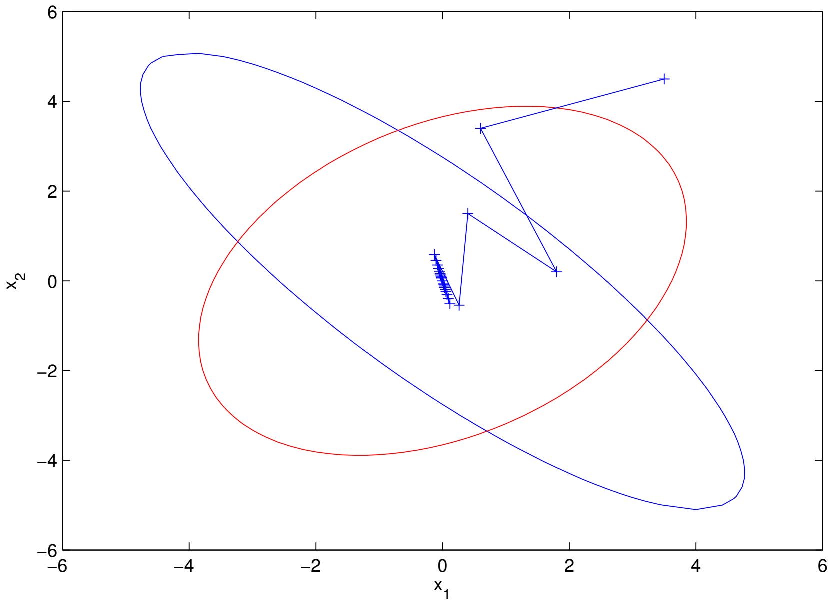

Figure 6.2: The terminal set X_f given by the intersection of the two ellipsoids and the closed-loop state trajectory corresponding to the MPC law κ_3 for the initial state $x = [3.5 \;\; 4.5]^T$.

obtain the following feasible solution (applying Algorithm 6.1.7):

$$F_{1,1} = F_{3,1} = [-0.7162 \quad -0.9662], \; F_{2,1} = F_{4,1} = [0.7657 \quad -0.4762]$$

$$P_{1,1} = P_{3,1} = \begin{bmatrix} 0.1589 & 0.1235 \\ 0.1235 & 0.1408 \end{bmatrix}, \; P_{2,1} = P_{4,1} = \begin{bmatrix} 0.0834 & -0.0207 \\ -0.0207 & 0.0815 \end{bmatrix}$$

$$S_1 = S_3 = \begin{bmatrix} 19.5829 & -17.1677 \\ -17.1677 & 22.1358 \end{bmatrix}, \; S_2 = S_4 = \begin{bmatrix} 12.1854 & 2.9662 \\ 2.9662 & 12.9486 \end{bmatrix}$$

$$U_{11} = \begin{bmatrix} 0.0046 & 0.0265 \\ 0.0265 & 0.0122 \end{bmatrix}, \; U_{12} = \begin{bmatrix} 0.0040 & 0.0301 \\ 0.0301 & 0.0065 \end{bmatrix},$$

$$U_{22} = \begin{bmatrix} 0.0001 & 0.0010 \\ 0.0010 & 0.0158 \end{bmatrix}, \; U_{21} = \begin{bmatrix} 0.0001 & 0.0022 \\ 0.0022 & 0.0154 \end{bmatrix}.$$

We note that the terminal set $\mathcal{E}_1 = \{x \in \mathbb{R}^2 : x^T P_{i,1} x \leq 1, \; i = 1, 2\}$ is small. Therefore, we use again the backward procedure to enlarge the terminal set. Using Remark 6.2.5, we obtain for $M = 4$ the following terminal set: $X_f = \{x \in \mathbb{R}^2 : x^T P_{i,4} x \leq 1, \; i = 1, 2\}$, where

$$P_{1,4} = P_{3,4} = \begin{bmatrix} 0.1405 & 0.1125 \\ 0.1125 & 0.1228 \end{bmatrix}, \; P_{2,4} = P_{4,4} = \begin{bmatrix} 0.0687 & -0.0292 \\ -0.0292 & 0.0689 \end{bmatrix}.$$

The terminal cost has the following expression: $V_f(x) = x^T P_i x \;\; \text{if} \;\; x \in \mathcal{C}_i$, where

$$P_1 = P_3 = \begin{bmatrix} 4.8284 & 1.5050 \\ 1.5050 & 0.8351 \end{bmatrix}, \; P_2 = P_4 = \begin{bmatrix} 4.4540 & 0.4351 \\ 0.4351 & 1.2127 \end{bmatrix}.$$

Applying the MPC for this terminal set and cost we obtain the trajectory from Figure 6.2. We have again asymptotic stability.

6.3 Robust MPC for PWA systems

In the previous sections we have considered deterministic PWA systems. However, in practice disturbances are always present and thus the designed controller must be robust. As in conventional linear systems we assume that the disturbances enter additively in the system equations. A PWA system with additive disturbance is defined as:

$$\begin{array}{rcl} x(k+1) & = & A_i x(k) + B_i u(k) + a_i + w(k) \\ y(k) & = & C_i x(k) + c_i \end{array} \quad \text{if } x(k) \in \mathcal{C}_i, \tag{6.61}$$

where $i \in \mathcal{I}$ and the disturbance w takes on values from a polytope $W = \{w \in \mathbb{R}^q : \Omega w \leq s\}$. Moreover, we assume that $0 \in W$. We should note that if the nominal PWA system corresponding to (6.61) (i.e. $w = 0$) is continuous and $B_i = B$ for all $i \in \mathcal{I}$, then the uncertain PWA system (6.61) can be written equivalently as an uncertain MMPS system (5.8) and MPC as was done in Section 5.2 can be applied to this class of systems.

Although the assumption of knowing the nominal model might seem restrictive, the description of the uncertainty by additive terms that are known to lie in a bounded set is a reasonable choice, as shown in the recent literature on robust control and identification [12, 105]. We use the notation:

$$f_{\text{PWA}}(x, u, w) = A_i x + B_i u + a_i + w \ \text{ if } \ x \in \mathcal{C}_i.$$

We also assume that the state and the input are constrained in some polytopes X and U that contain the origin in their interior. However, we do not restrict the polytopes X and U to have necessarily the form (6.3)–(6.4). Note that we can consider also mixed state-input constraints $\{[x^T \ u^T]^T : Hx + Gu \leq h\}$. The partition $\{\mathcal{C}_i\}_{i \in \mathcal{I}}$ is defined as in Section 6.1.

The PWL dynamics of the uncertain PWA system (6.61) are given by

$$\begin{array}{rcl} x(k+1) & = & A_i x(k) + B_i u(k) + w(k) \\ y(k) & = & C_i x(k) \end{array} \quad \text{if } x(k) \in \mathcal{C}_i, \tag{6.62}$$

where $i \in \mathcal{I}_0$ (we recall that $\mathcal{I}_0 \subseteq \mathcal{I}$).

The objective of this section is to design a feedback min-max MPC law that steers the state of the uncertain PWA system (6.61) as close as possible to the origin while satisfying the state and input constraints for all admissible disturbances. Clearly, the presence of a bounded disturbance acting on the system means that it is not possible to guarantee asymptotic stability and the most that can be achieved is to steer the state trajectory of the closed-loop system to a neighborhood of the origin and to keep it there.

6.3.1 RPI sets for uncertain PWL systems

In the sequel we assume that we have determined a PWL feedback controller (see Assumption **A5**)

$$\kappa_{\text{f}}(x) = F_i x \ \text{ if } \ x \in \mathcal{C}_i$$

that asymptotically stabilizes the nominal PWL dynamics

$$\begin{array}{rcl} x(k+1) & = & A_i x(k) + B_i u(k) \\ y(k) & = & C_i x(k) \end{array} \quad \text{if } x(k) \in \mathcal{C}_i.$$

The matrices F_i can be computed using the approach described in Section 6.1.2. We recall that $A_{F_i} = A_i + B_i F_i$ for all $i \in \mathcal{I}_0$. Using the feedback controller κ_{f} the PWL system with additive disturbance (6.62) becomes:

$$\begin{array}{rcl} x(k+1) & = & A_{F_i}x(k) + w(k) \\ y(k) & = & C_i x(k) \end{array} \quad \text{if } x(k) \in \mathcal{C}_i. \tag{6.63}$$

We define the following set:

$$X_F = \cup_{i \in \mathcal{I}_0} \{x \in \mathcal{C}_i : x \in X,\ F_i x \in U\}.$$

We recall the Definition 4.4.2 of an *RPI set* adapted to the class of systems (6.63): a set $Z \subseteq X_F$ is an RPI set for the system (6.63) if for all $x \in Z \cap \mathcal{C}_i$, $A_{F_i}x + w \in Z$ for all $i \in \mathcal{I}_0$ and $w \in W$. The *maximal* (minimal) RPI set for the system (6.63) is defined as the largest (smallest, non-empty) with respect to inclusion RPI set for (6.63) contained in X_F.

It can be easily seen that both the minimal and the maximal RPI set associated to the system (6.63) is non-convex in general (it is a union of polyhedral sets [79]). Our aim is to compute a polyhedral RPI set, since we want to obtain only linear constraints for the robust MPC schemes that we propose in the sequel. For the PWL system (6.63) the evolution of the mode $i = i(k)$ depends on the state $x(k)$. Nevertheless, for ease of computation of a convex (polyhedral) RPI set for (6.63) this relation mode-state will be disregarded and we will consider that $i(k)$ evolves independently of $x(k)$ (i.e. any mode $i(k) \in \mathcal{I}_0$ can be active at any sample step k):

$$\begin{array}{rcl} x(k+1) & = & A_{F_{i(k)}}x(k) + w(k) \\ y(k) & = & C_{i(k)}x(k) \\ i(k+1) & \in & \mathcal{I}_0, \end{array} \tag{6.64}$$

where $i(\cdot)$ is a switching signal in $\mathcal{I}_0^{\mathbb{N}}$. Note however that all trajectories of the PWL system (6.63) are still covered by the trajectories of the free switching system (6.64). Moreover, this relaxation is considered only in this section, in the next section where we design an MPC strategy, we consider again the standard PWA system (6.61). We recall that this type of relaxation was also used in Section 4.5.1 in order to derive a stabilizing MPC law for switching MPL systems. Similar relaxations were used in [31, 89] in the context of MPC for deterministic systems.

Definition 6.3.1 *A set $\Omega \subseteq X_F$ is an RPI set for system* (6.64) *if for all $x \in \Omega$ we have that $A_{F_i}x + w \in \Omega$ for all possible switchings $i \in \mathcal{I}_0$ and all admissible disturbances $w \in W$.* ♢

In the sequel we will use the following notations: given two sets $Y, Z \subseteq \mathbb{R}^n$, the *Minkowski sum* of Y and Z is defined as

$$Y \boxplus Z := \{y + z : y \in Y, z \in Z\}$$

and the *Pontryagin difference* as

$$Y \boxminus Z = \{y \in \mathbb{R}^n : y \boxplus Z \subseteq Y\}.$$

In the sequel we construct the maximal RPI set for the system (6.64). Let X_{F_i} denote the set of states that satisfy the state and input constraints:

$$X_{F_i} = \{x \in \cup_{i \in \mathcal{I}_0} \mathcal{C}_i : x \in X, F_i x \in U\}.$$

Recall that $\cup_{i \in \mathcal{I}_0} \mathcal{C}_i$ is assumed to be a polytope and thus X_{F_i} are polytopes for all $i \in \mathcal{I}_0$. It follows that $\bigcap_{i \in \mathcal{I}_0} X_{F_i} \subseteq X_F$. We define the following set recursion:

$$\begin{array}{rcl} \mathcal{O}_0^i & = & X_{F_i} \\ \mathcal{O}_k^i & = & \{x \in X_{F_i} : A_{F_i}x \boxplus W \subseteq \cap_{j \in \mathcal{I}_0} \mathcal{O}_{k-1}^j\} \end{array} \tag{6.65}$$

for all $i \in \mathcal{I}_0$ and $k \geq 1$. The set $\mathcal{O}_k^i$ represents the set of initial states $x(0)$ for which under the closed-loop dynamics (6.64) the state and input constraints are satisfied up to sample step k assuming that initially $i(0) = i$. It is clear from the recursion (6.65) that $\mathcal{O}_{k+1}^i \subseteq \mathcal{O}_k^i$. Therefore, $\mathcal{O}_k^i$ converges to the set $\mathcal{O}_\infty^i$, i.e.

$$\begin{aligned} \mathcal{O}_\infty^i &= \lim_{k\to\infty} \mathcal{O}_k^i = \cap_{k\geq 0} \mathcal{O}_k^i \\ \mathcal{O}_\infty &= \cap_{i\in\mathcal{I}_0} \mathcal{O}_\infty^i. \end{aligned} \tag{6.66}$$

Theorem 6.3.2 *(i) The maximal RPI set included in* $\cap_{i\in\mathcal{I}_0} X_{F_i}$ *for the system* (6.64) *is the convex set* $\mathcal{O}_\infty$.

(ii) Any RPI set for the system (6.64) *is also an RPI set for the PWL system* (6.63)*. In particular* $\mathcal{O}_\infty$ *is an RPI set for the PWL system* (6.63).

Proof: (i) It is easy to observe that since the sets X, U and W are polytopes (described by linear inequalities), all the sets $\mathcal{O}_k^i$ are described by a finite number of linear inequalities. Therefore, all $\mathcal{O}_k^i$ are polyhedra for all $i \in \mathcal{I}_0$ and $k \geq 0$. Since $\mathcal{O}_\infty$ is the intersection of polyhedra (i.e. intersection of convex sets), it follows that $\mathcal{O}_\infty$ is convex.

For all $x \in \mathcal{O}_\infty$ we have $x \in \mathcal{O}_{k+1}^i$ for all $i \in \mathcal{I}_0$ and $k \geq 0$. According to (6.65) we have $A_{F_i} x \boxplus W \subseteq \cap_{j\in\mathcal{I}_0} \mathcal{O}_k^j$ for all $i \in \mathcal{I}_0$ and $k \geq 0$. Hence $A_{F_i} x \boxplus W \subseteq \mathcal{O}_\infty$ for all $i \in \mathcal{I}_0$. It follows that $\mathcal{O}_\infty$ is an RPI set for the system (6.64).

It is well-known [19, 82] that the maximal RPI set contained in $\cap_{i\in\mathcal{I}_0} X_{F_i}$ for a system is the set of all initial states in $\cap_{i\in\mathcal{I}_0} X_{F_i}$ for which the evolution of the system remains in $\cap_{i\in\mathcal{I}_0} X_{F_i}$. Due to the recursion (6.65) it is clear that $\mathcal{O}_\infty$ is the maximal RPI set for system (6.64) included in $\cap_{i\in\mathcal{I}_0} X_{F_i}$. Indeed, let $T \subseteq \cap_{i\in\mathcal{I}_0} X_{F_i}$ be an RPI set for the system (6.64) and let $x \in T$. Then, from the definition of an RPI set for the system (6.64) (see Definition 6.3.1) we have $A_{F_i} x \boxplus W \subseteq T \subseteq \cap_{i\in\mathcal{I}_0} X_{F_i} = \cap_{i\in\mathcal{I}_0} \mathcal{O}_0^i$ for all $i \in \mathcal{I}_0$. This implies that $x \in \mathcal{O}_1^i$ for all $i \in \mathcal{I}_0$ (according to the recursion (6.65)). Therefore, $T \subseteq \mathcal{O}_1^i$ for all $i \in \mathcal{I}_0$. By iterating this procedure we obtain that $T \subseteq \mathcal{O}_k^i$ for all $k \geq 0$ and $i \in \mathcal{I}_0$. In conclusion $T \subseteq \mathcal{O}_\infty$, i.e. $\mathcal{O}_\infty$ is maximal.

(ii) First we have that $\mathcal{O}_\infty \subseteq \cap_{i\in\mathcal{I}_0} X_{F_i} \subseteq X_F$. If $x \in \mathcal{O}_\infty \cap \mathcal{C}_j$, then $A_{F_j} x \boxplus W \subseteq \mathcal{O}_\infty$ for all $j \in \mathcal{I}_0$. In particular for $j = i$ we have $A_{F_i} x \boxplus W \subseteq \mathcal{O}_\infty$. Therefore, $\mathcal{O}_\infty$ is an RPI set for the system (6.63). For a general RPI set for the system (6.64) the reasoning is similar. $\diamondsuit$

Because the sets $\mathcal{O}_k^i$ are described by a finite number of linear inequalities, it is important to know whether the set $\mathcal{O}_\infty$ can be *finitely determined* (see Definition 4.3.3), i.e. whether there exists a finite t^* such that $\mathcal{O}_{t^*}^i = \mathcal{O}_{t^*+1}^i$ for all $i \in \mathcal{I}_0$. Then, $\mathcal{O}_\infty = \cap_{i\in\mathcal{I}_0} \mathcal{O}_{t^*}^i$ and thus $\mathcal{O}_\infty$ is a polyhedral set. In the sequel we give necessary conditions for finite determination. Using the recursion (6.65) and the commutativity property of intersection, we have:

$$\mathcal{O}_0 = \cap_{i\in\mathcal{I}_0} \mathcal{O}_0^i, \quad \mathcal{O}_k = \cap_{i\in\mathcal{I}_0} \mathcal{O}_k^i \;\; \forall k \geq 1.$$

Note that $\mathcal{O}_{k+1} \subseteq \mathcal{O}_k$ and $\mathcal{O}_\infty = \cap_{k\geq 0} \mathcal{O}_k$. Now, $\mathcal{O}_k$ can be written in terms of Pontryagin differences as:

$$\begin{aligned} Y_0 &= \cap_{i\in\mathcal{I}_0} X_{F_i}, \;\; \mathcal{O}_0 = Y_0 \\ Y_1 &= Y_0 \boxminus W, \;\; \mathcal{O}_1 = \cap_{i\in\mathcal{I}_0} \{x \in \mathcal{O}_0 : A_{F_i} x \in Y_1\} \\ Y_k &= \cap_{(i_1,\ldots i_{k-1})\in\mathcal{I}_0^{k-1}} (Y_{k-1} \boxminus A_{F_{i_1}} \ldots A_{F_{i_{k-1}}} W) \\ \mathcal{O}_k &= \cap_{(i_1,\ldots i_k)\in\mathcal{I}_0^k} \{x \in \mathcal{O}_{k-1} : A_{F_{i_1}} \ldots A_{F_{i_k}} x \in Y_k\}. \end{aligned} \tag{6.67}$$

It is clear that $Y_{k+1} \subseteq Y_k$ (since $0 \in W$). We denote with $Y_\infty = \cap_{k\geq 0} Y_k$. We have the following theorem:

Theorem 6.3.3 *Suppose the system* (6.64) *is asymptotically stable. Suppose also that there exists an index* $t_0 \geq 0$ *such that* $\mathcal{O}_{t_0}$ *is bounded and that* $0 \in \text{int}(Y_\infty)$*. Then,* $\mathcal{O}_\infty$ *is a polyhedral set.*

Proof: Since (6.64) is asymptotically stable, it follows that for all $(i_1, \cdots, i_k) \in \mathcal{I}_0^k$ we have

$$\begin{cases} A_{F_{i_1}} \cdots A_{F_{i_k}} x \to 0 \ \text{ as } k \to \infty \ \ \forall x \in \mathbb{R}^n \\ \mathcal{O}_{t_0} \text{ is bounded}, \ \ 0 \in \text{int}(Y_\infty). \end{cases}$$

Then, there exists a $t^* \geq t_0$ such that for all $(i_1, \cdots, i_{t^*+1}) \in \mathcal{I}_0^{t^*+1}$:

$$A_{F_{i_1}} \cdots A_{F_{i_{t^*+1}}} x \in Y_\infty \subseteq Y_{t^*+1} \ \ \forall x \in \mathcal{O}_{t_0}.$$

Since $\mathcal{O}_{t^*} \subseteq \mathcal{O}_{t_0}$, we have :

$$A_{F_{i_1}} \cdots A_{F_{i_{t^*+1}}} x \in Y_{t^*+1} \ \ \forall x \in \mathcal{O}_{t^*}.$$

Therefore, according to the recursion (6.67) $\mathcal{O}_{t^*} \subseteq \mathcal{O}_{t^*+1}$. But $\mathcal{O}_{t^*+1} \subseteq \mathcal{O}_{t^*}$. In conclusion, we have the equality $\mathcal{O}_{t^*} = \mathcal{O}_{t^*+1}$ and thus $\mathcal{O}_\infty$ is finitely determined, i.e. $\mathcal{O}_\infty = \mathcal{O}_{t^*}$. Since $\mathcal{O}_{t^*}$ is described by a finite number of linear inequalities, it follows that $\mathcal{O}_\infty$ is a polyhedral set. ♢

The conditions from Theorem 6.3.3 are similar with those corresponding to the linear case [82]. The algorithm for computing $\mathcal{O}_\infty$ stops once the following condition is met: there exists an index t^* such that $\mathcal{O}_{t^*} = \mathcal{O}_{t^*+1}$.

If t^* is large, the procedure for the computation of $\mathcal{O}_\infty$ might require too many iterations. We now propose an alternative check whether or not a given polyhedral set is an RPI set for the system (6.63). Let $Z = \{x \in \mathbb{R}^n : h_j^T x \leq 1 \ \forall j \in \mathbb{N}_{[1,n_Z]}\} \subseteq X_F$ be a polytope that contains the origin in its interior. Then, Z is an RPI set for the system (6.63) if for all $x \in Z \cap \mathcal{C}_i$ and $i \in \mathcal{I}_0$ we have $A_{F_i} x \boxplus W \subseteq Z$. This condition can be translated in terms of computing some linear programs. We denote with $h_j^0 = \max_{w \in W} h_j^T w$ (this is a linear program, because we assumed that W is a polytope) for all $j \in \mathbb{N}_{[1,n_Z]}$. For all $i \in \mathcal{I}_0$ and $j \in \mathbb{N}_{[1,n_Z]}$ we consider the following linear program:

$$\sigma_i^j = \max_x \{h_j^T A_{F_i} x + h_j^0 - 1 : h_k^T x \leq 1 \ \ \forall k \in \mathbb{N}_{[1,n_Z]}, \ x \in \mathcal{C}_i\}$$

From the above discussion we have the following consequence:

Corollary 6.3.4 *If for all* $i \in \mathcal{I}_0$ *and* $j \in \mathbb{N}_{[1,n_Z]}$ *the optimal values* $\sigma_i^j \leq 0$*, then* Z *is an RPI set for the system* (6.63).

Proof: For a fixed $i \in \mathcal{I}_0$, the condition $\sigma_i^j \leq 0$ for all $j \in \mathbb{N}_{[1,n_Z]}$ expresses the fact that $A_{F_i} x \boxplus W \subseteq Z$ for all $x \in Z \cap \mathcal{C}_i$. Therefore, Z is an RPI set for the system (6.63). ♢

If after a certain number of iterations $k_{\max}$ the algorithm for computing $\mathcal{O}_\infty$ does not stop, then we have available the set $\mathcal{O}_{k_{\max}} = \{x : H_{k_{\max}} x \leq h_{k_{\max}}\}$. Therefore, a starting point in searching for a set Z in Corollary 6.3.4 might be to take $Z = \{x : H_{t_{\max}} x \leq h\}$, and h should be chosen appropriately, i.e. such that $\sigma_i^j \leq 0$.

6.3.2 Feedback min-max MPC for PWA systems

In the sequel we develop a robustly stabilizing MPC scheme for the uncertain PWA system (6.61) based on a feedback min-max approach. In general, for deterministic systems an MPC strategy contains two ingredients: a terminal set and a terminal cost (see Section 2.3.2 or Section 6.2). If the system is uncertain, the stability and also the feasibility may be lost. In order to achieve robust stability the MPC must stabilize the system for all possible realizations of the disturbance along the prediction horizon. Different robust MPC schemes have been proposed for linear systems: some of them are based on a nominal prediction [5, 111], others are based on the worst-case disturbance as in dual-mode feedback min-max MPC formulation [80, 148].

In this section we also use a dual-mode feedback min-max MPC approach. We assume that we have computed a stabilizing controller for the nominal PWL dynamics $\kappa_{\mathrm{f}}(x) = F_i x$ if $x \in \mathcal{C}_i$ (where the matrices F_i are determined as in Section 6.1.2) and also we have available a polyhedral RPI set X_{f} (e.g. $X_{\mathrm{f}} = \mathcal{O}_\infty$, where $\mathcal{O}_\infty$ is obtained according to Section 6.3.1).

Let $\mathbf{w} = [w_0^T\ w_1^T \cdots w_{N-1}^T]^T$ denote a realization of the disturbance over the prediction horizon N. In this section we define the decision variable in the optimal control problem for a given initial condition x as a control policy $\pi = (\mu_0, \mu_1, \ldots, \mu_{N-1})$, where each $\mu_i : \mathbb{R}^n \to \mathbb{R}^m$ is a state feedback law. Also, let $x_k = \phi(k; x, \pi, \mathbf{w})$ denote the state solution of the uncertain PWA system (6.61) at step k when the initial state is x at step 0, the control is determined by the policy π and the disturbance sequence is $\mathbf{w}$. For each initial condition x we define the set of feasible policies π:

$$\Pi_N^{\mathrm{fb}}(x) = \{\pi : \mu_i \in U,\ x_i \in X\ \forall i \in \mathbb{N}_{[0,N-1]},\ x_N \in X_{\mathrm{f}},\ \forall \mathbf{w} \in \mathcal{W}\}, \tag{6.68}$$

where we recall that $\mathcal{W} = W^N$. Also, let X_N^{fb} denote the set of initial states for which a feasible policy exists, i.e.

$$X_N^{\mathrm{fb}} = \{x : \Pi_N^{\mathrm{fb}}(x) \neq \emptyset\}. \tag{6.69}$$

For a given initial state x, control policy π and disturbance realization $\mathbf{w}$, the cost $V_N(x, \pi, \mathbf{w})$ is:

$$V_N(x, \pi, \mathbf{w}) = \sum_{i=0}^{N-1} \ell(x_i, \mu_i), \tag{6.70}$$

where the stage cost ℓ is assumed to be convex and satisfies

$$\begin{array}{lcll} \ell(x,u) & \geq & \alpha(d(x, X_{\mathrm{f}})) & \text{if} \quad x \notin X_{\mathrm{f}} \\ \ell(x,u) & = & 0 & \text{if} \quad x \in X_{\mathrm{f}}, \end{array}$$

with α a $\mathscr{K}$ function. Here, $d(x, X_{\mathrm{f}})$ denotes the distance from x to the set X_{f} induced by the $1/2/\infty$-norm. Some examples of such stage cost are:

$$\ell(x,u) = \begin{cases} \|Qx\| + \|Ru\| & \text{if } x \notin X_{\mathrm{f}} \\ 0 & \text{if } x \in X_{\mathrm{f}} \end{cases} \tag{6.71}$$

with the matrices $Q \succ 0, R \succ 0$. Another stage cost was proposed in [80] (recall that in Section 4.4.2 we use a similar stage cost (4.54) in the context of robust MPC for MPL systems):

$$\ell(x,u) = \inf_{z \in X_{\mathrm{f}}} \|Q(x - z)\| + \|R(u - \kappa_{\mathrm{f}}(x))\|. \tag{6.72}$$

Note that the stage cost (6.72) is continuous on $\mathbb{R}^n \times \mathbb{R}^m$ while the stage cost defined in (6.71) is discontinuous. The reader should note that in this robust MPC scheme we consider a zero terminal cost, i.e. $V_{\mathrm{f}} = 0$.

