

Prepared for:

National Institute for Coastal and Marine
Management / RIKZ

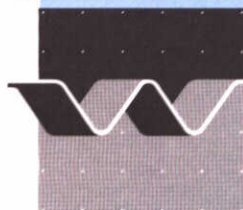
3D wave-current modelling

A model for secondary circulations

Report

June 1999

**THE KINEMATICS AND DYNAMICS
OF
WAVE-CURRENT INTERACTIONS**

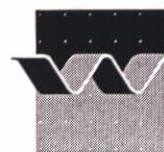


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3D wave-current modelling

A model for secondary circulations

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TITLE: 3D wave-current modelling; a model for secondary circulations.

ABSTRACT:

Surface waves propagating with a mean current can induce series of vortex pairs with horizontal axes. This phenomenon is called Langmuir circulation and it induces vertical transport of horizontal momentum as well as of dissolved or suspended matter in the top layer of lakes, reservoirs and seas. Langmuir circulation can be simulated numerically, see e.g. (Van Kester et al., 1996), but it demands significantly smaller grids than achievable in large-scale simulations of coastal seas. Therefore, Radder (1998) proposed a theoretical model for vertical momentum exchange, i.e. Reynold stresses, by Langmuir circulation. This model depends on mean-current and on surface-wave properties and the model is without free (calibration) coefficients. Radder's (1998) note is included as Appendix D.

The present report is concerned with a check the algebra as well as the suppositions of Radder's (1998) model. The main conclusion is that Radder derived his model correct, just minor alterations to his suppositions were considered. As a first step towards validation, this report recommends the addition of Radder's model to the code of Van Kester et al. (1996) and comparing the deformation of the mean current against simulations that resolve Langmuir circulation in laterally infinite fluid domains.

REFERENCES: Opdrachtbonnummer RIKZ/OS 995783

| VER. | ORIGINATOR | DATE | REMARKS | REVIEW | APPROVED BY |
|-------------------------|--------------------------------------|------------|--------------------------------|-------------------------------|--|
| | M.W. Dingemans <i>mw</i> | 28-06-1999 | | R.E. Uittenbogaard <i>REU</i> | T. Schilperoort <i>T.S.</i> A. Roelzema <i>AR</i> |
| | | | | | |
| | | | | | |
| PROJECT IDENTIFICATION: | | | | | |
| KEYWORDS: | | | | | |
| CONTENTS: | TEXT PAGES | 68 | TABLES | FIGURES | APPENDICES 4 |
| STATUS: | <input type="checkbox"/> PRELIMINARY | | <input type="checkbox"/> DRAFT | | <input checked="" type="checkbox"/> FINAL |

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Executive's summary

In this report the Craik-Leibovich equation is considered. The CL-equation is an equation for the mean current in which wave effects are accounted for, especially through the Stokes drift which results in a vortex force:

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + (\bar{\mathbf{u}} \cdot \text{grad}) \bar{\mathbf{u}} + \text{grad} \left(gz + \frac{\bar{p}}{\rho} + \left\langle \frac{1}{2} \tilde{\mathbf{u}} \cdot \tilde{\mathbf{u}} \right\rangle \right) = \bar{\mathbf{u}}^S \wedge \bar{\boldsymbol{\omega}} + \frac{1}{\rho} \text{div} \bar{\boldsymbol{\sigma}}',$$

with $\boldsymbol{\omega} = \text{curl} \bar{\mathbf{u}}$ and $\bar{\mathbf{u}}^S$ the Stokes drift and $\tilde{\mathbf{u}}$ is a contribution of the wave field. By means of an instability mechanism, this equation can describe the formation of Langmuir circulation cells. The spatial extent of these circulation cells is, in the direction perpendicular to the axis of the rolls, much smaller than the spatial scales of the current part itself, meaning that for a numerical solution a small mesh is needed in order to describe those cells. Radder (1998) devised a method to account for the effect of these cells on the current part, so that normal meshes for the flow computations can be used.

The purpose of this report is to check the algebra and the assumptions of Radder's approach. To that end the whole analysis is done anew and is written out in more detail than provided in Radder (1998). Our conclusion is that the analysis of Radder is correct.

First, the Craik-Leibovich model is considered and the principal assumptions are stated in Chapter 2. Because of an incorrect treatment in Van Kester et al. (1996), the Stokes drift itself is reconsidered in §2.2. It is shown that the part of the Stokes drift due to the current, is insignificant for both prototype and laboratory situations compared to the contribution due to the waves.

In Chapter 3 the instability mechanism is investigated. The mean current $\bar{\mathbf{u}}$ which figures in the CL-equation is written as the sum of a basic current \mathbf{U} plus a perturbation $\hat{\mathbf{u}}$. The same is done for the pressure. Radder (1998) used a constant eddy viscosity coefficient in his analysis. Because the eddy viscosity coefficient is a function of the velocity (it is certainly a function of the friction velocity), we also perturbed the eddy viscosity coefficient and wrote it as the sum of a mean part $\bar{\nu}_T$ and a perturbation $\hat{\nu}_T$. We supposed that $\bar{\nu}_T$ is a function of the slow spatial and temporal scales, while $\hat{\nu}_T$ is a function of the normal scale, i.e., the scales belonging to the circulation cells.

The CL-equation is subsequently split into a momentum equation for the basic state and one for the perturbed quantities. The equation for \mathbf{U} is simplified by assuming that \mathbf{U} has no vertical component, which essentially also means that the bottom has to be (nearly) horizontal. The momentum equation for the perturbation velocity is linearised in the perturbation.

In Chapter 4 the linear instability analysis is tackled and solutions of the perturbed momentum equations are found by expansion to a small parameter ε which is related to the Biot number. In these solutions still an unknown amplitude A_0 figures. A further simplification was introduced in that terms with $\hat{\nu}_T \partial_z U$ were neglected from the horizontal and vertical first-order momentum equations. This was argued to be

permitted because the CL-equation itself is only valid for small shear of the current profile.

By considering the change in mean kinetic energy for the perturbation velocities and using the solutions found in Chapter 4, a Landau-Stuart equation for the amplitude A_0 follows in Chapter 5:

$$\frac{dA_0^2}{dt} = 2\alpha A_0^2 - \ell A_0^4,$$

where $\ell > 0$. For a different type of expansion of \hat{u} such an equation was already found in §3.3. The coefficients of this equation depend on the solutions for the perturbed velocities, the basic velocities and the Stokes drift. For $\alpha > 0$ an equilibrium solution A_e for A_0 is possible: $A_e^2 = 2\alpha/\ell$, and secondary circulation cells are possible, i.e., Langmuir circulation cells are generated. For the case that $\alpha < 0$, we have $A_0 \rightarrow 0$ for $t \rightarrow \infty$, and no Langmuir circulation cells can be generated.

With the equilibrium solution A_e the full solution for the perturbation velocities is known and the Reynolds stress terms $\langle \hat{w}\hat{u} \rangle$ and $\langle \hat{w}\hat{v} \rangle$ can be expressed in known quantities. The consequence is that the CL-equation in \mathbf{U} , Eq. (3.13), can be solved numerically with a mesh which belongs to the current; it is not anymore necessary to use a fine mesh in order that the circulation cells are covered.

In Chapter 6 the discussion and recommendation follow. Radder's (1998) note is added to this report as Appendix D.

1 Introduction

By letter RIKZ/OS 987231 Rijkswaterstaat RIKZ asked WL|Delft Hydraulics to perform a study as part of project K2000*KOP. This study comprises a check on the secondary-circulation model of Radder (1998). The work consist of three parts:

1. Checking the note of Radder (1998), viz. *Pattern formation in a 3D wave-current interaction system; a subgrid model*, on correctness.
2. An assessment of the suppositions and the restrictions of the model.
3. A study concerning the application of the model in practical situations.

The problem is set by the so-called CL-equation, an equation for the mean current, which reads (i.e., Radder, 1994, Dingemans et al., 1996):

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + (\bar{\mathbf{u}} \cdot \text{grad}) \bar{\mathbf{u}} + \text{grad } \bar{\pi} = \bar{\mathbf{u}}^S \wedge \text{curl } \bar{\mathbf{u}}, \quad (1.1)$$

where the pressure term $\bar{\pi}$ is given by $\bar{\pi} = \frac{\bar{p}}{\rho} + gz + \left\langle \frac{1}{2} \tilde{\mathbf{u}} \cdot \tilde{\mathbf{u}} \right\rangle$, $\bar{\mathbf{u}}$ is the (Eulerian) mean velocity, $\bar{\mathbf{u}}^S$ is the Stokes drift and $\tilde{\mathbf{u}}$ is the wave part of the velocity.

Instability of the Craik-Leibovich equation may lead to secondary circulations, which can sometimes be identified as being Langmuir circulations. A first numerical approach has been described in Van Kester et al. (1996), see also Dingemans et al. (1996). This numerical approach was by necessity a rather approximate one, because a correct approach is one like large eddy simulation (LES) approaches, needing considerable numerical effort. The real effort lies in the fact that the numerics should be taylored to the need of resolving the perturbations of the instability of the CL equation. To resolve these Reynolds-like terms, Radder (1998) suggested an analytic model.

Radder now assumes the mean current $\bar{\mathbf{u}}$ to be composed of two quantities, a primary current \mathbf{U} and a secondary one, $\hat{\mathbf{u}}$, where the latter one is related to the vortex rolls which are formed by instability mechanism contained in the CL-equation:

$$\bar{\mathbf{u}} = \mathbf{U} + \hat{\mathbf{u}}. \quad (1.2)$$

By adopting a number of simplifying assumptions, and supposing that the secondary current component obeys a WKB-type of expansion with amplitude a product of a time-dependent and a vertical coordinate dependent function, respectively $A(t)$ and $v(z)$, Radder is able to derive for the time-dependent part of the amplitude a Landau-Stuart equation for $A^2(t)$:

$$\frac{dA^2}{dt} = \beta_1 A^2 - \beta_2 A^2 - \beta_3 A^4. \quad (1.3)$$

The problem is now to determine the constants β_i in the Landau-Stuart equation. These constants were determined by considering a linear theory of instability for the secondary current $\hat{\mathbf{u}}$ and adopting a long-wave expansion.

The Report is composed as follows. First, in Chapter 2, the CL-equation is discussed and estimates for the Stokes drift in waves and currents are given. This latter subject is addressed because of some errors appearing in Van Kester et al. (1996). It is observed that the part of the Stokes drift due to the current is insignificant compared to the contributions due to the waves.

The instability mechanism is described in Chapter 3 and simplification to the momentum equations for the basic state and the perturbed state are discussed. The energy equation is also looked into; this energy equation is used to obtain a description for the Reynolds-like terms in the perturbed velocities. This equation turns out to be a Landau-Stuart type of equation. A first derivation of this equation is given in Chapter 3.

In Chapter 4, a linear instability analysis of the momentum equation is given. By using expansions for the perturbed quantities, solutions up to these orders can be generated. These solutions are used in Chapter 5 to determine the coefficients of the Landau-Stuart equation. Using the equilibrium solution of this equation, it is possible to express the Reynolds stresses $\langle \hat{w}\hat{u} \rangle$ and $\langle \hat{w}\hat{v} \rangle$ in terms of the coefficients of the Landau-Stuart equation and the solutions of the perturbed quantities. These stresses do not depend on the size of the vortex cells and can therefore be used immediately in the mean-current equations.

The summary, discussion and recommendations follow in Chapter 6. Radder's (1998) note has been included in this report as Appendix D.

This research was funded in part by the Commission of the European Communities Directorate General for Science, Research and Development (contract no. MAS3-CT95-0011), MAST-project Kinematics and Dynamics of Wave-Current Interactions.

This study has been performed by M.W. Dingemans, who also drew up this report.

2 The Craik-Leibovich model

2.1 The basic model

When deriving the mean-current equations in so-called Generalised Lagrangian Mean (GLM) coordinates, the following equation is obtained (see Andrews and McIntyre, 1978a, Eq. (3.8), or Dingemans, 1997, Eq. (2.596)):

$$\overline{D}^L(\overline{u}_i^L) + \frac{1}{\rho_0} \frac{\partial \overline{p}^L}{\partial x_i} - \frac{\overline{\rho}^L}{\rho_0} g_i = \overline{D}^L(\overline{P}_i^L) + \frac{\partial}{\partial x_i} \left(\frac{1}{2} \overline{(u_m^L u_m^L)} \right) + \overline{P}_k^L \frac{\partial \overline{u}_k^L}{\partial x_i}, \quad (2.1)$$

where \overline{P}_i^L is the pseudo-momentum and $\overline{(\cdot)}^L$ is the Generalised Lagrangian mean, i.e. the mean over the perturbed position $\Xi(\mathbf{x}, t) = \mathbf{x} + \boldsymbol{\xi}(\mathbf{x}, t)$, with $\boldsymbol{\xi} = 0$. Notice that $\langle (\cdot) \rangle$ or $\overline{(\cdot)}$ denotes the Eulerian mean. Summation over repeated indices is used, i.e., the Einstein convention is applied.

Leibovich (1980) has shown that under mild conditions, Eq. (2.1) reduces to the so-called Craik-Leibovich equation in Eulerian coordinates, which can be written as (see also Radder, 1994 and Dingemans et al., 1996):

$$\frac{\partial \overline{\mathbf{u}}}{\partial t} + (\overline{\mathbf{u}} \cdot \text{grad}) \overline{\mathbf{u}} + \text{grad } \overline{\pi} = \overline{\mathbf{u}}^S \wedge \text{curl } \overline{\mathbf{u}}, \quad (2.2)$$

where the pressure term $\overline{\pi}$ is given by $\overline{\pi} = \frac{\overline{p}}{\rho} + gz + \left\langle \frac{1}{2} \tilde{\mathbf{u}} \cdot \tilde{\mathbf{u}} \right\rangle$, $\overline{\mathbf{u}}$ is the (Eulerian) mean velocity, $\overline{\mathbf{u}}^S$ is the Stokes drift and $\tilde{\mathbf{u}}$ is the wave part of the velocity. This CL-equation differs in two terms of the commonly used current equations, viz., in the terms $\left\langle \frac{1}{2} \tilde{\mathbf{u}} \cdot \tilde{\mathbf{u}} \right\rangle$ and the vortex force $\overline{\mathbf{u}}^S \wedge \text{curl } \overline{\mathbf{u}}$. The Stokes velocity $\overline{\mathbf{u}}^S$ is defined as the difference between the Lagrangian and the Eulerian mean velocity, i.e., $\langle \mathbf{u}(\mathbf{x}, t) \rangle^S = \langle \mathbf{u}(\mathbf{x}, t) \rangle^L - \langle \mathbf{u}(\mathbf{x}, t) \rangle$. Notice that with $\overline{\mathbf{u}}^S = (\overline{u}^S, \overline{v}^S, 0)^T$, $\overline{\mathbf{u}} = (\overline{u}, \overline{v}, \overline{w})^T$ and $\overline{\boldsymbol{\omega}} = \text{curl } \overline{\mathbf{u}}$ we have

$$\begin{aligned} \overline{\mathbf{T}} &= \overline{\mathbf{u}}^S \wedge \overline{\boldsymbol{\omega}} = \\ &= \left(\overline{v}^S \left(\frac{\partial \overline{v}}{\partial x} - \frac{\partial \overline{u}}{\partial y} \right), -\overline{u}^S \left(\frac{\partial \overline{v}}{\partial x} - \frac{\partial \overline{u}}{\partial y} \right), -\overline{u}^S \left(\frac{\partial \overline{w}}{\partial x} - \frac{\partial \overline{u}}{\partial z} \right) - \overline{v}^S \left(\frac{\partial \overline{w}}{\partial y} - \frac{\partial \overline{v}}{\partial z} \right) \right)^T \\ &\cong \left(0, \overline{u}^S \frac{\partial \overline{u}}{\partial y}, \overline{u}^S \frac{\partial \overline{u}}{\partial z} \right)^T \equiv \overline{\mathbf{T}}_0, \end{aligned} \quad (2.3)$$

The so-called mild conditions for which the approximations are valid, amount to the condition that the waves are primarily dominated by their irrotational part. This implies that either the mean shear or the mean current is relatively weak and we thus have to impose the condition that the current is small with respect to the phase velocity. Above approximation $\overline{\mathbf{T}}_0$ is valid for the case that the waves and the current are nearly aligned, as is the case in a wave flume. The vortex force in the form of $\overline{\mathbf{T}}_0$ has been used in Dingemans et al. (1996). The near-alignment approximation will not be used in the present work.

Of course, we also have the continuity equation:

$$\operatorname{div} \bar{\mathbf{u}} = 0. \quad (2.4)$$

In order to be able to ascertain the validity of the CL-approximation, the simplification of the GLM formulation into the CL-equation has to be considered in some detail. This is not carried out in this report.

2.2 Stokes drift

We investigate the Stokes drift in this section because the estimate of the Stokes drift as given in Van Kester et al. (1996), page 5-3, is wrong.

The Stokes drift is defined as

$$\langle \mathbf{u}(\mathbf{x}, t) \rangle^S = \langle \mathbf{u}(\mathbf{x}, t) \rangle^L - \langle \mathbf{u}(\mathbf{x}, t) \rangle. \quad (2.5)$$

A Taylor expansion of the Lagrangian mean of \mathbf{u} , $\langle \mathbf{u} \rangle^L$, yields (Andrews and McIntyre, 1978a, Eq. (2.27) or Dingemans, 1997, Eq. (2.617a)):

$$\bar{u}_m^S(\mathbf{x}, t) = \bar{u}_m^L - \bar{u}_m = \left\langle \xi_j \frac{\partial u'_m}{\partial x_j} \right\rangle + \frac{1}{2} \overline{(\xi_i \xi_j)} \frac{\partial^2 \bar{u}_m}{\partial x_i \partial x_j} + \mathcal{O}(a^3), \quad (2.6)$$

where \mathbf{u}' is defined through $\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}'$. Notice that, for this expression to be valid, the perturbation $\boldsymbol{\xi}$ should be of order a , i.e., $|\boldsymbol{\xi}| = \mathcal{O}(a)$.

To estimate the magnitude of the Stokes drift, we introduce a number of simplifications. Suppose that we consider the case of a relatively narrow wave flume, with the axis in the $x_1 = x$ direction. Because we also consider a flume with a horizontal bottom, the variation of the velocity in the $x_2 = y$ direction can be neglected as the current and waves are aligned in the x_1 -direction. We thus suppose that the Stokes drift in this case can be approximated by:

$$\begin{aligned} \bar{u}_m^S \cong & \left\langle \xi_1 \frac{\partial u'_m}{\partial x_1} \right\rangle + \left\langle \xi_3 \frac{\partial u'_m}{\partial x_3} \right\rangle + \frac{1}{2} \overline{\langle \xi_1^2 \rangle} \frac{\partial^2 \bar{u}_m}{\partial x_1^2} + \frac{1}{2} \overline{\langle \xi_3^2 \rangle} \frac{\partial^2 \bar{u}_m}{\partial x_3^2} + \\ & + \overline{\langle \xi_1 \xi_3 \rangle} \frac{\partial^2 \bar{u}_m}{\partial x_1 \partial x_3} + \mathcal{O}(a^3). \end{aligned} \quad (2.7)$$

From linear theory we have (e.g., Dingemans, 1997, Eqs. (2.14))

$$\mathbf{u}'(\mathbf{x}, z, t) = \frac{g}{\omega} a \mathbf{k} \frac{\cosh k(h+z)}{\cosh kh} \cos \chi \quad (2.8a)$$

$$w'(\mathbf{x}, z, t) = \frac{g}{\omega} a k \frac{\sinh k(h+z)}{\cosh kh} \sin \chi \quad (2.8b)$$

with

$$\chi = \mathbf{k} \cdot \mathbf{x} - \omega t. \quad (2.8c)$$

where now $\mathbf{x} = (x_1, x_2)^T$ and a is the amplitude of the free-surface elevation. Without variation in the x_2 -direction, we have $k_1 = k = |\mathbf{k}|$ and $\chi = kx - \omega t$.

