Analysis in Banach Spaces
Volume III. Harmonic Analysis and Spectral Theory

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DOI
10.1007/978-3-031-46598-7

Publication date
2023

Document Version
Final published version

Citation (APA)
Hytönen, T., van Neerven, J., Veraar, M., & Weis, L. (2023). *Analysis in Banach Spaces: Volume III. Harmonic Analysis and Spectral Theory*. (Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge; Vol. 76). Springer. https://doi.org/10.1007/978-3-031-46598-7

Ergebnisse der Mathematik und ihrer Grenzgebiete.
3. Folge / A Series of Modern Surveys in Mathematics 76

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Springer

Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics

Volume 76

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Analysis in Banach Spaces

Volume III: Harmonic Analysis and Spectral Theory

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ISSN 0071-1136 ISSN 2197-5655 (electronic)
Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics
ISBN 978-3-031-46597-0 ISBN 978-3-031-46598-7 (eBook)
https://doi.org/10.1007/978-3-031-46598-7

Mathematics Subject Classification (2020): 46Bxx, 35Kxx, 42Bxx

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The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

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Preface

Originally conceived to be the final volume of a trilogy on Analysis in Banach spaces, containing the applications of the infrastructure of Volumes I–II to Harmonic and Stochastic Analysis, it was eventually evident that the body of work that we wanted to discuss would never fit into one volume of comparable size. Thus it was decided to divide the topics over two volumes, of which the present one, subtitled Harmonic Analysis and Spectral Theory, will offer a systematic treatment of Banach space-valued singular integrals, Fourier transform, and function spaces; further develop and ramify the theory of functional calculus from Volume II; and culminate in applications of these notions and tools in the problem of maximal regularity of evolution equations. The subsequent Volume IV will then be dedicated to the stochastic counterparts of some of these topics.

Like the previous Volume II, the present Volume III has its time-wise centre of gravity firmly in the present century. At the same time, we always cover the necessary prerequisites from earlier developments, presenting a self-contained picture rather than just a modern uppermost layer. As one might expect, Banach spaces with the unconditional martingale differences (UMD) property, will again manifest themselves as the most useful class of spaces for our analysis, but many of the other Banach space properties discussed in Volumes I–II will also feature prominently.

Our discussion of singular integrals is thoroughly influenced by the recent notion of sparse domination, and we use this technology to prove the A_2 theorem on sharp weighted norm inequalities, only obtained in 2010/2012 and unforeseen at the time of starting our book project back in 2008. Our approach to this theorem is still more recent, a result of a sequence of simplifications and abstractions of the original argument achieved over the past decade. Another main result on singular integrals is the characterisation of their boundedness given by the $T(1)$ theorem, which goes back to the 1980's even in the vector-valued setting, but was only proved in the full operator-valued generality in 2006. Once again, several subsequent extensions of the argument have taken place, and we have tried to present a proof, while still non-trivial probably by necessity, that combines elements from several of the existing approaches, also highlighting intermediate results of independent interest.

The main theorem that we prove about the Fourier transform is the vector-valued Hausdorff–Young inequality due to Bourgain, again from the 1980's, but we have followed a more recent framework of the argument by Hinrichs, Pietsch and Wenzel from the late 1990's. Equipped with this tool, we can identify the operator-valued Fourier multipliers discussed in the first two volumes with some singular integrals treated in the present one, and thereby obtain results like weighted norm inequalities for these operators. Many of these corollaries are relatively recent in the literature. In the previous volumes, we have also seen that the UMD property is characterised by the boundedness of a few distinguished Fourier multipliers; we now extend such results to a much broader class covered by a 2010 result of Geiss, Montgomery-Smith and Saksman, which also allows us to complete some characterisations of UMD in terms of the equivalence of different function spaces.

The theory of vector-valued function spaces, already hinted at in several occasions in the previous volumes, is finally taken up here in a systematic way. With the complete scale of the relevant function spaces at hand, we can provide the final form with sharp end-point assumptions of several embedding theorems that were discussed in weaker or incomplete forms in the previous volumes. Some of these function spaces will also play an important role later in this volume, when we treat the maximal regularity problem of evolution equations.

The notion of functional calculus of sectorial operators was already developed at length in Volume II, but there is a lot more to add to this vast topic, both for intrinsic and applied interest. While the necessary background material on fractional powers dates back as far as the 1960's, we have followed a more modern approach of viewing these powers in the framework of the so-called extended calculus. Of special interest is the class of operators admitting bounded imaginary powers.

The theory of sums of operators and the operator-valued functional calculus offer significant extensions of the "vanilla" functional calculus that pave the way for the treatment of the maximal regularity problem. We also develop the perturbation theory of sectorial operators, which expands the list of examples of concrete operators for which we can check and hence apply the H^∞-calculus.

The penultimate chapter presents a treatment of the maximal regularity problem for evolution equations. The main characterisation result is from the turn of the millennium, and so are the related counterexamples by Kalton–Lancien, but for the latter we will follow a much simplified recent approach by Fackler from 2014–2016. In the final chapter we present a recent theory of parabolic nonlinear evolution equations in critical spaces based on maximal L^p and continuous regularity. This theory was developed during the last decade by LeCrone, Prüss, Simonett, and Wilke, and it has already turned out to lead to far-reaching improvements for several classes of parabolic partial differential equations.

*

The main stylistic conventions of the previous volumes are adopted in the present volume as well: Most of the time, we are quite explicit with the constants appearing in our estimates, and we especially try to keep track of the dependence on the main parameters involved. Some of these explicit quantitative formulations appear here for the first time. Where relevant, we also pay more attention than many texts to the impact of the underlying scalar field (real or complex) on the results under consideration, although a need for this perhaps appears slightly less in this volume than in the previous two. A notable instance is the distinction between the real and complex UMD constants, and their relation to various multiplier norms.

*

While the previous Volumes I and II of this series were written largely in parallel over the years 2008–2017, the major work on this Volume III only took place after the completion of the first two books. Critical to the progress of this endeavour was the possibility of intensive joint workshops that we held in the rural serenity of Stiftsgut Keysermühle in Klingenmünster (June 2018, March 2019) but also, due to other professional and personal commitments of some of us, in the urban beat of Delft (October 2018, January 2020) and Helsinki (January 2020). Shortly after the last two meetings, and with the completion of this volume already on the horizon, the global pandemic broke out. This changed our plans like so much else, and our progress on this work was essentially halted for two years. Only at the beginning of 2023 we were able to resume our 'live' writing sessions to finally bring it to completion.

Preliminary versions of parts of the material were presented in advanced courses and lecture series at various international venues and in seminars at our departments, and we would like to thank the students and colleagues who attended these events for feedback that shaped and improved the final form of the text. Special thanks go to Antonio Agresti, Sebastian Bechtel, Chenxi Deng, Emiel Lorist, Floris Roodenburg, Max Sauerbrey, Esmée Theewis, Joshua Willems, and Joris van Winden, who read in detail portions of this book. Needless to say, we take full responsibility for any remaining errors. Lists with errata for each of the three volumes are maintained on our personal websites.

During the writing of this book, we have benefited from external funding by the Academy of Finland / Research Council of Finland (grants 314829 and 346314 to T.H., and the Finnish Centre of Excellence in Randomness and Structures "FiRST", of which T.H. is a member), the Netherlands Organisation for Scientific research (NWO) (VIDI grant 639.032.427 and VICI grant 639.212.027 to M.V.), and the Deutsche Forschungsgemeinschaft (Project-ID 258734477 – SFB 1173 to L.W.).

Delft, Helsinki, and Karlsruhe, September 15, 2023

Contents

Symbols and notations

Sets

$\mathbb{N} = \{0, 1, 2, \dots\}$ - non-negative integers
$\mathbb{Z}$ - integers
$\mathbb{Q}$ - rational numbers
$\mathbb{R}$ - real numbers
$\mathbb{C}$ - complex numbers
$\mathbb{K}$ - scalar field ($\mathbb{R}$ or $\mathbb{C}$)
$\overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, \infty\}$ - extended integers
$\mathbb{R}_+ = (0, \infty)$ - positive real line
B_X - open unit ball
S_X - unit sphere
$B(x, r)$ - open ball centred at x with radius r
$\mathbb{D}$ - open unit disc
$\mathbb{S} = \{z \in \mathbb{C} : 0 < \Im z < 1\}$ - unit strip
Σ_ω = open sector of angle ω
$\Sigma_\omega^{\mathrm{bi}}$ = open bisector of angle ω
$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ - unit circle

Vector spaces

$B^s_{p,q}$ - Besov space
c_0 - space of null sequences
C - space of continuous functions
C_0 - space of continuous functions vanishing at infinity
C^α - space of Hölder continuous functions
C_{b} - space of bounded continuous functions
C_{c} - space of continuous functions with compact support
C_{c}^∞ - space of test functions with compact support
$\mathscr{C}^p$ - Schatten class
$\gamma(H, X)$ - space of γ-radonifying operators

$\gamma(S;X)$ - shorthand for $\gamma(L^2(S),X)$
$\gamma_\infty(H,X)$ - space of almost summing operators
$\gamma_\infty(S;X)$ - shorthand for $\gamma_\infty(L^2(S),X)$
E - space of primary functions
H - Hilbert space
$H^{s,p}$ - Bessel potential space
H^p - Hardy space
ℓ^p - space of p-summable sequences
ℓ^p_N - space of p-summable finite sequences
H^p - Hardy spaces on a sector or strip
L^p - Lebesgue space
$L^{p,q}$ - Lorentz space
$L^{p,\infty}$ - weak-L^p
$L^p_w(I;X)$ - weighted L^p
$\check{L}^1$ - space of inverse Fourier transforms
Lip - space of Lipschitz continuous functions
$\mathscr{L}(X,Y)$ - space of bounded linear operators
$\mathfrak{M}L^p$ - space of Fourier multipliers
$\mathfrak{M}$ - Mihlin class
$\mathscr{S}$ - space of Schwartz functions
$\mathscr{S}'$ - space of tempered distributions
$F^s_{p,q}$ - Triebel–Lizorkin space
$W^{k,p}$ - Sobolev space
$W^{s,p}$ - Sobolev-Slobodetskii space
X, Y, $\dots$- Banach spaces
$X_\mathbb{C}$ - complexification
$X^{\gamma,p}_\mathbb{C}$ - Gaussian complexification
X^*, Y^*, $\dots$- dual Banach spaces
$X^\odot$, $Y^\odot$, $\dots$- strongly continuous semigroup dual spaces
$X \otimes Y$ - tensor product
$[X_0,X_1]_\theta$ - complex interpolation space
$(X_0,X_1)_{\theta,p}$, $(X_0,X_1)_{\theta,p_0,p_1}$ - real interpolation spaces

Measure theory and probability

$\mathscr{A}$ - σ-algebra
$df_n = f_n - f_{n-1}$ - nth martingale difference
ϵ_n - signs in $\mathbb{K}$, i.e., scalars in $\mathbb{K}$ of modulus one
ε_n - Rademacher variables with values in $\mathbb{K}$
$\mathbb{E}$ - expectation
$\mathscr{F}$, $\mathscr{G}$, $\dots$- σ-algebras
$\mathscr{F}_f$ - collection of sets in $\mathscr{F}$ on which f is integrable
$\mathbb{E}(\cdot|\cdot)$ - conditional expectation
γ_n - Gaussian variables
h_I - Haar function

μ - measure
$\|\mu\|$ - variation of a measure
$(\Omega, \mathscr{A}, \mathbb{P})$ - probability space
$\mathbb{P}$ - probability measure
r_n - real Rademacher variables
$(S, \mathscr{A}, \mu)$ - measure space
$\sigma(f, g, \dots)$ - σ-algebra generated by the functions $f, g, \dots$
$\sigma(\mathscr{C})$ - σ-algebra generated by the collection $\mathscr{C}$
τ - stopping time
w_α - Walsh function

Norms and pairings

$|\cdot|$ - modulus, Euclidean norm
$\|\cdot\| = \|\cdot\|_X$ - norm in a Banach space X
$\|\cdot\|_p = \|\cdot\|_{L^p}$ - L^p-norm
$\langle\cdot,\cdot\rangle$ - duality
$(\cdot|\cdot)$ - inner product in a Hilbert space
$a \cdot b$ - inner product of $a, b \in \mathbb{R}^d$

Operators

A - closed linear operator
A^* - adjoint operator
$A^\odot$ - part of A^* in $X^\odot$
$\mathsf{D}(A)$ - domain of A
∇ - gradient
Δ - Laplace operator
$\gamma(\mathscr{T})$ - γ-bound of the operator family $\mathscr{T}$
$\gamma_p(\mathscr{T})$ - γ-bound of $\mathscr{T}$ with respect to the L^p-norm
$\mathscr{D}$ - dyadic system
$\partial_j = \partial/\partial x_j$ - partial derivative with respect to x_j
∂^α - partial derivative with multi-index α
$\mathbb{E}(\cdot|\cdot)$ - conditional expectation
$\mathscr{F}$ - Fourier transform
$\mathscr{F}^{-1}$ - inverse Fourier transform
H - Hilbert transform
$\widetilde{H}$ - periodic Hilbert transform
J_s - Bessel potential operator
$\ell^2(\mathscr{T})$ - ℓ^2-bound of the operator family $\mathscr{T}$
$\mathscr{L}(X, Y)$ - space of bounded operators from X to Y
$M_{\sigma,A}$ - sectoriality constant of A at angle σ
$\mathsf{N}(A)$ - null space of A
$\mathscr{R}(\mathscr{T})$ - R-bound of the operator family $\mathscr{T}$
$\mathscr{R}_p(\mathscr{T})$ - R-bound of $\mathscr{T}$ with respect to the L^p-norm
$\mathsf{R}(A)$ - range of A

R_j - jth Riesz transform
$R(\lambda, A) = (\lambda - A)^{-1}$ - resolvent of A at λ
$\varrho(A)$ - resolvent set of A
$\sigma(A)$ - spectrum of A
S, T, ...- bounded linear operators
$S(t)$, $T(t)$, ...- semigroup operators
$S^*(t)$, $T^*(t)$, ...- adjoint semigroup operators on the dual space X^*
$S^{\odot}(t)$, $T^{\odot}(t)$, ...- their parts in the strongly continuous dual $X^{\odot}$
T^* - adjoint of the operator T
$T^\star$ - Hilbert space (hermitian) adjoint of Hilbert space operator T
T_m - Fourier multiplier operator associated with multiplier m
$T \otimes I_X$ - tensor extension of T
$\omega(A)$ - angle of sectoriality of A
$\omega_R(A)$, $\omega_\gamma(A)$ - angles of R- and γ-sectoriality of A
$\omega_{\rm BIP}(A)$ - angle of bounded imaginary powers of A
$\omega_{H^\infty}(A)$ - angle of the H^∞-calculus of A
$\omega^{\rm bi}(A)$ - angle of bisectoriality of A

Constants and inequalities

$\alpha_{p,X}$ - Pisier contraction property constant
$\alpha^{\pm}_{p,X}$ - upper and lower Pisier contraction property constant
$\beta_{p,X}$ - UMD constant
$\beta^{\mathbb{R}}_{p,X}$ - UMD constant with signs ± 1
$\beta^{\pm}_{p,X}$ - upper and lower randomised UMD constant
$c_{q,X}$ - cotype q constant
$c^{\gamma}_{q,X}$ - Gaussian cotype q constant
$\varDelta_{p,X}$ - triangular contraction property constant
$\hbar_{p,X}$ - norm of the Hilbert transform on $L^p(\mathbb{R}; X)$
$K_{p,X}$ - K-convexity constant
$K^{\gamma}_{p,X}$ - Gaussian K-convexity constant
$\kappa_{p,q}$ - Kahane–Khintchine constant
$\kappa^{\mathbb{R}}_{p,q}$ - idem, for real Rademacher variables
$\kappa^{\gamma}_{p,q}$ - idem, for Gaussian sums
$\kappa_{p,q,X}$ - idem, for a fixed Banach space X
$\tau_{p,X}$ - type p constant
$\tau^{\gamma}_{p,X}$ - Gaussian type p constant
$\varphi_{p,X}(\mathbb{R}^d)$ - norm of the Fourier transform $\mathscr{F} : L^p(\mathbb{R}^d; X) \to L^{p'}(\mathbb{R}^d; X)$.

Miscellaneous

$\hookrightarrow$ - continuous embedding
$\mathbf{1}_A$ - indicator function
$a \lesssim b$ - $\exists C$ such that $a \leqslant Cb$
$a \lesssim_{p,P} b$ - $\exists C$, depending on p and P, such that $a \leqslant Cb$

C - generic constant
$\complement$ - complement
$d(x, y)$ - distance
$f^\star$ - maximal function
$\widetilde{f}$ - reflected function
$\widehat{f}$ - Fourier transform
$\widecheck{f}$ - inverse Fourier transform
$f * g$ - convolution
$\Im$ - imaginary part
Mf - Hardy–Littlewood maximal function
$p' = p/(p-1)$ - conjugate exponent
$p^* = \max\{p, p'\}$
$\wp$ - good set-bound
$\Re$ - real part
$s \wedge t = \min\{s, t\}$
$s \vee t = \max\{s, t\}$
x - generic element of X
x^* - generic element of X^*
$x \otimes y$ - elementary tensor
x^+, x^-, $|x|$ - positive part, negative part, and modulus of x
w - weight
w_α - power weight $t \mapsto t^\alpha$

Standing assumptions

Throughout this book, two conventions will be in force.

1. Unless stated otherwise, the scalar field $\mathbb{K}$ can be real or complex. Results which do not explicitly specify the scalar field to be real or complex are true over both the real and complex scalars.

2. In the context of randomisation, a *Rademacher variable* is a uniformly distributed random variable taking values in the set $\{z \in \mathbb{K} : |z| = 1\}$. Such variables are denoted by the letter ε. Thus, whenever we work over $\mathbb{R}$ it is understood that ε is a real Rademacher variable, i.e.,

$$\mathbb{P}(\varepsilon = 1) = \mathbb{P}(\varepsilon = -1) = \frac{1}{2},$$

and whenever we work over $\mathbb{C}$ it is understood that ε is a complex Rademacher variable (also called a *Steinhaus variable*), i.e.,

$$\mathbb{P}(a < \arg(\varepsilon) < b) = \frac{1}{2\pi}(b - a).$$

Occasionally we need to use real Rademacher variables when working over the complex scalars. In those instances we will always denote these with the letter r. Similar conventions are in force with respect to Gaussian random variables: a *Gaussian random variable* is a standard normal real-valued variable when working over $\mathbb{R}$ and a standard normal complex-valued variable when working over $\mathbb{C}$.

11

Singular integral operators

Various operators of Analysis, many of them already encountered in these volumes, take the generic form

$$Tf(s) = \int_{\mathbb{R}^d} K(s,t)f(t)\,\mathrm{d}t. \tag{11.1}$$

The mapping properties of T will of course heavily depend on the assumptions made on the *kernel* K that we will discuss in more detail in this chapter. A general feature of the different conditions is that the kernel is allowed to blow up on the 'diagonal' $\{(x,x) : x \in \mathbb{R}^d\}$, so that its natural domain of definition is the set

$$\dot{\mathbb{R}}^{2d} := \{(s,t) \in \mathbb{R}^d \times \mathbb{R}^d : s \neq t\}.$$

This blow-up is one of the reasons for referring to (11.1) as a *singular integral*; in general this formula requires a careful interpretation and will only be meaningful under restrictions on f and s.

In the prominent special case of a *convolution kernel* $K(s,t) = \mathfrak{K}(s-t)$, the operator (11.1) takes (at least formally, and under reasonable assumptions also rigorously) a simple representation "on the Fourier transform side":

$$\widehat{Tf}(\xi) = \widehat{\mathfrak{K} * f}(\xi) = \widehat{\mathfrak{K}}(\xi)\widehat{f}(\xi) =: m(\xi)\widehat{f}(\xi);$$

thus $T = T_m$ can be identified with a *Fourier multiplier*; they have been studied extensively in Chapter 5 and Section 8.3.

The motivations to investigate *singular integral operators* in the non-transformed representation (11.1) are at least threefold. First, it allows for a wider class of examples beyond those of the convolution form. Second, even when the alternative Fourier multiplier representation is available in principle, an operator may naturally arise in the form (11.1), and identifying or estimating the corresponding multiplier explicitly may not be feasible in practise, as the Fourier transform is not isomorphic between the natural function spaces for the kernel $\mathfrak{K}$ and the multiplier m. Finally, and perhaps most importantly,

T. Hytönen et al., *Analysis in Banach Spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics 76,
https://doi.org/10.1007/978-3-031-46598-7_1

even for multiplier operators, the point-of-view of singular integrals gives us access to new methods and conclusions.

An overarching theme is of this chapter is *extrapolation*: As soon as an operator (11.1), with natural assumptions on the kernel K, is bounded on a single space $L^{p_0}(\mathbb{R}^d;X)$, it will be automatically bounded on several more spaces, including $L^p(\mathbb{R}^d;X)$ for other exponents $p\in(1,\infty)$ (with certain substitute results at the end-points $p\in\{1,\infty\}$), and even their weighted versions $L^p(w;X)$, where w is an arbitrary weight in the Muckenhoupt class A_p (see Appendix J). These results will be used to deduce analogous extrapolation results for *maximal L^p-regularity* of the *abstract Cauchy problem* in Chapter 17.

In terms of Banach spaces, this chapter deals with relatively general results, most of which are valid without restrictions of the class of admissible spaces. Such restrictions, and notably the ubiquitous UMD condition, will reappear in the subsequent chapters, when searching for conditions to verify the boundedness of (11.1) on just one $L^{p_0}(\mathbb{R}^d;X)$, to serve as an input to the extrapolation results that we develop in the chapter at hand.

11.1 Local oscillations of functions

A characteristic feature of singular integrals, the main topic of this chapter, is that their boundedness properties depend not only naive size estimates but on rather delicate cancellations between different oscillatory components. Before we dwell into a deeper study of there operators, we dedicate this section to a general treatment of oscillations of functions per se; this will streamline the subsequent discussion, where the results of this section will be put into action in the context of operator norm estimates.

Given $f\in L^0(\mathbb{R}^d;X)$ and $\lambda>0$, we define the following measure of oscillation of f on a cube Q,

$$\operatorname{osc}_\lambda(f;Q):=\inf_{c\in X}\inf_{|E|\leqslant\lambda|Q|}\|(f-c)\mathbf{1}_{Q\setminus E}\|_\infty.$$

Here, and in many occasions below where we will use the same notation, it is understood that the supremum is taken over all measurable subsets E of Q satisfying the stated requirement that $|E|\leqslant\lambda|Q|$. The idea is to quantify how much f deviates from a constant, if we ignore its (possibly wild) behaviour on an exceptional set of controlled proportion. The above way of measuring oscillations is essentially 'minimal' in that it can be controlled by average L^q oscillations for any $q>0$:

Lemma 11.1.1. *For any $q\in(0,\infty)$, we have*

$$\operatorname{osc}_\lambda(f;Q)\leqslant\inf_{c\in X}\frac{\|(f-c)\mathbf{1}_Q\|_{L^{q,\infty}}}{(\lambda|Q|)^{1/q}}.$$

Proof. For a fixed c, let $g := (f - c)\mathbf{1}_Q$. If we choose $t := \|g\|_{L^{q,\infty}}/(\lambda|Q|)^{1/q}$, then

$$|E_t| := |\{\|g\| > t\}| \leqslant \frac{\|g\|_{L^{q,\infty}}^q}{t^q} = \lambda|Q|$$

But then it is clear that

$$\inf_{|E| \leqslant \lambda|Q|} \|g\mathbf{1}_{Q\setminus E}\|_\infty \leqslant \|g\mathbf{1}_{Q\setminus E_t}\|_\infty \leqslant t,$$

which is precisely the claimed bound. □

Given a *real-valued* $f \in L^0(\mathbb{R}^d;\mathbb{R})$, any $m \in \mathbb{R}$ such that

$$|Q \cap \{f \leqslant m\}| \geqslant \tfrac{1}{2}|Q|, \qquad |Q \cap \{f \geqslant m\}| \geqslant \tfrac{1}{2}|Q|$$

is called a *median* of f on the cube (or more general set of finite positive measure) $Q \subseteq \mathbb{R}^d$. One routinely checks that a median always exists but may fail to be unique.

Lemma 11.1.2. *If $\lambda \in (0, \frac{1}{2})$ and $m_f \in \mathbb{R}$ is a median of $f \in L^0(Q;\mathbb{R})$ on Q, then*

$$\inf_{|E| \leqslant \lambda|Q|} \|(f - m_f)\mathbf{1}_{Q\setminus E}\|_\infty \leqslant 2\,\mathrm{osc}_\lambda(f;Q).$$

Proof. Let $c \in \mathbb{R}$ be arbitrary. Then $f - m_f = f - c - (m_f - c)$ and hence

$$\inf_{|E| \leqslant \lambda|Q|} \|(f - m_f)1_{Q\setminus E}\|_\infty \leqslant \inf_{|E| \leqslant \lambda|Q|} \|(f - c)1_{Q\setminus E}\|_\infty + |m_f - c|.$$

Note that $m_f - c$ is a median of $g := f - c$ on Q. Hence it suffices to check that the median m_g always satisfies

$$|m_g| \leqslant \|g1_{Q\setminus E}\|_\infty$$

whenever $|E| \leqslant \lambda|Q|$ and $\lambda < \frac{1}{2}$. If $m_g \geqslant 0$, then

$$|Q \cap \{|g| \geqslant |m_g|\} \setminus E| \geqslant |Q \cap \{g \geqslant m_g\} \setminus E| \geqslant \tfrac{1}{2}|Q| - |E| \geqslant (\tfrac{1}{2} - \lambda)|Q| > 0$$

and thus $\|g\mathbf{1}_{Q\setminus E}\|_\infty \geqslant |m_g|$. If $m_g < 0$, the argument is the same, just replacing the second step above by $|Q \cap \{g \leqslant m_g\} \setminus E|$. □

The previous lemma motivates the following:

Definition 11.1.3. *Let X be a Banach space and $f \in L^0(Q;X)$. A vector $m \in X$ is called a λ-pseudomedian of f on Q if*

$$\inf_{|E| \leqslant \lambda|Q|} \|(f - m)\mathbf{1}_{Q\setminus E}\|_\infty \leqslant 2\,\mathrm{osc}_\lambda(f;Q).$$

Indeed, Lemma 11.1.2 says that the usual median is a λ-pseudomedian for every $\lambda \in (0, \frac{1}{2})$. Concerning existence in the general case, we have:

Lemma 11.1.4. *Let X be a Banach space, $f \in L^0(Q;X)$ and $\lambda \in (0,\frac{1}{2})$. Then f has a λ-pseudomedian on Q.*

Proof. If $\operatorname{osc}_\lambda(f;Q) > 0$, this is obvious, since we can always come within any positive distance from the infimum. So only the case $\operatorname{osc}_\lambda(f;Q) = 0$ needs attention. In this case, there we can find a sequence of vectors $c_n \in X$ and sets $E_n \subseteq Q$ with $|E_n| \leqslant \lambda|Q|$ such that $\|(f-c_n)\mathbf{1}_{Q\setminus E_n}\|_\infty \to 0$. Since $|E_n \cup E_m| \leqslant 2\lambda|Q| < |Q|$, any $Q\setminus(E_n\cup E_m)$ has positive measure, and thus

$$\begin{aligned}\|c_n - c_m\| &= \|(c_n-c_m)\mathbf{1}_{Q\setminus(E_n\cup E_m)}\|_\infty \\ &\leqslant \|(f-c_n)\mathbf{1}_{Q\setminus E_n}\|_\infty + \|(f-c_m)\mathbf{1}_{Q\setminus E_m}\|_\infty \to 0.\end{aligned}$$

Thus $(c_n)_{n\geqslant 1}$ is a Cauchy sequence and hence convergent to some $c\in X$. But then

$$\begin{aligned}\inf_{|E|\leqslant\lambda|Q|}\|(f-c)\mathbf{1}_{Q\setminus E}\|_\infty &\leqslant \liminf_{n\to\infty}\|(f-c)\mathbf{1}_{Q\setminus E_n}\|_\infty \\ &\leqslant \liminf_{n\to\infty}\Big(\|(f-c_n)\mathbf{1}_{Q\setminus E_n}\|_\infty + \|c_n - c\|\Big) = 0,\end{aligned}$$

and thus this limit c is a λ-pseudomedian. ☐

Lemma 11.1.5. *Let X be a Banach space, let $f\in L^0(\mathbb{R}^d;X)$ and $\lambda\in(0,\frac{1}{2})$, and let $m_f(Q)$ be a λ-pseudomedian of f on Q. Then*

$$E^0 := Q\cap\{\|f-m_f(Q)\| > 2\operatorname{osc}_\lambda(f;Q)\}$$

satisfies $|E^0|\leqslant\lambda|Q|$.

Proof. Suppose for contradiction that $|E^0|>\lambda|Q|$. Denoting

$$E^\varepsilon := Q\cap\{\|f-m_f(Q)\| > 2\operatorname{osc}_\lambda(f;Q)+\varepsilon\}$$

we have $E^0 = \bigcup_{n=1}^\infty E^{1/n}$, so that by continuity of measure, we also have $|E^\varepsilon|>\lambda|Q|$ for some $\varepsilon = 1/n > 0$.

Let $|E|\leqslant\lambda|Q|$. Then

$$\|(f-m_f(Q))\mathbf{1}_{Q\setminus E}\|_\infty \geqslant (2\operatorname{osc}_\lambda(f;Q)+\varepsilon)\|\mathbf{1}_{E^\varepsilon\setminus E}\|_\infty = 2\operatorname{osc}_\lambda(f;Q)+\varepsilon,$$

since $|E^\varepsilon\setminus E|\geqslant|E^\varepsilon|-|E|>\lambda|Q|-\lambda|Q|=0$. Taking the infimum over all $|E|\leqslant\lambda|Q|$, we contradict the definition of a λ-pseudomedian. ☐

11.1.a Sparse collections and Lerner's formula

Let us recall and expand the terminology related to dyadic cubes that we introduced in Chapter 3.

Definition 11.1.6. *A dyadic system of cubes on $\mathbb{R}^d$ is a collection $\mathscr{D} = \bigcup_{j\in\mathbb{Z}}\mathscr{D}_j$, where*

(i) *each* $\mathscr{D}_j$ *is a partition of* $\mathbb{R}^d$ *of the form*

$$\mathscr{D}_j = \Big\{ s_j + 2^{-j}(m + [0,1)^d) : m \in \mathbb{Z}^d \Big\},$$

(ii) *each* $\mathscr{D}_{j+1}$ *refines the previous* $\mathscr{D}_j$.

When $s_j = 0$ *for all* $j \in \mathbb{Z}$*, we refer to the corresponding* $\mathscr{D}$ *as the* standard dyadic system*, and denote it by* $\mathscr{D}^0$.

Remark 11.1.7. One might like to replace (i) in Definition 11.1.6 by the "more intrinsic"

(iii) each $\mathscr{D}_j$ is a partition of $\mathbb{R}^d$ consisting of left-closed, right-open cubes of side-length 2^{-j}.

When $d = 1$, one can check that (i) and (iii) are equivalent. But, for $d > 1$, condition (iii) is strictly more general. For instance

$$\mathscr{D}_j := \big\{ 2^{-j}\big(m + [0,1)^2\big) + (0, \alpha \mathbf{1}_{[0,\infty)}(m_1)) : m \in \mathbb{Z}^2 \big\}, \quad \alpha \in \mathbb{R},$$

where all cubes in the right half-plane are shifted in the y-direction by a fixed amount $\alpha \in \mathbb{R}$ relative to the standard dyadic cubes, would qualify for (iii) but not for (i). The preference over one or the other definition may be a question of taste; we choose to work with Definition 11.1.6 as stated.

We will work be working with an arbitrary dyadic system as in Definition 11.1.6. For many purposes, the reader who so wishes may think of the *standard dyadic system.*

$$\mathscr{D}^0 := \bigcup_{j \in \mathbb{Z}} \mathscr{D}^0_j, \qquad \mathscr{D}^0_j := \{ 2^{-j}([0,1)^d + k) : k \in \mathbb{Z}^d \}, \quad j \in \mathbb{Z},$$

but here and there we will also make use of other systems, which makes it convenient to deal with a generic system from the beginning. For any given cube, we may speak of its *dyadic subcubes*, by which we understand all cubes obtained by repeatedly bisecting the edges of Q. We will use the notation $\mathscr{D}(Q)$ for the collection of all dyadic subcubes of a cube Q. If Q belongs to a dyadic system $\mathscr{D}$, then

$$\mathscr{D}(Q) = \{ Q' \in \mathscr{D} : Q' \subseteq Q \}.$$

Definition 11.1.8. *A* quadrant *of a dyadic system* $\mathscr{D}$ *of* $\mathbb{R}^d$ *is the union of any strictly increasing sequence* $Q_1 \subsetneq Q_2 \subsetneq Q_3 \subsetneq \ldots$ *of cubes* $Q_i \in \mathscr{D}$.

Remark 11.1.9. The standard dyadic system $\mathscr{D}^0$ has 2^d quadrants of the form $S_1 \times \cdots \times S_d$, where $S_i \in \{(-\infty, 0), [0, infty)\}$ for each $i \in \{1, \ldots, d\}$. It is also easy to construct dyadic systems, where $\mathbb{R}^d$ is the only quadrant.

The *dyadic Hardy–Littlewood maximal function* is defined by

$$M_{\mathscr{D}}f(x) = \sup_{Q\in\mathscr{D}:x\in Q} \langle \|f\| \rangle_Q, \qquad \langle f\rangle_Q := \fint_Q f := \frac{1}{|Q|}\int_Q f,$$

where the supremum is taken over all dyadic cubes containing x. Here, and throughout this chapter, unless indicated otherwise, integrals are taken with respect to Lebesgue measure and are abbreviated in the above way to unburden notation. Thus, when g is an integrable function, $\int_Q g$ is shorthand for $\int_Q g(x)\,\mathrm{d}x$. When integrating over all of $\mathbb{R}^d$ we will even write $\int g$ for $\int_{\mathbb{R}^d} g$.

Definition 11.1.10. *A collection $\mathscr{S}$ of sets $S \subseteq \mathbb{R}^d$ of finite measure is called γ-sparse, if each $S \in \mathscr{S}$ has a measurable subset $E(S) \subseteq S$ of size $|E(S)| \geqslant \gamma|S|$ such that the sets $E(S)$ are pairwise disjoint.*

While the definition can be made for general measurable sets, we will be mostly concerned with the case when $\mathscr{S} \subseteq \mathscr{D}$ is a subcollection of the dyadic cubes of $\mathbb{R}^d$.

A disjoint collection is obviously 1-sparse with $E(S) = S$. The usefulness of general γ-sparse collections comes from the fact that, on the one hand, they are easier to create than genuinely disjoint collections while, on the other hand, for the purposes of L^p estimates they are essentially as good as disjoint ones. This is quantified by the following:

Proposition 11.1.11. *Let $\mathscr{S} \subseteq \mathscr{D}$ be a γ-sparse collection of dyadic cubes S with disjoint subsets $|E(S)| \geqslant \gamma|S|$.*

(1) *If $a_S \geqslant 0$, then for all $p \in (0,\infty)$,*

$$\Big\|\sum_{S\in\mathscr{S}} a_S \mathbf{1}_S\Big\|_p \leqslant c_{p,\gamma}\Big\|\sum_{S\in\mathscr{S}} a_S \mathbf{1}_{E(S)}\Big\|_p, \quad \textit{where } c_{p,\gamma} = \begin{cases}\gamma^{-1}p, & p\in[1,\infty),\\ \gamma^{-1/p}, & p\in(0,1).\end{cases}$$

(2) *If $f \geqslant 0$, then for all $p \in (1,\infty)$,*

$$\Big(\sum_{S\in\mathscr{S}} \langle f\rangle_S^p |S|\Big)^{1/p} \leqslant \gamma^{-1/p}\, p' \|f\|_p.$$

Proof of Proposition 11.1.11. If $p \in [1,\infty)$, we dualise the left side of (1) against $\phi \in L^{p'}$:

$$\begin{aligned}\int \Big(\sum_{S\in\mathscr{S}} a_S \mathbf{1}_S\Big)\phi &= \sum_{S\in\mathscr{S}} a_S |S| \fint_S \phi \leqslant \frac{1}{\gamma}\sum_{S\in\mathscr{S}} a_S |E(S)| \inf_S M_{\mathscr{D}}\phi \\ &\leqslant \frac{1}{\gamma}\int\Big(\sum_{S\in\mathscr{S}} a_S\mathbf{1}_{E(S)}\Big)M_{\mathscr{D}}\phi \leqslant \frac{1}{\gamma}\Big\|\sum_{S\in\mathscr{S}} a_S \mathbf{1}_{E(S)}\Big\|_p \|M_{\mathscr{D}}\phi\|_{p'},\end{aligned}$$

where $\|M_{\mathscr{D}}\phi\|_{p'} \leqslant p\|\phi\|_{p'}$ by Doob's maximal inequality (Theorem 3.2.2; cf. the explanations preceding Theorem 3.2.27).

If $p \in (0,1)$, then the left side of (1) can be estimated by

$$\int \Big(\sum_{S \in \mathscr{S}} a_S \mathbf{1}_S \Big)^p \leqslant \int \sum_{S \in \mathscr{S}} a_S^p \mathbf{1}_S = \sum_{S \in \mathscr{S}} a_S^p |S| \leqslant \frac{1}{\gamma} \sum_{S \in \mathscr{S}} a_S^p |E(S)|$$
$$= \frac{1}{\gamma} \int \sum_{S \in \mathscr{S}} a_S^p \mathbf{1}_{E(S)} = \frac{1}{\gamma} \int \Big(\sum_{S \in \mathscr{S}} a_S \mathbf{1}_{E(S)} \Big)^p,$$

and taking the pth root completes the proof of (1).

For (2), we use $\langle f \rangle_S \leqslant \inf_{z \in S} M_{\mathscr{D}} f(z)$ and $|S| \leqslant \gamma^{-1} |E(S)|$ to find that

$$\sum_{S \in \mathscr{S}} \langle f \rangle_S^p |S| \leqslant \frac{1}{\gamma} \sum_{S \in \mathscr{S}} \int_{E(S)} (M_{\mathscr{D}} f)^p \, \mathrm{d}x \leqslant \frac{1}{\gamma} \| M_{\mathscr{D}} f \|_p^p \leqslant \frac{1}{\gamma} (p')^p \| f \|_p^p,$$

again by Doob's inequality in the last step. □

The different notions introduced above come together in the following useful estimate, which is the main result of this section:

Theorem 11.1.12 (Lerner's formula). *Let X be a Banach space, $Q^0 \subseteq \mathbb{R}^d$ be a cube and $f \in L^0(Q^0; X)$. Then there is a $\frac{1}{2}$-sparse subcollection $\mathscr{S} \subseteq \mathscr{D}(Q^0)$ such that, almost everywhere,*

$$\mathbf{1}_{Q^0} \| f - m_f(Q^0) \| \leqslant 4 \sum_{S \in \mathscr{S}} \operatorname{osc}_\lambda(f; S) \mathbf{1}_S, \qquad \lambda = 2^{-2-d},$$

where $m_f(Q^0)$ is any λ-pseudomedian of f on Q^0.

By Lemma 11.1.2, if $X = \mathbb{R}$, we can take $m_f(Q^0)$ to be a usual median of f.

Proof. We begin with a preliminary observation. For any collection of disjoint sets $Q_j \in \mathscr{D}(Q^0)$, we have the identity

$$\begin{aligned} \mathbf{1}_{Q^0}(f - m_f(Q^0)) &= \mathbf{1}_{Q^0 \setminus \bigcup_j Q_j} (f - m_f(Q^0)) \\ &\quad + \sum_j \mathbf{1}_{Q_j} (m_f(Q_j) - m_f(Q^0)) \\ &\quad + \sum_j \mathbf{1}_{Q_j} (f - m_f(Q_j)). \end{aligned} \tag{11.2}$$

Turning to the actual proof, let

$$E^0 := Q^0 \cap \Big\{ \| f - m_f(Q^0) \| > 2 \operatorname{osc}_\lambda (f; Q^0) \Big\}$$

so that $|E^0| \leqslant \lambda |Q^0|$ by Lemma 11.1.5.

For $\alpha \in (0,1)$ to be chosen, let Q_j^1 be the maximal cubes in $\mathscr{D}(Q^0)$ such that $|Q_j^1 \cap E^0| > \alpha |Q_j^1|$. Since any two dyadic cubes are either disjoint, or one is contained in the other, dyadic cubes that are maximal with respect to some property are necessarily disjoint; hence our preliminary observation applies to $Q_j = Q_j^1$. Moreover, by definition of the dyadic maximal operator, we have $M_{\mathscr{D}}\mathbf{1}_{E^0}(x) > \alpha$, if and only if x is contained in some dyadic Q with $\langle \mathbf{1}_{E^0}\rangle_Q > \alpha$, if and only if it is contained in a maximal dyadic cube with this property. Hence

$$\bigcup_j Q_j^1 = \{M_{\mathscr{D}}\mathbf{1}_{E^0} > \alpha\},$$

so that by Doob's inequality

$$\sum_j |Q_j^1| \leqslant \frac{1}{\alpha}\|\mathbf{1}_{E^0}\|_1 \leqslant \frac{\lambda}{\alpha}|Q^0|.$$

By Lebesgue's differentiation theorem, almost every point of E is contained in some Q_j^1, and hence

$$\mathbf{1}_{Q^0\setminus\bigcup_j Q_j^1}\|f - m_f(Q^0)\| \leqslant \mathbf{1}_{Q^0\setminus\bigcup_j Q_j^1} 2\operatorname{osc}_\lambda(f;Q^0)$$

almost everywhere.

By the maximality of the Q_j^1, their parent cubes $\widehat{Q}_j^1$ satisfy the opposite bound $|\widehat{Q}_j^1 \cap E| \leqslant \alpha|\widehat{Q}_j^1|$. Hence in particular

$$|Q_j^1 \cap E^0| \leqslant |\widehat{Q}_j^1 \cap E^0| \leqslant \alpha|\widehat{Q}_j^1| = 2^d\alpha|Q_j^1|.$$

Let also

$$E_j^1 = Q_j^1 \cap \Big\{\|f - m_f(Q_j^1)\| > 2\operatorname{osc}_\lambda(f;Q_j^1)\Big\}$$

so that $|E_j^1| \leqslant \lambda|Q_j^1|$ by Lemma 11.1.5. It follows that

$$|Q_j^1 \cap (E^0 \cup E_j^1)| \leqslant (2^d\alpha + \lambda)|Q_j^1|.$$

If $2^d\alpha + \lambda < 1$, then $Q_j^1 \setminus (E^0 \cup E_j^1)$ has positive measure, and for any x in this set, we have both

$$\|f(x) - m_f(Q^0)\| \leqslant 2\operatorname{osc}_\lambda(f;Q^0), \quad \|f(x) - m_f(Q_j^1)\| \leqslant 2\operatorname{osc}_\lambda(f;Q_j^1).$$

Since such points x exist, it follows in particular that

$$\|m_f(Q_j^1) - m_f(Q^0)\| \leqslant 2\operatorname{osc}_\lambda(f;Q^0) + 2\operatorname{osc}_\lambda(f;Q_j^1).$$

Substituting this to (11.2), we have

$$\begin{aligned}\mathbf{1}_{Q^0}\|f-m_f(Q^0)\| &\leqslant \mathbf{1}_{Q^0\setminus\bigcup_j Q_j^1}2\,\mathrm{osc}_\lambda(f;Q^0)\\ &\qquad+\sum_j \mathbf{1}_{Q_j^1}\Big(2\,\mathrm{osc}_\lambda(f;Q^0)+2\,\mathrm{osc}_\lambda(f;Q^j)\Big)\\ &\qquad\qquad+\sum_j \mathbf{1}_{Q_j^1}\|f-m_f(Q_j^1)\| \\ &=\mathbf{1}_{Q^0}2\,\mathrm{osc}_\lambda(f;Q^0)+\sum_j \mathbf{1}_{Q_j^1}2\,\mathrm{osc}_\lambda(f;Q_j^1)\\ &\qquad+\sum_j \mathbf{1}_{Q_j^1}\|f-m_f(Q_j^1)\|,\end{aligned} \tag{11.3}$$

where each term in the last sum has exactly the same form as the left hand side and allows to iterate the same consideration.

Assuming that we have proved

$$\begin{aligned}\mathbf{1}_{Q^0}\|f-m_f(Q^0)\| \leqslant 4\sum_{n=0}^{N-1}\sum_j \mathbf{1}_{Q_j^n}\,\mathrm{osc}_\lambda(f;Q_j^n)+2\sum_j \mathbf{1}_{Q_j^N}\,\mathrm{osc}_\lambda(f;Q_j^N)\\ +\sum_j \mathbf{1}_{Q_j^N}\|f-m_f(Q_j^N)\|,\end{aligned}$$

where each Q_j^n is contained in some Q_i^{n-1} and

$$\sum_{j:Q_j^n\subseteq Q_i^{n-1}} |Q_j^n| \leqslant \frac{\lambda}{\alpha}|Q_i^{n-1}|, \tag{11.4}$$

applying (11.3) to each Q_j^N in place of Q^0 yields the analogue of the previous display with $N+1$ in place of N.

The support of the final error term has measure at most $\sum_j |Q_j^N| \leqslant (\lambda/\alpha)^N|Q^0|$, so if $\lambda/\alpha<1$, this error term tends to zero pointwise almost everywhere. Hence, in the limit, we have

$$\mathbf{1}_{Q^0}\|f-m_f(Q^0)\| \leqslant 4\sum_{n=0}^{\infty}\sum_j \mathbf{1}_{Q_j^n}\,\mathrm{osc}_\lambda(f;Q_j^n).$$

Choosing $\alpha=2\lambda$, (11.4) shows that the collection $\{Q_j^n\}_{n,j}$ is $\frac{1}{2}$-sparse, and with $\lambda=2^{-2-d}$, we also have $2^d\alpha+\lambda=(2^{d+1}+1)\lambda=2^{-1}+2^{-1-d}<1$, as required. This concludes the proof. □

11.1.b Almost orthogonality in L^p

In a Hilbert space H such as $H=L^2(\mathbb{R}^d)$, orthogonality of elements h_i implies the fundamental Pythagorean identity

$$\Big\| \sum_i h_i \Big\|_H = \Big(\sum_i \|h_i\|_H^2 \Big)^{1/2}.$$

As we have seen in the previous Volumes, L^p analogues of this identity tend to either take the form of a one-sided estimate only, or, insisting in a two-sided equivalence, require the introduction of some randomised norm. In contrast to this, it may come as a surprise that sparse collections lead to relatively simple constructions that allow almost complete L^p analogues of the Pythagorean identity in certain situations.

We introduce some additional notation. The following definition is meaningful for any subcollection $\mathscr{S} \subseteq \mathscr{D}$ of the dyadic cubes, but it will prove itself particularly useful when $\mathscr{S}$ is sparse.

Definition 11.1.13. *For any subcollection $\mathscr{S} \subseteq \mathscr{D}$ of dyadic cubes, we have the following notions:*

(1) *For each $S \in \mathscr{S}$, let $\mathrm{ch}_{\mathscr{S}}(S) \subseteq \mathscr{S}$ (the $\mathscr{S}$-children of S) denote the collection of all maximal $S' \in \mathscr{S}$ such that $S' \subsetneq S$.*
(2) *For each $S \in \mathscr{S}$, let $E_{\mathscr{S}}(S) := S \setminus \bigcup_{S' \in \mathrm{ch}_{\mathscr{S}}(S)} S'$.*
(3) *For each $Q \in \mathscr{D}$, let $\pi_{\mathscr{S}}(Q)$ denote the minimal $S \in \mathscr{S}$ such that $S \supseteq Q$.*

When $\mathscr{S} = \mathscr{D}$, we reproduce the familiar notion $\mathrm{ch}_{\mathscr{D}} = \mathrm{ch}$ of dyadic children. The other two notions above are uninteresting in this special case, as we simply have $E_{\mathscr{D}}(Q) = \varnothing$ and $\pi_{\mathscr{D}}(Q) = Q$ for all $Q \in \mathscr{D}$.

We begin with a one-sided estimate:

Proposition 11.1.14. *Let X be a Banach space and $p \in [1, \infty)$. Let $\mathscr{S} \subseteq \mathscr{D}$ be a γ-sparse collection of dyadic cubes. For each $S \in \mathscr{S}$, let $f_S \in L^p(\mathbb{R}^d; X)$ be a function supported on S and constant on each $S' \in \mathrm{ch}_{\mathscr{S}}(S)$. Then*

$$\Big\| \sum_{S \in \mathscr{S}} f_S \Big\|_{L^p(\mathbb{R}^d;X)} \leqslant (1 + \gamma^{-1/p'} p) \Big(\sum_{S \in \mathscr{S}} \|f_S\|^p_{L^p(\mathbb{R}^d;X)} \Big)^{1/p}.$$

Proof. We assume that the right-hand side is finite, for otherwise there is nothing to prove. We then assume without loss of generality that $\mathscr{S}$ is finite. In fact, once we have proved the result for finite families, in the infinite case it follows easily that the partial sums of the series $\sum_{S \in \mathscr{S}} f_S$ (with arbitrary enumeration) form a Cauchy sequence in $L^p(\mathbb{R}^d; X)$, from which we deduce the (unconditional) convergence of this series and the asserted norm bound.

Concentrating on the finite case, by dualising with $g \in L^{p'}(\mathbb{R}^d; X^*)$, it is equivalent to the estimate

$$\int \sum_S \langle f_S, g \rangle \, \mathrm{d}x \leqslant (1 + \gamma^{-1/p'} p) \Big(\sum_S \|f_S\|^p_{L^p(\mathbb{R}^d;X)} \Big)^{1/p} \|g\|_{L^{p'}(\mathbb{R}^d;X^*)}.$$

Since f_S is supported on S and constant on each $S' \in \mathrm{ch}_{\mathscr{S}}(S)$, and since S is partitioned by $\mathrm{ch}_{\mathscr{S}}(S) \cup \{E_{\mathscr{S}}(S)\}$, we have

$$\begin{aligned}\int \sum_{S\in\mathscr{S}} \langle f_S, g\rangle \,\mathrm{d}x &= \sum_{S\in\mathscr{S}} \int_S \langle f_S, g\rangle \,\mathrm{d}x \\ &= \sum_{S\in\mathscr{S}} \sum_{S'\in\mathrm{ch}_{\mathscr{S}}(S)} \langle\langle f_S\rangle_{S'}, \langle g\rangle_{S'}\rangle |S'| + \int \sum_{S\in\mathscr{S}} \mathbf{1}_{E_{\mathscr{S}}(S)} \langle f_S, g\rangle \,\mathrm{d}x.\end{aligned}$$

We can estimate the second term by Hölder's inequality and the pairwise disjointness of the sets $E_{\mathscr{S}}(S)$,

$$\begin{aligned}\Big|\int \sum_{S\in\mathscr{S}} \mathbf{1}_{E_{\mathscr{S}}(S)} \langle f_S, g\rangle \,\mathrm{d}x\Big| &\leqslant \Big\| \sum_{S\in\mathscr{S}} \mathbf{1}_{E_{\mathscr{S}}(S)} f_S \Big\|_{L^p(\mathbb{R}^d;X)} \|g\|_{L^{p'}(\mathbb{R}^d;X^*)} \\ &= \Big(\sum_{S\in\mathscr{S}} \|\mathbf{1}_{E_{\mathscr{S}}(S)} f_S\|^p_{L^p(\mathbb{R}^d;X)} \Big)^{1/p} \|g\|_{L^{p'}(\mathbb{R}^d;X^*)}.\end{aligned}$$

For the first term we argue as follows.

$$\begin{aligned}&\Big| \sum_{S\in\mathscr{S}} \sum_{S'\in\mathrm{ch}_{\mathscr{S}}(S)} \langle\langle f_S\rangle_{S'}, \langle g\rangle_{S'}\rangle |S'| \Big| \\ &\leqslant \Big(\sum_{S\in\mathscr{S}} \sum_{S'\in\mathrm{ch}_{\mathscr{S}}(S)} \|\langle f_S\rangle_{S'}\|^p_X |S'| \Big)^{1/p} \Big(\sum_{S\in\mathscr{S}} \sum_{S'\in\mathrm{ch}_{\mathscr{S}}(S)} \|\langle g\rangle_{S'}\|^{p'}_{X^*} |S'| \Big)^{1/p'} \\ &\leqslant \Big(\sum_{S\in\mathscr{S}} \|f_S\|^p_{L^p(\mathbb{R}^d;X)} \Big)^{1/p} \Big(\sum_{S'\in\mathscr{S}} \langle \|g\|_{X^*}\rangle^{p'}_{S'} |S'| \Big)^{1/p'},\end{aligned}$$

where in the second factor we rearranged the double sum into a single sum, observing that every $S' \in \mathscr{S}$ is counted at most once as a child of a unique $S \in \mathscr{S}$. The second factor is bounded by $\gamma^{-1/p'} p \|g\|_{p'}$ thanks to Proposition 11.1.11(2). Summing up the bounds, we complete the proof of the direct estimate. □

The following lemma describes useful projections and also provides prominent examples of the functions f_S featuring in Proposition 11.1.14.

Lemma 11.1.15. *For $S \in \mathscr{S} \subseteq \mathscr{D}$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d; X)$, let*

$$P_S f := \sum_{S'\in\mathrm{ch}_{\mathscr{S}}(S)} \mathbb{E}_{S'} f + \mathbf{1}_{E_{\mathscr{S}}(S)} f. \tag{11.5}$$

Then $\langle f\rangle_Q = \langle P_S f\rangle_Q$ for all $Q \in \mathscr{D}$ such that $\pi_{\mathscr{S}}(Q) = S$.

Proof. From definition, we have

$$\langle P_S f\rangle_Q = \sum_{S'\in\mathrm{ch}_{\mathscr{S}}} \langle f\rangle_{S'} \frac{|S'\cap Q|}{|Q|} + \frac{1}{|Q|} \int_{Q\cap E_{\mathscr{S}}(S)} f \,\mathrm{d}x.$$

Since $\pi_{\mathscr{S}}(Q) = S$, we have $Q \subseteq S$ and it is not possible that $Q \subseteq S' \in \mathrm{ch}_{\mathscr{S}}(S)$. Hence $S' \cap Q \in \{\varnothing, S'\}$ for all $S' \in \mathrm{ch}_{\mathscr{S}}(S)$ and Q is exactly partitioned by $Q \cap E_{\mathscr{S}}(S)$ and those $S' \in \mathrm{ch}_{\mathscr{S}}(S)$ with $S' \subsetneq Q$. Thus

$$\sum_{S'\in\mathrm{ch}_{\mathscr{S}}} \langle f\rangle_{S'} \frac{|S'\cap Q|}{|Q|} = \frac{1}{|Q|} \sum_{S'\in\mathrm{ch}_{\mathscr{S}},S'\subseteq Q} \langle f\rangle_{S'}|S'| = \frac{1}{|Q|} \sum_{S'\in\mathrm{ch}_{\mathscr{S}},S'\subseteq Q} \int_{S'} f\,\mathrm{d}x,$$

and

$$\langle P_S f\rangle_Q = \frac{1}{|Q|} \sum_{S'\in\mathrm{ch}_{\mathscr{S}},S'\subseteq Q} \int_{S'} f\,\mathrm{d}x + \frac{1}{|Q|}\int_{Q\cap E_{\mathscr{S}}(S)} f\,\mathrm{d}x = \frac{1}{|Q|}\int_Q f\,\mathrm{d}x,$$

confirming the lemma. □

A typical way in which a sparse collection arises is via the following basic construction:

Definition 11.1.16 (Principal cubes). *Let $Q_0 \in \mathscr{D}$ and $f \in L^1(Q_0;X)$. The collection of* principal cubes *of f in $\mathscr{D}(Q_0)$ with parameter $A>1$ is the family $\mathscr{S} = \bigcup_{k=0}^{\infty} \mathscr{S}_k$ constructed as follows:*

(1) $\mathscr{S}_0 := \{Q_0\}$.
(2) *If $\mathscr{S}_k$ is already defined for some $k\in\mathbb{N}$, then*
 (a) *for each $S\in\mathscr{S}_k$ we let*

$$\mathrm{ch}_{\mathscr{S}}(S) := \Big\{S' \in \mathscr{D}(S) \text{ maximal with } \langle \|f\|_X\rangle_{S'} > A\langle \|f\|_X\rangle_S\Big\},$$

 (b) *and then*

$$\mathscr{S}_{k+1} := \bigcup_{S\in\mathscr{S}_k} \mathrm{ch}_{\mathscr{S}}(S).$$

The first instance of the interplay of a function and its principal cubes is the following:

Lemma 11.1.17. *Let $f\in L^1(Q_0;X)$ and $\mathscr{S}$ be the principal cubes of f with parameter $A>1$. Then $\mathscr{S}$ is $(1-A^{-1})$-sparse, and in fact*

$$|E_{\mathscr{S}}(S)| \geqslant (1-\frac{1}{A})|S|. \tag{11.6}$$

If $P_S f$ is defined by (11.5), then $\|P_S f\|_{L^\infty(\mathbb{R}^d;X)} \leqslant 2^d A\langle \|f\|_X\rangle_S$.

Note that (11.6) is slightly more than the mere $(1-A^{-1})$-sparseness of $\mathscr{S}$: it says that the disjoint subsets $E(S)\subseteq S$ in the definition of sparseness may be chosen as $E(S) = E_{\mathscr{S}}(S)$, which is not always the case for an arbitrary sparse family. For instance, $\mathscr{S} = \{[0,1),[0,\frac12),[\frac12,1)\}$ is $\frac12$-sparse, and one can take for instance $E([0,1)) = [\frac14,\frac34)$, $E([0,\frac12)) = [0,\frac14)$ and $E([\frac12,1)) = [\frac34,1)$, but $E_{\mathscr{S}}([0,1)) = \varnothing$ in this case.

Proof. By maximality, the cubes $S'\in\mathrm{ch}_{\mathscr{S}}(S)$ are pairwise disjoint. From the defining condition it follows that

$$\sum_{S'\in \mathrm{ch}_{\mathscr{S}}(S)} |S'| \leqslant \sum_{S'\in \mathrm{ch}_{\mathscr{S}}(S)} \frac{\int_{S'} \|f\|_X \,\mathrm{d}x}{A\langle \|f\|_X\rangle_S} \leqslant \frac{\int_{S} \|f\|_X \,\mathrm{d}x}{A\langle \|f\|_X\rangle_S} = \frac{|S|}{A}$$

and hence

$$|E_{\mathscr{S}}(S)| = |S| - \sum_{S'\in \mathrm{ch}_{\mathscr{S}}(S)} |S'| \geqslant (1-\frac{1}{A})|S|.$$

If $x \in E_{\mathscr{S}}(S)$, then x is not contained in any $S' \in \mathrm{ch}_{\mathscr{S}}(S)$, and hence $\langle \|f\|_X\rangle_Q \leqslant A\langle \|f\|_X\rangle_S$ for all $Q \in \mathscr{D}(S)$ with $x \in Q$. As $\ell(Q) \to 0$, it follows from Lebesgue's Differentiation Theorem that $\|P_S f(x)\|_X = \|f(x)\|_X \leqslant A\langle \|f\|_X\rangle_S$ for almost every $x \in E_{\mathscr{S}}(S)$. If $x \in S' \in \mathrm{ch}_{\mathscr{S}}(S)$, then $f_S(x) = \langle f\rangle_{S'}$. By the maximality of S', its dyadic parent $\widehat{S}'$ satisfies the opposite inequality $\langle \|f\|_X\rangle_{\widehat{S}'} \leqslant A\langle \|f\|_X\rangle_S$, and hence

$$\begin{aligned}\|P_S f(x)\|_X \leqslant \langle \|f\|_X\rangle_{S'} &= \frac{1}{|S'|}\int_{S'} \|f\|_X \,\mathrm{d}x \\ &\leqslant \frac{2^d}{|\widehat{S}'|}\int_{\widehat{S}'} \|f\|_X \,\mathrm{d}x \leqslant 2^d A\langle \|f\|_X\rangle_S.\end{aligned}$$

These two cases confirm the upper bound $\|P_S f\|_{L^\infty(\mathbb{R}^d;X)} \leqslant 2^d A\langle \|f\|_X\rangle_S$. $\square$

11.1.c Maximal oscillatory norms for L^p spaces

Based on the oscillations studied above, we introduce the related *John–Strömberg maximal operator*

$$M^{\#}_{0,\lambda} f(x) := \sup_{Q\ni x} \mathrm{osc}_\lambda(f;Q),$$

where the supremum is taken over all cubes containing $x \in \mathbb{R}^d$; a dyadic version $M^{\#,\mathscr{D}}_{0,\lambda}$ is obtained by restricting the supremum to dyadic cubes $Q \in \mathscr{D}$ only. Via this maximal operator we can obtain a useful oscillatory characterisation of $L^p(\mathbb{R}^d;X)$, which we will prove in the rest of this section:

Theorem 11.1.18. *Let X be a Banach space, $p \in (0,\infty)$, $\lambda = 2^{-2-d}$, and $f \in L^0(\mathbb{R}^d;X)$. Then there is a constant $c \in X$ such that $f - c \in L^p(\mathbb{R}^d;X)$ if and only if $M^{\#}_{0,\lambda} f \in L^p(\mathbb{R}^d)$, and in this case*

$$c_d^{-1/p}\|M^{\#}_{0,\lambda} f\|_{L^p(\mathbb{R}^d)} \leqslant \|f-c\|_{L^p(\mathbb{R}^d;X)} \leqslant c_p\|M^{\#}_{0,\lambda} f\|_{L^p(\mathbb{R}^d)},$$

where $c_p = 8p$ for $p \in [1,\infty)$ and $c_p = 2^{2+1/p}$ for $p \in (0,1)$.

The result is also valid with $\mathbb{R}^d$ replaced by a cube $Q_0 \subseteq \mathbb{R}^d$ or a quadrant $S \subseteq \mathbb{R}^d$, and with the supremum in the maximal operator $M^{\#}_{0,\lambda}$ restricted to cubes $Q \subseteq Q_0$ or $Q \subseteq S$, respectively.

Remark 11.1.19. If we now *a priori* require that $f \in L^{p_0,\infty}(\mathbb{R}^d;X)$ for some $p_0 \in (0,\infty)$ (unrelated to the exponent p), then the constant $c \in X$ guaranteed by Theorem 11.1.18 is necessarily 0, and thus in fact $f \in L^p(\mathbb{R}^d;X)$.

Namely, if $f \in L^{p_0,\infty}(\mathbb{R}^d;X)$ and $f - c \in L^p(\mathbb{R}^d;X)$, it follows that $c = f-(f-c) \in L^{p_0,\infty}(\mathbb{R}^d;X) + L^p(\mathbb{R}^d;X)$, thus

$$|\{\|c\| > t\}| \leqslant |\{\|f\| > t/2\}| + |\{\|f-c\| > t/2\}| < \infty$$

for all $t > 0$, which would lead to a contradiction for $t \in (0, \|c\|)$.

By Lemma 11.1.1 for any $q \in (0,\infty)$, we have

$$\operatorname{osc}_\lambda(f;Q) \leqslant \inf_{c\in X} \frac{\|(f-c)\mathbf{1}_Q\|_{L^{q,\infty}}}{(\lambda|Q|)^{1/q}} \leqslant \frac{\|f\mathbf{1}_Q\|_{L^q}}{(\lambda|Q|)^{1/q}} = \lambda^{-1/q}\Big(\fint_Q \|f\|^q\Big)^{1/q}.$$

Taking the supremum over all cubes Q containing a given point, it follows that

$$M^{\#}_{0,\lambda} f \leqslant \lambda^{-1/q} M_q f, \qquad M_q f := \big(M(\|f\|^q)\big)^{1/q}, \tag{11.7}$$

where M is the Hardy–Littlewood maximal operator. The L^p boundedness of M_q is an easy combination of some estimates collected from Chapter 3:

Lemma 11.1.20. *For all* $0 < q < p < \infty$*, we have*

$$\max\Big(\|M_q\|_{L^p\to L^p}, \|M_q\|_{L^{p,\infty}\to L^{p,\infty}}\Big) \leqslant 3^{d/q+d/p}\Big(\frac{p}{p-q}\Big)^{1/q}.$$

Proof. The dyadic (in fact more general martingale) bounds for $M_q^{\mathscr{D}}$ on L^p and $L^{p,\infty}$ for $p \in (q,\infty)$, with norm bound $(p/(p-q))^{1/q}$ in each case, have been treated in Lemma 3.5.17. On the other hand, we recall from (3.36) that

$$Mf \leqslant 3^d \sup_{\alpha\in\{0,\frac13,\frac23\}^d} M^\alpha f,$$

thus

$$M_q f \leqslant 3^{d/q} \sup_{\alpha\in\{0,\frac13,\frac23\}^d} M_q^\alpha f \leqslant 3^{d/q}\Big(\sum_{\alpha\in\{0,\frac13,\frac23\}^d} [M_q^\alpha f]^p\Big)^{1/p}.$$

Hence

$$\|M_q f\|_p \leqslant 3^{d/q}\Big(\sum_{\alpha\in\{0,\frac13,\frac23\}^d} \|M_q^\alpha f\|_p^p\Big)^{1/p} \leqslant 3^{d/q+d/p}\Big(\frac{p}{p-q}\Big)^{1/q}\|f\|_p,$$

and, for every $\lambda > 0$,

$$\lambda|\{M_q f > \lambda\}|^{1/p} \leqslant \lambda\Big(\sum_\alpha |\{M_q^\alpha f > 3^{-d/q}\lambda\}|\Big)^{1/p} \leqslant 3^{d/q}\Big(\sum_\alpha \|M_q^\alpha f\|_{L^{p,\infty}}^p\Big)^{1/p},$$

after which the last step is exactly as in the strong-type case, now using the weak-type boundedness of the dyadic M_q^α instead. □

Proposition 11.1.21. *The operator $M^{\#}_{0,\lambda}$ is bounded from $L^p(\mathbb{R}^d;X)$ to $L^p(\mathbb{R}^d)$ and from $L^{p,\infty}(\mathbb{R}^d;X)$ to $L^{p,\infty}(\mathbb{R}^d)$, with norm at most $c_{d,\lambda}^{1/p}$, where $c_{d,\lambda}$ is a constant depending only on d and λ.*

The first half of Theorem 11.1.18 is immediate from this proposition (with the choice $\lambda = 2^{-2-d}$ so that $c_{d,\lambda} = c_d$), combined with the trivial observation that $M^{\#}_{0,\lambda}f = M^{\#}_{0,\lambda}(f-c)$ for any constant $c \in X$.

Proof. Let $Y \in \{L^p, L^{p,\infty}\}$. By (11.7) and Lemma 11.1.20, we have

$$\|M^{\#}_{0,\lambda}f\|_Y \leqslant \lambda^{-1/q}\|M_q f\|_Y \leqslant \lambda^{-1/q}3^{d/q+d/p}\Big(\frac{p}{p-q}\Big)^{1/q}\|f\|_Y.$$

With, say, $q = \frac{1}{2}p$, the right hand side takes the form $(\lambda^{-2}3^{3d}2^2)^{1/p}\|f\|_Y$. □

Towards the deduction of a global L^p estimate from local ones, we record:

Lemma 11.1.22. *Let X be a Banach space and $p \in (0,\infty)$. Suppose that $f \in L^p_{\rm loc}(\mathbb{R}^d;X)$ satisfies*

$$\|\mathbf{1}_Q(f - c_Q)\|_p \leqslant K$$

for some constants $c_Q \in X$ and all cubes $Q \subseteq \mathbb{R}^d$. Then there is a constant $c \in X$ such that $f - c \in L^p(\mathbb{R}^d;X)$ and

$$\|f - c\|_p \leqslant K.$$

Proof. Consider an increasing sequence of cubes $Q_1 \subseteq Q_2 \subseteq \dots$ such that $\bigcup_{n=1}^{\infty} Q_n = \mathbb{R}^d$. If $m \leqslant n$, then

$$\begin{aligned}\|c_{Q_m} - c_{Q_n}\| &= |Q_m|^{-1/p}\|\mathbf{1}_{Q_m}(c_{Q_m} - c_{Q_n})\|_p \\ &\leqslant |Q_m|^{-1/p}\Big(\|\mathbf{1}_{Q_m}(f - c_{Q_m})\|_p + \|\mathbf{1}_{Q_n}(f - c_{Q_n})\|_p\Big) \\ &\leqslant |Q_m|^{-1/p}2K \to 0 \qquad \text{as} \quad m \to \infty.\end{aligned}$$

Hence $(c_{Q_n})_{n\geqslant 1}$ is a Cauchy sequence and thus convergent to some $c \in X$. Now Fatou's lemma shows that

$$\int_{\mathbb{R}^d}\|f - c\|^p = \int_{\mathbb{R}^d}\lim_{n\to\infty}\mathbf{1}_{Q_n}\|f - c_{Q_n}\|^p \leqslant \liminf_{n\to\infty}\int_{Q_n}\|f - c_{Q_n}\|^p \leqslant K,$$

which completes the proof. □

We can now prove the remaining half of Theorem 11.1.18, which we restate as:

Proposition 11.1.23. *Let $f \in L^0(\mathbb{R}^d; X)$, $\lambda = 2^{-2-d}$, and suppose that $M^{\#}_{0,\lambda} f \in L^p(\mathbb{R}^d)$ for some $p \in (0, \infty)$. Then there is a constant $c \in X$ such that $f - c \in L^p(\mathbb{R}^d; X)$ and*

$$\|f - c\|_{L^p(\mathbb{R}^d;X)} \leqslant c_p \|M^{\#}_{0,\lambda} f\|_{L^p(\mathbb{R}^d)}, \qquad c_p = \begin{cases} 8p, & p \in [1, \infty), \\ 2^{2+1/p}, & p \in (0, 1). \end{cases}$$

The result also holds with $\mathbb{R}^d$ replaced by a cube $Q_0 \subseteq \mathbb{R}^d$ or a quadrant $S \subseteq \mathbb{R}^d$, and with the supremum in the maximal operator $M^{\#}_{0,\lambda}$ restricted to cubes contained in Q_0 or S, respectively.

Proof. Consider a fixed cube $Q^0 \subseteq \mathbb{R}^d$. By Lerner's formula (Theorem 11.1.12), there is a $\frac{1}{2}$-sparse subcollection $\mathscr{S} \subseteq \mathscr{D}(Q^0)$ such that

$$\mathbf{1}_{Q^0} \|f - m_f(Q^0)\| \leqslant 4 \sum_{S \in \mathscr{S}} \mathbf{1}_S \operatorname{osc}_\lambda(f; S),$$

whenever $m_f(Q^0)$ is a λ-pseudomedian of f on Q^0. Taking L^p norms and using Proposition 11.1.11 (with $\gamma = \frac{1}{2}$), we get

$$\begin{aligned} \|\mathbf{1}_{Q^0}(f - m_f(Q^0))\|_p &\leqslant 4 \Big\| \sum_{S \in \mathscr{S}} \mathbf{1}_S \operatorname{osc}_\lambda(f; S) \Big\|_p \\ &\leqslant 4 c_{p,\frac{1}{2}} \Big\| \sum_{S \in \mathscr{S}} \mathbf{1}_{E(S)} \operatorname{osc}_\lambda(f; S) \Big\|_p \leqslant 4 c_{p,\frac{1}{2}} \|M^{\#}_{0,\lambda} f\|_p. \end{aligned}$$

This estimate is uniform with respect to the choice of $Q^0 \subseteq \mathbb{R}^d$; hence we can apply Lemma 11.1.22 with $c_Q = m_f(Q)$ to complete the proof.

The variant in the case of a cube or a quadrant in place of $\mathbb{R}^d$ is immediate by inspection of the argument. □

We conclude this section with an end-point analogue of Theorem 11.1.18 for the space $\mathrm{BMO}(\mathbb{R}^d; X)$ in place of $L^p(\mathbb{R}^d; X)$. Recall that we have previously defined the space $\mathrm{BMO}(\mathbb{R}^d; X)$ of functions of *bounded mean oscillation* as the class of functions $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d; X)$ such that

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^d;X)} := \sup_Q \inf_{c \in X} \fint_Q \|f - c\| < \infty.$$

Proposition 11.1.24. *Let X be a Banach space, $\lambda = 2^{-2-d}$, and $f \in L^0(\mathbb{R}^d; X)$. Then $f \in \mathrm{BMO}(\mathbb{R}^d; X)$ if and only if $M^{\#}_{0,\lambda} f \in L^\infty(\mathbb{R}^d)$, and*

$$\lambda \|M^{\#}_{0,\lambda} f\|_{L^\infty(\mathbb{R}^d)} \leqslant \|f\|_{\mathrm{BMO}(\mathbb{R}^d;X)} \leqslant 8 \|M^{\#}_{0,\lambda} f\|_\infty.$$

Proof. From Lemma 11.1.1 it is immediate that

$$\operatorname{osc}_\lambda(f; Q) \leqslant \frac{1}{\lambda} \inf_{c \in X} \fint_Q \|f - c\|,$$

from which the first claimed inequality follows by taking the supremum over all cubes $Q \subseteq \mathbb{R}^d$.

In the other direction, given a cube $Q \subseteq \mathbb{R}^d$, Lerner's formula (Theorem 11.1.12) guarantees that

$$\fint_Q \|f - m_f(Q)\| \leqslant \frac{4}{|Q|} \sum_{S \in \mathscr{S}} |S| \operatorname{osc}_\lambda(f; S) \leqslant \frac{4}{|Q|} \sum_{S \in \mathscr{S}} 2|E(S)| \|M^{\#}_{0,\lambda} f\|_\infty \leqslant 8\|M^{\#}_{0,\lambda} f\|_\infty,$$

and taking the supremum over all cubes Q proves the second bound. □

11.1.d The dyadic Hardy space and BMO

Often an efficient way of capturing the relevant local oscillations of a function is in terms of the following notion:

Definition 11.1.25 (Atom). *A function $a : \mathbb{R}^d \to X$ is called a (normalised) $H^1_{\mathscr{D}}$-atom if*

(i) $\operatorname{supp} a \subseteq Q$ *for some* $Q \in \mathscr{D}$*;*
(ii) $a \in L^\infty(\mathbb{R}^d; X)$ *(and* $\|a\|_\infty \leqslant 1/|Q|$*);*
(iii) $\int_Q a = 0$.

It is immediate that a normalised atom satisfies $\|a\|_1 \leqslant 1$. If $a \neq 0$ is an atom supported on $Q \in \mathscr{D}$, then $\dfrac{a}{|Q|\|a\|_\infty}$ is a normalised atom. Out of these atoms we can then construct a useful subspace of $L^1(\mathbb{R}^d; X)$:

Definition 11.1.26 (Atomic Hardy space). *The atomic Hardy space*

$$H^1_{\mathscr{D},\mathrm{at}}(\mathbb{R}^d; X)$$

consists of all $f \in L^1(\mathbb{R}^d; X)$ that admit a representation

$$f = \sum_{k=1}^\infty \alpha_k \Big(= \sum_{k=1}^\infty \lambda_k a_k \Big),$$

absolutely convergent in $L^1(\mathbb{R}^d; X)$, where each α_k is an $H^1_{\mathscr{D}}$-atom supported in some $Q_k \in \mathscr{D}$ (or each a_k is a normalised $H^1_{\mathscr{D}}$-atom and $\lambda_k \in \mathbb{K}$) with

$$\sum_{k=1}^\infty \|\alpha_k\|_\infty |Q_k| < \infty \quad \Big(\sum_{k=1}^\infty |\lambda_k| < \infty \Big).$$

The norm in this space is defined as

$$\|f\|_{H^1_{\mathscr{D},\mathrm{at}}} := \inf \sum_{k=1}^\infty \|\alpha_k\|_\infty |Q_k| \Big(= \inf \sum_{k=1}^\infty |\lambda_k| \Big)$$

where the infimum is taken over all such representations.

It is immediate that the two versions of the definition are equivalent via the correspondence $\lambda_k = \|\alpha_k\|_\infty |Q_k|$ and $a_k = \lambda_k^{-1}\alpha_k$.

A disadvantage of this definition is the difficulty of checking the membership of a given function in $H^1_{\mathscr{D},\mathrm{at}}(\mathbb{R}^d;X)$, as doing this via the definition would require one to construct the atomic decomposition, which might not be an easy task. The following notion is much more amenable to this:

Definition 11.1.27 (Maximal Hardy space). *The maximal Hardy space*

$$H^1_{\mathscr{D},\max}(\mathbb{R}^d;X)$$

consists of all $f \in L^1(\mathbb{R}^d;X)$ for which also the (cancellative) dyadic maximal function

$$M_{\mathscr{D}}f(x) := \sup_{Q\in\mathscr{D}} \mathbf{1}_Q(x)\|\langle f\rangle_Q\|_X$$

satisfies $M_{\mathscr{D}}f \in L^1(\mathbb{R}^d)$. The norm in this space is defined as

$$\|f\|_{H^1_{\mathscr{D},\max}} := \|M_{\mathscr{D}}f\|_{L^1(\mathbb{R}^d)}.$$

Theorem 11.1.28. *Let X be a Banach space. The spaces $H^1_{\mathscr{D},\mathrm{at}}(\mathbb{R}^d;X)$ and $H^1_{\mathscr{D},\max}(\mathbb{R}^d;X)$ are equal with equivalent norms; in fact*

$$\|h\|_{H^1_{\mathscr{D},\max}(\mathbb{R}^d;X)} \leqslant \|h\|_{H^1_{\mathscr{D},\mathrm{at}}(\mathbb{R}^d;X)} \leqslant 6\cdot 2^d \cdot \|h\|_{H^1_{\mathscr{D},\max}(\mathbb{R}^d;X)}.$$

Proof. Suppose first that $a \in L^\infty(\mathbb{R}^d;X)$ satisfies $\operatorname{supp} a \subseteq Q$ for some dyadic cube and $\int a = 0$. Then $\langle a\rangle_R \neq 0$ only if $R \subsetneq Q$, and hence $\operatorname{supp} M_{\mathscr{D}}a \subseteq Q$ as well. It follows that

$$\|M_{\mathscr{D}}a\|_1 \leqslant |Q|\|M_{\mathscr{D}}a\|_\infty \leqslant |Q|\|a\|_\infty.$$

If $h = \sum_{i=1}^\infty a_i$ is a series of such function on cubes Q_i, then by sublinearity

$$\|h\|_{H^1_{\mathscr{D},\max}(\mathbb{R}^d;E)} = \|M_{\mathscr{D}}h\|_1 \leqslant \sum_{i=1}^\infty \|M_{\mathscr{D}}a_i\|_1 \leqslant \sum_{i=1}^\infty |Q_i|\|a_i\|_\infty,$$

and taking the infimum over all such representations of h shows that

$$\|h\|_{H^1_{\mathscr{D},\max}(\mathbb{R}^d;X)} \leqslant \|h\|_{H^1_{\mathscr{D},\mathrm{at}}(\mathbb{R}^d;X)}.$$

In the other direction, suppose that $h \in H^1_{\max}(\mathbb{R}^d;X)$. Given $\lambda > 0$, let $\mathscr{Q}_\lambda$ be the collection of maximal dyadic cubes Q such that $\|\langle h\rangle_Q\|_X > \lambda$. Then

$$\sum_{Q\in\mathscr{Q}_\lambda} |Q| = |\{M_{\mathscr{D}}h > \lambda\}| \leqslant \frac{1}{\lambda}\|M_{\mathscr{D}}h\|_{L^1(\mathbb{R}^d)} = \frac{1}{\lambda}\|h\|_{H^1_{\mathscr{D},\max}(\mathbb{R}^d;X)}.$$

Let $\widehat{\mathscr{Q}}_\lambda$ be the collection of maximal dyadic cubes that have a child in $\mathscr{Q}_\lambda$. Thus these cubes do not belong to $\mathscr{Q}_\lambda$ themselves. Hence $\|\langle h\rangle_Q\|_E \leqslant \lambda$ for $Q \in \widehat{\mathscr{Q}}_\lambda$, and also

$$\sum_{Q\in\widehat{\mathscr{D}}_\lambda}|Q| \leqslant \sum_{Q\in\mathscr{D}_\lambda}|\widehat{Q}| = \sum_{Q\in\mathscr{D}_\lambda}2^d|Q| = 2^d|\{Mh>\lambda\}|.$$

Let then

$$g_\lambda := 1_{\complement(\bigcup\widehat{\mathscr{D}}_\lambda)}h + \sum_{Q\in\widehat{\mathscr{D}}_\lambda}1_Q\langle h\rangle_Q, \qquad b_\lambda := \sum_{Q\in\widehat{\mathscr{D}}_\lambda}1_Q(h-\langle h\rangle_Q).$$

By definition of $M_{\mathscr{D}}$, we have $\|\langle h\rangle_Q\|_X \leqslant M_{\mathscr{D}}h(x)$ whenever $x\in Q\in\mathscr{D}$. As $\ell(Q)\to 0$, this gives $\|f(x)\|_X\leqslant M_{\mathscr{D}}h(x)$ at a.e. x by the Lebesgue Differentiation Theorem. Thus $\|g_\lambda(x)\|_\leqslant M_{\mathscr{D}}h(x)$ almost everywhere. On the other hand, we have $\|\langle h\rangle_Q\|_X\leqslant\lambda$ for $Q\in\widehat{\mathscr{D}}_\lambda$, and $M_{\mathscr{D}}h(x)\leqslant\lambda$ for $x\in\complement(\bigcup\widehat{\mathscr{D}}_\lambda)$; thus in fact $\|g_\lambda\|_X\leqslant\min(\lambda, M_{\mathscr{D}}h)$ almost everywhere, where $M_{\mathscr{D}}h\in L^1(\mathbb{R}^d)$. Moreover, $g_\lambda = h$ on $\{M_{\mathscr{D}}h\leqslant\lambda\}\to\mathbb{R}^d$ as $\lambda\to\infty$, and hence

$$g_\lambda \to \begin{cases} h, & \lambda\to\infty,\\ 0, & \lambda\to 0,\end{cases}$$

pointwise, and by dominated convergence also in $L^1(\mathbb{R}^d;X)$. Thus

$$\begin{aligned}
h &= \sum_{k\in\mathbb{Z}}(g_{2^{k+1}}-g_{2^k}) = \sum_{k\in\mathbb{Z}}(b_{2^k}-b_{2^{k+1}})\\
&= \sum_{k\in\mathbb{Z}}\Big(\sum_{Q\in\widehat{\mathscr{D}}_{2^k}}1_Q(h-\langle h\rangle_Q) - \sum_{R\in\widehat{\mathscr{D}}_{2^{k+1}}}1_R(h-\langle h\rangle_R)\Big)\\
&= \sum_{k\in\mathbb{Z}}\sum_{Q\in\widehat{\mathscr{D}}_{2^k}}\Big(1_{Q\setminus\bigcup\widehat{\mathscr{D}}_{2^{k+1}}}(h-\langle h\rangle_Q) + \sum_{\substack{R\in\widehat{\mathscr{D}}_{2^{k+1}}\\ R\subseteq Q}}1_R(\langle h\rangle_R-\langle h\rangle_Q)\Big)\\
&=: \sum_{k\in\mathbb{Z}}\sum_{Q\in\widehat{\mathscr{D}}_{2^k}} a_{k,Q}.
\end{aligned}$$

Here $\operatorname{supp} a_Q\subseteq Q$, $\int a_Q = 0$ and $\|a_Q\|_\infty\leqslant 2^{k+1}+2^k = 3\cdot 2^k$. Hence

$$\begin{aligned}
\|h\|_{H^1_{\mathscr{D},\mathrm{at}}} &\leqslant \sum_{k\in\mathbb{Z}}\sum_{Q\in\widehat{\mathscr{D}}_{2^k}}|Q|\|a_{k,Q}\|_\infty \leqslant \sum_{k\in\mathbb{Z}}3\cdot 2^k\sum_{Q\in\widehat{\mathscr{D}}_{2^k}}|Q|\\
&\leqslant \sum_{k\in\mathbb{Z}}3\cdot 2^k\cdot 2^d|\{M_{\mathscr{D}}h>2^k\}|\\
&\leqslant \sum_{k\in\mathbb{Z}}3\cdot 2\cdot 2^d\int_{2^{k-1}}^{2^k}|\{M_{\mathscr{D}}h>t\}|\,\mathrm{d}t\\
&= 6\cdot 2^d\|M_{\mathscr{D}}h\|_{L^1(\mathbb{R}^d)} = 6\cdot 2^d\|h\|_{H^1_{\mathscr{D},\max}(\mathbb{R}^d;X)}.
\end{aligned}$$

□

Corollary 11.1.29. *The space $H^1_{\mathscr{D},\mathrm{at}}(\mathbb{R}^d;X) = H^1_{\mathscr{D},\max}(\mathbb{R}^d;X)$ is complete.*

Proof. It in enough to prove this for $H^1_{\mathscr{D},\max}(\mathbb{R}^d;X)$. Since $\|f(x)\|_X \leqslant M_{\mathscr{D}}f(x)$ at a.e. $x \in \mathbb{R}^d$, we have $\|f\|_{L^1(\mathbb{R}^d;X)} \leqslant \|f\|_{H^1_{\mathscr{D},\max}(\mathbb{R}^d;X)}$. Hence, if $(f_n)_{n=1}^\infty$ is a Cauchy sequence in $H^1_{\mathscr{D},\max}(\mathbb{R}^d;X)$, it is also a Cauchy sequence in $L^1(\mathbb{R}^d;X)$ and thus $\|f_n - f\|_1 \to 0$ for some $f \in L^1(\mathbb{R}^d;X)$. Since $\langle\ \rangle_Q$ is continuous from $L^1(\mathbb{R}^d;X)$ to X, we have for all $x \in Q \in \mathscr{D}$ we have, for each $h \in H^1_{\mathscr{D},\max}(\mathbb{R}^d;X)$,

$$\|\langle f-h\rangle_Q\|_X = \lim_{n\to\infty} \|\langle f_n - h\rangle_Q\|_X \leqslant \liminf_{n\to\infty} M_{\mathscr{D}}(f_n - h)(x);$$

hence $M_{\mathscr{D}}(f-h)(x) \leqslant \liminf_{n\to\infty} M_{\mathscr{D}}(f_n-h)(x)$, and thus by Fatou's lemma

$$\|M_{\mathscr{D}}(f-h)\|_{L^1(\mathbb{R}^d)} \leqslant \liminf_{n\to\infty} \|M_{\mathscr{D}}(f_n-h)\|_{L^1(\mathbb{R}^d)}.$$

With $h=0$, this shows that $f \in H^1_{\mathscr{D},\max}(\mathbb{R}^d;X)$. With $h = f_m$, we find that

$$\lim_{m\to\infty} \|M_{\mathscr{D}}(f-f_m)\|_{L^1(\mathbb{R}^d)} \leqslant \lim_{m\to\infty} \liminf_{n\to\infty} \|M_{\mathscr{D}}(f_n-f_m)\|_{L^1(\mathbb{R}^d)} = 0,$$

and hence $f_m \to f$ in $H^1_{\mathscr{D},\max}(\mathbb{R}^d;X)$. □

Theorem 11.1.30. *Let X be a Banach space. The duality*

$$\langle b,h\rangle := \lim_{N\to\infty} \int \langle b_N, h\rangle = \sum_{i=1}^\infty \int \langle b, a_i\rangle, \quad b_N := \min\Big\{1, \frac{N}{\|b\|_{X^*}}\Big\} b$$

between $b \in \mathrm{BMO}_{\mathscr{D}}(\mathbb{R}^d;X^*)$ and $h \in H^1_{\mathscr{D},\mathrm{at}}(\mathbb{R}^d;X)$ is well defined, and realises $\mathrm{BMO}_{\mathscr{D}}(\mathbb{R}^d;X^*)$ *with the norm*

$$\|b\|_{\mathrm{BMO}_{\mathscr{D}}(\mathbb{R}^d;X)} := \sup_{Q\in\mathscr{D}} \inf_{c\in X} \fint_Q \|b-c\|_X$$

as an isometric subspace of $(H^1_{\mathscr{D},\mathrm{at}}(\mathbb{R}^d;X))^*$.

Proof. Since all norms BMO norms appearing in this proof are dyadic, we drop the subscript $\mathscr{D}$ for the benefit of slightly lighter notation.

Part 1: Estimating the dual norm by the BMO *norm*

If $\operatorname{supp} a_i \subseteq Q_i \in \mathscr{D}$ and $\int a_i = 0$, we have

$$\Big|\int \langle b, a_i\rangle\Big| = \Big|\int_{Q_i} \langle b-c, a_i\rangle\Big| \leqslant \fint_{Q_i} \|b-c\|_{X^*} |Q_i| \|a_i\|_\infty$$

for all $c \in E^*$. Taking the infimum over $c \in E^*$ it follows that

$$\Big|\int \langle b, a_i\rangle\Big| \leqslant \|b\|_{\mathrm{BMO}} |Q_i| \|a_i\|_\infty$$

and hence $\sum_{i=1}^{\infty}\int\langle b,a_i\rangle$ converges for $b\in \mathrm{BMO}(\mathbb{R}^d;E^*)$ and $\sum_{i=1}^{\infty}a_i\in H^1_{\mathrm{at}}(\mathbb{R}^d;E)$.

One checks that $\|b_N-c_N\|_{X^*}\leqslant 2\|b-c\|_{X^*}$, whence

$$\inf_{c\in E^*}\fint_Q\|b_N-c\|_{E^*}\leqslant\inf_{c\in E^*}\fint_Q\|b_N-c_N\|_{E^*}\leqslant 2\inf_{c\in E^*}\fint_Q\|b-c\|_{E^*},$$

so that $b_N\in(\mathrm{BMO}\cap L^\infty)(\mathbb{R}^d;X^*)$ and

$$\Big|\int\langle b_N,a_i\rangle\Big|\leqslant\|b_N\|_{\mathrm{BMO}}|Q_i|\|a_i\|_\infty\leqslant 2\|b\|_{\mathrm{BMO}}|Q_i|\|a_i\|_\infty.$$

Thus

$$\sum_{i=1}^{\infty}\int\langle b,a_i\rangle=\sum_{i=1}^{\infty}\lim_{N\to\infty}\int\langle b_N,a_i\rangle=\lim_{N\to\infty}\sum_{i=1}^{\infty}\int\langle b_N,a_i\rangle=\lim_{N\to\infty}\int\langle b_N,h\rangle,$$

where the first two identities use dominated convergence in $L^1(Q_i)$ and in ℓ^1, respectively, and the last one follows from the convergence of the series $h=\sum_{i=1}^{\infty}a_i$ in $L^1(\mathbb{R}^d;E)$, and the fact that $b_N\in L^\infty(\mathbb{R}^d;E^*)\subseteq(L^1(\mathbb{R}^d;E))^*$. This shows in particular that the pairing of $\langle b,h\rangle$ is independent of the particular series representation of h, and hence well defined. Taking the infimum over all representations in the estimate

$$|\langle b,h\rangle|\leqslant\sum_{i=1}^{\infty}\|b\|_{\mathrm{BMO}}|Q_i|\|a_i\|_\infty,$$

we find that

$$\|b\|_{(H^1_{\mathrm{at}}(\mathbb{R}^d;X))^*}\leqslant\|b\|_{\mathrm{BMO}(\mathbb{R}^d;X^*)}. \tag{11.8}$$

Part 2: Estimating the BMO *norm by the dual norm*

For the converse estimate, consider a cube Q and suppose first that $s\in L^1(Q;X^*)$ is a simple function, thus measurable with respect to a finite σ-algebra $\mathscr{F}$ of Q. The advantage of this setting is that, for a finite σ-algebra, we have the duality $(L^p(\mathscr{F};X))^*=L^{p'}(\mathscr{F};X^*)$ for an arbitrary Banach space X and for every $p\in[1,\infty]$, including in particular $p=\infty$. Now $\inf_{c\in E^*}\|s-c\|_{L^1(Q;X^*)}$ is the norm of the equivalence class $[s]\in L^1(\mathscr{F};X^*)/X^*$, where $L^1(\mathscr{F};X^*)=(L^\infty(\mathscr{F};X))^*$.

We claim that the quotient space above is the dual of the subspace $L^\infty_0(\mathscr{F};X)\subseteq L^\infty(\mathscr{F};X)$ of functions with mean zero. In fact, recall from Proposition B.1.4 that for any subspace $Y\subseteq Z$, we have the identification $Y^*=Z^*/Y^\perp$, the quotient of Z^* with the annihilator $Y^\perp$ of Y in Z^*. Now $Z=L^\infty(\mathscr{F};X)$ for a finite σ-algebra $\mathscr{F}$, in which case $Z^*=L^1(\mathscr{F};X^*)$. To identify $Y^\perp$ for $Y=L^\infty_0(\mathscr{F};X)$, it is easy to check that the only functions $f\in L^1(\mathscr{F};X^*)$ for which $\int\langle f,g\rangle=0$ for all $g\in L^\infty_0(\mathscr{F};X)$ are the constant functions. Thus indeed $L^1(\mathscr{F};X^*)/X^*=(L^\infty_0(\mathscr{F};X))^*$, and hence

$$\inf_{c\in X^*}\|s-c\|_{L^1(Q;X^*)}=\|[s]\|_{L^1(\mathscr{F};X^*)/X^*}=\sup_{\substack{g\in L^\infty_0(\mathscr{F};X)\\ \|g\|_\infty\leqslant 1}}\Big|\int\langle s,g\rangle\Big|.$$

Now, given $b\in \mathrm{BMO}(\mathbb{R}^d;X^*)$ and a cube Q, we choose a simple $s\in L^1(Q;X^*)$ such that $\|b-s\|_{L^1(Q;X^*)}\leqslant\varepsilon$. Then

$$\begin{aligned}\inf_{c\in X^*}\|b-c\|_{L^1(Q;X^*)}&\leqslant\inf_{c\in X^*}\|s-c\|_{L^1(Q;X^*)}+\varepsilon\\ &\leqslant\sup_{\substack{g\in L^\infty_0(Q;X)\\ \|g\|_\infty\leqslant 1}}\Big|\int\langle s,g\rangle\Big|+\varepsilon\leqslant\sup_{\substack{g\in L^\infty_0(Q;X)\\ \|g\|_\infty\leqslant 1}}\Big|\int\langle b,g\rangle\Big|+2\varepsilon.\end{aligned}$$

But each $g\in L^\infty_0(Q;X)$ is an $H^1_{\mathscr{D}}$-atom, and hence

$$\Big|\int\langle b,g\rangle\Big|\leqslant\|b\|_{(H^1_{\mathscr{D},\mathrm{at}}(\mathbb{R}^d;X))^*}\|g\|_{H^1_{\mathscr{D},\mathrm{at}}(\mathbb{R}^d;X)}\leqslant\|b\|_{(H^1_{\mathscr{D},\mathrm{at}}(\mathbb{R}^d;X))^*}\|g\|_\infty|Q|.$$

Dividing by $|Q|$ and letting $\varepsilon\to 0$, we obtain

$$\inf_{c\in X^*}⨍_Q\|b-c\|_{X^*}\leqslant\|b\|_{(H^1_{\mathscr{D},\mathrm{at}}(\mathbb{R}^d;X))^*},$$

and hence the estimate converse to (11.8). □

11.2 Singular integrals and extrapolation of L^{p_0} bounds

In this section we study a fairly broad class of kernels satisfying a relatively general integrability condition first introduced by Hörmander. Nevertheless, this condition turns out to be strong enough to yield a fundamental extrapolation property of singular integral operators: once bounded on one L^{p_0} space, they remain bounded on the full scale of L^p spaces for $p\in(1,\infty)$, together with appropriate end-point estimates for $p=1$ and $p=\infty$.

The precise classes of kernels relevant are described in the following definition. We recall that $\dot{\mathbb{R}}^{2d}=\mathbb{R}^{2d}\setminus\{(t,t):\,t\in\mathbb{R}^d\}$.

Definition 11.2.1. *Let X and Y be Banach spaces, $p_0\in[1,\infty]$, and consider*

$$K:\dot{\mathbb{R}}^{2d}\to\mathscr{L}(X,Y),\qquad T\in\mathscr{L}(L^{p_0}(\mathbb{R}^d;X),L^{p_0,\infty}(\mathbb{R}^d;Y)).$$

(1) *We say that T* has kernel *K, or that K* is the kernel of *T, if for every $f\in L^{p_0}_{\mathrm{c}}(\mathbb{R}^d;X)$ and almost every s at a positive distance from* $\operatorname{supp} f$ *the following holds: for every functional $y^*\in Y^*$, the function $t\mapsto\langle K(s,t)f(t),y^*\rangle$ is integrable, and*

$$\langle Tf(s),y^*\rangle=\int\langle K(s,t)f(t),y^*\rangle\,\mathrm{d}t.$$

(2) *We say that* K *is a* Hörmander (resp. operator-Hörmander) kernel*, or satisfies the* Hörmander (resp. operator-Hörmander) condition*, if the following estimate holds for all* $x \in X$ *and* $t, t' \in \mathbb{R}^d$ *with a fixed constant* c *independent of these quantities:*

$$\int_{|s-t|>2|t-t'|} \|[K(s,t) - K(s,t')]x\|_Y \, \mathrm{d}s \leqslant c\|x\|_X$$
$$\Big(\text{resp.} \quad \int_{|s-t|>2|t-t'|} \|K(s,t) - K(s,t')\|_{\mathscr{L}(X,Y)} \, \mathrm{d}x \leqslant c\Big). \tag{11.9}$$

The smallest admissible c *is denoted by* $\|K\|_{\text{Hör}}$ *(resp.* $\|K\|_{\text{Hör}_{\text{op}}}$*).*

(3) *We say that* K *is a* dual Hörmander (resp. dual operator-Hörmander) kernel*, or satisfies the* dual Hörmander (resp. dual operator-Hörmander) condition*, if the following estimate holds for every* $y^* \in Y^*$ *and* $s, s' \in \mathbb{R}^d$ *with a fixed constant* c' *independent of these quantities:*

$$\int_{|t-s|>2|s-s'|} \|[K(s,t)^* - K(s',t)^*]y^*\|_{X^*} \, \mathrm{d}t \leqslant c'\|y^*\|_{Y^*}$$
$$\Big(\text{resp.} \quad \int_{|t-s|>2|s-s'|} \|K(s,t) - K(s',t)\|_{\mathscr{L}(X,Y)} \, \mathrm{d}t \leqslant c'\Big) \tag{11.10}$$

The smallest admissible c' *is denoted by* $\|K\|_{\text{Hör}^*}$ *(resp.* $\|K\|_{\text{Hör}^*_{\text{op}}}$*).*

(4) *If* $Q \subseteq \mathbb{R}^d$ *is a cube or a quadrant, we make analogous definitions with each occurrence of* $\mathbb{R}^d$ *replaced by* Q*; in particular, with* $\dot{\mathbb{R}}^{2d}$ *by* $\{(s,t) \in Q \times Q : s \neq t\}$*, and the integrals extended over* Q *only, while keeping the other integrations conditions in force. In this situation, we say that* K *is a (dual/operator) Hörmander kernel on* Q*, respectively.*

Remark 11.2.2. If K is a (dual/operator) Hörmander kernel, then its restriction to $\{(s,t) \in Q \times Q : s \neq t\}$ is a (dual/operator) Hörmander kernel on Q.

Example 11.2.3. A kernel $K(x,y)$ that only depends on the difference $x - y$, i.e., $K(x-y) = k(x-y)$ for some function k, is called a *convolution kernel.* For such kernels, after simple changes of variables, the Hörmander and dual Hörmander conditions take the forms

$$\int_{|s|>2|t|} \|[k(s-t) - k(s)]x\|_Y \, \mathrm{d}s \leqslant c\|x\|_X,$$
$$\int_{|s|>2|t|} \|[k(s-t)^* - k(s)^*]y^*\|_{X^*} \, \mathrm{d}s \leqslant c'\|y^*\|_{Y^*},$$

and similar reformulations of the operator Hörmander conditions are obvious.

The role of these conditions in the extrapolation of L^p-boundedness is summarised in the next theorem. Before stating the result, we make a remark concerning the extension of the action of operators from $L^{p_0}(\mathbb{R}^d;X)$ to $L^\infty(\mathbb{R}^d;X)$. An inherent obstacle here is that the intersection $L^{p_0}(\mathbb{R}^d;X) \cap L^\infty(\mathbb{R}^d;X)$ is not dense in $L^\infty(\mathbb{R}^d;X)$. As a substitute we have:

Lemma 11.2.4. *Let X be a Banach space. The closure of $L^p(\mathbb{R}^d;X) \cap L^\infty(\mathbb{R}^d;X)$ in $L^\infty(\mathbb{R}^d;X)$ is independent of $p \in (0,\infty)$, and it coincides with*

$$\bar{L}^\infty_{\text{fin}}(\mathbb{R}^d;X) := \overline{L^\infty_{\text{fin}}(\mathbb{R}^d;X)}^{L^\infty(\mathbb{R}^d;X)}, \quad \textit{where}$$
$$L^\infty_{\text{fin}}(\mathbb{R}^d;X) := \{f \in L^\infty(\mathbb{R}^d;X) : |\{f \neq 0\}| < \infty\}.$$

Proof. It is clear that $L^\infty_{\text{fin}}(\mathbb{R}^d;X) \subseteq L^p(\mathbb{R}^d;X) \cap L^\infty(\mathbb{R}^d;X)$, and taking the closures of both sides proves one side of the claim.

Conversely, let $p \in (0,\infty)$, a function $f \in L^p(\mathbb{R}^d;X) \cap L^\infty(\mathbb{R}^d;X)$, and $\varepsilon > 0$ be given. Now

$$F_\varepsilon := \{\|f(\cdot)\|_X > \varepsilon\} \leqslant \varepsilon^{-p}\|f\|^p_{L^p(\mathbb{R}^d;X)} < \infty,$$

and hence $f_\varepsilon := \mathbf{1}_{F_\varepsilon} f \in L^\infty_{\text{fin}}(\mathbb{R}^d;X)$. On the other hand, it is clear that

$$\|f - f_\varepsilon\|_{L^\infty(\mathbb{R}^d;X)} = \|1_{\complement F_\varepsilon} f\|_{L^\infty(\mathbb{R}^d;X)} \leqslant \varepsilon.$$

Since this can be done for any $\varepsilon > 0$, we find that f belongs to the $L^\infty(\mathbb{R}^d;X)$-closure of $L^\infty_{\text{fin}}(\mathbb{R}^d;X)$. Since $f \in L^p(\mathbb{R}^d;X) \cap L^\infty(\mathbb{R}^d;X)$, this whole intersection belongs to the said closure, and then so does the closure of this intersection. This completes the proof. □

Theorem 11.2.5 (Calderón–Zygmund). *Let X and Y be Banach spaces and $p_0 \in [1,\infty]$. Let*

$$T \in \mathscr{L}(L^{p_0}(\mathbb{R}^d;X), L^{p_0,\infty}(\mathbb{R}^d;Y))$$

(where $L^{\infty,\infty} := L^\infty$) with norm $A_0 := \|T\|_{\mathscr{L}(L^{p_0}(\mathbb{R}^d;X),L^{p_0,\infty}(\mathbb{R}^d;Y))}$.

(1) *If T has a Hörmander kernel K, then*
   (a) *T extends uniquely to $T \in \mathscr{L}(L^p(\mathbb{R}^d;X), L^p(\mathbb{R}^d;Y))$ for all $p \in (1,p_0)$, and*

$$\|T\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \leqslant c_d\Big(\frac{p_0-1}{(p_0-p)(p-1)}\Big)^{1/p}(A_0 + \|K\|_{\text{Hör}});$$

   (b) *T extends uniquely to $T \in \mathscr{L}(L^1(\mathbb{R}^d;X), L^{1,\infty}(\mathbb{R}^d;Y))$ and*

$$\|T\|_{\mathscr{L}(L^1(\mathbb{R}^d;X),L^{1,\infty}(\mathbb{R}^d;Y))} \leqslant c_d(A_0 + \|K\|_{\text{Hör}}).$$

(2) *If T has a dual Hörmander kernel K, then*
   (a) *T extends uniquely to $T \in \mathscr{L}(L^p(\mathbb{R}^d;X), L^p(\mathbb{R}^d;Y))$ for all $p \in (p_0,\infty)$, and*

$$\|T\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \leqslant c_d p\Big(\frac{p_0}{p-p_0}\Big)^{1/p}(A_0 + \|K\|_{\text{Hör}^*});$$

(b) *T extends uniquely to $T \in \mathscr{L}(\bar{L}^\infty_{\text{fin}}(\mathbb{R}^d; X), \text{BMO}(\mathbb{R}^d; Y))$, where the space $\bar{L}^\infty_{\text{fin}}(\mathbb{R}^d; X)$ is as in Lemma 11.2.4, and*

$$\|T\|_{\mathscr{L}(\bar{L}^\infty_{\text{fin}}(\mathbb{R}^d;X),\text{BMO}(\mathbb{R}^d;Y))} \leqslant c_d(A_0 + \|K\|_{\text{Hör}})\|f\|_{L^\infty(\mathbb{R}^d;X)}$$

for all f in this space.

(3) *If T has a kernel K that satisfies both the Hörmander and the dual Hörmander conditions, then for all $p \in (1, \infty)$, T extends uniquely to $T \in \mathscr{L}(L^p(\mathbb{R}^d; X), L^p(\mathbb{R}^d; Y))$, and*

$$\|T\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \leqslant c_d \cdot pp' \cdot (A_0 + \|K\|_{\text{Hör}} + \|K\|_{\text{Hör}^*}).$$

(4) *All claims remain valid when $\mathbb{R}^d$ is replaced either by a cube or a quadrant throughout. In this case, it suffices to relax the Hörmander conditions accordingly, as in Definition 11.2.1(4).*

The rest of this section is dedicated to a case-by-case proof of the different assertions of Theorem 11.2.5. For the proof of (1), we introduce the fundamental *Calderón–Zygmund decomposition* in Proposition 11.2.6. The proof of (2), in turn, depends on the notion of local oscillations developed in Section 11.1. The result of (2b) does not directly allow the extension of T to all of $L^\infty(\mathbb{R}^d; X)$ since $L^{p_0}(\mathbb{R}^d; X) \cap L^\infty(\mathbb{R}^d; X)$ is not dense in this space; see Theorem 11.2.9 for results in this direction. The proof of (3) is essentially a combination of (1) and (2), but note that this case provides additional information about $p = p_0$ (bootstrapping the initial weak-type bound into a strong-type one) and improves the quantitative estimates for p close to p_0, where the bounds provided by (1) and (2) blow up as $p \to p_0$. Finally, the claims (4) will be dealt with by indicating the relevant modifications in the proofs of (1) through (3). As it turns out, these modifications are fairly minor, although in the case of (1) they might not be entirely obvious.

11.2.a Calderón–Zygmund decomposition and case $p \in (1, p_0)$

The key to extrapolating in this range is the following classical result:

Proposition 11.2.6 (Calderón–Zygmund decomposition). *Let X be a Banach space. Given $f \in L^1(\mathbb{R}^d; X)$ and $\lambda > 0$, there exists a decomposition $f = g + b$, where*

$$\|g\|_\infty \leqslant 2^d\lambda, \qquad \|g\|_1 \leqslant \|f\|_1,$$

and $b = \sum_i b_i$, where

$$\operatorname{supp} b_i \subseteq Q_i, \quad \int b_i = 0, \quad \sum_i |Q_i| \leqslant \frac{1}{\lambda}\|f\|_1, \quad \sum_i \|b_i\|_1 \leqslant 2\|f\|_1$$

for some disjoint dyadic cubes Q_i. If f is simple, then all b_i are also simple.

If $f \in L^1(Q_0; X)$ for some cube $Q_0 \subseteq \mathbb{R}^d$ and $\lambda \geqslant 2^{-d} \fint_{Q_0} \|f\|$, then the cubes Q_i can be chosen as dyadic subcubes of the initial Q_0, and the function g to be supported on Q_0.

If $f \in L^1(S; X)$ for some quadrant of $\mathbb{R}^d$, then we have $Q_i \subseteq S$.

Proof. Let $Q_i \in \mathscr{D}$ be the maximal dyadic cubes such that $\fint_{Q_i} \|f\| > \lambda$. Then they are pairwise disjoint, and

$$\sum_i |Q_i| = |\{M_{\mathscr{D}} f > \lambda\}| \leqslant \frac{1}{\lambda}\|f\|_1.$$

We define $b_i := 1_{Q_i}(f - \langle f \rangle_{Q_i})$ (which is clearly simple if f is), whence the first two properties of b_i are clear, and it remains to estimate

$$\sum_i \|b_i\|_1 \leqslant \sum_i (\|1_{Q_i} f\|_1 + |Q_i| \|\langle f \rangle_{Q_i}\|) \leqslant \sum_i 2 \int_{Q_i} \|f\| \leqslant 2\|f\|_1$$

by the disjointness of the cubes. To ensure that $f = g + b$, we must then define

$$g := 1_{\complement(\bigcup_i Q_i)} f + \sum_i 1_{Q_i} \langle f \rangle_{Q_i},$$

where the terms are disjointly supported. If $x \in \complement(\bigcup_i Q_i)$, then all dyadic cubes $Q \ni x$ satisfy $\fint_Q |f| \leqslant \lambda$, and thus

$$\|g(x)\| = \|f(x)\| = \lim_{\substack{Q \ni x \\ \ell(Q) \to 0}} \fint_Q \|f\| \leqslant \lambda$$

at almost every such x by the Lebesgue Differentiation Theorem 2.3.4 (or in fact just the scalar-valued version, since we apply it to the function $\|f(\cdot)\|$ rather than f itself). On the other hand, the maximality of Q_i implies that its dyadic parent $\widehat{Q}_i$ satisfies the opposite inequality, $\fint_{\widehat{Q}_i} |f| \leqslant \lambda$. Thus

$$\|g(x)\|_X = \|\langle f \rangle_{Q_i}\|_X \leqslant \frac{1}{|Q_i|} \int_{Q_i} \|f\|_X \leqslant \frac{|\widehat{Q}_i|}{|Q_i|} \cdot \frac{1}{|\widehat{Q}_i|} \int_{\widehat{Q}_i} \|f\|_X \leqslant 2^d \cdot \lambda$$

for $x \in Q_i$, and we see that $\|g(x)\| \leqslant 2^d \lambda$ in both cases. Moreover,

$$\|g\|_1 = \int_{\complement(\bigcup_i Q_i)} \|f\| + \sum_i |Q_i| \|\langle f \rangle_{Q_i}\| \leqslant \int_{\complement(\bigcup_i Q_i)} \|f\| + \sum_i \int_{Q_i} \|f\| = \|f\|_1$$

by the disjointness of the cubes.

If $f \in L^1(Q_0; X)$ and $\lambda \geqslant \fint_{Q_0} \|f\|$, then the maximal dyadic subcubes Q_i of Q_0 with $\fint_{Q_i} \|f\| > \lambda$, are necessarily strict subcubes of Q_0, and the same proof produces a decomposition with the claimed additional properties. If $\lambda \in [2^{-d}, 1) \fint_{Q_0} \|f\|$, then we let the family $\{Q_i\}_i$ consist of the initial cube Q_0 only, so that $g := \langle f \rangle_{Q_0} \mathbf{1}_{Q_0}$ and $b := (f - \langle f \rangle_{Q_0}) \mathbf{1}_{Q_0}$. Then $\|g\|_\infty = \|\langle f \rangle_{Q_0}\| \leqslant 2^d \lambda$ and $\sum_i |Q_i| = |Q_0| \leqslant \lambda^{-1} \|f\|_1$ by the two assumed bounds on λ. The last claim of the theorem is obvious. □

We can now give:

Proof of Theorem 11.2.5(1). Our plan is to first prove the weak-type result (1b), and then obtain the strong-type bound (1a) via the Marcinkiewicz Interpolation Theorem 2.2.3.

For $f \in L^{p_0}(\mathbb{R}^d; X) \cap L^1(\mathbb{R}^d; X)$ and $\lambda > 0$, we estimate $\lambda|\{\|Tf\| > \lambda\}|$.

Let $f = g + b$ the Calderón–Zygmund decomposition of f at level $\alpha\lambda$ (instead of λ), where α is to be determined. Then

$$\|g\|_{p_0} \leqslant \|g\|_\infty^{1/p_0'} \|g\|_1^{1/p_0} \leqslant (2^d \alpha\lambda)^{1/p_0'} \|f\|_1^{1/p_0},$$

so in particular $g \in L^{p_0}(\mathbb{R}^d; X)$, and thus $b = f - g \in L^{p_0}(\mathbb{R}^d; X)$. Since $b = \sum_i b_i$ and the b_i are disjointly supported, it follows that each b_i also belongs to $L^{p_0}(\mathbb{R}^d; X)$ and the identity $b = \sum_i b_i$ also holds in the sense of convergence in $L^{p_0}(\mathbb{R}^d; X)$. The assumption that $T \in \mathscr{L}(L^{p_0}(\mathbb{R}^d; X), L^{p_0,\infty}(\mathbb{R}^d; Y))$ then implies that

$$Tf = T(g + b) = Tg + Tb, \qquad Tb = T\sum_i b_i = \sum_i Tb_i.$$

If Q_i are the corresponding cubes, let B_i be the concentric ball of twice the diameter and $O^* := \bigcup_i B_i$. Then

$$|\{\|Tf\| > \lambda\}| \leqslant |\{\|Tg\| > \lambda/2\}| + |\{\|Tb\| > \lambda/2\} \setminus O^*| + |O^*|, \tag{11.11}$$

where the last term satisfies

$$|O^*| \leqslant \sum_i |B_i| = \sum_i c_d |Q_i| \leqslant \frac{c_d}{\alpha\lambda} \|f\|_1.$$

For the middle term, we have

$$|\{\|Tb\| > \lambda/2\} \setminus O^*| \leqslant \int_{\complement O^*} \frac{\|Tb\|}{\lambda/2} \leqslant \frac{2}{\lambda} \sum_i \int_{\complement O^*} \|Tb_i\| \leqslant \frac{2}{\lambda} \sum_i \int_{\complement B_i} \|Tb_i\|.$$

In order to estimate the ith term here, we denote by z_i the common centre of the cube Q_i and the ball B_i. Now the integral representation of $Tb_i(s)$ is available at $s \in \complement B_i$. Explicitly, for each $y^* \in Y^*$,

$$\langle Tb_i(s), y^* \rangle = \int \langle K(s,t) b_i(t), y^* \rangle \,\mathrm{d}t = \int \langle [K(s,t) - K(s,z_i)] b_i(t), y^* \rangle \,\mathrm{d}t,$$

where the last step follows from the fact that $\int b_i(t) \,\mathrm{d}t = 0$. Thus

$$\|Tb_i(s)\|_Y \leqslant \int_{Q_i} \|[K(s,t) - K(s,z_i)] b_i(t)\|_Y \,\mathrm{d}t$$

and hence

$$\int_{\complement B_i} \|Tb_i(s)\|_Y \,\mathrm{d}s \leqslant \int_{Q_i} \int_{\complement B_i} \|[K(s,t) - K(s,z_i)] b_i(t)\|_Y \,\mathrm{d}s \,\mathrm{d}t$$

$$\leqslant \int_{Q_i} \|K\|_{\mathrm{Hör}} \|b_i(t)\|_X \,\mathrm{d}t,$$

since $|s - z_i| \geqslant 2\,\mathrm{diam}(Q_i) \geqslant 2|t - z_i|$ for $s \in \complement B_i$ and $t \in Q_i$. Substituting back, it follows that

$$\frac{2}{\lambda}\sum_i \int_{\complement B_i} \|Tb_i\| \leqslant \frac{2}{\lambda}\|K\|_{\mathrm{Hör}} \sum_i \int_{Q_i} \|b_i\| = \frac{2}{\lambda}\|K\|_{\mathrm{Hör}}\|b\|_1 \leqslant \frac{4}{\lambda}\|K\|_{\mathrm{Hör}}\|f\|_1.$$

It remains to estimate $|\{\|Tg\| > \lambda/2\}|$. If $p_0 < \infty$, we have

$$|\{\|Tg\| > \lambda/2\}| \leqslant \frac{A_0^{p_0}}{(\lambda/2)^{p_0}}\|g\|_{p_0}^{p_0} \leqslant \frac{2^{p_0}}{\lambda^{p_0}} A_0^{p_0} \cdot (2^d\alpha\lambda)^{p_0-1}\|f\|_1,$$

so that altogether

$$|\{\|Tf\| > \lambda\}| \leqslant \Big(\frac{(2A_0 \cdot 2^d\alpha)^{p_0}}{2^d\alpha} + 4\|K\|_{\mathrm{Hör}} + \frac{c_d}{\alpha}\Big)\frac{\|f\|_1}{\lambda},$$

where we are still free to choose $\alpha > 0$. Taking

$$\alpha = 2^{-d-1}/A_0 \tag{11.12}$$

leads to

$$|\{\|Tf\| > \lambda\}| \leqslant (c_d A_0 + 4\|K\|_{\mathrm{Hör}})\frac{\|f\|_1}{\lambda}. \tag{11.13}$$

If $p_0 = \infty$, we observe that $\|Tg\|_\infty \leqslant A_0\|g\|_\infty \leqslant A_0 2^d\alpha\lambda$, so that the same choice of α guarantees that $|\{\|Tg\| > \lambda/2\}| = 0$. Thus, in this case, we only need to estimate the last two terms in (11.11), and these have exactly the same bounds in the case $p_0 < \infty$ that was already handled.

We have hence confirmed (11.13) for all $f \in L^{p_0}(\mathbb{R}^d; X) \cap L^1(\mathbb{R}^d; X)$ and $\lambda > 0$, and this proves the existence of a unique bounded extension $T \in \mathscr{L}(L^1(\mathbb{R}^d; X), L^{1,\infty}(\mathbb{R}^d; X))$ by the density of $L^{p_0}(\mathbb{R}^d; X) \cap L^1(\mathbb{R}^d; X)$ in $L^1(\mathbb{R}^d; X)$. This completes the proof of (1b).

(1b) *in case* (4)*:* Let then $\mathbb{R}^d$ be replaced by a cube Q_0. Note that

$$\|Tf\|_{L^{1,\infty}(Q_0;Y)} := \sup_{\lambda>0} \lambda|Q_0 \cap \{|Tf| > \lambda\}|.$$

If $\lambda \leqslant 2A_0 \fint_{Q_0} \|f\|$, then

$$\lambda|Q_0 \cap \{|Tf| > \lambda\}| \leqslant 2A_0 \fint_{Q_0} \|f\| \times |Q_0| = 2A_0\|f\|_1 \tag{11.14}$$

If $\lambda > 2A_0 \fint_{Q_0} \|f\|$ and α is as in (11.12), then

$$\alpha\lambda > 2^{-d} \fint_{Q_0} \|f\|$$

is in the admissible range to have Calderón–Zygmund decomposition at level $\alpha\lambda$ fully localised within the cube Q_0 (Proposition 11.2.6). Thus, the earlier argument for the full space $\mathbb{R}^d$ localises to Q_0 to produce the same conclusion (11.13), but with the integral defining $\|K\|_{\text{Hör}}$ restricted to Q_0 only. A combination with (11.14) shows that this estimate holds for all $\lambda > 0$, and hence we have the desired weak-type bound on Q_0.

The case of a quadrant S is an immediate variant of the case of $\mathbb{R}^d$, since Proposition 11.2.6 guarantees that the Calderón–Zygmund decomposition is localised to this quadrant for all values of the level parameter.

(1a)*:* A direct application of Marcinkiewicz Interpolation Theorem 2.2.3 (with 1 in place of p_0, and p_0 in place of p_1) shows that

$$\|T\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y)} \leqslant c(\theta,1,p_0)\Big(\frac{c_d(A_0+\|K\|_{\text{Hör}})}{1-\theta}\Big)^{1-\theta}\Big(\frac{A_0}{\theta}\Big)^{\theta},$$

where $\theta \in (0,1)$ is such that $1/p = (1-\theta)/1+\theta/p_0$,

$$c(\theta,1,p_0) = \Big\{p_0^{\frac{p-1}{p_0-1}}\frac{p_0-p}{(p_0-p)(p-1)}\Big\}^{\frac{1}{p}}$$

if $p_0 \in (1,\infty)$, and $c(\theta,1,\infty) = (p-1)^{-\frac{1}{p}}$. By the arithmetic–geometric mean inequality, we have

$$\Big(\frac{1}{1-\theta}\Big)^{1-\theta}\Big(\frac{1}{\theta}\Big)^{\theta} \leqslant 1-\theta\frac{1}{1-\theta}+\theta\frac{1}{\theta} = 2, \tag{11.15}$$

and by elementary calculus one verifies that $p_0^{\frac{1}{p_0-1}} \leqslant e$ for $p_0 \in (1,\infty)$. Substituting these estimates, we obtain

$$\|T\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y)} \leqslant 2e\cdot c_d\cdot\Big\{\frac{p_0-p}{(p_0-p)(p-1)}\Big\}^{\frac{1}{p}}(A_0+\|K\|_{\text{Hör}}),$$

which coincides with the claim after redefining c_d. Since the Marcinkiewicz Interpolation Theorem 2.2.3 is valid for general measure spaces, the same argument applies equally well in the case of a cube or a quadrant as the underlying domain. □

11.2.b Local oscillations of Tf and case $p \in (p_0,\infty)$

We next turn to the study of extrapolation of the boundedness to $p > p_0$, which will involve the dual Hörmander condition. A reader familiar with the scalar-valued counterpart of the theory might expect a duality argument at this point. While this might not be strictly out of question here, either, one should note that at least some number of technicalities would have to be tackled by such an approach. To begin with, the adjoint of $T \in \mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))$ would be an operator

$$T^* \in \mathscr{L}(L^p(\mathbb{R}^d;Y)^*, L^p(\mathbb{R}^d;X)^*),$$

where each $L^p(\mathbb{R}^d;Z)^*$ is in general a larger space than $L^{p'}(\mathbb{R}^d;Z^*)$, unless additional assumptions are imposed on Z^* (see Section 1.3). Rather than dwelling into such issues, we prefer a direct approach within the original spaces of X and Y valued functions that we are interested in.

We still need to settle a technical issue about the validity of the integral representation of $Tf(x)$ for certain non-compactly supported functions f:

Lemma 11.2.7. *Let X and Y be Banach spaces and $p_0 \in [1,\infty]$. Let $T \in \mathscr{L}(L^{p_0}(\mathbb{R}^d;X), L^{p_0,\infty}(\mathbb{R}^d;Y))$ be an operator with dual Hörmander kernel K. If $B \subseteq \mathbb{R}^d$ is a ball and $f \in L^{p_0}(\mathbb{R}^d;X) \cap L^\infty(\mathbb{R}^d;X)$ is supported in $\complement B$, then for almost all $s, s' \in \frac{1}{2}B$, we have*

$$\langle Tf(s) - Tf(s'), y^* \rangle = \int_{\complement B} \langle [K(s,t) - K(s',t)]f(t), y^* \rangle \,\mathrm{d}t \qquad \forall y^* \in Y^*.$$

Proof. Consider an increasing sequence of balls $B_1 \subseteq B_2 \subseteq \ldots$ such that $\bigcup_{n=1}^\infty B_n = \mathbb{R}^d$, and let $f_n := \mathbf{1}_{B_n} f$. Since $f_n = \mathbf{1}_{\complement B} f_n$ is compactly supported away from B, for almost every $s \in \frac{1}{2}B$ we have

$$\langle Tf_n(s), y^* \rangle = \int_{\complement B} \langle K(s,t) f_n(t), y^* \rangle \,\mathrm{d}t \qquad \forall y^* \in Y^*.$$

Thus, for almost every $s, s' \in \frac{1}{2}B$, the following holds for every $y^* \in Y^*$:

$$\langle Tf_n(s) - Tf_n(s'), y^* \rangle = \int_{\complement B} \langle f_n(t), [K(s,t)^* - K(s',t)^*] y^* \rangle \,\mathrm{d}t. \tag{11.16}$$

Now consider the limit $n \to \infty$. Since $f_n \to f$ in $L^{p_0}(\mathbb{R}^d;X)$ and $T \in \mathscr{L}(L^{p_0}(\mathbb{R}^d;X), L^{p_0,\infty}(\mathbb{R}^d;Y))$, we have $Tf_n \to Tf$ in $L^{p_0,\infty}(\mathbb{R}^d;Y)$. Hence a subsequence, which we keep denoting simply by f_n, also satisfies $Tf_n(s) \to Tf(s)$ at almost every $s \in \frac{1}{2}B$. This means that

$$LHS(11.16) \to \langle Tf(s) - Tf(s'), y^* \rangle.$$

It is also clear that $f_n(t) \to f(t)$ pointwise. On the other hand, the integrand in (11.16) is pointwise dominated by

$$(\|[K(s,t)^* - K(z_B,t)^*]y^*\|_{Y^*} + \|[K(s',t)^* - K(z_B,t)^*]y^*\|_{Y^*})\|f\|_\infty,$$

which is integrable over $t \in \complement B$ (thus $|t - z_B| \geqslant r_B \geqslant 2\max\{|s - z_B|, |s' - z_B|\}$) by the dual Hörmander condition. Hence

$$RHS(11.16) \to \int_{\complement B} \langle f(t), [K(s,t)^* - K(s',t)^*] y^* \rangle \,\mathrm{d}t$$

by dominated convergence. The equality of the limits is what we claimed. □

Recall the John–Strömberg maximal function and the local oscillations

$$M^{\#}_{0,\lambda} f(x) = \sup_{Q \ni x} \operatorname{osc}_\lambda(f;Q), \quad \operatorname{osc}_\lambda(f;Q) := \inf_{c\in X} \inf_{|E|\leqslant \lambda|Q|} \|(f-c)\mathbf{1}_{Q\setminus E}\|_\infty.$$

The following lemma contains the technical core of the upper extrapolation:

Lemma 11.2.8. *Under the assumptions of Theorem 11.2.5(2), for all* $f \in L^{p_0}(\mathbb{R}^d;X)\cap L^\infty(\mathbb{R}^d;X)$ *we have*

$$\|M^{\#}_{0,\lambda}(Tf)\|_\infty \leqslant (c_{d,\lambda}^{1/p_0} A_0 + 2\|K\|_{\text{Hör}^*})\|f\|_\infty.$$

If $\mathbb{R}^d$ *is replaced by a cube* $Q_0 \subseteq \mathbb{R}^d$ *or a quadrant* $S \subseteq \mathbb{R}^d$*, the conclusion remains valid with the following modifications:*

(a) *in the maximal operator* $M^{\#}_{0,\lambda}$*, the supremum is restricted to cubes* Q *contained in the initial cube* Q_0 *or the quadrant* S*;*
(b) *in the Hörmander norm* $\|K\|_{\text{Hör}^*}$*, the variables and the integrals are again restricted to* Q_0 *or* S*.*

Proof. Let $f \in L^{p_0}(\mathbb{R}^d;X)\cap L^\infty(\mathbb{R}^d;X)$ and let $Q\subseteq\mathbb{R}^d$ be a cube. Let B be a ball with the same centre and three time the diameter. We decompose

$$Tf = T(\mathbf{1}_B f) + [T(\mathbf{1}_{\complement B} f) - T(\mathbf{1}_{\complement B} f)(z)] + c,$$

where $c = T(\mathbf{1}_{\complement B} f)(z)$, and $z \in Q$ is fixed as one of the (almost all) points of Q where the conclusion of Lemma 11.2.7 is valid for the function $\mathbf{1}_{\complement B} f$. Thus

$$\|(Tf-c)\mathbf{1}_{Q\setminus E}\|_\infty \leqslant \|T(\mathbf{1}_B f)\mathbf{1}_{Q\setminus E}\|_\infty + \|[T(\mathbf{1}_{\complement B} f) - T(\mathbf{1}_{\complement B} f)(z)]\mathbf{1}_Q\|_\infty.$$

For the first term, we observe that

$$\|T(\mathbf{1}_B f)\|_{L^{p_0,\infty}} \leqslant A_0\|\mathbf{1}_B f\|_{p_0} \leqslant A_0|B|^{1/p_0}\|f\|_\infty,$$

and hence

$$|E_\Lambda| := |\{\|T(\mathbf{1}_B f)\| > \Lambda\}| \leqslant c_d\Big(\frac{A_0\|f\|_\infty}{\Lambda}\Big)^{p_0}|Q| \leqslant \lambda|Q|$$

if we choose $\Lambda := (c_d/\lambda)^{1/p_0} A_0\|f\|_\infty$. We conclude that

$$\|T(\mathbf{1}_B f)\mathbf{1}_{Q\setminus E_\Lambda}\|_\infty \leqslant (c_d/\lambda)^{1/p_0} A_0\|f\|_\infty.$$

For the other term, we estimate pointwise at almost every $s\in Q$ where the conclusion of Lemma 11.2.7 is valid. Recalling that $z \in Q$ was also chosen in this way and dualising against $y^*\in Y^*$, we get

$$|\langle T(\mathbf{1}_{\complement B} f)(s) - T(\mathbf{1}_{\complement B} f)(z), y^*\rangle| = \Big|\int_{\complement B}\langle f(t), [K(s,t)^* - K(z,t)^*]y^*\rangle \,\mathrm{d}t\Big|$$

$$\begin{aligned}&\leqslant \int_{\complement B} \|[K(s,t)^* - K(z,t)^*]y^*\|_{X^*}\,\mathrm{d}t\|f\|_\infty \\ &\leqslant 2\|K\|_{\mathrm{H\ddot{o}r}^*}\|y^*\|_{X^*}\|f\|_\infty.\end{aligned}$$

Taking the supremum over y^* in the unit ball of Y^* and the essential supremum over $s \in Q$, we arrive at

$$\|\mathbf{1}_Q[T(\mathbf{1}_{\complement B}f) - T(\mathbf{1}_{\complement B}f)(z)]\|_\infty \leqslant 2\|K\|_{\mathrm{H\ddot{o}r}^*}\|f\|_\infty.$$

Hence altogether

$$\mathrm{osc}_\lambda(Tf;Q) \leqslant \|(Tf-c)\mathbf{1}_{Q\setminus E_A}\|_\infty \leqslant (c_d/\lambda)^{1/p_0}A_0\|f\|_\infty + 2\|K\|_{\mathrm{H\ddot{o}r}^*}\|f\|_\infty,$$

and taking the supremum over all $Q \subseteq \mathbb{R}^d$ proves the lemma.

The modifications in the case of a cube Q_0 or a quadrant S in place of $\mathbb{R}^d$ are immediate by inspection. We note that the balls B featuring in the argument may extend beyond Q_0 or S; one simply thinks of $B \cap Q_0$ or $B \cap S$ in this case, while the complement $\complement B$ will be replaced by $Q_0 \setminus B$ or $S \setminus B$, respectively. □

Proof of Theorem 11.2.5(2a). Let us first consider the mapping properties of the sub-linear operator $M^\#_{0,\lambda} \circ T$, where $\lambda = 2^{-2-d}$.

By assumption, $T : L^{p_0}(\mathbb{R}^d;X) \to L^{p_0,\infty}(\mathbb{R}^d;Y)$ is bounded (with norm A_0), and Proposition 11.1.21 gives the boundedness of $M^\#_{0,\lambda} : L^{p_0,\infty}(\mathbb{R}^d;Y) \to L^{p_0,\infty}(\mathbb{R}^d)$ (with norm bounded by $c_{d,\lambda}^{1/p_0} \leqslant c_d$, since λ depends only on d, and $1/p_0 \leqslant 1$); thus the composition $M^\#_{0,\lambda} \circ T : L^{p_0}(\mathbb{R}^d;X) \to L^{p_0,\infty}(\mathbb{R}^d)$ is also bounded (with norm at most c_dA_0).

On the other hand, the previous Lemma 11.2.8 says that $M^\#_{0,\lambda} \circ T : L^{p_0}(\mathbb{R}^d;X) \cap L^\infty(\mathbb{R}^d;X) \to L^\infty(\mathbb{R}^d)$ is bounded (with norm at most $c_{d,\lambda}^{1/p_0}A_0 + \|K\|_{\mathrm{H\ddot{o}r}^*} \leqslant c_dA_0 + \|K\|_{\mathrm{H\ddot{o}r}^*}$), where the subspace $L^{p_0}(\mathbb{R}^d;X) \cap L^\infty(\mathbb{R}^d;X) \subseteq L^\infty(\mathbb{R}^d;X)$ is equipped with the norm of $L^\infty(\mathbb{R}^d;X)$.

This is essentially a setting to apply the Marcinkiewicz Interpolation Theorem 2.2.3: by inspection, one checks that the relaxed assumption

$$M^\#_{0,\lambda} \circ T : L^{p_0}(\mathbb{R}^d;X) \cap L^\infty(\mathbb{R}^d;X) \to L^\infty(\mathbb{R}^d)$$

(in place of $M^\#_{0,\lambda} \circ T : L^\infty(\mathbb{R}^d;X) \to L^\infty(\mathbb{R}^d)$) allows us to deduce the relaxed conclusion

$$M^\#_{0,\lambda} \circ T : L^{p_0}(\mathbb{R}^d;X) \cap L^p(\mathbb{R}^d;X) \to L^p(\mathbb{R}^d), \qquad p \in (p_0,\infty), \tag{11.17}$$

where $L^{p_0}(\mathbb{R}^d;X) \cap L^p(\mathbb{R}^d;X) \subseteq L^p(\mathbb{R}^d;X)$ is equipped with the norm of $L^p(\mathbb{R}^d;X)$. In fact, the proof of the Marcinkiewicz Interpolation Theorem 2.2.3 is based on decomposing a function f in the domain space into the two truncations, at varying level t,

$$\widetilde{f}^t := \Big(f - t\frac{f}{\|f\|}\Big) \cdot \mathbf{1}_{\{\|f\|>t\}},$$
$$\widetilde{f}_t := f \cdot \mathbf{1}_{\{\|f\|\leqslant t\}} + t\frac{f}{\|f\|} \cdot \mathbf{1}_{\{\|f\|>t\}},$$

and it is immediate to verify that, if $f \in L^{p_0}(\mathbb{R}^d;X)$, these remain in the space $L^{p_0}(\mathbb{R}^d;X)$, in addition to the other function space memberships used in the proof of Theorem 2.2.3.

If $\theta \in (0,1)$ is such that $1/p = (1-\theta)/p_0 + \theta/\infty = (1-\theta)/p_0$, the Marcinkiewicz Interpolation Theorem 2.2.3 shows that the norm of the operator in (11.17) is at most

$$\begin{aligned}
c(\theta,p_0,\infty)&\Big(\frac{\|M^{\#}_{0,\lambda}\circ T\|_{L^{p_0}(\mathbb{R}^d;X)\to L^{p_0,\infty}(\mathbb{R}^d)}}{1-\theta}\Big)^{1-\theta}\times \\
&\times\Big(\frac{\|M^{\#}_{0,\lambda}\circ T\|_{(L^{p_0}\cap L^\infty)(\mathbb{R}^d;X)\to L^\infty(\mathbb{R}^d)}}{\theta}\Big)^{\theta} \\
&\leqslant c(\theta,p_0,\infty)\Big(\frac{c_dA_0}{1-\theta}\Big)^{1-\theta}\Big(\frac{c_dA_0+\|K\|_{\text{Hör}^*}}{\theta}\Big)^{\theta} \\
&\leqslant c(\theta,p_0,\infty)\cdot 2\cdot (c_dA_0+\|K\|_{\text{Hör}^*})
\end{aligned}$$

by the arithmetic–geometric mean inequality (11.15) in the last step. Moreover, still from Theorem 2.2.3 and the identity $\Gamma(x+1) = x\Gamma(x)$,

$$c(\theta,p_0,\infty) = \Big\{\frac{\Gamma(p-p_0)\Gamma(p_0+1)}{\Gamma(p)}\Big\}^{1/p} = \{p_0B(p-p_0,p_0)\}^{1/p},$$

where the beta function is

$$\begin{aligned}
B(p-p_0,p_0) = \frac{\Gamma(p-p_0)\Gamma(p_0)}{\Gamma(p)} &= \int_0^1 u^{p-p_0-1}(1-u)^{p_0-1}\,\mathrm{d}u \\
&\leqslant \int_0^1 u^{p-p_0-1}\,\mathrm{d}u = \frac{1}{p-p_0},
\end{aligned}$$

since $p_0 \geqslant 1$ here. Substituting back (and redefining c_d), we find that the norm of the operator in (11.17) is at most

$$\Big(\frac{p_0}{p-p_0}\Big)^{1/p}(c_dA_0 + 2\|K\|_{\text{Hör}^*}).$$

Now Theorem 11.1.18, together with Remark 11.1.19 and the *a priori* condition that $Tf \in L^{p_0,\infty}(\mathbb{R}^d;X)$, show that

$$\begin{aligned}
\|Tf\|_{L^p(\mathbb{R}^d;Y)} &\leqslant 8p\|M^{\#}_{0,\lambda}(Tf)\|_{L^p(\mathbb{R}^d)} \\
&\leqslant p\Big(\frac{p_0}{p-p_0}\Big)^{1/p}(c_dA_0+16\|K\|_{\text{Hör}^*})\|f\|_{L^p(\mathbb{R}^d;X)}
\end{aligned}$$

for all $f \in L^p(\mathbb{R}^d;X) \cap L^{p_0}(\mathbb{R}^d;X)$ and $p \in (p_0,\infty)$. Since this is a dense subspace of $L^p(\mathbb{R}^d;X)$, the operator T has a unique extension to this space, with the same norm estimate above.

The case of a cube or a quadrant in place of $\mathbb{R}^d$ follows by the same argument, since all results quoted are also valid in these settings. □

It is also immediate from Lemma 11.2.8 and Proposition 11.1.24 that

$$\|Tf\|_{\mathrm{BMO}(\mathbb{R}^d;X)} \leqslant 8\|M^{\#}_{0,\lambda}(Tf)\|_{L^\infty(\mathbb{R}^d)} \leqslant (c_d A_0 + 8\|K\|_{\mathrm{H\ddot{o}r}^*})\|f\|_{L^\infty(\mathbb{R}^d;Y)}$$

for all $f \in L^{p_0}(\mathbb{R}^d;X) \cap L^\infty(\mathbb{R}^d;X)$. Since this is *not* a dense subspace of $L^\infty(\mathbb{R}^d;X)$, extending this estimate, and indeed the very meaning of "Tf", to all $f \in L^\infty(\mathbb{R}^d;X)$ requires an additional effort, to which we turn in Section 11.2.c below.

Proof of Theorem 11.2.5(3). We now assume that K satisfies both Hörmander and dual-Hörmander conditions, and hence we have access to both cases (1) and (2) that we already proved. By Theorem 11.2.5(1b), we have

$$\|T\|_{\mathscr{L}(L^1(\mathbb{R}^d;X),L^{1,\infty}(\mathbb{R}^d;Y)} \leqslant c_d(A_0 + \|K\|_{\mathrm{H\ddot{o}r}}).$$

We now use this estimate (rather than the original assumption) as input to Theorem 11.2.5(2a), i.e., we apply the latter with 1 in place of p_0 and $c_d(A_0 + \|K\|_{\mathrm{H\ddot{o}r}})$ in place of A_0. This gives, for all $p \in (1,\infty)$, the estimate

$$\begin{aligned}\|T\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y)} &\leqslant c_d p\Big(\frac{1}{p-1}\Big)^{1/p}\Big(c_d(A_0 + \|K\|_{\mathrm{H\ddot{o}r}}) + \|K\|_{\mathrm{H\ddot{o}r}^*}\Big)\\ &\leqslant c_d^2 pp'(A_0 + \|K\|_{\mathrm{H\ddot{o}r}} + \|K\|_{\mathrm{H\ddot{o}r}^*}),\end{aligned}$$

where we estimated

$$\Big(\frac{1}{p-1}\Big)^{1/p} \leqslant \Big(\frac{p}{p-1}\Big)^{1/p} = (p')^{1/p} \leqslant p'.$$

The conclusion agrees with the claim, after redefining c_d.

The case of a cube or a quadrant in place of $\mathbb{R}^d$ is immediate, since both (1) and (2) of the theorem, which we used above, were already proved in these cases as well. □

11.2.c The action of singular integrals on L^∞

The goal of this section is to establish the following theorem, in which indistinguishability of $\mathrm{BMO}(\mathbb{R}^d;X)$ functions only differing by an additive constant manifests itself.

Theorem 11.2.9. *Let X and Y be Banach spaces, $p_0 \in (1,\infty)$, and $T \in \mathscr{L}(L^{p_0}(\mathbb{R}^d;X), L^{p_0}(\mathbb{R}^d;Y))$ be an operator with a dual Hörmander kernel K. Suppose, moreover, at least one of the following:*

(1) *Y does not contain a copy of c_0, or*
(2) *K is a dual operator-Hörmander kernel.*

Then there is an operator $\widetilde{T} \in \mathscr{L}(L^\infty(\mathbb{R}^d; X), \mathrm{BMO}^{p_0}(\mathbb{R}^d; Y)/Y)$ of norm at most $(c_d A_0 + \|K\|_{\mathrm{H\ddot{o}r}^*})$ such that

(a) *for all $f \in L^{p_0}(\mathbb{R}^d; X) \cap L^\infty(\mathbb{R}^d; X)$, we have $Tf \equiv \widetilde{T}f$ modulo constants,*
(b) *for all $f \in L^\infty(\mathbb{R}^d; X)$ and $g \in L^\infty_{c,0}(\mathbb{R}^d; Y^*)$ (compactly supported with vanishing integral), we have*

$$\langle \widetilde{T}f, g\rangle = \lim_{n\to\infty} \langle T(\mathbf{1}_{E_n} f), g\rangle \tag{11.18}$$

for any bounded measurable sets $E_n \subseteq \mathbb{R}^d$ such that $\operatorname{dist}(\complement E_n, 0) \to \infty$.

Remark 11.2.10.

(1) By the John–Nirenberg inequality, the target space $\mathrm{BMO}^p(\mathbb{R}^d; Y)/Y$ of $\widetilde{T}$ is independent of the value of $p \in [1, \infty)$; however, the estimate for the operator norm need not be, and we specifically state it with $p = p_0$.
(2) The left-hand side of (11.18) could be more pedantically written as "$\langle h, g\rangle$, where $h \in [\widetilde{T}f]$ is arbitrary": the vanishing integral of g guarantees that this expression is independent of the choice of h.
(3) The boundedness requirement on T in Theorem 11.2.9 may seem stronger than in Theorem 11.2.5(2) (where it was only assumed that T maps boundedly into the larger space $L^{p_0,\infty}(\mathbb{R}^d; Y)$ and for some p_0 in the larger range $[1, \infty]$), but this is only superficial, as we can always arrange ourselves to be in the situation of Theorem 11.2.9 even under the apparently weaker boundedness hypothesis:
First, if $p_0 = \infty$, there is nothing to prove, as we can simply take $\widetilde{T} = T$, which already maps into $L^\infty(\mathbb{R}^d; Y) \subseteq \mathrm{BMO}(\mathbb{R}^d; Y)$. If, on the other hand, $p_0 \in [1, \infty)$, Theorem 11.2.5(2a) guarantees that $T \in \mathscr{L}(L^p(\mathbb{R}^d; X), L^p(\mathbb{R}^d; Y))$ for all $p \in (p_0, \infty) \subseteq (1, \infty)$, and choosing one such p as a new p_0, we are in the situation assumed in Theorem 11.2.9.

To deal with the equivalence classes modulo additive constants, it is convenient to make the following preliminary observation:

Lemma 11.2.11. *Let S be a set and X be a Banach spaces. There is a bijective linear correspondence between the following two classes of objects:*

(1) *equivalence classes $[b]$ of functions $b : S \to X$, where*

$$[b] = \{f : S \to X; s \mapsto f(s) - b(s) \text{ is constant on } S\},$$

(2) *functions $\varDelta : S \times S \to X$ with the property*

$$\varDelta(s,t) + \varDelta(t,u) = \varDelta(s,u) \qquad \forall\ s,t,u \in S. \tag{11.19}$$

This correspondence is realised by

$$[s \mapsto b(s)] \qquad \leftrightarrow \qquad (s,t) \mapsto \Delta(s,t) = b(s) - b(t).$$

Proof. To every $[b]$, we associate $\Delta(s,t) := b(s) - b(t)$, and it is clear that this is independent of the chosen representative of the equivalence class.

For the other direction, it is convenient to first record some additional algebraic relations automatically satisfied by Δ. Taking $s = t = u$, we have $2\Delta(s,s) = \Delta(s,s)$, and hence $\Delta(s,s) = 0$ for all $s \in S$. Then taking $u = s$, we have $\Delta(s,t) + \Delta(t,s) = \Delta(s,s) = 0$, and hence $\Delta(s,t) = -\Delta(t,s)$ for all $s,t \in S$. Now, to every Δ, we associate $[\Delta(\cdot,t)]$, where each $t \in S$ defines the same equivalence class. Indeed,

$$\Delta(s,t) - \Delta(s,u) = \Delta(u,s) + \Delta(s,t) = \Delta(u,t),$$

which is constant as a function of $s \in S$. It is immediate to verify that these operations sending $[b]$ to Δ, and Δ to $[b]$, are inverses of each other. □

For $S \subseteq \mathbb{R}^d$ (where we are mainly interested in the case that $S = \mathbb{R}^d$ or one of its dyadic quadrants), we define

$$\widetilde{\mathrm{BMO}}^p(S;X) := \Big\{ \Delta \in L^1_{\mathrm{loc}}(S \times S; X) \text{ with property (11.19)}, \\ \|\Delta\|_{*,p} := \sup_{\substack{Q \subseteq S \\ \text{cube}}} \Big(\fint_{Q \times Q} \|\Delta(s,t)\|_X^p \,\mathrm{d}s\,\mathrm{d}t \Big)^{1/p} < \infty \Big\}$$

and $\widetilde{\mathrm{BMO}}^p_{\mathscr{D}}(S;X)$ by replacing "$Q \subseteq S$ cube" by "$Q \in \mathscr{D}(S)$".

Lemma 11.2.12. *Under the correspondence $[b] \leftrightarrow \Delta$ of functions as in Lemma 11.2.11, we have the correspondence of spaces:*

$$\mathrm{BMO}^p(\mathbb{R}^d;X)/X \simeq \widetilde{\mathrm{BMO}}^p(\mathbb{R}^d;X),$$

with the equivalence of norms

$$\|b\|_{\mathrm{BMO}^p(\mathbb{R}^d;X)} \leqslant \|\Delta\|_{*,p} \leqslant 2\|b\|_{\mathrm{BMO}^p(\mathbb{R}^d;X)}. \tag{11.20}$$

The similar correspondence is valid with any of the dyadic quadrants S in place of $\mathbb{R}^d$ and the dyadic $\mathrm{BMO}^p_{\mathscr{D}}$ (both with and without tilde) in place of BMO^p.