The finite-horizon feedback min-max MPC problem for the class of uncertain PWA systems (6.61) is defined as:

$$V_N^{0,\text{fb}}(x) = \inf_{\pi \in \Pi_N^{\text{fb}}(x)} \max_{\mathbf{w} \in \mathcal{W}} V_N(x, \pi, \mathbf{w}). \tag{6.73}$$

For linear systems the optimization problem (6.73) can be solved efficiently using the extreme disturbance realizations [80, 148]. In our settings, due to the nonlinearities of the PWA system, this approach cannot be applied directly. To overcome this problem we propose a new method, namely to restrict the admissible control policies π to only those that guarantee that for every value of the disturbance the mode of the system $i(k)$ is the same at each sample step k, i.e. for all k and x there exists an $i(k)$ such that

$$\phi(k; x, \pi, \mathbf{w}) \in \mathcal{C}_{i(k)} \quad \forall \mathbf{w} \in \mathcal{W}. \tag{6.74}$$

Therefore, we restrict the PWA system only to the admissible control policies that guarantee the mode of the system is "certain" at sample step k while the state is not known exactly. This extra constraint (6.74), which expresses the fact that $i(k)$ is independent of the disturbance realization is not too restrictive since a cautious action may avoid uncertainty in the mode (at least in the case where the disturbances are not too large and the control inputs are not constrained too much). It can be easily observed that imposing (6.74) to the system (6.61) the state set generated by the disturbance at each sample step k is convex:

$$\phi(k; x, \pi, W^k) = \phi(k; x, \pi, 0) \boxplus X(k; i(0), \cdots, i(k-1), W^k), \tag{6.75}$$

where the first term expresses the nominal trajectory corresponding to the PWA system (6.61) (i.e. $w = 0$) and the second term represents a convex uncertainty set associated with the state, which depends on the switching mode sequence $i(0), \cdots, i(k-1)$ and on the set W^k. In this new setting, i.e. with the extra constraints (6.74), the set of feasible policies becomes:

$$\Pi_N^{\text{km}}(x) := \{\pi : \text{constraint (6.74)}, x_i \in X, \mu_i \in U \ \forall i \in \mathbb{N}_{[0,N-1]}, x_N \in X_\text{f}, \ \forall \mathbf{w} \in \mathcal{W}\}.$$

Now, the finite-horizon feedback min-max MPC problem becomes:

$$V_N^{0,\text{km}}(x) = \inf_{\pi \in \Pi_N^{\text{km}}(x)} \max_{\mathbf{w} \in \mathcal{W}} V_N(x, \pi, \mathbf{w}). \tag{6.76}$$

The optimization problem (6.76) has infinite dimension, but in the sequel we will show that (6.76) can be reduced to a finite dimensional optimization problem. Since W is a polytope with n_v vertexes, we denote with $\mathcal{L}_v^N$ the set of indexes l such that $\mathbf{w}^l = [(w_0^l)^T \ (w_1^l)^T \cdots (w_{N-1}^l)^T]^T$ takes values only on the vertexes of $\mathcal{W}$. It is clear that $\mathcal{L}_v^N$ is a finite set with the cardinality n_v^N. Further, let $\mathbf{u}^l = [(u_0^l)^T \ (u_1^l)^T \cdots (u_{N-1}^l)^T]^T$ denote a control sequence associated with the l^{th} disturbance realization $\mathbf{w}^l$ and let $x_k^l = \phi(k; x, \mathbf{u}^l, \mathbf{w}^l)$ be the solution of the PWA system (6.61) with the additional constraint (6.74). We consider a variable horizon $N \in \mathbb{N}_{[1,N_{\max}]}$, where $N_{\max}$ is a positive integer. Since the stage cost ℓ is convex in (x, u) and since the state set (6.75) generated by the disturbance is also convex (by imposing the constraint (6.74)), the optimization problem (6.76) is reduced to the following finite dimensional optimization problem which considers only the disturbance realizations that take on values at the vertexes of the disturbance set (see also [148]):

$$V_N^{0,\text{km}}(x) = \inf_{(\mathbf{u}^l, N \in \mathbb{N}_{[1,N_{\max}]})} \max_{l \in \mathcal{L}_v^N} V_N(x, \mathbf{u}^l, \mathbf{w}^l)$$

subject to :

$$\begin{cases} \text{constraint (6.74)}; \ x_0^l = x; \ x_i^l \in X \ \forall i \in \mathbb{N}_{[1,N-1]}; \ x_N^l \in X_\text{f} \ \ \forall l \in \mathcal{L}_v^N \\ u_i^l \in U \ \ \forall i \in \mathbb{N}_{[0,N-1]}, l \in \mathcal{L}_v^N; \ x_i^{l_1} = x_i^{l_2} \ \Rightarrow \ u_i^{l_1} = u_i^{\ell_2} \ \ \forall l_1, l_2 \in \mathcal{L}_v^N. \end{cases} \tag{6.77}$$

The last constraint is the *causality constraint* [148] and expresses the fact that the control law at step i for the state x_i^l is independent of the control and disturbance sequence taken to reach that state. The constraint (6.74) is imposed only to the states x_i^l with $i \in \mathbb{N}_{[1,N-1]}$ and not to x_N^l. The only constraint on the final state x_N^l is the terminal constraint: $x_N^l \in X_\mathrm{f}$.

Let $(\mathbf{u}^{0,l}(x), N^0(x))$ be an optimizer of (6.77) whenever the infimum is attained, where $\mathbf{u}^{0,l}(x) = [(u_0^{0,l}(x))^T \ (u_1^{0,l}(x))^T \cdots (u_{N-1}^{0,l}(x))^T]^T$. Note that $u_0^{0,l}(x) = u_0^0(x)$ for all $l \in \mathcal{L}_v^N$ (according to the causality constraint). At event step (k, x) we denote with $N_k = N^0(x)$.

We define an MPC scheme using a dual-mode approach. At event step (k, x) the MPC law is given by the following algorithm:

Algorithm 6.3.5

(i) if $x \in X_\mathrm{f} \cap \mathcal{C}_i$, then $\kappa_{N_k}(x) = F_i x \quad \forall i \in \mathcal{I}_0$

(ii) otherwise, solve (6.77) and set $\kappa_{N_k}(x) = u_0^0(x)$.

We now study robust stability for the closed-loop system:

$$\begin{array}{rcl} x(k+1) & = & A_i x(k) + B_i \kappa_{N_k}(x(k)) + a_i + w(k) \\ y(k) & = & C_i x(k) + c_i \end{array} \quad \text{if } \ x(k) \in \mathcal{C}_i, \tag{6.78}$$

where $i \in \mathcal{I}$. We show in the sequel that X_f is robustly asymptotically stable with respect to the closed-loop system (6.78). We first recall the definition of robust stability (see also Section 2.3.3). The set X_f is *robustly stable* with respect to (6.78) if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $d(x, X_\mathrm{f}) \leq \delta$ implies $d(\phi(k; x, \kappa_{N_k}, \mathbf{w}), X_\mathrm{f}) \leq \epsilon$ for all $k \geq 0$ and all admissible disturbance sequences $\mathbf{w}$. The set X_f is *robustly asymptotically (finite time) attractive* with domain of attraction X if for all $x \in X$, $\lim_{k\to\infty} d(\phi(k; x, \kappa_{N_k}, \mathbf{w}), X_\mathrm{f}) = 0$ (there exists a finite time k_T such that $\phi(k; x, \kappa_{N_k}, \mathbf{w}) \in X_\mathrm{f}$ for all $k \geq k_T$) for all admissible disturbance sequences. The set X_f is *robustly asymptotically (finite) time stable* with the domain of attraction X if it is robustly stable and robustly asymptotically (finite time) attractive with domain of attraction X.

Theorem 6.3.6 *(i) The set X_f is robustly asymptotically stable with respect to the closed-loop system* (6.78) *with a region of attraction X_{N_max}.*

(ii) Suppose that $\ell(x, u) \geq \alpha(\|x\|)$ for all $x \notin X_\mathrm{f}$, where α is a $\mathscr{K}$ function. Then, the set X_f is robustly finite time stable with respect to the closed-loop system (6.78) *with a region of attraction X_{N_max}.*

Proof: (i) First let us show robust feasibility. Let $x \in X_{N_\mathrm{max}} \cap \mathcal{C}_{i_0}$ for some $i_0 \in \mathcal{I}_0$. Then, there exists an $N \in \mathbb{N}_{[1,N_\mathrm{max}]}$ such that the optimization problem (6.77) has an optimal input sequence $\mathbf{u}^{0,l}(x) = [(u_0^{0,l}(x))^T \ (u_1^{0,l}(x))^T \cdots (u_{N-1}^{0,l}(x))^T]^T$ for the l^{th} disturbance realization, satisfying the constraints (6.74), therefore producing the fixed switching sequence $i_0, i_1, \cdots, i_{N-1}$. Let $\mathbf{x}^{0,l} = [x^T \ (x_1^{0,l})^T \cdots (x_N^{0,l})^T]^T$ be the corresponding optimal state trajectory. We recall that from the causality constraints we have: $u_0^{0,l_1}(x) = u_0^{0,l_2}(x) = u_0^0(x)$ for all $l_1 \neq l_2 \in \mathcal{L}_v^N$. Now, according to the receding horizon principle the input $\kappa_N(x) = u_0^0(x)$ is applied and the disturbance takes a certain value $w = \sum_{\ell \in \mathcal{L}_v^N} \eta_l w^l \in W$, where w^l is a vertex of W and η_l are appropriate convex scalar weights. Therefore, the next state is given by

$$f_\mathrm{PWA}(x, \kappa_N(x), w) = A_{i_0} x + B_{i_0} \kappa_N(x) + a_{i_0} + w = \sum_{l \in \mathcal{L}_v^N} \eta_l x_1^l,$$

where $x_1^l := A_{i_0}x + B_{i_0}\kappa_N(x) + a_{i_0} + w^l$, i.e. $f_{\text{PWA}}(x, \kappa_N(x), w)$ lies in the convex hull $\text{conv}\{x_1^l : l \in \mathcal{L}_v^N\}$. Now, at the next step we consider the prediction horizon $N-1$ and the following control sequence:

$$\mathbf{u}^{\text{f}} = \big[(\sum_{l \in \mathcal{L}_v^{N-1}} \eta_l u_1^{0,l}(x))^T \cdots (\sum_{l \in \mathcal{L}_v^{N-1}} \eta_l u_{N-1}^{0,l}(x))^T\big]^T. \quad (6.79)$$

Under this control policy, the state and input predictions over a prediction window of length $N-1$ evolve in the convex hulls generated by $[(x_1^{0,l})^T \cdots (x_N^{0,l})^T]^T$ and $[(u_1^{0,l}(x))^T \cdots (u_{N-1}^{0,l}(x))^T]$. Moreover, the switching sequence $i_1, ..., i_{N-1}$ is fixed (we used here that all sets X, U, X_{f} and W are polytopes). In conclusion, $f_{\text{PWA}}(x, \kappa_N(x), w) \in X_{N_{\max}}$, i.e. we have robust feasibility.

Robust asymptotic stability follows from the fact that the conditions $\mathcal{F}1^w - \mathcal{F}3^w$ and $\mathcal{S}1^w$ given in Section 2.3.3 are satisfied in this case for $\kappa_{\text{f}}, X_{\text{f}}$ and ℓ defined in this section (see also [80, 148]).

(ii) Since X_{f} is bounded, there exist an $\eta > 0$ such that $\|x\| \geq \eta$ for all $x \in X \setminus X_{\text{f}}$. Then, for all $x \in X_{N_{\max}}$ we have

$$\begin{aligned} &V_{N_{k+1}}^0(\phi(x; k+1, \kappa_{N_{k+1}}, \mathbf{w})) - V_{N_k}^0(\phi(x; k, \kappa_{N_k}, \mathbf{w})) \leq \\ &- \ell(\phi(x; k, \kappa_{N_k}, \mathbf{w}), \kappa_{N_k}(\phi(x; k, \kappa_{N_k}, \mathbf{w}))) \leq -\alpha(\|\phi(x; k, \kappa_{N_k}, \mathbf{w})\|) \leq -\alpha(\eta) \end{aligned}$$

if $\phi(x; k, \kappa_{N_k}, \mathbf{w}) \notin X_{\text{f}}$. Now, assume that for $k \to \infty$, $\phi(x; k, \kappa_{N_k}, \mathbf{w}) \notin X_{\text{f}}$. Then, $0 \leq V_{N_k}^0(\phi(x; k, \kappa_{N_k}, \mathbf{w})) \leq V_{N_0}^0(x) - k\alpha(\eta) \to -\infty$ as $k \to \infty$, i.e. a contradiction. Therefore, the state trajectory enters X_{f} in finite time and then the subsequent trajectory is maintained in this set according to Algorithm 6.3.5. ♢

Remark 6.3.7 An interesting case is when $\mathcal{I}_0$ contains only one element $\mathcal{I}_0 = \{1\}$, i.e. the origin is contained in the interior of $\mathcal{C}_1$ (this implies that we have only one PWL dynamic defined on a polyhedral set that contains the origin in its interior). As it is done in the linear case, we can construct a stabilizing controller and an RPI set for the PWL dynamic (i.e. for the linear subsystem). Then, we can formulate the feedback min-max MPC problem (6.77) with a *fixed* prediction horizon N, since in that case the control sequence defined as:

$$\mathbf{u}^{\text{f}} = [(\sum_{l \in \mathcal{L}_v^{N-1}} \eta_l u_1^{0,l}(x))^T \cdots (\sum_{l \in \mathcal{L}_v^{N-1}} \eta_l u_{N-1}^{0,l}(x))^T \ (\sum_{l \in \mathcal{L}_v^{N-1}} \eta_l F_1 x_N^{0,l})^T]^T$$

is feasible for the next step in the proof of Theorem 6.3.6 and it keeps the next N modes fixed. The same robust properties are valid in this particular case as in Theorem 6.3.6. ♢

From computational point of view, the optimization problem (6.77) can be recast as a mixed-integer linear program, provided that the stage cost is piecewise affine (e.g. for stage costs (6.71) or (6.72) based on $1/\infty$-norm).

6.4 Robust MPC for PWL systems

In this section we derive an MPC law for the uncertain PWL system (6.62), i.e.

$$\begin{aligned} x(k+1) &= A_i x(k) + B_i u(k) + w(k) \\ y(k) &= C_i x(k) \end{aligned} \quad \text{if } x(k) \in \mathcal{C}_i, \quad (6.80)$$

where now $i \in \mathcal{I}_0 = \mathcal{I}$ and $\bar{\mathcal{C}}_i = \{x \in \mathbb{R}^n : E_i x \geq 0\}$. As in the previous sections, we assume that there is available a stabilizing PWL controller

$$\kappa_{\mathrm{f}}(x) = F_i x \ \text{ if } \ x \in \mathcal{C}_i,$$

where the matrices F_i are computed as in Section 6.1.2, and a polyhedral RPI set X_{f} (e.g. $X_{\mathrm{f}} = \mathcal{O}_\infty$, where $\mathcal{O}_\infty$ is obtained according to Section 6.3.1).

The maximal RPI set $\tilde{\mathcal{O}}_\infty$ included in X_F associated to the PWL system (6.63) is in general not a convex set. Given an initial state $x \in \tilde{\mathcal{O}}_\infty$, it follows that the subsequent state trajectory of the system (6.63) remains in this set, as close as possible to the origin. However, the maximal RPI set $\tilde{\mathcal{O}}_\infty$, for which the PWL controller κ_{f} is feasible, is in general small. Now, we derive a robustly stable MPC scheme that uses in the MPC optimization problem control sequences that do not correspond to fixed state feedback control laws. Therefore, we enlarge the set of initial states that can be steered to a target set. We introduce a new control variable v using the so-called closed-loop paradigm [145] by considering a *semi-feedback* control which combines a local control law with an open-loop correction in order to guarantee that the constraints are satisfied, i.e.

$$\kappa(x) = F_i x + v \ \text{ if } \ x \in \mathcal{C}_i. \tag{6.81}$$

6.4.1 Semi-feedback MPC for PWL systems

We now provide a new MPC strategy for perturbed PWL systems such that we solve on-line a single quadratic optimization problem. It consists of two steps:

Off-line step: In this step we compute off-line the set of initial states and input correction sequences that steer these states to the RPI set $X_{\mathrm{f}} = \mathcal{O}_\infty$ in N steps, using the controller (6.81), where N is the prediction horizon. This set is obtained recursively as follows:

$$\begin{aligned} &\mathcal{X}_0^i = \mathcal{O}_\infty^i \ \ \forall i \in \mathcal{I}, \\ &\mathcal{X}_{k+1}^i = \left\{ \begin{bmatrix} x \\ v \\ \tilde{v} \end{bmatrix} \in \mathbb{R}^{n+m(k+1)} : \begin{array}{c} \begin{bmatrix} A_{F_i} x + B_i v \boxplus W \\ \tilde{v} \end{bmatrix} \in \bigcap_{j \in \mathcal{I}} \mathcal{X}_k^j \\ x \in X, \ F_i x + v \in U \end{array} \right\} \end{aligned} \tag{6.82}$$

for all $k \in \mathbb{N}_{[0,N-1]}$ and $i \in \mathcal{I}$. Note that a similar recursion was proposed also in [31] in the context of gain scheduling for nonlinear systems. The dimension of the sets $\mathcal{X}_k^i$ increases as k increases. Clearly $\mathcal{X}_N^i \subseteq \mathbb{R}^{n+mN}$. We denote with $X_k^i = \mathrm{Proj}_n \mathcal{X}_k^i$, i.e. the projection of $\mathcal{X}_k^i$ into the state space $\mathbb{R}^n$. In conclusion, the set of initial states for the PWL system (6.80) that can be steered to X_{f} in N steps while satisfying the constraints, using the semi-feedback controller (6.81), is given by:

$$X_N = \cup_{i \in \mathcal{I}} (X_N^i \cap \mathcal{C}_i).$$

Because X, U and W are polytopes and initially $\mathcal{X}_0^i = \mathcal{O}_\infty^i$ (recall that $\mathcal{O}_\infty^i$ are polytopes) we obtain that the sets $\mathcal{X}_k^i$ are also polytopes for all $k \geq 0$. As a consequence, the sets X_N^i are polytopes for all $i \in \mathcal{I}$. Therefore, X_N is a union of polytopes, but not necessarily convex.

On-line step: At event step (k, x), where $x \in \mathcal{C}_i$ solve the following quadratic program:

$$V_N^0(x) = \inf_{\mathbf{v}_N} \{\mathbf{v}_N^T \mathbf{v}_N : [x^T \ \mathbf{v}_N^T]^T \in \mathcal{X}_N^i\}, \tag{6.83}$$

where $\mathbf{v}_N = [v_0^T \ \ v_1^T \cdots v_{N-1}^T]^T$. Note that the infimum is attained in (6.83) (since the cost function is continuous and the feasible set is compact) and let $\mathbf{v}_N^0(x) = [(v_0^0(x))^T \ (v_1^0(x))^T \cdots (v_{N-1}^0(x))^T]^T$ be an optimizer. The MPC law is given by

$$\kappa_N(x) = F_i x + v_0^0(x) \ \text{ if } \ x \in \mathcal{C}_i.$$

Note that this MPC law has the form (6.81). We now study the behavior of the closed-loop system obtained from applying this MPC law to the PWL system (6.80), i.e.

$$\begin{array}{rcl} x(k+1) & = & A_{F_i}x(k) + B_i v_0^0(x(k)) + w(k) \\ y(k) & = & C_i x(k) \end{array} \quad \text{if } x(k) \in \mathcal{C}_i. \tag{6.84}$$

Theorem 6.4.1 *(i) The maximal RPI set $\tilde{\mathcal{O}}_\infty$ is asymptotically attractive for* (6.84) *with a domain of attraction X_N.*

(ii) Suppose that the matrices A_{F_i} are asymptotically stable for all $i \in \mathcal{I}$. Then, the RPI set $X_\mathrm{f} = \mathcal{O}_\infty$ is asymptotically attractive for (6.84) *with a domain of attraction X_N.*

Proof: (i) If $x \in X_N$, then the quadratic program (6.83) has an optimal solution $\mathbf{v}_N^0(x)$. Moreover, there exists an $i_0 \in \mathcal{I}$ such that $x \in \mathcal{C}_{i_0}$. Let us denote with $\mathbf{v}^\mathrm{f} = [(v_1^0(x))^T \ (v_2^0(x))^T \cdots (v_{N-1}^0(x))^T \ 0^T]^T$. Since X_f is an RPI set, it follows immediately that for each $w \in W$ there exists a $j \in \mathcal{I}$ such that $f_{\mathrm{PWL}}(x, \kappa_N(x) \in \mathcal{C}_j$ and $[(f_{\mathrm{PWL}}(x, \kappa_N(x), w))^T \ (\mathbf{v}^\mathrm{f})^T]^T \in \mathcal{X}_N^j$. Therefore, $\mathbf{v}^\mathrm{f}$ is feasible at the next step. Moreover, the following inequality holds:

$$V_N^0(f_{\mathrm{PWL}}(x, \kappa_N(x), w)) - V_N^0(x) \leq -(v_0^0(x))^T v_0^0(x) \quad \forall w \in W. \tag{6.85}$$

Let $\phi(k; x, \kappa_N, \mathbf{w})$ denote the state solution of (6.84) at step k when the initial state is x, the MPC law κ_N is employed and $\mathbf{w}$ is a disturbance sequence. Then, the sequence $\{V_N^0(\phi(k; x, \kappa_N, \mathbf{w}))\}_{k\geq 0}$ is non-increasing and bounded from below by 0 and thus it is convergent for all admissible disturbance sequences $\mathbf{w}$. Summing the relation (6.85) from $k = 0$ to ∞ we obtain that the series $\sum_{k\geq 0}(v_0^0(\phi(k; x, \kappa_N, \mathbf{w})))^T v_0^0(\phi(k; x, \kappa_N, \mathbf{w}))$ is also convergent. This leads to:

$$\lim_{k\to\infty} v_0^0(\phi(k; x, \kappa_N, \mathbf{w})) = 0 \tag{6.86}$$

for all admissible disturbance sequences $\mathbf{w}$. As a consequence, it follows that $d(\phi(k; x, \kappa_N, \mathbf{w}), \tilde{\mathcal{O}}_\infty) \to 0$ as $k \to \infty$, because $\tilde{\mathcal{O}}_\infty$ is the maximal set of states for which the PWL feedback controller κ_f is feasible and the closed-loop state trajectory satisfies the state constraints.

(ii) We now show that $d(\phi(k; x, \kappa_N, \mathbf{w}), \mathcal{O}_\infty) \to 0$ as $k \to \infty$, provided that the matrices A_{F_i} are asymptotically stable for all $i \in \mathcal{I}$. First, let us note that the closed-loop state trajectory is given by:

$$\begin{aligned} \phi(k+1; x, \kappa_N, \mathbf{w}) =& A_{F_{i(k)}} \ldots A_{F_{i(0)}} x + \\ & \sum_{j=1}^{k+1} A_{F_{i(k+1)}} \ldots A_{F_{i(j)}} (B_{i(j-1)} v_0^0(\phi(j-1; x, \kappa_N, \mathbf{w})) + w_{j-1}), \end{aligned} \tag{6.87}$$

where $A_{F_{i(k+1)}} = I$ and $i(0), \ldots, i(k)$ is a feasible switching sequence. Now, given $x \in X_N$ there exists an $x_0^o \in \mathcal{O}_\infty$ such that $d(x, \mathcal{O}_\infty) = \|x - x_0^o\|$ (recall that $\mathcal{O}_\infty$ is a compact set). Now $\phi(1; x, \kappa_N, \mathbf{w}) = A_{F_{i(0)}} x + B_{i(0)} v_0^0(x) + w_0$. Let us define $x_1^o = A_{F_{i(0)}} x_0^o + w_0$. From the definition of $\mathcal{O}_\infty$ it is clear that $x_1^o \in \mathcal{O}_\infty$. Therefore, we obtain:

$$d(\phi(1; x, \kappa_N, \mathbf{w}), \mathcal{O}_\infty) \leq \|\phi(1; x, \kappa_N, \mathbf{w}) - x_1^o\| \leq \|A_{F_{i(0)}}\| \, \|x - x_0^o\| + \|B_{i(0)} v_0^0(x)\|,$$

where $\|A\|$ denotes the induced norm of the matrix A. By induction, using (6.87), we can prove that

$$d(\phi(k+1;x,\kappa_N,\mathbf{w}),\mathcal{O}_\infty) \le \|\phi(k+1;x,\kappa_N,\mathbf{w}) - x^o_{k+1}\| \le \tag{6.88}$$
$$\|A_{F_{i(k)}}\|...\|A_{F_{i(0)}}\| \; \|x - x^o_0\| + \sum_{j=1}^{k+1} \|A_{F_{i(k+1)}}\|...\|A_{F_{i(j)}}\|\|\|B_{i(j-1)}v^0_0(\phi(j-1;x,\kappa_N,\mathbf{w}))\|,$$

where $x^o_{k+1} := A_{F_{i(k)}} x^o_k + w_k \in \mathcal{O}_\infty$. Since A_{F_i} are asymptotically stable for all $i \in \mathcal{I}$, there exists a constant $0 < \delta < 1$ and $L > 0$ such that

$$\|A_{F_{i(k)}}\|...\|A_{F_{i(j)}}\| \le L\delta^{k-j}. \tag{6.89}$$

Using now (6.89) and (6.86) in (6.88), we obtain

$$\lim_{k\to\infty} d(\phi(k;x,\kappa_N,\mathbf{w}),\mathcal{O}_\infty) = 0.$$

$\diamondsuit$

Note that we are able to show asymptotic attractiveness. But, in general, nothing can be said about robust stability. The MPC scheme derived in this section is more difficult to be implemented to PWA systems, due to the special construction of the set X_N in the off-line step. Of course, if we are able to construct a polyhedral RPI set for all dynamics of the PWA system (not only for the PWL dynamics), then the MPC scheme presented in this section can be extended also to PWA systems with additive disturbances.