Estimates for the horizontal particle path ξ_1 and the vertical one, ξ_3 , follow from

$$\begin{aligned}\xi_1 &= \int^t dt' u'(x + \xi_1, y + \xi_2, z + \xi_3, t') \cong \int^t dt' u'(x, y, z, t') = \\ &= -\frac{g a k_1}{\omega^2} \frac{\cosh k(h+z)}{\cosh kh} \sin \chi = -a \frac{\cosh k(h+z)}{\sinh kh} \sin \chi,\end{aligned}\quad (2.9)$$

and

$$\begin{aligned}\xi_3 &= \int^t dt' w'(x + \xi_1, y + \xi_2, z + \xi_3, t') \cong \int^t dt' w'(x, y, z, t') = \\ &= \frac{g a k}{\omega^2} \frac{\sinh k(h+z)}{\cosh kh} \cos \chi = a \frac{\sinh k(h+z)}{\sinh kh} \cos \chi.\end{aligned}\quad (2.10)$$

We see that $\langle \xi_1 \xi_3 \rangle = 0$ because $\cos \chi$ and $\sin \chi$ are in quadrature. For the current, we may neglect derivatives in the x_1 direction, while that is not permitted for the waves. The expression for the Stokes drift thus is given by:

$$\bar{u}_m^S \cong \left\langle \xi_1 \frac{\partial u'_m}{\partial x_1} \right\rangle + \left\langle \xi_3 \frac{\partial u'_m}{\partial x_3} \right\rangle + \frac{1}{2} \langle \xi_3^2 \rangle \frac{\partial^2 \bar{u}_m}{\partial x_3^2} + O(a^3).\quad (2.11)$$

We only need an expression for \bar{u}_1^S . With

$$\frac{\partial u'_1}{\partial x} = -\frac{g a k^2}{\omega} \frac{\cosh k(h+z)}{\cosh kh} \sin \chi \quad \text{and} \quad \frac{\partial u'_1}{\partial z} = \frac{g a k^2}{\omega} \frac{\sinh k(h+z)}{\cosh kh} \cos \chi$$

we obtain:

$$\left\langle \xi_1 \frac{\partial u'_1}{\partial x_1} \right\rangle = \frac{1}{2} a^2 k \omega \frac{\cosh^2 k(h+z)}{\sinh^2 kh} \quad \text{and} \quad \left\langle \xi_3 \frac{\partial u'_1}{\partial x_3} \right\rangle = \frac{1}{2} a^2 k \omega \frac{\sinh^2 k(h+z)}{\sinh^2 kh},\quad (2.12)$$

where the factors 1/2 are due to $\langle \cos^2 \chi \rangle$ and $\langle \sin^2 \chi \rangle$. Similarly we obtain:

$$\langle \xi_3^2 \rangle = \frac{1}{2} a^2 \sinh^2 k(h+z) \sinh^2 kh.\quad (2.13)$$

Close to the bed (i.e., for $z+h \cong 0$) a Taylor expansion of ξ_3 gives:

$$|\xi_3| \cong a \frac{k(h+z)}{\sinh kh} + \frac{1}{6} a \frac{\{k(h+z)\}^3}{\sinh kh} + \dots.\quad (2.14)$$

2.2.1 The mean current

For the mean current we assume that the eddy-viscosity concept may be used, resulting in (e.g., see Dingemans (1997, pp. 307-308):

$$\rho \nu_T \frac{\partial \bar{u}}{\partial z} = -\tau_c^b \frac{z}{h}, \quad -h \leq z \leq 0,\quad (2.15)$$

where τ_c^b is the bottom shear stress belonging to the part of the current. We choose a function for ν_T which is quadratic over the depth:

$$\nu_T = -\kappa \frac{(z+h)z}{h} \bar{u}^* \quad , \quad -h + \delta_w \leq z \leq 0 \quad , \quad (2.16)$$

with $\kappa \cong 0.40$ Von Karman's constant, δ_w the thickness of the boundary layer due to the wave motion and \bar{u}^* the friction velocity for the mean current, also defined by $\tau_c^b = \rho (\bar{u}^*)^2$. Integration of (2.15) to z then gives

$$\bar{u} = \frac{\bar{u}^*}{\kappa} \log(z+h) + c_1 \quad (2.17)$$

with \log being the natural logarithm and c_1 an integration constant. This constant has to be determined experimentally (see Fredsøe and Deigaard, 1992, p. 21). For the case of a rough wall (the situation for which the thickness of the viscous sublayer is smaller than the Nikuradse roughness k_N), Nikuradse's method leads to a constant

$$c_1 = 8.5 - \frac{1}{\kappa} \log(k_N) \quad , \quad (2.18)$$

and thus,

$$\begin{aligned} \bar{u} &= \frac{\bar{u}^*}{\kappa} \log\left(\frac{z+h}{k_N/30}\right) \\ &= \frac{\bar{u}^*}{\kappa} \log\left(\frac{z+h}{z_0}\right) \quad \text{with} \quad z_0 = \frac{k_N}{30} \quad \text{for} \quad -h + \delta_w \leq z \leq 0 \quad . \end{aligned} \quad (2.19)$$

We note that the logarithmic velocity profile is only valid outside the bottom-boundary layer, whose thickness may be estimated by (see Dingemans, 1997, Eq. (3.168c)):

$$\delta = 0.072 \left(A^3 k_N \right)^{1/4} \quad , \quad (2.20)$$

where A is the length of the semi-axis of the bottom excursion and k_N is the Nikuradse length scale. The bottom velocity u^b (valid just outside the bottom boundary layer) follows from (2.8) and $\chi = kx - \omega t$ as

$$u^b = \frac{agk}{\omega \cosh kh} \cos \chi = \frac{a\omega}{\sinh kh} \cos \chi = A\omega \cos \chi \quad , \quad (2.21)$$

so that

$$A = \frac{a}{\sinh kh} \quad . \quad (2.22)$$

We now have:

$$\left| \frac{\partial u'_1}{\partial z} \right| = \frac{gak^2}{\omega} \frac{\sinh k(h+z)}{\cosh kh} \quad \text{with} \quad \max_{-h \leq z \leq 0} \left| \frac{\partial u'_1}{\partial z} \right| = \omega ak \quad . \quad (2.23a)$$

$$\left| \frac{\partial \bar{u}_1}{\partial z} \right| = \frac{\bar{u}_*}{\kappa(h+z)} \quad (2.23b)$$

$$\left| \frac{\partial^2 \bar{u}_1}{\partial z^2} \right| = \frac{\bar{u}_*}{\kappa (h+z)^2}. \quad (2.23c)$$

2.2.2 Estimate of the Stokes drift

We can now estimate the terms in the right-hand member of Eq. (2.7). Using the expressions (2.10) and (2.23), we obtain:

$$\begin{aligned} \bar{u}_1^S &\cong \left\langle \xi_1 \frac{\partial u_1'}{\partial x} \right\rangle + \left\langle \xi_3 \frac{\partial u_1'}{\partial z} \right\rangle + \left\langle \frac{1}{2} \xi_3^2 \frac{\partial^2 \bar{u}_1}{\partial z^2} \right\rangle = \\ &= \frac{1}{2} a^2 k \omega \frac{\cosh^2 k(h+z)}{\sinh^2 kh} + \frac{1}{2} a^2 k \omega \frac{\sinh^2 k(h+z)}{\sinh^2 kh} + \\ &\quad + \frac{1}{4} \frac{a^2 \bar{u}^*}{\kappa (h+z)^2} \frac{\sinh^2 k(h+z)}{\sinh^2 kh}. \end{aligned} \quad (2.24)$$

Close to the bed we have $k+h \cong 0$. In the limit for $z \rightarrow -h$, a Taylor expansion in $(h+z)$ of (2.24) yields

$$\begin{aligned} \bar{u}_1^S &\cong \frac{1}{2} \frac{a^2 k \omega}{\sinh^2 kh} \{1 + k^2 (h+z)^2 + \dots\} + \frac{1}{2} \frac{a^2 k \omega}{\sinh^2 kh} \{k^2 (h+z)^2 + \dots\} + \\ &\quad + \frac{1}{4\kappa} \frac{(ak)^2 \bar{u}^*}{\sinh^2 kh} \left\{1 + \frac{1}{3} k^2 (h+z)^2 + \dots\right\}, \end{aligned} \quad (2.25)$$

and thus,

$$\lim_{h+z \rightarrow 0} \bar{u}_1^S = \frac{1}{2} \frac{(ak)^2}{\sinh^2 kh} \left[\frac{\omega}{k} + \frac{\bar{u}^*}{2\kappa} \right]. \quad (2.26)$$

The friction velocity \bar{u}^* can be estimated from

$$\tau_c^b = \rho (\bar{u}^*)^2 \quad \Rightarrow \quad \bar{u}^* = \sqrt{\frac{\tau_c^b}{\rho}}, \quad (2.27a)$$

where the bottom shear stress for the current is estimated by (see Soulsby et al., 1993):

$$\tau_c^b = \rho C_D \bar{u}^2 \quad \text{with} \quad C_D = \left(\frac{\kappa}{\log\left(\frac{h}{z_0}\right) - 1} \right)^2. \quad (2.27b)$$

It should be recognised that the bottom shear stress due to the current, τ_c^b , is modified by the presence of the waves. In Soulsby et al. (1993) parametrisations are given for a number of models for shear stress due to waves and currents acting simultaneously. Optimised parametrisations have been presented by Soulsby (1995), see also Soulsby (1997, pp. 68-70). The maximum shear stress is usually different from $|\tau_c^b + \tau_w^b|$, as is also found from the two examples furtheron. However, for an estimate of the Stokes drift due to the presence of a current, the estimate (2.27) will suffice.

We consider two numerical examples, with parameters as given in Table 2.1. We consider a so-called sea situation. We suppose a d_{50} distribution of sand of measure

250 μm and compute the ripple height of the sand bed by a method of Nielsen (1979), see Dingemans (1997, p. 304). The Nikuradse roughness parameter k_N is taken equal to the ripple height and z_0 subsequently follows by $z_0 = k_N/30$. This ripple height is usually larger than the grain size diameter d_{50} . The usual Nikuradse number is taken to be $2.5d_{50}$.

For the laboratory measurement is taken the situation as measured by Klopman (1994). In this case the bottom roughness parameter z_0 was estimated as 0.04 mm (see Klopman, 1994, p. 31).

The bottom shear stress due to the waves, τ_w^b , is estimated by the usual quadratic friction law

$$\tau_w^b = \frac{1}{2} \rho f_w (u^b)^2, \quad (2.28)$$

where the friction parameter f_w is estimated from a formula due to Soulsby (1995) (see also Dingemans, 1997, Eq. (3.174)):

$$f_w = 1.39 \left(\frac{A}{z_0} \right)^{-0.52}. \quad (2.29)$$

The total and mean bottom shear stresses are estimated from the parametrisation of Soulsby et al. (1993) while using the parametrisation for Fredsøe's model. The mean current in these models is the current averaged over the vertical. Because we already had a mean current averaged over horizontal space, the further average over vertical space is denoted by double bar.

| example | a [m] | T [s] | h [m] | \bar{u} [m/s] | z_0 [m] | k [m ⁻¹] | \bar{u}^* [cm/s] | u^b [m/s] | A [m] |
|---------|------------|------------|------------|--------------------|-----------------------|---------------------------|-----------------------|----------------|------------|
| sea (1) | 2.00 | 7.00 | 10 | 0.50 | 7.93×10^{-5} | .1050 | 1.86 | 1.43 | 1.59 |
| lab (2) | 0.06 | 1.44 | 0.5 | 0.16 | 4×10^{-5} | 2.350 | .759 | .179 | .041 |

Table 2.1: Parameters for the numerical examples

| example | τ_c^b [N/m ²] | τ_w^b [N/m ²] | τ_{tot}^b [N/m ²] | $\langle \tau^b \rangle$ [N/m ²] | f_w - | C_D - |
|---------|-----------------------------------|-----------------------------------|---------------------------------------|---|------------|------------|
| sea (1) | 0.3559 | 8.236 | 10.007 | 1.036 | 0.0080 | 0.00139 |
| lab (2) | 0.0576 | .6041 | 0.0355 | .6041 | 0.0378 | 0.00225 |

Table 2.2: Further parameters for the numerical examples

We plot the contribution from the waves and the mean current separately, together with the sum of these contributions.

It is clear from these examples that the contribution of \bar{u} to the Stokes drift is very small and may well be neglected in comparison to the contribution due to the waves. For the sea condition, with $\bar{u} = 0.50$ m/s, the contribution to the Stokes drift from the current is 0.046 cm/s at the free surface and 0.033 cm/s at the bottom. It can be argued that a current of 0.50 m/s is not very large in coastal areas, but in the

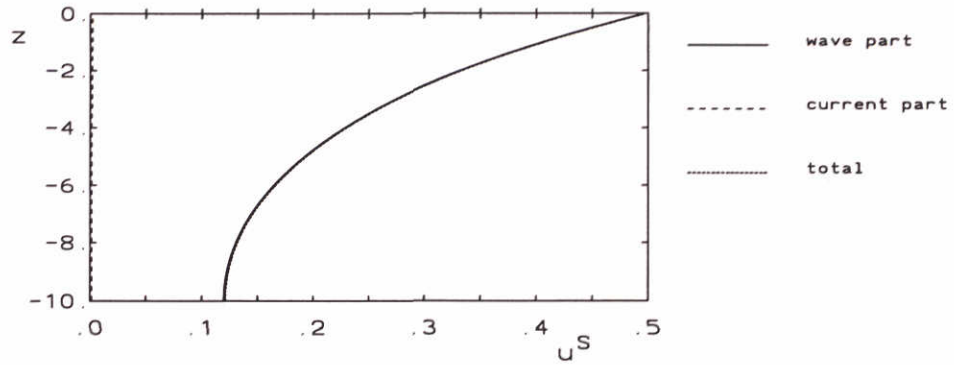


Figure 2.1: \bar{u}^S as function of z , sea situation.

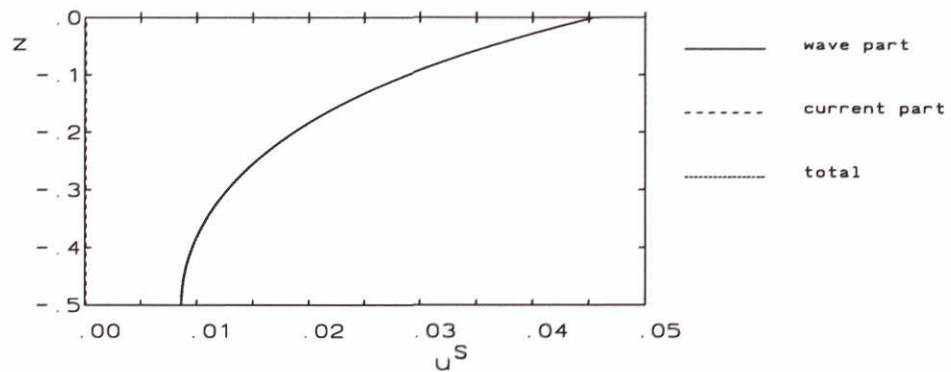


Figure 2.2: \bar{u}^S as function of z , the laboratory situation.

North sea currents are typically not much larger than 1 m/s. Redoing the analysis with $\bar{u} = 2$ m/s results in a friction velocity $\bar{u}^* = 7.45$ cm/s, roughly 4 times larger than was obtained for $\bar{u} = 0.50$ m/s, which was to be expected. The contribution of the current to the Stokes drift is now 0.19 cm/s at $z = 0$ and 0.13 cm/s at $z = -h$. Compared with a maximum contribution of 50 cm/s contribution from the waves, the current contribution to the Stokes drift remains insignificant.

3 Instability mechanism

3.1 Investigation of the momentum equations

One of the explanations of Langmuir circulations rests upon the supposition that an instability mechanism in the CL equation is responsible. Radder (1998) now supposes that the mean current $\bar{\mathbf{u}}$ is disturbed. We then have the situation that $\bar{\mathbf{u}}$ can be written as

$$\bar{\mathbf{u}} = \mathbf{U} + \hat{\mathbf{u}}, \quad (3.1)$$

where $\hat{\mathbf{u}}$ is the disturbance which is responsible for the formation of the vortex rolls and \mathbf{U} is the velocity of the basic state. In many instability problems it is common to suppose that the perturbations have some periodic nature. For many problems this is a natural choice because the (linearised) perturbation equations permit periodic solutions. The choice of Radder (1998) that $\hat{\mathbf{u}}$ obeys a WKBJ-type of behaviour is therefore natural, also in view of the resulting (periodic) vortex-roll motions.

Consider first the CL equation again. Using the notation of (2.3) the CL equation (2.2) reads

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + (\bar{\mathbf{u}} \cdot \text{grad}) \bar{\mathbf{u}} + \text{grad } \bar{\pi} = \bar{\mathbf{u}}^S \wedge \bar{\boldsymbol{\omega}}, \quad (3.2)$$

Although no viscosity is taken into account in the usual CL-equation-formulations, it is advantageous to do so. This has to do with the so-called Large Eddy Simulation (LES) programs. Viscosity in these equations is needed for obtaining shear in the mean-current equations, which, in its turn, is needed to generate the vorticity force term. Equation (3.2) then is extended to

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + (\bar{\mathbf{u}} \cdot \text{grad}) \bar{\mathbf{u}} + \text{grad} \left(gz + \frac{\bar{p}}{\rho} + \left\langle \frac{1}{2} \tilde{\mathbf{u}} \cdot \tilde{\mathbf{u}} \right\rangle \right) = \bar{\mathbf{u}}^S \wedge \bar{\boldsymbol{\omega}} + \frac{1}{\rho} \text{div } \bar{\boldsymbol{\sigma}}', \quad (3.3)$$

where $\text{div } \bar{\boldsymbol{\sigma}}' \equiv \partial \bar{\sigma}'_{ik} / \partial x_k$. We take the (eddy) viscosity coefficient to be isotropic because of the scales on which the flow occurs here. Applying the Boussinesq-hypothesis, the stresses $\bar{\sigma}'_{ik}$ are approximated as¹

$$\frac{1}{\rho} \bar{\sigma}'_{ik} = \nu_T \left(\frac{\partial \bar{u}_i}{\partial x_k} + \frac{\partial \bar{u}_k}{\partial x_i} \right), \quad (3.4)$$

while the eddy viscosity ν_T has still to be determined.

For the other quantities $\bar{\pi}$ and $\bar{\boldsymbol{\omega}}$ in the CL equation the same kind of perturbations are assumed to exist, viz. a basic state (denoted with capitals) and a perturbed state (denoted by hatted variables). Because the eddy viscosity is a function of the velocity

¹We write σ'_{ik} with the prime, denoting that this is the part of the stress tensor without the pressure, see also Landau and Lifchitz (1989, Eq. (15.2)) or Dingemans (1997, Eq. (1.14)).

\bar{u} and the depth z , a perturbation of ν_T is also necessary. An independent perturbation is not envisaged, only a perturbation through the velocity. There are several ways in which we can proceed with the splitting of the eddy viscosity. The first is one in which a formal way is followed where Taylor expansion is used. In the second one, use is made of the fact that the eddy viscosity is in fact a direct function of the friction velocity. The relation with the velocity itself comes only through the relation between friction velocity and velocity. The third way is one in which the eddy viscosity is first related to the flow properties and splitted afterwards. Here the relation with channel flow is used.

- **First, formal, method**

From a Taylor expansion we have:

$$\nu_T(\mathbf{U} + \hat{\mathbf{u}}, z) = \nu_T(\mathbf{U}, z) + \hat{\mathbf{u}} \cdot \frac{\partial \nu_T(\mathbf{U}, z)}{\partial \mathbf{U}} + \dots \quad (3.5)$$

We now define the basic state and the perturbation of the eddy viscosity as:

$$\bar{\nu}_T = \nu_T(\mathbf{U}, z) \quad (3.6a)$$

$$\hat{\nu}_T = \hat{\mathbf{u}} \cdot \frac{\partial \bar{\nu}_T(\mathbf{U}, z)}{\partial \mathbf{U}} \quad (3.6b)$$

We remark that the basic state eddy viscosity is only a function of the basic velocity \mathbf{U} , but the perturbed eddy viscosity depends on both the basic velocity and the perturbed velocity.

- **Second, formal, method**

The definition of eddy viscosity rests, in fact, on the friction velocity. Therefore, supposing ν_T to be a direct function of \bar{u}^* , a Taylor expansion would give:

$$\nu_T(U^* + \hat{u}^*, z) = \nu_T(U^*, z) + \hat{u}^* \frac{\partial \nu_T(U^*, z)}{\partial U^*} + \dots \quad (3.7)$$

- **Third method, related to channel flow**

In channel flow, the eddy viscosity is sometimes defined by (e.g., Rodi, 1980, Eq. (2.21)):

$$\nu_T = C \bar{u}^* h, \quad (3.8)$$

where C is a constant depending on the channel geometry, having the value 0.135 for wide channels and h is the water depth². This formula is valid when the turbulence is mainly bed-generated. For this situation it is particularly simple to split the eddy viscosity into a basic and perturbed quantity:

$$\nu_T = C U^* h + C \hat{u}^* h \equiv \bar{\nu}_T + \hat{\nu}_T \quad (3.9)$$

²For problems involving heat, C has to be divided by the Prandtl number, and for problems involving mass transport division by the Schmidt number is required.

Note that the z -dependence in (3.9) is lost. This is not really bothersome because the z -dependence is usually invented afterwards; in situations with waves and currents, a quadratic form in z is taken for the current, over the total water depth, while the influence of the waves is restricted to the boundary layers only. Examples can be found in Dingemans (1997, pp. 312-313) and in the references mentioned there. Important to note is that these chosen eddy viscosity coefficients are all linear in the friction velocity.

The dependence of the eddy viscosity on the current is still undefined in (3.6) and (3.7). When also here a linear dependence on the current is taken, then the representations (3.6) and (3.7) do not differ much with the representation (3.8), except the dependence on the friction velocity in the last ones. We therefore will take a representation of the form (3.8), equipped with some vertical distribution in z , possibly constant. We thus take for the eddy viscosity the representation

$$\nu_T(\bar{\mathbf{u}}^*, z) = \bar{\nu}_T(U^*, z) + \hat{\nu}_T(\hat{\mathbf{u}}^*, z) = c_1(z)U^* + c_2(z)\hat{\mathbf{u}}^* , \quad (3.10)$$

where the friction velocity $\hat{\mathbf{u}}^*$ is periodic with the same period as that of $\hat{\mathbf{u}}$.

For the other quantities we obtain:

$$\bar{\pi} = \Pi + \hat{\pi} = gz + \frac{\bar{P}}{\rho} + \left\langle \frac{1}{2} \tilde{\mathbf{u}} \cdot \tilde{\mathbf{u}} \right\rangle + \frac{\hat{p}}{\rho} \Rightarrow \hat{\pi} = \frac{\hat{p}}{\rho} \quad (3.11a)$$

$$\bar{\boldsymbol{\omega}} = \boldsymbol{\Omega} + \hat{\boldsymbol{\omega}} \quad (3.11b)$$

$$\bar{\mathbf{u}} = \mathbf{U} + \hat{\mathbf{u}} \quad (3.11c)$$

$$\bar{\boldsymbol{\sigma}}' = \boldsymbol{\Sigma}' + \hat{\boldsymbol{\sigma}}' , \quad (3.11d)$$

where

$$\bar{\Sigma}'_{ik} = \rho \bar{\nu}_T \left(\frac{\partial U_i}{\partial x_k} + \frac{\partial U_k}{\partial x_i} \right) + \rho \hat{\nu}_T \left(\frac{\partial \hat{u}_i}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial x_i} \right) \quad (3.11e)$$

and

$$\hat{\sigma}'_{ik} = \rho \hat{\nu}_T \left(\frac{\partial U_i}{\partial x_k} + \frac{\partial U_k}{\partial x_i} \right) + \rho \bar{\nu}_T \left(\frac{\partial \hat{u}_i}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial x_i} \right) . \quad (3.11f)$$

We also note that, due to the supposition of the existence of periodic perturbations, we have that the mean of the perturbations is zero, e.g., $\langle \hat{\mathbf{u}} \rangle = 0$, or, otherwise stated, $\langle \bar{\mathbf{u}} \rangle = \mathbf{U}$ and similarly for the other quantities.

Notice that the wave-related quantities $\tilde{\mathbf{u}}$ and $\bar{\mathbf{u}}^S$ are not perturbed, only the current-type quantities. In the sequel we write:

$$\bar{\mathbf{u}}^S = \mathbf{U}^S . \quad (3.11g)$$

The quantities (3.11) are to be inserted in the CL-equation (3.3). The result, without imposing any approximation at this stage, is

$$\begin{aligned} \frac{\partial}{\partial t} (\mathbf{U} + \hat{\mathbf{u}}) + [(\mathbf{U} + \hat{\mathbf{u}}) \cdot \text{grad}] (\mathbf{U} + \hat{\mathbf{u}}) + \text{grad} (\Pi + \hat{\pi}) = \\ = \mathbf{U}^S \wedge (\boldsymbol{\Omega} + \hat{\boldsymbol{\omega}}) + \frac{1}{\rho} \text{div} (\boldsymbol{\Sigma}' + \hat{\boldsymbol{\sigma}}') . \end{aligned} \quad (3.12)$$

This equation should be split into an equation for the basic state (i.e. the current) and one for the perturbation. It is here that the supposition that the perturbations are of a periodic nature is used a first time. We take the mean of Eq. (3.12), where the mean is taken over a period and length which are large compared to the characteristic period and wave length of the perturbations, but short compared to those of the basic state. Terms linear in the perturbation then average out and the equation for the basic state reads:

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \text{grad}) \mathbf{U} + \langle (\hat{\mathbf{u}} \cdot \text{grad}) \hat{\mathbf{u}} \rangle + \text{grad } \Pi = \mathbf{U}^S \wedge \boldsymbol{\Omega} + \frac{1}{\rho} \text{div } \boldsymbol{\Sigma}' , \quad (3.13)$$

Notice that $\langle (\hat{\mathbf{u}} \cdot \text{grad}) \hat{\mathbf{u}} \rangle$ are the Reynolds stresses. We furthermore remark that to obtain this result it is necessary to suppose that $\hat{\nu}$ is periodic; this is a reasonable assumption due to the relation (3.6b) which can be viewed as a definition of $\hat{\nu}$.

