Delft University of Technology

SUBJECT

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Interactive particle systems with pair-wise interactions: linear (in)stability analysis and simulations

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Abstract

In this project we study the stability of stationary solutions of interactive particle systems with short-range repulsion and long-range attraction. Firstly we describe the model and discuss ring steady state solutions. We give a comprehensive and detailed proof of *Theorem 2.1* done in [2]. This theorem gives the conditions for the (in)stability of stationary ring solutions. Omitting details of the computations, we present *Theorem 3.1* in [2] to find the conditions for (un)stable stationary solutions in the situation when the mode is sufficiently large. Having these conditions altogether we find the region of the parameters in which the stationary solutions are stable. Finally, we use a steepest descent method for the simulations of the model. Figures of equilibrium states of the particles corresponding to various families of interaction forces will be shown.

Contents

1 Introduction and motivation of the problem

The collective behavior of a large number of animals in biology, the dynamic evolution of granular gases or the self-assembly of nanoparticles in physics, or many others, might be seen as an interactive particle system and can be modelled mathematically with help of partial differential equations [8, 9, 13]. An interactive particle system is a process that describes the collective behavior of interacting components, such process can either be stochastic [11] or deterministic [10], also the individuals can either be influenced by each other or not [1]. The interactive particles in this system form naturally intriguing patterns that inspire researchers in the field of modern technologies such as autonomous vehicles, imaging techniques, biosensors, biomedical sciences etc [10, 14].

With help of a mathematical model, predictions of the collective behavior of a large number of individuals mentioned above can be made. For example, a two-zone model or a three-zone model, can be used to predict such collective behavior. A two-zone model is a model with two interaction forces, repulsion and attraction. As the name suggests, a two-zone model has two zones in which either repulsion force or attraction dominates the other force. A three-zone model has one more zone of interaction called orientation where individuals mimic the behavior of other nearby individuals. As a result, a group of particles could arrive at a consensus around some coherent structures such as vortices or milling as shown in [6, 7, 12]. Analysing the models mentioned above leads to a better understanding of the development of the interactive particle system. The interactive particle system that we analyse is nonlinear, non-local, and has short-range repulsion and long-range attraction. We study the linear (in)stability of ring solutions of such interactive particle systems arising in aggregation models with some specific form such as rings, annuli and uniform circular patches [3, 4, 10]. So the following questions are immediately raised: When are the solutions (un)stable? How do the solutions appear when they are (un)stable?

Specifically, in this project we will study the linear stability of the stationary ring solutions by reproducing the computations leading to Theorem 2.1 in [2], for model (2) described below. We will simulate the model and find the stationary states by using a numerical method.

1.1 Aggregation model and ring solutions

Consider an interaction particle system with N particles, where N is a large number. Let $E(X_1, \dots, X_N)$ be the total energy associated with the system. Assume that the potential energy of any two particles only depends on the distance between them and let $P : \mathbb{R} \to \mathbb{R}$ be such a function that returns the potential energy between those two particles. Since $E(X_1, \ldots, X_N)$ is the total amount of energy, we let it be the normalized sum of the aggregated potential energy between any two particles pairwisely so that it has the form

$$E(X_1, \ldots, X_N) = \frac{1}{N^2} \sum_{j=1}^{N} \sum_{\substack{k=1 \\ k \neq j}}^{N} P(\|X_j - X_k\|). \tag{1}$$

We have

$$\begin{aligned}
-\nabla_{X_j} E &= -\frac{1}{N} \sum_{\substack{k=1 \\ k \neq j}}^{N} \nabla_{X_j} P\left(\|X_j - X_k\|\right) \\
&= -\frac{1}{N} \sum_{\substack{k=1 \\ k \neq j}}^{N} P'\left(\|X_j - X_k\|\right) \nabla_{X_j} \left(\|X_j - X_k\|\right) \\
&= -\frac{1}{N} \sum_{\substack{k=1 \\ k \neq j}}^{N} P'\left(\|X_j - X_k\|\right) \left(\frac{(X_j - X_k)}{\|X_j - X_k\|} \right).
\end{aligned}$$

In the computation above we have use the fact that given a map $g : \mathbb{R}^d \mapsto \mathbb{R}$ defined by $g(V) = \|V\|$, where $\mathbb{R}^d$ is the d-dimensional real vector space, we have $\nabla g = \frac{V}{\|V\|}$. Hence, $\nabla_{X_j} (\|X_j - X_k\|) = \frac{(X_j - X_k)}{\|X_j - X_k\|}$ by letting $X_j - X_k = V$.

Now define $f(r) = F(r)/r$ and let $F(r) = -P'(r)$ as the force associated to our potential P for simplicity. With these notations we have that

$$\begin{aligned}
-\nabla_{X_j} E &= \frac{1}{N} \sum_{\substack{k=1 \\ k \neq j}}^{N} F\left(\|X_j - X_k\|\right) \left(\frac{(X_j - X_k)}{\|X_j - X_k\|} \right) \\
&= \frac{1}{N} \sum_{\substack{k=1 \\ k \neq j}}^{N} f\left(\|X_j - X_k\|\right) \left(X_j - X_k\right),
\end{aligned}$$

and we define the aggregation model given by the associated gradient flow to the interaction energy

$$\frac{dX_j}{dt} = \frac{1}{N} \sum_{\substack{k=1 \\ k \neq j}} f\left(\|X_j - X_k\|\right) \left(X_j - X_k\right). \tag{2}$$

In order to analyse the stationary solution when $\frac{dX_j}{dt} = 0$, we consider, for simplicity, the situation when X_j and X_k are two equally spaced particles that lie on a ring of radius R. With the computations performed below we will be able to compute R.

We can simplify equation (2) as follows. We first rewrite X_j and X_k in polar of form $X_j = Re^{2\pi i j/N}$ and $X_k = Re^{2\pi i k/N}$. There are N possible distances between X_k and X_j if $k = 1, \ldots, N$. To obtain the distances, we can, without loss of generality, fix $X_j = Re^{\frac{2\pi}{N} 0} = R$ and let X_k equal $Re^{\frac{2\pi k}{N} i}$ for $k = 1, \ldots, N$. Therefore, the distances are:

$$
\begin{aligned}
\|X_j - X_k\| &= \left\| R - Re^{\frac{2\pi k}{N}i} \right\| \\
&= R\left\| 1 - e^{\frac{2\pi k}{N}i} \right\| \\
&= R\sqrt{\left(1 - e^{\frac{2\pi k}{N}i}\right)\overline{\left(1 - e^{\frac{2\pi k}{N}i}\right)}} \\
&= R\sqrt{\left(1 - e^{\frac{2\pi k}{N}i}\right)\left(1 - e^{\frac{-2\pi k}{N}i}\right)} \\
&= R\sqrt{1 - e^{\frac{-2\pi k}{N}i} - e^{\frac{2\pi k}{N}i} + 1} \\
&= R\sqrt{2\left(1 - \cos\left(\frac{2\pi}{N}k\right)\right)} \\
&= R\sqrt{2\left(2\sin^2\left(\frac{2\pi}{N}k\right)\right)} \\
&= 2R\left|\sin\left(\frac{\pi}{N}k\right)\right|.
\end{aligned}
$$