Proof. For each cube $Q \subseteq \mathbb{R}^d$, we have

$$\begin{aligned} \inf_{c \in X} \Big(\fint_Q \|b(s) - c\|_X^p \,\mathrm{d}s \Big)^{1/p} &\leqslant \Big(\fint_Q \Big\| b(s) - \fint_Q b(t)\,\mathrm{d}t \Big\|_X^p \,\mathrm{d}s \Big)^{1/p} \\ &\leqslant \Big(\fint_Q \fint_Q \|b(s) - b(t)\|_X^p \,\mathrm{d}s\,\mathrm{d}t \Big)^{1/p} \\ &= \Big(\fint_Q \fint_Q \Big\| (b(s) - c) - (b(t) - c) \Big\|_X^p \,\mathrm{d}s\,\mathrm{d}t \Big)^{1/p} \leqslant 2 \Big(\fint_Q \|b(s) - c\|_X^p \,\mathrm{d}s \Big)^{1/p} \end{aligned}$$

and taking the infimum over $c \in X$ on the right, and then the supremum over all cubes $Q \subseteq \mathbb{R}^d$ of the whole chain, we derive (11.20). The dyadic case follows by taking the supremum over $Q \in \mathscr{D}(S)$ instead. $\square$

In view of Lemma 11.2.12, the construction of an extension

$$\widetilde{T} \in \mathscr{L}(L^\infty(\mathbb{R}^d; X), \mathrm{BMO}(\mathbb{R}^d; Y))$$

of $T \in \mathscr{L}(L^{p_0}(\mathbb{R}^d; X), L^{p_0}(\mathbb{R}^d; Y))$ is reduced to the construction of $\Delta_T \in \mathscr{L}(L^\infty(\mathbb{R}^d; X), \widetilde{\mathrm{BMO}}(\mathbb{R}^d; X))$ such that

$$\Delta_T f(s,u) = Tf(s) - Tf(u) \qquad \forall\ f \in L^{p_0}(\mathbb{R}^d; X) \cap L^\infty(\mathbb{R}^d; X).$$

It is convenient to define this as *a priori* mapping into Y^{**}-valued functions:

Lemma 11.2.13. *For $f \in L^\infty(\mathbb{R}^d; X)$, $y^* \in Y^*$ and $s, u \in \mathbb{R}^d$, the expression*

$$\begin{aligned} \langle y^*, \Delta_T f(s,u)\rangle := \langle T(\mathbf{1}_B f)(s) - T(\mathbf{1}_B f)(u), y^*\rangle \\ + \int_{\complement B} \langle [K(s,t) - K(u,t)] f(t), y^*\rangle \,\mathrm{d}t, \end{aligned} \tag{11.21}$$

is independent of the auxiliary ball B with $s, u \in \frac{1}{2}B$.

Proof. With f, y^*, s, u fixed, let us temporarily denote the expression of interest by $\delta(B)$. If B and B' are two such balls, we can choose a third such ball B'' that contains both of them. So it is enough to prove the equality $\delta(B) = \delta(B')$ for balls $B \subseteq B'$, hence $\complement B' \subseteq \complement B$. Note that $(\complement B) \setminus (\complement B') = B' \setminus B$. Then

$$\begin{aligned} \delta(B') - \delta(B) = \langle T(\mathbf{1}_{B'\setminus B} f)(s) - T(\mathbf{1}_{B'\setminus B} f)(u), y^*\rangle \\ + \Big(\int_{\complement B'} - \int_{\complement B}\Big) \langle [K(s,t) - K(u,t)] f(t), y^*\rangle \,\mathrm{d}t, \end{aligned}$$

where the difference of the integrals is

$$\int_{B'\setminus B} \langle [K(u,t) - K(s,t)] f(t), y^*\rangle \,\mathrm{d}t = \langle T(\mathbf{1}_{B'\setminus B} f)(u) - T(\mathbf{1}_{B'\setminus B} f)(s), y^*\rangle,$$

which exactly cancels out the first term in the formula of $\delta(B') - \delta(B)$. $\square$

Let us then check how Δ_T compares with the original T on the intersection of their domains of definition:

Lemma 11.2.14. *If $f \in L^{p_0}(\mathbb{R}^d; X) \cap L^\infty(\mathbb{R}^d; X)$, then*

$$\Delta_T f(s,u) = Tf(s) - Tf(u).$$

Proof. Under the stated assumptions, Lemma 11.2.7 guarantees that

$$\int_{\complement B} \langle [K(s,t) - K(u,t)]f(t), y^* \rangle \, dt = \langle T(\mathbf{1}_{\complement B} f)(s) - T(\mathbf{1}_{\complement B} f)(u), y^* \rangle$$

for almost all $s, u \in \frac{1}{2}B$ and all $y^* \in Y^*$, and hence

$$\begin{aligned} \langle y^*, \Delta_T f(s,u) \rangle &= \langle T(\mathbf{1}_B f)(s) - T(\mathbf{1}_B f)(u), y^* \rangle \\ &\quad + \langle T(\mathbf{1}_{\complement B} f)(s) - T(\mathbf{1}_{\complement B} f)(u), y^* \rangle = \langle Tf(s) - Tf(u), y^* \rangle \end{aligned}$$

Since this is true for all $y^* \in Y^*$, the claimed identity follows. □

To justify that the *a priori* Y^{**}-valued function $\Delta_T f$ actually takes values in Y, we invoke the following corollary of the Bessaga–Pełczyński Theorem 1.2.40. This is where the condition $c_0 \not\subseteq Y$ comes to use:

Proposition 11.2.15. *Let Y be a Banach space that does not contain an isomorphic copy of c_0. If $y_j \in Y$ satisfy*

$$\sum_{j=1}^{\infty} |\langle y_j, y^* \rangle| < \infty \qquad \forall y^* \in Y^*, \tag{11.22}$$

then the series $\sum_{j=1}^{\infty} y_j$ converges in norm in Y.

Proof. Let us first note that the condition (11.22) says that $y^* \mapsto (\langle y_j, y^* \rangle)_{j=1}^{\infty}$ defines a linear operator from Y^* into ℓ^1, which is easily seen to be closed, and therefore bounded. Thus the closed graph theorem improves (11.22) to

$$\sum_{j=1}^{\infty} |\langle y_j, y^* \rangle| \leqslant C \|y^*\|_{Y^*} \qquad \forall y^* \in Y^*.$$

If $\sum_{j=1}^{\infty} y_j$ does not converge, then the partial sums $\sum_{j=1}^{n} y_j$ fail the Cauchy criterion, and hence we can find $m_1 < n_1 < m_2 < \ldots$ and $\delta > 0$ such that

$$\|v_k\|_Y \geqslant \delta > 0, \qquad v_k := \sum_{j=m_k}^{n_k} y_j. \tag{11.23}$$

On the other hand, for any $\epsilon_k = \pm 1$ and any $y^* \in Y^*$, we also have

$$\Big| \Big\langle \sum_{k=1}^{K} \epsilon_k v_k, y^* \Big\rangle \Big| \leqslant \sum_{k=1}^{K} |\langle v_k, y^* \rangle| \leqslant \sum_{j=1}^{\infty} |\langle y_j, y^* \rangle| \leqslant C \|y^*\|_{Y^*};$$

hence

$$\Big\| \sum_{k=1}^{K} \epsilon_k v_k \Big\|_Y \leqslant C. \tag{11.24}$$

But the two conditions (11.23) and (11.24) are precisely those of the Bessaga–Pełczyński Theorem 1.2.40 that guarantee the containment of an isomorphic copy of c_0 in $\overline{\operatorname{span}}(v_k)_{k=1}^{\infty} \subseteq Y$. This contradicts the assumption on Y. □

After this interlude, we return to the main topic of this section:

Lemma 11.2.16. *Under the assumptions of Theorem 11.2.9, for every* $f \in L^\infty(\mathbb{R}^d; X)$*, the function* $\Delta_T f$ *in* (11.21) *is well defined, takes values in* $Y \subseteq Y^{**}$*, is strongly measurable, and satisfies*

$$\|\Delta_T f\|_{L^{p_0}(Q\times Q;Y)} \leqslant (c_d A_0 + \|K\|_{\text{Hör}^*})\|f\|_\infty |Q|^{2/p_0}$$

for every cube $Q \subseteq \mathbb{R}^d$.

Proof. Let B be the ball concentric with Q and with twice the diameter of Q; hence $Q \subseteq \frac{1}{2}B$. From the assumption that $T \in \mathscr{L}(L^{p_0}(\mathbb{R}^d; X), L^{p_0}(\mathbb{R}^d; Y))$ and $f \in L^\infty(\mathbb{R}^d; X)$, it is immediate that $T(\mathbf{1}_B f) \in L^{p_0}(\mathbb{R}^d; Y)$ and

$$\|T(\mathbf{1}_B f)\|_{p_0} \leqslant A_0 \|\mathbf{1}_B f\|_{p_0} \leqslant A_0 |B|^{1/p_0} \|f\|_\infty,$$

so that

$$\begin{aligned}&\|(s,u) \mapsto T(\mathbf{1}_B f)(s) - T(\mathbf{1}_B f)(u)\|_{L^{p_0}(Q\times Q;Y)}\\ &\leqslant 2|Q|^{1/p_0}\|T(\mathbf{1}_B f)\|_{L^{p_0}(\mathbb{R}^d;Y)} \leqslant c_d A_0 |Q|^{2/p_0}\|f\|_\infty,\end{aligned}$$

The more delicate matter is the integral in (11.21). Certainly this integral exists, since the dual Hörmander condition guarantees that $[K(s,t)^* - K(u,t)^*]y^*$ is jointly measurable and belongs to $L^1(\complement B, \mathrm{d}t; Y^*)$ uniformly in $(s,u) \in Q$, while $f \in L^\infty(\mathbb{R}^d; Y)$ by assumption. An immediate estimate with the dual Hörmander condition shows that this integral is bounded by $\|K\|_{\text{Hör}^*}\|f\|_\infty \|y^*\|_{Y^*}$, uniformly in $x \in Q$, and hence defines a Y^{**}-valued function $h(s,u)$ with the pointwise bound

$$\|h(s,u)\|_{Y^{**}} \leqslant \|K\|_{\text{Hör}^*}\|f\|_\infty. \tag{11.25}$$

What remains is to justify the Y-valuedness and the strong measurability of this weakly defined function. To this end, we write $f_n = \mathbf{1}_{2^n B\setminus 2^{n-1}B} f$, so that $\mathbf{1}_{\complement B} f = \sum_{n\geqslant 1} f_n$, say pointwise. Since each $f_n \in L^{p_0}(\mathbb{R}^d; X) \cap L^\infty(\mathbb{R}^d; X)$, we can apply Lemma 11.2.7 to see that

$$\begin{aligned}&\int_{\complement B} \langle [K(s,t) - K(u,t)] f_n(t), y^*\rangle \,\mathrm{d}t\\ &= \langle T f_n(s) - T f_n(u), y^*\rangle =: \langle h_n(s,u), y^*\rangle\end{aligned}$$

is the pairing of y^* with a Y-valued, strongly measurable function $h_n(s,u)$.

If we denote by h the *a priori* Y^{**}-valued function defined by

$$\langle y^*, h(s,u)\rangle := \int_{\complement B} \langle [K(s,t) - K(u,t)] f(t), y^*\rangle \,\mathrm{d}t,$$

then

$$\langle y^*, h(s,u)\rangle = \sum_{n=1}^{\infty} \langle h_n(s,u), y^*\rangle \qquad \forall y^* \in Y^*. \tag{11.26}$$

If K satisfies the dual operator-Hörmander condition, then

$$\sum_{n=1}^{\infty} \|h_n(s,u)\| \leqslant \int_{\complement B} \|K(s,t) - K(u,t)\| \|f\|_\infty \,\mathrm{d}t \leqslant 2\|K\|_{\mathrm{Hör}^*_{op}} \|f\|_\infty,$$

so the series $\sum_{n=1}^{\infty} h_n(s,u)$ converges absolutely, and hence in norm. Under the mere dual Hörmander condition, but with the assumption that Y does not contain an isomorphic copy of c_0, the needed norm convergence of $\sum_{n=1}^{\infty} h_n(s,u)$ follows by Proposition 11.2.15 and the bound

$$\begin{aligned}\sum_{n=1}^{\infty} |\langle h_n(s,u), y^*\rangle| &\leqslant \int_{\complement B} |\langle f(y), [K(s,t)^* - K(u,t)^*]y^*\rangle| \,\mathrm{d}t \\ &\leqslant 2\|K\|_{\mathrm{Hör}^*} \|y^*\|_{Y^*} \|f\|_\infty < \infty \qquad \forall y^* \in Y^*.\end{aligned}$$

In both cases, by (11.26), the limit of $\sum_{n=1}^{\infty} h_n(s,u)$ must be $h(s,u)$. Thus, as a pointwise limit of Y-valued strongly measurable functions, h itself must be both Y-valued and strongly measurable. Once these qualitative properties are verified, the quantitative $L^{p_0}(Q \times Q; Y)$ estimate is immediate by integrating over $Q \times Q$ the already observed pointwise bound (11.25). □

Now we are prepared to complete:

Proof of Theorem 11.2.9. The operator $\Delta_T : L^\infty(\mathbb{R}^d; X) \to L^{p_0}_{\mathrm{loc}}(\mathbb{R}^{d+d}; Y)$ is well defined by Lemma 11.2.16 and satisfies

$$\|\Delta_T f\|_{*,p_0} \leqslant (c_d A_0 + \|K\|_{\mathrm{Hör}^*}) \|f\|_\infty$$

for the norm defined in Lemma 11.2.12. By Lemma 11.2.12, we obtain a bounded linear operator $\widetilde{T} \in \mathscr{L}(L^\infty(\mathbb{R}^d; X), \mathrm{BMO}^{p_0}(\mathbb{R}^d; Y)/Y)$, with the same norm bound, by setting

$$\widetilde{T} f := [\Delta_T f(\cdot, u)] \quad \text{(the equivalence class modulo constants)}, \tag{11.27}$$

where the choice of $u \in \mathbb{R}^d$ is irrelevant. By Lemma 11.2.14, we have $\Delta_T f(s,u) = Tf(s) - Tf(u)$ for $f \in L^{p_0}(\mathbb{R}^d; X) \cap L^\infty(\mathbb{R}^d; X)$, and hence $\widetilde{T} f = [Tf]$ in this case. This completes the proof of Claim (a) of the theorem.

As for Claim (b), we note that pairing a $g \in L^\infty_{c,0}(\mathbb{R}^d; Y^*)$ with an element of $\mathrm{BMO}^{p_0}(\mathbb{R}^d; Y)/Y$ is well defined, and independent of the representative of the equivalence class, since the integral of g against any constant $c \in Y$ will vanish. By the assumptions on E_n, we can choose balls $B_n := B(0, r_n) := B(0, \mathrm{dist}(\complement E_n, 0)) \subseteq E_n$ with $r_n \to \infty$. Let n be so large that $\operatorname{supp} g \subseteq \frac{1}{2} B_n$. Since $\widetilde{T}$ is linear, we have

$$\langle \widetilde{T} f, g\rangle = \langle \widetilde{T}(\mathbf{1}_{E_n} f), g\rangle + \langle \widetilde{T}(\mathbf{1}_{\complement E_n} f), g\rangle =: I_n + II_n.$$

By Claim (a), which we already proved, we have

$$I_n = \langle T(\mathbf{1}_{E_n} f), g\rangle.$$

For II_n, recalling the construction of $\widetilde{T}$ from (11.27) with $u = 0$, and then the definition of $\Delta_T f(s,u)$ from (11.21) with $B = B_n$, we have

$$\begin{aligned}
II_n &= \langle \Delta_T(\mathbf{1}_{\complement E_n} f)(\cdot, 0), g\rangle \\
&= \int_{\mathbb{R}^d} \langle \Delta_T(\mathbf{1}_{\complement E_n} f)(s, 0), g(s)\rangle \,\mathrm{d}s \\
&= \int_{\mathbb{R}^d} \langle T(\mathbf{1}_{B_n}\mathbf{1}_{\complement E_n} f)(s) - T(\mathbf{1}_{B_n}\mathbf{1}_{\complement E_n} f)(0), g(s)\rangle \,\mathrm{d}s \\
&\qquad + \int_{\mathbb{R}^d}\int_{\complement B_n} \langle [K(s,t) - K(0,t)](\mathbf{1}_{\complement E_n} f)(t), g(s)\rangle \,\mathrm{d}t\,\mathrm{d}s \\
&=: III_n + IV_n = 0 + IV_n,
\end{aligned}$$

since $B_n \subseteq E_n$. Finally,

$$|IV_n| \leqslant \|f\|_{L^\infty(\mathbb{R}^d;X)} \int_{\mathbb{R}^d}\int_{\complement B_n} \|[K(s,t) - K(0,t)]^* g(s)\|_{X^*} \,\mathrm{d}t\,\mathrm{d}s.$$

For every fixed $s \in \operatorname{supp} g \subseteq \frac{1}{2}B_n$, the inner integral is bounded by $\|K\|_{\text{Hör}^*}\|g(s)\|_{Y^*}$, and, as $n \to \infty$, it converges to 0 by dominated convergence; the same is also true for $s \notin \operatorname{supp} g$, since both the integral and the upper bound vanish in this case. Thus also the double integral converges to 0 by another application of dominated convergence.

Altogether, we have seen that

$$\langle \widetilde{T} f, g\rangle - \langle T(\mathbf{1}_{E_n} f), g\rangle = II_n = IV_n \to 0 \quad \text{as} \quad n \to \infty,$$

which concludes the proof of the remaining Claim (b) of Theorem 11.2.9. □

11.3 Calderón–Zygmund operators and sparse bounds

The goal of this section is to derive a powerful pointwise domination of Calderón–Zygmund operators by simple averaging operators over sparse families of dyadic cubes; from this domination, norm estimates for Calderón–Zygmund operators in various different spaces follow almost instantly.

The assumptions that we have to make on the kernel of the operator in order to carry out this programme are somewhat stronger than those needed for the L^p extrapolation of the previous section:

Definition 11.3.1 (Calderón–Zygmund kernel). *Let Z be a Banach space, and $K : \dot{\mathbb{R}}^{2d} \to Z$. We define the quantities*

$$c_K := \sup\{|s-t|^d \cdot \|K(s,t)\| : (s,t) \in \dot{\mathbb{R}}^{2d}\},$$

and, for $u \in [0, \frac{1}{2}]$,

$$\omega_K^1(u) := \sup\Big\{|s-t|^d\|K(s,t) - K(s',t)\| : |s-s'| \leqslant u|s-t|\Big\},$$
$$\omega_K^2(u) := \sup\Big\{|s-t|^d\|K(s,t) - K(s,t')\| : |t-t'| \leqslant u|s-t|\Big\},$$
$$\omega_K(u) := \max_{i=1,2} \omega_K^i(u).$$

For $K \in C^1(\dot{\mathbb{R}}^{2d}; Z)$, *let further*

$$c_K^1 := \sup\{|s-t|^{d+1}\|\nabla_s K(s,t)\| : s \neq t\},$$
$$c_K^2 := \sup\{|s-t|^{d+1}\|\nabla_t K(s,t)\| : s \neq t\}.$$

We say that a kernel K *with* $c_K < \infty$ *is*

(i) *a* standard kernel *if* $\omega_K(u) \leqslant c_\delta u^\delta$ *for some* $\delta \in (0,1]$,
(ii) *a* Dini kernel *if* ω_K *satisfies the* Dini condition

$$\|\omega_K\|_{\text{Dini}} := \int_0^{1/2} \omega_K(u) \frac{\mathrm{d}u}{u} < \infty,$$

(iii) *a* C^1-Calderón–Zygmund kernel *if* $K \in C^1(\dot{\mathbb{R}}^{2d}; Z)$ *and* $c_K^i < \infty$, $i = 1,2$,
(iv) *an* ω-Calderón–Zygmund kernel *if* $\omega_K \leqslant \omega$,
(v) *an* (ω_1, ω_2)-Calderón–Zygmund kernel *if* $\omega_K^i \leqslant \omega_i$, $i = 1,2$.

We also apply these notions to kernels K *defined on* $\{(s,t) : s,t \in S, s \neq t\}$, *where* S *is either a cube or a quadrant of* $\mathbb{R}^d$*; in this case, each supremum above is taken only over the respective domain of definition.*

It is immediate that a standard kernel is a Dini kernel with $\|\omega\|_{\text{Dini}} \leqslant \delta^{-1} c_\delta$.

Remark 11.3.2. For a convolution kernel $K(x,y) = k(x-y)$, we have

$$c_K = \sup\{|s|^d\|k(s)\| : s \neq 0\},$$
$$c_K^i = \sup\{|s|^{d+1}\|\nabla k(s)\| : s \neq 0\}, \qquad i = 1,2,$$
$$\omega_K(u) = \omega_K^i(u) = \sup\{|s|^d\|k(s) - k(s-t)\| : |t| \leqslant u|s|\}, \qquad i = 1,2,$$

with no difference between $i = 1$ and $i = 2$ in the last two formulas.

Lemma 11.3.3.

$$\omega_K^i(\frac{1}{2}) \leqslant (1+2^d)c_K, \qquad \sum_{k=2}^\infty \omega_K^i(2^{-k}) \leqslant \frac{1}{\log 2}\|\omega_K^i\|_{\text{Dini}}.$$

Proof. If $|t-t'| \leqslant \frac{1}{2}|s-t|$, then $|s-t| \leqslant |s-t'| + |t-t'| \leqslant |s-t'| + \frac{1}{2}|s-t|$, and hence $|s-t| \leqslant 2|s-t'|$. Thus

$$|s-t|^d \|K(s,t) - K(s,t')\| \leqslant c_K + 2^d |s-t'|^d \|K(s,t')\| \leqslant (1+2^d)c_K,$$

and hence $\omega_K^2(\frac{1}{2}) \leqslant (1+2^d)c_K$. The proof for ω_K^1 is entirely similar.

If ω is increasing, which is obviously the case with $\omega = \omega_K^i$, it follows that

$$\omega(2^{-k-1}) \leqslant \omega(u), \quad u \in (2^{-k-1}, 2^{-k}),$$

hence

$$\omega(2^{-k-1}) \log 2 \leqslant \int_{2^{-k-1}}^{2^{-k}} \omega(u) \frac{\mathrm{d}u}{u},$$

and thus

$$\sum_{k=2}^{\infty} \omega(2^{-k}) = \sum_{k=1}^{\infty} \omega(2^{-1-k}) \leqslant \frac{1}{\log 2} \int_0^{1/2} \omega(u) \frac{\mathrm{d}u}{u}.$$

□

Lemma 11.3.4. *For $K : \dot{\mathbb{R}}^{2d} \to Z = \mathscr{L}(X,Y)$, we have:*

(1) *If $\|\omega_K^1\|_{\mathrm{Dini}} < \infty$, then K is a dual operator-Hörmander kernel, and*

$$\|K\|_{\mathrm{Hör}_{\mathrm{op}}^*} \leqslant \sigma_{d-1} \|\omega_K^1\|_{\mathrm{Dini}}.$$

(2) *If $\|\omega_K^2\|_{\mathrm{Dini}} < \infty$, then K is an operator-Hörmander kernel, and*

$$\|K\|_{\mathrm{Hör}_{\mathrm{op}}} \leqslant \sigma_{d-1} \|\omega_K^2\|_{\mathrm{Dini}}.$$

(3) *Every standard kernel is a Dini kernel with*

$$\|\omega\|_{\mathrm{Dini}} \leqslant 2^{d+1} \frac{c_K}{\delta} \Big(1 + \log_+ \frac{c_\delta}{2^{d+1} c_K}\Big).$$

(4) *Every C^1-Calderón–Zygmund kernel is a standard kernel with*

$$\omega_K^i(u) \leqslant 2^{d+1} c_K^i \cdot u$$

and a Dini kernel with

$$\|\omega_K^i\|_{\mathrm{Dini}} \leqslant 2^{d+1} c_K \Big(1 + \log_+ \frac{c_K^i}{c_K}\Big).$$

Here σ_{d-1} is the $(d-1)$-dimensional measure of the unit sphere in $\mathbb{R}^d$. The same conclusions hold with $\dot{\mathbb{R}}^{2d}$ replaced by $\dot{S}^2 := \{(s,t) : s,t \in S, s \neq t\}$, where S is either a cube or a quadrant of $\mathbb{R}^d$, and both the Dini and the Hörmander conditions are modified by restricting the variables to the respective domain of definition.

Note that, in concrete situations, the constants c_δ or c_K^i are often much larger than c_K. The point of the bounds in parts (3) and (4) is that these larger constants contribute to the Dini bounds only logarithmically.

Proof. We will first prove (2); the proof of (1) is analogous.

$$\begin{aligned}
&\int_{|x-y|>2|y-y'|} \|K(x,y)-K(x,y')\|\,\mathrm{d}x \\
&\leqslant \int_{|x-y|>2|y-y'|} \omega_K^2\Big(\frac{|y-y'|}{|x-y|}\Big)\frac{1}{|x-y|^d}\,\mathrm{d}x \\
&= \sigma_{d-1}\int_{2|y-y'|}^{\infty} \omega_K^2\Big(\frac{|y-y'|}{r}\Big)\frac{\mathrm{d}r}{r} = \sigma_{d-1}\int_0^{\frac12}\omega_K^2(t)d\frac{\mathrm{d}t}{t} = \sigma_{d-1}\|\omega_K^2\|_{\mathrm{Dini}}
\end{aligned}$$

and this is the required bound.

For the remaining claims, we begin with the following observation. For $|x-x'|\leqslant u|x-y|$ and $v\in[0,1]$, we have

$$|x+v(x'-x)-y| \geqslant |x-y|-|x'-x| \geqslant (1-u)|x-y| \geqslant \frac12|x-y|.$$

This implies the crude bound

$$\|K(x',y)-K(x,y)\| \leqslant \frac{c_K}{|x'-y|^d}+\frac{c_K}{|x-y|^d} \leqslant (2^d+1)\frac{c_K}{|x-y|^d} \leqslant \frac{2^{d+1}c_K}{|x-y|^d}.$$

This shows that $\omega_K^i(u)\leqslant 2^{d+1}c_K$ for all $u\in[0,\frac12]$ and $i=1$, and the proof for $i=2$ is similar.

(3): By the previous observation, denoting $c_0 := 2^{d+1}c_K$, the standard estimate $\omega(u)\leqslant c_\delta u^\delta$ bootstraps to $\omega(u)\leqslant\min\{c_0,c_\delta u^\delta\}$. If $c_0\leqslant c_\delta$, then

$$\begin{aligned}
\|\omega\|_{\mathrm{Dini}} &\leqslant \int_0^{(c_0/c_\delta)^{1/\delta}} c_\delta u^\delta\frac{\mathrm{d}u}{u} + \int_{(c_0/c_\delta)^{1/\delta}}^1 c_0\frac{\mathrm{d}u}{u} \\
&= \frac{c_\delta}{\delta}\frac{c_0}{c_\delta} + c_0\log\Big(\frac{c_\delta}{c_0}\Big)^{1/\delta} = \frac{c_0}{\delta}\Big(1+\log\frac{c_\delta}{c_0}\Big).
\end{aligned}$$

If $c_0>c_\delta$, we simply estimate $\|\omega\|_{\mathrm{Dini}}\leqslant\int_0^1 c_\delta u^{\delta-1}\,\mathrm{d}u = c_\delta/\delta\leqslant c_0/\delta$. Hence, in each case, we have

$$\|\omega\|_{\mathrm{Dini}} \leqslant \frac{c_0}{\delta}\Big(1+\log_+\frac{c_\delta}{c_0}\Big).$$

We will prove (4) in the case $i=1$, the case of $i=2$ is analogous. Hence

$$\begin{aligned}\|K(x',y)-K(x,y)\| &= \Big\|\int_{v=0}^{1} K(x+u(x'-x),y)\Big\| \\ &= \Big\|\int_0^1 (x'-x)\cdot\nabla_x K(x+v(x'-x),y)\,\mathrm{d}v\Big\| \\ &\leqslant |x'-x|\int_0^1 \frac{c_K^1}{|x+v(x'-x)-y|^{d+1}}\,\mathrm{d}v \\ &\leqslant u|x-y|\int_0^1 \frac{c_K^1}{(\frac{1}{2}|x-y|)^{d+1}}\,\mathrm{d}v = u\frac{2^{d+1}c_K^1}{|x-y|^d}.\end{aligned}$$

This is the claimed standard estimate, and the Dini estimate follows from part (3) with $\delta = 1$ and $c_\delta = 2^{d+1}c_K^i$.

The version with a cube or a quadrant follows with the same argument by simply restricting all the variables and the integrals to the relevant domain of definition. □

In particular, Dini kernels satisfy both Hörmander and dual Hörmander conditions, and hence all the results of the previous section apply to them:

Corollary 11.3.5 (Calderón–Zygmund). *Let X and Y be Banach spaces and $p_0 \in [1,\infty]$. Let $T \in \mathscr{L}(L^{p_0}(\mathbb{R}^d;X), L^{p_0,\infty}(\mathbb{R}^d;Y))$ be an operator with a Calderón–Zygmund kernel K. Then all conclusions of Theorem 11.2.5 hold with $\|K\|_{\text{Hör}}$ replaced by $\|\omega_K^2\|_{\text{Dini}}$ and $\|K\|_{\text{Hör}^*}$ by $\|\omega_K^1\|_{\text{Dini}}$ in the estimates.*

Proof. This follows at once from Theorem 11.2.5, where the same conclusions are deduced for Hörmander and/or dual Hörmander kernels K, and Lemma 11.3.4, where these assumptions are verified for under the Dini conditions. □

11.3.a An abstract domination theorem

We will first present an abstract form of the domination theorem, i.e., we postulate the relevant properties of the operator needed to carry out the proof, and only then return to the question of checking these properties in the concrete case of Calderón–Zygmund operators.

We will formulate the theorem for *positive sub-linear* operators mapping a linear space of X-valued functions into $L^0(\mathbb{R}^d;\mathbb{R}_+)$. By this we mean that for all functions f and g we have that $Tf \geqslant 0$ is a non-negative function, $T(\alpha f) = |\alpha| Tf$ for constants α, and $T(f+g) \leqslant Tf + Tg$ for all f, g in the domain of T. Note that if T is a *linear* operator mapping into $L^0(\mathbb{R}^d;Y)$, then the operator $f \mapsto \|Tf(\cdot)\|_Y$ is a positive sub-linear one, and this is the way that such operators will be naturally covered by the theory.

Theorem 11.3.6 (Abstract sparse domination). *Let X be a Banach space, let T be a positive sub-linear operator from $L^1(\mathbb{R}^d;X)$ into $L^0(\mathbb{R}^d;\mathbb{R}_+)$, and consider the associated maximal operator*

$$M_T^{\#} f(x) := \sup_{Q\ni x}\sup_{y,z\in Q} |T(\mathbf{1}_{\complement(5Q)}f)(y) - T(\mathbf{1}_{\complement(5Q)}f)(z)|. \tag{11.28}$$

Suppose that both T and $M_T^{\#}$ are bounded from $L^1(\mathbb{R}^d;X)$ to $L^{1,\infty}(\mathbb{R}^d)$. Then for every boundedly supported $f \in L^1(\mathbb{R}^d)$ and $\varepsilon \in (0,1)$, there is a $(1-\varepsilon)$-sparse family $\mathscr{S}$ of dyadic cubes such that, almost everywhere,

$$Tf \leqslant \frac{8 \cdot 10^d \cdot c_T}{\varepsilon} \sum_{S \in \mathscr{S}} \mathbf{1}_S \fint_{5S} \|f\|,$$

where

$$c_T := \|T\|_{1\to 1,\infty} + \|M_T^{\#}\|_{1\to 1,\infty}. \tag{11.29}$$

The heart of Theorem 11.3.6 is contained in the following lemma:

Lemma 11.3.7. *Under the assumptions of Theorem 11.3.6, for any $f \in L^1_{\text{loc}}(\mathbb{R}^d;X)$, any cube Q_0 and $\varepsilon \in (0,1)$, there are disjoint subcubes $Q_j \in \mathscr{D}(Q_0)$ such that*

$$\sum_j |Q_j| \leqslant \varepsilon |Q_0| \tag{11.30}$$

and, almost everywhere,

$$\mathbf{1}_{Q_0} T(\mathbf{1}_{5Q_0} f) \leqslant \frac{4 \cdot 10^d c_T}{\varepsilon} \Big(\mathbf{1}_{Q_0} \fint_{5Q_0} \|f\| + \sum_j \mathbf{1}_{Q_j} \fint_{5Q_j} \|f\| \Big) + \sum_j \mathbf{1}_{Q_j} T(\mathbf{1}_{5Q_j} f),$$

where c_T was defined in (11.29).

Proof. Given a cube Q_0, consider any disjoint family of its subcubes $Q_j \in \mathscr{D}(Q_0)$. Then we have

$$\begin{aligned} \mathbf{1}_{Q_0} T(\mathbf{1}_{5Q_0} f) &= \mathbf{1}_{Q_0 \setminus \bigcup_j Q_j} T(\mathbf{1}_{5Q_0} f) + \sum_j \mathbf{1}_{Q_j} T(\mathbf{1}_{5Q_0} f) \\ &\leqslant \mathbf{1}_{Q_0 \setminus \bigcup_j Q_j} T(\mathbf{1}_{5Q_0} f) + \sum_j \mathbf{1}_{Q_j} T(\mathbf{1}_{5Q_0 \setminus 5Q_j} f) + \sum_j \mathbf{1}_{Q_j} T(\mathbf{1}_{5Q_j} f), \end{aligned} \tag{11.31}$$

and

$$\begin{aligned} \mathbf{1}_{Q_j} T(\mathbf{1}_{5Q_0 \setminus 5Q_j} f) &\leqslant \mathbf{1}_{Q_j} [\inf_{Q_j} M_T^{\#}(\mathbf{1}_{5Q_0} f) + \inf_{Q_j} T(\mathbf{1}_{5Q_0 \setminus 5Q_j} f)] \\ &\leqslant \mathbf{1}_{Q_j} [\inf_{Q_j} M_T^{\#}(\mathbf{1}_{5Q_0} f) + \inf_{Q_j} \{T(\mathbf{1}_{5Q_0} f) + T(\mathbf{1}_{5Q_j} f)\}] \end{aligned} \tag{11.32}$$

where we used sublinearity and the definition of $M_T^{\#}$ to get the estimates. Note that no convergence issues arise when viewing the above lines in the pointwise sense.

The last term in (11.31) already has the correct form, and it remains to choose the cubes Q_j in such a way that we have (11.30) as well as

$$\mathbf{1}_{Q_0 \setminus \bigcup_j Q_j} T(\mathbf{1}_{5Q_0} f) + \sum_j \mathbf{1}_{Q_j} T(\mathbf{1}_{5Q_0 \setminus 5Q_j} f) \leqslant \mathbf{1}_{Q_0} \frac{c_d c_T}{\varepsilon} \fint_{5Q_0} \|f\|.$$

For a $\lambda > 0$ to be chosen and every $Q \in \mathscr{D}(Q_0)$, we define $F(Q) \subseteq Q$ by

$$F(Q) := Q \cap [\{T(\mathbf{1}_{5Q} f) > \lambda \langle \|f\| \rangle_{5Q}\} \cup \{M_T^{\#}(\mathbf{1}_{5Q} f) > \lambda \langle \|f\| \rangle_{5Q}\}].$$

Thus, by the assumed $L^1(\mathbb{R}^d; X)$ to $L^{1,\infty}(\mathbb{R})$ bounds,

$$\begin{aligned} |F(Q)| &\leqslant |\{T(\mathbf{1}_{5Q} f) > \lambda \langle \|f\| \rangle_{5Q}\}| + |\{M_T^{\#}(\mathbf{1}_{5Q} f) > \lambda \langle \|f\| \rangle_{5Q}\}| \\ &\leqslant (\|T\|_{1\to 1,\infty} + \|M_T\|_{1\to 1,\infty}) \frac{\|\mathbf{1}_{5Q} f\|_1}{\lambda \langle \|f\| \rangle_{5Q}} = \frac{5^d}{\lambda} c_T \cdot |Q|. \end{aligned} \tag{11.33}$$

Let then $Q_j \in \mathscr{D}(Q_0)$ be the maximal dyadic subcubes such that

$$\frac{|Q_j \cap F(Q_0)|}{|Q_j|} > 2^{-d-1}.$$

The cubes Q_j are disjoint, so that

$$\sum_j |Q_j| \leqslant \sum_j \frac{|Q_j \cap F(Q_0)|}{2^{-d-1}} \leqslant 2^{d+1} |F(Q_0)| \leqslant \frac{2 \cdot 10^d}{\lambda} c_T \cdot |Q_0| = \varepsilon |Q_0|,$$

which is (11.30), if we choose

$$\lambda := \frac{2 \cdot 10^d}{\varepsilon} c_T.$$

Substituting back to (11.33), this choice gives in particular that

$$|F(Q)| \leqslant 2^{-d-1} |Q|.$$

Since $\mathbf{1}_{F(Q_0)} \leqslant M_{\mathscr{D}}(\mathbf{1}_{F(Q_0)})$ almost everywhere, we see that $F(Q_0)$ is contained in $\bigcup_j Q_j = \{M_{\mathscr{D}}(\mathbf{1}_{F(Q_0)}) > 2^{-d-1}\}$, except perhaps for a subset of measure zero. In particular, we have (a.e.)

$$\mathbf{1}_{Q_0 \setminus \bigcup_j Q_j} T(\mathbf{1}_{5Q_0} f) \leqslant \mathbf{1}_{Q_0 \setminus \bigcup_j Q_j} \lambda \langle \|f\| \rangle_{5Q_0}. \tag{11.34}$$

On the other hand, the maximality of Q_j implies that its dyadic parent $\widehat{Q}_j$ satisfies the opposite inequality, and hence

$$\frac{|Q_j \cap F(Q_0)|}{|Q_j|} \leqslant \frac{|\widehat{Q}_j \cap F(Q_0)|}{2^{-d} |\widehat{Q}_j|} \leqslant \frac{2^{-d-1}}{2^{-d}} = \frac{1}{2}.$$

But also $|F(Q_j)| \leqslant 2^{-d-1} |Q_j| \leqslant \frac{1}{4} |Q_j|$, and hence

$$|Q_j \setminus [F(Q_0) \cup F(Q_j)]| \geqslant (1 - \frac{1}{2} - \frac{1}{4}) |Q_j| > 0.$$

With any z_j in the non-empty set $Q_j \setminus [F(Q_0) \cup F(Q_j)]$, we can now complete the estimation of (11.32) as follows:

$$\begin{aligned}\mathbf{1}_{Q_j} T(\mathbf{1}_{5Q_0 \setminus 5Q_j} f) &\leqslant \mathbf{1}_{Q_j} [M_T^{\#}(\mathbf{1}_{5Q_0} f)(z_j) + T(\mathbf{1}_{5Q_0} f)(z_j) + T(\mathbf{1}_{5Q_j} f)(z_j)] \\ &\leqslant \mathbf{1}_{Q_j} [\lambda \langle \|f\| \rangle_{5Q_0} + \lambda \langle \|f\| \rangle_{5Q_0} + \lambda \langle \|f\| \rangle_{5Q_j}],\end{aligned}$$

where we used the bounds for $M_T^{\#}(\mathbf{1}_{5Q_0} f)$ and $T(\mathbf{1}_{5Q_0} f)$ on $\complement F(Q_0)$ that follow directly from the definition of these sets, and the analogous bound for $T(\mathbf{1}_{5Q_j} f)$ on $\complement F(Q_j)$. Hence

$$\sum_j \mathbf{1}_{Q_j} T(\mathbf{1}_{5Q_0 \setminus 5Q_j} f) \leqslant \mathbf{1}_{\bigcup_j Q_j} 2\lambda \langle \|f\| \rangle_{5Q_0} + \sum_j \mathbf{1}_{Q_j} \lambda \langle \|f\| \rangle_{5Q_j},$$

and together with (11.31), (11.34) and the choice of λ, this completes the proof of the lemma. □

Iterating the previous lemma, we obtain:

Lemma 11.3.8. *Under the assumptions of Theorem 11.3.6, for any cube Q_0 and $f \in L^1_{\text{loc}}(\mathbb{R}^d; X)$ and $\varepsilon \in (0,1)$, there is a $(1-\varepsilon)$-sparse subcollection $\mathscr{S}(Q_0) \subseteq \mathscr{D}(Q_0)$ such that, almost everywhere,*

$$\mathbf{1}_{Q_0} T(\mathbf{1}_{5Q_0} f) \leqslant \frac{8 \cdot 10^d c_T}{\varepsilon} \sum_{S \in \mathscr{S}(Q_0)} \mathbf{1}_S \fint_{5S} \|f\|.$$

Proof. By Lemma 11.3.7, almost everywhere we have

$$\mathbf{1}_{Q_0} T(\mathbf{1}_{5Q_0} f) \leqslant \frac{c_d c_T}{\varepsilon} \Big(\mathbf{1}_{Q_0} \fint_{5Q_0} \|f\| + \sum_j \mathbf{1}_{Q_j} \fint_{5Q_j} \|f\| \Big) + \sum_j \mathbf{1}_{Q_j^1} T(\mathbf{1}_{5Q_j^1} f)$$

for disjoint subcubes $Q_j^1 \in \mathscr{D}(Q_0)$ such that

$$\sum_j |Q_j^1| \leqslant \varepsilon |Q_0|,$$

and $c_d = 4 \cdot 10^d$ Applying the same estimate to each Q_j^1 in place of Q_0, and continuing by induction, almost everywhere we obtain

$$\begin{aligned}\mathbf{1}_{Q_0} T(\mathbf{1}_{5Q_0} f) \leqslant \frac{c_d c_T}{\varepsilon} \Big(\mathbf{1}_{Q_0} \fint_{5Q_0} \|f\| + 2 \sum_{n=1}^{N-1} \sum_j \mathbf{1}_{Q_j^n} \fint_{5Q_j^n} \|f\| \\ + \sum_k \mathbf{1}_{Q_k^N} \fint_{5Q_k^n} \|f\| \Big) + \sum_k \mathbf{1}_{Q_k^N} T(\mathbf{1}_{5Q_k^N} f),\end{aligned} \tag{11.35}$$

where the Q_j^n are dyadic subcubes of some Q_i^{n-1} in such that

$$\sum_{j : Q_j^n \subseteq Q_i^{n-1}} |Q_j^n| \leqslant \varepsilon |Q_i^{n-1}|.$$

In particular,

$$\sum_j |Q_j^n| \leqslant \varepsilon \sum_i |Q_i^{n-1}| \leqslant \ldots \leqslant \varepsilon^n |Q_0|,$$

so that the support of the last term in (11.35) becomes negligible in the limit $N \to \infty$. Thus, almost everywhere, we have

$$\mathbf{1}_{Q_0} T(\mathbf{1}_{5Q_0} f) \leqslant 2\frac{c_d c_T}{\varepsilon} \sum_{n=0}^{\infty} \sum_j \mathbf{1}_{Q_j^n} \fint_{5Q_j^n} \|f\|, \tag{11.36}$$

where the pairwise disjoint subsets

$$E_j^n := Q_j^n \setminus \bigcup_k Q_k^{n+1}$$

have measure $|E_j^n| \geqslant (1-\varepsilon)|Q_j^n|$. In other words, the cubes Q_j^n form a $(1-\varepsilon)$-sparse subcollection $\mathscr{S}(Q_0) \subseteq \mathscr{D}(Q_0)$, and (11.36) is precisely the estimate asserted in the lemma. □

In order to pass from the local Lemma 11.3.8 to the global Theorem 11.3.6, we use:

Lemma 11.3.9. *Let $E \subseteq \mathbb{R}^d$ satisfy $0 < \mathrm{diam}(E) < \infty$. Then there is a partition $\mathscr{Q}$ of $\mathbb{R}^d$ by dyadic cubes Q such that $E \subseteq 5Q$ for every $Q \in \mathscr{Q}$.*

Proof. Consider all dyadic cubes $Q \in \mathscr{D}$ with the property that $E \nsubseteq 2Q$. Clearly all cubes with $\mathrm{diam}(Q) < \frac{1}{2}\mathrm{diam}(E)$ will satisfy this condition. On the other hand, every cube $Q \in \mathscr{D}$ is contained in some $\widetilde{Q} \in \mathscr{D}$ such that $E \subseteq 2\widetilde{Q}$: if we fix some $x \in Q$ and then $r > 0$ large enough so that $E \subseteq B(x, r)$, then it suffices to take $\widetilde{Q} \supseteq Q$ with $\ell(\widetilde{Q}) > 2r$, since then $2\widetilde{Q} \supseteq B(x, \frac{1}{2}\ell(\widetilde{Q})) \supseteq E$.

Let $\mathscr{Q}$ be the collection of *maximal* dyadic cubes with the property that $E \nsubseteq 2Q$. Maximality implies disjointness, and from what we just checked, it follows that every $x \in \mathbb{R}^d$ is contained in some $Q \in \mathscr{Q}$, so these cubes form a partition of $\mathbb{R}^d$.

Since Q is maximal, its dyadic parent $\widehat{Q}$ satisfies $E \subseteq 2\widehat{Q}$. It remains to observe that $2\widehat{Q} \subseteq 5Q$ to complete the proof. □

We now return to:

Proof of Theorem 11.3.6. If $f \equiv 0$, there is nothing to prove, so fix a non-zero, compactly supported $f \in L^1_c(\mathbb{R}^d; X)$. Thus $E = \operatorname{supp} f$ satisfies $0 < \mathrm{diam}(E) < \infty$ as required to apply Lemma 11.3.9. This lemma produces a partition $\mathscr{Q} \subseteq \mathscr{D}$ of $\mathbb{R}^d$ such that $\operatorname{supp} f \subseteq 5Q$, and thus $\mathbf{1}_{5Q} f = f$, for every $Q \in \mathscr{Q}$. This means that

$$Tf = \sum_{Q \in \mathscr{Q}} \mathbf{1}_Q Tf = \sum_{Q \in \mathscr{Q}} \mathbf{1}_Q T(\mathbf{1}_{5Q} f).$$

Now Lemma 11.3.8 applies to each term on the right, producing $(1-\varepsilon)$-sparse subcollections $\mathscr{S}(Q) \subseteq \mathscr{D}(Q)$ for each $Q \in \mathscr{Q}$, and

$$\sum_{Q\in\mathscr{Q}} \mathbf{1}_Q T(\mathbf{1}_{5Q} f) \leqslant \sum_{Q\in\mathscr{Q}} \frac{c_d c_T}{\varepsilon} \sum_{S\in\mathscr{S}(Q)} \mathbf{1}_S \fint_{5S} \|f\| = \frac{c_d c_T}{\varepsilon} \sum_{S\in\mathscr{S}} \mathbf{1}_S \fint_{5S} \|f\|,$$

where $\mathscr{S} := \bigcup_{Q\in\mathscr{Q}} \mathscr{S}(Q)$ and $c_d = 8 \cdot 10^d$. It is immediate that this union of disjointly supported sparse collections remains sparse, as the same pairwise disjoint subsets $E(S) \subseteq S$ remain pairwise disjoint also among all $S \in \mathscr{S}$. □

11.3.b Sparse operators and domination

With Theorem 11.3.6 at our disposal, the following notion should not appear too alien to the reader:

Definition 11.3.10 (Sparse operator). *Given a sparse collection of sets $\mathscr{S} \subseteq \mathscr{D}$, the associated* sparse operator *is*

$$A_{\mathscr{S}} f := \sum_{S\in\mathscr{S}} \mathbf{1}_S \fint_S f.$$

More generally, with a dilation factor $\varrho \geqslant 1$, we define

$$A^{\varrho}_{\mathscr{S}} f := \sum_{S\in\mathscr{S}} \mathbf{1}_S \fint_{\varrho S} f.$$

In contrast to most other operators that we encounter, the boundedness properties of the sparse operators tend to be extremely easy. As a first illustration, we check the L^p boundedness of $A_{\mathscr{S}}$ by dualising against $g \in L^{p'}$:

$$\begin{aligned}\int A_{\mathscr{S}} f \cdot g &= \sum_{S\in\mathscr{S}} \fint_S f \cdot \fint_S g \cdot |S| \leqslant \sum_{S\in\mathscr{S}} \inf_S M_{\mathscr{D}} f \cdot \inf_S M_{\mathscr{D}} g \cdot \frac{|E(S)|}{\gamma} \\ &\leqslant \frac{1}{\gamma} \int M_{\mathscr{D}} f \cdot M_{\mathscr{D}} g \leqslant \frac{1}{\gamma} \|M_{\mathscr{D}} f\|_p \cdot \|M_{\mathscr{D}} g\|_{p'} \leqslant \frac{1}{\gamma} p' \|f\|_p \cdot p \|g\|_{p'}.\end{aligned}$$

This shows that $\|A_{\mathscr{S}}\|_{p\to p} \leqslant \gamma^{-1} p p'$, where γ is the sparseness parameter; since $A_{\mathscr{S}}$ is manifestly positive, it suffices to consider positive functions above, and the same bound persists for vector-valued functions.

Looking back at the statement of Theorem 11.3.6, it *almost* says that $Tf \leqslant c \cdot A_{\mathscr{S}} \|f\|$ under the assumptions of the theorem, but the presence of the expanded cubes $5S$ prevents this from being strictly true in the stated form. While the variant of a sparse operator implicitly appearing in Theorem 11.3.6 would be almost as good as $A_{\mathscr{S}}$ for many purposes, the use of the more symmetric (indeed, self-dual) operators $A_{\mathscr{S}}$ as in Definition 11.3.10 is often preferred.

A trivial way to achieve this in Theorem 11.3.6 is to dominate $\mathbf{1}_S \leqslant \mathbf{1}_{5S}$, after which the same cube $5S$ will appear in both the indicator and the integral. These cubes will still be sparse, if only with a smaller parameter $\gamma = 5^{-d}(1-\varepsilon)$, since the disjoint major subsets $E(S) \subseteq S \subseteq 5S$ satisfy $|E(S)| \geqslant (1-\varepsilon)|S| = (1-\varepsilon)5^{-d}|5S|$ and hence also qualify for the disjoint major subsets of the expanded cubes $5S$. An apparent loss in this construction is the fact that these $5S$ are no longer *dyadic* cubes. Even this problem, however, can be fixed, by a variant of the shifted dyadic cubes that we introduced in Definition 3.2.25. Recall that the standard dyadic system is

$$\mathscr{D}^0 := \bigcup_{j\in\mathbb{Z}} \mathscr{D}^0_j, \qquad \mathscr{D}^0_j := \{2^{-j}([0,1)^d + m) : \ m \in \mathbb{Z}^d\}.$$

We will need the case $N = 5$ of the following statement, but we record the general formulation for convenience of reference, as the case $N = 3$ also features in various applications.

Proposition 11.3.11 (Dilated dyadic cubes). *Let $N \in \mathbb{Z}_+$ be odd. Then the collection of N-fold concentric dilations $\{NQ : Q \in \mathscr{D}(\mathbb{R}^d)\}$ can be partitioned into N^d subcollections $\mathscr{D}^{n;N}$, $n \in \mathbb{Z}^d_N$, each of which has the same covering and nestedness properties as $\mathscr{D}$, namely,*

$$\mathscr{D}^{n;N} = \bigcup_{j\in\mathbb{Z}} \mathscr{D}^{n;N}_j,$$

where for each $j \in \mathbb{Z}$:

(1) *$\mathscr{D}^{n;N}_j$ is a partition of $\mathbb{R}^d$ consisting cubes of side-length $N \cdot 2^{-j}$, and*
(2) *$\mathscr{D}^{n;N}_{j+1}$ is a refinement of $\mathscr{D}^{n;N}_j$.*

Proof. Since $\mathscr{D}_j(\mathbb{R}^d) = \{I_1 \times \cdots \times I_d : I_i \in \mathscr{D}_j(\mathbb{R})\}$ and $N(I_1 \times \cdots \times I_d) = NI_1 \times \cdots \times NI_d$, it suffices to verify the case $d = 1$. In the calculation that follows, we will need to dilate an interval $I = [c-r, c+r)$ both by the algebraic multiplication $a \cdot I = \{a \cdot t : t \in I\} = [ac - ar, ac + ar)$ and by the concentric dilation, for which we temporarily adopt the heavier notation $a \odot I = [c - ar, c + ar)$ for the sake of distinction.

With these notations fixed, we have

$$\begin{aligned}
\{N \odot I : I \in \mathscr{D}_j\} &= \{N \odot 2^{-j}([0,1) + m) : m \in \mathbb{Z}\} \\
&= \{2^{-j}([-N', N'+1) + m) : m \in \mathbb{Z}\} \quad \big(N := 2N'+1\big) \\
&= \{2^{-j}([0, N) + m - N') : m \in \mathbb{Z}\} \\
&= \{2^{-j}([0, N) + m) : m \in \mathbb{Z}\} \\
&= \Big\{N2^{-j}\Big([0,1) + \frac{m}{N}\Big) : m \in \mathbb{Z}\Big\}.
\end{aligned}$$

The sought-after partition of this collection is now achieved as follows: For each $n \in \mathbb{Z}_N = \{0, 1, \ldots, N-1\}$ and $j \in \mathbb{Z}$, we define

$$\mathscr{D}_j^{n;N} := \Big\{ N2^{-j}\Big([0,1)+k+\frac{\alpha(n,j)}{N}\Big) : k\in\mathbb{Z}\Big\} \tag{11.37}$$

for appropriate $\alpha(n,j)\in\mathbb{Z}_N$ to be shortly determined. It is clear that each $\mathscr{D}_j^{n;N}$ satisfies (1) from the statement of the Proposition, no matter how we choose $\alpha(n,j)$. To ensure (2), it suffices to check that the left (or equivalently right) half of any $I\in\mathscr{D}_j^{n;N}$ belongs to $\mathscr{D}_{j+1}^{n;N}$. For a generic I as written above, the left half will be

$$N2^{-j}\Big([0,\tfrac{1}{2})+k+\frac{\alpha(n,j)}{N}\Big) = N2^{-j-1}\Big([0,1)+2k+\frac{2\alpha(n,j)}{N}\Big).$$

For this to be in $\mathscr{D}_{j+1}^{n;N}$, it is necessary and sufficient that

$$2\alpha(n,j)\equiv\alpha(n,j+1) \mod N \tag{11.38}$$

If we specify $\alpha(n,0):=n$, all other $\alpha(n,j)$, $j\in\mathbb{Z}\setminus\{0\}$ will be uniquely determined by (11.38), since 2 has a multiplicative inverse in $\mathbb{Z}_N$ for odd N. Indeed, the solution is given by

$$\alpha(n,j)\equiv 2^j n \mod N, \tag{11.39}$$

where the negative powers are interpreted in the sense of the multiplicative inverse mod N.

For each $j\in\mathbb{Z}$, the map $n\mapsto 2^j n \mod N$ is a bijection on $\mathbb{Z}_N$, and thus

$$\begin{aligned}\bigcup_{n=0}^{N}\mathscr{D}_j^{n;N} &= \Big\{N2^{-j}\Big([0,1)+k+\frac{a}{N}\Big) : k\in\mathbb{Z}, a\in\mathbb{Z}_N\Big\} \\ &= \Big\{N2^{-j}\Big([0,1)+\frac{m}{N}\Big) : m\in\mathbb{Z}\Big\} = \{N\odot I : I\in\mathscr{D}_j\},\end{aligned}$$

so indeed $\{N\odot I : I\in\mathscr{D}\}$ is a disjoint union of the collections $\mathscr{D}^{n;N}$, $n\in\mathbb{Z}_N$, and we already checked that each $\mathscr{D}^{n;N}$ has the properties (1) and (2). □

Remark 11.3.12 (Shifted dyadic cubes). The cube families $\mathscr{D}^{n;N}$ constructed above are close relatives of the *shifted dyadic cubes* of Definition 3.2.25, and they satisfy a variant of the Covering Lemma 3.2.26:

> Given an odd $N\in\mathbb{Z}_+$, for every cube $Q\subseteq\mathbb{R}^d$, there exist a vector $n\in\mathbb{Z}_N^d$ and a cube $D\in\mathscr{D}^{n;N}$ such that
>
> $$\frac{N}{N-1}\ell(Q) < \ell(D) \leqslant \frac{2N}{N-1}\ell(Q) \quad\text{and}\quad Q\subseteq D. \tag{11.40}$$

In fact, let $R\in\mathscr{D}$ be a cube of side-length $\ell(R)\in(\ell(Q)/(2N'),\ell(Q)/N']$ that contains the centre z_Q of Q, where $N=2N'+1$ as before. Then $D=NR\in\mathscr{D}^{n;N}$ for some $n\in\mathbb{Z}_N^d$, and D contains the cube of side-length $2N'\ell(R)>\ell(Q)$ centred at z_Q; thus $D\supseteq Q$, and $\ell(D)=N\ell(R)$ lies exactly in the range asserted in (11.40).

Also note that both the partition and refinement properties (1) and (2) of Proposition 11.3.11 of each $\mathscr{D}^{n;N}$, as well as the covering property of every cube $Q \subseteq \mathbb{R}^d$ by a cube in some $\mathscr{D}^{n;N}$, remain invariant if we drop the algebraic dilation factor N in (11.37), so as to be back to cubes of side-length 2^{-j}. When $N = 3$, this reproduces precisely the shifted dyadic cubes of Definition 3.2.25; since $2 \equiv -1 \mod 3$, (11.39) reduces in this case to the simpler form $\alpha(n, j) = (-1)^j n$, where reference to modular arithmetic can be avoided.

It is now easy to show that the sparse operators with a dilation, $A^{\varrho}_{\mathscr{S}}$, may always be dominated by a finite number of the simple sparse operators $A_{\mathscr{S}^n}$. It is technically convenient to take an odd integer N for the dilation factor. This causes little loss of generality since, choosing $N \geqslant \varrho$, we can always dominate

$$\fint_{\varrho Q} f \leqslant \Big(\frac{N}{\varrho}\Big)^d \fint_{NQ} f$$

and hence $A^{\varrho}_{\mathscr{S}} f \leqslant (N/\varrho)^d A^{N}_{\mathscr{S}} f$ for $f \geqslant 0$.

Lemma 11.3.13. *Let $\mathscr{S} \subseteq \mathscr{D}$ be ε-sparse for some $\varepsilon \in (0, 1)$, and $N \in \mathbb{Z}_+$ be odd. Then there are $N^{-d}\varepsilon$-sparse collections $\mathscr{S}^n \subseteq \mathscr{D}^{n;N}$ for each $n \in \mathbb{Z}^d_N$ such that, for every non-negative $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$,*

$$A^N_{\mathscr{S}} f \leqslant \sum_{n \in \mathbb{Z}^d_N} A_{\mathscr{S}^n} f$$

Proof. We note that the collection $\{5Q : Q \in \mathscr{S}\}$ is $N^{-d}\varepsilon$-sparse, with the same disjoint subsets $E(Q) \subseteq Q \subseteq NQ$ that satisfy $|E(Q)| \geqslant \varepsilon|Q| = \varepsilon N^{-d}|NQ|$. By Proposition 11.3.11, we have a decomposition $\{NQ : Q \in \mathscr{D}\} = \bigcup_{n \in \mathbb{Z}^d_N} \mathscr{D}^{n;N}$ into dyadic systems $\mathscr{D}^{n;N}$. We then define $\mathscr{S}^n := \{NQ : Q \in \mathscr{S}\} \cap \mathscr{D}^{n;5}$. Thus

$$\begin{aligned} A^N_{\mathscr{S}} f &= \sum_{Q \in \mathscr{S}} \mathbf{1}_Q \fint_{NQ} f \leqslant \sum_{Q \in \mathscr{S}} \mathbf{1}_{NQ} \fint_{NQ} f = \sum_{n \in \mathbb{Z}^d_N} \sum_{\substack{Q \in \mathscr{S} \\ NQ \in \mathscr{D}^{n;N}}} \mathbf{1}_{NQ} \fint_{NQ} f \\ &= \sum_{n \in \mathbb{Z}^d_N} \sum_{Q' \in \mathscr{S}^n} \mathbf{1}_{Q'} \fint_{Q'} f = \sum_{n \in \mathbb{Z}^d_N} A_{\mathscr{S}^n} f. \end{aligned}$$

□

We can now reformulate Theorem 11.3.6 in terms of sparse operators:

Theorem 11.3.14 (Abstract sparse domination II). *Let X be a Banach space, and let T be a positive sub-linear operator from $L^1(\mathbb{R}^d; X)$ into $L^0(\mathbb{R}^d; \mathbb{R}_+)$, and consider the associated Lerner's maximal operator*

$$M^{\#}_T f(x) := \sup_{Q \ni x} \sup_{y,z \in Q} |T(\mathbf{1}_{\complement(5Q)} f)(y) - T(\mathbf{1}_{\complement(5Q)} f)(z)|.$$

Suppose that both T and M_T are bounded from $L^1(\mathbb{R}^d;X)$ to $L^{1,\infty}(\mathbb{R}^d)$. Then for every boundedly supported $f\in L^1(\mathbb{R}^d)$, there is a 5^{-1}-sparse collection $\mathscr{S}\subseteq\mathscr{D}$ and, for every $n\in\mathbb{Z}_5^d$, a 5^{-d-1}-sparse collection $\mathscr{S}^n\subseteq\mathscr{D}^{n;5}$ of the dyadic systems as in Proposition 11.3.11, such that almost everywhere

$$Tf\leqslant 10^{d+1}c_T A^5_{\mathscr{S}}\|f\|\leqslant 10^{d+1}c_T\sum_{n\in\mathbb{Z}_5^d}A_{\mathscr{S}^n}\|f\|,$$

where $c_T:=\|T\|_{1\to 1,\infty}+\|M_T^{\#}\|_{1\to 1,\infty}$.

Proof. Choosing $\varepsilon=4/5$ in Theorem 11.3.6, we find a $\frac{1}{5}$-sparse collection $\mathscr{S}\subseteq\mathscr{D}$ such that

$$Tf\leqslant\frac{8\cdot 10^d\cdot c_T}{4/5}\sum_{S\in\mathscr{S}}\mathbf{1}_S\fint_{5S}\|f\|=10^{d+1}c_T A^5_{\mathscr{S}}\|f\|.$$

This is the first claim, and the second one follows from Lemma 11.3.13. $\square$

11.3.c Sparse domination of Calderón–Zygmund operators

The goal of this section is to specialise the abstract Theorem 11.3.14 to the case of Calderón–Zygmund operators in the following form:

Theorem 11.3.15 (Sparse domination of singular integrals). *Let X and Y be Banach spaces, $p_0\in[1,\infty]$, and let*

$$T\in\mathscr{L}(L^{p_0}(\mathbb{R}^d;X),L^{p_0,\infty}(\mathbb{R}^d;Y))$$

be an operator with a Dini kernel K. Then for every boundedly supported $f\in L^1(\mathbb{R}^d)$, there is a 5^{-1}-sparse collection $\mathscr{S}\subseteq\mathscr{D}$ and, for every $n\in\mathbb{Z}_5^d$, a 5^{-d-1}-sparse collection $\mathscr{S}^n\subseteq\mathscr{D}^{n;5}$ of the dyadic systems as in Proposition 11.3.11, such that almost everywhere

$$\|Tf\|_Y\leqslant c_{d,T}A^5_{\mathscr{S}}\|f\|_X\leqslant c_{d,T}\sum_{n\in\mathbb{Z}_5^d}A_{\mathscr{S}^n}\|f\|,$$

where

$$c_{d,T}\leqslant c_d\big(\|T\|_{L^{p_0}(\mathbb{R}^d;X)\to L^{p_0,\infty}(\mathbb{R}^d;Y)}+c_K+\|\omega\|_{\mathrm{Dini}}\big)$$

with c_K and ω as in Definition 11.3.1.

The result remains true if $\mathbb{R}^d$ is systematically replaced by a cube or a quadrant of $\mathbb{R}^d$, both in the function spaces where the boundedness is considered, and in the definition of the kernel bounds c_K and $\|\omega_K\|_{\mathrm{Dini}}$.

Proof. By Theorem 11.3.14, applied to the positive sub-linear operator $U:f\mapsto\|Tf(\cdot)\|_Y$, the result follows if we can estimate $\|U\|_{L^1\to L^{1,\infty}}$ and $\|M_U\|_{L^1\to L^{1,\infty}}$ by the bound for $c_{d,T}$ given above. For the former, this is immediate from the Calderón–Zygmund Theorem 11.2.5 and Lemma 11.3.4, which show that

$$\begin{aligned}\|U\|_{L^1(\mathbb{R}^d;X)\to L^{1,\infty}(\mathbb{R}^d)} &= \|T\|_{L^1(\mathbb{R}^d;X)\to L^{1,\infty}(\mathbb{R}^d;Y)} \\ &\leqslant c_d\big(\|T\|_{L^{p_0}(\mathbb{R}^d;X)\to L^{p_0,\infty}(\mathbb{R}^d;Y)} + \|K\|_{\text{Hör}}\big) \\ &\leqslant c_d\big(\|T\|_{L^{p_0}(\mathbb{R}^d;X)\to L^{p_0,\infty}(\mathbb{R}^d;Y)} + \|K\|_{\text{Dini}}\big).\end{aligned}$$

For M_U, we first observe that, by the triangle inequality,

$$\begin{aligned}|U(\mathbf{1}_{\complement 5Q}f)(y) - U(\mathbf{1}_{\complement 5Q}f)(z)| &= |\|T(\mathbf{1}_{\complement 5Q}f)(y)\|_Y - \|T(\mathbf{1}_{\complement 5Q}f)(z)\|_Y| \\ &\leqslant \|T(\mathbf{1}_{\complement 5Q}f)(y) - T(\mathbf{1}_{\complement 5Q}f)(z)\|_Y.\end{aligned}$$

Hence, taking the supremum over $y, z \in Q$ and then over cubes $Q \ni x$, it follows that

$$M_U f(x) \leqslant M_T^{\#} f(x) := \sup_{Q\ni x}\sup_{y,z\in Q} \|T(\mathbf{1}_{\complement(5Q)}f)(y) - T(\mathbf{1}_{\complement(5Q)}f)(z)\|_Y.$$

The norm estimate of the latter is the content of the following lemma. □

Lemma 11.3.16. *Let X and Y be Banach spaces, $p_0 \in [1,\infty]$, and let T be an operator with a Dini kernel $K : \dot{\mathbb{R}}^{2d} \to \mathscr{L}(X,Y)$. Then the maximal operator*

$$M_T^{\#} f(x) := \sup_{Q\ni x}\sup_{y,z\in Q} \|T(\mathbf{1}_{\complement(5Q)}f)(y) - T(\mathbf{1}_{\complement(5Q)}f)(z)\|_Y$$

satisfies

$$M_T^{\#} f(x) \leqslant c_d(c_K + \|\omega_K\|_{\text{Dini}})Mf(x)$$

and

$$\|M_T^{\#}\|_{L^1(\mathbb{R}^d;X)\to L^{1,\infty}(\mathbb{R}^d)} \leqslant c_d(c_K + \|\omega_K\|_{\text{Dini}}).$$

The result remains true if $\mathbb{R}^d$ is systematically replaced by a cube $Q_0 \subseteq \mathbb{R}^d$ or a quadrant $S \subseteq \mathbb{R}^d$, both in the function spaces where the boundedness is considered, and in the definition of the kernel bounds c_K and $\|\omega_K\|_{\text{Dini}}$.

Proof. For $x, x_0, x_1 \in Q$, we have

$$\begin{aligned}&T(\mathbf{1}_{\complement(5Q)}f)(x_0) - T(\mathbf{1}_{\complement(5Q)}f)(x_1) \\ &\qquad= \sum_{j=0}^{1}(-1)^j[T(\mathbf{1}_{\complement(5Q)}f)(x_j) - T(\mathbf{1}_{\complement(5Q)}f)(x)] \\ &\qquad= \sum_{j=0}^{1}(-1)^j\int_{\complement(5Q)}[K(x_j,y) - K(x,y)]f(y)\,\mathrm{d}y,\end{aligned}$$

where

$$
\begin{aligned}
&\Big\| \int_{\complement(5Q)} [K(x_j,y)-K(x,y)]f(y)\,\mathrm{d}y \Big\| \\
&\leqslant \int_{\complement B(x,4\sqrt{d}\ell(Q))} \|[K(x_j,y)-K(x,y)]f(y)\|\,\mathrm{d}y \\
&\qquad + \int_{B(x,4\sqrt{d}\ell(Q))\setminus(5Q)} \|[K(x_j,y)-K(x,y)]f(y)\|\,\mathrm{d}y =: I + II
\end{aligned}
$$

where, observing that $|x_j - x| < \sqrt{d}\ell(Q) \leqslant \frac{1}{4}|x-y|$ for $x, x_j \in Q$ and $y \in \complement B(x, 4\sqrt{d}\ell(Q))$,

$$
\begin{aligned}
I &\leqslant \int_{\complement B(x,4\sqrt{d}\ell(Q))} \omega_K^1\Big(\frac{|x_j-x|}{|x-y|}\Big)\frac{1}{|x-y|^d}\|f(y)\|\,\mathrm{d}y \\
&\leqslant \sum_{k=2}^{\infty} \int_{2^k\sqrt{d}\ell(Q)\leqslant|y-x|<2^{k+1}\sqrt{d}\ell(Q)} \omega_K^1\Big(\frac{\sqrt{d}\ell(Q)}{2^k\sqrt{d}\ell(Q)}\Big)\frac{\|f(y)\|\,\mathrm{d}y}{(2^k\sqrt{d}\ell(Q))^d} \\
&\leqslant \sum_{k=2}^{\infty} \omega_K^1(2^{-k})c_d \fint_{B(x,2^{k+1}\sqrt{d}\ell(Q))} \|f(y)\|\,\mathrm{d}y \\
&\leqslant c_d Mf(x) \sum_{k=2}^{\infty} \omega_K^1(2^{-k}) \leqslant c_d Mf(x)\|\omega_K^1\|_{\mathrm{Dini}},
\end{aligned}
$$

by Lemma 11.3.3 in the last step. On the other hand, since $|x_j - y|, |x-y| \geqslant 2\ell(Q)$ for $x, x_j \in Q$ and $y \notin 5Q$, we obtain

$$
\begin{aligned}
II &\leqslant \int_{B(x,2\sqrt{d}\ell(Q))\setminus(5Q)} c_K \frac{2}{(2\ell(Q))^d}\|f(y)\|\,\mathrm{d}y \\
&\leqslant c_K c_d \fint_{B(x,2\sqrt{d}\ell(Q))} \|f(y)\|\,\mathrm{d}y \leqslant c_K c_d Mf(x).
\end{aligned}
$$

These bounds give the pointwise estimate for $M_T^{\#}f(x)$, and the norm estimate is then immediate from the corresponding bound for the Hardy–Littlewood maximal operator M.

The case of a cube or a quadrant in place of $\mathbb{R}^d$ follows by inspection of the same argument: if all variables under consideration are restricted like this, it is evident that only the corresponding restrictions of the kernel conditions will be needed to make the estimates. □

11.3.d Weighted norm inequalities and the A_2 theorem

We are now ready to provide the main application of the sparse domination of Calderón–Zygmund operators: their weighted norm inequalities with an optimal dependence of the weight. A function $w \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ is called a *weight* if $w(x) \in (0, \infty)$ almost everywhere. We recall from Appendix J the following definition, which we now extend to the local situation as well:

Definition 11.3.17. *For $p \in (1,\infty)$ the Muckenhoupt A_p characteristic of a weight w is defined by*

$$[w]_{A_p} := \sup_Q \Big(\fint_Q w(x)\,\mathrm{d}x \Big) \Big(\fint_Q w^{1-p'}(x)\,\mathrm{d}x \Big)^{p-1},$$

where the supremum is over all (axes-parallel) cubes $Q \subseteq \mathbb{R}^d$. We say that w is an A_p weight if $[w]_{A_p} < \infty$.

For a cube or quadrant $Q_0 \subseteq \mathbb{R}^d$, we define the local weight characteristic $[w]_{A_p(Q_0)}$ and the weight class $A_p(Q_0)$ in a similar way, but restricting the supremum to cubes $Q \subseteq Q_0$ only.

For the treatment of weighted norm inequalities, it is useful to introduce the following simple but far-reaching idea:

Remark 11.3.18 (Dual weight trick). Given an operator T, a weight w and an exponent $p \in (1,\infty)$, consider an inequality of the form

$$\|T(h)\|_{L^p(w)} \leqslant C\|h\|_{L^p(w)} \qquad \forall h \in L^p(w). \tag{11.41}$$

If σ is another weight, we observe that $h = f\sigma$ is in $L^p(w)$ if and only if $f \in L^p(\sigma^p w)$. With this substitution, the previous estimate becomes

$$\|T(f\sigma)\|_{L^p(w)} \leqslant C\|f\sigma\|_{L^p(w)} = C\|f\|_{L^p(\sigma^p w)} \qquad \forall f \in L^p(\sigma^p w).$$

Equating the weights inside the operator and on the right hand side, we want to arrange that $\sigma = \sigma^p w$, i.e., that $\sigma = w^{-1/(p-1)}$; this is called the (L^p-)*dual weight* of w. With this choice, the previous display reduces to

$$\|T(f\sigma)\|_{L^p(w)} \leqslant C\|f\|_{L^p(\sigma)} \quad \forall f \in L^p(\sigma), \quad \sigma := w^{-1/(p-1)}. \tag{11.42}$$

Applying duality in $L^p(w)$, yet another equivalent condition is given by the conveniently symmetric formulation

$$\int T(f\sigma)\cdot gw \leqslant C\|f\|_{L^p(\sigma)}\|g\|_{L^{p'}(w)} \quad \forall f \in L^p(\sigma), g \in L^{p'}(w). \tag{11.43}$$

Thus all three formulations (11.41), (11.42) and (11.43) are equivalent.

We now give the A_2 theorem for the sparse operators $A_{\mathscr{S}}$. The simplicity of this argument is a manifestation of the usefulness of dominating other operators by the sparse ones.

Theorem 11.3.19 (Cruz-Uribe–Martell–Pérez). *Let $\varepsilon \in (0,1)$ and $\mathscr{S} \subseteq \mathscr{D}$ be ε-sparse. Let $N \in \mathbb{Z}_+$ be odd. If $w \in A_2$, then the sparse operator $A^N_{\mathscr{S}}$ is bounded on $L^2(w)$, and*

$$\|A^N_{\mathscr{S}}\|_{\mathscr{L}(L^2(w))} \leqslant \frac{4}{\varepsilon} N^{2d} [w]_{A_2}.$$

Proof. By the dual weight trick (Remark 11.3.18), with $\sigma := w^{-1}$ we need to prove that

$$\int A^N_{\mathscr{S}}(f\sigma)\cdot gw \leqslant \frac{4}{\varepsilon}N^{2d}[w]_{A_2}\|f\|_{L^2(\sigma)}\|g\|_{L^2(w)} \qquad \forall f\in L^2(\sigma), g\in L^2(w).$$

Since $A_{\mathscr{S}}$ is a positive operator, both g and h may be taken to be positive, and there are no subtle convergence issues in the computation that follows. We first observe that

$$\langle f\sigma\rangle_Q = \frac{1}{|Q|}\int_Q f\sigma = \frac{\sigma(Q)}{|Q|}\frac{1}{\sigma(Q)}\int_Q f\sigma = \langle\sigma\rangle_Q\langle f\rangle^\sigma_Q,$$

where $\sigma(Q)=\int_Q\sigma$ and $\langle f\rangle^\sigma_Q$ is the average of f with respect to the measure induced by the weight σ. We denote the corresponding dyadic maximal operator by $M^\sigma_{\mathscr{D}}f := \sup_{Q\in\mathscr{D}}\mathbf{1}_Q\langle f\rangle^\sigma_Q$; this operator is bounded on $L^2(\sigma)$ with norm 2 according to Doob's maximal inequality (Theorem 3.2.2, cf. explanations preceding Theorem 3.2.27) with $p=p'=2$.

We can then estimate, using that $[w]_{A_2}=\sup_Q\langle w\rangle_Q\langle\sigma\rangle_Q$ by definition,

$$\begin{aligned}\int A^N_{\mathscr{S}}(f\sigma)\cdot gw &= \sum_{Q\in\mathscr{S}}\langle f\sigma\rangle_{NQ}\int_{\mathbb{R}^d}\mathbf{1}_Q\cdot gw\\ &= \sum_{Q\in\mathscr{S}}\langle f\sigma\rangle_{NQ}\langle gw\rangle_Q|Q|\\ &= \sum_{Q\in\mathscr{S}}\langle\sigma\rangle_{NQ}\langle w\rangle_Q\langle f\rangle^\sigma_{NQ}\langle g\rangle^w_Q|Q|,\end{aligned}$$

where

$$\langle\sigma\rangle_{NQ}\langle w\rangle_Q \leqslant \langle\sigma\rangle_{NQ}\langle w\rangle_{NQ}N^d \leqslant [w]_{A_2}N^d.$$

Hence

$$\int A^N_{\mathscr{S}}(f\sigma)\cdot gw \leqslant N^d[w]_{A_2}\sum_{Q\in\mathscr{S}}\langle f\rangle^\sigma_{NQ}\langle g\rangle^w_Q\frac{|E(Q)|}{\varepsilon},$$

where

$$\langle g\rangle^w_Q \leqslant \inf_{z\in Q}M^w_{\mathscr{D}}g(z)$$

by definition of the dyadic maximal operator. As for $\langle f\rangle^\sigma_{NQ}$, we observe by Proposition 11.3.11 that the dilated cube NQ belongs to one of the N^d dyadic system $\mathscr{D}^{n;N}$, where $n\in\mathbb{Z}^d_N$, and the average over NQ is then something that appears in the corresponding maximal operator $M_{\mathscr{D}^{n;N}}$. Hence

$$\begin{aligned}&\sum_{Q\in\mathscr{S}}\langle f\rangle^\sigma_{NQ}\langle g\rangle^w_Q\frac{|E(Q)|}{\varepsilon} \qquad (11.44)\\ &= \sum_{n\in\mathbb{Z}^d_N}\sum_{\substack{Q\in\mathscr{S}\\ NQ\in\mathscr{D}^{n;N}}}\langle f\rangle^\sigma_{NQ}\langle g\rangle^w_Q\frac{|E(Q)|}{\varepsilon}\end{aligned}$$

$$\begin{aligned}
&\leqslant \sum_{n\in\mathbb{Z}_N^d} \sum_{\substack{Q\in\mathscr{S}\\ NQ\in\mathscr{D}^{n;N}}} \inf_Q M^\sigma_{\mathscr{D}^{n;N}} f \cdot \inf_Q M^w_{\mathscr{D}} g \cdot \frac{|E(Q)|}{\varepsilon} \\
&\leqslant \frac{1}{\varepsilon} \sum_{n\in\mathbb{Z}_N^d} \sum_{\substack{Q\in\mathscr{S}\\ NQ\in\mathscr{D}^{n;N}}} \int_{E(Q)} M^\sigma_{\mathscr{D}^{n;N}} f \cdot M^w_{\mathscr{D}} g \\
&\leqslant \frac{1}{\varepsilon} \sum_{n\in\mathbb{Z}_N^d} \int_{\mathbb{R}^d} M^\sigma_{\mathscr{D}^{n;N}} f \cdot M^w_{\mathscr{D}} g \cdot \sigma^{1/2} w^{1/2} \\
&\leqslant \frac{1}{\varepsilon} \sum_{n\in\mathbb{Z}_N^d} \|M^\sigma_{\mathscr{D}^{n;N}} f\|_{L^2(\sigma)} \|M^w_{\mathscr{D}} g\|_{L^2(w)} \\
&\leqslant \frac{1}{\varepsilon} \sum_{n\in\mathbb{Z}_N^d} 2\|f\|_{L^2(\sigma)} \cdot 2\|g\|_{L^2(w)} \\
&= \frac{4}{\varepsilon} N^d \|f\|_{L^2(\sigma)} \|g\|_{L^2(w)}.
\end{aligned}$$

Substituting back, this gives the claimed bound for $\|A_{\mathscr{S}}\|_{\mathscr{L}(L^2(w))}$. □

Corollary 11.3.20. *Let $\varepsilon \in (0,1)$ and $\mathscr{S} \subseteq \mathscr{D}$ be ε-sparse, and let $Q_0 \in \mathscr{D}$. If $N \in \mathbb{Z}_+$ is odd, $p \in (1,\infty)$, and $w \in A_p$, then the sparse operator $A^N_{\mathscr{S}}$ is bounded on $L^p(w)$, and*

$$\|A^N_{\mathscr{S}}\|_{\mathscr{L}(L^p(w))} \leqslant c_{d,p} \frac{N^{2d}}{\varepsilon} [w]_{A_p}^{\max(1,\frac{1}{p-1})}.$$

Proof. This is an immediate consequence of Theorem 11.3.19 and Rubio de Francia's Extrapolation Theorem J.2.1. (In the latter, ϕ_{pr} and c_{pr} should be replaced by ϕ_{dpr} and c_{dpr}; the omission of dependence on d is a systematic typo in Theorem J.2.1 and its proof. This explains a need a constant $c_{d,p}$ rather than just c_p in the statement of the corollary.) □

It is also useful to record the following localised version:

Proposition 11.3.21. *Let $\varepsilon \in (0,1)$ and $\mathscr{S} \subseteq \mathscr{D}$ be ε-sparse, and let $Q_0 \in \mathscr{D}$. If $N \in \mathbb{Z}_+$ is odd and $w \in A_2(Q_0)$, then the sparse operator $A^N_{\mathscr{S}}$ is bounded on $L^2(Q_0, w)$, and*

$$\|A^N_{\mathscr{S}}\|_{\mathscr{L}(L^2(Q_0,w))} \leqslant \Big(\frac{4}{\varepsilon} N^{2d} + 1\Big)[w]_{A_2(Q_0)}.$$

The same result is true if the cube Q_0 is replaced by a quadrant of $\mathbb{R}^d$.

We start with a simple:

Lemma 11.3.22. *For every $Q \in \mathscr{D}(Q_0)$, there exists a cube $\widetilde{Q}$ such that $NQ \cap Q_0 \subseteq \widetilde{Q} \subseteq Q_0$ and $\ell(\widetilde{Q}) = \min\{N\ell(Q), \ell(Q_0)\}$.*

Proof. If $NQ \subseteq Q_0$, we take $\widetilde{Q} := NQ$, and if $N\ell(Q) \geqslant \ell(Q_0)$, we define $\widetilde{Q} := Q_0$.

Let us finally consider $Q \in \mathscr{D}(Q_0)$ such that $NQ \not\subseteq Q_0$ but $N\ell(Q) < \ell(Q_0)$. Let first $d = 1$, so that both $Q_0 = [a, b)$ and Q are intervals. If NQ extends to the left of a, then $\widetilde{Q} := [a, a + N\ell(Q))$ satisfies the desired properties. If NQ extends to the right of b, then $\widetilde{Q} := [b - N\ell(Q), b)$ works. For general $d \geqslant 1$ with $Q = I_1 \times \cdots \times I_d$ and $Q_0 = J_1 \times \cdots \times J_d$, we take $\widetilde{Q} := \widetilde{I}_1 \times \cdots \times \widetilde{I}_d$, where each $\widetilde{I}_i$ is built relative to the respective interval J_i as in the one-dimensional construction just given. This completes the proof. □

Proof of Proposition 11.3.21. The norm on the left is the $L^2(w)$-norm of the operator $f \mapsto \mathbf{1}_{Q_0} A^N_{\mathscr{S}}(\mathbf{1}_{Q_0} f)$, i.e., both the domain and the range of the operator is restricted to functions supported on Q_0. Since $Q_0 \in \mathscr{D}$, each $Q \in \mathscr{D} \subseteq \mathscr{S}$ that contributes to $\mathbf{1}_{Q_0} A_{\mathscr{S}}(\mathbf{1}_{Q_0} f)$ satisfies either $Q \subseteq Q_0$ or $Q \supsetneq Q_0$. Letting $\mathscr{S}' := \{Q \in \mathscr{S} : Q \subseteq Q_0\}$, we hence have

$$\begin{aligned}\int_{Q_0} & A^N_{\mathscr{S}}(\mathbf{1}_{Q_0} f\sigma) \cdot gw \\ &\leqslant \int_{Q_0} A^N_{\mathscr{S}'}(\mathbf{1}_{Q_0} f\sigma) \cdot gw + \int_{Q_0} \sum_{Q \supsetneq Q_0} \langle \mathbf{1}_{Q_0} f\rangle_{NQ} \cdot gw =: I + II.\end{aligned}$$

By the dual weight trick with $\sigma = w^{-1}$, estimating the left-hand side uniformly over $f \in L^2(Q_0, \sigma)$ and $g \in L^2(Q_0, w)$ of unit norm is equivalent to bounding $\|A^N_{\mathscr{S}}\|_{\mathscr{L}(L^2(Q_0,w))}$.

Term II is dominated by

$$\sum_{Q \supsetneq Q_0} \fint_{NQ} (\mathbf{1}_{Q_0} f) = \sum_{Q \supsetneq Q_0} \frac{|Q_0|}{|Q|} \fint_{NQ_0} (\mathbf{1}_{Q_0} f) = \sum_{k=1}^{\infty} 2^{-kd} \fint_{NQ_0} (\mathbf{1}_{Q_0} f),$$

where $\sum_{k=1}^{\infty} 2^{-kd} \leqslant \sum_{k=1}^{\infty} 2^{-k} = 1$. Thus

$$II \leqslant \left\| \mathbf{1}_{Q_0} \langle \mathbf{1}_{Q_0} f\rangle_{NQ_0} \right\|_{L^2(w)} \|g\|_{L^2(w)},$$

where

$$\begin{aligned}\left\| \mathbf{1}_{Q_0} \langle \mathbf{1}_{Q_0} f\rangle_{NQ_0} \right\|_{L^2(w)} &= \frac{w(Q_0)^{1/2}}{|NQ_0|} \int_{Q_0} f w^{1/2} \sigma^{1/2} \\ &\leqslant \frac{w(Q_0)^{1/2}}{|Q_0|} \|f\|_{L^2(Q_0,w)} \sigma(Q_0)^{1/2} \\ &\leqslant [w]^{1/2}_{A_2(Q_0)} \|f\|_{L^2(Q_0,w)} \leqslant [w]_{A_2(Q_0)} \|f\|_{L^2(Q_0,w)}.\end{aligned}$$

We then turn to the main part I involving $\mathscr{S}' := \{Q \in \mathscr{S} : Q \subseteq Q_0\}$. We can largely follow the proof of Theorem 11.3.19, but some care is needed to ensure that we only apply the A_2 condition to cubes contained in Q_0, which need not be the case with the dilated cubes NQ. We start with

$$\begin{aligned} I &= \sum_{Q\in\mathscr{S}'} \langle \mathbf{1}_{Q_0} f\sigma\rangle_{NQ} \int_{Q_0} \mathbf{1}_Q \cdot gw \\ &= \sum_{Q\in\mathscr{S}'} \frac{1}{|NQ|} \int_{NQ\cap Q_0} f\sigma \cdot \int_Q gw \\ &= \sum_{Q\in\mathscr{S}'} \frac{\sigma(NQ\cap Q_0)}{|NQ|} \frac{w(Q)}{|Q|} \langle f\rangle^{\sigma}_{NQ\cap Q_0} \cdot \langle g\rangle^w_Q |Q| \end{aligned}$$

By Lemma 11.3.22, for every $Q \in S' \subseteq \mathscr{D}(Q_0)$, there is a cube $\widetilde{Q}$ such that $Q \subseteq NQ \cap Q_0 \subseteq \widetilde{Q} \subseteq Q_0$ and $\ell(\widetilde{Q}) \leqslant N\ell(Q)$. Thus

$$\sigma(NQ\cap Q_0) \leqslant \sigma(\widetilde{Q}), \quad w(Q) \leqslant w(\widetilde{Q}), \quad |\widetilde{Q}| \leqslant |NQ| = N^d|Q|.$$

Hence

$$\frac{\sigma(NQ\cap Q_0)}{|NQ|} \frac{w(Q)}{|Q|} \leqslant \frac{\sigma(\widetilde{Q})}{|\widetilde{Q}|} \frac{w(\widetilde{Q})}{|\widetilde{Q}|} N^d \leqslant [w]_{A_2(Q_0)} N^d,$$

since $\widetilde{Q}$ is a cube contained in Q_0. Substituting back, and using sparseness, it follows that

$$\int_{Q_0} A^N_{\mathscr{S}'}(\mathbf{1}_{Q_0} f\sigma) \cdot gw \leqslant N^d [w]_{A_2(Q_0)} \sum_{Q\in\mathscr{S}'} \langle f\rangle^{\sigma}_{NQ\cap Q_0} \cdot \langle g\rangle^w_Q \frac{|E(Q)|}{\varepsilon}.$$

As in the proof of Theorem 11.3.19, we have $\langle g\rangle^w_Q \leqslant \inf_{z\in Q} M^w_{\mathscr{D}(Q_0)} g$. Also, using Proposition 11.3.11, each NQ belongs to one of the dilated dyadic systems $\mathscr{D}^{n;N}$, where $n \in \mathbb{Z}^d_N$. A key observation is that then also

$$\mathscr{C}^{n;N} := \{NQ \cap Q_0 : Q \in \mathscr{D}, NQ \in \mathscr{D}^{n;N}\}$$

is a nested family with *set-theoretic* (if not geometric) properties matching those of $\mathscr{D}(Q_0)$: Each of the subfamilies

$$\mathscr{C}^{n;N}_k := \{NQ\cap Q_0 \in \mathscr{C}^{n;N} : \ell(Q) = 2^{-k}\ell(Q_0)\}$$

is a partition of Q_0, and each $\mathscr{C}^{n;N}_{k+1}$ refines the previous $\mathscr{C}^{n;N}_k$. Thus, the corresponding maximal operators

$$M^{\sigma}_{\mathscr{C}^{n;N}} f := \sup_{R\in\mathscr{C}^{n;N}} \mathbf{1}_R \langle f\rangle^{\sigma}_R$$

are still instances of the Doob maximal operator with respect on abstract filtered spaces. Repeating the computation (11.44) *mutatis mutandis*, we then obtain

$$\begin{aligned}\sum_{Q\in\mathscr{S}'}&\langle f\rangle^{\sigma}_{NQ\cap Q_0}\langle g\rangle^{w}_{Q}\frac{|E(Q)|}{\varepsilon}\\&\leqslant\frac{1}{\varepsilon}\sum_{n\in\mathbb{Z}^d_N}\|M^{\sigma}_{\mathscr{C}^{n;N}}f\|_{L^2(\sigma)}\|M^{w}_{\mathscr{D}}g\|_{L^2(w)}\\&\leqslant\frac{1}{\varepsilon}\sum_{n\in\mathbb{Z}^d_N}2\|f\|_{L^2(\sigma)}\cdot 2\|g\|_{L^2(w)}=\frac{4}{\varepsilon}N^d\|f\|_{L^2(\sigma)}\|g\|_{L^2(w)}.\end{aligned}$$

Hence

$$I\leqslant N^d[w]_{A_2(Q_0)}\cdot\frac{4}{\varepsilon}N^d\|f\|_{L^2(\sigma)}\|g\|_{L^2(w)}.$$

In combination with the bound

$$II\leqslant[w]_{A_2(Q_0)}\|f\|_{L^2(\sigma)}\|g\|_{L^2(w)}.$$

Recalling that

$$\int_{Q_0}A^N_{\mathscr{S}}(\mathbf{1}_{Q_0}f\sigma)\cdot gw\leqslant\int_{Q_0}A^N_{\mathscr{S}'}(\mathbf{1}_{Q_0}f\sigma)\cdot gw+\int_{Q_0}\sum_{Q\supsetneq Q_0}\langle\mathbf{1}_{Q_0}f\rangle_{NQ}\cdot gw$$

and the dual weight trick, we conclude the proof in the case of a cube.

If Q_0 is replaced by a quadrant S, we note by density that it suffices to consider the integrals above compactly supported f and g. But then, if Q_0 is a sufficiently large cube contained in the quadrant and having one corner at the corner of the quadrant, then such f and g will be supported in Q_0. Thus the previous considerations apply and give a bound in terms of $[w]_{A_2(Q_0)}$, which is clearly dominated by $[w]_{A_2(S)}$. □

An extension of Proposition 11.3.21 to $p\neq 2$ follows, in principle, by Rubio de Francia's Extrapolation Theorem J.2.1 just like Corollary 11.3.20 from Theorem 11.3.19. Since Theorem J.2.1 was formulated for global $A_p(\mathbb{R}^d)$ weights only, we include some remarks about its local version. As a rule, all dyadic considerations carry over without any change. However, one needs to play a little attention to the interplay of dyadic and non-dyadic cubes in the local setting. The following is a local variant of the Covering Lemma 3.2.26:

Lemma 11.3.23. *For cubes $Q\subseteq Q_0\subseteq\mathbb{R}^d$, there exist a vector $\alpha\in\{0,\frac{1}{3},\frac{2}{3}\}^d$ and a dyadic cube*

$$D\in\mathscr{D}^{\alpha}(Q_0):=\{P+\alpha(-1)^{\log_2\frac{\ell(P)}{\ell(Q_0)}}\ell(P):P\in\mathscr{D}(Q_0)\}\tag{11.45}$$

(the shifted dyadic cubes from Definition 3.2.25) such that

$$\ell(D)\leqslant 3\ell(Q)\quad\textit{and}\quad Q\subseteq D\subseteq Q_0.$$

In (11.45), the point of the factor $(-1)^{\log_2\frac{\ell(P)}{\ell(Q_0)}}$ is simply to alternate between ±1 with each consecutive generation of the dyadic cubes. We refer the reader to the discussion preceding Lemma 3.2.26 for why such a factor is needed.

Proof. If $3\ell(Q) \geqslant \ell(Q_0)$, then clearly $D := Q_0 \in \mathscr{D}(Q_0) = \mathscr{D}^0(Q_0)$ satisfies the required properties.

Let then $3\ell(Q) < \ell(Q_0)$. By Lemma 3.2.26 (a global version of the lemma that we are proving), there exists a cube D as asserted, expect that we do not know whether $D \subseteq Q_0$ or not. If yes, then we are done, so suppose that $D \not\subseteq Q_0$. We will check that an appropriate shift of D will be a cube that we are looking for.

Let first $d = 1$ so that $Q_0 = [a, b)$ as well as Q and D are just intervals. If D extends to the left of a, then we can take $D' := [a, a + \ell(D)) \in \mathscr{D}(Q_0)$, and if D extends to the right of b, then we can take $D' := [b - \ell(D), b) \in \mathscr{D}(Q_0)$.

Let then $d \geqslant 1$ be arbitrary, $Q = I_1 \times \cdots \times I_d \subseteq D = J_1 \times \cdots \times J_d \in \mathscr{D}^\alpha(\mathbb{R}^d)$, and $Q_0 := K_1 \times \cdots \times K_d$. For each $i \in \{1, \ldots, d\}$, we run the previous construction: If $J_i \subseteq K_i$, we let $J_i' := J_i \in \mathscr{D}^{\alpha_i}(\mathbb{R})$. If $J_i \not\subseteq K_i$, we let J_i' be the interval of lengths $\ell(J_i)$ that meets the same end-point of K_i as J_i. Then $J_i' \in \mathscr{D}(K_i)$. Defining $D' := J_1' \times \cdots \times J_d'$, we have $D' \in \mathscr{D}^{\alpha'}$, where $\alpha_i' = \alpha_i$ if $J_i \subseteq K_i$ and $\alpha_i' = 0$ otherwise. This D' in place of D satisfies the claimed properties, and the proof of the lemma is complete. □

As in (3.36), we can now easily dominate the local maximal operator

$$M_{Q_0} f(x) := \sup_{\substack{Q \subseteq Q_0 \\ \text{cube}}} \mathbf{1}_Q(x) \fint_Q \|f(y)\| \, \mathrm{d}y$$

by the local dyadic maximal operators

$$M_{Q_0}^\alpha f(x) := \sup_{\substack{P \in \mathscr{D}^\alpha(Q_0) \\ P \subseteq Q_0}} \mathbf{1}_P(x) \fint_P \|f(y)\| \, \mathrm{d}y, \quad \alpha \in \{0, \tfrac{1}{3}, \tfrac{2}{3}\}^d$$

with

$$M_{Q_0} f \leqslant 3^d \max_{\alpha \in \{\{0, \frac{1}{3}, \frac{2}{3}\}^d} M_{Q_0}^\alpha f \leqslant 3^d \sum_{\alpha \in \{\{0, \frac{1}{3}, \frac{2}{3}\}^d} M_{Q_0}^\alpha f. \tag{11.46}$$

Proposition 11.3.24. *Let $p, r \in (1, \infty)$ and cube $Q_0 \subseteq \mathbb{R}^d$ be a cube. Then*

(1) $\|M_{Q_0} f\|_{L^p(Q_0, w)} \leqslant c_d p' [w]_{A_p}^{1/(p-1)} \|f\|_{L^p(Q_0, w)}$;

(2) *if a pair of functions (f, h) satisfies*

$$\|h\|_{L^r(Q_0, w)} \leqslant \phi_r([w]_{A_r(Q_0)}) \|f\|_{L^r(Q_0, w)}$$

for all $w \in A_r(Q_0)$, where ϕ_r is a non-negative increasing function, then

$$\|h\|_{L^p(Q_0, w)} \leqslant \phi_{dpr}([w]_{A_p}) \|f\|_{L^p(Q_0, w)}$$

for all $w \in A_p(Q_0)$, where each ϕ_{dpr} is a non-negative increasing function. In particular, if $\phi_r(t) = c_r t^\tau$, then $\phi_{dpr}(t) \leqslant c_{dpr} t^{\tau \max\{\frac{r-1}{p-1}, 1\}}$.

Proof. (1) follows by repeating the proof of Theorem J.1.1: the dyadic considerations are unchanged, and in the last step of the proof, one replaces an application of (3.36) by its localised version (11.46).

The proof of (2) is the same as the proof of Theorem J.2.1, except that the all references to the maximal operator M are replaced by the local version M_{Q_0} and, accordingly, all applications of Theorem J.1.1 by case (1) of the proposition that we already proved. (We note that the ϕ_{pr} and c_{pr} should be replaced by ϕ_{dpr} and c_{dpr} already in Theorem J.2.1; the omission of the dependence on d is a systematic typo in Theorem J.2.1 and its proof.) □

Corollary 11.3.25. *Let $\varepsilon \in (0,1)$ and $\mathscr{S} \subseteq \mathscr{D}$ be ε-sparse, and let $Q_0 \in \mathscr{D}$. If $N \in \mathbb{Z}_+$ is odd, $p \in (1,\infty)$, and $w \in A_p(Q_0)$, then the sparse operator $A^N_{\mathscr{S}}$ is bounded on $L^p(Q_0, w)$, and*

$$\|A^N_{\mathscr{S}}\|_{\mathscr{L}(L^p(Q_0,w))} \leqslant c_{d,p}\Big(\frac{4}{\varepsilon}N^{2d}+1\Big)[w]_{A_p(Q_0)}^{\max(1,\frac{1}{p-1})}.$$

The same result is true if Q_0 is replaced by a quadrant of $\mathbb{R}^d$.

Proof. The case of a cube is immediate from case $p=2$ established in Proposition 11.3.21 and extrapolation established in Proposition 11.3.24(2). The case of a quadrant follows from this by the same considerations as in the last paragraph of the proof of Proposition 11.3.21. □

Thanks to sparse domination, we also obtain the corresponding results for Calderón–Zygmund operators:

Theorem 11.3.26 (A_2 theorem). *Let X and Y be Banach spaces, $p_0 \in [1,\infty]$, and let*

$$T \in \mathscr{L}(L^{p_0}(\mathbb{R}^d;X), L^{p_0,\infty}(\mathbb{R}^d;Y))$$

with norm N_0 be an operator with a Dini kernel K. Then for every $p \in (1,\infty)$ and every $w \in A_p$, the operator T extends uniquely to

$$T \in \mathscr{L}(L^p(w;X), L^p(w;Y))$$

with norm estimate

$$\|T\|_{\mathscr{L}(L^p(w;X),L^p(w;Y))} \leqslant c_{d,p}\Big(N_0 + c_K + \|\omega_K\|_{\rm Dini}\Big)[w]_{A_p}^{\max(1,\frac{1}{p-1})}$$

where c_K, ω_K are as in Definition 11.3.1.

The result remain true if $\mathbb{R}^d$ is systematically replaced by a cube $Q_0 \subseteq \mathbb{R}^d$ or a quadrant $S \subseteq \mathbb{R}^d$, as the domain of the function spaces, in the definition of the Calderón–Zygmund constants c_K and $\|\omega_K\|_{\rm Dini}$, as well as in the definition of the weight class A_p.

Proof. Let us first consider the global case. Let $f \in L^p_{\rm c}(w;X)$ be supported on a compact set F. Denoting by $\sigma = w^{-1/(p-1)}$ the dual weight, we have

$$\int \|f\| = \int_K \|f\| w^{1/p}\sigma^{1/p'} \leqslant \|f\|_{L^p(w)}\sigma(K)^{1/p'} < \infty,$$

so that $f \in L^1_{\rm c}(\mathbb{R}^d;X)$ as well, and Tf is well defined by the Calderón–Zygmund theorem 11.2.5. Then Theorem 11.3.15 guarantees the existence of a $\frac{1}{5}$-sparse collection $\mathscr{S} \subseteq \mathscr{D}$ such that, pointwise almost everywhere,

$$\|Tf(x)\|_Y \leqslant c_d c_T (A^5_{\mathscr{S}}\|f\|_X)(x), \qquad c_T = N_0 + c_K + \|\omega\|_{\rm Dini}.$$

Thus, by Corollary 11.3.20, we have

$$\begin{aligned}\|Tf(x)\|_{L^p(w;Y)} &\leqslant c_d c_T \|A^5_{\mathscr{S}}(\|f\|_X)\|_{L^p(w)} \\ &\leqslant c_d c_T c_{d,p}[w]_{A_p}^{\max(1,\frac{1}{p-1})}\Big\|\|f\|_X\Big\|_{L^p(w)} \\ &= c_{d,p}c_T[w]_{A_p}^{\max(1,\frac{1}{p-1})}\|f\|_{L^p(w;X)}.\end{aligned} \tag{11.47}$$

Recalling the definition of c_T, this is the required norm estimate for T restricted to $L^p_{\rm c}(w;X)$; since this subspace is dense in $L^p(w;X)$, it allows to uniquely extend T to the whole space with the same norm.

The proof in the case of a cube or a quadrant in place of $\mathbb{R}^d$ remains the same, just using the local Corollary 11.3.25 in place of Corollary 11.3.20 to replace (11.47) by

$$\begin{aligned}\|Tf(x)\|_{L^p(Q_0,w;Y)} &\leqslant c_d c_T \|A^5_{\mathscr{S}}(\|f\|_X)\|_{L^p(Q_0,w)} \\ &\leqslant c_d c_T c_{d,p}[w]_{A_p(Q_0)}^{\max(1,\frac{1}{p-1})}\Big\|\|f\|_X\Big\|_{L^p(Q_0,w)} \\ &= c_{d,p}c_T[w]_{A_p(Q_0)}^{\max(1,\frac{1}{p-1})}\|f\|_{L^p(Q_0,w;X)}.\end{aligned}$$

□

Corollary 11.3.27 (A_2 theorem for the Hilbert transform). *Let X be a UMD space, $p \in (1,\infty)$ and $w \in A_p(\mathbb{R})$. Then the Hilbert transform*

$$Hf(s) = \lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{|s-t|>\varepsilon}\frac{f(t)}{s-t}\,{\rm d}t$$

extends uniquely to $H \in \mathscr{L}(L^p(w;X))$ with

$$\|H\|_{\mathscr{L}(L^p(w;X))} \leqslant c_p[w]_{A_p}^{\max(1,\frac{1}{p-1})}\hbar_{2,X}, \quad \hbar_{2,X} := \|H\|_{\mathscr{L}(L^2(\mathbb{R};X))}.$$

Proof. Recall that the Hilbert transform is bounded on $L^2(\mathbb{R};X)$ when X is a UMD space (Theorem 5.1.13). In particular, taking $T = H$ and $p_0 = 2$

in Theorem 11.3.26, we have $N_0 \leqslant \hbar_{2,X} < \infty$, using the notation from the statement of that theorem. The kernel of the Hilbert transform is $K(s,t) = \frac{1}{s-t}$, so that $c_K = 1$ qualifies for the constant in Definition 11.3.1. Moreover,

$$\Big|\frac{1}{s-t} - \frac{1}{s'-t}\Big| = \Big|\frac{s'-s}{(s-t)(s'-t)}\Big| \leqslant 2\frac{|s'-t|}{|s-t|^2} \quad \forall |s-s'| \leqslant \frac{1}{2}|s-t|,$$

so that we can take the modulus of continuity $\omega_1(u) = 2u$ in Definition 11.3.1. Checking that $\omega_2(u) = 2u$ also works in entirely similar. Thus $\|\omega\|_{\mathrm{Dini}} = \int_0^1 2u \frac{\mathrm{d}u}{u} = 2$. Finally, it is easy to check that the norm $\hbar_{2,X} = \|H\|_{\mathscr{L}(L^2(\mathbb{R};X))}$ is *at least* 1, say by Proposition 5.2.2, which says that H acts as multiplication by $-i$ on functions with Fourier transform supported on $\mathbb{R}_+$. Thus $N_0 + c_K + \|\omega\|_{\mathrm{Dini}} \leqslant \hbar_{2,X} + 1 + 2 \leqslant 4\hbar_{2,X}$. Substituting this into the result of Theorem 11.3.26 gives the claimed bound for $\|H\|_{\mathscr{L}(L^p(w;X))}$. □

11.3.e Sharpness of the A_2 theorem

Already in the scalar-valued case $X = \mathbb{K}$, Corollary 11.3.27, and hence Theorem 11.3.26, is sharp in its dependence on the weight characteristic $[w]_{A_p}$. In order to see this, we need to know about the behaviour of $[w]_{A_p}$ for some concrete examples of weights, for which we can also estimate the weighted norm of the Hilbert transform. The following important power weights will serve this purpose:

Example 11.3.28 (Power weights). Let $\alpha \in \mathbb{R}$, $p \in (1,\infty)$, $w(x) = |x|^\alpha$ for $x \in \mathbb{R}^d$, and $\sigma(x) = w(x)^{-1/(p-1)} = |x|^{-\alpha/(p-1)}$. Then

$$w \in A_p(\mathbb{R}^d) \quad \Leftrightarrow \quad w, \sigma \in L^1_{\mathrm{loc}}(\mathbb{R}^d) \quad \Leftrightarrow \quad -d < \alpha < d(p-1),$$

and if these equivalent conditions holds, then

$$c_{d,p}[w]_{A_p} \leqslant \frac{1}{1+\alpha}\Big(\frac{1}{p-1-\alpha}\Big)^{p-1} \leqslant C_{d,p}[w]_{A_p}.$$

To verify the claims of this example, we make use of the following:

Lemma 11.3.29. *If $Q \subseteq \mathbb{R}^d$ is any cube, and $\tilde{Q}$ is a cube of the same size centred at the origin, then*

$$\fint_Q |x|^{-\gamma}\,\mathrm{d}x \leqslant \fint_{\tilde{Q}} |x|^{-\gamma}\,\mathrm{d}x \eqsim_d \frac{\ell(Q)^{-\gamma}}{d-\gamma}, \qquad \gamma \in [0,d),$$
$$\fint_Q |x|^{\gamma}\,\mathrm{d}x \geqslant \fint_{\tilde{Q}} |x|^{\gamma}\,\mathrm{d}x \eqsim_{d,\Gamma} \ell(Q)^{\gamma}, \qquad \gamma \in [0,\Gamma], \qquad \Gamma > 0.$$

Proof. Let $Q = \prod_{i=1}^d I_i$ and $\tilde{Q} = \prod_{i=1}^d \tilde{I}_d$. Then Q is the disjoint union of the sets $Q_{\mathscr{I}} := \prod_{i\in\mathscr{I}}(I_i \cap \tilde{I}_i) \times \prod_{i\in\complement\mathscr{I}}(I_j \setminus \tilde{I}_j)$, where $\mathscr{I}$ ranges over all subsets of $\{1,\dots,d\}$, and $\complement\mathscr{I} := \{1,\dots,d\}\setminus\mathscr{I}$. Of course $\tilde{Q}$ is a similar union over $\tilde{Q}_{\mathscr{I}}$, defined by interchanging the roles of I_j and $\tilde{I}_j$ in $Q_{\mathscr{I}}$.

Since $\ell(I_j)=\ell(\tilde{I}_j)$ is the common side-length of Q and $\tilde{Q}$, it follows that also $|I_j\setminus\tilde{I}_j|=|\tilde{I}_j\setminus I_j|$. Since $\tilde{I}_j$ is centred at the origin, if $x_j\in I_j\setminus\tilde{I}_j$ and $\tilde{x}_j\in\tilde{I}_j\setminus I_j$, then $|\tilde{x}_j|\leqslant|x_j|$.

Now all $x=(x_i)_{i=1}^d\in Q_{\mathscr{I}}$ are in measure-preserving correspondence with $\tilde{x}=(\tilde{x}_i)_{i=1}^d\in\tilde{Q}_{\mathscr{I}}$, such that $|x_i|=|\tilde{x}_i|$ for all $i\in\mathscr{I}$, and $|x_j|\geqslant|\tilde{x}_j|$ for all $j\in\complement\mathscr{I}$; hence altogether $|x|\geqslant|\tilde{x}|$.

This implies inequalities like the first ones on each line of the lemma, for $Q_{\mathscr{I}}$ and $\tilde{Q}_{\mathscr{I}}$ in place of Q and $\tilde{Q}$, and thus also these inequalities as claimed, by summing over all $\mathscr{I}\subseteq\{1,\dots,d\}$.

To estimate the integrals over $\tilde{Q}$, we note that $B(0,\frac{1}{2}\ell(Q))\subseteq\tilde{Q}\subseteq B(0,\frac{1}{2}\sqrt{d}\ell(Q))$, where, for $\alpha>-d$,

$$\int_{B(0,c_d\ell(Q))}|x|^\alpha\,\mathrm{d}x=\int_0^{c_d\ell(Q)}r^\alpha r^{d-1}\sigma_{d-1}\,\mathrm{d}r=\frac{(c_d\ell(Q))^{d+\alpha}}{d+\alpha}\sigma_{d-1},$$

thus

$$2^{-d-\alpha}\sigma_{d-1}\frac{\ell(Q)^\alpha}{d+\alpha}\leqslant\fint_{\tilde{Q}}|x|^\alpha\,\mathrm{d}x\leqslant(2^{-1}\sqrt{d})^{d+\alpha}\sigma_{d-1}\frac{\ell(Q)^\alpha}{d+\alpha}.$$

For $\alpha=-\gamma\in(-d,0]$, the quantities multiplying $\ell(Q)^\alpha/(d+\alpha)=\ell(Q)^{-\gamma}/(d-\gamma)$ are clearly uniformly bounded from above and away from zero, with bounds depending on d only. Similarly, for $\alpha=\gamma\in[0,\Gamma]$, the quantities multiplying $\ell(Q)^\alpha=\ell(Q)^\gamma$ have this property, with bounds depending on d and Γ only. □

Proof of Example 11.3.28. The second $\Leftrightarrow$ in the claim is immediate.

Note that at least one of w and σ is $|x|$ to a non-negative exponent, and therefore locally integrable with a strictly positive integral over every cube Q. Thus, in order that $[w]_{A_p}$ is finite, it is necessary that the other of the two functions is locally integrable as well, showing the first $\Rightarrow$ in the claim.

It remains to check that $-d<\alpha<d(p-1)$ implies that $w\in A_p(\mathbb{R}^d)$, together with the claimed estimate for $[w]_{A_p}$.

Let first $\alpha\geqslant0$, and denote $\delta_Q:=\operatorname{dist}(Q,0)/\ell(Q)$. For $x\in Q$, we have $|x|\leqslant(\delta_Q+\sqrt{d})\ell(Q)$, and thus $\fint_Q w\leqslant(\delta_Q+\sqrt{d})^\alpha\ell(Q)^\alpha$. If $\delta_Q>0$, we also have $|x|^{-1}\leqslant\delta_Q^{-1}\ell(Q)^{-1}$, and hence $(\fint_Q\sigma)^{p-1}\leqslant\delta_Q^{-\alpha}\ell(Q)^{-1}$. Thus

$$\sup_{Q:\delta_Q\geqslant\delta}\fint_Q w\Big(\fint_Q\sigma\Big)^{p-1}\leqslant\sup_{Q:\delta_Q\geqslant\delta}(\delta_Q+\sqrt{d})^\alpha\delta_Q^{-\alpha}=\Big(1+\frac{\sqrt{d}}{\delta}\Big)^\alpha$$

On the other hand, for any cube Q, it follows from Lemma 11.3.29 that

$$\begin{aligned}\sup_{Q:\delta_Q\leqslant\delta}\int_Q w\Big(\fint_Q\sigma\Big)^{p-1}&\leqslant\sup_{Q:\delta_Q\leqslant\delta}(\delta_Q+\sqrt{d})^\alpha\ell(Q)^\alpha\Big(\frac{2^d d}{d-\frac{\alpha}{p-1}}\ell(Q)^{-\frac{\alpha}{p-1}}\Big)^{p-1}\\&=(\delta+\sqrt{d})^\alpha\Big(\frac{2^d d}{d-\frac{\alpha}{p-1}}\Big)^{p-1}\end{aligned}$$

Fixing some $\delta = \delta_{d,p}$, it is then immediate that

$$[w]_{A_p} \leqslant \frac{c_{d,p}}{(d - \frac{\alpha}{p-1})^{p-1}} = \frac{c'_{d,p}}{[d(p-1) - \alpha]^{p-1}}$$

For a matching lower bound, it is enough to consider just the unit cube Q, in which case the estimates of Lemma 11.3.29 apply with $\Gamma = d(p-1)$ to give that

$$[w]_{A_p} \geqslant \fint_Q w\Big(\fint_Q \sigma\Big)^{p-1} \eqsim_{d,p} 1 \cdot \Big(\frac{1}{d - \frac{\alpha}{p-1}}\Big)^{p-1} \eqsim_{d,p} \Big(\frac{1}{d(p-1) - \alpha}\Big)^{p-1}.$$

This completes the proof for $\alpha \in [0, d(p-1))$, noting that $\frac{1}{1+\alpha} \eqsim_{d,p} 1$ in this case.

For $\alpha = -\gamma < 0$, we note that

$$\begin{aligned}[][|x|^{-\gamma}]_{A_p} = [|x|^{\frac{\gamma}{p-1}}]_{A_{p'}}^{p-1} &\eqsim_{d,p} \Big\{\Big(\frac{1}{d(p'-1) - \frac{\gamma}{p-1}}\Big)^{p'-1}\Big\}^{p-1} \\ &= \frac{p-1}{d-\gamma} \eqsim_{d,p} \frac{1}{d+\alpha}\end{aligned}$$

by applying the previous case to $\frac{\gamma}{p-1} \geqslant 0$ and p' in place of α and p, and noting that $(p-1)(p'-1) = 1$. □

We are now fully equipped to confirm the sharpness of Corollary 11.3.27.

Proposition 11.3.30 (Buckley). *Fix $p \in (1, \infty)$, and suppose that $\phi : [1, \infty) \to [1, \infty)$ is an increasing function such that*

$$\|H\|_{\mathscr{L}(L^p(w))} \leqslant \phi([w]_{A_p}) \qquad \forall w \in A_p,$$

or even just for all power weights in A_p. Then

$$\phi(t) \geqslant c_p \cdot t^{\max(1, \frac{1}{p-1})} \qquad \forall t \geqslant 1.$$

Proof. Let $\sigma = w^{-1/(p-1)}$ denote the dual weight. Using the dualised formulation (11.43) of the $L^p(w)$-boundedness of $T = H$, and choosing f and g with positively separated compact supports, so that the kernel representation is available, we have

$$\frac{1}{\pi} \iint \frac{f(y)\sigma(y)g(x)w(x)}{x-y} \,\mathrm{d}x \,\mathrm{d}y \leqslant \phi([w]_{A_p}) \|f\|_{L^p(\sigma)} \|g\|_{L^{p'}(w)} \tag{11.48}$$

for all such f and g. If these functions are non-negative with $\operatorname{supp} f \subseteq \mathbb{R}_-$ and $\operatorname{supp} g \subseteq \mathbb{R}_+$, then the integrand is non-negative, and by monotone convergence (11.48) persists even if the supports of f and g meet at the origin.

The crucial point in bounding the Hilbert transform form below is the following observation: if $h(y) = |y|^{-\alpha}\mathbf{1}_{(-1,0)}(y)$, then for $x \in (0, 1)$,

$$Hh(x) = \frac{1}{\pi}\int_0^1 \frac{y^{-\alpha}}{x+y}\,\mathrm{d}y \geqslant \frac{1}{\pi}\int_0^x \frac{y^{-\alpha}}{2x}\,\mathrm{d}y = \frac{1}{2\pi}\frac{x^{-\alpha}}{1-\alpha}, \tag{11.49}$$

which is essentially h again, but with a factor $\frac{1}{1-\alpha}$ that blows up as $\alpha \to 1-$.

We now "test" (11.48) with two choices of (f, g, σ, w), so that $(f\sigma, gw)$ is either $(|y|^{-\alpha}\mathbf{1}_{(-1,0)}, \mathbf{1}_{(0,1)})$ or $(\mathbf{1}_{(-1,0)}, |y|^{-\alpha}\mathbf{1}_{(0,1)})$, with $\alpha \in [0, 1)$. In either case (11.49) shows that

$$LHS(11.48) \geqslant \frac{1}{2\pi}\int_0^1 \frac{x^{-\alpha}}{1-\alpha}\,\mathrm{d}x = \frac{1}{2\pi}\frac{1}{(1-\alpha)^2},$$

where we have accumulated a quadratic blow-up.

To estimate the right hand side of (11.48), we need to specify the individual functions, not just the products $f\sigma$ and gw. In the first case, let $f = \mathbf{1}_{(-1,0)}$ and $\sigma(y) = w(y)^{-1/(p-1)} = |y|^{-\alpha}$; thus $w(y) = |y|^{\alpha(p-1)}$ and $g(y) = \mathbf{1}_{(0,1)}(y)w(y)^{-1} = \mathbf{1}_{(0,1)}(y)|y|^{-\alpha(p-1)}$. Then

$$\begin{aligned}\|f\|_{L^p(\sigma)}\|g\|_{L^{p'}(w)} &= \Big(\int_0^1 x^{-\alpha}\,\mathrm{d}x\Big)^{1/p}\Big(\int_0^1 x^{\alpha(p-1)(1-p')}\,\mathrm{d}x\Big)^{1/p'} \\ &= 1/(1-\alpha).\end{aligned} \tag{11.50}$$

noting that $(p-1)(p'-1) = 1$, and Example 11.3.28 shows that $[w]_{A_p} \leqslant c_p/(1-\alpha)^{p-1}$. Thus, altogether, we have

$$\frac{1}{2\pi}\frac{1}{(1-\alpha)^2} \leqslant (11.48) \leqslant \phi\Big(\frac{c_p}{(1-\alpha)^{p-1}}\Big)\frac{1}{1-\alpha}. \tag{11.51}$$

Denoting $t = c_p/(1-\alpha)^{p-1}$, this reduces to

$$\phi(t) \geqslant \tilde{c}_p t^{1/(p-1)} \qquad \forall t \geqslant c_p. \tag{11.52}$$

Since $H^2 = -I$, it is clear that $\|H\|_{\mathscr{L}(L^p(w))} \geqslant 1$, and hence $\phi(t) \geqslant 1 \geqslant c'_p t^{1/(p-1)}$ for $t \in [1, c_p)$ as well.

In the second case, we take $g = \mathbf{1}_{(0,1)}$ and $w(x) = \sigma(x)^{1-p} = |x|^{-\alpha}$; thus $\sigma(x) = |x|^{\alpha/(p-1)} = |x|^{\alpha(p'-1)}$ and $f(x) = \mathbf{1}_{(-1,0)}(x)|x|^{-\alpha(p'-1)}$. A computation like (11.50) gives exactly the same final result, only with a slightly different intermediate step, and Example 11.3.28 shows that $[w]_{A_p} \leqslant c_p/(1-\alpha)$. With this quantity inside ϕ in (11.51), the substitution $t = c_p/(1-\alpha)$ then gives

$$\phi(t) \geqslant \tilde{c}_p t \qquad \forall t \geqslant c_p, \tag{11.53}$$

and the same bound for $t \in [1, c_p)$ follows from $H^2 = -I$ as before. The two lower bounds (11.52) and (11.53) together prove the proposition. □

11.4 Notes

Given the emphasis of these volumes in analysis of functions having their *range* in a Banach space, we have chosen to keep the consideration related to the *domain* of the functions relatively simple, concentrating on the canonical case of the Euclidean space $\mathbb{R}^d$ and, with specific applications in the later chapters in mind, its rather special subdomains—cubes and quadrants—only. However, much of this theory could be developed on far more general domains, notably on *spaces of homogeneous type* (espaces de nature homogène) introduced by Coifman and Weiss [1971] and extensively studied ever since. Since our treatment is heavily based on the dyadic cubes on $\mathbb{R}^d$, we recall that analogous constructions are also available in the mentioned generality. The construction of a fixed family of sets, sharing the essential properties of the standard dyadic cubes of $\mathbb{R}^d$, is due to Christ [1990]. We also make use of "adjacent" and "random" families of dyadic cubes; a reasonably comprehensive account of their analogues in spaces of homogeneous type is provided by Hytönen and Kairema [2012] with several variants and elaborations due to Auscher and Hytönen [2013], Hytönen and Martikainen [2012], Hytönen and Tapiola [2014], and Nazarov, Reznikov, and Volberg [2013].

Section 11.1

This section deals with relatively classical topics but with some modern flavour. In particular, the local oscillation decomposition of Theorem 11.1.12 dates essentially back to Lerner [2010] in the scalar-valued case. The vector-valued generalisation, introducing the notion of λ-pseudomedian, was first found by Hänninen and Hytönen [2014]. Our present proof streamlines the original one.

Proposition 11.1.14 was proved by Katz and Pereyra [1999] in the scalar-valued case via a multilinear estimate, and by Hänninen and Hytönen [2016] as stated.

Theorem 11.1.30 on the vector-valued H^1–BMO duality is essentially from Bourgain [1986], although the present proof is different. In this circle of ideas, we have only covered the relatively elementary part of the theory that does not require any assumptions on the underlying Banach space. Note that Theorem 11.1.30 says that $\mathrm{BMO}_{\mathscr{D}}(\mathbb{R}^d; X^*)$ can be identified with an isometric subspace of $(H^1_{\mathscr{D},\mathrm{at}}(\mathbb{R}^d; X))^*$. The same proof works in the non-dyadic case, where arbitrary cubes are allowed both in the definition of BMO and of the Hardy space atoms. To describe the full dual $(H^1_{\mathrm{at}}(\mathbb{R}^d; X))^*$, Blasco [1988] defines a class of Banach space Y-valued measures $\mathscr{BMO}(\mathbb{R}^d; Y)$. Among other things, he shows that $(H^1_{\mathrm{at}}(\mathbb{R}^d; X))^* = \mathscr{BMO}(\mathbb{R}^d; X^*)$ for every Banach space X, whereas $\mathscr{BMO}(\mathbb{R}^d; Y) = \mathrm{BMO}(\mathbb{R}^d; Y)$, if and only if Y has the Radon–Nikodým property. A recent account with more information on the Banach space valued H^1 and BMO can be found in Chapter 7 of Pisier [2016].

Section 11.2

The material of this section is predominantly classical, and most of the results would have been available in essentially the present form by the 1980's, if not earlier, even in the Banach space valued setting. The scalar-valued origins, of course, date much further back.

The essence of Theorem 11.2.5 comes from Calderón and Zygmund [1952], who consider the scalar-valued case ($X = Y = \mathscr{L}(X,Y) = \mathbb{C}$) and Dini kernels of the special form $K(x,y) = K(x-y) = |x-y|^{-d}\Omega\big(\frac{x-y}{|x-y|}\big)$, where moreover $\int_{S^{n-1}} \Omega \, \mathrm{d}\sigma = 0$. In contrast to Theorem 11.2.5, which extrapolates other L^p-bounds from an assumed *a priori* L^{p_0}-bound, Calderón and Zygmund [1952] obtained their L^p-boundedness conclusions unconditionally, i.e., they also deduce the initial L^{p_0}-bound for $p_0 = 2$ from their special assumptions on the kernel. Once this is achieved, the extrapolation to other L^p-bounds is carried out in much the same way as in the present treatment, particularly in the case $p < p_0$. The fact that the extrapolation part of Calderón and Zygmund [1952] argument remains valid under more general assumptions on the kernel was observed by Hörmander [1960], who introduced the conditions, now bearing his name, in Definition 11.2.1 in the case of scalar-valued convolution kernels $K(x,y) = \mathfrak{K}(x-y)$. What we have called the (operator-)Hörmander class Hör was designated as K^1 by Hörmander [1960], who also defines a family of related conditions K^a with a parameter $a \in [1,\infty]$. Just like Hör $= K^1$ is relevant for the extrapolation of L^p-boundedness, the condition K^a permits the extrapolation of L^p-to-L^q boundedness from one pair (p,q) with $\frac{1}{p} - \frac{1}{q} = 1 - \frac{1}{a}$ to other such pairs.

The first Banach space-valued generalisations, which used the operator-Hörmander conditions, were found by Schwartz [1961] and, apparently independently, by Benedek, Calderón, and Panzone [1962]. According to García-Cuerva and Rubio de Francia [1985], the fact that the mere Hörmander condition (involving integrals of $\|K(s,t)x - K(s',t)x\|_Y$ rather than $\|K(s,t) - K(s',t)\|_{\mathscr{L}(X,Y)}$) is sufficient for results like Theorem 11.2.5 "should have been observed by anyone trying to adapt the proof of [the Calderón–Zygmund theorem] to the vector valued case", yet they "do not emphasize very much the interest of this weaker condition since, in most of the applications of vector valued singular integrals, [the operator Hörmander condition] does hold." Rubio de Francia, Ruiz, and Torrea [1986] provided, in their own words, an "updated review" of Benedek et al. [1962], incorporating several new developments in singular integrals into the vector-valued theory, and in particular explicitly dealing with two-variable kernels $K(s,t)$, as we have done here. Our considerations related to c_0 in Theorem 11.2.9 were inspired by Girardi and Weis [2004].

A version of Theorem 11.2.5 for convolution kernels $K(s,t) = \mathfrak{K}(s-t)$ is also presented by Grafakos [2008], where (in contrast to our approach) the upper extrapolation is achieved by a duality argument, and the interested reader is referred to this work for details of that approach. Grafakos [2008] is also explicit about the norm estimate in Theorem 11.2.5(3); this is certainly well known, but often not spelled out in many references.

Section 11.3

The main body of this section consists of results from the 2010's. Since the discovery of the original forms of many of these results, there has been significant activity in generalising and streamlining their proofs, as well as developing entirely new approaches. As a result, our order of presentation deviates from the historical timeline in favour of a smoother mathematical story. A main result of this section is certainly the A_2 Theorem 11.3.26, but the various Sparse Domination Theorems 11.3.6, 11.3.14, and 11.3.15, originally developed as tools for proving the A_2 Theorem 11.3.26, have by now established themselves as results of intrinsic value and models for desirable type of domination to search for in other situations.

Prehistory of the A_2 theorem

In its scalar-valued and qualitative form (i.e., saying that T is bounded on $L^p(w)$, but without tracking the estimate for the operator norm), the result goes back to Hunt, Muckenhoupt, and Wheeden [1973] in the special case that T is the Hilbert transform (as in Corollary 11.3.27) and to Coifman and Fefferman [1974] for all standard Calderón–Zygmund operators of convolution type. The question of sharp dependence of the weighted operator norms $\|T\|_{\mathscr{L}(L^p(w))}$ on the weights constant $[w]_{A_p}$ was raised by Buckley [1993], who settled the case of the Hardy–Littlewood maximal operator (Theorem J.1.1) and obtained non-matching upper and lower bounds for Calderón–Zygmund operators. In particular, Proposition 11.3.30 saying that an estimate for $\|T\|_{\mathscr{L}(L^p(w))}$ can be no better than $[w]_{A_p}^{\max(1,\frac{1}{p-1})}$, is essentially from Buckley [1993]. In many papers, results of this type a stated in a slightly weaker form along the lines that "the power of $[w]_{A_p}$ can be no better than $\max(1,\frac{1}{p-1})$". However, in some related questions, the sharp estimate is known to exhibit behaviour different from a pure power law.

The question of Buckley [1993] gained new interest through the work of Astala, Iwaniec, and Saksman [2001], who considered the following problem: Let $\mathcal{O} \subseteq \mathbb{C}$ be a domain and $k \in (0,1)$. What is the minimal q such that all functions $f \in W^{1,q}_{\mathrm{loc}}(\mathcal{O})$ with $|\bar{\partial} f| \leqslant k|\partial f|$ (referred to as *weakly quasiregular*) must in fact belong to $f \in W^{1,2}_{\mathrm{loc}}(\mathcal{O})$ (and then be called simply *quasiregular*)? By results of Astala [1994], $q > 1 + k$ suffices; by examples due to Iwaniec and Martin [1993], $q < 1 + k$ does not, leaving $q = 1 + k$ as the critical case. Astala, Iwaniec, and Saksman [2001] proved that $q = 1+k$ is still sufficient for the said self-improvement, under their *conjecture* that the Beurling–Ahlfors transform

$$Bf(z) := -\frac{1}{\pi} \lim_{\varepsilon\to 0} \int_{\mathbb{C}\setminus D(z,\varepsilon)} \frac{f(y)\,\mathrm{d}A(y)}{(z-y)^2}, \quad D(z,\varepsilon) := \{y \in \mathbb{C} : |y - z| < \varepsilon\}$$

satisfies the upper bound

$$\|B\|_{\mathscr{L}(L^p(w))} \leqslant c_p [w]_{A_p}, \qquad p \in [2, \infty). \tag{11.54}$$

Special cases of the A_2 theorem

Shortly after being posed, the conjecture of Astala et al. [2001] was verified by Petermichl and Volberg [2002], and another proof was found by Dragičević and Volberg [2003]. Already Petermichl and Volberg [2002] observed that (11.54) as stated may be derived from its special case $p = 2$ by keeping track of the constants in the proof of Rubio de Francia's extrapolation theorem as presented, e.g., by García-Cuerva and Rubio de Francia [1985]. This idea was systematised by Dragičević, Grafakos, Pereyra, and Petermichl [2005], whose results were treated in Appendix J and applied in the section under discussion.

The positive results for the Beurling–Ahlfors transform inspired the question of sharp weighted bounds for other operators, and the special role of the exponent $p = 2$ as the critical case for extrapolation gave rise to the name "A_2 conjecture", several further cases of which were settled over the next few years. In particular, the Hilbert transform (the scalar-valued case of Corollary 11.3.27) and the Riesz transforms were handled by Petermichl [2007, 2008], a general class of sufficiently smooth odd kernels on $\mathbb{R}$ by Vagharshakyan [2010], and powers of the Beurling–Ahlfors operator by Dragičević [2011]. All these results relied on

(A) *ad hoc* representation formulas of special singular integrals in terms of simple "dyadic shifts" as in the representation of Petermichl [2000] for the Hilbert transform (see Theorem 5.1.13 and (5.20)), and
(B) Bellman function techniques for sharp weighted bounds of these shifts.

The component (B) behind these results was first challenged by Lacey, Petermichl, and Reguera [2010], who replaced it with

(C) "corona decompositions" to verify the "testing conditions" in a
(D) dyadic two-weight $T(1)$ theorem of Nazarov, Treil, and Volberg [2008].

Shortly after, a much simpler alternative to either (B) or (C)–(D) was found by Cruz-Uribe, Martell, and Pérez [2010], who in turn replaced it by methods largely similar to the ones that we have used here:

(E) domination of dyadic shifts from (A) (not yet of singular integrals directly) by the sparse operators $A_{\mathscr{S}}$, and
(F) estimating $\|A_{\mathscr{S}}\|_{\mathscr{L}(L^2(w))}$ as in Theorem 11.3.19, whose proof follows closely the original one from Cruz-Uribe et al. [2010],

However, component (A) of the original proofs remained unchallenged and, being somewhat *ad hoc* for the specific singular integrals considered thus far, restricted their extension to wider classes of operators.

The general A_2 theorem

These limitations of (A) were overcome by Hytönen [2012], who found

(G) a general *dyadic representation formula* (a variant of which will be presented in Theorem 12.4.27) of all standard Calderón–Zygmund operators in terms of a series of dyadic shifts of increasing complexity.

Moreover, (C) and (D) had to be replaced by

(C′) refinements of (C) to control the general shifts produced by (G), and
(D′) a difficult two-weight $T(1)$ theorem of Pérez, Treil, and Volberg [2010] about the singular integral itself, rather than the dyadic shifts as in (D).

A combination of (G), (C′), and (D′) gave the first proof of the A_2 Theorem 11.3.26 for all standard Calderón–Zygmund operators in the scalar case.

In a matter of months since the announcement of Hytönen [2012] in 7/2010, several variants and extensions were found. Streamlined versions and certain improvements of the original approach were obtained in Hytönen, Pérez, Treil, and Volberg [2014], Hytönen and Pérez [2013], and Hytönen [2017], which appeared in arXiv in 10/2010, 3/2011, and 8/2011, respectively. At the same time, alternatives to (C′) and (D′) by

(B′) elaborations of (B) with good control on the shift complexity

were obtained by Nazarov and Volberg [2013] (arXiv 4/2011) and Treil [2013] (arXiv 5/2011), and these were used by Nazarov, Reznikov, and Volberg [2013] (arXiv 6/2011) to give an extension of the A_2 theorem to doubling metric space domains in place of $\mathbb{R}^d$. (Thus, the versions with a cube or a quadrant that we have stated in Theorem 11.3.26 are but very particular instances of the general domains in which the result may be formulated.)

Still over the same hectic months, Hytönen, Lacey, Martikainen, Orponen, Reguera, Sawyer, and Uriarte-Tuero [2012] (arXiv 3/2011) combined the approach of Hytönen [2012] with input from the time–frequency techniques of Lacey and Thiele [2000] to extend the A_2 theorem to *maximally truncated Calderón–Zygmund operators*

$$T_{\#}f(x) = \sup_{\varepsilon>0} \|T_\varepsilon f(x)\|, \qquad T_\varepsilon f(x) = \int_{|x-y|>\varepsilon} K(x,y)f(y)\,\mathrm{d}y. \tag{11.55}$$

However, these results were shortly superseded by Hytönen and Lacey [2012] (arXiv 6/2011) by a new approach combining (G) with elaborations of (E) and (F) from the approach of Cruz-Uribe et al. [2010]:

(E′) domination of the general dyadic shifts from (G) by operators (essentially like) $A^N_{\mathscr{S}}$, where arbitrarily large N appear, and
(F′) estimating $\|A^N_{\mathscr{S}}\|_{\mathscr{L}(L^2(w))}$ with bounds polynomial in $\log N$ (which requires much more delicate analysis than Theorem 11.3.19).

As a curiosity, the term "sparse" in its present usage seems to have been introduced by Hytönen and Lacey [2012] (line below ($*$) on page 2042). This was pointed out by Andrei Lerner in his survey talk at the "AIM Workshop on sparse domination of singular integrals" in San José, California, in 10/2017.

Simpler proofs

The difficulties with arbitrarily high shift complexity N, which seemed unavoidable in the general A_2 theorem until this point, were finally eliminated by Lerner [2013a,b] (arXiv 2/2012). These papers provide two different proofs of the same main result, stating that

$$\|T_{\#}f\|_F \leqslant c_{d,T} \sup_{\mathscr{D},\mathscr{S}} \|A_{\mathscr{S}}f\|_F, \tag{11.56}$$

where $T_{\#}$ is the maximal truncation (11.55) of a standard Calderón–Zygmund operator, F is any Banach function space of $\mathbb{R}^d$, and the supremum is taken over all dyadic systems $\mathscr{D}$ and their sparse subcollections $\mathscr{S}$. With T in place of $T_{\#}$, this is slightly weaker than the pointwise estimate of Theorem 11.3.15 but, taking $F = L^p(w)$, quite sufficient for bounding T (or $T_{\#}$) on $L^p(w)$.

The first proof of (11.56) by Lerner [2013a] still started with (G) and (E′), but then proceeded with the key new idea of

(H) domination of the adjoints $(A^N_{\mathscr{S}})^*$ by the simple operators $A_{\mathscr{S}} = A^*_{\mathscr{S}}$.

(The fact that the argument passes through the adjoint is where the Banach function space F is needed, while everything else can be estimated pointwise.) The A_2 estimate can then be completed by the simple step (F).

At the same time, Hytönen, Lacey, and Pérez [2013] found a way of replacing the initial steps (G) and (E′) by

(I) direct domination of the singular integral by an infinite series of operators (essentially like) $A^N_{\mathscr{S}}$ with arbitrarily large N.

Thus, a self-contained proof of the A_2 theorem is obtained by concatenating the steps (I), (H), and (F), and these constitute the *simple proof of the A_2 conjecture* presented by Lerner [2013b]. As soon as things started falling into the right place, the progress was very fast, and the preprints of the just discussed papers appeared in the arXiv essentially over a weekend in February 2012: Lerner [2013a] on Thursday 9th, Hytönen et al. [2013] on Friday 10th, and Lerner [2013b] on Monday 13th.

The simple proof of Lerner [2013b] also admitted the first extension of the A_2 theorem to the weighted Bochner space $L^p(w; X)$ by Hänninen and Hytönen [2014]. At the time, the main difficulty with this Banach space valued extension was the dependence of the sparse domination (I), via its use of Lerner's local oscillation formula (Theorem 11.1.12), on the notion of median. Thus, a workable vector-valued version of this concept had to be developed; it is reproduced in Section 11.1.

Pointwise sparse domination

Although not a necessity for proving the A_2 theorem, the possibility of replacing (11.56) by pointwise domination presented itself as a natural question, which attracted some interest. This was independently achieved by Conde-Alonso and Rey [2016] (arXiv 9/2014) and Lerner and Nazarov [2019] (also announced and circulated around the same time in 2014, although in arXiv only in 8/2015). These results still slightly deviated from Theorem 11.3.15 by requiring a stronger form of the Dini condition,

$$\int_0^{1/2} \omega(t) \log_2 \Big(\frac{1}{t}\Big) \frac{\mathrm{d}t}{t} < \infty.$$

All Dini kernels were first covered by the "elementary" (but not so easy) proof of Lacey [2017] (arXiv 1/2015), which was further quantified (in terms of dependence on $\|\omega\|_{\mathrm{Dini}}$) by Hytönen, Roncal, and Tapiola [2017] (arXiv 10/2015) and remarkably simplified again by Lerner [2016] (arXiv 12/2015). In proving Theorem 11.3.15, we have followed the further simplification due to Lerner and Ombrosi [2020]. One advantage of their approach is a reduction of the prerequisites from classical Calderón–Zygmund theory necessary to run their argument. On the technical level, this is achieved by replacing the maximal operator

$$M_T f(x) = \sup_{Q \ni x} \sup_{y \in Q} T(\mathbf{1}_{\complement 5Q} f)(y)$$

of Lerner [2016] by its "sharp" version $M_T^{\#}$ defined in (11.28). While $M_T^{\#}$ can be estimated relatively directly, bounding the larger $M_T f$ originally required non-trivial classical results about the maximal truncations (11.55). However, it was later observed by Almeida, Betancor, Fariña, and Rodríguez-Mesa [2022] that the bounds for the two operators are actually equivalent under general assumptions only involving the bounds for T that are used in the theory anyway. Although not explicitly discussed by Lerner and Ombrosi [2020], the present vector-valued extensions of their results, leading to Theorems 11.3.15 and 11.3.26, involved little additional effort; this is in contrast to the first vector-valued A_2 theorem by Hänninen and Hytönen [2014]. Further abstractions are due to Lorist [2021] and Lerner, Lorist, and Ombrosi [2022]; the latter work also explicitly addresses the vector-valued case.

Routes to sharpness in weighted estimates

There are some alternative routes to see the sharpness result of Proposition 11.3.30, which goes back to Buckley [1993] well before the matching upper bounds were known. Luque, Pérez, and Rela [2015] made the curious observation that this can also be achieved without exhibiting any explicit examples in the weighted situation, but studying instead the asymptotics of the *unweighted* norms $\|T\|_{L^p \to L^p}$ as $p \to 1$ and $p \to \infty$. This depends on a variant of Rubio de Francia's Extrapolation Theorem J.2.1, where one keeps track of the p-dependence in the estimates for $\|T\|_{L^p \to L^p}$ given by extrapolating a bound of the type

$$\|T\|_{L^{p_0}(w) \to L^{p_0}(w)} \leqslant \phi([w]_{A_{q_0}}),$$

where q_0 can also be different from p_0. Via contraposition, a lower bound for $\|T\|_{L^p \to L^p}$ imposes a lower bound for ϕ. This quantitative weighted-to-unweighted extrapolation was already used earlier by Fefferman and Pipher [1997] in the "positive" direction to obtain sharp unweighted L^p-norm asymptotics for some operators by studying their weighted behaviour. They also obtained a certain predecessor of the A_2 Theorem 11.3.26 with $\|T\|_{\mathscr{L}(L^2(w))} \leqslant c_{d,T}[w]_{A_1}$, where

$$\begin{aligned} [w]_{A_1} := \|Mw/w\|_\infty &= \sup_Q \fint_Q w \Big(\operatorname*{ess\,sup}_Q w^{-1} \Big) \\ &\geqslant \sup_Q \fint_Q w \Big(\fint_Q w^{-1/(p-1)} \Big)^{p-1} = [w]_{A_p} \quad \forall p \in (1, \infty). \end{aligned}$$

Further results

For a while, it might have seemed that the new sharp weighted technology was essentially restricted to the class of Calderón–Zygmund operators. A certain discouragement against further extensions came from an observation of Orponen [2013] that *if* an operator T has a dyadic representation (G) in the sense of Hytönen [2012], *then* T must necessarily be a Calderón–Zygmund operator. However, as soon as the role of (G) in the A_2 theorem was challenged by other methods, the door was also open for extensions beyond the standard Calderón–Zygmund realm. Nevertheless, few could probably have expected how far this theory could indeed be extended.

As an application of the sharp weighted estimates for Dini kernels discussed above, Hytönen, Roncal, and Tapiola [2017] (arXiv 10/2015) showed that rough homogeneous singular integrals

$$T_\Omega f(s) := \text{p. v.} \int_{\mathbb{R}^d} \frac{\Omega(t/|t|)}{|t|^d} f(s-t) \, \mathrm{d}t, \quad \Omega \in L_0^\infty(S^{d-1}).$$

satisfy the weighted norm inequality

$$\|T_\Omega\|_{\mathscr{L}(L^2(w))} \leqslant c_d \|\Omega\|_\infty \phi([w]_{A_2})$$

with $\phi(u) \leqslant u^2$. Although dealing with a class of operators outside the direct scope of the sparse domination technology of the time, this result may nevertheless be seen as stretching those methods, rather than introducing genuinely new ones, in that the operator T_Ω was decomposed into a series of pieces in the scope of the previously available tools by following a classical approach to qualitative versions of similar results by Duoandikoetxea and Rubio de Francia

[1986], and Watson [1990]. A more intrinsic approach has been subsequently developed by Conde-Alonso, Culiuc, Di Plinio, and Ou [2017], but $\phi(u) \leqslant u^2$ seems to remain the best available bound at the time of writing. In the other direction, Honzík [2023] constructed examples of symbols Ω and weights w to show that $\phi(u) \geqslant u^{3/2}$; hence the quantitative behaviour of T_Ω is definitely different from the linear A_2 theorem for standard Calderón–Zygmund operators, but their precise bounds remain open.

Already a few weeks before Hytönen, Roncal, and Tapiola [2017] (late 10/2015 in arXiv), a far-reaching approach to sparse domination of a wide class of operators had been revealed by Bernicot, Frey, and Petermichl [2016] (early 10/2015 in arXiv). They observed that several operators that act boundedly in L^p only in some range $(p_0, q_0) \subsetneq (1, \infty)$ (and thus are definitely outside the Calderón–Zygmund class by Theorem 11.2.5) can be proved to possess *sparse form domination* of the type

$$|\langle Tf, g\rangle| \leqslant C \sum_{Q\in\mathscr{S}} |Q| \Big(\fint_{5Q} |f|^{p_0}\Big)^{1/p_0} \Big(\fint_{5Q} |g|^{q_0'}\Big)^{1/q_0'}.$$

This in turn implies weighted norm inequalities of the form

$$\|Tf\|_{L^p(w)} \leqslant C\big([w]_{A_{p/p_0}} [w]_{\mathrm{RH}_{(q_0/p)'}}\big)^{\alpha} \|f\|_{L^p(q)}, \quad p \in (p_0, q_0),$$

where $[w]_{\mathrm{RH}_t}$ is the best constant in the *reverse Hölder inequality*

$$\Big(\fint_Q w^t\Big)^{1/t} \leqslant C \fint_Q w,$$

and $\alpha = \alpha(p_0, q_0, p)$ is a certain explicit exponent depending on the indicated quantities only.

Typical examples in the scope of the theory of Bernicot, Frey, and Petermichl [2016] are various "singular non-integral operators" arising in harmonic analysis adapted to operators other than the classical Laplacian, e.g., generalised Riesz transforms $\nabla L^{-1/2}$, where L could be a second-order divergence-form operator $L = -\operatorname{div}(A\nabla)$ with bounded coefficient matrix A, or a Schrödinger operator $L = -\Delta + V$ with some potential V.

After the key observation that it is possible to go beyond Calderón–Zygmund theory at all, sparse domination results and weighted norm inequalities, as a corollary, for several different types of operators have been obtained:

- rough singular integrals (Conde-Alonso, Culiuc, Di Plinio, and Ou [2017], Di Plinio, Hytönen, and Li [2020a]);
- Bochner–Riesz multipliers (Benea, Bernicot, and Luque [2017], Conde-Alonso et al. [2017], Lacey, Mena, and Reguera [2019]);
- oscillatory integrals (Lacey and Spencer [2017], Krause, Lacey, and Wierdl [2019]);
- bilinear Hilbert transforms and related phase-space objects (Culiuc, Di Plinio, and Ou [2018a], Di Plinio, Do, and Uraltsev [2018]);

- singular integrals along curves, Radon transforms (Cladek and Ou [2018], Culiuc, Kesler, and Lacey [2019], Oberlin [2019], Anderson, Hu, and Roos [2021]);
- spherical maximal operators both on $\mathbb{R}^d$ (Lacey [2019], Beltran, Oberlin, Roncal, Seeger, and Stovall [2022a], Borges, Foster, Ou, Pipher, and Zhou [2023]) and on the Heisenberg group (Bagchi, Hait, Roncal, and Thangavelu [2021], Ganguly and Thangavelu [2021]);
- pseudo-differential operators (Beltran and Cladek [2020]).

A relatively general theory has been developed by Beltran, Roos, and Seeger [2022b], who also explicitly discuss Banach space valued operators.

Product space theory

A related direction, in which a weighted theory of singular integrals is well developed since the works of Fefferman and Stein [1982] and Fefferman [1987, 1988], yet the sparse domination technology has met obstacles, consists of the theory of product space or multi-parameter singular integrals modelled after the product Hilbert transform $H_1 \otimes H_2$ (where H_i denotes the Hilbert transform in the ith variable of $\mathbb{R}^2$). Natural maximal operators in this theory are the *strong maximal operator*

$$M_* f(s) := \sup_{R \text{ rectangle}} \mathbf{1}_R(s) \fint_R \|f(t)\| \, \mathrm{d}t.$$

and its dyadic version, where the rectangles are restricted to be dyadic (i.e., products of dyadic intervals). Barron, Conde-Alonso, Ou, and Rey [2019] have shown that it is impossible to dominate the strong dyadic maximal operator by sparse forms based on rectangles with sides parallel to the axes, which presents an obstacle to sparse techniques in this setting. While the most obvious extension of sparse domination is thus excluded, it was shown by Barron and Pipher [2017] that one can still obtain a workable substitute by replacing the dominating averages $\fint_R |f|$ of f with the averages $\fint_R Sf$ of its dyadic square function Sf on the right-hand side.