The equation for the perturbation is obtained by subtraction of (3.13) from (3.12). The result is

$$\begin{aligned} \frac{\partial \hat{\mathbf{u}}}{\partial t} + (\hat{\mathbf{u}} \cdot \text{grad}) \mathbf{U} + (\mathbf{U} \cdot \text{grad}) \hat{\mathbf{u}} + (\hat{\mathbf{u}} \cdot \text{grad}) \hat{\mathbf{u}} - \langle (\hat{\mathbf{u}} \cdot \text{grad}) \hat{\mathbf{u}} \rangle + \text{grad } \hat{\pi} = \\ = \mathbf{U}^S \wedge \hat{\boldsymbol{\omega}} + \frac{1}{\rho} \text{div } \hat{\boldsymbol{\sigma}}' . \end{aligned} \quad (3.14)$$

We remark that Eqs. (3.13) and (3.14) correspond with Eqs. (3) and (11) of Radder (1998). The current equation (3.13) can be solved in principle once the Reynolds stresses are known. A complication is the dependence of $\boldsymbol{\Sigma}'$ on the perturbation velocities $\hat{\mathbf{u}}$, see (3.11e). That means that the perturbation velocities $\hat{\mathbf{u}}$ have to be determined. Because a direct determination of these Reynolds stresses has to be carried out on a smaller scale than the computational mesh which is needed for the current-computations, a sub-grid model is advantageous³. The method in which Radder (1998) develops such a sub-grid model is addressed in the next section.

The continuity equation (2.4) splits in one for the basic flow \mathbf{U} and one for the perturbed flow $\hat{\mathbf{u}}$:

$$\text{div } \mathbf{U} = 0 \quad (3.15a)$$

and

$$\text{div } \hat{\mathbf{u}} = 0 . \quad (3.15b)$$

³Such a sub-grid model should not be confused with sub-grid models used in turbulence research, in which case sub-grid modelling means that the processes occurring on scales which are smaller than the smallest computational mesh are accounted for in some way.

3.2 The simplification of the mean-momentum equations (3.13)

The following approximations are introduced by Radder (1998):

1. The mean current component \mathbf{U} is uniform in the horizontal directions: $\mathbf{U} = \mathbf{U}(z, t)$ and, moreover, no vertical component exists:

$$\mathbf{U} = (U(z, t), V(z, t), 0)^T . \quad (3.16)$$

This means that (nearly) horizontal nearly-uniform shear flows are considered. As pointed out in Dingemans (1997, pp. 193 and 201), the vertical component of the mean current can only be neglected when the bottom is (nearly) horizontal. It is therefore also supposed that *the bottom is horizontal*, i.e.⁴

$$\nabla h(x, y) = 0 . \quad (3.17)$$

2. The wave-induced perturbation $\hat{\mathbf{u}}$ is supposed to be single-periodic in one specific direction θ . With θ the angle between the positive x -axis and the path of propagation s , one has $x = s \cos \theta - n \sin \theta$, $y = s \sin \theta + n \cos \theta$ and thus, (e.g., Dingemans, 1997, §2.3.3)

$$\frac{\partial}{\partial s} = \frac{\partial x}{\partial s} \frac{\partial}{\partial x} + \frac{\partial y}{\partial s} \frac{\partial}{\partial y} = \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y} \quad (3.18a)$$

$$\frac{\partial}{\partial n} = \frac{\partial x}{\partial n} \frac{\partial}{\partial x} + \frac{\partial y}{\partial n} \frac{\partial}{\partial y} = -\sin \theta \frac{\partial}{\partial x} + \cos \theta \frac{\partial}{\partial y} . \quad (3.18b)$$

3. It is supposed now that

$$\frac{\partial}{\partial n} \hat{\mathbf{u}}(x, y, z, t) \equiv 0 . \quad (3.19)$$

Remark

In our opinion the condition $\partial \hat{\mathbf{u}} / \partial n \equiv 0$ is much stronger than the condition that the perturbations are single-periodic in a specific direction. A WKBJ-type expression of the wave-induced perturbation can be written as

$$\begin{aligned} \hat{\mathbf{u}}(x, y, z, t) &= f(z, h) \hat{\mathbf{u}}'(x, y, t) = \\ &= \operatorname{Re} \left\{ f(z, h) a(\mathbf{X}, Y, T) \exp \left[\frac{i}{\delta} \chi(\mathbf{X}, T) \right] \right\} , \end{aligned} \quad (3.20)$$

where χ is the phase function, $\mathbf{k} = (k_1, k_2)^T = \partial \chi / \partial \mathbf{X}$, $\omega = -\partial \chi / \partial T$, $\mathbf{X} = (X, Y)^T = (\delta x, \delta y)^T$ is the slow spatial scale, $T = \delta t$ is the slow time, and

⁴For two-dimensional cases, when $\mathbf{x} = (x_1, x_2)^T \equiv (x, y)^T$, we write the gradient operator as ∇ , while for the three-dimensional case we use grad , see also the footnote 2 on page 2 in Dingemans (1997).

$f(z, h)$ denotes the vertical structure, which reads for linear waves:

$$f(z, h) = \frac{\cosh[k(h+z)]}{\cosh kh}, \quad (3.21)$$

as is also the case for the mild-slope approximation.

The condition $\partial a / \partial n \equiv 0$ in fact means that a ray-approximation for the propagation of the perturbations is taken. Only variations along the ray are accounted for. Note that the ray need not be straight in (x, y) space. This approximation can be compared with the usual approximations of the parabolic approximation and the mild-slope one. In the parabolic approximation some variation perpendicular to the wave propagation direction is permitted and in the mild-slope approximation the variation on all directions is supposed to be of the same order of magnitude. Notice moreover that the propagation direction of the perturbations need not to coincide with the wave direction.

4. The eddy viscosity ν_T and the Stokes drift \mathbf{U}^S are supposed to be only a function of the depth, i.e.,

$$\nu_T = \nu_T(z) \geq 0 \quad \text{and} \quad \mathbf{U}^S = \mathbf{U}^S(z) = \left(U^S(z), V^S(z), 0 \right)^T. \quad (3.22)$$

Remark

We have several comments on this proposition. We first comment on the situation that the eddy viscosity is not perturbed, which is the situation considered by Radder (1998). This leads to the following comment.

- In view of the supposition of a horizontal depth, the fact that ν_T and \mathbf{U}^S are independent of x and y is justified. The supposition that they also do not depend on time seems not to be defensible so easily. In the first place, it has been shown by for example Trowbridge and Madsen (1984a,b) that a time-varying eddy viscosity makes much difference in the resulting dynamics of the flow. Secondly, the supposition that the Stokes drift is independent of time (otherwise put: it is stationary) can only be true when the waves themselves are also stationary (in a fixed frame, which is the frame we use here). The time-variation of the Stokes drift can be taken to be slower than that of the waves proper, but it seems not to be justified without further analysis to neglect all time-variation in the Stokes drift. For the moment we will only suppose that ν_T and \mathbf{U}^S are functions of the slow time $T = \delta t$:

$$\nu_T = \nu_T(z, T) \geq 0 \quad \text{and} \quad (3.23a)$$

$$\mathbf{U}^S = \mathbf{U}^S(z, T) = \left(U^S(z, T), V^S(z, T), 0 \right)^T. \quad (3.23b)$$

- As discussed before, we are of the opinion that it is necessary to also perturb the eddy viscosity and the form (3.10) has been proposed. Via the friction velocities \bar{u}^* and U^* also the dependence on \mathbf{x} and t is

present. In view of the horizontal bottom, it seems to be an acceptable approximation to suppose that U^* is only a function of the slow space and time, i.e., $U^* = U^*(\mathbf{X}, T)$ with $\mathbf{X} = \delta \mathbf{x}$ and $T = \delta t$ and $\delta \ll 1$. Because \hat{v} has been supposed to be periodic on the same scale as is \hat{u} , the dependence is here on the normal scales, not on the slow scales, but it may be supposed that $\hat{v}_T \ll \bar{v}_T$, or, which amounts to the same, $\hat{u}^* \ll U^*$. We thus have

$$\bar{v}_T = \bar{v}_T(U^*(\mathbf{X}, T), z) \quad \text{and} \quad \hat{v}_T = \hat{v}_T(\hat{u}^*(\mathbf{x}, t), z). \quad (3.24)$$

We will use the representation (3.24) in the following and it is supposed that both \bar{v}_T and \hat{v}_T are strictly positive.

3.2.1 Analysis of the stress terms

We will now analyse the terms $\text{div } \Sigma' \equiv \partial \Sigma'_{ik} / \partial x_k$ and $\text{div } \hat{\sigma}'$ which occur in the momentum equations for the basic state, Eq. (3.13) and that for the perturbed velocities, Eq. (3.14). We have

$$\frac{\partial \Sigma'_{ik}}{\partial x_k} = \frac{\partial}{\partial x_k} \left\{ \rho \bar{v}_T \left(\frac{\partial U_i}{\partial x_k} + \frac{\partial U_k}{\partial x_i} \right) + \rho \hat{v}_T \left(\frac{\partial \hat{u}_i}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial x_i} \right) \right\}. \quad (3.25)$$

With the simplifications

$$\mathbf{U} = (U(z, t), V(z, t), 0)^T \equiv \mathbf{U}^h \quad (3.26a)$$

$$\bar{v}_T = \bar{v}_T(U^*(\delta \mathbf{x}, \delta t, z)) = c_1(z)U^* \quad (3.26b)$$

$$\hat{v}_T = \hat{v}_T(\mathbf{x}, t, z) = c_2(z)u^* \quad (3.26c)$$

we obtain

$$\begin{aligned} \frac{1}{\rho} \frac{\partial \Sigma'_{ik}}{\partial x_k} &= \frac{\partial}{\partial x_k} \left\{ \bar{v}_T \left(\frac{\partial U_i}{\partial x_k} + \frac{\partial U_k}{\partial x_i} \delta_{i3} \right) + \hat{v}_T \left(\frac{\partial \hat{u}_i}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial x_i} \right) \right\} = \\ &= \frac{\partial}{\partial z} \left(\bar{v}_T \frac{\partial U_i}{\partial z} \right) + \frac{\partial \bar{v}_T}{\partial x_k} \frac{\partial U_k}{\partial x_i} \delta_{i3} + \frac{\partial}{\partial x_k} \left\{ \hat{v}_T \left(\frac{\partial \hat{u}_i}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial x_i} \right) \right\}, \end{aligned} \quad (3.27)$$

with δ_{ik} being Kronecker's delta ($\delta_{ik} = 1$ when $i = k$ and $\delta_{ik} = 0$ when $i \neq k$). The magnitude of the three terms in the right-hand member of (3.27) is now considered. Because (3.26b) shows that \bar{v}_T is a function of the slow coordinate \mathbf{X} and z , the second term is of $\mathcal{O}(\delta)$. The third term consists of a product of \hat{v}_T and the perturbed velocities. Because we have $\hat{v}_T \ll \bar{v}_T$ and $\hat{u}_i \ll U_i$ we can estimate \hat{v}_T / \bar{v}_T as being of $\mathcal{O}(\mu)$, with $\mu \ll 1$ some ordering parameter and similarly for $|\hat{u}/U|$. To be specific we make the order of each term explicit by introducing scaled variables:

$$\frac{\partial \bar{v}_T}{\partial x_j} = \delta \frac{\partial \bar{v}_T}{\partial X_j}, \quad \frac{\partial \hat{u}_i}{\partial x_k} = \mu \frac{\partial \hat{u}'_i}{\partial x_k} \quad \text{and} \quad \hat{v}_T = \mu \hat{v}'_T. \quad (3.28)$$

Then we have

$$\frac{1}{\rho} \frac{\partial \Sigma'_{ik}}{\partial x_k} = \frac{\partial}{\partial z} \left(\bar{v}_T \frac{\partial U_i}{\partial z} \right) + \delta \frac{\partial \bar{v}_T}{\partial X_k} \frac{\partial U_k}{\partial x_i} \delta_{i3} + \mu^2 \frac{\partial}{\partial x_k} \left\{ \hat{v}'_T \left(\frac{\partial \hat{u}'_i}{\partial x_k} + \frac{\partial \hat{u}'_k}{\partial x_i} \right) \right\}. \quad (3.29)$$

We notice that the second, $\mathcal{O}(\delta)$, term only features in the vertical mean-momentum equation, which is not considered further. The third term of (3.28) is then of $\mathcal{O}(\mu^2)$. It will appear later (in §4.3) that the solutions \hat{u}'_j , $j = 1, 2$, depend on the slow scale \mathbf{X} and not on z , whereas \hat{w} is of still higher order. For the expansions used in this Report this third $\mathcal{O}(\mu^2)$ term is not used. The first term is thus the leading term and we have

$$\frac{\partial \Sigma'_{ik}}{\partial x_k} = \rho \frac{\partial}{\partial z} \left(\bar{\nu}_T \frac{\partial U_i}{\partial z} \right) + \mathcal{O}(\delta, \mu^2). \quad (3.30)$$

For the perturbed stress term $\text{div } \hat{\sigma}'$ we obtain:

$$\begin{aligned} \frac{1}{\rho} \frac{\partial \hat{\sigma}'_{ik}}{\partial x_k} &= \frac{\partial}{\partial x_k} \left\{ \hat{\nu}_T \left(\frac{\partial U_i}{\partial x_k} + \frac{\partial U_k}{\partial x_i} \right) + \bar{\nu}_T \left(\frac{\partial \hat{u}_i}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial x_i} \right) \right\} = \\ &= \frac{\partial}{\partial z} \left(\hat{\nu}_T \frac{\partial U_i}{\partial z} \right) + \frac{\partial}{\partial x_j} \left(\hat{\nu}_T \frac{\partial U_j}{\partial x_i} \delta_{i3} \right) + \frac{\partial}{\partial x_k} \left\{ \bar{\nu}_T \left(\frac{\partial \hat{u}_i}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial x_i} \right) \right\} = \\ &= \frac{\partial}{\partial z} \left(\hat{\nu}_T \frac{\partial U_i}{\partial z} \right) + \left(\frac{\partial \hat{\nu}_T}{\partial x_j} \right) \left(\frac{\partial U_j}{\partial x_i} \delta_{i3} \right) + \bar{\nu}_T \frac{\partial}{\partial x_k} \left(\frac{\partial \hat{u}_i}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial x_i} \right) + \\ &\quad + \frac{\partial \bar{\nu}_T}{\partial x_k} \left(\frac{\partial \hat{u}_i}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial x_i} \right), \quad \text{with } i, k = 1, 2, 3 \quad \text{and } j = 1, 2. \end{aligned} \quad (3.31)$$

Making the magnitudes of the terms explicit by introducing the scalings (3.28), we have:

$$\begin{aligned} \frac{1}{\rho} \frac{\partial \hat{\sigma}'_{ik}}{\partial x_k} &= \mu \frac{\partial}{\partial z} \left(\hat{\nu}'_T \frac{\partial U_i}{\partial z} \right) + \mu \left(\frac{\partial \hat{\nu}'_T}{\partial x_j} \right) \left(\frac{\partial U_j}{\partial x_i} \delta_{i3} \right) + \mu \frac{\partial}{\partial z} \left[\bar{\nu}_T \left(\frac{\partial \hat{u}'_i}{\partial z} + \frac{\partial \hat{w}'}{\partial x_i} \right) \right] + \\ &\quad + \mu \bar{\nu}_T \frac{\partial}{\partial x_j} \left(\frac{\partial \hat{u}'_i}{\partial x_j} + \frac{\partial \hat{u}'_j}{\partial x_i} \right) + \delta \mu \frac{\partial \bar{\nu}_T}{\partial X_j} \left(\frac{\partial \hat{u}'_i}{\partial x_j} + \frac{\partial \hat{u}'_j}{\partial x_i} \right). \end{aligned} \quad (3.32)$$

It is clear that the leading-order terms are of $\mathcal{O}(\mu)$. The first four terms in the right-hand member of (3.31) are $\mathcal{O}(\mu)$ and the fifth term is $\mathcal{O}(\delta\mu)$. In leading order, we then obtain

$$\begin{aligned} \frac{1}{\rho} \frac{\partial \hat{\sigma}'_{ik}}{\partial x_k} &= \mu \frac{\partial}{\partial z} \left(\hat{\nu}'_T \frac{\partial U_i}{\partial z} \right) + \mu \left(\frac{\partial \hat{\nu}'_T}{\partial x_j} \right) \left(\frac{\partial U_j}{\partial x_i} \delta_{i3} \right) + \mu \frac{\partial}{\partial z} \left[\bar{\nu}_T \left(\frac{\partial \hat{u}'_i}{\partial z} + \frac{\partial \hat{w}'}{\partial x_i} \right) \right] + \\ &\quad + \mu \bar{\nu}_T \frac{\partial}{\partial x_j} \left(\frac{\partial \hat{u}'_i}{\partial x_j} + \frac{\partial \hat{u}'_j}{\partial x_i} \right) + \mathcal{O}(\delta\mu), \\ &\quad \text{with } i, k = 1, 2, 3 \quad \text{and } j = 1, 2. \end{aligned} \quad (3.33)$$

3.2.2 The simplified mean-momentum equations

With the simplification (3.16) the vortex force $\bar{\mathbf{T}}$ of (2.3) simplifies further to

$$\bar{\mathbf{T}}_a \cong \left(0, 0, U^S \cdot \frac{\partial U^h}{\partial z} \right)^T = \left(0, 0, U^S \frac{\partial U}{\partial z} + V^S \frac{\partial V}{\partial z} \right)^T. \quad (3.34)$$

Application of the simplifications

$$\mathbf{U} = (U(z, t), V(z, t), 0)^T \equiv \mathbf{U}^h \quad (3.35a)$$

$$\mathbf{U}^S = (U^S(z, T), V^S(z, T), 0)^T \quad (3.35b)$$

$$\bar{v}_T = \bar{v}_T(U^*(\delta \mathbf{x}, \delta t), z) = c_1(z)U^* \quad (3.35c)$$

$$\hat{v}_T = \hat{v}_T(\hat{\mathbf{u}}^*(\mathbf{x}, t), z) = c_2(z)\hat{\mathbf{u}}^* \quad (3.35d)$$

$$\frac{\partial \Sigma'_{ik}}{\partial x_k} = \rho \frac{\partial}{\partial z} \left(\bar{v}_T \frac{\partial U_i}{\partial z} \right) + \mathcal{O}(\delta, \mu^2) \quad (3.35e)$$

$$\begin{aligned} \frac{1}{\rho} \frac{\partial \hat{\sigma}'_{ik}}{\partial x_k} &= \mu \frac{\partial}{\partial z} \left(\hat{v}'_T \frac{\partial U_i}{\partial z} \right) + \mu \left(\frac{\partial \hat{v}'_T}{\partial x_j} \right) \left(\frac{\partial U_j}{\partial x_i} \delta_{i3} \right) + \mu \frac{\partial}{\partial z} \left[\bar{v}_T \left(\frac{\partial \hat{u}'_i}{\partial z} + \frac{\partial \hat{w}'}{\partial x_i} \right) \right] \\ &+ \mu \bar{v}_T \frac{\partial}{\partial x_j} \left(\frac{\partial \hat{u}'_i}{\partial x_j} + \frac{\partial \hat{u}'_j}{\partial x_i} \right) + \mathcal{O}(\delta \mu). \end{aligned} \quad (3.35f)$$

and (3.34) for the vortex force, the simplified horizontal momentum equation (3.13) becomes

$$\frac{\partial \mathbf{U}^h}{\partial t} + \langle (\hat{\mathbf{u}} \cdot \text{grad}) \hat{\mathbf{u}} \rangle^h + \nabla \Pi_0 = \frac{\partial}{\partial z} \left(\bar{v}_T \frac{\partial \mathbf{U}^h}{\partial z} \right) \quad (3.36)$$

where $\Pi_0 = \bar{P}/\rho + \frac{1}{2} \langle \hat{\mathbf{u}} \cdot \hat{\mathbf{u}} \rangle$. We note that in the present approximation the vortex force has only a vertical component and therefore plays no role in the horizontal mean momentum equations. We have

$$\left(\hat{u}_j \frac{\partial}{\partial x_j} \right) \hat{u}_i = \frac{\partial}{\partial x_j} (\hat{u}_j \hat{u}_i) - \hat{u}_i \frac{\partial \hat{u}_j}{\partial x_j} = \frac{\partial}{\partial x_j} (\hat{u}_j \hat{u}_i),$$

because of the continuity equation for the perturbed velocities $\hat{\mathbf{u}}$. Equation (3.36) can therefore also be written in the equivalent form

$$\frac{\partial \mathbf{U}^h}{\partial t} + \left\langle \frac{\partial}{\partial x_j} (\hat{u}_j \hat{\mathbf{u}}) \right\rangle^h + \nabla \Pi_0 = \frac{\partial}{\partial z} \left(\bar{v}_T \frac{\partial \mathbf{U}^h}{\partial z} \right). \quad (3.37)$$

Notice that Radder (1998, Eqs. (7)) writes this as

$$\frac{\partial \mathbf{U}^h}{\partial t} + \frac{\partial}{\partial z} \langle \hat{w} \hat{\mathbf{u}} \rangle^h + \nabla \Pi_0 = \frac{\partial}{\partial z} \left(\bar{v}_T \frac{\partial \mathbf{U}^h}{\partial z} \right), \quad (3.38)$$

which implies that he neglects the contributions $\partial_x (\hat{u} \hat{\mathbf{u}})$ and $\partial_y (\hat{v} \hat{\mathbf{u}})$, which seems not to be in line with the simplifications (3.35). The reason for the neglect of the x and y dependence of the mean of the perturbation velocities $\langle \hat{u} \hat{\mathbf{u}} \rangle$ and $\langle \hat{v} \hat{\mathbf{u}} \rangle$ is that these are supposed to be single-periodic in one direction. The consequence is that the averages give zero. For the moment, we will not apply this approximation but keep the x and y dependence in consideration.