6.4.2 Example

We consider the following example taken from [11], but this time with an additive term to take also into account disturbances:

$$x(k+1) = 0.8 \begin{bmatrix} \cos\alpha(k) & -\sin\alpha(k) \\ \sin\alpha(k) & \cos\alpha(k) \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) + w(k),$$
$$\alpha(k) = \begin{cases} \pi/3 \ \text{ if } \ x(k) \in \mathcal{C}_1 \\ -\pi/3 \ \text{ if } \ x(k) \in \mathcal{C}_2, \end{cases}$$

where $\mathcal{C}_1 = \{x \in \mathbb{R}^n : [1 \;\; 0]x(k) \ge 0\}$, $\mathcal{C}_2 = \{x \in \mathbb{R}^n : [1 \;\; 0]x(k) < 0\}$ and the following constraints:

$$X = \{x \in \mathbb{R}^2 : \|x\|_\infty \le 10\}, \; U = \{u \in \mathbb{R} : |u| \le 1\}.$$

We assume that the disturbance set is given by:

$$W = \{w \in \mathbb{R}^2 : w_1 = w_2, \; \|w\|_\infty \le 0.1\}.$$

We get the following PWL feedback controller: $\kappa_\mathrm{f}(x) = F_i x$ if $x \in \mathcal{C}_i$, where $\mathcal{I} = \{1,2\}$ and the matrices F_i are determined in Section 6.1.2, i.e.

$$F_1 = [-0.692 \;\; -0.4], \; F_2 = [0.866 \;\; -0.5].$$

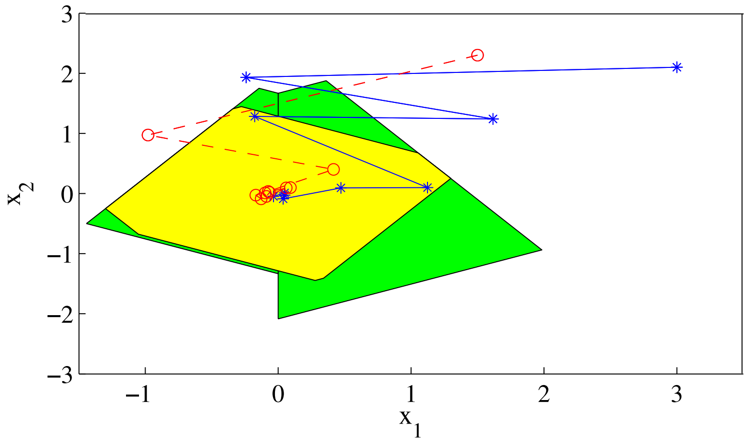

Figure 6.3: The closed-loop state trajectories corresponding to the robust MPC scheme from Section 6.3.2 (full line) and from Section 6.4.1 (dotted line). The inner polytope represents the RPI set $\mathcal{O}_\infty$ and the outer polygon is the maximal RPI set $\tilde{\mathcal{O}}_\infty$.

We see that the matrices A_{F_i} are strictly stable. Therefore, we can apply Theorem 6.3.3, the RPI set $\mathcal{O}_\infty$ being determined after 2 iterations (i.e. $t^* = 2$):

$$\mathcal{O}_\infty = \left\{ x \in \mathbb{R}^2 : \begin{bmatrix} -0.866 & -0.5 \\ 0.866 & 0.5 \\ 0.866 & -0.5 \\ -0.866 & 0.5 \\ 0.499 & -0.866 \\ -0.499 & 0.866 \\ 0.500 & 0.866 \\ -0.500 & -0.866 \end{bmatrix} x \leq \begin{bmatrix} 1.25 \\ 1.25 \\ 1 \\ 1 \\ 1.3906 \\ 1.3906 \\ 1.1125 \\ 1.1125 \end{bmatrix} \right\}$$

which is a polytope that contains the origin in its interior.

Applying the robust feedback min-max MPC scheme with known mode proposed in Section 6.3.2, with initial state $x = [3 \;\; 2.1]$, $Q = I, R = I$, together with the stage cost (6.71) defined by the ∞-norm, initial prediction horizon $N = 3$ and the terminal set being $\mathcal{O}_\infty$, we get the full line in Figure 6.3. We also apply the robust MPC scheme proposed in Section 6.4.1 for the initial state $x = [1.5 \;\; 2.1]$ with the same prediction horizon $N = 3$, the corresponding closed-loop state trajectory is displayed as dotted line. The inner polytope represents the RPI set $\mathcal{O}_\infty$ and the outer polygon is the maximal RPI set $\tilde{\mathcal{O}}_\infty$. We remark that once the trajectory enters $\mathcal{O}_\infty$ it remains there in both schemes.

6.5 Computational complexity

In this section we discuss the computational complexity of the MPC schemes derived in this chapter. The mixed logical dynamical framework represents one of the main tools for computing optimal control for PWA systems [11]. First let us note that the deterministic MPC scheme proposed in Section 6.2.1 using an ellipsoidal terminal set is based on solving on-line the non-convex optimization problem (6.51): the objective function is convex subject to linear and convex inequality constraints and nonlinear equality constraints (except the case $N = 1$ when the optimization problem (6.51) becomes a convex optimization problem). In [104] an algorithm based on feasible switching sequences is proposed that can be adapted to solve (6.51). Using a polyhedral terminal set determined as in Section 6.2.4 the optimization problem (6.51) becomes a mixed-integer quadratic program.

In the case of the feedback min-max MPC scheme derived in Section 6.3.2 the optimization problem (6.77) can be recast as a mixed-integer linear program, provided that the stage cost is piecewise affine (e.g. for stage costs (6.71) or (6.72) based on $1/\infty$-norm). The result is not surprising, since in Chapter 5 the min-max MPC law for perturbed MMPS systems, using similar stage costs, is also computed by solving a set of linear programs that can be seen as a mixed-integer linear program. For the semi-feedback MPC scheme proposed in Section 6.4.1 we have to solve on-line *only* a quadratic program. This is possible since we have removed some computations in the off-line step.

6.6 Example: adaptive cruise controller for a Smart

This application was motivated by the design of an adaptive cruise controller for a Smart. The objective is to follow as well as possible a leading vehicle in a highway environment. In order to meet realistic conditions several constraints on kinematic and dynamical entities are introduced, fulfilling safety, comfort and environmental issues. We assume that the reference trajectory, transmitted by the leading vehicle, will eventually reach a stationary speed, around which we guarantee stability and feasibility of the controller, even in presence of bounded disturbances. In general this stationary value is arbitrary, but to make the problem interesting in the hybrid framework we choose it along the switching manifold. Additionally piecewise constant references may be tracked under mild additional assumptions.

The design of the control law is split into two phases: during the transient of the reference trajectory we consider *tracking* of the speed of the leading vehicle. When the reference has reached its steady state we enforce *stability* of the closed-loop system (which is not guaranteed a priori) in the regulation. To this aim we compute a terminal cost and a terminal set, by means of the methods described in Sections 6.1.2 and 6.2. The disturbance w, due to the PWA approximation of the real system and to the measurement errors, may give rise to infeasibilities. For this reason we also implement *robust* MPC as derived in Section 6.3.

6.6.1 Cruise controller setup and simulations

The goal of a cruise controller for a road vehicle is to track the velocity of the vehicle in front, to guarantee secure driving, smoothness of platoons traffic [58], comfort and optimal usage of the engine/brake system. The descriptive scenario is shown in Figure 6.5.a, where two cars are driving after another in a string. We consider here platoons formed of only two vehicles, but the extension to the general case is also possible. We first describe the general setup, then we implement *deterministic* MPC as proposed in Section 6.2. We observe that due to disturbances, infeasibilities occur, motivating thus the use of *robust* MPC described in Section 6.3. When the reference reaches a stationary value we implement for both cases the stabilizing methods described in Section 6.2 and 6.3, by plugging into the MPC scheme the corresponding terminal cost and terminal set.

Model We consider a nonlinear *viscous* friction and a *road-tire* static friction, proportional to the mass m of the vehicle. Braking will be simulated by applying a negative input. The

m	Mass of vehicle	800 kg
c	Viscous coefficient	0.5 kg/m
μ	Coulomb friction coefficient (dry asphalt)	0.01
b	Traction force	3700 N
g	Gravity acceleration	9.8 m/s^2

Table 6.1: Definitions and values of the entries of equation (6.90).

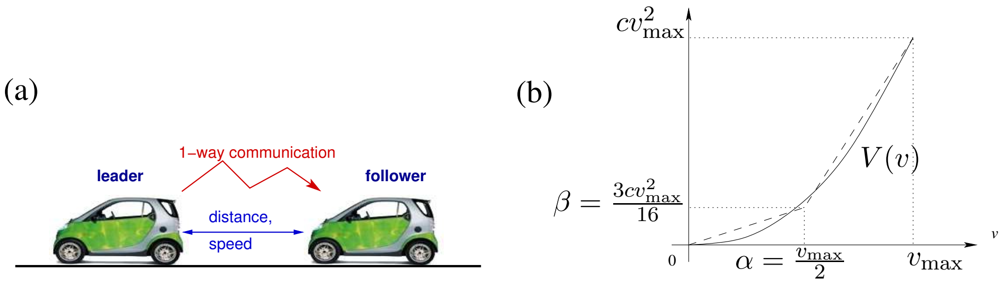

Figure 6.4: (a) Adaptive cruise control set up and (b) nonlinear to PWA approximation.

differential equation for positive velocity of the following vehicle is:

$$m\dot{x}(t) + cx^2(t) + \mu mg = bu(t), \tag{6.90}$$

where $x(t)$ is the speed of the following vehicle, $bu(t)$ is the traction/brake force, proportional to the input $u(t)$. Numerical values are listed in Table 6.6.1.

A least squares approximation (Figure 6.4.b) of the nonlinear friction curve $V(v) = cv^2$ leads to the continuous-time PWA system:

$$\begin{array}{l} m\dot{x}(t) + c_1 x(t) + f_1 = bu(t) \;\; \text{if} \;\; x < \alpha \\ m\dot{x}(t) + c_2 x(t) + f_2 = bu(t) \;\; \text{if} \;\; x \geq \alpha, \end{array}$$

where the coefficients c_1, c_2, f_1, f_2 are derived using the data shown in Figure 6.4.b[7]. The sampling time is $T = 1s$ and the discrete time uncertain PWA model has the following form:

$$\begin{array}{l} x(k+1) = A_1 x(k) + B_1 u(k) + a_1 + w(k) \;\; \text{if} \;\; x < \alpha \\ x(k+1) = A_2 x(k) + B_2 u(k) + a_2 + w(k) \;\; \text{if} \;\; x \geq \alpha \end{array} \tag{6.91}$$

with

$$A_1 = 0.9912,\; B_1 = 4.6047,\; a_1 = -0.0976,\; A_2 = 0.9626,\; B_2 = 4.5381,\; a_2 = 0.4428.$$

Comparing the true model with its PWA approximation we chose the disturbance set $W = [-0.5 \;\; 0.5]$.

Constraints Safety, comfort and economy or environmental issues result in defining constraints on the state x and the control input u. In particular we consider limitations on the velocity, acceleration, on the control input $u(k)$ and on its variation $u(k+1) - u(k)$. We require for all k

$$\begin{array}{c} x_{\min} \leq x(k) \leq x_{\max} \\ -u_{\max} \leq u(k) \leq u_{\max} \\ a_{\text{dec}} T \leq x(k+1) - x(k) \leq a_{\text{acc}} T \\ -\Delta_u T \leq u(k+1) - u(k) \leq \Delta_u T. \end{array} \tag{6.92}$$

[7] A finer approximation is also possible, by setting more than one breakpoint.

T	Sampling time	1 s
$x_{\min}$	Minimum velocity	5.0 m/s
$x_{\max}$	Maximum velocity	37.5 m/s
a_{acc}	Comfort acceleration	2.5 m/s^2
a_{dec}	Comfort deceleration	-1 m/s^2
$u_{\max}$	Maximum throttle/brake	1
Δ_u	Maximum throttle/brake variation	0.2
α	Switching velocity	18.75 m/s

Table 6.2: Values of the parameters specifying the constraints.

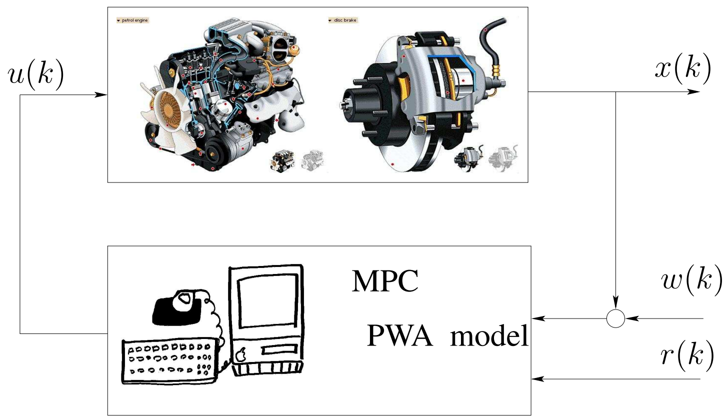

Figure 6.5: MPC controller setup: PWA prediction model versus nonlinear simulation model and disturbance injection.

Numerical values are listed in Table 6.6.1. Note that the first three constraints in (6.92) can be recast as constraints of the form (6.3)–(6.4). Therefore, we determine a feedback controller satisfying the first three constraints using the methods from Section 6.1.2 and then we check whether this controller fulfills also the fourth constraint. Note that although some of these constraints may be violated without causing major damages, i.e. collision or engine breakdown, we decided to consider all of them as *hard constraints*.

Tracking and regulation The weight matrices are chosen as $Q = 1$ and $R = 0.01$. The length of the prediction horizon is set to $N = 4$. The overall goal is to tune an on-line controller κ_N that tracks the reference velocity depicted in Figure 6.6.a (dashed), and within the constraints (6.92). The initial velocity is $x = 6\ m/s$. Figure 6.5 shows a block diagram of the simulation setup. At each step k the controller receives the N steps ahead prediction of the reference speed of the front vehicle.

Then, by measuring the speed, the MPC scheme computes the best control action using the prediction model (6.91) and feeds the first sample to the car's actuators, modeled here by equation (6.90). In the framework of the hybrid systems it is relevant to study the behavior of the stabilizing controller around the switching velocity $x_{\text{e}} = \alpha$, which becomes the state equilibrium. For this reason, without loss of general applicability, we use a reference signal the steady state of which is $x_{\text{e}} = \alpha$.

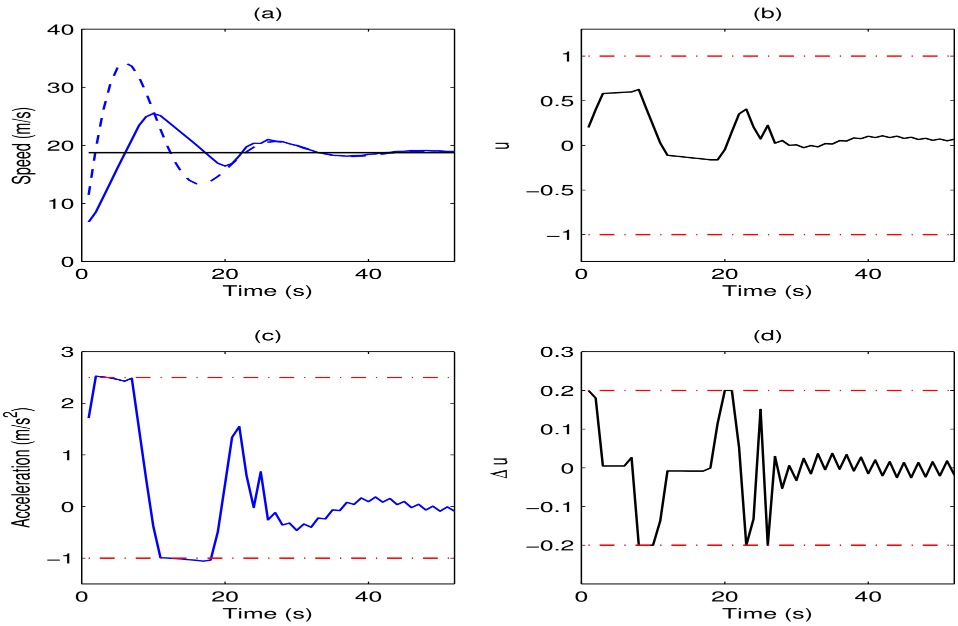

Figure 6.6: Simulation results for deterministic MPC without external noise. (a) System (solid) and reference (dashed) velocities, (b) Control input and constraints, (c) Acceleration and constraint (note the violations), (d) Variation of control input and constraints.

6.6.2 Simulations using deterministic MPC

The stabilizing controller obtained via the Lyapunov arguments described in Section 6.1.2 is $\kappa_{\mathrm{f}}(x) = -0.072x + 1.411$, and we observe that it is common for both subsystems. Additionally, we obtain the terminal cost $V_{\mathrm{f}}(x) = 1.766(x - x_{\mathrm{e}})^2$ and a polyhedral PI set corresponding to the controller κ_{f} and the constraints (6.92) given by $X_{\mathrm{f}} = \{x \in \mathbb{R} : 16.01 \leq x \leq 21.48\}$. We use these terminal values only when the reference signal of the front vehicle has reached its stationary value. This decision is taken by computing the standard deviation and the average of the last reference samples and by establishing some thresholds.

We show first the results obtained when the disturbance $w(k) = 0$. In Figure 6.6.(a-d) we show velocity, control, acceleration and Δu, obtained with the 2-norm used in the stage cost. From these figures we wish to point out that even in the absence of external disturbances there is a minor violation of the acceleration constraints. This is in fact due to the mismatch between the prediction and the simulation model.

In Figure 6.7 we show the solution offered by the deterministic MPC when the disturbance signal is active[8]. The large infeasibilities motivate the use of the robust MPC.

6.6.3 Simulations using robust MPC

We show in Figure 6.8 the results obtained, under the same scenario of the previous simulations, with the use of robust MPC described in Section 6.3. This robust MPC scheme is implemented using the stage cost (6.71) induced by the 1-norm, leading to a mixed-integer linear program. We immediately observe the benefits of applying robust MPC which eliminates the infeasibilities due to disturbances. For this robust rejection we use the terminal cost $V_{\mathrm{f}} = 0$ along all simulation

[8]The disturbance signal $w(k)$ is generated randomly, but we use the same one in all simulations.

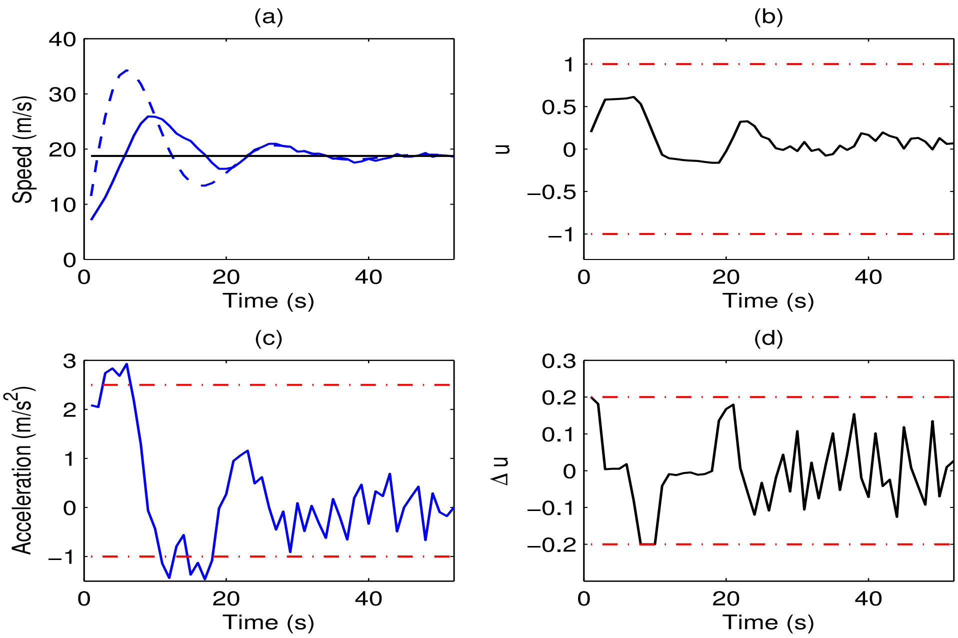

Figure 6.7: Simulation results for deterministic MPC with external noise. (a) System (solid) and reference (dashed) velocities, (b) Control input and constraints, (c) Acceleration and constraints (note the significant violations), (d) Variation of control input and constraints.

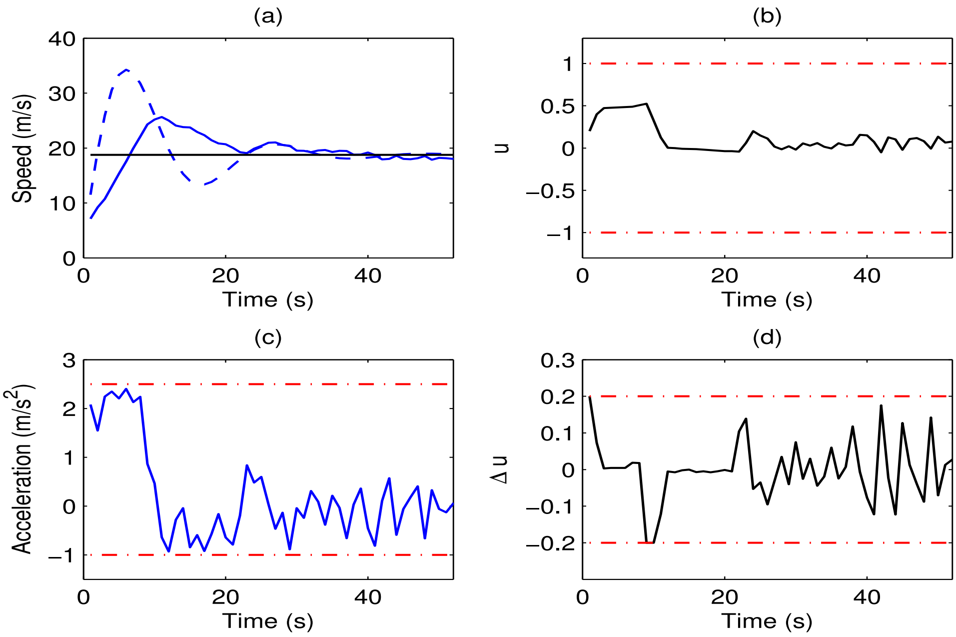

Figure 6.8: Simulation results for robust MPC. (a) System (solid) and reference (dashed) velocities, (b) Control input and constraints, (c) Acceleration and constraints, (d) Variation of control input and constraints. Note that no infeasibilities occur and we also have robust stability.

time. By means of the arguments of Section 6.3, we construct, in the vicinity of the reference stationary state, an RPI set which serves as a terminal set $X_{\mathrm{f}} = \{x : 16.9 \leq x \leq 20.63\}$, that guarantees robust asymptotic stability of the closed-loop system despite the added disturbances.

6.7 Conclusions

In the first part of this chapter we have derived a local PWL feedback controller corresponding to the PWL dynamics of a PWA system by means of LMIs. We have taken into account the piecewise linear structure of the system by using a relaxation procedure called the S-procedure. We have derived a stabilizing MPC strategy for the class of PWA systems that uses the standard ingredients: a terminal cost constructed from an upper bound on the infinite-horizon quadratic cost and a terminal set derived from the backward procedure which can be either convex or polyhedral. It is worth to note that although the PWA system might be discontinuous we have shown that the optimal value function of the MPC optimization problem is continuous at the equilibrium and can serve as a Lyapunov function for the closed-loop system.

In the second part we have developed a robustly stable MPC strategy for the class of PWA systems with additive disturbances. We use the dual-mode paradigm and the benefits of incorporating feedback to derive a robust MPC law based on solving a min-max control problem. In order to preserve convexity we have imposed that the mode is fixed at each step over the prediction horizon. This allows us to consider only the disturbance realizations that take on values at the vertexes of the disturbance polytope. Finally, for uncertain PWL systems we have derived a robust MPC scheme which removes the constraint that the mode is known and uses semi-feedback control policies. This scheme combines a local control law with an open-loop correction in order to guarantee satisfaction of the constraints. As an application, we have studied the problem of designing an adaptive cruise controller by means of MPC, which enabled us to tackle the control problem in a mixed-integer linear/quadratic program formulation. Once the reference speed has reached a steady state value we have also considered the problem of guaranteeing stability, via the construction of terminal cost and terminal set as explained in Section 6.2. Moreover we have considered the robust MPC as in Section 6.3 that allowed us to prevent infeasibility due to disturbances and still preserve robust stability.

Chapter 7

Conclusions and future research

In this final chapter we present a summary of the contributions that have been made in the preceding chapters. We also discuss some interesting open problems and possible future directions that are related to the research presented in this thesis.

7.1 Conclusions

The main focus of this Ph.D. thesis was to develop structured control design methods for specific classes of hybrid systems and DES that are industrially relevant. Among different existing control methods we chose the optimal control framework and its receding horizon implementation often referred to as the MPC due to their attractive features that make these control approaches also interesting and relevant for extension to hybrid systems and DES. The classes of systems studied in this thesis are MPL systems and switching MPL systems (corresponding to DES), MMPS systems (corresponding to DES and hybrid systems) and PWA systems (corresponding to hybrid systems). The main contributions of this thesis are summarized below.

Optimal control for MPL systems

- We have derived a solution to a class of finite-horizon optimal control problems for constrained MPL systems where the performance is measured via a cost function that may, in particular, be chosen to provide a just-in-time controller. We have determined sufficient conditions under which the optimization problem becomes a linear program. In the absence of constraints and for a particular stage cost, that provides a trade-off between minimizing the due date error and a just-in-time control, we have obtained an analytic solution for the optimal control problem.

- The robustification of the finite-horizon optimal control problem has been also considered. We have analyzed the solutions to three classes of finite-horizon min-max control problems for uncertain MPL systems subject to hard state and input constraints, depending on the nature of the control sequence over which we optimize: open-loop control sequences, disturbance feedback policies, and state feedback policies. Despite the fact that the controlled system is nonlinear, we were able to provide sufficient conditions, that are usually satisfied in practice, such that convexity of the optimal value function and its domain is preserved and consequently, the min-max control problems can be recast as a linear program or solved via N parametric linear programs, where N is the prediction horizon.

MPC for MPL systems

- We have introduced the notion of Lyapunov stability for MPL systems and we have found connections with the classical definition of stability for DES in terms of boundedness of the buffer levels. We have designed an MPC strategy for unconstrained MPL systems that guarantees a priori stability of the closed-loop system. In this particular case for proving stability we did not follow the classical approach based on a terminal set and a terminal cost, but rather taking advantage of the special properties that MPL systems posses, we have shown that by a proper tuning of the design parameters stability can be guaranteed.

- We have further extended the MPC framework to the class of constrained MPL systems. We have followed a similar finite-horizon MPC approach as for conventional, time-driven systems that uses a terminal set and a terminal cost as basic ingredients. However, the extension from classical time-driven systems to discrete event MPL systems is not trivial since many concepts from system theory have to be adapted adequately. In particular, we have introduced the notion of positively invariant set for a *normalized* MPL systems and the main properties were derived for such a set. Closed-loop stability was demonstrated using Lyapunov arguments.

- We have also considered robust stability of the MPC law corresponding to an uncertain MPL system. Based on the assumptions that a robustly positively invariant set is available and the stage cost has a particular representation we were able to prove robust stability of the closed-loop system.

- Finally, sufficient conditions for guaranteeing closed-loop stability for an MPC law applied to a switching MPL system have been derived. We have shown that under the boundedness assumption on the MPC law, the closed-loop state trajectory is also bounded.