Note that $\sin\left(\frac{\pi}{N}k\right)$ stays positive if k ranges from 1 to N, which means $\left|\sin\left(\frac{\pi}{N}k\right)\right| = \sin\left(\frac{\pi}{N}k\right)$. Substituting $\|X_j - X_k\| = 2R\left(\sin\left(\frac{\pi}{N}k\right)\right)$ and $X_j - X_k = R(1 - e^{\frac{2\pi k}{N}i})$ into (2) gives us

$$
\begin{aligned}
\frac{dX_j}{dt} &= \frac{1}{N}\sum_{j=1}^{N} f\left(2R\sin\left(\frac{\pi}{N}j\right)\right)\left(1 - e^{i2\pi j/N}\right) \\
&= \frac{1}{N}\sum_{j=1}^{N-1} f\left(2R\sin\left(\frac{\pi}{N}j\right)\right)\left(1 - e^{i2\pi j/N}\right).
\end{aligned}
$$

as $1 - e^{i2\pi j/N} = 0$ when $j = N$. Stationary solutions can be found by considering the equilibrium when $\frac{dX_j}{dt} = 0$. This is $\frac{1}{N}\sum_{j=1}^{N-1} f\left(2R\sin\left(\frac{\pi}{N}j\right)\right)\left(1 - e^{i2\pi j/N}\right) = 0$ in the case of ring solutions.

To find a more convenient formula to compute R, we may take $N \to \infty$ and let $\theta = \frac{\pi j}{N}$. We notice that θ ranges from 0 to $\pi - \frac{\pi}{N}$ such that

$$
\begin{aligned}
1 - e^{\frac{2\pi j}{N}i} &= 1 - e^{2\theta i} \\
&= 1 - \cos(2\theta) - i\sin 2\theta \\
&= 2\sin^2(\theta) - i\sin(2\theta).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
0 &= \lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N-1} f\left(2R\sin\left(\frac{\pi}{N}j\right)\right)\left(1 - e^{i2\pi j/N}\right) \\
&= \frac{1}{\pi}\int_0^{\pi} f(2R\sin(\theta))\, 2\sin^2(\theta)\, d\theta + \frac{1}{\pi}\int_0^{\pi} f(2R\sin(\theta))\, i\sin(2\theta)\, d\theta \\
&= \int_0^{\frac{\pi}{2}} f(2R\sin(\theta))\sin^2(\theta)\, d\theta + \int_0^{\pi/2} f(2R\sin(\theta))\, i\sin(2\theta)\, d\theta + \int_{\pi/2}^{\pi} f(2R\sin(\theta))\, i\sin(2\theta)\, d\theta \\
&= \int_0^{\frac{\pi}{2}} f(2R\sin(\theta))\sin^2(\theta)\, d\theta + \int_0^{\pi/2} f(2R\sin(\theta))\, i\sin(2\theta)\, d\theta - \int_0^{\pi/2} f(2R\sin(\theta))\, i\sin(2\theta)\, d\theta \\
&= \int_0^{\frac{\pi}{2}} f(2R\sin(\theta))\sin^2(\theta)\, d\theta. \qquad (3)
\end{aligned}
$$

Equation (3) characterizes the stationary ring solutions since it provides a way to compute R.

2 Analysis of the stability of ring solutions

Stability of ring solutions plays an important role when we are analysing the interactive particle systems, it tells us whether the ring solutions actually appear like a ring or have some other disconnected form. The ring solutions are stable if any perturbation of the particles does not lead to an incoherent state of the ring. Otherwise it is unstable. In this section we determine the stability of the ring solutions by analysing the corresponding perturbed system.

In the previous section we have seen that solutions of the form

$$X_j = Re^{2\pi ij/N}, \quad j = 1, \ldots, N, \tag{4}$$

are stationary solutions for certain R given by (3). In this section, we want to analyse the local stability of such solutions. To that end, we consider perturbations of the form

$$\tilde{X}_j = Re^{2\pi ij/N}(1 + h_j), \quad j = 1, \ldots, N, \tag{5}$$

where h_j is a function of t with $\|h_j\| \ll 1$.

First, we evaluate $\frac{d\tilde{X}_j}{dt}$ by substituting $\tilde{X}_j - \tilde{X}_k$ and $\|\tilde{X}_j - \tilde{X}_k\|$ into equation (2). To do that, consider

$$\begin{aligned}\tilde{X}_j &= Re^{2\pi ij/N}(1 + h_j),\\ \tilde{X}_k &= Re^{2\pi ik/N}(1 + h_k),\end{aligned}$$

and let $\phi = 2\pi(k - j)/N$. The difference between $\tilde{X}_j$ and $\tilde{X}_k$ is

$$\begin{aligned}\tilde{X}_j - \tilde{X}_k &= Re^{2\pi ij/N}\left(1 + h_j\right) - Re^{2\pi ik/N}\left(1 + h_k\right)\\ &= Re^{2\pi ij/N}\left(1 - e^{i\phi} + h_j - e^{i\phi}h_k\right).\end{aligned}$$

Next, we compute the modulus of this difference by using the fact that for any complex number x, $\|x\| = \sqrt{x\overline{x}}$. So we have

$$\begin{aligned}\|\tilde{X}_j - \tilde{X}_k\| &= \sqrt{\left(\tilde{X}_j - \tilde{X}_k\right)\overline{\left(\tilde{X}_j - \tilde{X}_k\right)}}\\ &= \sqrt{R^2\left(1 - e^{i\phi} + h_j - e^{i\phi}h_k\right)\left(1 - e^{-i\phi} + \overline{h_j} - e^{-i\phi}\overline{h_k}\right)}\\ &= R\sqrt{1 - e^{-i\phi} + \overline{h_j} - e^{-i\phi}h_k - e^{i\phi} + 1 - e^{i\phi}\overline{h_j} + \overline{h_k} + h_j - e^{-i\phi}h_j}\\ &\qquad \overline{+h_j\overline{h_j} - h_je^{-i\phi}\overline{h_k} - e^{i\phi}\overline{h_k} + h_k - e^{-i\phi}h_k\overline{h_j} + h_k\overline{h_k}}.\end{aligned}$$

Removing all the terms with higher orders than 1 yields

$$\begin{aligned}\|\tilde{X}_j - \tilde{X}_k\| &\sim R\sqrt{2 - e^{-i\phi} + \overline{h_j} - e^{-i\phi}\overline{h_k} - e^{i\phi} - e^{i\phi} - e^{i\phi}\overline{h_j} + \overline{h_k} + h_j - e^{-i\phi}h_j - e^{i\phi}h_k + h_k}\\ &= R\sqrt{2 - e^{-i\phi} - e^{i\phi} + \overline{h_j}\left(1 - e^{i\phi}\right) + \overline{h_k}\left(1 - e^{-i\phi}\right) + h_j\left(1 - e^{-i\phi}\right) + h_k\left(1 - e^{i\phi}\right)}\\ &= R\sqrt{2 - 2\cos\left(\phi\right) + \left[\left(\overline{h_k} + h_k\right)\left(1 - e^{-i\phi}\right) + \left(h_k + \overline{h_j}\right)\left(1 - e^{i\phi}\right)\right]}\\ &= R\sqrt{4\sin^2\left(\phi/2\right) + \left[\left(\overline{h_k} + h_k\right)\left(1 - e^{-i\phi}\right) + \left(h_k + \overline{h_j}\right)\left(1 - e^{i\phi}\right)\right]}\\ &= 2R\left|\sin\phi/2\right|\sqrt{1 + \frac{\left(\overline{h_k} + h_k\right)\left(1 - e^{-i\phi}\right) + \left(h_k + \overline{h_j}\right)\left(1 - e^{i\phi}\right)}{4\sin^2\left(\phi/2\right)}}.\end{aligned}$$