3.3 The energy equation and its simplification

To obtain an estimate of the Reynolds stresses in $\hat{\mathbf{u}}$, we need solutions of the perturbed velocities. To this end an energy equation for the perturbed velocities is needed. An energy equation is obtained from the momentum equation by scalar multiplication with the velocity vector. The energy equation for the perturbed velocities $\hat{\mathbf{u}}$ then results from (3.14) by scalar multiplication with $\hat{\mathbf{u}}$. At this stage, it is advantageous to work in the component notation and therefore we write (3.14) as⁵:

$$\frac{\partial \hat{u}_k}{\partial t} + \hat{u}_m \frac{\partial U_k}{\partial x_m} + U_m \frac{\partial \hat{u}_k}{\partial x_m} + \frac{1}{\rho} \frac{\partial}{\partial x_m} (\hat{\pi} \delta_{km}) = \epsilon_{ijk} U_i^S \hat{\omega}_j + \frac{1}{\rho} \frac{\partial \hat{\sigma}'_{km}}{\partial x_m}, \quad (3.39)$$

with δ_{mk} Kronecker's delta. Multiplication of this equation with \hat{u}_k then leads to the energy balance equation:

$$\begin{aligned} \hat{u}_k \frac{\partial \hat{u}_k}{\partial t} + \hat{u}_k \hat{u}_m \frac{\partial U_k}{\partial x_m} + \hat{u}_k U_m \frac{\partial \hat{u}_k}{\partial x_m} + \frac{1}{\rho} \hat{u}_k \frac{\partial}{\partial x_m} (\hat{\pi} \delta_{km}) = \\ = \hat{u}_k \epsilon_{ijk} U_i^S \hat{\omega}_j + \frac{1}{\rho} \hat{u}_k \frac{\partial \hat{\sigma}'_{km}}{\partial x_m}. \end{aligned} \quad (3.40)$$

Introducing the mean kinetic energy by

$$\mathcal{K} = \iiint dxdydz \frac{1}{2} \hat{\mathbf{u}} \cdot \hat{\mathbf{u}} \equiv \int_{-h}^0 dz \left\langle \frac{1}{2} \hat{\mathbf{u}} \cdot \hat{\mathbf{u}} \right\rangle \quad (3.41)$$

it follows from (3.40) (see Joseph, 1976, pp. 11-12) that the total change in mean kinetic energy is given by

$$\frac{d\mathcal{K}}{dt} = - \iiint d\mathbf{x} \left(\frac{1}{2} \hat{u}_i \hat{u}_j \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) + \bar{v}_T \left(\frac{\partial \hat{u}_i}{\partial x_j} \right)^2 + \mathbf{U}^S \cdot [\hat{\mathbf{u}} \wedge (\boldsymbol{\Omega} + \boldsymbol{\omega})] \right). \quad (3.42)$$

Application of the simplifications (3.35) results in a change of kinetic energy of the perturbed velocities:

$$\frac{d\mathcal{K}}{dt} = - \int_{-h}^0 dz \left\{ \langle \hat{u}_i \hat{u}_j \rangle \frac{\partial}{\partial z} (U_j^h + U_j^S) \right\} - \int_{-h}^0 dz \left\{ \bar{v}_T \left\langle \left(\frac{\partial \hat{u}_i}{\partial x_j} \right)^2 \right\rangle \right\}. \quad (3.43)$$

Radder (1998, Eq. (8)) now considers solutions for $\hat{\mathbf{u}}$ of the following form:

$$\hat{\mathbf{u}}(x, y, z, t) = \text{Re} \{ A(t) \mathbf{v}(z) \exp [i\chi(x, y, t)] \} + \text{HOT}, \quad (3.44a)$$

$$= \frac{1}{2} A(t) \mathbf{v}(z) \exp [i\chi(x, y, t)] + CC \quad (3.44b)$$

with

⁵Introduction of the antisymmetrical tensor ϵ_{ijk} permits us to write the cross product $\mathbf{a} \wedge \mathbf{b}$ as $\epsilon_{ijk} a_i b_j$. Notice that ϵ_{ijk} equals 1 for all different indices and even numbers of permutations of the indices; it equals -1 for all different indices and uneven permutations of the indices; it equals 0 if two arbitrary indices are equal, see Hinze (1975, p. 775). The values can be memorized by: $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = +1$ and $\epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1$. In component notation the term $\mathbf{U}^S \wedge \boldsymbol{\omega}$ then becomes $\epsilon_{ijk} U_i^S \hat{\omega}_j$.

$$\chi(x, y, t) = \mathbf{k} \cdot \mathbf{x} + \phi(t) = (k_1x + k_2y) + \phi(t) \quad (3.44c)$$

where $A(t)$ and $\phi(t)$ are real and $\mathbf{v}(z)$ is complex. Radder now applies Stuart's (1958) method, which consists of the following suppositions:

1. The flow is near a *local* equilibrium.

This means that in a first approximation, the mean flow is independent of time. The simplified mean-flow equation (3.37) then simplifies further to

$$\left\langle \frac{\partial}{\partial x_j} (\hat{u}_j \hat{u}_i) + \frac{\partial}{\partial z} (\hat{w} \hat{u}_i) \right\rangle + \nabla \Pi_0 = \frac{\partial}{\partial z} \left(\bar{\nu}_T \frac{\partial U_i^h}{\partial z} \right) \quad i, j = 1, 2. \quad (3.45)$$

Notice that we have the terms $\partial_{x_j} (\hat{u}_j \hat{u}_i)$, $i, j = 1, 2$, extra compared to Radder (1998, Eqs. (7)).

An integration over depth, from $z = -h$ to the arbitrary level z of equations (3.45) yields

$$\int_{-h}^z dz' \left\langle \frac{\partial}{\partial x_j} (\hat{u}_j \hat{u}_i) \right\rangle + \langle \hat{w} \hat{u}_i \rangle|_{z'=-h}^z + \int_{-h}^z dz' \nabla \Pi_0 = \left(\bar{\nu}_T \frac{\partial U_i^h}{\partial z'} \right) \Big|_{z'=-h}^z. \quad (3.46)$$

To evaluate these results, boundary conditions have to be specified at the bottom. In Dingemans (1997, Eq. (1.44)) the kinematic bottom condition is given by

$$\tau_k^b |\text{grad } B| = \sigma'_{jk} \frac{\partial h}{\partial x_j} + \sigma'_{3k} \quad k = 1, 2, 3 \quad \text{at } z = -h(\mathbf{x}), \quad (3.47)$$

where $B = z + h(\mathbf{x})$, $\mathbf{x} = (x_1, x_2)^T$, $\text{grad } B = (\partial h / \partial x, \partial h / \partial y, 1)^T$ and τ^b is the bottom shear stress, of which the sign is chosen in such a way that the action of the fluid on the bed is considered to be positive. For a horizontal bottom we have $|\text{grad } B| = 1$ and the bottom shear stress is defined by

$$\tau_k^b = \sigma'_{3k} \quad , \quad k = 1, 2, 3 \quad \text{at } z = -h. \quad (3.48a)$$

It follows from (3.11) that the mean and perturbed part of the bottom shear stress are given by:

$$\begin{aligned} \bar{\tau}_k^b = \Sigma'_{3k} &= \rho \bar{\nu}_T \left(\frac{\partial W}{\partial x_k} + \frac{\partial U_k}{\partial z} \right) + \rho \hat{\nu}_T \left(\frac{\partial \hat{w}}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial z} \right) = \\ &= \rho \bar{\nu}_T \frac{\partial U_k}{\partial z} + \mathcal{O}(\mu^2) \quad \text{at } z = -h \end{aligned} \quad (3.48b)$$

$$\hat{\tau}_k^b = \hat{\sigma}'_{3k} = \rho \hat{\nu}_T \frac{\partial U_k}{\partial z} + \rho \bar{\nu}_T \left(\frac{\partial \hat{w}}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial z} \right) \quad \text{at } z = -h, \quad (3.48c)$$

where (3.48b) and (3.48c) follow from the assumption that the mean velocities are (near)-horizontal and therefore $W \cong 0$.

The dynamic bottom conditions, in presence of viscosity, demand that the velocities at the bottom are zero:

$$\hat{u}_i = 0 \quad , \quad i = 1, 2 \quad \text{and} \quad \hat{w} = 0 . \tag{3.49}$$

Because use is made of reversing the order of differentiation and integration, the general expression for some function $F(x, z)$ is given below:

$$\frac{\partial}{\partial x} \int_{\alpha(x)}^{\beta(x)} F(x, z) dz = \int_{\alpha(x)}^{\beta(x)} \frac{\partial}{\partial x} F(x, z) dz + F(x, \beta) \frac{\partial \beta}{\partial x} - F(x, \alpha) \frac{\partial \alpha}{\partial x} . \tag{3.50}$$

We then have for a horizontal bottom

$$\int_{-h}^z dz' \left\langle \frac{\partial}{\partial x_j} (\hat{u}_j \hat{u}_i) \right\rangle = \frac{\partial}{\partial x_j} \int_{-h}^z dz' \langle \hat{u}_j \hat{u}_i \rangle . \tag{3.51a}$$

Using the representation (3.44) for \hat{u} , results in

$$\begin{aligned} \langle \hat{u}_i \hat{u}_j \rangle &= \frac{1}{4} A^2(t) \left\langle (v_i e^{i\chi} + v_i^* e^{-i\chi}) (v_j e^{i\chi} + v_j^* e^{i\chi}) \right\rangle \\ &= \frac{1}{4} A^2(t) [v_i v_j^* + CC] = \frac{1}{2} A^2(t) \text{Re} \{ v_i v_j^* \} \end{aligned} \tag{3.51b}$$

We notice now that all dependence on the horizontal coordinates x and y has disappeared in the expression for $\langle \hat{u}_i \hat{u}_j \rangle$. The result only depends on $A(t)$ and $v(z)$. Subsequent differentiation to x_j then gives zero contribution. We stress that this is only true for the representation (3.44). Were the representation (3.20) chosen, then the contribution $(\partial_{x_j} \langle \hat{u}'_j \hat{u}'_i \rangle) \int_{-h}^z f^2(z') dz'$ would also be part of \mathcal{T}_i in (3.52). In that case, the term with the \hat{u}'_j and \hat{u}'_i is extra compared to the expression given by Radder (1998, Eqs. (9)). However, with the condition of uni-periodicity in one direction, this extra term disappears.

Using (3.51a) and (3.48) in Eq. (3.46) then results in:

$$\begin{aligned} \bar{v}_T \frac{\partial U_i^h}{\partial z} &= \langle \hat{w} \hat{u}_i \rangle + \bar{\tau}_i^b + \int_{-h}^z dz' \nabla \Pi_0 \\ &\equiv \langle \hat{w} \hat{u}_i \rangle + \mathcal{T}_i . \end{aligned} \tag{3.52}$$

2. The dominant interaction is that between the mean flow and the *first harmonic* of the disturbance velocity, i.e., the higher-order terms in (3.44) are neglected. This approximation implies that only the effects on the first harmonic are investigated. A possible generation of higher harmonics is not accounted for.
3. The *shape* assumption is involved. This implies that the disturbance \hat{u} is similar in *shape* to the solution of linear theory. In our view this condition is already implied in the previous one.

We now use expressions (3.44) and (3.52) in the expression for the change of simplified kinetic energy equation (3.43). It is supposed now that an equilibrium flow exists and that \mathbf{U} can be distorted by the perturbations $\hat{\mathbf{u}}$ such that $dK/dt = 0$ (Stuart, 1958, p. 8).

First we compute $\langle \hat{w}\hat{u}_j \rangle$. We have

$$\begin{aligned}\hat{w}\hat{u}_j &= \frac{1}{2} (Av_3 e^{i\chi} + Av_3^* e^{-i\chi}) \cdot \frac{1}{2} (Av_j e^{i\chi} + Av_j^* e^{-i\chi}) = \\ &= \frac{1}{4} A^2 (v_j v_3 e^{i2\chi} + v_j v_3^*) + CC.\end{aligned}\quad (3.53a)$$

leading to⁶

$$\langle \hat{w}\hat{u}_j \rangle = \frac{1}{2} A^2 \text{Re} \{v_j v_3^*\}.\quad (3.53b)$$

The term $\partial_z U^h$ is given by Eq. (3.52), while U^S is taken to be known. Rest us to find an expression for $\langle (\partial\hat{u}_i/\partial x_j)^2 \rangle$. Substitution of expression (3.44), shows that

$$\frac{\partial\hat{u}_i}{\partial z} = \frac{1}{2} A(t) \frac{\partial v_i}{\partial z} e^{i\chi} + CC\quad (3.53c)$$

$$\frac{\partial\hat{u}_i}{\partial x_j} = \frac{1}{2} A(t) v_i(z) i k_j e^{i\chi(\mathbf{x},t)} + CC, \quad j = 1, 2.\quad (3.53d)$$

This leads to

$$\left\langle \left(\frac{\partial\hat{u}_i}{\partial z} \right)^2 \right\rangle = \frac{1}{2} A^2 \frac{\partial v_i}{\partial z} \frac{\partial v_i^*}{\partial z} = \frac{1}{2} A^2 \left| \frac{\partial v_i}{\partial z} \right|^2\quad (3.53e)$$

and

$$\left\langle \left(\frac{\partial\hat{u}_i}{\partial x_j} \right)^2 \right\rangle = \frac{1}{2} A^2 k_j^2 v_i v_i^* = \frac{1}{2} A^2 k_j^2 |v_i|^2, \quad j = 1, 2.\quad (3.53f)$$

The left-hand side of (3.43) reads $\int dz \frac{1}{2} \langle (\hat{u}_i)^2 \rangle$. We have

$$\left\langle \frac{1}{2} \hat{\mathbf{u}} \cdot \hat{\mathbf{u}} \right\rangle = \frac{1}{4} A^2 |v_i|^2.\quad (3.53g)$$

Substitution of expressions (3.53) in the equation for the change of kinetic energy, Eq. (3.43), yields:

$$\begin{aligned}\frac{d}{dt} \int_{-h}^0 dz \frac{1}{4} A^2 |v_i|^2 &= \\ &= - \int_{-h}^0 dz \frac{1}{2} A^2 \text{Re} \{v_i v_3^*\} \left[\frac{\partial U_i^S}{\partial z} + \frac{1}{\bar{\nu}_T} \left(\frac{1}{2} A^2 \text{Re} \{v_i^* v_3\} + \mathcal{T}_i \right) \right] \\ &\quad - \int_{-h}^0 dz \bar{\nu}_T \frac{1}{2} A^2 \left(\left| \frac{\partial v_i}{\partial z} \right|^2 + k_j^2 |v_i|^2 \right).\end{aligned}\quad (3.54)$$

⁶Because the average implies integration over horizontal space and the shape functions v_j are only functions of z , we have $\langle v_j \rangle = v_j$ and averages over the shape functions need not to be taken.

Because A is a function of t only and $\mathbf{v} = \mathbf{v}(z)$, this equation can be written as

$$\gamma_1 \frac{dA^2}{dt} = -\gamma_2 A^2 - \gamma_3 A^4 - \gamma_4 A^2 - \gamma_5 A^2, \quad (3.55)$$

where the coefficients γ_m are given by the integrals

$$\gamma_1 = \int_{-h}^0 dz \frac{1}{4} |v_i|^2 \quad (3.56a)$$

$$\gamma_2 = \int_{-h}^0 dz \frac{1}{2} \operatorname{Re} \{v_i v_3^*\} \frac{\partial U_i^S}{\partial z} \quad (3.56b)$$

$$\gamma_3 = \int_{-h}^0 dz \frac{1}{4\bar{\nu}_T} |v_i v_3|^2 \quad (3.56c)$$

$$\gamma_4 = \int_{-h}^0 dz \frac{1}{2\bar{\nu}_T} \operatorname{Re} \{v_i v_3^*\} + \mathcal{T}_i \quad (3.56d)$$

$$\gamma_5 = \int_{-h}^0 dz \frac{\bar{\nu}_T}{2} \left(\left| \frac{\partial v_i}{\partial z} \right|^2 + k_j^2 |v_i|^2 \right). \quad (3.56e)$$

Introducing new coefficients β_m by

$$\beta_1 = -\frac{\gamma_2 + \gamma_4}{\gamma_1} \quad (3.57a)$$

$$\beta_2 = \frac{\gamma_5}{\gamma_1} > 0 \quad (3.57b)$$

$$\beta_3 = \frac{\gamma_3}{\gamma_1} > 0. \quad (3.57c)$$

Notice that the sign of β_1 is not clear at this stage. The coefficient γ_2 denotes the effect of the mean flow and the Stokes drift, while γ_4 represents the effect of the mean flow and the Reynolds stresses. The coefficient γ_5 gives the dissipation of the mean flow.

Introducing the coefficients (3.57), the amplitude equation then reads

$$\frac{dA^2}{dt} = \beta_1 A^2 - \beta_2 A^2 - \beta_3 A^4 \quad (3.58)$$

which is the form given by Stuart (1958) and Radder (1998, Eq. (10)). The coefficient β_3 is the so-called Landau coefficient. This equation is known as *Landau-Stuart* equation. A similar equation is found in Landau and Lifchitz (1989, Eq. (26.7)); it is then written as

$$\frac{d|A|^2}{dt} = 2\sigma |A|^2 - \ell |A|^4, \quad (3.59)$$

which is also given in Drazin and Reid (1981, Eq. (49.3)). We take A to be positive.

3.4 Some solutions of the Landau-Stuart equation

We investigate the Landau-Stuart equation in the form (3.59) so as to be as close as possible to the analysis of Landau and Lifchitz (1989, §26) and Drazin and Reid (1981,

§49). Rewriting Eq. (3.59) for A^2 in terms of a linear differential equation for A^{-2} :

$$\frac{dA^{-2}}{dt} + 2\sigma A^{-2} = \ell \quad (3.60)$$

which has the general solution

$$A^{-2} = \frac{\ell}{2\sigma} + C_1 e^{-2\sigma t}, \quad (3.61)$$

with C_1 some integration constant. With the initial condition $A(0) = A_0$ we have (Drazin and Reid, 1981, p. 371)

$$A^{-2} = \frac{\ell}{2\sigma} + \left(\frac{1}{A_0^2} - \frac{\ell}{2\sigma} \right) e^{-2\sigma t} \quad (3.62a)$$

and therefore

$$A^2 = \frac{1}{\frac{\ell}{2\sigma} + \left(\frac{1}{A_0^2} - \frac{\ell}{2\sigma} \right) e^{-2\sigma t}}. \quad (3.62b)$$

The limiting behaviour for $t \rightarrow \infty$ depends on the sign of σ . For $\sigma > 0$ we have $A^2 \rightarrow (2\sigma/\ell) \equiv A_e^2$, independent of the initial condition. A_e is an equilibrium solution and the flow is stable for $\sigma > 0$. For $\sigma < 0$ we obtain $A^2 \rightarrow 0$.

Notice that the coefficients in the Landau-Stuart equation still depend on the vertical shape functions $\mathbf{v}(z)$, which are part of the description of the perturbation velocities $\hat{\mathbf{u}}$. These shape functions $\mathbf{v}(z)$ are determined from the momentum equation for the perturbation velocities, Eq. (3.14). We therefore consider this equation first.

3.5 Simplification and analysis of the perturbed velocity momentum equations (3.14)

We suppose that the perturbation velocity $\hat{\mathbf{u}}$ is so small that linearisation of (3.14) is permitted. Eq. (3.14) then simplifies to:

$$\frac{\partial \hat{\mathbf{u}}}{\partial t} + (\hat{\mathbf{u}} \cdot \text{grad}) \mathbf{U} + (\mathbf{U} \cdot \text{grad}) \hat{\mathbf{u}} + \text{grad } \hat{\pi} = \mathbf{U}^S \wedge \hat{\boldsymbol{\omega}} + \frac{1}{\rho} \text{div } \hat{\boldsymbol{\sigma}}', \quad (3.63)$$

where $\rho^{-1} \text{div } \hat{\boldsymbol{\sigma}}'$ is given by (3.33).

We now also use the simplification that the basic flow \mathbf{U} and the Stokes drift \mathbf{U}^S are (nearly) horizontal, i.e., conditions (3.35) apply. The momentum equation for the perturbation velocities then further simplify into:

$$\frac{\partial \hat{\mathbf{u}}^h}{\partial t} + (\hat{\mathbf{u}} \cdot \text{grad}) \mathbf{U}^h + (\mathbf{U}^h \cdot \nabla) \hat{\mathbf{u}}^h + \nabla \hat{\pi} = (\mathbf{U}^S \wedge \hat{\boldsymbol{\omega}})^h + \frac{1}{\rho} (\text{div } \hat{\boldsymbol{\sigma}}')^h \quad (3.64)$$

where the superscript h denotes two-dimensional (horizontal) vectors, i.e., $\hat{\mathbf{u}}^h = (\hat{u}, \hat{v})^T$. Using also the fact that the basic current does not depend on the horizontal coordinates, see conditions (3.35), Eq. (3.64) simplifies further to

$$\frac{\partial \hat{\mathbf{u}}^h}{\partial t} + \hat{w} \frac{\partial \mathbf{U}^h}{\partial z} + (\mathbf{U}^h \cdot \nabla) \hat{\mathbf{u}}^h + \nabla \hat{\pi} = (\mathbf{U}^S \wedge \hat{\boldsymbol{\omega}})^h + \frac{1}{\rho} (\text{div } \hat{\boldsymbol{\sigma}}')^h, \quad (3.65a)$$

where

$$\begin{aligned} \frac{1}{\rho} (\text{div } \hat{\boldsymbol{\sigma}}')^h &= \frac{1}{\rho} \frac{\partial \hat{\sigma}'_{ik}}{\partial x_k} = \frac{\partial}{\partial z} \left(\hat{\nu}_T \frac{\partial U_i}{\partial z} \right) + \frac{\partial}{\partial z} \left(\bar{\nu}_T \left(\frac{\partial \hat{u}_i}{\partial z} + \frac{\partial \hat{w}}{\partial x_i} \right) \right) \\ &\quad + \bar{\nu}_T \frac{\partial}{\partial x_j} \left(\frac{\partial \hat{u}_i}{\partial x_j} + \frac{\partial \hat{u}_j}{\partial x_i} \right) \\ &\text{with } k = 1, 2, 3 \text{ and } i, j = 1, 2. \end{aligned} \quad (3.65b)$$

The vertical momentum equation becomes:

$$\begin{aligned} \frac{\partial \hat{w}}{\partial t} + (\mathbf{U}^h \cdot \nabla) \hat{w} + \frac{\partial \hat{\pi}}{\partial z} &= (U^S \hat{\omega}_2 - V^S \hat{\omega}_1) + 2 \frac{\partial}{\partial z} \left(\bar{\nu}_T \left(\frac{\partial \hat{w}}{\partial z} \right) \right) + \\ &\quad + \frac{\partial \hat{\nu}_T}{\partial x_j} \frac{\partial U_j^h}{\partial z} + \bar{\nu}_T \frac{\partial}{\partial x_j} \left(\frac{\partial \hat{w}}{\partial x_j} + \frac{\partial \hat{u}_j}{\partial z} \right) \\ &\text{with } j = 1, 2. \end{aligned} \quad (3.65c)$$

Notice that the vortex force for the perturbation, $\hat{\mathbf{T}}$, becomes, in the present approximation of near-horizontal basic flow:

$$\hat{\mathbf{T}}_a = (V^S \hat{\omega}_3, -U^S \hat{\omega}_3, U^S \hat{\omega}_2 - V^S \hat{\omega}_1)^T \quad (3.65d)$$

with

$$\hat{\boldsymbol{\omega}} = (\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3)^T = \text{curl } \hat{\mathbf{u}} = \left(\frac{\partial \hat{w}}{\partial y} - \frac{\partial \hat{v}}{\partial z}, \frac{\partial \hat{u}}{\partial z} - \frac{\partial \hat{w}}{\partial x}, \frac{\partial \hat{v}}{\partial x} - \frac{\partial \hat{u}}{\partial y} \right)^T, \quad (3.65e)$$

and thus,

$$(\mathbf{U}^S \wedge \hat{\boldsymbol{\omega}})^h = \left(V^S \left(\frac{\partial \hat{v}}{\partial x} - \frac{\partial \hat{u}}{\partial y} \right), -U^S \left(\frac{\partial \hat{v}}{\partial x} - \frac{\partial \hat{u}}{\partial y} \right) \right)^T. \quad (3.65f)$$

4 Linear instability analysis

4.1 Introduction

The solution of Eqs. (3.65) is the subject of this Chapter. Following Cox (1997), Radder (1998) assumes that these equations can be solved asymptotically by applying a long-wave expansion. This expansion is based on the observation that Langmuir circulations have a much larger horizontal extent (in the direction perpendicular to the circulation) than the extent of the circulation cells. The small parameter is a Biot number which arises in the boundary conditions on the horizontal components of the stress. The boundary conditions for the perturbed velocities in this case become:

$$\frac{\partial \hat{u}}{\partial z} = \frac{\partial \hat{v}}{\partial z} = w = 0 \quad \text{at } z = 0 \quad (4.1a)$$

and

$$\frac{\partial \hat{u}}{\partial z} = \frac{\partial \hat{v}}{\partial z} = w = 0 \quad \text{at } z = -h \quad (4.1b)$$

It is noted that the conditions (4.1b) do not comply with the no-slip conditions, which should apply for viscous flow as is considered here. It seems reasonable to limit this stability analysis to the bulk of the fluid, just outside the bottom boundary layer.