MPC for MMPS systems

- We have derived an efficient algorithm for solving an open-loop MPC optimization problem for uncertain MMPS systems subject to hard input constraints and an MMPS (or PWA) stage cost, based on solving a set of linear programs.

- We have extended the feedback min-max MPC framework to the class of uncertain MMPS systems using disturbances feedback policies and we have derived an efficient algorithm for solving the corresponding min-max control problem.

MPC for PWA systems

- We have derived LMI conditions for the stabilization of a PWL system using a PWL feedback controller and a piecewise quadratic Lyapunov function. We have taken into account the structure of the system and different levels of conservatism from applying the S-procedure have been discussed. We have given a detailed discussion to the solution of the LMIs.

- We have extended the MPC formulation for PWA systems with a terminal equality constraint to a new MPC strategy based on a terminal *inequality* constraint corresponding to the PWL dynamics. Using an upper bound on the infinite-horizon quadratic cost as a terminal cost and deriving also a convex terminal set we have proved the stabilizing properties of the MPC law applied to a PWA system subject to input and output constraints. Note that although the PWA system might be discontinuous we have shown that the optimal value function of the MPC optimization problem is continuous at the origin and can serve as a Lyapunov function for the closed-loop system. We have derived an algorithm for enlarging the terminal set based on backward procedure, which in particular provides also a method to derive a polyhedral terminal set. In this way we overcome the drawback of the algorithms based on infinite recursive methods for constructing a terminal set.

- The robustification of the standard MPC via the addition of a robustness constraint was discussed. A new sufficient condition that enables us to preserve convexity of the state set evolution is presented and a state feedback min-max MPC scheme based on a dual-mode approach, which incorporates this condition, is derived together with its main features, in particular robust stability.

Examples

- As applications we have considered the design of a controller for a temperature control system in a room and of an adaptive cruise controller for a road vehicle by means of MPC strategies developed in this thesis. The main message resulting from these applications is the need for robust hybrid control and for efficient algorithms for solving such optimal control problems.

Structural properties of a traffic model

- In Appendix A we will present some work that has been done in the first year of the Ph.D. research. The central idea behind this chapter is to study the main properties of a macroscopic traffic flow model, called Helbing traffic flow model. For the first time it has been shown that this model does not give rise to negative flow and density. We will also derive the main properties such as the formulas for the shock and rarefaction waves, and the solution of the Riemann problem for the Helbing traffic flow model.

7.2 Directions for future research

Some possible suggestions for future research are outlined below. First we discuss some necessary improvements of our results and then we list several relevant research problems, which can be considered as future work.

Optimal control and MPC for MPL systems

- The number of inequalities that describe the feasible set of a min-max control problem from Section 3.3 is, in the worst case, exponential. The relaxation introduced in Section 3.2.2 for the deterministic case is not applicable for the robust case. Therefore, finding new relaxation methods that reduce the number of inequalities need to be developed. Some initial steps in this direction were already made in Section 3.4.2 where in some particular cases of the uncertainty description the robust control problem is recast as a deterministic one.

- Possible extensions of the robust control results obtained in Section 3.3 to a larger class of systems than the class of MPL systems should be investigated. We conjecture that these results can be extended to the class of monotone convex systems extensively used in system biology.
- Efficient algorithms for solving robust optimal control problems for MPL systems with stochastic disturbances need to be developed.
- Timing issues, i.e. availability of the state of an MPL system at a certain time, need to be investigated in more detail. Some initial ideas were already presented in Sections 3.2.3 and 3.3.5.
- The case of stability and feasibility of MPC with output feedback needs to be investigated. The simultaneous design of an observer and model predictive controller might prove beneficial in enlarging the region of attraction of the closed-loop MPL system.
- An interesting topic in the context of DES is the investigation of different notions of stability (e.g. Lyapunov stability, boundedness in terms of buffer levels, etc.) and their connections.

Optimal control and MPC for PWA systems

- The RPI sets for PWA systems are in general non-convex. We have proposed an algorithm for computing an RPI set for the PWL dynamics of the PWA system. Extension of this algorithm to the full PWA dynamics can be a challenge.
- Efficient robust MPC strategies for uncertain PWA systems by means of robustness constraints and more results regarding the robust stability and feasibility of these strategies need to be developed. Contraction constraints can be considered as an alternative to robustly invariant terminal sets.
- Extension of the MPC framework, for both deterministic and uncertain PWA systems, to output tracking should also be investigated. A first step in extending the results of Chapter 6 is to consider proving a tracking MPC strategy is stabilizing.
- Characterization of robust stability using less restrictive criteria need to be investigated. Integral quadratic constraints could be embedded in robust MPC schemes in order to keep the performance as close as possible to the optimal performance while still guaranteeing robust stability. Other types of stability criteria could be considered, e.g. input-to-state stability.

Other research topics

- The theoretical problems tackled in this Ph.D. thesis are “hard” in a mathematical sense, i.e. the computational complexity – in the worst case – grows exponentially with the problem size. This motivates further research for developing approximate control schemes (e.g. MPC) which provides good, not necessarily optimal answers for control problems with specific structures and where the computational complexity grows only polynomially.

- Relaxation procedures in robust control and characterization of the conservatism, in particular verification of the relaxation exactness are interesting topics. There exists a strong need to exploit the system theoretic structure in computations.

- Extension of the MPC framework to other classes of hybrid systems that are practically relevant should be considered: e.g. systems with uncertain switching surfaces, continuous PWA systems with state reset, PWA systems with interval uncertainty, etc.

- Extension of MPC strategies to complex stochastic nonlinear and hybrid systems using randomized algorithms or Markov chain Monte-Carlo methods could also be an interesting topic for future research.

- Extensive case studies, in particular focusing on implementation aspects in industrial environments need to be performed in more depth.

- Distributed control for large-scale systems. Up to now, most control methods for hybrid and discrete event system are based on a centralized control paradigm and/or on ad-hoc techniques. However, centralized control of large-scale systems is often not feasible in practice due to computational complexity, communication overhead, and lack of scalability. Furthermore, a structured control design method is also lacking. Therefore, there is a need for developing structured and tractable design methodologies for control of large-scale hybrid and discrete event systems.

Hoc erat in votis!

Appendix A

Structural properties of the Helbing traffic flow model

In the first year of my Ph.D. I was involved in a project whose main topic was traffic control. In this appendix we present some results obtained during that year on this topic.

We analyze the structural properties of the shock and rarefaction wave solutions of a macroscopic, second-order non-local continuum traffic flow model, namely *the Helbing model*. We will show that this model has two families of characteristics for the shock wave solutions: one characteristic is slower, and the other one is faster than the average vehicle speed. Corresponding to the slower characteristic we have 1-shocks and 1-rarefaction waves, the behavior of which is similar to that of shocks and rarefaction waves in the first-order model of Lighthill-Whitham-Richards. Corresponding to the faster characteristic there are 2-shocks and 2-rarefaction waves, which behave differently from the previous one, in the sense that the information in principle travels faster than average vehicle speed, but – as we shall see – in the Helbing model this inconsistency is solved via the addition of a non-local term. We also proved that for the Helbing model the shocks do not produce negative states as other second-order models do. Moreover, we derive the formulas for the solution of the Riemann problem associated with this model in the equilibrium case.

A.1 Introduction

Many researchers consider that traffic behavior on a freeway at a given point in time-space is only affected by the conditions of traffic in a neighborhood of that point, proposing different models based on partial differential equations. In this context, one of the most well known traffic flow models is the Lighthill-Whitham-Richards (LWR) model [96, 142, 166], which is a first-order model. In [135, 166] a second-order traffic model is proposed. In this appendix we discuss yet another macroscopic second-order model, which is based on gas-kinetic equations with a non-local term as proposed in [67,69,70,78]. In this model, traffic is described macroscopically as if it were a fluid with the cars as molecules, obtaining the traffic equations from a gas-kinetic level of description. The Helbing model is based on statistical kinetic theory, where macroscopic laws are obtained from integration of molecular properties such as positions, collisions, overtaking, and velocities.

As an introduction to our discussion and to make the appendix self-contained, a brief review of the Helbing model is presented in the next section. The new contributions start with Section A.3, where we discuss the structural properties of the shock wave solution. In Section A.4 we present the structural properties of the rarefaction waves solution, and in Section A.5 we discuss the solution of the Riemann problem associated with the Helbing model. This appendix is based on [116].

A.2 Helbing traffic flow model

In general, continuum macroscopic traffic models contain two independent variables: location x, and time t. There usually are three states: *density* ρ, *average speed* V, and *flow* Q with $Q = \rho V$. Because the number of vehicles is conserved, all macroscopic traffic flow models are based on the continuity equation, which expresses the relation between the rates of change of the density $\rho(x,t)$ with respect to t and of the flow $Q(x,t)$ with respect to x:

$$\frac{\partial \rho}{\partial t} + \frac{\partial Q}{\partial x} = 0.$$

Here, $\frac{\partial \rho}{\partial t}$ denotes the partial derivatives of ρ with respect to t. To describe time-varying and spatially varying average velocities $V(x,t)$ such as those that occur in traffic jams or stop-and-go traffic we need a dynamic velocity equation. Gas-kinetic equations for the average velocity have been proposed in a number of publications such as [71, 134, 136]. Because we are interested in macroscopic quantities we can integrate those equations to derive formulas for the first moment. For all these models after integration, the equation for average velocity can be written as

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + \frac{1}{\rho}\frac{\partial P}{\partial x} = \frac{V_{\mathrm{e}} - V}{\tau},$$

where P is the traffic pressure, defined as $P(x,t) = \rho(x,y)\theta(x,t)$ with θ the velocity variance (see also equation (A.2) below), and where V_{e} is the dynamical equilibrium velocity towards which the average velocity of vehicles relaxes. The macroscopic traffic equations of the Helbing model are based on the gas-kinetic traffic equations of [134] and a method analogous to the derivation of the Euler equations for ordinary fluids (i.e. the Chapman-Enskog expansion). Compared to the other models, in the Helbing model the dynamical equilibrium velocity V_{e} also depends on the density and average velocity at an interaction point that is advanced by about the safe distance. More specifically, the following Euler-like equation with a non-local term for the average vehicle velocity is considered:

$$\frac{\partial V}{\partial t} + \underbrace{V\frac{\partial V}{\partial x}}_{\text{transport}} + \underbrace{\frac{1}{\rho}\frac{\partial P}{\partial x}}_{\text{pressure}} = \underbrace{\frac{V_0 - V}{\tau}}_{\text{acceleration}} - \underbrace{\frac{V_0(\theta + \theta_{\mathrm{a}})}{\tau A(\rho_{\max})}\left(\frac{\rho_{\mathrm{a}} T}{1 - \rho_{\mathrm{a}}/\rho_{\max}}\right)^2 B(\delta_{\mathrm{v}})}_{\text{braking}}. \tag{A.1}$$

So, the change in time of the average velocity V is given by: a *transport term* originating from the propagation of the velocity profile with the average velocity V, a *pressure term* that has a dispersion effect due to a finite variance of the vehicle velocities, an *acceleration term* describing the acceleration towards the average desired velocity V_0 of the drivers with relaxation time τ, and finally a *braking term*: this is a non-local term that models braking in response to traffic situation downstream at the interaction point $x_{\mathrm{a}} = x + \gamma(1/\rho_{\max} + TV)$ with $1 < \gamma < 2$ a model parameter, $\rho_{\max}$ is the maximum density, and T is the average time headway. In equation (A.1) we also have a Boltzmann factor of the form

$$B(\delta_{\mathrm{v}}, S) = \delta_{\mathrm{v}}\frac{e^{-z^2/2}}{\sqrt{2\pi}} + (1 + \delta_{\mathrm{v}}^2)\int_{-\infty}^{\delta_{\mathrm{v}}} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz,$$

with $\delta_{\mathrm{v}} = \frac{V - V_{\mathrm{a}}}{\sqrt{\theta - \theta_{\mathrm{a}}}}$, which takes into account the velocity and variance at the actual position x and the interaction point x_{a} respectively. This non-local term around location x_{a} expresses that interactions between vehicles are forwardly directed, since drivers mainly react to the traffic situation in front of them until a certain distance. In this way the Helbing model remedies an inconsistency of previous models that was criticized in [40], namely that although the fluid particles respond to stimuli from ahead and from behind, a car is an anisotropic particle that responds to frontal stimuli (i.e. we require an anisotropic model). The shift term $d = \gamma(1/\rho_{\max} + TV)$ is taken from a car-following model and expresses the velocity-dependent safe distance.

Based on empirical data it was observed that the velocity variance θ (which appears in the definition of the traffic pressure $P = \rho\theta$) is a density-dependent fraction $A(\rho)$ of the squared velocity:

$$\theta(x,t) = A(\rho(x,t))V^2(x,t), \tag{A.2}$$

where $A(\rho)$ is the Fermi function:

$$A(\rho) = A_0 + \Delta A \left(1 + \tanh\left(\frac{\rho - \rho_{\mathrm{c}}}{\Delta\rho}\right)\right), \tag{A.3}$$

where A_0 and $A_0 + 2\Delta A$ are about the variance factors for free and congested traffic, ρ_{c} is of the order of the critical density for the transition from free to congested traffic, and $\Delta\rho$ is the width of the transition.

To summarize, the equations of the Helbing model are:

$$\frac{\partial \rho}{\partial t} + \frac{\partial Q}{\partial x} = 0 \tag{A.4}$$

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + \frac{1}{\rho}\frac{\partial P}{\partial x} = \frac{V_{\mathrm{e}} - V}{\tau} \tag{A.5}$$

$$Q = \rho V, \tag{A.6}$$

where the equilibrium velocity is written as

$$V_{\mathrm{e}} = V_0 \left(1 - \frac{\theta + \theta_{\mathrm{a}}}{A(\rho_{\max})} \left(\frac{\rho_{\mathrm{a}} T}{1 - \rho_{\mathrm{a}}/\rho_{\max}}\right)^2 B(\delta_{\mathrm{v}})\right). \tag{A.7}$$

Readers interested in an empirical validation of this model are referred to [68].

A.3 Hugoniot locus and shocks

In this section we show that the Helbing model can be written in a conservative form, and then we study the shocks arising from this model and we derive conditions under which a pair of states can be connected by a shock (i.e. we determine the Hugoniot locus). We will show that the shocks do not produce negative states as other second-order models do (see e.g. [40]). Therefore, for this model the Riemann problem is physically well-posed, as we will see in Section A.5.

Using $Q = \rho V$ and $P = \rho\theta = \rho A(\rho)V^2$, we can write

$$\rho V^2 + P = \frac{Q^2}{\rho}(1 + A(\rho)).$$

Then using previous formulas we see that a desirable property of the Helbing model equations (A.4)–(A.7) is that they can be formulated in terms of a system of *conservation equations* (i.e. a time-dependent system of nonlinear partial differential equations with a particular simple structure) but with a source term:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = S(u) \tag{A.8}$$

with state variables

$$u = \begin{bmatrix} \rho \\ Q \end{bmatrix},$$

flux function

$$f(u) = \begin{bmatrix} Q \\ \frac{Q^2}{\rho}(1 + A(\rho)) \end{bmatrix},$$

and source term

$$S(u) = \begin{bmatrix} 0 \\ \frac{\rho V_\mathrm{e} - Q}{\tau} \end{bmatrix}.$$

In matrix representation using the Jacobian $J(u) := \frac{\partial f(u)}{\partial u}$ we have:

$$\frac{\partial}{\partial t}\begin{bmatrix} \rho \\ Q \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{Q^2}{\rho^2} + \frac{\partial P}{\partial \rho} & 2\frac{Q}{\rho} + \frac{\partial P}{\partial Q} \end{bmatrix}}_{J(u)} \frac{\partial}{\partial x}\begin{bmatrix} \rho \\ Q \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\rho V_\mathrm{e} - Q}{\tau} \end{bmatrix}.$$

In our case pressure has the form $P = \rho A(\rho) V^2 = \frac{Q^2}{\rho} A(\rho)$, which implies that

$$J(u) = \begin{bmatrix} 0 & 1 \\ -\frac{Q^2}{\rho^2}\left(1 + A(\rho) - \rho \frac{d}{d\rho} A(\rho)\right) & 2\frac{Q}{\rho}(1 + A(\rho)) \end{bmatrix}.$$

When we compute the eigenvalues of the Jacobian, and using again the relation $V = \frac{Q}{\rho}$, we get

$$\lambda_{1,2}(u) = V\left(1 + A(\rho) \pm \sqrt{A^2(\rho) + A(\rho) + \rho \frac{d}{d\rho} A(\rho)}\right). \tag{A.9}$$

Using the fact that $A(\rho)$ is a Fermi function, it can be proved that for physical values of ρ the radical in equation (A.9) is well-defined, and that the eigenvalues are real and distinct. Therefore, we have a *strictly hyperbolic* system (since for any value of u the eigenvalues of the Jacobian are real and distinct). We see that λ_1 is smaller than the average vehicle velocity V, but λ_2 is larger than V. This is a drawback of this kind of models because this means that information travels faster than average vehicles speed, which was criticized in [40, 47]. However, note that V is an average vehicle speed, so there may exist vehicles that travel faster or slower than V.

Corresponding to the two distinct eigenvalues given by equation (A.9) we have two linearly independent eigenvectors

$$r_{1,2}(u) = \begin{bmatrix} 1 \\ \lambda_{1,2}(u) \end{bmatrix},$$

which makes the Jacobian matrix diagonalizable.

A6: As recommended in [69] for qualitative considerations, $A(\rho)$ can be chosen to be constant. We adopt this assumption henceforth because it simplifies our computations. We choose for $A(\rho)$ the value $c := A_0 + \Delta A \approx 0.028$ (which is the value around critical density where we have large oscillations of the speed). $\diamondsuit$

Remark A.3.1 We would like to stress that using this assumption the model remains anisotropic since we do not change the structure of the non-local term $V_\mathrm{e} = V_0 \left(1 - (V^2 + V_\mathrm{a}^2)\left(\frac{\rho_\mathrm{a} T}{1-\rho_\mathrm{a}/\rho_\mathrm{max}}\right)^2 B(\delta_\mathrm{v})\right)$ (cf. equation (A.7)) and since δ_v, which is the argument of Boltzmann factor, also depends on the interaction point x_a through V_a. $\diamondsuit$

With assumption **A6** the formulas for pressure P, flux f, and the eigenvalues λ_p ($p = 1, 2$) are

$$
\begin{aligned}
P &= c\rho V^2 = c\frac{Q^2}{\rho} \\
f(u) &= \begin{bmatrix} Q \\ \frac{Q^2}{\rho}(1+c) \end{bmatrix} \\
\lambda_{1,2}(u) &= V\left(1 + c \pm \sqrt{c^2+c}\right) = c_{1,2}V,
\end{aligned}
$$

where we denote $c_1 := 1 + c - \sqrt{c^2+c} \in (0,\ 1)$ and $c_2 := 1 + c + \sqrt{c^2+c} > 1$. Note that $\lambda_1 < \lambda_2$. Using the weak formulation [92] we can expand the class of solutions of the hyperbolic system (A.8) so as to include discontinuous solutions called *shocks*. Now let us study different kinds of shocks arising from the system and determine and characterize the conditions under which a pair of states $\hat{u} = [\hat{\rho}\ \ \hat{Q}]^T$, $\tilde{u} = [\tilde{\rho}\ \ \tilde{Q}]^T$ can be connected by a single shock.

First, note that in *short* time intervals the shocks arising from (A.8) are the same as those arising from

$$
\frac{\partial}{\partial t}\begin{bmatrix} \rho \\ Q \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -\frac{Q^2}{\rho^2}(1+c) & 2\frac{Q}{\rho}(1+c) \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} \rho \\ Q \end{bmatrix} = 0, \tag{A.10}
$$

i.e. the source term becomes zero (this can be done when traffic operations are in equilibrium but also because the relaxation term $\frac{\rho V_\mathrm{e} - Q}{\tau}$ is finite, so that its effect in short time intervals can be neglected in comparison with the effect caused by the infinite space derivatives of ρ and Q at the shock). However, the cumulative effect of the source term in long term cannot be ignored, this having smooth properties similar to those of a viscosity term therefore when $t \to \infty$ the solutions of (A.8) approach those of (A.10).

Because we have two characteristics (eigenvalues), two kinds of shocks arise from (A.10): we call them *1-shock* and *2-shock* respectively. Let us fix a state $\hat{u} = [\hat{\rho}\ \ \hat{Q}]^T$, and determine the set of states $\tilde{u}$ that can be connected by a discontinuity (called *Hugoniot locus*) to the point $\hat{u}$. For this, the Rankine-Hugoniot jump condition [92] must hold:

$$
f(\tilde{u}) - f(\hat{u}) = s(\tilde{u} - \hat{u}), \tag{A.11}
$$

where s is the propagation speed of the discontinuity along the road (known in traffic flow engineering as *congestion velocity*). The condition (A.11) expresses the fact that the propagation of a shock depends on both flow and density in the neighboring (upstream and downstream) region of a shock.

Furthermore, we should also take into account whether a given discontinuity is physically relevant. To this extent in [88] an entropy condition is introduced: the jump in the p^{th} field (from state $\hat{u}$ to $\tilde{u}$) is admissible only if

$$\lambda_p(\hat{u}) > s > \lambda_p(\tilde{u}).$$

Let us consider the Rankine-Hugoniot jump condition (A.11). Filling out the expression for f results in the following system of equations:

$$\begin{aligned}
&\tilde{Q} - \hat{Q} = s(\tilde{\rho} - \hat{\rho}) \\
&\frac{\tilde{Q}^2}{\tilde{\rho}}(1+c) - \frac{\hat{Q}^2}{\hat{\rho}}(1+c) = s(\tilde{Q} - \hat{Q}).
\end{aligned}$$

Writing down the solutions in terms of $\tilde{\rho}$ yields

$$\tilde{Q}_{1,2} = \hat{Q}\frac{1 \pm (\tilde{\rho} - \hat{\rho})\sqrt{\frac{c^2+c}{\tilde{\rho}\hat{\rho}}}}{1 - \frac{\tilde{\rho}-\hat{\rho}}{\tilde{\rho}}(1+c)}, \tag{A.12}$$

and the corresponding shock speed

$$s_{1,2} = \hat{Q}\frac{\frac{1+c}{\tilde{\rho}} \pm \sqrt{\frac{c^2+c}{\tilde{\rho}\hat{\rho}}}}{1 - \frac{\tilde{\rho}-\hat{\rho}}{\tilde{\rho}}(1+c)}, \tag{A.13}$$

where the $\pm$ signs give two solutions, one for each family of characteristic fields.

Now let us see what sign we should choose in formula (A.12) for the 1-shock and for the 2-shock respectively. Since $\tilde{Q}$ can be expressed in terms of $\tilde{\rho}$, we can parameterize these curves by taking, e.g. $\tilde{\rho}_p(\xi;\hat{u}) = \hat{\rho}(\xi+1)$ for $p = 1, 2$ with $\xi > -1$. Then from equations (A.12) and (A.13) we obtain:

$$\tilde{u}_p(\xi;\hat{u}) = \begin{bmatrix} \hat{\rho}(1+\xi) \\ \hat{Q}\frac{1\pm\xi\sqrt{\frac{c^2+c}{\xi+1}}}{1-\frac{\xi(c+1)}{\xi+1}} \end{bmatrix}, \qquad s_p(\xi;\hat{u}) = \frac{\hat{Q}}{\hat{\rho}}\frac{\frac{1+c}{1+\xi} \pm \sqrt{\frac{c^2+c}{1+\xi}}}{1 - \frac{\xi(1+c)}{1+\xi}}.$$

The choice of sign for each family is determined by the behavior as $\xi \to 0$ where the following relations must hold (see [92] for details):

1. $\frac{\partial}{\partial \xi}\tilde{u}_p(0;\hat{u})$ is a scalar multiple of the eigenvector $r_p(\hat{u})$: so $\frac{\partial}{\partial\xi}\tilde{u}_p(0;\hat{u}) = \hat{\rho}r_p(\hat{u})$ in our case;

2. $s_p(0;\hat{u}) = \lambda_p(\hat{u})$ for $p = 1, 2$.

Using these relations we find that for the 1-shock we must choose the minus sign and for the 2-shock the plus sign.

Remark A.3.2 We can see that each of the characteristic fields is *genuinely nonlinear*, which means that

$$\nabla^T \lambda_p(u) \cdot r_p(u) = c_p(c_p - 1)\frac{Q}{\rho^2} \neq 0 \quad \forall u = [\rho \;\; Q]^T \neq 0,$$

where

$$\nabla \lambda_p := \begin{bmatrix} \dfrac{\partial \lambda_p}{\partial \rho} \\ \dfrac{\partial \lambda_p}{\partial Q} \end{bmatrix}$$

is the gradient of λ_p ($p = 1, 2$). ◊

Now suppose we connect $\hat{u}$ to $\tilde{u}$ by a 1-shock and using the entropy condition mentioned above, we get:

$$c_1 \frac{\hat{Q}}{\hat{\rho}} > s > c_1 \frac{\tilde{Q}}{\tilde{\rho}}.$$

Replacing $s = \frac{\tilde{Q}-\hat{Q}}{\tilde{\rho}-\hat{\rho}}$ in the above inequality and using $c_1 = 1 + c - \sqrt{c^2 + c}$, we obtain

$$\frac{\hat{Q}}{\hat{\rho}} - s + (c - \sqrt{c^2+c})\frac{\hat{Q}}{\hat{\rho}} > 0 > \frac{\tilde{Q}}{\tilde{\rho}} - s + (c - \sqrt{c^2+c})\frac{\tilde{Q}}{\tilde{\rho}},$$

which after few steps leads to

$$\frac{\hat{Q}\tilde{\rho} - \tilde{Q}\hat{\rho}}{\tilde{\rho} - \hat{\rho}} < -\tilde{Q}(c - \sqrt{c^2+c})$$
$$\frac{\hat{Q}\tilde{\rho} - \tilde{Q}\hat{\rho}}{\tilde{\rho} - \hat{\rho}} > -\hat{Q}(c - \sqrt{c^2+c}).$$

Combining the last two inequalities we obtain

$$-\hat{Q}(c - \sqrt{c^2+c}) < -\tilde{Q}(c - \sqrt{c^2+c}) \quad \text{and thus} \quad \hat{Q} < \tilde{Q}.$$

So for the 1-shock we have obtained the following: $\hat{Q} < \tilde{Q}$, and we should take the minus sign in formulas (A.12) and (A.13). Combining these two conditions we can show that we must have $\tilde{\rho} > \hat{\rho}$. Indeed, we distinguish two cases:

1. The denominator in (A.12) is positive: $1 - \frac{\tilde{\rho}-\hat{\rho}}{\tilde{\rho}}(1+c) > 0$. Hence, $\tilde{\rho} < \hat{\rho}(1 + \frac{1}{c})$ and thus $\tilde{Q} = \hat{Q}\frac{1-(\tilde{\rho}-\hat{\rho})\sqrt{\frac{c^2+c}{\tilde{\rho}\hat{\rho}}}}{1-\frac{\tilde{\rho}-\hat{\rho}}{\tilde{\rho}}(1+c)} > \hat{Q}$ if and only if $(\tilde{\rho}-\hat{\rho})\sqrt{\frac{c^2+c}{\tilde{\rho}\hat{\rho}}} < \frac{\tilde{\rho}-\hat{\rho}}{\tilde{\rho}}(1+c)$ or $\tilde{\rho} > \hat{\rho}$, since for the inverse inequality we get a contradiction;
2. The denominator is negative: $1 - \frac{\tilde{\rho}-\hat{\rho}}{\tilde{\rho}}(1+c) < 0$, or $\tilde{\rho} > \hat{\rho}(1 + \frac{1}{c}) > \hat{\rho}$, and thus $\tilde{\rho} > \hat{\rho}$, and we can check that also $\tilde{Q} > \hat{Q}$ is satisfied.