Using the fact that the Taylor polynomial of degree 1 of $\sqrt{1+x}$ around $x = 0$ equals $1 + \frac{1}{2}x$, we obtain the following:

$$
\begin{aligned}
&2R\left|\sin\phi/2\right|\sqrt{1+\frac{\left(\overline{h_k}+h_k\right)\left(1-e^{-i\phi}\right)+\left(h_k+\overline{h_j}\right)\left(1-e^{i\phi}\right)}{4\sin^2\left(\phi/2\right)}}\\
=&2R\left|\sin\phi/2\right|\left(1+\frac{1}{2}\frac{\left(\overline{h_k}+h_k\right)\left(1-e^{-i\phi}\right)+\left(h_k+\overline{h_j}\right)\left(1-e^{i\phi}\right)}{4\sin^2\left(\phi/2\right)}\right)\\
=&2R\left|\sin\phi/2\right|+\frac{R}{4\left|\sin\phi/2\right|}\left[\left(\overline{h_k}+h_k\right)\left(1-e^{-i\phi}\right)+\left(h_k+\overline{h_j}\right)\left(1-e^{i\phi}\right)\right].
\end{aligned}
$$

For simplicity, we let $2R\left|\sin\phi/2\right|=\alpha$ and $\frac{R}{4\left|\sin\phi/2\right|}\left[\left(\overline{h_k}+h_k\right)\left(1-e^{-i\phi}\right)+\left(h_k+\overline{h_j}\right)\left(1-e^{i\phi}\right)\right]=\beta$. Substituting (5) into (2) leads to

$$
\begin{aligned}
\frac{d\tilde{X}_j}{dt}&=\frac{1}{N}\sum_k f\left(\|\tilde{X}_j-\tilde{X}_k\|\right)\left(\tilde{X}_j-\tilde{X}_k\right)\\
&=\frac{1}{N}Re^{2\pi ij/N}\sum_k f\left(\alpha+\beta\right)\left(1-e^{i\phi}+h_j-e^{i\phi}h_k\right).
\end{aligned}
$$

On the other hand, $\frac{d\tilde{X}_j}{dt}=Re^{2\pi ij/N}\frac{dh_j}{dt}$. Then

$$
\begin{aligned}
\frac{dh_j}{dt}&=\frac{1}{Re^{2\pi ij/N}}\left(\frac{1}{N}Re^{2\pi ij/N}\sum_k f\left(\alpha+\beta\right)\left(1-e^{i\phi}+h_j-e^{i\phi}h_k\right)\right)\\
&=\frac{1}{N}\sum_k f\left(\alpha+\beta\right)\left(1-e^{i\phi}+h_j-e^{i\phi}h_k\right)\\
&\sim\frac{1}{N}\sum_k \left(f\left(\alpha\right)+\beta f'\left(\alpha\right)\right)\left(1-e^{i\phi}+h_j-e^{i\phi}h_k\right),
\end{aligned}
$$

by using the first-order Taylor series of $f(\alpha+\beta)$ around α.

Hence, we have the following expression after expanding the brackets:

$$
\begin{aligned}
\frac{dh_j}{dt}\sim&\frac{1}{N}\left(\sum_k\left(f\left(\alpha\right)+\beta f'\left(\alpha\right)\right)\left(h_j-e^{i\phi}h_k\right)+\sum_k\left(f\left(\alpha\right)+\beta f'\left(\alpha\right)\right)\left(1-e^{i\phi}\right)\right)\\
=&\frac{1}{N}\left(\sum_k f\left(\alpha\right)\left(h_j-e^{i\phi}h_k\right)+\sum_k\beta f'\left(\alpha\right)\left(1-e^{i\phi}\right)+\sum_k\beta f'\left(\alpha\right)\left(h_j-e^{i\phi}h_k\right)+\sum_k f\left(\alpha\right)\left(1-e^{i\phi}\right)\right)\\
=&\frac{1}{N}\left(\sum_k f\left(\alpha\right)\left(h_j-e^{i\phi}h_k\right)+\sum_k\beta f'\left(\alpha\right)\left(1-e^{i\phi}\right)\right)\\
&+\frac{1}{N}\left(\sum_k f'\left(\alpha\right)\left(\beta\left(h_j-e^{i\phi}h_k\right)\right)+\sum_k f\left(\alpha\right)\left(1-e^{i\phi}\right)\right).
\end{aligned}
$$

Note that $\beta=O\left(h_k\right)$ and $\left(h_j-e^{i\phi}h_k\right)=O\left(h_k\right)$, but $f'\left(\alpha\right)\left(\beta\left(h_j-e^{i\phi}h_k\right)\right)=O\left(h_k^2\right)$. Since $\|h_k\|<<1$, we neglect this last term. In addition, $\sum_k f\left(\alpha\right)\left(1-e^{i\phi}\right)=0$ as it satisfies the equilibrium condition.

Thus, we have

$$
\frac{dh_j}{dt}=\frac{1}{N}\left(\sum_k f\left(\alpha\right)\left(h_j-e^{i\phi}h_k\right)+\sum_k\beta f'\left(\alpha\right)\left(1-e^{i\phi}\right)\right).
$$

If we substitute back $\alpha=2R\left|\sin\left(\phi/2\right)\right|$ and $\beta=\frac{R}{4\left|\sin(\phi/2)\right|}\left[\left(\overline{h_k}+h_k\right)\left(1-e^{-i\phi}\right)+\left(h_k+\overline{h_j}\right)\left(1-e^{i\phi}\right)\right]$ into the equation and, we obtain