Cox (1997) introduces the small parameter ε by

$$\varepsilon = (2\alpha_\ell + \alpha_b)^{1/4} \quad (4.2a)$$

where the Biot numbers α_ℓ and α_b are defined by

$$0 < \alpha_\ell = \frac{u_* R_*}{S} \ll 1 \quad (4.2b)$$

where S is the wind velocity and u_* is the friction velocity, defined by $\rho_w u_*^2 = C_m \rho_a S^2$ and R_* is a Reynold number based on the friction velocity: $R_* = u_* h / \nu_T$. The Biot number α_b is defined by

$$0 < \alpha_b = \frac{h w_c}{\nu_T} \ll 1, \quad (4.2c)$$

where w_c is a small speed of entrainment of abyssal fluid into the mixed layer.

The most unstable rolls have wave lengths $\mathcal{O}(\varepsilon^{-1})$. Cox (1997) further notices that for α_ℓ and α_b equal to zero, rolls of infinite width are predicted. For small, but positive values of α_ℓ and α_b circulation cells are predicted with much larger width than depth, contrary to the case we look for. However, previous numerical experiments of Cox and Leibovich (1993) have shown that nevertheless rolls with comparable horizontal and vertical dimensions are found.

4.2 The expansions

We now assume a slow growth rate σ and the expansions of $\hat{\mathbf{u}}$ and $\hat{\pi}$ are:

$$\hat{\mathbf{u}} = (\hat{\mathbf{u}}_0(z) + \varepsilon \hat{\mathbf{u}}_1(z) + \dots) \exp \left[i\varepsilon \tilde{\mathbf{k}} \cdot \mathbf{x} + \varepsilon (\sigma_1 + \varepsilon^2 \sigma_2 + \dots) t \right] \quad (4.3a)$$

$$\hat{\pi} = (\hat{\pi}_0(z) + \varepsilon \hat{\pi}_1(z) + \dots) \exp \left[i\varepsilon \tilde{\mathbf{k}} \cdot \mathbf{x} + \varepsilon (\sigma_1 + \varepsilon^2 \sigma_2 + \dots) t \right] \quad (4.3b)$$

with $\tilde{\mathbf{k}} = (\tilde{k}_1, \tilde{k}_2)^T$ the scaled wave number vector: $\tilde{\mathbf{k}} = \varepsilon \mathbf{k}$ with $|\tilde{\mathbf{k}}| = \mathcal{O}(1)$. In this way one focusses attention to the most unstable wave numbers \mathbf{k} , which are $\mathcal{O}(\varepsilon)$. Comparison with (3.44) shows that for $A(t)$ is written now $\exp(\sigma t)$.

In view of next analysis, we also write expansions (4.3) in the following short-hand notation:

$$\hat{\mathbf{u}}(\mathbf{x}, z, t) = \mathbf{u}'(z) e^{\vartheta} \quad (4.4a)$$

$$\hat{\pi}(\mathbf{x}, z, t) = \pi'(z) e^{\vartheta} \quad (4.4b)$$

with

$$\vartheta(\mathbf{x}, t) = i\varepsilon \tilde{\mathbf{k}} \cdot \mathbf{x} + \varepsilon \sigma t \quad (4.4c)$$

$$\sigma = \sigma_1 + \varepsilon \sigma_2 + \dots \quad (4.4d)$$

$$\mathbf{u}'(z) = \hat{\mathbf{u}}_0(z) + \varepsilon \hat{\mathbf{u}}_1(z) + \varepsilon^2 \hat{\mathbf{u}}_2(z) + \dots \quad (4.4e)$$

$$\pi'(z) = \hat{\pi}_0(z) + \varepsilon \hat{\pi}_1(z) + \varepsilon^2 \hat{\pi}_2(z) + \dots \quad (4.4f)$$

Substitution of the expansions (4.4) in the linearised momentum equations for the perturbed velocities (3.65) gives for each power of ε a set of equations:

$$\begin{aligned} \varepsilon \sigma \mathbf{u}' + w' \frac{\partial \mathbf{U}^h}{\partial z} + i\varepsilon (\tilde{\mathbf{k}} \cdot \mathbf{U}^h) \mathbf{u}' + i\varepsilon \tilde{\mathbf{k}} \pi' &= i\varepsilon (\tilde{k}_1 v' - \tilde{k}_2 u') \begin{pmatrix} V^S \\ -U^S \end{pmatrix} \\ + \frac{\partial}{\partial z} \left(\bar{\nu}_T \frac{\partial \mathbf{u}'}{\partial z} \right) + \frac{\partial}{\partial z} \left(\hat{\nu}_T \frac{\partial \mathbf{U}^h}{\partial z} \right) + i\varepsilon \mathbf{k} \frac{\partial}{\partial z} (\bar{\nu}_T w') \\ - \varepsilon^2 \bar{\nu}_T \left(|\tilde{\mathbf{k}}|^2 \mathbf{u}' + (\tilde{\mathbf{k}} \cdot \mathbf{u}') \tilde{\mathbf{k}} \right), \end{aligned} \quad (4.5a)$$

and

$$\begin{aligned} \varepsilon \sigma w' + i\varepsilon (\tilde{\mathbf{k}} \cdot \mathbf{U}^h) w' + \frac{\partial \pi'}{\partial z} &= U^S \cdot \frac{\partial \mathbf{u}'}{\partial z} - i\varepsilon (\tilde{\mathbf{k}} \cdot \mathbf{U}^S) w' + \\ + 2 \frac{\partial}{\partial z} \left(\bar{\nu}_T \frac{\partial w'}{\partial z} \right) + \nabla \hat{\nu}_T \cdot \frac{\partial \mathbf{U}^h}{\partial z} + i\varepsilon \bar{\nu}_T \tilde{\mathbf{k}} \cdot \frac{\partial \mathbf{u}'}{\partial z} - \varepsilon^2 \bar{\nu}_T |\tilde{\mathbf{k}}|^2 w'. \end{aligned} \quad (4.5b)$$

Remark

In the expansion used to obtain Eqs. (4.5) no provision was made for the ordering of the mean and perturbed eddy viscosity. As stated earlier, we have $\hat{\nu}_T / \bar{\nu}_T = \mathcal{O}(\mu)$ and to make this explicit we can write

$$\hat{\nu}_T = \mu \hat{\nu}'_T \quad \text{with} \quad \hat{\nu}'_T = \mathcal{O}(1). \quad (4.6)$$

Because we have

$$\hat{v}_T = c_2(z)\hat{u}^* \quad (4.7)$$

and \hat{u}^* is linked to $\hat{\mathbf{u}}$, also an expansion of the friction velocity \hat{u}^* and therefore also of \hat{v}_T should be envisaged. Then we would have

$$\begin{aligned} \hat{v}_T = \mu \hat{v}'_T &= \mu [\nu_0 + \varepsilon \nu_1 + \dots] e^{i\vartheta} \\ &= \mu [c_2(z)u_0^* + \varepsilon c_2(z)u_1^* + \dots] e^{i\vartheta}, \end{aligned} \quad (4.8)$$

with u_0^* and u_1^* constants. Applying the principle of least simplification, we suppose μ and ε to be of the same order of magnitude, $\mu \sim \varepsilon$. We will be writing μ and ε in the sequel for easy recognition. We then have

$$\begin{aligned} \varepsilon \sigma \mathbf{u}' + w' \frac{\partial \mathbf{U}^h}{\partial z} + i\varepsilon (\tilde{\mathbf{k}} \cdot \mathbf{U}^h) \mathbf{u}' + i\varepsilon \tilde{\mathbf{k}} \pi' &= i\varepsilon (\tilde{k}_1 v' - \tilde{k}_2 u') \begin{pmatrix} V^S \\ -U^S \end{pmatrix} \\ + \frac{\partial}{\partial z} \left(\bar{\nu}_T \frac{\partial \mathbf{u}'}{\partial z} \right) + \mu \frac{\partial}{\partial z} \left(\hat{v}'_T \frac{\partial \mathbf{U}^h}{\partial z} \right) + i\varepsilon \mathbf{k} \frac{\partial}{\partial z} (\bar{\nu}_T w') \\ - \varepsilon^2 \bar{\nu}_T \left(|\tilde{\mathbf{k}}|^2 \mathbf{u}' + (\tilde{\mathbf{k}} \cdot \mathbf{u}') \tilde{\mathbf{k}} \right), \end{aligned} \quad (4.9a)$$

and

$$\begin{aligned} \varepsilon \sigma w' + i\varepsilon (\tilde{\mathbf{k}} \cdot \mathbf{U}^h) w' + \frac{\partial \pi'}{\partial z} &= \mathbf{U}^S \cdot \frac{\partial \mathbf{u}'}{\partial z} - i\varepsilon (\tilde{\mathbf{k}} \cdot \mathbf{U}^S) w' + \\ + 2 \frac{\partial}{\partial z} \left(\bar{\nu}_T \frac{\partial w'}{\partial z} \right) + \mu \nabla \hat{v}'_T \cdot \frac{\partial \mathbf{U}^h}{\partial z} + i\varepsilon \bar{\nu}_T \tilde{\mathbf{k}} \cdot \frac{\partial \mathbf{u}'}{\partial z} - \varepsilon^2 \bar{\nu}_T |\tilde{\mathbf{k}}|^2 w'. \end{aligned} \quad (4.9b)$$

We also still have the continuity equation for the perturbed velocities. Inserting the expansions (4.4) in Eq. (3.15b) yields:

$$\varepsilon i (\tilde{k}_1 u' + \tilde{k}_2 v') + \frac{\partial w'}{\partial z} = 0. \quad (4.9c)$$

4.3 The various order equations

4.3.1 The zeroth-order equations

In zeroth order (i.e., for the ε^0 terms) Eqs. (4.9) yield

$$w_0 \partial_z U = \partial_z (\bar{\nu}_T \partial_z u_0) \quad (4.10a)$$

$$w_0 \partial_z V = \partial_z (\bar{\nu}_T \partial_z v_0) \quad (4.10b)$$

$$\partial_z \pi_0 = U^S \partial_z u_0 + V^S \partial_z v_0 + 2 \partial_z (\bar{\nu}_T \partial_z w_0). \quad (4.10c)$$

From the continuity equation (4.9c) we obtain

$$\partial_z w_0 = 0 \quad \Rightarrow \quad w_0 = \text{constant}. \quad (4.10d)$$

From the boundary conditions (4.1) it follows that $w_0 = 0$ at the boundary, and therefore, with (4.10d) $w_0 = 0$ everywhere. The zeroth-order equations (4.10a) and (4.10b) then simplify to:

$$\bar{v}_T \partial_z u_0 = \bar{v}_T \partial_z v_0 = \text{constant} . \quad (4.11)$$

Because the boundary conditions (4.1) show that $\partial_z u_0 = \partial_z v_0 = 0$ at $z = -h$ and $z = 0$, it follows that u_0 and v_0 are constant in the fluid domain. It now also follows that $\partial_z \pi_0 = 0$ and thus, π_0 is also constant.

4.3.2 First-order equations

At first order we obtain from (4.9) the following momentum equations:

$$\begin{aligned} \sigma_1 u_0 + w_1 \partial_z U + i (\tilde{\mathbf{k}} \cdot \mathbf{U}) u_0 + i k_1 \pi_0 = i (\tilde{k}_1 v_0 - \tilde{k}_2 u_0) V^S + \\ + \partial_z (\hat{v}'_T \partial_z U) + \partial_z (\bar{v}_T \partial_z u'_1) , \end{aligned} \quad (4.12a)$$

$$\begin{aligned} \sigma_1 v_0 + w_1 \partial_z V + i (\tilde{\mathbf{k}} \cdot \mathbf{U}) v_0 + i k_2 \pi_0 = \\ = -i (\tilde{k}_1 v_0 - \tilde{k}_2 u_0) U^S + 2 \partial_z (\bar{v}_T \partial_z v_1) , \end{aligned} \quad (4.12b)$$

and

$$\partial_z \pi_1 = U^S \partial_z u_1 + V^S \partial_z v_1 + \partial_z (\bar{v}_T \partial_z w_1) + \nabla \hat{v}'_T \cdot \frac{\partial \mathbf{U}^h}{\partial z} . \quad (4.12c)$$

From the continuity equation we obtain

$$i (\tilde{k}_1 u_0 + \tilde{k}_2 v_0) + \partial_z w_1 = 0 . \quad (4.12d)$$

The boundary conditions at this order are

$$\partial_z u_1 = \partial_z v_1 = w_1 = 0 \quad \text{at} \quad z = -h \quad \text{and} \quad z = 0 . \quad (4.13)$$

Integration of the continuity equation (4.12d) shows that

$$w_1 = (\tilde{k}_1 u_0 + \tilde{k}_2 v_0) z + c_3 , \quad (4.14a)$$

with c_3 some integration constant. Because $w_1 = 0$ at $z = 0$, we have $c_3 = 0$. The bottom condition yields

$$\tilde{k}_1 u_0 + \tilde{k}_2 v_0 = 0 , \quad (4.14b)$$

and, consequently,

$$w_1 \equiv 0 . \quad (4.14c)$$

Notice that relation (4.14b) gives a condition between the unknown constants u_0 and v_0 .

The horizontal momentum equations (4.12a) and (4.12b) are integrated over depth between the bottom ($z = -h$) and the free surface ($z = 0$). In this integration we note that

$$\int_{-h}^0 \partial_z (\bar{v}_T \partial_z u_1) dz = \bar{v}_T \partial_z u_1 \Big|_{-h}^0 = 0$$

because of the boundary conditions (4.13). We also have

$$\int_{-h}^0 \partial_z (\hat{v}'_T \partial_z U) dz = \hat{v}'_T \partial_z U \Big|_{-h}^0. \quad (4.15)$$

Although the quantity above is definitely not equal to zero, it can be argued that it will not be large. It has to be remembered that the CL-approximation is valid only for situations in which the shear of the mean current is not very large. We therefore neglect this contribution for the moment. For the same reason then also the last term in (4.12c) can be neglected, which we will do.

Using also $w_1 = 0$, the first vertically integrated horizontal momentum equation becomes:

$$\begin{aligned} \sigma_1 u_0 + i \tilde{k}_1 \pi_0 &= -i \left\{ \tilde{k}_1 \left(\frac{1}{h} \int_{-h}^0 U dz \right) + \tilde{k}_2 \left(\frac{1}{h} \int_{-h}^0 V dz \right) \right\} u_0 + \\ &+ i \left(\tilde{k}_1 v_0 - \tilde{k}_2 u_0 \right) \left(\frac{1}{h} \int_{-h}^0 V^S dz \right) \end{aligned}$$

We now introduce the vertically-averaged quantities, denoted by a double overbar:

$$\overline{\overline{U}} = (\overline{\overline{U}}, \overline{\overline{V}})^T = \frac{1}{h} \int_{-h}^0 \mathbf{U}(z) dz \quad (4.16a)$$

$$\overline{\overline{U^S}} = (\overline{\overline{U^S}}, \overline{\overline{V^S}})^T = \frac{1}{h} \int_{-h}^0 \mathbf{U}^S(z) dz. \quad (4.16b)$$

The vertically-integrated horizontal momentum equations then lead to

$$\sigma_1 u_0 + i \tilde{k}_1 \pi_0 = -i \left(\tilde{k}_1 \overline{\overline{U}} + \tilde{k}_2 \overline{\overline{V}} \right) u_0 + i \left(\tilde{k}_1 v_0 - \tilde{k}_2 u_0 \right) \overline{\overline{V^S}} \quad (4.17a)$$

$$\sigma_1 v_0 + i \tilde{k}_2 \pi_0 = -i \left(\tilde{k}_1 \overline{\overline{U}} + \tilde{k}_2 \overline{\overline{V}} \right) v_0 - i \left(\tilde{k}_1 v_0 - \tilde{k}_2 u_0 \right) \overline{\overline{U^S}} \quad (4.17b)$$

It is not immediately clear whether σ_1 is real or imaginary. Suppose it is real. It then follows immediately that $\sigma_1 \mathbf{u}_0 = 0$ and because $u_0 \neq 0$ and $v_0 \neq 0$, σ_1 has to be zero. We therefore suppose that σ_1 is imaginary, and we write for that purpose $\sigma_1 = i \sigma_1^{(i)}$ with $\sigma_1^{(i)}$ real. Solving Eqs. (4.17) for $\sigma_1^{(i)}$ and π_0 , it is found that

$$\pi_0 = u_0 \overline{\overline{U^S}} + v_0 \overline{\overline{V^S}} \equiv \mathbf{u}_0 \cdot \overline{\overline{U^S}} \quad (4.18a)$$

$$\sigma_1^{(i)} = -\tilde{k}_1 \left(\overline{\overline{U}} + \overline{\overline{U^S}} \right) - \tilde{k}_2 \left(\overline{\overline{V}} + \overline{\overline{V^S}} \right) \equiv -\tilde{\mathbf{k}} \cdot \left(\overline{\overline{U}} + \overline{\overline{U^S}} \right). \quad (4.18b)$$

We note that the solution for σ_1 is the same as given in Radder (1998, Appendix), but the solution for π_0 is seemingly different. The solution given in (4.18a) resembles the

one given by Cox (1997, p. 157); the difference with Cox is that Cox did not consider the velocity V^S . The solution for σ_1 also complies with the one given by Cox (1997, Eq. (15)), apart from the term with $\overline{V^S}$. Radder gave for π_0 the solution:

$$\pi_0 = \frac{1}{|\tilde{\mathbf{k}}|^2} \left(\tilde{k}_2 \overline{U^S} - \tilde{k}_1 \overline{V^S} \right) \left(\tilde{k}_2 u_0 - \tilde{k}_1 v_0 \right). \quad (4.18c)$$

Using the relation $\tilde{k}_1 u_0 + \tilde{k}_2 v_0 = 0$, it is readily shown that (4.18c) can be written in the form (4.18a).

4.3.3 Second-order equations

In second order we obtain from (4.9a) the horizontal momentum equations

$$\begin{aligned} i \left(\sigma_1^{(i)} + \tilde{\mathbf{k}} \cdot \mathbf{U} \right) \mathbf{u}_1 + \left(\sigma_2 + \bar{v}_T |\tilde{\mathbf{k}}|^2 \right) \mathbf{u}_0 + w_2 \frac{\partial \mathbf{U}}{\partial z} + i \tilde{\mathbf{k}} \pi_1 = \\ = i \left(\tilde{k}_1 v_1 - \tilde{k}_2 u_1 \right) \begin{pmatrix} V^S \\ -U^S \end{pmatrix} + \partial_z \left(\bar{v}_T \partial_z \mathbf{u}_2 \right). \end{aligned} \quad (4.19a)$$

The vertical momentum equation (4.9b) yields, using the findings that $w_0 = w_1 = 0$:

$$\frac{\partial \pi_2}{\partial z} = U^S \cdot \frac{\partial \mathbf{u}_2}{\partial z} + \frac{\partial}{\partial z} \left(\bar{v}_T \frac{\partial w_2}{\partial z} \right). \quad (4.19b)$$

From the continuity equation follows:

$$i \tilde{\mathbf{k}} \cdot \mathbf{u}_1 + \frac{\partial w_2}{\partial z} = 0. \quad (4.19c)$$

The boundary conditions are:

$$\partial_z u_2 = \partial_z v_2 = w_2 = 0 \quad \text{at} \quad z = -h \quad \text{and} \quad z = 0. \quad (4.19d)$$

Consider Eqs. (4.19c) and the first order horizontal momentum equations (4.12a) and (4.12b). By differentiation we obtain from the second-order continuity equation (4.19c):

$$i \tilde{k}_1 \partial_z \left(\bar{v}_T \partial_z u_1 \right) + i \tilde{k}_2 \partial_z \left(\bar{v}_T \partial_z v_1 \right) + \partial_z \left(\bar{v}_T \partial_z^2 w_2 \right) = 0. \quad (4.20a)$$

From (4.12a) and (4.12b) we obtain the expressions for $\partial_z (\nu_T \partial_z u_i)$ to be substituted in (4.20a). In first instance we obtain:

$$\begin{aligned} - \left(\sigma_1^{(i)} + \tilde{\mathbf{k}} \cdot \mathbf{U} \right) \left(\tilde{k}_1 u_0 + \tilde{k}_2 v_0 \right) - |\tilde{\mathbf{k}}|^2 \pi_0 + \\ + \left(\tilde{k}_1 v_0 - \tilde{k}_2 u_0 \right) \left(\tilde{k}_1 V^S - k_2 U^S \right) + \partial_z \left(\bar{v}_T \partial_z^2 w_2 \right) = 0. \end{aligned} \quad (4.20b)$$

Because we already found in (4.14b) that $\tilde{k}_1 u_0 + \tilde{k}_2 v_0 = 0$, we obtain after substituting the solution $\pi_0 = \mathbf{u}_0 \cdot \overline{\mathbf{U}^S}$:

$$\frac{\partial}{\partial z} \left(\bar{\nu}_T \frac{\partial^2 w_2}{\partial z^2} \right) = |\tilde{\mathbf{k}}|^2 \mathbf{u}_0 \cdot \overline{\mathbf{U}^S} + (\tilde{k}_1 v_0 - \tilde{k}_2 u_0) (\tilde{k}_2 U^S - \tilde{k}_1 V^S) . \quad (4.21a)$$

This differential equation seems different from the one given in Radder (1998, Appendix). Using Radder's solution for π_0 in Eq. (4.20b), it is found that

$$\frac{\partial}{\partial z} \left(\bar{\nu}_T \frac{\partial^2 w_2}{\partial z^2} \right) = \{ \tilde{k}_2 (\overline{U^S} - U^S) - \tilde{k}_1 (\overline{V^S} - V^S) \} (\tilde{k}_2 u_0 - \tilde{k}_1 v_0) . \quad (4.21b)$$

which is the same equation is found as given in Radder (1998). The difference in appearance is therefore wholly attributable to different solutions for π_0 .