We obtain the following conditions for a 1-shock:

$$\text{S1}: \quad \tilde{Q} = \hat{Q}\frac{1-(\tilde{\rho}-\hat{\rho})\sqrt{\frac{c^2+c}{\tilde{\rho}\hat{\rho}}}}{1-\frac{\tilde{\rho}-\hat{\rho}}{\tilde{\rho}}(1+c)}, \quad \tilde{\rho} > \hat{\rho}, \quad \tilde{Q} > \hat{Q} \tag{A.14}$$

with the corresponding speed of propagation

$$s_1 = \hat{Q}\frac{\frac{1+c}{\tilde{\rho}} - \sqrt{\frac{c^2+c}{\tilde{\rho}\hat{\rho}}}}{1-\frac{\tilde{\rho}-\hat{\rho}}{\tilde{\rho}}(1+c)}. \tag{A.15}$$

Now let us see what the interpretation is of a 1-shock. Do the drivers on the average really behave as described by S1 in equation (A.14)?

If we consider the *fundamental diagram* that relates speed and density then we see that the condition $\tilde{\rho} > \hat{\rho}$ implies that $\tilde{V} < \hat{V}$, i.e. the drivers that enter that shock reduce their speed abruptly, which coincides with real-life behavior (see also [2]).

In a similar way as for the 1-shock we can show that for a 2-shock we have:

$$\text{S2}: \quad \tilde{Q} = \hat{Q}\frac{1 + (\tilde{\rho} - \hat{\rho})\sqrt{\frac{c^2+c}{\tilde{\rho}\hat{\rho}}}}{1 - \frac{\tilde{\rho}-\hat{\rho}}{\tilde{\rho}}(1+c)}, \quad \tilde{\rho} < \hat{\rho}, \quad \tilde{Q} < \hat{Q}$$

and the corresponding speed of propagation

$$s_2 = \hat{Q}\frac{\frac{1+c}{\tilde{\rho}} + \sqrt{\frac{c^2+c}{\tilde{\rho}\hat{\rho}}}}{1 - \frac{\tilde{\rho}-\hat{\rho}}{\tilde{\rho}}(1+c)}.$$

Let us now study the sign for the propagation speeds s_1 and s_2 of the discontinuity. We distinguish two cases:

1. If the denominator $1 - \frac{\tilde{\rho}-\hat{\rho}}{\tilde{\rho}}(1+c)$ is larger than 0, then $\tilde{\rho} < \hat{\rho}(1+\frac{1}{c})$, and we obtain that $0 < s_1 < s_2$, i.e. the speed of propagation of the 1-shock is less than the speed of the 2-shock, but both are positive (since the discontinuity moves downstream).

2. If the denominator is less than 0, then $\tilde{\rho} > \hat{\rho}(1+\frac{1}{c})$ and $s_2 < 0 < s_1$, i.e. the speed for the 1-shock is positive and it moves downstream, but the speed for the 2-shock is negative, and it moves upstream.

Now we can sketch the Hugoniot locus in the phase plane, retaining only the points $\tilde{u}$ that can be connected to $\hat{u}$ by an entropy-satisfying shock, discarding the entropy-violating shocks (dotted lines in Figure A.1). Any right state $u_\mathrm{r} = [\rho_\mathrm{r} \quad Q_\mathrm{r}]^T$ can be connected to a left state $u_\mathrm{l} = [\rho_\mathrm{l} \quad Q_\mathrm{l}]^T$ by a 1-shock if the right state falls on the S1 curve that passes through $[\rho_\mathrm{l} \quad Q_\mathrm{l}]^T$ and similarly by a 2-shock if the right state falls on the S2 curve that passes through $[\rho_\mathrm{l} \quad Q_\mathrm{l}]^T$. We can see from Figure A.1 that the Hugoniot locus terminates at the origin and there are no states with $u_\mathrm{r} < 0$ that can be connected to u_l by a propagating discontinuity; therefore, the model does not produce negative density and flow at the point of discontinuity (as others models that do so, see [40] for details), so it makes physical sense to discuss about Riemann problem associated with this model (as we will do in Section A.5).

A.4 Rarefaction waves

For the LWR model it is known that when the left characteristic is slower than the right characteristic a fan of rarefaction waves results. In this section we show that the Helbing model also has this property, deriving the rarefaction curves corresponding to this model. We will see again that we cannot connect negative states through this kind of rarefaction waves. We will use this result when we discuss about Riemann problem.

If the two characteristic fields satisfy

$$\lambda_p(u_\mathrm{l}) < \lambda_p(u_\mathrm{r}) \quad \text{for } p = 1, 2\,, \tag{A.16}$$

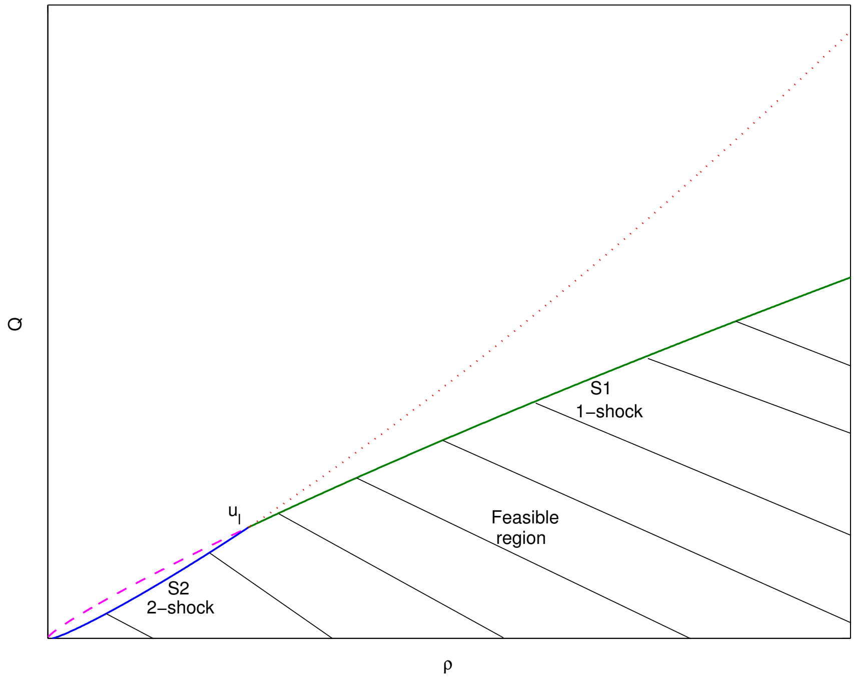

Figure A.1: Representation of the states u_r that can be connected to u_l by an entropy-satisfying shock. State u_r can be connected to u_l by a 1-shock if u_r lies on curve S1 passing through u_l, and by a 2-shock if u_r lies on curve S2 passing through u_l. The dotted and dashed curves represent entropy-violating points.

two families of smooth solutions, called *1-rarefaction waves* and *2-rarefaction waves* exist. Similar to the analysis of shock curves we shall derive the phase curves for both families of rarefaction waves. One can write equation (A.10) as:

$$\frac{\partial u}{\partial t} + J(u)\frac{\partial u}{\partial x} = 0 \quad \text{with } u = [\rho \quad Q]^T, \;\; f = \left[Q \quad \frac{Q^2}{\rho}(1+c)\right]^T, \;\; J(u) = \frac{\partial f}{\partial u}. \tag{A.17}$$

If $u(x,t)$ is a solution of the system (A.17), then we can show that $u(ax, at)$ is also a solution, where a is a scalar, i.e. the solutions are scaling-invariant. Therefore, the solution depends on (x,t) in the form $\xi = x/t$. A rarefaction wave solution to the system of equations takes the form:

$$u(x,t) = \begin{cases} u_\mathrm{l} & \text{if } \;\; x \leq \xi_1 t \\ w(x/t) & \text{if } \;\; \xi_1 t < x < \xi_2 t \\ u_\mathrm{r} & \text{if } \;\; x \geq \xi_2 t\,, \end{cases} \tag{A.18}$$

with $w(\cdot)$ smooth and $w(\xi_1) = u_\mathrm{l}$ and $w(\xi_2) = u_\mathrm{r}$. We will now prove that starting at each point u_l there are two curves consisting of points u_r that can be connected to u_l by a rarefaction wave, namely a subset of the integral curve of $r_p(u_\mathrm{l})$. An integral curve for $r_p(u)$ is a curve that has the property that the tangent to the curve at any point u lies in the direction $r_p(u)$. In order to determine explicitly the function $w(x/t)$ we differentiate $u(x,t) = w(x/t)$:

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) &= -\frac{x}{t^2} w'(x/t) \\ \frac{\partial u}{\partial x}(x,t) &= \frac{1}{t} w'(x/t), \end{aligned}$$

where $w'(\cdot)$ represents the derivative of $w(\cdot)$. Replacing these expressions in (A.17) with $\xi = x/t$ we get

$$J(w(\xi))\, w'(\xi) = \xi w'(\xi), \tag{A.19}$$

which means that $w'(\xi)$ is proportional to some eigenvector $r_p(w(\xi))$ of the Jacobian $J(w(\xi))$:

$$w'(\xi) = \alpha(\xi) r_p(w(\xi)),$$

i.e. $w(\xi)$ lies along some integral curve of r_p and ξ is an eigenvalue of the Jacobian.

Let us compute w, using the fact that our model is genuinely nonlinear, as was shown in Section A.3. Recall that (A.19) implies that ξ is an eigenvalue of $J(w(\xi))$. Differentiating $\xi = \lambda_p(w(\xi))$ with respect to ξ results in

$$1 = \nabla^T \lambda_p(w(\xi)) w'(\xi) = \nabla^T \lambda_p(w(\xi)) \alpha(\xi) r_p(w(\xi)).$$

Hence,

$$\alpha(\xi) = \frac{1}{\nabla^T \lambda_p(w(\xi)) r_p(w(\xi))},$$

which results in the differential equation

$$w'(\xi) = \frac{r_p(w(\xi))}{\nabla^T \lambda_p(w(\xi)) r_p(w(\xi))} \quad \text{for } \xi_1 \leq \xi \leq \xi_2$$

with initial condition

$$w(\xi_1) = u_{\mathrm{l}},\ \xi_1 = \lambda_p(u_{\mathrm{l}}) < \xi_2 = \lambda_p(u_{\mathrm{r}}).$$

For 1-rarefaction we have:

$$\lambda_1 = c_1 \frac{Q}{\rho} = c_1 V, \quad r_1 = \begin{bmatrix} 1 & c_1 \frac{Q}{\rho} \end{bmatrix}^T, \quad \nabla^T \lambda_1 r_1 = c_1(c_1 - 1) \frac{Q}{\rho^2} \neq 0,$$

and thus

$$\frac{d}{d\xi} \rho(\xi) = \frac{\rho^2(\xi)}{Q(\xi)} \frac{1}{c_1^2 - c_1} \qquad \text{with } \rho(\xi_1) = \rho_{\mathrm{l}} \tag{A.20}$$

$$\frac{d}{d\xi} Q(\xi) = \rho(\xi) \frac{1}{c_1 - 1} \qquad \text{with } Q(\xi_1) = Q_{\mathrm{l}},\ \xi_1 = \lambda_1(u_{\mathrm{l}}) = c_1 \frac{Q_{\mathrm{l}}}{\rho_{\mathrm{l}}}, \tag{A.21}$$

which is a system of two ordinary nonlinear differential equations. We see that (A.20) can be written as

$$\frac{d}{d\xi}\left(\frac{1}{\rho}\right) = -\frac{1}{Q} \frac{1}{c_1^2 - c_1}.$$

Denoting $\eta = \frac{1}{\rho}$ we get the system

$$Q \frac{d\eta}{d\xi} = -\frac{1}{c_1^2 - c_1}$$
$$\eta \frac{dQ}{d\xi} = \frac{1}{c_1 - 1}.$$

We add both equations obtaining a relation between states: $Q(\xi) = \frac{1}{c_1}\xi\rho(\xi)$, and finally after some computations we obtain the following solution:

$$\rho(\xi) = \left(\frac{\rho_l^{c_1}}{c_1 Q_l}\xi\right)^{\frac{1}{c_1-1}}$$
$$Q(\xi) = \frac{\xi}{c_1}\left(\frac{\rho_l^{c_1}}{c_1 Q_l}\xi\right)^{\frac{1}{c_1-1}} .$$

If we want to obtain an explicit expression for the integral curves in the phase plane, we eliminate ξ:

$$\rho^{c_1-1} = \xi\frac{\rho_l^{c_1}}{c_1 Q_l} \quad \Rightarrow \quad \xi = \frac{c_1 Q_l}{\rho_l^{c_1}}\rho^{c_1-1} \quad \Rightarrow \quad Q(\rho) = Q_l\left(\frac{\rho}{\rho_l}\right)^{c_1} .$$

We can construct the 2-rarefaction wave in exactly the same manner obtaining

$$\rho(\xi) = \left(\frac{\rho_l^{c_2}}{c_2 Q_l}\xi\right)^{\frac{1}{c_2-1}}$$
$$Q(\xi) = \frac{\xi}{c_2}\left(\frac{\rho_l^{c_2}}{c_2 Q_l}\xi\right)^{\frac{1}{c_2-1}} ,$$

and in the phase plane 2-rarefaction is given by

$$Q(\rho) = Q_l\left(\frac{\rho}{\rho_l}\right)^{c_2} .$$

Now two states u_l and u_r can be connected by a rarefaction wave provided that they lie on the same integral curve and $\lambda_p(u_l) < \lambda_p(u_r)$, which for 1-rarefaction results in

$$c_1\frac{Q_l}{\rho_l} < c_1\frac{Q_r}{\rho_r}, \qquad c_1 \in (0,1),$$

with

$$Q_r = Q_l\left(\frac{\rho_r}{\rho_l}\right)^{c_1} ,$$

and thus

$$\frac{1}{\rho_l} < \frac{\rho_r^{c_1-1}}{\rho_l^{c_1}} .$$

Hence, $\rho_l^{c_1-1} < \rho_r^{c_1-1}$ or $\rho_r < \rho_l$ since $c_1 \in (0,1)$. Therefore, we obtain the following expression for the 1-rarefaction curve:

$$\text{R1}: \quad Q_r = Q_l\left(\frac{\rho_r}{\rho_l}\right)^{c_1} , \quad \rho_r < \rho_l. \tag{A.22}$$

The 2-rarefaction curve is given by

$$\text{R2}: \quad Q_r = Q_l\left(\frac{\rho_r}{\rho_l}\right)^{c_2} , \quad \rho_r > \rho_l. \tag{A.23}$$

Figure A.2 shows the states u_r that can be connected to u_l by a 1-rarefaction wave, namely the states lying on the curve R1 passing through u_l. Furthermore, the states u_r lying on the curve

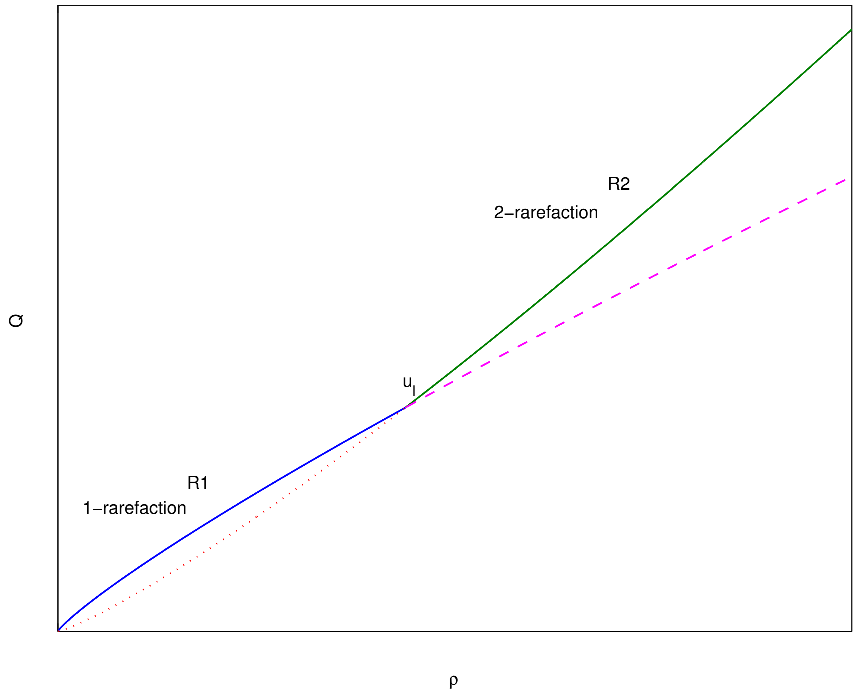

Figure A.2: Representation of the states u_r that can be connected to u_l by a rarefaction wave. State u_r can be connected to u_l by a 1-rarefaction if u_r lies on curve R1 passing through u_l, and by a 2-rarefaction wave if u_r lies on curve R2 passing through u_l. The dotted and dashed curves represent points that do not satisfy the rarefaction condition (A.16).

R2 passing through u_l can be connected to u_l by a 2-rarefaction wave. We can observe that the integral curves R1 and R2 are very similar to the Hugoniot locus. Moreover, locally near the point u_l they must in fact be very close to each other, because each of these curves is tangent to $r_p(u_\mathrm{l})$ at u_l. Therefore, locally around u_l the rarefaction waves are similar with the shock waves (we can see that a 1-rarefaction wave is similar to a 2-shock wave, and that a 2-rarefaction is similar to a 1-shock wave). Note that this does not imply non-existence of rarefaction wave solutions for the Helbing model, because this similarity is valid only locally and when we solve Riemann problem the intermediate states u_m can be given by the intersection of a shock curve with a rarefaction curve (see also Section A.5). Again we see that we do not connect negative states to u_l, which is a very important feature of the Helbing model, and we will use this result when we discuss the Riemann problem. An interpretation in terms of driver behavior of the rarefaction waves is similar with that of entropy-satisfying shock.

A.5 General solution of the Riemann problem

In this section we discuss the Riemann problem associated with the Helbing model, and based on the results of the two previous sections we will show that solutions of the Riemann problem with density and flow non-negative in the initial condition on either side of the discontinuity cannot give rise to negative flows or densities later on. Also we will see that for Riemann problem we can find more than one solution, and the condition for uniqueness is to select the entropy-satisfying weak solution, which results in a unique, physically valid solution.

A conservation law together with piecewise constant initial data having a single discontinuity results in a so-called *Riemann problem* (see [92, 130, 153] for more details). For instance, the

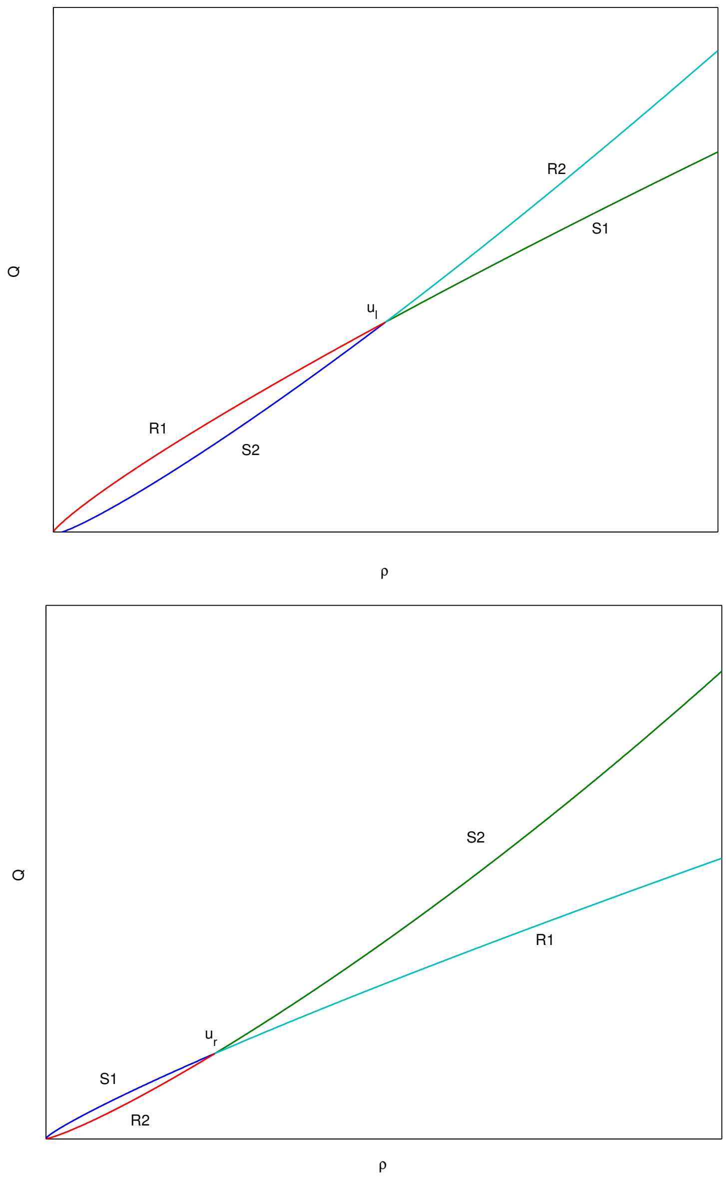

Figure A.3: Representation of the states u_r that can be connected to u_l by a shock or a rarefaction wave (top), and of the states u_l that can be connected to u_r by a shock or a rarefaction wave (bottom).

system (A.17) with initial condition

$$u(x,0) = \begin{cases} u_l & \text{if } x < 0 \\ u_r & \text{if } x > 0, \end{cases}$$

where u_l and u_r are given constants, is a Riemann problem.

If we combine Figures A.1 and A.2 we obtain a plot that shows us all points u_r that can be connected to a given point u_l by an entropy-satisfying wave (see Figure A.3 – top), either a shock wave or a rarefaction wave (u_r lies on one of the curves S1, S2, R1 or R2), and the states u_l that can be connected to a given u_r (see Figure A.3 – bottom). Therefore, when initial data u_l and u_r both lay on these curves then this discontinuity simply propagates with speed $s = \frac{Q_r - Q_l}{\rho_r - \rho_l}$ along the road.

But what happens if u_r does not reside on one of those curves passing through u_l? To solve this question, just as in the linear case, we can attempt to find a way to split this jump as a sum of two jumps, across each of which the Rankine-Hugoniot condition holds, i.e. we must find an intermediate state u_m such that u_l and u_m are connected by a discontinuity satisfying the Rankine-Hugoniot condition and so are u_m and u_r, which intuitively means to superimpose the appropriate plots and look for the intersections. When we want to determine analytically the intermediate state u_m, we must first determine whether each wave is a shock or a rarefaction, and then use the expressions relating ρ and Q determined in Sections A.3 and A.4 along each curve to solve for the intersection. When we solve the equation given by the intersection, we can get more than one solution for u_m, but only one gives a physically valid solution to the Riemann problem since the jump from u_l to u_m must travel more slowly than the jump from u_m to u_r (due to $\lambda_1 < \lambda_2$), therefore the condition for uniqueness is to pick up the weak solution that satisfies the above condition. Using the same parametrization $\rho_\mathrm{l} = \rho_\mathrm{m}(1+\xi_1)$ and $\rho_\mathrm{r} = \rho_\mathrm{m}(1+\xi_2)$, and replacing in equation (A.13) we get that the speeds of shock from u_l to u_m and from u_m to u_r are given by:

$$s_{\mathrm{l,m}} = \frac{Q_\mathrm{m}}{\rho_\mathrm{m}} \frac{\frac{1+c}{1+\xi_1} \pm \sqrt{\frac{c^2+c}{1+\xi_1}}}{1-\frac{\xi_1(1+c)}{1+\xi_1}}, \quad s_{\mathrm{m,r}} = \frac{Q_\mathrm{m}}{\rho_\mathrm{m}} \frac{\frac{1+c}{1+\xi_2} \pm \sqrt{\frac{c^2+c}{1+\xi_2}}}{1-\frac{\xi_2(1+c)}{1+\xi_2}}.$$

Now depending on what values we choose for u_l and u_r we can determine the sign in the previous formulas such that $s_{\mathrm{l,m}} < s_{\mathrm{m,r}}$ and thus we know what waves (1-wave or 2-wave) give the intersection. We can distinguish the following cases:

Case 1: Both curves are shocks.

Graphically this means to draw the Hugoniot locus for each of the states u_l and u_r and to look for the intersection. To obtain the correct value for $u_\mathrm{m} = [\rho_\mathrm{m} \quad Q_\mathrm{m}]^T$ we have to impose $s_{\mathrm{l,m}} < s_{\mathrm{m,r}}$. Let us consider an example; e.g. assume that u_m is connected to u_l by a 1-shock and to u_r by a 2-shock:

$$Q_\mathrm{m} = Q_\mathrm{l} \frac{1-(\rho_\mathrm{m}-\rho_\mathrm{l})\sqrt{\frac{c^2+c}{\rho_\mathrm{m}\rho_\mathrm{l}}}}{1-\frac{\rho_\mathrm{m}-\rho_\mathrm{l}}{\rho_\mathrm{m}}(1+c)}, \quad Q_\mathrm{m} = Q_\mathrm{r} \frac{1+(\rho_\mathrm{m}-\rho_\mathrm{r})\sqrt{\frac{c^2+c}{\rho_\mathrm{m}\rho_\mathrm{r}}}}{1-\frac{\rho_\mathrm{m}-\rho_\mathrm{r}}{\rho_\mathrm{m}}(1+c)}. \tag{A.24}$$

Equating the two right-hand sides gives a single equation for ρ_m. If we set $y = \sqrt{\rho_\mathrm{m}}$, we get a 4th degree polynomial equation in y, which can either be solved analytically (using Ferrari's method) or numerically (using, e.g. Newton's method or Laguerre's algorithm). After we obtain ρ_m we replace it in one of the previous equalities (A.24) to obtain Q_m. Using a reasoning that is similar to the one of [88], it can be shown that the equation in ρ_m always has a solution when u_l and u_r are sufficiently close.