$$
\begin{aligned}
\frac{dh_j}{dt} =& \frac{1}{N}\sum_k f'\left(2R\left|\sin\left(\phi/2\right)\right)\right| \frac{R}{4\left|\sin\left(\phi/2\right)\right|}\left[\left(1-e^{i\phi}\right)^2\left(h_k+\overline{h_j}\right)+\left(1-e^{i\phi}\right)\left(1-e^{-i\phi}\right)\left(\overline{h_k}+h_j\right)\right] \\
&+\frac{1}{N}\sum_k f\left(2R|\sin\left(\phi/2\right)|\right)\left(h_j-e^{i\phi}h_k\right) \\
=& \frac{1}{N}\sum_k f'\left(2R|\sin\left(\phi/2\right)|\right)\frac{R}{4\left|\sin\left(\phi/2\right)\right|}\left[-4\sin^2\left(\phi/2\right)e^{i\phi}\left(h_k+\overline{h_j}\right)+4\sin^2\left(\phi/2\right)\left(\overline{h_k}+h_j\right)\right] \\
&+\frac{1}{N}\sum_k f\left(2R|\sin\left(\phi/2\right)|\right)\left(h_j-e^{i\phi}h_k\right) \\
=& \frac{1}{N}\sum_k f'\left(2R|\sin\left(\phi/2\right)|\right)\frac{R}{4\left|\sin\left(\phi/2\right)\right|}\left[4\sin^2\left(\phi/2\right)\left(h_j-e^{i\phi}h_k\right)+4\sin^2\left(\phi/2\right)\left(\overline{h_k}-e^{i\phi}\overline{h_j}\right)\right] \\
&+\frac{1}{N}\sum_k f\left(2R|\sin\left(\phi/2\right)|\right)\left(h_j-e^{i\phi}h_k\right) \\
=& \frac{1}{N}R\sum_k f'\left(2R|\sin\left(\phi/2\right)|\right)\left|\sin\left(\phi/2\right)\right|\left(h_j-e^{i\phi}h_k\right) \\
&+\frac{1}{N}\sum_k f\left(2R|\sin\left(\phi/2\right)|\right)\left(h_j-e^{i\phi}h_k\right) \\
&+\frac{1}{N}R\sum_k f'\left(2R|\sin\left(\phi/2\right)|\right)\left|\sin\left(\phi/2\right)\right|\left(\overline{h_k}-e^{i\phi}\overline{h_j}\right) \\
=& \left(\frac{1}{N}R\sum_k f'\left(2R|\sin\left(\phi/2\right)|\right)\left|\sin\left(\phi/2\right)\right|+\frac{1}{N}\sum_k f\left(2R|\sin\left(\phi/2\right)|\right)\right)\left(h_j-e^{i\phi}h_k\right) \\
&+\frac{1}{N}R\sum_k f'\left(2R|\sin\left(\phi/2\right)|\right)\left|\sin\left(\phi/2\right)\right|\left(\overline{h_k}-e^{i\phi}\overline{h_j}\right) \\
=& \sum_k G_1\left(\phi/2\right)\left(h_j-e^{i\phi}h_k\right)+G_2\left(\phi/2\right)\left(\overline{h_k}-e^{i\phi}\overline{h_j}\right).
\end{aligned}
\tag{6}
$$

with

$$
\begin{aligned}
G_1(\phi) &= \frac{1}{N}R\, f'\left(2R|\sin\left(\phi\right)|\right)\left|\sin\left(\phi\right)\right|+\frac{1}{N}f\left(2R|\sin\left(\phi\right)|\right); \\
G_2(\phi) &= \frac{1}{N}Rf'\left(2R|\sin\left(\phi\right)|\right)\left|\sin\left(\phi\right)\right|\left(\overline{h_k}-e^{i\phi}\overline{h_j}\right).
\end{aligned}
$$

In the computation above we may use the fact that $\left(1-e^{i\phi}\right)^2=-4\sin^2\left(\frac{\phi}{2}\right)e^{i\phi}$ and $\left(1-e^{i\phi}\right)\left(1-e^{-i\phi}\right)=4\sin^2\left(\frac{\phi}{2}\right)$, these are derived in Appendix A.

Substituting the ansatz $h_j=\xi_+(t)e^{im\theta}+\xi_-(t)e^{-im\theta}, \theta=2\pi j/N, m\in\mathbb{N}$ into the equation (6) leads to

$$
\begin{aligned}
&\xi_+^{'}e^{im\theta}+\xi_-^{'}e^{-im\theta} \\
=& \sum_{k,k\neq j} G_1\left(\theta/2\right)\left(\xi_+e^{im\theta}+\xi_-e^{-im\theta}-\xi_+e^{im\theta}e^{i(m+1)\phi}-\xi_-e^{-im\theta}e^{i(-m+1)\phi}\right) \\
&+\sum_{k,k\neq j} G_2\left(\theta/2\right)\left(\overline{\xi}_+e^{im\theta}e^{im\phi}+\overline{\xi}_-e^{im\theta}e^{im\phi}-\overline{\xi}_+e^{im\theta}e^{i\phi}-\overline{\xi}_-e^{im\theta}e^{i\phi}\right) \\
=& \xi_+\sum_{k,k\neq j} G_1\left(\phi/2\right)\left(e^{im\theta}-e^{im\theta}e^{i(m+1)\phi}\right)+\overline{\xi}_-\sum_{k,k\neq j} G_2\left(\phi/2\right)\left(e^{im\theta}e^{im\phi}-e^{im\theta}e^{i\phi}\right) \\
&+\xi_-\sum_{k,k\neq j} G_1\left(\phi/2\right)\left(e^{-im\theta}-e^{-im\theta}e^{i(-m+1)\phi}\right)+\overline{\xi}_+\sum_{k,k\neq j} G_2\left(\phi/2\right)\left(e^{-im\theta}e^{-im\phi}-e^{-im\theta}e^{i\phi}\right).
\end{aligned}
$$

After separating the terms consisting of $e^{im\theta}$ and $e^{-im\theta}$ respectively, we obtain

$$\xi_+' e^{im\theta} = \xi_+ \sum_{k,k\neq j} G_1(\phi/2)\left(e^{im\theta} - e^{im\theta}e^{i(m+1)\phi}\right) + \overline{\xi}_- \sum_{k,k\neq j} G_2(\phi/2)\left(e^{im\theta}e^{im\phi} - e^{im\theta}e^{i\phi}\right)$$

$$\xi_-' e^{-im\theta} = \xi_- \sum_{k,k\neq j} G_1(\phi/2)\left(e^{-im\theta} - e^{-im\theta}e^{i(-m+1)\phi}\right) + \overline{\xi}_+ \sum_{k,k\neq j} G_2(\phi/2)\left(e^{-im\theta}e^{-im\phi} - e^{-im\theta}e^{i\phi}\right).$$

And this implies

$$\begin{aligned}\xi_+' &= \xi_+ \sum_{k,k\neq j} G_1(\phi/2)\left(1 - e^{i(m+1)\phi}\right) + \overline{\xi}_- \sum_{k,k\neq j} G_2(\phi/2)\left(e^{im\phi} - e^{i\phi}\right)\\ &= \xi_+ I_1(m) + \overline{\xi}_- I_2(m) \qquad (7)\\ \xi_-' &= \xi_- \sum_{k,k\neq j} G_1(\phi/2)\left(1 - e^{i(-m+1)\phi}\right) + \overline{\xi}_+ \sum_{k,k\neq j} G_2(\phi/2)\left(e^{-im\phi} - e^{i\phi}\right)\\ &= \xi_- I_1(-m) + \overline{\xi}_+ I_2(-m), \qquad (8)\end{aligned}$$

where we have divided both of the left and the right hand side of (7) and (8) by $e^{im\theta}$ and $e^{-im\theta}$ respectively, and we defined $I_1(m) := \sum_{k,k\neq j} G_1(\phi/2)\left(1 - e^{i(m+1)\phi}\right)$ and $I_2(m) := \sum_{k,k\neq j} G_2(\phi/2)\left(e^{im\phi} - e^{i\phi}\right)$ for convenience.