On the other hand, the original dyadic representation (G), while largely superseded by sparse technology in applications to standard Calderón–Zygmund operators, remains available, after natural modifications, in the product space theory, as first proved by Martikainen [2012b] in the two-parameter case and extended to arbitrarily many parameters by Ou [2017]. A vector-valued approach to this theory has been developed by Hytönen, Martikainen, and Vuorinen [2019a].

Sparse domination versus causality

While the current mainstream in sparse domination, evidenced by the previous list, consists of proving and applying domination for ever wider classes of operators, one may also pose a somewhat opposite question: Suppose that a given (say, standard Calderón–Zygmund) operator T possesses some additional properties. Can this be reflected in the dominating sparse operator as well? A concrete instance of such an additional property is causality. Suppose for simplicity that $d = 1$, and that $K(s,t)$ is non-zero only if $s > r$; thus $Tf(s)$ depends only on the "past" values $f(t)$ with $t < s$. If T is a Calderón–Zygmund operator, then it satisfies the sparse domination $Tf(s) \leqslant c_T A^5_{\mathscr{S}} f(s)$ by the general theory. However, the dominating sparse operator $A^5_{\mathscr{S}}$ is no longer causal. Is it possible to exploit the causality of T to obtain a sharper form of sparse domination, where this causality is preserved also in the right-hand side? Some partial (but far from complete) results in this direction have been obtained by Hytönen and Rosén [2023].

Aimar, Forzani, and Martín-Reyes [1997] have shown that causal Calderón–Zygmund operators remain bounded on the weighted space $L^p(w)$ for the larger class of *one-sided A_p weights*, defined by the finiteness of

$$[w]_{A_p^-} := \sup_{-\infty<a<b<c<\infty} \frac{1}{(c-a)^p}\Big(\int_b^c w\Big)\Big(\int_a^b w^{-\frac{1}{p-1}}\Big)^{p-1},$$

but the optimal bound for the operator norm $\|T\|_{\mathscr{L}(L^p(w))}$ in terms of $[w]_{A_p^-}$ remains open. In analogy with the A_2 Theorem 11.3.26, it is natural to make:

Conjecture 11.4.1 (One-sided A_2 conjecture of Chen, Han, and Lacey [2020]). For all causal Calderón–Zygmund operators,

$$\|T\|_{\mathscr{L}(L^p(w))} \leqslant c_T([w]_{A_p^-})^{\max(1,\frac{1}{p-1})}.$$

Partial results for Haar multipliers (see Section 12.1.a) in place of Calderón–Zygmund operators are obtained by Chen et al. [2020], but beyond that the conjecture remains open.

Causal operators appear very naturally; e.g., the operator-valued kernel

$$K(s,t) = \mathbf{1}_{\mathbb{R}_+}(s-t)Ae^{-(s-t)A},$$

of relevance to the maximal regularity problem studied in Chapter 17, has this form. A theory of one-sided singular integrals applicable to this operator-valued situation has been developed by Chill and Król [2018].

Matrix weighted spaces and convex body domination

Let $W : \mathbb{R}^d \to \mathbb{R}^{N\times N}$ be a *matrix weight*, i.e., measurable and positive definite almost everywhere, and $f : \mathbb{R}^d \to \mathbb{R}^N$ be measurable. The norm

$$\begin{aligned}\|f\|^2_{L^2(W)} &:= \int_{\mathbb{R}^d} \langle W(t)f(t), f(t)\rangle \,\mathrm{d}t \\ &= \int_{\mathbb{R}^d} |W(t)^{\frac12} f(t)|^2 \,\mathrm{d}t = \|W^{\frac12} f\|^2_{L^2(\mathbb{R}^d;\mathbb{R}^N)},\end{aligned}$$

appears naturally from the prediction theory for multivariate stationary stochastic processes $n \in \mathbb{Z} \mapsto \xi_n \in L^2(\Omega;\mathbb{R}^N)$ developed by Wiener and Masani [1958], where stationarity means that $\Gamma_{n-k} := \mathbb{E}\xi_n\xi_k^T \in \mathbb{R}^{N\times N}$ depends only on the difference of the discrete times $n, k \in \mathbb{Z}$. If W is the density of the spectral measure of the process, i.e., $\Gamma_k = \widehat{W}(k)$ are the Fourier coefficients of $W \in L^1(\mathbb{T};\mathbb{R}^{N\times N})$, the boundedness of the Hilbert transform on $L^2(W)$ is equivalent to a positive angle between the past and the future of the process. Even for $N = 1$, this problem was only solved 15 years later by Hunt, Muckenhoupt, and Wheeden [1973], who characterised this boundedness in terms of the A_2 condition. For $N > 1$, it took over 20 more years before the solution was obtained by Treil and Volberg [1997], who identified the correct analogue of the A_2 condition in the matrix-valued case:

$$[W]_{A_2} := \sup_Q |\langle W\rangle_Q^{1/2}\langle W^{-1}\rangle_Q^{1/2}|^2,$$

where $|\ |$ is (say) the operator norm on $\mathscr{L}(\mathbb{R}^N)$ (but the choice of the norm on $\mathbb{R}^{N\times N}$ is irrelevant, as they are all equivalent).

With the natural definition

$$\|f\|_{L^p(W)} := \|W^{\frac{1}{p}}f\|_{L^p(\mathbb{R}^d;\mathbb{R}^N)},$$

one is led to inquire about the boundedness of the Hilbert transform on $L^p(W)$. The characterising matrix-A_p condition, identified via different approaches by Nazarov and Treil [1996] and Volberg [1997], is less intuitive for $p \neq 2$. It is perhaps most easily formulated with the help of the classical theorem of John [1948], which guarantees that every norm on $\mathbb{R}^N$ is equivalent (with constants depending only on N) to a Euclidean norm, whose unit ball is a linear transformation of the standard unit ball. If W is a matrix weight and $V := W^{\frac{1}{p}}$, it is easy to see that

$$e \in \mathbb{R}^n \mapsto \Big(\fint_Q |V(t)e|^p\,\mathrm{d}t\Big)^{1/p}$$

is a norm, and hence, by the theorem of John [1948], there is a positive definite *reducing operator* $[V]_{Q,p} \in \mathbb{R}^{N\times N}$, such that

$$|[V]_{Q,p}e| \leqslant \Big(\fint_Q |V(t)e|^p\,\mathrm{d}t\Big)^{1/p} \leqslant \sqrt{N}\cdot|[V]_{Q,p}e|.$$

The matrix-A_p condition may then be defined by the finiteness of the constant

$$[W]_{A_p} := \sup_Q |[V]_{Q,p}[V^{-1}]_{Q,p'}|^p, \quad V := W^{\frac{1}{p}}.$$

The reader is invited to check that $[V]_{Q,p} = \langle V^p\rangle_Q^{\frac{1}{p}}$ if $N = 1$ or $p = 2$ (but not in general otherwise), so that the different definitions of A_p are consistent. It is possible to give an equivalent definition of the matrix A_p condition without reference to reducing operators, but one would still need them to prove anything interesting, which is why we prefer to state the definition as above.

While the qualitative boundedness of the Hilbert transform, and in fact of more general Calderón–Zygmund operators, on $L^p(W)$ was settled in the mentioned papers, the proof of the scalar-valued A_2 theorem raised the natural question of its extension to the matrix-weighted case. This remains open, but several related results have been achieved.

While sparse domination is perfectly applicable to vector-valued (even Banach space valued) functions, as we have seen in this chapter, it loses essential directional information, which makes it ill-suited for matrix-weighted considerations. To address this drawback, Nazarov, Petermichl, Treil, and Volberg [2017] invented a refined notion of *convex body domination*, where the averages $\langle \|f\| \rangle_Q$ are replaced by the related convex bodies

$$\Big\{ \langle \phi f \rangle_Q : \|\phi\|_{L^\infty(Q)} \leqslant 1 \Big\} \subseteq \mathbb{R}^N, \quad f \in L^1(Q; \mathbb{R}^N).$$

Convex body domination of T is most easily stated in its bilinear form, as an elaboration of the sparse form domination

$$\begin{aligned} |\langle Tf, g\rangle| &\leqslant c_{d,T} \sum_{Q\in\mathscr{S}} |Q| \langle |f| \rangle_{5Q} \langle |g| \rangle_{5Q} \\ &= c'_{d,T} \sum_{Q\in\mathscr{S}} \frac{1}{|Q|} \iint_{5Q\times 5Q} |f(s)||g(t)| \,\mathrm{d}s\,\mathrm{d}t. \end{aligned} \tag{11.57}$$

Convex body domination of T can now be stated in the form

$$|\langle Tf, g\rangle| \leqslant \sum_{Q\in\mathscr{S}} \frac{c_{d,N,T}}{|Q|} \sup_{\substack{\|\phi\|_\infty \leqslant 1 \\ \|\psi\|_\infty \leqslant 1}} \Big| \iint_{5Q\times 5Q} \phi(s)(s) \cdot \psi(t) g(t) \,\mathrm{d}s\,\mathrm{d}t \Big|, \tag{11.58}$$

with the important difference that we take the dot product of $f(s), g(t) \in \mathbb{R}^n$ first, and only then the absolute value of the result; this allows for critical directional cancellation compared to (11.57).

The proof of Nazarov, Petermichl, Treil, and Volberg [2017] (arXiv 1/2017), that standard Calderón–Zygmund operators satisfy (11.58), follows the same lines as the proof of Theorem 11.3.15 but with important elaborations at a few selected points, making again use of the ellipsoid theorem of John [1948]. On the other hand, with (11.58) available, Nazarov et al. [2017] can prove the bound

$$\|T\|_{\mathscr{L}(L^2(W))} \leqslant c_{d,T} [W]_{A_2}^{3/2},$$

which remains the best available matrix-weighted estimate for Calderón–Zygmund operators (or even just for the Hilbert transform) at the time of writing. A variant of the same results was also obtained by Culiuc, Di Plinio, and Ou [2018b], seemingly earlier (arXiv 10/2016) but not independently; according to their acknowledgment, the concept of domination by convex body averages was introduced to these authors by Sergei Treil during his seminar talk at Brown University in the Spring of 2016.

Since then, further applications and extensions of convex body domination have been explored by Cruz-Uribe, Isralowitz, and Moen [2018], Di Plinio, Hytönen, and Li [2020a], Isralowitz, Pott, and Rivera-Ríos [2021], Isralowitz, Pott, and Treil [2022], and Muller and Rivera-Ríos [2022]. Importantly, Bownik and Cruz-Uribe [2022] extended the Rubio de Francia algorithm (Proposition J.2.2), and its key application to weighted extrapolation (Theorem J.2.1), to matrix-valued weights, by further development of the convex body philosophy.

An abstract framework for convex body domination has been proposed by Hytönen [2023], allowing also Banach space valued functions in the theory. While genuinely operator-valued weights in infinite dimensions seem to be out of reach, this framework allows the treatment of $\mathbb{R}^{N\times N}$-valued weights on spaces of X^N-valued functions. In particular, the following simultaneous extensions of the boundedness of the Hilbert transform on the Banach space valued $L^2(\mathbb{R};X)$ by Burkholder [1983], and on the matrix-weighted $L^2(W)$ by Treil and Volberg [1997], is obtained there.

Theorem 11.4.2. *Let X be a UMD space, and $W:\mathbb{R}^d\to\mathbb{R}^{N\times N}$ be a matrix A_2 weight. Then the Hilbert transform H extends boundedly to*

$$L^2(W;X^N):=\Big\{f:\mathbb{R}\to X^N:\|f\|_{L^2(W;X^N)}:=\|W^{\frac12}f\|_{L^2(\mathbb{R};X^N)}<\infty\Big\}$$

and satisfies $\|H\|_{\mathscr{L}(L^2(W;X^N))}\leqslant c_N\hbar_{2,X}[W]_{A_2}^{3/2}\leqslant c_N\beta_{2,X}^2[W]_{A_2}^{3/2}$, where $\hbar_{2,X}=\|H\|_{L^2(\mathbb{R};X)}$ and $\beta_{2,X}$ is the UMD constant.

The stated quantitative formulation in terms of $\hbar_{2,X}$ is not explicit in Hytönen [2023], but can be tracked in the proof, in a similar way as in Corollary 11.3.27 in the text.

A summary of sharp weighted bounds for classical operators

Our discussion above has been focused on norms of Calderón–Zygmund singular integrals and their various extensions, viewed as operators on a weighted $L^p(w)$ (or matrix-weighted $L^p(W)$) space; these are referred to as strong-type bounds. We will briefly summarise results in two closely related directions. First, one may inquire about the corresponding weak-type bounds, i.e., operator norms in $\mathscr{L}(L^p(w),L^{p,\infty}(w))$. These are obviously dominated by the strong-type norms, but the point is that the optimal weak-type norms may be significantly smaller in some cases, which gives these questions an independent interest. Second, one may pose the same questions for various square-functions, which could be viewed as part of the extended family of (vector-valued, when suitably interpreted) Calderón–Zygmund operators; however, it turns out that these operators are actually slightly "better" in terms of the dependence of their norms on the weight constant. A basic example is the *dyadic square function*

$$Sf(x) := \Big(\sum_{Q \in \mathscr{D}} |\mathbb{D}_Q f(x)|^2 \Big)^{1/2},$$

(where the operators $\mathbb{D}_Q$ are defined in (12.1) and discussed extensively in Chapter 12), but several other classical square functions satisfy the same weighted bounds; we refer the reader to the papers quoted below for details.

A summary of the sharp bounds known for these operators is as follows:

Singular integrals:

For $p \in (1, \infty)$ and $w \in A_p$, the sharp estimates in $L^p(w)$ are:

(1) the strong-type bound is $[w]_{A_p}^{\max(1, \frac{1}{p-1})}$ (Hytönen [2012]);
(2) the weak-type bound is $[w]_{A_p}$ (Hytönen, Lacey, Martikainen, Orponen, Reguera, Sawyer, and Uriarte-Tuero [2012]);
(3) the weak-type $L^1(w)$ bound is $[w]_{A_1}(1 + \log[w]_{A_1})$ (the upper bound was proved by Lerner, Ombrosi, and Pérez [2009], its sharpness is due to Lerner, Nazarov, and Ombrosi [2020]).

A speculative linear-in-$[w]_{A_1}$ bound in (3) was known as the A_1 conjecture, or the weak Muckenhoupt–Wheeden conjecture. The original conjecture, disproved by Reguera [2011] and Reguera and Thiele [2012], was about the boundedness of $T : L^1(Mw) \to L^{1,\infty}(w)$ for any weight w. This holds for M in place of T (Theorem 3.2.27), which motivated the conjecture.

Square functions:

For the range of p as specified and $w \in A_p$, the sharp estimates in $L^p(w)$ are:

(4) the strong-type bound is $[w]_{A_p}^{\max(\frac{1}{2}, \frac{1}{p-1})}$ for $p \in (1, \infty)$ (Lerner [2011]);
(5) the weak-type bound is $[w]_{A_p}^{\max(\frac{1}{2}, \frac{1}{p})}$ for $p \in [1, \infty) \setminus \{2\}$ ($p = 1$: Chanillo and Wheeden [1987], Wilson [2007, 2008]; $p \in (1, 2)$: Lacey and Scurry [2012]; $p > 2$: Hytönen and Li [2018]);
(6) the weak-type $L^2(w)$ bound is at most $[w]_{A_2}^{\frac{1}{2}}(1 + \log[w]_{A_1})^{\frac{1}{2}}$ (Domingo-Salazar, Lacey, and Rey [2016]), but its sharpness seems to remain open (see Ivanisvili and Volberg [2018] for partial related results).

In contrast to singular integrals, the bounds at $p = 1$ above are consequences of the stronger statement that $S : L^1(Mw) \to L^{1,\infty}(w)$ is bounded for any weight w, i.e., the Muckenhoupt–Wheeden conjecture holds for square functions. This also explains the (implicit) appearance of sharp weighted bounds in Chanillo and Wheeden [1987], long before this became a fashionable topic.

For matrix-weights, the only known sharp estimates among these examples, at the time of writing, seem to be the square function bounds (4) for $p \in (1, 2]$; this was proved by Hytönen, Petermichl, and Volberg [2019b] for $p = 2$ and extended by Isralowitz [2020] to $p \in (1, 2)$.

12

Dyadic operators and the $T(1)$ theorem

In Chapter 11, we have mainly dealt with a situation, where a bounded linear operator on some $L^{p_0}(\mathbb{R}^d;X)$ space is given, and we have then explored its bounded extensions to other spaces including $L^p(\mathbb{R}^d;X)$ for $p \neq p_0$. We now turn to a somewhat different (and often more difficult) question of recognising such bounded operators to begin with.

Before addressing this question for the Calderón–Zygmund type operators of the kind studied in Chapter 11, we investigate a number of related objects in a simpler dyadic model. Besides serving as an introduction to some of the key techniques, it turns out that these dyadic operators can be, and will be, also used as building blocks of the proper singular integral operators towards the end of the chapter.

The dyadic operators will be of two essentially different types. The first class, which we vaguely refer to as "dyadic singular integrals" in Section 12.1, consist of a somewhat diverse family of relatives of the prototype dyadic shifts encountered in Chapter 5, where they we used to represent the prototype singular integral given by the Hilbert transform. It is thus only natural that a family of dyadic operators generalising this basic dyadic shift will serve as building block of the Calderón–Zygmund family of singular integrals generalising the Hilbert transform. Martingale techniques vaguely reminiscent of those in Section 5.1, but of somewhat higher complexity probably by necessity, will feature in the argument to put the UMD property of the underlying Banach space into action.

The second class of dyadic operators consists of so-called *paraproducts*, which we discuss in Section 12.2. These are new creatures of the non-convolution realm that we have entered and they will vanish (as we will see) as soon as we occasionally specialise our considerations to singular integral of the convolution form. However, for the representation the full class of Calderón–Zygmund operators they will turn out be quite essential.

The chapter will culminate in a lengthy treatment of the so-called $T(1)$ theorem, a general criterion for boundedness of singular integral operators. We will first discuss a version for abstract bilinear form in Section 12.3, and

T. Hytönen et al., *Analysis in Banach Spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics 76,
https://doi.org/10.1007/978-3-031-46598-7_2

only then, in the final Section 12.4, turn to the task of checking the assumptions of the abstract result for singular integral operators with a Calderón–Zygmund kernel, of the kind that we met Chapter 11. However, in order to establish boundedness on $L^p(\mathbb{R}^d;X)$ from scratch, rather than extrapolating it from another $L^{p_0}(\mathbb{R}^d;X)$ space where it was already known (as in Chapter 11), somewhat stronger versions of the Calderón–Zygmund conditions will be needed, and the notion of R-boundedness from Chapter 8 will, once again, play a prominent role. While the results of this chapter will generically be established in arbitrary UMD spaces, it turns out that additional information about type and cotype, as studied in Chapter 7 can be traded against the precise kernel conditions, so that slightly larger classes of kernels are admissible under conditions of type and cotype of the underlying space.

12.1 Dyadic singular integral operators

In this section, we introduce and study a family of dyadic models of singular integrals, starting from the simplest case of Haar multipliers and proceeding to their more complicated relatives. All these operators will eventual come together as parts of a decomposition of general singular integral operators towards the end of the chapter.

Since our aim is not to assume, but to prove, the L^p-boundedness of the relevant operators, we will first define their action on appropriate spaces of test functions only.

Definition 12.1.1 (Classes of simple functions). *For a collection $\mathscr{C}$ of bounded Borel subsets of $\mathbb{R}^d$, let*

$$\begin{aligned} S(\mathscr{C};X) &:= \operatorname{span}\Big\{\mathbf{1}_C\otimes x : C\in\mathscr{C}, x\in X\Big\},\\ S_0(\mathscr{C};X) &:= \Big\{f\in S(\mathscr{C};X) : \int_{\mathbb{R}^d} f(t)\,\mathrm{d}t = 0\Big\},\\ S_{\mathrm{loc}}(\mathscr{C};X) &:= \{f\in L^1_{\mathrm{loc}}(\mathbb{R}^d;X) : \mathbf{1}_C f\in S(\mathscr{C};X) \textit{ for all } C\in\mathscr{C}\},\\ S_\infty(\mathscr{C};X) &:= S_{\mathrm{loc}}(\mathscr{C};X)\cap L^\infty(\mathbb{R}^d;X). \end{aligned}$$

It is easy to see that $S(\mathscr{C};X)\subseteq L^p(\mathbb{R}^d;X)$ for all $p\in[1,\infty]$, and that

$$S_0(\mathscr{C};X)\subseteq S(\mathscr{C};X)\subseteq S_\infty(\mathscr{C};X)\subseteq S_{\mathrm{loc}}(\mathscr{C};X).$$

Our primary case of interest will be when $\mathscr{C}=\mathscr{D}$ is a collection of dyadic cubes of $\mathbb{R}^d$ in the sense of Definition 11.1.6. In this case, $S(\mathscr{C};X)$ is dense in $L^p(\mathbb{R}^d;X)$ for all $p\in[1,\infty)$. In (12.2) below, we will add yet another space $S_{00}(\mathscr{D};X)\subseteq S_0(\mathscr{D};X)$ to this list, but its introduction requires some preliminaries.

12.1.a Haar multipliers

We begin with what is arguably the simplest class of operators deserving the name of "dyadic singular integrals". In essence, we have encountered these operators already, at least implicitly on the one-dimensional domain space $\mathbb{R}^1$, where we dealt with operators of the form

$$f \mapsto \sum_{I\in\mathscr{D}} \epsilon_I \langle f, h_I\rangle h_I$$

and showed their uniform boundedness on $L^p(\mathbb{R}; X)$ for arbitrary unimodular coefficients ϵ_I, assuming that $p \in (1,\infty)$ and X is a UMD space (see Theorem 4.2.13). We now wish to extend these consideration to the general Euclidean domain $\mathbb{R}^d$. This hardly presents any new challenges, and mainly serves as a warm-up for the subsequent considerations.

We first recall and extend the notation related to conditional expectations and martingale differences over the dyadic filtration of $\mathbb{R}^d$. For any cube

$$Q = a_Q + \ell(Q)[0,1)^d,$$

with sidelength $\ell(Q) > 0$ and "lower left" corner $a_Q \in \mathbb{R}^d$, we denote by

$$\operatorname{ch}(Q) := \Big\{ a_Q + \frac{1}{2}\ell(Q)([0,1)^d + \alpha) : \alpha \in \{0,1\}^d \Big\}$$

the collection of its 2^d "children" obtained by bisecting each of the intervals in the Cartesian product defining Q. In particular, for

$$Q \in \mathscr{D}_k := \{2^{-k}([0,1)^d + n) : n \in \mathbb{Z}^d\},$$

we have

$$\operatorname{ch}(Q) = \{Q' \in \mathscr{D}_{k+1} : Q' \subseteq Q\}.$$

For every cube Q, we define the conditional expectation and martingale difference projections (acting on $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d; X)$)

$$\mathbb{E}_Q f := \mathbf{1}_Q \fint_Q f \,\mathrm{d}x, \qquad \mathbb{D}_Q f := \sum_{Q'\in\operatorname{ch}(Q)} \mathbb{E}_{Q'} f - \mathbb{E}_Q f. \tag{12.1}$$

Then for every $k \in \mathbb{Z}$, we let

$$\mathbb{E}_k f := \mathbb{E}(f|\sigma(\mathscr{D}_k)) = \sum_{Q\in\mathscr{D}_k} \mathbb{E}_Q f,$$

$$\mathbb{D}_k f := \mathbb{E}_{k+1} f - \mathbb{E}_k f = \sum_{Q\in\mathscr{D}_k} \mathbb{D}_Q f.$$

We still want to express the martingale difference projections $\mathbb{D}_Q$ in terms of vector-valued extensions of rank-one operators on scalar-valued functions.

In dimension $d = 1$, the operators already have this form, as we recall from Lemma 4.2.11 and the preceding discussion:

$$\mathbb{D}_I f = \langle f, h_I \rangle h_I, \qquad h_I = |I|^{-1/2}(\mathbf{1}_{I_-} - \mathbf{1}_{I_+}),$$

where h_I is called the *Haar function* associated with the interval I.

In higher dimensions, there are various ways of constructing analogues of the Haar functions. For the present purposes, a standard tensor construction suffices. In $d = 1$, we denote

$$h_I^1 := h_I, \qquad h_I^0 := |I|^{-1/2}\mathbf{1}_I.$$

Lemma 12.1.2. *In general dimension $d \geqslant 1$, the (tensor-)Haar functions*

$$h_Q^\alpha(x) = h_{I_1\times\cdots\times I_d}^{(\alpha_1,\ldots,\alpha_d)}(x_1,\ldots,x_d) := \prod_{i=1}^d h_{I_i}^{\alpha_i}(x_i), \quad \alpha = (\alpha_1,\ldots,\alpha_d) \in \{0,1\}^d.$$

satisfy the following identity for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d; X)$:

$$\mathbb{D}_Q f = \sum_{\alpha\in\{0,1\}^d\setminus\{0\}} \langle f, h_Q^\alpha \rangle h_Q^\alpha =: \sum_{\alpha\in\{0,1\}^d\setminus\{0\}} \mathbb{D}_Q^\alpha f.$$

Proof. From the (obvious) orthogonality of one-dimensional Haar functions, it follows that

$$\langle h_Q^\alpha, h_Q^\beta \rangle = \prod_{i=1}^d \langle h_{I_i}^{\alpha_i}, h_{I_i}^{\beta_i} \rangle = \prod_{i=1}^d \delta_{\alpha_i,\beta_i} = \delta_{\alpha,\beta}.$$

Let H_Q be the space of scalar-valued functions supported on Q, constant on each dyadic child of Q, and of mean zero. Clearly $\dim H_Q = (2^d - 1)$ and $h_Q^\alpha \in H_Q$ for each $\alpha \in \{0,1\}^d \setminus \{0\}$. Since these h_Q^α are orthonormal and their number is equal to $\dim H_Q$, they must form an orthonormal basis of H_Q. On the other hand, one easily verifies that D_Q is the orthogonal projection of $L^2(\mathbb{R}^d)$ onto H_Q, so in particular $D_Q f = f$ for all $f \in H_Q$. Since the h_Q^α form an orthonormal basis, the claimed identity is true for all $f \in H_Q$. If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d; X)$ and $x^* \in X^*$, then $\langle D_Q f, x^* \rangle \in H_Q$ and thus

$$\langle \mathbb{D}_Q f, x^* \rangle = \sum_{\alpha\in\{0,1\}^d\setminus\{0\}} \langle \langle \mathbb{D}_Q f, x^* \rangle, h_Q^\alpha \rangle h_Q^\alpha = \Big\langle \sum_{\alpha\in\{0,1\}^d\setminus\{0\}} \langle \mathbb{D}_Q f, h_Q^\alpha \rangle h_Q^\alpha, x^* \Big\rangle,$$

where

$$\begin{aligned}\langle \mathbb{D}_Q f, h_Q^\alpha \rangle &= \sum_{Q'\in\mathrm{ch}(Q)} \langle \mathbb{E}_{Q'} f, h_Q^\alpha \rangle - \langle \mathbb{E}_Q f, h_Q \rangle \\ &= \sum_{Q'\in\mathrm{ch}(Q)} \langle f, \mathbb{E}_{Q'} h_Q^\alpha \rangle = \langle f, h_Q^\alpha \rangle.\end{aligned}$$

The claimed identity follows, since the functionals $x^* \in X^*$ separate the points $x \in X$ by the Hahn–Banach theorem. □

The functions h_Q^α, with $\alpha \in \{0,1\}^d \setminus \{0\}$, are referred to as *cancellative Haar functions*, as they all have vanishing mean. In contrast, $h_Q^0 = |Q|^{-1/2}\mathbf{1}_Q$ is the *non-cancellative Haar function* on Q. In the wavelet literature, the cancellative Haar functions are special cases of *mother wavelets*, while the non-cancellative Haar function is the *father wavelet*.

Lemma 12.1.3. *Let X be a Banach space and $p \in (1,\infty)$. Then the space of finite linear combinations of cancellative Haar functions with X-coefficients,*

$$S_{00}(\mathscr{D};X) := \operatorname{span}\Big\{h_Q^\alpha \otimes x : Q \in \mathscr{D}, \alpha \in \{0,1\}^d \setminus \{0\}, x \in X\Big\}, \qquad (12.2)$$

is dense in $L^p(\mathbb{R}^d;X)$.

Proof. The filtration generated by the dyadic cubes, $(\mathscr{F}_k)_{k\in\mathbb{Z}} := (\sigma(\mathscr{D}_k))_{k\in\mathbb{Z}}$ is σ-finite with respect to the Lebesgue measure on $\mathbb{R}^d$, and $\mathscr{F}_\infty := \sigma\big(\bigcup_{k\in\mathbb{Z}}\mathscr{F}_k\big)$ is the Borel σ-algebra of $\mathbb{R}^d$. Hence $E_k f \to f$ in $L^p(\mathbb{R}^d;X)$ as $k \to \infty$ for all $f \in L^p(\mathbb{R}^d;X)$ by the forward convergence of generated martingales (Theorem 3.3.2). On the other hand, $\mathscr{F}_{-\infty} := \bigcap_{k\in\mathbb{Z}}\mathscr{F}_k$ contains only sets of Lebesgue measure 0 (the empty set) or ∞ (the quadrants, and their unions), which means (by definition) that the Lebesgue measure is *purely infinite* on $\mathscr{F}_{-\infty}$. Thus $E_k f \to 0$ in $L^p(\mathbb{R}^d;X)$ as $k \to -\infty$ for all $f \in L^p(\mathbb{R}^d;X)$ by the backward convergence of martingales (Theorem 3.3.5).

Combining these observation about the limits at $\pm\infty$, it follows that functions of the form $\mathbb{E}_M f - \mathbb{E}_m f = \sum_{k=m}^{M-1}\mathbb{D}_k f$ are dense in $L^p(\mathbb{R}^d;X)$. Next, we make the following observations about each $\mathbb{D}_k$ appearing in this expansion. First, for any $P \in \mathscr{D}_m$, multiplication with $\mathbf{1}_P$ commutes with $\mathbb{D}_k$; second, $\mathbf{1}_P\mathbb{D}_k f$ is a finite linear combination of some $\mathbb{D}_Q f$, and finally, if $(P_i)_{i=1}^\infty$ is an enumeration of $\mathscr{D}_m$, then $\sum_{i=1}^N \mathbf{1}_{P_i} f \to f$ in $L^p(\mathbb{R}^d;X)$ as $N \to \infty$. Thus finite linear combinations of $\mathbb{D}_Q f$ are dense in $L^p(\mathbb{R}^d;X)$. Finally, Lemma 12.1.2 shows that $\mathbb{D}_Q f \in S_{00}(\mathscr{D};X)$, and completes the proof. □

Remark 12.1.4. One can check that

$$S_{00}(\mathscr{D};X) = \Big\{f \in S(\mathscr{D};X) : \int_D f = 0 \text{ for each quadrant } D \text{ of } \mathbb{R}^d\Big\}.$$

In particular, if $\mathscr{D}$ is a connected tree of dyadic cubes (i.e., every two cubes are contained in a common bigger dyadic cube), then $S_{00}(\mathscr{D};X) = S_0(\mathscr{D};X)$. Making this connectedness assumption would slightly simplify some considerations, but have the disadvantage of excluding the standard dyadic system (cf. Remark 11.1.9).

After these preparatory considerations, we are in a position to prove the first non-trivial estimates for operators of dyadic singular integral type. As one expects, the UMD property is used, but in this first estimate still in a relatively straightforward manner.

Proposition 12.1.5. *Let X be a UMD space, $p \in (1,\infty)$, and $f \in S_{00}(\mathscr{D};X)$. For any any $\alpha \in \{0,1\}^d \setminus \{0\}$ and coefficients $\lambda_Q \in \mathbb{K}$, we have the estimates*

$$\Big\| \sum_{Q\in\mathscr{D}} \lambda_Q \langle h_Q^\alpha, f\rangle h_Q^\alpha \Big\|_{L^p(\mathbb{R}^d;X)} \leqslant \beta_{p,X} \sup_{Q\in\mathscr{D}} |\lambda_Q| \|f\|_{L^p(\mathbb{R}^d;X)},$$
$$\Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q \langle h_Q^\alpha, f\rangle h_Q^\alpha \Big\|_{L^p(\Omega\times\mathbb{R}^d;X)} \leqslant \beta_{p,X}^+ \|f\|_{L^p(\mathbb{R}^d;X)}.$$

Proof. Let us denote

$$\mathbb{D}_Q^\alpha f := \langle h_Q^\alpha, f\rangle h_Q^\alpha, \qquad \mathbb{D}_Q^{-\alpha} f := \mathbb{D}_Q f - \mathbb{D}_Q^\alpha f = \sum_{\gamma\in\{0,1\}^d\setminus\{0,\alpha\}} \langle h_Q^\gamma, f\rangle h_Q^\gamma.$$

Then $(\mathbb{D}_Q^\alpha f, \mathbb{D}_Q^{-\alpha} f)$ is a martingale difference sequence on Q, as each h_Q^γ with $\gamma \notin \{0,\alpha\}$ has average zero on the sets where h_Q^α is constant. Appropriately enumerated, $(\mathbb{D}_Q^\alpha f, \mathbb{D}_Q^{-\alpha} f)_{Q\in\mathscr{D}}$ also forms a martingale difference sequence. Estimating its martingale transform by a multiplying sequence of 0's and 1's, we obtain

$$\begin{aligned} \Big\| \sum_{Q\in\mathscr{D}} \lambda_Q \mathbb{D}_Q^\alpha f \Big\|_{L^p(\mathbb{R}^d;X)} &= \Big\| \sum_{Q\in\mathscr{D}} \Big(\lambda_Q \cdot \mathbb{D}_Q^\alpha f + 0 \cdot \mathbb{D}_Q^{-\alpha} f\Big) \Big\|_{L^p(\mathbb{R}^d;X)} \\ &\leqslant \beta_{p,X} \Big\| \sum_{Q\in\mathscr{D}} \Big(\mathbb{D}_Q^\alpha f + \mathbb{D}_Q^{-\alpha} f\Big) \Big\|_{L^p(\mathbb{R}^d;X)}, \end{aligned}$$

For the other claim, we argue by the contraction principle and the randomised UMD inequality to see that

$$\begin{aligned} \Big\| \sum_{Q\in\mathscr{F}} \varepsilon_Q \mathbb{D}_Q^\alpha f \Big\|_{L^p(\Omega\times\mathbb{R}^d;X)} &\leqslant \Big\| \sum_{Q\in\mathscr{D}} \Big(\varepsilon_Q \mathbb{D}_Q^\alpha f + \varepsilon'_Q \mathbb{D}_Q^{-\alpha} f\Big) \Big\|_{L^p(\Omega\times\mathbb{R}^d;X)} \\ &\leqslant \beta_{p,X}^+ \Big\| \sum_{Q\in\mathscr{D}} \Big(\mathbb{D}_Q^\alpha f + \mathbb{D}_Q^{-\alpha} f\Big) \Big\|_{L^p(\mathbb{R}^d;X)}, \end{aligned}$$

and in both cases we conclude by observing that

$$\sum_{Q\in\mathscr{D}} \Big(\mathbb{D}_Q^\alpha f + \mathbb{D}_Q^{-\alpha} f\Big) = \sum_{Q\in\mathscr{D}} \mathbb{D}_Q f = f.$$

□

For operator-valued coefficients $\lambda_Q \in \mathscr{L}(X,Y)$, the following variants of R-boundedness turn out to be relevant:

Definition 12.1.6. *For $p \in (1,\infty)$ and an operator family $\lambda = (\lambda_Q)_{Q\in\mathscr{C}} \subseteq \mathscr{L}(X,Y)$ indexed by a collection $\mathscr{C}$ of bounded Borel subsets of $\mathbb{R}^d$, we denote by $\mathscr{DR}_p(\lambda)$ and $\mathscr{ER}_p(\lambda)$ the smallest admissible constants such that the* *following estimates hold for all finitely non-zero families* $(x_Q)_{Q\in\mathscr{C}} \subseteq X$ *and* $(y_Q^*)_{Q\in\mathscr{C}} \subseteq Y^*$*:*

$$\begin{aligned}\sum_{Q\in\mathscr{C}} &|Q||\langle\lambda_Q x_Q, y_Q^*\rangle| \\ &\leqslant \mathscr{DR}_p(\lambda)\Big\|\sum_{Q\in\mathscr{C}}\varepsilon_Q x_Q \mathbf{1}_Q\Big\|_{L^p(\Omega\times\mathbb{R}^d;X)}\Big\|\sum_{Q\in\mathscr{C}}\varepsilon_Q y_Q^* \mathbf{1}_Q\Big\|_{L^{p'}(\Omega\times\mathbb{R}^d;Y^*)},\end{aligned}$$

and

$$\Big\|\sum_{Q\in\mathscr{C}}\varepsilon_Q \lambda_Q x_Q \mathbf{1}_Q\Big\|_{L^p(\Omega\times\mathbb{R}^d;Y)} \leqslant \mathscr{ER}_p(\lambda)\Big\|\sum_{Q\in\mathscr{C}}\varepsilon_Q x_Q \mathbf{1}_Q\Big\|_{L^p(\Omega\times\mathbb{R}^d;X)}.$$

We refer to $\mathscr{DR}_p(\lambda)$ *as the* $\mathscr{DR}_p$*-bound of* λ*, and say that* λ *is* $\mathscr{DR}_p$*-bounded if* $\mathscr{DR}_p(\lambda) < \infty$*. The same convention applies to* $\mathscr{ER}_p$ *in place of* $\mathscr{DR}_p$*.*

Remark 12.1.7. The primary case of interest will be when $\mathscr{C} = \mathscr{D}$ is a system of dyadic cubes. In this case, it is useful to observe at once that the defining inequality of $\mathscr{ER}_p(\lambda)$ immediately extends to Haar functions h_Q^α in place of the indicators $\mathbf{1}_Q$:

$$\Big\|\sum_{Q\in\mathscr{D}}\varepsilon_Q \lambda_Q x_Q h_Q^\gamma\Big\|_{L^p(\Omega\times\mathbb{R}^d;Y)} \leqslant \mathscr{ER}_p(\lambda)\Big\|\sum_{Q\in\mathscr{D}}\varepsilon_Q x_Q h_Q^\alpha\Big\|_{L^p(\Omega\times\mathbb{R}^d;X)}.$$

Proof. We have $h_Q^\alpha = \operatorname{sgn}(h_Q^\alpha)|Q|^{-1/2}\mathbf{1}_Q$ and hence, by the contraction principle,

$$\Big\|\sum_{Q\in\mathscr{D}}\varepsilon_Q z_Q h_Q^\alpha\Big\|_{L^p(\Omega\times\mathbb{R}^d;Z)} = \Big\|\sum_{Q\in\mathscr{D}}\varepsilon_Q |Q|^{-1/2} z_Q \mathbf{1}_Q\Big\|_{L^p(\Omega\times\mathbb{R}^d;Z)}$$

for both $(z_Q, Z) = \{(x_Q, X), (\lambda_Q x_Q, Y)\}$. Using this twice, with both α and γ, and in between the defining inequality of $\mathscr{ER}_p(\lambda)$ for $|Q|^{-1/2}x_Q$ in place x_Q, yields the claim. □

These notions are weaker than R-boundedness; we will shortly see that the converse fails in general.

Lemma 12.1.8. *For all Banach spaces* X *and* Y*, all operator families* $\lambda = (\lambda_Q)_{Q\in\mathscr{C}} \subseteq \mathscr{L}(X,Y)$ *and their adjoints* $\lambda^* := (\lambda_Q^*)_{Q\in\mathscr{D}} \subseteq \mathscr{L}(Y^*, X^*)$*, and all* $p \in (1,\infty)$*, we have*

$$\begin{aligned}\sup_{Q\in\mathscr{C}} \|\lambda_Q\| \leqslant \mathscr{DR}_p(\lambda) &\leqslant \min\{\mathscr{ER}_p(\lambda), \mathscr{ER}_{p'}(\lambda^*)\}, \\ \mathscr{ER}_p(\lambda) &\leqslant \|x \mapsto \mathscr{R}_p(\{\lambda_Q : Q \ni x\})\|_{L^\infty(\mathbb{R}^d)} \leqslant \mathscr{R}_p(\lambda).\end{aligned}$$

Proof. The last two estimates are immediate. The first estimate follows by testing the defining condition of $\mathscr{DR}_p$ with only one non-zero pair (x_Q, y_Q^*) at a time. To see that $\mathscr{DR}_p(\lambda) \leqslant \mathscr{ER}_p(\lambda)$, for suitable scalars $|\eta_Q| = 1$, we have

$$\begin{aligned}\sum_{Q\in\mathscr{C}}|Q||\langle\lambda_Q x_Q,y_Q^*\rangle| &= \sum_{Q\in\mathscr{C}}\int\eta_Q\langle\lambda_Q x_Q\mathbf{1}_Q,y_Q^*\mathbf{1}_Q\rangle\\ &=\mathbb{E}\int\Big\langle\sum_{Q\in\mathscr{C}}\varepsilon_Q\eta_Q\lambda_Q x_Q\mathbf{1}_Q,\sum_{R\in\mathscr{C}}\varepsilon_R y_R^*\mathbf{1}_R\Big\rangle\\ &\leqslant\Big\|\sum_{Q\in\mathscr{C}}\varepsilon_Q\eta_Q\lambda_Q x_Q\mathbf{1}_Q\Big\|_{L^p(\Omega\times\mathbb{R}^d;X)}\Big\|\sum_{R\in\mathscr{C}}\varepsilon_R y_R^*\mathbf{1}_R\Big\|_{L^{p'}(\Omega\times\mathbb{R}^d;Y^*)}\\ &\leqslant\mathscr{E}\mathscr{R}_p(\lambda)\Big\|\sum_{Q\in\mathscr{C}}\varepsilon_Q x_Q\mathbf{1}_Q\Big\|_{L^p(\Omega\times\mathbb{R}^d;X)}\Big\|\sum_{R\in\mathscr{C}}\varepsilon_R y_R^*\mathbf{1}_R\Big\|_{L^{p'}(\Omega\times\mathbb{R}^d;Y^*)},\end{aligned}$$

where we used Kahane's contraction principle and the definition of $\mathscr{E}\mathscr{R}_p(\lambda)$ to pull out the scalar η_Q and the operators λ_Q in the last step. Since $\langle\lambda_Q x_Q,y_Q^*\rangle=\langle x_Q,\lambda_Q^*y_Q^*\rangle$, and $\mathscr{D}\mathscr{R}_{p'}(\lambda)$ is defined by testing the expressions on the right over a more general choice of $x_Q^{**}\in X^{**}$ in place of $x_Q\in X$, it follows that

$$\mathscr{D}\mathscr{R}_p(\lambda)\leqslant\mathscr{D}\mathscr{R}_{p'}(\lambda^*)\leqslant\mathscr{E}\mathscr{R}_{p'}(\lambda^*)$$

by using what we already proved, but with λ^* in place of λ. □

Corollary 12.1.9. *If $\lambda=(\lambda_Q)_{Q\in\mathscr{C}}\subseteq\mathscr{L}(X)$ consists of scalar multiples of the identity, then*

$$\sup_{Q\in\mathscr{C}}|\lambda_Q|=\mathscr{D}\mathscr{R}_p(\lambda)=\mathscr{E}\mathscr{R}_p(\lambda)=\mathscr{R}_p(\lambda).$$

Proof. Lemma 12.1.9 shows that we have this chain with "$\leqslant$" in place of "$=$" throughout. On the other hand, Kahane's contraction principle guarantees that $\mathscr{R}_p(\lambda)=\sup_{Q\in\mathscr{C}}|\lambda_Q|$. Thus we have equality throughout. □

The following example of $\mathscr{D}\mathscr{R}_p$-bounded families will play a role in our investigation of criteria for boundedness of singular integral operators; the uniform boundedness of the quantities $|Q|^{-1}\langle T\mathbf{1}_Q,\mathbf{1}_Q\rangle$ is classically known as the *weak boundedness property* of the operator T.

Example 12.1.10. Suppose that $T\in\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))$, and define $\langle T(\mathbf{1}_Q),\mathbf{1}_Q\rangle\in\mathscr{L}(X,Y)$ by

$$\langle T\mathbf{1}_Q,\mathbf{1}_Q\rangle : x\mapsto\langle T(\mathbf{1}_Q x),\mathbf{1}_Q\rangle=\int_Q T(\mathbf{1}_Q x)\in Y.$$

For any collection $\mathscr{C}$ of bounded Borel subsets of $\mathbb{R}^d$, it follows that

$$\mathscr{D}\mathscr{R}_p\Big(\Big\{\frac{\langle T\mathbf{1}_Q,\mathbf{1}_Q\rangle}{|Q|}\Big\}_{Q\in\mathscr{C}}\Big)\leqslant\|T\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))}.$$

Proof. With suitable scalars $|\eta_Q|=1$, we have

$$\sum_{Q\in\mathscr{C}}|Q|\Big|\Big\langle\frac{\langle T\mathbf{1}_Q,\mathbf{1}_Q\rangle}{|Q|}x_Q,y_Q^*\Big\rangle\Big| = \sum_{Q\in\mathscr{C}}\eta_Q\langle T(\mathbf{1}_Q x_Q),\mathbf{1}_Q y_Q^*\rangle$$
$$= \mathbb{E}\Big\langle T\sum_{Q\in\mathscr{C}}\varepsilon_Q\eta_Q x_Q\mathbf{1}_Q,\sum_{R\in\mathscr{C}}\varepsilon_R y_R^*\mathbf{1}_R\Big\rangle$$
$$\leqslant \Big\|T\sum_{Q\in\mathscr{C}}\varepsilon_Q\eta_Q x_Q\mathbf{1}_Q\Big\|_{L^p(\Omega\times\mathbb{R}^d;Y)}\Big\|\sum_{R\in\mathscr{C}}\varepsilon_R y_R^*\mathbf{1}_R\Big\|_{L^{p'}(\Omega\times\mathbb{R}^d;Y)},$$

where

$$\Big\|T\sum_{Q\in\mathscr{C}}\varepsilon_Q\eta_Q x_Q\mathbf{1}_Q\Big\|_{L^p(\Omega\times\mathbb{R}^d;Y)}$$
$$\leqslant \|T\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))}\Big\|\sum_{Q\in\mathscr{C}}\varepsilon_Q x_Q\mathbf{1}_Q\Big\|_{L^p(\Omega\times\mathbb{R}^d;X)}$$

by the assumed boundedness of T and Kahane's contraction principle with the coefficients η_Q. □

While Example 12.1.10 will only play a role later, the weakening of R-boundedness has the following immediate application:

Theorem 12.1.11 (Haar multipliers). *Let X and Y be UMD spaces and $p\in(1,\infty)$. For $\alpha,\gamma\in\{0,1\}^d\setminus\{0\}$ and $\lambda=(\lambda_Q)_{Q\in\mathscr{D}}\subseteq\mathscr{L}(X,Y)$, consider the operator*

$$\mathfrak{H}_\lambda^{\alpha\gamma}: f\mapsto\sum_{Q\in\mathscr{D}}\lambda_Q\langle f,h_Q^\alpha\rangle h_Q^\gamma, \tag{12.3}$$

initially mapping $S_{00}(\mathscr{D};X)$ into $S_{00}(\mathscr{D};Y)$. Then $\mathfrak{H}_\lambda^{\alpha\gamma}$ extends to a bounded operator on $L^p(\mathbb{R}^d;X)$ if and only if $\mathscr{D}\mathscr{R}_p(\lambda)<\infty$, and in this case

$$\frac{\mathscr{D}\mathscr{R}_p(\lambda)}{\beta_{p,X}^-\beta_{p',Y^*}^-}\leqslant\|\mathfrak{H}_\lambda^{\alpha\gamma}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))}\leqslant\beta_{p,X}^+\beta_{p',Y^*}^+\mathscr{D}\mathscr{R}_p(\lambda). \tag{12.4}$$

Proof. Dualising $\mathfrak{H}_\lambda^{\alpha\gamma}f\in S_{00}(\mathscr{D};X)\subseteq L^p(\mathbb{R}^d;X)$ with $g\in S_{00}(\mathscr{D};Y^*)\subseteq L^{p'}(\mathbb{R}^d;Y^*)$, we arrive at

$$\begin{aligned}|\langle\mathfrak{H}_\lambda^{\alpha\gamma}f,g\rangle| &= \Big|\sum_{Q\in\mathscr{D}}\Big\langle\lambda_Q\langle f,h_Q^\alpha\rangle,\langle h_Q^\gamma,g\rangle\Big\rangle\Big|\\ &= \Big|\sum_{Q\in\mathscr{D}}|Q|\Big\langle\lambda_Q\frac{\langle f,h_Q^\alpha\rangle}{|Q|^{1/2}},\frac{\langle h_Q^\gamma,g\rangle}{|Q|^{1/2}}\Big\rangle\Big|\\ &\leqslant\mathscr{D}\mathscr{R}_p(\lambda)\Big\|\sum_{Q\in\mathscr{D}}\varepsilon_Q\langle f,h_Q^\alpha\rangle\frac{\mathbf{1}_Q}{|Q|^{1/2}}\Big\|_{L^p(\Omega\times\mathbb{R}^d;X)}\times\\ &\quad\times\Big\|\sum_{Q\in\mathscr{D}}\varepsilon_Q\langle g,h_Q^\alpha\rangle\frac{\mathbf{1}_Q}{|Q|^{1/2}}\Big\|_{L^{p'}(\Omega\times\mathbb{R}^d;Y^*)}.\end{aligned} \tag{12.5}$$

For a fixed $s \in \mathbb{R}^d$, the sequences $(\varepsilon_Q \mathbf{1}_Q(s)/|Q|^{1/2})_{Q\in\mathscr{D}}$ and $(\varepsilon_Q h_Q^\alpha)_{Q\in\mathscr{D}}$ have equal distribution; thus

$$\begin{aligned}\Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q \langle f, h_Q^\alpha\rangle \frac{\mathbf{1}_Q}{|Q|^{1/2}} \Big\|_{L^p(\Omega\times\mathbb{R}^d;X)} &= \Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q \langle f, h_Q^\alpha\rangle h_Q^\alpha \Big\|_{L^p(\Omega\times\mathbb{R}^d;X)} \\ &\leqslant \beta_{p,X}^+ \|f\|_{L^p(\mathbb{R}^d;X)}\end{aligned}$$

by Proposition 12.1.5 in the last step. Similarly, the last term in (12.5) is dominated by $\beta_{p',Y^*}^+ \|g\|_{L^{p'}(\mathbb{R}^d;Y^*)}$. Hence

$$|\langle \mathfrak{H}_\lambda^{\alpha\gamma} f, g\rangle| \leqslant \mathscr{D}\mathscr{R}_p(\lambda)\beta_{p,X}^+ \beta_{p',Y^*}^+ \|f\|_{L^p(\mathbb{R}^d;X)} \|g\|_{L^{p'}(\mathbb{R}^d;Y^*)},$$

which proves the second estimate in (12.4).

Conversely, for finitely non-zero families $(x_Q)_{Q\in\mathscr{D}} \subseteq X$ and $(y_Q^*)_{Q\in\mathscr{D}} \subseteq Y^*$, we choose scalar $|\eta_Q| = 1$ such that $|\langle \lambda_Q x_Q, y_Q^*\rangle| = \eta_Q \langle \lambda_Q x_Q, y_Q^*\rangle$ and consider the functions

$$f := \sum_{Q\in\mathscr{D}} |Q|^{1/2} \eta_Q x_Q h_Q^\alpha \in S_{00}(\mathscr{D};X), \quad g := \sum_{Q\in\mathscr{D}} |Q|^{1/2} y_Q^* h_Q^\gamma \in S_{00}(\mathscr{D};Y^*).$$

Then

$$\begin{aligned}\mathfrak{H}_\lambda^{\alpha\gamma} f &= \sum_{Q\in\mathscr{D}} |Q|^{1/2} \eta_Q \lambda_Q x_Q h_Q^\gamma, \\ \langle \mathfrak{H}_\lambda^{\alpha\gamma} f, g\rangle &= \sum_{Q\in\mathscr{D}} |Q| \eta_Q \langle \lambda_Q x_Q, y_Q^*\rangle,\end{aligned}$$

and hence

$$\sum_{Q\in\mathscr{D}} |Q| |\langle \lambda_Q x_Q, y_Q^*\rangle| \leqslant \|\mathfrak{H}_\lambda^{\alpha\gamma}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X), L^p(\mathbb{R}^d;Y))} \|f\|_{L^p(\mathbb{R}^d;X)} \|g\|_{L^{p'}(\mathbb{R}^d;Y^*)},$$

where

$$\begin{aligned}\|f\|_{L^p(\mathbb{R}^d;X)} &\leqslant \beta_{p,X}^- \Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q |Q|^{1/2} \eta_Q x_Q h_Q^\alpha \Big\|_{L^p(\Omega\times\mathbb{R}^d;X)} \\ &= \beta_{p,X}^- \Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q x_Q \mathbf{1}_Q \Big\|_{L^p(\Omega\times\mathbb{R}^d;X)}\end{aligned}$$

by a similar equidistribution property as before. Similarly, we have

$$\|g\|_{L^{p'}(\mathbb{R}^d;Y^*)} \leqslant \beta_{p',Y^*}^- \Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q y_Q^* \mathbf{1}_Q \Big\|_{L^{p'}(\Omega\times\mathbb{R}^d;Y^*)},$$

and combining the bounds, we have proved the first estimate in (12.4). □

Remark 12.1.12. Under stronger assumptions on the coefficients λ, one can improve the dependence on the UMD constants:

(1) If $X = Y$, $\alpha = \gamma$, and $\lambda \subseteq \mathbb{K} \cdot I_X$ is bounded, then $\mathfrak{H}_\lambda^{\alpha\alpha}$ extends to a bounded operator on $L^p(\mathbb{R}^d; X)$ of norm at most

$$\|\mathfrak{H}_\lambda^{\alpha\alpha}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} \leqslant \beta_{p,X}\|\lambda\|_\infty.$$

(2) If $\lambda \subseteq \mathscr{L}(X,Y)$ is R-bounded, then $\mathfrak{H}_\lambda^{\alpha\gamma}$ extends to a bounded operator from $L^p(\mathbb{R}^d; X)$ to $L^p(\mathbb{R}^d; Y)$ of norm at most

$$\|\mathfrak{H}_\lambda^{\alpha\gamma}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \leqslant \beta^-_{p,Y}\beta^+_{p,X}\mathscr{E}\mathscr{R}_p(\lambda),$$

where a partial advantage over Theorem 12.1.11 comes from $\beta^-_{p,Y} \leqslant \beta^+_{p',Y^*}$.

Proof. (1): This is a restatement of the first estimate in Proposition 12.1.5.

(2): Since $(h_Q^\gamma)_{Q\in\mathscr{D}}$ is a martingale difference sequence, using the defining properties of various constants and the definition of $\mathscr{E}\mathscr{R}_p(\lambda)$ via Remark 12.1.7, we have

$$\begin{aligned}
&\Big\|\sum_{Q\in\mathscr{D}} \lambda_Q\langle f, h_Q^\alpha\rangle h_Q^\gamma\Big\|_{L^p(\mathbb{R}^d;Y)} \\
&\leqslant \beta^-_{p,Y}\Big\|\sum_{Q\in\mathscr{D}} \varepsilon_Q\lambda_Q\langle f, h_Q^\alpha\rangle h_Q^\gamma\Big\|_{L^p(\Omega\times\mathbb{R}^d;X)} \\
&\leqslant \beta^-_{p,Y}\mathscr{E}\mathscr{R}_p(\lambda)\Big\|\sum_{Q\in\mathscr{D}} \varepsilon_Q\langle f, h_Q^\alpha\rangle h_Q^\alpha\Big\|_{L^p(\Omega\times\mathbb{R}^d;Y)} \\
&\leqslant \beta^-_{p,Y}\mathscr{E}\mathscr{R}_p(\lambda)\beta^+_{p,X}\|f\|_{L^p(\mathbb{R}^d;X)},
\end{aligned}$$

where, in the last step, we used the second estimate in Proposition 12.1.5. □

Here is a nice class of examples of coefficients satisfying the dyadic R-boundedness condition:

Proposition 12.1.13. *Let Y be a UMD space and $p \in (1,\infty)$. Let $b \in L^\infty(\mathbb{R}^d; \mathscr{L}(X,Y))$, let $a = (a_Q)_{Q\in\mathscr{D}} \in \ell^\infty(\mathscr{D}; L^\infty(\mathbb{R}^d))$, and*

$$\lambda := (\lambda_Q)_{Q\in\mathscr{D}} := (\langle a_Q b\rangle_Q)_{Q\in\mathscr{D}}.$$

Then

$$\mathscr{E}\mathscr{R}_p((\langle a_Q b\rangle_Q)_{Q\in\mathscr{D}}) \leqslant \beta^+_{p,Y}\|a\|_{\ell^\infty(L^\infty)}\|b\|_{L^\infty(\mathbb{R}^d;\mathscr{L}(X,Y))}$$

Thus, for $\alpha,\gamma \in \{0,1\}^d\backslash\{0\}$, the Haar multiplier $\mathfrak{H}_\lambda^{\alpha\gamma}$ extends to a bounded operator from $L^p(\mathbb{R}^d; X)$ to $L^p(\mathbb{R}^d; Y)$ of norm at most

$$\|\mathfrak{H}_\lambda^{\alpha\gamma}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \leqslant \beta^-_{p,Y}\beta^+_{p,Y}\beta^+_{p,X}\|a\|_{\ell^\infty(L^\infty)}\|b\|_{L^\infty(\mathbb{R}^d;\mathscr{L}(X,Y))}.$$

Proof. The second claim is immediate from the first one in combination with Remark 12.1.12(2), so we concentrate on the first one. We may assume by scaling that $\|a_Q\|_{L^\infty(\mathbb{R}^d)} \leqslant 1$. Then

$$\begin{aligned}\Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q \langle a_Q b\rangle_Q x_Q \mathbf{1}_Q \Big\|_{L^p(\mathbb{R}^d;Y)} &= \Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q \mathbb{E}_Q (a_Q b x_Q \mathbf{1}_Q) \Big\|_{L^p(\mathbb{R}^d;Y)} \\ &\leqslant \beta^+_{p,Y} \Big\| b \sum_{Q\in\mathscr{D}} \varepsilon_Q x_Q \mathbf{1}_Q \Big\|_{L^p(\mathbb{R}^d;Y)} \\ &\leqslant \beta^+_{p,Y} \|b\|_{L^\infty(\mathbb{R}^d;\mathscr{L}(X,Y))} \Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q x_Q \mathbf{1}_Q \Big\|_{L^p(\mathbb{R}^d;Y)},\end{aligned}$$

where, in the first estimate, we applied Stein's inequality (Theorem 4.2.23) followed by Kahane's contraction principle with the scalar coefficients a_Q. □

The following result shows that result of Proposition 12.1.13 cannot be improved to usual R-boundedness; thus the notions $\mathscr{DR}_p$ and $\mathscr{ER}_p$ represent genuine relaxations:

Proposition 12.1.14. *For non-zero Banach spaces X and Y, the following are equivalent:*

(1) *X has type 2 and Y has cotype 2;*
(2) *for every $b \in L^\infty(0,1;\mathscr{L}(X,Y))$, the set $\{\langle b\rangle_Q : Q \in \mathscr{D}([0,1))\}$ is R-bounded;*
(3) *for every $b \in L^\infty(0,1;\mathscr{L}(X,Y))$, the function*

$$x \mapsto \mathscr{R}\Big(\big\{\langle b\rangle_Q : x \in Q \in \mathscr{D}([0,1))\big\}\Big)$$

is essentially bounded.

Proof. (1)⇒(2): For $b \in L^\infty(0,1;\mathscr{L}(X,Y))$, it is clear that the $\{\langle b\rangle_Q : Q \in \mathscr{D}([0,1))\}$ is uniformly bounded. Under the assumption (1), this implies R-boundedness by Proposition 8.6.1.

(2)⇒(3): This is clear.

(3)⇒(1): From the definition of R-boundedness, it is immediate that $\mathscr{R}(\mathscr{T}) = \sup(\{\mathscr{R}(\mathscr{F}) : \mathscr{F} \subseteq \mathscr{T} \text{ finite}\})$. So if some collection $\mathscr{T}$ is not R-bounded, it has finite subcollections $\mathscr{F}_n$ with $\mathscr{R}(\mathscr{F}_n) \to \infty$. Then the countable collection $\bigcup_{n=1}^\infty \mathscr{F}_n \subseteq \mathscr{T}$ also fails to be R-bounded.

If (1) is not satisfied, then Proposition 8.6.1 says that the unit ball of $\bar{B}_{\mathscr{L}(X,Y)}$ of $\mathscr{L}(X,Y)$ is not R-bounded. By what we just observed, this means that we can find a sequence $\{u_k\}_{k=0}^\infty \subseteq \bar{B}_{\mathscr{L}(X,Y)}$ that fails to be R-bounded. Let $v_k := \frac{4}{3}u_k - \frac{1}{3}u_{k+1}$ and

$$b := \sum_{j=0}^\infty v_j \mathbf{1}_{[4^{-j-1},4^{-j})}.$$

Then $b \in L^\infty(\mathbb{R}^d;\mathscr{L}(X,Y))$ and $\|b\|_\infty = \sup_k \|v_k\| \leqslant \frac{5}{3}\sup_k \|u_k\| = \frac{5}{3}$. Moreover,

$$\langle b\rangle_{[0,4^{-k})} = 4^k \sum_{j=k}^{\infty} \frac{3}{4} 4^{-j} v_j = 4^k \frac{3}{4} \Big(\sum_{j=k}^{\infty} 4^{-j} \frac{4}{3} u_j - \sum_{j=k}^{\infty} 4^{-j} \frac{1}{3} u_{j+1} \Big) = u_k.$$

Then for each n, we have

$$\begin{aligned} &\| x \mapsto \mathscr{R}(\{\langle b\rangle_I : x \in I \in \mathscr{D}([0,1))\}) \|_{L^\infty(0,1)} \\ &\geqslant \mathscr{R}(\{\langle b\rangle_I : [0,4^{-n}) \in I \in \mathscr{D}([0,1))\}) \geqslant \mathscr{R}(\{\langle b\rangle_{[0,4^{-k})}\}_{k=0}^n) = \mathscr{R}(\{u_k\}_{k=0}^n), \end{aligned}$$

and hence

$$\begin{aligned} \infty = \mathscr{R}(\{u_k\}_{k=0}^\infty) &= \sup_{n\in\mathbb{N}} \mathscr{R}(\{u_k\}_{k=0}^n) \\ &\leqslant \| x \mapsto \mathscr{R}(\{\langle b\rangle_I : x \in I \in \mathscr{D}([0,1))\}) \|_{L^\infty(0,1)}. \end{aligned}$$

Thus (3) fails, and by contraposition this proves the claimed implication. ☐

Comparison of $\mathscr{D}\mathscr{R}_p$ and $\mathscr{E}\mathscr{R}_p$

In the rest of this section, we make a further comparison of the two relaxed notions of R-boundedness from Definition 12.1.6.. When Y is a UMD space—an assumption that we make a good part of the time—, these notions turn out to be equivalent. The universal bound $\mathscr{D}\mathscr{R}_p(\lambda) \leqslant \mathscr{E}\mathscr{R}_p(\lambda)$ was already observed in Lemma 12.1.8. The reverse estimate could be achieved essentially by concatenating a couple of results that we have treated earlier in these volumes, but it turns out that a slightly sharper quantitative bound can be achieved by also revisiting their proofs to establish the following proposition:

Proposition 12.1.15. *Let Y be a UMD space and $p \in (1,\infty)$. Let $\mathscr{E}_0 := \{\varnothing, \Omega\}$ be the trivial σ-algebra of a probability space $(\Omega, \mathscr{A}, \mathbb{P})$ supporting a Rademacher sequence $(\varepsilon_n)_{n=1}^N$, and $(\mathscr{F}_n)_{n=1}^N$ be a σ-finite filtration of some measure space $(S, \mathscr{F}, \mu)$. Then, for all $f \in L^p(\Omega \times S; Y)$, we have*

$$\Big\| \sum_{n=1}^N \varepsilon_n \mathbb{E}(\varepsilon_n f | \mathscr{E}_0 \times \mathscr{F}_n) \Big\|_{L^p(\Omega\times S;Y)} \leqslant \beta^+_{p,Y} \|f\|_{L^p(\Omega\times S;Y)}.$$

Proof. Let $\mathscr{E}_n := \sigma(\varepsilon_1, \dots, \varepsilon_n)$ for $n = 1, \dots, N$. Then

$$\begin{aligned} \mathbb{E}(\varepsilon_n f | \mathscr{E}_0 \times \mathscr{F}_n) &= \mathbb{E}(\mathbb{E}(\varepsilon_n f | \mathscr{E}_n \times \mathscr{F}_n) | \mathscr{E}_0 \times \mathscr{F}_n) \\ &= \mathbb{E}(\varepsilon_n \mathbb{E}(f | \mathscr{E}_n \times \mathscr{F}_n) | \mathscr{E}_0 \times \mathscr{F}_N), \end{aligned}$$

where in the last step we note that for both $k \in \{n, N\}$, the conditional expectation of the function inside, given $\mathscr{E}_0 \times \mathscr{F}_k$, is obtained by simply integrating out the dependence on $\omega \in \Omega$. On the other hand, we have

$$\begin{aligned} &\mathbb{E}(\varepsilon_n \mathbb{E}(f | \mathscr{E}_{n-1} \otimes \mathscr{F}_n) | \mathscr{E}_0 \times \mathscr{F}_N) \\ &= \mathbb{E}(\mathbb{E}(\varepsilon_n \mathbb{E}(f | \mathscr{E}_{n-1} \times \mathscr{F}_n) | \mathscr{E}_{n-1} \times \mathscr{F}_N) | \mathscr{E}_0 \times \mathscr{F}_N) \\ &= \mathbb{E}(\mathbb{E}(\varepsilon_n | \mathscr{E}_{n-1} \times \mathscr{F}_N) \mathbb{E}(f | \mathscr{E}_{n-1} \times \mathscr{F}_n) | \mathscr{E}_0 \times \mathscr{F}_N) \\ &= \mathbb{E}(0 \cdot \mathbb{E}(f | \mathscr{E}_{n-1} \times \mathscr{F}_n) | \mathscr{E}_0 \times \mathscr{F}_N) = 0. \end{aligned}$$

Thus

$$\begin{aligned}\mathbb{E}(\varepsilon_n f|\mathscr{E}_0 \times \mathscr{F}_n) &= \mathbb{E}(\varepsilon_n[\mathbb{E}(f|\mathscr{E}_n \times \mathscr{F}_n) - \mathbb{E}(f|\mathscr{E}_{n-1} \times \mathscr{F}_n)]|\mathscr{E}_0 \times \mathscr{F}_N) \\ &= \mathbb{E}(\varepsilon_n[\mathbb{E}(f|\mathscr{G}_{2n}) - \mathbb{E}(f|\mathscr{G}_{2n-1})]|\mathscr{E}_0 \times \mathscr{F}_N) \\ &= \mathbb{E}(\varepsilon_n d_{2n}|\mathscr{E}_0 \times \mathscr{F}_N),\end{aligned}$$

where

$$d_k := \begin{cases}\mathbb{E}(f|\mathscr{G}_k) - \mathbb{E}(f|\mathscr{G}_{k-1}), & k = 2, \dots, 2N, \\ \mathbb{E}(f|\mathscr{G}_1), & k = 1,\end{cases}$$

are martingale differences relative to a filtration $(\mathscr{G}_k)_{k=1}^{2N}$ on $\Omega \times S$ defined by

$$\mathscr{G}_{2n} := \mathscr{E}_n \times \mathscr{F}_n, \qquad \mathscr{G}_{2n-1} := \mathscr{E}_{n-1} \times \mathscr{F}_n.$$

Then, noting that $\mathbb{E}(\ |\mathscr{E}_0 \times \mathscr{F}_N)$ is constant in $\omega \in \Omega$, and denoting by $(\varepsilon'_k)_{k=1}^{2N}$ another Rademacher sequence on some $(\Omega', \mathscr{A}', \mathbb{P}')$, we have

$$\begin{aligned}&\Big\| \sum_{n=1}^N \varepsilon_n \mathbb{E}(\varepsilon_n f|\mathscr{E}_0 \times \mathscr{F}_n)\Big\|_{L^p(\Omega\times S;Y)} \\ &= \Big\| \sum_{n=1}^N \varepsilon'_{2n} \mathbb{E}(\varepsilon_n d_{2n}|\mathscr{E}_0 \times \mathscr{F}_N)\Big\|_{L^p(\Omega'\times\Omega\times S;Y)} \\ &= \Big\| \mathbb{E}(\sum_{n=1}^N \varepsilon'_{2n}\varepsilon_n d_{2n}|\mathscr{E}_0 \times \mathscr{F}_N)\Big\|_{L^p(\Omega'\times\Omega\times S;Y)} \\ &\leqslant \Big\| \sum_{n=1}^N \varepsilon'_{2n}\varepsilon_n d_{2n}\Big\|_{L^p(\Omega'\times\Omega\times S;Y)} = \Big\| \sum_{n=1}^N \varepsilon'_{2n} d_{2n}\Big\|_{L^p(\Omega'\times\Omega\times S;Y)} \\ &\leqslant \Big\| \sum_{k=1}^{2N} \varepsilon'_k d_k\Big\|_{L^p(\Omega'\times\Omega\times S;Y)} \leqslant \beta^+_{p,Y}\Big\| \sum_{k=1}^{2N} d_k\Big\|_{L^p(\Omega\times S;Y)} \\ &= \beta^+_{p,Y}\|\mathbb{E}(f|\mathscr{G}_{2N})\|_{L^p(\Omega\times S;Y)} \leqslant \beta^+_{p,Y}\|f\|_{L^p(\Omega\times S;Y)},\end{aligned}$$

where, in the four estimates, we applied the contractivity of conditional expectation on L^p, Kahane's contraction principle with coefficients $\{0, 1\}$, the definition of the UMD constant $\beta^+_{p,Y}$, and again the contractivity of conditional expectation on L^p. □

Remark 12.1.16. Proposition 12.1.15 is a simultaneous generalisation of Stein's inequality (Theorem 4.2.23),

$$\Big\| \sum_{n=1}^N \varepsilon_n \mathbb{E}(f_n|\mathscr{F}_n)\Big\|_{L^p(\Omega\times S;Y)} \leqslant \beta^+_{p,Y}\Big\| \sum_{n=1}^N \varepsilon_n f_n\Big\|_{L^p(\Omega\times S;Y)}, \tag{12.6}$$

for all $f_n \in L^p(S;Y)$, and the K-convexity inequality for UMD spaces (Proposition 4.3.10),

$$\Big\| \sum_{n=1}^{N} \varepsilon_n \mathbb{E}(\varepsilon_n f) \Big\|_{L^p(\Omega;Y)} \leqslant K_{p,Y} \|f\|_{L^p(\Omega;Y)}, \quad K_{p,Y} \leqslant \beta^+_{p,Y}, \tag{12.7}$$

for all $f \in L^p(\Omega;Y)$.

Namely, (12.6) is obtained from Proposition 12.1.15 by taking $f = \sum_{k=1}^N \varepsilon_k \otimes f_k$, in which case $\mathbb{E}(\varepsilon_n f|\mathscr{E}_0 \times \mathscr{F}_n) = \mathbb{E}(f_n|\mathscr{F}_n)$, while (12.7) is the special case where $S = \{s\}$ is a singleton, or in other words f is independent of $s \in S$. Moreover, Proposition 12.1.15 shows that (12.7) holds equally well with real or complex Rademacher variable ε_n, provided only that we use the UMD constant $\beta^-_{p,Y}$ defined in terms of the same variables; in contrast, the proof of Proposition 4.3.10 was written for the real Rademacher variables r_n and made some explicit (although not essential) use of this choice.

Qualitatively, Proposition 12.1.15 could also be derived from the said two results, but with a quantitatively weaker conclusion; namely,

$$\begin{aligned}
&\Big\| \sum_{n=1}^{N} \varepsilon_n \mathbb{E}(\varepsilon_n f|\mathscr{E}_0 \times \mathscr{F}_n) \Big\|_{L^p(\Omega\times S;Y)} \\
&= \Big\| \sum_{n=1}^{N} \varepsilon'_n \mathbb{E}(\mathbb{E}(\varepsilon_n f|\mathscr{E}_0 \times \mathscr{F})|\mathscr{E}_0 \times \mathscr{F}_n) \Big\|_{L^p(\Omega'\times\Omega\times S;Y)} \\
&\leqslant \beta^+_{p,Y} \Big\| \sum_{n=1}^{N} \varepsilon'_n \mathbb{E}(\varepsilon_n f|\mathscr{E}_0 \times \mathscr{F}) \Big\|_{L^p(\Omega'\times\Omega\times S;Y)} \\
&= \beta^+_{p,Y} \Big\| \sum_{n=1}^{N} \varepsilon_n \mathbb{E}(\varepsilon_n f|\mathscr{E}_0) \Big\|_{L^p(S;L^p(\Omega;Y))} \leqslant \beta^+_{p,Y} K_{p,Y} \|f\|_{L^p(S;L^p(\Omega;Y))},
\end{aligned}$$

using the K-convexity inequality in $L^p(\Omega;Y)$, pointwise at each $s \in S$, in the last step.

Corollary 12.1.17. *If Y is a UMD space and $\lambda = (\lambda_Q)_{Q\in\mathscr{D}} \subseteq \mathscr{L}(X,Y)$, then*

$$\mathscr{D}\mathscr{R}_p(\lambda) \leqslant \mathscr{E}\mathscr{R}_p(\lambda) \leqslant \beta^+_{p',Y^*} \mathscr{D}\mathscr{R}_p(\lambda).$$

Proof. We already proved the first inequality in Lemma 12.1.8. For the second inequality, we first note that, by Fubini's theorem,

$$\Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q z_Q \mathbf{1}_Q \Big\|_{L^p(\Omega\times\mathbb{R}^d;Z)} = \Big\| \sum_{Q\in\mathscr{D}} \varepsilon_{n(Q)} z_Q \mathbf{1}_Q \Big\|_{L^p(\Omega\times\mathbb{R}^d;Z)}, \tag{12.8}$$

where $n(Q) \in \mathbb{Z}$ is such that $Q \in \mathscr{D}_n$: This is because, pointwise at each $s \in \mathbb{R}^d$, there is exactly one dyadic $Q \ni s$ of each generation $n \in \mathbb{Z}$, and we can replace the sequence $(\varepsilon_Q)_{Q\ni s}$ by the equidistributed sequence $(\varepsilon_n)_{n\in\mathbb{Z}} = (\varepsilon_{n(Q)})_{Q\ni s}$. For $z_Q = \lambda_Q x_Q$ and $Z = Y$, we then dualise the right-hand side of (12.8) with $G \in L^{p'}(\Omega \times \mathbb{R}^d; Y^*)$:

$$\begin{aligned}\Big|\Big\langle \sum_{Q\in\mathscr{D}} \varepsilon_{n(Q)}\lambda_Q x_Q \mathbf{1}_Q, G\Big\rangle\Big| &= \Big|\sum_{Q\in\mathscr{D}_n} \Big\langle \lambda_Q x_Q, \langle \mathbb{E}(\varepsilon_{n(Q)}G)\rangle_Q\Big\rangle |Q|\Big| \\ &\leqslant \mathscr{D}\mathscr{R}_p(\lambda)\Big\|\sum_{Q\in\mathscr{D}} \varepsilon_{n(Q)} x_Q \mathbf{1}_Q\Big\|_{L^p(\Omega\times\mathbb{R}^d;X)}\times \\ &\qquad \times\Big\|\sum_{Q\in\mathscr{D}} \varepsilon_{n(Q)}\langle \mathbb{E}(\varepsilon_{n(Q)}G)\rangle_Q \mathbf{1}_Q\Big\|_{L^{p'}(\Omega\times\mathbb{R}^d;Y^*)}.\end{aligned} \tag{12.9}$$

In the $L^{p'}(\Omega\times\mathbb{R}^d;Y^*)$ norm on the right, we write

$$\begin{aligned}\sum_{Q\in\mathscr{D}} \varepsilon_{n(Q)}\langle \mathbb{E}(\varepsilon_{n(Q)}G)\rangle_Q \mathbf{1}_Q &= \sum_{n\in\mathbb{Z}} \varepsilon_n \sum_{Q\in\mathscr{D}_n} \mathbb{E}(\mathbb{E}(\varepsilon_n G)|\sigma(\mathscr{D}_n))\mathbf{1}_Q \\ &= \sum_{n\in\mathbb{Z}} \varepsilon_n \mathbb{E}(\mathbb{E}(\varepsilon_n G)|\sigma(\mathscr{D}_n)) = \sum_{n\in\mathbb{Z}} \varepsilon_n \mathbb{E}(\varepsilon_n G|\{\varnothing,\Omega\}\times\sigma(\mathscr{D}_n)).\end{aligned}$$

Thus, by a direct application of Proposition 12.1.15 in the UMD space Y^* in place of Y, it follows that

$$\Big\|\sum_{Q\in\mathscr{D}} \varepsilon_{n(Q)}\langle \mathbb{E}(\varepsilon_{n(Q)}G)\rangle_Q \mathbf{1}_Q\Big\|_{L^{p'}(\Omega\times\mathbb{R}^d;Y^*)} \leqslant \beta^+_{p',Y^*}\|G\|_{L^{p'}(\Omega\times\mathbb{R}^d;Y^*)}.$$

Substituting back to (12.9), it follows by duality that

$$\begin{aligned}&\Big\|\sum_{Q\in\mathscr{D}} \varepsilon_{n(Q)}\lambda_Q x_Q \mathbf{1}_Q\Big\|_{L^p(\Omega\times\mathbb{R}^d;Y)} \\ &\qquad\leqslant \beta^+_{p',Y^*}\mathscr{D}\mathscr{R}_p(\lambda)\Big\|\sum_{Q\in\mathscr{D}} \varepsilon_{n(Q)} x_Q \mathbf{1}_Q\Big\|_{L^p(\Omega\times\mathbb{R}^d;X)},\end{aligned}$$

and we can replace $n(Q)$ by Q on both sides according to (12.8) to obtain the claimed result. □

12.1.b Nested collections of unions of dyadic cubes

Before proceeding to more complicated dyadic singular integrals, we devote this intermediate section to elementary, although not entirely trivial, geometric–combinatorial considerations related to the dyadic cubes. Collecting the relevant auxiliary results here for easy reference will allow our subsequent analysis to flow with a nice tempo without annoying interruptions.

Definition 12.1.18 (Nestedness). *We say that two set E, F are* nested *if $E\cap F\in\{\varnothing, E, F\}$. A collection $\mathscr{E}$ of sets is called nested if any $E, F\in\mathscr{E}$ have this property.*

The fact that the collection $\mathscr{D}$ of dyadic cubes enjoys this property underlies many considerations that we have encountered in these volumes.

In the dyadic analysis of a singular integral operators that we undertake in this section, we will also need to deal with unions $Q_1 \cup Q_2$ of two dyadic cubes of the same size. A moment's thought confirms that two such sets will not be nested in general, yet quite frequently they still enjoy this property. Accordingly, a key to the related considerations will be the decomposition of collections of pairs of dyadic cubes into controllably many subcollections, where the nestedness of the unions $Q_1 \cup Q_2$ is valid.

Definition 12.1.19 (Strong nestedness). *Let $Q_1 \cup Q_2$ and $R_1 \cup R_2$ be two unions of some $Q_i, R_i \in \mathscr{D}$ with $\ell(Q_1) = \ell(Q_2)$ and $\ell(R_1) = \ell(R_2)$. We say that E and F are* strongly nested *if they are equal, or disjoint, or one of them, say $Q_1 \cup Q_2$, is contained not just in $R_1 \cup R_2$ but in a dyadic child of R_1 or R_2. A collection of such unions is called strongly nested if any two of its members have this property.*

Note that the dyadic cubes themselves, contained in this definition as a degenerate case with $Q_2 = Q_1$, clearly satisfy this strong nestedness. This notion is relevant for considerations dealing with Haar functions which, as we recall, are constant on the dyadic children of their supporting dyadic cubes; thus, if $Q_1 \cup Q_2$ and $R_1 \cup R_2$ are strongly nested, unequal but intersecting, then the smaller union is entirely contained in a set of constant value for any Haar function related to the larger union.

Our first (relatively simple) decomposition into strongly nested subcollections is the following:

Lemma 12.1.20. *Suppose that, for some $n \in \mathbb{N}$:*

(a) *$\mathscr{F} \subseteq \mathscr{D}$ is a finite subcollection;*
(b) *$\phi : \mathscr{F} \to \mathscr{D}$ is an injection with $\ell(\phi(Q)) = \ell(Q)$ for all $Q \in \mathscr{F}$;*
(c) *if $Q, R \in \mathscr{F}$ and $\ell(Q) < \ell(R)$, then $\ell(Q) < 2^{-n}\ell(R)$;*

and

$$\phi(Q) \subseteq Q^{(n)} \qquad \forall\, Q \in \mathscr{F}. \tag{12.10}$$

Then $\mathscr{F}$ can be partitioned into 3 subcollections $\mathscr{F}_i$ such that each collection $\{Q \cup \phi(Q) : Q \in \mathscr{F}_i\}$ is strongly nested.

Proof. Step 1 – Let all assumptions of the lemma be in force until further notice. For each $Q \in \mathscr{F} \cup \phi(\mathscr{F})$, we define a *label* $r(Q) \in \{0, 1, 2\}$ such that $r(Q) \neq r(\phi(Q))$ for every $Q \in \mathscr{F}$ unless $\phi(Q) = Q$. This ensures that $Q \cup \phi(Q)$ and $R \cup \phi(R)$ are disjoint whenever $Q, R \in \mathscr{F}$ are two different cubes with $r(Q) = r(R)$ and $\ell(Q) = \ell(R)$.

Indeed, $Q \neq R$ implies $\phi(Q) \neq \phi(R)$. Since different dyadic cubes of equal size are disjoint, this implies that $Q \cap R = \varnothing = \phi(Q) \cap \phi(R)$. If $\phi(Q) = Q$ or $\phi(R) = R$, this already shows that $Q \cup \phi(Q)$ and $R \cup \phi(R)$ are disjoint. If $\phi(Q) \neq Q$ and $\phi(R) \neq R$, then $r(\phi(Q)) \neq r(Q) = r(R)$ implies $\phi(Q) \neq R$ and similarly $\phi(R) \neq Q$. By equal size again, this implies that $\phi(Q) \cap R = \varnothing = Q \cap \phi(R)$, giving the (strong) nestedness when $\ell(Q) = \ell(R)$.

To define such $r(R)$, let us denote $\phi^{\circ 0}(Q) = Q$, $\phi^{\circ k}(Q) = \phi(\phi^{\circ(k-1)}(Q))$ for $k \geqslant 1$. An *orbit* of ϕ is a set $\{\phi^{\circ k}(Q) : k = 0, \ldots, K\}$, where $Q \in \mathscr{F}$ and either $\phi^{\circ(K+1)}(Q) = Q$ (in this case the orbit is *cyclic*), or $Q \notin \phi(\mathscr{F})$ and $\phi^{\circ K}(Q) \notin \mathscr{F}$. For all $Q \in \mathscr{F} \cup \phi(\mathscr{F})$, we define $r(Q) \in \{0, 1, 2\}$ by alternating the values 0 and 1 on both non-cyclic orbits and cyclic orbits of even length, and in addition using the value 2 once on cyclic orbits of odd length. In this way, we ensure that $r(Q) \neq r(\phi(Q))$ for any $Q \in \mathscr{F}$ unless $Q = \phi(Q)$.

Step 2 – It remains to check the strong nestedness in the case of $Q, R \in \mathscr{F}$ with $\ell(Q) < \ell(R)$, hence $\ell(Q) < 2^{-n}\ell(R)$. If $Q \cup \phi(Q)$ intersect $R \cup \phi(R)$, then one of $P \in \{Q, \phi(Q)\}$ intersects one of $S \in \{R, \phi(R)\}$. Since $\ell(P) < 2^{-n}\ell(S)$ and the cubes are dyadic, this implies that $P^{(n)} \subsetneq S$. Since $\phi(Q) \subseteq Q^{(n)}$, we have $\phi(Q)^{(n)} = Q^{(n)}$, and hence $Q \cup \phi(Q) \subseteq Q^{(n)} \subsetneq S$, confirming strong nestedness in the case of $\ell(Q) < \ell(R)$. □

In the lack of (12.10), the situation is somewhat more complicated. Suitable substitute conditions are provided in the following:

Lemma 12.1.21. *Assume conditions* (a) *through* (c) *as well as:*

(d) $\phi(Q) \subseteq 3Q^{(n)}$ *for all* $Q \in \mathscr{F}$*;*
(e) $3Q \subseteq Q^{(n)}$ *for all* $Q \in \mathscr{F} \cup \phi(\mathscr{F})$.

Then $\mathscr{F}$ *can be partitioned into nine subcollections* $\mathscr{F}_i$ *such that each collection* $\{Q \cup \phi(Q) : Q \in \mathscr{F}_i\}$ *is strongly nested.*

Proof. Step 1 – We define the label $r(Q) \in \{0, 1, 2\}$ exactly as in the proof of Lemma 12.1.20 to ensure that $r(Q) \neq r(\phi(Q))$ unless $Q = \phi(Q)$. This gives the nestedness of the sets $Q \cup \phi(Q)$ for cubes of a fixed sidelength, as before.

Step 2 – We claim that, for each $Q \in \mathscr{F} \cup \phi(\mathscr{F})$, there can be at most one $R \in \mathscr{F} \cup \phi(\mathscr{F})$ such that

$$Q \subsetneq R, \qquad 3Q^{(n)} \not\subseteq R_Q, \tag{12.11}$$

where R_Q is the unique dyadic child of R that contains $Q \subsetneq R$.

In fact, let R be as above, and $Q \subsetneq R \subsetneq S \in \mathscr{F} \cup \phi(\mathscr{F})$, thus $Q^{(n)} \subsetneq R$, $R^{(n)} \subsetneq S$ by (c). By (e) applied to the cube R, we then have $3Q^{(n)} \subseteq 3R \subseteq R^{(n)} \subseteq S_R$, so indeed S will not satisfy the condition (12.11) that R does, and this proves the uniqueness of R.

Step 3 – For each $P \in \mathscr{F}$, we define a second label $s(P) \in \{0, 1, 2\}$ in such a way that if $(r(P), s(P)) = (r(S), s(S))$, then (12.11) does not hold for either $R = S$ or $R = \phi(S)$. This will ensure strong nestedness for the subcollection with constant pairs of labels $(r(P), s(P))$.

Indeed, suppose that $P, S \in \mathscr{F}$ have $(r(P), s(P)) = (r(S), s(S))$ where $\ell(P) < \ell(S)$ and $P \cup \phi(P)$ intersects $S \cup \phi(S)$. Hence (at least) one of $Q \in \{P, \phi(P)\}$ intersects (at least) one of $R \in \{S, \phi(S)\}$ and thus $Q \subsetneq R$. By (d) and the failure of (12.11), we have $P \cup \phi(P) \subseteq (1 + 2^{n+1}Q) \subseteq R_Q$.

The required second label $s(P)$ is defined for each $P \in \mathscr{F}$ as follows. For all $P \in \mathscr{F}$ of maximal size, let $s(P) := 0$. Recursively, we proceed to the unlabelled cubes $P \in \mathscr{F}$ of maximal size. For these cubes, we first check whether (12.11) occurs with either $Q = P$ or $Q = \phi(P)$, and some $R \in \mathscr{F} \cup \phi(\mathscr{F})$. It could happen that $R \in \mathscr{F}$, or $R = \phi(S)$ with $S \in \mathscr{F}$, or both. We then require that $s(P)$ is chosen so that $(r(P), s(P)) \notin \{(r(R), s(R)), (r(S), s(S))\}$. If $S = R$, this is clearly one restriction on $S(P)$. But if $S \neq R = \phi(S)$, then $r(R) \neq r(S)$ by the alternating choice of r along the orbits, and we still get at most one restriction of the possible value of $s(P)$. Since different R and S may arise from the case $Q = P$ and $Q = \phi(P)$ we get altogether at most two restrictions on $s(P)$, and we can declare that $s(P)$ is the smallest remaining number in $\{0, 1, 2\}$. □

The next result relaxes the assumptions even further, at the cost of complicating the conclusions:

Lemma 12.1.22. *Assume conditions* (a) *through* (d)*. Then* $\mathscr{F}$ *can be partitioned into* 144 *subcollections* $\mathscr{F}_i$*, and on each of them we have injections* $\phi_{i,j} : \mathscr{F}_i \to \mathscr{D}$*,* $j = 0, 1, 2, 3$*, where* $\phi_{i,0}(Q) = Q$ *and* $\phi_{i,3}(Q) = \phi(Q)$ *such that each collection*

$$\{\phi_{i,j}(Q) \cup \phi_{i,j+1}(Q) : Q \in \mathscr{F}_i\} \tag{12.12}$$

is strongly nested.

Proof. The idea is to combine the special cases treated in the two previous Lemmas 12.1.20 and 12.1.21, which had the additional assumptions (12.10) and (e), respectively; neither is assumed now.

For every $R \in \mathscr{D}$, consider the 2^{nd} cubes $Q \in \mathscr{D}$ with $Q^{(n)} = R$. Among them, there are $(2^n - 2)^d$ off-boundary cubes Q with $3Q \subseteq R$, while the number of boundary cubes is then

$$2^{nd} - (2^n - 2)^d = 2^{nd}[1 - (1 - 2^{1-n})^d] \leqslant 2^{nd} \cdot 2^{1-n} d \leqslant \frac{1}{2} 2^{nd}$$

if $n \geqslant \log_2(4d)$. When this is the case, we can define a permutation $\psi : \mathscr{D} \to \mathscr{D}$ with $\ell(\psi(Q)) = \ell(Q)$, $\psi(Q) \subseteq Q^{(n)}$ (as in (12.10)) such that $\psi(Q)$ is an off-boundary cube in $Q^{(n)}$ whenever Q is a boundary cube in $Q^{(n)}$.

Let us first divide $\mathscr{F}$ into four subcollection $\mathscr{F}_{u,v}$, where $u, v \in \{0, 1\}$, so that $Q \in \mathscr{F}_{u,v}$ is a boundary cube in $Q^{(n)}$ if and only if $u = 1$, whereas $\phi(Q)$ is a boundary cube in $\psi(Q)^{(n)}$ if and only if $v = 1$.

Case $\mathscr{F}_{0,0}$: By Lemma 12.1.21, we can divide $\mathscr{F}_{0,0}$ into nine subcollections $\mathscr{F}_i$ such that $\{Q \cup \phi(Q) : Q \in \mathscr{F}_i\}$ is strongly nested. Letting $\phi_{i,1} = \phi_{i,2} = \phi_{i,3} = \phi$ in this case, we trivially have the strong nestedness of $\{\phi_{i,j}(Q) \cup \phi_{i,j+1}(Q) : Q \in \mathscr{F}_i\}$ for $j = 1, 2$ (since the collection is simply $\phi(\mathscr{F}_i) \subseteq \mathscr{D}$ in this case.

Case $\mathscr{F}_{0,1}$: On the collection $\mathscr{F}_{0,1}$, we consider the map $\psi \circ \phi$ and observe that it also satisfies (d); indeed, $\phi(Q) \subseteq 3Q^{(n)}$ lies inside one of the dyadic neighbours of $Q^{(n)}$, and ψ keeps it inside this same nth generation ancestor. Since $\phi(Q)$ is a boundary cube in $\phi(Q)^{(n)}$ for $Q \in \mathscr{F}_{0,1}$ by definition of this collection, $\psi(\phi(Q))$ is off-boundary in $\phi(Q)^{(n)} = \psi(\phi(Q))^{(n)}$ by definition of ψ, and hence $(\mathscr{F}_{0,1}, \psi \circ \phi)$ also satisfies (e) in place of $(\mathscr{F}, \phi)$. Then Lemma 12.1.21 shows that $\mathscr{F}_{0,1}$ can be divided into nine subcollections $\mathscr{F}'_a$ such that each $\{Q \cup \psi(\phi(Q)) : Q \in \mathscr{F}'_a\}$ is strongly nested. On the other hand, we can write

$$\{\psi(\phi(Q)) \cup \phi(Q) : Q \in \mathscr{F}_{0,1}\} = \{R \cup \psi(R) : R \in \phi(\mathscr{F}_{0,1})\}.$$

Here $(\mathscr{F}_{0,1}, \psi)$ satisfies the assumptions of Lemma 12.1.20, and hence $\phi(\mathscr{F}_{0,1})$ can be divided into three subcollections $\mathscr{G}_b$ such that $\{R \cup \psi(R) : R \in \mathscr{G}_b)\}$ is strongly nested. This since ϕ is injective, this induces a decomposition of $\mathscr{F}_{0,1}$ into three subcollections where $\mathscr{F}''_b$ such that $\{\psi(\phi(Q)) \cup \phi(Q) : Q \in \mathscr{F}''_b\}$ is strongly nested. Then, defining $\mathscr{F}_i = \mathscr{F}'_a \cap \mathscr{F}''_b$ for $i = (a, b)$, we find that both

$$\{Q \cup \psi(\phi(Q)) : Q \in \mathscr{F}_i\}, \qquad \{\psi(\phi(Q)) \cup \phi(Q) : Q \in \mathscr{F}_i\}$$

are strongly nested, and there is in total $9 \cdot 3$ such collections $\mathscr{F}_i$ decomposing $\mathscr{F}_{0,1}$. So taking $\phi_{i,1} = \psi \circ \phi$ and $\phi_{i,2} = \phi_{i,3}$, we have the strong nestedness of the collections in (12.12), the case $j = 2$ for trivial reasons as in case $\mathscr{F}_{0,0}$.

Case $\mathscr{F}_{1,0}$: Similarly, on the collection $\mathscr{F}_{1,0}$, Lemma 12.1.20 applies to the mapping ψ to provide three subcollection $\mathscr{F}'_a$ such that $\{Q \cup \psi(Q) : Q \in \mathscr{F}'_a\}$ is strongly nested. And Lemma 12.1.21 applies to $(\psi(\mathscr{F}_{1,0}), \phi \circ \psi^{-1})$ to provide nine subcollections $\mathscr{F}''_b$ such that $\{\psi(Q) \cup \phi(Q) : Q \in \mathscr{F}''_b\}$ is strongly nested. So altogether we have $3 \cdot 9$ subcollection $\mathscr{F}_i = \mathscr{F}'_a \cap \mathscr{F}''_b$ such that

$$\{Q \cup \psi(Q) : Q \in \mathscr{F}_i\}, \qquad \{\psi(Q) \cup \phi(Q) : Q \in \mathscr{F}_i\}$$

are strongly nested. We can hence define $\phi_{i,1} = \psi$, $\phi_{i,2} = \phi_{i,3} = \phi$ to get the claimed conclusions.

Case $\mathscr{F}_{1,1}$: Finally, on the collection $\mathscr{F}_{1,1}$, Lemma 12.1.20 applies to both $(\mathscr{F}_{1,1} : \psi)$ and to $(\psi \circ \phi(\mathscr{F}_{1,1} : \psi^{-1})$ to provide three subcollections $\mathscr{F}'_a$ and three other $\mathscr{F}''_b$ such that $\{Q \cup \psi(Q) : Q \in \mathscr{F}'_a\}$ and $\{\psi(\phi(Q)) \cup \phi(Q) : Q \in \mathscr{F}''_b\}$ are strongly nested. And we check that Lemma 12.1.21 applies to $(\psi(\mathscr{F}_{1,1}), \psi \circ \phi \circ \psi^{-1})$ to provide nine subcollections $\mathscr{F}'''_c$ such $\{\psi(Q) \cup \psi(\phi(Q)) : Q \in \mathscr{F}'''_c\}$ is strongly nested. Then with $\mathscr{F}_i = \mathscr{F}'_a \cap \mathscr{F}''_b \cap \mathscr{F}'''_c$ we obtain $3^2 \cdot 9$ subcollections such that $\{Q \cup \psi(Q) : Q \in \mathscr{F}_i\}$, $\{\psi(Q) \cup \psi(\phi(Q)) : Q \in \mathscr{F}_i\}$, and $\{\psi(\phi(Q)) \cup \phi(Q) : Q \in \mathscr{F}_i\}$ are strongly nested, and we can define $\phi_{i,1} = \psi$, $\phi_{i,2} = \psi \circ \phi$, $\phi_{i,3} = \phi$ in this case.

In total we have divided $\mathscr{F}$ into $9 + 2 \cdot 9 \cdot 3 + 9 \cdot 3^2 = 144$ subcollections $\mathscr{F}_i$ with required properties. □

Another variant of the conclusion with the same assumptions is as follows:

Lemma 12.1.23. *Assume conditions* (a) *through* (d). *Then* $\mathscr{F}$ *can be partitioned into* 3^{3d+1} *subcollections* $\mathscr{F}_i$ *such that each collection*

$$\{Q^{[m(i)]} \cup \phi(Q)^{[m(i)]} : Q \in \mathscr{F}_i\} \subseteq \mathscr{D}^{m(i);3}$$

is strongly nested, where

(1) $\mathscr{D}^{m(i);3}$ *is one of the dilated dyadic systems from Proposition 11.3.11;*
(2) *for each* $P \in \mathscr{D}$, *we denote by* $P^{[m]}$ *the unique*

$$P^{[m]} \in \mathscr{D}^{m;3} \text{ with } P^{[m]} \supseteq P \text{ and } \ell(P^{[m]}) = 3\ell(P). \tag{12.13}$$

Proof. We have $Q \cup \phi(Q) \subseteq (1 + 2^{n+1})Q \subseteq 3Q^{(n)}$, where $Q^{(n)}$ is the nth generation dyadic ancestor of Q. Recall that the cubes $3R$, $R \in \mathscr{D}$, can be split into 3^d new dyadic-like systems $\mathscr{D}^{m;3}$ by Proposition 11.3.11. For each $Q \in \mathscr{F}$, let m_Q be the index such that $3Q^{(n)} \in \mathscr{D}^{m_Q;3}$, and let $Q' = Q^{[m_Q]}$, $Q'' = \phi(Q)^{[m_Q]}$ be as in (12.13). (Thus Q' is the three-fold expansions of one of the neighbours of Q; any of these contains Q, and exactly one of them belongs to the correct $\mathscr{D}^{m_Q;3}$; the same remark applies to Q'' and $\phi(Q)$ in place of Q' and Q.) Note that the same Q' can arise from 3^d different cubes Q, and the same Q'' from 3^d different $\phi(Q)$; however, by dividing $\mathscr{F}$ into 9^d subcollections $\mathscr{F}^a$, we ensure that Q is uniquely determined by Q', and $\phi(Q)$ by Q'', within each $\mathscr{F}^a$.

Let us then consider the collections $\mathscr{F}^{a,m} = \{Q' : Q \in \mathscr{F}^a, m_Q = m\} \subseteq \mathscr{D}^{m;3}$ for the 3^d different values of m. We can define $\Phi : \mathscr{F}^{a,m} \to \mathscr{D}^{m;3}$ by $\Phi(Q') = Q''$; this is well-defined since Q' uniquely determines Q, which determines $\phi(Q)$ and then Q''. The map Φ is also injective, since Q'' uniquely determines $\phi(Q)$, which (since ϕ is injective) determines Q and then Q'. Moreover, we have

$$\ell(\Phi(Q')) = \ell(Q'') = 3\ell(\phi(Q)) = 3\ell(Q) = \ell(Q').$$

Thus $\mathscr{F}^{a,m} \subseteq \mathscr{D}^{m;3}$ and Φ satisfy properties (a) and (b) in place of $\mathscr{F} \subseteq \mathscr{D}$ and ϕ, and the scale-separation property (c) is clearly inherited by Φ from ϕ. Moreover, the nth $\mathscr{D}^{m;3}$-ancestor of both $\Phi(Q') = Q''$ and Q' is clearly $3Q^{(n)}$ by construction, and hence Φ satisfies condition (12.10) of Lemma 12.1.20. The said lemma guarantees that $\mathscr{F}^{a,m}$ can be split into 3 subcollections $\mathscr{F}^{a,m}_j$, so that each

$$\{Q' \cup \Phi(Q') : Q' \in \mathscr{F}^{a,m}_j\} \subseteq \mathscr{D}^{m;3}$$

is strongly nested. Writing $i = (a, m, j)$, and defining

$$\mathscr{F}_i := \{Q \in \mathscr{F} : m_Q = m, Q^{[m]} \in \mathscr{F}^{a,m}_j\},$$

these are precisely the collections that we wanted to construct. Since a takes 9^d values, m takes 3^d values, and j takes 3 values, the number of these collections is $9^d \cdot 3^d \cdot 3 = 3^{3d+1}$, as claimed. □

Remark 12.1.24. In each of the Lemmas 12.1.20 through 12.1.23, we can drop assumption (c) at the cost of multiplying the required number of decomposing subcollections $\mathscr{F}_i$ by $n+1$.

Proof. For any $\mathscr{F} \subseteq \mathscr{D}$, consider the $n+1$ subcollection $\mathscr{F}^k := \{Q \in \mathscr{F} : \log_2 \ell(Q) \equiv k \mod (n+1)\}$. Each of these clearly satisfies (c). Moreover, any of the other properties (a) through (e) as well as (12.10), if valid for $\mathscr{F}$, is clearly inherited by each $\mathscr{F}^k$. Thus, if $\mathscr{F}$ satisfies the assumptions of any of these lemmas with the possible exception of (c), then each $\mathscr{F}^k$ satisfies all of the relevant assumptions, and the lemma in question provides a decomposition of $\mathscr{F}^k$ into some $\mathscr{F}_i^k$ with appropriate nestedness conditions. The required decomposition of the original $\mathscr{F}$ is then obtained simply as $\mathscr{F} = \bigcup_{k=0}^n \bigcup_i \mathscr{F}_i^k$, and clearly the number of collections in this decomposition is $n+1$ times as many as in the decompositions $\mathscr{F}^k = \bigcup_i \mathscr{F}_i^k$ given by the lemmas. □

12.1.c The elementary operators of Figiel

We will now study another family of dyadic singular integral operators with more complicated interactions between Haar functions at different locations. The first class of these operators combines the action of a Haar multiplier with a translation of the Haar functions. One might be tempted to refer to such operators as dyadic or Haar "shifts", but this name has been adopted for a somewhat different class of operators in the literature.

While the parameter n attached with these operators may appear like a technical detail at this point, it is essential for subsequent applications that one obtains a good dependence on n.

Theorem 12.1.25 (Figiel). *Let $\phi : \mathscr{D} \to \mathscr{D}$ be an injection with $\ell(\phi(Q)) = \ell(Q)$ and $\phi(Q) \subseteq 3Q^{(n)}$ for some $n \in \mathbb{N}$. Let X and Y be a UMD spaces, and let $p \in (1,\infty)$. Let $\lambda = (\lambda_Q)_{Q\in\mathscr{D}} \subseteq \mathscr{L}(X,Y)$. Consider the mapping*

$$T_{\phi\lambda}^{\alpha\gamma} f = \sum_{Q\in\mathscr{D}} \lambda_Q \langle f, h_Q^\alpha \rangle h_{\phi(Q)}^\gamma, \tag{12.14}$$

initially from $S_{00}(\mathscr{D};X)$ to $S_{00}(\mathscr{D};Y)$. Let $A_d := 6 \cdot (81)^d$.

(0) *If λ is R-bounded, or more generally if*

$$\min\{\mathscr{E}\mathscr{R}_p(\lambda), \mathscr{E}\mathscr{R}_p(\lambda_{\phi^{-1}})\} < \infty, \quad (\lambda_{\phi^{-1}})_Q := \begin{cases} \lambda_{\phi^{-1}(Q)}, & Q \in \phi(\mathscr{D}), \\ 0, & \text{else}, \end{cases}$$

then $T_{\phi\lambda}^{\alpha\gamma}$ extends boundedly from $L^p(\mathbb{R}^d;X)$ to $L^p(\mathbb{R}^d;Y)$, with norm

$$\|T_{\phi\lambda}^{\alpha\gamma}\| := \|T_{\phi\lambda}^{\alpha\gamma}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \leqslant A_d(n+1)\beta_{p,Y}^{-}\beta_{p,X}^{+} C(X,Y,p;\lambda),$$

where

$$\begin{aligned} C(X,Y,p;\lambda) &:= \min\{\beta_{p,X}^{+} \cdot \mathscr{E}\mathscr{R}_p(\lambda_{\phi^{-1}}), \quad \beta_{p,Y}^{+} \cdot \mathscr{E}\mathscr{R}_p(\lambda)\} \\ &\leqslant \min\{\beta_{p,X}^{+}, \beta_{p,Y}^{+}\}\mathscr{R}_p(\lambda); \end{aligned}$$

(1) *If, in addition, Y has type $t \in [1,p]$ and X has cotype $q \in [p,\infty]$, or one of them has both, then we also have the estimate*

$$\|T^{\alpha\gamma}_{\phi\lambda}\| \leqslant A_d (n+1)^{1/t-1/q} \beta^-_{p,Y} \beta^+_{p,X} C(X,Y,p,q,t;\lambda)$$

where

$$\begin{aligned} C(X,Y,p,q,t;\lambda) := \min \Big\{ & \tau_{t,X;p} \cdot \beta^+_{p,X} \cdot c_{q,X;p} \cdot \mathscr{E}\mathscr{R}_p(\lambda_{\phi^{-1}}), \\ & \tau_{t,Y;p} \cdot \beta^+_{p,X} \cdot c_{q,X;p} \cdot \mathscr{E}\mathscr{R}_p(\lambda_{\phi^{-1}}), \\ & \tau_{t,Y;p} \cdot \beta^+_{p,Y} \cdot c_{q,X;p} \cdot \mathscr{E}\mathscr{R}_p(\lambda), \\ & \tau_{t,Y;p} \cdot \beta^+_{p,Y} \cdot c_{q,Y;p} \cdot \mathscr{E}\mathscr{R}_p(\lambda) \Big\} \\ \leqslant C(X,Y,p,q,t) \cdot \mathscr{R}_p(\lambda), \end{aligned}$$

and

$$\begin{aligned} C(X,Y,p,q,t) := \min \Big\{ & \tau_{t,X;p} \beta^+_{p,X} c_{q,X;p}, \ \tau_{t,Y;p} \beta^+_{p,X} c_{q,X;p}, \\ & \tau_{t,Y;p} \beta^+_{p,Y} c_{q,X;p}, \ \tau_{t,Y;p} \beta^+_{p,Y} c_{q,Y;p} \Big\}. \end{aligned} \tag{12.15}$$

(2) *If, in addition, $\lambda_Q \neq 0$ only when $\phi(Q) \subseteq Q^{(n)}$, then we have the alternative norm estimate*

$$\|T^{\alpha\gamma}_{\phi\lambda}\| \leqslant 3 \cdot \beta_{p,Y} \beta^+_{p,X} \min\{c_{q,X;p}, c_{q,Y;p}\} (n+1)^{1/q'} \mathscr{E}\mathscr{R}_p(\lambda).$$

(3) *For all $f \in L^p(\mathbb{R}^d; X)$ and $g \in L^{p'}(\mathbb{R}^d; Y^*)$, the extended operator has the absolutely convergent representation*

$$\langle T^{\alpha\gamma}_{\phi\lambda} f, g \rangle = \sum_{Q \in \mathscr{D}} \Big\langle \lambda_Q \langle f, h^\alpha_Q \rangle, \langle g, h^\gamma_{\phi(Q)} \rangle \Big\rangle.$$

When $\|f\|_{L^p(\mathbb{R}^d;X)} \leqslant$ and $\|g\|_{L^{p'}(\mathbb{R}^d;Y^*)} \leqslant 1$, the corresponding absolute value series is dominated by the same upper bounds as those given for $\|T^{\alpha\gamma}_{\phi\lambda}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X), L^p(\mathbb{R}^d;Y))}$ above.

Remark 12.1.26. (1) In the prominent special case that $X = Y$, we have

$$\begin{aligned} C(X,X,p,q,t;\lambda) &= C(X,X,p,q,t) \cdot \min\{\mathscr{E}\mathscr{R}_p(\lambda_{\phi^{-1}}), \mathscr{E}\mathscr{R}_p(\lambda)\}, \\ C(X,X,p,q,t) &= \tau_{t,X;p} \cdot \beta^+_{p,X} \cdot c_{q,X;p}. \end{aligned}$$

(2) Case (0) of Theorem 12.1.25 is a special case of (1) using the trivial type and cotype exponents $t = 1$, $q = \infty$ with corresponding constants equal to one. The role of non-trivial type and cotype is to relax the dependence on the parameter n. The estimate obtained in case (2) is not strictly comparable to the other two bounds; its main advantage over the other two is achieving a quadratic bound in terms of the UMD constants, in contrast to the cubic bound in the other cases.

(3) Recalling the Haar multipliers $\mathfrak{H}_\lambda^{\alpha\gamma}$ from Theorem 12.1.11, one can check that, for any $\theta \in \{0,1\}^d \setminus \{0\}$,

$$T_{\phi\lambda}^{\alpha\gamma} = T_{\phi\mathbf{1}}^{\theta\gamma} \circ \mathfrak{H}_\lambda^{\alpha\theta} = \mathfrak{H}_{\lambda_{\phi^{-1}}}^{\theta\gamma} \circ T_{\phi\mathbf{1}}^{\alpha\theta}$$

where $\mathbf{1}$ is the constant sequence of all ones. Hence, for the qualitative conclusion of Theorem 12.1.25, it would suffice to consider just $X = Y$ and $\lambda = \mathbf{1}$, and then combine this special case with Theorem 12.1.11; however, the reader will quickly realise that this approach would produce a higher power of the UMD constants in the quantitative conclusion.

Before going into the proof, let us still formulate a corollary in the important special case when $\phi : \mathscr{D} \to \mathscr{D}$ is a bijection:

Corollary 12.1.27. *Let $\phi : \mathscr{D} \to \mathscr{D}$ be a bijection with $\ell(\phi(Q)) = \ell(Q)$ and $\phi(Q) \subseteq 3Q^{(n)}$ for some $n \in \mathbb{N}$. Let X and Y be a UMD spaces, and let $p \in (1,\infty)$. Suppose that Y has type $t \in [1,p]$ and X has cotype $q \in [p,\infty]$, or one of them has both. Let $\lambda = (\lambda_Q)_{Q\in\mathscr{D}} \subseteq \mathscr{L}(X,Y)$ be R-bounded, consider the mapping $T_{\phi\lambda}^{\alpha\gamma}$ as in (12.14), and let*

$$\|T_{\phi\lambda}^{\alpha\gamma}\| := \|T_{\phi\lambda}^{\alpha\gamma}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))}.$$

(1) *We have the norm estimate*

$$\|T_{\phi\lambda}^{\alpha\gamma}\| \leqslant 6\cdot 3^{4d}\beta_{p,X}\beta_{p,Y}(n+1)^{1/t-1/q}\min\{C\mathscr{R}_p(\lambda), C^*\mathscr{R}_{p'}^*(\lambda)\}$$

where

$$C = C_{(12.15)}(X,Y,p,q,t), \qquad C^* := C_{(12.15)}(Y^*,X^*,p',t',q').$$

(2) *If, in addition, $\lambda_Q \neq 0$ only when $\phi(Q) \subseteq Q^{(n)}$, then we have the alternative norm estimate*

$$\|T_{\phi\lambda}^{\alpha\gamma}\| \leqslant 3\cdot\beta_{p,X}\beta_{p,Y}\min\Big\{C(n+1)^{1/q'}\mathscr{R}_p(\lambda), C^*(n+1)^{1/t}\mathscr{R}_{p'}^*(\lambda)\Big\}.$$

where

$$C = \min\{c_{q,X;p}, c_{q,Y;p}\}, \qquad C^* = \min\{c_{t',Y^*;p'}, c_{t',X^*;p'}\}$$

Proof. The first versions of both bounds (i.e, using the first item of the respective minimums) above are simply those of Theorem 12.1.25, cases (1) and (2), where we estimated all UMD constants by $\beta_{p,Z}^{\pm} \leqslant \beta_{p,Z}$. The second versions of both bounds then follow by duality: When $\phi : \mathscr{D} \to \mathscr{D}$ is a bijection, one directly verifies that

$$(T_{\phi\lambda}^{\alpha\gamma})^* = T_{\phi^{-1},\lambda^*_{\phi^{-1}}}^{\gamma\alpha}$$

is an operator of the same form, acting from $\mathscr{Q}_{00}(\mathbb{R}^d;Y^*)$ to $\mathscr{Q}_{00}(\mathbb{R}^d;X^*)$ and eventually from $L^{p'}(\mathbb{R}^d;Y^*)$ to $L^{p'}(\mathbb{R}^d;X^*)$. If $Z \in \{X,Y\}$ has type t, then Z^*

has cotype t' with $c_{t',Z^*;p'} \leqslant \tau_{t,Z;p}$. (See Proposition 7.1.13; it is formulated for $p = t$, but the same short argument is easily modified to give the general statement.) If a UMD space Z has cotype q, then it has martingale type q (Proposition 4.3.13), hence Z^* has martingale cotype q' (Proposition 3.5.29), and thus cotype q' (as observed right before Proposition 4.3.13). Thus we can apply the case already handled, with (Y^*, X^*, p', t', q') in place of (X, Y, p, q, t), to get

$$\begin{aligned}\|T^{\alpha\gamma}_{\phi\lambda}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} &= \|T^{\gamma\alpha}_{\phi^{-1},\lambda_{\phi^{-1}}}\|_{\mathscr{L}(L^{p'}(\mathbb{R}^d;Y^*),L^{p'}(\mathbb{R}^d;X^*))}\\ &\leqslant 6\cdot 3^{4d}\beta_{p',Y^*}\beta_{p',X^*}(n+1)^{1/q'-1/t'}C(Y^*,X^*,p',t',q')\mathscr{R}_{p'}(\lambda^*).\end{aligned}$$

The claim then follows from $\beta_{p',Z^*} = \beta_{p,Z}$ and $1/q' - 1/t' = 1/t - 1/q$.

The second version of the second bound is obtained from the first version in the entirely similar way by duality. □

Proof of Theorem 12.1.25. Claim (0) is the special case $t = 1$, $q = \infty$ of (1), so we only need to prove the latter of the two. Let $\mathscr{F}$ be a finite collection of dyadic cubes. Then $\mathscr{F}$ and ϕ satisfy the assumptions of Lemma 12.1.23, except possibly the scale separation (c). By Remark 12.1.24, the lemma still applies to produce $3^{3d+1}(n+1)$ subcollections $\mathscr{F}_i \subseteq \mathscr{F}$ with the properties given in Lemma 12.1.23. Let us write $x_Q = \langle f, h^\alpha_Q\rangle$. Since the functions $(h^\gamma_Q)_{Q\in\mathscr{F}}$ form a martingale difference sequence, we have

$$\Big\|\sum_{Q\in\mathscr{F}}\lambda_Q x_Q h^\gamma_{\phi(Q)}\Big\|_{L^p(\mathbb{R}^d;Y)} \leqslant \beta^-_{p,Y}\Big\|\sum_{Q\in\mathscr{F}}\varepsilon_Q\lambda_Q x_Q h^\gamma_{\phi(Q)}\Big\|_{L^p(\Omega\times\mathbb{R}^d;Y)}.$$

From this point on, we have some flexibility as to when we want to "pull out" the coefficients λ_Q. For this reason, let us write $z_Q \in Z$ for a generic choice of either $z_Q = \lambda_Q x_Q \in Y$ or $z_Q = x_Q \in X$. We then continue with

$$\begin{aligned}\Big\|\sum_{Q\in\mathscr{F}}\varepsilon_Q z_Q h^\gamma_{\phi(Q)}\Big\|_{L^p(\Omega\times\mathbb{R}^d;Z)} &= \Big\|\sum_i\sum_{Q\in\mathscr{F}_i}\varepsilon'_i\varepsilon_Q z_Q h^0_{\phi(Q)}\Big\|_{L^p(\Omega'\times\Omega\times\mathbb{R}^d;Z)}\\ &\leqslant \tau_{t,Z;p}\Big(\sum_i\Big\|\sum_{Q\in\mathscr{F}_i}\varepsilon_Q x_Q h^0_{\phi(Q)}\Big\|^t_{L^p(\Omega\times\mathbb{R}^d;Z)}\Big)^{1/t},\end{aligned}$$

where, in the two steps above, we used the facts that

1. when multiplied by the random sign ε_Q, both the independent random sign ε'_i and the possible difference of the signs of $h^\alpha_{\phi(Q)}(t)$ and $h^0_{\phi(Q)}(t)$ are invisible to the norm; and
2. whenever Z has type $t \in [1,p]$, then so has $L^p(S;Z)$ (here: $S = \Omega\times\mathbb{R}^d$), and $\tau_{t,L^p(S;Z);p} \leqslant \tau_{t,Z;p}$ by Proposition 7.1.4.

For $Q \in \mathscr{F}_i$, let us denote by $E(Q) = Q^{[m(i)]} \cup \phi(Q)^{[m(i)]}$ the sets provided by Lemma 12.1.23 that form a strongly nested family, as guaranteed by the said lemma. In particular $E(Q) \supseteq Q \cup \phi(Q)$ and $|E(Q)| \leqslant 2\cdot 3^d|Q|$. (The inequality is due to the fact that the cubes $Q^{[m(i)]}$ and $\phi(Q)^{[m(i)]}$ are not necessarily different.) Hence

$$\mathbf{1}_{\phi(Q)} \leqslant \mathbf{1}_{\phi(Q)} \frac{2 \cdot 3^d}{|E(Q)|} |Q| = \mathbf{1}_{\phi(Q)} 2 \cdot 3^d \fint_{E(Q)} \mathbf{1}_Q \leqslant 2 \cdot 3^d \mathbb{E}_{E(Q)} \mathbf{1}_Q,$$

where the $\mathbb{E}_{E(Q)}$ are conditional expectations associated with a nested family, and hence with a filtration. This allows us to use Stein's inequality (Theorem 4.2.23) to the effect that

$$\begin{aligned}
&\Big\| \sum_{Q \in \mathscr{F}_i} \varepsilon_Q z_Q h^0_{\phi(Q)} \Big\|_{L^p(\Omega \times \mathbb{R}^d; Z)} \\
&\leqslant 2 \cdot 3^d \Big\| \sum_{Q \in \mathscr{F}_i} \varepsilon_Q z_Q \mathbb{E}_{E(Q)} h^0_Q \Big\|_{L^p(\Omega \times \mathbb{R}^d; Z)} \\
&\leqslant 2 \cdot 3^d \cdot \beta^+_{p,Z} \Big\| \sum_{Q \in \mathscr{F}_i} \varepsilon_Q z_Q h^0_Q \Big\|_{L^p(\Omega \times \mathbb{R}^d; Z)}
\end{aligned} \tag{12.16}$$

Then

$$\begin{aligned}
&\Big(\sum_i \Big\| \sum_{Q \in \mathscr{F}_i} \varepsilon_Q z_Q h^0_Q \Big\|^t_{L^p(\Omega \times \mathbb{R}^d; Z)} \Big)^{1/t} \\
&\leqslant (3^{3d+1}(n+1))^{1/t-1/q} \Big(\sum_i \Big\| \sum_{Q \in \mathscr{F}_i} \varepsilon_Q z_Q h^0_Q \Big\|^q_{L^p(\Omega \times \mathbb{R}^d; Z)} \Big)^{1/q} \\
&\leqslant (3^{3d+1}(n+1))^{1/t-1/q} c_{q,Z;p} \Big\| \sum_i \varepsilon'_i \sum_{Q \in \mathscr{F}_i} \varepsilon_Q z_Q h^0_Q \Big\|_{L^p(\Omega' \times \Omega \times \mathbb{R}^d; Z)},
\end{aligned}$$

where, in the two steps above, we used

1. Hölder's inequality and counting of terms in the other factor; and
2. an application of the cotype q property of Z, recalling that this implies cotype q for $L^p(S; Z)$ (here: $S = \Omega \times \mathbb{R}^d$) with $c_{q,L^p(S;Z);p} \leqslant c_{q,Z;p}$ when $q \in [p, \infty]$ by Proposition 7.1.4.

By the invisibility of signs multiplying a random ε_Q, the last norm here is

$$\Big\| \sum_i \varepsilon'_i \sum_{Q \in \mathscr{F}_i} \varepsilon_Q z_Q h^0_Q \Big\|_{L^p(\Omega' \times \Omega \times \mathbb{R}^d; Z)} = \Big\| \sum_{Q \in \mathscr{F}} \varepsilon_Q z_Q h^\alpha_Q \Big\|_{L^p(\Omega \times \mathbb{R}^d; Z)}.$$

If we did not already pull out the coefficients λ_Q, we do it at this point, after which we are left with

$$\Big\| \sum_{Q \in \mathscr{F}} \varepsilon_Q x_Q h^\alpha_Q \Big\|_{L^p(\Omega \times \mathbb{R}^d; X)} \leqslant \beta^+_{p,X} \|f\|_{L^p(\mathbb{R}^d; X)},$$

where the last step was a direct application of Proposition 12.1.5.

It remains to collect the various coefficients that we accumulated. In any case, the first estimate gave $\beta^-_{p,Y}$ and the last one $\beta^+_{p,X}$, but depending on where we pull out the coefficients λ_Q, we may use the constant of the space X or Y in place of the generic Z.

If we pull out the λ_Q before the application of Stein's inequality in (12.16), then λ_Q is the coefficient of $h^\gamma_{\phi(Q)}$, hence the coefficient of h^γ_R is $\lambda_{\phi^{-1}(R)}$, and thus an application of Remark 12.1.7 produces the factor $\mathscr{E}\mathscr{R}_p(\lambda_{\phi^{-1}})$. On the other hand, pulling out the λ_Q only after (12.16) leads to a "direct" application of Remark 12.1.7 and the factor $\mathscr{E}\mathscr{R}_p(\lambda)$.

Aside from the numerical factors $2 \cdot 3^d$ and $(3^{3d+1}(n+1))^{1/t-1/q}$, we get one of the following:

$$
\begin{aligned}
&\mathscr{E}\mathscr{R}_p(\lambda_{\phi^{-1}}) \times \tau_{t,X;p} \times \beta^+_{p,X} \times c_{q,X;p}, \\
&\tau_{t,Y;p} \times \mathscr{E}\mathscr{R}_p(\lambda_{\phi^{-1}}) \times \beta^+_{p,X} \times c_{q,X;p}, \\
&\tau_{t,Y;p} \times \beta^+_{p,Y} \times \mathscr{E}\mathscr{R}_p(\lambda) \times c_{q,X;p}, \\
&\tau_{t,Y;p} \times \beta^+_{p,Y} \times c_{q,Y;p} \times \mathscr{E}\mathscr{R}_p(\lambda),
\end{aligned}
$$

where the order of the constants reflects the order of applying the related estimates: Before pulling out the coefficients λ_Q, we apply estimates on the Y side, and after that on the X side. Taking the minimum of the four terms, we arrive at the assertion of the theorem.

The alternative estimate (2)*:* In order to make efficient use of the additional assumption $\phi(Q) \subseteq Q^{(n)}$ when $\lambda_Q \neq 0$, we will need to modify the preceding considerations at various points.

Let $\mathscr{F}$ be a finite collection of dyadic cubes, and $\mathscr{F}^\lambda := \{Q \in \mathscr{F} : \lambda_Q \neq= 0\}$. Then $\mathscr{F}^\lambda$ and ϕ satisfy the assumptions of Lemma 12.1.20, except possibly the scale separation (c). By Remark 12.1.24, the lemma still applies to produce $3(n+1)$ subcollections $\mathscr{F}^\lambda_i \subseteq \mathscr{F}^\lambda$ with the properties given in Lemma 12.1.20. Let us write $x_Q = \langle f, h^\alpha_Q \rangle$. In the first step, we simply use the triangle inequality:

$$
\Big\| \sum_{Q \in \mathscr{F}} \lambda_Q x_Q h^\gamma_{\phi(Q)} \Big\|_{L^p(\mathbb{R}^d;Y)} \leqslant \sum_i \Big\| \sum_{Q \in \mathscr{F}^\lambda_i} \lambda_Q x_Q h^\gamma_{\phi(Q)} \Big\|_{L^p(\mathbb{R}^d;Y)}.
$$

The more interesting deviations from the previous case begin now.

Note that $h^\alpha_Q = |Q|^{-1/2}(\mathbf{1}_{Q^+_\alpha} - \mathbf{1}_{Q^-_\alpha})$ for suitable subsets $Q^\pm_\alpha \subseteq Q$ with $|Q^\pm_\alpha| = \frac{1}{2}|Q|$. If $Q \neq \phi(Q)$, we see that

$$
\begin{aligned}
d^+_Q &:= \frac{1}{2}(h^\alpha_Q + h^\gamma_{\phi(Q)}) = \frac{1}{2}|Q|^{-1/2}(\mathbf{1}_{Q^+_\alpha \cup \phi(Q)^+_\gamma} - \mathbf{1}_{Q^-_\alpha \cup \phi(Q)^-_\gamma}), \\
d^-_Q &:= \frac{1}{2}(h^\alpha_Q - h^\gamma_{\phi(Q)}) = \frac{1}{2}|Q|^{-1/2}(\mathbf{1}_{Q^+_\alpha \cup \phi(Q)^-_\gamma} - \mathbf{1}_{Q^-_\alpha \cup \phi(Q)^+_\gamma})
\end{aligned}
$$

form a martingale difference sequence (in either order) on $Q \cup \phi(Q)$, since either function has average zero on the sets where the other one is constant.

If $Q = \phi(Q)$ but $\alpha \neq \gamma$, then each of the sets $Q_\alpha^\pm \cap Q_\gamma^\pm$ has measure $\frac{1}{4}|Q|$, and once again

$$\begin{aligned} d_Q^+ &:= \frac{1}{2}(h_Q^\alpha + h_Q^\gamma) = |Q|^{-1/2}(\mathbf{1}_{Q_\alpha^+ \cap Q_\gamma^+} - \mathbf{1}_{Q_\alpha^- \cap Q_\gamma^-}), \\ d_Q^- &:= \frac{1}{2}(h_Q^\alpha - h_Q^\gamma) = |Q|^{-1/2}(\mathbf{1}_{Q_\alpha^+ \cap Q_\gamma^-} - \mathbf{1}_{Q_\alpha^- \cap Q_\gamma^+}) \end{aligned}$$

form a martingale difference sequence (in either order) on $Q \cup \phi(Q) = Q$, since either function has average zero on the sets where the other one is constant.

Finally, if $Q = \phi(Q)$ and $\alpha = \gamma$, then the same definition gives $d_Q^+ = h_Q^\alpha$, $d_Q^- = 0$, which is also a (rather trivial) martingale difference sequence.

The conclusion of Lemma 12.1.20, that each $\{Q \cup \phi(Q) : Q \in \mathscr{F}_i^\lambda\}$ is strongly nested, guarantees that the whole collection $\{d_Q^+, d_Q^-\}_{Q \in \mathscr{F}_i^\lambda}$ can be organised into a martingale difference sequence. Hence

$$\begin{aligned} &\Big\| \sum_{Q \in \mathscr{F}_i^\lambda} \lambda_Q x_Q h_{\phi(Q)}^\gamma \Big\|_{L^p(\mathbb{R}^d;Y)} \\ &= \Big\| \sum_{Q \in \mathscr{F}_i^\lambda} \lambda_Q x_Q (d_Q^+ - d_Q^-) \Big\|_{L^p(\mathbb{R}^d;Y)} \\ &\leqslant \beta_{p,Y} \Big\| \sum_{Q \in \mathscr{F}_i^\lambda} \varepsilon_Q \lambda_Q x_Q (d_Q^+ + d_Q^-) \Big\|_{L^p(\Omega \times \mathbb{R}^d;Y)} \\ &= \beta_{p,Y} \Big\| \sum_{Q \in \mathscr{F}_i^\lambda} \varepsilon_Q \lambda_Q x_Q h_Q^\alpha \Big\|_{L^p(\Omega \times \mathbb{R}^d;Y)}, \end{aligned} \tag{12.17}$$

where we used the definition of UMD with signs $\pm\varepsilon_Q$ multiplying the martingale differences $d_Q^\pm$, followed by taking an average over the ε_Q. (It might appear at first glance that we could have used just the one-sided UMD$^-$ property to arrive at the same conclusion with the smaller constant $\beta_{p,Y}^-$, but this is not the case: an application of the one-sided UMD$^-$ property would give us independent random signs, say $\varepsilon_Q^\pm$, in front of each $d_Q^\pm$, and this is not what we want.)

For $z_Q \in \{x_Q, \lambda_Q x_Q\}$ and $Z \in \{X, Y\}$ we then have

$$\begin{aligned} &\sum_i \Big\| \sum_{Q \in \mathscr{F}_i^\lambda} \varepsilon_Q z_Q h_Q^\alpha \Big\|_{L^p(\Omega \times \mathbb{R}^d;Z)} \\ &\leqslant (3(n+1))^{1/q'} \Big(\sum_i \Big\| \sum_{Q \in \mathscr{F}_i^\lambda} \varepsilon_Q z_Q h_Q^\alpha \Big\|_{L^p(\Omega \times \mathbb{R}^d;Z)}^q \Big)^{1/q} \\ &\leqslant (3(n+1))^{1/q'} c_{q,Z;p} \Big\| \sum_i \varepsilon_i' \sum_{Q \in \mathscr{F}_i^\lambda} \varepsilon_Q z_Q h_Q^\alpha \Big\|_{L^p(\Omega \times \mathbb{R}^d;Z)} \\ &= (3(n+1))^{1/q'} c_{q,Z;p} \Big\| \sum_{Q \in \mathscr{F}^\lambda} \varepsilon_Q z_Q h_Q^\alpha \Big\|_{L^p(\Omega \times \mathbb{R}^d;Z)}, \end{aligned} \tag{12.18}$$

using in the last step the fact that the $\mathscr{F}^\lambda = \bigcup_i \mathscr{F}_i^\lambda$ is a disjoint partition, so the independent random signs ε_Q with $Q \in \mathscr{F}^\lambda$ do not "see" the multiplying signs ε_i'. Hence, pulling out the λ_Q either at the beginning or at the end of (12.18) (but in any case only after having replaced the translated $h^\gamma_{\phi(Q)}$ by h^α_Q in (12.17), which in contrast to what happened in the previous case of the proof), we obtain

$$\begin{aligned}\Big\| \sum_{Q\in\mathscr{F}} \lambda_Q x_Q h^\gamma_{\phi(Q)} \Big\|_{L^p(\mathbb{R}^d;Y)} &\leqslant \beta_{p,Y} \sum_i \Big\| \sum_{Q\in\mathscr{F}_i^\lambda} \varepsilon_Q \lambda_Q x_Q h^\alpha_Q \Big\|_{L^p(\mathbb{R}^d;Y)} \\ &\leqslant \beta_{p,Y} \mathscr{E}\mathscr{R}_p(\lambda)(3(n+1))^{1/q'} \min\{c_{q,X;p}, c_{q,Y;p}\} \Big\| \sum_{Q\in\mathscr{F}^\lambda} \varepsilon_Q x_Q h^\alpha_Q \Big\|_{L^p(\mathbb{R}^d;X)}.\end{aligned}$$

Finally, recalling that $x_Q = \langle f, h^\alpha_Q\rangle$ and using the contraction principle to replace $\mathscr{F}^\lambda \subseteq \mathscr{F}$ by the finite set $\mathscr{F} = \{Q \in \mathscr{D} : \langle f, h^\alpha_Q\rangle \neq 0\}$, we obtain from Proposition 12.1.5 that

$$\Big\| \sum_{Q\in\mathscr{F}^\lambda} \varepsilon_Q x_Q h^\alpha_Q \Big\|_{L^p(\mathbb{R}^d;X)} \leqslant \Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q x_Q h^\alpha_Q \Big\|_{L^p(\mathbb{R}^d;X)} \leqslant \beta^+_{p,X} \|f\|_{L^p(\mathbb{R}^d;X)},$$

which concludes the estimate.

The representation (3)*:* Let first $\mathscr{F} \subseteq \mathscr{D}$ be finite. For suitable $\eta_Q \in \mathbb{K}$ with $|\eta_Q| = 1$, we have

$$\begin{aligned}\sum_{Q\in\mathscr{F}} \Big|\Big\langle \lambda_Q \langle f, h^\alpha_Q\rangle, \langle g, h^\gamma_{\phi(Q)}\rangle \Big\rangle\Big| &= \Big\langle \sum_{Q\in\mathscr{F}} \eta_Q \lambda_Q \langle f, h^\alpha_Q\rangle h^\gamma_{\phi(Q)}, g \Big\rangle \\ &= \langle T^{\alpha\gamma}_{\eta\lambda,\phi} P_{\mathscr{F}} f, g\rangle,\end{aligned}$$

where $(\eta\lambda)(Q) := \eta_Q \lambda_Q$, and

$$P_{\mathscr{F}} f := \sum_{\substack{Q\in\mathscr{F} \\ \theta\in\{0,1\}^d\setminus\{0\}}} \langle f, h^\theta_Q\rangle h^\theta_Q \in S_{00}(\mathscr{D}; X)$$

is a Haar projection of f; the action of $T^{\alpha\gamma}_{\eta\lambda,\phi}$ is thus well-defined via the initial definition on this space. From the previous part of the theorem that we already proved, we have

$$\begin{aligned}&\sum_{Q\in\mathscr{F}} \Big|\Big\langle \lambda_Q \langle f, h^\alpha_Q\rangle, \langle g, h^\gamma_{\phi(Q)}\rangle \Big\rangle\Big| \\ &\qquad \leqslant \|T^{\alpha\gamma}_{\eta\lambda,\phi}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X), L^p(\mathbb{R}^d;Y))} \|P_{\mathscr{F}} f\|_{L^p(\mathbb{R}^d;X)} \|g\|_{L^p(\mathbb{R}^d;Y^*)}.\end{aligned}$$

We now apply this estimate with the increasing sequence of finite sets

$$\mathscr{F}_N := \{Q \in \mathscr{D} : 2^{-N} < \ell(Q) \leqslant 2^N, \operatorname{dist}_\infty(Q, 0) \leqslant 2^N\},$$

whose union is $\bigcup_{N=1}^{\infty} \mathscr{F}_N = \mathscr{D}$. The corresponding projection can be expressed as

$$P_{\mathscr{F}_N} f = \mathbf{1}_{F_N}(\mathbb{E}_N - \mathbb{E}_{-N})f, \quad F_N := \bigcup_{\substack{Q\in\mathscr{D}_{-N} \\ \mathrm{dist}_\infty(Q,0)\leqslant 2^N}} Q,$$

and this is seen to satisfy $\|P_{\mathscr{F}_N} f\|_{L^p(\mathbb{R}^d;X)} \leqslant 2\|f\|_{L^p(\mathbb{R}^d;X)}$ and $P_{\mathscr{F}_N} \to f$ in $L^p(\mathbb{R}^d;X)$ as $N\to\infty$. Thus

$$\begin{aligned}
&\sum_{Q\in\mathscr{D}} \Big|\Big\langle \lambda_Q \langle f, h_Q^\alpha\rangle, \langle g, h_{\phi(Q)}^\gamma\rangle\Big\rangle\Big| \\
&\qquad = \lim_{N\to\infty} \sum_{Q\in\mathscr{F}_N} \Big|\Big\langle \lambda_Q \langle f, h_Q^\alpha\rangle, \langle g, h_{\phi(Q)}^\gamma\rangle\Big\rangle\Big| \\
&\qquad \leqslant \|T^{\alpha\gamma}_{\eta\lambda,\phi}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \lim_{N\to\infty} \|P_{\mathscr{F}} f\|_{L^p(\mathbb{R}^d;X)} \|g\|_{L^p(\mathbb{R}^d;Y^*)} \\
&\qquad = \|T^{\alpha\gamma}_{\eta\lambda,\phi}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \|f\|_{L^p(\mathbb{R}^d;X)} \|g\|_{L^p(\mathbb{R}^d;Y^*)},
\end{aligned}$$

where $T^{\alpha\gamma}_{\eta\lambda,\phi}$ has the same norm estimate as $T^{\alpha\gamma}_{\lambda,\phi}$, since

$$\mathscr{E}\mathscr{R}_p(\eta\lambda) = \mathscr{E}\mathscr{R}_p(\lambda), \qquad \mathscr{E}\mathscr{R}_p((\eta\lambda)_{\phi^{-1}}) = \mathscr{E}\mathscr{R}_p(\lambda_{\phi^{-1}})$$

by the contraction principle.

Thus we have shown the claimed absolute convergence, and hence the bilinear form

$$\mathfrak{t}^{\alpha\gamma}_{\lambda\phi}(f,g) := \sum_{Q\in\mathscr{D}} \langle \lambda_Q \langle f, h_Q^\alpha\rangle, \langle g, h_Q^\gamma\rangle\rangle$$

is well-defined and bounded from $L^p(\mathbb{R}^d;X)\times L^p(\mathbb{R}^d;Y)$ to $\mathbb{K}$. So is the bilinear form $\langle T^{\alpha\gamma}_{\lambda\phi} f, g\rangle$, where $T^{\alpha\gamma}_{\lambda\phi}$ denotes the bounded extension of the operator initially defined on $S_{00}(\mathscr{D};X)$. Moreover, these bilinear forms clearly coincide when $f\in S_{00}(\mathscr{D};X)$ and $g\in S_{00}(\mathscr{D};Y^*)$. By density, they must coincide for all f and g, and the proof is complete. □

The second class of operators that we deal with in this section have the additional twist of "tearing apart" the supports of Haar functions. The relevance of this feature will be justified in the appearance of this type of operators in the proof of the $T(1)$ theorem further below.

Theorem 12.1.28 (Figiel). *Let $\phi:\mathscr{D}\to\mathscr{D}$ be an injection with $\ell(\phi(Q)) = \ell(Q)$ and $\phi(Q)\subseteq 3Q^{(n)}$ for some $n\in\mathbb{N}$. Let X and Y be a UMD spaces and $p\in(1,\infty)$. Let $\lambda = (\lambda_Q)_{Q\in\mathscr{D}} \subseteq \mathscr{L}(X,Y)$, and consider the mapping*

$$U^\gamma_{\phi\lambda} : f \mapsto \sum_{Q\in\mathscr{D}} \lambda_Q \langle f, h_Q^\gamma\rangle (h^0_{\phi(Q)} - h^0_Q), \tag{12.19}$$

initially from $S_{00}(\mathscr{D};X)$ to $S_0(\mathscr{D};Y)$. Let $B_d := 5200\cdot(81)^d$.

(0) *If $\lambda \subseteq \mathscr{L}(X,Y)$ is R-bounded, or more generally if $\mathscr{E}\mathscr{R}_p(\lambda) < \infty$, then $U^\gamma_{\phi\lambda}$ extends boundedly from $L^p(\mathbb{R}^d;X)$ to $L^p(\mathbb{R}^d;Y)$ with norm*

$$\begin{aligned}\|U^\gamma_\phi\| &:= \|U^\gamma_\phi\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \\ &\leqslant B_d \cdot (n+1) \cdot \beta^-_{p,Y} \cdot \beta^+_{p,X} \cdot \min\{\beta^+_{p,X}\mathscr{R}_p(\lambda),\ \beta^+_{p,Y}\mathscr{E}\mathscr{R}_p(\lambda)\}.\end{aligned}$$

(1) *If, in addition, X or Y has cotype $q \in [p,\infty]$, then we also have*

$$\|U^\gamma_\phi\| \leqslant B_d(n+1)^{1-1/q}\beta^-_{p,Y}\beta^+_{p,X}\begin{cases}C(X,Y,p,q)\cdot\mathscr{R}_p(\lambda),\\ \beta^+_{p,Y}\cdot\min\{c_{q,X;p},\ c_{q,Y;p}\}\cdot\mathscr{E}\mathscr{R}_p(\lambda),\end{cases}$$

where

$$\begin{aligned}C(X,Y,p,q) &:= \min\Big\{\beta^+_{p,X}c_{q,X;p},\quad \beta^+_{p,Y}c_{q,X;p},\quad \beta^+_{p,Y}c_{q,Y;p}\Big\} \\ &= C_{(12.15)}(X,Y,p,q,1).\end{aligned} \tag{12.20}$$

(2) *If, in addition, we have $\lambda_Q \neq 0$ only when $\phi(Q) \subseteq Q^{(n)}$, then we have the alternative norm estimate*

$$\|U^\gamma_\phi\| \leqslant 6\cdot(n+1)^{1-1/q}\cdot\beta_{p,Y}\cdot\beta^+_{p,X}\cdot\min\{c_{q,X;p},c_{q,Y;p}\}\cdot\mathscr{E}\mathscr{R}_p(\lambda).$$

(3) *For all $f \in L^p(\mathbb{R}^d;X)$ and $g \in L^{p'}(\mathbb{R}^d;Y^*)$, the extended operator has the absolutely convergent representation*

$$\langle U^\gamma_{\phi\lambda}f,g\rangle = \sum_{Q\in\mathscr{D}}\Big\langle \lambda_Q\langle f,h^\gamma_Q\rangle, \langle g, h^0_{\phi(Q)} - h^0_Q\rangle\Big\rangle.$$

When $\|f\|_{L^p(\mathbb{R}^d;X)} \leqslant 1$ and $\|g\|_{L^{p'}(\mathbb{R}^d;Y^*)} \leqslant 1$, the corresponding absolute value series is dominated by the same upper bounds as those given for $\|U^\gamma_{\phi\lambda}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))}$ above.

Remark 12.1.29. We have observations analogous to Remark 12.1.26:

(1) When $X = Y$, we have $C(X,X,p,q) = \beta^+_{p,X}c_{q,X;p}$.
(2) Case (0) of Theorem 12.1.28 is a special case of (1) using the cotype exponent $q = \infty$ with corresponding constant equal to one. The role of finite cotype is to relax the dependence on the parameter n. As in Theorem 12.1.25(2), the main point of the alternative bound (2) to improve the cubic dependence on the UMD constants to a quadratic one; in contrast to the situation in Theorem 12.1.25(2), when $X = Y$, the present alternative bound (2) is a strict improvement of (1), in view of the fact that $\beta_{p,X} \leqslant \beta^-_{p,X}\beta^+_{p,X}$ (Proposition 4.2.3).
(3) Recalling the Haar multipliers $\mathfrak{H}^{\alpha\gamma}_\lambda$ from Theorem 12.1.11, one can check that, for any $\theta \in \{0,1\}^d \setminus \{0\}$,

$$U^\gamma_{\phi\lambda} = U^\theta_{\phi\mathbf{1}} \circ \mathfrak{H}^{\gamma\theta}_\lambda,$$

where $\mathbf{1}$ is the constant sequence of all ones. Hence, for the qualitative conclusion of Theorem 12.1.28, it would suffice to consider just $X = Y$ and $\lambda = \mathbf{1}$, and then combine this special case with Theorem 12.1.11; however, the reader will quickly realise that this approach would produce a higher power of the UMD constants in the quantitative conclusion.

(4) In contrast to Theorem 12.1.25, our proof of Theorem 12.1.28 does not allow replacing the assumptions on λ by $\mathscr{E}\mathscr{R}_p(\lambda_{\phi^{-1}}) < \infty$. The related issue of when in the argument, and under what assumptions, we may pull out the coefficients λ_Q, is shortly discussed inside the proof.

Proof of Theorem 12.1.28. Claim (0) is the special case $q = \infty$ of (1), so it suffices to consider the latter of these two claims. Let $\mathscr{F} \subseteq \mathscr{D}$ be finite. An additional challenge compared to the proof of Theorem 12.1.25 is that, unlike the Haar functions $h^\alpha_{\phi(Q)}$, the functions $h^0_{\phi(Q)} - h^0_Q$ do not necessarily form a martingale difference sequence, preventing a straightforward introduction of the random signs in the initial step. Instead, a decomposition of $\mathscr{F}$ is necessary from the beginning.

Let us denote by $\mathscr{F}^k = \{Q \in \mathscr{F} : \log_2 \ell(Q) \equiv k \mod (n+1)\}$ the scale-separated subcollections of $\mathscr{F}$ as in Remark 12.1.24. Then $\mathscr{F}^k$ and ϕ satisfy the assumptions of both Lemmas 12.1.22 and 12.1.23. Let us denote the decomposing subcollections of $\mathscr{F}_k$ provided by Lemma 12.1.22 by $\mathscr{A}^k_a$ and those provided by Lemma 12.1.23 by $\mathscr{B}^k_b$, let $\mathscr{F}^k_i = \mathscr{A}^k_a \cap \mathscr{B}^k_b$ for $i = (a, b)$, and let $\mathscr{F}_i$ consists of an enumeration of all these $\mathscr{F}^k_i$. The total number of these $\mathscr{F}_i$ is then $144 \cdot 3^{3d+1} \cdot (n+1)$, and they satisfy the conclusions of both Lemmas 12.1.22 and 12.1.23.

We first make use of Lemma 12.1.22. For $Q \in \mathscr{F}_i$, we have

$$h^0_{\phi(Q)} - h^0_Q = h^0_{\phi_{i,3}(Q)} - h^0_{\phi_{i,0}(Q)} = \sum_{j=0}^{2} (h^0_{\phi_{i,j+1}(Q)} - h^0_{\phi_{i,j}(Q)}),$$

where each collection $\{\phi_{i,j}(Q) \cup \phi_{i,j+1}(Q) : Q \in \mathscr{F}_i\}$ is strongly nested. But this implies that each

$$(h^0_{\phi_{i,j+1}(Q)} - h^0_{\phi_{i,j}(Q)})_{Q \in \mathscr{F}_i}$$

is (or can be enumerated as) a martingale difference sequence. Note that here it is important that a smaller union $\phi_{i,j+1}(Q) \cup \phi_{i,j}(Q)$ is not just contained in a larger $\phi_{i,j+1}(R) \cup \phi_{i,j}(R)$, but entirely in (a dyadic child of) one of $\phi_{i,j+1}(R)$ or $\phi_{i,j}(R)$, where the function $h^0_{\phi_{i,j+1}(R)} - h^0_{\phi_{i,j}(R)}$ is constant.