Remark A.5.1 We may wonder whether the equation in ρ_m has a solution anyway? The answer is always positive when u_l and u_r are sufficiently close. The idea of proof is given here, following [88]. We know that from u_l we can reach an intermediate state $u_\mathrm{m}(\xi_1)$ through a 1-shock. From $u_\mathrm{m}(\xi_1)$ we can reach another state $u_\mathrm{m}(\xi_1, \xi_2)$ through a 2-shock. Moreover, we know that:

$$\frac{\partial u_\mathrm{m}}{\partial \xi_p}(0,0) = \alpha(u_\mathrm{l}) \cdot r_p(u_\mathrm{l}) \quad \text{for } p = 1, 2.$$

where α is a scalar. But our system is hyperbolic, which means these vectors are linearly independent and hence the Jacobian of the map $u_\mathrm{m} : \mathbb{R}^2 \to \mathbb{R}^2, (\xi_1, \xi_2) \to u_\mathrm{m}(\xi_1, \xi_2)$ is

nonsingular. Therefore, from the *inverse function theorem* it follows that the map u_{m} is bijective in a neighborhood of the origin. Hence, for any u_{r} sufficiently close to u_{l} there is a unique set of parameters ξ_1, ξ_2 such that $u_{\mathrm{r}} = u_{\mathrm{m}}(\xi_1, \xi_2)$ and $u_{\mathrm{m}} = u_{\mathrm{m}}(\xi_1, 0)$.
When u_{m} is connected to u_{l} by a 2-shock and to u_{r} by a 1-shock we proceed similarly. We would like to point out that in general not for all u_{l} and u_{r} the weak solution previously constructed is a physically correct solution as it may be possible that one of the resulting shocks violate the entropy condition. In particular, for any u_{l} the feasible u_{r} lie in a bounded region formed by horizontal axis (ρ) and the curves S1 and S2 (indicated by the hashed region in Figure A.1. ♢

Case 2: Both curves are rarefactions.
If we assume that the intermediate state is connected to u_{l} by a 1-rarefaction and to u_{r} by a 2-rarefaction, then u_{m} must satisfy

$$Q_{\mathrm{m}} = Q_{\mathrm{l}} \left(\frac{\rho_{\mathrm{m}}}{\rho_{\mathrm{l}}}\right)^{c_1}, \quad Q_{\mathrm{m}} = Q_{\mathrm{r}} \left(\frac{\rho_{\mathrm{m}}}{\rho_{\mathrm{r}}}\right)^{c_2}. \tag{A.25}$$

Equating again we get an equation in ρ_{m} with solution

$$\rho_{\mathrm{m}} = \left(\frac{Q_{\mathrm{l}}}{Q_{\mathrm{r}}} \frac{\rho_{\mathrm{r}}^{c_2}}{\rho_{\mathrm{l}}^{c_1}}\right)^{\frac{1}{c_2 - c_1}},$$

and then we obtain Q_{m} from (A.25). We proceed similarly when we consider the opposite case: u_{m} is connected to u_{l} by a 2-rarefaction and to u_{r} by a 1-rarefaction.

Case 3: The solution consist of one shock and one rarefaction wave.
Again if we consider the case when the intermediate state u_{m} is connected to u_{l} by a 1-rarefaction and to u_{r} by a 2-shock, then we must solve for ρ_{m} and Q_{m} from the equations:

$$Q_{\mathrm{m}} = Q_{\mathrm{l}} \frac{1 + (\rho_{\mathrm{m}} - \rho_{\mathrm{l}}) \sqrt{\frac{c^2 + c}{\rho_{\mathrm{m}} \rho_{\mathrm{l}}}}}{1 - \frac{\rho_{\mathrm{m}} - \rho_{\mathrm{l}}}{\rho_{\mathrm{m}}}(1 + c)}, \quad Q_{\mathrm{m}} = Q_{\mathrm{r}} \left(\frac{\rho_{\mathrm{m}}}{\rho_{\mathrm{r}}}\right)^{c_1}.$$

We would like to point out that in general not for all u_{l} and u_{r} the weak solution previously constructed is a physically correct solution as it may be possible that one of the resulting shocks violate the entropy condition. In particular, for any u_{l} the *feasible* (i.e. entropy-satisfying) u_{r} lie in a bounded region formed by horizontal axis (ρ) and the curves S1 and S2 (indicated by the hashed region in Figure A.1).

Figure A.4 shows a plot for the Riemann problem with initial conditions $u_{\mathrm{l}} = [140 \quad 400]^T$ and $u_{\mathrm{r}} = [5 \quad 50]^T$, which corresponds, e.g. to a scenario such as the situation of traffic in front of a semaphore when it was red and then becomes green. The full curves represent the states that can be connected to u_{l}, and the dotted curves represent the states that can be connected to u_{r}. The intersection gives two points: the intermediate state u_{m} is obtained by intersection of R1 with S2, and u_{m}^* by intersection of R2 with S1. So, the Riemann problem has more than one solution in this case (this happens also for other traffic flow models), but only one is a physically valid solution because we should have $s_{\mathrm{l,m}} < s_{\mathrm{m,r}}$ (due to $\lambda_1 < \lambda_2$). If we do the computations, we get that in this case u_{m}^* is the solution that satisfies the entropy condition, i.e. u_{m}^* is the physically valid solution.

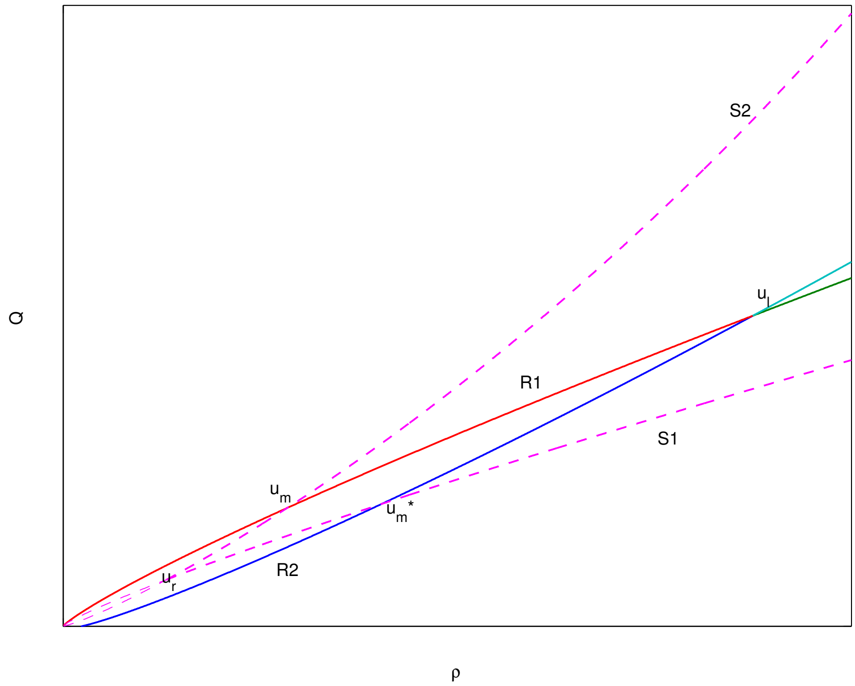

Figure A.4: Construction of the solution for the Riemann problem. We obtain two intermediate states u_{m} and u_{m}^{*}, but only u_{m}^{*} is a physically valid solution.

Remark A.5.2 Note that the model presented above is in continuous form, and an analytic solution cannot be obtained explicitly (even for the Riemann problem). Of course, if one wants to use it in practice (for simulation or control purposes) one needs to approximate the solution using numerical schemes. In the literature many researchers many different kinds of schemes to approximate the solution of a hyperbolic system have been proposed. It is known that for hyperbolic systems with the eigenvalues having the same sign (in our case both are positive) upwind methods give good results. However, when we have discontinuities (e.g. when we want to solve a Riemann problem considered in Section A.5), these methods do not perform well, they produce oscillations in the neighborhood of shocks which is not in accord with real traffic. In that case, Godunov-type methods [153] can be applied since they are based on the exact or approximate solution of a Riemann problem at cell interfaces (like the *van Leer splitting* scheme [153, 163] or the *Harten-Lax-van Leer* scheme [62], which approximate the Riemann problem using Roe's approximation). ♢

A.6 Conclusions

In this appendix we have discussed some properties of the Helbing traffic flow model. More specifically, we have derived the formulas for shocks and rarefaction waves. By selecting the states that satisfy the Lax entropy condition, we saw that we cannot connect to negative states. Finally, we have considered the Riemann problem associated with the Helbing model, based on the results in connection with the shocks and rarefaction waves. In particular, we have proved that when we have a Riemann problem with non-negative densities and flows on either side of discontinuity in the initial condition, the Helbing model cannot give rise to negative flows and density later on.

Appendix B

Linear matrix inequalities

In this chapter we collect some basic results in linear matrix inequalities (LMIs) [24], such as the Schur complement, the S-procedure and the elimination lemma.

B.1 Introduction

We define the subspace of symmetric matrices in $\mathbb{R}^{n\times n}$:

$$\mathbb{S}^n := \{Q \in \mathbb{R}^{n\times n} : Q = Q^T\}$$

and the *semidefinite cone*[1]

$$\mathbb{S}^n_+ := \{Q \in \mathbb{S}^n : x^T Q x \geq 0 \ \ \forall x \in \mathbb{R}^n\}.$$

A matrix $Q \in \mathbb{S}^n$ is called *positive semidefinite*, denoted as $Q \succcurlyeq 0$, when $Q \in \mathbb{S}^n_+$. A matrix $Q \in \mathbb{S}^n$ is called *positive definite*, denoted as $Q \succ 0$, when $Q \in \text{int}(\mathbb{S}^n_+)$, where $\text{int}(\cdot)$ denotes the interior of a set. In other words $Q \succ 0$ if and only if $x^T Q x > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$. For a symmetric matrix $Q \in \mathbb{S}^n$ the following are equivalent:

(i): $Q \succcurlyeq 0$

(ii): Q has only nonnegative eigenvalues

(ii): $Q = R^T R$ for some R (not necessarily square).

A *linear matrix inequality* is an expression of the form

$$Q_0 + x_1 Q_1 + \cdots + x_m Q_m \succcurlyeq 0, \tag{B.1}$$

where

- $x = [x_1 \cdots x_m]^T \in \mathbb{R}^m$ are the decision variables
- $Q_0, \cdots, Q_m \in \mathbb{S}^n$ (i.e. they are symmetric matrices)

An LMI can be formulated as a convex optimization problem, referred to as *semidefinite program*:

$$\inf_x \{c^T x : Q_0 + x_1 Q_1 + \cdots + x_m Q_m \succcurlyeq 0\}.$$

Semidefinite programs can be regarded as an extension of linear programming where the component-wise inequalities between vectors are replaced by matrix inequalities, or, equivalently, the nonnegative orthant is replaced by the semidefinite cone.

[1] For a finite-dimensional normed space $\mathcal{Y}$ a set $\mathcal{K} \subseteq \mathcal{Y}$ is a cone if $y \in \mathcal{K}$ implies $\alpha y \in \mathcal{K}$ for all $\alpha \geq 0$.

B.2 The Schur complement formula

The Schur complement formula is a very useful tool for manipulating matrix inequalities. It is often used to transform nonlinear matrix inequalities into LMIs. Let $Q \in \mathbb{S}^n$, $R \in \mathbb{S}^m$ and $S \in \mathbb{R}^{n\times m}$. The Schur complement formula for positive semidefinite matrices reads as follows. The following matrix inequalities are equivalent:

$$\begin{array}{ll} (a) & \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \succcurlyeq 0 \\ (b) & R \succcurlyeq 0,\ Q - SR^{\dagger}S^T \succcurlyeq 0,\ S(I - RR^{\dagger}) = 0 \\ (c) & Q \succcurlyeq 0,\ R - S^TQ^{\dagger}S \succcurlyeq 0,\ S^T(I - QQ^{\dagger}) = 0, \end{array} \tag{B.2}$$

where $R^{\dagger}$ and $Q^{\dagger}$ denote the pseudo-inverse of R and Q, respectively.

For the positive definite case, the following are equivalent:

$$\begin{array}{ll} (i) & \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \succ 0 \\ (ii) & R \succ 0,\ Q - SR^{-1}S^T \succ 0 \\ (iii) & Q \succ 0,\ R - S^TQ^{-1}S \succ 0 \end{array} \tag{B.3}$$

Note that the last two formulas $(ii) - (iii)$ in (B.3) are nonlinear matrix inequalities in S while the first formula (i) is an LMI in S.

B.3 The S-procedure

The S-procedure is frequently use in system theory to derive stability and performance results for certain classes of nonlinear systems. First, let us recall a basic LMI result. Let us consider the quadratic function $q(x,y) = x^TQx + 2s^Txy + ry^2$, where $Q \in \mathbb{S}^n$, $s \in \mathbb{R}^n$ and $r \in \mathbb{R}$. Then $q(x,y) \geq 0$ for all $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$ if and only if $q(x,1) \geq 0$ for all $x \in \mathbb{R}^n$. In matrix notation this means that

$$\begin{bmatrix} Q & s \\ s^T & r \end{bmatrix} \succcurlyeq 0 \tag{B.4}$$

if and only if

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} Q & s \\ s^T & r \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \geq 0 \ \ \forall x \in \mathbb{R}^n. \tag{B.5}$$

We use a continuity argument to prove this equivalence. It is clear that (B.4) implies (B.5). Let us show the other implication. Since $q(x,1) \geq 0$ for all $x \in \mathbb{R}^n$, it follows that[2]

$$q(x,y) \geq 0 \ \ \forall x \in \mathbb{R}^n, y \in \mathbb{R}, y \neq 0 \tag{B.6}$$

It remains to prove that $q(x,0) = x^TQx \geq 0$ for all $x \in \mathbb{R}^n$. Let us assume that there exists an $x_0 \in \mathbb{R}^n$ such that $x_0^TQx_0 < 0$. Since the function $g : \mathbb{R} \to \mathbb{R}$, $g(y) = 2s^Tx_0y + ry^2$ is continuous and $g(0) = 0$ it follows that we can choose y_0 arbitrarily close to 0 but $y_0 \neq 0$ such that $q(x_0, y_0) < 0$. But this is a contradiction with (B.6). Note that the equivalence does not hold if we replace the inequalities (B.4)– (B.5) with strict inequalities (since the continuity argument cannot be used anymore).

The basic idea behind the S-procedure is trivial (see also [77]): let $q_k : \mathbb{R}^n \to \mathbb{R}, k \in \mathbb{N}_{[0,N]}$ be real valued functions and consider the following two conditions

[2] We can divide $q(x,y)$ by y and by redefining x/y as x we obtain $q(x,1)$.

$\mathcal{C}1:$ $q_0(x) \geq 0$ for all $x \in \mathcal{F}$, where $\mathcal{F} = \{x \in \mathbb{R}^n : q_k(x) \geq 0 \ \forall k \in \mathbb{N}_{[1,N]}\}$

$\mathcal{C}2:$ There exist scalars $\lambda_k \geq 0$ for all $k \in \mathbb{N}_{[1,N]}$ such that $q_0(x) - \sum_{k=1}^N \lambda_k q_k(x) \geq 0$ for all $x \in \mathbb{R}^n$.

It is clear that $\mathcal{C}2$ implies $\mathcal{C}1$. Therefore, the condition $\mathcal{C}1$ is relaxed[3] to a more conservative condition $\mathcal{C}2$. When q_k are quadratic functions condition $\mathcal{C}2$ can be recast as an LMI. Indeed, we consider

$$q_k(x) = x^T Q_k x + 2s_k^T x + r_k,$$

where $Q_k \in \mathbb{S}^n, s_k \in \mathbb{R}^n$ and $r_k \in \mathbb{R}$ for all $k \in \mathbb{N}_{[0,N]}$. If the functions q_k are non-convex, then $\mathcal{C}1$ reduces at checking if the minimum of a non-convex function over a non-convex set is nonnegative (a NP hard problem in general). But, condition $\mathcal{C}2$ reduces at solving an LMI (a convex optimization problem). Indeed, $\mathcal{C}2$ can be written equivalently as: there exist $\lambda_k \geq 0$ for all $k \in \mathbb{N}_{[1,N]}$ such that

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} Q_0 + \sum_{k=1}^N \lambda_k Q_k & s_0 + \sum_{k=1}^N \lambda_k s_k \\ s_0^T + \sum_{k=1}^N \lambda_k s_k^T & r_0 \sum_{k=1}^N \lambda_k r_k \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \geq 0 \ \ \forall x \in \mathbb{R}^n.$$

or in matrix notation (using the equivalence (B.4)– (B.5)) as: there exists $\lambda_k \geq 0$ for all $k \in \mathbb{N}_{[1,N]}$ such that

$$\begin{bmatrix} Q_0 + \sum_{k=1}^N \lambda_k Q_k & s_0 + \sum_{k=1}^N \lambda_k s_k \\ s_0^T + \sum_{k=1}^N \lambda_k s_k^T & r_0 + \sum_{k=1}^N \lambda_k r_k \end{bmatrix} \succcurlyeq 0.$$

It is well known [77] that for $N = 1$ and quadratic functions q_0 and q_1 such that there exists an $x_0 \in \mathbb{R}^n$ satisfying[4] $q_1(x_0) > 0$ the S-procedure is exact (i.e. there is no conservatism by replacing condition $\mathcal{C}1$ by condition $\mathcal{C}2$).

It often happens (see Chapter 5) that we require strict inequality for q_0. In that case the following implication holds:

$\mathcal{C}1':$ $q_0(x) > 0$ for all $x \in \mathcal{F}, x \neq 0$, where $\mathcal{F} = \{x \in \mathbb{R}^n : q_k(x) \geq 0 \ \forall k \in \mathbb{N}_{[1,N]}\}$

$\mathcal{C}2':$ There exist scalars $\lambda_k \geq 0$ for all $k \in \mathbb{N}_{[1,N]}$ such that $q_0(x) - \sum_{k=1}^N \lambda_k q_k(x) > 0$ for all $x \in \mathbb{R}^n, x \neq 0$.

Then, $\mathcal{C}2'$ implies $\mathcal{C}1'$. Moreover, the S-procedure is exact (i.e. $\mathcal{C}1'$ is equivalent to $\mathcal{C}2'$) provided that $N = 1$, $q_0(x) = x^T Q_0 x, \ q_1(x) = x^T Q_1 x$ and there exists an $x_0 \in \mathbb{R}^n$ such that $q_1(x_0) > 0$. The reason for requiring that q_0, q_1 contain only quadratic terms is clear since the equivalence (B.4)–(B.5) does not hold for strict inequalities, as we mentioned before.

B.4 Elimination lemma

Given a matrix $S \in \mathbb{R}^{n \times m}$ then the kernel of S is defined as

$$\ker(S) = \{x \in \mathbb{R}^m : Sx = 0\}$$

and the column space of S is defined as the image of S, i.e.

$$\text{Im}(S) = \{Sx : x \in \mathbb{R}^m\}.$$

[3]Note that the S-procedure is a consequence of Lagrange duality [25].

[4]This is similar to Slater's constraint qualification from Lagrange duality.

Let as assume that the rank of S is r. The orthogonal complement of S is the matrix $S^{\perp} \in \mathbb{R}^{(n-r)\times n}$ such that $S^{\perp}S = 0$ and $S^{\perp}S^{\perp,T} \succ 0$. Note that such a matrix $S^{\perp}$ exists if and only if S has linearly dependent rows (i.e. $r < n$). It is known [72] that the orthogonal complement of the column space of S is the null space of S^T. We give here a simple version of the elimination lemma. Given a symmetric matrix $Q \in \mathbb{S}^n$, the following two relations are equivalent:

$$\begin{array}{ll} (i) & S^{\perp}QS^{\perp,T} \prec 0 \\ (ii) & Q \prec \lambda_0\, SS^T \text{ for some } \lambda_0 \in \mathbb{R}. \end{array} \tag{B.7}$$

B.5 Bilinear matrix inequalities

A *bilinear matrix inequality* (BMI) is an expression of the form

$$Q_0 + \sum_{i=1}^{m} x_i Q_i + \sum_{i,j=1}^{m} x_i x_j Q_{ij} \succcurlyeq 0, \tag{B.8}$$

where

- $x = [x_1 \cdots x_m]^T \in \mathbb{R}^m$ are the decision variables
- $Q_0, Q_i, Q_{ij} \in \mathbb{S}^n$ for all $i, j \in \mathbb{N}_{[1,m]}$.

A BMI can be formulated as a non-convex optimization problem:

$$\inf_x \{c^T x : Q_0 + \sum_{i=1}^{m} x_i Q_i + \sum_{i,j=1}^{m} x_i x_j Q_{ij} \succcurlyeq 0\},$$

which, in general, is an NP-hard problem.

Appendix C

Lyapunov stability

We summarize some basic results on Lyapunov stability. Our presentation here follows a similar approach as in [146, 165], but we try to be as general as possible.

C.1 Lyapunov stability for general nonlinear systems

We consider the autonomous discrete-time system

$$x(k+1) = f(x(k)), \tag{C.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfying $f(0) = 0$ (i.e. the origin is an equilibrium point for (C.1)). Let $\phi(k; x)$ denote the solution of (C.1) at step k when the initial state at step 0 is x. In the sequel the symbol $\|\cdot\|$ denotes a norm on $\mathbb{R}^n$.

We now introduce the so-called $\mathscr{K}$ functions: a continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a $\mathscr{K}$ function if: (1) $\alpha(0) = 0$, (2) $\alpha(z) > 0$ for all $z > 0$, and (3) α is strictly increasing.

Recall the definition of a positively invariant (PI) set given in Section 2.3.2: the set $\mathcal{X}$ is PI set for the dynamic system (C.1) if every system trajectory which starts from a point in $\mathcal{X}$ remains in $\mathcal{X}$ for all future times. In other words $f(\mathcal{X}) \subseteq \mathcal{X}$. An example of such a set for the system (C.1) is the zero solution $\mathcal{X} = \{0\}$.

We now give the definition of stability in terms of Lyapunov:

Definition C.1.1 *The origin is stable with respect to the system* (C.1) *if for any $\epsilon > 0$ there exists a $\delta > 0$ (depending on ϵ) such that $\|x\| \leq \delta$ implies $\|\phi(k; x)\| \leq \epsilon$ for all $k \geq 0$.*

If $\lim_{k\to\infty} \phi(k; x) = 0$ for all $x \in \mathcal{X}$, then the origin is asymptotically attractive with respect to the system (C.1) *with a region of attraction $\mathcal{X}$.*

When both conditions are satisfied we refer to the origin as asymptotically stable with respect to the system (C.1) *with a region of attraction $\mathcal{X}$.* $\diamondsuit$

Note that stability implies automatically that f should be continuous at the origin.

Theorem C.1.2 *Let $\mathcal{X}$ be a bounded PI set for the system* (C.1) *containing the origin in its interior. Then, the origin is stable with respect to the system* (C.1) *if and only if there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ such that*

(i): $V(0) = 0$, V continuous at the origin,

(ii): $V(x) \geq \alpha(\|x\|)$ for all $x \in \mathcal{X}$, where α is a $\mathscr{K}$ function,

(iii): $V(f(x)) - V(x) \leq 0$ *for all* $x \in \mathcal{X}$.

Proof: (*Sufficiency*) Let $\epsilon > 0$. Since V is continuous at the origin, there exists a $\delta > 0$ such that for any x satisfying $\|x\| \leq \delta$ we have:

$$V(x) \leq \alpha(\epsilon).$$

Moreover, since $\mathcal{X}$ contains the origin in its interior, we can choose δ such that the ball $\{x \in \mathbb{R}^n : \|x\| \leq \delta\} \subseteq \mathcal{X}$. Suppose there exists an x_0 satisfying $\|x_0\| \leq \delta$ and k_0 a finite index such that:

$$\|\phi(k_0; x_0)\| = \epsilon' > \epsilon.$$

Since V is decreasing along the trajectories starting in $\mathcal{X}$ and α is a strictly increasing function, the following inequalities can be deduced:

$$\alpha(\epsilon) < \alpha(\epsilon') \leq V(\phi(k_0; x_0)) \leq V(x_0) \leq \alpha(\epsilon)$$

which is a contradiction.

(*Necessity*) For any $x \in \mathbb{R}^n$ we define

$$V(x) = \sup_{k \geq 0} \{\|\phi(k; x)\|\}.$$

We show that V defined above satisfies conditions *(i)–(iii)*. First, it is clear that $V(0) = 0$ (since $0 = f(0)$) and V is continuous at the origin (since the system (C.1) is stable). Second, $V(x) \geq \|x\|$ and thus we can choose the $\mathscr{K}$ function α as $\alpha(y) = y$. Third, for any $x \in \mathcal{X}$, $V(x)$ is finite since $\mathcal{X}$ is a bounded PI set and thus

$$V(f(x)) - V(x) = \sup_{k \geq 1} \{\|\phi(k; x)\|\} - \sup_{k \geq 0} \{\|\phi(k; x)\|\} \leq 0$$

◇

Definition C.1.3 *The function V satisfying the conditions (i)–(iii) from Theorem C.1.2 is called a Lyapunov function.*

The following corollary, which is a simple consequence of Theorem C.1.2, provides sufficient conditions for asymptotic stability:

Corollary C.1.4 *Let $\mathcal{X}$ be a PI set for the system* (C.1) *containing the origin in its interior. Assume that there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ such that*

(i): $V(0) = 0$, *V continuous at the origin,*

(ii): $V(x) \geq \alpha(\|x\|)$ *for all* $x \in \mathcal{X}$, *where α is a $\mathscr{K}$ function,*

(iii'): $V(f(x)) - V(x) \leq -\beta(\|x\|)$ *for all* $x \in \mathcal{X}$, *where β is a $\mathscr{K}$ function.*

Then the origin is asymptotically stable with respect to the system (C.1) *with a region of attraction $\mathcal{X}$.*

Proof : Stability follows from Theorem C.1.2. Note that for proving sufficiency in Theorem C.1.2 we do not need $\mathcal{X}$ to be a bounded set. It remains to prove attractiveness. Let us note that V is bounded from below by 0, i.e. $V(x) \geq \alpha(\|x\|) \geq 0$. Moreover, from *(iii')* it follows that V decreases along the trajectories starting in $\mathcal{X}$. We conclude that for all $x \in \mathcal{X}$ the sequence $\{V(\phi(k;x))\}_{k\geq 0}$ is convergent:

$$\lim_{k\to\infty} V(\phi(k;x)) = V^*(x),$$

where $V^*(x)$ is a nonnegative but finite constant. It follows that

$$\lim_{k\to\infty} V(\phi(k;x)) - V(\phi(k+1;x)) = 0 \ \ \forall x \in \mathcal{X}.$$

On the other hand, we have:

$$0 \leq \beta(\|\phi(k;x)\|) \leq V(\phi(k;x)) - V(\phi(k+1;x)) \ \ \forall x \in \mathcal{X}.$$

We conclude that

$$\lim_{k\to\infty} \beta(\|\phi(k;x)\|) = 0 \ \ \forall x \in \mathcal{X}.$$

From the definition of a $\mathscr{K}$ function it follows that

$$\lim_{k\to\infty} \phi(k;x) = 0 \ \ \forall x \in \mathcal{X}.$$

This concludes our proof. $\diamondsuit$

It is important to note that we require for the Lyapunov function V to be continuous *only* at the origin and not on the entire set $\mathcal{X}$ as it is necessary in the continuous time case. This requirement has proved to be crucial in our results on stability for PWA systems.