We obtain that

$$\overline{\xi}_-' = \xi_- I_1(-m) + \overline{\xi}_+ I_2(-m) \qquad (9)$$

by applying the conjugate of (8) in both left and right hand sides. After that, we may write equations (7) and (9) in matrix form. That is

$$\begin{pmatrix}\xi_+' \\ \overline{\xi}_-'\end{pmatrix} = M \begin{pmatrix}\xi_+ \\ \overline{\xi}_-,\end{pmatrix} \qquad (10)$$

where $M = \begin{pmatrix} I_1(m) & I_2(m) \\ I_2(m) & I_1(-m)\end{pmatrix}$.

If we let $b_\pm$ be any real constant and substitute $\xi_\pm = b_\pm e^{\lambda t}$ into equation (10), we see that λ is an eigenvalue of the matrix $M = \begin{pmatrix} I_1(m) & I_2(m) \\ I_2(m) & I_1(-m)\end{pmatrix}$. If $\lambda \leq 0$, then $e^{\lambda t} \to 0$ as $t \to \infty$, so that $\xi_\pm \to 0$ as t becomes large. If this is the case, then the system is stable, it is unstable otherwise (when $\lambda \geq 0$). Since m is an arbitrary integer, λ must be non-positive for any given integer m to ensure the stability.

Moreover, we can rewrite $I_1(m)$ and $I_2(m)$ into simpler forms. Note that $k - j$ varies from 1 to N. We can set $j = 0$ and let k vary from 1 to N without loss of generality. As a result, the angle ϕ varies from $\frac{2\pi}{N}$ to 2π. Then the expression $I_1(m)$ can be rewritten as follows

$$
\begin{aligned}
I_1(m) &= \sum_{k,k\neq j} G_1(\phi/2)\left(1-e^{i(m+1)\phi}\right)\\
&= \sum_{k=1}^{N} G_1\left(\frac{\pi k}{N}\right)\left(1-e^{i(m+1)\frac{2\pi k}{N}}\right)\\
&= \sum_{k=1}^{N} G_1\left(\frac{\pi k}{N}\right)\left(1-\cos\left(\frac{(m+1)\,2\pi k}{N}\right)-i\sin\left(\frac{(m+1)\,2\pi k}{N}\right)\right)\\
&= \sum_{k=1}^{N} G_1\left(\frac{\pi k}{N}\right)\left(1-\cos\left(\frac{(m+1)\,2\pi k}{N}\right)\right)-\sum_{k=1}^{N} G_1\left(\frac{\pi k}{N}\right)i\sin\left(\frac{(m+1)\,2\pi k}{N}\right)\\
&= 2\sum_{k=1}^{N} G_1\left(\frac{\pi k}{N}\right)\sin^2\left(\frac{(m+1)\,\pi k}{N}\right)\\
&\quad-\left(\sum_{k=1}^{N/2} G_1\left(\frac{\pi k}{N}\right)i\sin\left(\frac{(m+1)\,2\pi k}{N}\right)+\sum_{k=N/2}^{N} G_1\left(\frac{\pi k}{N}\right)i\sin\left(\frac{(m+1)\,2\pi k}{N}\right)\right)\\
&= 2\sum_{k=1}^{N} G_1\left(\frac{\pi k}{N}\right)\sin^2\left(\frac{(m+1)\,\pi k}{N}\right)\\
&\quad-\left(\sum_{k=1}^{N/2} G_1\left(\frac{\pi k}{N}\right)i\sin\left(\frac{(m+1)\,2\pi k}{N}\right)+\sum_{k=N/2}^{N} G_1\left(\frac{\pi k}{N}\right)i\sin\left(\frac{(m+1)\,2\pi k}{N}\right)\right)\\
&= 2\sum_{k=1}^{N} G_1\left(\frac{\pi k}{N}\right)\sin^2\left(\frac{(m+1)\,\pi k}{N}\right)\\
&= 4\sum_{k=1}^{N/2} G_1\left(\frac{\pi k}{N}\right)\sin^2\left(\frac{(m+1)\,\pi k}{N}\right),
\end{aligned}
\tag{11}
$$

as the angle between 0 to π is just the inverse of the angle between π to 2π.

We use a similar computation for $I_2(m)$:

$$
\begin{aligned}
I_2(m) &= \sum_{k,k\neq j} G_2(\phi/2)\left(e^{im\phi}-e^{i\phi}\right)\\
&= \sum_{k=1}^{N} G_2(\phi/2)\left(\cos(m\phi)+i\sin(m\phi)-\cos(\phi)-i\sin(\phi)\right)\\
&= \sum_{k,k\neq j} G_2(\phi/2)\left(\cos(m\phi)-\cos(\phi)\right)+\sum_{k,k\neq j} G_2(\phi/2)\left(i\sin(m\phi)-i\sin(\phi)\right).
\end{aligned}
$$

Note that

$$
\begin{aligned}
G_2(\phi/2) &= Rf'\left(2R\left|\sin\left(\frac{\pi k}{N}\right)\right|\right)\left|\sin\left(\frac{\pi k}{N}\right)\right|,\\
&= Rf'\left(2R\left|\sin\left(\frac{\pi(N-k)}{N}\right)\right|\right)\left|\sin\left(\frac{\pi(N-k)}{N}\right)\right|,
\end{aligned}
\tag{12}
$$

and

$$
\begin{aligned}
&i\sin(m\phi)-i\sin(\phi)\\
=&i\sin\left(m\frac{2k\pi}{N}\right)-i\sin\left(\frac{2k\pi}{N}\right)\\
=&-\left(i\sin\left(m\frac{2(N-k)\pi}{N}\right)-i\sin\left(\frac{2(N-k)\pi}{N}\right)\right)
\end{aligned}
\tag{13}
$$

for k ranges from 1 to N by using the symmetry of the sine function. As a result, equations (12) and (13) together imply

$$\begin{aligned}&\sum_{k=1}^{N/2} G_2(\phi/2)(i\sin(m\phi) - i\sin(\phi))\\&= -\sum_{k=N/2+1}^{N} G_2(\phi/2)(i\sin(m\phi) - i\sin(\phi)).\end{aligned}$$

Adding up with all possible $k, k \neq j$ makes this summation $\sum_{k,k\neq j} G_2(\phi/2)(i\sin(m\phi) - i\sin(\phi))$ equal 0. Now we have

$$\begin{aligned}I_2(m) &= \sum_{k,k\neq j} G_2(\phi/2)\left(1 - 2\sin^2(m\phi/2) - 1 + 2\sin^2(\phi/2)\right)\\&= 2\sum_{k,k\neq j} G_2(\phi/2)\left(\sin^2(\phi/2) - \sin^2(m\phi/2)\right)\\&= 2\sum_{k=1}^{N} G_2\left(\frac{\pi k}{N}\right)\left(\sin^2\left(\frac{\pi k}{N}\right) - \sin^2\left(\frac{m\pi k}{N}\right)\right)\\&= 4\sum_{k=1}^{N/2} G_2\left(\frac{\pi k}{N}\right)\left(\sin^2\left(\frac{\pi k}{N}\right) - \sin^2\left(\frac{m\pi k}{N}\right)\right).\end{aligned} \tag{14}$$