Using this martingale difference property, we can then proceed as in the proof of Theorem 12.1.25. Let us abbreviate $x_Q := \langle f, h^\gamma_Q \rangle \in X$ and $y_Q := \lambda_Q x_Q \in Y$.

$$\begin{aligned}
&\Big\| \sum_{Q\in\mathscr{F}} y_Q(h^0_{\phi(Q)} - h^0_Q)\Big\|_{L^p(\mathbb{R}^d;Y)} \\
&\leqslant \sum_{j=0}^{2}\sum_i \Big\| \sum_{Q\in\mathscr{F}_i} y_Q(h^0_{\phi_{i,j+1}(Q)} - h^0_{\phi_{i,j}(Q)})\Big\|_{L^p(\mathbb{R}^d;Y)} \\
&\leqslant \sum_{j=0}^{2}\sum_i \beta^-_{p,Y}\Big\| \sum_{Q\in\mathscr{F}_i} \varepsilon_Q y_Q(h^0_{\phi_{i,j+1}(Q)} - h^0_{\phi_{i,j}(Q)})\Big\|_{L^p(\Omega\times\mathbb{R}^d;Y)} \\
&\leqslant \beta^-_{p,Y}\sum_{j=0}^{3}\alpha_j\sum_i \Big\| \sum_{Q\in\mathscr{F}_i} \varepsilon_Q y_Q h^0_{\phi_{i,j}(Q)}\Big\|_{L^p(\Omega\times\mathbb{R}^d;Y)}, \qquad \begin{cases}\alpha_0=\alpha_3=1,\\ \alpha_1=\alpha_2=2,\end{cases}
\end{aligned}$$

where the first and the last steps were simply triangle inequalities.

As in the proof of Theorem 12.1.25, we have some flexibility on when to pull out the coefficients λ_Q, and we again proceed with a generic choice of $z_Q \in Z$ for either $y_Q \in Y$ or $x_Q \in X$. The norm to be estimated has exactly the same form as what we estimated (12.16) in the proof of Theorem 12.1.25 (using Lemma 12.1.23 in this step), and we can there read the bound

$$\begin{aligned}
&\Big\| \sum_{Q\in\mathscr{F}_i} \varepsilon_Q z_Q h^0_{\phi_{i,j}(Q)}\Big\|_{L^p(\Omega\times\mathbb{R}^d;Z)} \\
&\qquad \leqslant 2\cdot 3^d\cdot\beta^+_{p,Z}\Big\| \sum_{Q\in\mathscr{F}_i} \varepsilon_Q z_Q h^0_Q\Big\|_{L^p(\Omega\times\mathbb{R}^d;Z)}.
\end{aligned} \tag{12.21}$$

By Hölder's inequality, we have

$$\begin{aligned}
&\sum_i \Big\| \sum_{Q\in\mathscr{F}_i} \varepsilon_Q z_Q h^0_Q\Big\|_{L^p(\Omega\times\mathbb{R}^d;Z)} \\
&\leqslant \big(144\cdot 3^{3d+1}\cdot(n+1)\big)^{1/q'}\Big(\sum_i \Big\| \sum_{Q\in\mathscr{F}_i} \varepsilon_Q z_Q h^0_Q\Big\|^q_{L^p(\Omega\times\mathbb{R}^d;Z)}\Big)^{1/q}.
\end{aligned}$$

Invoking cotype q of Z, and recalling that this implies cotype q of $L^p(S;Z)$ (here: $S = \Omega\times\mathbb{R}^d$) with constant $c_{q,L^p(S;Z);p} \leqslant c_{q,Z;p}$ when $q\in[p,\infty]$ by Proposition 7.1.4, we continue with

$$\begin{aligned}
&\Big(\sum_i \Big\| \sum_{Q\in\mathscr{F}_i} \varepsilon_Q z_Q h^0_Q\Big\|^q_{L^p(\Omega\times\mathbb{R}^d;Z)}\Big)^{1/q} \\
&\leqslant c_{q,L^p(Z);p}\Big\| \sum_i \varepsilon'_i \sum_{Q\in\mathscr{F}_i} \varepsilon_Q z_Q h^0_Q\Big\|_{L^p(\Omega'\times\Omega\times\mathbb{R}^d;Z)} \\
&= c_{q,L^p(Z);p}\Big\| \sum_{Q\in\mathscr{F}} \varepsilon_Q z_Q h^\gamma_Q\Big\|_{L^p(\Omega\times\mathbb{R}^d;Z)}.
\end{aligned}$$

It is no later than here that we should to pull out the coefficients λ_Q, after which we are left with the final step, based on Proposition 12.1.5, that

$$\Big\| \sum_{Q\in\mathscr{F}} \varepsilon_Q x_Q h_Q^{\gamma} \Big\|_{L^p(\Omega\times\mathbb{R}^d;X)} \leqslant \beta^+_{p,X} \|f\|_{L^p(\mathbb{R}^d;X)}.$$

Under the assumption of R-boundedness of λ, depending on the moment of pulling out the coefficients λ_Q, the constants that we accumulate in the various steps with the option of estimating in $Z \in \{X, Y\}$ produce, aside from the numerical factors $2\cdot 3^d$ and $\big(144\cdot 3^{3d+1}\cdot (n+1)\big)^{1/q'}$, one of the products

$$\begin{aligned}
&\mathscr{R}_p(\lambda)\cdot \beta^+_{p,X}\cdot c_{q,X;p},\\
&\beta^+_{p,Y}\cdot \mathscr{R}_p(\lambda)\cdot c_{q,X;p},\\
&\beta^+_{p,Y}\cdot c_{q,Y;p}\cdot \mathscr{R}_p(\lambda).
\end{aligned}$$

In the latter two versions, i.e., pulling out the λ_Q only after making the step (12.21) with $Z = Y$, we might as well replace $\mathscr{R}_p(\lambda)$ by $\mathscr{E}\mathscr{R}_p(\lambda)$, thus leading to the possible upper bounds

$$\begin{aligned}
&\beta^+_{p,Y}\cdot \mathscr{E}\mathscr{R}_p(\lambda)\cdot c_{q,X;p},\\
&\beta^+_{p,Y}\cdot c_{q,Y;p}\cdot \mathscr{E}\mathscr{R}_p(\lambda).
\end{aligned}$$

(On the other hand, if we wanted to pull out the λ_Q before step (12.21), and thus apply (12.21) with $Z = X$, the coefficient λ_Q would be multiplying a Haar function $h^0_{\phi_{i,j}(Q)}$; this would lead to a constant of the type $\mathscr{E}\mathscr{R}_p(\lambda_{\phi^{-1}_{i,j}})$, where $\phi_{i,j}$ need not be the original ϕ from the assumptions of the theorem, but one of the auxiliary mappings produced by Lemma 12.1.22. This would lead to an unreasonably technical formulation of probably little practical value, which is why we have not included the resulting alternative upper bound in the statement of the theorem.)

Altogether, choosing the best of the possible alternative estimates, we arrive at

$$\begin{aligned}
&\Big\| \sum_{Q\in\mathscr{F}} x_Q (h^0_{\phi(Q)} - h^0_Q) \Big\|_{L^p(\mathbb{R}^d;X)} \|f\|^{-1}_{L^p(\mathbb{R}^d;X)}\\
&\leqslant \beta^-_{p,X} \sum_{j=0}^{3} \alpha_j (2\cdot 3^d)\big(144\cdot 3^{3d+1}\cdot (n+1)\big)^{1/q'} \beta^+_{p,X}\times\\
&\qquad \times \begin{cases} C(X,Y,p,q)\mathscr{R}_p(\lambda),\\ \beta^+_{p,Y}\min\{c_{q,X;p}, c_{q,Y;p}\}\mathscr{E}\mathscr{R}_p(\lambda),\end{cases}
\end{aligned}$$

where $C(X,Y,p,q)$ is as in the statement of the Theorem, and $\sum_{j=0}^{3}\alpha_j = 1+2+2+1 = 6$.

The alternative estimate (2)*:* As in the previous proof of Theorem 12.1.25(2), we construct some auxiliary martingale differences. The initial considerations are identical:

Let again $\mathscr{F}$ be a finite collection of dyadic cubes, and $\mathscr{F}^\lambda := \{Q \in \mathscr{F} : \lambda_Q \neq 0\}$. Then $\mathscr{F}^\lambda$ and ϕ satisfy the assumptions of Lemma 12.1.20, except possibly the scale separation (c). By Remark 12.1.24, the lemma still applies to produce $3(n+1)$ subcollections $\mathscr{F}_i^\lambda \subseteq \mathscr{F}^\lambda$ with the properties given in Lemma 12.1.20. Let us write $x_Q = \langle f, h_Q^\alpha \rangle$. In the first step, we simply use the triangle inequality:

$$\Big\| \sum_{Q\in\mathscr{F}} \lambda_Q x_Q (h^0_{\phi(Q)} - h^0_Q) \Big\|_{L^p(\mathbb{R}^d;Y)} \leqslant \sum_i \Big\| \sum_{Q\in\mathscr{F}_i^\lambda} \lambda_Q x_Q (h^0_{\phi(Q)} - h^0_Q) \Big\|_{L^p(\mathbb{R}^d;Y)}.$$

The slight symmetry break between h_Q^α and $h^0_{\phi(Q)} - h^0_Q$ is also reflected in the construction of the auxiliary martingale differences. As in the proof of Theorem 12.1.25(2), we denote $Q_\alpha^\pm := Q \cap \{\operatorname{sgn}(h_Q^\alpha) = \pm 1\}$. If $\phi(Q) \neq Q$, we choose

$$\begin{aligned} d_Q^1 &:= \frac{1}{3}|Q|^{-1/2}(\mathbf{1}_{\phi(Q)\cup Q_\alpha^+} - 3\cdot \mathbf{1}_{Q_\alpha^-}), \\ d_Q^2 &:= \frac{1}{3}|Q|^{-1/2}(-\mathbf{1}_{\phi(Q)} + 2\cdot \mathbf{1}_{Q_\alpha^+}), \end{aligned}$$

where d_Q^2 has average zero on the sets where d_Q^1 is constant; note that, unlike in the proof of Theorem 12.1.25(2), the order matters now. Moreover, we can recover the original functions by

$$\begin{aligned} d_Q^1 + d_Q^2 &= \frac{1}{3}|Q|^{-1/2}\big((1-1)\mathbf{1}_{\phi(Q)} + (1+2)\mathbf{1}_{Q_\alpha^+} - 3\cdot\mathbf{1}_{Q_\alpha^-}\big) = h_Q^\alpha, \\ d_Q^1 - 2d_Q^2 &= \frac{1}{3}|Q|^{-1/2}\big((1+2)\mathbf{1}_{\phi(Q)}) + (1-4)\mathbf{1}_{Q_\alpha^+} - 3\cdot\mathbf{1}_{Q_\alpha^-}\big) = h^0_{\phi(Q)} - h^0_Q. \end{aligned}$$

If $\phi(Q) = Q$, then $h^0_{\phi(Q)} - h^0_Q = 0$, and we can simply set $d_Q^1 := h_Q^\alpha$ and $d_Q^2 = 0$, and the original functions are recovered by

$$h_Q^\alpha = d_Q^1 = d_Q^1 + d_Q^2, \qquad h^0_{\phi(Q)} - h^0_Q = 0 = 0\cdot d_Q^1 - 2d_Q^2.$$

The conclusion of Lemma 12.1.20, that $\{Q \cup \phi(Q) : Q \in \mathscr{F}_i^\lambda\}$ is strongly nested, ensures that the full collection $\{d_Q^1, d_Q^2\}_{Q\in\mathscr{F}_i^\lambda}$, appropriately enumerated, is a martingale difference sequence. Hence

$$\begin{aligned} &\Big\| \sum_{Q\in\mathscr{F}_i^\lambda} \lambda_Q x_Q (h^0_{\phi(Q)} - h^0_Q) \Big\|_{L^p(\mathbb{R}^d;Y)} \\ &= \Big\| \sum_{Q\in\mathscr{F}_i^\lambda} \lambda_Q x_Q \big((1-\delta_{Q,\phi(Q)})d_Q^+ - 2\cdot d_Q^-\big) \Big\|_{L^p(\mathbb{R}^d;Y)} \\ &\leqslant 2\beta_{p,Y} \Big\| \sum_{Q\in\mathscr{F}_i^\lambda} \varepsilon_Q \lambda_Q x_Q (d_Q^+ + d_Q^-) \Big\|_{L^p(\Omega\times\mathbb{R}^d;Y)} \\ &= 2\beta_{p,Y} \Big\| \sum_{Q\in\mathscr{F}_i^\lambda} \varepsilon_Q \lambda_Q x_Q h_Q^\alpha \Big\|_{L^p(\Omega\times\mathbb{R}^d;Y)} \end{aligned} \tag{12.22}$$

as an application of the definition of UMD via martingale transforms with a multiplying sequences of numbers $\{0, 1, -2\} \times \varepsilon_Q$, and averaging over independent random ε_Q.

Except for the factor 2, the right side of (12.22) coincides with the right side of (12.17) from the proof of Theorem 12.1.25(2). Hence the rest of the estimate can be concluded by repeating the said proof *verbatim*.

The representation (3)*:* This is proved in the same way as the corresponding part of Theorem 12.1.25. □

12.2 Paraproducts

The notion of paraproducts arises from a number of considerations. Here we choose a point of departure that also motivates their name: they are objects that arise from a decomposition of the ordinary pointwise product of functions. While paraproducts certainly look more complicated than the regular product, it turns out that in certain respects they actually behave better. Another motivation is the key role that these objects play in the $T(1)$ theorem in Section 12.3. Some further connections will be discussed in the Notes.

Proposition 12.2.1. *Let $b \in L^1_{\rm loc}(\mathbb{R}^d; \mathscr{L}(X,Y))$, where X and Y are Banach spaces, and let $f \in S_{00}(\mathscr{D}; X)$. Then*

$$bf = \sum_{\alpha,\gamma \in \{0,1\}^d \setminus \{0\}} \mathfrak{H}_b^{\alpha\gamma} f + \Pi_b f + \Pi_b^* f, \tag{12.23}$$

where $\mathfrak{H}_b^{\alpha\gamma}$ are Haar multipliers of the form

$$\mathfrak{H}_b^{\alpha\gamma} f := \sum_{Q \in \mathscr{D}} \langle \operatorname{sgn}(h_Q^\alpha h_Q^\gamma) b \rangle_Q \langle f, h_Q^\alpha \rangle h_Q^\gamma,$$

and the remaining terms are the paraproducts

$$\Pi_b f := \sum_{\substack{Q \in \mathscr{D} \\ \alpha \in \{0,1\}^d \setminus \{0\}}} \langle b, h_Q^\alpha \rangle \langle f \rangle_Q h_Q^\alpha,$$

$$\Pi_b^* f := \sum_{\substack{Q \in \mathscr{D} \\ \alpha \in \{0,1\}^d \setminus \{0\}}} \langle b, h_Q^\alpha \rangle \langle f, h_Q^\alpha \rangle \frac{\mathbf{1}_Q}{|Q|},$$

where the series of $\Pi_b^* f$ is finitely non-zero, and the non-zero terms in $\Pi_b f$ are attached to cubes contained in finitely many maximal ones, and the series converges (at least) conditionally along any decreasing order of the dyadic cubes contained in these maximal ones.

The notation Π_b^* is motivated by the easily verified duality relation

$$\langle \Pi_b^* f, g\rangle = \langle f, \Pi_{b^*} g\rangle, \qquad f \in S_{00}(\mathscr{D}; X), \quad g \in S_{00}(\mathscr{D}; Y),$$

where $b^* \in L^\infty(\mathbb{R}^d; \mathscr{L}(Y^*, X^*))$ is the pointwise adjoint of b.

Remark 12.2.2. The diagonal $\alpha = \gamma$ of the sum in (12.23) is

$$\sum_{\alpha\in\{0,1\}^d\setminus\{0\}} \mathfrak{H}_b^{\alpha\alpha} f = \sum_{\substack{Q\in\mathscr{D} \\ \alpha\in\{0,1\}^d\setminus\{0\}}} \langle b\rangle_Q \langle f, h_Q^\alpha\rangle h_Q^\alpha$$

This has formally the same structure as $\Pi_b f$, but with the roles of b and f reversed, and hence (12.23) could be also written in the form

$$bf = \sum_{\substack{\alpha,\gamma\in\{0,1\}^d\setminus\{0\} \\ \alpha\neq\gamma}} \mathfrak{H}_b^{\alpha\gamma} f + \Pi_f b + \Pi_b f + \Pi_b^* f,$$

where the summation is empty in dimension $d = 1$ (since there is only one possible value of $\alpha \in \{0,1\}\setminus\{0\}$). It is also evident that $\Pi_b^* f$ is symmetric in b and f, and hence a more symmetric notation could also be preferred. However, we shall not pursue this point of view any further, since the roles played by the two functions b and f will be quite different in our main applications, so that such symmetries would be only misleading.

Proof of Proposition 12.2.1. It suffices to prove this for $f = x \otimes h_R^\theta$. Then

$$bf = (b - \langle b\rangle_R)f + \langle b\rangle_R f = \sum_{\substack{Q\subseteq R \\ \alpha\in\{0,1\}^d\setminus\{0\}}} \langle b, h_Q^\alpha\rangle x \otimes h_Q^\alpha h_R^\theta + \langle b\rangle_R x \otimes h_R^\theta,$$

where the series converges (at least) conditionally along any decreasing order of the dyadic cubes $Q \subseteq R$, by the Martingale Converge Theorem 3.3.2, since this is a martingale difference expansion of the function $\mathbf{1}_R(b - \langle b\rangle_R)x \in L^1(\mathbb{R}^d; Y)$.

We observe that

$$h_Q^\alpha h_R^\theta = h_Q^\alpha \langle h_R^\theta\rangle_Q \qquad \forall Q \subsetneq R,$$

whereas

$$h_R^\theta h_R^\theta = \frac{\mathbf{1}_R}{|R|}, \qquad h_R^\alpha h_R^\theta \frac{h_R^{\alpha+\theta}}{|R|^{1/2}}, \quad \forall \alpha \neq \theta,$$

where we use modulo 2 addition in $\{0,1\}^d$. Hence

$$\sum_{\substack{Q\subsetneq R \\ \alpha\in\{0,1\}^d\setminus\{0\}}} \langle b, h_Q^\alpha\rangle x \otimes h_Q^\alpha h_R^\theta = \sum_{\substack{Q\in\mathscr{D} \\ \alpha\in\{0,1\}^d\setminus\{0\}}} \langle b, h_Q^\alpha\rangle \langle f\rangle_Q \otimes h_Q^\alpha = \Pi_b f,$$

observing that $\langle f\rangle_Q = \langle h_R^\theta\rangle_Q x = 0$ unless $Q \subsetneq R$. Moreover,

$$\begin{aligned}
&\sum_{\alpha\in\{0,1\}^d\setminus\{0\}} \langle b, h_R^\alpha\rangle x \otimes h_R^\alpha h_R^\theta + \langle b\rangle_R x \otimes h_R^\theta \\
&= \Big(\langle b, h_R^\theta\rangle x \otimes \frac{\mathbf{1}_R}{|R|} + \sum_{\alpha\in\{0,1\}^d\setminus\{0,\theta\}} \langle b, h_R^\alpha\rangle x \otimes \frac{h_R^{\alpha+\theta}}{|R|^{1/2}}\Big) + \frac{\langle b, h_R^0\rangle}{|R|^{1/2}} x \otimes h_R^\theta \\
&= \langle b, h_R^\theta\rangle x \otimes \frac{\mathbf{1}_R}{|R|} + \sum_{\alpha\in\{0,1\}^d\setminus\{\theta\}} \langle b, h_R^\alpha\rangle x \otimes \frac{h_R^{\alpha+\theta}}{|R|^{1/2}}.
\end{aligned}$$

Using the orthogonality of the Haar functions, we see that

$$\langle b, h_R^\theta\rangle x \otimes \frac{\mathbf{1}_R}{|R|} = \sum_{\substack{Q\in\mathscr{D}\\ \alpha\in\{0,1\}^d\setminus\{0\}}} \langle b, h_Q^\alpha\rangle\langle f, h_Q^\alpha\rangle \otimes \frac{\mathbf{1}_Q}{|Q|} = \varPi_b^* f.$$

Finally, with the change of variable $\gamma := \alpha + \theta$

$$\begin{aligned}
&\sum_{\alpha\in\{0,1\}^d\setminus\{\theta\}} \langle b, h_R^\alpha\rangle x \otimes \frac{h_R^{\alpha+\theta}}{|R|^{1/2}} = \sum_{\gamma\in\{0,1\}^d\setminus\{0\}} \langle b, h_R^{\gamma+\theta}\rangle x \otimes \frac{h_R^\gamma}{|R|^{1/2}} \\
&= \sum_{\gamma\in\{0,1\}^d\setminus\{0\}} \langle b \operatorname{sgn}(h_R^\gamma h_R^\theta)\rangle_R x \otimes h_R^\gamma = \sum_{\gamma\in\{0,1\}^d\setminus\{0\}} \langle a_R^{\theta\gamma} b\rangle_R x \otimes h_R^\gamma \\
&= \sum_{\alpha,\gamma\in\{0,1\}^d\setminus\{0\}} \sum_{Q\in\mathscr{D}} \langle a_Q^{\alpha\gamma} b\rangle_Q \langle f, h_Q^\alpha\rangle \otimes h_Q^\gamma = \sum_{\alpha,\gamma\in\{0,1\}^d\setminus\{0\}} \mathfrak{H}_b^{\alpha\gamma} f,
\end{aligned}$$

again by the orthogonality of the Haar functions in the penultimate step. □

Proposition 12.2.3. *Let X and Y be UMD spaces and $p \in (1,\infty)$. Let $b \in L^\infty(\mathbb{R}^d; \mathscr{L}(X,Y))$. Then $\varLambda_b := \varPi_b + \varPi_b^*$, initially defined on $S_{00}(\mathscr{D}; X)$, extends to a bounded operator from $L^p(\mathbb{R}^d; X)$ to $L^p(\mathbb{R}^d; Y)$ of norm*

$$\|\varLambda_b\|_{\mathscr{L}(L^p(\mathbb{R}^d;X), L^p(\mathbb{R}^d;Y))} \leqslant \Big(1 + (2^d-1)^2 \beta_{p,Y}^- \beta_{p,Y}^+ \beta_{p,X}^+\Big) \|b\|_{L^\infty(\mathbb{R}^d;\mathscr{L}(X,Y))},$$

and we have the identity

$$bf = \sum_{\alpha,\gamma\in\{0,1\}^d\setminus\{0\}} \mathfrak{H}_b^{\alpha\gamma} f + \varLambda_b f \qquad \forall\, f \in L^p(\mathbb{R}^d; X).$$

We will obtain a far better estimate in Theorem 12.2.25, but it seems worthwhile recording this relatively simple bound as an illustration of the techniques that we have developed thus far.

Proof of Proposition 12.2.3. It is clear that pointwise multiplication by $b \in L^\infty(\mathbb{R}^d; \mathscr{L}(X,Y))$ defines a bounded operator from $L^p(\mathbb{R}^d; X)$ to $L^p(\mathbb{R}^d; Y)$, for any Banach spaces X, Y and all $p \in [1, \infty]$. Moreover, the Haar multiplier $\mathfrak{H}_b^{\alpha\gamma}$ featuring in Proposition 12.2.3 have exactly the form considered in Proposition 12.1.13, and hence

$$\|\mathfrak{H}_b^{\alpha\gamma} f\|_{L^p(\mathbb{R}^d;Y)} \leqslant \beta_{p,Y}^{-}\beta_{p,Y}^{+}\beta_{p,X}^{+}\|b\|_{L^\infty(\mathbb{R}^d;\mathscr{L}(X,Y))}\|f\|_{L^p(\mathbb{R}^d;X)}.$$

By triangle inequality, it then follows from (12.23) that

$$\begin{aligned}\|\Lambda_b f\|_{L^p(\mathbb{R}^d;Y)} &\leqslant \|bf\|_{L^p(\mathbb{R}^d;Y)} + \sum_{\alpha,\gamma\in\{0,1\}^d\setminus\{0\}} \|\mathfrak{H}_b^{\alpha\gamma} f\|_{L^p(\mathbb{R}^d;Y)} \\ &\leqslant \|b\|_{L^\infty(\mathbb{R}^d;\mathscr{L}(X,Y))}\Big(1 + (2^d-1)^2\beta_{p,Y}^{-}\beta_{p,Y}^{+}\beta_{p,X}^{+}\Big)\|f\|_{L^p(\mathbb{R}^d;X)}\end{aligned}$$

for all $f \in S_{00}(\mathscr{D}; X)$, and hence Λ_b extends to a bounded operator from $L^p(\mathbb{R}^d; X)$ to $L^p(\mathbb{R}^d; Y))$ with the asserted norm estimate. Since the claimed identity holds (by Proposition 12.2.1) for all $f \in S_{00}(\mathscr{D}; X)$, and each term is continuous with respect to the $L^p(\mathbb{R}^d; X)$ norm of f (as we just showed), it is immediate that this identity extends to all $f \in L^p(\mathbb{R}^d; X)$. □

As we shall see later, the operator Λ_b is not only as good as, but actually *better* than the pointwise product $f \mapsto bf$, in the sense that it remains a bounded operator for a broader class of functions b than just the bounded ones. As the reader will have guessed from the introduced notation, we will also be interested in the mapping properties of the individual paraproducts Π_b and Π_b^*.

While the paraproduct Π_b arouse from our analysis of the pointwise product with a multiplier b, in other considerations we will encounter similar series

$$\Pi f = \sum_{\substack{Q\in\mathscr{D} \\ \alpha\in\{0,1\}^d\setminus\{0\}}} \pi_Q^\alpha \langle f\rangle_Q \otimes h_Q^\alpha$$

with some coefficient π_Q^α replacing the Haar coefficients $\langle b, h_Q^\alpha\rangle$ of a function b above. Formally, we always have $\pi_Q^\alpha = \langle b, h_Q^\alpha\rangle$ by choosing

$$\text{“}\quad b := \Pi(1) = \sum_{\substack{Q\in\mathscr{D} \\ \alpha\in\{0,1\}^d\setminus\{0\}}} \pi_Q^\alpha \otimes h_Q^\alpha \quad\text{”},$$

but giving a precise meaning for this series requires non-trivial considerations in general, and it is hence useful not to insist in the a priori existence of function b generating the coefficients in this way.

12.2.a Necessary conditions for boundedness

As we already saw in the analysis of the pointwise product bf, and we will see again in the $T(1)$ theorem below, paraproducts frequently appear in pairs of the form $\Pi_1 + \Pi_2^*$, where Π_1 is a paraproduct as in the previous section, and Π_2^* is the formal adjoint of another paraproduct. In other words, we are concerned with the operator

$$\Lambda f := \sum_{\substack{Q\in\mathscr{D} \\ \alpha\in\{0,1\}^d\setminus\{0\}}} \Big(\pi_Q^{1,\alpha}\langle f\rangle_Q h_Q^\alpha + \pi_Q^{2,\alpha}\langle f, h_Q^\alpha\rangle \frac{\mathbf{1}_Q}{|Q|}\Big). \tag{12.24}$$

Of course this covers both Π_1 and Π_2^* as special cases, by simply setting some of the coefficients $\pi_Q^{i,\alpha}$ equal to zero.

Compared to the operator Λ_b featuring in Proposition 12.2.3, we now allow possibly different coefficients $\pi_Q^{1,\alpha}$ and $\pi_Q^{2,\alpha}$ in the first and second term above, as this will be relevant in the $T(1)$ theorem. Via the duality relations

$$\langle \Lambda f, g\rangle = \langle f, \Lambda^* g\rangle = \mathfrak{L}(f,g), \qquad f\in S_{00}(\mathscr{D};X),\ g\in S_{00}(\mathscr{D};Y^*),$$

we define the formal adjoint

$$\Lambda^* g := \sum_{\substack{Q\in\mathscr{D} \\ \alpha\in\{0,1\}^d\setminus\{0\}}} \Big((\pi_Q^{1,\alpha})^*\langle g, h_Q^\alpha\rangle \frac{\mathbf{1}_Q}{|Q|} + (\pi_Q^{2,\alpha})^*\langle g\rangle_Q h_Q^\alpha\Big) \tag{12.25}$$

which has exactly the same form as Λ, only with different coefficients, and the associated bilinear form

$$\mathfrak{L}(f,g) := \sum_{\substack{Q\in\mathscr{D} \\ \alpha\in\{0,1\}^d\setminus\{0\}}} \Big(\Big\langle \pi_Q^{1,\alpha}\langle f\rangle_Q, \langle h_Q^\alpha, g\rangle\Big\rangle + \Big\langle \pi_Q^{2,\alpha}\langle f, h_Q^\alpha\rangle, \langle g\rangle_Q\Big\rangle\Big). \tag{12.26}$$

Lemma 12.2.4. *The series* (12.26) *is finitely non-zero whenever*

$$(f,g)\in \big(S_{00}(\mathscr{D};X)\times S(\mathscr{D};Y^*)\big)\cup\big(S(\mathscr{D};X)\times S_{00}(\mathscr{D};Y^*)\big).$$

In particular, we have

$$\mathfrak{L}(x\otimes \mathbf{1}_R, y^*\otimes h_R^\beta) = \langle \pi_R^{1,\beta}x, y^*\rangle, \quad \mathfrak{L}(x\otimes h_R^\beta, y^*\otimes \mathbf{1}_R) = \langle \pi_R^{2,\beta}x, y^*\rangle$$

for all $x\in X$, $y^*\in Y^*$, $R\in\mathscr{D}$ and $\beta\in\{0,1\}^d\setminus\{0\}$.

Proof. By symmetry, it is enough to consider $(f,g)\in S_{00}(\mathscr{D};X)\times S(\mathscr{D};Y^*)$. We may further assume that

$$f = x\otimes h_P^\beta, \qquad g = y^*\otimes \mathbf{1}_R$$

for some $x\in X, y^*\in Y^*$, $P,R\in\mathscr{D}$ and $\beta\in\{0,1\}^d\setminus\{0\}$, since general f and g are finite linear combinations of such functions.

For such f and g, the (Q,α) term in (12.26), is given by

$$\langle \pi^{1,\alpha}x, y^*\rangle\langle h_P^\beta\rangle_Q\langle h_Q^\alpha, \mathbf{1}_R\rangle + \langle \pi^{2,\alpha}x, y^*\rangle\langle h_P^\beta, h_Q^\alpha\rangle\langle \mathbf{1}_R\rangle_Q,$$

where $\langle h_P^\beta\rangle_Q \neq 0$ only if $Q \subsetneq P$, while $\langle h_Q^\alpha, \mathbf{1}_R\rangle \neq 0$ only if $R \subsetneq Q$; finally, $\langle h_P^\beta, h_Q^\alpha\rangle \neq \delta_{P,Q}\delta_{\alpha,\beta}$. Thus

$$\mathfrak{L}(x \otimes \mathbf{1}_P, y^* \otimes h_R^\beta) := \sum_{\substack{Q\in\mathscr{D} \\ R\subsetneq Q\subsetneq P \\ \alpha\in\{0,1\}^d\setminus\{0\}}} \langle \pi_Q^{1,\alpha} x, y^*\rangle \langle h_P^\beta\rangle_Q \langle h_Q^\alpha, \mathbf{1}_R\rangle + \langle \pi_P^{2,\beta} x, y^*\rangle \langle \mathbf{1}_R\rangle_P,$$

which is clearly a finite sum. When $P = R$, the sum above is void, and we get

$$\mathfrak{L}(x \otimes \mathbf{1}_R, y^* \otimes h_R^\beta) = \langle \pi_R^{2,\beta} x, y^*\rangle.$$

The other case follows by symmetry. □

Although our main concern is L^p boundedness, we formulate the following lemma slightly more generally, since the additional generality comes essentially for free.

Lemma 12.2.5. *Let $p, q \in [1,\infty)$. A necessary condition for $\mathfrak{L}$ to satisfy*

$$|\mathfrak{L}(f,g)| \leqslant C\|f\|_{L^p(\mathbb{R}^d;X)}\|g\|_{L^{q'}(\mathbb{R}^d;Y^*)},$$

uniformly for all (f,g) of the form $(x\otimes \mathbf{1}_Q, y^*\otimes h_Q^\alpha)$ and $(x\otimes h_Q^\alpha, y^*\otimes \mathbf{1}_Q)$, is that

$$\|\pi_Q^{i,\alpha}\|_{\mathscr{L}(X,Y)} \leqslant C|Q|^\gamma, \qquad \gamma := \frac{1}{p} - \frac{1}{q} + \frac{1}{2} < \frac{3}{2}. \tag{12.27}$$

On the other hand, assuming the coefficient bound (12.27), the defining series (12.26) of $\mathfrak{L}(f,g)$ converges absolutely for all

$$(f,g) \in S(\mathscr{D}; X) \times S(\mathscr{D}; Y^*)$$

Proof. We have

$$\begin{aligned} |\langle \pi_Q^{1,\alpha} x, y^*\rangle| &= |\mathfrak{L}(x\otimes \mathbf{1}_Q, y^*\otimes h_Q^\alpha)| \\ &\leqslant C\|x\otimes \mathbf{1}_Q\|_{L^p(\mathbb{R}^d;X)}\|y^*\otimes h_Q^\alpha\|_{L^{q'}(\mathbb{R}^d;Y^*)} \\ &= C\|x\|_X|Q|^{1/p}\|y^*\|_{Y^*}|Q|^{1/q'-1/2} \\ &= C\|x\|_X\|y^*\|_{Y^*}|Q|^{1/p-1/q+1/2} \end{aligned}$$

and taking the supremum over $\|y^*\|_{Y^*} \leqslant 1$ and $\|x\|_X \leqslant 1$ proves the estimate for $i = 1$. The case $i = 2$ is entirely symmetric. Finally, note that $1/p, 1/q \in (0,1]$ so that $1/p - 1/q < 1$.

To prove the convergence, it is enough to consider $f = x\otimes \mathbf{1}_P$, $g = y^*\otimes \mathbf{1}_R$, and moreover, by symmetry, just the first half of $\mathscr{L}(f,g)$ with coefficients $\pi_Q^{1,\alpha}$. Now

$$|\langle \pi_Q^{1,\alpha}\langle f\rangle_Q, \langle g, h_Q^\alpha\rangle\rangle| = |\langle \pi_Q^{1,\alpha} x, y^*\rangle||\langle \mathbf{1}_P\rangle_Q||\langle \mathbf{1}_R, h_Q^\alpha\rangle|,$$

where

$$|\langle \pi_Q^{1,\alpha} x, y^* \rangle| \leqslant C|Q|^\gamma \|x\|_X \|y^*\|_{Y^*}, \quad \langle \mathbf{1}_P \rangle_Q \leqslant \frac{|P|}{|Q|}, \quad |\langle \mathbf{1}_R, h_Q^\alpha \rangle| \leqslant \frac{|R|}{|Q|^{1/2}},$$

and moreover the last pairing is non-zero only if $Q \supsetneq R$. Hence the absolute convergence of the series follows from the convergence of

$$\sum_{\substack{Q \in \mathscr{D} \\ Q \supsetneq R}} |Q|^{\gamma - 3/2} = |R|^{\gamma - 3/2} \sum_{k=1}^{\infty} 2^{kd(\gamma - 3/2)} < \infty,$$

since this is as a geometric series with $\gamma - 3/2 < 0$. □

Lemma 12.2.6. *Suppose that the series defining Λf converges (even just conditionally) in $L^p(\mathbb{R}^d; Y)$ for some $f = \mathbf{1}_R \otimes x$, where $R \in \mathscr{D}$ and $x \in X$. Then*

$$(\mathbf{1}_R - E_R)\Lambda(\mathbf{1}_R \otimes x) = \sum_{\substack{Q \subseteq R \\ \alpha \in \{0,1\}^d \setminus \{0\}}} \pi_Q^{1,\alpha} x \otimes h_Q^\alpha$$

Proof. We have

$$\mathbf{1}_R \Big(\pi_Q^{1,\alpha} \langle \mathbf{1}_R \otimes x \rangle_Q h_Q^\alpha + \pi_Q^{2,\alpha} \langle \mathbf{1}_R \otimes x, h_Q^\alpha \rangle \frac{\mathbf{1}_Q}{|Q|} \Big) = \begin{cases} \pi_Q^{1,\alpha} x \otimes h_Q^\alpha + 0, & Q \subseteq R, \\ y_{Q,R}^\alpha \otimes \mathbf{1}_R, & Q \not\subseteq R, \end{cases}$$

for some $y_{Q,R}^\alpha \in Y$, which is not difficult to find explicitly, but it is irrelevant for the present purposes. The assumed convergence in $L^p(\mathbb{R}^d; Y)$, and the boundedness of the conditional expectation E_R and the pointwise multiplier $\mathbf{1}_R$ on $L^p(\mathbb{R}^d; Y)$ guarantee that we can move $(\mathbf{1}_R - E_R)$ inside the defining series. Since $E_R(y_{Q,R}^\alpha \otimes \mathbf{1}_R) = y_{Q,R}^\alpha \otimes \mathbf{1}_R$, we have

$$(\mathbf{1}_R - E_R)\Lambda(\mathbf{1}_R \otimes x) = \sum_{\substack{Q \subseteq R \\ \alpha \in \{0,1\}^d \setminus \{0\}}} \pi_Q^{1,\alpha} x \otimes h_Q^\alpha,$$

as claimed. □

Lemma 12.2.7. *Let Y be a Banach space, and $p \in [1, \infty)$. Let $y_Q^\alpha \in Y$ for all $Q \in \mathscr{D}$, $\alpha \in \{0,1\}^d \setminus \{0\}$. For each $R \in \mathscr{D}$ and $n \in \mathbb{N}$, consider the sum*

$$B_R^n := \sum_{\substack{Q \subseteq R \\ \ell(Q) > 2^{-n}\ell(R) \\ \alpha \in \{0,1\}^d \setminus \{0\}}} y_Q^\alpha \otimes h_Q^\alpha$$

Suppose that, for every $R \in \mathscr{D}$, we have one of the following

(1) $B_R := \lim_{n \to \infty} B_R^n$ *exists in $L^p(\mathbb{R}^d; Y)$, or*

(2) *Y has the Radon–Nikodým property, and* $\sup_{n\in\mathbb{N}} \|B_R^n\|_{L^p(\mathbb{R}^d;Y)} < \infty$.

Then there exists a function $b \in L^p_{\rm loc}(\mathbb{R}^d;Y)$ *such that*

$$\mathbf{1}_R(b - \langle b\rangle_R) = B_R, \quad \langle b, h_R^\alpha\rangle = y_R^\alpha, \quad \forall R \in \mathscr{D},\ \alpha \in \{0,1\}^d \setminus \{0\}.$$

If, moreover, the supremum below is finite, then $b \in \mathrm{BMO}_{\mathscr{D}}(\mathbb{R}^d;Y)$ *and*

$$\sup_{\substack{Q\in\mathscr{D}\\ \alpha\in\{0,1\}^d\setminus\{0\}}} \frac{\|y_Q^\alpha\|_Y}{|Q|^{1/2}} \leqslant \|b\|_{\mathrm{BMO}^p_{\mathscr{D}}(\mathbb{R}^d;Y)} = \sup_{R\in\mathscr{D}} \inf_{c\in Y} \frac{\|B_R - c\|_{L^p(\mathbb{R}^d;Y)}}{|R|^{1/p}} \tag{12.28}$$

Proof. It is immediate to verify that $(B_R^n)_{n=0}^\infty$ is a martingale in $L^p(\mathbb{R}^d;Y)$. By the Martingale Convergence Theorem 3.3.16, it follows that (2) implies (1). Hence it suffices to prove the lemma under assumption (1).

We construct the function b via the correspondence established in Lemma 11.2.11. It is enough to construct $b|_S$ separately for each quadrant $S \subseteq \mathbb{R}^d$. So we fix a quadrant $S \subseteq \mathbb{R}^d$, and let

$$\Delta(s,t) := \sum_{\substack{Q\in\mathscr{D}(S)\\ \alpha\in\{0,1\}^d\setminus\{0\}}} (h_Q^\alpha(s) - h_Q^\alpha(t))y_Q^\alpha,$$

where we need to justify the convergence of this series in some sense. We will prove that it converges in $L^p_{\rm loc}(S\times S;Y)$. To this end, note that any bounded subset of $S\times S$ is contained in $R\times R$ for some $R \in \mathscr{D}(S)$. For $s,t\in R$, only $Q \in \mathscr{D}(S)$ with $Q\cap R \neq \varnothing$ can contribute to the series; moreover, if $Q \supsetneq R$, then h_Q^α is constant on R, and hence $h_Q^\alpha(s) - h_Q^\alpha(t) = 0$ for $s,t\in R$. Thus

$$\begin{aligned}(\mathbf{1}_{R\times R}\Delta)(s,t) &= \mathbf{1}_{R\times R}(s,t) \sum_{\substack{Q\in\mathscr{D}(R)\\ \alpha\in\{0,1\}^d\setminus\{0\}}} (h_Q^\alpha(s) - h_Q^\alpha(t))y_Q^\alpha \\ &= \mathbf{1}_{R\times R}(s,t)(B_R(s) - B_R(t)),\end{aligned} \tag{12.29}$$

where the (conditional) convergence in $L^p(R\times R, \mathrm{d}s\,\mathrm{d}t;Y)$ follows by Fubini's theorem from the assumed (conditional) convergence of each B_R in $L^p(R;Y)$.

Now that the convergence of the defining series of $\Delta(s,t)$ has been justified, it is immediate from the defining formula that $\Delta(s,t) + \Delta(t,u) = \Delta(s,u)$ for $s,t,u\in S$. By Lemma 11.2.11, we have the existence of $b : S \to Y$ such that $\Delta(s,t) = b(s) - b(t)$. substituting this into (12.29), we obtain

$$b(s) - b(t) = B_R(s) - B_R(t), \qquad \text{for } s,t\in R,$$

and hence $b(\cdot) = B_R(\cdot) + (b(t) - B_R(t)) \in L^p(R;Y) \subseteq L^1(R;Y)$. Taking the average over $t\in R$, it follows that

$$b(s) - \langle b\rangle_R = B_R(s) - \langle B_R\rangle_R = B_R(s), \qquad \text{for } s\in R,$$

observing that $\langle h_Q^\alpha\rangle_R = 0$ for all $Q\subseteq R$ that appear in the series of B_R. Then it also follows that

$$\langle b, h_R^\alpha\rangle = \langle \mathbf{1}_R(b-\langle b\rangle_R), h_R^\alpha\rangle = \langle B_R, h_R^\alpha\rangle = y_R^\alpha.$$

This also implies, for any $c\in Y$, that

$$\frac{\|y_Q^\alpha\|_Y}{|Q|^{1/2}} = \Big\|\Big\langle B_Q - c, \frac{h_Q^\alpha}{|Q|^{1/2}}\Big\rangle\Big\|_Y \leqslant \int_Q \|B_Q - c\|_Y \frac{1}{|Q|} \leqslant \Big(\fint_Q \|B_Q - c\|_Y^p\Big)^{1/p},$$

and (12.28) follows from the identity $B_Q = \mathbf{1}_Q(b-\langle b\rangle_Q)$, which implies that

$$\begin{aligned}\|b\|_{\mathrm{BMO}^p_{\mathscr{D}}(\mathbb{R}^d;Y)} &:= \sup_{Q\in\mathscr{D}}\inf_{c\in Y}\|\mathbf{1}_Q(b-c)\|_{L^p(Q;Y)} \\ &= \sup_{Q\in\mathscr{D}}\inf_{c'\in Y}\|\mathbf{1}_Q(B_Q-c')\|_{L^p(Q;Y)}\end{aligned}$$

by a simple change of variable. □

Proposition 12.2.8. *Let X and Y be Banach spaces and $p\in(1,\infty)$. Let $\pi_Q^{i,\alpha}\in\mathscr{L}(X,Y)$, and let Λ be defined by the formal series in (12.24).*

(1) *If, for some $x\in X$m the series (12.24) defining Λf converges (even just conditionally) in $L^p(\mathbb{R}^d;Y)$ whenever $f=\mathbf{1}_R\otimes x$ for $R\in\mathscr{D}$, and these satisfy the testing condition*

$$\|\mathbf{1}_R\Lambda(\mathbf{1}_R\otimes x)\|_{L^p(\mathbb{R}^d;Y)} \leqslant \mathscr{T}_\Lambda^x|R|^{1/p},$$

then $\|\pi_Q^{1,\alpha}x\|_Y \leqslant \mathscr{T}_\Lambda^x|Q|^{1/2}$ and there is $b_1^x\in\mathrm{BMO}_{\mathscr{D}}(\mathbb{R}^d;Y)$ of norm $\|b_1^x\|_{\mathrm{BMO}^p_{\mathscr{D}}(\mathbb{R}^d;Y)}\leqslant\mathscr{T}_\Lambda^x$ such that $\pi_Q^{1,\alpha}x = \langle b_1^x, h_Q^\alpha\rangle$.

(2) *If, in addition to (1), we have $X=Y$ and $\pi_Q^{1,\alpha}\in\mathbb{K}$, then $b_1^x = b_1\otimes x$ for some $b_1\in\mathrm{BMO}_{\mathscr{D}}(\mathbb{R}^d)$ that is independent of x.*

(3) *If, for some $y^*\in Y^*$, the series (12.25) defining $\Lambda^* g$ converges (even just conditionally) in $L^{p'}(\mathbb{R}^d;Y^*)$ whenever $g=\mathbf{1}_R\otimes y^*$ for $R\in\mathscr{D}$, and these satisfy the testing condition*

$$\|\mathbf{1}_R\Lambda^*(\mathbf{1}_R\otimes y^*)\|_{L^{p'}(\mathbb{R}^d;X^*)} \leqslant \mathscr{T}_{\Lambda^*}^{y^*}|R|^{1/p'},$$

then $\|(\pi_Q^{2,\alpha})^*y^*\|_{X^*}\leqslant\mathscr{T}_{\Lambda^*}^{y^*}|Q|^{1/2}$ and there is $b_2^{y^*}\in\mathrm{BMO}_{\mathscr{D}}(\mathbb{R}^d;X^*)$ with $\|b_2^{y^*}\|_{\mathrm{BMO}^{p'}_{\mathscr{D}}(\mathbb{R}^d;X^*)}\leqslant\mathscr{T}_{\Lambda^*}^{y^*}$ and $(\pi_Q^{2,\alpha})^*y^* = \langle b_2^{y^*}, h_Q^\alpha\rangle$.

(4) *If, in addition to (3), we have $X=Y$ and $\pi_Q^{2,\alpha}\in\mathbb{K}$, then $b_2^{y^*} = b_2\otimes y^*$ for some $b_2\in\mathrm{BMO}_{\mathscr{D}}(\mathbb{R}^d)$.*

Proof. (1): Let us fix an $x\in X$. By assumption and Lemma 12.2.6, the series

$$B_R^x := \sum_{\substack{Q\subseteq R\\ \alpha\in\{0,1\}^d\setminus\{0\}}} \pi_Q^{1,\alpha}x\otimes h_Q^\alpha = (\mathbf{1}_R - E_R)\Lambda(\mathbf{1}_R\otimes x)$$

converges (conditionally) in $L^p(\mathbb{R}^d;Y)$. Since $E_R\Lambda(\mathbf{1}_R\otimes x)$ is constant on R, we have the uniform estimate

$$\inf_{c\in Y}\|B_R^x-c\|_{L^p(\mathbb{R}^d;Y)}\leqslant\|\mathbf{1}_R\Lambda(\mathbf{1}_R\otimes x)\|_{L^p(\mathbb{R}^d;Y)}\leqslant\mathscr{T}_\Lambda^x|R|^{1/p}.$$

By Lemma 12.2.7, there is a function $b_1^x\in\mathrm{BMO}_{\mathscr{D}}(\mathbb{R}^d;Y)$ such that

$$\|b_1^x\|_{\mathrm{BMO}^p_{\mathscr{D}}(\mathbb{R}^d;Y)}\leqslant\mathscr{T}_\Lambda^x,\quad\langle b_1^x,h_Q^\alpha\rangle=\pi_Q^\alpha x$$

for all $Q\in\mathscr{D}$ and $\alpha\in\{0,1\}^d\setminus\{0\}$,and $\|\pi_Q^\alpha x\|_Y\leqslant\mathscr{T}_\Lambda|Q|^{1/2}\|x\|_X$, from which the claimed bound for $\|\pi_Q^\alpha\|_{\mathscr{L}(X,Y)}$ is immediate.

(2): Under the assumptions of this case, an inspection of the previous argument shows that all auxiliary functions in the construction of b_1^x have the form $\phi\otimes x$ for different scalar functions ϕ, and hence this form also remains in the final result.

(3)–(4) follow by repeating the proof of (1)–(2) on the dual side. □

Remark 12.2.9. In the setting of Proposition 12.2.8, if we know *a priori* that $\pi_Q^{1,\alpha}x=\langle b_1(\cdot)x,h_Q^\alpha\rangle$ for some $b_1\in L^1_{\mathrm{loc,so}}(\mathbb{R}^d;\mathscr{L}(X,Y))$, then our conclusion on b_1^x implies that $b_1\in\mathrm{BMO}_{\mathscr{D},\mathrm{so}}(\mathbb{R}^d;\mathscr{L}(X,Y))$ and

$$\|b_1\|_{\mathrm{BMO}^p_{\mathscr{D},\mathrm{so}}(\mathbb{R}^d;Y)}\leqslant\mathscr{T}_\Lambda.$$

According to Proposition 12.2.8, the following is a natural necessary condition for the L^p boundedness of paraproducts, even when restricted to very special functions only.

Definition 12.2.10. *We say that a paraproduct Λ as in (12.24) satisfies the* weak coefficient bound *if there is a finite constant C such that*

$$\|\pi_Q^{i,\alpha}\|_{\mathscr{L}(X,Y)}\leqslant C|Q|^{1/2}\tag{12.30}$$

for all $i=1,2$, $\alpha\in\{0,1\}^d\setminus\{0\}$ and $Q\in\mathscr{D}$.

While rather far from being a sufficient condition for any interesting boundedness results, this weak coefficient bound nevertheless allows us to make sense of the defining series of the paraproduct on a sufficiently rich class of functions for our subsequent purposes.

We have the following useful convergence result for *truncated* paraproducts:

Lemma 12.2.11. *Suppose that $\pi_Q^\alpha\in\mathscr{L}(X,Y)$ satisfy (12.30). Let $p\in(1,\infty)$ and $f\in L^p(\mathbb{R}^d;X)$ be boundedly supported, and consider the truncated paraproduct*

$${}_m\Pi f:=\sum_{\substack{Q\in\mathscr{D}\\ \ell(Q)>2^{-m}\\ \alpha\in\{0,1\}^d\setminus\{0\}}}\pi_Q^\alpha\langle f\rangle_Q h_Q^\alpha.$$

(1) *For any* $m \in \mathbb{Z}$, *the series defining* ${}_m\Pi f$ *converges absolutely in* $L^p(\mathbb{R}^d;Y)$.
(2) *For* $2^{-m} \geqslant \operatorname{diam}(\operatorname{supp} f)$, *we have*

$$\|{}_m\Pi f\|_{L^p(\mathbb{R}^d;Y)} \leqslant c_{d,p} C \|E_m f\|_{L^p(\mathbb{R}^d;X)}, \tag{12.31}$$

and if $g \in L^{p'}(\mathbb{R}^d;Y^*)$, *then*

$$|\langle {}_m\Pi f, g\rangle| \leqslant c_{d,p} C \|E_m f\|_{L^p(\mathbb{R}^d;X)} \|E_m g\|_{L^{p'}(\mathbb{R}^d;Y^*)} \underset{m\to-\infty}{\longrightarrow} 0, \tag{12.32}$$

where C *is the constant in* (12.30) *and* $c_{d,p} = \dfrac{2^d-1}{1-2^{-d/p'}}$.

Proof. Let us first consider (2). When $2^{-m} \geqslant \operatorname{diam}(\operatorname{supp} f)$, the support $\operatorname{supp} f$ is contained in at most 2^d cubes $R_i \in \mathscr{D}$. Then in ${}_m\Pi f$, we only need to consider $Q \in \mathscr{D}$ with $Q \supsetneq R_i$ for some (not necessarily unique) $i = 1, \ldots, 2^d$. Then

$$\|{}_m\Pi f\|_{L^p(\mathbb{R}^d;Y)} = \sum_{i=1}^{2^d} \sum_{\substack{Q\in\mathscr{D} \\ Q\supsetneq R_i \\ \alpha\in\{0,1\}^d\setminus\{0\}}} \|\pi_Q^\alpha \langle f\rangle_Q h_Q^\alpha\|_{L^p(\mathbb{R}^d;Y)},$$

where

$$\begin{aligned}\|\pi_Q^\alpha \langle f\rangle_Q h_Q^\alpha\|_{L^p(\mathbb{R}^d;Y)} &\leqslant \|\pi_Q^\alpha\|_{\mathscr{L}(X,Y)} \|\langle f\rangle_Q\|_X \|h_Q^\alpha\|_{L^p(\mathbb{R}^d)} \\ &= \|\pi_Q^\alpha\|_{\mathscr{L}(X,Y)} \frac{1}{|Q|} \Big\| \int_{R_i} f \Big\|_X \frac{|Q|^{1/p}}{|Q|^{1/2}} \leqslant \frac{C}{|Q|^{1/p'}} \Big\| \int_{R_i} f \Big\|_X,\end{aligned}$$

and hence

$$\begin{aligned}\|{}_m\Pi f\|_{L^p(\mathbb{R}^d;Y)} &\leqslant \sum_{i=1}^{2^d} (2^d-1) C \Big\| \int_{R_i} f \Big\|_X \sum_{Q\supsetneq R_i} \frac{1}{|Q|^{1/p'}} \\ &= \sum_{i=1}^{2^d} \frac{(2^d-1)C}{|R_i|^{1/p'}} \Big\| \int_{R_i} f \Big\|_X \sum_{k=1}^{\infty} 2^{-kd/p'} \\ &= \frac{2^d-1}{2^{d/p'}-1} C \sum_{i=1}^{2^d} |R_i|^{1/p} \Big\| \fint_{R_i} f \Big\|_X,\end{aligned}$$

where

$$\sum_{i=1}^{2^d} |R_i|^{1/p} \Big\| \fint_{R_i} f \Big\|_X \leqslant 2^{d/p'} \Big(\sum_{i=1}^{2^d} |R_i| \Big\| \fint_{R_i} f \Big\|_X^p \Big)^{1/p} = 2^{d/p'} \|E_m f\|_{L^p(\mathbb{R}^d;X)}.$$

This proves both the convergence of the series and the claimed bound (12.31).

Each term in the series defining ${}_m\Pi f$ is constant on cubes $R \in \mathscr{D}_m$. Hence ${}_m\Pi f = E_m({}_m\Pi f)$, and thus

$$|\langle {}_m\Pi f, g\rangle| = |\langle {}_m\Pi f, E_m g\rangle| \leqslant \|{}_m\Pi f\|_{L^p(\mathbb{R}^d;Y)} \|E_m g\|_{L^{p'}(\mathbb{R}^d;Y^*)},$$

so that (12.32) follows from (12.31).

Concerning (1), it only remains to consider the part of the series with $2^{-m} < \ell(Q) \leqslant \operatorname{diam}(\operatorname{supp} f)$. But there are only finitely many cubes Q of fixed side-length that intersect $\operatorname{supp} f$, and hence only finitely many cubes altogether that contribute to this remaining sub-series. Thus the absolute convergence is trivial. □

Corollary 12.2.12. *Suppose that $\pi_Q^\alpha \in \mathscr{L}(X,Y)$ satisfy (12.30). Then the series*

$$\sum_{\substack{Q\in\mathscr{D}\\ \alpha\in\{0,1\}^d\setminus\{0\}}} \Big\langle \pi_Q^\alpha \langle f\rangle_Q, \langle h_Q^\alpha, g\rangle \Big\rangle$$

defining $\langle \Pi f, g\rangle$ converges absolutely for all $f \in S(\mathscr{D};X)$ and $g \in S(\mathscr{D};Y^*)$, and

$$\langle \Pi f, E_m g\rangle = \langle {}_m\Pi f, g\rangle.$$

Proof. Let $v := g - E_m g$. Then $v \in S_{00}(\mathscr{D};Y^*)$, and hence only finitely many of the terms $\langle h_Q^\alpha, v\rangle$ are non-zero. Hence it is enough to prove the convergence with $E_m g$ in place of g. Since $\langle h_Q^\alpha, E_m g\rangle = 0$ when $\ell(Q) \leqslant 2^{-m}$, this coincides with the series of $\langle {}_m\Pi f, E_m g\rangle$. Since $f \in S(\mathscr{D};X) \subseteq L^p(\mathbb{R}^d;X)$ is boundedly supported, the series defining ${}_m\Pi f$ converges absolutely in $L^p(\mathbb{R}^d;Y)$ by Lemma 12.2.11. Since $E_m g \in S(\mathscr{D};Y^*) \subseteq L^{p'}(\mathbb{R}^d;Y^*) \subseteq (L^p(\mathbb{R}^d;Y))^*$, the series of the bilinear form converges absolutely in $\mathbb{K}$.

The last identity follows by observing that

$$\langle h_Q^\alpha, E_m g\rangle = \begin{cases} \langle h_Q^\alpha, g\rangle, & \ell(Q) > 2^{-m}, \\ 0, & \ell(Q) \leqslant 2^{-m}, \end{cases}$$

and the proof is complete. □

Corollary 12.2.13. *Suppose that $\pi_Q^{i,\alpha} \in \mathscr{L}(X,Y)$ satisfy (12.30). Then the two series*

$$\sum_{\substack{Q\in\mathscr{D}\\ \alpha\in\{0,1\}^d\setminus\{0\}}} \Big\langle \pi_Q^{1,\alpha} \langle f\rangle_Q, \langle h_Q^\alpha, g\rangle \Big\rangle + \sum_{\substack{Q\in\mathscr{D}\\ \alpha\in\{0,1\}^d\setminus\{0\}}} \Big\langle \pi_Q^{2,\alpha} \langle h_Q^\alpha, f\rangle, \langle g\rangle_Q \Big\rangle$$

defining $\langle \Lambda f, g\rangle$ converge absolutely for all $f \in S(\mathscr{D};X)$ and $g \in S(\mathscr{D};Y^*)$.

Proof. The convergence of the first series is the content of Corollary 12.2.12. The convergence of the second series follows by permuting the roles of $f \in S(\mathscr{D};X)$ and $g \in S(\mathscr{D};Y^*)$, and transposing $\pi_Q^{2,\alpha}$ to the dual side, since $(\pi_Q^{2,\alpha})^* \in \mathscr{L}(Y^*, X^*)$ satisfies the same estimate. □

12.2.b Sufficient conditions for boundedness

We will then turn to exploring conditions that ensure the boundedness of the full paraproduct Π. The obtained necessary conditions serve as a model for the type of sufficient conditions that we are looking for.

It is convenient to begin with a reduction to finite series. When Y is reflexive, we have

$$L^p(\mathbb{R}^d;Y) = L^p(\mathbb{R}^d;Y^{**}) \simeq (L^{p'}(\mathbb{R}^d;Y^*))^*.$$

Since $S_{00}(\mathscr{D};Y^*)$ is dense in $L^{p'}(\mathbb{R}^d;Y^*)$, it is enough to show that the action of Πf is bounded on $S_{00}(\mathscr{D};Y^*)$ with respect to the norm of $L^{p'}(\mathbb{R}^d;Y^*)$, uniformly for f in the unit ball of $L^p(\mathbb{R}^d;X)$. Since any fixed $g \in S_{00}(\mathscr{D};Y^*)$ only "sees" a finite part of Πf, it is enough to prove a uniform $L^p(\mathbb{R}^d;Y)$ estimate for the finite sums $\sum \pi_Q^\alpha \langle f\rangle_Q h_Q^\alpha$. A key initial estimate in this direction is the following:

Lemma 12.2.14. *Let X be a Banach space, Y be a UMD space, and $p \in (1,\infty)$. Let $\mathscr{F}$ be a finite collection of dyadic cubes. For all $f \in L^p(\mathbb{R}^d;X)$ and $\pi_Q^\alpha \in \mathscr{L}(X,Y)$, we then have*

$$\Big\| \sum_{Q\in\mathscr{F}} \pi_Q^\alpha \langle f\rangle_Q h_Q^\alpha \Big\|_{L^p(\mathbb{R}^d;Y)} \leqslant \beta_{p,Y}^- \beta_{p,Y}^+ \Big\| \Big(\sum_{Q\in\mathscr{F}} \varepsilon_Q \pi_Q^\alpha h_Q^0 \Big) f \Big\|_{L^p(\mathbb{R}^d;Y)}.$$

Proof. Since $(\pi_Q^\alpha \langle f\rangle_Q h_Q^\alpha)_{Q\in\mathscr{F}}$ is a martingale difference sequence in $L^p(\mathbb{R}^d;Y)$, we have

$$\Big\| \sum_{Q\in\mathscr{F}} \pi_Q^\alpha \langle f\rangle_Q h_Q^\alpha \Big\|_{L^p(\mathbb{R}^d;Y)} \leqslant \beta_{p,Y}^- \Big\| \sum_{Q\in\mathscr{F}} \varepsilon_Q \pi_Q^\alpha \langle f\rangle_Q h_Q^\alpha \Big\|_{L^p(\mathbb{R}^d\times\Omega;Y)}.$$

Rewriting the L^p norm on the product $\mathbb{R}^d \times \Omega$ with the help of Fubini's theorem, we observe that at each fixed $t \in \mathbb{R}^d$, the sequence of random variables

$$\varepsilon_Q \pi_Q^\alpha \langle f\rangle_Q h_Q^\alpha(t)$$

has the same joint distribution as

$$\varepsilon_Q \pi_Q^\alpha \langle f\rangle_Q h_Q^0(t) = \varepsilon_Q \mathbb{E}_Q(\pi_Q^\alpha f h_Q^0)(t),$$

since the possibly different sign of $h_Q^\alpha(t)$ and $h_Q^0(t)$ is invisible after multiplication by ε_Q. Using this and Stein's inequality (Theorem 4.2.23), we conclude that

$$\begin{aligned} \Big\| \sum_{Q\in\mathscr{F}} \varepsilon_Q \pi_Q^\alpha \langle f\rangle_Q h_Q^\alpha \Big\|_{L^p(\mathbb{R}^d\times\Omega;Y)} &= \Big\| \sum_{Q\in\mathscr{F}} \varepsilon_Q \mathbb{E}_Q(\pi_Q^\alpha f h_Q^0) \Big\|_{L^p(\mathbb{R}^d\times\Omega;Y)} \\ \leqslant \beta_{p,Y}^+ \Big\| \sum_{Q\in\mathscr{F}} \varepsilon_Q \pi_Q^\alpha f h_Q^0 \Big\|_{L^p(\mathbb{R}^d\times\Omega;Y)} &= \beta_{p,Y}^+ \Big\| \Big(\sum_{Q\in\mathscr{F}} \varepsilon_Q h_Q^0 \pi_Q^\alpha \Big) f \Big\|_{L^p(\mathbb{R}^d\times\Omega;Y)}. \end{aligned}$$

□

The previous lemma motivates the following. A background for the nomenclature will be discussed in the Notes.

Definition 12.2.15. *Let $p \in (1,\infty)$. For an indexed family $(\pi_Q)_{Q\in\mathscr{D}}$ in a Banach space Z, we define the* Carleson norm

$$\|(\pi_Q)\|_{\mathrm{Car}^p(\mathbb{R}^d;Z)} := \sup_{Q_0\in\mathscr{D}} \sup_{\substack{\mathscr{F}\subseteq\mathscr{D}\\ \text{finite}}} \frac{1}{|Q_0|^{1/p}} \Big\| \sum_{\substack{Q\subseteq Q_0\\ Q\in\mathscr{F}}} \varepsilon_Q h_Q^0 \pi_Q \Big\|_{L^p(Q_0\times\Omega;Z)}.$$

With the help of Theorem 3.2.17 (the John–Nirenberg inequality for adapted sequences), one can check that any these Carleson norms are actually equivalent for different values of p. We will not need this observation, since the following proof directly shows that we can use any of these norms in our upper bound, as we like. Our first sufficient condition for paraproduct boundedness is stated in terms of this notion as follows:

Proposition 12.2.16 (Paraproducts vs. Carleson norms). *Let X be a Banach space, Y be a UMD space, and $p,q \in (1,\infty)$. Let Π be the paraproduct defined by an indexed family $(\pi_Q^\alpha)_{Q\in\mathscr{D},\alpha\in\{0,1\}^d\setminus\{0\}}$. In order that Π is bounded from $L^p(\mathbb{R}^d;X)$ to $L^p(\mathbb{R}^d;Y)$, it is sufficient that $(\pi_Q^\alpha)_{Q\in\mathscr{D}}$ satisfies the* Car^p *condition for every $\alpha\in\{0,1\}^d\setminus\{0\}$. Moreover, we have the bound*

$$\|\Pi\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \leqslant 32\cdot 4^d pp'\beta_{q,Y}^-\beta_{q,Y}^+ \sum_{\alpha\in\{0,1\}^d\setminus\{0\}} \|(\pi_Q^\alpha)\|_{\mathrm{Car}^q(\mathbb{R}^d;\mathscr{L}(X,Y))}.$$

Proof. We are going to estimate

$$\langle \Pi^\alpha f, g\rangle = \sum_{Q\in\mathscr{D}} \Big\langle \pi_Q^\alpha \langle f\rangle_Q, \langle h_Q^\alpha, g\rangle \Big\rangle \tag{12.33}$$

for $f\in S_{00}(\mathbb{R}^d;X)$ and $g\in S_{00}(\mathbb{R}^d;Y^*)$; the latter guarantees that the sum is finitely nonzero. We may thus replace π_Q^α by $\mathbf{1}_{\mathscr{F}}(Q)\pi_Q^\alpha$ for some finite set $\mathscr{F}\subseteq\mathscr{D}$, but we do not indicate this explicitly in the notation.

Part I: Construction of principal cubes

Let $\mathscr{P}_0$ be the maximal cubes appearing in this sum. We then construct cube families $\mathscr{P}_n$ inductively as follows. For each $P\in\mathscr{P}_n$, let $\mathrm{ch}_{\mathscr{P}}(P)$ be the maximal dyadic subcubes P' of F such that either

$$\fint_{P'} \|f\|_X > 4\fint_P \|f\|_X \qquad \text{or} \qquad \fint_{P'} \|g\|_{Y^*} > 4\fint_P \|g\|_{Y^*}.$$

For each such P', we have

$$|P'| \leqslant \frac{1}{4}\max\Big\{\frac{\int_{P'}\|f\|_X}{\fint_P \|f\|_X}, \frac{\int_{P'}\|g\|_{Y^*}}{\fint_P \|g\|_{Y^*}}\Big\}.$$

Since these P' are pairwise disjoint, we have

$$\sum_{P'\in\mathrm{ch}_{\mathscr{P}}(P)}\int_{P'}\|f\|_X\leqslant\int_P\|f\|_X=|P|\fint_P\|f\|_X$$

and similarly with g, and hence

$$\sum_{P'\in\mathrm{ch}_{\mathscr{P}}(P)}\leqslant\frac{1}{4}(|P|+|P|)=\frac{1}{2}|P|.$$

Thus

$$E_{\mathscr{P}}(P):=P\setminus\bigcup_{P'\in\mathrm{ch}_{\mathscr{P}}(P)}\qquad\text{satisfies}\quad|E_{\mathscr{P}}(P)|\geqslant\frac{1}{2}|P|.$$

Then we let

$$\mathscr{P}_{n+1}:=\bigcup_{P\in\mathscr{P}_n}\mathrm{ch}_{\mathscr{P}}(P),\quad\mathscr{P}:=\bigcup_{n=0}^{\infty}\mathscr{P}_n,$$

and the sets $E_{\mathscr{P}}(P)$, $P\in\mathscr{P}$, are seen to be pairwise disjoint.

Now every Q with a nonzero contribution to (12.33) will be contained in some $P\in\mathscr{P}_0\subseteq\mathscr{P}$. Let $\mathrm{par}_{\mathscr{P}}P\in\mathscr{P}$ be the minimal such P. By construction, it follows that

$$\fint_Q\|f\|_X\leqslant 4\fint_P\|f\|_X,\quad\fint_Q\|g\|_{Y^*}\leqslant 4\fint_P\|g\|_{Y^*},\quad\text{if}\quad\mathrm{par}_{\mathscr{P}}Q=P.$$

For $P\in\mathscr{P}$, let

$$\mathbb{P}_Ph:=\sum_{P'\in\mathrm{ch}_{\mathscr{P}}(P)}\mathbf{1}_{P'}\langle h\rangle_{P'}+\mathbf{1}_{E_{\mathscr{P}}(P)}h.$$

Let $h\in\{f,g\}$. If $u\in E_{\mathscr{P}}$ be a Lebesgue point of h, then all Q with $u\in Q\subseteq P$ fail the stopping criterion, and hence

$$\|\mathbb{P}_Ph(u)\|=\|h(u)\|=\lim_{Q\to u}\|\langle h\rangle_Q\|\leqslant 4\langle\|h(\cdot)\|\rangle_P.$$

On the other hand, if $u\in P'\in\mathrm{ch}_{\mathscr{P}}(P)$, then its dyadic parent $\widehat{P'}$ fails the stopping criterion, and hence

$$\|\mathbb{P}_Ph(u)\|=\|\langle h\rangle_{P'}\|\leqslant\langle\|h(\cdot)\|\rangle_{P'}\leqslant 2^d\langle\|h(\cdot)\|\rangle_{\widehat{P'}}\leqslant 2^d\cdot 4\langle\|h(\cdot)\|\rangle_P.$$

Hence we conclude that

$$\|\mathbb{P}_Ph(u)\|\leqslant 4\cdot 2^d\cdot\mathbf{1}_P(u)\langle\|h\|\rangle_P,\qquad h\in\{f,g\}.$$

If $\mathrm{par}_{\mathscr{P}}Q=P$ and $Q'\in\mathrm{ch}_{\mathscr{D}}Q$, then each $P'\in\mathrm{ch}_{\mathscr{P}}P$ is either disjoint from Q (thus *a fortiori* from Q'), or strictly contained in Q, hence contained in Q'. Thus

$$\int_{Q'} \mathbb{P}_P h = \sum_{\substack{P' \in \operatorname{ch}_{\mathscr{P}}(P) \\ P' \subsetneq Q'}} |P'| \langle h \rangle_{P'} + \int_{Q' \cap E_{\mathscr{P}}(P)} h = \int_{Q'} h, \quad \operatorname{par}_{\mathscr{P}} Q = P.$$

Since both $\mathbf{1}_Q$ and h_Q^α are linear combination of $Q' \in \operatorname{ch}_{\mathscr{D}} Q$, this implies in particular that

$$\langle f \rangle_Q = \langle \mathbb{P}_P f \rangle_Q, \quad \langle h_Q^\alpha, g \rangle = \langle h_Q^\alpha, \mathbb{P}_P g \rangle, \quad \operatorname{par}_{\mathscr{P}} Q = P.$$

Part II: Estimates under the principal cubes

With the principal cubes $P \in \mathscr{P}$ just constructed, we can now rearrange the sum (12.33) as

$$\langle \Pi^\alpha f, g \rangle = \sum_{P \in \mathscr{P}} \Big\langle \sum_{\substack{Q \in \mathscr{D} \\ \operatorname{par}_{\mathscr{P}} Q = P}} \pi_Q^\alpha \langle \mathbb{P}_P f \rangle_Q h_Q^\alpha, \mathbb{P}_P g \Big\rangle =: \sum_{P \in \mathscr{P}} I_P.$$

By Lemma 12.2.14 at the key step introducing the UMD constants, and applications of Hölder's inequality and the properties of the principal cubes elsewhere,

$$\begin{aligned}
I_P &\leqslant \Big\| \sum_{\substack{Q \in \mathscr{D} \\ \operatorname{par}_{\mathscr{P}} Q = P}} \pi_Q^\alpha \langle \mathbb{P}_P f \rangle_Q h_Q^\alpha \Big\|_{L^q(\mathbb{R}^d;Y)} \| \mathbb{P}_P g \|_{L^{q'}(\mathbb{R}^d;Y^*)} \\
&\leqslant \beta_{q,Y}^- \beta_{q,Y}^+ \Big\| \Big(\sum_{\substack{Q \in \mathscr{D} \\ \operatorname{par}_{\mathscr{P}} Q = P}} \varepsilon_Q \pi_Q^\alpha h_Q^0 \Big) \mathbb{P}_P f \Big\|_{L^q(\mathbb{R}^d \times \Omega;Y)} \| \mathbb{P}_P g \|_{L^{q'}(\mathbb{R}^d;Y^*)} \\
&\leqslant \beta_{q,Y}^- \beta_{q,Y}^+ \Big\| \sum_{\substack{Q \in \mathscr{D} \\ \operatorname{par}_{\mathscr{P}} Q = P}} \varepsilon_Q \pi_Q^\alpha h_Q^0 \Big\|_{L^q(\mathbb{R}^d \times \Omega;\mathscr{L}(X,Y))} \| \mathbb{P}_P f \|_{L^\infty(\mathbb{R}^d;X)} \times \\
&\qquad \times \| \mathbb{P}_P g \|_{L^\infty(\mathbb{R}^d;Y^*)} |P|^{1/q'} \\
&\leqslant \beta_{q,Y}^- \beta_{q,Y}^+ \| (\pi_Q^\alpha) \|_{\operatorname{Car}^q(\mathbb{R}^d;\mathscr{L}(X,Y))} |P|^{1/q} \times 4 \cdot 2^d \langle \| f \|_X \rangle_P \times \\
&\qquad \times 4 \cdot 2^d \langle \| g \|_{Y^*} \rangle_P |P|^{1/q'} \\
&= 16 \cdot 4^d \cdot \beta_{q,Y}^- \beta_{q,Y}^+ \| (\pi_Q^\alpha) \|_{\operatorname{Car}^2(\mathbb{R}^d;\mathscr{L}(X,Y))} \langle \| f \|_X \rangle_P \langle \| g \|_{Y^*} \rangle_P |P| \\
&=: 16 \cdot 4^d \cdot \beta_{q,Y}^- \beta_{q,Y}^+ \| (\pi_Q^\alpha) \|_{\operatorname{Car}^2(\mathbb{R}^d;\mathscr{L}(X,Y))} \times II_P.
\end{aligned}$$

(Note that, in the step that lead to the appearance of the Carleson norm, we made use of our implicit replacement of π_Q^α by $\mathbf{1}_{\mathscr{F}}(Q)\pi_Q^\alpha$, for some finite $\mathscr{F} \subseteq \mathscr{D}$, in the beginning of the proof.)

Finally,

$$
\begin{aligned}
\sum_{P\in\mathscr{P}} II_P &\leqslant 2\sum_{P\in\mathscr{P}} \langle\|f\|_X\rangle_P\langle\|g\|_{Y^*}\rangle_P|E_{\mathscr{P}}(P)| \\
&\leqslant 2\sum_{P\in\mathscr{P}}\int_{E_{\mathscr{P}}(P)} M_{\mathscr{D}}f\cdot M_{\mathscr{D}}g \\
&\leqslant 2\int_{\mathbb{R}^d} M_{\mathscr{D}}f\cdot M_{\mathscr{D}}g \\
&\leqslant 2\|M_{\mathscr{D}}f\|_{L^p(\mathbb{R}^d)}\|M_{\mathscr{D}}g\|_{L^{p'}(\mathbb{R}^d)} \\
&\leqslant 2\cdot p'\|f\|_{L^p(\mathbb{R}^d;X)}\cdot p\|g\|_{L^{p'}(\mathbb{R}^d;Y^*)},
\end{aligned}
$$

by Doob's maximal inequality in the last step.

A combination of the estimates proves the proposition. □

To compare the necessary and sufficient conditions for paraproduct boundedness, we have the following relation between bounded mean oscillation and Carleson norms.

Proposition 12.2.17 (Carleson norms vs. BMO). *Let Z be a UMD space, and $p\in(1,\infty)$. If $b\in\mathrm{BMO}_{\mathscr{D}}(\mathbb{R}^d;Z)$, then $(\pi_Q^\alpha)_{Q\in\mathscr{D}}:=(\langle b,h_Q^\alpha\rangle)_{Q\in\mathscr{D}}$ satisfies the Car^p condition for each $\alpha\in\{0,1\}^d\setminus\{0\}$, and*

$$
\max_{\alpha\in\{0,1\}^d\setminus\{0\}}\|(\pi_Q^\alpha)\|_{\mathrm{Car}^p(\mathbb{R}^d;Z)}\leqslant\beta_{p,Z}^+\|b\|_{\mathrm{BMO}_{\mathscr{D}}^p(\mathbb{R}^d;Z)}.
$$

This estimate also has a converse, but since it has no immediate use in the present discussion, we leave the details to an interested reader.

Proof. This is a direct computation

$$
\begin{aligned}
&\Big\|\sum_{\substack{Q\subseteq Q_0\\ Q\in\mathscr{F}}}\varepsilon_Q h_Q^0\langle b,h_Q^\alpha\rangle\Big\|_{L^p(Q_0\times\Omega;Z)} \\
&\leqslant\inf_{c\in Z}\Big\|\sum_{\substack{Q\subseteq Q_0\\ Q\in\mathscr{F}}}\varepsilon_Q h_Q^\alpha\langle\mathbf{1}_{Q_0}(b-c),h_Q^\alpha\rangle\Big\|_{L^p(Q_0\times\Omega;Z)} \\
&\leqslant\inf_{c\in Z}\beta_{p,Z}^+\|\mathbf{1}_{Q_0}(b-c)\|_{L^p(\mathbb{R}^d;Z)} \qquad\text{by Proposition 12.1.5} \\
&\leqslant\beta_{p,Z}^+|Q_0|^{1/p}\|b\|_{\mathrm{BMO}_{\mathscr{D}}^p(\mathbb{R}^d;Z)}.
\end{aligned}
$$

Taking the supremum over finite $\mathscr{F}\subseteq\mathscr{D}$ and $Q_0\in\mathscr{D}$, the claimed bound follows from the definition of Car^p. □

We can now formulate conditions for the boundedness of a paraproduct Π_b in terms of function space properties of b:

Theorem 12.2.18. *Let X be a Banach space, Y be a UMD space, and $p\in(1,\infty)$. Let $b\in L^1_{\mathrm{loc,so}}(\mathbb{R}^d;\mathscr{L}(X,Y))$, and let Π_b be the paraproduct defined by the operators $\pi_Q^\alpha: x\mapsto\langle b(\cdot)x,h_Q^\alpha\rangle$. In order that Π_b is bounded from $L^p(\mathbb{R}^d;X)$ to $L^p(\mathbb{R}^d;Y)$, it is*

(1) *necessary that* $b \in \mathrm{BMO}_{\mathscr{D},\mathrm{so}}(\mathbb{R}^d;\mathscr{L}(X,Y))$, *and*
(2) *sufficient that* $b \in \mathrm{BMO}_{\mathscr{D}}(\mathbb{R}^d;Z)$ *for some subspace* $Z \hookrightarrow \mathscr{L}(X,Y)$ *with the UMD property.*

Moreover, we have the quantitative bounds

$$\begin{aligned}\|b\|_{\mathrm{BMO}_{\mathscr{D},\mathrm{so}}(\mathbb{R}^d;\mathscr{L}(X,Y))} &\leqslant \|\Pi\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \\ &\leqslant 32\cdot 8^d pp'\beta^-_{q,Y}\beta^+_{q,Y}\|j\|_{\mathscr{L}(Z,\mathscr{L}(X,Y))}\beta^+_{q,Z}\|b\|_{\mathrm{BMO}^q_{\mathscr{D}}(\mathbb{R}^d;Z)},\end{aligned}$$

where $j: Z \to \mathscr{L}(X,Y)$ *is the embedding map and* $q \in (1,\infty)$ *is arbitrary.*

Proof. The necessary condition and the lower bound for $\|\Pi\|$ are just restatements of Proposition 12.2.8 and Remark 12.2.9.

For the sufficient condition, from Proposition 12.2.16 we obtain

$$\|\Pi\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \leqslant 32\cdot 4^d pp'\beta^-_{q,Y}\beta^+_{q,Y}\sum_{\alpha\in\{0,1\}^d\setminus\{0\}}\|(\pi^\alpha_Q)\|_{\mathrm{Car}^q(\mathbb{R}^d;\mathscr{L}(X,Y))},$$

and the assumed embedding followed by Proposition 12.2.17 give us

$$\begin{aligned}\|(\pi^\alpha_Q)\|_{\mathrm{Car}^q(\mathbb{R}^d;\mathscr{L}(X,Y))} &\leqslant \|j\|_{\mathscr{L}(Z,\mathscr{L}(X,Y))}\|(\pi^\alpha_Q)\|_{\mathrm{Car}^q(\mathbb{R}^d;Z)} \\ &\leqslant \|j\|_{\mathscr{L}(Z,\mathscr{L}(X,Y))}\beta^+_{q,Z}\|b\|_{\mathrm{BMO}^q_{\mathscr{D}}(\mathbb{R}^d;Z)}.\end{aligned}$$

The estimate is concluded by noting that $\#(\{0,1\}^d\setminus\{0\}) = 2^d - 1 < 2^d$. □

For paraproducts defined by scalar-valued coefficients, we now obtain a complete characterisation of their boundedness on UMD spaces. For $p = q$, the equivalence (1) ⇔ (4) provides a partial solution of the L^p extension problem, discussed in Section 2.1, in the particular case of the paraproducts. Note, however, it does not exclude the possibility of $L^p(\mathbb{R}^d)$-bounded paraproducts extending boundedly to other classes of spaces besides UMD.

Corollary 12.2.19. *Let* X *be a UMD space, and* $p,q \in (1,\infty)$. *Let* Π_1, Π_2^* *and* $\Lambda := \Pi_1 + \Pi_2^*$ *be paraproducts with scalar coefficients* $\pi^{1,\alpha}_Q, \pi^{2,\alpha}_Q \in \mathbb{K}$. *Then the following are equivalent:*

(1) $\Lambda \in \mathscr{L}(L^p(\mathbb{R}^d;X))$;
(2) *both* $\Pi_1, \Pi_2^* \in \mathscr{L}(L^p(\mathbb{R}^d;X))$;
(3) *for some* $b_i \in \mathrm{BMO}(\mathbb{R}^d)$, *we have*

$$\pi^{1,\alpha}_Q = \langle b_1, h^\alpha_Q\rangle,\quad \pi^{2,\alpha}_Q = \langle b_2, h^\alpha_Q\rangle,\quad \forall Q\in\mathscr{D},\alpha\in\{0,1\}^d\setminus\{0\};$$

(4) $\Lambda \in \mathscr{L}(L^q(\mathbb{R}^d))$.

Under these equivalent conditions, we have the estimates

$$\begin{aligned}\max_{i=1,2}\|b_i\|_{\mathrm{BMO}^{p_i}_{\mathscr{D}}(\mathbb{R}^d)} &\leqslant \|\Lambda\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))}, \\ \|\tilde{\Pi}_i\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} &\leqslant 32\cdot 8^d\cdot pp'\cdot\beta^2_{q,X}\cdot\beta_{q,\mathbb{K}}\cdot\|b_i\|_{\mathrm{BMO}^{q_i}_{\mathscr{D}}(\mathbb{R}^d)}, \\ \|\Lambda\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} &\leqslant 64\cdot 8^d\cdot pp'\cdot\beta^2_{q,X}\cdot\beta_{q,\mathbb{K}}\cdot\|\Lambda\|_{\mathscr{L}(L^q(\mathbb{R}^d))}.\end{aligned}$$

where $\tilde{\Pi}_1 := \Pi_1$, $\tilde{\Pi}_2 := \Pi_2^*$, $p_1 := p$, $p_2 := p'$, $q_1 := q$, $q_2 := q'$.

Proof. (1) ⇒ (3): The assumed boundedness (1) and duality clearly implies the testing conditions

$$\begin{aligned}\|\Lambda(\mathbf{1}_Q \otimes x)\|_{L^p(\mathbb{R}^d,X)} &\leqslant \|\Lambda\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))}\|\mathbf{1}_Q \otimes x\|_{L^p(\mathbb{R}^d;X)},\\ \|\Lambda^*(\mathbf{1}_Q \otimes x^*)\|_{L^{p'}(\mathbb{R}^d,X^*)} &\leqslant \|\Lambda\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))}\|\mathbf{1}_Q \otimes x^*\|_{L^{p'}(\mathbb{R}^d;X^*)}.\end{aligned}$$

Condition (3) then follows from Proposition 12.2.8, which also provides the bounds

$$\max\Big(\|b_1\|_{\mathrm{BMO}^p_{\mathscr{D}}(\mathbb{R}^d)}, \|b_2\|_{\mathrm{BMO}^{p'}_{\mathscr{D}}(\mathbb{R}^d)}\Big) \leqslant \|\Lambda\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))}.$$

(3) ⇒ (2): We use Theorem 12.2.18 with $Y = X$ and $Z = \mathbb{K}\cdot I_X$, which clearly embeds into $\mathscr{L}(X)$ with constant one. With this choice, the theorem shows that

$$\begin{aligned}\|\Pi_1\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} &\leqslant 32\cdot 8^d \cdot pp'\beta^-_{q,X}\beta^+_{q,X}\beta^+_{q,\mathbb{K}}\|b_1\|_{\mathrm{BMO}^q_{\mathscr{D}}(\mathbb{R}^d)}.\\ &\leqslant 32\cdot 8^d \cdot pp'\beta^2_{q,X}\beta_{q,\mathbb{K}}\|b_1\|_{\mathrm{BMO}^p_{\mathscr{D}}(\mathbb{R}^d)},\end{aligned}$$

where we also used $\beta^{\pm}_{p,X} \leqslant \beta_{p,X}$. Similarly, we have

$$\|\Pi_2^*\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} = \|\Pi_2\|_{\mathscr{L}(L^{p'}(\mathbb{R}^d;X^*))} \leqslant 32\cdot 8^d\cdot pp'\beta^2_{q,X}\beta_{q,\mathbb{K}}\|b_2\|_{\mathrm{BMO}^{q'}_{\mathscr{D}}(\mathbb{R}^d)},$$

using the same bound on the dual side and recalling that $\beta_{q',X^*} = \beta_{q,X}$.

(2) ⇒ (1): This is trivial by the triangle inequality.

(3) ⇔ (4): This is the already established equivalence (3) ⇔ (1) specialised to $X = \mathbb{K}$. The final quantitative bound follows by combining the bounds already established:

$$\begin{aligned}\|\Lambda\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} &\leqslant \sum_{i=1}^2 \|\tilde{\Pi}_i\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))}\\ &\leqslant \sum_{i=1}^2 32\cdot 8^d\cdot pp'\beta^2_{q,X}\beta_{q,\mathbb{K}}\|b_i\|_{\mathrm{BMO}^{q_i}_{\mathscr{D}}(\mathbb{R}^d)},\\ &\leqslant \sum_{i=1}^2 32\cdot 8^d\cdot pp'\beta^2_{q,X}\beta_{q,\mathbb{K}}\|\Lambda\|_{\mathscr{L}(L^q(\mathbb{R}^d))}\end{aligned}$$

and $\sum_{i=1}^2 32 = 64$. □

12.2.c Symmetric paraproducts

In this section, we will take a closer look at the special case of the symmetric paraproduct Λ_b with equal coefficients $\pi_Q^{i,\alpha} = \langle b, h_Q^\alpha\rangle$ for both $i = 1, 2$. Our goal is to obtain a qualitative improvement of the earlier Proposition 12.2.3. This will require developing modest prerequisites about the projective tensor product of Banach spaces, and we first turn to this task.

Definition 12.2.20. *For two Banach spaces X and Z, and a bilinear form $\lambda : X \times Z \to \mathbb{K}$, we define*

$$\|\lambda\|_{\mathscr{B}(X,Z)} := \sup\Big\{|\lambda(x,z)| : \|x\|_X \leqslant 1, \|z\|_Z \leqslant 1\Big\},$$
$$\mathscr{B}(X,Z) := \Big\{\lambda : X \times Z \to \mathbb{K} \text{ bilinear } \Big| \|\lambda\|_{\mathscr{B}(X,Z)} < \infty\Big\}.$$

Lemma 12.2.21. $\mathscr{B}(X,Z) \simeq \mathscr{L}(X,Z^*) \simeq \mathscr{L}(Z,X^*)$.

Proof. For $u \in \mathscr{L}(X,Z^*)$, we see that

$$\mathrm{Form}(u) : X \times Z \to \mathbb{K}, (x,z) \mapsto \langle ux, z\rangle$$

defines $\mathrm{Form}(u) \in \mathscr{B}(X,Z)$ of norm at most $\|u\|_{\mathscr{L}(X,Z^*)}$. For $\lambda \in \mathscr{B}(X,Z)$, we see that $\mathrm{Op}(\lambda) : X \to Z^* : x \mapsto \lambda(x,\cdot)$ defines $\mathrm{Op}(\lambda) \in \mathscr{L}(X,Z^*)$ of norm at most $\|\lambda\|_{\mathscr{B}(X,Z)}$. Both $\mathrm{Form} : \mathscr{L}(X,Z^*) \to \mathscr{B}(X,Z)$ and $\mathrm{Op} : \mathscr{B}(X,Z) \to \mathscr{L}(X,Z^*)$ are clearly linear and we just saw that they are contractive. Since both $\mathrm{Form} \circ \mathrm{Op}$ and $\mathrm{Op} \circ \mathrm{Form}$ are identities of the respective spaces, they must in fact be isometries. This proves the first identification, and $\mathscr{B}(X,Z) \simeq \mathscr{L}(Z,X^*)$ follows by symmetry, since clearly $\mathscr{B}(X,Z) \simeq \mathscr{B}(Z,X)$. ☐

Definition 12.2.22. *For two Banach spaces X and Z, and elements $x \in X$ and $z \in Z$, we define $x \otimes x \in \mathscr{B}(X,Z)^*$ by*

$$x \otimes z : \mathscr{B}(X,Z) \to \mathbb{K} : \lambda \mapsto \lambda(x,z).$$

Let further

$$X \otimes Z := \mathrm{span}\{x \otimes z : x \in X, z \in Z\} \subseteq \mathscr{B}(X,Z)^*,$$

and, for all $v \in X \otimes Z$,

$$\|v\|_{X\widehat{\otimes}Z} := \inf\Big\{\sum_{i=1}^n \|x_i\|_X\|z_i\|_Z : v = \sum_{i=1}^n x_i \otimes z_i\Big\},$$

where the infimum is over all possible representations of v of any length n. Finally, let $X\widehat{\otimes}Z$ be the completion of $X \otimes Z$ with respect to this norm.

Proposition 12.2.23. *For all Banach spaces X and Z, we have*

$$(X\widehat{\otimes}Z)^* = \mathscr{B}(X,Z),$$

in the following sense: For all $v \in X \otimes Z$ and $\lambda \in \mathscr{B}(X,Z)$, the pairing

$$\langle v, \lambda\rangle := \sum_{i=1}^n \lambda(x_i,z_i), \quad \text{if} \quad v = \sum_{i=1}^n x_i \otimes z_i,$$

is well defined and extends by continuity to all $v \in X\widehat{\otimes}Z$. Conversely, every element of $(X \otimes Z)^*$ has this form, and

$$\|\lambda\|_{(X\otimes Z)^*} = \|\lambda\|_{\mathscr{B}(X,Z)}.$$

Proof. To check that $\langle v,\lambda\rangle$ is well-defined, we need to verify that two different representations $v=\sum_{i=1}^{n_a} x_i^a\otimes z_i^a$, $a=1,2$, result in the same right-hand side. To see this, pick a basis $(x_j^0)_{j=1}^p$ for $\operatorname{span}\{x_i^a : 1\leqslant i\leqslant n_a, a=1,2\}$ and a basis $(z_k^0)_{k=1}^q$ for $\operatorname{span}\{z_i^a : 1\leqslant i\leqslant n_a, a=1,2\}$ and expand all x_i^a and z_i^a in the respective basis. With the help of the Hahn–Banach theorem, pick $x_m^*\in X^*$ and $z_n^*\in Z^*$ such that $\langle x_j^0, x_m^*\rangle=\delta_{j,m}$ and $\langle z_k^0, z_n^*, =\rangle\delta_{k,n}$, and consider the forms $\lambda_{m,n}(\,\cdot_1,\,\cdot_2)=\langle\,\cdot_1, x_m^*\rangle\langle\,\cdot_2, z_n^*\rangle\in\mathscr{B}(X,Z)$ to see that $x_j^0\otimes z_k^0$ are linearly independent in $\mathscr{B}(X,Z)^*$. Hence their coefficients must be equal in the two expansions of v. Make the same expansions on the right-hand side, using the bilinearity of λ, to find that both expansions lead to linear combinations with equal coefficients of the values $\lambda(x_j^0,z_k^0)$.

Having verified that the action of λ on $X\otimes Z$ is well defined, its linearity is clear. Moreover,

$$\sum_{i=1}^n |\lambda(x_i,z_i)|\leqslant\|\lambda\|_{\mathscr{B}(X,Z)}\sum_{i=1}^n\|x_i\|_X\|z_i\|_Z,$$

and taking the infimum over all representations of v shows that

$$|\langle v,\lambda\rangle|\leqslant\|\lambda\|_{\mathscr{B}(X,Z)}\|v\|_{X\widehat{\otimes}Z}$$

for all $v\in X\otimes Z$. From this estimate, we can uniquely extend the action of λ to all $v\in X\widehat{\otimes}Z$ by density, with the estimate

$$\|\lambda\|_{(X\otimes Z)^*}\leqslant\|\lambda\|_{\mathscr{B}(X,Z)}.$$

On the other hand, we also have

$$|\lambda(x,z)|=|\langle x\otimes z,\lambda\rangle|\leqslant\|x\otimes z\|_{X\widehat{\otimes}Z}\|\lambda\|_{(X\otimes Z)^*}\leqslant\|x\|_X\|z\|_Z\|\lambda\|_{(X\otimes Z)^*};$$

thus $\|\lambda\|_{\mathscr{B}(X,Z)}\leqslant\|\lambda\|_{(X\otimes Z)^*}$, and hence in fact there is equality.

Conversely, if $\xi\in(X\otimes Z)^*$, we can define $\lambda\in\mathscr{B}(X,Z)$ by $\lambda(x,z):=\langle x\otimes z,\xi\rangle$. From the previous construction, it is then clear that $\langle v,\lambda\rangle=\langle v,\xi\rangle$ for all $v\in(X\otimes Z)$, and hence every $\xi\in(X\otimes Z)^*$ arises from the previous construction. □

Corollary 12.2.24. $\|x\otimes z\|_{X\widehat{\otimes}Z}=\|x\|_X\|z\|_Z$.

Proof. We compute the norm by duality:

$$\begin{aligned}\|x\otimes z\|_{X\widehat{\otimes}Z}&=\sup\Big\{|\langle x\otimes z,\xi\rangle| : \|\xi\|_{(X\widehat{\otimes}Z)^*}\leqslant 1\Big\}\\&=\sup\Big\{|\lambda(x,z)| : \|\lambda\|_{\mathscr{B}(X,Z)}\leqslant 1\Big\}.\end{aligned}$$

It is clear from the definition that $|\lambda(x,z)|\leqslant\|x\|_X\|z\|_Z$ for any λ as in the last supremum. On the other hand, the Hahn–Banach theorem guarantees the existence of $x^*\in X^*$ and $z^*\in Z^*$ of norm one such that $\langle x,x^*\rangle=\|x\|_X$ and $\langle z,z^*\rangle=\|z\|_Z$. Then clearly $\lambda(\,\cdot_1,\,\cdot_2)=\langle\,\cdot_1,x^*\rangle\langle\,\cdot_2,z^*\rangle$ has $\|\lambda\|_{\mathscr{B}(X,Z)}\leqslant 1$ and gives $\lambda(x,z)=\|x\|_X\|z\|_Z$. □

We are now ready to prove the following improvement of Proposition 12.2.3:

Theorem 12.2.25. *Let X and Y be UMD spaces and $p \in (1,\infty)$. For every function $b \in \mathrm{BMO}_{\mathscr{D}}(\mathbb{R}^d;\mathscr{L}(X,Y))$, the symmetric paraproduct Λ_b defines a bounded operator from $L^p(\mathbb{R}^d;X)$ to $L^p(\mathbb{R}^d;Y)$ of norm*

$$\begin{aligned}\|\Lambda_b\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} &\leqslant 6\cdot 2^d\cdot(pp'+\beta^+_{p,X}\beta^+_{p',Y^*})\|b\|_{\mathrm{BMO}_{\mathscr{D}}(\mathbb{R}^d;\mathscr{L}(X,Y))}\\ &\leqslant 30\cdot 2^d\cdot\beta_{p,X}\beta_{p,Y}\|b\|_{\mathrm{BMO}_{\mathscr{D}}(\mathbb{R}^d;\mathscr{L}(X,Y))}\end{aligned}$$

Proof. By density, it suffices to consider the action of Λ_b on $f\in S_{00}(\mathscr{D};X)$, paired with $g\in S_{00}(\mathscr{D};Y^*)$. We will rewrite this pairing with the help of the projective tensor product duality between $X\widehat{\otimes}Y^*$ and $\mathscr{B}(X,Y^*)\simeq\mathscr{L}(X,Y^{**})=\mathscr{L}(X,Y)$, recalling that UMD spaces are reflexive (Theorem 4.3.3). In the following computation, the summation is always over $Q\in\mathscr{D}$ and $\alpha\in\{0,1\}^d\setminus\{0\}$.

$$\begin{aligned}\langle\Lambda_b f,g\rangle &= \sum\Big\{\Big\langle\langle b,h_Q^\alpha\rangle\langle f\rangle_Q,\langle h_Q^\alpha,g\rangle\Big\rangle_{X,Y^*}+\Big\langle\langle b,h_Q^\alpha\rangle\langle f,h_Q^\alpha\rangle,\langle g\rangle_Q\Big\rangle_{X,Y^*}\Big\}\\ &=\sum\Big\langle\langle b,h_Q^\alpha\rangle,\langle f\rangle_Q\otimes\langle h_Q^\alpha,g\rangle+\langle f,h_Q^\alpha\rangle\otimes\langle g\rangle_Q\Big\rangle_{\mathscr{L}(X,Y),X\widehat{\otimes}_\pi Y^*}\\ &=\Big\langle b,\sum h_Q^\alpha\Big[\langle f\rangle_Q\otimes\langle h_Q^\alpha,g\rangle+\langle f,h_Q^\alpha\rangle\otimes\langle g\rangle_Q\Big]\Big\rangle=:\langle b,h\rangle.\end{aligned}$$

On the last line, we are using the H^1–BMO-duality from Theorem 11.1.30; for $f\in S_{00}(\mathscr{D};X)$ and $g\in S_{00}(\mathscr{D};Y^*)$, the summation is finite, and thus $h\in L^\infty_c(\mathbb{R}^d;X\widehat{\otimes}_\pi Y^*)$. Since $b\in\mathrm{BMO}_{\mathscr{D}}(\mathbb{R}^d;\mathscr{L}(X,Y))\subseteq L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathscr{L}(X,Y))$, the pointwise duality product $\langle b(u),h(u)\rangle$ is integrable, and one find by dominated convergence in the defining formula of Theorem 11.1.30 that the duality can be computed simply as the integral of $\langle b(u),h(u)\rangle$ over $\mathbb{R}^d$. Thus, an application of Theorem 11.1.30 followed by Theorem 11.1.28 shows that

$$\begin{aligned}|\langle\Lambda_b f,g\rangle| &\leqslant\|b\|_{\mathrm{BMO}(\mathbb{R}^d;\mathscr{L}(X,Y))}\|h\|_{H^1_{\mathrm{at}}(\mathbb{R}^d;X\otimes Y^*)}\\ &\leqslant\|b\|_{\mathrm{BMO}(\mathbb{R}^d;\mathscr{L}(X,Y))}\cdot 6\cdot 2^d\cdot\|h\|_{H^1_{\max}(\mathbb{R}^d;X\otimes Y^*)},\end{aligned}$$

and it remains to estimate the H^1 norm here. Recall that

$$\|h\|_{H^1_{\max}(\mathbb{R}^d;X\otimes Y^*)}=\|M_{\mathscr{D}}h\|_{L^1(\mathbb{R}^d)}=\Big\|\sup_{R\in\mathscr{D}}\mathbf{1}_R\|\langle h\rangle_R\|_{X\otimes Y^*}\Big\|_{L^1(\mathbb{R}^d)}.$$

By the properties of Haar functions, we find that

$$\begin{aligned}\langle h\rangle_R &=\sum_{Q\supsetneq R}\sum_{\alpha\in\{0,1\}^d\setminus\{0\}}\Big\langle h_Q^\alpha\Big[\langle f\rangle_Q\otimes\langle h_Q^\alpha,g\rangle+\langle f,h_Q^\alpha\rangle\otimes\langle g\rangle_Q\Big]\Big\rangle_R\\ &=\sum_{Q\supsetneq R}\Big[\langle f\rangle_Q\otimes(\langle g\rangle_{Q_R}-\langle g\rangle_Q)+(\langle f\rangle_{Q_R}-\langle f\rangle_Q)\otimes\langle g\rangle_Q\Big],\end{aligned}$$

where Q_R is the unique dyadic child of Q that contains R.

Next, we make the following algebraic observation:

$$\begin{aligned}
&\langle f\rangle_{Q_R}\otimes\langle g\rangle_{Q_R}-\langle f\rangle_Q\otimes\langle g\rangle_Q \\
&=(\langle f\rangle_{Q_R}-\langle f\rangle_Q+\langle f\rangle_Q)\otimes(\langle g\rangle_{Q_R}-\langle g\rangle_Q+\langle g\rangle_Q)-\langle f\rangle_Q\otimes\langle g\rangle_Q \\
&=\langle f\rangle_Q\otimes(\langle g\rangle_{Q_R}-\langle g\rangle_Q)+(\langle f\rangle_{Q_R}-\langle f\rangle_Q)\otimes\langle g\rangle_Q \\
&\qquad+(\langle f\rangle_{Q_R}-\langle f\rangle_Q)\otimes(\langle g\rangle_{Q_R}-\langle g\rangle_Q).
\end{aligned}$$

Thus

$$\begin{aligned}
\langle f\rangle_R=\sum_{Q\supsetneq R}&\Big[\langle f\rangle_{Q_R}\otimes\langle g\rangle_{Q_R}-\langle f\rangle_Q\otimes\langle g\rangle_Q\Big] \\
&+\sum_{Q\supsetneq R}(\langle f\rangle_{Q_R}-\langle f\rangle_Q)\otimes(\langle g\rangle_{Q_R}-\langle g\rangle_Q)=:I_R+II_R.
\end{aligned}$$

The sum I_R is telescopic and, since $f\in S_{00}(\mathscr{D};X)$ (we don't even need the similar property of g at this point), its terms vanish for all large enough Q. Thus in fact

$$I_R=\langle f\rangle_R\otimes\langle g\rangle_R,\qquad \|I_R\|_{X\widehat{\otimes}_\pi Y^*}=\|\langle f\rangle_R\|_X\|\langle g\rangle_R\|_{Y^*}$$

and

$$\begin{aligned}
\Big\|\sup_{R\in\mathscr{D}}\mathbf{1}_R\|I_R\|_{X\widehat{\otimes}_\pi Y^*}\Big\|_{L^1(\mathbb{R}^d)}&\leqslant\|M_{\mathscr{D}}f\cdot M_{\mathscr{D}}g\|_{L^1(\mathbb{R}^d)} \\
&\leqslant\|M_{\mathscr{D}}f\|_{L^p(\mathbb{R}^d)}\|M_{\mathscr{D}}g\|_{L^{p'}(\mathbb{R}^d)} \\
&\leqslant p'\|f\|_{L^p(\mathbb{R}^d;X)}\cdot p\|g\|_{L^{p'}(\mathbb{R}^d;Y^*)}
\end{aligned}$$

by Doob's maximal inequality in the last step.

Turning to II_R, we note that $\langle f\rangle_{Q_R}-\langle f\rangle_Q$ is the constant value of $\mathbb{D}_Q f(u)$ for any $u\in R$, and similarly for g. As before, the summation in II_R is finitely non-zero, and we can disentangle it with the help of a Rademacher sequence $(\varepsilon_Q)_{Q\in\mathscr{D}}$ as

$$II_R=\mathbb{E}\Big(\sum_{P\supsetneq R}\varepsilon_P\mathbb{D}_P f(u)\Big)\otimes\Big(\sum_{Q\supsetneq R}\overline{\varepsilon}_Q\mathbb{D}_Q g(u)\Big).$$

Thus

$$\begin{aligned}
\|II_R\|_{X\widehat{\otimes}Y^*}&\leqslant\mathbb{E}\Big\|\sum_{P\supsetneq R}\varepsilon_P\mathbb{D}_P f(u)\Big\|_X\Big\|\sum_{Q\supsetneq R}\overline{\varepsilon}_Q\mathbb{D}_Q g(u)\Big\|_{Y^*} \\
&\leqslant\Big\|\sum_{P\supsetneq R}\varepsilon_P\mathbb{D}_P f(u)\Big\|_{L^p(\Omega;X)}\Big\|\sum_{Q\supsetneq R}\overline{\varepsilon}_Q\mathbb{D}_Q g(u)\Big\|_{L^{p'}(\Omega;Y^*)} \\
&\leqslant\Big\|\sum_{P\in\mathscr{D}}\varepsilon_P\mathbb{D}_P f(u)\Big\|_{L^p(\Omega;X)}\Big\|\sum_{Q\in\mathscr{D}}\overline{\varepsilon}_Q\mathbb{D}_Q g(u)\Big\|_{L^{p'}(\Omega;Y^*)},
\end{aligned}$$

where the last step was an application of the contraction principle. Thus

$$\sup_{R\ni u}\|II_R\|_{X\widehat{\otimes}Y^*}\leqslant\Big\|\sum_{P\in\mathscr{D}}\varepsilon_P\mathbb{D}_Pf(u)\Big\|_{L^p(\Omega;X)}\Big\|\sum_{Q\in\mathscr{D}}\bar{\varepsilon}_Q\mathbb{D}_Qg(u)\Big\|_{L^{p'}(\Omega;Y^*)}$$

and

$$\begin{aligned}\Big\|\sup_{R\in\mathscr{D}}\mathbf{1}_R\|II_R\|_{X\widehat{\otimes}Y^*}\Big\|_{L^1(\mathbb{R}^d)}&\\ &\leqslant\Big\|\sum_{P\in\mathscr{D}}\varepsilon_P\mathbb{D}_Pf\Big\|_{L^p(\mathbb{R}^d\times\Omega;X)}\Big\|\sum_{Q\in\mathscr{D}}\bar{\varepsilon}_Q\mathbb{D}_Qg\Big\|_{L^{p'}(\mathbb{R}^d\times\Omega;Y^*)}\\ &\leqslant\beta^+_{p,X}\|f\|_{L^p(\mathbb{R}^d;X)}\cdot\beta^+_{p',Y^*}\|g\|_{L^{p'}(\mathbb{R}^d;Y^*)}.\end{aligned}$$

A combination of the estimates of I_R and II_R shows that

$$\begin{aligned}\|h\|_{H^1_{\max}(\mathbb{R}^d;X\widehat{\otimes}Y^*)}&\\ &\leqslant\Big\|\sup_{R\in\mathscr{D}}\mathbf{1}_R\|I_R\|_{X\widehat{\otimes}Y^*}\Big\|_{L^1(\mathbb{R}^d)}+\Big\|\sup_{R\in\mathscr{D}}\mathbf{1}_R\|I_R\|_{X\widehat{\otimes}Y^*}\Big\|_{L^1(\mathbb{R}^d)}\\ &\leqslant(pp'+\beta^+_{p,X}\beta^+_{p',Y^*})\|f\|_{L^p(\mathbb{R}^d;X)}\|g\|_{L^{p'}(\mathbb{R}^d;Y^*)},\end{aligned}$$

and altogether we have proved the first estimate claimed in the theorem.

The final estimate is seen as follows: First, we have $\beta^+_{p,X}\leqslant\beta_{p,X}$ and $\beta^+_{p',Y^*}\leqslant\beta_{p',Y^*}=\beta_{p,Y}$ by the observation after Proposition 4.2.3, and Proposition 4.2.17(2). Second, denoting $p^*=\max(p,p')\geqslant 2$, we have $\beta_{p,Z}\geqslant\beta_{p,\mathbb{R}}=p^*-1\geqslant\frac{1}{2}p^*$ by Theorem 4.5.7, and hence $pp'\leqslant(p^*)^2\leqslant 4\beta_{p,X}\beta_{p,Y}$. □

12.2.d Mei's counterexample: no simple sufficient conditions

The following theorem shows the impossibility of obtaining simple upper bounds for operator-valued paraproducts in infinite-dimensional spaces, even by considering Hilbert spaces only, and even by replacing the bounded mean oscillation conditions by the stronger L^∞ norm.

Theorem 12.2.26 (Mei). *Let ϕ be a function such that*

$$\|\Pi_b\|_{\mathscr{L}(L^2(\mathbb{R};\ell^2_N))}\leqslant\phi(N)\|b\|_{L^\infty(\mathbb{R};\mathscr{L}(\ell^2_N))}\quad\textit{for all } b\in L^\infty(\mathbb{R};\mathscr{L}(\ell^2_N)).$$

Then

$$\phi(N)\geqslant\|\triangle_N\|_{\mathscr{L}(\mathscr{L}(\ell^2_N))}\geqslant\frac{1}{\pi}(\log N-1),$$

where $\triangle_N:\mathscr{L}(\ell^2_N)\to\mathscr{L}(\ell^2_N)$ is the lower triangle projection defined by

$$\triangle_N(e_i\otimes e_j):=\begin{cases}e_i\otimes e_j, & \textit{if } i>j,\\ 0, & \textit{else}\end{cases}$$

and extended by linearity.

Proof. For $a \in \mathscr{L}(\ell^2_N)$ and $u, v \in \ell^2_N$, we have the tensor product $u \otimes v \in \mathscr{L}(\ell^2_N)$, and the trace duality $\langle a, u \otimes v\rangle = \langle au, v\rangle$.

Let $b \in L^\infty(\mathbb{R}; \mathscr{L}(\ell^2_N))$, and $f, g \in L^2(\mathbb{R}; \ell^2_N)$. We can then write

$$\langle \Pi_b f, g\rangle = \Big\langle \sum_{I\in\mathscr{D}} \langle b, h_I\rangle\langle f\rangle_I h_I, g\Big\rangle = \Big\langle b, \sum_{I\in\mathscr{D}} \langle f\rangle_I \otimes \langle h_I, g\rangle h_I\Big\rangle =: \langle b, \Pi_{\otimes g} f\rangle,$$

where suggestive notation $\Pi_{\otimes g}$ is defined by the last identity. In the two rightmost expressions, the duality is that between $L^\infty(\mathbb{R}; \mathscr{L}(\ell^2_N))$ and its predual $L^1(\mathbb{R}; \mathscr{C}^1(\ell^2_N))$. (We recall from Theorem D.2.6 that $(\mathscr{C}^1(H))^* = \mathscr{L}(H)$ for any Hilbert space H and from Theorem 1.3.10 that $(L^1(\mathbb{R}; X))^* = L^\infty(\mathbb{R}; X^*)$ when X^* has the Radon–Nikodým property, which the finite-dimensional (hence reflexive) $X = \mathscr{L}(\ell^2_N)$ does by Theorem 1.3.21.)

Thus we deduce that

$$\begin{aligned}\|\Pi_{\otimes g} f\|_{L^1(\mathbb{R};\mathscr{C}^1(\ell^2_N))} &= \sup\Big\{|\langle b, \Pi_{\otimes g} f\rangle| : \|b\|_{L^\infty(\mathbb{R};\mathscr{L}(\ell^2_N))} \leqslant 1\Big\} \\ &= \sup\Big\{|\langle \Pi_b f, g\rangle| : \|b\|_{L^\infty(\mathbb{R};\mathscr{L}(\ell^2_N))} \leqslant 1\Big\} \\ &\leqslant \phi(N)\|f\|_{L^2(\mathbb{R};\ell^2_N)}\|g\|_{L^2(\mathbb{R};\ell^2_N)}.\end{aligned}$$

We now apply this to a special choice of $f, g \in L^2(\mathbb{R}; \ell^2_N)$. Let $(r_i)_{i=1}^N$ be the standard realisation of a Rademacher sequence on $[0, 1)$, i.e., $r_i(t) := \mathbf{1}_{[0,1)}(t)\,\mathrm{sgn}(\sin(2^i\pi t))$. With $u, v \in \ell^2_N$, we take $f = \sum_{i=1}^N r_i\langle u, e_i\rangle e_i$ and $g = \sum_{i=1}^N r_i\langle v, e_i\rangle e_i$, where $(e_i)_{i=1}^N$ is the standard orthonormal basis of ℓ^2_N. Then

$$\begin{aligned}\Pi_{\otimes g} f(t) &= \sum_{j=1}^N \sum_{i=1}^{j-1} r_i(t)\langle u, e_i\rangle e_i \otimes r_j(t)\langle v, e_j\rangle e_j \\ &= D_{r(t)}\Big(\sum_{1\leqslant i<j\leqslant N} \langle u, e_i\rangle\langle v, e_j\rangle e_i \otimes e_j\Big) D_{r(t)} \\ &= D_{r(t)}\Big(T_N \sum_{i,j=1}^N \langle u, e_i\rangle\langle v, e_j\rangle e_i \otimes e_j\Big) D_{r(t)} = D_{r(t)}\big(T_N(u \otimes v)\big) D_{r(t)}\end{aligned}$$

where $D_{r(t)} = \sum_{i=1}^N r_i(t) e_i \otimes e_i$ and $\tilde{\triangle}_N$ is the upper triangle projection defined by

$$\tilde{\triangle}_N(e_i \otimes e_j) := \begin{cases} e_i \otimes e_j, & \text{if } i < j, \\ 0, & \text{else}\end{cases}$$

and extended by linearity. Since $D_{r(t)}$ is unitary for every $t \in [0, 1)$, it follows that

$$\|\Pi_{\otimes g} f\|_{L^1(\mathbb{R};\mathscr{C}^1(\ell^2_N))} = \|\tilde{\triangle}_N(u \otimes v)\|_{L^1([0,1);\mathscr{C}^1(\ell^2_N))} = \|\tilde{\triangle}_N(u \otimes v)\|_{\mathscr{C}^1(\ell^2_N)}.$$

Dy Lemma D.1.1 and the definition of the Schatten class, every $s \in \mathscr{C}^1(\ell^2_N)$ has a singular value decomposition

$$s = \sum_{k=1}^{n} a_k(s) u_k \otimes v_k, \quad \|u_k\|_{\ell^2_N} = \|v_k\|_{\ell^2_N} = 1, \quad \sum_{k=1}^{n} a_k(s) = \|s\|_{\mathscr{C}^1(\ell^2_N)}$$

where $a_k(s) \geqslant 0$ are the approximation numbers of s. Letting $f_k, g_k \in L^2(\mathbb{R}; \ell^2_N)$ of norm one be the functions corresponding to u_k, v_k, we find that

$$\begin{aligned} \|\tilde{\triangle}_N s\|_{\mathscr{C}^1(\ell^2_N)} &\leqslant \sum_{k=1}^{n} a_k(s) \|\tilde{\triangle}_N (u_k \otimes v_k)\|_{\mathscr{C}^1(\ell^2_N)} \\ &= \sum_{k=1}^{n} a_k(s) \|\Pi_{\otimes g_k} f_k\|_{L^1(\mathbb{R}; \mathscr{C}^1(\ell^2_N))} \\ &\leqslant \sum_{k=1}^{n} a_k(s) \phi(N) = \phi(N) \|s\|_{\mathscr{C}^1(\ell^2_N)}. \end{aligned}$$

Noting that the lower triangle projection $\triangle_N$ on $\mathscr{L}(\ell^2_N) = (\mathscr{C}^1(\ell^2_N))^*$ is the adjoint of the upper triangle projection $\tilde{\triangle}_N$ on $\mathscr{C}^1(\ell^2_N))$, this implies that

$$\|\triangle_N\|_{\mathscr{L}(\mathscr{L}(\ell^2_N))} = \|\tilde{\triangle}_N\|_{\mathscr{L}(\mathscr{C}^1(\ell^2_N))} \leqslant \phi(N),$$

which is the first claimed inequality.

The final bound is essentially Lemma 7.5.12, where a variant

$$T_N(e_i \otimes e_j) := \begin{cases} e_i \otimes e_j, & \text{if } i \geqslant j, \\ 0, & \text{else} \end{cases}$$

was considered instead. However, the lower bound for the norm of this operator was achieved by testing with the Hilbert matrix $A_N = (\mathbf{1}_{\{i \neq j\}} (i-j)^{-1})_{i,j=1}^N$ with vanishing diagonal; hence $\Delta_N(A_N) = T_N(A_N)$, and the same lower bound follows for Δ_N as well. □

12.3 The $T(1)$ theorem for abstract bilinear forms

In Sections 11.2 and 11.3, the leading theme was extrapolating the boundedness of a singular integral operator from $L^{p_0}(\mathbb{R}^d; X)$ to $L^p(\mathbb{R}^d; X)$, with a different exponent p, or even to $L^p(w; X)$, with a different weight w. A question that was largely left open in these sections was how to verify the assumed boundedness on some $L^{p_0}(\mathbb{R}^d; X)$ to begin with. In the spirit of the L^p-extension problem discussed in Section 2.1, we here obtain the following useful answer that allows us to extrapolate the vast existing information about scalar-valued singular integrals to the UMD-valued situation:

Theorem 12.3.1. *Let $p_0 \in (1, \infty)$, and let $T \in \mathscr{L}(L^{p_0}(\mathbb{R}^d))$ be an operator associated with a Calderón–Zygmund standard kernel $K : \dot{\mathbb{R}}^{2d} \to \mathbb{K}$. Let X be a UMD space and $p \in (1, \infty)$. Then $T \otimes I_X$ extends to a bounded linear operator on $L^p(\mathbb{R}^d; X)$.*

In fact, this result will be obtained as a corollary of general criteria, known as "$T(1)$ theorems", for the boundedness of operators associated with Calderón–Zygmund kernels; and we will also obtain versions dealing with operator-valued kernels. However, the very statement of these results requires some preparations that we take up next. Concerning the proofs, we only mention at this point that the dyadic singular integral operators and paraproducts, whose boundedness we already studied in Sections 12.1 and 12.2, will play a significant role; indeed, our general strategy is to decompose a Calderón–Zygmund operator into a convergent series of dyadic singular integral operators and paraproducts. Thus, this final section brings together several of the themes developed in this chapter.

12.3.a Weakly defined bilinear forms

In order to make a non-tautological study of the question of boundedness of an operator, we need to give a meaning to the notion of an "operator" before its boundedness has been established. As usual, this will involve postulating the action of the operator on a dense class of test functions from which we wish to extend this action to the full space under consideration. For a dyadic analysis of singular integral operators, it is convenient to adopt the following framework:

Definition 12.3.2. *For a Banach space Z, a Z-valued bilinear form on $S(\mathscr{D})$ is a bilinear mapping*

$$\mathfrak{t} : S(\mathscr{D})^2 \to Z.$$

If $Z = \mathscr{L}(X, Y)$, we extend the action of such a mapping to

$$\mathfrak{t} : S(\mathscr{D}; X) \times S(\mathscr{D}; Y^*) \to \mathbb{K}$$

by letting

$$\mathfrak{t}(\phi \otimes x, \psi \otimes y^*) := \langle \mathfrak{t}(\phi, \psi)x, y^* \rangle \in \mathbb{K}, \qquad \phi, \psi \in S(\mathscr{D}), \quad x \in X, y^* \in Y^*,$$

and extending by bilinearity, observing that $S(\mathscr{D}; X) = S(\mathscr{D}) \otimes X$.

Remark 12.3.3 ($S(\mathscr{D})$ vs. $S_{00}(\mathscr{D})$ in the definition). Since $S_{00}(\mathscr{D}; X)$ is already dense in $L^p(\mathbb{R}^d; X)$, in order to construct a bounded bilinear form on $L^p(\mathbb{R}^d; X) \times L^{p'}(\mathbb{R}^d; Y)$, it would be sufficient to have an *a priori* estimate on $S_{00}(\mathscr{D}; X) \times S_{00}(\mathscr{D}; Y^*)$. However, for the type of theorems that we have in mind, we also like to make assumptions on the action of our bilinear forms on functions like $\mathbf{1}_Q \in S(\mathscr{D}) \setminus S_{00}(\mathscr{D})$, and hence we need to have our initial bilinear form defined on the larger product $S(\mathscr{D}; X) \times S(\mathscr{D}; Y^*)$. This gives rise to the following problem, where we take $X = Y = \mathbb{K}$ for simplicity, since the issue is already present in this case:

Suppose that we have a bilinear form $\mathfrak{t} : S(\mathscr{D})^2 \to \mathbb{K}$ that satisfies the estimate

$$|\mathfrak{t}(f,g)| \leqslant C\|f\|_p\|g\|_{p'} \quad \forall (f,g)\in S_{00}(\mathscr{D})^2.$$

Thus there exists $T\in\mathscr{L}(L^p(\mathbb{R}^d))$ such that $\mathfrak{t}(f,g)=\langle Tf,g\rangle$ whenever $(f,g)\in S_{00}(\mathscr{D})^2$. Does it follow that $\mathfrak{t}(f,g)=\langle Tf,g\rangle$ for all $(f,g)\in S(\mathscr{D})^2$?

Perhaps unexpectedly, the answer is “no”: Consider the bilinear form

$$\mathfrak{t}(f,g) := \int_{\mathbb{R}^d} f \cdot \int_{\mathbb{R}^d} g, \quad (f,g)\in S(\mathscr{D})^2.$$

If $(f,g)\in S_{00}(\mathscr{D})^2$, we have the a priori bound $|\mathfrak{t}(f,g)|=0$, and hence the unique operator $T\in\mathscr{L}(L^p(\mathbb{R}^d))$ is given by $T=0$. But of course $\mathfrak{t}$ is not identically zero on $S(\mathscr{D})^2$. It is also clear that there cannot possible be any $T\in\mathscr{L}(L^p(\mathbb{R}^d))$ with $\langle Tf,g\rangle=\mathfrak{t}(f,g)$ for all $(f,g)\in S(\mathscr{D})^2$.

To avoid this problem, we make sure to get our *a priori* estimates on the full set $S(\mathscr{D};X)\times S(\mathscr{D};Y^*)$.

Definition 12.3.4. *A bilinear form $\mathfrak{t}: S(\mathscr{D})^2\to\mathscr{L}(X,Y)$ is said to determine a bounded operator $T\in\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))$ provided that this operator T satisfies*

$$\mathfrak{t}(f,g)=\langle Tf,g\rangle$$

for all $(f,g)\in S(\mathscr{D};X)\times S(\mathscr{D};Y^*)$.

In the case of reflexive spaces, the last-mentioned condition can be characterised by an *a priori* estimate. Finding sufficient conditions for such an estimate will be our primary concern below. The assumption of reflexivity is not a serious restriction at this stage, since the deeper related considerations that we shall encounter below will have much stronger assumptions, anyway.

Lemma 12.3.5. *Let X and Y be reflexive Banach spaces, and let $X_0\subseteq X$ and $Y^0\subseteq Y^*$ be dense. Consider a bilinear form*

$$\mathfrak{t}: S(\mathscr{D};X_0)\times S(\mathscr{D};Y^0)\to\mathbb{K}.$$

Let $C\geqslant 0$ be a constant and $p\in(1,\infty)$. Then the following conditions, each to hold for every choice of $(f,g)\in S(\mathscr{D};X_0)\times S(\mathscr{D};Y^0)$, are equivalent:

(1) *There is $T\in\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))$ of norm at most C such that*

$$\langle Tf,g\rangle=\mathfrak{t}(f,g).$$

(2) *There is $T^*\in\mathscr{L}(L^{p'}(\mathbb{R}^d;Y^*),L^{p'}(\mathbb{R}^d;X^*))$ of norm at most C such that*

$$\langle f,T^*g\rangle=\mathfrak{t}(f,g).$$

(3) *There is a uniform estimate*

$$|\mathfrak{t}(f,g)|\leqslant C\|f\|_{L^p(\mathbb{R}^d;X_0)}\|g\|_{L^{p'}(\mathbb{R}^d;Y^0)}.$$

Proof. (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (3) are immediate.

(3) $\Rightarrow$ (1): Fix $f \in \mathcal{Q}(\mathbb{R}^d; X_0)$. Then $g \mapsto \mathfrak{t}(f,g)$ defines a bounded linear functional on a dense subspace of $L^{p'}(\mathbb{R}^d; Y^*)$, and hence on $L^{p'}(\mathbb{R}^d; Y^*)$. Thus there is $\Lambda_f \in (L^{p'}(\mathbb{R}^d; Y^*))^*$ such that

$$\mathfrak{t}(f,g) = \langle \Lambda_f, g\rangle.$$

Moreover, since $Y = Y^{**}$ is reflexive, it has the Radon–Nikodým property by Theorem 1.3.21, and hence $\Lambda_f \in (L^{p'}(\mathbb{R}^d; Y^*))^* \simeq L^p(\mathbb{R}^d; Y)$ by Theorem 1.3.10.

From the linearity of the left side in f, one deduces that $f \mapsto \Lambda_f$ is a linear map from $S(\mathscr{D}; X) \subseteq L^p(\mathbb{R}^d; X)$ to $L^p(\mathbb{R}^d; Y)$, and (3) shows that it is bounded. Hence there is a bounded extension $T \in \mathscr{L}(L^p(\mathbb{R}^d; X), L^p(\mathbb{R}^d; Y))$ with the required identity for $(f,g) \in S(\mathscr{D}; X_0) \times S(\mathscr{D}; Y^0)$.

(3) $\Rightarrow$ (2): This can be proved either similarly to the previous case, or using the already proven implication (3) $\Rightarrow$ (1) and the general existence result of an adjoint

$$T^* \in \mathscr{L}((L^p(\mathbb{R}^d; Y))^*, (L^p(\mathbb{R}^d; X))^*) \simeq \mathscr{L}(L^{p'}(\mathbb{R}^d; Y^*), L^{p'}(\mathbb{R}^d; X^*)),$$

where the identification of the spaces was again based on the assumed reflexivity via Theorems 1.3.21 and 1.3.10. By definition, the adjoint satisfies

$$\langle f, T^* g\rangle = \langle Tf, g\rangle$$

for all (f,g) in $L^p(\mathbb{R}^d; X) \times L^{p'}(\mathbb{R}^d; Y^*) \supseteq S(\mathscr{D}; X_0) \times S(\mathscr{D}; Y^0)$. □

The very formulation of the conditions that give rise to the name "$T(1)$ theorem" requires us to slightly extend the initial domain of weakly defined singular integral operators.

Definition 12.3.6. *For a bilinear* $\mathfrak{t} : S(\mathscr{D})^2 \to Z$, *we say that* $\mathfrak{t}(h_Q^\alpha, \mathbf{1})$ *is well-defined if the series*

$$\mathfrak{t}(h_Q^\alpha, \mathbf{1}) := \sum_{\substack{R \in \mathscr{D} \\ \ell(R) = \ell(Q)}} \mathfrak{t}(h_Q^\alpha, \mathbf{1}_R)$$

converges absolutely. We say that $\mathfrak{t}(\cdot, \mathbf{1})$ *is well-defined if* $\mathfrak{t}(h_Q^\alpha, \mathbf{1})$ *is well-defined for every* $Q \in \mathscr{D}$ *and* $\alpha \in \{0,1\}^d \setminus \{0\}$.

We define $\mathfrak{t}(\mathbf{1}, h_Q^\alpha)$ *and* $\mathfrak{t}(\mathbf{1}, \cdot)$ *analogously.*

Lemma 12.3.7. *If* $\mathfrak{t}(h_Q^\alpha, \mathbf{1})$ *is well-defined, then*

(1) *for every* $k \in \mathbb{Z}$ *with* $2^{-k} \geqslant \ell(Q)$, *we have*

$$\mathfrak{t}(h_Q^\alpha, \mathbf{1}) = \sum_{R \in \mathscr{D}_k} \mathfrak{t}(h_Q^\alpha, \mathbf{1}_R),$$

where the series converges absolutely in the weak operator topology;

(2) *for every* $f \in S_{00}(\mathscr{D})$, *the series*

$$\mathfrak{t}(f, \mathbf{1}) := \sum_{R \in \mathscr{D}_k} \mathfrak{t}(f, \mathbf{1}_R)$$

converges absolutely at least for all sufficiently negative $k \in \mathbb{Z}$; *moreover, the value of the series is independent of* $k \in \mathbb{Z}$, *as long as it converges absolutely.*

The analogous statements hold for $\mathfrak{t}(\mathbf{1}, \cdot)$.

Proof. (1): Let $\ell(Q) = 2^{-j}$. For $k = j$, the claim of the lemma is just the definition. For $2^{-k} > 2^{-j}$ and $R \in \mathscr{D}_k$, we have

$$\mathfrak{t}(h_Q^\alpha, \mathbf{1}_R) = \mathfrak{t}\Big(h_Q^\alpha, \sum_{\substack{S \in \mathscr{D}_j \\ S \subseteq R}} \mathbf{1}_S\Big) = \sum_{\substack{S \in \mathscr{D}_j \\ S \subseteq R}} \mathfrak{t}(h_Q^\alpha, \mathbf{1}_S).$$

With $f = h_Q^\alpha$, we then have

$$\mathfrak{t}(f, \mathbf{1}) = \sum_{S \in \mathscr{D}_j} \mathfrak{t}(f, \mathbf{1}_S) = \sum_{R \in \mathscr{D}_k} \sum_{\substack{S \in \mathscr{D}_j \\ S \subseteq R}} \mathfrak{t}(f, \mathbf{1}_S) = \sum_{R \in \mathscr{D}_k} \mathfrak{t}(f, \mathbf{1}_R), \tag{12.34}$$

where the first equality holds by assumption, and the assumed absolute convergence allows to make the rearrangements and to get the absolute convergence also in the subsequent steps.

(2): Each $f \in S_{00}(\mathscr{D})$ is a linear combination of terms of the form $h_{Q_i}^{\alpha_i}$, where $i \in \mathscr{F}$ for some finite index set $\mathscr{F}$. If $Q_0 \in \mathscr{D}_{j_0}$ is the largest cube appearing here, then by the previous part of the lemma we know that

$$\sum_{R \in \mathscr{D}_k} \mathfrak{t}(h_{Q_i}^{\alpha_i}, \mathbf{1}_R)$$

converges absolutely for each $k \leqslant j_0$. Hence also

$$\sum_{R \in \mathscr{D}_k} \mathfrak{t}(f, \mathbf{1}_R) = \sum_{i \in \mathscr{F}} \langle f, h_{Q_i}^{\alpha_i} \rangle \sum_{R \in \mathscr{D}_k} \mathfrak{t}(h_{Q_i}^{\alpha_i}, \mathbf{1}_R)$$

converges absolutely. If the absolute convergence holds for some j and k, the equality of the corresponding series follows from (12.34).

The case of $\mathfrak{t}(\mathbf{1}, \cdot)$ is entirely analogous. □

As we shall see later, the forms $\mathfrak{t}(\mathbf{1}, \cdot)$ and $\mathfrak{t}(\cdot, \mathbf{1})$ are closely related to paraproducts. Since the boundedness of paraproducts is tricky, it is useful to be able identify situations, when they can be avoided, i.e., when $\mathfrak{t}(\mathbf{1}, \cdot) = 0 = \mathfrak{t}(\cdot, \mathbf{1})$.

With this goal in mind, we will now discuss an important case of *translation-invariant* bilinear forms. We first check that some natural candidates for the definition are equivalent:

Lemma 12.3.8. *Let Z be a Banach space. The following conditions are equivalent for a bilinear form $\mathfrak{t} : S(\mathscr{D})^2 \to Z$:*

(1) $\mathfrak{t}(\mathbf{1}_Q, \mathbf{1}_R) = \mathfrak{t}(\mathbf{1}_{Q\dot{+}m}, \mathbf{1}_{R\dot{+}m})$ *for all $Q, R \in \mathscr{D}$ with $\ell(Q) = \ell(R)$, and all $m \in \mathbb{Z}^d$, where $Q\dot{+}m := Q + m\ell(Q)$.*

(2) $\mathfrak{t}(f, g) = \mathfrak{t}(\tau_h f, \tau_h g)$ *for all $f, g \in S(\mathscr{D})$ and all dyadic rational vectors h, i.e., all h of the form $h = m2^{-k}$ for some $m \in \mathbb{Z}^d$ and $k \in \mathbb{Z}$, where $\tau_h f(s) := f(s - h)$.*

If $Z = \mathscr{L}(X, Y)$, these are also equivalent to a variant of (2) for all $f \in S(\mathscr{D}; X)$ and $g \in S(\mathscr{D}; Y^*)$ instead.

Proof. (2)⇒(1): This is immediate by taking $f = \mathbf{1}_Q$, $g = \mathbf{1}_R$ and $h = m\ell(Q)$, or $f = \mathbf{1}_Q \otimes x$, $g = \mathbf{1}_R \otimes y^*$ for arbitrary $x \in X$ and $y^* \in Y^*$ in the variant with $Z = \mathscr{L}(X, Y)$.

(1)⇒(2): By definition, each f, g is a linear combination of some indicators $\mathbf{1}_Q$ (or $\mathbf{1}_Q \otimes x$ resp. $\mathbf{1}_Q \otimes y^*$) with $Q \in \mathscr{D}$ (and $x \in X$, $y^* \in Y^*$), and we have $h = m_h 2^{-k_h}$ for some $m_h \in \mathbb{Z}^d$ and $k_h \in \mathbb{Z}$. Since any dyadic cube is an exact union of dyadic cubes of any given smaller size, and h can be expressed in a similar form $h = (2^{(k-k_h)})2^{-k}$ for any $k \geqslant k_h$, we may assume that we have $Q \in \mathscr{D}_k$ and $h = m2^{-k}$ for the same $k \in \mathbb{Z}$ to begin with. By bilinearity of both sides of the claim in (2), we thus need to verify that $\mathfrak{t}(\mathbf{1}_Q, \mathbf{1}_R) = \mathfrak{t}(\tau_h \mathbf{1}_Q, \tau_h \mathbf{1}_R) = (\mathbf{1}_{Q\dot{+}m}, \mathbf{1}_{R\dot{+}m})$ for each $Q, R \in \mathscr{D}_k$, but this is exactly what we assumed in (1). □

Definition 12.3.9. *A bilinear form $\mathfrak{t} : S(\mathscr{D})^2 \to Z$ is called* translation-invariant, *if it satisfies the equivalent conditions of Lemma 12.3.8.*

Formally, it is easy to see that $\mathfrak{t}(\mathbf{1}, \cdot) = 0 = \mathfrak{t}(\cdot, \mathbf{1})$ if $\mathfrak{t}$ is translation invariant. Namely, if $Q \in \mathscr{D}$, and Q_1 is the "lower left quadrant" of Q, then

$$Q = \bigcup_{\gamma \in \{0,1\}^d} (Q_1 \dot{+} \gamma), \qquad h_Q^\alpha = \sum_{\gamma \in \{0,1\}^d} \langle h_Q^\alpha \rangle_{Q_1 \dot{+} \gamma} \mathbf{1}_{Q_1 \dot{+} \gamma},$$

where the coefficients $\langle h_Q^\alpha \rangle_{Q_1 \dot{+} \gamma}$ are equal to $\pm |Q|^{-1/2}$, with equally many of each sign. Now, *formally*, we have

$$\text{“}\quad \mathfrak{t}(\mathbf{1}, \mathbf{1}_{Q_1 \dot{+} \gamma}) = \mathfrak{t}(\tau_{\gamma \ell(Q_1)} \mathbf{1}, \tau_{\gamma \ell(Q)} \mathbf{1}_{Q_1}) = \mathfrak{t}(\mathbf{1}, \mathbf{1}_{Q_1}), \quad \text{”}$$

and hence

$$\begin{aligned} \text{“}\quad (\mathbf{1}, h_Q^\alpha) &= \sum_{\gamma \in \{0,1\}^d} \langle h_Q^\alpha \rangle_{Q_1 \dot{+} \gamma} \mathfrak{t}(\mathbf{1}, \mathbf{1}_{Q_1 \dot{+} \gamma}) \\ &= \sum_{\gamma \in \{0,1\}^d} \langle h_Q^\alpha \rangle_{Q_1 \dot{+} \gamma} \mathfrak{t}(\mathbf{1}, \mathbf{1}_{Q_1}) = 0 \cdot \mathfrak{t}(\mathbf{1}, \mathbf{1}_{Q_1}) = 0. \quad \text{”} \end{aligned}$$

Problems with this computation are:

(1) While we defined $\mathfrak{t}(\mathbf{1}, h_Q^\alpha)$ for cancellative Haar functions h_Q^α, the expressions "$\mathfrak{t}(\mathbf{1}, \mathbf{1}_{Q_1 \dot{+} \gamma})$" above need not even be defined; i.e., even if the series defining the former converges, an analogous series for the latter need not.
(2) The assumption that $\mathfrak{t}$ is translation invariant was made on the class of functions $S(\mathscr{D})$ only, and the constant function $\mathbf{1}$ is not in this class.

Nevertheless, under a mild decay assumption, and some care with limits, we can bootstrap the above heuristics into a solid argument:

Proposition 12.3.10. *Suppose that $\mathfrak{t} : S(\mathscr{D})^2 \to Z$ is translation-invariant and satisfies the decay assumption, for all $Q \in \mathscr{D}$ and $m \geqslant M_Q$, that*

$$\|\mathfrak{t}(\mathbf{1}_Q, \mathbf{1}_{Q\dot{+}m})\| + \|\mathfrak{t}(\mathbf{1}_{Q\dot{+}m}, \mathbf{1}_Q)\| \leqslant c_Q |m|^{-d}. \tag{12.35}$$

Then $\mathfrak{t}(\mathbf{1}, \cdot) = 0 = \mathfrak{t}(\cdot, \mathbf{1})$.

Proof. We fix some $Q \in \mathscr{D}_k$ and $\alpha \in \{0,1\}^d \setminus \{0\}$. By definition, we have

$$\begin{aligned}
\mathfrak{t}(\mathbf{1}, h_Q^\alpha) &= \sum_{m \in \mathbb{Z}^d} \mathfrak{t}(\mathbf{1}_{Q\dot{+}m}, h_Q^\alpha) = \lim_{M\to\infty} \sum_{\substack{m \in \mathbb{Z}^d \\ |m|_\infty \leqslant M}} \mathfrak{t}(\mathbf{1}_{Q\dot{+}m}, h_Q^\alpha) \\
&= \lim_{M\to\infty} \sum_{\beta,\gamma \in \{0,1\}^d} \langle h_Q^\alpha \rangle_{Q_1 \dot{+} \gamma} \sum_{\substack{m \in \mathbb{Z}^d \\ |m|_\infty \leqslant M}} \mathfrak{t}(\mathbf{1}_{Q_1 \dot{+} \beta \dot{+} 2m}, \mathbf{1}_{Q_1 \dot{+}\gamma}),
\end{aligned}$$

where rearranging the order of the finite sums inside the limit presents no issues. Here

$$\mathfrak{t}(\mathbf{1}_{Q_1 \dot{+} \beta \dot{+} 2m}, \mathbf{1}_{Q_1 \dot{+}\gamma}) = \mathfrak{t}(\mathbf{1}_{Q_1 \dot{+} (\beta-\gamma) \dot{+} 2m}, \mathbf{1}_{Q_1}),$$

and hence, noting that $\beta - \gamma \in \{-1, 0, 1\}^d$,

$$\begin{aligned}
\sum_{\substack{m \in \mathbb{Z}^d \\ |m|_\infty \leqslant M}} \mathfrak{t}(\mathbf{1}_{Q_1 \dot{+} \beta \dot{+} 2m}, \mathbf{1}_{Q_1 \dot{+}\gamma}) &= \sum_{\substack{n \in \mathbb{Z}^d \\ n \in [-2M, 2M]^d + (\beta-\gamma)}} \mathfrak{t}(\mathbf{1}_{Q_1 \dot{+} n}, \mathbf{1}_{Q_1}) \\
&= \Big(\sum_{\substack{n \in \mathbb{Z}^d \\ n \in [-(2M-1), 2M-1]^d}} + \sum_{\substack{n \in \mathbb{Z}^d \\ n \in [-2M, 2M]^d + (\beta-\gamma) \\ n \notin [-(2M-1), 2M-1]^d}} \Big) \mathfrak{t}(\mathbf{1}_{Q_1 \dot{+} n}, \mathbf{1}_{Q_1}) \\
&=: I_M + II_M^{\beta-\gamma}.
\end{aligned}$$

In $II_M^{\beta-\gamma}$, we note that at least one component n_i of n must satisfy $|n_i| \geqslant 2M$, and hence the decay assumption (12.35) ensures that

$$\|\mathfrak{t}(\mathbf{1}_{Q_1 \dot{+} n}, \mathbf{1}_{Q_1})\| \leqslant c_{Q_1} (1 + 2M)^{-d}.$$

On the other hand, we have $n \in [-(2M+1), 2M+1]^d \setminus [-(2M-1), (2M-1)]^d$, and the total number of such $n \in \mathbb{Z}^d$ is

$$\begin{aligned}(1+2(2M+1))^d-(1+2(2M-1))^d &= (4M+3)^d-(4M-1)^d\\ &\leqslant 4d(4M+3)^{d-1},\end{aligned}$$

and hence

$$\|II_M^{\beta-\gamma}\| \leqslant 4d(4M+3)^{d-1}\times c_{Q_1}(1+2M)^{-d}\leqslant c_d c_{Q_1} M^{-1}.$$

Substituting back, it follows that

$$\begin{aligned}\mathfrak{t}(\mathbf{1},h_Q^\alpha) &= \lim_{M\to\infty}\sum_{\beta,\gamma\in\{0,1\}^d}\langle h_Q^\alpha\rangle_{Q_1\dot{+}\gamma}(I_M+II_M^{\beta-\gamma})\\ &= \lim_{M\to\infty}\sum_{\beta,\gamma\in\{0,1\}^d}\langle h_Q^\alpha\rangle_{Q_1\dot{+}\gamma}II_M^{\beta-\gamma}=\lim_{M\to\infty}O(M^{-1})=0.\end{aligned}$$

The computation for $\mathfrak{t}(h_Q^\alpha,\mathbf{1})$ is entirely similar. □

Remark 12.3.11. It is easy to see from the proof that the decay assumption (12.35) could be somewhat weakened. We have not strived for maximal generality at this point, but stated a condition that is both relatively simple to formulate and easy to verify in our main application to Calderón–Zygmund singular integrals.

12.3.b The BCR algorithm and Figiel's decomposition

In order to analyse $\mathfrak{t}(f,g)$, we will use the auxiliary operators

$$E_kf=\sum_{Q\in\mathscr{D}_k}E_Qf=\sum_{Q\in\mathscr{D}_k}\langle f\rangle_Q 1_Q,\qquad \mathscr{D}_k=\{Q\in\mathscr{D}:\ell(Q)=2^{-k}\}.$$

$$D_kf=E_{k+1}f-E_kf=\sum_{Q\in\mathscr{D}_k}\Big(\sum_{Q'\in\mathrm{ch}(Q)}E_{Q'}f-E_Qf\Big)=\sum_{Q\in\mathscr{D}_k}D_Qf.$$

Our starting point for the analysis of a bilinear form is the following useful identity:

Lemma 12.3.12 (Beylkin–Coifman–Rokhlin (BCR) algorithm). *Let X,Y be Banach spaces, and let $\mathfrak{t}:S(\mathscr{D})^2\to\mathscr{L}(X,Y)$ be bilinear. Suppose that $f\in S(\mathscr{D};X)$ and $g\in S(\mathscr{D};Y^*)$ are constant on all $Q\in\mathscr{D}_M$. Then for all integers $m<M$,*

$$\begin{aligned}\mathfrak{t}(f,g)=\sum_{m\leqslant k<M}\mathfrak{t}(D_kf,D_kg)+\sum_{m\leqslant k<M}\mathfrak{t}(D_kf,E_kg)\\ +\sum_{m\leqslant k<M}\mathfrak{t}(E_kf,D_kg)+\mathfrak{t}(E_mf,E_mg).\end{aligned}\tag{12.36}$$

Proof. That f is constant on all $Q \in \mathscr{D}_M$ means that $f = E_M f$, and similarly $g = E_M g$. Thus we have

$$\begin{aligned}\mathfrak{t}(f,g) - \mathfrak{t}(E_m f, E_m g) &= \mathfrak{t}(E_M f, E_M g) - \mathfrak{t}(E_m f, E_m g) \\ &= \sum_{m \leqslant k < M} (\mathfrak{t}(E_{k+1} f, E_{k+1} g) - \mathfrak{t}(E_k f, E_k g)),\end{aligned}$$

where

$$\begin{aligned}\mathfrak{t}(E_{k+1} f, E_{k+1} g) &= \mathfrak{t}((D_k + E_k) f, (D_k + E_k) g) \\ &= \mathfrak{t}(D_k f, D_k g) + \mathfrak{t}(D_k f, E_k g) + \mathfrak{t}(E_k f, D_k g) + \mathfrak{t}(E_k f, E_k g),\end{aligned}$$

and hence

$$\begin{aligned}&\mathfrak{t}(E_{k+1} f, E_{k+1} g) - \mathfrak{t}(E_k f, E_k g) \\ &\qquad = \mathfrak{t}(D_k f, D_k g) + \mathfrak{t}(D_k f, E_k g) + \mathfrak{t}(E_k f, D_k g).\end{aligned}$$

□

Remark 12.3.13. The upper bound $k < M$ imposed on the summation variables above is redundant: the condition that f and g are constant on all $Q \in \mathscr{D}_M$ implies that $D_k f = 0 = D_k g$ for $k \geqslant M$, so that the right side would remain unchanged if we allow the summations to run to infinity.