Bibliography

[1] F. Allgöwer, T.A. Badgwell, J.S. Qin, J.B. Rawlings, and S.J. Wright. Nonlinear predictive control and moving horizon estimation – An introductory overview. In P.M. Frank, editor, *Advances in Control: Highlights of ECC '99*, pages 391–449. London, UK: Springer, 1999.

[2] R. Ansorge. What does the entropy condition mean in traffic flow theory. *Transportation Research B*, 24(2):133–143, 1990.

[3] P.J. Antsaklis. A brief introduction to the theory and applications of hybrid systems. *Proceedings of the IEEE, Special Issue on Hybrid Systems: Theory and Applications*, 88(7):879–886, 2000.

[4] F. Baccelli, G. Cohen, G.J. Olsder, and J.P. Quadrat. *Synchronization and Linearity*. John Wiley & Sons, New York, 1992.

[5] T.A. Badgwell. Robust model predictive control of stable linear systems. *International Journal of Control*, 68(4):797–818, 1997.

[6] A. Bemporad, F. Borrelli, and M. Morari. Optimal controllers for hybrid systems: stability and piecewise linear explicit form. In *Proceedings of the 2000 IEEE Conference on Decision and Control*, pages 1810–1815, Sydney, Australia, December 2000.

[7] A. Bemporad, F. Borrelli, and M. Morari. Model predictive control based on linear programming – The explicit solution. *IEEE Transactions on Automatic Control*, 47(12):1974–1985, December 2002.

[8] A. Bemporad, F. Borrelli, and M. Morari. Min-max control of constrained uncertain discrete-time linear systems. *IEEE Transactions on Automatic Control*, 48(9):1600–1606, September 2003.

[9] A. Bemporad, G. Ferrari-Trecate, and M. Morari. Observability and controllability of piecewise affine and hybrid systems. *IEEE Transactions on Automatic Control*, 45(10):1864–1876, 2000.

[10] A. Bemporad, W.P.M.H. Heemels, and B. De Schutter. On hybrid systems and closed-loop MPC systems. *IEEE Transactions on Automatic Control*, 47(5):863–869, May 2002.

[11] A. Bemporad and M. Morari. Control of systems integrating logic, dynamics, and constraints. *Automatica*, 35(3):407–427, March 1999.

[12] A. Bemporad and M. Morari. Robust model predictive control: A survey. In A. Garulli, A. Tesi, and A. Vicino, editors, *Identification and Control*, volume 245 in Lecture Notes in Control and Information Sciences, pages 207–226. Springer-Verlag, 1999.

[13] A. Bemporad, M. Morari, V. Dua, and E.N. Pistikopoulos. The explicit linear quadratic regulator for constrained systems. *Automatica*, 38(1):3–20, January 2002.

[14] A. Ben-Tal, A. Goryashko, E. Guslitzer, and A. Nemirovski. Adjustable robust solutions of uncertain linear programs. *Mathematical Programming: Series A and B*, 99(2):351–376, 2004.

[15] D.P. Bertsekas. *Dynamic Programming and Optimal Control (Second edition), Vol I and II*. Athena Scientific, 2001.

[16] D.P. Bertsekas and I.B. Rhodes. On the minmax reachability of target sets and target tubes. *Automatica*, 7:233–247, 1971.

[17] D.P. Bertsekas and I.B. Rhodes. Sufficiently informative functions and the minimax feedback control of uncertain dynamic systems. *IEEE Transactions on Automatic Control*, 18(2):117–124, April 1973.

[18] F. Blanchini. Minimum-time control for uncertain discrete-time linear systems. In *Proceedings of the 31th IEEE Conference on Decision and Control*, pages 2629–2634, Tucson AZ, USA, 1992.

[19] F. Blanchini. Set invariance in control. *Automatica*, 35:1747–1767, 1999.

[20] T.S. Blyth and M.F. Janowitz. *Residuation Theory*. Oxford: Pergamon Press, 1972.

[21] R.K. Boel, B. De Schutter, G. Nijsse, J.M. Schumacher, and J.H. van Schuppen. Approaches to modelling, analysis, and control of hybrid systems. *Journal A*, 40(4):16–27, December 1999.

[22] F. Borrelli. *Constrained Optimal Control of Linear and Hybrid Systems*, volume 290 of *Lecture Notes in Control and Information Sciences*. Springer, Berlin, 2003.

[23] F. Borrelli, A. Bemporad, and M. Morari. A geometric algorithm for multi-parametric linear programming. *Journal of Optimization Theory and Applications*, 118(3):515–540, September 2003.

[24] S. Boyd, L.E. Ghaoui, E. Feron, and V. Balakrishna. *Linear Matrix Inequalities in System and Control Theory*. SIAM, Studies in Applied Mathematics, Philadelphia, 1994.

[25] S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.

[26] M.S. Branicky. *Studies in Hybrid Systems: Modeling, Analysis, and Control*. PhD thesis, Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, Cambridge, Massachusetts, June 1995.

[27] E.F. Camacho and C. Bordons. *Model Predictive Control in the Process Industry*. Springer-Verlag, Berlin, Germany, 1995.

[28] C.G. Cassandras and S. Lafortune. *Introduction to Discrete Event Systems*. Kluwer Academic Publishers, Boston, 1999.

[29] C.G. Cassandras, D.L. Pepyne, and Y. Wardi. Optimal control of a class of hybrid systems. *IEEE Transactions on Automatic Control*, 46(3):398–415, March 2001.

[30] H. Chen and F. Allgower. A quasi-infinite horizon nonlinear predictive control scheme with guaranteed stability. *Automatica*, 34:1205–1217, 1998.

[31] L. Chisci, P. Falugi, and G. Zappa. Gain-scheduling MPC of nonlinear systems. *International Journal of Robust and Nonlinear Control*, 13:295–308, March-April 2003.

[32] D.D. Cofer and V.K. Garg. A generalized max-algebra model for performance analysis of timed and untimed discrete event systems. In *Proceedings of the 1993 American Control Conference*, pages 2288–2292, San Francisco, California, June 1993.

[33] D.D. Cofer and V.K. Garg. Supervisory control of real-time discrete-event systems using lattice theory. *IEEE Transactions on Automatic Control*, 41(2):199–209, February 1996.

[34] G. Cohen, D. Dubois, J.P. Quadrat, and M. Viot. A linear-system-theoretic view of discrete-event processes and its use for performance evaluation in manufacturing. *IEEE Transactions on Automatic Control*, 30(3):210–220, March 1985.

[35] D. Corona, I. Necoara, B. De Schutter, and T.J.J. van den Boom. Robust hybrid MPC applied to the design of an adaptive cruise controller for a road vehicle. In *Proceedings of the 45th IEEE Conference on Decision and Control*, San Diego, USA, December 2006.

[36] B. Cottenceau, L. Hardouin, J. Boimond, and J. Ferrier. Model reference control for timed event graphs in dioid. *Automatica*, 37(8):1451–1458, 2001.

[37] R.A. Cuninghame-Green. *Minimax Algebra*, volume 166 of *Lecture Notes in Economics and Mathematical Systems*. Springer-Verlag, Berlin, Germany, 1979.

[38] C.R. Cuttler and B.C. Ramaker. Dynamic matrix control–a computer control algorithm. In *Proceedings of the 1980 American Control Conference*, volume WP5-B, San Francisco, USA, 1980.

[39] J. Daafouz and J. Bernussou. Parameter dependent Lyapunov functions for discrete time systems with time varying parametric uncertainties. *Systems & Control Letters*, 43(5):355–359, 2001.

[40] C.F. Daganzo. Requiem for second-order fluid approximations of traffic flow. *Transportation Research Part B*, 29(4):277–286, August 1995.

[41] B. De Schutter. *Max-Algebraic System Theory for Discrete Event Systems*. PhD thesis, Faculty of Applied Sciences, K.U.Leuven, Leuven, Belgium, February 1996.

[42] B. De Schutter. Optimal control of a class of linear hybrid systems with saturation. *SIAM Journal on Control and Optimization*, 39(3):835–851, 2000.

[43] B. De Schutter and B. De Moor. The extended linear complementarity problem. *Mathematical Programming*, 71(3):289–325, December 1995.

[44] B. De Schutter and T.J.J van den Boom. On model predictive control for max-min-plus-scaling discrete event systems. Technical Report bds:00-04, Control Systems Engineering, Faculty of Information Technology and Systems, Delft University of Technology, Delft, The Netherlands, June 2000.

[45] B. De Schutter and T.J.J. van den Boom. Model predictive control for max-plus-linear discrete event systems. *Automatica*, 37(7):1049–1056, July 2001.

[46] B. De Schutter and T.J.J. van den Boom. MPC for continuous piecewise-affine systems. *Systems & Control Letters*, 52:179–192, 2004.

[47] J.M. del Castillo. A formulation for the reaction time of traffic flow models. In C. Daganzo, editor, *Transportation and Traffic Theory*, pages 133–144, 1993.

[48] M.C. Delfour and S.K. Mitter. Reachability of perturbed systems and min sup problems. *SIAM Journal of Control*, 47(4):521–533, 1969.

[49] J. Ezzine and A.H. Haddad. Controllability and observability of hybrid systems. *International Journal of Control*, 49(6):2045–2055, July 1989.

[50] G. Ferrari-Trecate, F.A. Cuzzola, D. Mignone, and M. Morari. Analysis and control with performance of piecewise affine and hybrid systems. In *Proceedings of the 2001 American Control Conference*, pages 200–205, Arlington, Virginia, June 2001.

[51] A.V. Fiacco. *Introduction to sensitivity and stability analysis in nonlinear programming*. Academic Press, London, U.K., 1983.

[52] T. Gal. *Postoptimal Analyses, Parametric Programming, and Related Topics*. de Gruyter, Berlin, 1995.

[53] C.E. Garcia and A.M. Morshedi. Quadratic programming solution of dynamic matrix control (QDMC). *Chemical Engeneering Communications*, 46:73–87, 1986.

[54] C.E. García, D.M. Prett, and M. Morari. Model predictive control: Theory and practice — A survey. *Automatica*, 25(3):335–348, May 1989.

[55] S. Gaubert. *Théorie des Systèmes Linéaires dans les Dioïdes*. PhD thesis, Ecole Nationale Supérieure des Mines de Paris, France, July 1992.

[56] M.J. Gazarik and B. E.W. Kamen. Reachability and observability of linear systems over max-plus. *Kybernetika*, 35(1):2–12, January 1999.

[57] E.G. Gilbert and K. T. Tan. Linear systems with state and control constraints:the theory and applications of maximal output admissible sets. *IEEE Transactions on Automatic Control*, 36(9):1008–1020, September 1991.

[58] D. Godbole and J. Lygeros. Longitudinal control of the lead car of a platoon. *IEEE Transactions on Automatic Control*, 43(11):1125–1135, November 1994.

[59] J.E. Goodman and J. O'Rourke. *Handbook of Discrete and Computational Geometry*. Discrete Mathematics and Its Applications, Boca Raton, 1997.

[60] P.J. Goulart, E.C. Kerrigan, and J.M. Maciejowski. Optimization over state feedback policies for robust control with constraints. *Automatica*, 42(4):523–533, 2005.

[61] J. Gunawardena. Min-max functions. *Discrete Event Dynamic Systems: Theory and Applications*, 4(4):377–407, September 1994.

[62] A. Harten, P.D. Lax, and B. van Leer. On upstream differencing and Godunov-type schemes for hyperbolic conservation laws. *SIAM Review*, 25:35–61, 1983.

[63] S. Hedlund and A. Rantzer. Convex dynamic programming for hybrid systems. *IEEE Transactions on Automatic Control*, 47(9):1536–1540, September 2002.

[64] W.P.M.H. Heemels, B. De Schutter, and A. Bemporad. Equivalence of hybrid dynamical models. *Automatica*, 37(7):1085–1091, July 2001.

[65] W.P.M.H. Heemels, J.M. Schumacher, and S. Weiland. Linear complementarity systems. *SIAM Journal on Applied Mathematics*, 60(4):1234–1269, 2000.

[66] B. Heidergott, G.J. Olsder, and J. van der Woude. *Max Plus at Work*. Princeton University Press, Princeton, 2006.

[67] D. Helbing. High-fidelity macroscopic traffic equations. *Physica A*, 219:391–407, 1995.

[68] D. Helbing. Derivation and empirical validation of a refined traffic flow model. *Physica A*, 233:253–282, 1996.

[69] D. Helbing. Derivation, properties, and simulation of a gas-kinetic-based, nonlocal traffic model. *Physical Review E*, 59(1):239–253, 1999.

[70] D. Helbing. Micro- and macro-simulation of freeway traffic. *Mathematical and Computer Modelling*, 35:517–547, 2002.

[71] S.P. Hoogendoorn and P.H.L. Bovy. Generic gas-kinetic traffic systems modeling with applications to vehicular traffic flow. *Transportation Research Part B*, 35(4):317–336, May 2001.

[72] R.A. Horn and C.R. Johnson. *Matrix Analysis*. Cambridge University Press, Cambridge, United Kingdom, 1985.

[73] T. Iwasaki and R.E. Skelton. The X-Y centering algorithm for the dual LMI problem: a new approach to fixed order control design. *International Journal of Control*, 62(6):1257–1272, 1995.

[74] M. Johansson. *Piecewise Linear Control Systems*, volume 284 of *Lecture Notes in Control and Information Sciences*. Springer, Berlin, Germany, 2003.

[75] M. Johansson and A. Rantzer. Computation of piecewise quadratic Lyapunov functions for hybrid systems. *IEEE Transactions on Automatic Control*, 43(4):555–559, April 1998.

[76] C.N. Jones. *Polyhedral Tools for Control*. PhD thesis, Department of Engineering, University of Cambridge, Cambridge, UK, 2005.

[77] U.T. Jonsson. A lecture on the S-procedure. KTH Lecture Notes, Royal Institute of Technology, 10044 Stockolm, Sweden, May 2001.

[78] B.S. Kerner and P. Konhauser. Structure and parameters of clusters in traffic flow. *Physical Review E*, 50(1):54–83, 1994.

[79] E.C. Kerrigan. *Robust Constraint Satisfaction: Invariant Sets and Predictive Control*. PhD thesis, Department of Engeneering, University of Cambridge, Cambridge, UK, 2000.

[80] E.C. Kerrigan and J.M. Maciejowski. Feedback min-max MPC using a single linear program: Robust stability and the explicit solution. *International Journal of Robust and Nonlinear Control*, 13(3–4):1–18, 2003.

[81] E.C. Kerrigan and D.Q. Mayne. Optimal control of constrained, piecewise affine systems with bounded disturbances. In *Proceedings of the 2002 IEEE Conference on Decision and Control*, pages 1552–1557, Las Vegas, Nevada, December 2002.

[82] I. Kolmanovsky and E.G. Gilbert. Theory and computations of disturbance invariant sets for discrete-time linear systems. *Mathematical Problems in Engineering: Theory, Methods and Applications*, 4:317–363, 1998.

[83] M.V. Kothare, V. Balakrishnan, and M. Morari. Robust constrained model predictive control using linear matrix inequalities. *Automatica*, 32:1361–1379, 1996.

[84] B. Kouvaritakis, J.A. Rossiter, and J. Schuurmans. Efficient robust predictive control. *IEEE Transactions on Automatic Control*, 45(8):1545–1549, August 2000.

[85] R. Kumar and V.K. Garg. Extremal solutions of inequations over lattices with applications to supervisory control. In *Proceedings of the 33rd IEEE Conference on Decision and Control*, pages 3636–3641, Orlando, Florida, December 1994.

[86] M. Kvasnica, P. Grieder, and M. Baotić. Multi-Parametric Toolbox (MPT), 2005. http://control.ee.ethz.ch/˜mpt/.

[87] H. Kwakernaak and R. Sivan. *Modern Signals and Systems*. Prentice–Hall, Englewood Cliffs, New Jersey, USA, 1991.

[88] P.D. Lax. *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*. Society for Industrial and Applied Mathematics, Philadelphia, PA, 1973.

[89] M. Lazar, W.P.M.H. Heemels, S. Weiland, and A. Bemporad. Stabilization conditions for model predictive control of constrained PWA systems. In *Proceedings of the 43rd IEEE Conference on Decision and Control*, pages 4595–4600, Paradise Island, The Bahamas, December 2004.

[90] M. Lazar, W.P.M.H. Heemels, S. Weiland, and A. Bemporad. Stabilizing receding horizon control of piecewise linear systems: an LMI approach. In *Proceedings of the 16th Mathematical Theory of Networks and Systems*, Leuven, Belgium, June 2004.

[91] J.H. Lee and Z. Yu. Worst-case formulations of MPC for systems with bounded parameters. *Automatica*, 33(5):763–781, 1997.

[92] R.J. LeVeque. *Numerical Methods for Conservation Laws*. Lectures in Mathematics. Birkhauser, 1992.

[93] M. Lhommeau, L. Hardouin, B. Cottenceau, and L. Jaulin. Interval analysis in dioid: application to robust controller design for timed-event graphs. *Automatica*, 40(11):1923–1930, November 2004.

[94] L. Libeaut and J.J. Loiseau. Admissible initial conditions and control of timed event graphs. In *Proceedings of the 34th IEEE Conference on Decision and Control*, pages 2011–2016, New Orleans, Louisiana, December 1995.

[95] L. Libeaut and J.J. Loiseau. Model matching for timed event graphs. In *Proceedings of the 13th IFAC World Congress (IFAC'96)*, pages 383–388, San Francisco, California, June–July 1996.

[96] M.J. Lighthill and G.B. Whitham. On kinematic waves: II. A theory of traffic flow on long crowded roads. *Proceedings of the Royal Society of London*, 299A:317–345, May 1955.

[97] J. Lofberg. Approximations of closed-loop MPC. In *Proceedings of the 42th IEEE Conference on Decision and Control*, pages 1438–1442, Maui, Hawaii, USA, December 2003.

[98] Y. Lu and Y. Arkun. Quasi-min-max MPC algorithms for LPV systems. *Automatica*, 36(4):527–540, 2000.

[99] J. Lygeros, C. Tomlin, and S. Sastry. Controllers for reachability specifications for hybrid systems. *Automatica*, 35(3):349–370, March 1999.

[100] J.M. Maciejowski. *Multivariable Feedback Design*. Addison–Wesley, Wokingham, 1989.

[101] J.M. Maciejowski. *Predictive Control with Constraints*. Prentice Hall, Harlow, England, 2002.

[102] C.A. Maia, L. Hardouin, R. Santos-Mendes, and B. Cottenceau. Optimal closed-loop control of timed event graphs in dioids. *IEEE Transactions on Automatic Control*, 48(12):2284–2287, December 2003.

[103] D.Q. Mayne. Control of constrained dynamic systems. *European Journal of Control*, 7:87–99, 2001. Survey paper.

[104] D.Q. Mayne and S. Rakovic. Model predictive control of constrained PWA discrete-time systems. *International Journal of Robust and Nonlinear Control*, 13(2):261–279, 2003.

[105] D.Q. Mayne, J.B. Rawlings, C.V. Rao, and P.O.M. Scokaert. Constrained model predictive control: Stability and optimality. *Automatica*, 36(7):789–814, June 2000.

[106] D.Q. Mayne and W.R. Schroeder. Robust time-optimal control of constrained linear systems. *Automatica*, 33(12):2103–2118, 1997.

[107] E. Menguy, J.L. Boimond, and L. Hardouin. A feedback control in max-algebra. In *Proceedings of the European Control Conference (ECC'97)*, Brussels, Belgium, paper 487, July 1997.

[108] E. Menguy, J.L. Boimond, and L. Hardouin. Optimal control of discrete event systems in case of updated reference input. In *Proceedings of the IFAC Conference on System Structure and Control (SSC'98)*, pages 601–607, Nantes, France, July 1998.

[109] E. Menguy, J.L. Boimond, L. Hardouin, and J.L. Ferrier. A first step towards adaptive control for linear systems in max algebra. *Discrete Event Dynamic Systems: Theory and Applications*, 10(4):347–367, 2000.

[110] E. Menguy, J.L. Boimond, L. Hardouin, and J.L. Ferrier. Just-in-time control of timed event graphs : Update of reference input, presence of uncontrollable input. *IEEE Transactions on Automatic Control*, 45(9):2155–2159, September 2000.

[111] H. Michalska and D.Q. Mayne. Robust receding horizon control of constrained nonlinear systems. *IEEE Transactions on Automatic Control*, 38(11):1623–1633, November 1993.

[112] D. Mignone, G. Ferrari-Trecate, and M. Morari. Stability and stabilization of piecewise affine and hybride systems: An LMI approach. In *Proceedings of the 39rd IEEE Conference on Decision and Control*, pages 504–509, Sydney, Australia, December 2000.

[113] T.S. Motzkin, H. Raiffa, G.L. Thompson, and R.M. Thrall. The double description method. In H.W. Kuhn and A.W. Tucker, editors, *Contributions to the Theory of Games*, number 28 in *Annals of Mathematics Studies*, pages 51–73. Princeton University Press, Princeton, New Jersey, 1953.

[114] I. Necoara, B. De Schutter, W.P.M.H. Heemels, S. Weiland, M. Lazar, and T.J.J. van den Boom. Control of PWA systems using a stable receding horizon method. In *Proceedings of the 16th IFAC World Congress*, Prague, Czech Republic, July 2005. Paper 2794 / Tu-E21-TO/2.

[115] I. Necoara, B. De Schutter, and H. Hellendoorn. On structural properties of Helbing's gas-kinetic traffic flow Model. In *Proceedings of the 2004 American Control Conference (ACC 2004)*, pages 5508–5513, Boston, Massachusetts, June–July 2004.

[116] I. Necoara, B. De Schutter, and J. Hellendoorn. Structural properties of Helbing's traffic flow model. *Transportation Research Record*, (1883):21–30, 2004.

[117] I. Necoara, B. De Schutter, and T.J.J. van den Boom. Model predictive control for perturbed piecewise affine systems. In *Proceedings of the 5th EUROMECH Nonlinear Dynamics Conference (ENOC-2005)*, pages 231–240, Eindhoven, The Netherlands, August 2005.

[118] I. Necoara, B. De Schutter, T.J.J. van den Boom, and H. Hellendoorn. Model predictive control for perturbed continuous piecewise affine systems with bounded disturbances. Technical Report 04-025, Delft Center for Systems and Control, Delft University of Technology, Delft, The Netherlands, April 2005. Provisionally accepted for International Journal of Control.

[119] I. Necoara, B. De Schutter, T.J.J. van den Boom, and J. Hellendoorn. Model predictive control for perturbed continuous piecewise affine systems with bounded disturbances. In *Proceedings of the 43rd IEEE Conference on Decision and Control*, pages 1848–1853, Paradise Island, The Bahamas, December 2004.

[120] I. Necoara, B. De Schutter, T.J.J. van den Boom, and J. Hellendoorn. Robustly stabilizing MPC for perturbed PWL systems. In *Proceedings of the 44th IEEE Conference on Decision and Control, and the European Control Conference 2005 (CDC-ECC'05)*, pages 3759–3764, Seville, Spain, December 2005.

[121] I. Necoara, B. De Schutter, T.J.J. van den Boom, and J. Hellendoorn. Min-max model predictive control for uncertain max-min-plus-scaling systems. In *8th International Workshop on Discrete Event Systems (WODES'06)*, pages 439–444, Ann Arbor, Michigan, USA, July 2006.

[122] I. Necoara, B. De Schutter, T.J.J. van den Boom, and J. Hellendoorn. Robust control of constrained max-plus-linear systems. Technical Report 06-024, Delft Center for Systems and Control, Delft University of Technology, Delft, The Netherlands, May 2006. Submitted to a journal.

[123] I. Necoara, B. De Schutter, T.J.J. van den Boom, and J. Hellendoorn. Stable model predictive control for constrained max-plus-linear systems. Technical Report 06-023, Delft Center for Systems and Control, Delft University of Technology, Delft, The Netherlands, March 2006. Submitted to a journal.

[124] I. Necoara, B. De Schutter, T.J.J. van den Boom, and J. Hellendoorn. Stable receding horizon control for constrained max-plus-linear systems. In *Proceedings of the 2006 American Control Conference (ACC 2006)*, pages 4055–4060, Minneapolis, Minnesota, USA, July 2006.

[125] I. Necoara, E.C. Kerrigan, B. De Schutter, and T.J.J. van den Boom. Finite-horizon min-max control of max-plus-linear systems. Technical Report 06-002, Delft Center for Systems and Control, Delft University of Technology, Delft, The Netherlands, December 2005. Provisionally accepted for IEEE Transaction on Automatic Control.

[126] I. Necoara, E.C. Kerrigan, B. De Schutter, and T.J.J. van den Boom. Worst-case optimal control of uncertain max-plus-linear systems. In *Proceedings of the 45th IEEE Conference on Decision and Control*, San Diego, USA, December 2006.

[127] I. Necoara, T.J.J. van den Boom, B. De Schutter, and J. Hellendoorn. Stabilization of max-plus-linear systems using receding horizon control — The unconstrained case. In *Proceedings of the 2nd IFAC Conference on Analysis and Design of Hybrid Systems (ADHS'06)*, pages 148–153, Alghero, Italy, June 2006.

[128] I. Necoara, T.J.J. van den Boom, B. De Schutter, and J. Hellendoorn. Stabilization of MPL systems using model predictive control: The unconstrained case. Technical Report 06-006, Delft Center for Systems and Control, Delft University of Technology, Delft, The Netherlands, January 2006. Provisionally accepted for Automatica.

[129] G.J. Olsder. On structural properties of min-max systems. In G. Cohen and J.P. Quadrat, editors, *Proceedings of the 11th International Conference on Analysis and Optimization of Systems* (Sophia-Antipolis, France, June 1994), volume 199 of *Lecture Notes in Control and Information Sciences*, pages 237–246. London: Springer-Verlag, 1994.

[130] S. Osher. Riemann solvers, the entropy condition and difference approximation. *SIAM Journal on Numerical Analysis*, 21:217–235, 1984.

[131] S. Ovchinnikov. Max-min representation of piecewise linear function. *Beitrage zur Algebra und Geometrie/Contributions to Algebra and Geometry*, 43(1):297–302, 2002.

[132] P.M. Pardalos and M.G.C. Resende, editors. *Handbook of Applied Optimization*. Oxford University Press, Oxford, UK, 2002.

[133] K.M. Passino and K.L. Burgess. *Stability Analysis of Discrete Event Systems*. John Wiley & Sons, New York, 1998.

[134] S.L. Paveri-Fontana. On Boltzmann-like treatments for traffic flow. A critical review of the basic model and an alternative proposal for dilute traffic analysis. *Transportation Research*, 9:225–235, 1975.

[135] H.J. Payne. Models of freeway traffic and control. In G.A. Bekey, editor, *Mathematical Models of Public Systems*, volume 1, no. 1 of *Simulation Council Proceedings Series*, pages 51–61. La Jolla, California, 1971.

[136] I. Prigogine and R. Hermann. *Kinetic Theory Of Vehicular Traffic*. Elsevier, New York, 1971.

[137] S.J. Qin and T.A. Badgwell. Nonlinear Model Predictive Control. In *Progress in Systems and Control Theory*, volume 26, chapter :An overview of nonlinear model predictive control applications, pages 369–392. Birkhauser Verlag, 2000.