Eq. (4.21a) can also be written differently. Using the property that $\tilde{k}_1 u_0 + \tilde{k}_2 v_0 = 0$, we can also write:

$$\begin{aligned} \frac{\partial}{\partial z} \left(\bar{\nu}_T \frac{\partial^2 w_2}{\partial z^2} \right) &= (\tilde{k}_1^2 + \tilde{k}_2^2) \{ u_0 (\overline{U^S} - U^S) + v_0 (\overline{V^S} - V^S) \} \\ &\equiv |\tilde{\mathbf{k}}|^2 \mathbf{u}_0 \cdot (\overline{\mathbf{U}^S} - \mathbf{U}^S) . \end{aligned} \quad (4.21c)$$

The right-hand side is thus zero when no shear is present (i.e., when \mathbf{U}^S is constant over the depth).

By working out the right-hand side of (4.21b) and using the property $\tilde{k}_1 u_0 + \tilde{k}_2 v_0 = 0$, it can be readily shown that Eqs. (4.21c) and (4.21b) are in fact equivalent.

5 The Landau-Stuart equation

5.1 Determination of the coefficients in the Landau-Stuart equation

The results of last section should be used in the coefficients of the Landau-Stuart equation. In deriving the Landau-Stuart equation (3.58) we used the expansion (3.44), which is repeated below:

$$\hat{\mathbf{u}}(\mathbf{x}, z, t) = \text{Re} \left\{ A(t) \mathbf{v}(z) e^{i\chi(\mathbf{x}, t)} \right\} \quad \text{with} \quad \chi(\mathbf{x}, t) = \mathbf{k} \cdot \mathbf{x} + \phi(t). \quad (5.1)$$

In the linear instability analysis, the expansions (4.3) have been used. Of these expansions, only the leading-order terms are used and we have

$$\begin{aligned} \hat{\mathbf{u}}(\mathbf{x}, z, t) &= \frac{1}{2} \begin{pmatrix} u_0 \\ v_0 \\ \varepsilon^2 w_2(z) \end{pmatrix} e^{\vartheta} + CC \quad \text{with} \quad \vartheta = i\varepsilon \tilde{\mathbf{k}} \cdot \mathbf{x} + \varepsilon \sigma_1 t \\ &= \begin{pmatrix} u_0 \\ v_0 \\ \varepsilon^2 w_2(z) \end{pmatrix} \text{Re} \left\{ e^{\vartheta} \right\} \end{aligned} \quad (5.2a)$$

and, for the pressure,

$$\hat{\pi} = \frac{1}{2} \pi_0 e^{\vartheta} + CC = \pi_0 \text{Re} \left\{ e^{\vartheta} \right\}. \quad (5.2b)$$

We already have $\mathbf{k} = \varepsilon \tilde{\mathbf{k}}$. The following relations were found in the instability analysis:

$$\tilde{k}_1 u_0 + \tilde{k}_2 v_0 = 0 \quad (5.3a)$$

$$\pi_0 = \mathbf{u}_0 \cdot \overline{\mathbf{U}^S} \quad (5.3b)$$

$$\sigma_1 = -i\tilde{\mathbf{k}} \cdot \left(\overline{\mathbf{U}} + \overline{\mathbf{U}^S} \right) \quad (5.3c)$$

and for w_2 the following differential equation was obtained:

$$\frac{d}{dz} \left(\bar{\nu}_T \frac{d^2 w_2}{dz^2} \right) = |\tilde{\mathbf{k}}|^2 \mathbf{u}_0 \cdot \left(\overline{\mathbf{U}^S} - \mathbf{U}^S \right). \quad (5.3d)$$

These expansions are equivalent under the following conditions:

$$A(t) v_1(z) = u_0 \quad (5.4a)$$

$$A(t) v_2(z) = v_0 \quad (5.4b)$$

$$A(t) v_3(z) = \varepsilon^2 w_2(z) \quad (5.4c)$$

From these equations it follows that the shape functions v_1 and v_2 are necessarily independent of z , and are therefore constants. Radder (1998, p. 6) introduces a new amplitude, A_0 , by means of

$$A_0 = \sqrt{u_0^2 + v_0^2}. \quad (5.5)$$

We thus have

$$A_0^2 = A^2 (v_1^2 + v_2^2) . \tag{5.6}$$

Using (5.3a), it follows from (5.5) that

$$u_0 = \frac{\tilde{k}_2}{\tilde{k}} A_0 \quad \text{and} \quad v_0 = -\frac{\tilde{k}_1}{\tilde{k}} A_0 , \tag{5.7}$$

where $\tilde{k} = |\tilde{\mathbf{k}}| = \sqrt{\tilde{k}_1^2 + \tilde{k}_2^2}$. We also write¹

$$\varepsilon^2 w_2 = \varepsilon^2 \tilde{k}^2 m_2(z) = k^2 m_2(z) , \tag{5.8}$$

and m_2 satisfies:

$$A_0 \frac{d}{dz} \left(\bar{\nu}_T \frac{d^2 m_2}{dz^2} \right) = \mathbf{u}_0 \cdot \left(\overline{U^S} - U^S \right) , \tag{5.9}$$

with boundary conditions²

$$m_2(z) = \partial_z^2 m_2(z) = 0 \quad \text{at} \quad z = -h \quad \text{and} \quad z = 0 . \tag{5.10}$$

Instead of the expansion (5.2a) we now have the expansion

$$\hat{\mathbf{u}}(\mathbf{x}, z, t) = \frac{1}{2} \begin{pmatrix} \frac{\tilde{k}_2}{\tilde{k}} \\ -\frac{\tilde{k}_1}{\tilde{k}} \\ \varepsilon^2 \tilde{k}^2 m_2(z) \end{pmatrix} A_0 e^{\vartheta} + CC . \tag{5.11}$$

with

$$\vartheta = i\varepsilon \tilde{\mathbf{k}} \cdot \mathbf{x} + i\varepsilon \sigma_1^{(i)} t .$$

It should be recognised that in (5.9) the quantity m_2 also encompasses the phase function ϑ , as does \mathbf{u}_0 . When only the amplitude $A_0 m_2$ is considered, together with the amplitudes $(\tilde{k}_2/\tilde{k})A_0$ for u_0 and $-(\tilde{k}_1/\tilde{k})A_0$ for v_0 , the differential equation for $m_2(z)$ reads

$$\frac{d}{dz} \left(\bar{\nu}_T(z) \frac{d^2 m_2}{dz^2} \right) = \left(\frac{\tilde{k}_2}{\tilde{k}} , -\frac{\tilde{k}_1}{\tilde{k}} \right) \begin{pmatrix} \overline{U^S} - U^S \\ \overline{V^S} - V^S \end{pmatrix}$$

¹We write m_2 to stress the fact that it stands for a second-order solution; Radder (1998) writes m_0 .

²We use the short-hand notation ∂_z for $\partial/\partial z$ or d/dz as is the case here; thus $\partial_z m_2$ stands for dm_2/dz and $\partial_z^2 m_2 \equiv d^2 m_2/dz^2$.

and thus,

$$\frac{d}{dz} \left(\bar{\nu}_T(z) \frac{d^2 m_2}{dz^2} \right) = \frac{1}{\bar{k}} \left[\bar{k}_2 \left(\bar{U}^S - U^S \right) - \bar{k}_1 \left(\bar{V}^S - V^S \right) \right]. \quad (5.12)$$

We now consider the simplified energy equation (3.43). Following Stuart (1958), we now suppose the amplitude A_0 to be a function of time, $A_0 = A_0(t)$. Using the representation (3.53g), this equation becomes:

$$\begin{aligned} \frac{d\mathcal{K}}{dt} = \frac{d}{dt} \left\{ \int_{-h}^0 dz \left\langle \frac{1}{2} \hat{\mathbf{u}} \cdot \hat{\mathbf{u}} \right\rangle \right\} &= - \int_{-h}^0 dz \left\{ \langle \hat{w} \hat{u}_j \rangle \frac{\partial}{\partial z} \left(U_j^h + U_j^S \right) \right\} \\ &\quad - \int_{-h}^0 dz \left\{ \bar{\nu}_T \left\langle \left(\frac{\partial \hat{u}_i}{\partial x_j} \right)^2 \right\rangle \right\}, \end{aligned} \quad (5.13)$$

where

$$\mathcal{K} = \int_{-h}^0 \left\langle \frac{1}{2} \hat{\mathbf{u}} \cdot \hat{\mathbf{u}} \right\rangle dz = \int_{-h}^0 dz \left\{ \frac{1}{\mathcal{A}} \iint_{\mathcal{A}} dx dy \left(\frac{1}{2} \hat{\mathbf{u}} \cdot \hat{\mathbf{u}} \right) \right\}$$

and $i, j = 1, 2, 3$ have been used; \mathcal{A} is the horizontal space over which the average is taken. The representation (5.11) has to be substituted in this equation. Information regarding the terms $\partial_z U_j^h$ is obtained from (3.52). A Landau-Stuart-type equation of the following form results (see Appendix A):

$$\gamma_1 \frac{dA_0^2}{dt} = -\gamma_2 A_0^2 - \gamma_3 A_0^2 - \gamma_4 A_0^2 - \gamma_5 A_0^4 \quad (5.14)$$

with the coefficients γ_i given by:

$$\gamma_1 = \frac{1}{4} \int_{-h}^0 dz \left(1 + \varepsilon^4 \bar{k}^4 m_2^2(z) \right) \quad (5.15a)$$

$$\gamma_2 = \frac{1}{2} \varepsilon^2 \bar{k} \int_{-h}^0 dz \left(\bar{k}_2 \partial_z U^S(z) - \bar{k}_1 \partial_z V^S(z) \right) m_2(z) \quad (5.15b)$$

$$\gamma_3 = \frac{1}{2} \varepsilon^2 \bar{k} \int_{-h}^0 dz \frac{m_2(z)}{\bar{\nu}_T(z)} \left(\bar{k}_2 \mathcal{T}_1 - \bar{k}_1 \mathcal{T}_2 \right) \quad (5.15c)$$

$$\gamma_4 = \frac{1}{2} \varepsilon^2 \bar{k}^2 \int_{-h}^0 dz \bar{\nu}_T(z) \left(1 + \varepsilon^2 \bar{k}^2 \left(\frac{dm_2}{dz} \right)^2 + \varepsilon^4 \bar{k}^4 m_2^2(z) \right) \quad (5.15d)$$

$$\gamma_5 = \frac{1}{4} \varepsilon^4 \bar{k}^4 \int_{-h}^0 dz \frac{m_2^2(z)}{\bar{\nu}_T(z)}. \quad (5.15e)$$

Some simplification of the coefficients seems possible. Notwithstanding the fact that it is difficult now to carry out order arguments in a strictly organised way, we conclude that the integrands in γ_1 and γ_4 are of the form $(1 + \mathcal{O}(\varepsilon^4))$ and $(1 + \mathcal{O}(\varepsilon^2))$ respectively. For the numerical accuracy of the coefficients these $\mathcal{O}(\varepsilon^2, \varepsilon^4)$ terms are of minor importance and it therefore seems justified to ignore them. Instead of γ_1 and γ_4 we then have:

$$\gamma'_1 = \frac{1}{4} h \quad (5.16a)$$

$$\gamma'_4 = \frac{1}{2} \varepsilon^2 \bar{k}^2 \int_{-h}^0 dz \bar{\nu}_T(z). \quad (5.16b)$$

It should be recognised that $\mathcal{O}(\varepsilon^4)$ terms cannot be totally ignored, otherwise coefficient γ_5 would be zero and an equation of totally different properties would result. We do take into account the *leading order* of each coefficient. The same procedure has been followed with the solutions of \hat{u} : here also the leading-order solutions have been considered, that is, $\mathcal{O}(1)$ solutions for \hat{u} and \hat{v} and $\mathcal{O}(\varepsilon^2)$ solution for \hat{w} because that was the first term unequal to zero.

Introducing coefficients β_i by:

$$\beta_1 = -\frac{\gamma_2 + \gamma_3}{\gamma'_1} \quad (5.17a)$$

$$\beta_2 = \frac{\gamma'_4}{\gamma'_1} > 0 \quad (5.17b)$$

$$\beta_3 = \frac{\gamma_5}{\gamma'_1} > 0, \quad (5.17c)$$

the Landau-Stuart equation (5.14) can be written in the form:

$$\frac{dA_0^2}{dt} = \beta_1 A_0^2 - \beta_2 A_0^2 - \beta_3 A_0^4. \quad (5.18)$$

The coefficients β_i are given by:

$$\beta_1 = -2\varepsilon^2 \frac{\tilde{k}}{h} \int_{-h}^0 dz m_2(z) \left[\tilde{k}_2 \left(\partial_z U^S + \frac{\mathcal{T}_1}{\bar{v}_T} \right) - \tilde{k}_1 \left(\partial_z V^S + \frac{\mathcal{T}_2}{\bar{v}_T} \right) \right] \quad (5.19a)$$

$$\beta_2 = 2\varepsilon^2 \frac{\tilde{k}^2}{h} \int_{-h}^0 dz \bar{v}_T(z) > 0 \quad (5.19b)$$

$$\beta_3 = \varepsilon^4 \frac{\tilde{k}^4}{h} \int_{-h}^0 dz \frac{m_2^2(z)}{\bar{v}_T(z)} > 0. \quad (5.19c)$$

To compare these coefficients with the ones given by Radder (1998), his equations (21), we notice that we write $\varepsilon^2 m_2(z)$ for the function $m_0(z)$ and \tilde{k} for κ as used by Radder. It is then immediately clear that β_2 and β_3 from (5.19) correspond with the corresponding coefficients of Radder. To be sure about β_1 some more effort is needed. Thereto we first introduce the shorthand notation (as did Radder):

$$Z_1 = \left(\partial_z U^S + \frac{\mathcal{T}_1}{\bar{v}_T} \right) \quad \text{and} \quad Z_2 = \left(\partial_z V^S + \frac{\mathcal{T}_2}{\bar{v}_T} \right). \quad (5.20)$$

The coefficient β_1 is then given as:

$$\beta_1 = -2\varepsilon^2 \frac{\tilde{k}}{h} \int_{-h}^0 dz m_2 \left[\tilde{k}_2 Z_1 - \tilde{k}_1 Z_2 \right]. \quad (5.21a)$$

We start now with the expression for β_1 as given by Radder (1998) and write it in our variables; we denote it as $\beta_1^{(r)}$:

$$\beta_1^{(r)} = -2\varepsilon^2 \frac{\tilde{k}^2}{h} \int_{-h}^0 dz m_2 \frac{Z_1 - q Z_2}{\sqrt{1 + q^2}} \quad \text{with} \quad q = \frac{\tilde{k}_1}{\tilde{k}_2}. \quad (5.21b)$$

Because

$$\frac{1}{\sqrt{1+q^2}} = \frac{1}{\sqrt{\frac{\tilde{k}_2^2 + \tilde{k}_1^2}{k_2^2}}} = \frac{\tilde{k}_2}{\tilde{k}}$$

and

$$Z_1 - qZ_2 = \frac{1}{\tilde{k}_2} (\tilde{k}_2 Z_1 - \tilde{k}_1 Z_2)$$

we have for $\beta_1^{(r)}$:

$$\beta_1^{(r)} = -2\varepsilon^2 \frac{\tilde{k}}{h} \int_{-h}^0 dz m_2 (\tilde{k}_2 Z_1 - \tilde{k}_1 Z_2) . \quad (5.21c)$$

Our conclusion therefore is that also the coefficient α in Eq. (21a) of Radder (1998) is correct.

5.2 Analysis of the Landau-Stuart equation

Introducing the coefficients $2\alpha = \beta_1 - \beta_2$ and $\ell = \beta_3$, the Landau-Stuart equation (5.18) is written as:

$$\frac{dA_0^2}{dt} = 2\alpha A_0^2 - \ell A_0^4 . \quad (5.22)$$

We have $\ell > 0$ and $\beta_2 > 0$, but the sign of β_1 and thus also the sign of α is not clear beforehand. As shown in §3.4 an exact solution is given by

$$A_0^2 = \frac{1}{\frac{\ell}{2\alpha} + \left(\frac{1}{A_0^2} - \frac{\ell}{2\alpha}\right) e^{-2\alpha t}} . \quad (5.23)$$

When $\alpha > 0$, the solution (5.23) approaches the equilibrium solution, $A_0^2 \rightarrow A_e^2 = 2\alpha/\ell$ for $t \rightarrow \infty$. When $\alpha < 0$, $A_0 \rightarrow 0$ for $t \rightarrow \infty$.

5.3 Solution of $m_2(z)$

We have for m_2 the differential equation (5.12), which is repeated below for ease of exposition:

$$\frac{d}{dz} \left(\bar{v}_T(z) \frac{d^2 m_2}{dz^2} \right) = \frac{1}{\tilde{k}} \left[\tilde{k}_2 (\bar{U}^S - U^S) - \tilde{k}_1 (\bar{V}^S - V^S) \right] . \quad (5.24)$$

Integrating once to z results in

$$\frac{d^2 m_2}{dz^2} = \frac{1}{\bar{v}_T} \int_{-h}^z dz' \frac{1}{\tilde{k}} \left[\tilde{k}_2 (\bar{U}^S - U^S) - \tilde{k}_1 (\bar{V}^S - V^S) \right] . \quad (5.25)$$

As boundary condition is given in (5.10) $\partial_z^2 m_2 = 0$ for $z = 0$ and $z = -h$. For $z = -h$ we indeed have $\partial_z^2 m_2$ be zero because the integration interval in (5.25) equals zero. That $\partial_z^2 m_2 = 0$ also for $z = 0$ is seen as follows. $\int_{-h}^0 dz (\bar{U} - U(z)) = h\bar{U} - h\bar{U} = 0$ and similarly for the V -terms.

A further integration of (5.25) to z gives

$$\begin{aligned} \frac{dm_2}{dz} &= \int_0^z dz' \frac{1}{\bar{v}_T(z')} \int_{-h}^{z'} dz'' \frac{1}{\tilde{k}} \left[\tilde{k}_2 (\bar{U}^S - U^S(z'')) - \tilde{k}_1 (\bar{V}^S - V^S(z'')) \right] + c_4 \\ &\equiv F(z, \tilde{k}_2) + c_4, \end{aligned} \quad (5.26)$$

with c_4 an integration constant, to be determined by the boundary condition.

A final integration to z yields

$$m_2(z) = \int_0^z d\hat{z} F(\hat{z}, \tilde{k}_2) + c_4 z + c_5.$$

Application of the boundary condition $m_2(0) = 0$ yields $c_5 = 0$ and the boundary condition $m_2(-h) = 0$ yields

$$c_4 = \frac{1}{h} \int_0^{-h} d\hat{z} F(\hat{z}, \tilde{k}_2).$$

The solution for $m_2(z)$ can finally be written as:

$$m_2(z; \tilde{k}_2) = \int_0^z d\hat{z} F(\hat{z}, \tilde{k}_2) + \frac{z}{h} \int_0^{-h} d\hat{z} F(\hat{z}, \tilde{k}_2), \quad (5.27)$$

where has been written $m_2(z; \tilde{k}_2)$ to stress the fact that m_2 still depends on the parameter \tilde{k}_2 .

Radder (1998) chooses instead of \tilde{k}_1 or \tilde{k}_2 the parameter $q = \tilde{k}_1/\tilde{k}_2$. With $\tilde{k}_1 = \tilde{k} \cos \varphi$ and $\tilde{k}_2 = \tilde{k} \sin \varphi$ we have $q = 1/\tan \varphi$ where φ is the direction of the alignment of the vortex rolls. To see the difference in the various representations, we consider the function G given in the right-hand member of the differential equation for m_2 , Eq. (5.12):

$$G = \frac{1}{\tilde{k}} \left[\tilde{k}_2 (\bar{U}^S - U^S) - \tilde{k}_1 (\bar{V}^S - V^S) \right]. \quad (5.28)$$

G can be written as

$$\begin{aligned} G &= \frac{\tilde{k}_2}{\tilde{k}} \left[(\bar{U}^S - U^S) - q (\bar{V}^S - V^S) \right] \\ &= \frac{1}{\sqrt{1+q^2}} \left[(\bar{U}^S - U^S) - q (\bar{V}^S - V^S) \right], \end{aligned} \quad (5.29)$$

and $F(z; \tilde{k}_2)$ can also be written as

$$f(z; q) = \int_0^z dz' \frac{1}{\bar{v}_T(z')} \int_{-h}^{z'} dz'' G(z''; q). \quad (5.30)$$

Radder (1998) uses the function $f(z; q)$.

To make the dependence of m_2 on q explicit, we write G as

$$G(z; q) = \frac{1}{\sqrt{1+q^2}} (G_1(z) - qG_2(z)) \quad (5.31a)$$

with

$$G_1(z) = \overline{U}^S - U^S(z) \quad \text{and} \quad G_2(z) = \overline{V}^S - V^S(z). \quad (5.31b)$$

Then the function $f(z; q)$ is written as

$$f(z; q) = \frac{f_1(z) - qf_2(z)}{\sqrt{1+q^2}}. \quad (5.31c)$$

Introducing furthermore the notation

$$M_1(z) = \int_0^z d\hat{z} f_1(\hat{z}) + \frac{z}{h} \int_0^{-h} d\hat{z} f_1(\hat{z}) \quad (5.31d)$$

$$M_2(z) = \int_0^z d\hat{z} f_2(\hat{z}) + \frac{z}{h} \int_0^{-h} d\hat{z} f_2(\hat{z}), \quad (5.31e)$$

the solution of $m_2(z; q)$ can be written short as

$$m_2(z; q) = \frac{M_1(z) - qM_2(z)}{\sqrt{1+q^2}}. \quad (5.32)$$

5.4 The alignment of the vortex rolls

5.4.1 The principle of exchange of stability

To obtain the direction of the axis of the vortex rolls, Radder (1998) suggests that the *principle of exchange of stability* (PES) may be used. A few remarks on PES are collected in Appendix B. We have investigated perturbations of the form (4.4a), $\hat{\mathbf{u}}(\mathbf{x}, z, t) = \mathbf{u}' e^{\vartheta(\mathbf{x}, t)}$ and the stability of these perturbations was subject of investigation. Here is $\vartheta = i\epsilon \tilde{\mathbf{k}} \cdot \mathbf{x} + \epsilon \sigma t$, indicating that in horizontal space the solution is periodic and in time growth or decay of solutions may occur. The question of stability is only occurring in the dimension of time. It was found that the exponent σ has zero real part. When the imaginary part is unequal to zero, just periodic (neutrally stable) perturbations result. When the imaginary part is also zero for one or more of the eigenvalues (i.e., solutions σ), then a bifurcation of the basic flow into a secondary motion may result. This secondary motion may be stable or unstable, depending on the prevailing conditions. Because we look for the generation of secondary currents due to instability of the basic current, this principle may well be valid in our case. It is not a simple matter to prove its validity in case the waves and current are not aligned (or, otherwise stated, when \mathbf{U} and \mathbf{U}^S have different directions). PES has not been applied much for unbounded flows (see Herron, 1985,1996), most applications are for confined flows. It follows from (4.18b) that $\sigma = i\sigma_1^i = 0$ occurs for:

$$\sigma_1^i = -\tilde{\mathbf{k}}_c \cdot (\overline{\mathbf{U}} + \overline{\mathbf{U}^S}) = 0 \quad (5.33)$$

where $\tilde{\mathbf{k}}_c$ is the solution of $\tilde{\mathbf{k}}$ for which this condition is true. The condition (5.33) can also be written as

$$q_c \left(\overline{\mathbf{U}} + \overline{\mathbf{U}}^S \right) + \left(\overline{\mathbf{V}} + \overline{\mathbf{V}}^S \right) = 0. \quad (5.34)$$

It is seen from (5.33) that the direction of $\tilde{\mathbf{k}}$ is perpendicular to the combined direction of the vertically averaged current $\overline{\mathbf{U}}$ and ditto Stokes drift $\overline{\mathbf{U}}^S$. This of course also follows from (5.34) by noting that $q_c = \tan \varphi_c$ (see above Eq. (5.28)) and writing $\overline{\mathbf{U}} = |\overline{\mathbf{U}}| (\cos \varphi_u, \sin \varphi_u)^T$ and ditto for $\overline{\mathbf{U}}^S$ with the direction φ_s . Condition (5.34) then can be written in the following form:

$$\tan \varphi_c = - \frac{|\overline{\mathbf{U}}| \cos \varphi_u + |\overline{\mathbf{U}}^S| \cos \varphi_s}{|\overline{\mathbf{U}}| \sin \varphi_u + |\overline{\mathbf{U}}^S| \sin \varphi_s}. \quad (5.35)$$

Suppose that φ_u and φ_s are in the first quadrant (i.e. are between 0 and $\pi/2$ radians). Because then $\tan \varphi < 0$, φ lies either in the second or fourth quadrant.