The final term in the expansion 12.36 is an error term, and can be controlled under the following mild conditions, which are obviously necessary for $\mathfrak{t}$ to define a bounded operator on L^p:

Definition 12.3.14. *We say that a bilinear* $\mathfrak{t} : S(\mathscr{D})^2 \to Z$ *satisfies*

(1) *the* weak boundedness property *if*

$$\|\mathfrak{t}(\mathbf{1}_Q, \mathbf{1}_Q)\|_Z \leqslant \|\mathfrak{t}\|_{wbp} |Q| \qquad \forall Q \in \mathscr{D};$$

(2) *the* adjacent weak boundedness property *if*

$$\|\mathfrak{t}(\mathbf{1}_Q, \mathbf{1}_{Q \dot{+} n})\|_Z \leqslant \|\mathfrak{t}\|_{awbp} |Q| \quad \forall Q \in \mathscr{D}, \ \forall n \in \{-1, 0, 1\}^d. \tag{12.37}$$

Lemma 12.3.15. *Let X, Y be Banach spaces, and let a bilinear $\mathfrak{t} : S(\mathscr{D})^2 \to \mathscr{L}(X, Y)$ satisfy the adjacent weak boundedness property. Then for all $f \in S(\mathscr{D}; X)$ and $g \in S(\mathscr{D}; Y)$, and all negative enough m, we have*

$$|\mathfrak{t}(E_m f, E_m g)| \leqslant 2^d \|\mathfrak{t}\|_{awbp} \|E_m f\|_{L^p(\mathbb{R}^d; X)} \|E_m g\|_{L^{p'}(\mathbb{R}^d; Y^*)} \underset{m \to -\infty}{\longrightarrow} 0.$$

Proof. We choose m so negative that the (bounded) supports of $f \in S(\mathscr{D}; X)$ and $g \in S(\mathscr{D}; Y^*)$ are both contained in the union of at most 2^d cubes $Q \in \mathscr{D}_m$ such that any two of them are related by $R = Q \dot{+} n$ for some $n \in \{-1, 0, 1\}^d$. We then have

$$\mathfrak{t}(E_m f, E_m g) = \sum_{Q,R\in\mathscr{D}_m} \mathfrak{t}(E_Q f, E_R g) = \sum_{Q,R\in\mathscr{D}_m} \mathfrak{t}(\langle f\rangle_Q \mathbf{1}_Q, \langle g\rangle_R \mathbf{1}_R),$$

and thus

$$\begin{aligned}
|\mathfrak{t}(E_m f, E_m g)| &\leqslant \sum_{Q,R\in\mathscr{D}_m} \|\mathfrak{t}(\mathbf{1}_Q, \mathbf{1}_R)\|_{\mathscr{L}(X,Y)} \|\langle f\rangle_Q\|_X \|\langle g\rangle_R\|_{Y^*} \\
&\leqslant \sum_{Q,R\in\mathscr{D}_m} \|\mathfrak{t}\|_{awbp} |Q| \|\langle f\rangle_Q\|_X \|\langle g\rangle_R\|_{Y^*} \\
&= \|\mathfrak{t}\|_{awbp} \sum_{Q\in\mathscr{D}_m} |Q|^{1/p} \|\langle f\rangle_Q\|_X \sum_{R\in\mathscr{D}_m} |R|^{1/p'} \|\langle g\rangle_R\|_{Y^*} \\
&\leqslant \|\mathfrak{t}\|_{awbp} 2^{d/p'} \Big(\sum_{Q\in\mathscr{D}_m} |Q| \|\langle f\rangle_Q\|_X^p \Big)^{1/p} 2^{d/p} \Big(\sum_{R\in\mathscr{D}_m} |R| \|\langle g\rangle_R\|_{Y^*}^{p'} \Big)^{1/p'} \\
&= 2^d \|\mathfrak{t}\|_{awbp} \|E_m f\|_{L^p(\mathbb{R}^d;X)} \|E_m g\|_{L^p(\mathbb{R}^d;Y^*)},
\end{aligned}$$

which is the claimed bound. □

The other terms in (12.36) can be identified with the various operators that we have studied in the previous sections:

Definition 12.3.16. *Let X, Y be Banach spaces, let $\mathfrak{t} : S(\mathscr{D})^2 \to \mathscr{L}(X, Y)$ be a bilinear form, and let $\mathfrak{t}(\cdot, \mathbf{1})$ and $\mathfrak{t}(\mathbf{1}, \cdot)$ be well-defined. We define the following operators associated with $\mathfrak{t}$:*

$$\mathfrak{H}_{\mathfrak{t}} := \sum_{\alpha,\gamma} \mathfrak{H}^{\alpha\gamma}_{\mathfrak{t}_0^{\alpha,\gamma}}, \quad \textit{where } \mathfrak{H}^{\alpha\gamma}_{\mathfrak{t}_0^{\alpha,\gamma}} \textit{ are Haar multipliers } (12.3),$$

$$T_{n,\mathfrak{t}} := \sum_{\alpha,\gamma\in\{0,1\}^d\setminus\{0\}} T^{\alpha\gamma}_{\phi_n,\mathfrak{t}_n^{\alpha,\gamma}}, \quad \textit{where } T^{\alpha\gamma}_{\phi_n,\mathfrak{t}_n^{\alpha,\gamma}} \textit{ are Figiel's operators } (12.14)$$

$$\textit{with } \begin{cases} \phi_n(Q) := Q\dot{+}n := Q + n\ell(Q), \\ \mathfrak{t}_n^{\alpha,\gamma}(Q) := \mathfrak{t}(h_Q^\alpha, h^\gamma_{Q\dot{+}n}), \end{cases}$$

$$U^i_{n,\mathfrak{t}} := \sum_{\alpha\in\{0,1\}^d\setminus\{0\}} U^\alpha_{\phi_n,\mathfrak{u}_n^{i,\alpha}}, \quad \textit{where } U^\alpha_{\phi_n,\mathfrak{u}_n^{i,\alpha}} \textit{ are Figiel's operators } (12.19),$$

$$\textit{with } \mathfrak{u}_n^{i,\alpha}(Q) := \begin{cases} \mathfrak{t}_n^{1,\alpha}(Q)^* := \mathfrak{t}(h^0_{Q\dot{+}n}, h^\alpha_Q)^*, & i = 1, \\ \mathfrak{t}_n^{2,\alpha}(Q) := \mathfrak{t}(h^\alpha_Q, h^0_{Q\dot{+}n}), & i = 2. \end{cases}$$

We also define the related paraproducts:

$\Pi^1_{\mathfrak{t}}$:= *paraproduct with coefficients* $\mathfrak{t}(\mathbf{1}, h_Q^\alpha)$,

$\Pi^2_{\mathfrak{t}}$:= *paraproduct with coefficients* $\mathfrak{t}(h_Q^\alpha, \mathbf{1})^*$,

$\Lambda_{\mathfrak{t}}$:= *bi-paraproduct with coefficients* $\pi_Q^{\alpha,1} = \mathfrak{t}(\mathbf{1}, h_Q^\alpha)$ *and* $\pi_Q^{\alpha,2} = \mathfrak{t}(h_Q^\alpha, \mathbf{1})$,

$\mathfrak{l}_{\mathfrak{t}}$:= *the bilinear form of* $\Lambda_{\mathfrak{t}}$.

We may drop the subscript $\mathfrak{t}$ from these notations if it is obvious from the context.

Remark 12.3.17. Our indexing of the operators $U^i_{n,\mathfrak{t}}$ may appear counterintuitive at first sight, as one might like to think of the operators $U^2_{n,\mathfrak{t}}$, which act on $f \in L^p(\mathbb{R}^d; X)$ with coefficients $\mathfrak{t}(h^\alpha_Q, h^0_{Q\dot{+}n}) \in \mathscr{L}(X, Y)$, as deserving to be the "primary" ones, rather than $U^1_{n,\mathfrak{t}}$, which act on the dual side $g \in L^{p'}(\mathbb{R}^d; Y^*)$ with adjoint coefficients $\mathfrak{t}(h^0_{Q\dot{+}n}, h^\alpha_Q)^* \in \mathscr{L}(Y^*, X^*)$. However, this indexing is chosen, since the operators $U^i_{n,\mathfrak{t}}$ naturally arise in parallel with the paraproducts Π_i of the same index $i \in \{1, 2\}$—see (12.42) and (12.43) below—, and it turns out to have some other advantages in the sequel.

With this notation, we can formula Figiel's decomposition of a bilinear form:

Proposition 12.3.18 (Figiel). *Let X, Y be Banach spaces, let $\mathfrak{t} : S(\mathscr{D})^2 \to \mathscr{L}(X, Y)$ be a bilinear form, and let $\mathfrak{t}(\cdot, \mathbf{1})$ and $\mathfrak{t}(\mathbf{1}, \cdot)$ be well-defined. For all*

$$f \in S(\mathscr{D}; X), \qquad g \in S(\mathscr{D}; Y^*), \qquad m \in \mathbb{Z},$$

denoting

$$u := (I - E_m)f \in S_{00}(\mathscr{D}; X), \qquad v := (I - E_m)g \in S_{00}(\mathscr{D}; Y^*),$$

we have the following identity with absolute convergence:

$$\begin{aligned}\mathfrak{t}(f, g) = \langle \mathfrak{H}_\mathfrak{t} u, g\rangle + \langle \Pi^1_\mathfrak{t} f, v\rangle + \langle u, \Pi^2_\mathfrak{t} g\rangle + \mathfrak{t}(E_m f, E_m g) +& \\ + \sum_{\substack{n\in\mathbb{Z}^d\\ n\neq 0}} \Big\{ \langle T_{n,\mathfrak{t}} u, g\rangle + \langle f, U^1_{n,\mathfrak{t}} v\rangle + \langle U^2_{n,\mathfrak{t}} u, g\rangle \Big\},& \end{aligned} \tag{12.38}$$

where the operators on the right are as in Definition 12.3.16. If these coefficients satisfy

$$\|\mathfrak{t}(\mathbf{1}, h^\alpha_Q)\|, \|\mathfrak{t}(h^\alpha_Q, \mathbf{1})\| \leqslant C|Q|^{1/2}, \tag{12.39}$$

then we have the further identity, with all terms below well defined:

$$\langle \Pi^1_\mathfrak{t} f, v\rangle + \langle u, \Pi^2_\mathfrak{t} g\rangle = \langle \Lambda_\mathfrak{t} f, g\rangle - \langle {}_m\Pi^1_\mathfrak{t} f, g\rangle - \langle f, {}_m\Pi^2_\mathfrak{t} g\rangle. \tag{12.40}$$

Remark 12.3.19. Since $\mathfrak{H}^{\alpha\gamma}_\lambda = T^{\alpha\gamma}_{\phi_0,\lambda}$, we could have incorporated the Haar multiplier into the second line of (12.38) as $\langle \mathfrak{H}_\mathfrak{t} u, g\rangle = \langle T_{0,\mathfrak{t}} u, g\rangle$. But we prefer to keep it separate, since its treatment will involve some differences compared to the rest of the $T_{n,\mathfrak{t}}$.

Proof of Proposition 12.3.18. We start with the identity (12.36) of Lemma 12.3.12. Since the sums are finitely nonzero, we are free rearrange as follows, observing that dyadic cubes Q, R of the same size are necessarily integer (times side-length) translates of each other:

$$\begin{aligned}\sum_{k\geqslant m} \mathfrak{t}(D_k f, D_k g) &= \sum_{k\geqslant m} \sum_{Q,R\in\mathscr{D}_k} \mathfrak{t}(D_Q f, D_R g) \\ &= \sum_{k\geqslant m} \sum_{Q\in\mathscr{D}_k} \sum_{n\in\mathbb{Z}^d} \mathfrak{t}(D_Q f, D_{Q\dot{+}n} g) = \sum_{\substack{Q\in\mathscr{D}\\ \ell(Q)\leqslant 2^{-m}}} \sum_{n\in\mathbb{Z}^d} \mathfrak{t}(D_Q f, D_{Q\dot{+}n} g)\end{aligned}$$

and we can also switch the order of the last two sums. Observing that $u = (I-E_m)f$ satisfies $D_Q u = D_Q f$ for $\ell(Q) \leqslant 2^{-m}$ and $D_Q u = 0$ for $\ell(Q) > 2^{-m}$, we find that, replacing f by u (and/or g by v) we can drop the restriction $\ell(Q) \leqslant 2^{-m}$ in the sum. Moreover, using the convention that summations over α and γ are always over the set $\{0,1\}^d \setminus \{0\}$,

$$\begin{aligned}\sum_{Q\in\mathscr{D}} \mathfrak{t}(D_Q u, D_{Q\dot{+}n} g) &= \sum_{\alpha,\gamma}\sum_{Q\in\mathscr{D}} \Big\langle \mathfrak{t}(h_Q^\alpha, h_{Q\dot{+}n}^\gamma)\langle h_Q^\alpha, u\rangle, \langle h_{Q\dot{+}n}^\gamma, g\rangle\Big\rangle \\ &= \sum_{\alpha,\gamma} \langle T^{\alpha,\gamma}_{\phi_n, \mathfrak{t}_n^{\alpha\gamma}} u, g\rangle = \langle T_n u, g\rangle.\end{aligned}$$

Hence

$$\sum_{k\geqslant m} \mathfrak{t}(D_k f, D_k g) = \sum_{n\in\mathbb{Z}^d} \langle T_n u, g\rangle = \langle \mathfrak{H} u, g\rangle + \sum_{\substack{n\in\mathbb{Z}^d\\ n\neq 0}} \langle T_n u, g\rangle \tag{12.41}$$

For the terms involving E_k, we begin in the same way but then introduce an additional twist to force some cancellation:

$$\begin{aligned}\sum_{k\geqslant m} \mathfrak{t}(D_k f, E_k g) = &\sum_{\substack{Q\in\mathscr{D}\\ \ell(Q)\leqslant 2^{-m}}} \sum_{n\in\mathbb{Z}^d} \mathfrak{t}(D_Q f, E_{Q\dot{+}n} g) \\ = &\sum_{\substack{Q\in\mathscr{D}\\ \ell(Q)\leqslant 2^{-m}}} \sum_{n\in\mathbb{Z}^d} \Big(\mathfrak{t}(D_Q f, \mathbf{1}_{Q\dot{+}n}(\langle g\rangle_{Q\dot{+}n} - \langle g\rangle_Q)) + \mathfrak{t}(D_Q f, \mathbf{1}_{Q\dot{+}n}\langle g\rangle_Q))\Big).\end{aligned}$$

The assumption that $\mathfrak{t}(\cdot, \mathbf{1})$ is well-defined guarantees the absolute convergence of

$$\sum_{n\in\mathbb{Z}^d} \mathfrak{t}(D_Q f, \mathbf{1}_{Q\dot{+}n}\langle g\rangle_Q) =: \mathfrak{t}(D_Q f, \langle g\rangle_Q).$$

Recalling that only finitely many $D_Q f$ with $\ell(Q) \leqslant 2^{-m}$ are non-zero, we also get the absolute convergence of

$$\sum_{\substack{Q\in\mathscr{D}\\ \ell(Q)\leqslant 2^{-m}}} \sum_{n\in\mathbb{Z}^d} \mathfrak{t}(D_Q f, \mathbf{1}_{Q\dot{+}n}\langle g\rangle_Q) = \sum_{\substack{Q\in\mathscr{D}\\ \ell(Q)\leqslant 2^{-m}}} \mathfrak{t}(D_Q f, \langle g\rangle_Q) =: \mathfrak{p}_m(f,g),$$

and hence, by triangle inequality, that of

$$\sum_{n\in\mathbb{Z}^d} \mathfrak{t}(D_Q f, \mathbf{1}_{Q\dot{+}n}(\langle g\rangle_{Q\dot{+}n} - \langle g\rangle_Q)).$$

Thus we can make the rearrangements

$$\sum_{k\geqslant m} \mathfrak{t}(D_k f, E_k g) = \sum_{\substack{n\in\mathbb{Z}^d\\ n\neq 0}} \sum_{\substack{Q\in\mathscr{D}\\ \ell(Q)\leqslant 2^{-m}}} \mathfrak{t}(D_Q f, \mathbf{1}_{Q\dot{+}n}(\langle g\rangle_{Q\dot{+}n} - \langle g\rangle_Q)) + \mathfrak{p}_m(f,g)$$

where adding the summation condition $n \neq 0$ was for free, since the factor $\langle g\rangle_{Q\dot{+}n} - \langle g\rangle_Q$ evidently vanishes when $n = 0$. Again, replacing f by u allows us to drop the restrictions to $\ell(Q) \leqslant 2^{-m}$ both in the sum spelled out above and in $\mathfrak{p}_m(f,g)$. Moreover,

$$\begin{aligned}
&\sum_{Q\in\mathscr{D}} \mathfrak{t}(D_Q u, \mathbf{1}_{Q\dot{+}n}(\langle g\rangle_{Q\dot{+}n} - \langle g\rangle_Q)) \\
&\qquad = \sum_{\alpha}\sum_{Q\in\mathscr{D}} \Big\langle \mathfrak{t}(h_Q^\alpha, h_{Q\dot{+}n}^0)\langle h_Q^\alpha, u\rangle, \langle h_{Q\dot{+}n}^0 - h_Q^0, g\rangle\Big\rangle \\
&\qquad = \sum_{\alpha} \langle U^\alpha_{\phi_n,\mathfrak{t}_n^{\alpha,0}} u, g\rangle = \langle U^2_{n,\mathfrak{t}} u, g\rangle.
\end{aligned}$$

Directly from the definitions, we also have

$$\begin{aligned}
\mathfrak{p}_m(f,g) &= \sum_{Q\in\mathscr{D}} \mathfrak{t}(D_Q u, \langle g\rangle_Q) \\
&= \sum_{Q\in\mathscr{D}}\sum_{\alpha\in\{0,1\}^d\setminus\{0\}} \mathfrak{t}(h_Q^\alpha\langle h_Q^\alpha, u\rangle, \langle g\rangle_Q) \\
&= \sum_{Q\in\mathscr{D}}\sum_{\alpha\in\{0,1\}^d\setminus\{0\}} \Big\langle \mathfrak{t}(h_Q^\alpha, \mathbf{1})\langle h_Q^\alpha, u\rangle, \langle g\rangle_Q\Big\rangle \\
&= \sum_{Q\in\mathscr{D}}\sum_{\alpha\in\{0,1\}^d\setminus\{0\}} \Big\langle \langle h_Q^\alpha, u\rangle, \mathfrak{t}(h_Q^\alpha, \mathbf{1})^*\langle g\rangle_Q\Big\rangle = \langle u, \Pi^2_{\mathfrak{t}} g\rangle.
\end{aligned}$$

In the computation above, the fact that $u \in S_{00}(\mathscr{D}; X)$ guarantees that all summations are finite, and the last step is simply the definition of the paraproduct via its action of the finitely non-zero Haar expansions in the dual space. Hence we have verified that

$$\sum_{k\geqslant m} \mathfrak{t}(D_k f, E_k g) = \sum_{\substack{n\in\mathbb{Z}^d\\ n\neq 0}} \langle U^2_{n,\mathfrak{t}} u, g\rangle + \langle u, \Pi^2_{\mathfrak{t}} g\rangle, \tag{12.42}$$

and the proof that

$$\sum_{k\geqslant m} \mathfrak{t}(E_k f, D_k g) = \sum_{\substack{n\in\mathbb{Z}^d\\ n\neq 0}} \langle f, U^1_{n,\mathfrak{t}} v\rangle + \langle \Pi^1_{n,\mathfrak{t}} f, v\rangle \tag{12.43}$$

is entirely analogous. Substituting the previous two identities and (12.41) into (12.36), we obtain the claimed (12.38).

Under the additional assumption (12.39), we know from Corollary 12.2.12 that $\langle \Pi^1_{\mathfrak{t}} f, g\rangle$ is well-defined and bilinear in $(f,g) \in S(\mathscr{D}; X) \times S(\mathscr{D}; Y^*)$, and hence

$$\langle \Pi^1_{\mathfrak{t}} f, v\rangle = \langle \Pi^1_{\mathfrak{t}} f, g\rangle - \langle \Pi^1_{\mathfrak{t}} f, E_m g\rangle = \langle \Pi^1_{\mathfrak{t}} f, g\rangle - \langle {}_m\Pi^1_{\mathfrak{t}} f, g\rangle.$$

Similarly, $\langle u, \Pi^2_{\mathfrak{t}} g\rangle = \langle f, \Pi^2_{\mathfrak{t}} g\rangle - \langle f, {}_m\Pi^2_{\mathfrak{t}} g\rangle$, and the previous two identities combine to give (12.40), noting that $\langle \Pi^1_{\mathfrak{t}} f, g\rangle + \langle f, \Pi^2_{\mathfrak{t}} g\rangle = \langle \Lambda_{\mathfrak{t}} f, g\rangle$. □

12.3.c Figiel's $T(1)$ theorem

The previous section culminated in Proposition 12.3.18, which established a decomposition of a generic bilinear form $\mathfrak{t} : S(\mathscr{D})^2 \to \mathscr{L}(X,Y)$ in terms of various fundamental operators. This is as far as it seems useful to proceed with identities, and we now turn to conditions that allow us to make meaningful estimates of the terms in the obtained decomposition. For this purpose, we introduce a certain family of norms. For a smooth discussion of a couple of closely related variants, it is convenient to adopt the following general framework.

Definition 12.3.20. *Let Z be a Banach space, and $\mathscr{P}(Z)$ the collection of all subsets of Z. We say that $\wp : \mathscr{P}(Z) \to [0,\infty]$ is a good set-bound on Z, if it satisfies the following properties for all $\mathscr{S}, \mathscr{T} \subseteq Z$:*

(1) *If $\mathscr{S} \subseteq \mathscr{T}$, then $\wp(\mathscr{S}) \leqslant \wp(\mathscr{T})$.*
(2) $\wp(\mathscr{S} \cup \mathscr{T}), \wp(\mathscr{S} + \mathscr{T}) \leqslant \wp(\mathscr{S}) + \wp(\mathscr{T})$.
(3) *If $\mathscr{Z} \subseteq \mathbb{K}$, then $\wp(\mathscr{Z}\mathscr{T}) \leqslant \sup_{z\in\mathscr{Z}} |z| \times \wp(\mathscr{T})$.*
(4) $\wp(\mathscr{T}) = \wp(\operatorname{conv} \mathscr{T}) = \wp(\operatorname{abs\,conv} \mathscr{T})$.
(5) $\wp(\mathscr{T}) = \wp(\overline{\mathscr{T}})$, *where $\overline{\mathscr{T}}$ denotes the norm-closure of $\mathscr{T}$.*

We primarily have in mind the following three cases:

Lemma 12.3.21. *Let X and Y be Banach spaces and $p \in [1,\infty)$. Then each of the following $\wp$ is a good set-bound on $Z = \mathscr{L}(X,Y)$:*

(a) $\wp = \mathscr{U}$, *where $\mathscr{U}(\mathscr{T}) := \sup\{\|T\| : T \in \mathscr{T}\}$,*
(b) $\wp = \mathscr{R}_p$, *the R-bound of order p,*
(c) $\wp = \mathscr{R}_p^*$, *the dual R-bound defined by*

$$\mathscr{R}_p^*(\mathscr{T}) := \mathscr{R}_p(\mathscr{T}^*), \qquad \mathscr{T}^* := \{T^* \in \mathscr{L}(Y^*, X^*) : T \in \mathscr{T}\}.$$

Proof. (a): The verification of the properties is immediate.

(b): Properties (1) and (2) for $\wp = \mathscr{R}_p$ are contained in the items with same numbers in Proposition 8.1.19. Property (3) follows from

$$\mathscr{R}_p(\mathscr{Z}\mathscr{T}) \leqslant \mathscr{R}_p(\mathscr{Z})\mathscr{R}_p(\mathscr{T}), \quad \mathscr{R}_p(\mathscr{Z}) = \sup_{z\in\mathscr{Z}} |z|,$$

where the first estimate is Proposition 8.1.19(3) and the second is immediate from Kahane's contraction principle (cf. the discussion right before Definition 8.1.1 of R-boundedness). Finally, properties (4) and (5) are contained in Propositions 8.1.21 and 8.1.22, respectively.

(c): All properties are direct corollaries of the corresponding properties in (b), since all set operations involved in these properties are well-behaved under the adjoint operation:

(1) $\mathscr{S} \subseteq \mathscr{T}$ if and only if $\mathscr{S}^* \subseteq \mathscr{T}^*$,

(2) $(\mathscr{S} \cup \mathscr{T})^* = \mathscr{S}^* \cup \mathscr{T}^*$ and $(\mathscr{S} + \mathscr{T})^* = \mathscr{S}^* + \mathscr{T}^*$,
(3) if $\mathscr{Z} \subseteq \mathbb{K}$, then $(\mathscr{Z}\mathscr{T})^* = \mathscr{Z}\mathscr{T}^*$,
(4) $(\operatorname{conv} \mathscr{T})^* = \operatorname{conv}(\mathscr{T}^*)$ and $(\operatorname{abs\,conv} \mathscr{T})^* = \operatorname{abs\,conv}(\mathscr{T}^*)$,
(5) $(\overline{\mathscr{T}})^* = \overline{\mathscr{T}^*}$.

□

Definition 12.3.22 (Figiel norms of a bilinear form). *For a bilinear form $\mathfrak{t} : S(\mathscr{D})^2 \to \mathscr{L}(X, Y)$, let $\mathfrak{t}_n^{\alpha\gamma}, \mathfrak{t}_n^{i,\alpha} : \mathscr{D} \to \mathscr{L}(X, Y)$ be the associated functions appearing in Proposition 12.3.18. For $s \geqslant 0$ and a good set-bound $\wp$ on $\mathscr{L}(X, Y)$, we define*

$$\begin{aligned}
\|\mathfrak{t}^\theta\|_{\mathrm{Fig}^s(\wp)} &:= \sum_{\substack{n\in\mathbb{Z}^d\\ n\neq 0}} (2+\log_2|n|)^s \wp(\mathfrak{t}_n^\theta), \qquad \theta \in \{(\alpha,\gamma),(i,\alpha)\},\\
\|\mathfrak{t}^{(0)}\|_{\mathrm{Fig}^s(\wp)} &:= \sum_{\alpha,\gamma\in\{0,1\}^d\setminus\{0\}} \|\mathfrak{t}^{\alpha\gamma}\|_{\mathrm{Fig}^s(\wp)},\\
\|\mathfrak{t}^{(i)}\|_{\mathrm{Fig}^s(\wp)} &:= \sum_{\alpha\in\{0,1\}^d\setminus\{0\}} \|\mathfrak{t}^{i,\alpha}\|_{\mathrm{Fig}^s(\wp)}, \qquad i \in \{1,2\},\\
\|\mathfrak{t}\|_{\mathrm{Fig}^s(\wp)} &:= \sum_{i=0}^{2} \|\mathfrak{t}^{(i)}\|_{\mathrm{Fig}^s(\wp)}.
\end{aligned}$$

When $\wp = \mathscr{U}$ is as in Lemma 12.3.21(a), we write $\mathrm{Fig}^s(\infty) := \mathrm{Fig}^s(\mathscr{U})$.

Remark 12.3.23. Referring to Proposition 12.3.18, one observes that the Figiel norms impose control on pairings $\mathfrak{t}(h_Q^\alpha, h_Q^\gamma)$, where at least one of the Haar functions is cancellative, i.e., $(\alpha, \gamma) \neq (0, 0)$. This is in contrast to the decay condition (12.35), where $\alpha = \gamma = 0$.

Since we also encountered the adjoint function $\mathfrak{u}_n^{1,\alpha}(Q) := (\mathfrak{t}_n^{1,\alpha}(Q))^*$, we recall the following results from the previous volumes:

Proposition 12.3.24. *Let X and Y be Banach spaces, $\mathscr{T} \subseteq \mathscr{L}(X, Y)$, and $p \in (1, \infty)$. If X is K-convex (resp. a UMD space), then*

$$\mathscr{R}_{p'}^*(\mathscr{T}) \leqslant K_{p,X}\mathscr{R}_p(\mathscr{T})\Big(\leqslant \beta_{p,X}^+ \mathscr{R}_p(\mathscr{T})\Big).$$

If Y is K-convex (resp. a UMD space), then

$$\mathscr{R}_p(\mathscr{T}) \leqslant K_{p,Y}\mathscr{R}_{p'}^*(\mathscr{T})\Big(\leqslant \beta_{p,Y}^+ \mathscr{R}_{p'}^*(\mathscr{T})\Big).$$

In particular, if both X and Y are K-convex (resp. UMD spaces), the set-bounds $\mathscr{R}_p$ and $\mathscr{R}_{p'}^*$ are equivalent on $\mathscr{L}(X, Y)$.

Proof. The first inequalities in both chains are restatements of bounds in Proposition 8.4.1, and we have $K_{p,Z} \leqslant \beta_{p,Z}^+$ by Proposition 4.3.10. □

Thanks to Proposition 12.3.24, we would not need to distinguish (when working in UMD spaces) between direct and adjoint R-boundedness conditions, as such assumptions are actually equivalent. Nevertheless, we choose to do so, for twofold reasons. First, as far as quantitative conclusions are concerned, we would lose a constant each time we pass to the dual side, whereas in many applications, verifying the R-boundedness of concrete operators is just as easy (or difficult) directly on the dual side, so that applying the general duality result for R-boundedness is unnecessary. Second, writing the adjoint bounds explicitly, where they are relevant, will hopefully better clarify the role of the different assumptions in the estimates.

In the following lemma, we observe that Figiel norm estimates, of the type we will need to assume any way, will also guarantee the well-definedness of $\mathfrak{t}(\cdot,\mathbf{1})$ and $\mathfrak{t}(\mathbf{1},\cdot)$), which allows us to drop these as separate assumptions in the sequel.

Lemma 12.3.25. *Let X and Y be Banach spaces, and let $\mathfrak{t} : S(\mathscr{D})^2 \to \mathscr{L}(X,Y)$ be a bilinear form. If $\|\mathfrak{t}^{(2)}\|_{\mathrm{Fig}^0(\infty)} < \infty$ (resp. $\|\mathfrak{t}^{(1)}\|_{\mathrm{Fig}^0(\infty)} < \infty$), then $\mathfrak{t}(\cdot,\mathbf{1})$ (resp. $\mathfrak{t}(\mathbf{1},\cdot)$) is well defined, and*

$$\begin{aligned} &\|\mathfrak{t}(h_Q^\alpha,\mathbf{1})\| \leqslant \|\mathfrak{t}^{2,\alpha}\|_{\mathrm{Fig}^0(\infty)}|Q|^{1/2}, \\ &\Big(\|\mathfrak{t}(\mathbf{1},h_Q^\alpha)\| \leqslant \|\mathfrak{t}^{1,\alpha}\|_{\mathrm{Fig}^0(\infty)}|Q|^{1/2}\Big). \end{aligned} \tag{12.44}$$

Proof. For every $Q \in \mathscr{D}$ and $\alpha \in \{0,1\}^d \setminus \{0\}$, we have

$$\begin{aligned} \sum_{\substack{R\in\mathscr{D}\\ \ell(R)=\ell(Q)}} \|\mathfrak{t}(h_Q^\alpha,\mathbf{1}_R)\| &= \sum_{n\in\mathbb{Z}^d} \|\mathfrak{t}(h_Q^\alpha,h^0_{Q\dot{+}n})\|\,|Q|^{1/2} \\ &\leqslant \sum_{n\in\mathbb{Z}^d} \|\mathfrak{t}_n^{\alpha,0}(Q)\|\,|Q|^{1/2} = \|\mathfrak{t}^{2,\alpha}\|_{\mathrm{Fig}^0(\infty)}|Q|^{1/2} < \infty, \end{aligned}$$

which shows both that $\mathfrak{t}(\cdot,\mathbf{1})$ is well defined and the related bound. The case of $\mathfrak{t}(\mathbf{1},\cdot)$ is analogous. □

Theorem 12.3.26 ($T(1)$ theorem for bilinear forms). *Let $p \in (1,\infty)$ and $1 \leqslant t_i \leqslant p \leqslant q_i \leqslant \infty$, $i = 0,1,2$, where $q_1 = \infty$ and $t_2 = 1$. Consider the following conditions:*

(i) *X and Y are UMD spaces;*
(ii) *X has cotype q_i and Y has type t_i, or one of them has both, for each $i = 0,1,2$,*
(iii) *$\mathfrak{t} : S(\mathscr{D})^2 \to \mathscr{L}(X,Y)$ is a bilinear form with*

$$\sum_{\alpha,\gamma} \mathscr{D}\mathscr{R}_p(\mathfrak{t}_0^{\alpha\gamma}) + \sum_{i=0}^{2} \|\mathfrak{t}^{(i)}\|_{\mathrm{Fig}^{\sigma_i}(\mathscr{R}_p)} < \infty,$$

where $\sigma_i := 1/t_i - 1/q_i$,

(iv) $\mathfrak{t}$ *satisfies the adjacent weak boundedness property.*

Under assumptions (i) *through* (iv)*, the bilinear form* $\mathfrak{t} - \mathfrak{l}_{\mathfrak{t}}$ *defines a bounded operator* $T - \Lambda_{\mathfrak{t}} \in \mathscr{L}(L^p(\mathbb{R}^d; X), L^p(\mathbb{R}^d; Y))$ *that satisfies*

(a) *the norm estimate:*

$$\|T - \Lambda_{\mathfrak{t}}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \leqslant \beta_{p,X}\beta_{p,Y}\Big\{\sum_{\alpha,\gamma} \mathscr{D}\mathscr{R}_p(\mathfrak{t}_0^{\alpha,\gamma}) + \\ + A_d \min_{i=1,2} C_{0,i}\|\mathfrak{t}^{(0)}\|_{\mathrm{Fig}^{\sigma_0}(\wp_i)} + B_d \sum_{i=1}^{2} C_{i,i}\|\mathfrak{t}^{(i)}\|_{\mathrm{Fig}^{\sigma_i}(\wp_i)}\Big\}$$

where $A_d := 6 \cdot (81)^d$*,* $B_d := 5\,200 \cdot (81)^d$*,* $\wp_1 := \mathscr{R}^*_{p'}$*,* $\wp_2 := \mathscr{R}_p$*, and*

$$C_{i,2} := C_{(12.15)}(X, Y, p, q_i, t_i), \qquad C_{i,1} := C_{(12.15)}(Y^*, X^*, p', t_i', q_i'),$$

(b) *the representation formula, with absolute convergence for all* $f \in L^p(\mathbb{R}^d; X)$ *and* $g \in L^{p'}(\mathbb{R}^d; Y^*)$*:*

$$\langle (T - \Lambda_{\mathfrak{t}})f, g\rangle = \langle \mathfrak{H}_{\mathfrak{t}} f, g\rangle + \sum_{\substack{n\in\mathbb{Z}^d\\ n\neq 0}} \Big(\langle T_{n,\mathfrak{t}} f, g\rangle + \langle f, U^1_{n,\mathfrak{t}} g\rangle + \langle U^2_{n,\mathfrak{t}} f, g\rangle\Big), \tag{12.45}$$

where the operators on the right are as in Definition 12.3.16.

Under assumptions (i) *through* (iii)*, the following conditions are equivalent:*

(1) $\mathfrak{t}$ *defines a bounded operator* $T \in \mathscr{L}(L^p(\mathbb{R}^d; X), L^p(\mathbb{R}^d; Y))$*;*
(2) $\mathfrak{t}$ *satisfies* (iv)*, and* $\mathfrak{l}_{\mathfrak{t}}$ *defines a bounded* $\Lambda_{\mathfrak{t}} \in \mathscr{L}(L^p(\mathbb{R}^d; X), L^p(\mathbb{R}^d; Y))$*.*

Under these equivalent conditions, we have both (a) *and* (b)*.*

Remark 12.3.27. The assumptions of Theorem 12.3.26 allow a certain trade-off between the Figiel norms that one imposes on the bilinear form $\mathfrak{t}$ on the one hand, and (co)type assumptions (and the size of the related constants) on the spaces X and Y on the other hand. Indeed, the norms $\|\ \|_{\mathrm{Fig}^{\sigma_i}}$ become smaller with decreasing $\sigma_i = 1/t_i - 1/q_i$, thus with increasing type t_i or decreasing cotype q_i, but at the same time the related constants $C_{(12.15)}$ may increase.

Let $1 \leqslant t \leqslant p \leqslant q \leqslant \infty$ and suppose that X has cotype q and Y has type t, or one of them has both. In Theorem 12.3.26, we will then choose $(t_1, q_1) = (t, \infty)$ and $(t_2, q_2) = (1, q)$; thus $\sigma_1 = 1/t$ and $\sigma_2 = 1/q'$. However, there are three prominent choices of the exponents t_0 and q_0:

(0) With $(t_0, q_0) = (t, q)$, we have

$$\sigma_0 = \frac{1}{t} - \frac{1}{q} \leqslant \min_{i=1,2} \sigma_i,$$

with strict inequality if both t and q are chosen to be non-trivial (as one always can for UMD spaces X and Y by Proposition 7.3.15). This shows that a strictly weaker condition is required on $\mathfrak{t}^{(0)}$ than on $\mathfrak{t}^{(i)}$ with $i = 1, 2$, but this seems to be largely a curiosity.

(1) With $(t_0, q_0) = (t_1, q_1) = (t, \infty)$, we have $\sigma_0 = \sigma_1$. Thus, we impose a stronger norm of $\mathfrak{t}^{(0)}$ than in case (0), but we achieve the following better constants in Theorem 12.3.26(a) under this choice:

$$C_{0,1} = C_{(12.15)}(Y^*, X^*, p', t', 1) = C_{1,1},$$

while an inspection of (12.15) shows that $C_{0,1}$ is larger than $C_{1,1}$ in general.

(2) Similarly, with $(t_0, q_0) = (t_2, q_2) = (1, q)$, we get

$$C_{0,2} = C_{(12.15)}(X, Y, p, q, 1) = C_{2,2}.$$

Using either choice (1) or (2) in Theorem 12.3.26, its key norm estimate admits the following form, under the assumption (we recall) that X has cotype q and Y has type t, or one of them has both,

$$\|T - \Lambda_{\mathfrak{t}}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \leqslant \beta_{p,X}\beta_{p,Y}\Big\{ \sum_{\alpha,\gamma} \mathscr{D}\mathscr{R}_p(\mathfrak{t}_0^{\alpha,\gamma}) + \\ + \sum_{i=1}^{2} C_i \Big(A_d \|\mathfrak{t}^{(0)}\|_{\mathrm{Fig}^{\sigma_i}(\wp_i)} + B_d \|\mathfrak{t}^{(i)}\|_{\mathrm{Fig}^{\sigma_i}(\wp_i)} \Big) \Big\},$$

where $\sigma_1 = 1/t$, $\sigma_2 = 1/q'$, and

$$C_1 := C_{(12.15)}(Y^*, X^*, p', t', 1), \quad C_2 := C_{(12.15)}(X, Y, p, q, 1).$$

Proof of Theorem 12.3.26. The core of the proof will consist of establishing claims (a) and (b) under the full set of assumptions (i) through (iv). Assuming that this is already done, let us see how to conclude the rest of the proof.

The equivalence of (1) and (2) is asserted under the assumptions (i) through (iii) only. However, the adjacent weak boundedness property (iv) is clearly necessary for (1) and it is explicitly assumed in (2), so we can assume that this condition is satisfied in any case, and so we are in fact working under the full set of assumptions (i) through (iv) also in this remaining part of the proof. Thus the consequences (a) and (b) of this assumption are valid. In particular, since the bilinear form $\mathfrak{t} - \mathfrak{l}$ defines a bounded operator under this assumption, it is clear that $\mathfrak{t}$ defines a bounded operator if and only if $\mathfrak{l}$ does.

We then turn to the actual proof of (a) and (b) under the assumptions (i) through (iv). From Lemma 12.3.25, we get that $\mathfrak{t}(\cdot, \mathbf{1})$ and $\mathfrak{t}(\mathbf{1}, \cdot)$, and hence the two paraproducts, are well defined, and their coefficients satisfy the bounds (12.44). For $f \in S(\mathscr{D}; X)$ and $g \in S(\mathscr{D}; Y^*)$, we then have both identities (12.38) and (12.40) provided by Proposition 12.3.18. Combined together, they read as

$$\mathfrak{t}(f, g) = \langle \mathfrak{H} u_m, g\rangle + \langle \Lambda f, g\rangle + \mathscr{E}_m(f, g) + \\ + \sum_{\substack{n\in\mathbb{Z}^d\\ n\neq 0}} \Big\{ \langle T_n u_m, g\rangle + \langle f, U_n^1 v_m\rangle + \langle U_n^2 u_m, g\rangle \Big\}, \tag{12.46}$$

where $u_m := (I - E_m)f \in S_{00}(\mathscr{D}; X)$, $v_m := (I - E_m)g \in S_{00}(\mathscr{D}; Y^*)$, and the error term

$$\mathscr{E}_m(f, g) = \langle {}_m\Pi_1 f, g\rangle + \langle f, {}_m\Pi_2 g\rangle + \mathfrak{t}(E_m f, E_m g)$$

satisfies

$$\begin{aligned} |\mathscr{E}_m(f, g)| \leqslant \Big(c_{d,p} \sum_{i=1}^{2} \|\mathfrak{t}^{(i)}\|_{\mathrm{Fig}^0(\infty)} + 2^d \|\mathfrak{t}\|_{awbp}\Big) \times \\ \times \|E_m f\|_{L^p(\mathbb{R}^d;X)} \|E_m g\|_{L^p(\mathbb{R}^d;Y^*)} \underset{m\to-\infty}{\longrightarrow} 0 \end{aligned} \tag{12.47}$$

by Lemmas 12.2.11 and 12.3.25 for the paraproduct terms and Lemma 12.3.15 for both the final term and the limit.

Directly from Theorem 12.1.11, we deduce that

$$\begin{aligned} |\langle \mathfrak{H} u_m, g\rangle| &\leqslant \sum_{\alpha,\gamma} |\langle \mathfrak{H}^{\alpha\gamma}_{\mathfrak{t}_0^{\alpha,\gamma}} u_m, g\rangle| \\ &\leqslant \beta^+_{p,X} \beta^+_{p',Y^*} \sum_{\alpha,\gamma} \mathscr{D}\mathscr{R}_p(\mathfrak{t}_0^{\alpha,\gamma}) \|u_m\|_p \|g\|_p, \end{aligned} \tag{12.48}$$

where, and in the rest of the proof, we abbreviate

$$\|\ \|_p := \|\ \|_{L^p(\mathbb{R}^d;X)}, \qquad \|\ \|_{p'} := \|\ \|_{L^{p'}(\mathbb{R}^d;Y^*)}.$$

Note that $\phi_n(Q) := Q \dot{+} n$ satisfies $\phi_n(Q) \subseteq 3Q^{(N)}$ provided that $|n| \leqslant 2^N$; thus in particular for $N = \lceil \log_2^+ |n| \rceil$; this is relevant in view of applying Corollary 12.1.27 and Theorem 12.1.28. From Corollary 12.1.27, we deduce that

$$\begin{aligned} |\langle T_n u_m, g\rangle| &\leqslant \sum_{\alpha,\gamma} |\langle T^{\alpha\gamma}_{\phi_n, \mathfrak{t}_n^{\alpha,\gamma}} u_m, g\rangle| \\ &\leqslant A_d \beta_{p,X} \beta_{p,Y} (2 + \log_2 |n|)^{1/t_0 - 1/q_0} \min_{i=1,2} C_{0,i} \wp_i(\mathfrak{t}_n^{\alpha,\gamma}) \|u_m\|_p \|g\|_{p'} \end{aligned}$$

using the notation of the statement of the theorem that we are proving. Hence

$$\sum_{\substack{n\in\mathbb{Z}^d \\ n\neq 0}} |\langle T_n u_m, g\rangle| \leqslant A_d \beta_{p,X} \beta_{p,Y} \min_{i=1,2} C_{0,i} \|\mathfrak{t}^{(0)}\|_{\mathrm{Fig}^{1/t_0 - 1/q_0}(\wp_i)} \|u_m\|_p \|g\|_{p'}$$

Similarly, recalling that $t_2 := 1$, Theorem 12.1.28 guarantees that

$$\begin{aligned} |\langle U_n^2 u_m, g\rangle| &\leqslant \sum_{\alpha} |\langle U^{\alpha}_{\phi_n, \mathfrak{t}_n^{2,\alpha}} u_m, g\rangle| \\ &\leqslant B_d \beta_{p,X} \beta_{p,Y} (2 + \log_2 |n|)^{1/t_2 - 1/q_2} \sum_{\alpha} C_{2,2} \wp_2(\mathfrak{t}_n^{2,\alpha}) \|u_m\|_p \|g\|_{p'} \end{aligned}$$

in the notation of the theorem, and hence

$$\sum_{\substack{n\in\mathbb{Z}^d\\ n\neq 0}} |\langle U_n^2 u_m, g\rangle| \leqslant B_d \beta_{p,X}\beta_{p,Y} C_{2,2} \|\mathfrak{t}^{(2)}\|_{\mathrm{Fig}^{1/t_2-1/q_2}(\wp_2)} \|u_m\|_p \|g\|_{p'}.$$

For the term $\langle f, U_n^1 v_m\rangle$, we again apply Theorem 12.1.28 but on the dual side, with X, Y, p replaced by Y^*, X^*, p'. By assumption, Y has type $t_1 \leqslant p$, and hence Y^* has cotype $t_1' \geqslant p'$ by Proposition 7.1.13. So we can indeed apply Theorem 12.1.28 with X, Y, p, q replaced by Y^*, X^*, p', t_1'. Recalling that $q_1 := \infty$, and noting that $1 - 1/t_1' = 1/t_1 = 1/t_1 - 1/q_1$, this gives

$$\begin{aligned} |\langle f, U_n^1 v_m\rangle| &\leqslant \sum_\alpha |\langle f, U^\alpha_{\phi_n,(\mathfrak{t}_n^{1,\alpha})^*} v_m\rangle| \\ &\leqslant B_d \beta_{p',X^*}\beta_{p',Y^*}(2+\log_2|n|)^{1/t_1-1/q_1} \|f\|_p \|v_m\|_{p'} \times \\ &\qquad \times C(Y^*, X^*, p', t_1') \mathscr{R}_{p'}((\mathfrak{t}_n^{1,\alpha})^*), \end{aligned}$$

where $\beta_{p',X^*}\beta_{p',Y^*} = \beta_{p,X}\beta_{p,Y}$ and

$$C(Y^*, X^*, p', t_1') \mathscr{R}_{p'}((\mathfrak{t}_n^{1,\alpha})^*) = C_{1,1} \mathscr{R}^*_{p'}(\mathfrak{t}_n^{1,\alpha}) = C_{1,1}\wp_1(\mathfrak{t}_n^{1,\alpha})$$

in the notation of the theorem. Hence

$$\sum_{\substack{n\in\mathbb{Z}^d\\ n\neq 0}} |\langle f, U_n^1 v_m\rangle| \leqslant B_d \beta_{p,X}\beta_{p,Y} C_{1,1} \|\mathfrak{t}^{(1)}\|_{\mathrm{Fig}^{1/t_1-1/q_1}(\wp_1)} \|f\|_p \|v_m\|_{p'}.$$

Noting that $\|u_m\|_p \leqslant 2\|f\|_p$ and $\|v_m\|_{p'} \leqslant 2\|g\|_{p'}$, and using the assumption about $\|\mathfrak{t}^{(i)}\|_{\mathrm{Fig}^{1/t_i-1/q_i}(\mathscr{R}_p)}$ (combined with Proposition 12.3.24 in the case of $\mathscr{R}_{p'}((\mathfrak{t}_n^{1,\alpha})^*)$), it follows that the series in (12.46) are term-wise and uniformly in m dominated by absolutely convergent series. This allows us to pass to the limit $m \to -\infty$ in (12.46) with dominated convergence to deduce that

$$(\mathfrak{t} - \mathfrak{l})(f, g) = RHS(12.45) \qquad \forall\ f \in S(\mathscr{D}; X),\ g \in S(\mathscr{D}; Y^*). \tag{12.49}$$

Taking the same limit in the term-wise bounds above, we obtain

$$\begin{aligned} |(\mathfrak{t} - \mathfrak{l})(f, g)| &= |\mathfrak{t}(f, g) - \langle \Lambda f, g\rangle| \\ &\leqslant \beta_{p,X}\beta_{p,Y}\Big\{ \sum_{\alpha,\gamma} \mathscr{D}\mathscr{R}_p(\mathfrak{t}_0^{\alpha,\gamma}) + A_d \min_{i=1,2} C_{0,i} \|\mathfrak{t}^{(0)}\|_{\mathrm{Fig}^{1/t_0-1/q_0}(\wp_i)} \\ &\qquad + B_d \sum_{i=1}^2 C_{i,i} \|\mathfrak{t}^{(i)}\|_{\mathrm{Fig}^{1/t_i-1/q_i}(\wp_i)} \Big\} \|f\|_{L^p(\mathbb{R}^d;X)} \|g\|_{L^{p'}(\mathbb{R}^d;Y^*)} \end{aligned} \tag{12.50}$$

again for all $f \in S(\mathscr{D}; X)$ and $g \in S(\mathscr{D}; Y^*)$, where A_d, B_d and C_i are as in the statement of the Theorem.

This estimate shows that the bilinear form $\mathfrak{t} - \mathfrak{l}$ satisfies a relevant *a priori* bound, and hence defines an operator $T - \Lambda \in \mathscr{L}(L^p(\mathbb{R}^d; X), L^p(\mathbb{R}^d; Y))$. By density, it is immediate that (12.50) remains valid with general $f \in L^p(\mathbb{R}^d; X)$ and $g \in L^{p'}(\mathbb{R}^d; Y^*)$, and this proves the claimed norm bound (a) for $T - \Lambda$.

We can then replace $(\mathfrak{t}-\mathfrak{l})(f,g)$ by $\langle (T-\Lambda)f,g\rangle$ in (12.49). Approximating general $f \in L^p(\mathbb{R}^d;X)$ and $g \in L^{p'}(\mathbb{R}^d;Y^*)$ by functions as in (12.49), and using dominated convergence and the term-wise bounds recorded above, this proves the representation (b). This completes the proof of the claims under the assumption that $\mathfrak{t}$ satisfies the adjacent weak boundedness property. □

12.3.d Improved estimates via random dyadic cubes

A feature of Theorem 12.3.26 is that it deals with a bilinear form adapted to a fixed system of dyadic cubes $\mathscr{D}$. This is an advantage in applications to questions of intrinsically dyadic nature. But it is also a certain limitation in view of applications to non-dyadic questions, in that the assumptions of Theorem 12.3.26 fail to take advantage of possible information about non-dyadic cubes. For example, with some effort, one could use Theorem 12.3.26 to re-derive the boundedness of the Hilbert transform on $L^p(\mathbb{R};X)$, which we proved in a different way in Theorem 5.1.13. However, the conclusion derived from Theorem 12.3.26 would be quantitatively weaker, in terms of the dependence on the UMD constant $\beta_{p,X}$, which was quadratic in Theorem 5.1.13. For $X=Y$, Theorem 12.3.26 also features the explicit factor $\beta_{p,X}^2$, but there is another $\beta_{p,X}$ implicit in the constants $C_{(12.15)}$. On the other hand, it is evident that, for $\mathfrak{t}(f,g) := \langle Hf,g\rangle$, there is no difference in estimating $\mathfrak{t}(h_I^\alpha, h_K^\gamma)$ for dyadic or non-dyadic intervals I,J. But Theorem 12.3.26, as formulated, makes no use of this additional information.

We now wish derive to variant of Theorem 12.3.26 to address these issues. First of all, we need a straightforward generalisation to $\mathbb{R}^d$ of the random dyadic systems that we used in the one-dimensional case in Section 5.1.

Lemma 12.3.28. *Let $\mathscr{D}$ be a fixed dyadic system on $\mathbb{R}^d$, in the sense of Definition 11.1.6.*

(1) *For every $\omega = (\omega_j)_{j\in\mathbb{Z}^d} \in (\{0,1\}^d)^{\mathbb{Z}}$,*

$$\mathscr{D}^\omega := \{Q\dot{+}\omega : Q \in \mathscr{D}\}$$

is another dyadic system on $\mathbb{R}^d$, where

$$Q\dot{+}\omega := Q + \ell(Q,\omega), \quad \ell(Q,\omega) := \sum_{j:2^{-j}<\ell(Q)} 2^{-j}\omega_j.$$

(2) *Conversely, every dyadic system $\mathscr{D}'$ has this form for some $\omega \in (\{0,1\}^d)^{\mathbb{Z}}$.*

Proof. Let $\mathscr{D}^0$ be the standard dyadic system, and consider a family of shifts $s_j + \mathscr{D}_j^0$. These clearly satisfy property (i) of Definition 11.1.6. A necessary and sufficient condition for them to satisfy (ii) of Definition 11.1.6 is that $s_j - s_{j+1} \in 2^{-j-1}\mathbb{Z}^d$.

If $\mathscr{D}$ is a dyadic system defined by shifts s_j, then $\mathscr{D}^\omega$ is defined by the shifts $s_j + \omega_{(j)}$, where

$$\omega_{(j)} := \sum_{k>j} \omega_k 2^{-k}.$$

These satisfy $(s_j+\omega_{(j)})-(s_{j+1}-\omega_{(j+1)}) = (s_j-s_{j+1})+\omega_{j+1}2^{-j-1} \in 2^{-j-1}\mathbb{Z}^d$, and hence $\mathscr{D}^\omega$ is also a dyadic system, as claimed in (1).

Then suppose that $\mathscr{D}$ and $\mathscr{D}'$ are two dyadic systems defined by shifts s_j and s'_j, respectively. It is clear that the family $\mathscr{D}_j = s_j + \mathscr{D}_j^0$ only depends on $s_j \mod 2^{-j}$, and hence we may assume without loss of generality that both $s_j \in [0, 2^{-j})^d$ and $t_j := s'_j - s_j \in [0, 2^{-j})^d$. Since both $s_j - s_{j+1} \in 2^{-j-1}\mathbb{Z}^d$ and $s'_j - s'_{j+1} \in 2^{-j-1}\mathbb{Z}^d$, it follows that also $t_j - t_{j+1} \in 2^{-j-1}\mathbb{Z}^d$. Together with the fact that $t_j \in [0, 2^{-j})^d$ and $t_{j+1} \in [0, 2^{-j-1})^d$, one finds that in fact $t_j - t_{j+1} \in 2^{-j-1}\{0,1\}^d$. Denoting $\omega_{j+1} := 2^{j+1}(t_j - t_{j+1}) \in \{0,1\}^d$, we obtain

$$t_j = t_{j+1} + 2^{-j-1}\omega_{j+1} = \ldots = \sum_{k>j} 2^{-k}\omega_k = \omega_{(j)},$$

and then

$$\mathscr{D}'_j = s'_j + \mathscr{D}_j^0 = t_j + s_j + \mathscr{D}_j^0 = \omega_{(j)} + \mathscr{D}_j = \mathscr{D}_j^\omega,$$

as claimed in (2), and this completes the proof. □

Definition 12.3.29. *For $\omega = (\omega_j)_{j\in\mathbb{Z}} \in (\{0,1\}^d)^{\mathbb{Z}}$, let*

$$j_\omega := \sup\{j \in \mathbb{Z} : \omega_j \neq 0\} \in \mathbb{Z} \cup \{-\infty, \infty\},$$
$$(\{0,1\}^d)_0^{\mathbb{Z}} := \Big\{\omega \in (\{0,1\}^d)^{\mathbb{Z}} : j_\omega < \infty\Big\}.$$

We say that $\omega \in (\{0,1\}^d)^{\mathbb{Z}}$ is eventually zero *if $\omega \in (\{0,1\}^d)_0^{\mathbb{Z}}$.*

Lemma 12.3.30. *For every $\omega \in (\{0,1\}^d)_0^{\mathbb{Z}}$, we have*

$$S(\mathscr{D}^\omega) = S(\mathscr{D}), \qquad S_0(\mathscr{D}^\omega) = S_0(\mathscr{D}).$$

Moreover, there exists an $\omega \in (\{0,1\}^d)_0^{\mathbb{Z}}$ such that $S_{00}(\mathscr{D}^\omega) = S_0(\mathscr{D})$.

Proof. Recall that $S(\mathscr{D})$ is the span of indicators $\mathbf{1}_Q$ of $Q \in \mathscr{D}$. Since every $Q \in \mathscr{D}_j$ can be written as a union of smaller cubes $Q' \in \mathscr{D}_k$, for any $k > j$, we see that, for any given $j_0 \in \mathbb{Z}$, the space $S(\mathscr{D})$ only depends on $\bigcup_{j>j_0} \mathscr{D}_j$. On the other hand, if ω is eventually zero, and j_ω is as in the definition of this property, then $\mathscr{D}_j^\omega = \mathscr{D}_j$ for $j > j_\omega$. The first claimed identity thus follows.

The second identity follows by restricting to functions of vanishing integral on both sides.

Finally, it is easy to choose $\omega \in (\{0,1\}^d)_0^{\mathbb{Z}}$ in such a way that $\mathscr{D}^\omega$ contains an increasing sequence of cubes that exhausts all $\mathbb{R}^d$. Then, given any $f \in S(\mathscr{D})$, we can find some $Q_0 \in \mathscr{D}^\omega$ that contains the support of f. If, in addition, $f \in S_0(\mathscr{D}) = S_0(\mathscr{D}^\omega)$, then f can be expanded in terms of finitely many Haar functions h_Q^α with $Q \subseteq Q_0$, and thus $f \in S_{00}(\mathscr{D}^\omega)$. Since this holds for every $f \in S_0(\mathscr{D})$, we obtain the final identity. □

Remark 12.3.31. Without the assumption of eventually zero, the conclusion of Lemma 12.3.30 fails in general. For instance, the indicator of the shifted dyadic interval $\frac{1}{3}+[0,1)$ cannot be expressed as a finite linear combination of standard dyadic intervals.

Thanks to Lemma 12.3.30, any bilinear form $\mathfrak{t} : S(\mathscr{D})^2 \to Z$ may also be regarded as a bilinear form $\mathfrak{t} : S(\mathscr{D}^\omega)^2 \to Z$ for every eventually zero ω. Although the objects in fact coincide, it will be convenient to denote the latter by $\mathfrak{t}^\omega$. This is particularly relevant when considering the various auxiliary objects derived from the bilinear form. In particular, extending the notation from Proposition 12.3.18, we have

$$\mathfrak{t}_n^{\omega;\alpha,\gamma}(R) := \mathfrak{t}(h_R^\alpha, h_{R\dot{+}n}^\gamma), \qquad R = Q\dot{+}\omega \in \mathscr{D}^\omega$$

$$\mathfrak{u}_n^{\omega;i,\alpha}(R) := \begin{cases} \mathfrak{t}_n^{\omega;1,\alpha}(R)^* := \mathfrak{t}(h_{R\dot{+}n}^0, h_R^\alpha)^*, & i=1, \\ \mathfrak{t}_n^{\omega;2,\alpha}(R) := \mathfrak{t}(h_R^\alpha, h_{R\dot{+}n}^0)^*, & i=2. \end{cases}$$

The advantage of considering several dyadic systems $\mathscr{D}^\omega$ is that this allows us to dispense with some of the cubes within each $\mathscr{D}^\omega$.

Definition 12.3.32. *For a dyadic system $\mathscr{D}$ and $k \in \mathbb{Z}_{\geqslant 2}$, a cube $Q \in \mathscr{D}$ is called k-good (in $\mathscr{D}$) if*

$$\operatorname{dist}(R, \complement R^{(k)}) \geqslant \frac{1}{4}\ell(R^{(k)}) = 2^{k-2}\ell(R),$$

where $R^{(k)}$ is the kth dyadic ancestor of R in $\mathscr{D}$.

Lemma 12.3.33. *Consider a random choice of $\omega \in (\{0,1\}^d)^{\mathbb{Z}_{\geqslant M}}$ with respect to the uniform probability on this space. For every $Q \in \mathscr{D}$ with $\ell(Q) \geqslant 2^{-M}$,*

(1) *the random set $Q\dot{+}\omega$ and the event $\{Q\dot{+}\omega$ is k-good in $\mathscr{D}^\omega\}$ are independent;*
(2) $\mathbb{P}(Q\dot{+}\omega$ is k-good in $\mathscr{D}^\omega) = 2^{-d}$.

Proof. (1) follows by observing that $Q\dot{+}\omega$ depends only on ω_j with $2^{-M} \leqslant 2^{-j} < \ell(Q)$, whereas $\{Q\dot{+}\omega$ is k-good in $\mathscr{D}^\omega\}$ depends on the relative position of $Q\dot{+}\omega$ with respect to cubes $R\dot{+}\omega$ with $\ell(R) = 2^k\ell(Q)$, which in turn depends on ω_j with $\ell(Q) \leqslant 2^{-j} < 2^k\ell(Q)$.

(2): When all ω_j with $\ell(Q) \leqslant 2^{-j} < 2^k\ell(Q)$ are independently chosen from $\{0,1\}^d$, it is easy to see that the probability of $\{Q\dot{+}\omega$ is k-good in $\mathscr{D}^\omega\}$ is equal to the geometric probability (i.e., the relative volume) of the "good region"

$$R_{\text{good}} := \Big\{ s \in R : \operatorname{dist}(s, \complement R) \geqslant \frac{1}{4}\ell(R) \Big\} = \frac{1}{2}\bar{R}$$

of the $\mathscr{D}^\omega$-ancestor R of Q, and this is simply

$$\frac{|R_{\text{good}}|}{|R|} = \frac{|\frac{1}{2}\bar{R}|}{|R|} = 2^{-d}.$$

□

Definition 12.3.34. *For $\theta \in \{(\alpha,\gamma),(i,\alpha)\}$, and $n \in \mathbb{Z}^d \setminus \{0\}$, we define*

$$\begin{aligned} \mathfrak{t}^{\omega;\theta}_{n,\mathrm{good}}(R) &:= \mathbf{1}_{\{R \text{ is } k(n)\text{-good in } \mathscr{D}^\omega\}} \mathfrak{t}^{\omega;\theta}_n(R), \\ k(n) &:= 2 + \lceil \log_2 |n| \rceil. \end{aligned}$$

We define Figiel's operators $T^{\mathrm{good}}_{n,\mathfrak{t}^\omega}$ and $U^{i,\mathrm{good}}_{m,\mathfrak{t}^\omega}$ as in Definition 12.3.16, but with $\mathfrak{t}^{\omega;\theta}_{n,\mathrm{good}}$ in place of the respective $\mathfrak{t}^\theta_n$

For $n \in \mathbb{Z}^d \setminus \{0\}$, we have $k(n) \geqslant 2$, and hence the notion of "$k(n)$-good" is well-defined. For $n = 0$ we would formally get $k(0) = -\infty$, and "$-\infty$-good" reduces to the triviality $\mathrm{dist}(R, \complement R) \geqslant 0$; accordingly, for definiteness, we let

$$\mathfrak{t}^{\omega;\theta}_{0,\mathrm{good}}(R) := \mathfrak{t}^{\omega;\theta}_0(R).$$

Replacing all quantities in Definition 12.3.22 by their "good" restrictions, we have a natural definition of the Figiel norms

$$\begin{aligned} &\|\mathfrak{t}^{\omega;\theta}_{\mathrm{good}}\|_{\mathrm{Fig}^s(\wp)}, \quad \theta \in \{(\alpha,\gamma),(i,\alpha)\}, \\ &\|\mathfrak{t}^{\omega;(i)}_{\mathrm{good}}\|_{\mathrm{Fig}^s(\wp)}, \quad i = 1,2, \qquad \|\mathfrak{t}^{\omega}_{\mathrm{good}}\|_{\mathrm{Fig}^s(\wp)}. \end{aligned}$$

As we are about to see, these good parts will suffice to control a bounded extension of the form $\mathfrak{t}$, and this also allows us to obtain a better dependence on the UMD constants. Here is the precise statement:

Theorem 12.3.35 ($T(1)$ theorem for bilinear forms, random version). *Let $p \in (1,\infty)$ and $1 \leqslant t \leqslant p \leqslant q \leqslant \infty$, and consider the conditions:*

(i) *X and Y are UMD spaces,*
(ii) *X has cotype q and Y has type t, or one of them has both,*
(iii) *$\mathfrak{t} : S(\mathscr{D})^2 \to \mathscr{L}(X,Y)$ is a bilinear form with*

$$\sum_{\alpha,\gamma \in \{0,1\}^d \setminus \{0\}} \mathscr{D}\mathscr{R}_p(\mathfrak{t}^{\omega;\alpha,\gamma}_0) + \min_{i=1,2} \|\mathfrak{t}^{\omega;(0)}\|_{\mathrm{Fig}^{\sigma_i}(\mathscr{R}_p)} + \sum_{i=1}^2 \|\mathfrak{t}^{\omega;(i)}\|_{\mathrm{Fig}^{\sigma_i}(\mathscr{R}_p)} \leqslant C,$$

uniformly in $\omega \in (\{0,1\}^d)^{\mathbb{Z}}_0$, where $\sigma_1 = 1/t$ and $\sigma_2 = 1/q'$.

(iv) *the forms $\mathfrak{t}^\omega$ satisfy the adjacent weak boundedness property $\|\mathfrak{t}^\omega\|_{awbp} \leqslant C$ uniformly in $\omega \in (\{0,1\}^d)^{\mathbb{Z}}_0$,*

Under assumptions (i) through (iii), the following conditions are equivalent:

(1) *$\mathfrak{t}$ defines a bounded linear operator $T \in \mathscr{L}(L^p(\mathbb{R}^d;X), L^p(\mathbb{R}^d;Y))$;*
(2) *$\mathfrak{t}$ satisfies (iv) and the paraproducts $\Lambda_{\mathfrak{t}^\omega}$ are uniformly bounded.*

Under these equivalent conditions, we have:

(a) *the norm estimate:*

$$\begin{aligned}\|T\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))}&\\ \leqslant \sup_{\omega}\|\varLambda_{\mathfrak{t}^\omega}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))}&+\beta_{p,X}\beta_{p,Y}\Big\{\sup_{\omega}\sum_{\alpha,\gamma}\mathscr{D}\mathscr{R}_p(\mathfrak{t}_0^{\omega;\alpha,\gamma})\\ &+12\cdot 2^d\sup_{\omega}\Big(\min_{i=1,2}c_i\|\mathfrak{t}_{\mathrm{good}}^{\omega;(0)}\|_{\mathrm{Fig}^{\sigma_i}(\wp_i)}+\sum_{i=1}^{2}c_i\|\mathfrak{t}_{\mathrm{good}}^{\omega;(i)}\|_{\mathrm{Fig}^{\sigma_i}(\wp_i)}\Big)\Big\},\end{aligned}$$

where the suprema are over $\omega\in(\{0,1\}^d)_0^{\mathbb{Z}}$*, and*

$$\begin{aligned}&\wp_1:=\mathscr{R}_{p'}^*,\quad \wp_2:=\mathscr{R}_p,\quad \sigma_1:=1/t,\quad \sigma_2:=1/q',\\ &c_1:=\min_{Z=X,Y}c_{t',Z^*;p'},\quad c_2:=\min_{Z=X,Y}c_{q,Z;p};\end{aligned}\tag{12.51}$$

(b) *the representation formula*

$$\begin{aligned}\langle Tf,g\rangle=\mathbb{E}\Big(\langle\mathfrak{H}_{\mathfrak{t}^\omega}f,g\rangle+\langle\varLambda_{\mathfrak{t}^\omega}f,g\rangle+2^d\sum_{\substack{n\in\mathbb{Z}^d\\ n\neq 0}}\Big\{\langle T_{n,\mathfrak{t}^\omega}^{\mathrm{good}}f,g\rangle+&\\ +\langle f,U_{n,\mathfrak{t}^\omega}^{1,\mathrm{good}}g\rangle+\langle U_{n,\mathfrak{t}^\omega}^{2,\mathrm{good}}f,g\rangle\Big\}\Big),&\end{aligned}\tag{12.52}$$

with absolute convergence for all $f\in S(\mathscr{D};X)$ *and* $g\in S(\mathscr{D};Y^*)$*, where* $\mathbb{E}$ *is the expectation over* $\omega\in(\{0,1\}^d)^{\mathbb{Z}_{\leqslant M}}$*, and* $M\in\mathbb{Z}$ *is any large enough number such that* f *and* g *are constant on all* $Q\in\mathscr{D}_M$*.*

Proof. We begin by observing that, according to Lemma 12.3.30, assumptions (i) through (iii) of the present theorem imply assumption (i) through (iii) of Theorem 12.3.26 uniformly for every $\omega\in(\{0,1\}^d)_0^{\mathbb{Z}}$. Thus the qualitative statement (1)⇔(2) is just an application of Theorem 12.3.26 to each $\mathscr{D}^\omega$ in place of $\mathscr{D}$, observing the uniformity just mentioned.

The more interesting part consist of the new quantitative conclusions that we obtain for the implication (2)⇒(1). This requires revisiting some details of the proof of Theorem 12.3.26.

Let $f\in S(\mathscr{D};X)$ and $g\in S(\mathscr{D};Y^*)$, and let us specifically assume that both f and g are constant on all $Q\in\mathscr{D}_M$ for some (in general large) $M\in\mathbb{Z}$. We identify $(\{0,1\}^d)^{\mathbb{Z}_{\leqslant M}}$ with $\{\omega=(\omega_j)_{j\in\mathbb{Z}}\in(\{0,1\}^d)^{\mathbb{Z}}:\omega_j=0 \text{ for } j>M\}$.

For each $\omega\in(\{0,1\}^d)^{\mathbb{Z}_{\leqslant M}}$, we have $\mathscr{D}_M^\omega=\mathscr{D}_M$, and hence f and g have the same piecewise constancy property with respect to these dyadic systems. For each $m\leqslant M$ and $\omega\in(\{0,1\}^d)^{\mathbb{Z}_{\leqslant M}}$, we then write an analogue of (12.46),

$$\begin{aligned}\mathfrak{t}(f,g)=\langle\mathfrak{H}_{\mathfrak{t}^\omega}u_m^\omega,g\rangle+\mathfrak{l}_{\mathfrak{t}^\omega}(f,g)+\mathscr{E}_m^\omega(f,g)+&\\ +\sum_{\substack{n\in\mathbb{Z}^d\\ n\neq 0}}\Big\{\langle T_{n,\mathfrak{t}^\omega}u_m^\omega,g\rangle+\langle f,U_{n,\mathfrak{t}^\omega}^1v_m^\omega\rangle+\langle U_{n,\mathfrak{t}^\omega}^2u_m^\omega,g\rangle\Big\},&\end{aligned}\tag{12.53}$$

where all symbols have the same meaning as in (12.46), but with $\mathscr{D}^\omega$ in place of $\mathscr{D}$. In particular,

$$u_m^\omega = (I - E_m^\omega)f, \qquad v_m^\omega = (I - E_m^\omega)g,$$

where $E_m^\omega = \mathbb{E}(\ |\mathscr{D}_m^\omega)$ satisfy $\|u_m^\omega\|_p \leqslant 2\|f\|_p$ and $\|v_m^\omega\|_{p'} \leqslant 2\|g\|_{p'}$.

The first and third terms on the right of (12.53) are estimated as in the proof Theorem 12.3.26. As in (12.47), we have

$$|\mathscr{E}_m^\omega(f,g)| \leqslant \Big(c_{d,p}\sum_{i=1}^{2}\|\mathfrak{t}^{\omega;(i)}\|_{\mathrm{Fig}^0(\infty)} + 2^d\|\mathfrak{t}^\omega\|_{awbp}\Big)\|E_m^\omega f\|_p\|E_m^\omega g\|_{p'} \to 0$$

when $m \to -\infty$; note that this convergence is bounded by (iii), (iv), and the easy estimates $\|E_m^\omega f\|_p \leqslant \|f\|_p$ and $\|E_m^\omega g\|_{p'} \leqslant \|g\|_{p'}$. Then, as in (12.48), from Theorem 12.1.11 we get

$$|\langle \mathfrak{H}_{\mathfrak{t}^\omega} u_m^\omega, g\rangle| \leqslant \beta_{p,X}\beta_{p,Y}\sum_{\alpha,\gamma}\mathscr{D}\mathscr{R}_p(\mathfrak{t}_0^{\omega;\alpha,\gamma})\|u_m^\omega\|_p\|g\|_{p'}.$$

The second term on the right of (12.53) is directly estimated by the uniform boundedness of the paraproducts $\Lambda_{\mathfrak{t}^\omega}$.

We then turn to the more interesting part on the second line of (12.53), where we begin with some observations. Due to the presence of the truncation parameter m, all dyadic operators in (12.53) involve cubes of side-length at most 2^{-m}. On the other hand, due to the constancy of f and g on $Q \in \mathscr{D}_M = \mathscr{D}_M^\omega$, their martingale differences are non-zero only on cubes of side-length strictly larger than 2^{-M}. Hence the right-hand side of (12.53) actually depends on $(\omega_j)_{m<j\leqslant M}$ only, rather than the infinite sequence $(\omega_j)_{j\leqslant M}$. Nevertheless, it will be convenient to also refer to this latter sequence, as we are about to see.

We compute the expectation of (12.53) with respect to the choice of $\omega \in (\{0,1\}^d)^{\mathbb{Z}_{\leqslant M}}$. As we just observed, this is actually just an arithmetic average over a finite set of $2^{d(M-m)}$ elements, so no integrability or measurability issues arise at this point.

We wish to manipulate this average a little. We note that each of the terms on the second line of (12.53) take the generic form

$$\sum_{Q\in\mathscr{D}}^{*}\Phi(Q\dot{+}\omega),$$

where

$$\Phi(R) \in \Big\{\sum_{\alpha,\gamma}\Big\langle \mathfrak{t}(h_R^\alpha, h_{R\dot{+}n}^\gamma)\langle f, h_R^\alpha\rangle, \langle g, h_{R\dot{+}n}^\gamma\rangle\Big\rangle,$$
$$\sum_{\alpha}\Big\langle \mathfrak{t}(h_R^\alpha, h_{R\dot{+}n}^0)\langle f, h_R^\alpha\rangle, \langle g, h_{R\dot{+}n}^0 - h_R^0\rangle\Big\rangle,$$

$$\sum_{\gamma}\Big\langle \mathfrak{t}(h^0_{R\dot{+}n},h^\gamma_R)\langle f,h^0_{R\dot{+}n}-h^0_R\rangle,\langle g,h^\gamma_R\rangle\Big\rangle\Big\},$$

and the notation $\sum^*$ suppresses not only the size condition that $2^{-M}<\ell(Q)\leqslant 2^{-m}$ but also an implicit restriction to a fixed finite family of cubes of each size, depending on the supports of f and g.

Inserting $1=2^d\cdot\mathbb{E}(\mathbf{1}_{\{Q\dot{+}\omega\text{ is }k\text{-good}\}})$, it hence follows, using in particular the independence property established in Lemma 12.3.33(1), that

$$\begin{aligned}\mathbb{E}\sum_{Q\in\mathscr{D}}^{*}\Phi(Q\dot{+}\omega)&=\sum_{Q\in\mathscr{D}}^{*}2^d\cdot\mathbb{E}(\mathbf{1}_{\{Q\dot{+}\omega\text{ is }k\text{-good}\}})\mathbb{E}\Phi(Q\dot{+}\omega)\\&=2^d\sum_{Q\in\mathscr{D}}^{*}\mathbb{E}\big(\mathbf{1}_{\{Q\dot{+}\omega\text{ is }k\text{-good}\}}\Phi(Q\dot{+}\omega)\big)\\&=2^d\cdot\mathbb{E}\sum_{\substack{Q\in\mathscr{D}:\\ Q\dot{+}\omega\text{ is }k\text{-good}}}^{*}\Phi(Q\dot{+}\omega).\end{aligned}$$

Thus, at the cost of the factor 2^d, we can reduce the summation to k-good cubes only.

Taking the expectation of (12.53) and applying the above observation to the terms on the second line, with $k=k(n)$ as in Definition 12.3.34, we obtain

$$\begin{aligned}\mathfrak{t}(f,g)&=\mathbb{E}\Big(\langle\mathfrak{H}_{\mathfrak{t}^\omega}u^\omega_m,g\rangle+\mathfrak{l}_{\mathfrak{t}^\omega}(f,g)+\mathscr{E}^\omega_m(f,g)+\\&+2^d\sum_{\substack{n\in\mathbb{Z}^d\\ n\neq 0}}\Big\{\langle T^{\text{good}}_{n,\mathfrak{t}^\omega}u^\omega_m,g\rangle+\langle f,U^{1,\text{good}}_{n,\mathfrak{t}^\omega}v^\omega_m\rangle+\langle U^{2,\text{good}}_{n,\mathfrak{t}^\omega}u^\omega_m,g\rangle\Big\}\Big),\end{aligned}\tag{12.54}$$

where the various "good" operators are defined in Definition 12.3.34.

When $k=k(n)$ is as in Definition 12.3.34, and $R=Q\dot{+}\omega$ is k-good, it follows directly from Definition 12.3.32 that

$$\operatorname{dist}(R,\complement R^{(k,\omega)})\geqslant 2^{k-2}\ell(R)\geqslant|n|\ell(R),$$

and hence $R\dot{+}n\subseteq R^{(k,\omega)}$. Thus the operators on the right of (12.54) are in the scope of the sharper special cases of Figiel's estimates, Corollary 12.1.27(2) and Theorem 12.1.28(2).

An application of these estimates to (12.54), in the case of $U^{\omega,1}_n$ on the dual side and otherwise directly as in Corollary 12.1.27(2) and Theorem 12.1.28(2), gives

$$\begin{aligned}|\langle f,U^{1,\text{good}}_{n,\mathfrak{t}^\omega}v^\omega_m\rangle|&\leqslant\sum_\alpha 6\beta_{p,X}\beta_{p,Y}c_1(1+k(n))^{\sigma_1}\wp_1(\mathfrak{t}^{\omega;1,\alpha}_{n,\text{good}})\|f\|_p\|v^\omega_m\|_{p'},\\|\langle U^{2,\text{good}}_{n,\mathfrak{t}^\omega}u^\omega_m,g\rangle|&\leqslant\sum_\alpha 6\beta_{p,X}\beta_{p,Y}c_2(1+k(n))^{\sigma_2}\wp_2(\mathfrak{t}^{\omega;2,\alpha}_{n,\text{good}})\|u^\omega_m\|_p\|g\|_{p'},\end{aligned}$$

$$|\langle T^{\text{good}}_{n,\mathfrak{t}^\omega} u^\omega_m, g\rangle| \leqslant \sum_{\alpha,\gamma} 3\beta_{p,X}\beta_{p,Y} \min_{i=1,2} c_i(1+k(n))^{\sigma_i}\wp_i(\mathfrak{t}^{\omega;\alpha,\gamma}_{n,\text{good}})\|u^\omega_m\|_p\|g\|_{p'}.$$

It follows from Definition 12.3.34 that

$$k(n)+1 \leqslant 4+\log_2|n| \leqslant 2(2+\log_2|n|),$$

and hence

$$\sum_{\substack{n\in\mathbb{Z}^d\\ n\neq 0}}\sum_{\alpha}(1+k(n))^{\sigma_i}\wp_i(\mathfrak{t}^{\omega;i,\alpha}_{n,\text{good}}) \leqslant 2\|\mathfrak{t}^{\omega;(i)}_{\text{good}}\|_{\text{Fig}^{\sigma_i}(\wp_i)},$$

$$\sum_{\substack{n\in\mathbb{Z}^d\\ n\neq 0}}\sum_{\alpha,\gamma}(1+k(n))^{\sigma_i}\wp_i(\mathfrak{t}^{\omega;\alpha,\gamma}_{n,\text{good}}) \leqslant 2\|\mathfrak{t}^{\omega;(0)}_{\text{good}}\|_{\text{Fig}^{\sigma_i}(\wp_i)}.$$

We have thus estimated all terms on the right of (12.54). Let us further recall that $\|u^\omega_m\|_p \leqslant 2\|f\|_p$ and $u^\omega_m \to f$ in $L^p(\mathbb{R}^d;X)$ as $m\to-\infty$, with similar results for v^ω_m, g and p' in place of u^ω_m, f and p. We can thus pass to the limit $m\to-\infty$ in (12.54) and apply dominated convergence to deduce the claimed representation formula (12.52). Applying the same estimates above to (12.52) in place of (12.54), we deduce the claimed norm estimate (a). This completes the proof of Theorem 12.3.35. □

12.4 The $T(1)$ theorem for singular integrals

A natural question arising from the Theorems 12.3.26 and 12.3.35 above is whether their assumptions are verified by some familiar operators. In particular, what is the relation of these conditions to the Calderón–Zygmund operators discussed in Chapter 11? We will address this question in the present section. Recall from Definition 11.3.1 that

$$c_K := \sup\{|s-t|^d\|K(s,t)\| : (s,t)\in\dot{\mathbb{R}}^{2d}\}.$$

Definition 12.4.1 (Weakly defined singular integral operator). *Let Z be a Banach space, and $\mathscr{C}$ be a collection of bounded Borel subsets of $\mathbb{R}^d$. We say that a bilinear form $\mathfrak{t} : S(\mathscr{C})^2 \to Z$ is a weakly defined singular integral with associated kernel $K : \dot{\mathbb{R}}^{2d} \to Z$, if $c_K < \infty$ and*

$$\mathfrak{t}(\mathbf{1}_Q,\mathbf{1}_R) = \iint_{\mathbb{R}^{2d}} K(s,t)\mathbf{1}_Q(t)\mathbf{1}_Q(s)\,\mathrm{d}s\,\mathrm{d}t \tag{12.55}$$

whenever $Q,R\in\mathscr{C}$ are disjoint.

As usual, the main case of interest will be $\mathscr{C}=\mathscr{D}$.

The following lemma, which will also play a role later, shows that the integral in (12.55) is well defined under the assumption that $c_K<\infty$: While in (12.55) we do not require the cubes to have equal size, we can always dominate the integral with such a case by passing to a dyadic ancestor of the smaller cube, if necessary.

Lemma 12.4.2. *For disjoint cubes $Q, R \subseteq \mathbb{R}^d$ of equal size $\ell(Q) = \ell(R)$, we have*

$$\iint_{Q\times R} \frac{1}{|s-t|^d}\,\mathrm{d}s\,\mathrm{d}t \leqslant \Big(1+\frac{dv_d}{2}\Big)|Q| < 18\cdot|Q|,$$

where v_d is the volume of the unit ball in $\mathbb{R}^d$.

Proof. We first write

$$\begin{aligned}\iint_{Q\times R} \frac{1}{|s-t|^d}\,\mathrm{d}s\,\mathrm{d}t &= \iint_{Q\times R} d\int_{|s-t|}^{\infty} r^{-d-1}\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}t \\ &= d\int_0^\infty |\{(s,t)\in Q\times R : |s-t|<r\}| r^{-d-1}\,\mathrm{d}r.\end{aligned}$$

Denoting by v_d is the volume of the unit ball in $\mathbb{R}^d$, we have

$$\begin{aligned}&|\{(s,t)\in Q\times R : |s-t|<r\}| = \int_{\{s\in Q:\mathrm{dist}(s,R)<r\}} |\{t\in R : |s-t|<r\}|\,\mathrm{d}s \\ &\leqslant |\{s\in Q:\mathrm{dist}(s,R)<r\}|\Big(\frac{v_d r^d}{2}\wedge|R|\Big) \leqslant (r\wedge \ell(Q))\frac{|Q|}{\ell(Q)}\Big(\frac{v_d r^d}{2}\wedge|R|\Big),\end{aligned}$$

where we used the geometric observation that, for $s \in Q \subseteq \complement R$, at least half of any ball of centre s lies in $\complement R$. Hence

$$\begin{aligned}\iint_{Q\times R} \frac{1}{|s-t|^d}\,\mathrm{d}s\,\mathrm{d}t &\leqslant d\int_0^{\ell(Q)} r\frac{|Q|}{\ell(Q)}\cdot\frac{v_d r^d}{2}\cdot r^{-d-1}\,\mathrm{d}r \\ &\quad + d\int_{\ell(Q)}^\infty |Q|\cdot|R| r^{-d-1}\,\mathrm{d}r = \frac{dv_d}{2}|Q|+|R|,\end{aligned}$$

where $|R| = |Q|$, since $\ell(R) = \ell(Q)$.

Finally, $dv_d/2 = \pi^{d/2}/\Gamma(d/2) =: f(d/2)$. From the functional equation $\Gamma(x+1) = x\Gamma(x)$, we find that $f(x+1)/f(x) = \pi/x$, so that $\max\{f(n) : n \in \mathbb{N}\} = f(4)$ and $\max\{f(n+\frac{1}{2}) : n\in\mathbb{N}\} = f(7/2)$. Computing these two values, one checks that $\max\{f(d/2) : d\in\mathbb{N}\} = f(7/2) = \frac{8}{15}\pi^3 < 17$. □

For weakly defined singular integrals, some properties imposed as assumptions on general bilinear forms are automatically satisfied:

Lemma 12.4.3. *Let Z be a Banach space and $\mathfrak{t} : \dot{\mathbb{R}}^{2d} \to Z$ a weakly defined singular integral operator with kernel K. Then $\mathfrak{t}$ satisfies the adjacent weak boundedness property if and only if it satisfies the weak boundedness property, and moreover*

$$\|\mathfrak{t}\|_{wbp} \leqslant \|\mathfrak{t}\|_{awbp} \leqslant \max\{\|\mathfrak{t}\|_{wbp}, 18\cdot c_K\}.$$

Proof. The "only if" part is obvious. For "if", it suffices to estimate $\mathfrak{t}(\mathbf{1}_Q, \mathbf{1}_R)$ for $R = Q \dot{+} n$ and $n \in \{-1,0,1\}^d \setminus \{0\}$. Then $Q \cap R = \varnothing$, so that we have access to the kernel representation (12.55), and Lemma 12.4.2 provides us with the bound

$$\|\mathfrak{t}(\mathbf{1}_Q, \mathbf{1}_R)\| \leqslant \iint_{Q\times R} \frac{c_K}{|s-t|^d}\,\mathrm{d}s\,\mathrm{d}t \leqslant 18 \cdot |Q| \cdot c_K.$$

□

Proposition 12.4.4. *Let Z be a Banach space and $\mathfrak{t} : \dot{\mathbb{R}}^{2d} \to Z$ a weakly defined singular integral operator. If $\mathfrak{t}$ is translation-invariant (in the sense of Definition 12.3.9), then $\mathfrak{t}(\mathbf{1},\cdot) = 0 = \mathfrak{t}(\cdot,\mathbf{1})$.*

Proof. By Proposition 12.3.10, it suffices to verify that $\mathfrak{t}$ satisfies the decay condition (12.35). Let $Q \in \mathscr{D}$ and $m \in \mathbb{Z}^d \setminus \{-1,0,1\}^d$. Then, for $s \in Q$ and $t \in Q \dot{+} m$, and denoting by z_Q the centre of Q, we have

$$\begin{aligned}|s-t| \geqslant |s-t|_\infty &\geqslant |m\ell(Q)|_\infty - |s-z_Q|_\infty - |t-(z_Q+m\ell(Q))|_\infty \\ &\geqslant |m|_\infty \ell(Q) - \frac{1}{2}\ell(Q) - \frac{1}{2}\ell(Q) \geqslant \frac{1}{2}|m|_\infty \ell(Q) \geqslant \frac{|m|\ell(Q)}{2\sqrt{d}},\end{aligned}$$

and hence

$$\begin{aligned}\|\mathfrak{t}(\mathbf{1}_Q, \mathbf{1}_{Q\dot{+}m})\| &\leqslant \int_Q \int_{Q\dot{+}m} \frac{c_K}{|s-t|^d}\,\mathrm{d}s\,\mathrm{d}t \\ &\leqslant |Q|^2 c_K \Big(\frac{2\sqrt{d}}{|m|\ell(Q)}\Big)^d = |Q| c_K (2\sqrt{d})^d |m|^{-d}.\end{aligned}$$

This is one half of the decay condition (12.35). The estimate for $\mathfrak{t}(\mathbf{1}_{Q\dot{+}m}, \mathbf{1}_Q)$ is entirely similar. □

Despite the simple observations above, in order to make serious conclusions about weakly defined singular integrals, we will need the following elaboration of the earlier Definition 11.3.1:

Definition 12.4.5 ($\wp$–Calderón–Zygmund kernel). *Let Z be a Banach space, $\wp$ a good set-bound on Z, and $K : \dot{\mathbb{R}}^{2d} \to Z$. We define the quantities*

$$c_K(\wp) := \wp(\{|s-t|^d K(s,t) : s \neq t\}),$$

and, for $u \in [0, \frac{1}{2}]$,

$$\omega_K^1(\wp; u) := \wp\Big(\Big\{|s-t|^d\big(K(s,t) - K(s',t)\big) : |s-s'| \leqslant u|s-t|\Big\}\Big), \quad (12.56)$$

$$\omega_K^2(\wp; u) := \wp\Big(\Big\{|s-t|^d\big(K(s,t) - K(s,t')\big) : |t-t'| \leqslant u|s-t|\Big\}\Big). \quad (12.57)$$

Remark 12.4.6. (1) We recover Definition 11.3.1 by taking $\wp(\mathscr{T}) = \mathscr{U}(\mathscr{T}) := \sup\{\|T\| : T \in \mathscr{T}\}$. Our main interest now will be $\wp \in \{\mathscr{R}_p, \mathscr{R}^*_{p'}\}$.

(2) In analogy with Lemma 11.3.3, one can check that

$$\omega^i_K(\wp; \frac{1}{2}) \leqslant (1 + 2^d) c_K(\wp).$$

(3) If $K(s,t) = \mathfrak{K}(s-t)$ for some $\mathfrak{K} : \mathbb{R}^d \setminus \{0\} \to Z$, then

$$c_K(\wp) = \wp(\{|s|^d \mathfrak{K}(s) : s \neq 0\}) =: \tilde{c}_{\mathfrak{K}}(\wp),$$

and, for both $i \in \{1,2\}$,

$$\omega^i_K(\wp; u) = \wp\Big(\Big\{|s|^d\big(\mathfrak{K}(s) - \mathfrak{K}(s')\big) : |s - s'| \leqslant u|s|\Big\}\Big) =: \tilde{\omega}_{\mathfrak{K}}(\wp; u).$$

Such a K (or $\mathfrak{K}$) is referred to as a *convolution kernel.*

If $\mathfrak{t}$ is a weakly defined singular integral with $\wp$–Calderón–Zygmund kernel K, the conditions of Definition 12.4.5 only provide control away from the diagonal $s = t$. To compensate for this, we also need the following assumption directly on the bilinear form $\mathfrak{t}$:

Definition 12.4.7 (Weak $\mathscr{D}\mathscr{R}_p$-boundedness property). *Letting* $\mathfrak{t} : S(\mathscr{D})^2 \to \mathscr{L}(X,Y)$ *be a bilinear form, we define*

$$\|\mathfrak{t}\|_{wbp(\mathscr{D}\mathscr{R}_p)} := \mathscr{D}\mathscr{R}_p\Big(\Big\{\frac{\mathfrak{t}(\mathbf{1}_Q, \mathbf{1}_Q)}{|Q|}\Big\}_{Q \in \mathscr{D}}\Big)$$

Our goal in this section will be to use these assumptions to control the Haar coefficients $\mathfrak{t}(h^\alpha_Q, h^\gamma_R)$, where $R = Q + \ell(Q)n$, in the way that was assumed in the Theorems 12.3.26 and 12.3.35 on bilinear forms. Using the defining condition (12.55) and bilinearity (noting that h^α_Q is a linear combination of $\mathbf{1}_{Q'}$ for $Q' \in \mathrm{ch}(Q)$, and likewise h^γ_R), we have in particular that

$$\mathfrak{t}(h^\alpha_Q, h^\gamma_R) = \iint_{Q \times R} K(s,t)\,\mathrm{d}s\,\mathrm{d}t, \qquad Q \cap R = \varnothing.$$

If K is a $\wp$-Calderón–Zygmund kernel, we can establish the following estimates:

Lemma 12.4.8. *Let Z be a Banach space and $\wp$ a good set-bound on Z. Let $\mathfrak{t} : S(\mathscr{D})^2 \to Z$ be a weakly defined singular integral with kernel $K : \dot{R}^{2d} \to Z$. Then for all $\alpha, \gamma \in \{0,1\}^d$, we have, for all $n \in \mathbb{Z}^d \setminus \{0\}$,*

$$\wp\Big\{\mathfrak{t}(h^\alpha_Q, h^\gamma_{Q\dot{+}n}) : Q \in \mathscr{D}\Big\} \leqslant 18 \cdot 2^d \cdot c_K(\wp), \tag{12.58}$$

and, for $|n| \geqslant \frac{3}{2}\sqrt{d}$,

$$\wp\Big\{\mathfrak{t}(h^\alpha_{Q\dot{+}n}, h^\gamma_Q) : Q \in \mathscr{D}\Big\} \leqslant (\frac{3}{2})^d \cdot |n|^{-d} \cdot \omega^1_K(\wp; \frac{\frac{3}{4}\sqrt{d}}{|n|}) \quad \textit{if} \quad \gamma \neq 0. \quad (12.59)$$

$$\wp\Big\{\mathfrak{t}(h^\alpha_Q, h^\gamma_{Q\dot{+}n}) : Q \in \mathscr{D}\Big\} \leqslant (\frac{3}{2})^d \cdot |n|^{-d} \cdot \omega^2_K(\wp; \frac{\frac{3}{4}\sqrt{d}}{|n|}) \quad \textit{if} \quad \alpha \neq 0, \quad (12.60)$$

Proof. Including momentarily also $n = 0$ for later use, we have the expansion

$$\begin{aligned} \mathfrak{t}(h^\alpha_Q, h^\gamma_{Q\dot{+}n}) &= \sum_{\substack{R\in \mathrm{ch}(Q) \\ S\in \mathrm{ch}(Q\dot{+}n)}} \mathfrak{t}(\mathbf{1}_R, \mathbf{1}_S)\langle h^\alpha_Q\rangle_R \langle h^\gamma_{Q\dot{+}n}\rangle_S \\ &= \delta_{n,0} \sum_{R\in \mathrm{ch}(Q)} \mathfrak{t}(\mathbf{1}_R, \mathbf{1}_R)\langle h^\alpha_Q\rangle_R \langle h^\gamma_{Q\dot{+}n}\rangle_S \\ &\qquad + \sum_{\substack{R\in \mathrm{ch}(Q) \\ S\in \mathrm{ch}(Q\dot{+}n) \\ R\neq S}} \mathfrak{t}(\mathbf{1}_R, \mathbf{1}_S)\langle h^\alpha_Q\rangle_R \langle h^\gamma_{Q\dot{+}n}\rangle_S =: I_Q + II_Q. \end{aligned} \quad (12.61)$$

(The summation condition $R \neq S$ in II_Q is automatic for $n \neq 0$, but it makes no harm to include it). Since

$$\sum_{\substack{R\in \mathrm{ch}(Q) \\ S\in \mathrm{ch}(Q\dot{+}n)}} |R||\langle h^\alpha_Q\rangle_R \langle h^\gamma_{Q\dot{+}n}\rangle_S| = \sum_{\substack{R\in \mathrm{ch}(Q) \\ S\in \mathrm{ch}(Q\dot{+}n)}} |R|\frac{1}{|Q|} = \sum_{S\in \mathrm{ch}(Q\dot{+}n)} 1 = 2^d,$$

we see that

$$II_Q \in 2^d \operatorname{abs\,conv}\Big(\Big\{\frac{\mathfrak{t}(\mathbf{1}_U, \mathbf{1}_V)}{|U|} : U, V \in \mathscr{D},\ U \cap V = \varnothing,\ \ell(U) = \ell(V)\Big\}\Big),$$

where

$$\begin{aligned} \mathfrak{t}(\mathbf{1}_U, \mathbf{1}_V) &= \iint_{U\times V} K(s,t)\,\mathrm{d}s\,\mathrm{d}t = \iint_{U\times V} |s-t|^d K(s,t)\frac{\mathrm{d}s\,\mathrm{d}t}{|s-t|^d} \\ &\in 18 \cdot |U| \cdot \overline{\operatorname{abs\,conv}}\Big(\Big\{|u-v|^d K(u,v) : (u,v) \in \dot{\mathbb{R}}^{2d}\Big\}\Big), \end{aligned}$$

by Proposition 1.2.12 and Lemma 12.4.2 in the last step. Combining the above inclusions with the defining properties of good set-bounds (Definition 12.3.20), we obtain

$$\wp(\{II_Q : Q \in \mathscr{D}\}) \leqslant 18 \cdot 2^d \cdot c_K(\wp), \quad (12.62)$$

which coincides with (12.58) when $n \neq 0$.

For large values of n, we want to obtain a decay, which is not present in the uniform estimate just established. In this case we apply the kernel representation combined with the vanishing mean of h^α_Q (when $\alpha \neq 0$), to the result that

$$
\begin{aligned}
\mathfrak{t}(h_Q^\alpha, h_{Q\dot{+}n}^\gamma) &= \iint K(s,t) h_Q^\alpha(t) h_{Q\dot{+}n}^\gamma(s) \,\mathrm{d}s \,\mathrm{d}t \\
&= \iint [K(s,t) - K(s, z_Q)] h_Q^\alpha(t) h_{Q\dot{+}n}^\gamma(s) \,\mathrm{d}s \,\mathrm{d}t,
\end{aligned}
$$

where z_Q is the centre of Q. For $t \in Q$ and $s \in Q\dot{+}n$, we have $|t - z_Q| \leqslant \frac{1}{2}\sqrt{d}\ell(Q)$, whereas

$$
|s - z_Q| \geqslant |z_{Q\dot{+}n} - z_Q| - |s - z_{Q\dot{+}n}| \geqslant (|n| - \frac{1}{2}\sqrt{d})\ell(Q),
$$

and hence

$$
\frac{|t - z_Q|}{|s - z_Q|} \leqslant \frac{\frac{1}{2}\sqrt{d}}{|n| - \frac{1}{2}\sqrt{d}} \leqslant \frac{1}{2} \quad \text{if} \quad |n| \geqslant \frac{3}{2}\sqrt{d}.
$$

In this case we have

$$
\begin{aligned}
&\mathfrak{t}(h_Q^\alpha, h_{Q\dot{+}n}^\gamma) \in \iint \frac{1}{|s - z_Q|^d} |h_Q^\alpha(t) h_{Q\dot{+}n}^\gamma(s)| \,\mathrm{d}s \,\mathrm{d}t \\
&\quad \times \overline{\operatorname{abs\,conv}}\Big(\Big\{|u - v|^d [K(u,v) - K(u,v')] : |v - v'| \leqslant \frac{\frac{1}{2}\sqrt{d}}{|n| - \frac{1}{2}\sqrt{d}} |u - v|\Big\}\Big),
\end{aligned}
$$

and hence, by estimate (12.56) of a Calderón–Zygmund kernel (Definition 12.4.5) and the defining properties of good set-bounds (Definition 12.3.20), we arrive at

$$
\begin{aligned}
&\wp\Big(\Big\{\mathfrak{t}(h_Q^\alpha, h_{Q\dot{+}n}^\gamma) : Q \in \mathscr{D}\Big\}\Big) \\
&\qquad \leqslant \frac{1}{|Q|} \iint_{Q \times (Q\dot{+}n)} \frac{1}{|s - z_Q|^d} \,\mathrm{d}s \,\mathrm{d}t \times \omega_K^2\Big(\frac{\frac{1}{2}\sqrt{d}}{|n| - \frac{1}{2}\sqrt{d}}\Big) \\
&\qquad \leqslant \frac{1}{(|n| - \frac{1}{2}\sqrt{d})^d} \omega_K^2\Big(\frac{\frac{1}{2}\sqrt{d}}{|n| - \frac{1}{2}\sqrt{d}}\Big) \leqslant (\frac{3}{2})^d \cdot |n|^{-d} \omega_K^2\Big(\frac{\frac{3}{4}\sqrt{d}}{|n|}\Big)
\end{aligned}
$$

when $|n| \geqslant \frac{3}{2}\sqrt{d}$.

The estimate of $\mathfrak{t}(h_Q^\alpha, h_{Q\dot{+}n}^\gamma)$ with $\gamma \neq 0$ is entirely analogous to this, using regularity in the other variable instead. □

Concerning the diagonal $n = 0$, which was excluded in Lemma 12.4.8, we have the following estimate:

Lemma 12.4.9. *Let X and Y be Banach spaces and $p \in (1, \infty)$. Let $\mathfrak{t} : \dot{\mathbb{R}}^{2d} \to \mathscr{L}(X, Y)$ be a weakly defined singular integral with the weak $\mathscr{D}\mathscr{R}_p$-boundedness property. Then*

$$
\mathscr{D}\mathscr{R}_p(\{\mathfrak{t}(h_Q^\alpha, h_Q^\gamma)\}_{Q \in \mathscr{D}}) \leqslant \|\mathfrak{t}\|_{wbp(\mathscr{D}\mathscr{R}_p)} + 18 \cdot 2^d \cdot c_K(\wp), \quad \wp \in \{\mathscr{R}_p, \mathscr{R}_{p'}^*\}.
$$

Proof. We use the expansion (12.61) with $n = 0$,

$$\mathfrak{t}(h_Q^\alpha, h_Q^\gamma) = I_Q + II_Q,$$

where we now need to consider also the term I_Q. We estimate the expression in the definition of $\mathscr{DR}_p(\{I_Q\}_{Q\in\mathscr{D}})$:

$$\begin{aligned}\sum_{Q\in\mathscr{D}} |Q||\langle I_Q x_Q, y_Q^*\rangle| &\leqslant \sum_{Q\in\mathscr{D}} |Q| \sum_{R\in\mathrm{ch}(Q)} |\langle \mathfrak{t}(\mathbf{1}_R, \mathbf{1}_R)x_Q, y_Q^*\rangle||\langle h_Q^\alpha\rangle_R \langle h_Q^\gamma\rangle_R| \\ &= \sum_{Q\in\mathscr{D}} \sum_{R\in\mathrm{ch}(Q)} |\langle \mathfrak{t}(\mathbf{1}_R, \mathbf{1}_R)x_Q, y_Q^*\rangle| \\ &= \sum_{R\in\mathscr{D}} |\langle \mathfrak{t}(\mathbf{1}_R, \mathbf{1}_R)x_{R^{(1)}}, y_{R^{(1)}}^*\rangle| \\ &\leqslant \|\mathfrak{t}\|_{wbp(\mathscr{DR}_p)} \Big\| \sum_{R\in\mathscr{D}} \varepsilon_R x_{R^{(1)}} \mathbf{1}_R \Big\|_{L^p(\Omega\times\mathbb{R}^d;X)} \times \\ &\qquad \times \Big\| \sum_{R\in\mathscr{D}} \varepsilon_R y_{R^{(1)}}^* \mathbf{1}_R \Big\|_{L^{p'}(\Omega\times\mathbb{R}^d;Y^*)}.\end{aligned}$$

Using the usual observation that, by Fubini's theorem and the fact that only one $R \in \mathscr{D}$ of each generation is "seen" at each fixed $s \in \mathbb{R}^d$, we can replace the random ε_R by $\varepsilon_{n(R)}$ depending on the generation of R only, or further by the equidistributed sequence of $\varepsilon_{n(R^{(1)})}$, we have

$$\begin{aligned}\Big\| \sum_{R\in\mathscr{D}} \varepsilon_R z_{R^{(1)}} \mathbf{1}_R \Big\|_{L^p(\Omega\times\mathbb{R}^d;Z)} &= \Big\| \sum_{Q\in\mathscr{D}} \sum_{R\in\mathrm{ch}(Q)} \varepsilon_{n(Q)} z_Q \mathbf{1}_R \Big\|_{L^p(\Omega\times\mathbb{R}^d;Z)} \\ = \Big\| \sum_{Q\in\mathscr{D}} \varepsilon_{n(Q)} z_Q \mathbf{1}_Q \Big\|_{L^p(\Omega\times\mathbb{R}^d;Z)} &= \Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q z_Q \mathbf{1}_Q \Big\|_{L^p(\Omega\times\mathbb{R}^d;Z)}\end{aligned}$$

for both choices of $z_Q \in \{x_Q, y_Q^*\}$ and $Z \in \{X, Y\}$. Hence

$$\mathscr{DR}_p(\{I_Q\}_{Q\in\mathscr{D}}) \leqslant \|\mathfrak{t}\|_{wbp(\mathscr{DR}_p)},$$

and hence, by the obvious triangle inequality for $\mathscr{DR}_p$, and its domination by either $\wp \in \{\mathscr{R}_p, \mathscr{R}_{p'}^*\}$ according to Lemma 12.1.8, we have

$$\begin{aligned}\mathscr{DR}_p(\{\mathfrak{t}(h_Q^\alpha, h_Q^\gamma)\}_{Q\in\mathscr{D}}) &\leqslant \mathscr{DR}_p(\{I_Q\}_{Q\in\mathscr{D}}) + \mathscr{DR}_p(\{II_Q\}_{Q\in\mathscr{D}}) \\ &\leqslant \|\mathfrak{t}\|_{wbp(\mathscr{DR}_p)} + \wp(\{II_Q\}_{Q\in\mathscr{D}}) \\ &\leqslant \|\mathfrak{t}\|_{wbp(\mathscr{DR}_p)} + 18 \cdot c_K(\wp)\end{aligned}$$

by (12.62) in the last step. □

We can now give estimates for the Figiel norms featuring in the $T(1)$ theorems for bilinear forms:

Lemma 12.4.10. *Let Z be a Banach space and $\wp$ a good set-bound on Z. Let $\mathfrak{t} : S(\mathscr{D})^2 \to Z$ be a weakly defined singular integral with kernel $K : \dot{\mathbb{R}}^{2d} \to Z$. Then for all $s \in [0,1]$, we have the estimates*

$$\|\mathfrak{t}^{(0)}\|_{\mathrm{Fig}^s(\wp)}, \|\mathfrak{t}^{(i)}\|_{\mathrm{Fig}^s(\wp)} \leqslant a_d c_K(\wp) + b_d \|\omega_K^i(\wp)\|_{\mathrm{Dini}^s}, \qquad i = 1, 2,$$

where a_d, b_d depend only on the dimension d, and

$$\|\omega\|_{\mathrm{Dini}^s} := \int_0^{1/2} \omega(u)(\log_2 \frac{1}{u})^s \frac{\mathrm{d}u}{u}. \tag{12.63}$$

Remark 12.4.11. For $u \in (0, \frac{1}{2})$, we have $\frac{1}{u} \in (2, \infty)$, thus $\log_2 \frac{1}{u} \in (1, \infty)$. Hence $(\log_2 \frac{1}{u})^s$ and therefore $\|\omega\|_{\mathrm{Dini}^s}$ are increasing in s.

Proof of Lemma 12.4.10. From Definition 12.3.22 and Lemma 12.4.8, it follows that

$$\begin{aligned}
\|\mathfrak{t}^{(0)}\|_{\mathrm{Fig}^s(\wp)} &= \sum_{\alpha,\gamma \in \{0,1\}^d \setminus \{0\}} \|\mathfrak{t}^{\alpha\gamma}\|_{\mathrm{Fig}^s(\wp)}, \\
&= \sum_{\alpha,\gamma \in \{0,1\}^d \setminus \{0\}} \sum_{\substack{n \in \mathbb{Z}^d \\ n \neq 0}} (2 + \log_2 |n|)^s \wp(\{\mathfrak{t}(h_Q^\alpha, h_{Q\dot{+}n}^\gamma) : Q \in \mathscr{D}\}) \\
&\leqslant (2^d - 1)^2 \Big\{ \sum_{|n| < 3\sqrt{d}} (2 + \log_2(3\sqrt{d})) \cdot 18 \cdot 2^d \cdot c_K(\wp) + \\
&\qquad + \sum_{|n| \geqslant 3\sqrt{d}} (2 + \log_2 |n|)^s (\frac{3}{2})^d |n|^{-d} \omega_K^i\Big(\wp; \frac{\frac{3}{4}\sqrt{d}}{|n|}\Big) \Big\} \\
&=: (2^d - 1)^2 (I + II_i) \leqslant 4^d (I + II_i).
\end{aligned} \tag{12.64}$$

Since both $\alpha \neq 0 \neq \gamma$, one can apply either of the estimates (12.59) or (12.60) of Lemma 12.4.8, and thus take either $i \in \{1,2\}$ above. Similarly,

$$\begin{aligned}
\|\mathfrak{t}^{(2)}\|_{\mathrm{Fig}^s(\wp)} &= \sum_{\alpha \in \{0,1\}^d \setminus \{0\}} \|\mathfrak{t}^{2,\alpha}\|_{\mathrm{Fig}^s(\wp)}, \\
&= \sum_{\alpha \in \{0,1\}^d \setminus \{0\}} \sum_{\substack{n \in \mathbb{Z}^d \\ n \neq 0}} (2 + \log_2 |n|)^s \wp(\{\mathfrak{t}(h_Q^\alpha, h_{Q\dot{+}n}^0) : Q \in \mathscr{D}\}) \\
&\leqslant (2^d - 1)(I + II_2) \leqslant 4^d (I + II_2),
\end{aligned} \tag{12.65}$$

where we only have access to estimate (12.60), but not (12.59), of Lemma 12.4.8, now that the second Haar function $h_{Q\dot{+}n}^0$ is non-cancellative. The very last step in (12.65) is of course wasteful, but we make it in order to treat the right-hand sides of both (12.64) and (12.65) at the same time.

Finally, in complete analogy with (12.65), we also have

$$\|\mathfrak{t}^{(1)}\|_{\mathrm{Fig}^s(\wp)} = \sum_{\alpha\in\{0,1\}^d\setminus\{0\}} \|\mathfrak{t}^{1,\alpha}\|_{\mathrm{Fig}^s(\wp)} \leqslant 4^d(I + II_1), \tag{12.66}$$

as we now have access to estimate (12.59), but not (12.60), of Lemma 12.4.8.

It is immediate that

$$4^d I = a_d \cdot c_K(\wp), \quad a_d := 4^d \sum_{|n|<3\sqrt{d}} (2+\log_2(3\sqrt{d}))\cdot 18\cdot 2^d. \tag{12.67}$$

For the other term, we partition the summation over dyadic annuli, in which the summand is roughly a constant:

$$4^d II_i \leqslant 6^d \sum_{k=0}^\infty \sum_{\substack{3\cdot 2^k\sqrt{d}\leqslant|n|\\ <3\cdot 2^{k+1}\sqrt{d}}} (2+\log_2(3\sqrt{d})+k)^s (3\cdot 2^k\sqrt{d})^{-d}\omega_K^i(\wp;2^{-k-2}).$$

The unit-cubes Q_n with centres $n\in\mathbb{Z}^d$ are disjoint, and for $|n| < 3\cdot 2^{k+1}\sqrt{d}$, they are contained in $B(0,(3\cdot 2^{k+1}+\frac{1}{2})\sqrt{d})$. Thus

$$\sum_{|n|<3\cdot 2^{k+1}\sqrt{d}} 1 \leqslant v_d\Big((3\cdot 2^{k+1}+\frac{1}{2})\sqrt{d}\Big)^d \leqslant v_d(6.5)^d 2^{kd}\sqrt{d}^d, \tag{12.68}$$

where v_d is the volume of the unit ball, and hence

$$\begin{aligned} 4^d II_i &\leqslant 6^d \sum_{k=0}^\infty v_d (6.5)^d 2^{kd}\sqrt{d}^d (2+\log_2(3\sqrt{d})+k)^s(3\cdot 2^k\sqrt{d})^{-d}\omega_K^i(\wp;2^{-k-2}) \\ &\leqslant (13)^d v_d(2+\log_2(3\sqrt{d}))\sum_{k=0}^\infty (1+k)^s\omega_K^i(\wp;2^{-k-2}). \end{aligned}$$

Since $\omega_K^i(\wp;u)$ is non-decreasing, we can finally estimate

$$(1+k)^s\omega_K^i(\wp;2^{-k-2}) \leqslant \frac{1}{\log 2}\int_{2^{-k-2}}^{2^{-k-1}} (\log_2\frac{1}{u})^s\frac{\omega_K^i(\wp;u)}{\log 2}\frac{\mathrm{d}u}{u}, \quad k=0,1,\ldots,$$

and hence

$$4^d II_i \leqslant b_d\|\omega_K^i(\wp)\|_{\mathrm{Dini}^s}, \qquad b_d := \frac{(13)^d v_d(2+\log_2(3\sqrt{d}))}{\log 2}.$$

With (12.64), (12.65), (12.66), and (12.67), this concludes the proof. (An estimate similar to (12.68) could also be used to give a more explicit bound for the constant a_d in (12.67), if desired.) □

We have now everything prepared for proving the following:

Theorem 12.4.12 ($T(1)$ theorem for operator-valued kernels). *Let $p\in(1,\infty)$ and $1\leqslant t\leqslant p\leqslant q\leqslant\infty$, and suppose that:*

(i) *X and Y are UMD spaces.*
(ii) *X has cotype q and Y has type t, or one of them has both.*
(iii) *$\mathfrak{t} : S(\mathscr{D})^2 \to Z := \mathscr{L}(X, Y)$ is a weakly defined singular integral and the kernel $K : \dot{\mathbb{R}}^{2d} \to Z$ of $\mathfrak{t}$ satisfies the Calderón–Zygmund estimates*

$$c_K(\mathscr{R}_p) + \|\omega_K^1(\mathscr{R}_p)\|_{\mathrm{Dini}^{1/t}} + \|\omega_K^2(\mathscr{R}_p)\|_{\mathrm{Dini}^{1/q'}} < \infty. \tag{12.69}$$

Then the following conditions are equivalent:

(1) *$\mathfrak{t}$ defines a bounded operator $T \in \mathscr{L}(L^p(\mathbb{R}^d; X), L^p(\mathbb{R}^d; Y))$;*
(2) *$\mathfrak{t}$ satisfies the weak $\mathscr{D}\mathscr{R}_p$-boundedness property $\|\mathfrak{t}\|_{wbp(\mathscr{D}\mathscr{R}_p)} < \infty$, and the associated bi-paraproduct $\Lambda_{\mathfrak{t}}$ is bounded in $\mathscr{L}(L^p(\mathbb{R}^d; X), L^p(\mathbb{R}^d; Y))$;*
(3) *each $\mathfrak{t}^\omega$ satisfies the weak $\mathscr{D}\mathscr{R}_p$-boundedness property $\|\mathfrak{t}^\omega\|_{wbp(\mathscr{D}\mathscr{R}_p)} \leqslant C$, and the associated bi-paraproduct $\Lambda_{\mathfrak{t}^\omega}$ defines a bounded operator in $\mathscr{L}(L^p(\mathbb{R}^d; X), L^p(\mathbb{R}^d; Y))$, uniformly in $\omega \in (\{0,1\}^d)_0^{\mathbb{Z}}$.*

Under these equivalent conditions, we have

(a) *the first norm estimate:*

$$\begin{aligned}\|T - \Lambda_{\mathfrak{t}}\|&_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \\ &\leqslant \beta_{p,X}\beta_{p,Y}\Big\{4^d\|\mathfrak{t}\|_{wbp(\mathscr{D}\mathscr{R}_p)} + c_d\Big(C_1 c_K(\mathscr{R}_{p'}^*) + C_2 c_K(\mathscr{R}_p)\Big) + \\ &\qquad + c_d'\Big(C_1\|\omega_K^1(\mathscr{R}_{p'}^*)\|_{\mathrm{Dini}^{1/t}} + C_2\|\omega_K^2(\mathscr{R}_p)\|_{\mathrm{Dini}^{1/q'}}\Big)\Big\},\end{aligned}$$

where c_d, c_d' are constants that depend only on d, and

$$C_1 := C_{(12.15)}(Y^*, X^*, p', t', 1), \qquad C_2 := C_{(12.15)}(X, Y, p, q, 1);$$

(b) *the second norm estimate:*

$$\begin{aligned}\|T\|&_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} - \sup_\omega \|\Lambda_{\mathfrak{t}^\omega}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \\ &\leqslant \beta_{p,X}\beta_{p,Y}\Big\{4^d \sup_\omega \|\mathfrak{t}^\omega\|_{wbp(\mathscr{D}\mathscr{R}_p)} + c_d^0\Big(c_1 c_K(\mathscr{R}_{p'}^*) + c_2 c_K(\mathscr{R}_p)\Big) + \\ &\qquad + c_d^1\Big(c_1\|\omega_K^1(\mathscr{R}_{p'}^*)\|_{\mathrm{Dini}^{1/t}} + c_2\|\omega_K^2(\mathscr{R}_p)\|_{\mathrm{Dini}^{1/q'}}\Big)\Big\},\end{aligned}$$

where the suprema are over $\omega \in (\{0,1\}^d)_0^{\mathbb{Z}}$, the constants c_d, c_d' depend only on d, and

$$c_1 := \min_{Z=X,Y} c_{t',Z^*;p'}, \qquad c_2 := \min_{Z=X,Y} c_{q,Z;p}; \tag{12.70}$$

(c) *the representation formulas (12.45) and (12.52).*

Proof. The plan of the proof is to reduce the theorem at hand to Theorems 12.3.26 and 12.3.35 on abstract bilinear forms.

(1)⇔(2): This will be an application of Theorem 12.3.26 (and Remark 12.3.27). Assumption (i) is identical in both theorems. Next, as explained in

Remark 12.3.27, under the (co)type assumption (ii) of Theorem 12.4.12, the assumption (ii) of Theorem 12.3.26 are satisfied with

$$(t_1, q_1) := (t, \infty), \qquad (t_2, q_2) := (1, q),$$

and both choices of $(t_0, q_0) \in \{(t_i, q_i)\}_{i=1}^2$. Let $\sigma_1 := 1/t$ and $\sigma_2 := 1/q'$.

Concerning assumption (iii) on the bilinear form $\mathfrak{t}$, we need to check that the kernel assumptions (12.69) of the present theorem imply the assumptions on the Haar coefficients $\mathfrak{t}(h_Q^\alpha, h_R^\gamma)$ and the related Figiel norms of the bilinear form $\mathfrak{t}$. With the choices of (t_i, q_i) as just explained, and recalling that the set-bounds $\wp_1 := \mathscr{R}_{p'}^*$ and $\wp_2 := \mathscr{R}_p$ are equivalent in the spaces that we are considering, the assumption (12.69) can be equivalently written as

$$c_K(\wp_i) + \|\omega_K^i(\wp_i)\|_{\mathrm{Dini}^{\sigma_i}} < \infty, \quad i \in \{1, 2\}. \tag{12.71}$$

By Example 12.1.10, we know that

$$\|\mathfrak{t}\|_{wbp(\mathscr{D}\mathscr{R}_p)} \leqslant \|T\|_{\mathscr{L}(L^p(\mathbb{R}^d;X), L^p(\mathbb{R}^d;Y))}, \tag{12.72}$$

so in particular the weak $\mathscr{D}\mathscr{R}_p$-boundedness property is either assumed, or implied by the assumptions, in each case of Theorem 12.4.12.

From Lemma 12.4.9, we then have

$$\mathscr{D}\mathscr{R}_p(\{\mathfrak{t}(h_Q^\alpha, h_Q^\gamma)\}_{Q \in \mathscr{D}}) \leqslant \|\mathfrak{t}\|_{wbp(\mathscr{D}\mathscr{R}_p)},$$

whereas Lemma 12.4.10 guarantees, for both $i \in \{1, 2\}$, that

$$\begin{aligned} \|\mathfrak{t}^{(0)}\|_{\mathrm{Fig}^{\sigma_i}(\wp_i)} &\leqslant a_d c_K(\wp_i) + b_d \|\omega_K^i(\wp_i)\|_{\mathrm{Dini}^{\sigma_i}}, \\ \|\mathfrak{t}^{(i)}\|_{\mathrm{Fig}^{\sigma_i}(\wp_i)} &\leqslant a_d c_K(\wp_i) + b_d \|\omega_K^i(\wp_i)\|_{\mathrm{Dini}^{\sigma_i}}, \end{aligned} \tag{12.73}$$

where both right-hand sides of are finite by (12.71). With either choice of $(t_0, q_0) \in \{(t_i, q_i)\}_{i=1}^2$, the resulting finiteness of the left-hand sides coincides with the assumption on these quantities in (iii) of Theorem 12.3.26.

Summarising, assumptions (i) through (iii) of Theorem 12.4.12, together with the weak $\mathscr{D}\mathscr{R}_p$-boundedness property of $\mathfrak{t}$, which is either assumed or implied by the assumptions of each case of Theorem 12.4.12, imply the corresponding assumptions (i) through (iii) of Theorem 12.3.26. Moreover, the condition of adjacent weak boundedness property appearing in Theorem 12.3.26 also follows from these assumptions by Lemma 12.4.3 and the domination of uniform bounds by either $\mathscr{D}\mathscr{R}_p$-bounds or $\wp_i$-bounds:

$$\|\mathfrak{t}\|_{awbp} \leqslant \max\{\|\mathfrak{t}\|_{wbp}, 18 \cdot c_K\} \leqslant \max\{\|\mathfrak{t}\|_{wbp(\mathscr{D}\mathscr{R}_p)}, 18 \cdot c_K(\wp_i)\}.$$

Hence all assumptions, and thus all conclusions of Theorem 12.3.26 are valid under the assumptions of Theorem 12.4.12. This proves in particular the qualitative equivalence (1)⇔(2).

(a): For this quantitative estimate, we apply Remark 12.3.27, followed by (12.72) and (12.73), to get

$$\begin{aligned}
&\|T - \Lambda_{\mathfrak{t}}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \\
&\leqslant \beta_{p,X}\beta_{p,Y}\Big\{ \sum_{\alpha,\gamma} \mathscr{DR}_p(\mathfrak{t}_0^{\alpha,\gamma}) + \sum_{i=1}^{2} C_i\Big(A_d\|\mathfrak{t}^{(0)}\|_{\mathrm{Fig}^{\sigma_i}(\wp_i)} + B_d\|\mathfrak{t}^{(i)}\|_{\mathrm{Fig}^{\sigma_i}(\wp_i)}\Big)\Big\} \\
&\leqslant \beta_{p,X}\beta_{p,Y}\Big\{4^d\|\mathfrak{t}\|_{wbp(\mathscr{DR}_p)} + \sum_{i=1}^{2} C_i\Big(c_d c_K(\wp_i) + c_d'\|\omega_K^i(\wp_i)\|_{\mathrm{Dini}^{\sigma_i}}\Big)\Big\},
\end{aligned}$$

where $c_d := (A_d + B_d)a_d$ and $c_d' := (A_d + B_d)b_d$. This is readily recognised to coincide with the bound asserted in (a) of the theorem.

(1)⇔(3): This will be an application of Theorem 12.3.35. Assumptions (i) and (ii) are identical in both theorems.

Concerning assumption (iii), we need to check that the kernel assumptions (12.69) of the present theorem imply the estimates on Figiel norms of each bilinear form $\mathfrak{t}^\omega$, uniformly in $\omega \in (\{0,1\}^d)_0^{\mathbb{Z}}$. We already did this for $\mathfrak{t} = \mathfrak{t}^0$ above. However, all the lemmas of this section are stated for an arbitrary dyadic system $\mathscr{D}$, so we may in particular use them with any $\mathscr{D}^\omega$ in place of $\mathscr{D}$. Moreover, the constants in these estimates are explicit, and clearly independent of the particular ω. This proves the qualitative equivalence (1)⇔(3).

(b): For this quantitative estimate, we apply Theorem 12.3.35(a), followed by (12.72) and (12.73) with $\mathfrak{t}^\omega$ and $\mathscr{D}^\omega$ in place of $\mathfrak{t}$ and $\mathscr{D}$, to get

$$\begin{aligned}
&\Big(\|T\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} - \sup_\omega \|\Lambda_{\mathfrak{t}^\omega}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))}\Big)\frac{1}{\beta_{p,X}\beta_{p,Y}} \\
&\leqslant \sup_\omega \sum_{\alpha,\gamma} \mathscr{DR}_p(\mathfrak{t}_0^{\omega;\alpha,\gamma}) + 12\cdot 2^d \sup_\omega \sum_{i=1}^{2} c_i \sum_{j\in\{0,i\}} \|\mathfrak{t}_{\mathrm{good}}^{\omega;(j)}\|_{\mathrm{Fig}^{\sigma_i}(\wp_i)} \\
&\leqslant 4^d \sup_\omega \|\mathfrak{t}^\omega\|_{wbp(\mathscr{DR}_p)} + \sup_\omega \sum_{i=1}^{2} c_i\Big(c_d^0 c_K(\wp_i) + c_d^1\|\omega_K^i(\wp_i)\|_{\mathrm{Dini}^{\sigma_i}}\Big),
\end{aligned}$$

where $c_d^0 = 24\cdot 2^d\cdot a_d$ and $c_d^1 = 24\cdot 2^d\cdot b_d$. This is readily recognised to coincide with the bound asserted in (a) of the theorem.

(c): The representation formulas are immediate from Theorems 12.3.26 and 12.3.35, since we already verified that the assumptions of the said theorems are valid in the present setting. □

12.4.a Consequences of the $T(1)$ theorem

We will now explore various consequences of Theorem 12.4.12 to more particular classes of operators. While Theorem 12.4.12 gives a complete characterisation of the boundedness of an operator T, a drawback is the fact that this characterisation involves the boundedness of another operator $\Lambda_{\mathfrak{t}}$ that is not necessarily easy to check, as we found in Section 12.2. Thus, the following special case, in which these paraproducts are completely avoided, will be useful:

Corollary 12.4.13 ($T(1)$ theorem for convolution kernels). *Let $p \in (1,\infty)$ and $1 \leqslant t \leqslant p \leqslant q \leqslant \infty$, and suppose that:*

(i) *X and Y are UMD spaces;*
(ii) *X has cotype q and Y has type t, or one of them has both;*
(iii) *$\mathfrak{t} : S(\mathscr{D})^2 \to Z := \mathscr{L}(X,Y)$ is a weakly defined singular integral and the kernel $K : \dot{\mathbb{R}}^{2d} \to Z$ of $\mathfrak{t}$ has the convolution form $K(s,t) = \mathfrak{K}(s-t)$ and satisfies the Calderón–Zygmund estimates*

$$\tilde{c}_{\mathfrak{K}}(\mathscr{R}_p) + \|\tilde{\omega}_{\mathfrak{K}}(\mathscr{R}_p)\|_{\mathrm{Dini}^\sigma} < \infty, \qquad \sigma := \max(\frac{1}{t}, \frac{1}{q'}), \tag{12.74}$$

where $\tilde{c}_{\mathfrak{K}}$ and $\tilde{\omega}_{\mathfrak{K}}$ are as in Remark 12.4.6(3);
(iv) *$\mathfrak{t}(\mathbf{1}_Q, \mathbf{1}_Q) = (\mathbf{1}_{Q\dot{+}m}, \mathbf{1}_{Q\dot{+}m})$ for all $Q \in \mathscr{D}$ and $m \in \mathbb{Z}^d$.*

Then the following conditions are equivalent:

(1) *$\mathfrak{t}$ defines a bounded operator $T \in \mathscr{L}(L^p(\mathbb{R}^d;X), L^p(\mathbb{R}^d;Y))$;*
(2) *$\mathfrak{t}$ satisfies the weak $\mathscr{D}\mathscr{R}_p$-boundedness property $\|\mathfrak{t}\|_{wbp(\mathscr{D}\mathscr{R}_p)} < \infty$;*
(3) *each $\mathfrak{t}^\omega$ satisfies the weak $\mathscr{D}\mathscr{R}_p$-boundedness property $\|\mathfrak{t}^\omega\|_{wbp(\mathscr{D}\mathscr{R}_p)} \leqslant C$, uniformly in $\omega \in (\{0,1\}^d)_0^{\mathbb{Z}}$.*

Under these equivalent conditions, we have

(a) *the norm estimate*

$$\begin{aligned}&\|T\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))}\\ &\qquad\leqslant \beta_{p,X}\beta_{p,Y}\Big\{4^d \sup_\omega \|\mathfrak{t}^\omega\|_{wbp(\mathscr{D}\mathscr{R}_p)} + c_d^0\Big(c_1\tilde{c}_{\mathfrak{K}}(\mathscr{R}_{p'}^*) + c_2\tilde{c}_{\mathfrak{K}}(\mathscr{R}_p)\Big) +\\ &\qquad\qquad + c_d^1\Big(c_1\|\tilde{\omega}_{\mathfrak{K}}(\mathscr{R}_{p'}^*)\|_{\mathrm{Dini}^{1/t}} + c_2\|\tilde{\omega}_{\mathfrak{K}}(\mathscr{R}_p)\|_{\mathrm{Dini}^{1/q'}}\Big)\Big\},\end{aligned}$$

where the supremum is over $\omega \in (\{0,1\}^d)_0^{\mathbb{Z}}$, the constants c_d, c_d' depend only on d, and c_1, c_2 are as in (12.70);
(b) *the representation formulas (12.45) and (12.52) with $\Lambda_{\mathfrak{t}} = \Lambda_{\mathfrak{t}^\omega} = 0$.*

Proof. We will check that $\mathfrak{t}$ is translation-invariant in the sense of Definition 12.3.9, i.e., that it satisfies the condition of Lemma 12.3.8(1). The very assumption (iv) of the corollary already takes care of the case $Q = R$. On the other hand, if $Q \neq R$ are dyadic cubes of the same size, then $Q \cap R = \varnothing = (Q\dot{+}m) \cap (R\dot{+}m)$, and hence we have access to the kernel representation

$$\begin{aligned}\mathfrak{t}(\mathbf{1}_Q, \mathbf{1}_R) &= \int_R \int_Q K(s,t)\,\mathrm{d}s\,\mathrm{d}t = \int_R \int_Q \mathfrak{K}(s-t)\,\mathrm{d}s\,\mathrm{d}t \\ &= \int_R \int_Q \mathfrak{K}((s+m)-(t+m))\,\mathrm{d}s\,\mathrm{d}t \\ &= \int_{R\dot{+}m} \int_{Q\dot{+}m} \mathfrak{K}(s-t)\,\mathrm{d}s\,\mathrm{d}t = \mathfrak{t}(\mathbf{1}_{Q\dot{+}m}, \mathbf{1}_{R\dot{+}m}),\end{aligned}$$

which proves the condition of Lemma 12.3.8(1) for arbitrary $Q, R \in \mathscr{D}$ of equal size. Thus indeed $\mathfrak{t}$ is translation-invariant.

Next, we wish to have the same property for $\mathfrak{t}^\omega$, for every $\omega \in (\{0,1\}^d)^{\mathbb{Z}}_0$, and requires verifying the identity $\mathfrak{t}(\mathbf{1}_{Q'}, \mathbf{1}_{R'}) = \mathfrak{t}(\mathbf{1}_{Q'\dot{+}m}, \mathbf{1}_{R'\dot{+}m})$ for each $Q', R' \in \mathscr{D}^\omega$ of equal size. By Lemma 12.3.30, we have $S(\mathscr{D}^\omega) = S(\mathscr{D})$ whenever $\omega \in (\{0,1\}^d)^{\mathbb{Z}}_0$. If $Q' \in \mathscr{D}^\omega$, then clearly $f = \mathbf{1}_{Q'} \in S(\mathscr{D}^\omega) = S(\mathscr{D})$, and similarly with $g = \mathbf{1}_{R'}$ where $R' \in \mathscr{D}^\omega$ has the same size. Thus Lemma 12.3.8 guarantees that

$$\mathfrak{t}(\mathbf{1}_{Q'\dot{+}m}, \mathbf{1}_{R'\dot{+}m}) = \mathfrak{t}(\tau_{m\ell(Q')} f, \tau_{m\ell(Q')} g) = \mathfrak{t}(f,g) = \mathfrak{t}(\mathbf{1}_{Q'}, \mathbf{1}_{R'})$$

for all $Q', R' \in \mathscr{D}^\omega$ of the same size, and hence also $\mathfrak{t}^\omega$ is translation-invariant.

By Proposition 12.4.4, it then follows that $\mathfrak{t}^\omega(\mathbf{1}, \cdot) = 0 = \mathfrak{t}^\omega(\cdot, \mathbf{1})$, for every $\omega \in (\{0,1\}^d)^{\mathbb{Z}}_0$. Thus the conclusions of the corollary are immediate from Theorem 12.4.12 by setting all $\Lambda_{\mathfrak{t}}$ and $\Lambda_{\mathfrak{t}^\omega}$ to be zero. □

Lemma 12.4.14. *Let $Z = \mathscr{L}(X,Y)$ and $\Phi \in C_{\mathrm{b}}([0,\infty); Z) \cap C^1((0,\infty); Z)$, and suppose that*

(i) *$\mathfrak{K}(u) := \mathbf{1}_{(0,\infty)}(u)\Phi'(u)$ satisfies the Calderón–Zygmund estimate (12.74);*
(ii) *the range of Φ is R-bounded, $\mathscr{R}_p(\Phi) := \mathscr{R}_p(\{\Phi(u) : u \in [0,\infty)\}) < \infty$;*
(iii) *a bilinear form $\mathfrak{t} : S(\mathscr{D})^2 \to Z$ is defined, for all $f, g \in S(\mathscr{D})$, by*

$$\mathfrak{t}(f,g) := \lim_{\varepsilon\to 0} \iint_{|u-v|>\varepsilon} \mathfrak{K}(u-v) f(v) g(u)\,\mathrm{d}v\,\mathrm{d}u.$$

Then

(1) *$\mathfrak{t}$ is well-defined as a weakly defined singular integral with convolution kernel $K(u,v) = \mathfrak{K}(u-v)$;*
(2) *$\mathfrak{t}^\omega$ satisfies the weak $\mathscr{D}\mathscr{R}_p$-boundedness property*

$$\|\mathfrak{t}^\omega\|_{wbp(\mathscr{D}\mathscr{R}_p)} \leqslant \|\Phi(0)\| + \min\{\mathscr{R}_p(\Phi), \mathscr{R}^*_{p'}(\Phi)\};$$

(3) *$\mathfrak{t}(\mathbf{1}_I, \mathbf{1}_I) = (\mathbf{1}_{I\dot{+}m}, \mathbf{1}_{I\dot{+}m})$ for all $I \in \mathscr{D}$ and $m \in \mathbb{Z}$.*

Proof. (1): Clearly the integral inside the limit is well-defined, since we are cutting away the singularity. To show the existence of the limit, let first $f = 1_I$ and $g = 1_J$ for some intervals $I = [a_I, b_I)$ and $J = [a_J, b_J)$. Then

$$\begin{aligned}\int_{|u-v|>\varepsilon} \mathfrak{K}(u-v)f(v)\,\mathrm{d}v &= \int_{a_I}^{b_I} \mathbf{1}_{(\varepsilon,\infty)}(u-v)\Phi'(u-v)\,\mathrm{d}v \\ &= \mathbf{1}_{(a_I+\varepsilon,\infty)}(u)\int_{a_I}^{b_I\wedge(u-\varepsilon)} \Phi'(u-v)\,\mathrm{d}v \\ &= \mathbf{1}_{(a_I+\varepsilon,\infty)}(u)[\Phi((u-b_I)\vee\varepsilon)-\Phi(u-a_I)].\end{aligned}$$

Since Φ is continuous on $[0,\infty)$, we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{|u-v|>\varepsilon} \mathfrak{K}(u-v)f(v)\,\mathrm{d}v &= \mathbf{1}_{(a_I,\infty)}(u)[\Phi((u-b_I)_+)-\Phi(u-a_I)], \\ &= \Phi((u-b_I)_+)-\Phi((u-a_I)_+).\end{aligned}$$

Since Φ is bounded on $[0,\infty)$, we can apply dominated convergence to obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\iint_{|u-v|>\varepsilon} &\mathfrak{K}(u-v)\mathbf{1}_I(v)\mathbf{1}_J(u)\,\mathrm{d}v\,\mathrm{d}u \\ &= \int_J [\Phi((u-b_I)_+)-\Phi((u-a_I)_+)]\,\mathrm{d}u.\end{aligned}\tag{12.75}$$

In particular, the limit defining $\mathfrak{t}(f,g)$ exists for all f,g of the form $f=\mathbf{1}_I$ and $g=\mathbf{1}_J$. By (bi)linearity, it exists for all $f,g\in S(\mathscr{D})$.

If $f,g\in S(\mathscr{D})$ are disjointly supported, then $\mathfrak{K}(u-v)f(v)g(u)$ is integrable. Hence

$$\mathfrak{t}(f,g)=\iint \mathfrak{K}(u-v)f(v)g(u)\,\mathrm{d}v\,\mathrm{d}u$$

by dominated convergence, and thus $\mathfrak{t}$ is a weakly defined singular integral with kernel $K(u,v)=\mathfrak{K}(u-v)$.

(2): With $J=I\in\mathscr{D}^\omega$, noting that $a_I\leqslant u<b_I$ for all $u\in I$, the identity (12.75) shows that

$$\begin{aligned}\frac{\mathfrak{t}(\mathbf{1}_I,\mathbf{1}_I)}{|I|} &= \fint_I \big(\Phi(0)-\Phi(u-a_I)\big)\,\mathrm{d}u \\ &= \fint_0^{\ell(I)} \big(\Phi(0)-\Phi(u)\big)\,\mathrm{d}u \in \Phi(0)+\overline{\mathrm{abco}}(\Phi).\end{aligned}\tag{12.76}$$

Thus, by Lemma 12.1.8, we find that

$$\begin{aligned}\|t^\omega\|_{wbp(\mathscr{D}\mathscr{R}_p)} &:= \mathscr{D}\mathscr{R}_p\Big(\Big\{\frac{\mathfrak{t}(\mathbf{1}_I,\mathbf{1}_I)}{|I|}\Big\}_{I\in\mathscr{D}^\omega}\Big) \\ &\leqslant \min_{i=0,1}\wp_i\Big(\Big\{\frac{\mathfrak{t}(\mathbf{1}_I,\mathbf{1}_I)}{|I|}\Big\}_{I\in\mathscr{D}^\omega}\Big), \qquad \wp_0:=\mathscr{R}_p,\quad \wp_1:=\mathscr{R}^*_{p'}, \\ &\leqslant \|\Phi(0)\|+\min_{i=0,1}\wp_i(\Phi).\end{aligned}$$

(3): From (12.76) it is evident that $\mathfrak{t}(\mathbf{1}_I,\mathbf{1}_I)$ depends only on $\ell(I)$; since $\ell(I)=\ell(I\dot{+}m)$, it follows that $\mathfrak{t}(\mathbf{1}_I,\mathbf{1}_I)=\mathfrak{t}(\mathbf{1}_{I\dot{+}m},\mathbf{1}_{I\dot{+}m})$, as claimed. □

It often happens that kernels that we encounter satisfy standard Calderón–Zygmund estimates with the best possible Lipschitz modulus of continuity $\omega(u) = O(u)$ as $u \to 0$, but the implies constant in this estimate can be very large. At the same time, we also have a trivial bound $\omega(u) = O(1)$, where the implied constant may be much smaller. The following lemma provides a useful estimate of the Dini norms of ω in such cases, showing that the larger constant enters the estimates only via its logarithm:

Lemma 12.4.15. *Let* $0 < A \leqslant B < \infty$ *and* $\sigma \in [0,1]$. *If* $\omega(u) \leqslant \min(A, Bu)$, *then*

$$\|\omega\|_{\mathrm{Dini}^\sigma} \leqslant 3A\Big(1 + \log^{\sigma+1} \frac{B}{A}\Big).$$

Proof.

$$(\log 2)^\sigma \|\omega\|_{\mathrm{Dini}^\sigma} \leqslant \int_0^{A/B} B\Big(\log \frac{1}{u}\Big)^\sigma \,\mathrm{d}u + \int_{A/B}^1 A\Big(\log \frac{1}{u}\Big)^\sigma \frac{\mathrm{d}u}{u} =: I + II,$$

where

$$I \leqslant -B \int_0^{A/B} \log u \,\mathrm{d}u = -B(u \log u - u)\Big|_0^{A/B} = A(\log \frac{B}{A} + 1)$$

and

$$II = A \int_{A/B}^1 (-\log u)^\sigma \frac{\mathrm{d}u}{u} = -A \frac{(-\log u)^{\sigma+1}}{\sigma+1}\Big|_{A/B}^1 = \frac{A}{\sigma+1}\Big(\log \frac{B}{A}\Big)^{\sigma+1}.$$

Let $G := \log(B/A)$. Since

$$G = (G^{\sigma+1})^{1/(\sigma+1)} \cdot 1^{\sigma/(\sigma+1)} \leqslant \frac{1}{\sigma+1} G^{\sigma+1} + \frac{\sigma}{\sigma+1},$$

we obtain

$$I + II \leqslant \frac{2A}{\sigma+1} G^{\sigma+1} + A\Big(1 + \frac{\sigma}{\sigma+1}\Big) \leqslant 2A(G^{\sigma+1} + 1).$$

Since $(\log 2)^{-\sigma} \leqslant (\log 2)^{-1} < 3/2$, the claim follows. □

Example 12.4.16. Let $\omega \in [0, \pi/2]$, $\sigma \in [0,1]$, and suppose that

$$\Phi \in C([0,\infty), Z) \cap H^\infty(\Sigma_\omega; Z)$$

has an R-bounded range. Then $\Phi|_{[0,\infty)}$ and $\mathfrak{K}(u) = \mathbf{1}_{(0,\infty)}(u)\Phi'(u)$ satisfy the assumptions of Lemma 12.4.14 with

$$\tilde{c}_{\mathfrak{K}}(\wp) \leqslant \frac{\wp(\Phi)}{\sin \omega}, \quad \|\tilde{\omega}_{\mathfrak{K}}(\wp)\|_{\mathrm{Dini}^\sigma} \leqslant \frac{3\wp(\Phi)}{\sin \omega}\Big(1 + \log^{1+\sigma} \frac{4}{\sin \omega}\Big), \quad \wp \in \{\mathscr{R}_p, \mathscr{R}^*_{p'}\}.$$

A particular instance of such a Φ is (the negation of) an R-bounded holomorphic semigroup $\Phi(z) = -e^{-zA}$, in which case $\mathfrak{K}(u) = Ae^{-uA}$ is the kernel of the so-called maximal regularity operator.

Remark 12.4.17. The role of the parameter $\sigma \in [0,1]$ in Example 12.4.16 is relatively insignificant and only recorded for curiosity. First, it only affects the power of the logarithm. Second, for applying Lemma 12.4.14, it is necessary to take $\sigma \geqslant \max(1/t, 1/q') \geqslant \frac{1}{2}$, and it is always sufficient to take $\sigma = 1$, so that the power of the logarithm will always be in the range $[\frac{3}{2}, 2]$.

Proof of Example 12.4.16. Let $\wp \in \{\mathscr{R}_p, \mathscr{R}_p^*\}$. It is evident that

$$\wp(\{\Phi(u) : u \in [0,\infty)\}) = \wp(\overline{\{\Phi(u) : u \in (0,\infty)\}}) \leqslant \wp(\{\Phi(z) : z \in \Sigma_\omega\}).$$

By Cauchy's formula, we have

$$\Phi^{(j)}(u) = \frac{j!}{2\pi i} \oint_{|z-u|=u\sin\omega} \frac{f(z)}{(u-z)^{j+1}} |\mathrm{d}z|, \qquad u > 0.$$

Denoting $\wp(\Phi) := \wp(\Phi(z) : z \in \Sigma_\omega)$, we hence have

$$\wp(t^j \Phi^{(j)}(t) : t > 0) \leqslant \frac{j!}{2\pi} \wp(\Phi) \sup_{t>0} \oint_{|z-t|=t\sin\omega} \frac{t^j |\mathrm{d}z|}{(t\sin\omega)^{j+1}} = \frac{j!\wp(\Phi)}{(\sin\omega)^j}.$$

With $\mathfrak{K}(u) = \mathbf{1}_{(0,\infty)}(u)\Phi'(u)$, it follows that

$$\tilde{c}_{\mathfrak{K}}(\wp) = \wp(|u|\mathfrak{K}(u) : u \neq 0) = \wp(u\Phi'(u) : u > 0) \leqslant \frac{\wp(\Phi)}{\sin\omega}.$$

Moreover,

$$\begin{aligned} \tilde{\omega}_{\mathfrak{K}}(\wp; s) &= \wp(|u|[\mathfrak{K}(u) - \mathfrak{K}(u')] : |u-u'| \leqslant s|u|) \\ &= \wp\Big(u \int_{u'}^{u} \Phi''(v)\,\mathrm{d}v : |u-u'| \leqslant su\Big) \\ &\leqslant \frac{2\wp(\Phi)}{(\sin\omega)^2} \sup_{|u-u'|\leqslant su} \Big| \int_{u'}^{u} \frac{u}{v^2}\,\mathrm{d}v \Big|, \end{aligned}$$

where

$$\Big| \int_{u'}^{u} \frac{u}{v^2}\,\mathrm{d}v \Big| = u\Big|\frac{1}{u} - \frac{1}{u'}\Big| = \frac{|u-u'|}{u'} \leqslant \frac{su}{(1-w)u} = \frac{s}{1-s} \leqslant 2s$$

for $|u-u'| \leqslant wu$ and $s \in [0, \frac{1}{2}]$. Thus $\tilde{\omega}_{\mathfrak{K}}(\wp; s) \leqslant 4\wp(\Phi)(\sin\omega)^{-2}s$.

By Remark 12.4.6(2), we also have $\tilde{\omega}_{\mathfrak{K}}(\wp; s) \leqslant \tilde{c}_{\mathfrak{K}}(\wp) \leqslant \wp(\Phi)(\sin\omega)^{-1}$. Thus, an application of Lemma 12.4.15 with $0 < A = \wp(\Phi)(\sin\omega)^{-1} < 4\wp(\Phi)(\sin\omega)^{-2} = B < \infty$, we deduce that

$$\|\tilde{\omega}_{\mathfrak{K}}\|_{\mathrm{Dini}^\sigma} \leqslant \frac{3\wp(\Phi)}{\sin\omega}\Big(1 + \log^{1+\sigma} \frac{4}{\sin\omega}\Big).$$

This completes the proof. ∎

We proceed to further corollaries of Theorem 12.4.12.

Corollary 12.4.18 ($T(1)$ theorem for antisymmetric kernels). *Let $p \in (1,\infty)$ and $1 \leqslant t \leqslant p \leqslant q \leqslant \infty$, and suppose that:*

(i) *X and Y are UMD spaces.*
(ii) *X has cotype q and Y has type t, or one of them has both.*
(iii) *$K : \dot{\mathbb{R}}^{2d} \to Z := \mathscr{L}(X,Y)$ is an antisymmetric kernel, i.e.,*

$$K(s,t) = -K(t,s) \quad \text{for all } (s,t) \in \dot{\mathbb{R}}^{2d},$$

which satisfies the Calderón–Zygmund estimates

$$c_K(\mathscr{R}_p) + \|\omega_K^1(\mathscr{R}_p)\|_{\mathrm{Dini}^{\max(1/t,1/q')}} < \infty. \tag{12.77}$$

(iv) *A bilinear form $\mathfrak{t} : S(\mathscr{D})^2 \to Z$ is defined for all $f,g \in S(\mathscr{D})$ by*

$$\mathfrak{t}(f,g) := \frac{1}{2}\iint K(s,t)\big(f(t)g(s) - f(s)g(t)\big)\,\mathrm{d}t\,\mathrm{d}s. \tag{12.78}$$

Then $\mathfrak{t}$ is well-defined as a weakly defined singular integral with kernel K, and the following conditions are equivalent:

(1) *$\mathfrak{t}$ defines a bounded operator $T \in \mathscr{L}(L^p(\mathbb{R}^d;X), L^p(\mathbb{R}^d;Y))$;*
(2) *$\Lambda_{\mathfrak{t}}$ defines a bounded operator in $\mathscr{L}(L^p(\mathbb{R}^d;X), L^p(\mathbb{R}^d;Y))$;*
(3) *each $\Lambda_{\mathfrak{t}^\omega}$ defines a bounded operator in $\mathscr{L}(L^p(\mathbb{R}^d;X), L^p(\mathbb{R}^d;Y))$, uniformly in $\omega \in (\{0,1\}^d)_0^{\mathbb{Z}}$.*

Under these equivalent conditions, we have

(a) *the norm estimates as in parts (a) and (b) of Theorem 12.4.12, with $\|\mathfrak{t}\|_{wbp(\mathscr{D}\mathscr{R}_p)} = \|\mathfrak{t}^\omega\|_{wbp(\mathscr{D}\mathscr{R}_p)} = 0$;*
(b) *the representation formulas (12.45) and (12.52).*

Proof. To check that $\mathfrak{t}$ is well-defined, we need to verify that the integrals in (12.78) make sense. By linearity, it is enough to consider $f = \mathbf{1}_Q$ and $g = \mathbf{1}_R$ for some $Q, R \in \mathscr{D}$. If $Q \cap R = \varnothing$, then each of the two terms under the integral is separately integrable by Lemma 12.4.2, and hence so is their difference. Otherwise, we may assume by the nestedness of dyadic cubes and symmetry that, e.g., $Q \subseteq R$. We can then split

$$\begin{aligned} f(t)g(s) - f(s)g(t) &= \mathbf{1}_Q(t)\mathbf{1}_R(s) - \mathbf{1}_Q(s)\mathbf{1}_R(t) \\ &= \mathbf{1}_Q(t)\big(\mathbf{1}_Q(s) + \mathbf{1}_{R\setminus Q}(s)\big) - \mathbf{1}_Q(s)\big(\mathbf{1}_Q(t) + \mathbf{1}_{R\setminus Q}(t)\big) \\ &= \mathbf{1}_Q(t)\mathbf{1}_{R\setminus Q}(s) - \mathbf{1}_Q(s)\mathbf{1}_{R\setminus Q}(t), \end{aligned}$$

observing the cancellation of the two equal terms $\mathbf{1}_Q(s)\mathbf{1}_Q(t)$. We can divide $R \setminus Q$ into finitely many cubes $P \in \mathscr{D}$ of the same size as Q, and then the integrability of each of the terms on the left against $K(s,t)$ follows from

Lemma 12.4.2. Thus the formula defining $\mathfrak{t}$ as a bilinear form $\mathfrak{t} : S(\mathscr{D})^2 \to Z$ is meaningful.