Consider $I_1(m)$ given by (11). Note that the summands in $I_1(m)$ becomes neglectfully small when k is close to N. Therefore, for N large we can approximate $I_1(m)$ with its continuum limit by letting $N \to \infty$. If we let $\theta = \frac{\pi k}{N}$, we see that θ ranges from $\frac{\pi}{N}$ to $\frac{\pi}{2}$. Therefore, $\frac{\pi}{N}$ approaches 0 and $I_1(m)$ becomes

$$\begin{aligned}&\lim_{N\to\infty} I_1(m)\\&=4\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N/2}\left(Rf'\left(2R\left|\sin\left(\frac{\pi k}{N}\right)\right|\right)\left|\sin\left(\frac{\pi k}{N}\right)\right| + f\left(2R\left|\sin\left(\frac{\pi k}{N}\right)\right|\right)\right)\sin^2\left((m+1)\pi\frac{k}{N}\right)\\&=\frac{4}{\pi}\int_0^{\pi/2}\left(Rf'(2R|\sin(\theta)|)|\sin(\theta)| + f(2R|\sin(\theta)|)\right)\sin^2((m+1)\theta)\,d\theta\\&=\frac{4}{\pi}\int_0^{\pi/2}\left(Rf'(2R\sin(\theta))\sin(\theta) + f(2R\sin(\theta))\right)\sin^2((m+1)\theta)\,d\theta,\end{aligned}$$

as $|\sin(\theta)| = \sin(\theta)$ for $0 < \theta < \pi$.

Similarly, consider $I_2(m)$ given by (14), if we let $\theta = \frac{\pi k}{N}$, we observe that θ ranges from $\frac{\pi}{N}$ to $\frac{\pi}{2}$. Therefore, if $N \to \infty$, $\frac{\pi}{N}$ approaches 0 and $I_2(m)$ becomes

$$\begin{aligned}&\lim_{N\to\infty} I_2(m)\\&=\lim_{N\to\infty} 4\sum_{k=1}^{N/2} G_2\left(\frac{\pi k}{N}\right)\left(\sin^2\left(\frac{\pi k}{N}\right) - \sin^2\left(\frac{m\pi k}{N}\right)\right)\\&=\frac{4}{\pi}\int_0^{\pi/2} Rf'(2R|\sin(\theta)|)|\sin(\theta)|\left(\sin^2(\theta) - \sin^2(m\theta)\right)d\theta\\&=\frac{4}{\pi}\int_0^{\pi/2}\left(Rf'(2R\sin(\theta))\sin(\theta)\right)\left(\sin^2(\theta) - \sin^2(m\theta)\right)d\theta\end{aligned}$$

as $|\sin(\theta)| = \sin(\theta)$ for $0 < \theta < \pi$.

We summarize the results above into the following theorem.

Theorem 1 ([2], Theorem 2.1)**.** *In the continuum limit $N \to \infty$, consider the ring equilibrium of radius R given by equation (3) for the flow (2). Suppose that $f(r)$ is piece-wise C^1 for $r \geq 0$. Define*

$$I_1(m) := \frac{4}{\pi} \int_0^{\pi/2} \left(Rf'\left(2R\sin(\theta)\right)\sin(\theta) + f\left(2R\sin(\theta)\right)\right)\sin^2\left((m+1)\theta\right) d\theta$$
$$I_2(m) := \frac{4}{\pi} \int_0^{\pi/2} \left(Rf'\left(2R\sin(\theta)\right)\sin(\theta)\right)\left(\sin^2(\theta) - \sin^2(m\theta)\right) d\theta$$
$$M(m) := \begin{pmatrix} I_1(m) & I_2(m) \\ I_2(m) & I_1(-m) \end{pmatrix}.$$

If $\lambda \leq 0$ for all eigenvalues λ of $M(m)$ for all $m \in \mathbb{N}$ then the ring equilibrium is linearly stable. It is unstable otherwise. For finite N, the ring is stable if $\lambda \leq 0$ for all eigenvalues λ of $M(m)$ for all $m = 1, 2, \ldots N$, but with I_1, I_2 as given by equation (11) and equation (14).

2.1 High wave-number stability

We have seen above that the ring equilibrium is linearly stable if $\lambda \leq 0$ for all eigenvalues λ of $M(m)$ for all $m \in \mathbb{N}$, which means it must hold for large m as well. In this subsection we examine the stability of the ring equilibria when m is large. A ring is called *short-wave* stable if the eigenvalues of $M(m)$ corresponding to sufficiently large m have negative real parts and is called *short-wave* unstable otherwise (see [2]). Omitting the computational details, the following theorem can be derived:

Theorem 2 ([2], Theorem 3.1)**.** *Suppose that f(r) admits a generalised power series expansion of the form*

$$f(s) = a_0 s^{p_0} + a_1 s^{p_1} + \ldots, p_0 < p_1 < \ldots. \tag{15}$$

Moreover, suppose that $p_0 > -3, a_0 > 0$, and all constants $a_j, j = 1, 2, \ldots$ are non-zero. Let p_l be the smallest power which is not even. Then the following conditions are sufficient for the ring to be short-wave stable:

$$p_0 > -1;$$
$$\int_0^{\pi/2} (Rf'(2R\sin\theta)\sin\theta + f(2R\sin\theta))d\theta < 0; \tag{16}$$
$$\begin{aligned} &\textit{either } a_l > 0 \textit{ and } p_l \in (-1, 0) \cup (1, 2) \cup (4, 6) \ldots \\ &\textit{or } a_l < 0 \textit{ and } p_l \in (0, 1) \cup (2, 4) \cup (6, 8) \ldots. \end{aligned} \tag{17}$$

The ring is short-wave unstable if either $p_0 \leq -1$ or the inequality in either (16) or (17) is reversed.

Remark. *According to [2], the necessary condition for stability is that trace$(M(m)) < 0$ as $m \to \infty$, which is equivalent to (16). The sufficient condition is that det$(M(m)) > 0$ as $m \to \infty$.*

Therefore, for $f(r)$ that admits a generalised power series expansion of the form (15), the intersection of the regions corresponding to all $m \in \mathbb{N}$ in which all of the powers $p_0, p_1, \ldots$ satisfy the conditions elaborated in Theorem 1 and Theorem 2 is the region where the ring equilibria are stable. We present an example below.