[138] S.V. Rakovic, E.C. Kerrigan, and D.Q. Mayne. Optimal control of constrained piecewise affine systems with state- and input-dependent disturbances. In *Proceedings 16th International Symposium on Mathematical Theory of Networks and Systems*, Katholieke Universiteit Leuven, Belgium, July 2004.

[139] A. Rantzer and M. Johansson. Piecewise linear quadratic optimal control. *IEEE Transactions on Automatic Control*, 45(4):629–637, April 2000.

[140] J. Richalet. Industrial applications of model based predictive control. *Automatica*, 29(5):1251–1274, May 1993.

[141] J. Richalet, A. Rault, J.L. Testud, and J. Papon. Model predictive heuristic control: Applications to industrial processes. *Automatica*, 14(5):413–428, September 1978.

[142] P.I. Richards. Shock waves on the highway. *Operations Research*, 4:42–51, 1956.

[143] S.M. Robinson and R.H. Day. A sufficient condition for continuity of optimal sets in mathematical programming. *Journal of Mathematical Analysis and Applications*, 45:506–511, 1974.

[144] R.T. Rockafellar and R.J.-B. Wets. *Variational Analysis*. Springer-Verlag, Berlin, 2004.

[145] J.A. Rossiter, B. Kouvaritakis, and M.J. Rice. A numerically robust state-space approach to stable predictive control strategies. *Automatica*, 34(1):65–73, 1998.

[146] J.P. La Salle. *The Stability of Dynamical Systems*. Society for Industrial and Applied Mathematics, 1976.

[147] A. Schrijver. *Theory of Linear and Integer Programming*. John Wiley & Sons, Chichester, UK, 1986.

[148] P.O.M. Scokaert and D.Q. Mayne. Min-max feedback model predictive control for constrained linear system. *IEEE Transactions on Automatic Control*, 43(8):1136–1142, August 1998.

[149] P.O.M. Scokaert, D.Q. Mayne, and J.B. Rawlings. Suboptimal model predictive control (feasibility implies stability). *IEEE Transactions on Automatic Control*, 44(3):648–654, 1999.

[150] O. Slupphaug and B.A. Foss. Constrained quadratic stabilization of discrete-time uncertain non-linear multi-model systems using piecewise affine state-feedback. *International Journal of Control*, 72(7):686–701, 1999.

[151] E.D. Sontag. Nonlinear regulation: The piecewise linear approach. *IEEE Transactions on Automatic Control*, 26(2):346–357, April 1981.

[152] J. Spjøtvold, P. Tøndel, and T.A. Johansen. Unique polyhedral representations to continuous selections for convex multiparametric quadratic programs. In *Proceedings 2005 American Control Conference*, Portland, Oregon USA, June 2005.

[153] E.F. Toro. *Riemann Solvers and Numerical Methods for Fluid Dynamics — A Practical Introduction*. Springer Verlag, 2nd edition, 1999.

[154] H.D. Tuan, P. Apkarian, S. Hosoe, and H. Tuy. D.c. optimization approach to robust control: feasibility problems. *International Journal of Control*, 73(2):89–104, 2000.

[155] T.J.J. van den Boom and B. De Schutter. Model predictive control for perturbed max-plus-linear systems. *Systems & Control Letters*, 45(1):21–33, January 2002.

[156] T.J.J. van den Boom and B. De Schutter. Modelling and control of discrete event systems using switching max-plus-linear systems. In *Proceedings of the 7th International Workshop on Discrete Event Systems (WODES'04)*, pages 115–120, Reims, France, September 2004.

[157] T.J.J. van den Boom and B. De Schutter. Modelling and control of discrete event systems using switching max-plus-linear systems. *Control Engineering Practice*, 14(10):1199–1211, October 2006.

[158] T.J.J. van den Boom, B. De Schutter, and I. Necoara. On MPC for max-plus-linear systems: Analytic solution and stability. In *Proceedings of the 44th IEEE Conference on Decision and Control, and the European Control Conference 2005 (CDC-ECC'05)*, pages 7816–7821, Seville, Spain, December 2005.

[159] T.J.J. van den Boom, B. De Schutter, and I. Necoara. Stable MPC for switching MPL systems. Technical Report 06-031, DCSC, Delft University of Technology, Delft, The Netherlands, September 2006. URL: http://www.dcsc.tudelft.nl/~bdeschutter/pub.

[160] A.J. van der Schaft and J.M. Schumacher. Complementarity modeling of hybrid systems. *IEEE Transactions on Automatic Control*, 43(4):483–490, April 1998.

[161] A.J. van der Schaft and J.M. Schumacher. *An Introduction to Hybrid Dynamical Systems*, volume 251 of *Lecture Notes in Control and Information Sciences*. Springer-Verlag, London, 2000.

[162] R.J. van Egmond, G.J. Olsder, and H.J. van Zuylen. A new way to optimise network traffic control using maxplus algebra. In *Proceedings of the 78th Annual Meeting of the Transportation Research Board*, Washington, DC, January 1999. Paper 660.

[163] B. van Leer. Towards the ultimate conservative difference scheme — V. A second order sequel to Godunov's method. *Journal of Computational Physics*, 32:101–136, 1979. Reprinted in Vol. 135, pp. 229–248, 1997.

[164] L. Vandenberghe and S. Boyd. Semidefinite programming. *SIAM Review*, 38(1):49–95, March 1996.

[165] L. Weiss and L. Lam. Stability of non-linear discrete-time systems. *International Journal of Control*, 17(3):465–470, 1973.

[166] G.B. Whitham. *Linear and Nonlinear Waves*. Wiley, New York, 1974.

[167] H.S. Witsenhausen. A minmax control problem for sampled linear systems. *IEEE Transactions on Automatic Control*, 13(1):5–21, 1968.

[168] Y. Zhang and L. Gao. On numerical solution of the maximum volume ellipsoid problem. *SIAM Journal of Optimization*, 14(1):53–76, March 2003.

Notation

Here we list some of the symbols and acronyms that occur frequently in this thesis and with which the reader might not be familiar.

List of Symbols

Sets

$\emptyset$	empty set	
$\lvert X\rvert$	cardinality of the set X	
$\bar{X}$	closure of the set X	
$\mathrm{int}(X)$	interior of the set X	
$Z \setminus X$	complement of X contained in Z: $Z \setminus X := \{x \in Z : x \notin X\}$	
$X \subseteq Z$	X is a subset of Z	
$X \subset Z$	X is a proper subset of Z	
$\mathbb{R}$	set of real numbers	
$\mathbb{R}_+$	set of nonnegative real numbers	
$\mathbb{N}$	set of nonnegative integers: $\mathbb{N} = \{0, 1, 2, \ldots\}$	
$\mathbb{N}_{[k,l]}$	set of integers: $\mathbb{N}_{[k,l]} = \{k, k+1, \ldots, l\}$	14
$\mathcal{O}_k$	k^{th} iteration in the computation of positively invariant sets	80
$\mathcal{I}, \mathcal{J}$	finite index sets	10
X_f	terminal set	19
N	prediction horizon	19
$\Pi_N(x)$	set of feasible input sequences for the initial state x	20
X_N	set of feasible initial states: $X_N = \{x : \Pi_N(x) \neq \emptyset\}$	20
$\lfloor x \rfloor$	largest integer less than or equal to x	111

Functions

$f : D \to T$	function with domain D and co-domain T	
$\mathrm{dom} f$	effective domain of the function f	10
$\mathcal{F}_\mathrm{mps}$	set of max-plus-scaling functions	34
$\mathcal{F}^+_\mathrm{mps}$	set of max-plus-nonnegative-scaling functions	34
ℓ	stage cost	20
V_N	cost function	20
V_f	terminal cost	20
$\mathbf{u}$	control sequence	20
$\mathbf{w}$	disturbance sequence	25
$\phi(k; x, \mathbf{u})$	the state solution of a dynamic system at step k, when the initial state is x and the control sequence $\mathbf{u}$ is applied	19

Matrices and Vectors

$\mathbb{R}^{m\times n}$	set of m by n matrices with real entries	
$\mathbb{R}^n$	set of real column vectors with n components: $\mathbb{R}^n = \mathbb{R}^{n\times 1}$	
A^T	transpose of the matrix A	
I	identity matrix of appropriate dimensions	
0	zero matrix of appropriate dimensions	
a_i	i^{th} component of the vector a	
A_{ij}	entry of the matrix A on the i^{th} row and the j^{th} column	
$A_{i.}$	i^{th} row of the matrix A	
$A_{.j}$	j^{th} column of the matrix A	
$\|A\|$	induced norm of the matrix A	
$\|x\|$	norm of the vector x	
$x \geq y$	inequality over the nonnegative orthant $\mathbb{R}^n_+$, i.e. $x_i \geq y_i \ \forall i \in \mathbb{N}_{[1,n]}$	
$H \geq 0$	nonnegative matrix, i.e. $H_{ij} \geq 0 \ \forall i, j$	33
$Q \succcurlyeq 0$	inequality over the semidefinite cone, i.e. $x^T Q x \geq 0 \ \forall x$	118

Max-Plus Algebra

$\oplus$	max-plus addition: $x \oplus y = \max\{x, y\}$	14
$\otimes$	max-plus multiplication: $x \otimes y = x + y$	14
ε	zero element for $\otimes$: $\varepsilon = -\infty$	14
E	max-plus identity matrix of appropriate dimensions	30
$\mathcal{E}$	max-plus zero matrix of appropriate dimensions	30
$A^{\otimes^k}$	k^{th} max-plus power of the matrix A	30
$\mathbb{R}_\varepsilon$	$\mathbb{R}_\varepsilon = \mathbb{R} \cup \{-\infty\}$	14
$\oplus'$	min-plus operator: $x \oplus' y = \min\{x, y\}$	31
A^*	$A^* := \lim_{k\to\infty} E \oplus A \oplus \cdots \oplus A^{\otimes^k}$	30
λ^*	the largest max-plus eigenvalue of the matrix A	31

Miscellaneous

$\binom{n}{k}$	binomial coefficient: $\binom{n}{k} = \dfrac{n!}{(n-k)!\,k!}$	111

We use $\diamond$ to indicate the end of a proof or an example, etc.

Remark: The notation we use for the max-plus algebraic symbols corresponds to a large extent to that of [4], which is one of the basic references in the field of the max-plus algebra. Nevertheless, there are a few differences that are mainly caused by the fact that we use concepts from both conventional algebra and max-plus algebra in this thesis. The main differences are:

- We use $A^{\otimes^r}$ instead of A^r to denote the max-algebraic power. Furthermore, we never omit the $\oplus, \otimes$ signs in all equations.

- The operations '$+$' and '$\cdot$' denote the conventional summation and multiplication operators (only the conventional multiplication operator is omitted).

Acronyms

DES	Discrete event system	13
DP	Dynamic programming	52
LMIs	Linear matrix inequalities	115
MMPS	Max-min-plus-scaling	4
MPC	Model predictive control	3
MPL	Max-plus-linear	4
PWA	Piecewise affine	4
PWL	Piecewise linear	8
PI	Positively invariant	23
RPI	Robustly positively invariant	87

Summary

Model Predictive Control for Max-Plus-Linear and Piecewise Affine Systems

Increasing demands on modern technology have caused a significant interest in the study of nonlinear dynamical systems that are capable of exhibiting continuous dynamics and/or discrete event dynamics: hybrid systems and discrete event systems (DES). This type of systems abound in nature and they are not limited to engineering systems with logic controllers but they also arise naturally in manufacturing, planning, chemistry, biology, etc. The need for efficiency, safety, reliability makes it necessary to design controllers that ensure these systems meet certain requirements. Even though the academic community and industry recognize that a large number of engineering systems are hybrid or evolve in time by occurrence of events, these systems have been traditionally analyzed and designed using a purely continuous or a purely discrete formulation. The explanation for this fact is that only recently a theory is taking shape that captures the interaction between the continuous dynamics and the discrete events.

This Ph.D. thesis considers the development of new analysis and control techniques for special classes of hybrid systems and DES. Two particular classes of hybrid systems (piecewise affine (PWA) systems and max-min-plus-scaling (MMPS) systems), and two particular classes of DES (max-plus-linear (MPL) systems and switching MPL systems) are studied. Using the optimal control framework, model predictive control (MPC) schemes are designed that make use of the special structure of these systems and that incorporate conditions to guarantee a priori closed-loop asymptotic stability. Stability is obtained by deriving bounds on the tuning parameters or by imposing a terminal set constraint and using an appropriate terminal cost. Since the main topic of this thesis is optimal control and its receding horizon implementation called MPC, the thesis starts with an overview of optimal control, MPC, and some possible solutions to the main issues in MPC for general nonlinear systems (feasibility, robustness, and closed-loop stability).

Optimal control and MPC for MPL systems

Classical optimal control for MPL systems is based on two main ingredients, residuation theory and input-output models, and they lead to a just-in-time controller. Besides being based on an input-output model, in general the residuation approach is not able to cope with input and output constraints, the initial state is not included explicitly in the optimization problem, and stability cannot be guaranteed a priori.

To overcome these limitations new optimal control methods based on state-space models have been derived. Because MPL systems are nonlinear, non-convexity is clearly a problem if one seeks to develop efficient methods for solving optimal control problems for MPL systems. However, by employing recent results in polyhedral algebra and multi-parametric linear programming, we provide sufficient conditions that allow to preserve convexity of the optimal value function and its domain, and thus to compute optimal controllers for MPL systems in an

efficient way. Moreover, in the unconstrained case we show that for an appropriate cost function the just-in-time controller can be recovered and that it is also physically feasible compared to residuation-based just-in-time controller, which sometimes may be physically infeasible.

We also introduce the notion of Lyapunov stability for MPL systems, and we show connections with the classical definition of stability for DES in terms of boundedness of the buffer levels. Furthermore, we design new MPC strategies that guarantee a priori asymptotic stability for the corresponding closed-loop system. A similar finite-horizon MPC approach as for conventional, time-driven systems is followed. However, the extension from time-driven systems to discrete event MPL systems is not trivial since many concepts from system theory have to be adapted adequately. In particular, we introduce the notion of positively invariant set for a normalized MPL system, and the main properties are derived for such a set. The stability results are obtained either by deriving bounds on the tuning parameters or by using a terminal cost and a terminal set approach.

Min-max MPC for MMPS systems
MMPS systems represent a more general framework for modeling DES (e.g. MPL systems are a particular subclass), but they are also equivalent to some relevant subclasses of hybrid systems. In the literature on robust control for hybrid systems, a min-max framework and dynamic programming are used to derive a robust controller, in particular a feedback min-max MPC. However, the dynamic programming solution may be computationally intensive and is not able to cope with variable constraints on inputs, states and disturbances.

We present two alternative approaches to design a min-max MPC for uncertain MMPS systems that remove some of these drawbacks, depending on the nature of the input over which it is optimized: open-loop input sequences and disturbance feedback policies. We show that the corresponding min-max control problems can be recast as a finite sequence of linear programs or can be solved using an iterative procedure based on solving a finite sequence of linear programs.

MPC for PWA systems
Most of the literature on stability of the closed-loop MPC for PWA systems uses a terminal equality constraint approach. An Achilles' heel in this approach is that we need a long prediction horizon in order to guarantee feasibility of the optimization problem, which leads to an increased computational burden.

We extend the MPC formulation for PWA systems with a terminal equality constraint to a new MPC strategy based on a terminal inequality constraint corresponding to the piecewise linear dynamics. We use an upper bound on the infinite-horizon quadratic cost as a terminal cost and we also construct a convex terminal set, taking into account the structure of the system. Based on these two ingredients we derive an MPC scheme for PWA systems and we prove asymptotic stability for the closed-loop MPC. Despite the fact that the PWA system might be discontinuous, we are able to prove that the optimal value function of the MPC optimization problem is continuous at the origin and can serve as a Lyapunov function for the closed-loop system. We also derive an algorithm for enlarging the terminal set based on backward procedure, which in particular also provides a method to construct a polyhedral terminal set. Therefore, the drawback of the algorithms based on recursive methods for constructing a positively invariant set, which theoretically might require an infinite number of recursions, is overcome by this algorithm. By enlarging the terminal set the prediction horizon can be chosen shorter and thus the computational complexity decreases.

The robustification of the standard MPC via the addition of a robustness constraint is also discussed. We present a new sufficient condition that enables us to preserve convexity of the state set evolution for an uncertain PWA system. Making use of this condition we propose a state feedback min-max MPC scheme based on a dual-mode approach. We also derive the main features of this robust MPC scheme, in particular robust stability.

We conclude the Ph.D. thesis with a summary of the main contributions and an outlook on open problems and possible future research topics in the field.

Samenvatting

Modelgebaseerd Voorspellend Regelen van Max-Plus-Lineaire en Stuksgewijs-Affiene Systemen

De steeds stringenter wordende eisen ten aanzien van moderne technologie hebben een grote wetenschappelijke interesse gecreëerd in niet-lineaire dynamische systemen, die continue dynamica en/of discrete-gebeurtenissen dynamica vertonen: zogenaamde hybride systemen en discrete-gebeurtenissystemen (DGS). Dit type van systemen komen veelvuldig voor in de natuur en zijn niet alleen aanwezig in technische systemen met op logica gebaseerde regelaars, maar komen ook naar voren in productieprocessen, planning, chemie, biologie, enz. De roep om efficiëntie, veiligheid en betrouwbaarheid maakt het noodzakelijk om regelaars te ontwerpen die garanderen dat deze systemen aan bepaalde eisen voldoen. Hoewel de academische wereld en de industrie beide erkennen dat een groot aantal technische systemen verandert doorheen de tijd door toedoen van discrete gebeurtenissen, wordt dit type systemen gewoonlijk geanalyseerd en ontworpen vanuit een puur continu, of juist een puur discreet perspectief. De reden hiervan is dat a theorie die de interactie tussen continue en discrete gebeurtenissen kan vatten, pas sinds kort in volle ontwikkeling is.

Dit proefschrift beschouwt de ontwikkeling van nieuwe analyse- en regelmethoden voor een aantal specifieke klassen van hybride systemen en DGS. Twee klassen hybride systemen, met name stuksgewijs-affiene (afgekort als PWA (*piecewise affine*)) systemen en *max-min-plus-scaling* (MMPS) systemen, en twee specifieke klassen DGS, met name max-plus-lineaire (MPL) systemen en schakelende MPL systemen, worden hier onderzocht. Wij ontwikkelen methoden voor modelgebaseerde voorspellende regeling (afgekort als MPC (*model predictive control*)) binnen het kader van de optimale regeling. Deze MPC methoden houden expliciet rekening met de specifieke structuur van deze systemen en bevatten condities die asymptotische stabiliteit van de gesloten lus a priori garanderen. Stabiliteit wordt verkregen door het afleiden van grenzen op de MPC-instelparameters, of door het opleggen van een eindpuntbeperking. Omdat optimale regeling, en de verplaatsende-horizon implementatie ervan, d.w.z. MPC, de hoofdonderwerpen van dit proefschrift zijn, start het proefschrift met een overzicht van optimale regeling, MPC en enkele mogelijke oplossingen voor de belangrijke onderwerpen op het gebied van MPC voor generieke niet-lineaire systemen zoals oplosbaarheid, robuustheid en stabiliteit van het gesloten-lus systeem.

Optimale regeling en MPC voor MPL systemen
De klassieke optimale regeltheorie omvat twee componenten, met name residuatie-theorie en ingangs-uitgangsmodellen. Beide componenten leiden tot een zogenaamde *just-in-time* regelaar. Naast het feit dat de residuatie-theorie met ingangs-uitgangsmodellen werkt, is deze techniek niet in staat om rekening te houden met beperkingen op de ingangs- en uitgangsvariabelen. Daarnaast

wordt de begintoestand niet expliciet in het optimalisatieprobleem meegenomen en kan stabiliteit niet a-priori worden gegarandeerd.

Om deze beperkingen te overwinnen zijn nieuwe optimale regelmethoden afgeleid die zijn gebaseerd op toestandsruimte-modellen. Omdat MPL systemen niet-lineair zijn, is het gebrek aan convexiteit een centraal aspect in de zoektocht naar efficiënte methoden voor het oplossen van optimale regelproblemen voor deze systemen. Door toepassen van recente resultaten uit de polyhedrale algebra en het multi-parametrisch programmeren, kunnen we echter voldoende voorwaarden opstellen voor het behoud van convexiteit van de optimale-waarde functie en haar domein, zodanig dat we op efficiënte wijze optimale regelaars voor MPL systeem kunnen uitrekenen. Daarnaast kunnen we, in het geval er geen beperkingen op de ingangen en uitgangen aanwezig zijn, aantonen dat met een geschikte kostenfunctie een *just-in-time* regelaar kan worden berekend, die ook fysisch realiseerbaar is, terwijl de op residuatie gebaseerde *just-in-time* regelaar daarentegen niet altijd fysisch realiseerbaar is.

We introduceren ook het begrip Lyapunov stabiliteit voor MPL systemen en laten we de relatie zien tot de klassieke definitie van stabiliteit voor DGS in termen van de begrenzing van de bufferniveaus. Daarnaast ontwerpen we nieuwe MPC strategieën met een a-priori garantie voor stabiliteit van het gesloten-lus systeem. Daarbij wordt een soortgelijke aanpak gekozen als voor conventionele continue-dynamica systemen. Echter, de uitbreiding van continue dynamica naar een discrete-gebeurtenissen dynamica is geen triviale kwestie, omdat tal van concepten uit de systeemtheorie op adequate wijze aangepast dienen te worden. In het bijzonder introduceren we het concept van de positief-invariante verzameling voor een genormaliseerd MPL systeem en worden de eigenschappen van dergelijke verzamelingen afgeleid. De resultaten inzake stabiliteit kunnen worden verkregen door het afleiden van grenzen voor de MPC-instelparameters of door het gebruiken van een aanpak gebaseerd op de kostenfunctie in het eindpunt of op een eindpuntverzameling.

Min-max MPC voor MMPS systemen

MMPS systemen vertegenwoordigen een meer algemeen raamwerk voor het modeleren van DGS (MPL systemen zijn bijvoorbeeld een bijzondere subcategorie binnen de MMPS systemen), maar ze zijn ook equivalent aan sommige relevante subklassen van hybride systemen. In de literatuur over robuust regelen van hybride systemen, worden een min-max aanpak en dynamisch programmeren toegepast om een robuuste regelaar af te leiden, in het bijzonder een min-max MPC regelaar gebaseerd op terugkoppeling. De oplosmethode die gebruik maakt van dynamisch programmeren kan echter leiden tot aanzienlijke rekentijden en kan daarnaast geen rekening houden met variabele grenzen op de ingangsvariabelen, toestanden en verstoringen.

Wij stellen twee alternatieve manieren voor om een robuuste MPC regelaar te ontwerpen voor onzekere MMPS systemen die enkele van de genoemde bezwaren oplossen, afhankelijk van het type ingangssignaal dat geoptimaliseerd wordt: open-lus ingangssignalen en het terugkoppelen van verstoringen. We laten zien dat het overeenkomstige min-max regelprobleem kan worden vertaald in een eindige reeks van lineaire-programmeringsproblemen, of dat het probleem kan worden opgelost door gebruik te maken van een iteratieve aanpak die is gebaseerd op een eindige reeks lineaire-programmeringsproblemen.

MPC voor PWA systemen

Het overgrote deel van de literatuur met betrekking tot de stabiliteit van gesloten-lus MPC voor PWA systemen maakt gebruik van een strikte eindpuntbeperking. De Achilleshiel van deze aanpak is dat een lange horizon voor de voorspelling nodig is om de realiseerbaarheid van het regelprobleem te kunnen garanderen, hetgeen leidt tot een groei van de rekentijd.

Wij breiden de formulering van het MPC probleem voor PWA op basis van een eindpuntbeperking uit naar een nieuwe aanpak, die is gebaseerd op een eindpuntongelijkheid die overeenstemt met de stuksgewijs-lineaire dynamica. We maken daarbij gebruik van een bovengrens op de oneindige-horizon kwadratische kost als eindpuntkostenfunctie. We construeren eveneens een convexe eindpuntverzameling en houden daarbij rekening met de structuur van het systeem. Met deze twee ingrediënten leiden we een MPC aanpak af voor PWA systemen en bewijzen we de stabiliteit van het gesloten-lus systeem. Ondanks het feit dat het PWA systeem discontinu kan zijn, kunnen we bewijzen dat de optimale-waarde functie van het systeem continu is in de oorsprong en daarmee kan dienen als een Lyapunov functie voor het gesloten-lus systeem. Daarnaast leiden we een algoritme af voor het vergroten van de eindpuntverzameling gebaseerd op een achterwaartse procedure, die op haar beurt weer een methode biedt om een polyhedrale eindpuntverzameling te construeren. Dit algoritme lost het probleem op van de recursieve methoden voor het construeren van een positief-invariante verzameling, die theoretisch gezien oneindig veel recursies kunnen vergen. Door het vergroten van de eindpuntverzameling kan de voorspellingshorizon korter worden gekozen en daalt bijgevolg de complexiteit van de berekeningen.

Het robuust maken van standaard MPC-technieken door de toevoeging van een robuustheidsconditie is ook onderwerp van dit proefschrift. We presenteren een nieuwe voldoende voorwaarde die het mogelijk maakt om de convexiteit van de evolutie van de verzameling van toestanden voor een onzeker PWA systeem te behouden. Door gebruik te maken van deze voorwaarde kunnen we een min-max MPC aanpak gebaseerd op toestandsterugkoppeling presenteren die is gebruik maakt van een twee-mode benadering. Tevens leiden we de belangrijkste eigenschappen van deze robuuste MPC aanpak af, in het bijzonder de robuuste stabiliteit.

Het proefschrift wordt afgesloten met een samenvatting van de belangrijkste bijdragen en een blik voorwaarts op open problemen en toekomstige onderzoeksonderwerpen in het gebied.

Curriculum Vitae

Ion Necoara was born in Vrancea, Romania, on September 20, 1977. He received the B.Sc. degree in mathematics from the University of Bucharest, Faculty of Mathematics, and the M.Sc. degree in optimization and statistics from the same faculty in 2000 and 2002, respectively. During his M.Sc. studies he visited the Zentrum Mathematik, Technical University München for 6 months and he also worked in Risk Management Department of Hypovereinsbank München. For his B.Sc. thesis he carried out theoretical research on value at risk in risk management. His M.Sc. thesis was concerned with theoretical research on state-space approaches to transfer function modeling and was carried out at the Department of Probability, Statistics and Operations Research, University of Bucharest, under the supervision of Prof. Dr. M. Dumitrescu. After graduating he has been working as a Ph.D. student at the Delft Center for Systems and Control, Delft University of Technology, The Netherlands. In 2005 he visited for 3 months the Control Group of Prof. Dr. J. Maciejowski, University of Cambridge, UK, where he worked together with Dr. E. Kerrigan on robust control for max-plus-linear systems. Ion Necoara is a member of the Dutch Institute of Systems and Control. His research interests are optimization in control, physical modeling, and control techniques, especially robust control. In his spare time he likes to listen to music and to read, as well as to travel the world while exercising his filming skills.