To show that $\mathfrak{t}$ has associated kernel K, let $f, g \in S(\mathscr{D})$ be disjointly supported. As we already observed, in this case both terms under the integral are separately integrable, and we can write

$$\begin{aligned}\mathfrak{t}(f,g) &= \frac{1}{2}\iint K(s,t)\big(f(t)g(s) - f(s)g(t)\big)\,\mathrm{d}t\,\mathrm{d}s \\ &= \frac{1}{2}\iint K(s,t)f(t)g(s)\,\mathrm{d}t\,\mathrm{d}s - \frac{1}{2}\iint K(s,t)f(s)g(t)\,\mathrm{d}t\,\mathrm{d}s =: \frac{I - II}{2}.\end{aligned}$$

Using the antisymmetry of K and interchanging the names of the variables, and applying Fubini's theorem, we find that

$$-II = \iint K(t,s)f(s)g(t)\,\mathrm{d}t\,\mathrm{d}s = \iint K(s,ty)f(t)g(s)\,\mathrm{d}s\,\mathrm{d}t = I.$$

Hence

$$\mathfrak{t}(f,g) = \frac{I - II}{2} = I = \iint K(s,t)f(t)g(s)\,\mathrm{d}t\,\mathrm{d}s,$$

as required for $\mathfrak{t}$ to be a weakly defined singular integral with kernel K.

From the defining formula (12.78) it is immediate that $\mathfrak{t}(\mathbf{1}_Q, \mathbf{1}_Q) = 0$, and hence the quantities featuring in the weak boundedness property of $\mathfrak{t}$ vanish. With $Q \in \mathscr{D}^\omega$ (which still satisfies $\mathbf{1}_Q \in S(\mathscr{D})$ for $\omega \in (\{0,1\}^d)_0^{\mathbb{Z}}$, by Lemma 12.3.30, the same conclusion extends to $\mathfrak{t}^\omega$ for all $\omega \in (\{0,1\}^d)_0^{\mathbb{Z}}$. The rest of the corollary is then a direct consequence of Theorem 12.4.12, simply setting $\|\mathfrak{t}\|_{wbp(\mathscr{D}\mathscr{R}_p)} = \|\mathfrak{t}^\omega\|_{wbp(\mathscr{D}\mathscr{R}_p)} = 0$. We only need to note that $\omega_K^1(\wp) = \omega_K^2(\wp)$ when K is antisymmetric, which is why a seemingly weaker assumption suffices in (12.77). □

Corollary 12.4.19 ($T(1)$ theorem for antisymmetric convolutions). *Let $p \in (1,\infty)$ and $1 \leqslant t \leqslant p \leqslant q \leqslant \infty$, and suppose that:*

(i) *X and Y are UMD spaces.*
(ii) *X has cotype q and Y has type t, or one of them has both.*
(iii) *$K : \dot{\mathbb{R}}^{2d} \to Z := \mathscr{L}(X,Y)$ is an antisymmetric convolution kernel, i.e.,*

$$K(s,t) = \mathfrak{K}(s-t) = -\mathfrak{K}(t-s) \quad \textit{for all } (s,t) \in \dot{\mathbb{R}}^{2d},$$

which satisfies the Calderón–Zygmund estimates (12.74).

(iv) *A bilinear form $\mathfrak{t} : S(\mathscr{D})^2 \to Z$ is defined for all $f, g \in S(\mathscr{D})$ by*

$$\mathfrak{t}(f,g) := \frac{1}{2}\iint \mathfrak{K}(s-t)\big(f(t)g(s) - f(s)g(t)\big)\,\mathrm{d}t\,\mathrm{d}s.$$

Then $\mathfrak{t}$ is well-defined as a weakly defined singular integral with kernel K, which defines a bounded operator $T \in \mathscr{L}(L^p(\mathbb{R}^d;X), L^p(\mathbb{R}^d;Y))$ and satisfies

(a) *the norm estimate*

$$\|T\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \leqslant \beta_{p,X}\beta_{p,Y}\Big\{c_d^0\Big(c_1\tilde{c}_{\mathfrak{K}}(\mathscr{R}_{p'}^*)+c_2\tilde{c}_{\mathfrak{K}}(\mathscr{R}_p)\Big)+ + c_d^1\Big(c_1\|\tilde{\omega}_{\mathfrak{K}}(\mathscr{R}_{p'}^*)\|_{\mathrm{Dini}^{1/t}}+c_2\|\tilde{\omega}_{\mathfrak{K}}(\mathscr{R}_p)\|_{\mathrm{Dini}^{1/q'}}\Big)\Big\},$$

where the supremum is over $\omega \in (\{0,1\}^d)_0^{\mathbb{Z}}$, *the constants* c_d, c_d' *depend only on* d, *and* c_1, c_2 *are as in* (12.70).

(b) *the representation formulas* (12.45) *and* (12.52) *with* $\Lambda_{\mathfrak{t}} = \Lambda_{\mathfrak{t}^\omega} = 0$.

Proof. This is straightforward by combining (the proofs of) Corollaries 12.4.13 and 12.4.18. In particular, in the proof of Corollary 12.4.18 we observed that any bilinear form defined as in (iv) of the present corollary will satisfy $\mathfrak{t}(\mathbf{1}_Q,\mathbf{1}_Q)=0$ for all $Q\in\mathscr{D}$, and hence also $\mathfrak{t}(\mathbf{1}_{Q\dotplus m},\mathbf{1}_{Q\dotplus m})=0=\mathfrak{t}(\mathbf{1}_Q,\mathbf{1}_Q)$ for all $m\in\mathbb{Z}^d$. This is condition (iv) of Corollary 12.4.13 that was not explicitly assumed in the corollary that we are proving. □

Remark 12.4.20. As an immediate consequence of Corollary 12.4.13, we obtain another proof of the essence of Theorem 5.1.13 on the boundedness of the Hilbert transform H on $L^p(\mathbb{R};X)$ whenever $p\in(1,\infty)$ and X is a UMD space. Indeed, take $X=Y$, $t=1$, and $q=\infty$, so that the constants in (12.70) are simply $c_1=c_2=1$. Clearly the kernel $K(u,v)=\pi^{-1}(u-v)^{-1}$ of the Hilbert transform is an antisymmetric convolution kernel, and it is easy to check the Calderón–Zygmund estimates (12.74) with Dini^1 norms. Thus we obtain the estimate

$$\|H\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} \leqslant c\cdot\beta_{p,X}^2,$$

with the same quantitative form as (5.24), aside from the unspecified numerical factor above, in contrast to the explicit constant 2 in (5.24). This is quite natural, considering that (5.24) was obtained by an argument tailored for the very Hilbert transform, whereas the argument that we just sketched was a specialisation of a much more general argument to the particular case of H.

The following corollary provides a solution to the L^p extension problem from Section 2.1 for the important class of Calderón–Zygmund operators:

Theorem 12.4.21 ($T(1)$ theorem for scalar-valued kernels). *Let* $p,s\in(1,\infty)$ *and* $1\leqslant t\leqslant p\leqslant q\leqslant\infty$, *and suppose that:*

(i) X *is a UMD space with cotype* q *and type* t,
(ii) $\mathfrak{t}: S(\mathscr{D})^2\to\mathbb{K}$ *is a weakly defined singular integral, whose kernel* $K:\dot{\mathbb{R}}^{2d}\to\mathbb{K}$ *satisfies the Calderón–Zygmund estimates*

$$c_K+\sum_{i=1}^2\|\omega_K^i\|_{\mathrm{Dini}^{\sigma_i}}<\infty, \tag{12.79}$$

where $\sigma_1=1/t$ *and* $\sigma_2=1/q'$.

Then the following conditions are equivalent:

(1) $\mathfrak{t}$ *defines a bounded operator* $T \in \mathscr{L}(L^p(\mathbb{R}^d;X))$*;*
(2) $\mathfrak{t}$ *defines a bounded operator* $T \in \mathscr{L}(L^s(\mathbb{R}^d))$*;*
(3) $\|\mathfrak{t}^\omega\|_{wbp} \leqslant C$ *uniformly in* $\omega \in (\{0,1\}^d)_0^{\mathbb{Z}}$*, and for some* $b_i \in \mathrm{BMO}(\mathbb{R}^d)$*,*

$$\mathfrak{t}(\mathbf{1}, g) = \langle b_1, g\rangle, \quad \mathfrak{t}(f, \mathbf{1}) = \langle f, b_2\rangle \tag{12.80}$$

for all $f, g \in S_{00}(\mathscr{D}^\omega)$ *and* $\omega \in (\{0,1\}^d)_0^{\mathbb{Z}}$*;*
(4) $\|\mathfrak{t}\|_{wbp} < \infty$*, and* (12.80) *for some* $b_i \in \mathrm{BMO}_{\mathscr{D}}(\mathbb{R}^d)$ *and all* $f, g \in S_{00}(\mathscr{D})$*.*

Under these equivalent conditions, we have

$$\begin{aligned} &\|T\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} \leqslant \tilde{c}_d \beta_{p,X}^2 (c_1 + c_2) c_K + \\ &+ \tilde{c}_d \Big(\beta_{p,X}^2 + pp' \beta_{s,X}^2 \beta_{s,\mathbb{K}}\Big)\Big(\|T\|_{\mathscr{L}(L^s(\mathbb{R}^d))} + \sum_{i=1}^2 c_i \|\omega_K^i\|_{\mathrm{Dini}^{\sigma_i}}\Big), \end{aligned} \tag{12.81}$$

with a dimensional constant $\tilde{c}_d$ *and cotype constants*

$$c_1 = c_{t', X^*; p'}, \qquad c_2 = c_{q, X; p}.$$

In particular, every $L^p(\mathbb{R}^d)$*-bounded Calderón–Zygmund operator having kernel bounds* (12.80) *with* $\sigma_1 = \sigma_2 = 1$*, extends boundedly to* $L^p(\mathbb{R}^d; X)$ *for every UMD space* X*, and one can take* $c_1 = c_2 = 1$ *in the estimate* (12.81)*.*

Proof. (1) $\Rightarrow$ (2): For $s = p$, this is evident by restricting the action of the operator to a one-dimensional subspace of X. The case of general $s \in (1, \infty)$ follows from the Calderón–Zygmund Theorem 11.2.5 (or even just its classical scalar-valued version).

(2) $\Rightarrow$ (3): The weak boundedness property follows from Example 12.1.10:

$$\|\mathfrak{t}^\omega\|_{wbp} \leqslant \|T\|_{\mathscr{L}(L^s(\mathbb{R}^d))}, \tag{12.82}$$

and we turn to the construction of the functions b_i.

The operator $T \in \mathscr{L}(L^s(\mathbb{R}^d))$ is a Calderón–Zygmund operator with kernel K that satisfies in particular the Dini conditions in both variables, and hence both direct and dual (operator-)Hörmander conditions by Lemma 11.3.4. (The qualifier "operator" is redundant for scalar-valued kernels.) By (just the scalar-valued version of) Theorem 11.2.9, T has an extension $\widetilde{T} \in \mathscr{L}(L^\infty(\mathbb{R}^d), \mathrm{BMO}(\mathbb{R}^d)/\mathbb{K})$. By Theorem 11.2.9(b), for functions $\mathbf{1} \in L^\infty(\mathbb{R}^d)$ and $g \in S_{00}(\mathscr{D}^\omega) \subseteq L^\infty_{c,0}(\mathbb{R}^d)$, we have

$$\begin{aligned} \langle \widetilde{T}(\mathbf{1}), g\rangle &= \lim_{M\to\infty} \langle T(\mathbf{1}_{(1+2M)Q}), g\rangle \\ &= \lim_{M\to\infty} \sum_{\substack{m\in\mathbb{Z}^d \\ |m|_\infty \leqslant M}} \mathfrak{t}(\mathbf{1}_{Q\dot{+}m}, g) = \mathfrak{t}(\mathbf{1}, g). \end{aligned}$$

This is one of the claimed identities with $b_1 := \widetilde{T}(\mathbf{1}) \in \mathrm{BMO}(\mathbb{R}^d; Y)$, and Theorem 11.2.9, followed by Lemma 11.3.4, provide us with the estimates

$$\begin{aligned}\|b_1\|_{\mathrm{BMO}^s(\mathbb{R}^d;Y)} &= \|\widetilde{T}(\mathbf{1})\|_{\mathrm{BMO}^s(\mathbb{R}^d)} \\ &\leqslant (c_d\|T\|_{\mathscr{L}(L^s(\mathbb{R}^d)} + \|K\|_{\mathrm{H\ddot{o}r}^*})\|\mathbf{1}\|_{L^\infty(\mathbb{R}^d)} \\ &\leqslant (c_d\|T\|_{\mathscr{L}(L^s(\mathbb{R}^d)} + \sigma_{d-1}\|\omega_K^1\|_{\mathrm{Dini}}).\end{aligned} \tag{12.83}$$

The identity involving $b_2 := \widetilde{T^*}(\mathbf{1})$, and the estimate

$$\begin{aligned}\|b_2\|_{\mathrm{BMO}^{s'}(\mathbb{R}^d)} &= \|\widetilde{T^*}(\mathbf{1})\|_{\mathrm{BMO}^{s'}(\mathbb{R}^d)} \\ &\leqslant \big(c_d\|T^*\|_{\mathscr{L}(L^{s'}(\mathbb{R}^d))} + \\ &\qquad + \|(u,v)\mapsto K(u,v)^*\|_{\mathrm{H\ddot{o}r}^*}\big)\|\mathbf{1}\|_{L^\infty(\mathbb{R}^d)} \\ &\leqslant (c_d\|T\|_{\mathscr{L}(L^s(\mathbb{R}^d))} + \sigma_{d-1}\|\omega_K^2\|_{\mathrm{Dini}})\end{aligned} \tag{12.84}$$

are entirely analogous on the dual side.

(3) $\Rightarrow$ (4): This is obvious by restricting to $\omega = 0$ and noting that $\mathrm{BMO}(\mathbb{R}^d) \subseteq \mathrm{BMO}_{\mathscr{D}}(\mathbb{R}^d)$.

(4) $\Rightarrow$ (1): Under assumption (4), we see that the paraproducts related to $\mathfrak{t}$ are in fact $\varPi_{\mathfrak{t}}^i = \varPi_{b_i}$, where $b_i \in \mathrm{BMO}_{\mathscr{D}}(\mathbb{R}^d)$ by assumption. Thus Corollary 12.2.19 guarantees that

$$\begin{aligned}\|\varLambda_{\mathfrak{t}}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} &= \|\varPi_{b_1} + \varPi_{b_2}^*\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} \\ &\leqslant 64\cdot 8^d \cdot pp'\beta_{s,X}^2\beta_{s,\mathbb{K}}(\|b_1\|_{\mathrm{BMO}^s_{\mathscr{D}}(\mathbb{R}^d)} + \|b_2\|_{\mathrm{BMO}^{s'}_{\mathscr{D}}(\mathbb{R}^d)}).\end{aligned} \tag{12.85}$$

Our assumption (4) also involves $\|\mathfrak{t}\|_{wbp} < \infty$, and Corollary 12.1.9 guarantees that this coincides with the finiteness of $\|\mathfrak{t}\|_{wbp(\mathscr{D}\mathscr{R}_p)} = \|\mathfrak{t}\|_{wbp}$, when $\mathfrak{t}$ is scalar-valued. Thus both assumptions $\|\mathfrak{t}\|_{wbp(\mathscr{D}\mathscr{R}_p)} < \infty$ and $\|\varLambda_{\mathfrak{t}}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} < \infty$ of Theorem 12.4.12(2) are satisfied, hence also the equivalent condition of Theorem 12.4.12(1), and this coincides with condition (1) of the corollary that we are proving.

The quantitative estimates: While we have already closed the chain of implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1), the claimed quantitative bounds require a direct analysis of the implication (3) $\Rightarrow$ (1), which relates to the implication (3) $\Rightarrow$ (1) of Theorem 12.4.12.

As in the proof of "(4) $\Rightarrow$ (1)", under assumption (3), we see that the paraproducts related to $\mathfrak{t}^\omega$ are in fact $\varPi_{\mathfrak{t}^\omega}^i = \varPi_{b_i}^\omega$; while the function $b_i \in \mathrm{BMO}(\mathbb{R}^d) \subseteq \mathrm{BMO}_{\mathscr{D}^\omega}(\mathbb{R}^d)$ is independent of ω, the superscript of the paraproduct signifies the fact that the defining series involves Haar functions and averages related to $Q \in \mathscr{D}^\omega$. Thus, imitating (12.85) and substituting the bounds (12.83) and (12.84), we obtain, with $s_1 := s$ and $s_2 := s'$,

$$\begin{aligned}\|\Lambda_{\mathfrak{t}^\omega}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} &= \|\Pi^\omega_{b_1} + (\Pi^\omega_{b_2})^*\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} \\ &\leqslant 64\cdot 8^d\cdot pp'\beta^2_{s,X}\beta_{s,\mathbb{K}}\sum_{i=1}^2 \|b_i\|_{\mathrm{BMO}^{s_i}(\mathbb{R}^d)} \\ &\leqslant 64\cdot 8^d\cdot pp'\beta^2_{s,X}\beta_{s,\mathbb{K}}\sum_{i=1}^2 \Big(c_d\|T\|_{\mathscr{L}(L^s(\mathbb{R}^d))} + \sigma_{d-1}\|\omega^i_K\|_{\mathrm{Dini}}\Big).\end{aligned} \tag{12.86}$$

where we implicitly dominated $\|b_i\|_{\mathrm{BMO}^{s_i}_{\mathscr{D}^\omega}(\mathbb{R}^d)} \leqslant \|b_i\|_{\mathrm{BMO}^{s_i}(\mathbb{R}^d)}$ in the first estimate. We now substitute (12.86) and (12.82) into the second norm estimate in Theorem 12.4.12(b), noting that all R-bounds and $\mathscr{DR}_p$-bounds may be omitted, since they simply reduce to uniform bounds for scalar-valued functions:

$$\begin{aligned}\|T\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} \leqslant \sup_\omega \|\Lambda_{\mathfrak{t}^\omega}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} + \\ + \beta^2_{p,X}\Big\{4^d \sup_\omega \|\mathfrak{t}^\omega\|_{wbp} + c^0_d(c_1+c_2)c_K + \\ + c^1_d\Big(c_1\|\omega^1_K\|_{\mathrm{Dini}^{1/t}} + c_2\|\omega^2_K\|_{\mathrm{Dini}^{1/q'}}\Big)\Big\}.\end{aligned}$$

This gives the bound asserted in the corollary. □

Remark 12.4.22. If $b_1 = b_2$, the term $pp'\beta^2_{s,X}\beta_{s,\mathbb{K}}$ can be omitted in (12.81). This applies in particular if T is translation-invariant.

Proof. By inspection of the proof of Theorem 12.4.21, the said term only arises in the estimate of $\Lambda_{\mathfrak{t}^\omega}$ in (12.86). Under the assumption that $b_1 = b_2$, we have $\Lambda_{\mathfrak{t}^\omega} = \Lambda^\omega_{b_1}$, and we may replace (12.86) by an application of Theorem 12.2.25:

$$\|\Lambda_{\mathfrak{t}^\omega}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} = \|\Lambda^\omega_{b_1}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} \leqslant 30\cdot 2^d\cdot \beta^2_{p,X}\|b_1\|_{\mathrm{BMO}(\mathbb{R}^d)},$$

where

$$\|b_1\|_{\mathrm{BMO}(\mathbb{R}^d)} \leqslant \|b_1\|_{\mathrm{BMO}^s(\mathbb{R}^d)} \leqslant c_d\|T\|_{\mathscr{L}(L^s(\mathbb{R}^d))} + \sigma_{d-1}\|\omega^1_K\|_{\mathrm{Dini}}.$$

Substituting this alternative estimate into the proof of Theorem 12.4.21, we obtain the claimed modification of (12.81).

If T is translation-invariant, the paraproduct terms vanish, and hence we can take $b_1 = b_2 = 0$, which is indeed a special case of $b_1 = b_2$. Of course, in this case, we do not even need to use Theorem 12.2.25. □

12.4.b The dyadic representation theorem

The randomised dyadic representation (12.52) underlying the proof of $T(1)$ Theorem 12.3.26 can be further reorganised into a form that has proven to be useful for various extensions. Recalling Definition 12.3.34 of the good parts of Figiel's operators, and in particular the quantity $k(n) := 2 + \lceil \log_2 |n| \rceil$, we regroup the sum over $n \in \mathbb{Z}^d \setminus \{0\}$ in (12.52) according to a constant value of $k(n)$ as

$$\sum_{n\in\mathbb{Z}^d\setminus\{0\}} = \sum_{k=2}^{\infty} \sum_{\substack{n\in\mathbb{Z}^d \\ 2^{k-3}<|n|\leqslant 2^{k-2}}} .$$

We denote by $\mathrm{ch}^{(k)}(P)$ the collection of dyadic descendants of P of generation k, and define the operators

$$\begin{aligned} \mathbb{D}_P^{(k)} &:= \sum_{Q\in\mathrm{ch}^{(k)}(P)} \mathbb{D}_Q, \quad \mathbb{E}_P^{(k)} := \sum_{Q\in\mathrm{ch}^{(k)}(P)} \mathbb{E}_Q, \\ \mathbb{D}_P^{[0,k)} &:= \mathbb{E}_P^{(k)} - E_P^{(0)} = \sum_{j=0}^{k-1} \mathbb{D}_P^{(j)}. \end{aligned}$$

Lemma 12.4.23. *If* $\mathfrak{t} : S(\mathscr{D})^2 \to \mathscr{L}(X,Y) =: Z$ *is a weakly defined singular integral with kernel* $K : \dot{\mathbb{R}}^{2d} \to Z$*, then*

$$\begin{aligned} \mathscr{T}_k &:= \sum_{\substack{n\in\mathbb{Z}^d \\ 2^{k-3}<|n|\leqslant 2^{k-2}}} \langle T^{\mathrm{good}}_{n,\mathfrak{t}^\omega} f, g\rangle = \langle S^{(0,k)} f, g\rangle, \\ \mathscr{U}_k^1 &:= \sum_{\substack{n\in\mathbb{Z}^d \\ 2^{k-3}<|n|\leqslant 2^{k-2}}} \langle f, U^{1,\mathrm{good}}_{n,\mathfrak{t}^\omega} g\rangle = \langle S^{(1,k)} f, g\rangle, \\ \mathscr{U}_k^2 &:= \sum_{\substack{n\in\mathbb{Z}^d \\ 2^{k-3}<|n|\leqslant 2^{k-2}}} \langle U^{2,\mathrm{good}}_{n,\mathfrak{t}^\omega} f, g\rangle = \langle S^{(2,k)} f, g\rangle, \end{aligned}$$

where

$$S^{(i,k)} f = \sum_{P\in\mathscr{D}} A_P^{(i,k)} f, \qquad A_P^{(i,k)} f(s) = \fint_P a_P^{(i,k)}(s,t) f(t)\,\mathrm{d}t,$$

and these satisfy the identities

$$\begin{aligned} A_P^{(0,k)} &= \mathbb{D}_P^{(k)} A_P^{(0,k)} \mathbb{D}_P^{(k)}, \\ A_P^{(1,k)} &= \mathbb{D}_P^{(k)} A_P^{(0,k)} \mathbb{D}_P^{[0,k)}, \\ A_P^{(2,k)} &= \mathbb{D}_P^{[0,k)} A_P^{(0,k)} \mathbb{D}_P^{(k)}. \end{aligned} \tag{12.87}$$

For $i = 1, 2$*, we have the further splitting*

$$A_P^{(i,k)} f = A_{P;P}^{(i,k)} f - \sum_{R\in\mathrm{ch}^{(k)}} A_{P;R}^{(i,k)} f$$

where

$$A^{(i,k)}_{P;R} f(s) = \fint_R a^{(i,k)}_{P;R}(s,u) f(u)\,\mathrm{d}u, \qquad R \in \{P\} \cup \mathrm{ch}^{(k)}(P),$$

and these kernels have the bounds

$$\wp(\{a^{(0,k)}_P(s,u), a^{(i,k)}_{P,R}(s,u) : s,u \in R \in \{P\}\cup \mathrm{ch}^{(k)}(P), P \in \mathscr{D}\}) \leqslant c_d \begin{cases} c_K(\wp), & \text{if } 2^k \leqslant 12\sqrt{d}, \\ \omega^i_K(\wp; \frac{6\sqrt{d}}{2^k}), & \text{if } 2^k \geqslant 12\sqrt{d}, \end{cases}$$

Proof. By definition, the left-hand side of the claim is equal to

$$\mathscr{T}_k = \sum_{\substack{n\in\mathbb{Z}^d \\ 2^{k-3}<|n|\leqslant 2^{k-2}}} \sum_{\substack{Q\in\mathscr{D}_{k\text{-good}} \\ \alpha,\gamma\in\{0,1\}^d\setminus\{0\}}} \Big\langle \mathfrak{t}(h^\alpha_Q, h^\gamma_{Q\dot{+}n}) \langle f, h^\alpha_Q\rangle, \langle g, h^\gamma_{Q\dot{+}n}\rangle \Big\rangle,$$

where the k-goodness of Q guarantees that $R := Q\dot{+}n$, for $|n| \leqslant 2^{k-2}$, shares with Q the same kth dyadic ancestor $R^{(k)} = Q^{(k)} =: P \in \mathscr{D}$. Thus we can regroup this series under the ancestors P to get

$$\mathscr{T}_k = \sum_{P\in\mathscr{D}} \sum_{\substack{(Q,R)\in\mathscr{C}_k(P) \\ \alpha,\gamma\in\{0,1\}^d\setminus\{0\}}} \Big\langle \mathfrak{t}_{\mathrm{good}}(h^\alpha_Q, h^\gamma_R) \langle f, h^\alpha_Q\rangle, \langle g, h^\gamma_R\rangle \Big\rangle,$$

where

$$\mathscr{C}_k(P) := \Big\{(Q,R) : \quad Q,R \in \mathrm{ch}^{(k)}(P), \quad \frac{1}{8}\ell(P) < |z_Q - z_R| \leqslant \frac{1}{4}\ell(P)\Big\}.$$

The subseries under each $P \in \mathscr{D}$ takes the asserted form $\langle A^{(k)}_P f, g\rangle$ if we define

$$a^{(0,k)}_P(s,u) := |P| \sum_{\substack{(Q,R)\in\mathscr{C}_k(P) \\ \alpha,\gamma\in\{0,1\}^d\setminus\{0\}}} \mathfrak{t}_{\mathrm{good}}(h^\alpha_Q, h^\gamma_R) h^\alpha_Q(u) h^\gamma_R(s).$$

The cases of $\mathscr{U}^i_k$ are analogous, and lead to representations of the same form with

$$a^{(1,k)}_P(s,u) := |P| \sum_{\substack{(Q,R)\in\mathscr{C}_k(P) \\ \gamma\in\{0,1\}^d\setminus\{0\}}} \mathfrak{t}_{\mathrm{good}}(h^0_Q, h^\gamma_R)[h^0_Q(u) - h^0_R(u)] h^\gamma_R(s),$$

and

$$a^{(2,k)}_P(s,u) := |P| \sum_{\substack{(Q,R)\in\mathscr{C}_k(P) \\ \alpha\in\{0,1\}^d\setminus\{0\}}} \mathfrak{t}_{\mathrm{good}}(h^\alpha_Q, h^0_R) h^\alpha_Q(u)[h^0_R(s) - h^0_Q(s)],$$

The further splitting is then naturally defined with

$$a_{P;P}^{(1,k)}(s,u) := |P| \sum_{\substack{(Q,R)\in\mathscr{C}_k(P)\\ \gamma\in\{0,1\}^d\setminus\{0\}}} \mathfrak{t}_{\rm good}(h_Q^0, h_R^\gamma) h_Q^0(u) h_R^\gamma(s),$$

$$a_{P;R}^{(1,k)}(s,u) := |R| \sum_{\substack{Q:(Q,R)\in\mathscr{C}_k(P)\\ \gamma\in\{0,1\}^d\setminus\{0\}}} \mathfrak{t}_{\rm good}(h_Q^0, h_R^\gamma) h_R^0(u) h_R^\gamma(s), \quad R\in \operatorname{ch}^k(P),$$

where the last summation runs over all relevant $Q \in \operatorname{ch}^k(P)$, for fixed R. Observe that $a_{P;R}^{(1,k)}$ has the factor $|R|$ in front, instead of $|P|$, due to our definition of $A_{P;R}^{(1,k)} f(s)$ as the average integral $\fint_R a_{P;R}^{(1,k)}(s,u) f(u)\,\mathrm{d}u$.

The splitting of $a_P^{(2,k)}$ is entirely analogous; in particular,

$$a_{P;Q}^{(2,k)}(s,u) := |Q| \sum_{\substack{R:(Q,R)\in\mathscr{C}_k(P)\\ \alpha\in\{0,1\}^d\setminus\{0\}}} \mathfrak{t}_{\rm good}(h_Q^\alpha, h_R^0) h_Q^\alpha(u) h_Q^0(s), \quad Q\in \operatorname{ch}^k(P).$$

It remains to verify that these operators and their kernels satisfy the asserted properties. The identity $A_P^{(0,k)} = \mathbb{D}_P^{(k)} A_P^{(0,k)} \mathbb{D}_P^{(k)}$ is immediate from the orthogonality of the Haar functions, and the invariance of $A_P^{(i,k)}$ under composition by $\mathbb{D}_P^{(k)}$ on the side, where the cancellative Haar function appear in $a_P^{(i,k)}$ is justified similarly. Concerning the factors $\mathbb{D}_P^{[0,k)}$, we note that the are orthogonal projections onto functions supported on P, constant on each $Q \in \operatorname{ch}^{(k)}(Q)$, and integrating to zero. Noting the functions $h_Q^0 - h_R^0$ belong to this class then justifies the remaining parts of the claimed identities.

Concerning the claimed bounds, we note that any given $(s,u) \in P \times P$ is contained in exactly one $Q \times R$ with $Q, R \in \operatorname{ch}^{(k)}(P)$, and moreover,

$$|h_Q^\alpha \otimes h_R^\gamma| = \frac{1_{Q\times R}}{|Q|^{1/2}|R|^{1/2}} = \frac{2^{kd}}{|P|} 1_{Q\times R}.$$

The claimed $\wp$-bounds for $a_P^{(0,k)}(s,u)$, as well as for $a_{P;P}^{(i,k)}(s,u)$, then follow from Lemma 12.4.8, noting that the factor $|P|$ in the definition of these kernels cancels with the $\frac{1}{|P|}$ above.

For $a_{P;R}^{(1,k)}$ with $R \in \operatorname{ch}^{(k)}(P)$, all terms in the defining sum are supported on the same $1_{R\times R}$, and each individual summand can be estimates by Lemma 12.4.8. We now have the smaller factor $|R|$ in front, but at the same time there are up to 2^{kd} terms in the sum, all of which accumulate on the same support now. Since $2^{kd}|R| = |P|$, we get the same final bound as before. The case of $a_{P;Q}^{(1,k)}$ with $Q \in \operatorname{ch}^{(k)}(P)$ in entirely analogous, and completes the proof. □

Definition 12.4.24. *An operator $S : S_{00}(\mathscr{D};X) \to S_{00}(\mathscr{D};Y)$ is called a dyadic shift of type (i,k), where $i \in \{0,1,2\}$ and $k \in \{2,3,\ldots\}$, if*

$$S=\sum_{P\in\mathscr{D}}A_P,\qquad A_Pf(s)=A_{P;P}f(s)-\sum_{Q\in\operatorname{ch}^{(k)}(P)}A_{P;Q}f(s),$$

where

$$A_{P;R}f(s)=\fint_R a_{P;R}(s,u)f(u)\,\mathrm{d}u,\qquad R\in\{P\}\cup\operatorname{ch}^{(k)}(P),$$

$\operatorname{supp} a_{P;R}\subseteq R\times R$,

$$\|S\|_{\mathrm{Shift}(\wp)}:=\wp\Big(\Big\{a_{P;R}(s,u):s,u\in R\in\{P\}\cup\operatorname{ch}^{(k)}(P),P\in\mathscr{D}\Big\}\Big)<\infty$$

for $\wp=\mathscr{R}_2$*, and moreover, for every* $P\in\mathscr{D}$,

(0) *if* $i=0$*, then* $A_P=\mathbb{D}_K^{(k)}A_P\mathbb{D}_K^{(k)}$*, and* $A_{P;Q}=0$ *for all* $Q\in\operatorname{ch}^{(k)}(P)$*;*
(1) *if* $i=1$*, then* $A_P=\mathbb{D}_K^{(k)}A_P\mathbb{D}_K^{[0,k)}$*;*
(2) *if* $i=2$*, then* $A_P=\mathbb{D}_K^{[0,k)}A_P\mathbb{D}_K^{(k)}$*.*

We say that a shift has type $i\in\{0,1,2\}$*, if it has type* (i,k) *with some* k*.*

Remark 12.4.25. In the language of Definition 12.4.24, the operators $S^{(i,k)}$ of Lemma 12.4.23 are dyadic shifts of type (i,k), and we may further write

$$\|S^{(i,k)}\|_{\mathrm{Shift}(\wp)}\leqslant c_d\begin{cases}c_K(\wp), & \text{if } 2^k\leqslant 12\sqrt{d},\\ \omega_K^i(\wp;\frac{6\sqrt{d}}{2^k}), & \text{if } 2^k\geqslant 12\sqrt{d}.\end{cases}$$

The key boundedness properties of these dyadic shifts are contained in the following:

Theorem 12.4.26. *Let* X *and* Y *be UMD spaces, and* $p\in(1,\infty)$*. Suppose that* X *has cotype* q *and* Y *has type* t *for some* $1\leqslant t\leqslant p\leqslant q\leqslant\infty$*.*

Then for all $i\in\{0,1,2\}$ *and* $k\in\{2,3,\dots\}$*, all dyadic shifts* S *of type* (i,k) *extends to a bounded operator from* $L^p(\mathbb{R}^d;X)$ *to* $L^p(\mathbb{R}^d;Y)$*. Moreover, they satisfy the norm estimates*

$$\|S\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))}\leqslant 4\cdot\beta_{p,X}\beta_{p,Y}\times\begin{cases}\|S\|_{\mathrm{Shift}(\mathscr{R}_p)}c_{t',Y^*;p'}\cdot k^{1/t}, & i=1,\\ \|S\|_{\mathrm{Shift}(\mathscr{R}^*_{p'})}c_{q,X;p}\cdot k^{1/q'}, & i=2;\end{cases}$$

and the norm of a shift of type $(0,k)$ *is bounded by the minimum of these two bounds, but with* 6 *in place of* 4*.*

Proof. We divide the proof into case according to the type of the shift under consideration.

Shifts of type 1

Let us start with the case $i = 1$. For $f \in S_{00}(\mathbb{R}^d;X) \subseteq L^p(\mathbb{R}^d;X)$ and $g \in S_{00}(\mathbb{R}^d;Y) \subseteq L^{p'}(\mathbb{R}^d;Y^*)$, we expand the pairing $\langle Sf, g\rangle$ by separating the scales according to $\log_2 \ell(P) \mod k$:

$$\begin{aligned}
|\langle Sf,g\rangle| &= \Big|\sum_{j=0}^{k-1} \sum_{\substack{P\in\mathscr{D}\\ \log_2\ell(P)\equiv j\\ \bmod k}} \Big\langle \mathbb{D}_P^{(k)} A_P f, \mathbb{D}_P^{(k)} g\Big\rangle\Big| \\
&\leqslant \sum_{j=0}^{k-1} \Big\| \sum_{\substack{P\in\mathscr{D}\\ \log_2\ell(P)\equiv j\\ \bmod k}} \varepsilon_P \mathbb{D}_P^{(k)} A_P f\Big\|_{L^p(\Omega\times\mathbb{R}^d;Y)} \\
&\qquad \times \Big\| \sum_{\substack{P\in\mathscr{D}\\ \log_2\ell(P)\equiv j\\ \bmod k}} \varepsilon_P \mathbb{D}_P^{(k)} g\Big\|_{L^{p'}(\Omega\times\mathbb{R}^d;Y^*)} =: \sum_{j=0}^{k-1} I_j \times II_j.
\end{aligned}$$

In I_j, we write out $\mathbb{D}_P^{(k)} = \sum_{Q\in\mathrm{ch}^{(k)}(P)} \mathbb{D}_Q$ and note that, in a randomised sum like here, we are free to replace ε_Q by ε_P, since the difference is invisible to the $L^p(\Omega;Y)$ at a fixed $s\in\mathbb{R}^d$. This gives

$$I_j = \Big\| \sum_{\substack{P\in\mathscr{D}\\ \log_2\ell(P)\equiv j\\ \bmod k}} \sum_{Q\in\mathrm{ch}^{(k)}(P)} \varepsilon_Q \mathbb{D}_Q A_P f\Big\|_{L^p(\mathbb{R};Y)}.$$

Using the splitting of A_P, it then follows that

$$\begin{aligned}
I_j &\leqslant \Big\| \sum_{\substack{P\in\mathscr{D}\\ \log_2\ell(P)\equiv j\\ \bmod k}} \sum_{Q\in\mathrm{ch}^{(k)}(P)} \varepsilon_Q \mathbb{D}_Q A_{P;P} f\Big\|_{L^p(\Omega\times\mathbb{R};Y)} \\
&\qquad + \Big\| \sum_{\substack{P\in\mathscr{D}\\ \log_2\ell(P)\equiv j\\ \bmod k}} \sum_{Q\in\mathrm{ch}^{(k)}(P)} \varepsilon_Q \mathbb{D}_Q A_{P;Q} f\Big\|_{L^p(\Omega\times\mathbb{R};Y)} \\
&=: III_j + IV_j.
\end{aligned}$$

We first consider IV_j. Denoting by Q_s the unique dyadic child of Q that contains a given $s \in Q$, and with the understanding that $\mathbb{D}_Q$ acts in the s variable, we have

$$\begin{aligned}
\mathbb{D}_Q A_{P;Q} f(s) &= \fint_Q \mathbb{D}_Q a_{P;Q}(s,u) \mathbb{D}_P^{[0,k)} f(u)\,\mathrm{d}u \\
&= \mathbf{1}_Q(s) \fint_Q \Big(\langle a_{P;Q}(\cdot,u)\rangle_{Q_s} - \langle a_{P;Q}(\cdot,u)\rangle_Q\Big)(\langle f\rangle_Q - \langle f\rangle_P)\,\mathrm{d}u
\end{aligned}$$

$$=: \alpha_{P;Q}(s)\mathbf{1}_Q(s)(\langle f\rangle_Q - \langle f\rangle_P) = \alpha_{P;Q}(s)\mathbf{1}_Q(s)\mathbb{D}_P^{[0,k)}(s),$$

where

$$\alpha_{P;Q}(s) := \fint_Q \Big(\langle a_{P;Q}(\cdot,u)\rangle_{Q_s} - \langle a_{P;Q}(\cdot,u)\rangle_Q\Big)\,\mathrm{d}u$$

belongs to the two-fold multiple of the absolute convex hull of the set appearing in the definition of $\|S\|_{\mathrm{Shift}(\wp)}$. Thus

$$\begin{aligned} IV_j &= \Big\| \sum_{\substack{P\in\mathscr{D}\\ \log_2\ell(P)\equiv j\\ \mathrm{mod}\ k}} \sum_{Q\in\mathrm{ch}^{(k)}(P)} \varepsilon_Q\alpha_{P;Q}\mathbf{1}_Q\mathbb{D}_P^{[0,k)}f\Big\|_{L^p(\Omega\times\mathbb{R};Y)} \\ &\leqslant 2\|S\|_{\mathrm{Shift}(\mathscr{R}_p)}\Big\| \sum_{\substack{P\in\mathscr{D}\\ \log_2\ell(P)\equiv j\\ \mathrm{mod}\ k}} \sum_{Q\in\mathrm{ch}^{(k)}(P)} \varepsilon_Q\mathbf{1}_Q\mathbb{D}_P^{[0,k)}f\Big\|_{L^p(\Omega\times\mathbb{R};X)} \\ &\leqslant 2\|S\|_{\mathrm{Shift}(\mathscr{R}_p)}\Big\| \sum_{\substack{P\in\mathscr{D}\\ \log_2\ell(P)\equiv j\\ \mathrm{mod}\ k}} \varepsilon_P\mathbb{D}_P^{[0,k)}f\Big\|_{L^p(\Omega\times\mathbb{R};X)}, \end{aligned}$$

using the identity $\sum_{Q\in\mathrm{ch}^{(k)}(P)}\mathbf{1}_Q = \mathbf{1}_P$ and the interchangeability of ε_P and ε_Q in the random sum in the last step.

Observing that $(\mathbb{D}_P^{[0,k)}f)_{\log_2\ell(P)\equiv j \mod k}$ is a martingale difference decomposition of f for each $j\in\{0,\ldots,k-1\}$ to deduce directly from the definition of the UMD constants that

$$IV_j \leqslant 2\|S\|_{\mathrm{Shift}(\mathscr{R}_p)}\beta^+_{p,X}\|f\|_{L^p(\mathbb{R}^d;X)}.$$

We then turn to term III_j. By the exchangeability of ε_P and ε_Q again, this can be written as

$$III_j = \Big\| \sum_{\substack{P\in\mathscr{D}\\ \log_2\ell(P)\equiv j\\ \mathrm{mod}\ k}} \varepsilon_P\mathbb{D}_P^{(k)}A_{P;P}f\Big\|_{L^p(\Omega\times\mathbb{R};Y)},$$

where

$$\mathbb{D}_P^{(k)}A_{P;P}f(s) = \fint_P \mathbb{D}_P^{(k)}a_{P;P}(s,u)\mathbb{D}_P^{[0,k)}f(u)\,\mathrm{d}u,$$

and it is understood that $\mathbb{D}_P^{(k)}$ acts with respect to the s variable.

We will now make use of the tangent martingale construction as in Corollary 4.4.15 and explained just before the statement of the said result: For every $P\in\mathscr{D}$, let T_P be a copy of P equipped with the normalised measure $\nu_P := |P|^{-1}m|_P$, where m is the Lebesgue measure, and consider the product space $T := \prod_{P\in\mathscr{D}} T_P$ with probability measure $\nu := \otimes_{P\in\mathscr{D}}\nu_P$. Writing a typical element of T as $\mathbf{t} = (t_P)_{P\in\mathscr{D}}$, we then have

$$\mathbb{D}_P^{(k)} A_{P;P} f(s) = \int_T \mathbb{D}_P^{(k)} a_{P;P}(s, t_P) \mathbb{D}_P^{[0,k)} f(t_P) \, \mathrm{d}\nu(\mathbf{t}).$$

Hence (suppressing, as usual, the dependence of random functions on $\omega \in \Omega$),

$$\begin{aligned} III_j &= \Big\| s \mapsto \int_T \sum_{\substack{P \in \mathscr{D} \\ \log_2 \ell(P) \equiv j \\ \bmod k}} \varepsilon_P \mathbb{D}_P^{(k)} a_{P;P}(s, t_P) \mathbb{D}_P^{[0,k)} f(t_P) \, \mathrm{d}\nu(\mathbf{t}) \Big\|_{L^p(\Omega \times \mathbb{R}^d; Y)} \\ &\leqslant \Big\| (s, \mathbf{t}) \mapsto \sum_{\substack{P \in \mathscr{D} \\ \log_2 \ell(P) \equiv j \\ \bmod k}} \varepsilon_P \mathbf{1}_P(s) \mathbb{D}_P^{(k)} a_{P;P}(s, t_P) \mathbb{D}_P^{[0,k)} f(t_P) \Big\|_{L^p(\Omega \times \mathbb{R}^d \times T; Y)}. \end{aligned}$$

Here, $\mathbb{D}_P^{(k)} a_{P;P}(s, t_P)$ is the difference of two averages $\langle a_{P;P}(\cdot, t_P) \rangle_Q$, and hence in twice the absolute convex hull of the set in the definition of $\|S\|_{\mathrm{Shift}(\wp)}$. Thus, the definition of R-boundedness implies that

$$III_j \leqslant 2\|S\|_{\mathrm{Shift}(\mathscr{R}_p)} \Big\| (s, \mathbf{t}) \mapsto \sum_{\substack{P \in \mathscr{D} \\ \log_2 \ell(P) \equiv j \\ \bmod k}} \varepsilon_P \mathbf{1}_P(s) \mathbb{D}_P^{[0,k)} f(t_P) \Big\|_{L^p(\Omega \times \mathbb{R}^d \times T; X)}.$$

We are now in a position to apply Corollary 4.4.15. Indeed, the functions $\mathbb{D}_P^{[0,k)} f$ are "atoms" in the sense defined before that corollary: $\mathbb{D}_P^{[0,k)} f$ is supported on P, of average 0, and constant on all $P' \in \mathrm{ch}^{(k)}(P)$, which are the next smaller cubes in the scales-separated dyadic system $\{P \in \mathscr{D} : \log_2 \ell(P) \equiv j \mod k\}$. Thus, a direct application of Corollary 4.4.15 to

$$f = \sum_{\substack{P \in \mathscr{D} \\ \log_2 \ell(P) \equiv j \\ \bmod k}} \mathbb{D}_P^{[0,k)} f$$

shows that

$$\Big\| (s, \mathbf{t}) \mapsto \sum_{\substack{P \in \mathscr{D} \\ \log_2 \ell(P) \equiv j \\ \bmod k}} \varepsilon_P \mathbf{1}_P(s) \mathbb{D}_P^{[0,k)} f(t_P) \Big\|_{L^p(\Omega \times \mathbb{R}^d \times T; X)} \leqslant \beta_{p,X} \|f\|_{L^p(\mathbb{R}^d; X)},$$

and hence

$$III_j \leqslant 2\|S\|_{\mathrm{Shift}(\mathscr{R}_p)} \beta_{p,X} \|f\|_{L^p(\mathbb{R}^d; X)}.$$

Combining this with the estimate for term IV_j (and estimating the one-sided UMD constant by the basic UMD constant), we deduce that

$$I_j \leqslant III_j + IV_j \leqslant 4\|S\|_{\mathrm{Shift}(\mathscr{R}_p)} \beta_{p,X} \|f\|_{L^p(\mathbb{R}^d; X)}.$$

Hence

$$|\langle Sf,g\rangle| \leqslant \sum_{j=0}^{k-1} I_j \times II_j \leqslant 4\|S\|_{\mathrm{Shift}(\mathscr{R}_p)}\beta_{p,X}\|f\|_{L^p(\mathbb{R}^d;X)}\sum_{j=0}^{k-1} II_j,$$

where

$$\begin{aligned}\sum_{j=0}^{k-1} II_j &\leqslant k^{1/t}\Big(\sum_{j=0}^{k-1} II_j^{t'}\Big)^{\frac{1}{t'}} \\ &= k^{1/t}\Big(\sum_{j=0}^{k-1}\Big\| \sum_{\substack{P\in\mathscr{D}\\ \log_2\ell(P)\equiv j\\ \mathrm{mod}\ k}} \varepsilon_P \mathbb{D}_P^{(k)} g\Big\|_{L^{p'}(\Omega\times\mathbb{R}^d;Y^*)}^{t'}\Big)^{\frac{1}{t'}} \\ &\leqslant k^{1/t}\cdot c_{t',Y^*;p'}\Big\|\sum_{P\in\mathscr{D}}\varepsilon_P\mathbb{D}_P^{(k)} g\Big\|_{L^{p'}(\Omega\times\mathbb{R}^d;Y^*)} \\ &\leqslant k^{1/t}\cdot c_{t',Y^*;p'}\cdot\beta^+_{p',Y^*}\|g\|_{L^{p'}(\mathbb{R}^d;Y^*)}.\end{aligned}$$

Here $\beta^+_{p',Y^*} \leqslant \beta_{p',Y^*} = \beta_{p,Y}$ by Proposition 4.2.17(2), and $c_{t',Y^*;p'} \leqslant \tau_{t,Y;p}$ by Proposition 7.1.13 (or its easy extension to deal with the third index in these constants). This completes the proof for shift of type $(1,k)$.

Shifts of type 2

For a shift of type $(2,k)$, we note that its adjoint S^* is a shift of type $(1,k)$, and hence

$$\begin{aligned}\|S\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} &= \|S^*\|_{\mathscr{L}(L^{p'}(\mathbb{R}^d;Y^*),L^{p'}(\mathbb{R}^d;X^*))} \\ &\leqslant 4\|S^*\|_{\mathrm{Shift}(\mathscr{R}_{p'})}\beta_{p',Y^*}\beta_{p',X^{**}}c_{q,X;p}k^{1/q'} \\ &= 4\|S\|_{\mathrm{Shift}(\mathscr{R}^*_{p'})}\beta_{p,Y}\beta_{p,X}c_{q,X;p}k^{1/q'},\end{aligned}$$

which is the asserted bound in this case.

Shifts of type 0

Let finally S be a shift of type $(0,k)$. We can then proceed as in the case of type $(1,k)$ with slight modifications: In view of the eventual application of the tangent martingale estimate of Corollary 4.4.15, we now separate scales by $k+1$ levels instead of k, since $\mathbb{D}_P^{(k)} f$ is only guaranteed to be constant on $Q \in \mathrm{ch}^{(k+1)}(P)$. On the other hand, we now have $IV_j = 0$, and hence $I_j = III_j$.

Following the argument in the case of type $(1,k)$ leads to

$$III_j \leqslant 2\|S\|_{\mathrm{Shift}(\mathscr{R}_p)}\Big\|(s,\mathbf{t})\mapsto \sum_{\substack{P\in\mathscr{D}\\ \log_2\ell(P)\equiv j\\ \mathrm{mod}\ k+1}} \varepsilon_P \mathbf{1}_P(s)\mathbb{D}_P^{(k)} f(t_P)\Big\|_{L^p(\Omega\times\mathbb{R}^d\times T;X)}.$$

To complete the estimate, we will need a little additional trick compared to the previous cases. First, we observe that

$$\mathbb{D}_P^{(k)} = (I - \mathbb{E}_P^{(k)})\mathbb{D}_P^{[0,k+1)}.$$

Second, we have

$$\begin{aligned}\mathbb{E}_P^{(k)} f(t_P) &= \mathbb{E}(f|\sigma(\mathrm{ch}^{(k+1)}(P)))(t_P) \\ &= \mathbb{E}\Big(\mathbf{t} \mapsto f(t_P)\Big| \bigotimes_{Q\in\mathscr{D}} \sigma(\mathrm{ch}^{(k+1)}(Q))\Big) =: \mathbb{E}\Big(\mathbf{t}\mapsto f(t_P)\Big|\mathscr{G}_{k+1}\Big),\end{aligned}$$

where on the right-hand side we take a conditional expectation with respect to a product σ-algebra on the product probability space T, of a function that only depends on the "coordinate" t_P of $\mathbf{t} \in T$. The importance of this last formula comes from the fact that only the function inside the conditional expectation, but not the conditional expectation operator itself, depends on the dyadic cube P. Using the previous two formulas, it follows that

$$\begin{aligned}&\Big\| \sum_{\substack{P\in\mathscr{D} \\ \log_2 \ell(P)\equiv j \\ \mathrm{mod}\ k+1}} \varepsilon_P \mathbf{1}_P(s)\mathbb{D}_P^{(k)} f(t_P)\Big\|_{L^p(\Omega\times\mathbb{R}^d\times T;X)} \\ &\leqslant \Big\| \sum_{\substack{P\in\mathscr{D} \\ \log_2 \ell(P)\equiv j \\ \mathrm{mod}\ k+1}} \varepsilon_P \mathbf{1}_P(s)\mathbb{D}_P^{[0,k+1)} f(t_P)\Big\|_{L^p(\Omega\times\mathbb{R}^d\times T;X)} \\ &\qquad + \Big\|\mathbb{E}\Big(\sum_{\substack{P\in\mathscr{D} \\ \log_2 \ell(P)\equiv j \\ \mathrm{mod}\ k+1}} \varepsilon_P \mathbf{1}_P(s)\mathbb{D}_P^{[0,k+1)} f(t_P)\Big|\mathscr{G}_{k+1}\Big)\Big\|_{L^p(\Omega\times\mathbb{R}^d\times T;X)} \\ &\leqslant 2\Big\| \sum_{\substack{P\in\mathscr{D} \\ \log_2 \ell(P)\equiv j \\ \mathrm{mod}\ k+1}} \varepsilon_P \mathbf{1}_P(s)\mathbb{D}_P^{[0,k+1)} f(t_P)\Big\|_{L^p(\Omega\times\mathbb{R}^d\times T;X)}\end{aligned}$$

by the contractivity of the conditional expectation in the last step. This last expression has the same form as what we encountered with shifts of type $(1, k)$, only with $k + 1$ in place of k. Thus, by an application of the tangent martingale inequality of Corollary 4.4.15, we have

$$\Big\| \sum_{\substack{P\in\mathscr{D} \\ \log_2 \ell(P)\equiv j \\ \mathrm{mod}\ k+1}} \varepsilon_P \mathbf{1}_P(s)\mathbb{D}_P^{[0,k+1)} f(t_P)\Big\|_{L^p(\Omega\times\mathbb{R}^d\times T;X)} \leqslant \beta_{p,X}\|f\|_{L^p(\mathbb{R}^d;X)}.$$

Thus,

$$I_j = III_j \leqslant 4\|S\|_{\mathrm{Shift}(\mathscr{R}_p)}\beta_{p,X}\|f\|_{L^p(\mathbb{R}^d;X)},$$

which is the same bound as for the corresponding terms in the estimate of shifts of type $(1, k)$. The rest of the argument is exactly the same, only with $k + 1$ in place of k, and leads to the conclusion that

$$\|S\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \leqslant 4\|S\|_{\mathrm{Shift}(\mathscr{R}_p)}\beta_{p,X}\beta_{p,Y}c_{t',Y^*;p'}(k+1)^{1/t}.$$

Since the adjoint of a shift of type $(0,k)$ is another shift of the same type, we also obtain

$$\|S\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \leqslant 4\|S\|_{\mathrm{Shift}(\mathscr{R}^*_{p'})}\beta_{p,X}\beta_{p,Y}c_{q,X;p}(k+1)^{1/q'},$$

and we can take the minimum of the two bounds. Since $k \geqslant 2$, we can also make the trivial estimates $k+1 \leqslant \frac{3}{2}k$ and $4\cdot(\frac{3}{2})^{1/v} \leqslant 6$ for $v \in \{t, q'\}$ so that in case $v \geqslant 1$. □

With the help of the shifts, we can represent any weakly defined singular integral with appropriate kernel bounds as follows:

Theorem 12.4.27 (Dyadic Representation Theorem). *Let $p \in (1,\infty)$ and $1 \leqslant t \leqslant p \leqslant q \leqslant \infty$, and suppose that:*

(i) *X and Y are UMD spaces,*
(ii) *X has cotype q and Y has type t,*
(iii) *$\mathfrak{t} : S(\mathscr{D})^2 \to Z := \mathscr{L}(X,Y)$ is a weakly defined singular integral and the kernel $K : \dot{\mathbb{R}}^{2d} \to Z$ of $\mathfrak{t}$ satisfies the Calderón–Zygmund estimates*

$$c_K(\mathscr{R}_p) + \|\omega^1_K(\mathscr{R}_p)\|_{\mathrm{Dini}^{1/t}} + \|\omega^2_K(\mathscr{R}_p)\|_{\mathrm{Dini}^{1/q'}} < \infty.$$

Then the following conditions are equivalent:

(1) *$\mathfrak{t}$ defines a bounded operator $T \in \mathscr{L}(L^p(\mathbb{R}^d;X), L^p(\mathbb{R}^d;Y))$;*
(2) *each $\mathfrak{t}^\omega$ satisfies the weak $\mathscr{D}\mathscr{R}_p$-boundedness property $\|\mathfrak{t}^\omega\|_{wbp(\mathscr{D}\mathscr{R}_p)} \leqslant C$, and the associated bi-paraproduct $\Lambda_{\mathfrak{t}^\omega}$ defines a bounded operator in $\mathscr{L}(L^p(\mathbb{R}^d;X), L^p(\mathbb{R}^d;Y))$, uniformly in $\omega \in (\{0,1\}^d)^{\mathbb{Z}}_0$.*

Under these equivalent conditions, we have

(a) *the dyadic representation formula*

$$\langle Tf, g\rangle = \mathbb{E}\Big(\langle \mathfrak{H}_{\mathfrak{t}^\omega} f, g\rangle + \langle \Lambda_{\mathfrak{t}^\omega} f, g\rangle + \sum_{\substack{k=2\\ i\in\{0,1,2\}}}^{\infty} \langle S^{(i,k)}_\omega f, g\rangle\Big)$$

with absolute convergence for all $f \in S(\mathscr{D};X)$ and $g \in S(\mathscr{D};Y^*)$, where $\mathbb{E}$ is the expectation over $\omega \in (\{0,1\}^d)^{\mathbb{Z}_{\leqslant M}}$, and $M \in \mathbb{Z}$ is any large enough number such that f and g are constant on all $Q \in \mathscr{D}_M$; the operators $\mathfrak{H}_{\mathfrak{t}^\omega}$ and $\Lambda_{\mathfrak{t}^\omega}$ are a Haar multiplier and a paraproduct as in (12.52), and each $S^{(i,k)}_\omega$ is a dyadic shift of type (i,k) (Definition 12.4.24) with respect to the dyadic system $\mathscr{D}^\omega$ and with shift norms estimated by

$$\|S^{(i,k)}_\omega\|_{\mathrm{Shift}(\wp)} \leqslant c_d \begin{cases} c_K(\wp), & \text{if } 2^k \leqslant 12\sqrt{d},\\ \omega^i_K(\wp; \frac{6\sqrt{d}}{2^k}), & \text{if } 2^k \geqslant 12\sqrt{d};\end{cases}$$

(b) *the resulting norm estimate:*

$$\begin{aligned}\|T\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} - \sup_{\omega} \|\Lambda_{\mathfrak{t}^\omega}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \\ \leqslant \beta_{p,X}\beta_{p,Y}\Big\{4^d \sup_{\omega} \|\mathfrak{t}^\omega\|_{wbp(\mathscr{D}\mathscr{R}_p)} + c_d^0\Big(c_K(\mathscr{R}_p) + c_K(\mathscr{R}_{p'}^*)\Big) + \\ + c_d^1\Big(c_{t',Y^*;p'}\|\omega_K^1(\mathscr{R}_p)\|_{\mathrm{Dini}^{1/t}} + c_{q,X;p}\|\omega_K^2(\mathscr{R}_{p'}^*)\|_{\mathrm{Dini}^{1/q'}}\Big)\Big\},\end{aligned}$$

where the suprema are over $\omega \in (\{0,1\}^d)_0^{\mathbb{Z}}$, *and the constants* c_d^0, c_d^1 *depend only on* d

Proof. We note that the present assumptions coincide with those of Theorem 12.4.12, except that (ii) of the present theorem is slightly stronger than (ii) of Theorem 12.4.12. Thus the equivalence of (1) and (2) is just repetition from Theorem 12.4.12.

The first new claim is the dyadic representation formula (a). To see this, recall that Theorem 12.4.12 gave the representation formula (12.52), repeated for convenience as

$$\begin{aligned}\langle Tf,g\rangle = \mathbb{E}\Big(\langle \mathfrak{H}_{\mathfrak{t}^\omega} f,g\rangle + \langle \Lambda_{\mathfrak{t}^\omega} f,g\rangle + 2^d \sum_{\substack{n\in\mathbb{Z}^d\\ n\neq 0}} \Big\{\langle T_{n,\mathfrak{t}^\omega}^{\mathrm{good}} f,g\rangle + \\ + \langle f, U_{n,\mathfrak{t}^\omega}^{1,\mathrm{good}} g\rangle + \langle U_{n,\mathfrak{t}^\omega}^{2,\mathrm{good}} f,g\rangle\Big\}\Big),\end{aligned}$$

where f, g, and $\mathbb{E}$ have the same meaning as in the claimed formula (a). On the other hand, Lemma 12.4.23 and Remark 12.4.25 inform us that the summation of the three types of terms over $n \in \mathbb{Z}^d \setminus \{0\}$ can be rearranged into a sum over $k \geqslant 2$ and $i \in \{0,1,2\}$ exactly as in the assertion.

From the representation (a), we can then estimate

$$\begin{aligned}\|T\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} - \sup_{\omega} \|\Lambda_{\mathfrak{t}^\omega}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \\ \leqslant \sup_{\omega} \Big(\|\mathfrak{H}_{\mathfrak{t}^\omega}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \\ + \sum_{\substack{k=2\\ i\in\{0,1,2\}}}^{\infty} \|S_\omega^{(i,k)}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))}\Big).\end{aligned}$$

The first term here is estimated as in the proof of Theorem 12.4.12 by $4^d\|\mathfrak{t}\|_{wbp(\mathscr{D}\mathscr{R}_p)}$. For the remaining sum over shifts, we obtain from Theorem 12.4.26 (using this theorem with trivial type $t = 1$ for small k, and as stated for large k) that

$$\sum_{k=2}^{\infty} \|S_\omega^{(1,k)}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))}$$

$$\begin{aligned}
&\leqslant 4\cdot \beta_{p,X}\beta_{p,Y}\Big(\sum_{k:1\leqslant 2^k\leqslant 12\sqrt{d}} \|S_\omega^{(1,k)}\|_{\mathrm{Shift}(\mathscr{R}_p)}\cdot k \\
&\qquad\qquad + \sum_{k:2^k\geqslant 12\sqrt{d}} \|S_\omega^{(1,k)}\|_{\mathrm{Shift}(\mathscr{R}_p)} c_{t',Y^*;p'}\cdot k^{1/t}\Big) \\
&\leqslant c_d\beta_{p,X}\beta_{p,Y}\Big(\sum_{k:1\leqslant 2^k\leqslant 12\sqrt{d}} c_K(\mathscr{R}_p)k \\
&\qquad\qquad + c_{t',Y^*;p'}\sum_{k:2^k\geqslant 12\sqrt{d}} \omega_K^1(\mathscr{R}_p;\frac{6\sqrt{d}}{2^k})k^{1/t}\Big) \\
&\leqslant c_d'\beta_{p,X}\beta_{p,Y}\Big(c_K(\mathscr{R}_p)+c_{t',Y^*;p'}\|\omega_K^1(\mathscr{R}_p)\|_{\mathrm{Dini}^{1/t}}\Big)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\sum_{k=2}^\infty \|S_\omega^{(2,k)}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \\
&\qquad \leqslant c_d'\beta_{p,X}\beta_{p,Y}\Big(c_K(\mathscr{R}_{p'}^*)+c_{q,X;p}\|\omega_K^2(\mathscr{R}_{p'}^*)\|_{\mathrm{Dini}^{1/q'}}\Big).
\end{aligned}$$

Finally, The sum over shifts of type $(0,k)$ may be estimated by either of the two bounds above (the different numerical constant in Theorem 12.4.26 is in any case absorbed into the unspecified dimensional constant). □

Remark 12.4.28. The norm estimate obtained in Theorem 12.4.27 via the representation in terms of dyadic shifts is essentially the same as that in Theorem 12.4.12 obtained via Figiel's representation. While the proof of Theorem 12.4.27 partially relied on the proof of Theorem 12.4.12 to avoid repetition, a larger part of the machinery behind the proof of Theorem 12.4.12, relying in particular on Figiel's Theorems 12.1.25 and 12.1.28 concerning his elementary operators, was replaced in the proof of Theorem 12.4.27 by Theorem 12.4.26 on the dyadic shifts, which in turn was based on the tangent martingale bounds of Corollary 4.4.15.

12.5 Notes

Section 12.1

The Haar multipliers $\mathfrak{H}_\lambda = \mathfrak{H}_\lambda^{\alpha\alpha}$ are special cases of martingale transforms discussed extensively in Volume I; see in particular Sections 3.5 and 4.2.e. In this framework, the predictable sequences multiplying the martingale differences

$$\mathbb{D}_k^\alpha f := \sum_{Q\in\mathscr{D}_k}\langle f,h_Q^\alpha\rangle h_Q^\alpha, \qquad \mathbb{D}_k^{-\alpha} f := \sum_{Q\in\mathscr{D}_k}\mathbb{D}_Q^{-\alpha} f$$

are then

$$v_k^{\alpha} = \sum_{Q\in\mathscr{D}_k} \lambda_k \mathbf{1}_{Q_k} \in L^\infty(\sigma(\mathscr{D}_k); \mathscr{L}(X,Y)), \qquad v_k^{-\alpha} \equiv 0.$$

On the other hand, the Haar multipliers $\mathfrak{H}_\lambda^{\alpha\gamma}$ with $\alpha \neq \gamma$ already take a departure from the general theory, and this is even more so with the general operators of Figiel.

(Note that the conventional indices of dyadic analysis and martingale theory are off by one from each other. In martingale theory, it is customary to emphasise measurability, and hence the indices of martingale differences agree with those of the σ-algebra that makes them measurable, while predictable multipliers are measurable with respect to the "previous" σ-algebra with index $k-1$. In dyadic analysis, the emphasis is on the supporting dyadic cubes, and hence the "kth" martingale difference $\mathbb{D}_k f$ is the sum of the local martingale differences $\mathbb{D}_Q$ supported, and averaging to zero, on the dyadic cubes $Q \in \mathscr{D}_k$, but then they are actually measurable only with respect to the "next" $\sigma(\mathscr{D}_{k+1})$; at the same time, the "predictable" multipliers are then measurable with respect to the σ-algebra indicated by their index.)

The relaxed R-boundedness notion $\mathscr{D}\mathscr{R}_p$ of Definition 12.1.6 seems to be new, but the slightly stronger $\mathscr{E}\mathscr{R}_p$ appears implicitly in Di Plinio, Li, Martikainen, and Vuorinen [2020b, Remark 6.29], where it is shown that the family $|Q|^{-1}\langle T\mathbf{1}_Q, \mathbf{1}_Q\rangle$ of Example 12.1.10 has this property when $T \in \mathscr{L}(L^p(\mathbb{R}^d;X), L^p(\mathbb{R}^d;Y))$ and X and Y are UMD spaces; this also follows by combining our Example 12.1.10 (on the $\mathscr{D}\mathscr{R}_p$ property of this family) and Corollary 12.1.17 (the equivalence of $\mathscr{D}\mathscr{R}_p$ and $\mathscr{E}\mathscr{R}_p$ for UMD spaces). An advantage of the new $\mathscr{D}\mathscr{R}_p$ is that it allows Example 12.1.10 without any assumptions on the Banach spaces.

The exact characterisation of the boundedness of the Haar multipliers $\mathfrak{H}_\lambda$ in Theorem 12.1.11 is new; by Lemma 12.1.8 and Propositions 12.1.13 and 12.1.14, the characterising condition is strictly more general than the R-boundedness condition $\|x \mapsto \mathscr{R}(\{\lambda_Q : x \in Q \in \mathscr{D}\})\|_\infty < \infty$. This seems at first to contradict Girardi and Weis [2005], where the necessity of uniform pointwise R-boundedness for operator-valued martingale transforms is established. This apparent contradiction is resolved by observing that, in order to obtain this necessity of R-boundedness, Girardi and Weis [2005] actually assume that their transforming sequence $(v_k)_{k\geqslant 1}$ is allowed to multiply any subsequence $(df_{n_k})_{k\geqslant 1}$ of the martingale difference sequence $(df_k)_{k=1}^\infty$, i.e., they assume the boundedness of the family of operators $f \mapsto \sum_{k\geqslant 1} v_k df_{n_k}$ instead of just $f \mapsto \sum_{k\geqslant 1} v_k df_k$. In the case of Haar multipliers, this would mean that, for a given sequence $\lambda = (\lambda_Q)_{Q\in\mathscr{D}}$ we would consider a family of operators including in particular all

$$f \mapsto \sum_{Q\in\mathscr{D}} \lambda_{Q^{(k)}} \langle f, h_Q^\alpha\rangle h_Q^\alpha,$$

where $k \in \mathbb{N}$ and $Q^{(k)}$ is the k generations larger dyadic ancestor of Q. However, in particular situations like that of Propositions 12.1.13, each coefficient λ_Q is naturally associated to a unique cube Q only.

The underlying ideas of Section 12.1.b come from Figiel [1988], and they have been developed further by Hytönen [2006], but substantial details of the present treatment are new. Figiel [1988] also introduced the elementary operators T and U and proved the first versions of Theorems 12.1.25 and 12.1.28. A novelty of the present treatment, also reflected in the auxiliary considerations in Section 12.1.b, is to set up the argument in such a way as to obtain a reasonably efficient dependence of the estimates on the UMD constants, although we make no claims concerning sharpness. A technical point was to use the decomposition of Lemma 12.1.22 in such a way that the parts of the decomposition contribute additively, rather than multiplicatively, to the operator norms in Theorems 12.1.25 and 12.1.28; while this seems only natural in retrospect, it was not the case with earlier treatments of the analogous bounds by Figiel [1988] and Hytönen [2006]. This proof detail only affects the constants in the final estimates, which was not a concern in these earlier works.

Besides the "dyadic singular integrals" studied in this section, there are related classes of operators that might be regarded as "dyadic pseudo-differential operators", in that their symbol depends on both the spatial variable $s \in \mathbb{R}^d$ and the "dyadic frequency variable" $I \in \mathscr{D}$. These are the generalised Haar multipliers

$$\mathfrak{H}_{\lambda(s)} f(s) = \sum_{I \in \mathscr{D}} \lambda_I(s) \langle f, h_I \rangle h_I(s),$$

where each coefficient $\lambda_I(\cdot)$ is a function. A primary example considered by Katz and Pereyra [1999] consists of

$$\lambda_I(s) = w_I^t(s) := \Big(\frac{w(s)}{\langle w \rangle_I} \Big)^t,$$

where $t \in \mathbb{R}$ and w is in a (dyadic) A_p or (dyadic) reverse Hölder class. Given the close relation of their techniques to those of the present section, it seems likely that some of the results concerning the operators $\mathfrak{H}_{\lambda(\cdot)}$ could be generalised to functions taking values in a UMD space, but this line of research seems not to have been pursued so far.

Section 12.2

In analogy with the quote of Stein [1982] on square functions at the beginning of Chapter 9, also the concept of paraproduct is "not an idea in its pure form, but rather takes various shapes depending on the uses it is put to". A friendly overview to this variety of "shapes and uses" of paraproducts can be found in Bényi, Maldonado, and Naibo [2010]. Paraproducts were systematically introduced by Bony [1981], but Bényi et al. [2010] convincingly argue that their first version is already implicit in the treatment of commutators of singular integrals by Calderón [1965].

Our treatment concentrates on *dyadic* paraproducts. We are uncertain about the earliest appearance of this notion in the literature but it was certainly known to Figiel [1990]; according to this paper, the $L^p(\mathbb{R}^d;X)$-boundedness of the dyadic paraproduct with a scalar-valued $b \in \mathrm{BMO}_{\mathscr{D}}(\mathbb{R}^d)$ "relies on an estimate due to Jean Bourgain (October 1987, unpublished)". This argument was only presented in print much later by Figiel and Wojtaszczyk [2001]. In particular, Corollary 12.2.19 goes back to these works. The first results on the boundedness of operator-valued paraproducts on UMD spaces were obtained by Hytönen and Weis [2006b] for a Fourier-analytic cousin of the dyadic paraproduct that we have treated. A sufficient condition similar to Proposition 12.2.16, in terms of a version of the Carleson norm, was identified there under the name of "Littlewood–Paley–BMO" norm. The condition of Theorem 12.2.18, in terms of $\mathrm{BMO}(\mathbb{R}^d;Z)$ with values in a UMD subspace $Z \hookrightarrow \mathscr{L}(X,Y)$, is also implicit in Hytönen and Weis [2006b], and explicitly formulated by Hytönen [2006]. However, both Hytönen and Weis [2006b] and Hytönen [2006] also required an additional R-boundedness condition, most easily formulated by the requirement that the unit ball $\bar{B}_Z$ of Z should be an R-bounded subset of $\mathscr{L}(X,Y)$. This condition was found to be superfluous by Hytönen [2014] when revising the argument for an extension to non-doubling measures, a generality that we have not considered here. The details of the present approach are largely borrowed from Hänninen and Hytönen [2016], where several simplifications were found when specialising the considerations back to the case of the Lebesgue measure. A particular novelty of Hänninen and Hytönen [2016], which we have followed, was to estimate the vector-valued paraproduct directly in $L^p(\mathbb{R}^d;Y)$, in contrast to earlier arguments that achieved the L^p bounds only via interpolation from auxiliary end-point estimates between the Hardy space $H^1(\mathbb{R}^d;X)$ and $L^1(\mathbb{R}^d;Y)$ on the one hand, and between $L^\infty(\mathbb{R}^d;X)$ and $\mathrm{BMO}(\mathbb{R}^d;Y)$ on the other hand.

Theorem 12.2.25 on the boundedness of the symmetric paraproduct Λ_b is from Hytönen [2021]. The case when $p=2$ and $X=Y$ is a Hilbert space was obtained earlier by Blasco and Pott [2008], and extended to any $p \in (1,\infty)$ and any non-commutative $L^p(\mathcal{M})$ space (with the same p) by Mei [2010]. (Recall that $L^p(\mathcal{M})$ is a UMD space for $p \in (1,\infty)$—the case of Schatten classes, due to Bourgain [1986], is treated in Proposition 5.4.2, while the general case can be found in Berkson et al. [1986b]—so the mentioned result of Mei [2010] is indeed a special case of Theorem 12.2.25.) The auxiliary material on projective tensor products is classical; much more on this topic can be found in Ryan [2002].

Theorem 12.2.26 on the dimensional growth of the norms of operator-valued paraproducts is from Mei [2006]. The optimal dimensional dependence in the estimate

$$\|\Pi_b\|_{\mathscr{L}(L^2(\mathbb{R};\ell^2_N))} \leqslant \psi(N)\|b\|_{\mathrm{BMO}^{\mathrm{so}}_{\mathscr{D}}(\mathbb{R};\mathscr{L}(\ell^2_N))} := \psi(N) \sup_{u\in\bar{B}_{\ell^2_N}} \|b(\cdot)u\|_{\mathrm{BMO}_{\mathscr{D}}(\mathbb{R};\ell^2_N)}.$$

had been settled some years earlier: Independently, Katz [1997] and Nazarov, Treil, and Volberg [1997b] proved that $\psi(N) \lesssim 1 + \log N$, and the latter authors also obtained the preliminary lower bound $\psi(N) \gtrsim (1 + \log N)^{1/2}$. This was improved to $\psi(N) \gtrsim 1 + \log N$ by Nazarov, Pisier, Treil, and Volberg [2002a]. For a while, there were hopes in the air of obtaining a dimension-free estimate with $\mathrm{BMO}_{\mathscr{D}}(\mathbb{R}; \mathscr{L}(\ell^2_N))$ in place of $\mathrm{BMO}^{\mathrm{so}}_{\mathscr{D}}(\mathbb{R}; \mathscr{L}(\ell^2_N))$ on the right. Some indications that made this plausible are discussed in the introduction of Mei [2006] who, however, destroyed such hopes were by the main result of that paper, reproduced as Theorem 12.2.26. In combination with the upper bound by Katz [1997] and Nazarov et al. [1997b] just mentioned, it shows that $1 + \log N$ is the optimal upper bound for $\|\Pi_b\|_{\mathscr{L}(L^2(\mathbb{R};\ell^2_N))}/\|b\|_{F(\mathbb{R};\mathscr{L}(\ell^2_N))}$ for any of the choices $F \in \{\mathrm{BMO}^{\mathrm{so}}_{\mathscr{D}}, \mathrm{BMO}_{\mathscr{D}}, L^\infty\}$.

Further relations between various BMO-type quantities and the norms of related transformations in infinite-dimensional Hilbert spaces have been studied by Blasco and Pott [2008, 2010]. Analogous results in the context of the operator-valued BMOA space of analytic functions are due to Rydhe [2017].

Section 12.3

We refer the reader to the Notes of the following section for an account of the $T(1)$ theorem in its more traditional meaning as a boundedness criterion for Calderón–Zygmund operators (as in the title of David and Journé [1984]). The section under discussion presents a rather non-canonical approach to this theory, introduced and described by Figiel [1990] as follows:

> Our approach is indirect in the following sense. Rather than trying to prove that some "classical" operators are bounded, we start from considering certain rather new operators, which in our opinion have a basic nature. (All the "singularities" which can occur in our context are neatly packaged inside the basic operators.) Having established precise estimates for the norms of those basic operators, we can take up the "general case". We just look at the class of those operators which can be realised as the sum of an absolutely convergent (in the operator norm) operator series whose summands are simple compositions of our basic operators. Then it turns out that the choice was sufficiently efficient for that class to contain so-called generalised Calderón–Zygmund operators and much more.

A large part of this section, up to and including $T(1)$ Theorem 12.3.26, is an updated review of Figiel [1990], incorporating a few elaborations:

- the trade-off between the type and cotype properties of the underlying spaces and the minimal rate of convergence of the coefficients of the bilinear form, as in Theorem 12.3.26(ii) (which is implicit in the combination of Figiel [1988, 1990]);

- conditions involving R-boundedness to deal with operator-valued versions (first introduced into the context of $T(1)$ theorems at large by Hytönen and Weis [2006b] and into Figiel's approach by Hytönen [2006]);
- keeping track of, and optimising the argument for, the quantitative dependence on parameters like the UMD constants (which seems new for this "non-random" version of the $T(1)$ theorem, involving—in contrast to Theorem 12.3.35—one dyadic system $\mathscr{D}$ only).

The decomposition (12.36) of $\mathfrak{t}(f,g)$ into three *one-parameter* series, in contrast to the perhaps more obvious *two-parameter* decomposition

$$\mathfrak{t}(f,g) = \sum_{i,j} \mathfrak{t}(D_i f, D_j g),$$

was already used by Figiel [1990], but it is frequently referred to as the "BCR algorithm" after Beylkin, Coifman, and Rokhlin [1991]. They explored its advantages for the numerical evaluation of singular integrals, also making a connection with the $T(1)$ theorem but apparently independently of Figiel [1990]. A decade later in 2002, when two of the present authors started to investigate a Banach space valued $T(1)$ theorem (eventually published in Hytönen and Weis [2006b]), they were also initially unaware of the work of Figiel [1990], which was first brought to their attention by Hans-Olav Tylli. Ever since, the approach of Figiel [1990] has been highly influential for the development of the theory of Banach space valued singular integrals.

The second $T(1)$ Theorem 12.3.35, which makes use of a random choice of the dyadic system $\mathscr{D}^\omega$, has a history of its own. This method, referred to by its inventors as "pulling ourselves by hair", was introduced by Nazarov, Treil, and Volberg [1997a] to tackle the difficulties in estimating singular integrals with respect to a *non-doubling* measure μ, thus going beyond the established theory in spaces of homogeneous type due to Coifman and Weiss [1971]. Their original idea consisted of splitting a function into its "good" and "bad" parts, according to the "good" and "bad" cubes supporting the martingale differences $\mathbb{D}_Q f$:

$$f^\omega_{\rm good} := \sum_{Q\in\mathscr{D}^\omega_{\rm good}} \mathbb{D}_Q f, \quad f^\omega_{\rm bad} := \sum_{Q\in\mathscr{D}^\omega_{\rm bad}} \mathbb{D}_Q f,$$

and showing that the latter is small, "on average", with respect to a random choice of ω:

$$\mathbb{E}\|f^\omega_{\rm bad}\|_{L^2(\mu)} \leqslant \varepsilon\|f\|_{L^2(\mu)}.$$

As a result, it is enough to estimate (an *a priori* bounded) operator T of "good" functions only. Namely, if

$$|\langle Tf^\omega_{\rm good}, g^\omega_{\rm good}\rangle| \leqslant C\|f^\omega_{\rm good}\|_2\|g^\omega_{\rm good}\|_2 \leqslant C\|f\|_2\|g\|_2,$$

then

$$\begin{aligned}|\langle Tf,g\rangle| &\leqslant |\langle Tf^{\omega}_{\rm good},g^{\omega}_{\rm good}\rangle|+|\langle Tf^{\omega}_{\rm good},g^{\omega}_{\rm bad}\rangle|+|\langle Tf^{\omega}_{\rm bad},g\rangle|\\ &\leqslant C\|f\|_2\|g\|_2+\|T\|\,\|f\|_2\|g^{\omega}_{\rm bad}\|_2+\|T\|\,\|f^{\omega}_{\rm bad}\|_2\|g\|_2.\end{aligned}$$

Taking the expectations of both sides, it follows that

$$|\langle Tf,g\rangle|\leqslant C\|f\|_2\|g\|_2+2\varepsilon\|T\|\,\|f\|_2\|g\|_2,$$

hence

$$\|T\|\leqslant C+2\varepsilon\|T\|,\qquad \|T\|\leqslant\frac{C}{1-2\varepsilon}.$$

This method was successfully applied and further developed by Nazarov, Treil, and Volberg [2002b, 2003]. The latter work was extended to Banach space valued singular integrals with respect to non-doubling measures by Hytönen [2014]. The first arXiv version of this paper was posted already in 2008, and hence it was available to provide the backbone for the proof of the A_2 theorem in Hytönen [2012] (arXiv 2010); see the Notes of Chapter 11 for more on the latter. It was for the purposes of the A_2 theorem that a technical elaboration of the averaging method of Nazarov, Treil, and Volberg [1997a, 2002b, 2003] had to be invented: "on average", the bad part is not only small but completely absent. This allows the replacement of the estimates above by *identities* of the type

$$\langle Tf,g\rangle=\mathbb{E}\langle T^{\omega}_{\rm good}f,g\rangle.$$

The observation that one can combine this averaging method with Figiel's decomposition of singular integrals in order to simplify the latter, and thereby obtain sharper quantitative conclusions (notably, a quadratic dependence on the UMD constant), was then made in Hytönen [2012] (arXiv 2011), where a version of Theorem 12.3.35 (for scalar kernels and under vanishing paraproduct conditions) was first established. The question of obtaining a linear dependence on the UMD constant is an outstanding open problem already in the special case of the Hilbert transform (see Problem O.6); but of course a possible counterexample could be more feasible within the larger class of operators covered by Theorem 12.3.35. A positive answer has been obtained for sufficiently smooth *even* singular integrals on $L^p(\mathbb{R};X)$ by Pott and Stoica [2014]; their result depends on the same averaging trick and the resulting dyadic representation theorem, but then applies different techniques to complete the estimate.

While our approach to the "random" $T(1)$ Theorem 12.3.35 took a detour via the "non-random" $T(1)$ Theorem 12.3.26, we should emphasise that this is by no means necessary; rather, in many recent extensions of the $T(1)$ theorem, one starts with the randomised set-up from the beginning, and it is often not even clear whether this could be avoided. We will say more about some of these extensions later in these Notes. The reasons that we have chosen to present also the non-random $T(1)$ Theorem 12.3.26 are (at least) two-fold: On the one hand, we feel that there is some historical documentary value in providing (probably) the first detailed exposition of the original Banach space valued $T(1)$ theorem of Figiel [1990], considering also the number of other results in the literature relying on this in their proofs (although, in many cases, one could alternatively apply one or several of the more recent variants). On the other hand, the non-random $T(1)$ Theorem 12.3.26 is not in all respects subsumed by the random $T(1)$ Theorem 12.3.35, which makes the first one applicable in some situations where the latter one is not, and it might hence be useful for the reader to keep the original $T(1)$ Theorem 12.3.26 in their toolbox.

While we are not aware of many such applications, here is at least one: *Pseudo-localisation* of singular integrals refers to estimates of the form

$$\|\mathbf{1}_{\complement \Sigma_{f,s}} Tf\|_{L^p(\mathbb{R}^d;X)} \leqslant \phi(s)\|f\|_{L^p(\mathbb{R}^d;X)}, \qquad s \in \mathbb{N},$$

where

$$\Sigma_{f,s} := \bigcup\{9Q : Q \in \mathscr{D}, \mathbb{D}_Q^{(s)} f \not\equiv 0\}, \quad \mathbb{D}_Q^{(s)} f := \sum_{R \in \mathrm{ch}^s Q} \mathbb{D}_R f,$$

and the point is obtaining a quantitative decay $\phi(s) \to 0$ as $s \to \infty$. Case $p = 2$ was considered by Parcet [2009] for $X = \mathbb{K}$ and by Mei and Parcet [2009] for a Hilbert space X, with applications to non-commutative Calderón–Zygmund and Littlewood–Paley theory, respectively. An extension to $p \in (1, \infty)$ and a UMD space X was obtained by Hytönen [2011] using a version of the $T(1)$ Theorem 12.3.26. This leads to studying a bilinear form whose Haar coefficients satisfy a non-standard estimate of the form

$$|\mathfrak{t}(h_Q^\alpha, h_{Q\dot{+}m}^\gamma)| \lesssim |m|^{-(d+\varepsilon)}\mathbf{1}_{(2\cdot 2^s,\infty)}(|m|) + |m|^{-d}\mathbf{1}_{(4\cdot 2^s-2,4\cdot 2^s+2)}(|m|).$$

The first term on the right with decay $d + \varepsilon$ is typical, but the second one, without any ε, is not. However, this term is only supported in a relatively narrow region of values of the parameter $m \in \mathbb{Z}^d$, which still allows one to make favourable estimates of the Figiel norms of $\mathfrak{t}$.

A notable aspect of this application is that the construction of the set $\Sigma_{f,s}$ refers to a fixed dyadic system $\mathscr{D}$, which calls for a Haar expansion of the operator in terms of this same $\mathscr{D}$, as in the non-random $T(1)$ Theorem 12.3.26, and seems to prevent any effective application of the random systems $\mathscr{D}^\omega$, as in the random $T(1)$ Theorem 12.3.35. This suggests that, even after the successful recent (and very likely future) development of $T(1)$ theorems and other results based on random dyadic systems, the non-random $T(1)$ Theorem 12.3.26 might not become completely obsolete.

Section 12.4

The classical theory of Calderón and Zygmund [1952] had its focus on convolution operators. Their $L^2(\mathbb{R}^d)$ boundedness is amenable to methods of Fourier analysis, which then serves as a starting point for extrapolation to other $L^p(\mathbb{R}^d)$ and different function spaces, as discussed at length in Chapter 11. It was observed quite early, notably by Coifman and Weiss [1971], that these extrapolation aspects of the theory could be extended to much greater generality, certainly including non-convolution operators on $\mathbb{R}^d$ and much more. On the other hand, the boundedness of some prominent non-convolution operators was obtained by different methods over the years, including the commutators of Calderón [1965, 1977], and the *Cauchy integral on a Lipschitz graph*, which we give in the parametrised form

$$\mathscr{C}_A f(s) := \text{p.v.} \int_{-\infty}^{\infty} \frac{f(t)\,\mathrm{d}t}{s - t + i(A(s) - A(t))}.$$

The boundedness of $\mathscr{C}_A$ was first established, in the case of a small Lipschitz constant $\|A\|_{\mathrm{Lip}}$, by Calderón [1977], and eventually in full generality by Coifman, McIntosh, and Meyer [1982]. However, a general criterion for verifying the $L^2(\mathbb{R}^d)$ boundedness of any given Calderón–Zygmund operators was missing.

The first such general criterion was provided by the "$T(1)$ theorem" of David and Journé [1984]. In its original formulation, this theorem stated that an operator $T : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$, with a Calderón–Zygmund standard kernel, extends to a bounded operator on $L^2(\mathbb{R}^d)$, if and only if it satisfies the following three conditions, from which the name of the theorem (also introduced by David and Journé [1984] in the title of the first section of their paper) is derived:

(i) $T(1) \in \mathrm{BMO}(\mathbb{R}^d)$,
(ii) $T^*(1) \in \mathrm{BMO}(\mathbb{R}^d)$,
(iii) T has the weak boundedness property.

Despite being a complete and elegant characterisation, giving, e.g., the results of Calderón [1977] as a quick corollary, it turned out that it is not always feasible to use this theorem for some operators. As a prime example, the theorem of Coifman, McIntosh, and Meyer [1982] could not be directly recovered by David and Journé [1984], since $\mathscr{C}_A(1)$ does not admit an expression whose BMO norm could be easily estimated.

This shortcoming was fixed by the more general "$T(b)$ theorem" of David, Journé, and Semmes [1985], which replaced (i) and (ii) by the more flexible conditions

(i') $T(b_1) \in \mathrm{BMO}(\mathbb{R}^d)$,
(ii') $T^*(b_2) \in \mathrm{BMO}(\mathbb{R}^d)$,

where one is free to choose the pair of functions $b_i \in L^\infty(\mathbb{R}^d)$ subject only to the restriction that they be *accretive* (meaning $\Re b_i \geqslant \delta > 0$ almost everywhere) or just *para-accretive* (a technical generalisation, for which we refer the interested reader to the original paper). In particular, one can take $b_i = 1 + iA'$ for which the computation of (any finite truncations of) $\mathscr{C}_A(1 + iA')$ is easy.

While also this $T(b)$ theorem has been extended to UMD spaces by Hytönen [2006], the need for this is perhaps not as great as in the scalar-valued case, at least as far as the extension of the boundedness of scalar-valued Calderón–Zygmund operators to $L^p(\mathbb{R}^d;X)$ is concerned. The reason for this is that, while it might be difficult to check the $T(1)$ conditions (i) and (ii) *directly*, they can nevertheless be verified by the converse direction of the $T(1)$ theorem, provided that the $L^2(\mathbb{R}^d)$ boundedness of T is already known by some other method (such as the $T(b)$ theorem). This is, in essence, the point of the scalar-kernel $T(1)$ Theorem 12.4.21.

Corollary 12.5.1. *Let X be a UMD space, $p \in (1,\infty)$, and $A : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. Then the Cauchy integral on a Lipschitz graph $\mathscr{C}_A$ extends to a bounded operator on $L^2(\mathbb{R};X)$ and*

$$\|\mathscr{C}_A\|_{\mathscr{L}(L^p(\mathbb{R};X))} \leqslant c_A pp' \cdot \beta_{2,X}^2,$$

where c_A is a constant that depends on A only.

Sketch of proof. By the theorem of Coifman, McIntosh, and Meyer [1982], the operator $\mathscr{C}_A$ is bounded on $L^2(\mathbb{R})$. It is straightforward to verify that the kernel of $\mathscr{C}_A$ is a standard kernel, and hence verifies the assumptions of Theorem 12.4.21 with Dini[1] conditions (and associated constants depending only on A), in which case only trivial type and cotype is needed. Thus Theorem 12.4.21, with $s = p = 2$, proves the corollary for $p = 2$. While we could apply Theorem 12.4.21 with $s = 2$ and any $p \in (1,\infty)$, a better quantitative conclusion for $p \neq 2$ is obtained by using case $p_0 = 2$ as input to the Calderón–Zygmund theorem 11.2.5, which then yields the asserted bound for all $p \in (1,\infty)$. □

Corollary 12.5.1 seems to have been first stated in Hytönen [2006]; however, given that it is essentially a concatenation of its scalar case due to Coifman, McIntosh, and Meyer [1982], and the $T(1)$ theorem of Figiel [1990], it was probably "known to experts" much earlier. The case when X is a UMD lattice was established by a different method already by Rubio de Francia [1986].

In a similar way, the extension of the non-homogeneous $T(1)$ theorem of Nazarov, Treil, and Volberg [2003] to UMD spaces has the following consequence:

Theorem 12.5.2. *Let μ be a positive non-atomic Radon measure on $\mathbb{C}$. Then the following conditions are equivalent*

(1) *There is a constant $c < \infty$ such that, for every disk $D = D(z,r) \subseteq \mathbb{C}$, the measure μ satisfies*
  (a) *the* linear growth condition $\mu(D(z,r)) \leqslant cr$, *and*
  (b) *the* local curvature condition

$$\iiint_{D\times D\times D} \frac{\mathrm{d}\mu(u)\,\mathrm{d}\mu(v)\,\mathrm{d}\mu(z)}{R(u,v,z)} \leqslant c\mu(D),$$

where $R(u,v,z)$ is the radius of the circle through u,v,z (understood as ∞, if the points are collinear).

(2) *The Cauchy integral*

$$\mathscr{C}_\mu f(u) := \int_{\mathbb{C}} \frac{f(v)\,\mathrm{d}\mu(v)}{u-v}$$

defines a bounded operator on $L^2(\mu)$.

(3) *For every UMD space X and every $p \in (1,\infty)$, the Cauchy integral $\mathscr{C}_\mu$ defines a bounded operator on $L^p(\mu;X)$.*

Note that $\mathscr{C}_A$ is (equivalent to) the special case, where μ is the arc-length measure on the graph $\{(t,A(t)) : t \in \mathbb{R}\}$.

Sketch of proof. The implication (2)⇒(1a) is due to David [1991] and (2)⇒(1b) due to Melnikov and Verdera [1995] and Mattila, Melnikov, and Verdera [1996]. The sufficiency of these geometric conditions, (1)⇒(2), was proved by Tolsa [1999].

The implication ((1a) and (2))⇒(3) follows from an analogue of Theorem 12.4.21 for measures on $\mathbb{R}^d$ with the power growth bound $\mu(B(s,r)) \leqslant cr^n$ $(0 < n \leqslant d)$, which is one of the main results of Hytönen [2014]. The implication (3)⇒(2) is trivial. □

This proof sketch highlights the role of $T(1)$ theorems as a device for extending deep results about the boundedness of specific operators from scalar-valued to vector-valued spaces, without the need to revisit the details of the original arguments. Indeed, by using the scalar-valued result (2) as an intermediate step, the equivalence of (1) and (3) is obtained without ever having to deal with the local curvature condition (1b) in the context of vector-valued functions!

Our operator-kernel $T(1)$ Theorem 12.4.12 is the outcome of a line of evolution starting with the first such results obtained by Hytönen and Weis [2006b] and Hytönen [2006], and continued with several variants and extensions addressing

- non-homogeneous measures (Hytönen [2014] (arXiv 2008), Martikainen [2012a] (arXiv 2010), Hytönen and Vähäkangas [2015]);
- simplifications of the underlying decomposition of the operator (Hytönen [2012], Hänninen and Hytönen [2016]);
- sharper conclusions under additional symmetry assumptions (Pott and Stoica [2014], Hytönen [2021]);
- product-space/multiparameter singularities (Di Plinio and Ou [2018], Hytönen, Martikainen, and Vuorinen [2019a]);
- multilinear operators (Di Plinio, Li, Martikainen, and Vuorinen [2020b], Airta, Martikainen, and Vuorinen [2022]).

While these papers extend the theory into several directions that we have not considered here, many of them also provide valuable pieces of insight into the basic case of linear Calderón–Zygmund operators on $\mathbb{R}^d$ with the Lebesgue measure, which we have tried to incorporate into the present treatment. Despite this extensive background material, some aspects of our present $T(1)$ Theorem 12.4.12 appear to be new:

(1) For the first time, we are able to state an operator-valued $T(1)$ theorem that gives a *characterisation* (as in the scalar-valued $T(1)$ theorem of David and Journé [1984]), and not just a *sufficient condition* (as in all operator-valued papers cited above), for the boundedness of a Calderón–Zygmund operator with an operator-valued kernel. This depends on two recent ideas, the combination of which appears here for the first time:
   (a) Replacing the (sufficient but not necessary) weak R-boundedness property of most of the previous contributions by the correct weak $\mathscr{D}\mathscr{R}_p$-boundedness property. As discussed in the Notes of Section 12.1, this idea is from Di Plinio, Li, Martikainen, and Vuorinen [2020b].
   (b) Treating the bi-paraproduct $\Lambda = \Pi_{T(1)} + \Pi^*_{T^*(1)}$ as a single object, and making its boundedness into a condition in its own right, rather than trying (in vain) to force it into a form involving some operator-valued BMO space. This is implicit in Hytönen [2021].

(2) Recording the quantitative dependence of the estimate in terms of both the UMD and the (co)type constants, and optimising the argument for what seems to be the best possible bound currently available. This was available in important special cases (notably in Hytönen [2012]), and arguably implicit in some other works, but seems to be original as an explicit statement in the present generality.

Consequences of the $T(1)$ theorem

The "$T(1)$ theorem for convolution kernels", Corollary 12.4.13, is a somewhat untypical statement, in that convolution kernels have been usually treated by more traditional Fourier-analytic methods, rather than the $T(1)$ technology. As such, this very formulation seems to be new. However, essentially the same class of operators was considered with Fourier methods by Hytönen and Weis [2007]. (Despite the publication year, this paper was actually the first joint project of its authors, which they completed and submitted in 8/2002, before starting their follow-up work on the $T(1)$ theorem, Hytönen and Weis [2006b], later in the same year.) In place of the combinatorial estimates for Figiel's operators from Sections 12.1.b and 12.1.c, this proof employed analogous Fourier-analytic estimates due to Bourgain [1986]. Just like the combinatorial details of the $T(1)$ theorem can be simplified with the random dyadic systems, the proof of the key lemma of Bourgain [1986] was later simplified in Hytönen [2012] by the same technology.

While the direct comparison of Corollary 12.4.13 with the results of Hytönen and Weis [2007] is complicated by the presence in Corollary 12.4.13 of the (untypical in the classical theory) weak boundedness property, Corollary 12.4.19 on antisymmetric kernels comes rather close to some results of Hytönen and Weis [2007]. Indeed, in this special situation, one can completely avoid both paraproducts and the weak boundedness property, obtaining a boundedness criterion in terms of the Calderón–Zygmund kernel bounds alone.

Corollary 12.4.18 on antisymmetric but non-convolution kernels (where the weak boundedness is automatic but a paraproduct is present) is probably new in the operator-valued setting, but a rather straightforward adaptation of similar statements that are well known in the scalar-valued theory.

On minimal smoothness conditions

As one can see from $T(1)$ Theorem 12.4.12 and its corollaries, the minimal smoothness of the kernel involves a modulus of continuity $\|\omega\|_{\mathrm{Dini}^{\sigma}}$, where $\sigma = \max(1/t, 1/q')$ if X has cotype q and Y has type t, or one of them has both. In the scalar-valued (or more generally Hilbert space) case, this reduces to $\sigma = \frac{1}{2}$. Incidentally, this appears to be the minimal condition required to run any known proof of the $T(1)$ theorem, even in the scalar case. As Figiel [1990] puts it,

> it was a nice surprise that such austere methods could in fact lead to some results which were not less general than their counterparts established earlier with no restrictions on the range of admissible methods.

While the original $T(1)$ theorem of David and Journé [1984] and most of its successors are formulated for Calderón–Zygmund standard kernels, an extension to Dini-type conditions was obtained shortly after by Yabuta [1985], who proved the theorem under the condition that $\|\omega^{\frac{1}{3}}\|_{\mathrm{Dini}} < \infty$. It is not obvious at first sight how this compared to Figiel's condition $\|\omega\|_{\mathrm{Dini}^{\frac{1}{2}}} < \infty$. However, we may observe that any non-decreasing ω on $[0,1]$ satisfies

$$\begin{aligned}\int_0^1 \omega(t)\Big(\log\frac{1}{t}\Big)^{\alpha}\frac{\mathrm{d}t}{t} &= \int_0^1 \omega(t)^{\frac{1}{1+\alpha}}\omega(t)^{\frac{\alpha}{1+\alpha}}\Big(\int_t^1\frac{\mathrm{d}s}{s}\Big)^{\alpha}\frac{\mathrm{d}t}{t}\\ &\leqslant \int_0^1 \omega(t)^{\frac{1}{1+\alpha}}\Big(\int_t^1 \omega(s)^{\frac{1}{1+\alpha}}\frac{\mathrm{d}s}{s}\Big)^{\alpha}\frac{\mathrm{d}t}{t}\\ &\leqslant \int_0^1 \omega(t)^{\frac{1}{1+\alpha}}\frac{\mathrm{d}t}{t}\Big(\int_0^1 \omega(s)^{\frac{1}{1+\alpha}}\frac{\mathrm{d}s}{s}\Big)^{\alpha}\\ &= \Big(\int_0^1 \omega(s)^{\frac{1}{1+\alpha}}\frac{\mathrm{d}s}{s}\Big)^{1+\alpha}.\end{aligned}$$

With $\frac{1}{1+\alpha} = \frac{1}{3}$, we see that Yabuta's $\|\omega^{\frac{1}{3}}\|_{\mathrm{Dini}}$ dominates $\|\omega\|_{\mathrm{Dini}^{\alpha}}$ with $\alpha = 2$. (While the Dini^{s} norms were previously defined with $\log_2$ in place of log, and integrating over $[0, \frac{1}{2}]$ instead of $[0,1]$, the reader may easily verify that, extending ω from $[0, \frac{1}{2}]$ to $[0,1]$ by $\omega(t) := \omega(\min(t, \frac{1}{2}))$, these details affect at most the constants in the final conclusions.)

Subsequently, Meyer [1986] (according to Han and Hofmann [1993], but we have not been able to verify the original reference) relaxed the assumption to $\alpha = 1$ (plus a further weakening of the pointwise bounds to integral conditions rather closer to the Figiel conditions for bilinear forms as in our abstract $T(1)$ Theorems 12.3.26 and 12.3.35). Han and Hofmann [1993] obtained a further slight relaxation of the conditions of Meyer [1986], and Yang, Yan, and Deng [1997] proved the $T(1)$ theorem with assumptions essentially matching the special case $\alpha = \frac{1}{2}$ of the conditions of Figiel [1990] in the scalar-case. Later attempts to relax this condition were made by Grau de la Herrán and Hytönen [2018], who found that the same regularity is sufficient also for the non-homogeneous $T(1)$ theorem, but did not succeed in relaxing it even in the standard case. Thus, various different proof strategies all seem to meet this same threshold.

At the same time, it seems to remain unknown whether even the much weaker Hörmander conditions of Definition 11.2.1 could in principle be enough for a $T(1)$ theorem. A positive result in this direction seems out of reach with the presently available methods, but there does not seem to be any definitive counterexample to rule out this possibility. As very partial evidence for a counterexample, Yang, Yan, and Deng [1997] show that the $T(1)$ conditions for a Hörmander kernel are insufficient to guarantee the boundedness in some end-point spaces.

The dyadic representation theorem

A dyadic representation formula resembling Theorem 12.4.27 was first obtained by Hytönen [2012] as a key component of the original proof of the A_2 Theorem 11.3.26 for all standard Calderón–Zygmund operators in the scalar-valued case. Subsequent refinements and simplifications of the original representation were obtained by Hytönen, Pérez, Treil, and Volberg [2014], and Hytönen [2017]. The first version of both Theorems 12.4.26 and 12.4.27 for dyadic shifts and singular integrals on $L^p(\mathbb{R}^d; X)$ with operator-valued kernels were obtained by Hänninen and Hytönen [2016], by essentially the same techniques (notably, the tangent martingale estimates of Corollary 4.4.15) that we have followed. In all these contributions, like several other contemporary ones, the notion of dyadic shift was essentially that of Hytönen [2012], which is somewhat different from the present Definition 12.4.24. In the shifts of Hytönen [2012], the components A_K take the form

$$A_P f = \sum_{\substack{Q \in \mathrm{ch}^{(i)}(P) \\ R \in \mathrm{ch}^{(j)}(P)}} \alpha^P_{Q,R} \langle f, h^\alpha_Q \rangle h^\gamma_R,$$

with two independent complexity parameters $(i, j) \in \mathbb{N}^2$ in place of the single $k \geqslant 2$ in Theorem 12.4.27. The "new shifts" of Definition 12.4.24 were first introduced by Grau de la Herrán and Hytönen [2018]. Their Banach space valued theory, including Theorems 12.4.26 and 12.4.27 in essentially their present form, as well as multilinear extensions, has been developed by Airta, Martikainen, and Vuorinen [2022].

As far as proving the $T(1)$ theorem for Calderón–Zygmund operators on $L^p(\mathbb{R}^d; X)$ is concerned, the advantages of the Dyadic Representation Theorem 12.4.27 over (the randomised version of) Figiel's representation may be considered a question of mathematical taste (depending, among other things, on one's preference for the tangent martingales methods of Section 4.4 over the dyadic singular integrals of Section 12.1 or vice versa). However, these advantages become prominent in extensions of the $T(1)$ theory to other situations that we have not treated here. Roughly speaking, the decomposition of Figiel is essentially based on multi-scale versions of *translations*—reasonably well-behaved objects as far as translation-invariant spaces like $L^p(\mathbb{R}^d; X)$ are concerned, but somewhat unstable in more general situations. In contrast, the basic building block A_K of the dyadic shifts are essentially *averages*, which are much more stable operations. In particular, the averages $f \mapsto \mathbf{1}_Q \langle f \rangle_Q$ over arbitrary cubes $Q \subseteq \mathbb{R}^d$ are uniformly bounded on $L^p(w)$ if and only if $w \in A_p$, which partially explains the usefulness of such objects in the original context of proving the A_2 theorem. Averages are somewhat well-behaved even when taken with respect to non-doubling measures, which is the context in which a certain precursor of the dyadic representation of Hytönen [2012] (arXiv 2010) was established by Hytönen [2014] (arXiv 2008) in order to extend the non-homogeneous $T(1)$ theorem of Nazarov, Treil, and Volberg [2003] to the Banach space valued setting. Conversely, after the discovery of the Dyadic Representation Theorem, it was used by Volberg [2015] to give a new proof of the non-homogeneous $T(1)$ theorem.

An adaptation of the Dyadic Representation Theorem 12.4.27, by Hytönen, Li, H., and Vuorinen [2022], was instrumental in extending the $T(1)$ theory to singular integral operators adapted to so-called Zygmund dilations $(x_1, x_2, x_3) \mapsto (sx_1, tx_2, stx_3)$, where $s, t > 0$ are two independent parameters. Variants of the Dyadic Representation Theorem 12.4.27, with the Haar functions replaced by smoother wavelets, have been explored by Hytönen and Lappas [2022], Di Plinio, Wick, and Williams [2023c], and Di Plinio, Green, and Wick [2023b,a].

$T(1)$ theorems on other function spaces

The original $T(1)$ theorem of David and Journé [1984] was a characterisation of boundedness on $L^2(\mathbb{R}^d)$, while we have dealt with extensions of such results to $L^p(\mathbb{R}^d; X)$. However, the boundedness of a given (singular integral) operator is basic question arising in several other function spaces as well, and the $T(1)$ theorem has served as a model for similar results in other spaces. (See Chapter 14 for information about the functions spaces appearing in this discussion.) Extensions of the $T(1)$ theorems to Besov spaces $\dot{B}^s_{p,q}$ were obtained by Lemarié [1985] and to Triebel–Lizorkin spaces $\dot{B}^s_{p,q}$ and $\dot{F}^s_{p,q}$ by Frazier, Han, Jawerth, and Weiss [1989]. In these results, $p, q \in [1, \infty]$, and the smoothness parameter s was restricted by the Hölder exponent of the standard kernel of T. In order to cover a broader range of Besov and Triebel–Lizorkin spaces, where the smoothness index can take any value $s \in \mathbb{R}$, it is necessary to consider higher order Calderón–Zygmund estimates such as

$$|\partial^\alpha K(s,t)| \leqslant C|s-t|^{-d-|\alpha|}.$$

With appropriate assumptions of this type in place, Frazier, Torres, and Weiss [1988] and Torres [1991] obtained $T(1)$ criteria for the boundedness of Calderón–Zygmund operators on any Triebel–Lizorkin space $\dot{F}^s_{p,q}$, where $s \in \mathbb{R}$ and $p, q \in (0, \infty]$. The precise assumptions are necessarily somewhat technical, and the result splits into three cases, where $s < 0$; or $s \geqslant 0$ and $p, q \in [1, \infty]$, or $s \geqslant 0$ and $\min(p, q) \in (0, 1)$.

In a limited range of s again, $T(1)$ theorems on (scalar-)weighted Triebel–Lizorkin spaces $\dot{F}^s_{p,q}(w)$ were obtained by Han and Hofmann [1993], and on matrix-weighted Besov spaces $\dot{B}^s_{p,q}(W)$ by Roudenko [2003]. The full scale of both matrix-weighted Besov and Triebel–Lizorkin spaces $\dot{B}^s_{p,q}(W)$ and $\dot{F}^s_{p,q}(W)$ (as well as further generalisations with a fourth index) was covered by Bu, Hytönen, Yang, and Yuan [2023]. When restricted to the unweighted case, this last work even slightly simplifies the assumptions of Frazier, Torres, and Weiss [1988] and Torres [1991].

In all these mentioned works on $T(1)$ theorems beyond L^p spaces, the focus has been on special $T(1)$ theorems providing sufficient conditions for boundedness under vanishing paraproduct assumptions. General $T(1)$ theorems, providing a characterisation of boundedness on a given space, were obtained on Besov spaces $\dot{B}^s_{p,q}$ of positive smoothness $s > 0$ by Youssfi [1989], in terms of the weak boundedness property and the boundedness of higher order paraproducts. A far-reaching extension to Triebel–Lizorkin and other function spaces, including versions on quite general domains $\mathcal{O} \subseteq \mathbb{R}^d$, is due to Di Plinio, Green, and Wick [2023a].

For Banach space valued functions, special $T(1)$ theorems (i.e., with vanishing paraproduct assumptions) on Riesz potential spaces $\dot{H}^{s,p}(\mathbb{R}^d; X)$ and Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^d; X)$ were proved by Kaiser [2007, 2009], respectively. The results in $\dot{H}^{s,p}(\mathbb{R}^d; X)$ need the UMD property of X, but those in $\dot{B}^s_{p,q}(\mathbb{R}^d; X)$ do not. While we are not going to discuss these specific results in any further detail, the reader can witness a similar dichotomy—that the UMD property is needed to obtain results in certain function spaces, but not for analogous results in certain others—in our discussion of the theory of Banach space valued function spaces in Chapter 14.

13

The Fourier transform and multipliers

In this chapter, we complement the discussion of three major themes of Fourier analysis that we have studied in the previous Volumes. The first one is the Banach space valued Hausdorff–Young inequality

$$\|\widehat{f}\|_{L^{p'}(\mathbb{R}^d;X)} \leqslant C\|f\|_{L^p(\mathbb{R}^d;X)}. \tag{13.1}$$

As we recall from Section 2.4.b, this is a non-trivial condition, expressed by saying that the space X have *Fourier type* p. The basic theory around this notion was already developed in 2.4.b, but we now turn to the main result on this topic, Bourgain's Theorem 13.1.33, which says that (13.1) holds for some $p > 1$ if and only if X has some non-trivial type. Section 13.1 is dedicated to a detailed proof of this deep result.

The second theme is about connecting the Fourier multipliers $T_m : f \mapsto (m\widehat{f})^\vee$ from Chapter 5 and Section 8.3 with the Calderón–Zygmund theory of Chapter 11. In principle, we have

$$T_m f = (m\widehat{f})^\vee = \widehat{m} * f = k * f,$$

where the right-hand side has the formal structure of the operators studied in Chapter 11, but the question then becomes the correspondence of the conditions on the multiplier m and on the singular convolution kernel k. As we will see in Section 13.2.a, the function k will be a nice Calderón–Zygmund kernel, and hence $f \mapsto k * f$ will be in the scope of all results of Chapter 11 (notably, including those dealing with extrapolation of boundedness to the weighted $L^p(w;X)$ spaces), as soon as m satisfies assumptions like those in the Mihlin Multiplier Theorem 5.5.10 for sufficiently many derivatives $\partial^\alpha m$. Moreover, this result is very general in that it holds for multipliers taking values in arbitrary Banach spaces, and then in particular in $\mathscr{L}(X,Y)$ for any Banach spaces X and Y. However, the required number of derivatives on this level of generality is higher than that in the Mihlin Multiplier Theorem 5.5.10. Coping only with the same derivatives as in Mihlin's theorem turns out to be more delicate and require the use of a Banach space valued Hausdorff–Young inequality

T. Hytönen et al., *Analysis in Banach Spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics 76,
https://doi.org/10.1007/978-3-031-46598-7_3

(13.1). It will be convenient to know, thanks to Bourgain's Theorem 13.1.33, that this estimate is always available in the UMD spaces that we so frequently deal with (recalling that every UMD space has non-trivial type by Proposition 7.3.15). As we have already seen in a number of occasions (notably, Bourgain's Theorem 5.2.10 on the Hilbert transform, and Guerre-Delabrière's Theorem 10.5.1 on the imaginary powers $(-\Delta)^{is}$ of the Laplacian), the UMD condition is often necessary for the theory that we develop.

As the third topic of this chapter, we complement these result by Theorem 13.3.5 of Geiss, Montgomery-Smith, and Saksman, which significantly extends the previous examples of Fourier multipliers whose $L^p(\mathbb{R}^d; X)$ boundedness implies the UMD condition. As one of its consequences, in Corollary 13.3.9, we are able to compete the characterisation of situations in which there is a continuous embedding $H^{k,p}(\mathbb{R}^d; X) \hookrightarrow W^{k,p}(\mathbb{R}^d; X)$ between two classes of classical function spaces studied in the previous Volumes. This also provides a link with the following Chapter 14, where we undertake a systematic development of the theory of function spaces of Banach space valued functions.

Despite the interconnected themes of the three sections of this chapter, any of them can be studied independently of the other two by a reader interested in a particular topic.

13.1 Bourgain's theorem on Fourier type

Already in Section 2.4.b, we discussed in some detail the notion of Fourier type, or the extent to which the Hausdorff–Young inequality $\|\widehat{f}\|_{p'} \leqslant C\|f\|_p$ remains valid for the Fourier transform of vector-valued functions. In the Notes of Chapter 2, we also mentioned without proof the main theorem on this topic, due to Bourgain, stating that non-trivial type implies non-trivial Fourier type (and hence is equivalent to it, the other direction being a rather easier Proposition 7.3.6). The aim of this section is to prove this fundamental result, which will also play a role in the subsequent parts of the book.

We recall from Proposition 2.4.20 that the Fourier type $p \in [1, 2]$ of a Banach space X can be defined by any of the following equivalent conditions, where moreover any choice of $d \in \mathbb{Z}_+$ is equivalent by Proposition 2.4.11:

(1) The Fourier transform on $\mathbb{R}^d$, defined on $f \in L^1(\mathbb{R}^d; X)$ by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x\cdot\xi} \,\mathrm{d}\xi, \quad \xi \in \mathbb{R}^d,$$

extends to a bounded operator from $L^p(\mathbb{R}^d; X)$ to $L^{p'}(\mathbb{R}^d; X)$.

(2) The Fourier transform on $\mathbb{T}^d$, defined on $f \in L^1(\mathbb{T}^d; X)$ by

$$\widehat{f}(k) = \int_{\mathbb{T}^d} f(t) e^{-2\pi i t\cdot k} \,\mathrm{d}t, \quad k \in \mathbb{Z}^d,$$

restricts to a bounded operator from $L^p(\mathbb{T}^d)$ to $\ell^{p'}(\mathbb{Z}^d)$.

(3) The Fourier transform on $\mathbb{Z}^d$, defined on $x = (x_k)_{k\in\mathbb{Z}^d} \in \ell^1(\mathbb{Z}^d; X)$ by

$$\widehat{x}(t) = \sum_{k\in\mathbb{Z}} e^{-2\pi i k\cdot t} x_k, \quad t \in \mathbb{T}^d,$$

extends to a bounded operator from $\ell^p(\mathbb{Z}^d; X)$ to $L^{p'}(\mathbb{T}^d; X)$.

Denoting the norms of the respective extensions (or restrictions) by $\varphi_{p,X}(\mathbb{R}^d)$, $\varphi_{p,X}(\mathbb{T}^d)$ and $\varphi_{p,X}(\mathbb{Z}^d)$, we have:

Proposition 13.1.1. *Let X be a Banach space, $p \in (1, 2]$ and $d \in \mathbb{Z}_+$. Then*

$$\varphi_{p,\mathbb{C}}(\mathbb{R}^{d-1})\varphi_{p,X}(\mathbb{R}) \leqslant \varphi_{p,X}(\mathbb{R}^d) \leqslant \big(\varphi_{p,X}(\mathbb{R})\big)^d, \tag{13.2}$$

$$\varphi_{p,X}(\mathbb{R}^d) = \varphi_{p,X^*}(\mathbb{R}^d) \leqslant \begin{Bmatrix} \varphi_{p,X^*}(\mathbb{T}^d) = \varphi_{p,X}(\mathbb{Z}^d) \\ \varphi_{p,X}(\mathbb{T}^d) = \varphi_{p,X^*}(\mathbb{Z}^d) \end{Bmatrix} \leqslant \frac{\varphi_{p,X}(\mathbb{R}^d)}{\varphi_{p,\mathbb{C}}(\mathbb{R}^d)}. \tag{13.3}$$

It is actually known that $\varphi_{p,\mathbb{C}}(\mathbb{R}^d) = (p^{1/p}(p')^{-1/p'})^d$. For the purposes of deriving Proposition 13.1.1 with these explicit values, one only needs the easier lower bound $\varphi_{p,\mathbb{C}}(\mathbb{R}^d) \geqslant (p^{1/p}(p')^{-1/p'})^d$, which is readily deduced by computing the L^p norms of $\phi(x) = \widehat{\phi}(x) = e^{-\pi|x|^2}$.

As we shortly recall in more detail, most of the estimates of Proposition 13.1.1 have been proved in Section 2.4.b. To complete the picture with the final estimate in (13.3) (stated in Proposition 2.4.20 with a weaker constant), we begin with:

Lemma 13.1.2. *Let X be a Banach space and $p \in (1, \infty)$. Let $f \in L^p(\mathbb{T}^d; X)$ be a trigonometric polynomial, which we identify with its periodic extension to $\mathbb{R}^d$, and let $\phi \in \mathscr{S}(\mathbb{R}^d; X)$. Then*

$$\lim_{\varepsilon\downarrow 0} \|f(\cdot)\phi(\varepsilon\cdot)\varepsilon^{d/p}\|_{L^p(\mathbb{R}^d;X)} = \|f\|_{L^p(\mathbb{T}^d;X)}\|\phi\|_{L^p(\mathbb{R}^d)},$$
$$\lim_{\varepsilon\downarrow 0} \|\mathscr{F}[f(\cdot)\phi(\varepsilon\cdot)\varepsilon^{d/p}]\|_{L^{p'}(\mathbb{R}^d;X)} = \|\widehat{f}\|_{\ell^{p'}(\mathbb{Z}^d;X)}\|\widehat{\phi}\|_{L^{p'}(\mathbb{R}^d)}.$$

Proof. For the L^p norm we have

$$\begin{aligned}\|f(\cdot)\phi(\varepsilon\cdot)\varepsilon^{d/p}\|^p_{L^p(\mathbb{R}^d;X)} &= \int_{\mathbb{R}^d} \|f(t)\phi(\varepsilon t)\|^p_X \varepsilon^d \,\mathrm{d}t \\ &= \int_{\mathbb{T}^d} \|f(t)\|^p_X \Big(\sum_{k\in\mathbb{Z}^d} |\phi(\varepsilon(t+k))|^p \varepsilon^d\Big)\,\mathrm{d}t,\end{aligned}$$

where in parentheses we have a Riemann sum of $\int_{\mathbb{R}^d} |\phi(t)|^p \,\mathrm{d}t$.

For the $L^{p'}$ norm, let us write $f(t) = \sum_{k\in\mathbb{Z}^d} x_k e_k(t)$. Then

$$\mathscr{F}[f(\cdot)\phi(\varepsilon\cdot)\varepsilon^{d/p}](\xi) = \sum_{k\in\mathbb{Z}^d} x_k \int_{\mathbb{R}^d} \phi(\varepsilon t)\varepsilon^{d/p} e^{2\pi i k\cdot t} e^{-2\pi i \xi\cdot t}\,\mathrm{d}t$$

$$= \sum_{k\in\mathbb{Z}^d} x_k \widehat{\phi}(\varepsilon^{-1}(\xi-k))\varepsilon^{-d/p'}$$

Let us split this into two parts,

$$I := \sum_{k\in\mathbb{Z}^d} x_k \mathbf{1}_Q(\xi-k)\widehat{\phi}(\varepsilon^{-1}(\xi-k))\varepsilon^{-d/p'},$$
$$II := \sum_{k\in\mathbb{Z}^d} x_k \mathbf{1}_{\complement Q}(\xi-k)\widehat{\phi}(\varepsilon^{-1}(\xi-k))\varepsilon^{-d/p'},$$

where $Q = [-\frac{1}{2},\frac{1}{2})^d$. The terms in I are disjointly supported, and hence

$$\begin{aligned}\|I\|_{L^{p'}(\mathbb{R}^d;X)} &= \Big(\sum_{k\in\mathbb{Z}^d} \|x_k\|^{p'} \|\mathbf{1}_Q(\cdot-k)\widehat{\phi}(\varepsilon^{-1}(\cdot-k))\varepsilon^{-d/p'}\|^{p'}_{L^{p'}(\mathbb{R}^d)}\Big)^{1/p'} \\ &= \Big(\sum_{k\in\mathbb{Z}^d} \|x_k\|^{p'} \|\mathbf{1}_Q(\varepsilon\cdot)\widehat{\phi}\|^{p'}_{L^{p'}(\mathbb{R}^d)}\Big)^{1/p'} \\ &\to \Big(\sum_{k\in\mathbb{Z}^d} \|x_k\|^{p'} \|\widehat{\phi}\|^{p'}_{L^{p'}(\mathbb{R}^d)}\Big)^{1/p'}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|II\|_{L^{p'}(\mathbb{R}^d;X)} &\leqslant \sum_{k\in\mathbb{Z}^d} \|x_k\| \|\mathbf{1}_{\complement Q}(\cdot-k)\widehat{\phi}(\varepsilon^{-1}(\cdot-k))\varepsilon^{-d/p'}\|_{L^{p'}(\mathbb{R}^d)} \\ &\leqslant \sum_{k\in\mathbb{Z}^d} \|x_k\| \|\mathbf{1}_{\complement Q}(\varepsilon\cdot)\widehat{\phi}\|_{L^{p'}(\mathbb{R}^d)} \to 0.\end{aligned}$$

Thus $\|I+II\|_{L^{p'}(\mathbb{R}^d;X)}$ indeed converges to the claimed limit. □

Proof of Proposition 13.1.1. The second bound in (13.2) is contained in Proposition 2.4.11. The first bound is also there, but in a slightly different form, and the present formulation is obtained by repeating the same proof: Given $f \in L^p(\mathbb{R};X)$ and $\phi \in L^p(\mathbb{R}^{d-1})$, we have

$$\begin{aligned}\|\widehat{f}\|_{L^{p'}(\mathbb{R};X)}\|\widehat{\phi}\|_{L^{p'}(\mathbb{R}^{d-1})} &= \|\mathscr{F}(f\otimes\phi)\|_{L^{p'}(\mathbb{R}^d;X)} \\ &\leqslant \varphi_{p,X}(\mathbb{R}^d)\|f\otimes\phi\|_{L^p(\mathbb{R}^d;X)} = \varphi_{p,X}(\mathbb{R}^d)\|f\|_{L^p(\mathbb{R};X)}\|\phi\|_{L^p(\mathbb{R}^{d-1})}.\end{aligned}$$

Choosing f and ϕ that (almost) achieve equality in the definition of the constants $\varphi_{p,X}(\mathbb{R})$ and $\varphi_{p,\mathbb{C}}(\mathbb{R}^{d-1})$, we obtain the first bound in (13.2).

The first equality in (13.3) is Proposition 2.4.16. The first pair of inequalities and the two equalities in the middle of in (13.3) are all contained in Proposition 2.4.20 (either as stated or substituting X^* in place of X).

Concerning the last pair of inequalities in (13.3), it suffices to prove that

$$\varphi_{p,X}(\mathbb{T}^d) \leqslant \frac{\varphi_{p,X}(\mathbb{R}^d)}{\varphi_{p,\mathbb{C}}(\mathbb{R}^d)}, \tag{13.4}$$

since the other bound follows with X^* in place of X and using the first equality in (13.3). To this end, it follows from Lemma 13.1.2 that

$$\begin{aligned}\|\widehat{f}\|_{\ell^{p'}(\mathbb{Z}^d;X)}\|\widehat{\phi}\|_{L^{p'}(\mathbb{R}^d)} &= \lim_{\varepsilon\downarrow 0}\|\mathscr{F}[f(\cdot)\phi(\varepsilon\cdot)\varepsilon^{d/p}]\|_{L^{p'}(\mathbb{R}^d;X)} \\ &\leqslant \lim_{\varepsilon\downarrow 0}\varphi_{p,X}(\mathbb{R}^d)\|f(\cdot)\phi(\varepsilon\cdot)\varepsilon^{d/p}\|_{L^p(\mathbb{R}^d;X)} \\ &= \varphi_{p,X}(\mathbb{R}^d)\|f\|_{L^p(\mathbb{T}^d;X)}\|\phi\|_{L^p(\mathbb{R}^d)}.\end{aligned}$$

Choosing, again, f and ϕ that (almost) achieve equality in the definition of the constants $\varphi_{p,X}(\mathbb{T}^d)$ and $\varphi_{p,\mathbb{C}}(\mathbb{R}^d)$, we complete the proof of (13.4), and hence the Proposition. □

Proposition 13.1.1 at hand, in order to prove that a given Banach space has Fourier type p, we can pick any of the equivalent conditions amenable to our analysis. We will eventually achieve our goal with the constant $\varphi_{p,X}(\mathbb{T})$, but a major part of the work will take place on the dual group $\mathbb{Z}$. This has the advantage of presenting a convenient finite formulation as follows:

Definition 13.1.3. *Let X be a Banach space, $p,q\in[1,\infty]$ and $n\in\mathbb{Z}_+$. Then $\varphi_{p,X}^{(q)}(n)$ is the smallest admissible constant such that the inequality*

$$\Big\|\sum_{k=1}^n e_k x_k\Big\|_{L^q(\mathbb{T};X)} \leqslant \varphi_{p,X}^{(q)}(n)\Big(\sum_{k=1}^n\|x_k\|^p\Big)^{1/p}, \quad e_k(t):=e^{2\pi ikt} \quad (t\in\mathbb{T}),$$

holds for every choice of $x_1,\dots,x_n\in X$. We abbreviate $\varphi_{p,X}(n):=\varphi_{p,X}^{(p')}(n)$.

Although the case $q=p'$ is most directly linked with the Hausdorff–Young inequality on the infinite spaces $\mathbb{R}^d,\mathbb{T}^d$ and $\mathbb{Z}^d$, it turns out that our intermediate steps towards this final goal will also need to make use of the more general definition with "mismatched" exponents. Moreover, we will even need some further variations of this definition (e.g., involving other index sets F in place of $\{1,\dots,n\}$), but we postpone them until the point where they will be used. For the moment, we have the fairly obvious

Lemma 13.1.4. *Let X be a Banach space and $p,q\in[1,\infty]$. The sequence $(\varphi_{p,X}^{(q)}(n))_{n\geqslant 1}$ is increasing, and*

$$1\leqslant\varphi_{p,X}^{(q)}(n)\leqslant n^{1/p'}, \qquad \varphi_{p,X}(\mathbb{Z})=\lim_{n\to\infty}\varphi_{p,X}(n)\in[1,\infty].$$

Proof. That the sequence is increasing follows simply by extending a shorter sequence by additional zeroes. This also shows the existence of a (possibly infinite) limit $\lim_{n\to\infty}\varphi_{p,X}(n)$. The lower bound follows by taking $x_1\neq 0=x_k$ for $k\geqslant 2$, and the upper bound is also simply the triangle and Hölder's inequality

$$\Big\| \sum_{k=1}^{n} e_k x_k \Big\|_{L^q(\mathbb{T};X)} \leqslant \sum_{k=1}^{n} \|x_k\| \leqslant n^{1/p'} \Big(\sum_{k=1}^{n} \|x_k\|^p \Big)^{1/p}.$$

Given $(x_k)_{k=1}^n$, let $x = (x_k)_{k\in\mathbb{Z}}$ be its zero extension. The upper bound $\varphi_{p,X}(n) \leqslant \varphi_{p,X}(\mathbb{Z})$ follows by observing that $\sum_{k=1}^n e_k(t)x_k$ is simply $\widehat{x}(-t)$.

It only remains to check that $\varphi_{p,X}(\mathbb{Z}) \leqslant \lim_{n\to\infty} \varphi_{p,X}(n)$. Let $x = (x_k)_{k\in\mathbb{Z}}$ be finitely supported, i.e., $x_k = 0$ if $|k| \geqslant N$ for some finite N. Now

$$\widehat{x} = \sum_{|k|\leqslant N-1} e_{-k} x_k = \sum_{j=1}^{2N-1} e_{-N+j} x_{N-j} = e_{-N} \sum_{j=1}^{2N-1} e_j x_{N-j},$$

hence

$$\begin{aligned} \|\widehat{x}\|_{L^{p'}(\mathbb{T};X)} &= \Big\| \sum_{j=1}^{2N-1} e_j x_{N-j} \Big\|_{L^{p'}(\mathbb{T};X)} \leqslant \varphi_{p,X}(2N-1) \Big(\sum_{j=1}^{2N-1} \|x_{N-j}\|^p \Big)^{1/p} \\ &= \varphi_{p,X}(2N-1)\|x\|_{\ell^p(\mathbb{Z};X)} \leqslant \lim_{n\to\infty} \varphi_{p,X}(n)\|x\|_{\ell^p(\mathbb{Z};X)}. \end{aligned}$$

By the density of finitely supported sequences in $\ell^p(\mathbb{Z};X)$, this shows that $\varphi_{p,X}(\mathbb{Z}) \leqslant \lim_{n\to\infty} \varphi_{p,X}(n)$, and completes the proof. □

The task of proving that a space X has non-trivial Fourier type (assuming non-trivial type) is hence reduced, in principle, to showing the boundedness of the sequence $(\varphi_{p,X}(n))_{n\geqslant 1}$ for some $p > 1$. Although the proof that we are about to give is eventually set up slightly differently, this idea serves as a good motivation for a major part of the subsequent analysis. The proof that we will present can be roughly divided into the following main steps, treated in the next four sections:

1. Using type bounds on Sidon sets that partition $\{1, \dots, n\}$ gives a first mild improvement $\varphi_{2,X}(n) = o(n^{1/2})$ over the trivial estimate $\varphi_{2,X}(n) \leqslant n^{1/2}$.
2. Comparison with the finite Fourier transform on $\mathbb{Z}_n$ gives sub-multiplicativity and leads to $\varphi_{2,X}(n) = O(n^{1/r-1/2})$ for some $r > 1$.
3. By a delicate Lemma 13.1.25 of Bourgain, this gives a first uniform bound $\varphi^{(2)}_{s;X}(n) = O(1)$, but with mismatched exponents $s \in (1, r)$ and $2 \neq s'$.
4. Standard duality and interpolation, combined with repeating the same key Lemma 13.1.25 on the dual side, allow us to conclude with $p \in (1, r)$.

A thorough reader may recognise some conceptual similarity with the considerations encountered in Section 7.3.b in the context of deducing non-trivial type (and cotype) from the non-containment of certain subspaces. There we defined the finite type constant $\tau_{2,X}(n)$ as the best constant in the estimate

$$\Big\| \sum_{k=1}^{n} \varepsilon_k x_k \Big\|_{L^2(\Omega;X)} \leqslant \tau_{2,X}(n) \Big(\sum_{k=1}^{n} \|x_k\|^2 \Big)^{1/2} \quad \forall x_1, \dots, x_n \in X. \tag{13.5}$$

These numbers will play a role in the first proof step outlined above.

13.1.a Hinrichs's inequality: breaking the trivial bound

Recall that our goal is deriving non-trivial Fourier type from non-trivial type. Thus, from the knowledge that *random sums* $\sum_k \varepsilon_k x_k$ can be dominated by $\|(x_k)\|_{\ell^p}$, we would like to conclude that *trigonometric sums* $\sum_k e_k x_k$ can be similarly dominated (though possibly with a different p). An obvious idea that suggests itself is to try to dominate the trigonometric sum by the random sum. Indeed, we know from Section 6.5 that this can be done under particular circumstances if the trigonometric sum is restricted to a special set called a *Sidon set*. This leads to the following strategy: Given the initial sum over $k \in \{1, \ldots, N\}$, we want to partition this into sums over Sidon sets on which we can make estimates, and this partitioning should be done sufficiently economically so that it allows us to beat the trivial estimate. To carry out this idea, we need to be able to

1. efficiently recognise Sidon sets, and
2. decompose arbitrary sets into as few as possible Sidon sets.

We now turn to these tasks. Recall from Section 6.5 that a subset $\Lambda \subseteq \mathbb{Z}$ is called a *Sidon set* if the following estimate holds uniformly over all finitely non-zero sequences $(c_\lambda)_{\lambda\in\Lambda}$ of complex numbers:

$$\sum_{\lambda\in\Lambda} |c_\lambda| \leqslant C \Big\| \sum_{\lambda\in\Lambda} c_\lambda e_\lambda \Big\|_\infty .$$

The smallest admissible constant C is called the *Sidon constant* of Λ and is denoted by $S(\Lambda)$. However, this definition in itself is hardly helpful in checking whether or not a particular set actually satisfies this property. A first sufficient condition for a set to be a Sidon set was achieved in Proposition 6.5.3, showing in particular that $S(\{2^k : k \in \mathbb{N}\}) \leqslant 4$. For the present purposes, we require a more robust criterion, which is provided in the following:

Definition 13.1.5 (Quasi-independent set). *A subset $F \subseteq \mathbb{Z} \setminus \{0\}$ is called* quasi-independent *if $\alpha_k \equiv 0$ is the only finitely non-zero sequence such that $\alpha_k \in \{-1, 0, +1\}$ for all $k \in F$ and*

$$\sum_{k\in F} \alpha_k \cdot k = 0.$$

Example 13.1.6. The sequence $\{2^k : k \in \mathbb{N}\}$ is quasi-independent. In fact, if $\sum_{k=0}^\infty \alpha_k 2^k = 0$ for a finitely non-zero sequence $(\alpha_k)_{k=1}^\infty$, then

$$\sum_{k:\alpha_k=+1} 2^k = \sum_{k:\alpha_k=-1} 2^k .$$

It follows from the uniqueness of the binary expansion that $\{k : \alpha_k = +1\} = \{k : \alpha_k = -1\}$, and this is possible only if both sets are empty. Hence $\alpha_k \equiv 0$.

Proposition 13.1.7 (Bourgain). *Every quasi-independent set $F \subseteq \mathbb{Z} \setminus \{0\}$ is a Sidon set with*

$$S(F) \leqslant 16.$$

By Example 13.1.6, this gives another proof of the fact that $\{2^k : k \in \mathbb{N}\}$ is a Sidon set, but with a slightly weaker constant than Proposition 6.5.3.

Proof. This is based on a variant of the Riesz product method also used in the proof of Proposition 6.5.3, but the details are somewhat different, and we will provide a self-contained argument. By considering every finite subset of the original F, we may assume without loss of generality that F is finite to begin with. Given parameters $\varrho \in (0, 1]$ and $\xi = (\xi_k)_{k \in F} \in \mathbb{R}^F$, let then

$$\begin{aligned} R_\xi(t) &:= \prod_{k \in F} \big(1 + \varrho \cos(2\pi(kt + \xi_k))\big) \\ &= \prod_{k \in F} \Big(1 + \frac{\varrho}{2}(e_k(t)e_1(\xi_k) + e_{-k}(t)e_{-1}(\xi_k))\Big) \\ &= \sum_{\alpha \in \{-1,0,+1\}^F} 2^{-|\alpha|}\varrho^{|\alpha|} \exp\Big(2\pi i \sum_{k \in F} \alpha_k \cdot kt\Big) \exp\Big(2\pi i \sum_{k \in F} \alpha_k \xi_k\Big), \end{aligned}$$

where $|\alpha| := \sum_{k \in F} |\alpha_k|$ as usual for multi-indices,. (To relax the notation, we do not explicate the dependence of R_ξ on ϱ.)

From the assumption that F is quasi-independent, it follows that

$$\sum_{k \in F} \alpha_k \cdot k = 0 \quad \text{only if} \quad \alpha_k \equiv 0,$$

and hence $\widehat{R}_\xi(0) = 1$. It is also clear from the first line of the definition of $R_\xi(t)$ (recalling that $\varrho \in (0, 1]$) that $R_\xi(t) \geqslant 0$, and hence

$$\|R_\xi\|_{L^1(\mathbb{T})} = \int_0^1 R_\xi(t)\,\mathrm{d}t = \widehat{R}_\xi(0) = 1.$$

Let us further write

$$R_\xi^{(m)}(t) := \sum_{\substack{\alpha \in \{-1,0,+1\}^F \\ |\alpha| = m}} 2^{-|\alpha|} \exp\Big(2\pi i \sum_{k \in F} \alpha_k \cdot kt\Big) \exp\Big(2\pi i \sum_{k \in F} \alpha_k \xi_k\Big),$$

so that

$$R_\xi(t) = \sum_{m=0}^{\#F} \varrho^m R_\xi^{(m)}(t), \qquad \text{where}$$

$$R_\xi^{(0)}(t) = 1, \qquad R_\xi^{(1)}(t) = \frac{1}{2} \sum_{k \in F} \big(e_k(t)e_1(\xi_k) + e_{-k}(t)e_{-1}(\xi_k)\big).$$

From the orthogonality of the exponentials, for each $j \in F$, we have

$$\int_0^1 R_\xi^{(1)}(t)e_j(t)\,\mathrm{d}t = \frac{1}{2}\sum_{k\in F}\big(\delta_{k,-j}e_1(\xi_k)+\delta_{k,j}e_{-1}(\xi_k)\big) = \frac{1}{2}e_{-1}(\xi_j),$$

where we observed that $k=-j$ is not possible when k,j belong to the same quasi-independent set F, since $1\cdot k+1\cdot j=0$ is a direct violation of the defining condition. It is also immediate that $\int R_\xi^{(0)}e_j = 0$ for all $j\in F\subseteq\mathbb{Z}\setminus\{0\}$.

For

$$f=\sum_{j\in F}c_je_j,$$

we then conclude that

$$\begin{aligned}\int_0^1 R_\xi f &= \int_0^1\Big(\sum_{m=0}^{\#F}\varrho^m R_\xi^{(m)}\Big)\Big(\sum_{j\in F}c_je_j\Big)\\ &= 0+\frac{\varrho}{2}\sum_{j\in F}c_je_{-1}(\xi_j)+\sum_{j\in F}c_j\sum_{m\geqslant 2}\varrho^m\int_0^1 R_\xi^{(m)}e_j.\end{aligned}\tag{13.6}$$

Using again the orthogonality of the exponentials, we have

$$\begin{aligned}\Big|\int_0^1 R_\xi^{(m)}e_j\Big| &= \Big|\sum_{\substack{\alpha\in\{-1,0,+1\}^F\\|\alpha|=m\\ \sum_{k\in F}\alpha_k\cdot k=-j}}2^{-m}\exp\Big(2\pi i\sum_{k\in F}\alpha_k\xi_k\Big)\Big|\\ &\leqslant\sum_{\substack{\alpha\in\{-1,0,+1\}^F\\|\alpha|=m\\ \sum_{k\in F}\alpha_k\cdot k=-j}}2^{-m}=\int_0^1 R_0^{(m)}e_j,\end{aligned}$$

where $R_0^{(m)}$ is simply $R_\xi^{(m)}$ with $\xi=0$. It follows that

$$\sum_{m=0}^{\#F}\Big|\int_0^1 R_\xi^{(m)}e_j\Big|\leqslant\sum_{m=0}^{\#F}\int_0^1 R_0^{(m)}e_j=\int_0^1 R_0e_j\leqslant\|R_0\|_{L^1(\mathbb{T})}=1.$$

The last term in (13.6) can now be estimated by

$$\Big|\sum_{j\in F}c_j\sum_{m\geqslant 2}\varrho^m\int_0^1 R_\xi^{(m)}e_j\Big|\leqslant\sum_{j\in F}|c_j|\sum_{m\geqslant 2}\varrho^2\Big|\int_0^1 R_\xi^{(m)}e_j\Big|\leqslant\sum_{j\in F}|c_j|\varrho^2.$$

If we now choose ξ_j so that $c_je_{-1}(\xi_j)=|c_j|$, then (13.6) gives

$$\begin{aligned}\frac{\varrho}{2}\sum_{j\in F}|c_j| &= \int_0^1 R_\xi f-\sum_{j\in F}c_j\sum_{m\geqslant 2}\varrho^m\int_0^1 R_\xi^{(m)}e_j\\ &\leqslant\|R_\xi\|_{L^1(\mathbb{T})}\|f\|_{L^\infty(\mathbb{T};X)}+\varrho^2\sum_{j\in F}|c_j|,\end{aligned}$$

and hence

$$\Big(\frac{\varrho}{2}-\varrho^2\Big)\sum_{j\in F}|c_j|\leqslant \|f\|_{L^\infty(\mathbb{T};X)}=\Big\|\sum_{k\in F}c_k e_k\Big\|_{L^\infty(\mathbb{T};X)}.$$

Choosing finally $\varrho=\frac{1}{4}$ completes the proof. □

By the previous result, our initial task of decomposing arbitrary sets into Sidon sets is reduced to decomposing into quasi-independent sets. A first step in this direction is to know that every set has a quasi-independent subset of somewhat substantial size.

Lemma 13.1.8. *Any finite subset $F\subseteq\mathbb{Z}\setminus\{0\}$ has a quasi-independent subset $F_0\subseteq F$ of cardinality $\#F_0\geqslant\lceil\log_3\#F\rceil$.*

Proof. Let $F_0\subseteq F$ be a quasi-independent subset of maximal cardinality, and let

$$F_1:=\Big\{\sum_{k\in F_0}\alpha_k\cdot k:\alpha_k\in\{-1,0,+1\}\Big\}.$$

Clearly $F_1\supseteq F_0$, and we claim that in fact $F_1\supseteq F$. If not, let $k_0\in F\setminus F_1$. We will check that $F_0\cup\{k_0\}$ is quasi-independent, contradicting the maximality of F_0. Namely, suppose that

$$\sum_{k\in F_0\cup\{k_0\}}\alpha_k\cdot k=0,$$

where $\alpha_k\in\{-1,0,+1\}$. If $\alpha_{k_0}=\pm1$, then

$$k_0=\sum_{k\in F_0}(-\alpha_{k_0}\alpha_k)\cdot k\in F_1,$$

contradicting $k_0\notin F_1$. Thus $\alpha_{k_0}=0$, but then also $\alpha_k=0$ for all $k\in F_0$, since F_0 is quasi-independent, and this proves that $F_0\cup\{k_0\}$ is quasi-independent.

As explained above, this proves that $F_1\supseteq F$, and hence

$$\#F\leqslant\#F_1\leqslant 3^{\#F_0},$$

from which the proposition follows, since $\#F_0\geqslant\log_3\#F$ is necessarily an integer. □

By recursively removing big quasi-independent subsets, we arrive at the desired decomposition of the initial set:

Lemma 13.1.9. *For $N\in\mathbb{Z}_+$, let*

$$d(N):=\min\{k\in\mathbb{Z}_+:\textit{any subset } F\subseteq\mathbb{Z}\setminus\{0\}\textit{ of size }\#F\leqslant N\textit{ can be divided into at most }k\textit{ quasi-independent subsets}\}.$$

Then $d(3^n)\leqslant\dfrac{2\cdot3^n}{n+1}$ for all $n\in\mathbb{N}$. For all $n\geqslant1$, each of the partitioning quasi-independent subsets can be chosen to have size at most n.

Proof. Since clearly $d(3^n) \leqslant 3^n$ (as each singleton is quasi-independent), the claim is obvious for $n \leqslant 1$. For $3 < \#F \leqslant 9$, Lemma 13.1.8 guarantees a quasi-independent subset of size 2. Starting from a set of size 9 and repeatedly extracting 3 quasi-independent subsets of size 2, we are left with a subset of size 3 that trivially splits into 3 quasi-independent subsets of size 1. Hence $d(3^2) \leqslant 3 + 3 = 6 = 2 \cdot 3^2/(2+1)$. We then assume that, for some $n \geqslant 2$, any set of size 3^n can be divided into at most $2 \cdot 3^n/(n+1)$ quasi-independent subsets of size at most n, and we prove the same for $n+1$.

If $\#F = 3^{n+1}$, Lemma 13.1.8 guarantees that we can repeatedly extract quasi-independent subsets F_i $(i = 1, \dots, j)$ of size $n+1$, until

$$3^{n+1} - j(n+1) \leqslant 3^n < 3^{n+1} - (j-1)(n+1),$$

thus

$$\frac{2 \cdot 3^n}{n+1} \leqslant j < 1 + \frac{2 \cdot 3^n}{n+1}.$$

The remaining set of size at most 3^n can then be divided into at most $d(3^n)$ quasi-independent subsets, and by the induction assumption we have

$$d(3^{n+1}) \leqslant j + d(3^n) < \Big(1 + \frac{2 \cdot 3^n}{n+1}\Big) + \frac{2 \cdot 3^n}{n+1} = 1 + \frac{4 \cdot 3^n}{n+1}.$$

For $n \geqslant 2$, we have $1/(n+1) \leqslant \frac{4}{3}/(n+2)$ and $3^n/(n+2) \geqslant \frac{9}{4}$, and hence

$$d(3^{n+1}) \leqslant \Big(\frac{4}{9} + \frac{16}{3}\Big)\frac{3^n}{n+2} < 6\frac{3^n}{n+2} = \frac{2 \cdot 3^{n+1}}{(n+1)+1},$$

and this completes the induction step. Note that all quasi-independent subsets that we constructed in the induction step had either size $n+1$, or they came from the induction assumption, in which case their size is at most n. □

In the next remark, we indicate converses to the obtained bounds of Lemmas 13.1.8 and 13.1.9.

Remark 13.1.10. Let $F_0 \subseteq F := \{1, \dots, N\}$ be such that $N \geqslant 2$ and F_0 is quasi-independent. We claim that necessarily $\#F_0 \leqslant 2\log_2(N)$. Clearly, this implies $d(N) \geqslant \frac{N}{2\log_2(N)}$. Indeed, write $m = \#F_0$. It suffices to consider $m \geqslant 2$. Let $A \subseteq F_0$ be arbitrary. Then

$$0 \leqslant \sum_{a \in A} a \leqslant \sum_{a \in F_0} a < N^2.$$

Therefore, the number of different values can be estimated by

$$\#\Big\{\sum_{a \in A} a : A \subseteq F_0\Big\} \leqslant N^2.$$

One the other hand, if $A, B \subseteq F_0$ are such that $\sum_{a \in A} a = \sum_{b \in B} b$, then the quasi-independence of F_0 implies $A = B$. Therefore,

$$2^m = \#\{A \subseteq F_0\} \leqslant \#\Big\{ \sum_{a\in A} a : A \subseteq F_0 \Big\}.$$

We can conclude $2^m \leqslant N^2$ and thus the claim follows.

We now possess all the ingredients needed for the first estimate of Fourier type in terms of type, stated in terms of the finite versions of both properties. The reader may wish to compare the next proposition to Theorem 7.6.12 which gives a related inequality for the Walsh system.