2.1.1 Example with power-law potentials

Consider interaction forces of the form $F(r) = r^p - ar^q$ with $0 \leq p < q$. Fix $R = 1/2$ for convenience and substitute $f(r) = F(r)/r = r^{p-1} - ar^{q-1}$ into equation (3). Under these conditions $a = \frac{\Gamma(1+p/2)\Gamma(3/2+q/2)}{\Gamma(3/2+p/2)\Gamma(1+q/2)}$ (see page 963 of [2]). Moreover, if we substitute $f(r) = r^{p-1} - ar^{q-1}$ into $I_1(m), I_2(m), I_1(-m)$, we are able to find the stability boundaries for this kind of interaction forces, the stability boundaries for each mode m can be plotted as curves. Using the help of Maple we show these curves in Figure 1.

3 Visualisation of Stationary Solutions

This section presents visualisation of stationary solutions related to the stability region of the ring solutions. The numerical method that is used to obtain the ring solutions will be introduced. For the visualisation, we simulate the motion of the particles with various parameters from an interaction potential, this gives us an easy and direct way to see whether the stationary solutions are stable or not. Specifically we simulate the system

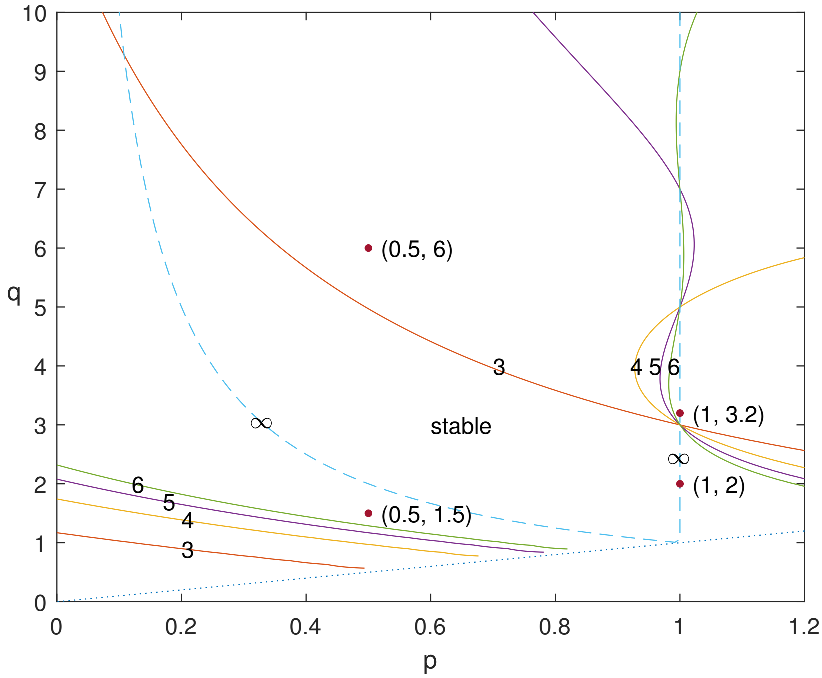

Figure 1: Stability region of a ring solution for the force law $F(r) = r^p - ar^q$. Each of the curves with a number on it corresponds to instability boundaries with mode $m = 2, 3, 4, 5, 6,$, and ∞. The dotted line denotes the boundary $p = q$, above which we have long-range attraction and short-range repulsion. The red dots denote the positions of parameters p, q in $F(r) = r^p - ar^q$ that corresponds to four different force laws in Figure 3.

described in (2) with interaction force according to power-law force, and we are able to confirm the results presented in the previous sections.

In order to find the stationary solution, it is natural to seek for the state where the total amount of energy (1) is minimized. We use here a *steepest descent method*. This method is a natural approach as it can be easily applied thanks to the gradient flow structure of our system. An advantage of this method is that although it converges only linearly to the solution, it usually converges even for poor initial conditions [5]. We use this method throughout the simulation process as we randomly generate the initial conditions. This method determines a local minimum for a multi-variable function of the form $g : \mathbb{R}^n \to \mathbb{R}$. Since we assume that the function P is one time continuously differentiable and since the energy we want to minimize is $E : \mathbb{R}^n \to \mathbb{R}$, the criteria of using this method is satisfied.

The method is as follows:

1. Evaluate the energy E_0 at the initial position $x^{(0)} = \left(x_1^{(0)}, ..., x_n^{(0)}\right)$.
2. Evaluate -$\nabla_{X_j} E$, this is given in equation (2).
3. Move an appropriate amount in the direction of -$\nabla_{X_j} E$ to obtain $x^{(1)}$ with $E_1 < E_0$.
4. Repeat steps 1 to 3.

The algorithm for finding the stationary solutions of the interactive particles is found in Algorithm 1.

Algorithm 1 Steepest descent method for the aggregation equation

Require: $N > 0$ number of particles
Require: $S_{\max} > 0$ integer (total number of loops to run).

```
dx ← 0.01 initial time step
TOL ← 1e − 8 (tolerance for the change in energy between time steps)
steps ← 0 (counter variable for loop steps)
pos_0 ← X_1^0, X_2^0, ... X_N^0 (the initial position of the particles)
E_0 ← E(pos_0) (energy of the initial state)
loop
  steps ← steps + 1
  for i ← 1; i ≤ N; i ← i + 1 do
    X_i^1 ← X_i^0 + dx Σ_{j=1, j≠i}^N f(‖X_i^0 − X_j^0‖)(X_i^0 − X_j^0)
  end for
  pos_1 ← X_1^1, X_2^1, ..., X_N^1
  E_1 ← E(pos_1)
  if E_1 < E_0 then
    if |E_1 − E_0| < TOL then
      break
    else
      E_0 ← E_1
      pos_0 ← pos_1
      dx ← 2 × dx
    end if
  else
    dx ← 1/2 × dx
  end if
  if steps > S_max then
    print Maximum number of steps reached
    break
  end if
end loop
```

Using Algorithm 1 we are able to find the stationary solution of the particle system with interaction forces of the form $F(r) = \tanh[(1-r)a] + b$ with $0 < a; -\tanh(a) < b < 1$, and $F(r) = r^p - r^q$ with $0 \leq p < q$. Those are presented in Figure 2 and Figure 3 respectively. The ring solutions in Figure 3 correspond to the red dots in Figure 1. We observe that for $p = 1$, $q = 2$ the ring solutions are stable but for other p and q the ring solutions are unstable and they appear in different forms, this is in accordance with Figure 1 as the only point that lies in the stable-region is $(p, q) = (1, 2)$. In Figure 2 we see that most ring solutions with $b \leq -0.3$ are stable. In general, the higher b is, the more likely for the ring solutions to be unstable.