Interpretation of the $\tilde{\mathbf{k}}$ vector

The vector $\tilde{\mathbf{k}}$ was introduced in the expansion (4.3) as a kind of wave number vector, but then to indicate the periodicity of the vortex cells. The extent of the periodic structure is inversely proportional to the component of $\tilde{\mathbf{k}}$. The vector $\tilde{\mathbf{k}}$ points in the direction with the smallest periodicity of the periodic structure. In the perpendicular direction, the component is zero (or, at least, very small), signifying that the extent of the periodicity is very large. The axis of the vortex rolls is thus in a direction perpendicular to that of $\tilde{\mathbf{k}}$. Above result that $\tilde{\mathbf{k}}$ is perpendicular to the vector $\left(\overline{\mathbf{U}} + \overline{\mathbf{U}}^S \right)$ thus means that the alignment of the vortex rolls is in the combined direction of these two current vectors. When the directions of these vertically averaged current contributions are the same, so is the direction of the axis of the vortex cells. This is a situation which is encountered in not too-wide flumes.

For the special case that $\overline{\mathbf{U}}$ and $\overline{\mathbf{U}}^S$ are in the x -direction so that $\tilde{k}_2 = 0$, we have $\tilde{k}_1 = 0$, signifying infinitely long rolls in the x -direction.

5.4.2 Maximal growth of the perturbations

A different method to find the critical direction given by q_c is by determining the maximum growth rate of the amplitude A_0 of the perturbations. This is carried out for infinitesimal perturbations so that it is sufficient to consider only part of the Landau-

Stuart equation³:

$$\max_q \frac{dA_0}{dt} = 2\alpha A^2. \quad (5.36)$$

Maximum growth is obtained for

$$\frac{d\alpha}{dq} = 0 \quad \text{together with} \quad \frac{d^2\alpha}{dq^2} < 0. \quad (5.37)$$

We have $\alpha = \beta_1 - \beta_2$ and the coefficients β_i have been given in Eqs. (5.19). We suppose that $\alpha > 0$ because in that case the equilibrium solution of the Landau-Stuart equation can be reached; for $\alpha < 0$ we have $A_0 \rightarrow 0$ for $t \rightarrow \infty$. Because β_2 does not depend on q , $\partial_q \alpha = \partial_q \beta_1$ and, using the representation (5.32) for m_2 , and also taking \tilde{k} fixed, we then have

$$-\int_{-h}^0 dz \frac{\partial}{\partial q} \left[\frac{(M_1(z) - qM_2(z))(Z_1(z) - qZ_2(z))}{1 + q^2} \right] = 0. \quad (5.38a)$$

$$\Rightarrow \int_{-h}^0 dz \left[(M_1 Z_2 + M_2 Z_1) (q^2 - 1) + 2(M_2 Z_2 - M_1 Z_1) q \right] = 0. \quad (5.38b)$$

Introducing the notation

$$Q_0 = \int_{-h}^0 dz (M_2 Z_2 - M_1 Z_1), \quad Q_1 = \int_{-h}^0 dz (M_1 Z_2 + M_2 Z_1), \quad Q = \frac{Q_1}{Q_0}, \quad (5.38c)$$

we obtain the condition

$$Qq^2 + 2q - Q = 0, \quad (5.38d)$$

with the solutions

$$q_1 = \frac{-1 + \sqrt{1 + Q^2}}{Q} \quad \text{and} \quad q_2 = \frac{-1 - \sqrt{1 + Q^2}}{Q}. \quad (5.38e)$$

Which one of these two solutions we should choose depends on the value of the second derivative of β_1 evaluated at these solutions. We obtain

$$\frac{\partial^2 \alpha}{\partial q^2} = -\frac{\partial}{\partial q} \left(\frac{Q_1 q^2 + 2Q_0 q - Q_1}{(1 + q^2)^2} \right) = -Q_0 \frac{\partial}{\partial q} \left(\frac{Qq^2 + 2q - Q}{(1 + q^2)^2} \right). \quad (5.39)$$

Substituting the two solutions q_1 and q_2 , we find:

$$\left. \frac{\partial^2 \alpha}{\partial q^2} \right|_{q_2} = -\frac{1}{4} Q_0 \frac{Q^6 (1 + Q^2) (-1 + \sqrt{1 + Q^2})}{(-Q^2 - 1 + \sqrt{1 + Q^2})^4} \quad (5.40a)$$

and

³The part $2\alpha A_0^2$ in the Landau-Stuart equation gives the initial linear growth of the amplitude A_0 of the instabilities, while the term ℓA_0^2 set a bound to the growth of the amplitudes, which bounding effect becomes effective after some growth has been taken place. Here we only consider maximum *initial* growth.

$$\left. \frac{\partial^2 \alpha}{\partial q^2} \right|_{q_1} = \frac{1}{4} Q_0 \frac{Q^6 (1 + Q^2) (1 + \sqrt{1 + Q^2})}{(Q^2 + 1 + \sqrt{1 + Q^2})^4}. \quad (5.40b)$$

The fractions with Q are positive and the sign of the second derivatives depends solely on the sign of Q_0 . The solution for q_c is either q_1 or q_2 , depending on the sign of Q_0 . The solution for q_c can now be written as

$$q_c = \frac{-1 - \text{sign}(Q_0) \sqrt{1 + Q^2}}{Q}. \quad (5.41)$$

This result is the same as Radder's (1998) result (27), when is recognised that our definition of Q_0 and Radders one, $Q_0^{(r)}$, relate like $Q_0 = -Q_0^{(r)}$.

5.5 The Reynolds stresses

We can now compute the Reynolds stresses $\langle \hat{w}\hat{u} \rangle$ and $\langle \hat{w}\hat{v} \rangle$. From the solutions (5.11), with \tilde{k}_2/\tilde{k} and \tilde{k}_1/\tilde{k} replaced by $1/\sqrt{1+q^2}$ and $q/\sqrt{1+q^2}$ respectively we obtain, when $2\alpha = \beta_1 - \beta_2 > 0$:

$$\langle \hat{w}\hat{u} \rangle = \frac{1}{2} \varepsilon^2 \tilde{k}^2 A_0^2 \frac{m_2}{\sqrt{1+q^2}} \quad \text{and} \quad \langle \hat{w}\hat{v} \rangle = \frac{1}{2} \varepsilon^2 \tilde{k}^2 A_0^2 \frac{qm_2}{\sqrt{1+q^2}} \quad (5.42)$$

When $\alpha < 0$, the Reynolds stresses are zero.

For the amplitude A_0 , we now use the equilibrium solution of the Landau-Stuart equations, A_e , given by:

$$A_e^2 = \frac{2\alpha}{\ell} \equiv \frac{\beta_1 - \beta_2}{\beta_3}. \quad (5.43)$$

From (5.19b) and (5.19c) for β_2 and β_3 and (5.21b) for β_1 it follows that we can write:

$$\beta_1 = -2\varepsilon^2 \frac{\tilde{k}^2}{h} B_1 \quad \text{with} \quad B_1 = \int_{-h}^0 dz m_2(z; q) \frac{Z_1(z) - qZ_2(z)}{\sqrt{1+q^2}} \quad (5.44a)$$

$$\beta_2 = 2\varepsilon^2 \frac{\tilde{k}^2}{h} B_2 \quad \text{with} \quad B_2 = \int_{-h}^0 dz \bar{\nu}_T(z) \quad (5.44b)$$

$$\beta_3 = \varepsilon^4 \frac{\tilde{k}^4}{h} B_3 \quad \text{with} \quad B_3 = \int_{-h}^0 dz \frac{m_2^2(z; q)}{\bar{\nu}_T(z)}. \quad (5.44c)$$

The equilibrium amplitude then is

$$A_e^2 = -\frac{2}{\tilde{k}^2} \frac{B_1(q) + B_2}{B_3(q)}, \quad (5.45)$$

where for q the critical solution q_c is to be used. Using this equilibrium solution in the expressions for the Reynolds stresses, the latter ones become:

$$\langle \hat{w}\hat{u} \rangle = -\varepsilon^2 \frac{(B_1(q_c) + B_2)}{B_3(q_c)} \cdot \frac{m_2(z; q_c)}{\sqrt{1 + q_c^2}} \quad (5.46a)$$

and

$$\langle \hat{w}\hat{v} \rangle = -\varepsilon^2 \frac{(B_1(q_c) + B_2)}{B_3(q_c)} \cdot \frac{q_c m_2(z; q_c)}{\sqrt{1 + q_c^2}}, \quad (5.46b)$$

from which it is clear that the Reynolds stresses do not depend on \tilde{k} , but only on the direction of $\tilde{\mathbf{k}}$, expressed by q_c . This finding means that the Reynolds stresses do not depend on the extent of the circulation cells (the size being proportional to $1/(\varepsilon k)$).

6 Discussion and recommendations

The present report is concerned with the question of modelling the Reynolds-type stresses of the perturbed velocities which occur due to an instability mechanism of the Craik-Leibovich (CL) equation. The CL-equation is an equation for the mean current in which some wave effects are accounted for, the most important one being the Stokes drift, resulting in the vortex force, which can be related to the pseudomomentum term in the GLM description. These instabilities are held responsible for the occurrence of so-called Langmuir circulation cells. The present report addresses a method for the analytic modelling of these stresses, proposed by Radder (1998).

The primary purpose of the present report is to check the derivations of Radder (1998). Secondly, his simplifications should be assessed. Thirdly, the applicability of these approximations should be assessed. It should be mentioned that the results presented by Radder (1998) are all found to be correct.

Because instabilities of the (mean) current in the CL-equation generate vortex cells of Langmuir type, it is a logical step to start at the outset with a description in which this mean current is perturbed to some basic flow plus perturbations. A solution for these perturbed velocities is sought for by a linear instability analysis of the momentum equations. For the solution only the leading-order terms are considered, which is an appropriate approximation at this stage. Using these solutions in the energy equation for the perturbations, a Landau-Stuart-type of equation can be derived. This equation predicts the growth of the perturbations and its final level because of the non-linear term in the equation. An exact solution can be found from which it follows that either the amplitudes of the perturbed velocities tend to an equilibrium value or go to zero as time passes. Which behaviour prevails depends on one of the coefficients in the equation ($\alpha = \beta_1 - \beta_2$) being positive. It cannot be stated generally when this occurs, but some examples considered in Radder (1998) suggest that the shear of the Stokes drift has to be large enough, or, otherwise stated, the waves should be large enough. This corresponds with the finding in the literature (see, e.g., the discussion in Appendix A of Van Kester et al., 1996) that the wind velocity should be large enough in order that Langmuir circulations can be generated.

One of the important approximations used in developing the theory is that the basic flow has no vertical component and, therefore, one has strictly only the case of a horizontal bottom. It is known from other applications at sea that neglect of bottom-slope terms often lead to quite acceptable results. One of the examples is given by the mean-flow equations, which are usually in a form as is for example given by Dingemans (1997, Eqs. (2.470)). These equations are repeated below:

$$\rho (h + \bar{\zeta}) \left[\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} + g \frac{\partial \bar{\zeta}}{\partial x_i} \right] + \frac{\partial S_{ij}}{\partial x_j} = \langle \tau_i^s \rangle - \langle \tau_i^b \rangle, \quad (6.1)$$

where S_{ij} is the radiation stress. Here all bottom-slope terms are neglected, but these equations are nevertheless used for many phenomena occurring in the near-shore region, inclusive surf beat.

Considering above examples, it seems to be not unduly optimistic to suppose that the present horizontal-bottom theory also will perform well in the nearshore region where (albeit small) bottom slopes different from zero occur.

One of the important features of the present results is that the modelled Reynolds stresses do not depend on the size of the circulation cells, meaning that these expressions can easily be introduced in the momentum equation for the basic state, Eq. (3.13), or its simplified form, Eq. (3.38). Numerical experiments have to be carried out to check the applicability of the present theory. Care should be exercised in the definition of the experiments in order that indeed the generation of Langmuir cells would be predicted. That is, care should be taken that indeed the coefficient α is positive and, to have some success with the numerical experiments, it seems a necessity to consider only cases in which the shear of U^S is large enough to be able to generate the vortex cells numerically. A short discussion of some numerical properties is in order now.

6.1 Numerical properties

In Appendix A of Van Kester et al. (1996) it is mentioned (p. 3) that a model of Mobley (1976) designed to test the generation of Langmuir circulations through the CL1 instability mechanism, did not work despite the fact that the circumstances which these computations simulated were favourable for the generation of these cells. That was attributed to the fact that the large eddy simulation code used treated the upward boundary as a fixed horizontal boundary, not as a free boundary. We conclude from this that the upward boundary should indeed be a free boundary.

In Van Kester et al. (1996), The CL equation (3.3) was solved numerically in the (y, z) plane, leaving out possible variations in the wave propagation direction x . It was found that one pair of circulation cells developed not because of instabilities in the flow field, but due to side-wall effects. This was shown by doing not only the computation to compare with measurements of Klopman (1994), but performing also computations for a very wide flume with the same vertical dimensions. Due to the way in which the boundary conditions with respect to the Stokes drift were handled, (by means of a constant Stokes drift \bar{u}^S over the cross-section, while to obtain the vortex force in the CL equation it was multiplied with the mean vorticity that increased strongly towards the sidewall boundary layers), this resulted in an overestimation of the vortex force near the side walls, leading to a possible overestimation of the strength of the vortex pair. This overestimation was later confirmed by further experiments of Klopman (1997) for the same wave and current conditions, in which not only was measured in the centre-line as in the 1994 measurements, but now also over half the cross-section (by symmetry, the other half could be inferred). We expect that possible instability effects are more or less drowned in the (too strong) generation of vortex cells due to the sidewall effects.

It is not clear beforehand that the method of Van Kester et al. (1996) would also be unsuccessful when the numerics are based on Eqs. (3.13) in which the Reynolds stresses of the perturbed velocities are made explicit and when the analytic model for these stresses is implemented. The difference between the CL-equations (3.3) and

(3.13) is in effect only the term with the Reynolds stresses in (3.13). It could be that the analytic model forces the effects of the instability enough, even when the numerics involved in itself is not good enough to handle the instabilities themselves. Because it is an easy exercise, we recommend a few computations with van Kester's model with the analytic model for the Reynolds stresses included. A short study would involve two computations based on the flume dimensions (following an opposing current) and two for a very wide flume.

It is our feeling that with a correct modelling of the side-wall boundaries it should also be possible to obtain circulation cell growth due to instability of the CL-equation. To make this happen in a numerical model, stringent conditions on the numerical procedure should be exercised; the numerical method should be of high-enough accuracy and the numerical dissipation should be very low so that the effects of the instability mechanism are not dissipated immediately. The numerical method used by Van Kester et al. (1996) is a low-order one with also too much dissipation and is therefore not suitable to use for a check on the present theory. Some code should be used which is both accurate and which treats the free surface in an accurate way. In this respect the method coined semi-Poisson method in Appendix A, p. 12 of Van Kester et al. (1996) could be a useful one. Casulli and Stelling (1998) have shown that tests with this quasi-hydrostatic method gave accurate results.

6.2 Recommendations

The recommendations will be summarised below.

1. The effect of the model of the Reynolds stresses of the perturbation velocities can be tested in the old code of Van Kester et al. (1996). The following steps are envisaged:
 - a. Better side-wall conditions have to be built in. This can be restricted at this stage at a more accurate representation of the Stokes drift near the boundary, e.g., in the way as is done by Groeneweg (1999, §5.2).
 - b. Redoing the wide-flume computations with the thus changed code.
 - c. Adding the analytic model for the Reynolds stresses in the (new) Van Kester code. Now the CL-type equation (3.13) is modelled. Tests for the validity of the approach should be built in the code. For example, it is necessary to check the sign of $\alpha = \beta_1 - \beta_2$ in the Landau-Stuart equation; only for $\alpha > 0$ the analytic model should be used, in other situations no equilibrium solution of the Landau equation exists.
 - d. Both a computation for the wide-flume case with a fine and a coarse mesh is to be performed. The coarse mesh should give the same results as the fine-mesh computation because of the sub-grid modelling of the effect of the Reynolds stresses.
2. A real test of the behaviour of the new model is obtained by performing a numerical test with an accurate numerical model.
 - a. One of the candidates for using is the semi-Poisson approach as used by Casulli and Stelling (1998).
 - b. Other methods, requiring much more numerical effort, may be some

large eddy-simulation, in which the free surface should be treated accurately or a direct numerical solution of the Navier-Stokes equations. Care should be taken that computations are carried out for situations in the correct parameter regime so that generation of Langmuir circulation cells is possible. The other methods can possibly be carried out at the Burgers Centre for Fluid Dynamics.

A Derivation of Landau-Stuart equation

The change in kinetic energy is given by

$$\begin{aligned} \frac{dK}{dt} = \frac{d}{dt} \left\{ \int_{-h}^0 dz \left\langle \frac{1}{2} \hat{\mathbf{u}} \cdot \hat{\mathbf{u}} \right\rangle \right\} &= - \int_{-h}^0 dz \left\{ \langle \hat{w} \hat{u}_j \rangle \frac{\partial}{\partial z} (U_j^h + U_j^S) \right\} \\ &\quad - \int_{-h}^0 dz \left\{ \bar{\nu}_T \left\langle \left(\frac{\partial \hat{u}_i}{\partial x_j} \right)^2 \right\rangle \right\}, \end{aligned} \tag{A.1}$$

The solutions for the perturbed velocities \hat{u}_j are given by (5.11), repeated below:

$$\hat{\mathbf{u}}(\mathbf{x}, z, t) = \frac{1}{2} \begin{pmatrix} \frac{\tilde{k}_2}{\tilde{k}} \\ -\frac{\tilde{k}_1}{\tilde{k}} \\ \varepsilon^2 \tilde{k}^2 m_2(z) \end{pmatrix} A_0 e^{\vartheta} + CC. \tag{A.2a}$$

with

$$\vartheta = i\varepsilon \tilde{\mathbf{k}} \cdot \mathbf{x} + i\varepsilon \sigma_1^{(i)} t \equiv i\chi. \tag{A.2b}$$

The z -derivative of the basic current U^h is given by (3.52), repeated below:

$$\begin{aligned} \bar{\nu}_T \frac{\partial U_i^h}{\partial z} &= \langle \hat{w} \hat{u}_i \rangle + \bar{\tau}_i^b + \int_{-h}^z dz' \nabla \Pi_0 \equiv \langle \hat{w} \hat{u}_i \rangle + \mathcal{T}_i \\ \Rightarrow \frac{\partial U_i^h}{\partial z} &= \frac{1}{\bar{\nu}_T} \langle \hat{w} \hat{u}_i \rangle + \frac{1}{\bar{\nu}_T} \mathcal{T}_i. \end{aligned} \tag{A.3}$$

Because \hat{u}_1 and \hat{u}_2 are independent of z , we now have

$$\begin{aligned} \left\langle \left(\frac{\partial \hat{u}_i}{\partial x_j} \right)^2 \right\rangle &= \left\langle \left(\frac{\partial \hat{u}}{\partial x} \right)^2 + \left(\frac{\partial \hat{u}}{\partial y} \right)^2 + \left(\frac{\partial \hat{v}}{\partial x} \right)^2 + \left(\frac{\partial \hat{v}}{\partial y} \right)^2 \right\rangle \\ &\quad + \left\langle \left(\frac{\partial \hat{w}}{\partial x} \right)^2 + \left(\frac{\partial \hat{w}}{\partial y} \right)^2 + \left(\frac{\partial \hat{w}}{\partial z} \right)^2 \right\rangle. \end{aligned}$$

It follows from (A.2) that

$$\frac{\partial \hat{u}}{\partial x} = \frac{1}{2} i\varepsilon \frac{\tilde{k}_1 \tilde{k}_2}{\tilde{k}} A_0 (e^{i\chi} - i e^{-i\chi})$$

and, noting that

$$\left\langle \left\{ i (e^{i\chi} - i e^{-i\chi}) \right\}^2 \right\rangle = 2 \tag{A.4}$$

we have

$$\left\langle \left(\frac{\partial \hat{u}}{\partial x} \right)^2 \right\rangle = \frac{1}{2} \varepsilon^2 A_0^2 \frac{\tilde{k}_1^2 \tilde{k}_2^2}{\tilde{k}^2}$$

and similarly for $\langle (\partial_y \hat{u})^2 \rangle$, $\langle (\partial_x \hat{v})^2 \rangle$ and $\langle (\partial_y \hat{v})^2 \rangle$. We then have:

$$\left\langle \left(\frac{\partial \hat{u}}{\partial x} \right)^2 + \left(\frac{\partial \hat{u}}{\partial y} \right)^2 + \left(\frac{\partial \hat{v}}{\partial x} \right)^2 + \left(\frac{\partial \hat{v}}{\partial y} \right)^2 \right\rangle = \frac{1}{2} \varepsilon^2 \tilde{k}^2 A_0^2. \quad (\text{A.5a})$$

In a similar way is obtained

$$\left\langle \left(\frac{\partial \hat{w}}{\partial x} \right)^2 + \left(\frac{\partial \hat{w}}{\partial y} \right)^2 \right\rangle = \frac{1}{2} \varepsilon^6 \tilde{k}^6 m_2^2 A_0^2 \quad (\text{A.5b})$$

and

$$\left\langle \left(\frac{\partial \hat{w}}{\partial z} \right)^2 \right\rangle = \frac{1}{2} \varepsilon^4 \tilde{k}^4 \left(\frac{dm_2}{dz} \right)^2 A_0^2. \quad (\text{A.5c})$$