Proposition 13.1.11 (Hinrichs's inequality). *For all $n \geqslant 1$ we have*

$$\frac{\varphi_{2,X}(3^n)}{\sqrt{3^n}} \leqslant 16\sqrt{2} \cdot \frac{\tau_{2,X}(n)}{\sqrt{n}}.$$

Proof. By Lemma 13.1.9, the set $\{1, \dots, 3^n\}$ can be divided into

$$A \leqslant 2 \cdot 3^n/(n+1)$$

quasi-independent subsets F_a of size $\#F_a \leqslant n$. By Proposition 13.1.7, each quasi-independent F_a is a Sidon set with $S(F_a) \leqslant 16$. By Pisier's Theorem 6.5.5, trigonometric series over a Sidon set is comparable in the L^p norm to the corresponding Rademacher series, up to the Sidon constant. Chaining these observations and using the definition of the type constants $\tau_{2,X}(n)$ and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
\Big\| \sum_{k=1}^{3^n} e_k x_k \Big\|_{L^2(\mathbb{T};X)} = \Big\| \sum_{a=1}^{A} \sum_{k\in F_a} e_k x_k \Big\|_{L^2(\mathbb{T};X)} &\leqslant \sum_{a=1}^{A} \Big\| \sum_{k\in F_a} e_k x_k \Big\|_{L^2(\mathbb{T};X)} \\
&\leqslant \sum_{a=1}^{A} 16 \Big\| \sum_{k\in F_a} \varepsilon_k x_k \Big\|_{L^2(\Omega;X)} \\
&\leqslant \sum_{a=1}^{A} 16 \cdot \tau_{2,X}(\#F_a) \Big(\sum_{k\in F_a} \|x_k\|^2 \Big)^{1/2} \\
&\leqslant 16 \cdot \max_{1\leqslant a\leqslant A} \tau_{2,X}(\#F_a) \sqrt{A} \Big(\sum_{a=1}^{A} \sum_{k\in F_a} \|x_k\|^2 \Big)^{1/2} \\
&\leqslant 16 \cdot \tau_{2,X}(n) \sqrt{\frac{2\cdot 3^n}{n+1}} \Big(\sum_{k=1}^{3^n} \|x_k\|^2 \Big)^{1/2},
\end{aligned}$$

from which the proposition follows. □

The following corollary gives the promised improvement over the trivial bound $\varphi_{2,X}(3^n) \leqslant \sqrt{3^n}$ as soon as n is large enough.

Corollary 13.1.12. *Let X be a Banach space of type $p \in (1,2]$. Then for all $n \geqslant 1$, we have*

$$\frac{\varphi_{2,X}(3^n)}{\sqrt{3^n}} \leqslant 16\sqrt{2} \cdot \tau_{p,X;2} \cdot n^{-1/p'}.$$

The type constant $\tau_{p,X;s}$ with a secondary parameter (above $s = 2$) was introduced right before Proposition 7.1.4 as the best constant in the inequality

$$\Big\| \sum_{k=1}^{K} \varepsilon_k x_k \Big\|_{L^s(\Omega;X)} \leqslant \tau_{p,X;s} \Big(\sum_{k=1}^{K} \|x_k\|^p \Big)^{1/p}, \tag{13.7}$$

where $x_1, \dots, x_K \in X$ and $K \in \mathbb{Z}_+$ are arbitrary. Recall that $\tau_{p,X} := \tau_{p,X;p}$.

Proof. From the definition of the type constants and Hölder's inequality, it is immediate that

$$\frac{\tau_{2,X}(n)}{\sqrt{n}} \leqslant \frac{\tau_{p,X;2} \cdot n^{1/p-1/2}}{\sqrt{n}} = \tau_{p,X;2} \cdot n^{-1/p'}.$$

In combination with Proposition 13.1.11, this gives the result. □

13.1.b The finite Fourier transform and sub-multiplicativity

Note that the improvement of Corollary 13.1.12 over the trivial bound is only very slight. Our first goal in bootstrapping this initial estimate is to obtain a power-type bound of the form $\varphi_{2,X}(N) = O(N^{1/2-\delta})$. As the reader can easily verify (perhaps referring to Lemma 7.3.19), this would readily follow from the established bound, *if* in addition we had a sub-multiplicative estimate

$$\varphi_{2,X}(nm) \overset{?}{\leqslant} \varphi_{2,X}(n)\varphi_{2,X}(m).$$

As we do not know whether this is true, we take a detour by comparing the sequence $\varphi_{2,X}(n)$ with the following discretised variant:

Definition 13.1.13. *Let X be a Banach space and $n \in \mathbb{Z}_+$. Then $\varphi_{p,X}^{(q)}(\mathbb{Z}_n)$ is the best constant in the following inequality with arbitrary $x_1, \dots, x_n \in X$:*

$$\Big(\frac{1}{n} \sum_{h=1}^{n} \Big\| \sum_{k=1}^{n} e_k(h/n) x_k \Big\|^q \Big)^{1/q} \leqslant \varphi_{p,X}^{(q)}(\mathbb{Z}_n) \Big(\sum_{k=1}^{n} \|x_k\|^p \Big)^{1/p}.$$

As the notation suggests, $\varphi_{p,X}^{(q)}(\mathbb{Z}_n)$ has an interpretation as the norm of the Fourier transform (thus, a Fourier type constant) of functions on the finite group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, but there is no need to insist too much on this point here. The difference of the defining inequalities of $\varphi_{p,X}^{(q)}(n)$ and $\varphi_{p,X}^{(q)}(\mathbb{Z}_n)$ is that the $L^p(\mathbb{T}; X)$ integral norm in the former is replaced by a finite Riemann sum approximation in the latter. We will next develop some tools for comparing the two kinds of norms. This will involve elements of some fairly classical Fourier analysis, and we begin with

Definition 13.1.14. *The* Dirichlet kernel *is defined by*

$$D_n(t) := \sum_{|k|\leqslant n} e_k(t), \quad t \in \mathbb{T},$$

the Fejér kernel *by*

$$F_n(t) := \frac{1}{n+1}\sum_{k=0}^{n} D_k(t) = \sum_{|k|\leqslant n}\Big(1 - \frac{|k|}{n+1}\Big)e_k(t), \quad t \in \mathbb{T},$$

and the de la Vallée–Poussin kernel *by*

$$V_n(t) := \frac{1}{n}\sum_{k=n}^{2n-1} D_k(t) = \sum_{|j|\leqslant n} e_j(t) + \sum_{n<|j|<2n}\Big(2 - \frac{|j|}{n}\Big)e_j(t), \quad t \in \mathbb{T}.$$

Lemma 13.1.15. *These kernels satisfy the identities*

$$D_n(t) = \frac{\sin(\pi(2n+1)t)}{\sin(\pi t)}, \qquad F_n(t) = \frac{1}{n+1}\frac{\sin^2(\pi(n+1)t)}{\sin^2(\pi t)} \geqslant 0,$$

$$V_n(t) = 2F_{2n-1}(t) - F_{n-1}(t).$$

Proof. The formula for D_n is the summation of a geometric series:

$$D_n(t) := \sum_{|k|\leqslant n} e^{2\pi i k t} = e^{-2\pi i n t}\frac{e^{2\pi i(2n+1)t} - 1}{e^{2\pi i t} - 1} = \frac{\sin(\pi(2n+1)t)}{\sin(\pi t)}.$$

Since

$$\begin{aligned}\sum_{k=0}^{n}\sin(\pi(2k+1)t) &= \Im\sum_{k=0}^{n} e^{i\pi t}e^{i2\pi k t} = \Im\Big(e^{i\pi t}\frac{e^{i2\pi(n+1)t} - 1}{e^{i2\pi t} - 1}\Big)\\ &= \Im\Big(e^{i\pi(n+1)t}\frac{\sin(\pi(n+1)t)}{\sin(\pi t)}\Big) = \frac{\sin^2(\pi(n+1)t)}{\sin(\pi t)},\end{aligned}$$

we obtain the formula for F_n by summing over the formula for D_k. Finally,

$$V_n = \frac{1}{n}\sum_{k=n}^{2n-1} D_k = \frac{1}{n}\Big(\sum_{k=0}^{2n-1} D_k - \sum_{k=0}^{n-1} D_k\Big) = \frac{1}{n}\Big(2nF_{2n-1} - nF_{n-1}\Big).$$

□

Lemma 13.1.16. *If f is a trigonometric polynomial with* $\deg(f) < n$*, then for all $s \in \mathbb{R}$ we have*

$$\int_0^1 f(t)\,\mathrm{d}t = \frac{1}{n}\sum_{h=1}^{n} f(s + h/n),$$

i.e., f can be integrated exactly by uniform Riemann sums of order n.

Proof. It is enough to consider $f(t) = e_k(t)$, where $|k| < n$. We observe that

$$\sum_{h=1}^{n} e^{2\pi i k h/n} = \begin{cases} e^{2\pi i k/n} \frac{e^{2\pi i k n/n}-1}{e^{2\pi i k/n}-1} = 0, & 0 < |k| < n, \\ n, & k = 0, \end{cases}$$

and hence

$$\frac{1}{n}\sum_{h=1}^{n} f(s+h/n) = \frac{e_k(s)}{n}\sum_{h=1}^{n} e^{2\pi i k h/n} = e_k(s)\delta_{k,0} = \delta_{k,0} = \int_0^1 e_k(t)\,\mathrm{d}t.$$

□

On the level of L^p norms, this leads to the following comparison result:

Proposition 13.1.17 (Marcinkiewicz inequality). *Let X be a Banach space and $p \in [1,\infty)$. Then for all $n \in \mathbb{Z}_+$ and $x_1, \dots, x_n \in X$, we have*

$$\Big(\frac{1}{n}\sum_{h=1}^{n}\Big\|\sum_{k=1}^{n} e_k(h/n)x_k\Big\|^p\Big)^{1/p} \leqslant 3\Big\|\sum_{k=1}^{n} e_k x_k\Big\|_{L^p(\mathbb{T};X)}.$$

With the usual modification, the result is also true (and entirely trivial) for $p = \infty$: of course the supremum over $\{j/n : j = 1, \dots, n\}$ is dominated by the supremum over all of $\mathbb{T}$!

Proof. Let

$$f(t) := \sum_{k=1}^{n} e_k(t)x_k, \qquad m := \lfloor n/2 \rfloor.$$

Then $(n-1)/2 \leqslant m \leqslant n/2$ and the function $e_{-(m+1)}f$ is a linear combination of e_k with

$$-m = 1-(m+1) \leqslant k \leqslant n-(m+1) \leqslant (2m+1)-(m+1) = m,$$

so $e_{-(m+1)}f$ is a trigonometric polynomial of degree m.

Since the de la Vallée–Poussin kernel V_m from Definition 13.1.14 has Fourier coefficients $\widehat{V}_m(k) = 1$ for all values $|k| \leqslant m$ on which the Fourier coefficients of $e_{-(m+1)}f$ are supported, we conclude that

$$\widehat{V}_m[e_{-(m+1)}f]\widehat{\ } = [e_{-(m+1)}f]\widehat{\ },$$

hence $V_m * (e_{-(m+1)}f) = e_{-(m+1)}f$. Thus

$$\begin{aligned} \|f(t)\| &= \|e_{-(m+1)}(t)f(t)\| = \|V_m * (e_{-(m+1)}f)(t)\| \\ &\leqslant \int_{\mathbb{T}} |V_m(t-s)|\|f(s)\|\,\mathrm{d}s \\ &\leqslant \Big(\int_{\mathbb{T}} |V_m(t-s)|\,\mathrm{d}s\Big)^{1/p'}\Big(\int_{\mathbb{T}} |V_m(t-s)|\|f(s)\|^p\,\mathrm{d}s\Big)^{1/p} \end{aligned} \tag{13.8}$$

By Lemma 13.1.15, we have

$$|V_m| = |2F_{2m-1} - F_{m-1}| \leqslant 2F_{2m-1} + F_{m-1}, \tag{13.9}$$

where $\int_{\mathbb{T}} F_k(t)\,\mathrm{d}t = \widehat{F}_k(0) = 1$, and hence

$$\int_{\mathbb{T}} |V_m(t-s)|\,\mathrm{d}s \leqslant 3.$$

Substituting into (13.8) and summing, we have

$$\sum_{h=1}^{n} \|f(h/n)\|^p \leqslant 3^{p/p'} \int_{\mathbb{T}} \sum_{h=1}^{n} |V_m(h/n-s)| \|f(s)\|^p\,\mathrm{d}s.$$

Since the right-hand side of (13.9) is a trigonometric polynomial of degree $2m-1 \leqslant n-1$, Lemma 13.1.16 guarantees that

$$\begin{aligned}\sum_{h=1}^{n} |V_m(h/n-s)| &\leqslant \sum_{h=1}^{n} (2F_{2m-1} + F_{m-1})(h/n-s) \\ &= n \int_{\mathbb{T}} (2F_{2m-1} + F_{m-1})(u)\,\mathrm{d}u = 3n.\end{aligned}$$

Substituting back, we conclude that

$$\frac{1}{n}\sum_{h=1}^{n} \|f(h/n)\|^p \leqslant 3^{p/p'} \int_{\mathbb{T}} 3\|f(s)\|^p\,\mathrm{d}s = 3^p \|f\|^p_{L^p(\mathbb{T};X)}.$$

□

We now have the desired comparison of the two finite Fourier type constants:

Lemma 13.1.18. *For any Banach space X and $n \in \mathbb{Z}_+$, we have*

$$\varphi^{(q)}_{p,X}(n) \leqslant \varphi^{(q)}_{p,X}(\mathbb{Z}_n) \leqslant 3 \cdot \varphi^{(q)}_{p,X}(n).$$

Proof. Substituting $e_k(t)x_k$ in place of x_k in Definition 13.1.13, we find that

$$\frac{1}{n}\sum_{h=1}^{n} \Big\| \sum_{k=1}^{n} e_k(t+h/n)x_k \Big\|^q \leqslant \big(\varphi^{(q)}_{p,X}(\mathbb{Z}_n)\big)^q \Big(\sum_{k=1}^{n} \|x_k\|^p \Big)^{q/p}$$

Integrating over $t \in \mathbb{T}$ and using the translation invariance

$$\|f(\cdot + h/n)\|_{L^q(\mathbb{T};X)} = \|f\|_{L^q(\mathbb{T};X)},$$

we obtain

$$\frac{1}{n}\sum_{h=1}^{n} \int_{\mathbb{T}} \Big\| \sum_{k=1}^{n} e_k(t)x_k \Big\|^q \,\mathrm{d}t \leqslant \big(\varphi^{(q)}_{p,X}(\mathbb{Z}_n)\big)^q \Big(\sum_{k=1}^{n} \|x_k\|^p \Big)^{q/p},$$

and hence $\varphi_{p,X}^{(q)}(n) \leqslant \varphi_{p,X}^{(q)}(\mathbb{Z}_n)$.

The other estimate follows at once from the Marcinkiewicz inequality (Proposition 13.1.17), which is the first step in

$$\Big(\frac{1}{n}\sum_{h=1}^{n}\Big\|\sum_{k=1}^{n} e_k(h/n)x_k\Big\|^q\Big)^{1/q} \leqslant 3\Big\|\sum_{k=1}^{n} e_k x_k\Big\|_{L^q(\mathbb{T};X)}$$
$$\leqslant 3\varphi_{p,X}^{(q)}(n)\Big(\sum_{k=1}^{n}\|x_k\|^p\Big)^{1/p}.$$

□

The following lemma is our reason for considering the quantities $\varphi_{p,X}^{(q)}(\mathbb{Z}_n)$:

Lemma 13.1.19. *For any Banach space X and $m, n \in \mathbb{Z}_+$, we have the sub-multiplicative estimate*

$$\varphi_{p,X}^{(q)}(\mathbb{Z}_{mn}) \leqslant \varphi_{p,X}^{(q)}(\mathbb{Z}_m)\varphi_{p,X}^{(q)}(\mathbb{Z}_n), \qquad 1 \leqslant p \leqslant q \leqslant \infty;$$

in particular

$$\varphi_{p,X}(\mathbb{Z}_{mn}) \leqslant \varphi_{p,X}(\mathbb{Z}_m)\varphi_{p,X}(\mathbb{Z}_n) \qquad \forall p \in [1,2].$$

Proof. The second estimate is an obvious special case with $q = p' \geqslant 2 \geqslant p$. For the proof of the general estimate, it is convenient to observe that, by simple reindexing and modular arithmetic, the condition defining $\varphi_{p,X}^{(q)}(\mathbb{Z}_n)$ is unchanged if instead of $\{1,\ldots,n\}$ we take all sums over $\{0,\ldots,n-1\}$. In the defining condition of the constant $\varphi_{p,X}^{(q)}(\mathbb{Z}_{mn})$, we should then sum over $\{0,\ldots,mn-1\}$, and the key trick of the proof is to use a non-symmetric reindexing of this range for the h and k sums, namely

$$\begin{aligned} h &= an+b: \quad a = 0,\ldots,m-1, \quad b = 0,\ldots,n-1,\\ k &= um+v: \quad u = 0,\ldots,n-1, \quad v = 0,\ldots,m-1. \end{aligned}$$

Then

$$hk = (an+b)(um+v) = aumn + avn + bum + bv,$$

and hence, noting that $e^{2\pi i a u} = 1$,

$$e_k(h/mn) = e_u(b/n)e_v(a/m)e_v(b/mn).$$

Thus

$$\Big\{\frac{1}{mn}\sum_{h=0}^{mn-1}\Big\|\sum_{k=0}^{mn-1} e_k(h/mn)x_k\Big\|^q\Big\}^{1/q}$$
$$= \Big\{\frac{1}{n}\sum_{b=0}^{n-1}\frac{1}{m}\sum_{a=0}^{m-1}\Big\|\sum_{v=0}^{m-1} e_v(a/m)y_v^{(b)}\Big\|^q\Big\}^{1/q},$$

$$y_v^{(b)} := e_v(b/mn)\sum_{u=0}^{n-1} e_u(b/n)x_{um+v},$$

$$\leqslant \Big\{\frac{1}{n}\sum_{b=0}^{n-1}\varphi_{p,X}^{(q)}(\mathbb{Z}_m)^q\Big(\sum_{v=0}^{m-1}\|y_v^{(b)}\|^p\Big)^{q/p}\Big\}^{1/q}$$

$$\leqslant \varphi_{p,X}^{(q)}(\mathbb{Z}_m)\Big\{\sum_{v=0}^{m-1}\Big(\frac{1}{n}\sum_{b=0}^{n-1}\Big\|\sum_{u=0}^{n-1}e_u(b/n)x_{um+v}\Big\|^q\Big)^{p/q}\Big\}^{1/p}$$

by Minkowski's inequality for $p \leqslant q$,

$$\leqslant \varphi_{p,X}^{(q)}(\mathbb{Z}_m)\Big\{\sum_{v=0}^{m-1}\varphi_{p,X}^{(q)}(\mathbb{Z}_n)^p\sum_{u=0}^{n-1}\|x_{um+v}\|^p\Big\}^{1/p}$$

$$= \varphi_{p,X}^{(q)}(\mathbb{Z}_m)\varphi_{p,X}^{(q)}(\mathbb{Z}_n)\Big\{\sum_{k=0}^{mn-1}\|x_k\|^p\Big\}^{1/p},$$

where we used the defining condition for $\varphi_{p,X}^{(q)}(\mathbb{Z}_m)$ with the sequences $(y_v^{(b)})_{v=0}^{m-1}$ for each fixed $b = 0,\dots,n-1$, and that for $\varphi_{p,X}^{(q)}(\mathbb{Z}_n)$ with the sequences $(x_{mu+v})_{u=0}^{n-1}$ for each fixed $v = 0,\dots,m-1$. □

Combining the above results with Corollary 13.1.12 of Hinrichs's inequality, we achieve the desired power-type improvement over the trivial estimate $\varphi_{2,X}(N) \leqslant N^{1/2}$. One could try to deduce this from Lemma 7.3.19 applied to $\varphi_{2,X}(\mathbb{Z}_n)$. However, this time that does not work since we do not know whether $\varphi_{2,X}(\mathbb{Z}_n)$ is increasing in n. Therefore, we adapt the proof of the lemma and use the facts that $\varphi_{2,X}(n)$ is increasing and that $\varphi_{2,X}(\mathbb{Z}_n)$ is submultiplicative. Our choice of notation r' below is indicative of the fact that this is the Hölder conjugate of a (small) exponent $r > 1$.

Corollary 13.1.20. *Let X be a Banach space of type $p \in (1,2]$. Then*

$$\varphi_{2,X}(N) \leqslant C\cdot N^{1/2-1/r'} = C\cdot N^{1/r-1/2},$$

where

$$r' := 3p'(68\cdot\tau_{p,X;2})^{p'}, \qquad C := e^{\frac{r'}{2p'}}. \tag{13.10}$$

Proof. Given $N, n \in \mathbb{Z}_+$, let $k \in \mathbb{Z}_+$ satisfy

$$3^{n(k-1)} \leqslant N < 3^{nk}. \tag{13.11}$$

Then

$$\begin{aligned}\varphi_{2,X}(N) &\leqslant \varphi_{2,X}(3^{nk}) && \text{since } \varphi_{2,X} \text{ is increasing by Lemma 13.1.4,}\\ &\leqslant \varphi_{2,X}(\mathbb{Z}_{3^{nk}}) && \text{by the comparison in Lemma 13.1.18,}\end{aligned}$$

$$\leqslant \varphi_{2,X}(\mathbb{Z}_{3^n})^k \quad \text{by sub-multiplicativity (Lemma 13.1.19),}$$
$$\leqslant (3 \cdot \varphi_{2,X}(3^n))^k \quad \text{by the comparison in Lemma 13.1.18.}$$

Therefore, by (13.11) for any $s \in (1,2]$ we find

$$\begin{aligned} N^{1/2-1/s}\varphi_{2,X}(N) &\leqslant 3^{n(k-1)(\frac{1}{2}-\frac{1}{s})}(3 \cdot \varphi_{2,X}(3^n))^k \\ &= 3^{n(\frac{1}{s}-\frac{1}{2})}[3^{n(\frac{1}{2}-\frac{1}{s})} \cdot 3 \cdot \varphi_{2,X}(3^n)]^k \end{aligned}$$

For appropriate n and s, we will show that the term within brackets satisfies $[\dots] \leqslant 1$. By Corollary 13.1.12, we can estimate

$$[\dots] \leqslant 3^{n(\frac{1}{2}-\frac{1}{s})} \cdot 3 \cdot 16\sqrt{2} \cdot \tau_{p,X;2} \cdot 3^{n/2} \cdot n^{-1/p'} =: 3^{n/s'} T n^{-1/p'},$$

where $T := 48\sqrt{2}\tau_{p,X;2}$. Therefore, setting $s' = (1 + eT^{p'})p' \log(3)$ and taking $eT^{p'} \leqslant n < eT^{p'} + 1$ we find that

$$3^{n/s'} T n^{-1/p'} \leqslant e^{1/p'} T \frac{e^{-1/p'}}{T} = 1.$$

From the above we conclude that

$$N^{1/2-1/s}\varphi_{2,X}(N) \leqslant 3^{n(\frac{1}{s}-\frac{1}{2})} \leqslant 3^{n/2} = e^{n\log(3)/2} \leqslant e^{s'/(2p')}.$$

The above trivially holds true if we replace s' by any $r' > s'$. Since $50 \leqslant T \leqslant 68\tau_{p,X;2}$ and $p' \geqslant 2$, one can check that

$$s' = (1 + eT^{p'})p' \log(3) = T^{p'}(T^{-p'} + e)p' \log(3) \leqslant (68\tau_{p,X;2})^{p'} 3p' =: r'.$$

Thus the statement follows. □

Before turning to some of the sophisticated constructions and estimates for Bourgain's theorem, we discuss a much simpler situation where one can obtain $\varphi_{2,X}(n) = O(n^{1/r-1/2})$ with $r \in (1,2)$. It does not play a role in the proof of Bourgain's theorem.

Proposition 13.1.21. *If X has type p and cotype q, then for all $n \geqslant 1$,*

$$\varphi_{2,X}(n) \leqslant \tau_{2,X}(n)c_{2,X}(n) \leqslant \tau_{p,X;2}c_{q,X;2}n^{1/p-1/q}.$$

Of course the latter bound is nontrivial only if $\frac{1}{p} - \frac{1}{q} < \frac{1}{2}$.

Proof. Let $(\gamma_h)_{h\geqslant 1}$ be a complex Gaussian sequence (i.e., standard independent Gaussian random variables). Also let

$$\widetilde{\gamma}_k = \frac{1}{\sqrt{n}} \sum_{h=1}^{n} \gamma_h e_k(\frac{h}{n}), \quad k = 1, \dots, n.$$

Then $(\widetilde{\gamma}_k)_{k=1}^n$ are also independent standard Gaussian random variables (see Section E.2). Hence, using the natural Gaussian analogue of the finite type and cotype constants,

$$\begin{aligned}\frac{1}{n}\sum_{h=1}^n\Big\|\sum_{k=1}^n x_k e_k(\frac{h}{n})\Big\|^2 &\leqslant \frac{1}{n}c_{2,X}^{\gamma}(n)^2\mathbb{E}\Big\|\sum_{h=1}^n\gamma_h\sum_{k=1}^n x_k e_k(\frac{h}{n})\Big\|^2\\ &= c_{2,X}^{\gamma}(n)^2\mathbb{E}\Big\|\sum_{k=1}^n\widetilde{\gamma}_k x_k\Big\|^2\\ &\leqslant c_{2,X}^{\gamma}(n)^2\tau_{2,X}^{\gamma}(n)^2\sum_{k=1}^n\|x_k\|^2.\end{aligned}$$

Since $\|\gamma\|_2 = 1$, Proposition 7.1.18 informs us that $\tau_{2,X}^{\gamma} \leqslant \tau_{2,X}$ and $c_{2,X}^{\gamma} \leqslant c_{2,X}$, and the analogous result for the finite constants $\tau_{2,X}^{\gamma}(n)$ etc. follows by the same argument. Finally, Hölder's inequality implies $\tau_{2,X}(n) \leqslant \tau_{p,X;2}n^{\frac{1}{p}-\frac{1}{2}}$ and $c_{2,X}(n) \leqslant c_{q,X;2}n^{\frac{1}{2}-\frac{1}{q}}$. □

13.1.c Key lemmas for an initial uniform bound

The core of this section consists of two delicate lemmas of Bourgain that allow us to bootstrap the power-type improvement over the trivial bound on the growth of $\varphi_{2,X}(N)$, as given in Corollary 13.1.20, into a uniform estimate for the constants $\varphi_{s,X}^{(2)}(N)$ with some $s > 1$. To streamline the presentation of the core arguments, we begin with the following classical identity:

Lemma 13.1.22. *Let $f = \sum_{j\in\mathbb{Z}}\widehat{f}(j)e_j$ with $(\widehat{f}(j))_{j\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$. Then*

$$\sum_{j\equiv n \bmod N}\widehat{f}(j) = \frac{1}{N}\sum_{h=1}^N e_{-1}(nh/N)f(h/N).$$

Proof. We first observe that

$$\frac{1}{N}\sum_{h=1}^N f(h/N) = \sum_{j\in\mathbb{Z}}\widehat{f}(j)\frac{1}{N}\sum_{h=1}^N e_j(h/N) = \sum_{j\equiv 0 \bmod N}\widehat{f}(j),$$

which is case $n = 0$ of the claim.

We apply this with f replaced by

$$e_{-n}f = \sum_{j\in\mathbb{Z}}\widehat{f}(j)e_{j-n} = \sum_{j\in\mathbb{Z}}\widehat{f}(j+n)e_j$$

to find that

$$\frac{1}{N}\sum_{h=1}^N(e_{-n}f)(h/N) = \sum_{j\equiv 0 \bmod N}\widehat{f}(j+n) = \sum_{j\equiv n \bmod N}\widehat{f}(j),$$

which is the general case. □

Lemma 13.1.23 (Bourgain). *Let $F \subseteq \mathbb{Z}$ be a finite subset with $\#F = N$. Then there exists $t_0 \in \mathbb{T}$ such that at least $\frac{1}{8}N$ of the pairwise disjoint intervals*

$$I_n := \frac{1}{N}[n - \frac{1}{2}, n + \frac{1}{2}), \qquad n = 1, \dots, N,$$

satisfy $t_0 k \in I_n + \mathbb{Z}$ for some $k \in F$.

Proof. We in fact show that this is true for the "average" choice of $t_0 \in \mathbb{T}$. For $t \in \mathbb{T}$ and $n = 1, \dots, N$, we denote

$$\begin{aligned} \nu_n(t) &:= \#\{k \in F : tk \in I_n + \mathbb{Z}\}, \\ N(t) &:= \#\{n = 1, \dots, N : \nu_n(t) > 0\}. \end{aligned}$$

The claim is then that $N(t_0) \geqslant \frac{1}{8}N$ for some $t_0 \in \mathbb{T}$, and we will prove that

$$\int_0^1 N(t) \, \mathrm{d}t \geqslant \frac{1}{8}N, \tag{13.12}$$

which clearly implies the existence of a desired t_0.

The strategy of the proof is as follows. Since each of the N different $k \in F$ satisfies $tk \in I_n + \mathbb{Z}$ for exactly one $n = 1, \dots, N$, we have

$$N = \sum_{n=1}^N \nu_n(t) = \sum_{\substack{1 \leqslant n \leqslant N \\ \nu_n(t) > 0}} \nu_n(t) \leqslant N(t)^{1/2} \Big(\sum_{n=1}^N \nu_n(t)^2 \Big)^{1/2}.$$

Integrating and using the Cauchy–Schwarz inequality, we obtain

$$N \leqslant \Big(\int_0^1 N(t) \, \mathrm{d}t \Big)^{1/2} \Big(\int_0^1 \sum_{n=1}^N \nu_n(t)^2 \, \mathrm{d}t \Big)^{1/2},$$

and (13.12) follows if we can prove that

$$\int_0^1 \sum_{n=1}^N \nu_n(t)^2 \, \mathrm{d}t \leqslant 8N. \tag{13.13}$$

Now

$$\nu_n(t) = \sum_{k \in F} \mathbf{1}_{I_n + \mathbb{Z}}(kt) = \sum_{k \in F} \mathbf{1}_{I_0 + \mathbb{Z}}(kt - n/N).$$

For the convenience of Fourier analysis, we replace the indicator

$$\mathbf{1}_{I_0 + \mathbb{Z}}(t) = \mathbf{1}_{[-\frac{1}{2N}, \frac{1}{2N})}(t), \qquad t \in [-\frac{1}{2}, \frac{1}{2}),$$

by the regularised version given by the 1-periodic extension of the "tent"

$$s(t) := (1 - N|t|)_+, \qquad t \in [-\frac{1}{2}, \frac{1}{2}).$$

An elementary computation of the Fourier coefficients shows that

$$\widehat{s}(j) = \frac{1}{N}\operatorname{sinc}^2(\pi j/N) = \begin{cases} 1/N, & j = 0, \\ 0, & 0 \neq j \equiv 0 \bmod N. \end{cases} \tag{13.14}$$

Note that the first equality above is valid for all $j \in \mathbb{Z}$, although in the second we only consider particular cases. Clearly $0 \leqslant \widehat{s}(j) = O(j^{-2})$, so that Lemma 13.1.22 applies to $f = s$. Since $s(h/N) = \mathbf{1}_{N\mathbb{Z}}(h)$ for $h \in \mathbb{Z}$, the conclusion of the lemma takes a particularly clean form, namely

$$\sum_{j \equiv n \bmod N} \widehat{s}(j) = \frac{1}{N} \qquad \forall n = 1, \dots, N. \tag{13.15}$$

We observe that $\mathbf{1}_{I_0+\mathbb{Z}}(t) \leqslant 2s(t)$, and hence

$$\begin{aligned} \nu_n(t) &\leqslant 2\sum_{k\in F} s(kt - n/N) = 2\sum_{k\in F}\sum_{j\in\mathbb{Z}} \widehat{s}(j) e_j(kt - n/N) \\ &= 2\sum_{h=1}^{N} e_h(-n/N) \sum_{\substack{j\equiv h \\ \bmod N}} \widehat{s}(j) \sum_{k\in F} e_j(kt). \end{aligned}$$

Substituting this into (13.13), we can now estimate

$$\begin{aligned} \int_0^1 \sum_{n=1}^{N} \nu_n^2 &\leqslant 4\int_0^1 \sum_{n=1}^{N} \Big| \sum_{h=1}^{N} e_h(-n/N) \sum_{\substack{j\equiv h \\ \bmod N}} \widehat{s}(j) \sum_{k\in F} e_j(kt) \Big|^2 \,\mathrm{d}t \\ &= 4\int_0^1 N \sum_{h=1}^{N} \Big| \sum_{\substack{j\equiv h \\ \bmod N}} \widehat{s}(j) \sum_{k\in F} e_j(kt) \Big|^2 \,\mathrm{d}t, \\ &\qquad \text{since the matrix } (N^{-1/2} e_h(-n/N))_{h,n=1}^{N} \text{ is unitary,} \\ &\leqslant 4N \sum_{h=1}^{N} \Big(\sum_{\substack{j\equiv h \\ \bmod N}} \widehat{s}(j) \Big\| \sum_{k\in F} e_{jk} \Big\|_{L^2(\mathbb{T})} \Big)^2 \\ &\leqslant 4N \Big\{ \sum_{h=1}^{N-1} \Big(\sum_{\substack{j\equiv h \\ \bmod N}} \widehat{s}(j) N^{1/2} \Big)^2 + \Big(\widehat{s}(0) N + \sum_{\substack{0\neq j\equiv 0 \\ \bmod N}} \widehat{s}(j) N^{1/2} \Big)^2 \Big\}, \\ &\qquad \text{since } \Big\| \sum_{k\in F} e_{jk} \Big\|_{L^2(\mathbb{T})} = \begin{cases} N, & j = 0, \\ N^{1/2}, & \text{otherwise,} \end{cases} \\ &= 4N \Big\{ \sum_{h=1}^{N-1} \Big(\frac{1}{N} N^{1/2} \Big)^2 + \Big(\frac{1}{N} N + 0 \Big)^2 \Big\}, \end{aligned}$$

$$\text{by (13.15) and (13.14),}$$
$$= 4N\{(N-1)N^{-1}+1\} = 4\{2N-1\} < 8N = \text{RHS (13.13)}.$$

This confirms (13.13) and hence, as explained in the beginning of the proof, the assertion of the Lemma. □

From the comparison between $\ell_N^p(X)$ and $\ell_N^\infty(X)$, it is immediate that

$$\varphi_{\infty,X}^{(q)}(N) \leqslant \varphi_{p,X}^{(q)}(N) \cdot N^{1/p}.$$

This triviality admits a crucial improvement, where on the left we have a similar quantity associated to an *arbitrary* subset $F \subseteq \mathbb{Z}$ of size N.

Definition 13.1.24. *Given $F \subseteq \mathbb{Z}$, we denote by $\varphi_{\infty,X}^{(q)}(F)$ be the best constant in the estimate*

$$\Big\| \sum_{k\in F} e_k x_k \Big\|_{L^q(\mathbb{T};X)} \leqslant \varphi_{\infty,X}^{(q)}(F) \sup_{k\in F} \|x_k\|,$$

which is to holds for arbitrary families $(x_k)_{k\in F}$ in X.

Clearly the previously considered $\varphi_{\infty,X}^{(q)}(N)$ is the special case $\varphi_{\infty,X}^{(q)}(N) = \varphi_{\infty,X}^{(q)}(\{1,\dots,N\})$ in this notation. In contrast to random sums with independent sequences of random variables, the particular choice of the indexing set F is very relevant here, since the joint distribution of $(e_k)_{k\in F}$ can be very different from that of $(e_k)_{k=1}^N$.

Lemma 13.1.25 (Bourgain). *For any Banach space X and exponents $p, q \in [1,\infty)$ we have*

$$\varphi_{\infty,X}^{(q)}(F) \leqslant A\varphi_{p,X}^{(q)}(N) \cdot N^{1/p}, \qquad A := \big(8p(\pi + 2^{1/q}\cdot 3)\big)^{1+1/q},$$

whenever $F \subseteq \mathbb{Z}$ is a subset of size $\#F = N$.

Remark 13.1.26. We only apply Lemma 13.1.25 with $p = 2 \leqslant q$. In this case

$$A \leqslant \big(16(\pi + \sqrt{2}\cdot 3)\big)^{3/2} < 1285.$$

Proof of Lemma 13.1.25. Since

$$\Big\| \sum_{k\in F} e_k x_k \Big\|_{L^q(\mathbb{T};X)} \leqslant \sum_{k\in F} \|x_k\| \leqslant N \max_{k\in F} \|x_k\|,$$

and $\varphi_{p,X}^{(q)}(N) \geqslant 1$, we have the trivial estimate

$$\varphi_{\infty,X}^{(q)}(F) \leqslant N \leqslant \varphi_{p,X}^{(q)}(N) N^{1/p'+1/p} \leqslant A\varphi_{p,X}^{(q)}(N) N^{1/p} \qquad \forall N \leqslant A^{p'}.$$

Suppose then, for induction, that $N > A^{p'}$, and moreover that the Lemma has been verified for all $N' < N$ in place of N. For $F \subseteq \mathbb{Z}_+$ of size N, we consider a splitting (with $\varnothing \neq F_0 \subsetneq F$ to be specified shortly)

$$\Big\| \sum_{k\in F} e_k x_k \Big\|_{L^q(\mathbb{T};X)} \leqslant \Big\| \sum_{k\in F_0} e_k x_k \Big\|_{L^q(\mathbb{T};X)} + \Big\| \sum_{k\in F\setminus F_0} e_k x_k \Big\|_{L^q(\mathbb{T};X)} =: I + II.$$

Since $F \setminus F_0 \subsetneq F$ is a strictly smaller set and $\varphi^{(q)}_{p,X}$ is clearly non-decreasing, the induction hypothesis applies to show that

$$\begin{aligned} II &\leqslant \varphi^{(q)}_{\infty,X}(F \setminus F_0) \max_{k\in F\setminus F_0} \|x_k\| \\ &\leqslant A\varphi^{(q)}_{p,X}(N)\#(F \setminus F_0)^{1/p} \max_{k\in F} \|x_k\|. \end{aligned} \tag{13.16}$$

Let us make a specific choice of $F_0 \subsetneq F$ as follows. By Lemma 13.1.23, there exist $t_0 \in T$ and $1 \leqslant n_1 < n_2 < \dots n_\ell \leqslant N$ with $\ell \geqslant \frac{1}{8}\#F$ such that each of the mutually disjoint sets

$$I_{n_j} + \mathbb{Z} = \frac{1}{N}[n - \frac{1}{2}, n + \frac{1}{2}) + \mathbb{Z}, \quad (j = 1, \dots, \ell),$$

intersects with the set $\{kt_0 : k \in F\}$. For each $j \in \{1, \dots, \ell\}$, we pick a $k_j \in F$ such that $k_j t_0 \in I_{n_j} + \mathbb{Z}$, and set $F_0 := \{k_j : j = 1, \dots, \ell\}$. Then $\#F_0 = \ell \geqslant \frac{1}{8}\#F$. The size bound on $\#F_0$ shows that (13.16) implies

$$II \leqslant A\varphi^{(q)}_{p,X}(N)\Big(\frac{7}{8}N\Big)^{1/p} \max_{k\in F} \|x_k\|. \tag{13.17}$$

Let $\psi : k_j \to n_j$ be the corresponding bijection from F_0 onto $\psi(F_0) \subseteq \{1, \dots, N\}$. Thus by definition that $k_j t_0 \in I_{\psi(k_j)} + \mathbb{Z} = I_{n_j} + \mathbb{Z}$ for all $j = 1, \dots, \ell$. For any $h \in \mathbb{Z}$, we then have

$$\begin{aligned} I &= \Big\| \sum_{k\in F_0} e_k(\cdot + ht_0) x_k \Big\|_{L^q(\mathbb{T};X)} \qquad \text{by translation invariance} \\ &\leqslant \Big\| \sum_{k\in F_0} [e_k(ht_0) - e_h(\frac{\psi(k)}{N})] e_k x_k \Big\|_{L^q(\mathbb{T};X)} + \Big\| \sum_{k\in F_0} e_h(\frac{\psi(k)}{N}) e_k x_k \Big\|_{L^q(\mathbb{T};X)} \\ &=: I_1(h) + I_2(h), \end{aligned}$$

where (using again the induction hypothesis, now with the smaller set $F_0 \subsetneq F$)

$$\begin{aligned} I_1(h) &\leqslant \varphi^{(q)}_{\infty,X}(F_0) \max_{k\in F_0} \Big| \exp(2\pi i k h t_0) - \exp(i2\pi h \frac{\psi(k)}{N}) \Big| \|x_k\| \\ &\leqslant A\varphi^{(q)}_{p,X}(\#F_0)^{1/p} \max_{k\in F_0} \Big(2\pi |h| \inf_{j\in\mathbb{Z}} \Big| kt_0 - \frac{\psi(k)}{N} - j \Big| \Big) \max_{k\in F_0} \|x_k\| \\ &\leqslant A\varphi^{(q)}_{p,X}(N) N^{1/p} \frac{\pi |h|}{N} \max_{k\in F} \|x_k\|, \end{aligned} \tag{13.18}$$

since $kt_0 \in I_{\psi(k)} + \mathbb{Z}$.

Having estimated both $I_1(h)$ and II in terms of the induction hypothesis, the serious work is left with $I_2(h)$, which we first average over a range $h = 1, \ldots, H \leqslant N$, where a favourable value of H is to be determined. We have

$$\begin{aligned}
\frac{1}{H}\sum_{h=1}^{H} I_2(h)^q &= \frac{N}{H}\int_{\mathbb{T}} \frac{1}{N}\sum_{h=1}^{H}\Big\| \sum_{j\in\psi(F_0)} e_h(j/N) e_{\psi^{-1}(j)}(t) x_{\psi^{-1}(j)}\Big\|^q \,\mathrm{d}t \\
&\leqslant \frac{N}{H}\int_{\mathbb{T}} \frac{1}{N}\sum_{h=1}^{N}\Big\|\sum_{j=1}^{N} e_h(j/N) y_j(t)\Big\|^q \,\mathrm{d}t \\
&\qquad y_j(t) := \begin{cases} e_{\psi^{-1}(j)}(t) x_{\psi^{-1}(j)}, & j \in \psi(F_0), \\ 0, & \text{else}, \end{cases} \\
&\leqslant \frac{N}{H}\int_{\mathbb{T}} \Big(\varphi_{p,X}^{(q)}(\mathbb{Z}_N)\Big[\sum_{j=1}^{N}\|y_j(t)\|^p\Big]^{1/p}\Big)^q \,\mathrm{d}t \\
&\qquad \text{by definition of } \varphi_{p,X}^{(q)}(\mathbb{Z}_N) \\
&\leqslant \frac{N}{H}\big(3\varphi_{p,X}^{(q)}(N)\big)^q \Big(\sum_{k\in F_0}\|x_k\|^p\Big)^{q/p} \qquad \text{by Lemma 13.1.18} \\
&\leqslant \frac{N}{H}\big(3\varphi_{p,X}^{(q)}(N)\big)^q (\#F_0)^{q/p} \max_{k\in F_0}\|x_k\|^q.
\end{aligned}$$

Combining the previous bound with (13.17) and (13.18), we have

$$\begin{aligned}
\Big\|\sum_{k\in F} e_k x_k\Big\|_{L^q(\mathbb{T};X)} &\leqslant I + II \leqslant \frac{1}{H}\sum_{h=1}^{H}(I_1(h) + I_2(h)) + II \\
&\leqslant \max_{1\leqslant h\leqslant H} I_1(h) + \Big(\frac{1}{H}\sum_{h=1}^{H} I_2(h)^q\Big)^{1/q} + II \\
&\leqslant \Big(A\frac{\pi H}{N} + 3\Big(\frac{N}{H}\Big)^{1/q} + A\Big(\frac{7}{8}\Big)^{1/p}\Big) N^{1/p}\varphi_{p,X}^{(q)}(N)\max_{k\in F}\|x_k\|,
\end{aligned} \tag{13.19}$$

where $N = \#F$, as we recall. We now choose H so as to essentially equate the first two terms:

$$H := \lfloor H' \rfloor, \qquad H' := A^{-q/(q+1)}N.$$

Since $A > 1$, we have $H \leqslant H' \leqslant N$. Recalling that $N > A^{p'}$, and noting that $p' > 1 > q/(q+1)$, we also observe that $H' \geqslant 1$, and hence $H \geqslant 1$. Thus this choice of H lies in the admissible range considered above. We also have $H' \leqslant H + 1 \leqslant 2H$, and thus

$$A\frac{\pi H}{N} + 3\Big(\frac{N}{H}\Big)^{1/q} \leqslant A\frac{\pi H'}{N} + 3\Big(\frac{2N}{H'}\Big)^{1/q} = (\pi + 2^{1/q}\cdot 3)A^{1/(q+1)}.$$

We also note that

$$\Big(\frac{7}{8}\Big)^{1/p} - 1 = \frac{1}{p}\xi^{1/p-1}\Big(\frac{7}{8}-1\Big) \leqslant -\frac{1}{8p}, \quad \text{for some } \xi \in \Big(\frac{7}{8}, 1\Big).$$

Substituting into (13.19), we hence have

$$\varphi^{(q)}_{\infty,X}(F) \leqslant \Big[(\pi + 2^{1/q}\cdot 3)\cdot A^{1/(q+1)} + \Big(1 - \frac{1}{8p}\Big)A\Big]\varphi^{(q)}_{p,X}(N)\cdot N^{1/p}.$$

To complete the induction step, it remains to check that the quantity in brackets is at most A, which after easy simplification is the same as

$$(\pi + 2^{1/q}\cdot 3)\cdot A^{1/(q+1)} \leqslant \frac{1}{8p}A.$$

Clearly this is the case with the choice of A stated in the Lemma. □

We are now ready for a first uniform bound on the finite Fourier type constants:

Corollary 13.1.27. *Let X be a Banach space, $r \in (1, 2]$, and suppose that*

$$\varphi_{2,X}(N) \leqslant C \cdot N^{1/r-1/2} \qquad \forall N \in \mathbb{Z}_+.$$

Then for all $s \in (1, r)$, we have

$$\varphi^{(2)}_{s,X}(N) \leqslant 3500\frac{Cr}{r-s} \qquad \forall N \in \mathbb{Z}_+.$$

Proof. By Lemma 13.1.25 and Remark 13.1.26 with $p = q = 2$, we have

$$\varphi^{(2)}_{\infty,X}(F) \leqslant 1285\cdot\varphi_{2,X}(N)\cdot N^{1/2} \leqslant 1285\cdot C\cdot N^{1/r} \tag{13.20}$$

whenever $F \subseteq \mathbb{Z}$ has size $\#F = N$.

Let $x = (x_k)_{k=1}^N \in \ell^s_N(X)$ have norm one. For $\alpha \in (0, 1)$ to be chosen, we denote

$$F_j := \{n \in \mathbb{Z} : \alpha^j < \|x_n\| \leqslant \alpha^{j-1}\}, \qquad x^{(j)} := (\mathbf{1}_{F_j}(k)\cdot x_k)_{k=1}^N.$$

Note that $F_j = \varnothing$ and $x^{(j)} = 0$ for $j \leqslant 0$, and

$$\#F_j \leqslant \#\{n \in \mathbb{Z} : \alpha^j < \|x_n\|\} \leqslant \alpha^{-js}\|x\|_{\ell^s_N(X)} = \alpha^{-js}, \qquad j \geqslant 1.$$

Thus

$$\Big\|\sum_{k=1}^N e_k x_k\Big\|_{L^2(\mathbb{T};X)} \leqslant \sum_{j=1}^\infty \Big\|\sum_{k\in F_j} e_k x_k^{(j)}\Big\|_{L^2(\mathbb{T};X)} \leqslant \sum_{j=1}^\infty \varphi^{(2)}_{\infty,X}(F_j)\max_{k\in F_j}\|x_k^{(j)}\|$$

$$\begin{aligned}
&\leqslant \sum_{j=1}^{\infty} 1285 \cdot \varphi_{2,X}^{(2)}(\#F_j) \cdot (\#F_j)^{1/2} \cdot \alpha^{j-1} \qquad \text{by (13.20)} \\
&\leqslant \sum_{j=1}^{\infty} 1285 \cdot C(\#F_j)^{1/r} \alpha^{j-1} \leqslant \sum_{j=1}^{\infty} 1285 \cdot C\alpha^{-js/r}\alpha^{j-1} \\
&= \frac{1285 \cdot C}{\alpha} \frac{\alpha^{1-s/r}}{1-\alpha^{1-s/r}}.
\end{aligned}$$

The choice $\alpha = (s/r)^{r/(r-s)}$ gives

$$\frac{1}{\alpha}\frac{\alpha^{1-s/r}}{1-\alpha^{1-s/r}} = \frac{\alpha^{-s/r}}{1-\alpha^{1-s/r}} = \frac{(r/s)^{s/(r-s)}}{1-s/r} \leqslant \frac{e}{1-s/r}$$

by an elementary optimisation in the last step. Substituting back, this gives

$$\Big\| \sum_{k=1}^{N} e_k x_k \Big\|_{L^2(\mathbb{T};X)} \leqslant 1285 \cdot C \cdot \frac{e}{1-s/r} = 1285 \cdot e \cdot \frac{Cr}{r-s} < 3500 \cdot \frac{Cr}{r-s}$$

for all $(x_k)_{k=1}^N \in \ell_N^s(X)$ of norm one, which is the claimed bound. □

13.1.d Conclusion via duality and interpolation

With the uniform bound of Corollary 13.1.27, we have already covered the core of the deep implication from non-trivial type to non-trivial Fourier type. The rest of the argument depends on the more routine techniques of duality and interpolation, but is still not entirely straightforward. We now turn our attention to giving these finishing touches to the proof. At the end of this section, a statement and proof of Bourgain's theorem will finally be given.

The first duality that we want to use is most elegantly expressed in terms of the Fourier type constants on the cyclic group $\mathbb{Z}_N$:

Lemma 13.1.28. *Let X be a Banach space, $N \in \mathbb{Z}_+$ and $p, q \in (1, \infty)$. Then*

$$N^{1/q}\varphi_{p,X}^{(q)}(\mathbb{Z}_N) = N^{1/p'}\varphi_{q',X^*}^{(p')}(\mathbb{Z}_N).$$

Proof. Since X is norming for X^*, Proposition 1.3.1 shows that $\ell_N^p(X)$ is norming for $\ell_N^{p'}(X^*)$, so that

$$\begin{aligned}
&\Big(\sum_{h=1}^{N} \Big\| \sum_{k=1}^{N} e_k(h/N) x_k^* \Big\|^{p'} \Big)^{1/p'} \\
&\qquad = \sup \Big\{ \sum_{h=1}^{N} \Big\langle x_h, \sum_{k=1}^{N} e_k(h/N) x_k^* \Big\rangle : \Big(\sum_{h=1}^{N} \|x_h\|^p \Big)^{1/p} \leqslant 1 \Big\},
\end{aligned}$$

where, observing the symmetry $e_k(h/N) = e^{2\pi i k h/N} = e_h(k/N)$,

$$\sum_{h=1}^{N}\Big\langle x_h,\sum_{k=1}^{N}e_k(h/N)x_k^*\Big\rangle=\sum_{k=1}^{N}\Big\langle\sum_{h=1}^{N}e_h(k/N)x_h,x_k^*\Big\rangle$$
$$\leqslant\Big(\sum_{k=1}^{N}\Big\|\sum_{h=1}^{N}e_h(k/N)x_h\Big\|^q\Big)^{1/q}\Big(\sum_{k=1}^{N}\|x_k^*\|^{q'}\Big)^{1/q'}$$
$$\leqslant N^{1/q}\varphi_{p,X}^{(q)}(\mathbb{Z}_N)\Big(\sum_{h=1}^{N}\|x_h\|^p\Big)^{1/p}\Big(\sum_{k=1}^{N}\|x_k^*\|^{q'}\Big)^{1/q'}.$$

Substituting back, this proves that

$$N^{1/p'}\varphi_{q',X^*}^{(p')}(\mathbb{Z}_N)\leqslant N^{1/q}\varphi_{p,X}^{(q)}(\mathbb{Z}_N).$$

Permuting the names of the exponents and using the isometric embedding of X into X^{**}, it also follows that

$$N^{1/q}\varphi_{p,X}^{(q)}(\mathbb{Z}_N)\leqslant N^{1/q}\varphi_{p,X^{**}}^{(q)}(\mathbb{Z}_N)\leqslant N^{1/p'}\varphi_{q',X^*}^{(p')}(\mathbb{Z}_N),$$

which proves the claimed equality. □

Corollary 13.1.29. *Let X be a Banach space, $r\in(1,2]$, and suppose that*

$$\varphi_{2,X}(N)\leqslant C\cdot N^{1/r-1/2}\qquad\forall N\in\mathbb{Z}_+.$$

Then for all $s\in(1,r)$ we have

$$\varphi_{\infty,X^*}^{(s')}(F)\leqslant 1.35\cdot 10^7\frac{Cr}{r-s}N^{1/s}\qquad\forall s\in(1,r),$$

whenever $F\subseteq\mathbb{Z}$ is a subset of size $\#F=N$.

Recall from Corollary 13.1.20 that if X has type $p\in(1,2]$, then the assumption is satisfied with C and r as in (13.10).

Proof. By using both estimates of Lemma 13.1.18 with Lemma 13.1.28 in between, and finally Corollary 13.1.27, we have

$$N^{1/s'}\varphi_{2,X^*}^{(s')}(N)\leqslant N^{1/s'}\varphi_{2,X^*}^{(s')}(\mathbb{Z}_N)=N^{1/2}\varphi_{s,X}^{(2)}(\mathbb{Z}_N)$$
$$\leqslant N^{1/2}\cdot 3\varphi_{s,X}^{(2)}(N)\leqslant N^{1/2}\cdot 3\cdot 3500\frac{Cr}{r-s}.$$

Then Lemma 13.1.25 and Remark 13.1.26 with $p=2\leqslant q=s'$ show that

$$\varphi_{\infty,X^*}^{(s')}(F)\leqslant 1285\cdot\varphi_{2,X^*}^{(s')}(N)\cdot N^{1/2}<1.35\cdot 10^7\frac{Cr}{r-s}\cdot N^{1/2+1/2-1/s'}.$$

whenever $F\subseteq\mathbb{Z}$ is a subset of size $\#F=N$. □

We now come to another form of duality, where we pass from the Fourier transform on $\mathbb{Z}$ to that on the circle $\mathbb{T}$, and it is in this latter setting that our argument will be completed.

Lemma 13.1.30. *Let X be a Banach space, $1 \leqslant s \leqslant \infty$, and suppose that*

$$\varphi^{(s')}_{\infty,X^*}(F) \leqslant K \cdot N^{1/s}$$

whenever $F \subseteq \mathbb{Z}$ is a subset of size $\#F = N$. Then the Fourier transform

$$\mathscr{F} : f \in L^1(\mathbb{T}; X) \mapsto (\widehat{f}(k))_{k\in\mathbb{Z}}, \qquad \widehat{f}(k) = \int_{\mathbb{T}} e_{-k}(t) f(t)\,\mathrm{d}t,$$

satisfies the weak-type estimate

$$\|\mathscr{F} f\|_{\ell^{s',\infty}(\mathbb{Z};X)} \leqslant K \|f\|_{L^s(\mathbb{T};X)}. \tag{13.21}$$

Proof. Let $f \in L^s(\mathbb{T}; X)$, let $\lambda > 0$, and let F be a finite subset of $\{k \in \mathbb{Z} : \|\widehat{f}(k)\| > \lambda\}$. (By a periodic analogue of the Riemann–Lebesgue Lemma 2.4.3, which has essentially the same proof, we could argue that this set is finite to begin with, but we do not need this here.) Then

$$\begin{aligned}
\#F &\leqslant \frac{1}{\lambda} \sum_{k\in F} \|\widehat{f}(k)\| = \frac{1}{\lambda} \sum_{k\in F} \langle \widehat{f}(k), x^*_{-k}\rangle \\
&\qquad \text{for suitable } x^*_{-k} \in X^* \text{ of norm one} \\
&= \frac{1}{\lambda} \int_{\mathbb{T}} f(t) \Big(\sum_{k\in F} e_{-k}(t) x^*_{-k} \Big)\,\mathrm{d}t \\
&\leqslant \frac{1}{\lambda} \|f\|_{L^s(\mathbb{T};X)} \Big\| \sum_{k\in -F} e_k x_k \Big\|_{L^{s'}(\mathbb{T};X^*)} \\
&\leqslant \frac{1}{\lambda} \|f\|_{L^s(\mathbb{T};X)} \varphi^{(s')}_{\infty,X^*}(-F) \leqslant \frac{1}{\lambda} \|f\|_{L^s(\mathbb{T};X)} K (\#F)^{1/s},
\end{aligned}$$

and hence

$$\lambda (\#F)^{1-1/s} \leqslant K \|f\|_{L^s(\mathbb{T};X)}.$$

Since this is true for any finite $F \subseteq \{k \in \mathbb{Z} : \|\widehat{f}(k)\| > \lambda\}$, it is also true for $F = \{k \in \mathbb{Z} : \|\widehat{f}(k)\| > \lambda\}$ (showing, *a posteriori*, the finiteness of this set). Then the supremum over $\lambda > 0$ of the left-hand side is precisely the $\ell^{s',\infty}(\mathbb{Z}; X)$ norm that we wanted to estimate. □

From (13.21) and the trivial fact that $\mathscr{F}$ is bounded from $L^1(\mathbb{T}; X) \to \ell^\infty(\mathbb{Z}; X)$, it seems apparent that we should conclude that $\mathscr{F}$ is bounded from $L^p(\mathbb{T}; X)$ to $\ell^{p'}(\mathbb{Z}; X)$ by interpolation. However, the version of the Marcinkiewicz Interpolation Theorem 2.2.3 covered in the text is not sufficient for this purpose, and we would need the generalisation stated in the Notes as Theorem 2.7.5. We will give a proof of a quantitative version of the special case relevant for the present application:

Lemma 13.1.31. *Let X be a Banach space such that (13.21) holds for some $s \in (1,2]$. Then*

$$\|\mathscr{F} f\|_{\ell^{t'}(\mathbb{Z};X)} \leqslant \frac{3K}{(s-t)^{1/t'}} \|f\|_{L^t(\mathbb{T};X)} \qquad \forall t \in (1,s).$$

Proof. By homogeneity we may assume that $\|f\|_{L^t(\mathbb{T};X)} = 1$. We have

$$\begin{aligned}\|\mathscr{F} f\|_{\ell^{t'}(\mathbb{Z};X)} &= \int_0^\infty t'\lambda^{t'-1}\#\{k : \|\widehat{f}(k)\| > \lambda\}\,\mathrm{d}\lambda \\ &\leqslant \int_0^\infty t'\lambda^{t'-1}\#\{k : \|\widehat{f_\lambda}(k)\| > \theta_0\lambda\}\,\mathrm{d}\lambda \\ &\qquad + \int_0^\infty t'\lambda^{t'-1}\#\{k : \|\widehat{f^\lambda}(k)\| > \theta_1\lambda\}\,\mathrm{d}\lambda,\end{aligned} \tag{13.22}$$

where $\theta_0 + \theta_1 = 1$ and, with parameters A and γ to be chosen shortly,

$$f_\lambda := f\cdot \mathbf{1}_{\{\|f\|_X \leqslant A\lambda^\gamma\}}, \qquad f^\lambda := f\cdot \mathbf{1}_{\{\|f\|_X > A\lambda^\gamma\}}.$$

Then

$$\|f^\lambda\|_{L^1(\mathbb{T};X)} = \int_{\{\|f\|_X > A\lambda^\gamma\}} \|f\|_X \leqslant (A\lambda^\gamma)^{1-t}\|f\|^t_{L^t(\mathbb{T};X)} = (A\lambda^\gamma)^{1-t}$$

and hence

$$\|\widehat{f^\lambda}\|_{\ell^\infty(\mathbb{Z};X)} \leqslant (A\lambda^\gamma)^{1-t} \leqslant \theta_1\lambda,$$

provided that we choose

$$\gamma = -1/(t-1), \qquad A = \theta_1^{-1/(t-1)}.$$

Then the second term on the right of (13.22) vanishes, and subsequently

$$\begin{aligned}\|\mathscr{F} f\|^{t'}_{\ell^{t'}(\mathbb{Z};X)} &\leqslant \int_0^\infty t'\lambda^{t'-1}\#\{k : \|\widehat{f_\lambda}(k)\| > \theta_0\lambda\}\,\mathrm{d}\lambda \\ &\leqslant \int_0^\infty t'\lambda^{t'-1}(\theta_0\lambda)^{-s'}K^{s'}\|f_\lambda\|^{s'}_{L^s(\mathbb{T};X)}\,\mathrm{d}\lambda \\ &\qquad \text{by Lemma 13.1.30} \\ &= t'\Big(\frac{K}{\theta_0}\Big)^{s'}\Big(\int_0^\infty \lambda^{t'-s'-1}\|f_\lambda\|^{s'}_{L^s(\mathbb{T};X)}\,\mathrm{d}\lambda\Big)^{s'/s'} \\ &\leqslant t'\Big(\frac{K}{\theta_0}\Big)^{s'}\Big\|\Big(\int_0^\infty \lambda^{t'-s'-1}\|f_\lambda\|^{s'}_X\,\mathrm{d}\lambda\Big)^{1/s'}\Big\|^{s'}_{L^s(\mathbb{T})} \\ &\qquad \text{by Minkowski's inequality with exponents } s \leqslant s' \\ &= t'\Big(\frac{K}{\theta_0}\Big)^{s'}\Big\|\Big(\int_0^{(A/\|f\|_X)^{t-1}} \lambda^{t'-s'-1}\|f\|^{s'}_X\,\mathrm{d}\lambda\Big)^{1/s'}\Big\|^{s'}_{L^s(\mathbb{T})}\end{aligned}$$

$$\text{keeping in mind the choice } \gamma = -1/(t-1)$$
$$= t'\Big(\frac{K}{\theta_0}\Big)^{s'} \Big\| \Big(\frac{1}{t'-s'} \Big[\frac{A}{\|f\|_X} \Big]^{(t-1)(t'-s')} \|f\|_X^{s'} \Big)^{1/s'} \Big\|_{L^s(\mathbb{T})}^{s'}$$
$$= t'\Big(\frac{K}{\theta_0}\Big)^{s'} \frac{1}{\theta_1^{t'-s'}} \frac{1}{t'-s'} \Big\| \Big(\|f\|_X^{s'-(t-1)(t'-s')} \Big)^{1/s'} \Big\|_{L^s(\mathbb{T})}^{s'},$$

where, observing that $tt' = t + t'$, we have

$$s' - (t-1)(t'-s') = s' - (t + t' - ts' - t' + s') = t(s'-1),$$

so that

$$\Big\| \Big(\|f\|_X^{s'-(t-1)(t'-s')} \Big)^{1/s'} \Big\|_{L^s(\mathbb{T})}^{s'} = \Big\| \|f\|_X^{t/s} \Big\|_{L^s(\mathbb{T})}^{s'} = \|f\|_{L^t(\mathbb{T};X)}^{ts'/s} = 1.$$

Taking $\theta_0 = \theta_1 = \frac{1}{2}$ and using $(t')^{1/t'} \leqslant e^{1/e} < \frac{3}{2}$, we obtain

$$\|\mathscr{F} f\|_{\ell^{t'}(\mathbb{Z};X)} \leqslant \frac{2(t')^{1/t'} K^{s'/t'}}{(t'-s')^{1/t'}} \leqslant \frac{3K^{s'/t'}}{(t'-s')^{1/t'}}.$$

Testing (13.21) with a constant function $f \equiv x$, with Fourier coefficients $\widehat{f}(k) = \delta_{k,0} x$, shows that $K \geqslant 1$ and hence $K^{s'/t'} \leqslant K$. Moreover,

$$t' - s' = \frac{t}{t-1} - \frac{s}{s-1} = \frac{t(s-1) - s(t-1)}{(s-1)(t-1)} = \frac{s-t}{(s-1)(t-1)} \geqslant s - t,$$

and hence

$$\|\mathscr{F} f\|_{\ell^{t'}(\mathbb{Z};X)} \leqslant \frac{3K}{(s-t)^{1/t'}}.$$

□

Lemma 13.1.32. *Let X be a Banach space, and suppose that there are constants C and $r \in (1,2]$ such that*

$$\varphi_{2,X}(N) \leqslant C \cdot N^{1/r - 1/2}$$

for all $N \in \mathbb{Z}_+$. Then for all $t \in (1,r)$, we have

$$\varphi_{t,X} \leqslant \frac{10^9 \cdot C}{(r-t)^{1+1/t'}}.$$

Proof. By Corollary 13.1.29, for all $s \in (1,r)$, we then have

$$\varphi_{\infty,X^*}^{(s')}(F) \leqslant 1.35 \cdot 10^7 \frac{Cr}{r-s} N^{1/s} =: K \cdot N^{1/s}$$

whenever $F \subseteq \mathbb{Z}$ is a subset of size $\#F = N$.

By Lemma 13.1.30, it follows that

$$\|\mathscr{F}\|_{\mathscr{L}(L^s(\mathbb{T};X),\ell^{s',\infty}(\mathbb{Z};X))} \leqslant K,$$

which by Lemma 13.1.31 implies

$$\varphi_{t,X}(\mathbb{T}) := \|\mathscr{F}\|_{\mathscr{L}(L^t(\mathbb{T};X),\ell^{t'}(\mathbb{Z};X))} \leqslant \frac{3K}{(s-t)^{1/t'}} \leqslant \frac{5 \cdot 10^7 \cdot C \cdot r}{(r-s)(s-t)^{1/t'}}$$

for all $1 < t < s < r$. Optimising the bound with respect to s in this range, we choose

$$s = \frac{t^2 + (t-1)r}{2t-1}.$$

With this choice, a computation shows that

$$r - s = \frac{t(r-t)}{2t-1} \geqslant \frac{1}{3}(r-t), \quad s - t = \frac{(r-t)(t-1)}{2t-1} \geqslant \frac{1}{3}(r-t)(t-1).$$

Substituting back,

$$\varphi_{t,X}(\mathbb{T}) \leqslant 5 \cdot 10^7 \cdot C \cdot r \frac{3^{1+1/t'}}{(r-t)^{1+1/t'}(t-1)^{1/t'}},$$

where $r \leqslant 2$ and $3^{1+1/t'} \leqslant 3^{3/2}$ and, for $t \in (1,2)$,

$$(t-1)^{1/t'} = [(t-1)^{t-1}]^{1/t} \geqslant [e^{-1/e}]^{1/t} \geqslant e^{-1/e}.$$

Thus

$$\varphi_{t,X}(\mathbb{T}) \leqslant 10^8 \cdot C \frac{3^{3/2} \cdot e^{1/e}}{(r-t)^{1+1/t'}} \leqslant \frac{10^9 \cdot C}{(r-t)^{1+1/t'}}.$$

□

We are finally ready for the main theorem:

Theorem 13.1.33 (Bourgain). *A Banach space X has non-trivial type if and only if it has non-trivial Fourier-type. Quantitatively:*

(1) *If X has Fourier-type $t \in (1,2]$, then it has type t with $\tau_{t,X} \leqslant \varphi_{t,X}(\mathbb{Z})$.*
(2) *If X has type $p \in (1,2]$ with related constant $\tau_{p,X;2}$ as defined in (13.7), then it has Fourier-type*

$$t = 1 + \frac{1}{6p'(68 \cdot \tau_{p,X;2})^{p'}}$$

with constants

$$\varphi_{t,X}(\mathbb{R}) \leqslant \varphi_{t,X}(\mathbb{T}) \leqslant \exp\big(2(68 \cdot \tau_{p,X;2})^{p'}\big).$$

Proof. (1): This is contained in Proposition 7.3.6.

(2): This is the main part of the proof, and depends on the results developed in the section. By Corollary 13.1.20, the assumptions imply that

$$\varphi_{2,X}(N) \leqslant C \cdot N^{1/r-1/2},$$

where, denoting $T := (68 \cdot \tau_{p,X;2})^{p'} \geqslant 68^2 > 4\,000$, we have

$$r' = 3p'T, \quad C = e^{\frac{r'}{2p'}} = e^{\frac{3}{2}T}.$$

Thus Lemma 13.1.32 shows that

$$\varphi_{t,X}(\mathbb{T}) \leqslant \frac{10^9 \cdot C}{(r-t)^{1+1/t'}}, \quad t \in (1,r),$$

where $r \geqslant 1 + (3p'T)^{-1}$. Hence, choosing $t := 1 + (6p'T)^{-1} \in (1,r)$, we have

$$r - t \geqslant (6p'T)^{-1}, \quad (r-t)^{1+1/t'} \leqslant (6p'T)^{\frac{3}{2}}.$$

Thus, noting that $p' \leqslant p' \log(68\tau_{p,X;2}) = \log T$, where $T \geqslant 68^2 > 4\,000$,

$$\begin{aligned} \varphi_{t,X}(\mathbb{T}) &\leqslant 10^9 \cdot e^{\frac{3}{2}T} \cdot (6p'T)^{\frac{3}{2}} \\ &= 10^9 \cdot 6^{\frac{3}{2}} \cdot (\log T)^{\frac{3}{2}} \cdot T^{\frac{3}{2}} \cdot e^{\frac{3}{2}T} \\ &\leqslant e^{\frac{1}{6}T} \cdot e^{\frac{1}{6}T} \cdot e^{\frac{1}{6}T} \cdot e^{\frac{3}{2}T} = e^{2T}. \end{aligned}$$

Finally, $\varphi_{t,X}(\mathbb{R}) \leqslant \varphi_{t,X}(\mathbb{T})$ is part of Propositions 13.1.1. □

Example 13.1.34. For each $r \in [2,\infty)$, the space $X = L^r(S)$ has type 2 with $\tau_{2,X;2} = \kappa_{r,2,\mathbb{K}}$ (the Kahane–Khintchine constant from the scalar-valued case of Theorem 6.2.4), but Fourier-type t if and only if $t \in [1,r']$. Hence, any estimate of the Fourier-type exponent in terms of the type of X must necessarily depend not only on the type exponent but also on the type constant of X.

Proof. The estimate $\tau_{2,X;2} \leqslant \kappa_{r,2,\mathbb{K}}$ follows from

$$\begin{aligned} \Big\| \sum_{n=1}^N \varepsilon_n x_n \Big\|_{L^2(\Omega;L^r(S))} &\leqslant \Big\| \sum_{n=1}^N \varepsilon_n x_n \Big\|_{L^r(\Omega;L^r(S))} = \Big\| \sum_{n=1}^N \varepsilon_n x_n \Big\|_{L^r(S;L^r(\Omega))} \\ &\leqslant \kappa_{r,2,\mathbb{K}} \Big\| \{x_n\}_{n=1}^N \Big\|_{L^r(S;\ell^2_N)} \leqslant \kappa_{r,2,\mathbb{K}} \Big\| \{x_n\}_{n=1}^N \Big\|_{\ell^2_N(L^r(S))}. \end{aligned}$$

For the reverse estimate, it suffices to pick some non-zero $\phi \in L^r(S)$ and observe that the type inequality for $x_n = a_n\phi \in X$ implies the Kahane–Khintchine inequality for $a_n \in \mathbb{K}$.

The fact that X has Fourier-type t if $t \in [1,r']$ follows from the scalar-valued Hausdorff–Young inequality and Minkowski's inequality:

$$\|\widehat{f}\|_{L^{t'}(\mathbb{R};L^r(S))} \leqslant \|\widehat{f}\|_{L^r(S;L^{t'}(\mathbb{R}))} \leqslant \|f\|_{L^r(S;L^t(\mathbb{R}))} \leqslant \|f\|_{L^t(\mathbb{R};L^r(S))}$$

We indicate two alternative proofs of the "only if" part:

(1) In Example 2.1.15, it is verified directly that the Fourier transform is not bounded from $L^p(\mathbb{R};\ell^{r'})$ to $L^{p'}(\mathbb{R};\ell^{r'})$ for $p \in (r',2]$. By duality, it is also not bounded from $L^p(\mathbb{R};\ell^r)$ to $L^{p'}(\mathbb{R};\ell^r)$.
(2) Proposition 7.3.6 says that if X has Fourier type p, then it has cotype p'. But Corollary 7.1.6 says that $L^r(S)$ has cotype p' only for $p' \in [r,\infty]$.

This concludes the verification of the example. □

We also record the following simpler variant, which is nevertheless sufficient for many purposes:

Proposition 13.1.35. *Let X have type p and cotype q, where $\frac{1}{p}-\frac{1}{q}<\frac{1}{2}$. Let*

$$\frac{1}{r} := \frac{1}{2}+\frac{1}{p}-\frac{1}{q} \in \Big[\frac{1}{2},1\Big).$$

Then X has every Fourier-type $t \in (1,r)$, and

$$\varphi_{t,X}(\mathbb{R}) \leqslant \varphi_{t,X}(\mathbb{T}) \leqslant 10^9 \frac{\tau_{p,X;2}c_{q,X;2}}{(r-t)^{1+1/t'}}$$

Proof. By Proposition 13.1.21, we have

$$\varphi_{2,X}(N) \leqslant N^{\frac{1}{p}-\frac{1}{q}} = N^{\frac{1}{r}-\frac{1}{2}}, \quad C := \tau_{p,X;2}c_{q,X;2}$$

Thus Lemma 13.1.32 implies the bound for $\varphi_{t,X}(\mathbb{T})$, and Proposition 13.1.1 the bound for $\varphi_{t,X}(\mathbb{R})$. □

Remark 13.1.36. The assumptions of Proposition 13.1.35 are satisfied by many "common" spaces of nontrivial type (and hence finite cotype). Namely, such space often have type *or* cotype 2, and hence either $\frac{1}{p}-\frac{1}{q}=\frac{1}{2}-\frac{1}{q}<\frac{1}{2}$ or $\frac{1}{p}-\frac{1}{q}=\frac{1}{p}-\frac{1}{2}<1-\frac{1}{2}=\frac{1}{2}$.

13.2 Fourier multipliers as singular integrals

The goal of this section is to see how the results on singular integrals proved above can be applied to the theory Fourier multipliers developed in Sections 5.3 and 5.5. Given $m \in L^\infty(\mathbb{R}^d;\mathscr{L}(X,Y))$, we recall that the operator T_m is a priori defined as $T_m : \check{L}^1(\mathbb{R}^d;X) \to \check{L}^1(\mathbb{R}^d;Y)$ by

$$T_m f(x) = \int_{\mathbb{R}^d} m(\xi)\widehat{f}(\xi)e^{2\pi i\xi\cdot x}\,\mathrm{d}\xi.$$

The notation $\mathfrak{M}L^p(\mathbb{R}^d;X,Y)$ stands for the space of all $m \in L^\infty(\mathbb{R}^d;\mathscr{L}(X,Y))$ for which T_m extends to a bounded linear operator from $L^p(\mathbb{R}^d;X)$ to $L^p(\mathbb{R}^d;Y)$. The connection of Fourier multipliers to integral operators is particularly simple in the following special case:

Proposition 13.2.1. *Let X, Y be Banach spaces and $m \in L^\infty_c(\mathbb{R}^d; \mathscr{L}(X,Y))$. Then for all $f \in L^1 \cap \check{L}^1(\mathbb{R}^d; X)$, we have*

$$T_m f(x) = \int_{\mathbb{R}^d} k(x-y) f(y)\,\mathrm{d}y,$$

where $k = \check{m} \in \check{L}^1(\mathbb{R}^d; \mathscr{L}(X,Y))$.

Proof. Under these assumptions, we can make a direct computation

$$\begin{aligned} T_m f(x) &= \int_{\mathbb{R}^d} m(\xi)\widehat{f}(\xi) e^{2\pi i x\cdot\xi}\,\mathrm{d}\xi \\ &= \int_{\mathbb{R}^d} m(\xi)\Big(\int_{\mathbb{R}^d} f(y) e^{-2\pi i y\cdot\xi}\,\mathrm{d}y\Big) e^{2\pi i x\cdot\xi}\,\mathrm{d}\xi \\ &= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} m(\xi) e^{2\pi i (x-y)\cdot\xi}\,\mathrm{d}\xi\Big) f(y)\,\mathrm{d}y = \int_{\mathbb{R}^d} \check{m}(x-y) f(y)\,\mathrm{d}y, \end{aligned}$$

where the first step is the definition of T_m for $f \in \check{L}^1(\mathbb{R}^d; X)$, the second is the definition of $\widehat{f}(\xi)$ for $f \in L^1(\mathbb{R}^d; X)$, the third is Fubini's theorem that applies since both $m \in L^1(\mathbb{R}^d; \mathscr{L}(X,Y))$ and $f \in L^1(\mathbb{R}^d; X)$, and the fourth is the definition of the inverse Fourier transform of $m \in L^1(\mathbb{R}^d; \mathscr{L}(X,Y))$. □

The compact support assumption on m in Proposition 13.2.1 is not as restrictive as it may seem at first sight, as one can often reduce considerations to this situation by simple limiting arguments that we shortly explain. Recall from Definition 5.5.20 that $\psi \in \mathscr{S}(\mathbb{R}^d)$ is called a smooth Littlewood–Paley function if

(i) $\widehat{\psi}$ is smooth, non-negative, and supported in $\{\xi \in \mathbb{R}^d : \frac{1}{2} \leqslant |\xi| \leqslant 2\}$;
(ii) $\sum_{j\in\mathbb{Z}} \widehat{\psi}(2^{-j}\xi) = 1$ for all $\xi \in \mathbb{R}^d \setminus \{0\}$.

Such functions exist by Lemma 5.5.21, whose proof also gives the identity $\widehat{\psi}(\xi) = \widehat{\varphi}(\xi) - \widehat{\varphi}(2\xi)$ and hence

$$\sum_{L<j\leqslant N} \widehat{\psi}(2^{-j}\xi) = \widehat{\varphi}(2^{-N}\xi) - \widehat{\varphi}(2^{-L}\xi)$$

for some $\widehat{\varphi} \in \mathscr{D}(\mathbb{R}^d)$ with $\widehat{\varphi}(0) = \int \varphi = 1$. Let

$$\begin{aligned} m_j(\xi) &:= \widehat{\psi}(2^{-j}\xi) m(\xi), \quad m^N(\xi) := \widehat{\varphi}(2^{-N}\xi) m(\xi), \\ m^N_L(\xi) &:= m^N(\xi) - m^L(\xi) = \sum_{L<j\leqslant N} m_j(\xi), \end{aligned} \tag{13.23}$$

and observe that $m^N \in L^\infty_c(\mathbb{R}^d; \mathscr{L}(X,Y))$, whereas

$$m_j, m^N \in L^\infty_c(\mathbb{R}^d \setminus \{0\}; \mathscr{L}(X,Y)),$$

i.e., these are supported away from both the origin and infinity. While the support away from zero is not required by Proposition 13.2.1, it is a convenience for forthcoming considerations due to the special role of the origin in various multiplier conditions. The next two lemmas describe a precise sense in which, for many purposes, it is "enough" to study the truncated multipliers m^N.

Lemma 13.2.2. *Let X, Y be Banach spaces and $m \in L^\infty(\mathbb{R}^d; \mathscr{L}(X,Y))$. For $p \in (1,\infty)$, we have $m \in \mathfrak{M}L^p(\mathbb{R}^d; X, Y)$, if and only if $m^N \in \mathfrak{M}L^p(\mathbb{R}^d; X, Y)$ uniformly in N, if and only if $m_L^N \in \mathfrak{M}L^p(\mathbb{R}^d; X, Y)$ uniformly in M and N.*

Proof. By the algebra of multipliers (Lemma 5.3.2), we have

$$T_{m^N} f = T_m(T_{\widehat{\varphi}(2^{-N}\cdot)} f) = T_m(\varphi_{2^{-N}} * f),$$

where $\varphi_t(x) = t^{-d}\varphi(t^{-1}x)$ and

$$\|\varphi_t * f\|_p \leqslant \|\varphi_t\|_1 \|f\|_p = \|\varphi\|_1 \|f\|_p,$$

so that $\|m^N\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)} \leqslant \|\varphi\|_1 \|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)}$, and thus

$$\|m_L^N\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)} \leqslant 2\|\varphi\|_1 \|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)}.$$

On the other hand, it is evident from property (ii) of Littlewood–Paley functions that $m^N(\xi) \to m(\xi)$ as $N \to \infty$ for every $\xi \in \mathbb{R}^d$, and $m_L^N(\xi) \to m(\xi)$ as $N \to \infty$ and $L \to -\infty$ for every $\xi \in \mathbb{R}^d \setminus \{0\}$. In particular, both limits hold for almost every $\xi \in \mathbb{R}^d$. Then Proposition 5.3.16 implies that

$$\|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)} \leqslant \liminf_{N\to\infty} \|m^N\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)},$$
$$\|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)} \leqslant \liminf_{\substack{N\to\infty \\ L\to-\infty}} \|m_L^N\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)}.$$

□

13.2.a Smooth multipliers have Calderón–Zygmund kernels

We will be mostly concerned with multipliers satisfying Mihlin-type conditions of the form

$$\|\partial^\alpha m(\xi)\| \leqslant M|\xi|^{-|\alpha|}, \quad \xi \in \mathbb{R}^d \setminus \{0\}, \tag{13.24}$$

for some set of multi-indices $\alpha \in \mathbb{N}^d$. Recall that the Mihlin class, introduced and used in Definitions 5.3.17 and 5.5.9 and Theorems 5.3.18 and 5.5.10 (in one and several variables, respectively) to deduce that $m \in \mathfrak{M}L^p(\mathbb{R}^d; X, Y)$ for all $p \in (1,\infty)$ without any *a priori* boundedness assumptions on T_m, featured stronger R-boundedness versions of such conditions. The difference in the present context is that we are willing to assume that $m \in \mathfrak{M}L^{p_0}(\mathbb{R}^d; X, Y)$ for some $p_0 \in (1,\infty)$ to begin with, and we wish to show that this *a priori* boundedness on one space can then be extrapolated to boundedness on other function spaces under conditions that are similar to those in Mihlin's theorems, but without the R-bounded aspects. As a matter of fact, these pointwise bounds can often be relaxed to weaker integrated versions, which is easily verified by inspecting the proofs, but for the clarity of the exposition we state the results under such pointwise assumptions. This is hardly a restriction for most applications.

The role of the multiplier conditions (13.24) for the kernel estimates is via careful use of the fundamental relation $\widehat{\partial_j f}(\xi) = 2\pi i \xi_j \widehat{f}(\xi)$. So as to make most efficient use of the relation, and to unburden the formulae from inessential constants, we introduce the abbreviation

$$\eth := \partial/2\pi i$$

so that

$$\widehat{\eth_j f}(\xi) = \xi_j \widehat{f}(\xi).$$

The deduction of the kernel estimates is easiest when sufficiently many derivatives are allowed in (13.24); as it turns out, this is somewhat more than the collection $\alpha \in \{0,1\}^d$ appearing in Mihlin's Theorem 5.5.10. We formulate several results for a generic Banach space Z instead of $\mathscr{L}(X,Y)$, as the operator structure plays no role here; this also makes the formulae slightly shorter. We say that a collection $\mathscr{A}$ of multi-indices is *convex*, if $\alpha \in \mathscr{A}$ implies $\beta \in \mathscr{A}$ whenever $0 \leqslant \beta \leqslant \alpha$.

Lemma 13.2.3. *If $m \in L^\infty(\mathbb{R}^d; Z)$ satisfies (13.24) for a convex set of multi-indices α, then each $m_j \in L^\infty_{\rm c}(B(0,2^{j+1}); Z)$ satisfies*

$$\|\eth^\alpha m_j\|_\infty \leqslant M 2^{-j|\alpha|}$$

for the same set of multi-indices, where M is the constant of (13.24).

Proof. By the Leibniz rule, we have

$$\eth^\alpha m_j(\xi) = \eth^\alpha[\widehat{\psi}(2^{-j}\xi)m(\xi)] = \sum_{\theta \leqslant \alpha} \binom{\alpha}{\theta} 2^{-j|\theta|} \eth^\theta \widehat{\psi}(2^{-j}\xi) \eth^{\alpha-\theta} m(\xi),$$

where each $\eth^{\alpha-\theta} m$ also satisfies (13.24) by convexity. Thus

$$\begin{aligned}
\|\eth^\alpha m_j(\xi)\| &\leqslant \sum_{\theta \leqslant \alpha} \binom{\alpha}{\theta} 2^{-j|\theta|} \mathbf{1}_{2^{j-1} \leqslant |\xi| \leqslant 2^{j+1}} M |\xi|^{-|\alpha-\theta|} \\
&\leqslant \sum_{\theta \leqslant \alpha} \binom{\alpha}{\theta} 2^{-j|\theta|} M (2^{j-1})^{-|\alpha|+|\theta|} \\
&= M 2^{-j|\alpha|} 2^{|\alpha|} \sum_{\theta \leqslant \alpha} \binom{\alpha}{\theta} 2^{|\alpha-\theta|} \cdot 1^{|\theta|}
\end{aligned}$$

$$= M2^{-j|\alpha|}2^{|\alpha|}(2+1)^{|\alpha|} = M2^{-j|\alpha|}6^{|\alpha|},$$

where the binomial formula was used in the second to last step. The result follows after dividing both sides by $(2\pi)^{|\alpha|} \geqslant 6^{|\alpha|}$. □

Lemma 13.2.4. *Let Z be a Banach space and $f \in L^\infty_c(B(0,A);Z)$ have distributional derivatives that satisfy*

$$\|\partial^\alpha f\|_\infty \leqslant A^{-|\alpha|}$$

for some $A > 0$ and all multi-indices α in some convex set. Then

$$\|x \mapsto \partial_x^\alpha[(e^{2\pi i y\cdot x} - 1)f(x)]\|_\infty \leqslant (6 + 2^{|\alpha|})A|y| \cdot A^{-|\alpha|}$$

for all $y \in \mathbb{R}^d$ with $|y| \leqslant A^{-1}$, and for the same set of multi-indices.

Proof. The derivatives are given by

$$\partial_x^\alpha[(e^{2\pi i y\cdot x} - 1)f(x)] = (e^{2\pi i y\cdot x} - 1)\partial^\alpha f(x) + \sum_{0\neq\gamma\leqslant\alpha} y^\gamma e^{2\pi i y\cdot x}\partial^{\alpha-\gamma} f(x),$$

and hence

$$\begin{aligned}\|\partial_x^\alpha[(e^{2\pi i y\cdot x} - 1)f(x)]\| &\leqslant 2\pi|y|A \cdot A^{-|\alpha|} + \sum_{0\neq\gamma\leqslant\alpha} |y|^{|\gamma|}A^{-|\alpha|+|\gamma|} \\ &\leqslant |y|A \cdot A^{-|\alpha|}\Big(2\pi + \sum_{0\neq\gamma\leqslant\alpha}(A|y|)^{|\gamma|-1}\Big).\end{aligned}$$

If $A|y| \leqslant 1$, then $(A|y|)^{|\gamma|-1} \leqslant 1$ and $\sum_{0\neq\gamma\leqslant\alpha} 1 = 2^{|\alpha|} - 1$. □

Lemma 13.2.5. *Let Z be a Banach space and $f \in L^\infty_c(B(0,A);Z)$ have distributional derivatives that satisfy*

$$\|\partial^\alpha f\|_\infty \leqslant A^{-|\alpha|} \qquad \forall|\alpha| \leqslant d+1$$

for some $A > 0$. Then for almost all $x, y \in \mathbb{R}^d$ with $|y| \leqslant \frac{1}{2}|x|$, we have

$$|x|^n|\widehat{f}(x)| \leqslant c_d A^{d-n}, \tag{13.25}$$

$$|x|^n|\widehat{f}(x-y) - \widehat{f}(x)| \leqslant c_d A^{d-n}\min\{A|y|, 1\} \tag{13.26}$$

for all $n = 0, 1, \ldots, d+1$. In particular, $\widehat{f} \in L^1(\mathbb{R}^d;Z)$ and

$$\|\widehat{f}\|_1 \leqslant c_d.$$

Proof. For $x \in B(0,A)$, we have

$$\|x^\alpha \widehat{f}\|_\infty \leqslant \|\partial^\alpha f\|_1 \leqslant \|\partial^\alpha f\|_\infty\|\mathbf{1}_{B(0,A)}\|_1 \leqslant A^{-|\alpha|}\omega_d A^d,$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$. With $\alpha = ne_i$, this shows that $|x_i|^n|\widehat{f}(x)| \leqslant \omega_d A^{d-n}$ for $i = 1, \ldots, d$, which readily gives (13.25).

We observe that $\widehat{f}(x-y) - \widehat{f}(x)$ is the Fourier transform of $(e^{2\pi i x\cdot y} - 1)f(x)$, which satisfies the same assumptions as f for $|y| \leqslant A^{-1}$, except for a multiplicative factor $(6+2^d)A|y|$, by Lemma 13.2.4. An application of (13.25) to this function in place of f hence gives

$$|x|^n|\widehat{f}(x-y) - \widehat{f}(x)| \leqslant c_d A^{d-n} A|y|$$

when $A|y| \leqslant 1$. On the other hand, if $A|y| > 1$, then we simply estimate $\widehat{f}(x-y)-\widehat{f}(x)$ by (13.25) and the triangle inequality, recalling the assumptions that $|y| \leqslant \frac{1}{2}|x|$ and $n \leqslant d+1$:

$$\begin{aligned} |\widehat{f}(x-y) - \widehat{f}(x)| &\leqslant |\widehat{f}(x-y)| + |\widehat{f}(x)| \leqslant c_d A^{d-n}(|x-y|^{-n} + |x|^{-n}) \\ &\leqslant c_d A^{d-n}(2^n+1)|x|^{-n} \leqslant c_d' A^{d-n}|x|^{-n}. \end{aligned}$$

The last two bounds are both seen to be dominated by the claimed bound in (13.26).

That $\widehat{f} \in L^1(\mathbb{R}^d; Z)$ is immediate from (13.25) by integrating the estimate

$$|\widehat{f}(x)| \leqslant c_d A^d \min\Big\{1, (A|x|)^{-d-1}\Big\}.$$

□

Proposition 13.2.6. *Let X, Y be Banach spaces and $m \in L^\infty(\mathbb{R}^d; \mathscr{L}(X,Y))$ satisfy*

$$\|\partial^\alpha m(\xi)\| \leqslant M|\xi|^{-|\alpha|} \qquad \forall |\alpha| \leqslant d+1.$$

Then each $K^N(x,y) = k^N(x-y) = \check{m}^N(x-y)$ is a Calderón–Zygmund kernel with the following bounds independent of the truncation N:

$$\|k^N(x)\| \leqslant \frac{c}{|x|^d}, \qquad \|k^N(x-y) - k^N(x)\| \leqslant \frac{1}{|x|^d}\omega\Big(\frac{|y|}{|x|}\Big),$$

for all $x, y \in \mathbb{R}^d$ with $|y| \leqslant \frac{1}{2}|x|$, where

$$c = c_d M, \qquad \omega(t) = c_d M \cdot t \cdot \Big(1 + \log\frac{1}{t}\Big).$$

Note that the modulus of continuity ω above is slightly "worse" (i.e., with slower decay as $t \to 0$) than the Lipschitz modulus $\omega_1(t) = t$, but "better" than any of the Hölder moduli $\omega_\delta(t) = t^\delta$ for $\delta \in (0,1)$.

Proof. By Lemma 13.2.3, the functions m_j satisfy the assumptions, and hence the conclusions, of Lemma 13.2.5 with $A = 2^{j+1}$ and a multiplicative factor $c_d M$. Thus,

$$\begin{aligned}|k^N(x)| &\leqslant \sum_{j\leqslant N}|k_j(x)| \leqslant \sum_{j\in\mathbb{Z}}\min_{0\leqslant h\leqslant d+1}\frac{c_d 2^{(j+1)(d-h)}}{|x|^h}M\\ &\leqslant \sum_{j:2^{j+1}\leqslant 1/|x|} c_d 2^{(j+1)d}M + \sum_{j:2^{j+1}\geqslant 1/|x|}\frac{c_d 2^{-(j+1)}}{|x|^{d+1}}M \leqslant c_d'|x|^{-d}M.\end{aligned}$$

Similarly, for $|y| \leqslant \frac{1}{2}|x|$,

$$\begin{aligned}&|k^N(x-y)-k^N(x)| \leqslant \sum_{j\in\mathbb{Z}}\min_{0\leqslant h\leqslant d+1}\frac{c_d 2^{(j+1)(d-h)}}{|x|^h}\min\{2^{j+1}|y|,1\}M\\ &\leqslant \sum_{j:2^{j+1}\leqslant 1/|x|} c_d 2^{(j+1)(d+1)}|y|M + \sum_{j:1/|x|\leqslant 2^{j+1}\leqslant 1/|y|}\frac{c_d}{|x|^{d+1}}|y|M\\ &\qquad + \sum_{j:2^{j+1}\geqslant 1/|y|}\frac{c_d 2^{-(j+1)}}{|x|^{d+1}}M\\ &\leqslant c_d'\frac{1}{|x|^{d+1}}|y|M + \frac{c_d'}{|x|^{d+1}}|y|\Big(1+\log\frac{|x|}{|y|}\Big)M + \frac{c_d'}{|x|^{d+1}}|y|M\\ &\leqslant \frac{c_d''}{|x|^{d+1}}|y|\Big(1+\log\frac{|x|}{|y|}\Big)M.\end{aligned}$$

This completes the proof. □

With the uniform pointwise bounds of Proposition 13.2.6 at hand, we can strengthen the sense in which the operator T_m with such bounds is associated with a Calderón–Zygmund kernel k:

Proposition 13.2.7. *Let X, Y be Banach spaces, $p \in [1,\infty)$, and $m \in \mathfrak{M}L^p(\mathbb{R}^d; X, Y)$ satisfy*

$$\|\partial^\alpha m(\xi)\| \leqslant M|\xi|^{-|\alpha|} \qquad \forall |\alpha| \leqslant d+1.$$

Then there is a kernel $k \in C(\mathbb{R}^d\setminus\{0\}; \mathscr{L}(X,Y))$ that satisfies the same bounds as k^N in Proposition 13.2.6 and such that

$$T_m f(x) = \int_{\mathbb{R}^d} k(x-y)f(y)\,\mathrm{d}y$$

for all $f \in L^p(\mathbb{R}^d; X)$ and almost all $x \in \mathbb{R}^d$ outside the support of f.

Proof. We split the proof into two cases:

Case $p \in (1,\infty)$: Let $f \in L^p(\mathbb{R}^d; X)$. Using the notation from the proof of Lemma 13.2.2 and the preceding discussion, we have

$$T_{m_L^N} f = T_m[(\varphi_{2^{-N}} * f) - (\varphi_{2^{-L}} * f)],$$

where $\varphi_{2^{-N}} * f \to f$ in $L^p(\mathbb{R}^d;X)$ as $N \to \infty$ by a standard mollifier result (e.g., Proposition 1.2.32). We also have $\|\varphi_R * f\|_p \leqslant \|\varphi_R\|_p \|f\|_1 = R^{-n/p'}\|f\|_1 \to 0$ as $R \to \infty$ if $f \in L^1(\mathbb{R}^d;X)$ and $\|\varphi_R * f\|_p \leqslant \|\varphi_R\|_1\|f\|_p = \|f\|_p$ uniformly in R. Since $(L^1 \cap L^p)(\mathbb{R}^d;X)$ is dense in $L^p(\mathbb{R}^d;X)$, it follows that $\varphi_{2^{-L}} * f \to 0$ in $L^p(\mathbb{R}^d;X)$ as $L \to -\infty$ for all $f \in L^p(\mathbb{R}^d;X)$.

Summarising this discussion, it follows that, for all $f \in L^p(\mathbb{R}^d;X)$, we have the convergence $T_{m_L^N} f \to f$ in $L^p(\mathbb{R}^d;X)$ as $N \to \infty$ and $L \to -\infty$. By passing to a subsequence if needed, we may assume that this convergence also takes place almost everywhere.

If $f \in \check{L}^1(\mathbb{R}^d;X) \cap L^1(\mathbb{R}^d;X) \subseteq L^p(\mathbb{R}^d;X)$, then Proposition 13.2.1 shows that

$$T_{m_L^N} f = k_L^N * f,$$

where $T_{m_L^N}$ is bounded from $L^p(\mathbb{R}^d;X)$ to $L^p(\mathbb{R}^d;Y)$ by Lemma 13.2.2. On the other hand, k_L^N is a finite sum of $k_j = \check{m}_j$, where the multipliers m_j are in the scope of Lemma 13.2.5, and hence $k_L^N \in L^1(\mathbb{R}^d;\mathscr{L}(X,Y))$. But then also $f \mapsto k_N * f$ is bounded from $L^p(\mathbb{R}^d;X)$ to $L^p(\mathbb{R}^d;Y)$, and the previous display must remain valid for all $f \in L^p(\mathbb{R}^d;X)$ by continuity. Combining these pieces, we obtain

$$T_m f(x) = \lim_{\substack{N\to\infty \\ L\to-\infty}} T_{m_L^N} f(x) = \lim_{\substack{N\to\infty \\ L\to-\infty}} \int_{\mathbb{R}^d} k_L^N(x-y) f(y)\,\mathrm{d}y$$

for all $f \in L^p(\mathbb{R}^d;X)$ and almost every $x \in \mathbb{R}^d$.

Let us finally consider $x \in \complement \operatorname{supp} f$. Since this set is open, we can pick an $\varepsilon > 0$ such that $B(x,\varepsilon) \subseteq \complement \operatorname{supp} f$. For such x and any $y \in \operatorname{supp} f$, the series

$$\sum_{j\in\mathbb{Z}} k_j(x-y) = \lim_{\substack{N\to\infty \\ L\to-\infty}} k_L^N(x-y)$$

converges absolutely by the proof of Proposition 13.2.6. We denote by $k(x-y)$ the limit. Moreover, the same proposition shows that

$$\|k_L^N(x-y)f(y)\| \leqslant \frac{c_d M}{|x-y|^d}\|f(y)\|,$$

which is integrable over $y \in \mathbb{R}^d$ by Hölder's inequality, since $f \in L^p(\mathbb{R}^d;X)$ and $[y \mapsto |x-y|^{-d}] \in L^{p'}(\complement B(x,\varepsilon))$. Thus

$$T_m f(x) = \lim_{\substack{N\to\infty \\ L\to-\infty}} \int_{\mathbb{R}^d} k_L^N(x-y) f(y)\,\mathrm{d}y = \int_{\mathbb{R}^d} k(x-y) f(y)\,\mathrm{d}y$$

by dominated convergence. The pointwise estimates of k_L^N are clearly inherited by k by the pointwise convergence. This completes the proof for $p \in (1,\infty)$.

Case $p = 1$: We can still make use of large parts of the preceding considerations, but some details require a modification. The standard mollifier result (Proposition 1.2.32) still applies to show that $\varphi_{2^{-N}} * f \to f$, and hence $T_{m^N} f \to T_m f$, in $L^1(\mathbb{R}^d; X)$ as $N \to \infty$, but it no longer guaranteed that $\varphi_{2^{-L}} * f$ should converge to 0 as $L \to -\infty$. Hence, we will separately deal with $T_{m^L} f$.

For $f \in \check{L}^1(\mathbb{R}^d; X) \cap L^1(\mathbb{R}^d; X)$, we have

$$\begin{aligned}\|T_{m^L} f(x)\| &= \Big\| \int_{\mathbb{R}^d} \varphi(2^{-L}\xi) m(\xi) \widehat{f}(\xi) \,\mathrm{d}\xi \Big\| \\ &\leqslant \int_{|\xi| \leqslant 2^{L+1}} \|m\|_\infty \|\widehat{f}(\xi)\| \,\mathrm{d}\xi \leqslant \omega_d (2^{L+1})^d \|m\|_\infty \|f\|_1.\end{aligned}$$

Hence T_{m^L} extends to a bounded operator from $L^1(\mathbb{R}^d; X)$ to $L^\infty(\mathbb{R}^d; Y)$ of norm at most $\omega_d 2^{(L+1)d} \|m\|_\infty \to 0$ as $L \to -\infty$.

For $f \in \check{L}^1(\mathbb{R}^d; X) \cap L^1(\mathbb{R}^d; X)$, we can now write

$$T_{m^N} f = T_{m_L^N} f + T_{m^L} f = k_{m_L^N} * f + T_{m^L} f.$$

Since all of the operators acting on f above are bounded from $L^1(\mathbb{R}^d; X)$ to $L^1(\mathbb{R}^d; Y) + L^\infty(\mathbb{R}^d; Y)$, the identity continues to hold for all $f \in L^1(\mathbb{R}^d; X)$. Taking the limits $N \to \infty$ and $L \to -\infty$, we have $T_{m^N} f \to T_m f$ in $L^1(\mathbb{R}^d; Y)$ and $T_{m^L} f \to 0$ in $L^\infty(\mathbb{R}^d; Y)$. Along suitable subsequences, we have both limits almost everywhere, and hence we arrive at the same pointwise limit

$$T_m f(x) = \lim_{\substack{N \to \infty \\ L \to -\infty}} \int_{\mathbb{R}^d} k_L^N(x - y) f(y) \,\mathrm{d}y$$

as in the case $p \in (1, \infty)$. The rest of the proof can then be concluded in the same way as before. Specifically, let us note that the final application of dominated convergence is justified simple because the product of $[y \mapsto |x - y|^{-d}] \in L^\infty(\complement B(x, \varepsilon))$ and $f \in L^1(\mathbb{R}^d; X)$ is integrable. □

Corollary 13.2.8. *Let X, Y be Banach spaces and $p_0 \in [1, \infty)$. Suppose that $m \in \mathfrak{M}L^{p_0}(\mathbb{R}^d; X, Y)$ satisfies*

$$\|\partial^\alpha m(\xi)\| \leqslant M |\xi|^{-|\alpha|} \qquad \forall |\alpha| \leqslant d + 1.$$

Then T_m extends to a bounded operator from $L^p(w; X)$ to $L^p(w; Y)$ for every $p \in (1, \infty)$ and every Muckenhoupt weight $w \in A_p$. Moreover,

$$\|T_m\|_{\mathscr{L}(L^p(w;X), L^p(w;Y))} \leqslant c_{d,p} (\|m\|_{\mathfrak{M}L^{p_0}(\mathbb{R}^d; X, Y)} + M) [w]_{A_p}^{\max\{1, \frac{1}{p-1}\}}.$$

Proof. By Proposition 13.2.7, the A_2 Theorem 11.3.26 applies to such an operator T_m, and this gives precisely the stated conclusions. □

Corollary 13.2.9. *Let X, Y be UMD spaces. Suppose that $m \in L^\infty(\mathbb{R}^d; X, Y)$ satisfies*

$$\|\partial^\alpha m(\xi)\| \leqslant M|\xi|^{-|\alpha|} \qquad \forall|\alpha| \leqslant d+1,$$

and in addition

$$\mathscr{R}\big(\big\{|\xi|^{|\alpha|}\partial^\alpha m(\xi) : \xi \in \mathbb{R}^d \setminus \{0\}\big\}\big) \leqslant \widetilde{M} \qquad \forall\alpha \in \{0,1\}^d,$$

Then T_m extends to a bounded operator from $L^p(w; X)$ to $L^p(w; Y)$ for every $p \in (1, \infty)$ and every Muckenhoupt weight $w \in A_p$. Moreover,

$$\|T_m\|_{\mathscr{L}(L^p(w;X),L^p(w;Y))} \leqslant c_{d,p}(\min(\hbar^d_{p,X}, \hbar^d_{p,Y})\beta_{p,X}\beta_{p,Y}\widetilde{M}+M)[w]_{A_p}^{\max\{1,\frac{1}{p-1}\}}.$$

Proof. By Mihlin's Multiplier Theorem 5.5.10, the assumptions imply that

$$\|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)} \leqslant c_d \min(\hbar^d_{p,X}, \hbar^d_{p,Y})\beta_{p,X}\beta_{p,Y}\widetilde{M}.$$

We then conclude with an application of Corollary 13.2.8. □

This proof displays a certain dichotomy between the multiplier conditions needed to get the boundedness of T_m to begin with, and the conditions needed to extrapolate this boundedness to other spaces. The former one needs the stronger R-boundedness assumptions, but only for a smaller number of derivatives, while the latter only needs usual pointwise bounds, but for a larger set of derivatives. This dichotomy disappears from sight in the following important special case:

Corollary 13.2.10. *Let X be a UMD space. Suppose that a scalar-valued $m \in L^\infty(\mathbb{R}^d)$ satisfies*

$$|\partial^\alpha m(\xi)| \leqslant M|\xi|^{-|\alpha|} \qquad \forall|\alpha| \leqslant d+1.$$

Then T_m extends to a bounded operator on $L^p(w; X)$ for every $p \in (1, \infty)$ and every Muckenhoupt weight $w \in A_p$. Moreover,

$$\|T_m\|_{\mathscr{L}(L^p(w;X)))} \leqslant c_{d,p}\hbar^d_{p,X}\beta^2_{p,X}M[w]_{A_p}^{\max\{1,\frac{1}{p-1}\}}.$$

Proof. The assumed pointwise bounds coincide with the R-bounds required by Corollary 13.2.10 in the case of a scalar-valued multiplier m. □

13.2.b Mihlin multipliers have Hörmander kernels

We now turn to the question of kernel estimates assuming only the multiplier conditions appearing in Mihlin's Theorem 5.5.10. It turns out that the maximal order of d derivatives is just on the border of what we need to make useful estimates, and in order to cope with this condition, we need to impose an additional assumption on the underlying Banach space X in terms of the notion of Fourier type discussed in Section 13.1.

The analogue of Lemma 13.2.5 in the present context is the following rather more complicated assertion.

Lemma 13.2.11. *Let X be a Banach space with Fourier type $p \in (1,2]$. Let $f \in L^\infty_c((-A,A)^d;X)$ satisfy*

$$\|\partial\!\!\!/^\alpha f\|_\infty \leqslant A^{-|\alpha|} \qquad \forall \alpha \in \{0,1\}^d$$

for some $A > 0$. Then $\widehat{f} \in L^1(\mathbb{R}^d;X)$ and, denoting

$$\Phi_{p,X} := 4p'(4 + \log_2^+ \varphi_{p,X}),$$

we have the estimates

$$\|\widehat{f}\|_1 \leqslant \Phi^d_{p,X}, \tag{13.27}$$

$$\|\mathbf{1}_{\complement[-R,R]^d}\widehat{f}\|_1 \leqslant \Phi^d_{p,X}\frac{4d\varphi_{p,X}}{(AR)^{1/p'}} \quad \forall R > 0, \tag{13.28}$$

$$\|\widehat{f}(\cdot - y) - \widehat{f}(\cdot)\|_1 \leqslant \Phi^d_{p,X} \cdot 4 \cdot 2^d A|y| \quad \forall y \in \mathbb{R}^d, \tag{13.29}$$

$$\|\mathbf{1}_{\complement B(0,3|y|)}[\widehat{f}(\cdot - y) - \widehat{f}(\cdot)]\|_1 \leqslant \Phi^d_{p,X} \min\Big\{2, \frac{8d^2\varphi_{p,X}}{(Ar)^{1/p'}}, 4\cdot 2^d Ar\Big\}. \tag{13.30}$$

Remark 13.2.12. Thanks to Bourgain's Theorem 13.1.33, the assumption on the Banach space X in Lemma 13.2.11 is simply that X has some non-trivial type $r \in (1,2]$. Namely, Theorem 13.1.33 guarantees that we can then take

$$p' = 1 + 6r'T, \quad \varphi_{p,X} \leqslant e^{2T}, \quad T := (68\tau_{r,X;2})^{r'} \geqslant 68^2 > 4\,000,$$

and hence

$$\begin{aligned}\Phi_{p,X} &\leqslant 4\big(1+6r'T\big)\Big(4 + \frac{2}{\log 2}T\Big) = \frac{48}{\log 2}\big(\frac{1}{6r'T}+1\big)\big(\frac{2\log 2}{T}+1\big)r'T^2 \\ &\leqslant 70 \cdot r'T^2 = 70r'(68\tau_{r,X;2})^{r'}.\end{aligned}$$

Proof of (13.27). For $k \in \mathbb{Z}^d$, let

$$D_k = \{x \in \mathbb{R}^d : x_i \in [2^{k_i}, 2^{k_i+1}) \ \forall i = 1,\dots,d\}$$

so that obviously

$$\|\widehat{f}\|_1 = \sum_{k\in\mathbb{Z}^d} \|\mathbf{1}_{D_k} f\|_1.$$

For each $k \in \mathbb{Z}^d$, we partition $\mathbf{1} = \alpha + \beta + \gamma$ for some $\alpha, \beta, \gamma \in \{0,1\}^d$ yet to be chosen. Then

$$D_k = D_k^\alpha \times D_k^\beta \times D_k^\gamma, \qquad D_k^\theta = \{(x_i)_{i:\theta_i=1} : x_i \in [2^{k_i}, 2^{k_i+1})\}.$$

Similarly,

$$\mathbb{R}^d = \mathbb{R}^\alpha \times \mathbb{R}^\beta \times \mathbb{R}^\gamma, \qquad \mathbb{R}^\theta = \{(x_i)_{i:\theta_i=1} : x_i \in \mathbb{R}\},$$

and we abbreviate $L^sL^t_\gamma := L^s(\mathbb{R}^\alpha \times \mathbb{R}^\beta; L^t(\mathbb{R}^\gamma;X))$.

For $x \in D_k$, we have $|x_i| \geqslant 2^{k_i}$, and hence $|x^{\beta+\gamma}| \geqslant 2^{k\cdot(\beta+\gamma)}$. We can now make the following estimate. At a critical point, passing from a norm of the Fourier transform $\widehat{f}$ to a norm of f itself, we apply the Fourier type assumption to $\mathscr{F} : L^p(\mathbb{R}^\gamma; X) \to L^{p'}(\mathbb{R}^\gamma; X)$, producing the constant $\varphi_{p,X}(\mathbb{R}^\gamma) \leqslant \varphi_{p,X}^{|\gamma|}$, and the trivial boundedness of the Fourier transform $\mathscr{F} : L^1(\mathbb{R}^{\alpha+\beta}; Z) \to L^\infty(\mathbb{R}^{\alpha+\beta}; Z)$, with $Z = L^q(\mathbb{R}^\gamma; X)$ for either $q = p$ or $q = p'$, depending on the (irrelevant) order in which we perform these two steps:

$$\begin{aligned}
\|\mathbf{1}_{D_k}\widehat{f}\|_{L^1} &\leqslant 2^{-k\cdot(\beta+\gamma)}\|\mathbf{1}_{D_k}x^{\beta+\gamma}\widehat{f}\|_{L^1} \\
&\leqslant 2^{-k\cdot(\beta+\gamma)}\|\mathbf{1}_{D_k}\|_{L^1L^p_\gamma}\|x^{\beta+\gamma}\widehat{f}\|_{L^\infty L^{p'}_\gamma} \\
&\leqslant 2^{-k\cdot(\beta+\gamma)}\cdot 2^d 2^{k\cdot(\alpha+\beta+\gamma/p)}\cdot\varphi_{p,X}^{|\gamma|}\|\partial^{\beta+\gamma}f\|_{L^1L^p_\gamma} \\
&\leqslant 2^d 2^{k\cdot(\alpha-\gamma/p')}\cdot\varphi_{p,X}^{|\gamma|}A^{-|\beta|-|\gamma|}2^d A^{|\alpha|+|\beta|+|\gamma|/p} \\
&\leqslant 4^d 2^{k\cdot(\alpha-\gamma/p')}\cdot\varphi_{p,X}^{|\gamma|}A^{|\alpha|-|\gamma|/p'} \\
&= 4^d\times\prod_{i:\alpha_i=1}(A2^{k_i})\times\prod_{i:\beta_i=1}1\times\prod_{i:\gamma_i=1}(\varphi_{p,X}(2^{k_i}A)^{-1/p'}).
\end{aligned}$$

Since the splitting $\mathbf{1} = \alpha+\beta+\gamma$ is free for us to choose, it is obvious that, for each i, we choose it to be in the first, second or third category according to which of the three numbers

$$A2^{k_i}, \qquad 1, \qquad \varphi_{p,X}(2^{k_i}A)^{-1/p'}$$

is the smallest. This gives us the estimate

$$\begin{aligned}
\|\widehat{f}\|_1 &= \sum_{k\in\mathbb{Z}^d}\|\mathbf{1}_{D_k}f\|_1 \\
&\leqslant 4^d\sum_{k\in\mathbb{Z}^d}\prod_{i=1}^d\min\{A2^{k_i}, 1, \varphi_{p,X}(2^{k_i}A)^{-1/p'}\} \\
&= 4^d\Big(\sum_{k\in\mathbb{Z}}\min\{A2^k, 1, \varphi_{p,X}(2^kA)^{-1/p'}\}\Big)^d \\
&\leqslant 4^d\Big(\sum_{k:A2^k\leqslant 1}A2^k + \sum_{k:1\leqslant A2^k\leqslant\varphi_{p,X}^{p'}}1 + \sum_{k:A2^k\geqslant\varphi_{p,X}^{p'}}\varphi_{p,X}(2^kA)^{-1/p'}\Big)^d \\
&\leqslant 4^d\Big(2 + (1+\log_2^+\varphi_{p,X}^{p'}) + \frac{\varphi_{p,X}(\varphi_{p,X}^{p'})^{-1/p'}}{1-2^{-1/p'}}\Big)^d \\
&\leqslant 4^d(3+p'\log_2^+\varphi_{p,X}+2p')^d \leqslant (4p')^d(4+\log_2^+\varphi_{p,X})^d
\end{aligned}$$

where we observed that $1-2^{-1/p'} \geqslant 1/(2p')$, since the function $g(u) = u/2 + 2^{-u}$ satisfies $g(u) \leqslant 1$ for $u = 1/p' \in [0, \frac{1}{2}]$, being convex with $g(0) = 1$ and $g(\frac{1}{2}) = 1/4 + 2^{-1/2} < 1$. □

Proof of (13.28). Making the same decomposition

$$\|\mathbf{1}_{\complement[-R,R]^d}\widehat{f}\|_1 = \sum_{k\in\mathbb{Z}^d} \|\mathbf{1}_{D_k}\mathbf{1}_{\complement[-R,R]^d}\widehat{f}\|_1$$

as in the proof of (13.27), we observe hat $\mathbf{1}_{D_k}\mathbf{1}_{\complement[-R,R]^d}$ is non-zero only if at least one k_i satisfies $2^{k_i+1} > R$. Thus

$$\begin{aligned}\|\mathbf{1}_{\complement[-R,R]^d}\widehat{f}\|_1 &\leqslant \sum_{i=1}^{d} \sum_{\substack{k\in\mathbb{Z}^d\\ 2^{k_i}>R/2}} \|\mathbf{1}_{D_k}\widehat{f}\|_1 \\ &\leqslant d\cdot 4^d\Big(\sum_{k\in\mathbb{Z}} \min\{A2^k, 1, \varphi_{p,X}(2^kA)^{-1/p'}\}\Big)^{d-1} \\ &\qquad\times\Big(\sum_{k:2^k>R/2} \min\{A2^k, 1, \varphi_{p,X}(2^kA)^{-1/p'}\}\Big),\end{aligned}$$

by inspection of the proof of (13.27). The factor raised to power $d-1$ is estimated as in the proof of (13.27) by

$$\Big(\sum_{k\in\mathbb{Z}} \min\{A2^k, 1, \varphi_{p,X}(2^kA)^{-1/p'}\}\Big)^{d-1} \leqslant (p')^{d-1}\Big(4+\log_2^+\varphi_{p,X}\Big)^{d-1}.$$

On the other hand, we have

$$\begin{aligned}&\sum_{k:2^k>R/2} \min\{A2^k, 1, \varphi_{p,X}(2^kA)^{-1/p'}\} \\ &\qquad\leqslant \sum_{k:2^k>R/2} \varphi_{p,X}(2^kA)^{-1/p'} \\ &\qquad\leqslant \frac{\varphi_{p,X}(AR/2)^{-1/p'}}{1-2^{-1/p'}} \leqslant 4p'\varphi_{p,X}(AR)^{-1/p'},\end{aligned}$$

again by recycling some estimates from the proof of (13.27). Collecting the bounds, the proof of (13.28) is complete. □

Proof of (13.29). We observe that $\widehat{f}(x-y)-\widehat{f}(x)$ is the Fourier transform of $f(x)e^{2\pi i x\cdot y}$, which verifies the same assumptions as f by Lemma 13.2.4, aside from the multiplicative factor $(6+2^d)A|y|$, provided that $A|y|\leqslant 1$. Applying (13.27) to this function gives (13.29) for $A|y|\leqslant 1$. But for $A|y|>1$, (13.29) is an immediate consequence of (13.27) by the triangle inequality. □

Proof of (13.30). This final bound is a certain synthesis of the other bounds. The first and third bounds in the minimum are obtained from (13.27) (with the triangle inequality) and from (13.29), respectively, ignoring the restriction to $\complement B$, which only increases the norm.

For the second bound, we also use the triangle inequality, but keeping the restriction to $\complement B$. Then

$$\begin{aligned}\|\mathbf{1}_{\complement B(0,3|y|)}\widehat{f}(\cdot-y)\|_1 &= \|\mathbf{1}_{\complement B(-y,3|y|)}\widehat{f}\|_1\\ &\leqslant \|\mathbf{1}_{\complement B(0,2|y|)}\widehat{f}\|_1 \leqslant \|\mathbf{1}_{\complement(-2r/\sqrt{d},2r/\sqrt{d})^d)}\widehat{f}\|_1,\end{aligned}$$

and the same bound is obvious for $\widehat{f}$ in place of $\widehat{f}(\cdot-y)$. Applying (13.28) with $R = 2r/\sqrt{d}$ produces the required bound. □

Proposition 13.2.13. *Let X, Y be Banach spaces, and suppose that $m \in L^\infty(\mathbb{R}^d; \mathscr{L}(X,Y))$ satisfies*

$$\|\partial^\alpha m(\xi)\| \leqslant M|\xi|^{-|\alpha|} \qquad \forall \alpha \in \{0,1\}^d. \tag{13.31}$$

Let $K^N(t,s) = k^N(t-s) = \check{m}^N(x-y)$ be the kernels related to the Littlewood–Paley truncations m^N of m as in (13.23).

(1) *If the space Y has Fourier type $p \in (1,2]$, then the kernels K^N satisfy the Hörmander condition uniformly in N, and quantitatively*

$$\int_{|t|>3|s|} \|(k^N(t-s) - k^N(t))x\|_Y \,\mathrm{d}t \leqslant (2\Phi_{p,Y})^{d+1} M \|x\|_X \quad \forall x \in X,$$

where $\Phi_{p,Y} = 4p'(4 + \log_2^+ \varphi_{p,Y})$.

(2) *If the space X has Fourier type $p \in (1,2]$, then the kernels K^N satisfy the dual Hörmander condition uniformly in N, and quantitatively*

$$\int_{|t|>3|s|} \|(k^N(t-s)^* - k^N(t)^*)y^*\| \,\mathrm{d}t \leqslant (2\Phi_{p,X})^{d+1} M \|y^*\|_{Y^*} \quad \forall y^* \in Y^*,$$

where $\Phi_{p,X} = 4p'(4 + \log_2^+ \varphi_{p,X})$.

Proof of (1). From Lemma 13.2.3 it follows that each Littlewood–Paley truncation $m_j \in L^\infty_{\rm c}(B(0,2^{j+1}); \mathscr{L}(X,Y))$ satisfies

$$\|\vartheta^\alpha m_j\|_\infty \leqslant 2^d M 2^{-(j+1)|\alpha|},$$

which is like the condition of Lemma 13.2.11 with $A = 2^{j+1}$ and an additional multiplicative constant $2^d M$.

Moreover, for $x \in X$, the function $m_j(\cdot)x \in L^\infty_{\rm c}(B(0,2^{j+1}); Y)$ satisfies the same assumption with constant $2^d M\|x\|$, and now the range Y also has Fourier type $p \in (1,2]$, as required to apply Lemma 13.2.11. In particular, from (13.30), we conclude that

$$\begin{aligned}\int_{|t|>3|s|} &\|(k_j(t-s) - k_j(t))x\|_Y \,\mathrm{d}t\\ &\leqslant \Phi^d_{p,Y} 2^d M \|x\| \min\Big\{2, \frac{8d^2\varphi_{p,Y}}{(2^{j+1}r)^{1/p'}}, 8\cdot 2^d 2^{j+1} r\Big\}.\end{aligned}$$

Since $m^N \in L^\infty_{\rm c}(\mathbb{R}^d;\mathscr{L}(X,Y)) \subseteq L^1(\mathbb{R}^d;\mathscr{L}(X,Y))$, the kernels $k^N = \check{m}^N \in C_0(\mathbb{R}^d;\mathscr{L}(X,Y))$ are well defined, and we can estimates

$$\begin{aligned}
&\int_{|t|>3|s|} \|(k^N(t-s)-k^N(t))x\|_Y \,\mathrm{d}t \\
&\quad\leqslant \sum_{j\leqslant N} \int_{\complement B} \|(k_j(t-s)-k_j(t))x\|_Y \,\mathrm{d}t \\
&\quad\leqslant \varPhi^d_{p,Y} 2^d M\|x\| \Big(\sum_{j:8\cdot 2^d 2^{j+1} r \leqslant 2^{-d-5}} 8\cdot 2^d 2^{j+1} r \\
&\qquad\qquad + \sum_{j:2^{-d-3}\leqslant 2^{j+1}r\leqslant (8d^2\varphi_{p,Y})^{p'}} 1 \\
&\qquad\qquad + \sum_{2^{j+1}r\geqslant (8d^2\varphi_{p,Y})^{p'}} \frac{8d^2\varphi_{p,Y}}{(2^{j+1}r)^{1/p'}} \Big) \\
&\quad\leqslant \varPhi^d_{p,Y} 2^d M\|x\| \Big(2 + (\log_2^+ (8d^2\varphi_{p,Y})^{p'} + d + 4) + \frac{1}{1-2^{-1/p'}}\Big) \\
&\quad\leqslant \varPhi^d_{p,Y} 2^d M\|x\| \Big(6 + 3d + \log_2^+ \varphi_{p,Y}\Big) p' \\
&\quad\leqslant \varPhi^d_{p,Y} 2^d M\|x\| \cdot d \cdot \varPhi_{p,Y} \leqslant (2\varPhi_{p,Y})^{d+1} M\|x\|.
\end{aligned}$$

□

Proof of (2). We note that (13.31) implies a similar bound for the pointwise adjoint function $m^* = m(\cdot)^* \in L^\infty(\mathbb{R}^d;\mathscr{L}(Y^*,X^*))$, while the assumption that X has Fourier type $p \in (1,2]$ implies that X^* has the same Fourier type with $\varphi_{p,X^*} = \varphi_{p,X}$ (Proposition 2.4.16). Thus case (2) follows from the already proven case (1) applied to (m^*,Y^*,X^*) in place of (m,X,Y). □

Corollary 13.2.14. *Let X,Y be Banach spaces with non-trivial Fourier type, let $p_0 \in [1,\infty)$, and suppose that $m \in \mathfrak{M}L^{p_0}(\mathbb{R}^d;X,Y)$ satisfies*

$$\|\partial^\alpha m(\xi)\| \leqslant M|\xi|^{-|\alpha|} \qquad \forall \alpha \in \{0,1\}^d.$$

Then $m \in \mathfrak{M}L^p(\mathbb{R}^d;X,Y)$ for all $p \in (1,\infty)$.

Proof. By Lemma 13.2.2, the Littlewood–Paley truncations of m satisfy $m^N \in \mathfrak{M}L^{p_0}(\mathbb{R}^d;X,Y)$ uniformly in $N \in \mathbb{Z}$. By Proposition 13.2.13, the kernels $k^N = \check{m}^N$ satisfy both Hörmander and dual Hörmander conditions uniformly in $N \in \mathbb{Z}$. On the other hand, by Lemma 13.2.11, the kernel $k_j = \check{m}_j$ satisfy $k_j(\cdot)x \in L^1(\mathbb{R}^d;Y)$ for all $x \in X$, uniformly in $\|x\| \leqslant 1$, and hence $k_j \in L^1_{\rm so}(\mathbb{R}^d;\mathscr{L}(X,Y))$.

It follows that the kernels k^N_L satisfy both Hörmander and dual Hörmander conditions uniformly in $L,N \in \mathbb{Z}$, and they belong to $L^1_{\rm so}(\mathbb{R}^d;\mathscr{L}(X,Y))$ (but in general *not* uniformly). Thus the convolution with k^N_L defines a bounded operator from $L^{p_0}(\mathbb{R}^d;X)$ to $L^{p_0}(\mathbb{R}^d;Y)$. So does $T_{m^N_L}$, and hence the identity

$$T_{m_L^N} f = k_L^N * f,$$

initially guaranteed by Proposition 13.2.1 for all $f \in L^1 \cap \check{L}^1(\mathbb{R}^d; X)$, extends by continuity and density to all $f \in L^{p_0}(\mathbb{R}^d; X)$. Since the operators are uniformly bounded on this space, and their kernels satisfy both Hörmander and dual Hörmander conditions uniformly, it follows from the Calderón–Zygmund Theorem 11.2.5 that they extend boundedly from $L^p(\mathbb{R}^d; X)$ to $L^p(\mathbb{R}^d; Y)$ for all $p \in (1, \infty)$, again uniformly in $L, N \in \mathbb{Z}$. This is the same as $m_L^N \in \mathfrak{M}L^{p_0}(\mathbb{R}^d; X, Y)$ uniformly in $L, N \in \mathbb{Z}$, which, by Lemma 13.2.2, implies that $m \in \mathfrak{M}L^p(\mathbb{R}^d; X, Y)$. ☐

The following corollary is just the operator-valued Mihlin Multiplier Theorem 5.5.10 in the special case of Hilbert spaces (in contrast to general UMD spaces covered by Theorem 5.5.10); we state it here for the sake of pointing out the alternative approach to this special case via the Calderón–Zygmund extrapolation theory developed in this chapter.

Corollary 13.2.15. *Let H_1, H_2 be Hilbert spaces and suppose that $m \in L^\infty(\mathbb{R}^d; \mathscr{L}(H_1, H_2))$ satisfies*

$$\|\partial^\alpha m(\xi)\| \leqslant M|\xi|^{-|\alpha|} \qquad \forall \alpha \in \{0, 1\}^d.$$

Then $m \in \mathfrak{M}L^p(\mathbb{R}^d; H_1, H_2)$ for all $p \in (1, \infty)$.

Proof. By Plancherel's theorem in both Hilbert spaces, we have

$$\|T_m f\|_{L^2(\mathbb{R}^d;H_2)} = \|m\widehat{f}\|_{L^2(\mathbb{R}^d;H_2)} \leqslant M\|\widehat{f}\|_{L^2(\mathbb{R}^d;H_1)} = M\|f\|_{L^2(\mathbb{R}^d;H_1)},$$

and thus $\|m\|_{\mathfrak{M}L^2(\mathbb{R}^d;H_1,H_2)} \leqslant M$. Since both H_i have Fourier type 2, Corollary 13.2.14 applies to give that $m \in \mathfrak{M}L^p(\mathbb{R}^d; H_1, H_2)$ for all $p \in (1, \infty)$. ☐

13.3 Necessity of UMD for multiplier theorems

In the previous sections, we have seen Fourier multiplier theorems of roughly two types:

1. If we already know the boundedness of such an operator on one $L^{p_0}(\mathbb{R}^d; X)$, then this boundedness can be extrapolated to other $L^p(\mathbb{R}^d; X)$ spaces under relatively mild (or even no) assumptions on the space X.
2. If we need to prove the boundedness "from scratch", then the required assumptions on X tend to be much stronger, and in particular involve the UMD property.

Let us also recall from the previous volumes that the need of the UMD property is not only imposed by the chosen proof strategies, but by the very nature of things: for prominent examples of multipliers like $-i\,\mathrm{sgn}(\xi)$ corresponding to the Hilbert transform (Theorem 5.2.10), or $|\xi|^{is}$ corresponding to imaginary powers of the Laplacian (Corollary 10.5.2), the UMD property is indeed necessary. The goal of this section is to continue this list by yet another class of Fourier multipliers whose boundedness requires UMD, and thereby close the circle of implications in a number of useful characterisations of UMD spaces. We start by discussing the types of multipliers that we are going to consider:

Definition 13.3.1. *We say that m is constant in the direction of $x \in \mathbb{R}^d \setminus \{0\}$ if $m(tx) = m(x)$ for all $t > 0$. We say that m is* stably constant *in the direction of $x \in \mathbb{R}^d \setminus \{0\}$ if, in addition, we have*

$$\lim_{t\to\infty} m(y+tx) = m(x) \qquad \forall y \in \mathbb{R}^d.$$

Note that if m is stably constant in the direction of x, then for every $s > 0$,

$$\lim_{t\to\infty} m(y+tsx) = \lim_{t\to\infty} m(y+tx) = m(x) = m(sx),$$

where the last step follows from the assumption (included in the definition of stably constant) that m is in particular constant in the direction of x.

Example 13.3.2. Suppose that $m \in C(\mathbb{R}^d \setminus \{0\})$ is homogeneous, $m(tx) = m(x)$ for all $t > 0$ and $x \in \mathbb{R}^d \setminus \{0\}$. Then m is stably constant in every direction. Indeed

$$\lim_{t\to\infty} m(y+tx) = \lim_{t\to\infty} m(t^{-1}y + x) = m(x)$$

simply by the continuity of m at x.

Example 13.3.3. Suppose that $m \in C^1(\mathbb{R}^d \setminus \{0\})$ satisfies the first order Mihlin condition $|\nabla m(x)| \leqslant M|x|^{-1}$ for all $x \in \mathbb{R}^d \setminus \{0\}$. If m is constant in the direction of some x, then m is stably constant in this direction. Indeed

$$\begin{aligned}|m(y+tx) - m(x)| = |m(y+tx) - m(tx)| &= \Big| \int_0^1 y \cdot \nabla m(ys+tx)\,\mathrm{d}s \Big| \\ &\leqslant |y| \int_0^1 \frac{M\,\mathrm{d}s}{|ys+tx|} \leqslant \frac{M|y|}{t|x| - |y|},\end{aligned}$$

and clearly this converges to 0 as $t \to \infty$.

Proposition 13.3.4 (Transference from $\mathbb{T}^d$ to $\mathbb{T}^{rd}$). *Let*

$$m \in C(\mathbb{R}^d \setminus \{0\}; \mathscr{L}(X)),$$

and suppose that it induces a periodic Fourier multiplier

$$T := \widetilde{T}_{(m(j))_{j\in\mathbb{Z}^n\setminus\{0\}}} \in \mathscr{L}(L^p_0(\mathbb{T}^d; X)).$$

If T_k is the extension of T to $L^p_0(\mathbb{T}^d; L^p(\mathbb{T}^{(k-1)d}; X))$ ($L^p(\mathbb{T}^0; X) := X$), then

$$\Big\| \sum_{k=1}^{r} T_k f_k \Big\|_{L^p(\mathbb{T}^{rd}, \mathrm{d}t_1 \dots \mathrm{d}t_r; X)} \leqslant \|T\|_{\mathscr{L}(L_0^p(\mathbb{T}^d;X))} \Big\| \sum_{k=1}^{r} f_k \Big\|_{L^p(\mathbb{T}^{rd}, \mathrm{d}t_1 \dots \mathrm{d}t_r; X)}$$

for all $f_k = f_k(t_1, \dots, t_k) \in L_0^p(\mathbb{T}^d, \mathrm{d}t_k; L^p(\mathbb{T}^{(k-1)d}, \mathrm{d}t_1 \dots \mathrm{d}t_{k-1}; X))$ *that have non-zero Fourier coefficients with respect to* t_k *only in the directions where* m *is stably constant.*

Proof. By the density of trigonometric polynomials in L^p, we may assume that

$$f_k(t_1, \dots, t_{k-1}, t_k) = f_k(\bar{t}_{k-1}, t_k) = \sum_{\substack{\ell \in \mathbb{Z}^n \\ 0 < |\ell| \leqslant B}} \sum_{\substack{j \in \mathbb{Z}^{(k-1)n} \\ |j| \leqslant B}} a_{j,\ell}^{(k)} e_j(\bar{t}_{k-1}) e_\ell(t_k),$$

where

$$\bar{t}_{k-1} = (t_1, \dots, t_{k-1}) \in (\mathbb{T}^d)^{k-1}, \quad t_k \in \mathbb{T}^d,$$
$$e_j(\bar{t}_{k-1}) := \exp(2\pi i j \cdot \bar{t}_{k-1}), \quad e_\ell(t_k) := \exp(2\pi i \ell \cdot t_k),$$

and we may choose the same B for all the f_k, since there are only finitely many of them. Then $T_k f_k$ has a similar expansion with the (j, ℓ) term multiplied by $m(\ell)$.

Let us fix some $\bar{t}_k := (\bar{t}_{k-1}, t_k) = (t_1, \dots, t_k) \in \mathbb{T}^{kd}$ for the moment, and

$$\bar{N}_k := (N_1, \dots, N_{k-1}, N_k) = (\bar{N}_{k-1}, N_k) \in \mathbb{Z}_+^k$$

to be chosen below.

We will shortly define an auxiliary function of the new variable $t \in \mathbb{T}^d$. For this we need to introduce a couple of product-like operations between vectors of different lengths. We set

$$\bar{N}_k \otimes t := (\bar{N}_{k-1} \otimes t, N_k t) = (N_1 t, \dots, N_k t) \in (\mathbb{T}^d)^k, \quad \bar{N}_k \in \mathbb{Z}^k,$$

$$\bar{N}_{k-1} \odot j := N_1 j_1 + \dots + N_{k-1} j_{k-1} \in \mathbb{Z}^d, \quad j = (j_1, \dots, j_{k-1}) \in (\mathbb{Z}^d)^{k-1}.$$

These operations satisfy the identity

$$j \cdot (\bar{N}_{k-1} \otimes t) = (\bar{N}_{k-1} \odot j) \cdot t, \quad \text{hence} \quad e_j(\bar{N}_{k-1} \otimes t) = e_{\bar{N}_{k-1} \odot j}(t),$$

where $\cdot$ stands for the usual Euclidean scalar product.

The new function is then defined by

$$\begin{aligned} \tilde{f}_k(t) &:= f_k(\bar{t}_k + \bar{N}_k \otimes t) \\ &= \sum_{\substack{\ell \in \mathbb{Z}^n \\ 0 < |\ell| \leqslant B}} \sum_{\substack{j \in \mathbb{Z}^{(k-1)n} \\ |j| \leqslant B}} a_{j,\ell}^{(k)} e_j(\bar{t}_{k-1}) e_\ell(t_k) e_{\bar{N}_{k-1} \odot j + N_k \ell}(t), \end{aligned} \tag{13.32}$$

The function $\widetilde{T_k f_k} : \mathbb{T}^n \to X$ is defined analogously.

We now want to compare $\widetilde{T_k f_k}$ with $T\tilde{f}_k$. They are both multiplier transforms of $\tilde{f}_k$, where in the first one the exponential $e_{\bar{N}_{k-1}\odot j+N_k\ell}$ is multiplied by $m(\bar{N}_{k-1}\odot j+N_k\ell)$, and in the second one by $m(\ell)$.

By the assumption on f_k, we know that m is stably constant in the direction of ℓ whenever $a_{j,\ell}^{(k)}\neq 0$, and therefore

$$\lim_{N_k\to\infty} m(\bar{N}_{k-1}\odot j+N_k\ell)=m(\ell).$$

Hence, assuming that $\bar{N}_{k-1}$ was already chosen, and recalling that $j\in\mathbb{Z}^{(k-1)d}$ and $\ell\in\mathbb{Z}^d$ with $|j|,|\ell|\leqslant B$ take only finitely many different values, we can choose N_k large enough so that

$$|m(\bar{N}_{k-1}\odot j+N_k\ell)-m(\ell)|\leqslant\varepsilon$$

for any preassigned $\varepsilon>0$ and all relevant values of j and ℓ.

In conclusion, denoting by $\|g\|_A$ the sum of the norms of the Fourier coefficients of a trigonometric polynomial g (on a torus of any dimension), we have

$$\|\widetilde{T_k f_k}-T\tilde{f}_k\|_p\leqslant\|\widetilde{T_k f_k}-T\tilde{f}_k\|_A\leqslant\varepsilon\|\tilde{f}_k\|_A\leqslant\varepsilon\|f_k\|_A.$$

Of course the $\|\cdot\|_A$ norms are finite since the functions above are all trigonometric polynomials.

Summing up, it follows that

$$\Big\|\sum_{k=1}^r\widetilde{T_k f_k}\Big\|_p\leqslant\Big\|T\sum_{k=1}^r\tilde{f}_k\Big\|_p+\varepsilon\sum_{k=1}^r\|f_k\|_A. \tag{13.33}$$

Here the L^p norms are taken with respect to the variable $t\in\mathbb{T}^d$, and we recall that the variables $t_1,\ldots,t_r\in\mathbb{T}^d$ were kept fixed until now. We now take the L^p norms of (13.33) with respect to $\bar{t}_r=(t_1,\ldots,t_r)\in\mathbb{T}^{rd}$ and use the triangle inequality to get

$$\begin{aligned}&\Big(\int_{\mathbb{T}^{rn}}\int_{\mathbb{T}^n}\Big\|\sum_{k=1}^r(T_kf_k)(\bar{t}_r+\bar{N}_r\otimes t)\Big\|_X^p\,\mathrm{d}t\,\mathrm{d}\bar{t}_r\Big)^{1/p}\\&\leqslant\|T\|_{\mathscr{L}(L_0^p(\mathbb{T}^n;X))}\Big(\int_{\mathbb{T}^{rn}}\int_{\mathbb{T}^n}\Big\|\sum_{k=1}^r f_k(\bar{t}_r+\bar{N}_r\otimes t)\Big\|_X^p\,\mathrm{d}t\,\mathrm{d}\bar{t}_r\Big)^{1/p}\\&\qquad+\varepsilon\sum_{k=1}^r\|f_k\|_A.\end{aligned}$$

Exchanging the order of the integrations on $\mathbb{T}^{rd}$ and $\mathbb{T}^d$, we find by translation invariance that the dependence on t and $\bar{N}_r$ disappears and we are left with

$$\Big\|\sum_{k=1}^r T_kf_k\Big\|_{L_0^p(\mathbb{T}^{rd};X)}\leqslant\|T\|_{\mathscr{L}(L_0^p(\mathbb{T}^d;X))}\Big\|\sum_{k=1}^r f_k\Big\|_{L_0^p(\mathbb{T}^{rd};X)}+\varepsilon\sum_{k=1}^r\|f_k\|_A.$$

Since there is no more explicit $\bar{N}_r$ dependence, we may take $\varepsilon \to 0$, and this gives the assertion. $\square$

Theorem 13.3.5 (Geiss–Montgomery-Smith–Saksman). *Let $d \geqslant 2$ and $m \in C(\mathbb{R}^d \setminus \{0\})$ be a multiplier that is stably constant in the directions of four vectors $\pm u_i$, $i = 1, 2$, where moreover*

$$m(-u_1) = m(u_1) \neq m(u_2) = m(-u_2).$$

If $m \in \mathfrak{M}L^p(\mathbb{R}^d; X)$, then X is a UMD space and

$$\beta^{\mathbb{R}}_{p,X} \leqslant \frac{2\|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)}}{|m(u_1) - m(u_2)|}. \tag{13.34}$$

To streamline the proof, we recall a transference result that we already observed and used in the proof of Corollary 10.5.2:

Lemma 13.3.6. *If $m \in C(\mathbb{R}^d \setminus \{0\}) \cap \mathfrak{M}L^p(\mathbb{R}^d; X)$, then $(m(k))_{k\in\mathbb{Z}^d\setminus\{0\}} \in \mathfrak{M}L^p_0(\mathbb{T}^d; X)$ and*

$$\|(m(k))_{k\in\mathbb{Z}^d\setminus\{0\}}\|_{\mathfrak{M}L^p_0(\mathbb{T}^d;X)} \leqslant \|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)}.$$

Proof. This is a slight variant of Proposition 5.7.1, which says that if every $k \in \mathbb{Z}^d$ is a Lebesgue point of $m \in L^\infty(\mathbb{R}^d)$, then $(m(k))_{k\in\mathbb{Z}^d}$ is a Fourier multiplier on $L^p(\mathbb{T}^d; X)$ of at most the norm of the Fourier multiplier m on $L^p(\mathbb{R}^d; X)$. A slight obstacle is that 0 may fail to be a Lebesgue point of our $m(\xi)$, no matter how we define $m(0)$. But, if we only consider the action of these operators on $L^p_0(\mathbb{T}^d; X)$, the 0th frequency never shows up, and one can check that the proof of Proposition 5.7.1 also applies, with trivial modifications, to the case that each $k \in \mathbb{Z}^d \setminus \{0\}$ is a Lebesgue point, giving exactly what we claimed. $\square$

Proof of Theorem 13.3.5. We begin by essentially the same reduction as in the proofs of both Theorems 5.2.10 and 10.5.1 (the necessity of UMD for the boundedness of the Hilbert transform and the imaginary powers of the Laplacian, respectively); but we repeat this short step for the reader's convenience: By Theorem 4.2.5 it suffices to estimate the dyadic UMD constant. In order to most conveniently connect this with Fourier analysis, we choose a model of the Rademacher system $(r_k)_{k=1}^n$, where the probability space is $\mathbb{T}^{dn} = \mathbb{T}^d_1 \times \cdots \times \mathbb{T}^d_n$ (each $\mathbb{T}^d_k$ is simply an indexed copy of $\mathbb{T}^d$), and $r_k = r_k(t_k)$ is a function of the kth coordinate $t_k \in \mathbb{T}^d_k$ only. Moreover, we are free to choose any instance of such function, as long as it takes both values ± 1 on subsets of $\mathbb{T}^d$ of measure $\frac{1}{2}$. Then it is sufficient to prove that

$$\Big\| \sum_{k=1}^n \epsilon_k f_k \Big\|_{L^p(\mathbb{T}^{dn};X)} \leqslant K \Big\| \sum_{k=1}^n f_k \Big\|_{L^p(\mathbb{T}^{dn};X)},$$

where K is the constant on the right of (13.34), for all signs $\epsilon_k = \pm 1$, for all f_k of the form $f_k = \phi_k(r_1, \dots, r_{k-1}) r_k$; these are precisely the martingale differences of Paley–Walsh martingales (see Proposition 3.1.10). We use the convention that $L^p(\mathbb{T}^0; X) := X$.

Let us then observe that, with suitable choice of the invertible matrices A_j, $j = 1, 2$, the multipliers $m_j(\xi) = m(A_j \xi)$ (of the same multiplier norm as the original m) are stably constant in the directions of $\pm e_k$, $k = 1, 2$, and moreover $m_j(\pm e_k) = m(u_1)$ if $j = k$ and $m_j(\pm e_k) = m(u_2)$ if $j \neq k$. Defining yet another multiplier $m' = \frac{1}{2}(m_1 - m_2)$ (of at most the same multiplier norm as m), we find that m' is also stably constant in the directions of $\pm e_k$, $k = 1, 2$, and moreover $m'(\pm e_1) = \frac{1}{2}(m(u_1) - m(u_2)) =: a$ and $m'(e_2) = -a$. If we can prove the claim with m', e_1, e_2 in place of the original m, u_1, u_2, then the original claim also follows from

$$\beta_{p,X} \leqslant \frac{2\|m'\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)}}{|m'(e_1) - m'(e_2)|} \leqslant \frac{2\|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)}}{|m(u_1) - m(u_2)|}.$$

Dropping the primes, we assume without loss of generality that $m(\pm e_1) = a = -m(\pm e_2)$, and m is stably constant in the directions of $\pm e_j$, $j = 1, 2$.

From Proposition 13.3.4 and Lemma 13.3.6 we know that, for suitable functions f_k,

$$\begin{aligned}\Big\| \sum_{k=1}^n \widetilde{T}_k f_k \Big\|_{L^p(\mathbb{T}^{dn};X)} &\leqslant \|m\|_{\mathfrak{M}L^p_0(\mathbb{T}^d;X)} \Big\| \sum_{k=1}^n f_k \Big\|_{L^p(\mathbb{T}^{dn};X)} \\ &\leqslant \|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)} \Big\| \sum_{k=1}^n f_k \Big\|_{L^p(\mathbb{T}^{dn};X)},\end{aligned}$$

where T_k is a copy $\widetilde{T}_{(m(j))_{j\in\mathbb{Z}^d\setminus\{0\}}}$ acting in the kth $\mathbb{T}^d_k$, thus

$$T_k f_k = \phi_k(r_1, \dots, r_{k-1}) \widetilde{T}_{(m(j))_{j\in\mathbb{Z}^d\setminus\{0\}}} r_k.$$

The required condition on f_k above is that its Fourier coefficients with respect to the variable t_k should be non-zero only in the directions, where m is stably constant, i.e., only in the directions $\pm e_1$ and $\pm e_2$. Given the product form of f_k, this means more simply that r_k should have non-zero Fourier coefficients only in these directions, which holds in particular if r_k is a function of only the first or only the second coordinate. Note that this gives still (more than) enough flexibility to make r_k equidistributed with a Rademacher variable.

Now, given a sequence $(\varepsilon_k)_{k=1}^r$, we choose r_k to be a function of the first coordinate if $\varepsilon_k = +1$, and of the second coordinate if $\varepsilon_k = -1$. It then follows that in either case $\widetilde{T}_{(m(j))_{j\in\mathbb{Z}^d\setminus\{0\}}} r_k = a\varepsilon_k r_k$, and we conclude that

$$\Big\|\sum_{k=1}^n \epsilon_k f_k\Big\|_{L^p(\mathbb{T}^{dn};X)} = \frac{1}{|a|}\Big\|\sum_{k=1}^n T_k f_k\Big\|_{L^p(\mathbb{T}^{dn};X)} \leqslant \frac{2}{|m(e_1)-m(e_2)|}\|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)}\Big\|\sum_{k=1}^n f_k\Big\|_{L^p(\mathbb{T}^{dn};X)},$$

which is what we claimed. □

For the sake of precise quantitative conclusions, we also record the following variant of Theorem 13.3.5. The assumptions of the next result are much stronger than those of Theorem 13.3.5, so that the qualitative conclusion that X is a UMD space is immediate from the previous theorem. The point of this variant is that under the stronger assumption we can directly estimate the complex UMD constant $\beta^{\mathbb{C}}_{p,X}$ of X. The result is not strictly a corollary of Theorem 13.3.5 itself, but follows by a modification of its proof, as we are about to see.

Corollary 13.3.7. *Let $d \geqslant 2$ and $m \in C(\mathbb{R}^d \setminus \{0\})$ be an even, homogeneous multiplier whose range contains the complex unit circle. If $m \in \mathfrak{M}L^p(\mathbb{R}^d;X)$, then X is a UMD space and*

$$\beta^{\mathbb{C}}_{p,X} \leqslant \|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)}.$$

Proof. By the same reductions and notation as in the proof of Theorem 13.3.5, we now need to check that

$$\Big\|\sum_{k=1}^n \sigma_k f_k\Big\|_{L^p(\mathbb{T}^{dn};X)} \leqslant \|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)}\Big\|\sum_{k=1}^n f_k\Big\|_{L^p(\mathbb{T}^{dn};X)},$$

for any $\sigma_k \in \mathbb{C}$ with $|\sigma_k| = 1$. By the assumption about the range of m, we can further write $\sigma_k = m(u_k)$ for some $u_k \in \mathbb{C}$ with $|u_k| = 1$.