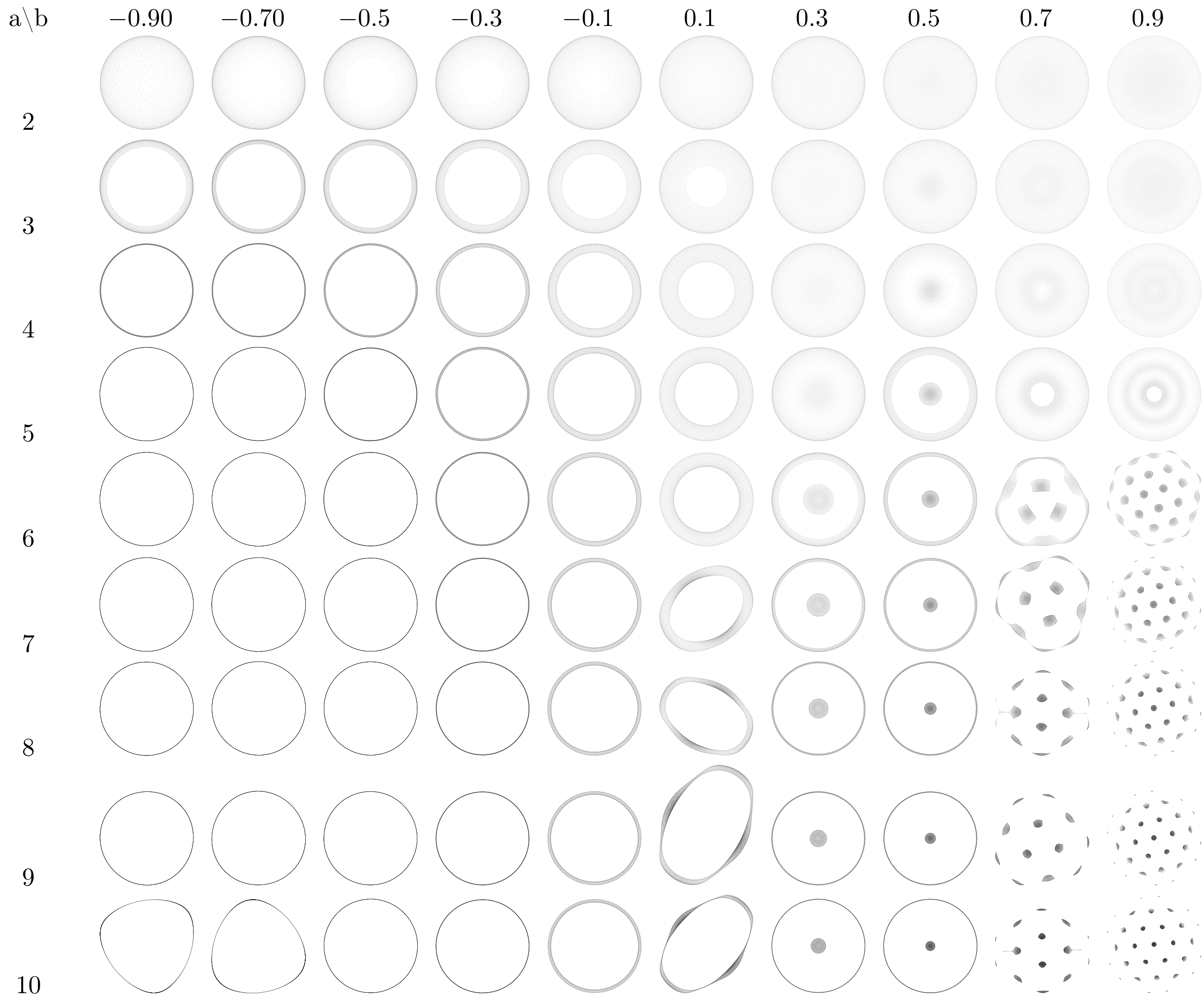

Figure 2: Positions of 5000 particles with force law $F(r) = \tanh[(1-r)a] + b$ with $0 < a$ and $-\tanh(a) < b < 1$.

A Useful Identities and computations

The following identities are used in Section 2:

$$\begin{aligned}
\left(1-e^{i\phi}\right)^2 &= 1 + e^{2\phi i} - 2e^{i\phi} \\
&= 1 + \cos(2\phi) + i\sin(2\phi) - 2\cos(\phi) - 2i\sin(\phi) \\
&= 1 + 2\cos^2(\phi) - 1 + 2i\sin(\phi)\cos(\phi) - 2\cos(\phi) - 2i\sin(\phi) \\
&= 2\cos(\phi)(\cos(\phi)-1) + 2i\sin(\phi)(\cos(\phi)-1) \\
&= 2(\cos(\phi) + i\sin(\phi))\left(-2\sin^2\left(\frac{\phi}{2}\right)\right) \\
&= -4\sin^2\left(\frac{\phi}{2}\right)e^{i\phi}
\end{aligned}$$

$$\begin{aligned}
\left(1-e^{i\phi}\right)\left(1-e^{-i\phi}\right) &= 1 - e^{-\phi i} - e^{i\phi} + 1 \\
&= 2 - e^{-i\phi} - e^{i\phi} \\
&= 2 - (\cos(\phi) - i\sin(\phi)) - (\cos(\phi) + i\sin(\phi)) \\
&= 2 - 2\cos(\phi) \\
&= 4\sin^2\left(\frac{\phi}{2}\right)
\end{aligned}$$

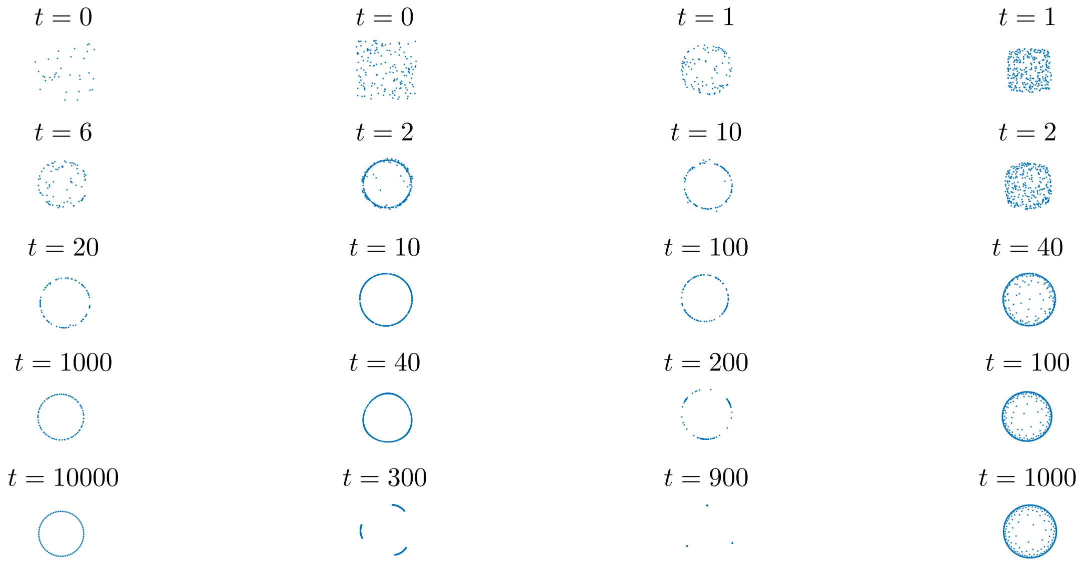

Figure 3: Positions of particles with force law $F(r) = r^p - r^q$ with $0 \leq p < q$. First column: $F(r) = r - r^2$, $N = 80$. Second column: $F(r) = r^{0.5} - r^6$, $N = 300$. Third column: $F(r) = r - r^{3.2}$, $N = 100$. Forth column: $F(r) = r^{0.5} - r^{1.5}$, $N = 300$. Reproduced from [10].

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