In a similar way we have

$$\langle \hat{w} \hat{u}_j \rangle \partial_z U_j^S = \frac{1}{2} i \varepsilon \tilde{k} A_0^2 m_2(z) (\tilde{k}_2 \partial_z U^S - \tilde{k}_1 \partial_z V^S) \quad (\text{A.6a})$$

and

$$\begin{aligned} \langle \hat{w} \hat{u}_j \rangle \partial_z U_j^h &= \frac{1}{\bar{\nu}_T} (\langle \hat{w} \hat{u}_j \rangle^2 + \langle \hat{w} \hat{u}_j \rangle \mathcal{T}_j) = \\ &= \frac{1}{\bar{\nu}_T} \left[\frac{1}{4} \varepsilon^4 \tilde{k}^4 m_2^2 A_0^4 + \frac{1}{2} \varepsilon^2 \tilde{k} m_2 A_0^2 (\tilde{k}_2 \mathcal{T}_1 - \tilde{k}_1 \mathcal{T}_2) \right]. \end{aligned} \quad (\text{A.6b})$$

Lastly we have

$$\left\langle \frac{1}{2} \hat{\mathbf{u}} \cdot \hat{\mathbf{u}} \right\rangle = \frac{1}{4} (1 + \varepsilon^4 \tilde{k}^4 m_2^2) A_0^2. \quad (\text{A.6c})$$

Substitution of relations (A.5) and (A.6) in Eq. (A.1) leads to the following Landau-Stuart equation:

$$\gamma_1 \frac{dA_0^2}{dt} = -\gamma_2 A_0^2 - \gamma_3 A_0^2 - \gamma_4 A_0^2 - \gamma_5 A_0^4 \quad (\text{A.7})$$

with the coefficients γ_i given by:

$$\gamma_1 = \frac{1}{4} \int_{-h}^0 dz (1 + \varepsilon^4 \tilde{k}^4 m_2^2(z)) \quad (\text{A.8a})$$

$$\gamma_2 = \frac{1}{2} \varepsilon^2 \tilde{k} \int_{-h}^0 dz (\tilde{k}_2 \partial_z U^S(z) - \tilde{k}_1 \partial_z V^S(z)) m_2(z) \quad (\text{A.8b})$$

$$\gamma_3 = \frac{1}{2} \varepsilon^2 \tilde{k} \int_{-h}^0 dz \frac{m_2(z)}{\bar{\nu}_T(z)} (\tilde{k}_2 \mathcal{T}_1 - \tilde{k}_1 \mathcal{T}_2) \quad (\text{A.8c})$$

$$\gamma_4 = \frac{1}{2} \varepsilon^2 \tilde{k}^2 \int_{-h}^0 dz \bar{\nu}_T(z) \left(1 + \varepsilon^2 \tilde{k}^2 \left(\frac{dm_2}{dz} \right)^2 + \varepsilon^4 \tilde{k}^4 m_2^2(z) \right) \quad (\text{A.8d})$$

$$\gamma_5 = \frac{1}{4} \varepsilon^4 \tilde{k}^4 \int_{-h}^0 dz \frac{m_2^2(z)}{\bar{\nu}_T(z)}. \quad (\text{A.8e})$$

B A few remarks on the principle of exchange of stability

The *principle of exchange of stability* (PES) is used by Radder (1998) as a possible method to find the direction of the critical rolls. A few illuminating remarks on the nature of the PES are found in Joseph (1976) and are cited below. It should be noted that exponential solutions of the form

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{f}(\mathbf{x}) e^{-\sigma t} \quad , \quad p(\mathbf{x}, t) = p(\mathbf{x}) e^{-\sigma t} \quad .$$

exist provided that there exist numbers σ for which the spectral problem

$$-\sigma \mathbf{f} + \mathcal{L}[\mathbf{U}, \nu] \mathbf{f} + \text{grad } p = 0 \quad , \quad \text{div } \mathbf{f} = 0 \quad , \quad \mathbf{f}|_S = 0$$

has nontrivial solutions. The special numbers σ are the *eigenvalues* of above set of equations and the non-trivial solutions (\mathbf{v}, p) are the *eigenfunctions* belonging to σ .

- From Joseph (1976, p. 26 and 27):
"A system of stability concepts is customarily defined relative to the spectral problem. Here, a flow is called *stable* if there are no eigenvalues such that $\text{Re}(\sigma) < 0$; *marginally* or neutrally stable if there is one eigenvalue with $\text{Re}(\sigma) = 0$ and $\text{Re}(\sigma) > 0$ for the other eigenvalues; and *unstable* if at least one eigenvalue has $\text{Re}(\sigma) < 0$.

Neutral disturbances are of two kinds. If, when $\text{Re}(\sigma) = 0$ one also has $\text{Im}(\sigma) = 0$, then the neutral solution is steady and a principle of exchange of stability is said to hold. The neutral solution is time-periodic if $\text{Im}(\sigma) \neq 0$ when $\text{Re}(\sigma) = 0$."

- From Joseph (1976, p. 55):
"Loss of stability of the basic flow when $\sigma = 0$ portends the bifurcation of the basic flow into a secondary steady motion which may itself be stable or unstable, depending on conditions."
- From Joseph (1976, p. 55):
"Finally, we note that the energy method of Stuart (1958) is also connected with the linear stability theory. This is an approximate method which assumes that the spatial form (shape) of the nonlinear disturbance is the same as the shape of marginal disturbances of the linearised theory, but with unknown amplitude A . This energy method yields interesting nonlinear results but does not yield sufficient conditions for stability of the form of the disturbance which increases initially at the largest viscosity."

In Herron (1985, 1996) PES is defined to be *the first unstable eigenvalue has imaginary part equal to zero*.

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- D Radder's (1998) note: Pattern formation in a 3D wave-current interaction system; a subgrid model

**PATTERN FORMATION IN A 3D WAVE-CURRENT INTERACTION SYSTEM.
A SUBGRID MODEL**

1. Introduction. The CL-theory

The interaction of sea waves and currents in the nearshore zone is relevant to the study of mixing processes and sediment-transport properties (see *e.g.* the review by Peregrine and Jonsson 1983; Klopman 1992). The influence of surface waves propagating on a (tidal-, or wind-generated) current in coastal seas (*c.q.* waves on a current in a wave flume) is threefold:

- (i) the mean-velocity profile $U(z)$ can be strongly affected by the presence of waves;
- (ii) the mean bed shear stress τ_0 increases specifically;
- (iii) coherent circulations, in the form of roll patterns, may appear, in the field (Langmuir circulations, *cf.* Leibovich 1983) as well as in the laboratory (secondary circulations, *cf.* van Kester, Uittenbogaard and Dingemans 1996; Klopman 1997; Melville, Shear and Veron 1998).

On the other hand, any velocity shear $\partial U/\partial z$ at the surface has effect on the wave propagation, *e.g.* via the dispersion relation (*cf.* Nepf and Monismith 1994); besides, apart from Doppler-shift effects, wind-drift currents induce scattering of the wave spectrum through nonlinear interactions (Shrira 1998).

The *generalized Lagrangian mean* (GLM)- theory is the appropriate method to describe the interaction between waves and mean flows (*e.g.* Dingemans 1997, § 2.10.6). This formulation was used lately by Groeneweg and Klopman (1998) to describe nonlinear changes in the vertical distribution of the mean velocity due to the presence of waves, in a wave-current channel problem. Whereas good agreement between GLM-results and experimental results of Klopman (1994) was found, the approach lacks a clear physical interpretation, and velocity variations in cross-direction of the channel are neglected.

Recently, the *Craik-Leibovich* (CL) equation, originally derived by Craik and Leibovich (1976), has been used as a model equation to describe wave-induced mean-flow variations in a wave flume (van Kester *et al.* 1996; Dingemans *et al.* 1996). Under rather mild conditions, the CL-theory can be derived simply from the GLM-theory (*e.g.* Leibovich 1983; Craik 1985; Radder 1994); for a further discussion on its limitations, in particular when applied in a wave flume, see van Kester *et al.* 1996, §5.3. In the present work, the theory is applied to the case where surface waves and shear flow have different directions, in a laterally unbounded domain in shallow water; under suitable circumstances, roll-patterns appear through an instability mechanism (these patterns are similar to wind-driven Langmuir circulations in the ocean; as a universal phenomenon, patterns in diverse forms occur in a wide variety of physical systems in the laboratory as well as in the field; see the review by Bowman and Newell 1998). Assuming the wave field to be known in advance, this leads to a fairly simple parametrization of the wave-induced stresses in the momentum equations for the mean flow.

3. The disturbance-amplitude equation

In order to develop a subgrid model, further assumptions must be made:

let (x,y,z) be a *local* coordinate system, then

- (i) the mean velocity is a function of the vertical coordinate z and time t only (*i.e.* nearly horizontal flow on a horizontal bottom at $z = -h$), $\mathbf{U} \equiv \{U(z,t), V(z,t), 0\}$;
- (ii) the wave-induced disturbance is single-periodic in some direction and does not depend on the perpendicular direction θ , $\mathbf{u} \equiv \{u(x,y,z,t), v(x,y,z,t), w(x,y,z,t)\}$, with $\nabla_{\theta} \mathbf{u} = 0$;
- (iii) the eddy viscosity and Stokes-drift velocity are functions of z only, $v_T \equiv v_T(z) > 0$ and $\mathbf{U}_S \equiv \{U_S(z), V_S(z), 0\}$.

The vortex force now becomes

$$\mathbf{U}_S \times (\boldsymbol{\Omega} + \boldsymbol{\omega}) \equiv \{\omega_z V_S, -\omega_z U_S, U_S (\partial U / \partial z + \omega_y) + V_S (\partial V / \partial z - \omega_x)\}, \quad (4)$$

with

$$\omega_x = \partial w / \partial y - \partial v / \partial z; \quad \omega_y = \partial u / \partial z - \partial w / \partial x; \quad \omega_z = \partial v / \partial x - \partial u / \partial y,$$

while the mean vortex force is given by

$$\langle \mathbf{U}_S \times (\boldsymbol{\Omega} + \boldsymbol{\omega}) \rangle \equiv \mathbf{U}_S \times \boldsymbol{\Omega} = \{0, 0, U_S \cdot \partial U / \partial z\}. \quad (5)$$

The energy-balance equation (2) reduces to

$$dK/dt = - \int_{-h}^0 dz \{ \langle wu \rangle \partial(U + U_S) / \partial z + \langle wv \rangle \partial(V + V_S) / \partial z \} - \int_{-h}^0 dz v_T \{ \langle u_x^2 \rangle + \dots + \langle w_z^2 \rangle \}. \quad (6)$$

The mean horizontal velocities U , V obey the equations

$$\partial U / \partial t + \partial \langle wu \rangle / \partial z + \partial \Pi_0 / \partial x = \partial (v_T \partial U / \partial z) / \partial z, \quad (7a)$$

$$\partial V / \partial t + \partial \langle wv \rangle / \partial z + \partial \Pi_0 / \partial y = \partial (v_T \partial V / \partial z) / \partial z, \quad (7b)$$

with $\Pi_0 = P/\rho + \frac{1}{2} \langle \mathbf{u} \cdot \mathbf{u} \rangle$. The equations (6) and (7) will be used in the following to simulate the growth of the disturbance \mathbf{u} and the formation of a pattern of helical vortices; to this end, a (linear) stability analysis is needed. The theory of wave-driven longitudinal-vortex instability is analogous, under certain conditions, to the theory of thermal-convection instability, and to the theory of instability of flow between rotating cylinders (*cf.* Craik 1977; 1985, §13.2). Here, we follow the simple treatment of the *supercritical* instability problem by Stuart (1958): due to the inherent nonlinearity an equilibrium state $dK/dt = 0$ is possible with a definite finite amplitude of the disturbance.

Accordingly, the disturbance velocity is written in the form:

$$\mathbf{u}(x,y,z,t) = \Re \{ A(t) \mathbf{v}(z) e^{i(kx+ly) + iB(t)} \} + H.O.T., \quad (8)$$

When the disturbance is sufficiently small, linearization is permitted, and for a *nearly horizontal* flow this yields:

$$\partial u/\partial t + U\partial u/\partial x + V\partial u/\partial y + w\partial U/\partial z + 1/\rho \partial p/\partial x = \nabla \cdot (\mathbf{v}_T \nabla \mathbf{u}) + V_S \omega_z, \quad (12a)$$

$$\partial v/\partial t + U\partial v/\partial x + V\partial v/\partial y + w\partial V/\partial z + 1/\rho \partial p/\partial y = \nabla \cdot (\mathbf{v}_T \nabla \mathbf{v}) - U_S \omega_x, \quad (12b)$$

$$\partial w/\partial t + U\partial w/\partial x + V\partial w/\partial y + 1/\rho \partial p/\partial z = \nabla \cdot (\mathbf{v}_T \nabla \mathbf{w}) + U_S \omega_y - V_S \omega_x, \quad (12c)$$

with continuity equation

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0. \quad (13)$$

The linear stability problem can be analysed using a *long-wave asymptotic expansion* (cf. Cox 1997); the boundary conditions on the velocities u, v, w are chosen correspondingly:

$$\partial u/\partial z = \partial v/\partial z = w = 0, \quad (14)$$

at the mean free surface $z = 0$ and at the bed $z = -h$ (note that the proposed long-wave expansion is not consistent with a 'no-slip' boundary condition at the rigid bottom $z = -h$).

Following Cox (1997), we introduce a small parameter ε such that the most unstable rolls have wavelength $\mathcal{O}(1/\varepsilon)$; assuming a slow growth rate σ , we set $A(t) \approx \exp(\sigma t)$ in equ. (8) and consider the expansion

$$(\mathbf{u}; p/\rho) \approx (\mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \dots; p_0 + \varepsilon p_1 + \dots) \exp\{i\varepsilon(kx + ly) + \varepsilon(\sigma_1 + \varepsilon\sigma_2 + \dots)t\}, \quad (15)$$

with $\mathbf{u}_i(z)$ and $p_i(z)$ functions of z alone, and k, l scaled cell wavenumbers, $\mathcal{O}(1)$.

Upon substituting (15) into (12)-(13) we are able to solve the equations that result at successive orders in ε , taking into account the boundary conditions (14) at each order in the expansion. Leaving the details to appendix A, we obtain the following results: u_0, v_0 (and p_0) are constants, generally $\neq 0$ and yet to be determined, satisfying the relation

$$ku_0 + lv_0 = 0, \quad (16)$$

$w_0 \equiv w_1 \equiv 0$, and w_2 is given by the solution of:

$$D(\mathbf{v}_T D^2 w_2) = [k(\underline{U}_S - U_S) - l(\underline{V}_S - V_S)](lu_0 - kv_0), \quad (17)$$

with $D \equiv \partial/\partial z$ and $\underline{U}_S, \underline{V}_S$ denotes an average over depth.

For the growth rate σ_1 we find

$$\sigma_1 = -i\{k(\underline{U} + \underline{U}_S) + l(\underline{V} + \underline{V}_S)\}. \quad (18)$$

The growth rate σ_2 is real, while u_1, v_1 and p_1 are purely imaginary (cf. Cox 1997); however, these expressions are not needed for the present purpose: only the *leading order terms*, significant in the limit $\kappa = \sqrt{(k^2 + l^2)} \rightarrow 0$,

of the stability spectrum: the most unstable mode has an eigenvalue with vanishing imaginary part, when P.E.S. holds. In the present case, this amounts to $\sigma_1 = 0$ for the orientation q_c of the critical rolls, and from (18):

$$q_c (\underline{U} + \underline{U}_s) + (\underline{V} + \underline{V}_s) = 0 . \quad (25)$$

This simple result means that the axes of the critical rolls must be oriented between the direction of the shear flow and the direction of the Stokes drift (*cf.* Gnanadesikan and Weller 1995; Cox 1997; Polonichko 1997). However, it has to be proved that P.E.S. will apply when mean flow and waves are *not* aligned; moreover, in this case, the supposition $\sigma_1 = 0$ is not compatible with the observed lateral drift of windrows (*cf.* Cox 1997 for further details).

- given the size of the cells (*i.e.* given the value of κ), we may calculate for infinitesimal values of the amplitude A_0 the maximum growth rate: $\max_{(q)} dA_0^2/dt$; this implies that q_c must satisfy the relations: $d(\alpha-\beta)/dq = 0$ and $d^2(\alpha-\beta)/dq^2 < 0$.

Define the ratio

$$Q \equiv Q_1/Q_0 \equiv \int_{-h}^0 dz (M_x Z_y + M_y Z_x) / \int_{-h}^0 dz (M_x Z_x - M_y Z_y) , \quad (26)$$

then the orientation q_c is given by

$$q_c = [1 + \delta \sqrt{1 + Q^2}] / Q , \quad (27)$$

with $\delta = \text{sign}(Q_0)$.

In the special case that mean flow and waves are aligned, *i.e.* $V_s/U_s = T_y/T_x = r$, with constant proportion r , we have $M_y/M_x = Z_y/Z_x = r$, $Q = 2r/(1-r^2)$ and $q_c = -r$; the same result would follow from (25). In more general cases, the results of (25) and (26), (27) for q_c differ.

5. Equilibrium solution. The wave-induced eddy diffusivity

The *Landau-Stuart* equation (10) has a non-trivial equilibrium solution

$$A_e^2 = (\beta_1 - \beta_2) / \beta_3 , \quad (28)$$

whenever $\beta_1 > \beta_2 > 0$, as $\beta_3 > 0$ (*cf.* the expressions for the coefficients in (21a,b,c)).

[The resulting coherent cell pattern, in the form of two-dimensional rolls, may undergo secondary instabilities due to effects of higher harmonics (Eckhaus instabilities, *cf.* Eckhaus 1965, Ch.8); this phenomenon will not be considered here, see *e.g.* Cox and Leibovich 1997, who extend the analysis to three-dimensional circulations].

The wave-induced Reynolds stresses now follow from (19), (21) and (28), in case of $\alpha > \beta$:

$$\langle wu \rangle = \frac{1}{2} [(\alpha-\beta)/\gamma] m_0(z) \sqrt{1+q^2} , \quad \langle wv \rangle = -\frac{1}{2} [(\alpha-\beta)/\gamma] q m_0(z) \sqrt{1+q^2} , \quad (29a,b)$$

where $m_0(z)$ is given by (22), (23) and $q = q_c$; otherwise, when $\alpha \leq \beta$, $\langle wu \rangle = \langle wv \rangle \equiv 0$.

The coefficients γ , β and α can be evaluated successively as:

$\gamma = \frac{1}{2} [\frac{3}{4} - \ln 2] h \mu_0^2 / \nu_T \approx h \mu_0^2 / (36 \nu_T)$, where $\ln 2 \approx 0.6931$ is approximated by $25/36 \approx 0.6944$;

$\beta = 2 h \nu_T$; whereas $\nu_T(0) = 0$, we infer from (9): $T_x = \int_0^z dz' \partial \Pi_Q / \partial x$, with $\partial \Pi_Q / \partial x = P_x / \rho$ assumed to be constant; using the shallow-water approximation $dU_S/dz \approx (1+z/h) \{dU_S/dz\}_{z=0}$, we obtain:

$\alpha \approx 1/6 h \mu_0 [\{dU_S/dz\}_0 - 7/6 h P_x / (\rho \nu_T)]$.

Since $P_x < 0$, the coefficient α is definitely positive, and consequently $\alpha > \beta$ when the Stokes-drift shear is sufficiently large, compared with the mean-flow shear, at the surface.

The wave-induced stress (29a) becomes, in the present approximation,

$$\langle wu \rangle \cong \mu_S z/h (1+z/h) . \quad (35)$$

where $\mu_S = \frac{1}{2} \mu_0 \max\{0, (\alpha-\beta)/\gamma\} \geq 0$.

A wave-induced eddy diffusivity ν_L can now be defined according to

$$\langle wu \rangle = -\nu_L dU/dz , \quad (36)$$

and from (32) and (35), we obtain

$$\nu_L(z) \cong -\frac{1}{2} \mu_S / \{dU/dz\}_0 z/h (1+z/h)^2 (2+z/h) . \quad (37)$$

with $\{dU/dz\}_0 = U_0/2h > 0$.

The ν_L -profile, with maximum value at $z/h = -1+1/\sqrt{2} \approx -0.293$, is outlined in figure 2.

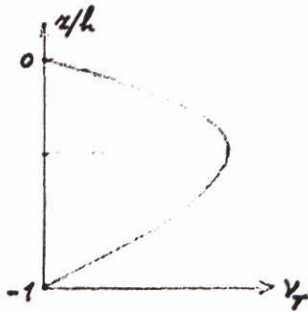


figure 1

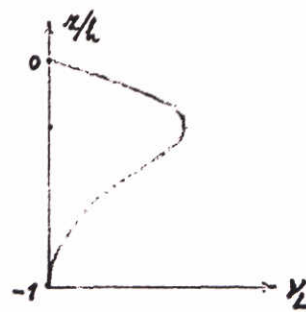


figure 2

The physical meaning of the expressions (36) and (37) becomes clear after substitution in the momentum equation (7a) : the effect of the waves consists of the additional shear stress exerted on the mean flow, while outside the bottom boundary layer the velocity shear dU/dz is reduced (see figure 3).

Appendix A. Order equations in the long-wave asymptotic expansion

At *zeroth* order, we have

$$\begin{aligned} Dw_0 &= 0, \\ D(v_T Du_0) &= D(v_T Dv_0) = 0, \\ Dp_0 &= U_S Du_0 + V_S Dv_0. \end{aligned}$$

Together with the boundary conditions (14), we obtain: $w_0 \equiv 0$; u_0 , v_0 and p_0 constants, generally $\neq 0$.

At *first* order:

$$\begin{aligned} i(ku_0 + lv_0) + Dw_1 &= 0, \\ \sigma_1 u_0 + i(kU + lV) u_0 + w_1 DU + ik p_0 &= D(v_T Du_1) - V_S i(lu_0 - kv_0), \\ \sigma_1 v_0 + i(kU + lV) v_0 + w_1 DV + il p_0 &= D(v_T Dv_1) + U_S i(lu_0 - kv_0), \\ Dp_1 &= D(v_T Dw_1) + U_S Du_1 + V_S Dv_1. \end{aligned}$$

Integrating the continuity equation gives: $w_1 = -i(ku_0 + lv_0)z + c_1$, and subsequently, in view of the boundary conditions (14), $c_1 = 0$, $ku_0 + lv_0 = 0$, and so $w_1 \equiv 0$. Averaging the momentum equations over depth, it is found that

$$\begin{aligned} p_0 &= (l\underline{U}_S - k\underline{V}_S) (lu_0 - kv_0)/\kappa^2, \\ \sigma_1 &= -i\{k(\underline{U} + \underline{U}_S) + l(\underline{V} + \underline{V}_S)\}, \end{aligned}$$

where $\kappa = \sqrt{(k^2 + l^2)}$.

At *second* order:

$$i(ku_1 + lv_1) + Dw_2 = 0.$$

Elimination of u_1 , v_1 from this equation and the first-order momentum equations yields:

$$D(v_T D^2 w_2) = [l(\underline{U}_S - U_S) - k(\underline{V}_S - V_S)] (lu_0 - kv_0),$$

with boundary conditions: $w_2(z) = 0$ and $D^2 w_2(z) = 0$, at $z = 0$ and $z = -h$.

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