Consider a large number $R > 0$. For each k, we can find an integer vector $n_k \in \mathbb{Z}^d$ such that $\|n_k - Ru_k\|_{\ell^\infty} \leqslant \frac{1}{2}$. Thus $\|u_k - R^{-1}n_k\|_{\ell^\infty} \leqslant \frac{1}{2R}$. Since m is continuous, by choosing R large enough we ensure that $|m(u_k) - m(R^{-1}n_k)| \leqslant \delta$ for each $k = 1, \dots, n$ and any given $\delta > 0$. Thus

$$\Big\|\sum_{k=1}^n \sigma_k f_k\Big\|_{L^p(\mathbb{T}^{dn};X)} = \Big\|\sum_{k=1}^n m(u_k) f_k\Big\|_{L^p(\mathbb{T}^{dn};X)} \leqslant \Big\|\sum_{k=1}^n m(n_k) f_k\Big\|_{L^p(\mathbb{T}^{dn};X)} + \sum_{k=1}^n \delta\|f_k\|_{L^p(\mathbb{T}^{dn};X)},$$

where we also used the homogeneity $m(R^{-1}n_k) = m(n_k)$.

We now come to our choice of the Rademachers functions r_k appearing in the martingale differences $f_k = \phi_k(r_1, \dots, r_{k-1})r_k$. Fixing any Rademacher function r on $\mathbb{T}$, we take $r_k(t) := r(n_k \cdot t)$ for $t \in \mathbb{T}^d$. Substituting $n_k \cdot t$ into the Fourier series of r, we find that

$$r_k(t) = \sum_{j\in\mathbb{Z}} \widehat{r}(j) e^{2\pi i j n_k \cdot t}$$

has a Fourier series involving only frequencies that are multiples of the vector n_k. By the homogeneity of m again, this means that

$$\widetilde{T}_{(m(j))_{j\in\mathbb{Z}^d\setminus\{0\}}} r_k = m(n_k) r_k,$$

and thus

$$\begin{aligned}\Big\| \sum_{k=1}^n m(n_k) f_k \Big\|_{L^p(\mathbb{T}^{dn};X)} &= \Big\| \sum_{k=1}^n \widetilde{T}_{(m(j))_{j\in\mathbb{Z}^d\setminus\{0\}}} f_k \Big\|_{L^p(\mathbb{T}^{dn};X)} \\ &\leqslant \|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)} \Big\| \sum_{k=1}^n f_k \Big\|_{L^p(\mathbb{T}^{dn};X)}.\end{aligned}$$

Collecting the estimates, we have checked that

$$\begin{aligned}\Big\| \sum_{k=1}^n \sigma_k f_k \Big\|_{L^p(\mathbb{T}^{dn};X)} \leqslant \|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)} &\Big\| \sum_{k=1}^n f_k \Big\|_{L^p(\mathbb{T}^{dn};X)} \\ &+ \delta \sum_{k=1}^n \|f_k\|_{L^p(\mathbb{T}^{dn};X)},\end{aligned}$$

or in other words

$$\begin{aligned}&\Big\| \sum_{k=1}^n \sigma_k r_k \phi_k(r_1,\dots,r_{k-1}) \Big\|_{L^p(\mathbb{T}^{dn};X)} \\ &\quad\leqslant \|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)} \Big\| \sum_{k=1}^n r_k \phi_k(r_1,\dots,r_{k-1}) \Big\|_{L^p(\mathbb{T}^{dn};X)} \\ &\qquad + \delta \sum_{k=1}^n \|r_k \phi_k(r_1,\dots,r_{k-1})\|_{L^p(\mathbb{T}^{dn};X)}.\end{aligned}$$

While the specific choice of the Rademacher functions r_k depended on the numbers n_k, which in turn depended on δ, it is clear that this last bound is true for any Rademacher sequence $(r_k)_{k=1}^n$, as soon as it is true for one. Once this observation is made, we see that everything is independent of δ, and taking the limit $\delta \to 0$, we obtain the required bound. □

Corollary 13.3.8. *Let X be a Banach space, $d \geqslant 2$ and $p \in (1,\infty)$. If any of the following operators is bounded on $L^p(\mathbb{R}^d;X)$, then X is a UMD space:*

(1) *a second-order Riesz transform $R_j R_k$, $1 \leqslant j,k \leqslant d$,*
(2) *their non-zero difference $R_j^2 - R_k^2$, $1 \leqslant j \neq k \leqslant d$,*
(3) *the Beurling transform $B = (R_2^2 - R_1^2) + i2R_1R_2$ $(d = 2)$.*

Moreover, we have the following estimates:

(1) $\beta^{\mathbb{R}}_{p,X} \leqslant 2\|R_j R_k\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))}$,
(2) $\beta^{\mathbb{R}}_{p,X} \leqslant \|R_j^2 - R_k^2\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))}$,
(3) $\beta^{\mathbb{C}}_{p,X} \leqslant \|B\|_{\mathscr{L}(L^p(\mathbb{R}^2;X))}$.

Proof. These operators correspond to the multipliers

$$m_{R_j R_k}(\xi) = -\frac{\xi_j \xi_k}{|\xi|^2}, \quad m_{R_j^2 - R_k^2}(\xi) = -\frac{\xi_j^2 - \xi_k^2}{|\xi|^2}, \quad m_B(\xi) = -\frac{\xi_1 - i\xi_2}{\xi_1 + i\xi_2},$$

each of which is even and homogeneous, in particular stably constant in all directions.

Writing $\xi_1 + i\xi_2$ in the polar coordinates as $re^{i\theta}$, it is clear that $m_B(\xi) = m_B(re^{i\theta}) = -e^{-i2\theta}$ takes all values in the complex unit circle. Hence the claims concerning B are immediate from Corollary 13.3.7.

For $R_j R_k$, we observe that $m_{R_j^2}(\xi) = -\xi_j^2/|\xi|^2$ is -1 for $\xi = e_j$ and 0 for $\xi = e_k$, $k \neq j$, whereas $m_{R_j R_k}(\xi) = -\frac{1}{2}$ for $\xi = (e_j + e_k)$ and $\frac{1}{2}$ for $\xi = (e_j - e_k)$ when $k \neq j$; in each case we have $|m(u_1) - m(u_2)| = 1$ for suitable vectors u_i. For $R_j^2 - R_k^2$, the multiplier is -1 for $\xi = e_j$ and $+1$ for $\xi = e_k$, so that $|m(e_j) - m(e_k)| = 2$. In each case, the claimed conclusion is immediate from Theorem 13.3.5. □

Corollary 13.3.8 allows us to complete a characterisation of a function space embedding that we studied in Section 5.6:

Corollary 13.3.9. *Let X be a Banach space, let $d, k \geqslant 1$ and $p \in (1, \infty)$. Then there is a constant C such that*

$$\|f\|_{W^{k,p}(\mathbb{R}^d;X)} \leqslant \|f\|_{H^{k,p}(\mathbb{R}^d;X)} \qquad \forall f \in \mathscr{S}(\mathbb{R}^d; X)$$

if and only if at least one of the following holds:

(1) *$d = 1$ and k is even, or*
(2) *X is a UMD space.*

Proof. The sufficiency of (1) has been established in Proposition 5.6.10 and the sufficiency of (2) in Theorem 5.6.11. Moreover, in Theorem 5.6.12, it has been shown that the UMD property is necessary when k is odd, and that the boundedness of the second-order Riesz transform R_1^2 is necessary when k is even and $d \geqslant 2$. By Corollary 13.3.8, the UMD property follows from this, and hence it is necessary in all cases except (1). □

In our final corollary to Theorem 13.3.5, we dispense with the evenness condition.

Corollary 13.3.10. *Let $d \geqslant 1$ and $m \in C(\mathbb{R}^d \setminus \{0\})$ be any positively homogeneous multiplier (i.e., $m(\lambda\xi) = m(\xi)$ for all $\xi \in \mathbb{R}^d \setminus \{0\}$ and $\lambda > 0$) that is not identically constant. If $m \in \mathfrak{M}L^p(\mathbb{R}^d; X)$, then X is a UMD space and*

$$\beta^{\mathbb{R}}_{p,X} \leqslant \min_{u_1,u_2 \in S^{d-1}} \frac{4\|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)}}{|m(u_1) + m(-u_1) - m(u_2) - m(-u_2)|},$$
$$\beta^{\mathbb{R}}_{p,X} \leqslant (\hbar_{p,X})^2 \leqslant \Big(\min_{u \in S^{d-1}} \frac{2\|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)}}{|m(u) - m(-u)|} \Big)^2,$$

where at least one of the right-hand sides is finite.

The assumption that m is not identically constant, rather than the perhaps expected "not identically zero", is necessary: the Fourier multiplier T_m with $m \equiv c$ coincides with the scalar multiplication $f \mapsto c \cdot f$, whose boundedness certainly needs no UMD.

Proof. As pointed out right before Proposition 5.3.7, the assumption that $m \in \mathfrak{M}L^p(\mathbb{R}^d; X)$ implies the same property for the reflected function $\widetilde{m}(\xi) := m(-\xi)$. Then, by the triangle inequality, the even and odd parts $m_{\rm even} := \frac{1}{2}(m + \widetilde{m})$ and $m_{\rm odd} := \frac{1}{2}(m - \widetilde{m})$ are also positively homogeneous multipliers of at most the same multiplier norm as m. Since m is not identically constant, and $m = m_{\rm even} + m_{\rm odd}$, at least one of $m_{\rm even}$ or $m_{\rm odd}$ is not identically constant.

If $m_{\rm even}$ is not identically constant, there are two directions $u_1, u_2 \in S^{d-1}$ such that $m_{\rm even}(u_1) \neq m_{\rm even}(u_2)$ and hence, by evenness,

$$m_{\rm even}(-u_1) = m_{\rm even}(u_1) \neq m_{\rm even}(u_2) = m_{\rm even}(-u_2).$$

By Example 13.3.2, the homogeneous $m_{\rm even} \in C(\mathbb{R}^d \setminus \{0\})$ is stably constant in every directions. Hence $m_{\rm even}$ satisfies the assumptions of the Geiss–Montgomery-Smith–Saksman Theorem 13.3.5, and the said theorem guarantees that, for any such $u_1, u_2 \in S^{d-1}$,

$$\beta^{\mathbb{R}}_{p,X} \leqslant \frac{2\|m_{\rm even}\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)}}{m_{\rm even}(u_1) - m_{\rm even}(u_2)} \leqslant \frac{4\|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)}}{m(u_1) + m(-u_1) - m(u_2) - m(-u_2)}.$$

(Note that the condition that $m_{\rm even}(u_1) \neq m_{\rm even}(u_2)$ is precisely the requirement that the denominator is non-zero, and hence can extend the previous display to all pairs of $u_1, u_2 \in S^{d-1}$; interpreting $1/0 = \infty$, as usual, this only amounts to adding the triviality $\beta^{\mathbb{R}}_{p,X} \leqslant \infty$.)

For the odd part $m_{\rm odd}$, being not identically constant is equivalent to being not identically zero. If this is the case, there is some direction $u \in S^{d-1}$ such that $m(-u) = -m(u) \neq 0$. Writing $\xi \in \mathbb{R}^d$ as $\xi = (\xi \cdot u)u + [\xi - (\xi \cdot u)u]$, we consider the invertible linear transformations $A_\lambda \xi = (\xi \cdot u)u + \lambda[\xi - (\xi \cdot u)u]$, where $\lambda > 0$. By Proposition 5.3.8, each $m_{\rm odd} \circ A_\lambda$ has the same multiplier norm as $m_{\rm odd}$. As $\lambda \to 0$, it is clear that $A_\lambda \xi \to (\xi \cdot u)u$ for all $\xi \in \mathbb{R}^d$ and thus, by the continuity of m and hence $m_{\rm odd}$,

$$m_{\rm odd} \circ A_\lambda(\xi) \to m_{\rm odd}((\xi \cdot u)u) = \operatorname{sgn}(\xi \cdot u) m_{\rm odd}(u).$$

A convergence result for multipliers, Proposition 5.3.16, then implies that

$$\begin{aligned}|m_{\rm odd}(u)| \|\xi \mapsto \operatorname{sgn}(\xi \cdot u)\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)} &\leqslant \liminf_{\lambda \to 0} \|m_{\rm odd} \circ A_\lambda\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)} \\ &= \|m_{\rm odd}\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)}.\end{aligned}$$

By another application of Proposition 5.3.8 with a rotation that sends u to e_1, it follows that

$$\begin{aligned}\|\xi \mapsto \operatorname{sgn}(\xi_1)\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)} &= \|\xi \mapsto \operatorname{sgn}(\xi \cdot u)\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)} \\ &\leqslant \frac{\|m_{\rm odd}\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)}}{|m_{\rm odd}(u)|} \leqslant \frac{2\|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)}}{|m(u) - m(-u)|}.\end{aligned}$$

(The bound remains valid for all $u \in S^{d-1}$, reducing to a triviality if $m(u) = m(-u)$.) By Fubini's theorem, we find that

$$\hbar_{p,X} := \|\xi \mapsto \operatorname{sgn}(\xi)\|_{\mathfrak{M}L^p(\mathbb{R};X)} \|\xi \mapsto \operatorname{sgn}(\xi_1)\|_{\mathfrak{M}L^p(\mathbb{R};X)}.$$

The bound between $\beta^{\mathbb{R}}_{p,X} \leqslant (\hbar_{p,X})^2$ is contained in Corollary 5.2.11. □

13.4 Notes

Section 13.1

The precise quantitative form of the final bound in the comparison of various Fourier-type constants in Proposition 13.1.1 seems to be new; we were not aware of this estimate at the time of completing Volume II, where a weaker version was given. The identity $\varphi_{p,\mathbb{C}}(\mathbb{R}^d) = (p^{1/p}(p')^{-1/p'})^d$ mentioned below the said proposition is due to Babenko [1961] in the special case that p' is an even integer, and due to Beckner [1975] in full generality.

The main result of this section, Theorem 13.1.33 is from Bourgain [1988a], with preliminary versions going back to Bourgain [1981, 1982]. The main theorem of Bourgain [1982] reads as follows: If X is a B-convex Banach space (which is equivalent to non-trivial type by Proposition 7.6.8), then there are $u, v \in (1,\infty)$ and $\delta, M \in (0,\infty)$ such that

$$\delta\Big(\sum_{\gamma\in\Gamma} \|x_\gamma\|_X^u\Big)^{1/u} \leqslant \Big\|\sum_{\gamma\in\Gamma} \gamma x_\gamma\Big\|_{L^2(G;X)} \leqslant M\Big(\sum_{\gamma\in\Gamma} \|x_\gamma\|_X^v\Big)^{1/v}, \qquad (13.35)$$

whenever $\{x_\gamma\}_{\gamma\in\Gamma}$ is a finitely non-zero sequence of elements of X and Γ is the spectrum of the compact abelian group G. This is a Hausdorff–Young inequality with mismatched exponents; our Corollary 13.1.27 is the special case of the right-hand inequality with $G = \mathbb{T}$ and $\Gamma = \{e_k\}_{k\in\mathbb{Z}}$. For these particular G and Γ, and under the stronger assumption that X be super-reflexive, (13.35) was proved in Bourgain [1981]. A further predecessor of such results is due to James [1972], who proved a bound like (13.35) with a super-reflexive Z in place of both X and $L^2(G;X)$, and $z_k \in Z$ in place of both x_γ and γx_γ, under the assumption that $(z_k)_{k=1}^\infty$ is a *basic sequence* in Z, i.e.,

$$\Big\| \sum_{k=1}^{K} a_k z_k \Big\|_Z \leqslant C \Big\| \sum_{k=1}^{L} a_k z_k \Big\|_Z \tag{13.36}$$

for all scalars a_k and integers $K \leqslant L$. Requiring (13.36) for $z_k = e_k x_k \in Z = L^2(\mathbb{T};X)$, uniformly in $x_k \in X$, is equivalent to the still stronger property that X be a UMD space, which is why additional work was required by Bourgain [1981] to obtain his result for trigonometric series in super-reflexive spaces. (The estimate (13.36) in the said special case is equivalent to the $L^2(\mathbb{T};X)$-boundedness of the periodic Hilbert transform by Proposition 5.2.7, and this is equivalent to the UMD property by Corollary 5.2.11. UMD spaces are super-reflexive by Corollary 4.3.8, but the converse is false. Various examples showing the last point are due to Pisier [1975], Bourgain [1983], Garling [1990], Geiss [1999], and Qiu [2012]. The example of Qiu [2012] is an infinitely iterated $L^p(L^q)$ space, which has been presented in Theorem 4.3.17, but the super-reflexivity of this space is not treated there.)

As in our treatment in the section under discussion, getting from estimate (13.35) with mismatched exponents to dual pairs requires further ideas. This was achieved by Bourgain [1988b], who proved that, for some $u_1, v_1 \in (1,\infty)$ and $\delta_1, M_1 \in (0,\infty)$, there further holds

$$\begin{aligned} \delta_1 \Big(\sum_{\gamma\in\Gamma} \|x_\gamma\|_X^{u_1'} \Big)^{1/u_1'} &\leqslant \Big\| \sum_{\gamma\in\Gamma} \gamma x_\gamma \Big\|_{L^{u_1}(G;X)} \\ &\leqslant \Big\| \sum_{\gamma\in\Gamma} \gamma x_\gamma \Big\|_{L^{v_1'}(G;X)} \leqslant M_1 \Big(\sum_{\gamma\in\Gamma} \|x_\gamma\|_X^{v_1} \Big)^{1/v_1}, \end{aligned} \tag{13.37}$$

when G is either $\mathbb{T}$ or the Cantor group $\{-1,1\}^{\mathbb{N}}$. For $G = \mathbb{T}$, the leftmost and rightmost estimates correspond, in our notation, to $\varphi_{u_1,X}(\mathbb{T}) \leqslant 1/\delta_1$ and $\varphi_{v_1,X}(\mathbb{Z}) \leqslant M_1$, respectively. The easy estimate $\varphi_{p,X}(\mathbb{R}) \leqslant \varphi_{p,X}(\mathbb{T})$ was also observed by Bourgain [1988b]. In contrast to the case of $\mathbb{T}$, a scaling argument (substituting $f(\lambda\cdot)$ in place of f and considering the limit $\lambda \to 0$ or $\lambda \to \infty$) shows that an estimate of the from $\|\widehat{f}\|_{L^q(\mathbb{R};X)} \leqslant C\|f\|_{L^p(\mathbb{R};X)}$ can only hold for $q' = p$; thus, in order to deduce any Hausdorff–Young inequality on $\mathbb{R}$ at all, the additional steps from the mismatched exponents of Bourgain [1982] to the dual exponents of Bourgain [1988b] seem to be necessary.

The second half of the argument leading to Bourgain's Theorem 13.1.33, as presented in Sections 13.1.c and 13.1.d, is close to the treatment of Bourgain

[1988b], although we have also benefited from the exposition of these steps by Pietsch and Wenzel [1998]. On the other hand, the first half of our treatment, in Sections 13.1.a and 13.1.b, is also based on Pietsch and Wenzel [1998] but deviates from the original approach of Bourgain [1982]. The beginning of the argument, leading to Proposition 13.1.11 on "breaking the trivial bound" is due to Hinrichs [1996], but it also uses a result of Bourgain [1985], Proposition 13.1.7, on the Sidon property of quasi-independent sets.

We have chosen this approach of Hinrichs [1996] and Pietsch and Wenzel [1998] due to an independent interest, in our opinion, of some of its intermediate steps, despite the fact that the original argument of Bourgain [1982, 1988b] seems slightly more efficient in terms of the final quantitative conclusions. In any case, the main result says that every Banach space of type $p \in (1,2]$ will have Fourier-type $r = 1 + (c\tau_{p,X;2})^{-p'}$, for some absolute constant c. (The additional factor $6p'$ in our formulation of Theorem 13.1.33 could obviously be absorbed by choosing a larger constant c.) The difference is in the numerical value of c, which is 68 in our formulation (up to the lower order factor just mentioned) and 17 in Bourgain [1982, 1988b].

In our approach, this constant comes from the proof of Corollary 13.1.20, where the estimate $48\sqrt{2}$ (≈ 67.88) $\leqslant 68$ is made. (Since we are clearly off Bourgain's constant at this point already, it would seem pointless to insist in the decimals here.) The constant $48\sqrt{2}$, in turn, is produced as

$$48\sqrt{2} = 16 \cdot \sqrt{2} \cdot 3, \qquad \text{where}$$

(i) 16 is the upper bound of the Sidon constants of quasi-independent sets from Proposition 13.1.7;
(ii) $\sqrt{2}$ comes from the factor in front of the upper bound of the number of quasi-independent sets required to partition a given set in Lemma 13.1.9; the root is due to the use of this number count after an application of the Cauchy–Schwarz inequality in the proof of Proposition 13.1.11;
(iii) 3 is the constant from the Marcinkiewicz inequality (Proposition 13.1.17), which enters into the estimate through an application of the Comparison Lemma 13.1.18 in the proof of Corollary 13.1.20.

One may speculate that the constant 16 (just below the 17 of Bourgain [1982]) is the heart of the matter, and the other two factors are only produced by secondary details that should be avoidable by more careful reasoning.

The approach of Bourgain [1982] is based on two abstract results (avoided in the present treatment) about the collection of tuples of functions

$$\mathscr{O} := \{\boldsymbol{\xi} = (\xi_i)_{i=1}^n \in L^2(\Omega;\mathbb{R})^n : \|\xi_i\|_\infty \leqslant 1, \int \xi_i = \int \xi_i\xi_j = 0 \quad \text{for all } 1 \leqslant i \neq j \leqslant n\}$$

on a probability space Ω; namely:

(1) The set $\mathscr{E}$ of extreme points of $\mathscr{O}$ consists of tuples of ± 1-valued functions.

(2) For each $\boldsymbol{\xi} \in \mathscr{O}$, there is a Borel probability measure μ on $\mathscr{E}$ such that

$$\xi_i = \int_{\mathscr{E}} \eta_i \, \mathrm{d}\mu(\boldsymbol{\eta}), \quad \text{for every } i = 1, \dots, n.$$

According to Bourgain [1982], the proof of (1) is "essentially contained in" Dor [1975], while (2) can be derived from a generalisation of Choquet's integral representation theorem due to Edgar [1976]. Combining these abstract tools with delicate hard analysis, Bourgain [1982] eventually arrives at his key technical estimate, which in our notation (and exchanging the roles of X and X^* compared to Bourgain [1982]) may be stated as

$$\varphi^{(2)}_{\infty,X^*}(F) \leqslant K \cdot N^{1/t}, \quad t' = (17 \cdot \tau_{p,X;2})^{p'}. \tag{13.38}$$

This is recognised as a close relative of Corollary 13.1.29, where the bound

$$\varphi^{(s')}_{\infty,X^*}(F) \leqslant K \cdot N^{1/s}, \quad s' > r' = 3p'(68 \cdot \tau_{p,X;2})^{p'}$$

is obtained. While the left-hand sides are not identical, (13.38) allows Bourgain [1982] to deduce the Hausdorff–Young inequality with mismatched exponents as in (13.35) (with X^* in place of X) for any $v \in (1,t)$, and finally, in Bourgain [1988b], also the classical Hausdorff–Young inequality (13.37) (again with X^* in place of X) with any $u_1 \in (1,v)$. Since $v \in (1,t)$ is arbitrary, one can reach any $u_1 \in (1,t)$, and thus in particular the r determined by

$$r' = (18 \cdot \tau_{p,X;2})^{p'} \tag{13.39}$$

is a Fourier type of X^*, and hence of X.

Remark 13.4.1 (A typo in the statement of Bourgain's theorem in König [1991]). It seems to be claimed by König [1991] that every space of type $p > 1$ would have Fourier-type r with $r' = c \cdot \tau^{p'}_{p,X;2}$ and $c = 18$ (forgetting brackets from (13.39)). As written, this is absurd for any absolute constant c:

It is straightforward to verify that, for every $p \in (1,2]$, the space $X = \ell^p$ has type p with constant $\tau_{p,X;2} = 1$:

$$\begin{aligned}\Big\| \sum_{n=1}^N \varepsilon_n x_n \Big\|_{L^2(\Omega;\ell^p)} &\leqslant \Big\| \sum_{n=1}^N \varepsilon_n x_n \Big\|_{\ell^p(L^2(\Omega))} = \Big\| \{x_n\}_{n=1}^N \Big\|_{\ell^p(\ell^2_N)} \\ &\leqslant \Big\| \{x_n\}_{n=1}^N \Big\|_{\ell^p(\ell^p_N)} = \Big(\sum_{n=1}^N \|x_n\|_X^p \Big)^{1/p}\end{aligned}$$

Thus, were the claim in the beginning of the remark true, all these spaces would have the Fourier-type $r = \frac{c}{c-1} > 1$, which is impossible for $p \in (1,r)$ by Example 2.1.15.

Since the numerical constant in (13.38) may be affected by an equivalent choice of the type constant, we note that Bourgain [1982] is not explicit about the precise definition of the constant that he denotes by C, but one can see in the proof of the first step of his Proposition 4 that $C = \tau_{p,X^*;2}$; recall that we exchanged the roles of X and X^* compared to Bourgain [1982].

More details on quasi-independent sets can be found in the monograph of Graham and Hare [2013]. Sometimes quasi-independent sets are called *dissociate sets*, but it seems that in more recent works this terminology is reserved for the slightly stronger property where one allows $\alpha_k \in \{-2,-1,0,1,2\}$ in Definition 13.1.5. In particular, one can find there that quasi-independent are Sidon sets with constant $6\sqrt{6} \approx 14.70$, which is slightly better than the constant 16 in Proposition 13.1.7. The converse bounds of Remark 13.1.10 have been shown to us by Dion Gijswijt. If one replaces the group $\mathbb{Z}$ by another group it was shown on page 203 in Pietsch and Wenzel [1998] that the bound of Lemma 13.1.8 is sharp.

The result of Proposition 13.1.21 states that type p and cotype q with $1/p - 1/q < 1/r - 1/2$ with $r \in (1,2)$ implies Fourier type r. In the limiting case of equality it is unknown what happens. However, the result is sharp in the sense that for every $r \in (1,2)$ and for every $p \in (r,2)$ there exists a Banach space X such that X has type p, cotype q, and Fourier type r with $\frac{1}{p} - \frac{1}{q} = \frac{1}{r} - \frac{1}{2}$, and none of the exponents (p,q,r) can be improved (see Bourgain [1988a] and García-Cuerva, Torrea, and Kazarian [1996]). This example was also used to show that the dependence on the type constant is necessary in Theorem 13.1.33. The following improvement was observed in García-Cuerva, Torrea, and Kazarian [1996] for Banach lattices X:

$$\begin{aligned}\sup\{p \in (1,2] :& X \text{ has Fourier type } p\} \\ &= \sup\{p \in (1,2] : X \text{ has type } p \text{ and cotype } p'\}.\end{aligned}$$

Section 13.2

In the scalar-valued case, considerations of the kind that we have presented in this section go back to Hörmander [1960] who used similar methods to rederive (a variant of) the multiplier theorem of Mihlin [1956, 1957] by transforming it into a form where the theory of Calderón and Zygmund [1952] could be applied. The methods of Hörmander [1960] are already very close to the ones in the Section 13.2.b, the key difference being that he can make use of the Plancherel theorem to pass between L^2 estimates in the space and frequency variables. For functions taking values in a general Banach space, the only available substitute is the elementary $L^1(\mathbb{R}^d;X)$-to-$L^\infty(\mathbb{R}^d;X)$ boundedness of the Fourier transform. This still allows essentially similar conclusions, at the cost of requiring estimates for a higher number of derivatives as input. On the other hand, as soon as we start imposing such stronger assumptions, we can also obtain stronger conclusions, namely, standard Calderón–Zygmund kernels rather than just Hörmander kernels, as in Section 13.2.a. Scalar-valued versions of such results are again well known; for example, a version of Proposition 13.2.7 with $d+2$ derivatives (instead of $d+1$ in the said proposition) appears in the book of Stein [1993]. Under this stronger assumption, Stein [1993] deduces that $k \in C^1(\mathbb{R}^d \setminus \{0\})$, while Proposition 13.2.6 gives the slightly weaker conclusion that k is just barely below Lipschitz, with a modulus of continuity $\omega(t) = O(t \cdot \log(1+1/t))$. This is still quite enough to derive like Corollaries 13.2.8, 13.2.9, and 13.2.10 on the boundedness of Fourier multipliers on weighted $L^p(w;X)$ spaces. Using the result from Stein [1993] in place of Proposition 13.2.7, a version of Corollary 13.2.10 assuming $d+2$ derivatives was formulated by Meyries and Veraar [2015]. In principle, variants of Propositions 13.2.6 and 13.2.7 sufficient for Corollaries 13.2.8 through 13.2.10 would only require smoothness of order $d+\varepsilon$, but such statements and proofs are bound to have additional technicalities due to the very formulation of fractional order smoothness conditions. Various results in this direction, involving kernel bounds for Fourier multipliers with close-to-critical fractional smoothness, were explored by Hytönen [2004].

To get rid of the $\varepsilon > 0$ altogether, i.e., to deduce useful (in view of Calderón–Zygmund extrapolation) kernel estimates for $k = \check{m}$ from just d derivatives of m, one needs to impose assumptions on the Fourier-type of the underlying spaces. While we have only dealt with the sufficiency of the Fourier-type assumption in Section 13.2.b, an early result involving both directions, in dimension $d = 1$, is the following:

Theorem 13.4.2 (König [1991]). *A Banach space X is K-convex if and only if every $f \in C^1(\mathbb{T}, X)$ has Fourier coefficients $(\widehat{f}(n))_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z}; X)$.*

Recall that K-convexity is equivalent to non-trivial type by Pisier's Theorem 7.4.23, and non-trivial type is equivalent to non-trivial Fourier-type by Bourgain's Theorem 13.1.33. The proof of "$\Rightarrow$" in Theorem 13.4.2 is then straightforward from non-trivial Fourier type. For the converse, König [1991] starts with a concrete counterexample when $X = L^1(\mathbb{T})$, and approximates this finite versions that can be represented in ℓ^1_N, with blow-up in the limit $N \to \infty$. By the Maurey–Pisier Theorem 7.3.8, if X does not have non-trivial type, then it contains subspaces isomorphic to ℓ^1_N uniformly, and hence the said finite examples can also be represented in X. Finally, the closed graph theorem guarantees that a sequence of examples with blow-up also guarantees the existence of a single $f \in C^1(\mathbb{T}, X)$ with $(\widehat{f}(n))_{n\in\mathbb{Z}} \notin \ell^1(\mathbb{Z}; X)$.

In our formulation of Proposition 13.2.13, the assumed Fourier-type $p \in (1,2]$ only affects the constant in the estimate. However, by more careful reasoning, one could show that also the number of the required derivative $\partial^\alpha m$ could be reduced as a function of p; roughly speaking, one needs only derivatives up to order $\lfloor d/p \rfloor + 1$, or more generally fractional smoothness of order $d/p + \varepsilon$, to obtain the same conclusions. Such results can be found in Hytönen [2004]. In the more general context of various function spaces, this phenomenon will be explored further in Chapter 14; see Proposition 14.5.3 and take $q = \infty$ there.

Our focus in the section under discussion has been exploring conditions that one needs to assume on a multiplier m in order that their associated kernel $k = \check{m}$ satisfies the assumptions of one of the extrapolation theorems of Chapter 11 (so that the *a priori* boundedness of T_m on one $L^{p_0}(\mathbb{R}^d; X)$ extends to other spaces), but similar considerations can also be used to reduce the required smoothness, as a function of the Fourier-type of the underlying spaces, in results like Mihlin's Multiplier Theorem 5.5.10 (where the boundedness of T_m on $L^p(\mathbb{R}^d; X)$ is deduced "from scratch"). Such results were pioneered by Girardi and Weis [2003b] and further elaborated by Hytönen [2004]. If m is scalar-valued, it is also possible to replace Fourier-type by quantitatively weaker assumptions on type or cotype; see Hytönen [2010].

Section 13.3

The main results of this section, notably Proposition 13.3.4, Theorem 13.3.5, and Corollary 13.3.8, are essentially from Geiss, Montgomery-Smith, and Saksman [2010], but we have incorporated some improvements, partially inspired by unpublished observations of Alex Amenta that he kindly shared with us.

These results may be seen as successors, in terms of both statement and proof, of Theorem 5.2.10 of Bourgain [1983] and Theorem 10.5.1 of Guerre-Delabrière [1991], which deal with the necessity of UMD for the boundedness of the Hilbert transform and the imaginary powers $(-\Delta)^{is}$ of the Laplacian, respectively. However, none of these three results contains any of the other two.

Certain elaborations of Corollary 13.3.8 are due to Castro and Hytönen [2016]. Namely, the identity $\partial_j \partial_k u = -R_j R_k \Delta u$ implies that

$$\|\partial_j \partial_k u\|_{L^p(\mathbb{R}^d;X)} \leqslant C \sum_{i=1}^{d} \|R_i^2 u\|_{L^p(\mathbb{R}^d;X)}, \tag{13.40}$$

where $C \leqslant \|R_j R_k\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))}$, but C could *a priori* be much smaller. However, Castro and Hytönen [2016] show that the seemingly weaker inequality (13.40) still implies the UMD property with the same control

$$\beta_{p,X} \leqslant 2C_{(13.40)} \tag{13.41}$$

as in Corollary 13.3.8 for $\|R_j R_k\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))}$. More generally, the same paper proves the necessity of UMD for any member of a family of inequalities of the form

$$\|\partial^\beta u\|_{L^p(\mathbb{R}^d;X)} \leqslant C \sum_{\alpha \in \mathscr{A}} \|\partial^\alpha u\|_{L^p(\mathbb{R}^d;X)},$$

but the relation between the constants is particularly clean in the example just mentioned.

It could be of interest to identify more general criteria (subsuming previous related results) for inequalities of classical/harmonic analysis to

(1) imply the UMD property of X (as in all mentioned results), or
(2) control the UMD constant $\beta_{p,X}$ linearly by the constant in the inequality (as in Theorems 10.5.1 and 13.3.5, but not Theorem 5.2.10).

While we have concentrated, in this section, on lower bounds of multiplier norms by the UMD constants, Geiss, Montgomery-Smith, and Saksman [2010] also treat the other direction. In particular, they show that the first two bounds of Corollary 13.3.8 are actually identities:

$$\|2R_jR_k\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} = \|R_j^2 - R_k^2\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} = \beta_{p,X}^{\mathbb{R}} \tag{13.42}$$

for all $1 \leqslant j \neq k \leqslant d$. The upper bounds for the norms are proved by representing and estimating the operators by means of stochastic integrals. Yaroslavstev [2018] obtained further variants of these estimates for related operators. We plan to detail this in a forthcoming Volume. By (13.41), a trivial bound, and (13.42), it follows that

$$\beta_{p,X} \leqslant 2C_{(13.40)} \leqslant 2\|R_jR_k\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} = \beta_{p,X},$$

and hence all these quantities must be equal. In particular, as observed by Castro and Hytönen [2016], it follows that

$$C_{(13.40)}^{X=\mathbb{R}} = \frac{1}{2}\beta_{p,\mathbb{R}} = \frac{1}{2}(\max(p,p') - 1),$$

using Burkholder's Theorem 4.5.7 for the last equality. We are not aware of another method than that of Geiss, Montgomery-Smith, and Saksman [2010] to determine the exact norms (13.42) or the sharp constant in (13.40), which highlights the benefits of martingale techniques even for questions of classical analysis.

In the third case of Corollary 13.3.8 concerning the Beurling–Ahlfors transform, the matching upper bound is an outstanding open problem even for $X = \mathbb{C}$ (see Problems O.1 and O.2).

More generally, Geiss, Montgomery-Smith, and Saksman [2010] prove that all real, even, and homogeneous (i.e., $m(t\xi) = m(\xi) \in \mathbb{R}$ for all $\xi \in \mathbb{R}^d \setminus \{0\}$ and $t \in \mathbb{R} \setminus \{0\}$) multipliers $m \in C^\infty(\mathbb{R}^d \setminus \{0\})$ satisfy the estimate

$$\|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)} \leqslant C_m \cdot \beta_{p,X},$$

where C_m depends only on m. Note in particular that the estimate is linear in $\beta_{p,X}$, improving on the quadratic estimate provided by $T(1)$ Theorem 12.4.21, or the still higher order dependence in the Mihlin Multiplier Theorem 5.5.10. By elaborations of the $T(1)$ technology, linear dependence has also been obtained for a class of even non-convolution operators on $L^p(\mathbb{R};X)$ (but only in dimension $d = 1$, as written) by Pott and Stoica [2014], but beyond that the availability of linear bounds in terms of $\beta_{p,X}$ remains open. In particular, a possible linear estimate between $\beta_{p,X}$ and the norm of the Hilbert transform $\hbar_{p,X} = \|H\|_{\mathscr{L}(L^p(\mathbb{R};X))}$, in either direction, is unknown (see Problem O.6).

Certain substitute results related to the latter are due to Domelevo and Petermichl [2023c,d]. They construct a new dyadic operator and show that its boundedness is equivalent to that of the Hilbert transform, with linear dependence between the respective norms in both directions. Analogous results for the Riesz transforms are obtained in Domelevo and Petermichl [2023a,b].

Further estimates between the Hilbert transform (and variants) and decoupling constants related to the UMD constant can be found in Osękowski and Yaroslavtsev [2021].

Corollary 13.3.9 characterises situations in which there is a continuous embedding $H^{k,p}(\mathbb{R}^d;X) \hookrightarrow W^{k,p}(\mathbb{R}^d;X)$. Several related results, including versions on domains $\mathcal{O} \subseteq \mathbb{R}^d$, are due to Arendt, Bernhard, and Kreuter [2020].

14

Function spaces

This chapter presents an in-depth study of several classes of vector-valued function spaces defined by smoothness conditions. In Volume I we have already encountered two such classes: the Sobolev spaces $W^{s,p}(\mathbb{R}^d; X)$ for $s \in \mathbb{N}$ and $s \in (0,1)$ (Chapter 2) and the Bessel potential spaces $H^{s,p}(\mathbb{R}^d; X)$ for $s \in \mathbb{R}$ (Chapter 5). Both classes are parametrised by an integrability parameter p and smoothness parameter s. The present chapter introduces two related classes of function spaces, the Besov spaces $B^s_{p,q}(\mathbb{R}^d; X)$ and the Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{R}^d; X)$. From the point of view of applications these spaces play an important role in the theory of partial differential equations, where they typically occur as trace spaces associated with initial value problems. What makes these spaces interesting from a mathematical point of view is the wealth of different characterisations of these classes: they can equivalently be introduced via Littlewood–Paley decompositions, difference norms, and interpolation.

In line with earlier developments, we introduce both Besov spaces and Triebel–Lizorkin spaces via their Littlewood–Paley decompositions. These involve a so-called inhomogeneous Littlewood–Paley sequence $(\varphi_k)_{k\geqslant 0}$ of Schwartz functions on $\mathbb{R}^d$ whose Fourier transforms behave, informally speaking, as a dyadic partition of unity radially. In terms of such sequences, the Besov and Triebel–Lizorkin norms are defined by

$$\|f\|_{B^s_{p,q}(\mathbb{R}^d;X)} = \big\|(2^k\varphi_k * f)_{k\geqslant 0}\big\|_{\ell^q(L^p(\mathbb{R}^d;X))}$$

and

$$\|f\|_{F^s_{p,q}(\mathbb{R}^d;X)} := \big\|(2^{ks}\varphi_k * f)_{k\geqslant 0}\big\|_{L^p(\mathbb{R}^d;\ell^q(X))},$$

in the sense that a tempered distribution $f \in \mathscr{S}'(\mathbb{R}^d; X)$ belongs to either one space if and only if the respective expression is well defined and finite. The third parameter q featuring in these definitions is often referred to as the *microscopic parameter*.

In both cases, the norms are independent of the Littlewood–Paley sequence up to a multiplicative constant independent of f. Accordingly, it will

T. Hytönen et al., *Analysis in Banach Spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics 76,
https://doi.org/10.1007/978-3-031-46598-7_4

be a standing assumption that *throughout the chapter we fix a Littlewood–Paley sequence* $(\varphi_k)_{k\geqslant 0}$ *once and for all* (Convention 14.2.8). Dependence of constants on this sequence will never be tracked.

Interestingly, the Bessel potential spaces studied in Chapter 5, and whose study is continued in the present chapter, admit a similar decomposition replacing ℓ^q-norms by Rademacher norms (Theorem 14.7.5) in case X has UMD:

$$\begin{aligned}\|f\|_{H^{s,p}(\mathbb{R}^d;X)} &\eqsim \big\|(\varepsilon_k 2^{ks}\varphi_k * f)_{k\geqslant 0}\big\|_{\varepsilon^p(L^p(\mathbb{R}^d;X))}\\ &= \big\|(\varepsilon_k 2^{ks}\varphi_k * f)_{k\geqslant 0}\big\|_{L^p(\mathbb{R}^d;\varepsilon^p(X))},\end{aligned}$$

using the notation for Rademacher spaces introduced Section 6.3; the equality of the latter two norms is obtained by repeating the proof of Theorem 9.4.8 for Rademacher sums. Comparing these norms with the previous two, it is also of interest to note that equivalent norms are obtained if the ε^p-norm is replaced by an ε^q-norm, by the Kahane–Khintchine inequalities.

In view of their very similar definitions, it comes as no surprise that the theories of Besov and Triebel–Lizorkin spaces largely parallel each other and resemble the theory of Bessel potential spaces to some extent. There are some notable differences however, due to the different orders in which the L^p-norm and ℓ^q-norm are taken; as we have already pointed out, the Triebel–Lizorkin norm is generally speaking more difficult to handle. The main advantage of the Besov and Triebel–Lizorkin over the Bessel potential spaces is that they are often easier to work with, and indeed many basic results for these spaces in the vector-valued setting do not rely on the geometry of the Banach space X. This is in stark contrast with the theory of Bessel potential spaces, where the corresponding results often require geometrical properties such as the UMD property of X or the Radon–Nikodým property of X^*, as we have seen in Chapter 5.

After establishing notation and proving some preliminary results in Section 14.1, the class of Besov spaces is introduced in Section 14.4 via their Littlewood–Paley decompositions. Several basic aspects of these spaces are discussed, such as their independence of the inhomogeneous Littlewood–Paley sequence used in the definition, the density of smooth functions, and Sobolev type embeddings. We continue with several more advanced results, including a difference norm characterisation, identification the complex and real interpolation spaces, and identification of the dual spaces. In Section 14.5 these results are used to prove embedding theorems for the spaces $\gamma(L^2(\mathbb{R}^d), X)$ introduced in Chapter 9 and to prove R-boundedness of the ranges of smooth operator-valued functions under type and cotype assumptions. In the same section we discuss Fourier multiplier results for Besov spaces under (co)type and Fourier type assumptions.

In Section 14.6 the Triebel–Lizorkin spaces are introduced. Proving the same basic properties as before is more complicated, especially for the important endpoint exponent $q = 1$, and requires the boundedness of the so-called Peetre maximal function and the boundedness of Fourier multiplier operators for functions with compact Fourier support in an $L^p(\mathbb{R}^d;\ell^q(X))$-setting. Most of the elementary and more advanced results discussed for Besov spaces have a counterpart for Triebel–Lizorkin spaces and indeed our treatment mirrors that of the Besov spaces. Some results, however, have a different flavour, such as the Sobolev embedding theorem (Theorem 14.6.14), the Gagliardo–Nirenberg inequalities (Proposition 14.6.15), and the embedding theorem of Franke and Jawerth (Theorem 14.6.26), all of which have an improvement in the microscopic parameter q. In some situations this improvement makes it possible to derive results for general Banach spaces X in an effective way. For instance, for any Banach space X one has continuous embeddings (here and below denoted by "$\hookrightarrow$")

$$F^s_{p,1}(\mathbb{R}^d;X) \hookrightarrow H^{s,p}(\mathbb{R}^d;X) \hookrightarrow F^s_{p,\infty}(\mathbb{R}^d;X) \tag{14.1}$$

for $p \in (1,\infty)$ and $s \in \mathbb{R}$. For Hilbert spaces X this can be improved to

$$H^{s,p}(\mathbb{R}^d;X) = F^s_{p,2}(\mathbb{R}^d;X)$$

with equivalent norms for all $p \in (1,\infty)$ and $s \in \mathbb{R}$; this identity characterises Hilbert spaces up to isomorphism (Theorem 14.7.9). The "sandwich result" (14.1) often makes it possible to prove results about $H^{s,p}(\mathbb{R}^d;X)$ without conditions on X by factoring through a Triebel–Lizorkin space. At the end of the section apply some of the obtained result to prove boundedness of pointwise multiplication by the function $\mathbf{1}_{\mathbb{R}_+}$ in Triebel–Lizorkin spaces and Besov spaces. Such results are non-trivial due to the non-smoothness of $\mathbf{1}_{\mathbb{R}_+}$, and are important in applications to interpolation with boundary conditions of vector-valued function spaces used for evolution equations.

In Section 14.7 we return to the study of Bessel potential spaces and discuss some basic properties not covered in the earlier volumes. These include improvements of (14.1) for UMD spaces X under type and cotype assumptions, as well as some advanced results on complex interpolation of Bessel potential spaces (Corollary 14.7.13). At the end of the section we prove the boundedness of pointwise multiplication by the function $\mathbf{1}_{\mathbb{R}_+}$ in Bessel potential spaces again for UMD spaces.

As we will be using Fourier techniques practically everywhere, it will be a further standing assumption that *throughout the chapter we work over the complex scalar field.* As usually is the case, the case of real Banach spaces can be treated by standard complexification arguments. In some cases one can argue directly on real Banach spaces (see Remark 14.2.6). Unless stated otherwise, X will always denote an arbitrary complex Banach space.

14.1 Summary of the main results

Scattered over this section a wealth of inclusion and interpolation results are developed. For the convenience of the reader, we include a concise overview of them here, with pointers to their location.

In all identities, unless otherwise no geometric restrictions apply to Banach spaces and the occurring indices are taken in the ranges

$$p_0, p_1, p \in [1,\infty], \quad q_0, q_1, q \in [1,\infty], \quad s_0, s_1, s \in \mathbb{R}, \quad k_0, k_1 \in \mathbb{N},$$

or subsets thereof. The interpolation results assume that $\theta \in (0,1)$ and where

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

and

$$s_\theta = (1-\theta)s_0 + \theta s_1, \quad k_\theta = (1-\theta)k_0 + \theta k_1.$$

The complex and real interpolation spaces of an interpolation couple (X_0, X_1) of Banach spaces are denoted by

$$X_\theta = [X_0, X_1]_\theta, \quad X_{\theta,p} = (X_0, X_1)_{\theta,p}$$

respectively.

Identities. Up to equivalent norms we have the following identifications. If $p \in [1,\infty)$, $s \in (0,1)$, then

$$W^{s,p}(\mathbb{R}^d; X) = B^s_{p,p}(\mathbb{R}^d; X) \qquad \text{(Corollary 14.4.25)}$$

and, if $s \in (0,\infty) \setminus \mathbb{N}$,

$$C^s_{\rm ub}(\mathbb{R}^d; X) = B^s_{\infty,\infty}(\mathbb{R}^d; X). \qquad \text{(Corollary 14.4.26)}$$

If H is a Hilbert space and $p \in (1,\infty)$, $s \in \mathbb{R}$, then

$$H^{s,p}(\mathbb{R}^d; H) = F^s_{p,2}(\mathbb{R}^d; H) \qquad \text{(Theorem 14.7.9)}$$

and, if $p \in (1,\infty)$ and $k \in \mathbb{N}$,

$$W^{k,p}(\mathbb{R}^d; H) = F^k_{p,2}(\mathbb{R}^d; H). \qquad \text{(Theorem 14.7.9)}$$

If X is a UMD space and $p \in (1,\infty)$, $k \in \mathbb{N}$, then

$$W^{k,r}(\mathbb{R}^d; X) = H^{k,r}(\mathbb{R}^d; X). \qquad \text{(Theorem 5.6.11)}$$

Embeddings. We have the following continuous embeddings:

$$\mathscr{S}(\mathbb{R}^d; X) \hookrightarrow B^s_{p,q}(\mathbb{R}^d; X) \hookrightarrow \mathscr{S}'(\mathbb{R}^d; X) \qquad \text{(Proposition 14.4.3)}$$
$$B^s_{p,1}(\mathbb{R}^d; X) \hookrightarrow B^s_{p,q}(\mathbb{R}^d; X) \hookrightarrow B^s_{p,\infty}(\mathbb{R}^d; X) \qquad \text{(Proposition 14.4.18)}$$
$$\mathscr{S}(\mathbb{R}^d; X) \hookrightarrow F^s_{p,q}(\mathbb{R}^d; X) \hookrightarrow \mathscr{S}'(\mathbb{R}^d; X) \qquad \text{(Proposition 14.6.8)}$$

$$F^s_{p,1}(\mathbb{R}^d;X) \hookrightarrow F^s_{p,q}(\mathbb{R}^d;X) \hookrightarrow F^s_{p,\infty}(\mathbb{R}^d;X) \qquad \text{(Proposition 14.6.13)}$$
$$F^k_{p,1}(\mathbb{R}^d;X) \hookrightarrow W^{k,p}(\mathbb{R}^d;X) \hookrightarrow F^k_{p,\infty}(\mathbb{R}^d;X) \qquad \text{(Proposition 14.6.13)}$$
$$F^s_{p,1}(\mathbb{R}^d;X) \hookrightarrow H^{s,p}(\mathbb{R}^d;X) \hookrightarrow F^s_{p,\infty}(\mathbb{R}^d;X) \qquad \text{(Proposition 14.6.13)}$$

and, if $p \in [1,\infty)$,

$$B^s_{p,p\wedge q}(\mathbb{R}^d;X) \hookrightarrow F^s_{p,q}(\mathbb{R}^d;X) \hookrightarrow B^s_{p,p\vee q}(\mathbb{R}^d;X). \qquad \text{(Proposition 14.6.8)}$$

Sobolev embedding theorem I: If (and only if) either one of the following three conditions holds: $p_0 = p_1$ and $s_0 > s_1$; $p_0 = p_1$ and $s_0 = s_1$ and $q_0 \leqslant q_1$; $p_0 < p_1$ and $q_0 \leqslant q_1$ and $s_0 - \frac{d}{p_0} \geqslant s_1 - \frac{d}{p_1}$; then

$$B^{s_0}_{p_0,q_0}(\mathbb{R}^d;X) \hookrightarrow B^{s_1}_{p_1,q_1}(\mathbb{R}^d;X). \qquad \text{(Theorem 14.4.19)}$$

Sobolev embedding theorem II: Let $p_0, p_1 \in [1,\infty)$. If (and only if) either one of the following three conditions holds: $p_0 = p_1$ and $s_0 > s_1$; $p_0 = p_1$ and $s_0 = s_1$ and $q_0 \leqslant q_1$; $p_0 < p_1$ and $s_0 - \frac{d}{p_0} \geqslant s_1 - \frac{d}{p_1}$ (no condition on q_0, q_1); then

$$F^{s_0}_{p_0,q_0}(\mathbb{R}^d;X) \hookrightarrow F^{s_1}_{p_1,q_1}(\mathbb{R}^d;X). \qquad \text{(Theorem 14.6.14)}$$

Sobolev embedding theorem III: Let $p_0, p_1 \in (1,\infty)$. If (and only if) either one of the following three conditions holds: $p_0 = p_1$ and $s_0 \geqslant s_1$; $p_0 < p_1$ and $s_0 - \frac{d}{p_0} \geqslant s_1 - \frac{d}{p_1}$; then

$$H^{s_0,p_0}(\mathbb{R}^d;X) \hookrightarrow H^{s_1,p_1}(\mathbb{R}^d;X) \qquad \text{(Theorem 14.7.1)}$$

and, if in addition $s_0, s_1 \in \mathbb{N}$, then the same necessary and sufficient conditions give

$$W^{s_0,p_0}(\mathbb{R}^d;X) \hookrightarrow W^{s_1,p_1}(\mathbb{R}^d;X). \qquad \text{(Theorem 14.7.1)}$$

For $k \in \mathbb{N}$,

$$B^k_{\infty,1}(\mathbb{R}^d;X) \hookrightarrow C^k_{\rm ub}(\mathbb{R}^d;X) \hookrightarrow B^k_{\infty,\infty}(\mathbb{R}^d;X). \qquad \text{(Proposition 14.4.18)}$$

If $p_0 \in [1,\infty]$ and $s_0, s_1 \geqslant 0$ satisfy $s_0 - \frac{d}{p_0} \geqslant s_1$, then

$$B^{s_0}_{p_0,1}(\mathbb{R}^d;X) \hookrightarrow C^{s_1}_{\rm ub}(\mathbb{R}^d;X) \qquad \text{(Proposition 14.4.27)}$$

and, if in addition $q \in [1,\infty]$ and $s_1 \notin \mathbb{N}$,

$$B^{s_0}_{p_0,q}(\mathbb{R}^d;X) \hookrightarrow C^{s_1}_{\rm ub}(\mathbb{R}^d;X). \qquad \text{(Proposition 14.4.27)}$$

Jawerth–Franke theorem: If $p_0 < p_1$, and $s_0 - \frac{d}{p_0} \geqslant s_1 - \frac{d}{p_1}$, then

$$F^{s_0}_{p_0,q}(\mathbb{R}^d; X) \hookrightarrow B^{s_1}_{p_1,p_0}(\mathbb{R}^d; X) \qquad \text{(Theorem 14.6.26)}$$

and, if $p_1 < \infty$,

$$B^{s_0}_{p_0,p_1}(\mathbb{R}^d; X) \hookrightarrow F^{s_1}_{p_1,q}(\mathbb{R}^d; X). \qquad \text{(Theorem 14.6.26)}$$

If $k \geqslant d$, then

$$F^k_{1,\infty}(\mathbb{R}^d; X) \hookrightarrow C^{k-d}_{\rm ub}(\mathbb{R}^d; X). \qquad \text{(Corollary 14.6.27)}$$

Embeddings under (co)type assumptions: If (and only if) X has type $p \in [1, 2]$,

$$B^{(\frac{1}{p}-\frac{1}{2})d}_{p,p}(\mathbb{R}^d; X) \hookrightarrow \gamma(L^2(\mathbb{R}^d), X). \qquad \text{(Theorem 14.5.1)}$$

If (and only if) X has cotype $q \in [2, \infty]$,

$$\gamma(L^2(\mathbb{R}^d), X) \hookrightarrow B^{(\frac{1}{q}-\frac{1}{2})d}_{q,q}(\mathbb{R}^d; X). \qquad \text{(Theorem 14.5.1)}$$

If X has type p_0, then for all $p \in [1, p_0)$ we have

$$H^{(\frac{1}{p}-\frac{1}{2})d,p}(\mathbb{R}^d; X) \hookrightarrow \gamma(L^2(\mathbb{R}^d), X). \qquad \text{(Corollary 14.7.7)}$$

If X has cotype q_0, then for all $q \in (q_0, \infty)$ we have

$$\gamma(L^2(\mathbb{R}^d), X) \hookrightarrow H^{(\frac{1}{q}-\frac{1}{2})d,q}(\mathbb{R}^d; X). \qquad \text{(Corollary 14.7.7)}$$

If X is a UMD Banach space with type $p_0 \in [1, 2]$ and cotype $q_0 \in [2, \infty]$, and if $p \in (1, \infty)$, $s \in \mathbb{R}$, then

$$F^s_{p,p_0}(\mathbb{R}^d; X) \hookrightarrow H^{s,p}(\mathbb{R}^d; X) \hookrightarrow F^s_{p,q_0}(\mathbb{R}^d; X). \qquad \text{(Proposition 14.7.6)}$$

Complex interpolation. Let (X_0, X_1) be an interpolation couple of Banach spaces. Let $p_0, p_1 \in [1, \infty]$ with $\min\{p_0, p_1\} < \infty$, $q_0, q_1 \in [1, \infty]$ with $\min\{q_0, q_1\} < \infty$, and $s_0, s_1 \in \mathbb{R}$ or $k_0, k_1 \in \mathbb{N}$. Under these assumptions:

$$[B^{s_0}_{p_0,q_0}(\mathbb{R}^d; X_0), B^{s_1}_{p_0,q_0}(\mathbb{R}^d; X_1)]_\theta = B^{s_\theta}_{p_\theta,q_\theta}(\mathbb{R}^d; X_\theta). \qquad \text{(Theorem 14.4.30)}$$

If $p_0, p_1 \in (1, \infty)$ and $q_0, q_1 \in (1, \infty]$,

$$[F^{s_0}_{p,q}(\mathbb{R}^d; X_0), F^{s_1}_{p,q}(\mathbb{R}^d; X_1)]_\theta = F^{s_\theta}_{p,q}(\mathbb{R}^d; X_\theta) \qquad \text{(Theorem 14.6.23)}$$

and, if in addition X is a UMD space, then

$$[W^{k_0,p_0}(\mathbb{R}^d;X_0),W^{k_1,p_1}(\mathbb{R}^d;X_1)]_\theta = H^{k_\theta,p_\theta}(\mathbb{R}^d;X_\theta) \qquad \text{(Corollary 14.7.13)}$$
$$[H^{s_0,p_0}(\mathbb{R}^d;X_0),H^{s_1,p_1}(\mathbb{R}^d;X_1)]_\theta = H^{s_\theta,p_\theta}(\mathbb{R}^d;X_\theta). \qquad \text{(Theorem 14.7.12)}$$

Real interpolation. Let (X_0, X_1) be an interpolation couple of Banach spaces and X be a Banach space. Let $p_0, p_1 \in [1,\infty]$ with $\min\{p_0,p_1\} < \infty$, $q_0, q_1 \in [1,\infty]$ with $\min\{q_0,q_1\} < \infty$, $s_0, s_1 \in \mathbb{R}$, and $k_0, k_1 \in \mathbb{N}$. Under these assumptions:

If $s_0 \neq s_1$, then

$$(B^{s_0}_{p,q_0}(\mathbb{R}^d;X),B^{s_1}_{p,q_1}(\mathbb{R}^d;X))_{\theta,q} = B^{s_\theta}_{p,q}(\mathbb{R}^d;X) \qquad \text{(Theorem 14.4.31)}$$
$$(H^{s_0,p}(\mathbb{R}^d;X),H^{s_1,p}(\mathbb{R}^d;X))_{\theta,q} = B^{s_\theta}_{p,q}(\mathbb{R}^d;X). \qquad \text{(Theorem 14.4.31)}$$

In addition, if $s_0, s_1 \in \mathbb{N}$ with $s_0 \neq s_1$, then

$$(W^{s_0,p}(\mathbb{R}^d;X),W^{s_1,p}(\mathbb{R}^d;X))_{\theta,q} = B^{s_\theta}_{p,q}(\mathbb{R}^d;X) \qquad \text{(Theorem 14.4.31)}$$

and if $s_0, s_1 \in (0,1)$ with $s_0 \neq s_1$ and $p \in [1,\infty)$, then

$$(W^{s_0,p}(\mathbb{R}^d;X),W^{s_1,p}(\mathbb{R}^d;X))_{\theta,q} = B^{s_\theta}_{p,q}(\mathbb{R}^d;X). \qquad \text{(Theorem 14.4.31)}$$

If $s_0, s_1 \in [0,\infty)$ satisfy $s_0 \neq s_1$, then

$$(C^{s_0}_{\rm ub}(\mathbb{R}^d;X),C^{s_1}_{\rm ub}(\mathbb{R}^d;X))_{\theta,\infty} = B^{s_\theta}_{\infty,\infty}(\mathbb{R}^d;X). \qquad \text{(Corollary 14.4.32)}$$

If $p \in [1,\infty)$ and $s_0 \neq s_1$, then

$$(F^{s_0}_{p,q_0}(\mathbb{R}^d;X),F^{s_1}_{p,q_1}(\mathbb{R}^d;X))_{\theta,q} = B^{s_\theta}_{p,q}(\mathbb{R}^d;X). \qquad \text{(Proposition 14.6.24)}$$

Duality. With respect to the natural duality pairing of $L^2(\mathbb{R}^d;X)$ and $L^2(\mathbb{R}^d;X^*)$, for $p, q \in [1,\infty)$ and $s \in \mathbb{R}$ we have, up to equivalent norms,

$$B^s_{p,q}(\mathbb{R}^d;X)^* = B^{-s}_{p',q'}(\mathbb{R}^d;X^*) \qquad \text{(Theorem 14.4.34)}$$

and, for $p, q \in (1,\infty)$ and $s \in \mathbb{R}$,

$$F^s_{p,q}(\mathbb{R}^d;X)^* = F^{-s}_{p',q'}(\mathbb{R}^d;X^*). \qquad \text{(Theorem 14.6.28)}$$

If X^* has the Radon-Nikodým property, $p \in [1,\infty)$, and $s \in \mathbb{R}$, then

$$H^{s,p}(\mathbb{R}^d;X)^* = H^{-s,p'}(\mathbb{R}^d;X^*). \qquad \text{(Proposition 5.6.7)}$$

14.2 Preliminaries

In this section we prepare some, mostly technical, results that will be of use in our treatments of both Besov and Triebel–Lizorkin spaces.

14.2.a Notation

We start by reviewing some notation that has been introduced in the two earlier volumes. We use the standard multi-index notation explained in Section 2.5. For the details we refer to the relevant sections (Section 2.4.c for Schwartz functions, 2.4.d for tempered distributions, 2.5.b and 2.5.d for Sobolev spaces, and 5.6.a for Bessel potential spaces).

Let X be a Banach space and let $d \geqslant 1$ be an integer. The *Schwartz space* $\mathscr{S}(\mathbb{R}^d; X)$ is the space of all $f \in C^\infty(\mathbb{R}^d; X)$ for which the seminorms

$$[f]_{\alpha,\beta} := \sup_{x\in\mathbb{R}^d} \|x^\beta \partial^\alpha f(x)\| \tag{14.2}$$

are finite for all multi-indices $\alpha, \beta \in \mathbb{N}^d$. These seminorms define a locally convex topology $\mathscr{S}(\mathbb{R}^d; X)$ in which sequential convergence $f_n \to f$ is equivalent to the convergence $[f - f_n]_{\alpha,\beta} \to 0$ for all multi-indices $\alpha, \beta \in \mathbb{N}^d$. This topology is metrisable by the metric

$$d(f, g) := \sum_{\alpha,\beta\in\mathbb{N}^d} 2^{-|\alpha|-|\beta|} \frac{[f - g]_{\alpha,\beta}}{1 + [f - g]_{\alpha,\beta}}$$

which turns $\mathscr{S}(\mathbb{R}^d; X)$ into a complete metric space. Thus $\mathscr{S}(\mathbb{R}^d; X)$ has the structure of a Fréchet space. As a consequence of Lemma 1.2.19 or Lemma 14.2.1, the space $C_c^\infty(\mathbb{R}^d) \otimes X$ is dense in $L^p(\mathbb{R}^d; X)$ for $1 \leqslant p < \infty$. We will prove in Lemma 14.2.1 that $C_c^\infty(\mathbb{R}^d) \otimes X$ is sequentially dense in both $C_c^\infty(\mathbb{R}^d; X)$ and $\mathscr{S}(\mathbb{R}^d; X)$.

The space of continuous linear operators

$$\mathscr{S}'(\mathbb{R}^d; X) := \mathscr{L}(\mathscr{S}(\mathbb{R}^d), X)$$

is called the space of *tempered distributions* with values in X.

Let D be an open subset of $\mathbb{R}^d$. For $1 \leqslant p \leqslant \infty$ and $k \in \mathbb{N}$ the *Sobolev space* $W^{k,p}(D; X)$ is the space of functions $f \in L^p(D; X)$ whose weak derivatives $\partial^\alpha f$ of order $|\alpha| \leqslant k$ exist and belong to $L^p(D; X)$. Recall that a function $g \in L^1_{\rm loc}(D)$ is said to be the *weak derivative* of order α of f if

$$\int_D f(x)\partial^\alpha \phi(x)\,\mathrm{d}x = (-1)^{|\alpha|} \int_D g(x)\phi(x)\,\mathrm{d}x \quad \text{for all } \phi \in C_c^\infty(D).$$

Such a function g, if it exists, is unique. With respect to the norm

$$\|f\|_{W^{k,p}(D;X)} := \sum_{|\alpha|\leqslant k} \|\partial^\alpha f\|_p,$$

the space $W^{k,p}(D;X)$ is a Banach space. For $1 \leqslant p < \infty$ and $0 < s < 1$, the *Sobolev–Slobodetskii space* $W^{s,p}(\mathbb{R}^d;X)$ is the space of all functions $f \in L^p(\mathbb{R}^d;X)$ for which the seminorm

$$[f]_{W^{s,p}(D;X)} := \Big(\int_D\int_D \frac{\|f(x)-f(y)\|^p}{|x-y|^{sp+d}}\,\mathrm{d}x\,\mathrm{d}y\Big)^{1/p}$$

is finite. With respect to the norm

$$\|f\|_{W^{s,p}(D;X)} := \|f\|_p + [f]_{W^{s,p}(D;X)},$$

the space $W^{s,p}(\mathbb{R}^d;X)$ is a Banach space. By Theorem 2.5.17, for $1 \leqslant p <\in \infty$ and $0 < s < 1$ the real interpolation method gives

$$(L^p(\mathbb{R}^d;X), W^{1,p}(\mathbb{R}^d;X))_{\theta,p} = W^{\theta,p}(\mathbb{R}^d;X)$$

with equivalent norms.

For $1 \leqslant p \leqslant \infty$ and $s \in \mathbb{R}$ the *Bessel potential space* $H^{s,p}(\mathbb{R}^d;X)$ consists of all $u \in \mathscr{S}'(\mathbb{R}^d;X)$ for which the tempered distribution $J_s u \in \mathscr{S}'(\mathbb{R}^d;X)$ defined by

$$J_s u := ((1+4\pi^2|\cdot|^2)^{s/2}\widehat{u})^{\smallsmile}$$

belongs to $L^p(\mathbb{R}^d;X)$. Recall that the Fourier transform of u is defined by $\widehat{u}(f) = u(\widehat{f})$ for $f \in \mathscr{S}(\mathbb{R}^d;X)$, where the Fourier transform of a function $f \in L^1(\mathbb{R}^d;X)$ is defined as

$$\widehat{f}(\xi) = \mathscr{F}f(\xi) := \int_{\mathbb{R}} f(x)e^{-2\pi i x\cdot\xi}\,\mathrm{d}x, \quad \xi \in \mathbb{R}^d.$$

The inverse Fourier transform of a tempered distribution is defined similarly. With respect to the norm

$$\|u\|_{H^{s,p}(\mathbb{R}^d;X)} := \|J_s u\|_{L^p(\mathbb{R}^d;X)},$$

$H^{s,p}(\mathbb{R}^d;X)$ is a Banach space. The following continuous embeddings hold, the first being dense if $1 \leqslant p < \infty$:

$$\mathscr{S}(\mathbb{R}^d;X) \hookrightarrow H^{s,p}(\mathbb{R}^d;X) \hookrightarrow \mathscr{S}'(\mathbb{R}^d;X).$$

By Theorem 5.6.1, complex interpolation gives

$$[L^p(\mathbb{R}^d;X), W^{k,p}(\mathbb{R}^d;X)]_\theta = H^{\theta k,p}(\mathbb{R}^d;X)$$

with equivalent norms, provided X is a UMD space, $1 < p < \infty$, and $k \geqslant 1$ is an integer. Under the same assumptions, Theorem 5.6.9 gives

$$[H^{s_0,p}(\mathbb{R}^d;X), H^{s_1,p}(\mathbb{R}^d;X)]_\theta = H^{s_\theta,p}(\mathbb{R}^d;X)$$

with equivalent norms, for $s_0, s_1 \in \mathbb{R}$ satisfying $s_0 < s_1$ and with $s_\theta = (1-\theta)s_0 + \theta s_1$. Still for UMD spaces X and $1 < p < \infty$, by Theorem 5.6.11 for all integers $k \geqslant 1$ we have

$$W^{k,p}(\mathbb{R}^d; X) = H^{k,p}(\mathbb{R}^d; X)$$

with equivalent norms. For $k = 0$ we have the trivial identities

$$W^{0,p}(\mathbb{R}^d; X) = H^{0,p}(\mathbb{R}^d; X) = L^p(\mathbb{R}^d; X),$$

valid for all Banach spaces X and $1 \leqslant p < \infty$.

For $k \in \mathbb{N}$ the space $C_{\mathrm{b}}^k(\mathbb{R}^d; X)$ consists of all k-times continuously differentiable functions $f : \mathbb{R}^d \to X$ whose partial derivatives $\partial^\alpha f$ are bounded for all multi-indices $\alpha \in \mathbb{N}^d$ satisfying $|\alpha| \leqslant k$. With respect to the norm

$$\|f\|_{C_{\mathrm{b}}^k(\mathbb{R}^d;X)} := \sup_{|\alpha| \leqslant k} \|\partial^\alpha f\|_\infty,$$

the space $C_{\mathrm{b}}^k(\mathbb{R}^d; X)$ is a Banach space. We denote by $C_{\mathrm{ub}}^k(\mathbb{R}^d; X)$ its closed subspace of functions for which $\partial^\alpha f$ is uniformly continuous for all $|\alpha| \leqslant k$.

For $\theta \in (0,1)$ the space of *Hölder continuous* functions $C_{\mathrm{b}}^\theta(\mathbb{R}^d; X)$ consists of all bounded continuous $f : \mathbb{R}^d \to X$ for which the seminorm

$$[f]_{C_{\mathrm{b}}^\theta(\mathbb{R}^d;X)} := \sup_{x,y \in \mathbb{R}^d, x \neq y} \frac{\|f(x) - f(y)\|}{|x-y|^\theta}$$

is finite. With respect to the norm

$$\|f\|_{C_{\mathrm{b}}^\theta(\mathbb{R}^d;X)} := \|f\|_\infty + [f]_{C_{\mathrm{b}}^\theta(\mathbb{R}^d;X)}$$

the space $C_{\mathrm{b}}^\theta(\mathbb{R}^d; X)$ is a Banach space. The Banach space obtained by taking $\theta = 1$ in these expressions is called the space of *Lipschitz continuous* functions and is denoted by $\mathrm{Lip}(\mathbb{R}^d; X)$.

For $s = k + \theta$, with $k \in \mathbb{N}$ and $\theta \in (0,1)$, the space $C_{\mathrm{b}}^s(\mathbb{R}^d; X)$ is defined as the space of all $f \in C_{\mathrm{b}}^k(\mathbb{R}^d; X)$ for which $\partial^\alpha f \in C_{\mathrm{b}}^\theta(\mathbb{R}^d; X)$ for all multi-indices satisfying $|\alpha| \leqslant k$. With the norm

$$\|f\|_{C_{\mathrm{b}}^s(\mathbb{R}^d;X)} := \sup_{|\alpha| \leqslant k} \|\partial^\alpha f\|_{C_{\mathrm{b}}^\theta(\mathbb{R}^d;X)},$$

this space is a Banach space. For all $s \in [0, \infty)$ we have continuous embeddings

$$\mathscr{S}(\mathbb{R}^d; X) \hookrightarrow C_{\mathrm{b}}^s(\mathbb{R}^d; X) \hookrightarrow \mathscr{S}'(\mathbb{R}^d; X).$$

The first embedding is not dense, as non-zero constant functions cannot be approximated by Schwartz functions. For non-integers $s > 0$ we will use the notation

$$C_{\mathrm{ub}}^s(\mathbb{R}^d; X) = C_{\mathrm{b}}^s(\mathbb{R}^d; X).$$

14.2.b A density lemma and Young's inequality

Let $U \subseteq \mathbb{R}^d$ be an open set. The elements of the space $C_c^\infty(U;X)$ will be referred to as X-valued *test functions*. Sequential convergence in $C_c^\infty(U;X)$ is defined by insisting that $f_n \to f$ in $C_c^\infty(U;X)$ if there exists a compact set K of U containing the support of all f_n and $\|\partial^\alpha f - \partial^\alpha f\|_\infty \to 0$ for all $\alpha \in \mathbb{N}^d$. Related sequential notions, such as Cauchy sequences, are defined similarly. Note that if $f_n \to f$ in $C_c^\infty(U;X)$, then also $f_n \to f$ in $\mathscr{S}(\mathbb{R}^d;X)$, provided we extend the functions identically zero outside U.

Lemma 14.2.1. *The space $C_c^\infty(\mathbb{R}^d) \otimes X$ is sequentially dense in $C_c^\infty(\mathbb{R}^d;X)$ and $\mathscr{S}(\mathbb{R}^d;X)$.*

Proof. We prove the lemma in two steps.

Step 1 – We first prove that $C_c^\infty(\mathbb{R}^d;X)$ is sequentially dense in $\mathscr{S}(\mathbb{R}^d;X)$.

Let $f \in \mathscr{S}(\mathbb{R}^d;X)$. Let $\zeta \in C^\infty(\mathbb{R}^d)$ satisfy $\zeta \equiv 1$ on $\{\xi \in \mathbb{R}^d : |\xi| \leqslant 1\}$ and $\zeta \equiv 0$ on $\{\xi \in \mathbb{R}^d : |\xi| \geqslant 2\}$, and put $f_n(x) := \zeta(x/n)f(x)$. Then $f_n \in C_c^\infty(\mathbb{R}^d;X)$. To prove that $f_n \to f$ in $\mathscr{S}(\mathbb{R}^d;X)$ it suffices to check that for all multi-indices $\alpha, \beta \in \mathbb{N}^d$ we have

$$\lim_{n\to\infty} \big\|(\cdot)^\beta \partial^\alpha[(1-\zeta(\cdot/n))f]\big\|_\infty = 0.$$

The elementary verification is left to the reader.

Step 2 – Let $f \in C_c^\infty(\mathbb{R}^d;X)$. Choose bounded open sets $O, U, V \subseteq \mathbb{R}^d$ such that $\operatorname{supp}(f) \subseteq U \subseteq \overline{U} \subseteq V \subseteq \overline{V} \subseteq O$. We first claim that for every $\varepsilon > 0$ there exists a $g \in C_c^\infty(V) \otimes X$ such that $\|f - g\|_\infty \leqslant \varepsilon$. Fix $\varepsilon > 0$. Since $f(\overline{U}) \subseteq X$ is compact, it follows that there exist $x_1, \dots, x_n \in X$ such that $f(\overline{U}) \subseteq B(x_1,\varepsilon) \cup \dots B(x_n,\varepsilon)$. The sets $U_0 = O \backslash \overline{U}$ and $U_j = f^{-1}(B(x_j,\varepsilon)) \cap V$ for $j = 1, \dots, n$ define an open cover $(U_j)_{j=0}^n$ of $\overline{V}$. Let $(\psi_j)_{j=0}^n$ be a smooth partition of unity subordinate to this cover, i.e., $\psi_j \in C_c^\infty(U_j)$, $0 \leqslant \psi_j \leqslant 1$, and $\sum_{j=0}^n \psi_j \equiv 1$ on $\overline{V}$. Letting $g := \sum_{j=0}^n \psi_j \otimes x_j$ with $x_0 = 0$, for all $u \in \mathbb{R}^d$ we have

$$\|f(u) - g(u)\| \leqslant \sum_{j=0}^n \psi_j(u)\|f(u) - x_j\| < \varepsilon.$$

which proves the claim.

Let $\phi \in C_c^\infty(\mathbb{R}^d)$ satisfy $\int_{\mathbb{R}^d} \phi(u)\,\mathrm{d}u = 1$ and put $\phi_j(u) := j^d\phi(ju)$. By compactness, the exists an index $j_0 \in \mathbb{N}$ such that for all $j \geqslant j_0$ and all $g \in C_c^\infty(V;X)$ we have $\phi_j * g \in C_c^\infty(O;X)$ and, for all multi-indices $\alpha, \beta \in \mathbb{N}^d$,

$$[\phi_j * g - g]_{\alpha,\beta} \leqslant C_{O,\beta}\|\phi_j * \partial^\alpha g - \partial^\alpha g\|_\infty \to 0$$

as $j \to \infty$, by the uniform continuity of $\partial^\alpha g$. We conclude that for all such g and $j \geqslant 0$ we have $\phi_j * g \to g$ in $\mathscr{S}(\mathbb{R}^d;X)$. In particular, this holds with $g = f$. By the claim, we can find a sequence $(g_k)_{k\geqslant 1}$ in $C_c^\infty(V) \otimes X$ such that $\|f - g_k\|_\infty \to 0$. Now for each $j \geqslant j_0$ the functions $g_{kj} := \psi_j * g_k$ belong to

$C_c^\infty(O)\otimes X$, and by the above we have $g_{kj} \to g_k$ in $\mathscr{S}(\mathbb{R}^d;X)$. For appropriate $j_k \geqslant j_0$ we find that $g_{kj_k} \to g$ in $\mathscr{S}(\mathbb{R}^d;X)$. Since $g_{kj_k} \in C_c^\infty(O)\otimes X$, this proves density in $C_c^\infty(\mathbb{R}^d;X)$.

To prove density in $\mathscr{S}(\mathbb{R}^d;X)$ let $f\in\mathscr{S}(\mathbb{R}^d;X)$. By Step 1 there exists a sequence $(f_n)_{n\geqslant 1}$ in $C_c^\infty(\mathbb{R}^d;X)$ such that $f_n\to f$ in $\mathscr{S}(\mathbb{R}^d;X)$. Using Step 2, for every $n\geqslant 1$ choose a sequence $(f_{n,k})_{k\geqslant 1}$ in $C_c^\infty(\mathbb{R}^d)\otimes X$ such that $f_{n,k}\to f_n$ in $C_c^\infty(\mathbb{R}^d;X)$. Then in particular, $f_{n,k}\to f_n$ in $\mathscr{S}(\mathbb{R}^d;X)$. Since convergence in $\mathscr{S}(\mathbb{R}^d;X)$ is governed by countably many seminorms, a standard diagonal argument allows us to find a subsequence such that $f_{n,k_n}\to f$ in $\mathscr{S}(\mathbb{R}^d;X)$. □

As a corollary to the above lemma we record:

Proposition 14.2.2. *For all $p\in[1,\infty)$ and $s\in\mathbb{R}$ the space $C_c^\infty(\mathbb{R}^d)\otimes X$ is dense in $H^{s,p}(\mathbb{R}^d;X)$.*

Proof. By Proposition 5.6.4, for all $p\in[1,\infty)$ and $s\in\mathbb{R}$ we have a dense embedding $\mathscr{S}(\mathbb{R}^d;X)\hookrightarrow H^{s,p}(\mathbb{R}^d;X)$. □

We will often make use of the following version of Young's inequality, which extends a special case already proven in Lemma 1.2.30.

Lemma 14.2.3 (Young's inequality). *Let $p,q,r\in[1,\infty]$ be such that $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$. If $f\in L^p(\mathbb{R}^d;\mathscr{L}(X,Y))$ and $g\in L^q(\mathbb{R}^d;X)$, then $f*g\in L^r(\mathbb{R}^d;Y)$ and*

$$\|f*g\|_{L^r(\mathbb{R}^d;Y)} \leqslant \|f\|_{L^p(\mathbb{R}^d;\mathscr{L}(X,Y))}\|g\|_{L^q(\mathbb{R}^d;X)}.$$

Proof. For $1\leqslant q<\infty$, by density it suffices to prove the estimate for $g\in C_c^\infty(\mathbb{R}^d)\otimes X$, and if $q=\infty$, then $p=1$ and $r=\infty$ and it suffices to prove the required estimate for $f\in C_c^\infty(\mathbb{R}^d)\otimes\mathscr{L}(X,Y)$. In either case, $f*g$ is strongly measurable and we have the bound $\|f*g\|\leqslant\|f\|*\|g\|$. The estimate then follows from the scalar version of Young's inequality. □

We recall from Section 1.3 that the *variation* of an operator-valued measure $\Phi:\mathscr{A}\to\mathscr{L}(X,Y)$, where $(S,\mathscr{A})$ is a measurable space, is the measure $\|\Phi\|:\mathscr{A}\to[0,\infty]$ given by

$$\|\Phi\|(A)=\sup_\pi\sum_{B\in\pi}\|\Phi(B)\|,$$

the supremum being taken over all finite disjoint partitions π of the set $A\in\mathscr{A}$; the is taken in $\mathscr{L}(X,Y)$. We say that Φ has *bounded variation* if $\|\Phi\|(S)<\infty$. For a strongly measurable function $f:S\to X$ such that

$$\int_S\|f(s)\|\,\mathrm{d}\|\Phi\|(s)<\infty,$$

the construction of the Bochner integral (see Section 1.2.a) can be repeated to define $\int_S f\,\mathrm{d}\Phi$ as an element of Y satisfying

$$\Big\| \int_S f\,\mathrm{d}\Phi \Big\| \leqslant \int_S \|f\|\,\mathrm{d}\|\Phi\|.$$

When $(S, \mathscr{A}, \mu)$ is a measure space, a simple example of an operator-valued measure with bounded variation is obtained by taking $\Phi(A) := \int_A \phi\,\mathrm{d}\mu$ with $\phi \in L^1(S, \mu; \mathscr{L}(X, Y))$. The total variation of this measure satisfies

$$\|\Phi\|(S) \leqslant \|\phi\|_{L^1(S,\mu;\mathscr{L}(X,Y))}.$$

Standard arguments show that $\int_S \|f\|_X\,\mathrm{d}\|\Phi\| < \infty$ if and only if $\phi f \in L^1(S; Y)$ and in that case

$$\int_S f\,\mathrm{d}\Phi = \int_S \phi f\,\mathrm{d}\mu.$$

Lemma 14.2.4 (Convolutions with measures). *Let $\Phi : \mathbb{R}^d \to \mathscr{L}(X, Y)$ be an operator-valued measure of bounded variation, and let $f \in L^p(\mathbb{R}^d; X)$. For almost all $x \in \mathbb{R}^d$ the integral $\int_{\mathbb{R}^d} f(x-y)\,\mathrm{d}\Phi(y)$ is well defined in the above sense, and the convolution*

$$\Phi * f(x) := \int_{\mathbb{R}^d} f(x-y)\,\mathrm{d}\Phi(y)$$

defines a function $\Phi * f \in L^p(\mathbb{R}^d; Y)$ which satisfies

$$\|\Phi * f\|_{L^p(\mathbb{R}^d;Y)} \leqslant \|\Phi\|(\mathbb{R}^d)\|f\|_{L^p(\mathbb{R}^d;X)}.$$

Proof. For $1 \leqslant p < \infty$, Minkowski's inequality (Proposition 1.2.22) implies

$$\begin{aligned}\Big\| x \mapsto \int_{\mathbb{R}^d} \|f(x-y)\|\,\mathrm{d}\|\Phi\|(y) \Big\|_{L^p(\mathbb{R}^d)} &\leqslant \int_{\mathbb{R}^d} \big\| x \mapsto f(x-y) \big\|_{L^p(\mathbb{R}^d;X)}\,\mathrm{d}\|\Phi\|(y) \\ &= \|f\|_{L^p(\mathbb{R}^d;X)}\|\Phi\|(\mathbb{R}^d).\end{aligned}$$

For $p = \infty$, the same holds for trivial reasons. It follows that for almost all $x \in \mathbb{R}^d$ the integral $\Phi * f(x) = \int_{\mathbb{R}^d} f(x-y)\,\mathrm{d}\Phi(y)$ is well defined in Y. By approximation with simple functions it is seen that $\Phi * f$ is strongly measurable, and since

$$\|\Phi * f(x)\| \leqslant \int_{\mathbb{R}^d} \|f(x-y)\|\,\mathrm{d}\|\Phi\|(y),$$

the required estimate also follows. □

14.2.c Inhomogeneous Littlewood–Paley sequences

We now introduce one of our main technical tools, which allows us to break up a function spectrally into pieces with control on their frequencies.

Let Φ denote the set of all Schwartz functions $\varphi \in \mathscr{S}(\mathbb{R}^d)$ with the following properties:

(i) $0 \leqslant \widehat{\varphi}(\xi) \leqslant 1, \quad \xi \in \mathbb{R}^d$,
(ii) $\widehat{\varphi}(\xi) = 1$ if $|\xi| \leqslant 1$,
(iii) $\widehat{\varphi}(\xi) = 0$ if $|\xi| \geqslant \frac{3}{2}$.

Such functions can be constructed in a similar way as in Lemma 5.5.21.

Remark 14.2.5. If $\phi \in \Phi$, the function $\psi \in \mathscr{S}(\mathbb{R}^d)$ given by

$$\widehat{\psi}(\xi) := \widehat{\varphi}(\xi) - \widehat{\varphi}(2\xi)$$

is a smooth Littlewood–Paley function in the sense of Definition 5.5.20, i.e.,

(i) $\widehat{\psi}$ is smooth, non-negative, and supported in $\{\xi \in \mathbb{R}^d : \frac{1}{2} \leqslant |\xi| \leqslant 2\}$;
(ii) $\sum_{k\in\mathbb{Z}} \widehat{\psi}(2^{-k}\xi) = 1$ for all $\xi \in \mathbb{R}^d \setminus \{0\}$.

Remark 14.2.6. It is possible to choose the function φ is real and even (or equivalently $\widehat{\varphi}$ real and even). In that case it would be possible to use real Banach spaces in several of the definitions and results of this chapter. For instance if $f \in L^p(\mathbb{R}^d; X)$ or even $\mathscr{S}'(\mathbb{R}^d; X)$, then $\varphi * f$ can be defined without using any complex structure.

Definition 14.2.7 (Inhomogeneous Littlewood–Paley sequence). *The* inhomogeneous Littlewood–Paley sequence *associated with a function $\varphi \in \Phi$ is the sequence $(\varphi_k)_{k\geqslant 0}$ in $\mathscr{S}(\mathbb{R}^d)$ given by*

$$\begin{aligned} \widehat{\varphi}_0(\xi) &:= \widehat{\varphi}(\xi), & k &= 0,\ \xi \in \mathbb{R}^d, \\ \widehat{\varphi}_k(\xi) &:= \widehat{\varphi}(2^{-k}\xi) - \widehat{\varphi}(2^{-k+1}\xi), & k &\geqslant 1,\ \xi \in \mathbb{R}^d. \end{aligned} \tag{14.3}$$

Note the scaling property

$$\widehat{\varphi}_k(\xi) = \widehat{\varphi}_1(2^{-k+1}\xi), \quad k \geqslant 1, \tag{14.4}$$

and the telescoping properties

$$\sum_{k=0}^{n} \widehat{\varphi}_k(\xi) = \widehat{\varphi}_0(2^{-n}\xi), \quad \sum_{k\geqslant 0} \widehat{\varphi}_k(\xi) = 1. \tag{14.5}$$

We will often use the simple L^1-norm identity

$$\Big\| \sum_{k=0}^{n} \varphi_k \Big\|_1 = \int_{\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} \widehat{\varphi}_0(2^{-n}\xi)\,\mathrm{d}\xi \Big|\,\mathrm{d}x = 2^n \int_{\mathbb{R}^d} |\varphi_0(2^n x)|\,\mathrm{d}x = \|\varphi_0\|_1, \tag{14.6}$$

which implies

$$\|\varphi_k\|_1 = \Big\| \sum_{k=0}^{n} \varphi_k - \sum_{k=0}^{n-1} \varphi_k \Big\|_1 \leqslant 2\|\varphi_0\|_1, \quad k \geqslant 1. \tag{14.7}$$

The adjective ‘inhomogeneous’ refers to the special role played by the function φ_0 whose support contains an open neighbourhood of the origin.

Inhomogeneous Littlewood–Paley sequences will be used to define the classes of Besov spaces and Triebel–Lizorkin spaces. Up to equivalent norms, the definitions of these spaces turn out to be independent of the particular inhomogeneous Littlewood–Paley sequence chosen. This allows us to fix an arbitrary such sequence once and for all and always work with that given sequence. In order to avoid endless repetitions we therefore make the following convention.

Convention 14.2.8. *Throughout this entire chapter, $(\varphi_k)_{k\in\mathbb{N}}$ denotes the inhomogeneous Littlewood–Paley sequence associated with a function $\varphi \in \Phi$ which we fix once and for all. Whenever this is useful, we extend the index set of the sequence to include the negative integers by setting*

$$\phi_k \equiv 0, \quad k = -1, -2, \ldots$$

Constants in estimates involving a Littlewood–Paley sequences or spaces defined by using them will often also depend on the generating function $\varphi \in \Phi$, but since it is considered to be fixed we will not express these dependencies in our estimates.

Let us collect some easy properties of inhomogeneous Littlewood–Paley sequences. It is immediate to check the Fourier support property

$$\widehat{\varphi}_k(\xi) \equiv 1 \ \text{ for } \ \frac{3}{4} \cdot 2^k \leqslant |\xi| \leqslant 2^k, \quad k \geqslant 1, \tag{14.8}$$

and

$$\operatorname{supp} \widehat{\varphi}_k \subseteq \{\xi \in \mathbb{R}^d : 2^{k-1} \leqslant |\xi| \leqslant 3 \cdot 2^{k-1}\}, \qquad k \geqslant 1. \tag{14.9}$$

In particular we have the disjointness property

$$\operatorname{supp} \widehat{\varphi}_j \cap \operatorname{supp} \widehat{\varphi}_k = \varnothing, \quad |j-k| \geqslant 2, \tag{14.10}$$

which implies the orthogonality properties

$$\widehat{\varphi}_j \widehat{\varphi}_k = 0 \ \text{ and } \ \varphi_j * \varphi_k = 0, \quad |j-k| \geqslant 2. \tag{14.11}$$

From (14.5) and (14.11) we infer

$$\sum_{j=-1}^{1} \widehat{\varphi}_{k+j} \equiv 1 \ \text{ on } \ \operatorname{supp}(\widehat{\varphi}_k), \quad k \geqslant 0, \tag{14.12}$$

using the convention $\varphi_{-1} = 0$ for the case $k = 0$.

By Proposition 2.4.32, for $\psi \in \mathscr{S}(\mathbb{R}^d)$ and $u \in \mathscr{S}'(\mathbb{R}^d; X)$ the convolution

$$\psi * u = \mathscr{F}^{-1}(\widehat{\psi}\widehat{f}) \tag{14.13}$$

is well defined as element of $C^\infty(\mathbb{R}^d; X)$ and as such it has at most polynomial growth. For later use we record the following useful consequence:

Lemma 14.2.9. *Every $f \in \mathscr{S}'(\mathbb{R}^d; X)$ with compact Fourier support belongs to $C^\infty(\mathbb{R}^d; X)$ and has at most polynomial growth.*

Proof. This follows from Proposition 2.4.32 by writing $f = f * g$ with $g \in \mathscr{S}(\mathbb{R}^d)$ satisfying $\widehat{g} \equiv 1$ on $\mathrm{supp}(f)$. □

Returning to the main line of development, by applying (14.13) to the convolutions $\varphi_k * u$, the latter can be identified with distributions in $\mathscr{S}'(\mathbb{R}^d; X)$ and we have the following result:

Lemma 14.2.10. *Let $E = \mathscr{S}(\mathbb{R}^d; X)$ or $E = \mathscr{S}'(\mathbb{R}^d; X)$. For all $f \in E$ we have*

$$f = \sum_{k\geqslant 0} \varphi_k * f = \sum_{\ell=-1}^{1} \sum_{k\geqslant 0} \varphi_{k+\ell} * \varphi_k * f$$

with convergence of the sums in E.

Proof. The second identity follows by applying the first twice and (14.11). It thus remains to prove the first identity.

By the second identity in (14.5), (14.13), and the continuity of the Fourier transform on E proved in Proposition 2.4.22, it suffices to show that $\sum_{k\geqslant 0} \widehat{\varphi}_k g = g$ for all $g \in E$, with convergence of the sum in E.

First suppose that $g \in \mathscr{S}(\mathbb{R}^d; X)$. In view of the first identity in (14.5) we must to show that, for arbitrary multi-indices α, β,

$$\lim_{n\to\infty} \big\| (\cdot)^\beta \partial^\alpha [(1 - \widehat{\varphi}(2^{-n}\cdot))g] \big\|_\infty = 0.$$

This is elementary and left to the reader.

Next suppose that $g \in \mathscr{S}'(\mathbb{R}^d; X)$. Fix a function $\psi \in \mathscr{S}(\mathbb{R}^d)$. We need to check that $\sum_{k\geqslant 0} g(\psi\widehat{\varphi}_k) = g(\psi)$. For this it suffices to check that $\sum_{k\geqslant 0} \psi\widehat{\varphi}_k = \psi$ in $\mathscr{S}(\mathbb{R}^d)$, which is the content of the previous case. □

As a first application of Littlewood–Paley sequence techniques we prove a lemma that will be useful for establishing Fourier multiplier results in later subsections. For its proof we recall from Volume I the space

$$\check{L}^1(\mathbb{R}^d; X) := \{f \in L^\infty(\mathbb{R}^d; X) : \mathscr{F}^{-1} f \in L^1(\mathbb{R}^d; X)\},$$

where the inverse Fourier transforms is viewed as an element of $\mathscr{S}'(\mathbb{R}^d; X)$. With respect to the norm

$$\|f\|_{\check{L}^1(\mathbb{R}^d;X)} = \|\widehat{f}\|_{L^1(\mathbb{R}^d;X)},$$

$\check{L}^1(\mathbb{R}^d; X)$ is a Banach space. It enjoys the scaling invariance property

$$\|f(\lambda\cdot)\|_{\check{L}^1(\mathbb{R}^d;X)} = \|f\|_{\check{L}^1(\mathbb{R}^d;X)}, \quad \lambda > 0, \tag{14.14}$$

which is proved by a simple change of variables.

Lemma 14.2.11 (Integrability of Fourier transforms – I). *Let $f \in C^{d+1}(\mathbb{R}^d; X)$, and suppose that there exists an $\varepsilon > 0$ such that*

$$C_{f,d,\varepsilon} := \max_{|\alpha|\leqslant d+1} \sup_{\xi\in\mathbb{R}^d} (1+|\xi|^{|\alpha|+\varepsilon})\|\partial^\alpha f(\xi)\| < \infty.$$

Then $\widehat{f} \in L^1(\mathbb{R}^d; X)$ and $\|\widehat{f}\|_{L^1(\mathbb{R}^d;X)} \lesssim_{d,\varepsilon} C_{f,d,\varepsilon}$.

Note that $C_{f,d,\varepsilon}$ is trivially finite (for all $\varepsilon > 0$) if $f \in C^{d+1}(\mathbb{R}^d; X)$ has compact support.

Proof. In view of (14.5) we have $\|f\|_{\check{L}^1(\mathbb{R}^d;X)} \leqslant \sum_{j\geqslant 0} \|\widehat{\varphi}_j f\|_{\check{L}^1(\mathbb{R}^d;X)}$, and therefore it is enough to show that for all $j \geqslant 0$ we have

$$\|\widehat{\varphi}_j f\|_{\check{L}^1(\mathbb{R}^d;X)} \lesssim_d 2^{-(j-1)\varepsilon} C_{f,d,\varepsilon}. \tag{14.15}$$

First we consider indices $j \geqslant 1$. Setting $B := \{\xi \in \mathbb{R}^d : |\xi| \leqslant 1\}$, by (14.4) and (14.14) we obtain

$$\begin{aligned}\|\widehat{\varphi}_j f\|_{\check{L}^1(\mathbb{R}^d;X)} &= \|\widehat{\varphi}_1(\cdot) f(2^{j-1}\cdot)\|_{\check{L}^1(\mathbb{R}^d;X)} \\ &= \|\mathscr{F}(\widehat{\varphi}_1(\cdot) f(2^{j-1}\cdot))\|_{L^1(B;X)} + \|\mathscr{F}(\widehat{\varphi}_1(\cdot) f(2^{j-1}\cdot))\|_{L^1(\mathbb{R}^d\setminus B;X)} \\ &=: T_1 + T_2.\end{aligned}$$

The first term is easy to handle. Indeed, since $\|\mathscr{F}\|_{L^1\to L^\infty} \leqslant 1$ and $0 \leqslant \widehat{\varphi}_1 \leqslant 1$,

$$\begin{aligned}T_1 &\leqslant |B|\,\|\mathscr{F}(\widehat{\varphi}_1(\cdot) f(2^{j-1}\cdot))\|_\infty \\ &\leqslant |B|\|\widehat{\varphi}_1(\cdot) f(2^{j-1}\cdot)\|_{L^1(\mathbb{R}^d;X)} \leqslant |B|\|f(2^{j-1}\cdot)\|_{L^1(3B\setminus B;X)},\end{aligned}$$

using that $\widehat{\varphi}_1$ is supported in $3B \setminus B$ in the last step. Together with the assumed bound for f with $\alpha = 0$, for $\xi \in 3B \setminus B$ we have

$$\|f(2^{j-1}\xi)\| \leqslant \frac{C_{f,d,\varepsilon}}{1+2^{(j-1)\varepsilon}|\xi|^\varepsilon} \leqslant 2^{-(j-1)\varepsilon} C_{f,d,\varepsilon}.$$

Combining this with the previous estimate, this gives the bound $T_1 \leqslant 2^{-(j-1)\varepsilon} C_{f,d,\varepsilon}|3B \setminus B||B|$.

For the second term we use the finiteness of $C_d := \int_{\mathbb{R}^d\setminus B} |x|^{-d-1}\,\mathrm{d}x$ to obtain

$$T_2 \leqslant C_d \big\|\xi \mapsto |\xi|^{d+1}\mathscr{F}(\widehat{\varphi}_1 f(2^{j-1}\cdot))(\xi)\big\|_\infty.$$

By the estimate $|\xi|^{d+1} \lesssim_d \sum_{|\alpha|=d+1} |\xi^\alpha|$ and the identity $(2\pi i)^{|\alpha|}\xi^\alpha \mathscr{F}(g)(\xi) = \mathscr{F}(\partial^\alpha g)(\xi)$, for each $\xi \in \mathbb{R}^d$ we can further estimate

$$\big\|\,|\xi|^{d+1}\mathscr{F}(\widehat{\varphi}_1 f(2^{j-1}\cdot))(\xi)\big\|_X \lesssim_d \sum_{|\alpha|=d+1} \big\|(2\pi\xi)^\alpha \mathscr{F}(\widehat{\varphi}_1 f(2^{j-1}\cdot))(\xi)\big\|_X$$

$$= \sum_{|\alpha|=d+1} \big\|\mathscr{F}(\partial^\alpha[\widehat{\varphi}_1 f(2^{j-1}\cdot)])(\xi)\big\|_X.$$

Using that $\widehat{\varphi}_1$ is compactly supported we obtain

$$\big\|\mathscr{F}(\partial^\alpha[\widehat{\varphi}_1 f(2^{j-1}\cdot)])\big\|_\infty \leqslant \big\|\partial^\alpha[\widehat{\varphi}_1 f(2^{j-1}\cdot)]\big\|_1 \lesssim_d \big\|\partial^\alpha[\widehat{\varphi}_1 f(2^{j-1}\cdot)]\big\|_\infty.$$

After an application of the Leibniz rule it remains to estimate terms of the form $\partial^\beta\widehat{\varphi}_1\partial^\gamma[f(2^{j-1}\cdot)]$ with $|\beta|+|\gamma| = |\alpha| = d+1$. By the assumptions and the fact that $\widehat{\varphi}_1$ is supported in $3B \setminus B$,

$$\|\partial^\beta\widehat{\varphi}_1\partial^\gamma[f(2^{j-1}\cdot)]\|_\infty \lesssim_d \sup_{1\leqslant|\xi|\leqslant 3} \|2^{(j-1)|\gamma|}\partial^\gamma f(2^{j-1}\xi)\| \leqslant 2^{-(j-1)\varepsilon}C_{f,d,\varepsilon}.$$

It follows that $T_2 \lesssim_d 2^{-(j-1)\varepsilon}C_{f,d,\varepsilon}$. This proves (14.15) for $j \geqslant 1$. The case $j=0$ can be shown in a similar way, skipping the dilation step. □

For later reference we state the following consequence of Lemma 14.2.11.

Lemma 14.2.12. *Let $\lambda \geqslant 0$ and suppose that $f \in C^{d+1+\lceil\lambda\rceil}(\mathbb{R}^d; X)$ has support in the ball $B_R = \{\xi \in \mathbb{R}^d : |\xi| \leqslant R\}$. Then $(1+|\cdot|)^\lambda \widehat{f}(\cdot) \in L^1(\mathbb{R}^d; X)$ and*

$$\|(1+|\cdot|)^\lambda \widehat{f}(\cdot)\|_{L^1(\mathbb{R}^d;X)} \leqslant C_{R,d}\|f\|_{C_{\rm b}^{d+1+\lceil\lambda\rceil}(\mathbb{R}^d;X)}.$$

Proof. Upon replacing λ by $\lceil\lambda\rceil$ we may assume that $\lambda \in \mathbb{N}$. By Lemma 14.2.11 we have $\widehat{f} \in L^1(\mathbb{R}^d; X)$. Therefore it suffices to prove the estimate with $(1+|\cdot|)^\lambda$ replaced by $|\cdot|^\lambda$.

Arguing as before, since $|x|^\lambda \lesssim_d \sum_{|\beta|=\lambda}|x^\beta|$,

$$\big\||\cdot|^\lambda \widehat{f}\big\|_{L^1(\mathbb{R}^d;X)} \lesssim_{d,R} \sum_{|\beta|=\lambda} \|\widehat{\partial^\beta f}\|_{L^1(\mathbb{R}^d;X)}.$$

Therefore, the required result follows from Lemma 14.2.11 applied to $\partial^\beta f$. □

14.3 Interpolation of L^p-spaces with change of weights

When $(S, \mathscr{A}, \mu)$ is σ-finite measure space, we call a measurable function $w : S \to [0,\infty]$ a *weight* if $w(x) \in (0,\infty)$ for almost all $x \in S$. On earlier occasions (e.g., in Appendix J and Chapter 11) we have considered the weighted spaces

$$L^q(w;X) := \Big\{f : S \to X \text{ strongly measurable,} \\ \|f\|_{L^q(w;X)} := \Big(\int_S \|f(x)\|_X^q w(x)\,\mathrm{d}\mu(x)\Big)^{1/q} < \infty\Big\}.$$

For the present purposes, it is more convenient to introduce the variant

$$L^q_w(S;X) := \Big\{ f : S \to X \text{ strongly measurable}, \\ \|f\|_{L^q_w(S;X)} := \Big(\int_S \|f(x)w(x)\|_X^q \, \mathrm{d}\mu(x) \Big)^{1/q} < \infty \Big\}.$$

For $q < \infty$, this is just another way of expressing the same spaces with a different normalisation of the weight, namely $L^q_w(S;X) = L^q(w^q;X)$. However, using the usual modification for $q = \infty$, the first version reduces to just $L^\infty(w;X) = L^\infty(S;X)$ (since $\mathrm{d}\mu$ and $w\,\mathrm{d}\mu$ share the same zero sets), whereas $L^\infty_q(S;X)$ with norm $\|f\|_{L^\infty_w(S;X)} = \|fw\|_{L^\infty(S;X)}$ is a new space with non-trivial dependence on the weight w.

14.3.a Complex interpolation

Our first main result concerning these spaces is the following:

Theorem 14.3.1 (Stein–Weiss). *Let (Y_0, Y_1) be an interpolation couple of Banach spaces, let $q_0, q_1 \in [1,\infty]$ satisfy $\min\{q_0, q_1\} < \infty$. Let $(S, \mathscr{A}, \mu)$ be a σ-finite measure space, let w_0, w_1 be two weight functions on S, and let $\theta \in (0,1)$. Then*

$$[L^{q_0}_{w_0}(S;Y_0), L^{q_1}_{w_1}(S;Y_1)]_\theta = L^q_w(S;[Y_0,Y_1]_\theta)$$

isometrically, where

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \qquad w = w_0^{1-\theta} w_1^\theta.$$

We first record the simple:

Lemma 14.3.2. *In the setting of Theorem 14.3.1, if $f_n \to f$ in the norm of $L^{q_0}_{w_0}(S;Y_0) + L^{q_1}_{w_1}(S;Y_1)$, then a subsequence converges almost everywhere in the norm of $Y_0 + Y_1$ to the same limit function.*

Proof. We assume that $\|f_n - f\|_{L^{q_0}_{w_0}(S;Y_0)+L^{q_1}_{w_1}(S;Y_1)} \to 0$. Hence, for every n, there is a decomposition $f_n - f = f_n^0 + f_n^1$, where $\|f_n^j\|_{L^{q_j}_{w_j}(S;Y_j)} \to 0$ for $j = 0,1$. By the well known version of the Lemma in just one L^p space, a subsequence of f_n^0 converges to 0 almost everywhere in the norm of Y_0. By the same result, a further subsequence of f_n^1 also converges to 0 almost everywhere in the norm if Y_1. Thus, along this last subsequence, $f_n - f = f_n^0 + f_n^1$ converges to 0 almost everywhere in the norm of $Y_0 + Y_1$. □

Proof of Theorem 14.3.1. The unweighted version ($w_0 = w_1 = w \equiv 1$) of this result is contained in Theorem 2.2.6. We will reduce the weighted version to this special case. Let us abbreviate $Y := [Y_0, Y_1]_\theta$. For $n \in \mathbb{Z}_+$, we denote $S_n := \{n^{-1} \leqslant w_0, w_1 \leqslant n\}$. Then $\bigcup_{n=1}^\infty S_n$ exhausts S, up to a set of measure zero, by definition of weights.

Step 1 – $L^q_w(S;[Y_0,Y_1]_\theta) \subseteq [L^{q_0}_{w_0}(S;Y_0), L^{q_1}_{w_1}(S;Y_1)]_\theta$:
Let $f \in L^q_w(S;Y)$, and assume first assume that $\{f \neq 0\}$ is contained in S_n for some $n \in \mathbb{N}$. Thus

$$\phi := fw \in L^q(S;Y) = [L^{q_0}(S;Y_0), L^{q_1}(S;Y_1)]_\theta,$$

where the equality of space is Theorem 2.2.6, and hence $\phi = \Phi(\theta)$ for some $\Phi \in \mathscr{H}(L^{q_0}(S;Y_0), L^{q_1}(S;Y_1))$, where this notation of holomorphic functions on the unit strip with appropriate boundary behaviour is defined in Section C.2. The relation $\phi = \Phi(\theta)$ remains valid if we replace $\Phi(z)$ by $\Phi(z)\mathbf{1}_{E_n}$, and hence all the subsequent considerations can be restricted to E_n. In particular, multiplication by any power of w_0 or w_1 is then a bounded operation on any of the (weighted or not) L^p spaces appearing in this argument. Now

$$f = \phi w^{-1} = \Phi(\theta) w_0^{-(1-\theta)} w_1^{-\theta} = F(\theta),$$

where $F(z) := \Phi(z) w_0^{-(1-z)} w_1^{-z} \in \mathscr{H}(L^{q_0}(w_0;Y_0), L^{q_1}(w_1;Y_1))$. Qualitatively, the last inclusion is easy from the corresponding relation for Φ and the restriction of the supports on E_n, where all multiplications by powers of w_i are bounded. Quantitatively, we have

$$\begin{aligned} \|F(j+it)\|_{L^{q_j}(w_j;Y_j)} &= \|\Phi(j+it) w_0^{-(1-j)} w_1^{-j}\|_{L^{q_j}(w_j;Y_j)} \\ &= \|\Phi(j+it)\|_{L^{q_j}(S;Y_j)}, \qquad j = 0,1, \end{aligned}$$

thus, recalling that $\|F\|_{\mathscr{H}(X_0,X_1)} := \max_{j=0,1} \sup_{t\in\mathbb{R}} \|F(j+it)\|_{X_j}$,

$$\|F\|_{\mathscr{H}(L^{q_0}_{w_0}(S;Y_0), L^{q_1}_{w_1}(S;Y_1))} = \|\Phi\|_{\mathscr{H}(L^{q_0}(S;Y_0), L^{q_1}(S;Y_1))}, \tag{14.16}$$

and hence

$$\begin{aligned} \|f\|_{[L^{q_0}_{w_0}(S;Y_0), L^{q_1}_{w_1}(S;Y_1)]_\theta} &\leqslant \|F\|_{\mathscr{H}(L^{q_0}_{w_0}(S;Y_0), L^{q_1}_{w_1}(S;Y_1))} \\ &= \|\Phi\|_{\mathscr{H}(L^{q_0}(S;Y_0), L^{q_1}(S;Y_1))}. \end{aligned}$$

Taking the infimum over all Φ in this space with $\phi = \Phi(\theta)$, we obtain

$$\begin{aligned} \|f\|_{[L^{q_0}_{w_0}(S;Y_0), L^{q_1}_{w_1}(S;Y_1)]_\theta} &\leqslant \|\phi\|_{[L^{q_0}(S;Y_0), L^{q_1}(S;Y_1)]_\theta} \\ &= \|\phi\|_{L^q(S;Y)} = \|f\|_{L^q_w(S;Y)}. \end{aligned}$$

Recall that the previous estimate was obtained under the assumption that $f \in L^q_w(S;Y)$ satisfies $\{f \neq 0\} \subseteq S_n$. For a general $f \in L^q_w(S;Y)$, this bound holds with either $\mathbf{1}_{S_n} f$ of $\mathbf{1}_{S_n} f - \mathbf{1}_{S_m} f$ in place of f. Since $\mathbf{1}_{S_n} f \to f$ in $L^q_w(S;Y)$ by dominated convergence, it follows that $\mathbf{1}_{S_n} f$ is a Cauchy sequence, and hence convergent, in the interpolation space $[L^{q_0}_{w_0}(S;Y_0), L^{q_1}_{w_1}(S;Y_1)]_\theta$ and thus in the sum space $L^{q_0}_{w_0}(S;Y_0) + L^{q_1}_{w_1}(S;Y_1)$ by Lemma C.2.5. By Lemma 14.3.2, a subsequence converges almost everywhere to the same limit function. But it is clear that the a.e. limit is f, and hence

$$
\begin{aligned}
\|f\|_{[L^{q_0}_{w_0}(S;Y_0),L^{q_1}_{w_1}(S;Y_1)]_\theta} &= \lim_{n\to\infty} \|\mathbf{1}_{S_n} f\|_{[L^{q_0}_{w_0}(S;Y_0),L^{q_1}_{w_1}(S;Y_1)]_\theta} \\
&\leqslant \lim_{n\to\infty} \|\mathbf{1}_{S_n} f\|_{L^q_w(S;Y)} = \|f\|_{L^q_w(S;Y)}.
\end{aligned}
$$

Step 2 – $[L^{q_0}_{w_0}(S;Y_0), L^{q_1}_{w_1}(S;Y_1)]_\theta \subseteq L^q_w(S;[Y_0,Y_1]_\theta)$:
Let $f = F(\theta) \in [L^{q_0}_{w_0}(S;Y_0), L^{q_1}_{w_1}(S;Y_1)]_\theta$, where

$$
F \in \mathscr{H}(L^{q_0}_{w_0}(S;Y_0), L^{q_1}_{w_1}(S;Y_1)).
$$

As before, we first assume that $\{f \neq 0\} \subseteq S_n$, and then without loss of generality (multiplying by $\mathbf{1}_{E_n}$ if necessary) that $F(z)$ has the same property for every z. We can then reverse the previous reasoning. Defining

$$
\Phi(z) := F(z) w_0^{(1-z)} w_1^z,
$$

we check the same relation (14.16), and hence

$$
\begin{aligned}
\|f\|_{L^q(w;Y)} &= \|F(\theta)w\|_{L^q(S;Y)} = \|\Phi(\theta)\|_{[L^{q_0}(S;Y_0),L^{q_1}(S;Y_0)]_\theta} \\
&\leqslant \|\Phi\|_{\mathscr{H}(L^{q_0}(S;Y_0),L^{q_1}(S;Y_0))} = \|F\|_{\mathscr{H}(L^{q_0}_{w_0}(S;Y_0),L^{q_1}_{w_1}(S;Y_0))}.
\end{aligned}
$$

Taking the infimum over the relevant F with $F(\theta) = f$, we get

$$
\|f\|_{L^q_w(S;Y)} \leqslant \|f\|_{[L^{q_0}_{w_0}(S;Y_0),L^{q_1}_{w_1}(S;Y_0)]_\theta}, \quad \{f \neq 0\} \subseteq S_n. \tag{14.17}
$$

Consider next a general $f \in [L^{q_0}_{w_0}(S;Y_0), L^{q_1}_{w_1}(S;Y_0)]_\theta$. Multiplication by $\mathbf{1}_{S_n}$ contracts all L^p spaces, including weighted ones, and hence also the interpolation space $[L^{q_0}_{w_0}(S;Y_0), L^{q_1}_{w_1}(S;Y_1)]_\theta$ by Theorem C.2.6. Now (14.17) holds with $\mathbf{1}_{S_n} f$ in place of f, and hence

$$
\|\mathbf{1}_{S_n} f\|_{L^q(w;Y)} \leqslant \|\mathbf{1}_{S_n} f\|_{[L^{q_0}_{w_0}(S;Y_0),L^{q_1}_{w_1}(S;Y_0)]_\theta} \leqslant \|f\|_{[L^{q_0}_{w_0}(S;Y_0),L^{q_1}_{w_1}(S;Y_0)]_\theta}.
$$

But then monotone convergence shows that

$$
\|f\|_{L^q(w;Y)} = \lim_{n\to\infty} \|\mathbf{1}_{S_n} f\|_{L^q_w(S;Y)} \leqslant \|f\|_{[L^{q_0}_{w_0}(S;Y_0),L^{q_1}_{w_1}(S;Y_0)]_\theta}.
$$

This completes the proof. □

For easy reference later in this chapter, we state the special case of the previous result for sequence space with the weights $w_s(k) = 2^{ks}$ on the integers.

Proposition 14.3.3 (Complex interpolation of the spaces $\ell^q_{w_s}(Y)$). *Let (Y_0, Y_1) be an interpolation couple of Banach spaces, let $q_0, q_1 \in [1,\infty]$ satisfy $\min\{q_0,q_1\} < \infty$, and let $s_0, s_1 \in \mathbb{R}$ and $\theta \in (0,1)$. Then*

$$
[\ell^{q_0}_{w_{s_0}}(Y_0), \ell^{q_1}_{w_{s_1}}(Y_1)]_\theta = \ell^q_{w_s}([Y_0,Y_1]_\theta)
$$

isometrically, where $s = (1-\theta)s_0 + \theta s_1$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

Proof. The condition $s = (1-\theta)s_0 + \theta s_1$ is equivalent to $w_s = w_{s_0}^{1-\theta} w_{s_1}^\theta$; whence the Proposition is a special case of Theorem 14.3.1. □

14.3.b Real interpolation

We next turn to the case of real interpolation. Recall that for a Banach couple (X_0, X_1), the real interpolation space $(X_0, X_1)_{\theta,p}$ with $p \in [1,\infty]$ and $\theta \in (0,1)$, was introduced in Section C.3. Also recall from Theorem C.3.14 that if $p_0, p_1 \in [1,\infty]$ satisfy $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, then $(X_0, X_1)_{\theta,p} = (X_0, X_1)_{\theta,p_0,p_1}$ with equivalent norms, where the latter denotes the Lions–Peetre interpolation of X_0 and X_1 (second mean method). The main result of this section is as follows.

Theorem 14.3.4 (Stein–Weiss, real version). *Let (Y_0, Y_1) be an interpolation couple of Banach spaces, let $q_0, q_1 \in [1,\infty]$ satisfy $\min\{q_0, q_1\} < \infty$. Let $(S, \mathscr{A}, \mu)$ be a σ-finite measure space, let w_0, w_1 be two weight functions on S, and let $\theta \in (0,1)$. Then*

$$(L^{q_0}_{w_0}(S; Y_0), L^{q_1}_{w_1}(S; Y_1))_{\theta,q_0,q_1} = L^q_w(S; (Y_0, Y_1)_{\theta,q_0,q_1})$$

isometrically, where

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \qquad w = w_0^{1-\theta} w_1^{\theta}.$$

In particular,

$$(L^{q_0}_{w_0}(S; Y_0), L^{q_1}_{w_1}(S; Y_1))_{\theta,q} = L^q_w(S; (Y_0, Y_1)_{\theta,q}),$$

with equivalent norms.

Proof. The unweighted version ($w_0 = w_1 = w \equiv 1$) of this result is contained in Theorem 2.2.10. We will reduce the weighted version to this special case. Let us abbreviate $Y := (Y_0, Y_1)_{\theta,q_0,q_1}$. As in the proof of Theorem 14.3.1 we denote $S_n := \{n^{-1} \leqslant w_0, w_1 \leqslant n\}$ for each $n \in \mathbb{Z}_+$, and observe that $\bigcup_{n=1}^{\infty} S_n$ exhausts S, up to a set of measure zero, by definition of weights.

Step 1 – $L^q_w(S; Y) \subseteq (L^{q_0}_{w_0}(S; Y_0), L^{q_1}_{w_1}(S; Y_1))_{\theta,q_0,q_1}$:
Let $f \in L^q(w; Y)$, and assume first that $\{f \neq 0\}$ is contained in S_n for some $n \in \mathbb{N}$. We also make the technical assumption that the weights w_j are discrete, in that they only take values of the form ρ^k, where $\rho > 1$ is a fixed number, and $k \in \mathbb{Z}$. This plays a role in the representation (14.18) below. Now

$$\phi := fw \in L^q(S; Y) = (L^{q_0}(S; Y_0), L^{q_1}(S; Y_1))_{\theta,q_0,q_1},$$

where the equality of spaces is Theorem 2.2.10. Hence, by Definition C.3.10 of the Lions–Peetre interpolation method $(\ ,\ )_{\theta,q_0,q_1}$, for some strongly measurable $\Phi : (0,\infty) \to L^{q_0}(S; Y_0) \cap L^{q_1}(S; Y_1)$, we have

$$\phi = \int_0^\infty \Phi(t) \frac{\mathrm{d}t}{t},$$

where $t^{j-\theta}\Phi(t) \in L^{q_j}(\,\mathrm{d}t/t; L^{q_j}(S;Y_j))$ for $j = 0,1$, and (as a consequence) the improper integral converges in $L^{q_0}(S;Y_0)+L^{q_1}(S;Y_1)$. Multiplying by $\mathbf{1}_{S_n}$ if necessary, we may assume that each $\Phi(t)$ is also supported on S_n.

Choosing the auxiliary weight $W := w_0^{-1}w_1$, we then have

$$f = \phi w^{-1} = \int_0^\infty \Phi(t)w^{-1}\frac{\mathrm{d}t}{t} = \int_0^\infty \Phi(tW)w^{-1}\frac{\mathrm{d}t}{t} =: \int_0^\infty F(t)\frac{\mathrm{d}t}{t}.$$

On S_n, both w_j are bounded from above and below. Due to the technical assumption on the discreteness of their ranges, both these weights, and hence W, only take finitely many possible value on S_n. Hence

$$F(t) = \Phi(tW)w^{-1} = \sum_{k=1}^K \mathbf{1}_{E_k}\Phi(t\alpha_k)\beta_k^{-1} \tag{14.18}$$

for some $\alpha_k, \beta_k \in (0,\infty)$ and sets $E_k \subseteq S_n$, from which it is immediate that also $F : (0,\infty) \to L^{q_0}(S;Y_0) \cap L^{q_1}(S;Y_1)$ is strongly measurable. This still remains true with each $L^{q_j}(S;Y_j)$ replaced by $L^{q_j}(w_j;Y_j)$ since the intersections of these spaces with functions supported on S_n coincide. With these qualitative issues out of the way, we make the quantitative observation

$$\begin{aligned}
&\int_0^\infty \|t^{j-\theta}F(t)\|^{q_j}_{L^{q_j}_{w_j}(S;Y_j)}\frac{\mathrm{d}t}{t} \\
&\quad= \int_0^\infty \|t^{j-\theta}\Phi(tW)w^{-1}w_j\|^{q_j}_{L^{q_j}(S;Y_j)}\frac{\mathrm{d}t}{t} \\
&\quad= \int_0^\infty \|W^{\theta-j}w^{-1}w_j t^{j-\theta}\Phi(t)\|^{q_j}_{L^{q_j}(S;Y_j)}\frac{\mathrm{d}t}{t} \\
&\quad= \int_0^\infty \|t^{j-\theta}\Phi(t)\|^{q_j}_{L^{q_j}(S;Y_j)}\frac{\mathrm{d}t}{t},
\end{aligned} \tag{14.19}$$

where in the last step our choice $W := w_0^{-1}w_1$ and the assumption $w = w_0^{1-\theta}w_1^{\theta}$ show that $W^{\theta-j}w^{-1}w_j \equiv 1$ for both $j = 0,1$ (and indeed having this identity dictates our choice of the auxiliary W).

Now, by the Lions–Peetre method, we have

$$\begin{aligned}
\|f\|_{(L^{q_0}_{w_0}(S;Y_0),L^{q_1}_{w_1}(S;Y_1))_{\theta,q_0,q_1}} &\leqslant \sup_{j=0,1}\|t \mapsto t^{j-\theta}F(t)\|_{L^{q_j}(\,\mathrm{d}t/t;L^{q_j}_{w_j}(S;Y_j))} \\
&= \sup_{j=0,1}\|t \mapsto t^{j-\theta}\Phi(t)\|_{L^{q_j}(\,\mathrm{d}t/t;L^{q_j}(S;Y_j))},
\end{aligned}$$

and taking the infimum over all such Φ shows that

$$\begin{aligned}
\|f\|_{(L^{q_0}_{w_0}(S;Y_0),L^{q_1}_{w_1}(S;Y_1))_{\theta,q_0,q_1}} &\leqslant \|\phi\|_{(L^{q_0}(S;Y_0),L^{q_1}(S;Y_1))_{\theta,q_0,q_1}} \\
&= \|\phi\|_{L^q(S;Y)} = \|f\|_{L^q_w(S;Y)}.
\end{aligned}$$

We proved this assuming that $\{f \neq 0\} \subseteq S_n$. For arbitrary $f \in L^q_w(S;Y)$, this is true with either $\mathbf{1}_{S_n}f$ or $\mathbf{1}_{S_n}f - \mathbf{1}_{S_m}f$ in place of f. It follows that $\mathbf{1}_{S_n}f$

is a Cauchy sequence, and hence convergent, in $(L^{q_0}_{w_0}(S;Y_0), L^{q_1}_{w_1}(S;Y_1))_{\theta,q_0,q_1}$, and thus in $L^{q_0}_{w_0}(S;Y_0) + L^{q_1}_{w_1}(S;Y_1)$ by the very Definition C.3.10 (recall that $f \in (X_0, X_1)_{\theta,q_0,q_1}$ is given by an integral that converges in $X_0 + X_1$). By Lemma 14.3.2, a subsequence converges almost everywhere to the same limit, and hence this limit must be f. Thus $f \in (L^{q_0}_{w_0}(S;Y_0), L^{q_1}_{w_1}(S;Y_1))_{\theta,q_0,q_1}$, and

$$\begin{aligned}\|f\|_{(L^{q_0}_{w_0}(S;Y_0),L^{q_1}_{w_1}(S;Y_1))_{\theta,q_0,q_1}} &= \lim_{n\to\infty} \|\mathbf{1}_{S_n} f\|_{(L^{q_0}_{w_0}(S;Y_0),L^{q_1}_{w_1}(S;Y_1))_{\theta,q_0,q_1}} \\ &\leqslant \lim_{n\to\infty} \|\mathbf{1}_{S_n} f\|_{L^q_w(S;Y)} = \|f\|_{L^q_w(S;Y)}.\end{aligned}$$

We still had the additional hypothesis on the discreteness of the ranges of both w_j. For arbitrary weights w_j and $\rho > 1$, we consider

$$w_{j,\rho} := \sup\{\rho^k : \rho^k \leqslant w_j, k \in \mathbb{Z}\},$$

which clearly satisfy the discreteness property, as well as $w_{j,\rho} \leqslant w_j \leqslant \rho w_{j,\rho}$. Hence

$$\|f\|_{L^{q_j}_{w_j}(S;Y_j)} \leqslant \rho \|f\|_{L^{q_j}_{w_j^\rho}(S;Y_j)}$$

and Theorem C.3.16 gives the first estimate in

$$\begin{aligned}\|f\|_{(L^{q_0}_{w_0}(S;Y_0),L^{q_1}_{w_1}(S;Y_1))_{\theta,q_0,q_1}} &\leqslant \rho^{1-\theta}\rho^\theta \|f\|_{(L^{q_0}_{w_{0,\rho}}(S;Y_0),L^{q_1}_{w_{1,\rho}}(S;Y_1))_{\theta,q_0,q_1}} \\ &= \rho \|f\|_{L^q_{w_{0,\rho}^{1-\theta} w_{1,\rho}^\theta}(S;Y)} \\ &\leqslant \rho \|f\|_{L^q(w;Y)}.\end{aligned}$$

Taking the limit $\rho \to 1$, we finally deduce

$$\|f\|_{(L^{q_0}_{w_0}(S;Y_0),L^{q_1}_{w_1}(S;Y_1))_{\theta,q_0,q_1}} \leqslant \|f\|_{L^q_w(S;Y)}$$

unconditionally.

Step 2 – $(L^{q_0}_{w_0}(S;Y_0), L^{q_1}_{w_1}(S;Y_1))_{\theta,q_0,q_1} \subseteq L^q_w(S;Y)$:
Let $f \in (L^{q_0}_{w_0}(S;Y_0), L^{q_1}_{w_1}(S;Y_1))_{\theta,q_0,q_1}$. We make the same initial assumptions on both f and the weights w_j as in the previous part. By definition, we have $f = \int_0^\infty F(t) \frac{\mathrm{d}t}{t}$ with $t^{j-\theta}F(t) \in L^{q_j}(\mathrm{d}t/t; L^{q_j}_{w_j}(S;Y_j))$. Working the previous computations backwards, we find that

$$\phi := fw = \int_0^\infty F(t) w \frac{\mathrm{d}t}{t} = \int_0^\infty F(tW^{-1}) w \frac{\mathrm{d}t}{t} =: \int_0^\infty \Phi(t) \frac{\mathrm{d}t}{t},$$

where Φ satisfies the relevant measurability conditions (by the structural assumptions on the weights) and the quantitative relation (14.19). We conclude that

$$\begin{aligned}\|\phi\|_{(L^{q_0}(S;Y_0),L^{q_1}(S;Y_1))_{\theta,q_0,q_1}} &\leqslant \sup_{j=0,1} \|t^{j-\theta}\Phi(t)\|_{L^{q_j}(\mathrm{d}t/t;L^{q_j}(S;Y_j))} \\ &= \sup_{j=0,1} \|t^{j-\theta}F(t)\|_{L^{q_j}(\mathrm{d}t/t;L^{q_j}_{w_j}(S;Y_j))},\end{aligned}$$

and taking the infimum over all relevant F,

$$\|f\|_{L^q_w(S;Y)} = \|\phi\|_{L^q(S;Y)} = \|\phi\|_{(L^{q_0}(S;Y_0),L^{q_1}(S;Y_1))_{\theta,q_0,q_1}} \leqslant \|f\|_{(L^{q_0}_{w_0}(S;Y_0),L^{q_1}_{w_1}(S;Y_1))_{\theta,q_0,q_1}}.$$

For a general f in the interpolation space, applying the previous conclusion to $\mathbf{1}_{S_n}f$ in place of f, we have

$$\|\mathbf{1}_{S_n}f\|_{L^q_w(S;Y)} \leqslant \|\mathbf{1}_{S_n}f\|_{(L^{q_0}_{w_0}(S;Y_0),L^{q_1}_{w_1}(S;Y_1))_{\theta,q_0,q_1}} \leqslant \|f\|_{(L^{q_0}_{w_0}(S;Y_0),L^{q_1}_{w_1}(S;Y_1))_{\theta,q_0,q_1}},$$

since multiplication by $\mathbf{1}_{S_n}$ is clearly contractive on each $L^{q_j}(w_j;Y_j)$, and hence on the interpolation space by Theorem C.3.16. It then follows from monotone convergence that

$$\|f\|_{L^q_w(S;Y)} = \lim_{n\to\infty}\|\mathbf{1}_{S_n}f\|_{L^q_w(S;Y)} \leqslant \|f\|_{(L^{q_0}_{w_0}(S;Y_0),L^{q_1}_{w_1}(S;Y_1))_{\theta,q_0,q_1}}.$$

Finally, the discreteness assumption on the weights can be removed by the same considerations as in the previous part: For general weights w_j and the auxiliary discrete $w_{j,\rho}$ as in the previous part, we have

$$\begin{aligned}\|f\|_{L^q_w(S;Y)} = \|f\|_{L^q_{w_0^{1-\theta}w_1^\theta}(S;Y)} &\leqslant \rho^{(1-\theta)+\theta}\|f\|_{L^q_{w_{0,\rho}^{1-\theta}w_{1,\rho}^\theta}(S;Y)}\\ &\leqslant \rho\|f\|_{((L^{q_0}_{w_{0,\rho}}(S;Y_0),L^{q_1}_{w_{1,\rho}}(S;Y_1))_{\theta,q_0,q_1}}\\ &\leqslant \rho\|f\|_{((L^{q_0}_{w_0}(S;Y_0),L^{q_1}_{w_1}(S;Y_1))_{\theta,q_0,q_1}},\end{aligned}$$

and taking the limit $\rho\to 1$ completes the proof. □

For applications of the real interpolation theorem to Besov spaces, it is useful to include a version that is genuine variant, rather than just a special case of the previous theorem. This version is concerned with the particular case of $S=\mathbb{N}$ or $S=\mathbb{Z}$ with the exponential weights $w_s(k)=2^{ks}$, and restricting to just one range space $Y_0=Y_1=Y$. Remarkably, under these circumstances the condition $\frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}$ of Theorem 14.3.4 can be omitted:

Proposition 14.3.5 (Real interpolation of the spaces $\ell^q_{w_s}(Y)$). *Let $p,q_0,q_1\in[1,\infty]$, let $s_0,s_1\in\mathbb{R}$ satisfy $s_0\neq s_1$, let $\theta\in(0,1)$, and let $s=(1-\theta)s_0+\theta s_1$. Then*

$$(\ell^{q_0}_{w_{s_0}}(Y),\ell^{q_1}_{w_{s_1}}(Y))_{\theta,p} = \ell^p_{w_s}(Y) \quad \textit{with equivalent norms,}$$

with constants in the norm estimates only depending on θ, p, s_0, s_1. Moreover, for all $y\in\ell^{q_0}_{w_{s_0}}(Y)\cap\ell^{q_1}_{w_{s_1}}(Y)$ we have

$$\|y\|_{\ell^p_{w_s}(Y)} \leqslant C\|y\|^{1-\theta}_{\ell^{q_0}_{w_{s_0}}(Y)}\|y\|^{\theta}_{\ell^{q_1}_{w_{s_1}}(Y)},$$

where C only depends on s_0, s_1, θ.

Proof. We will present the details for $S = \mathbb{N}$, as the case $S = \mathbb{Z}$ is proved in the same way. By interchanging the roles of $\ell^{q_0}_{w_{s_0}}(Y)$ and $\ell^{q_1}_{w_{s_1}}(Y)$ if necessary, without loss of generality we may assume that $s_0 > s_1$.

Since $\ell^{q_0}_{w_{s_0}}(Y) \hookrightarrow \ell^{\infty}_{w_{s_0}}(Y)$ and $\ell^{q_1}_{w_{s_1}}(Y) \hookrightarrow \ell^{\infty}_{w_{s_1}}(Y)$ continuously, real interpolation (Theorem C.3.3) gives $(\ell^{q_0}_{w_{s_0}}(Y), \ell^{q_1}_{w_{s_1}}(Y))_{\theta,p} \hookrightarrow (\ell^{\infty}_{w_{s_0}}(Y), \ell^{\infty}_{w_{s_1}}(Y))_{\theta,p}$ continuously. Hence to show that $(\ell^{q_0}_{w_{s_0}}(Y), \ell^{q_1}_{w_{s_1}}(Y))_{\theta,p}$ embeds into $\ell^{p}_{w_s}(Y)$ it suffices to consider the case $q_0 = q_1 = \infty$. If $y = y^{(0)} + y^{(1)} \in \ell^{\infty}_{w_{s_0}}(Y) + \ell^{\infty}_{w_{s_1}}(Y)$, then

$$\|y_k\| \leqslant \|y_k^{(0)}\| + \|y_k^{(1)}\| \leqslant 2^{-ks_0}\|y^{(0)}\|_{\ell^{\infty}_{w_{s_0}}(Y)} + 2^{-ks_1}\|y^{(1)}\|_{\ell^{\infty}_{w_{s_1}}(Y)}.$$

Multiplying with 2^{ks_0} and taking the infimum over all admissible pairs $(y^{(0)}, y^{(1)})$, we find

$$2^{ks_0}\|y_k\| \leqslant K(2^{k(s_0-s_1)}, y)$$

using the notation of Section C.3. In combination with the identity $\theta(s_1 - s_0) = s - s_0$ and the fact that the K-functional is non-decreasing, this gives

$$\begin{aligned}
\|y\|_{\ell^p_{w_s}} &\leqslant \Big(\sum_{k\geqslant 0} |2^{k(s-s_0)} K(2^{k(s_0-s_1)}, y)|^p\Big)^{1/p} \\
&\leqslant C_0\Big(\sum_{k\geqslant 0} \int_{2^{k(s_0-s_1)}}^{2^{(k+1)(s_0-s_1)}} |t^{-\theta} K(2^{k(s_0-s_1)}, y)|^p \,\frac{\mathrm{d}t}{t}\Big)^{1/p} \\
&\leqslant C_0\Big(\int_0^\infty |t^{-\theta} K(t, y)|^p \,\frac{\mathrm{d}t}{t}\Big)^{1/p} = C_0\|y\|_{(\ell^{q_0}_{w_{s_0}}(Y), \ell^{q_1}_{w_{s_1}}(Y))_{\theta,p}},
\end{aligned}$$

where $C_0 = \frac{(\theta p)^{1/p}}{(1-2^{-(s_0-s)p})^{1/p}}$ if $p < \infty$. A simple modification of this argument gives the same result with $C_0 = 1$ if $p = \infty$.

To prove the reverse inequality it suffices to consider the case $q_0 = q_1 = 1$. Discretising as before, we find

$$\begin{aligned}
\|y\|_{(\ell^{1}_{w_{s_0}}(Y), \ell^{1}_{w_{s_1}}(Y))_{\theta,p}} &\leqslant \Big(\sum_{k\geqslant 0} \int_{2^{(k-1)(s_0-s_1)}}^{2^{k(s_0-s_1)}} |t^{-\theta} K(t, y)|^p \,\frac{\mathrm{d}t}{t}\Big)^{1/p} \\
&\leqslant C_1\Big(\sum_{k\geqslant 0} |2^{-\theta k(s_0-s_1)} K(2^{k(s_0-s_1)}, y)|^p\Big)^{1/p},
\end{aligned}$$

where $C_1 = \frac{(2^{(s_0-s)p}-1)^{1/p}}{(\theta p)^{1/p}}$. If $p = \infty$ we consider the supremum norm in the above and take $C_1 = 2^{s_0-s}$. Splitting $y_m = y_m \mathbf{1}_{\{m\leqslant k\}} + y_m \mathbf{1}_{\{m>k\}}$, we estimate

$$K(2^{k(s_0-s_1)}, y) \leqslant \sum_{m=-\infty}^{k} 2^{ms_0}\|y_m\| + 2^{k(s_0-s_1)} \sum_{m=k+1}^{\infty} 2^{s_1 m}\|y_m\|.$$

Therefore, since $\theta(s_1 - s_0) = s - s_0$ and $(1-\theta)(s_1 - s_0) = s - s_1$,

$$2^{-\theta k(s_0-s_1)} K(2^{k(s_0-s_1)}, y) \leqslant \sum_{m=-\infty}^{k} 2^{(m-k)(s_0-s)} 2^{ms} \|y_m\| + \sum_{m=k+1}^{\infty} 2^{-(m-k)(s-s_1)} 2^{ms} \|y_m\|.$$

Taking ℓ^p-norms in k and using Young's inequality for convolutions we obtain

$$\Big(\sum_{k\geqslant 0} |2^{-\theta k(s_0-s_1)} K(2^{k(s_0-s_1)}, y)|^p\Big)^{1/p} \leqslant (C_2 + C_3)\|y\|_{\ell^p_{w_s}(Y)}.$$

where $C_2 = \sum_{k=0}^\infty 2^{-k(s_0-s)}$ and $C_3 = \sum_{k=1}^\infty 2^{-k(s-s_1)}$. This gives the inequality

$$\|y\|_{(\ell^1_{w_{s_0}}(Y), \ell^1_{w_{s_1}}(Y))_{\theta,p}} \leqslant C_1(C_2 + C_3)\|y\|_{\ell^p_{w_s}(Y)}.$$

The final assertion is immediate from the first assertion and the log-convexity inequality (L.2). □

14.4 Besov spaces

The various Littlewood–Paley decompositions encountered in Chapter 5 express the norm of a function $f \in L^p(\mathbb{R}^d; X)$ in terms of (sharp or smooth) dyadic cut-offs in the frequency domain. For instance, in Theorem 5.5.22 we have seen that if X is a UMD Banach space, $p \in (1, \infty)$, and ψ is a smooth Littlewood–Paley function,

$$\|f\|_{L^p(\mathbb{R}^d;X)} \eqsim \Big\| \sum_{k\in\mathbb{Z}} \varepsilon_k \psi_k * f \Big\|_{L^p(\Omega\times\mathbb{R}^d;X)}, \tag{14.20}$$

where $\psi_k(x) := 2^k \psi(2^k x)$ and $(\varepsilon_k)_{k\in\mathbb{Z}}$ is a Rademacher sequence. With an eye toward the ensuing discussion we also remark that we have an equivalence of norms

$$\|f\|_{L^p(\mathbb{R}^d;X)} \eqsim \Big\| \sum_{k\in\mathbb{N}} \varepsilon_k \varphi_k * f \Big\|_{L^p(\Omega\times\mathbb{R}^d;X)}, \tag{14.21}$$

where now $(\varphi_k)_{k\in\mathbb{N}}$ is an inhomogeneous Littlewood–Paley sequence as in (14.20). This follows from Theorem 14.7.5 below, but could already have been proved in Chapter 5 with the methods presented there.

The idea behind the Littlewood–Paley approach to Besov spaces is to take this representation as a starting point, introducing an additional smoothness parameter $s \in \mathbb{R}$, and trading the norm of the Rademacher sum for an $\ell^q_{w_s}$-sum. The possibility of having $p \neq q$ presents us with two possible definitions, utilising the spaces $\ell^q_{w_s}(L^p(\mathbb{R}^d; X))$ and $L^p(\mathbb{R}^d; \ell^q_{w_s}(X))$ respectively. For $p =$

q, these spaces are canonically isometric by Fubini's theorem. The two choices lead to the theory of Besov spaces and Triebel–Lizorkin spaces, respectively.

The choice $\ell^q_{w_s}(\mathbb{Z})$ with the (homogeneous) Littlewood–Paley sequence $(\psi_k)_{k\in\mathbb{Z}}$ as in (14.20) leads to the so-called *homogeneous* Besov and Triebel–Lizorkin spaces. Alternatively, the choice $\ell^q_{w_s}(\mathbb{N})$ and the use of Littlewood–Paley sequences $(\varphi_k)_{k\in\mathbb{N}}$ as introduced in Definition 14.2.7 leads to the inhomogeneous version of these spaces. In what follows we will only present in the inhomogeneous case. Both classes of spaces are used in applications to PDE. The advantage of inhomogeneous spaces is that, in the development of their theory, one can make effective use of Schwartz functions and tempered distributions. The theory of homogeneous spaces is technically more involved and requires the use of different classes of test functions and equivalence classes of tempered distributions modulo polynomials. Since we have already encountered Schwartz functions and tempered distributions in many places, we choose to only develop the theory of inhomogeneous spaces here. Homogeneous spaces have better scaling properties, and scaling often plays a crucial role in PDE, but for the purposes of the theory developed here homogeneous spaces are not essential.

The proofs of (14.20) and (14.21) require the Banach space X to be UMD. In contrast, in the theory of Besov and Triebel–Lizorkin spaces these norm equivalences *are promoted to definitions*, thus eliminating the need of imposing any conditions on X. By taking this approach, most of the fundamental results in the theory of Besov spaces and Triebel–Lizorkin spaces are true for arbitrary Banach spaces X. They come with their own versions of the Mihlin multiplier theorem which does not require the UMD property either, allowing multipliers without singularities at the origin in case of inhomogeneous spaces. The more general multipliers considered in Chapter 5 have corresponding versions for homogeneous Besov and Triebel–Lizorkin spaces. Perhaps more surprising is the fact that also for the duality theory of these spaces no geometrical conditions need to be imposed on X. This contrast the duality theory for the Bochner spaces, which requires that X^* have the Radon–Nikodým property.

14.4.a Definitions and basic properties

As anticipated in the above discussion, we now introduce scale of Besov spaces through a Littlewood–Paley decomposition.

Definition 14.4.1. *Let* $p, q \in [1, \infty]$ *and* $s \in \mathbb{R}$. *The* Besov space $B^s_{p,q}(\mathbb{R}^d; X)$ *is the space of all* $f \in \mathscr{S}'(\mathbb{R}^d; X)$ *for which* $\varphi_k * f \in L^p(\mathbb{R}^d; X)$ *for all* $k \geqslant 0$ *and for which the quantity*

$$\|f\|_{B^s_{p,q}(\mathbb{R}^d;X)} := \big\|(2^{ks}\varphi_k * f)_{k\geqslant 0}\big\|_{\ell^q(L^p(\mathbb{R}^d;X))}$$

is finite.

Here, $(\varphi_k)_{k\geqslant 0}$ is the inhomogeneous Littlewood–Paley sequence that has been fixed throughout the chapter (see Convention 14.2.8). By the discussion of (14.13), the tempered distribution $\varphi_k * f$ is a C^∞-function of polynomial growth, so that the L^p-norm in the above definition makes sense.

To see that $\|\cdot\|_{B^s_{p,q}(\mathbb{R}^d;X)}$ is indeed a norm, suppose that $\|f\|_{B^s_{p,q}(\mathbb{R}^d;X)} = 0$. Then $\widehat{\varphi}_k\widehat{f} = \mathscr{F}(\varphi_k * f) = 0$ for all $k \geqslant 0$, so $\varphi_k * f = 0$ for all $k \geqslant 0$, and therefore $f = 0$ by Lemma 14.2.10. All other properties of a norm can be deduced from the fact that $\|\cdot\|_{\ell^q(L^p(\mathbb{R}^d;X))}$ is a norm.

It is immediate from Young's inequality, applied term-wise with respect to the ℓ^q-sum, that $\psi * f \in B^s_{p,q}(\mathbb{R}^d;X)$ whenever $\psi \in L^1(\mathbb{R}^d)$ and $f \in B^s_{p,q}(\mathbb{R}^d;X)$, and more generally the analogue of Proposition 14.2.3 is valid.

Up to an equivalent norm the above definition is independent on the choice of the sequence $(\varphi_k)_{k\geqslant 0}$, as will be shown in Proposition 14.4.2.

From the continuous embedding $\ell^{q_0} \hookrightarrow \ell^{q_1}$ for $1 \leqslant q_0 \leqslant q_1 \leqslant \infty$ we obtain the continuous embedding

$$B^s_{p,q_0}(\mathbb{R}^d;X) \hookrightarrow B^s_{p,q_1}(\mathbb{R}^d;X). \tag{14.22}$$

For $1 \leqslant q_0, q_1 \leqslant \infty$ and $s_0 > s_1$ we have the continuous embedding

$$B^{s_0}_{p,q_0}(\mathbb{R}^d;X) \hookrightarrow B^{s_1}_{p,q_1}(\mathbb{R}^d;X). \tag{14.23}$$

Indeed, for $q_0 \leqslant q_1$ this follows from (14.22) and the inequality $2^{ks_0} \leqslant 2^{ks_1}$ for $k \geqslant 0$. For $q_0 > q_1$ this follows from Hölder's inequality with $\frac{1}{q_1} = \frac{1}{q_0} + \frac{1}{r}$ and using that $\sum_{k\geqslant 0} 2^{-k(s_0-s_1)r} < \infty$.

Proposition 14.4.2. *For all $p, q \in [1,\infty]$ and $s \in \mathbb{R}$, up to an equivalent norm the space $B^s_{p,q}(\mathbb{R}^d;X)$ is independent of the choice of inhomogeneous Littlewood–Paley sequence $(\varphi_k)_{k\geqslant 0}$.*

The proof will give explicit constants depending only on s and φ_0 (in one direction), respectively s and ψ_0 (in the other direction).

Proof. Suppose $(\psi_k)_{k\geqslant 0}$ is another inhomogeneous Littlewood–Paley sequence. Then the analogues of (14.10) and (14.11) hold with φ_j and ψ_k; in particular for all $j, k \geqslant 0$ with $|j-k| \geqslant 2$ we have $\varphi_k * \psi_j = 0$. Using (14.12) for the sequence $(\psi_k)_{k\geqslant 0}$, the triangle inequality, Young's inequality, and (14.7), we obtain

$$\begin{aligned}
&\big\|(2^{ks}\varphi_k * f)_{k\geqslant 0}\big\|_{\ell^q(L^p(\mathbb{R}^d;X))} \\
&\quad\leqslant \sum_{j=-1}^{1} \big\|(2^{ks}\varphi_k * \psi_{k+j} * f)_{k\geqslant 0}\big\|_{\ell^q(L^p(\mathbb{R}^d;X))} \\
&\quad\leqslant \|\varphi_k\|_1 \sum_{j=-1}^{1} 2^{|s|}\big\|(2^{(k+j)s}\psi_{k+j} * f)_{k\geqslant 0}\big\|_{\ell^q(L^p(\mathbb{R}^d;X))}
\end{aligned}$$

$$\leqslant 6\|\varphi_0\|_1 2^{|s|} \big\|(2^{ks}\psi_k * f)_{k\geqslant 0}\big\|_{\ell^q(L^p(\mathbb{R}^d;X))},$$

where we used (14.7). This gives the required estimate in one direction. The reverse estimate is obtained by reversing the rôles of φ_k and ψ_k. □

Proposition 14.4.3. *For all $p,q \in [1,\infty]$ and $s \in \mathbb{R}$ we have continuous embeddings*

$$\mathscr{S}(\mathbb{R}^d;X) \hookrightarrow B^s_{p,q}(\mathbb{R}^d;X) \hookrightarrow \mathscr{S}'(\mathbb{R}^d;X).$$

Moreover, if $1 \leqslant p,q < \infty$, then $C^\infty_c(\mathbb{R}^d)\otimes X$ is dense in $B^s_{p,q}(\mathbb{R}^d;X)$.

Proof. We split the proof into three steps.

Step 1 – For the first embedding, by (14.22) it is enough to prove that $\mathscr{S}(\mathbb{R}^d;X)$ embeds into $B^s_{p,1}(\mathbb{R}^d;X)$. For $f \in \mathscr{S}(\mathbb{R}^d;X)$ and $L = L_{p,d} \in \mathbb{N}$ so large that $(1+|2\pi\cdot|^{2L})^{-1} \in L^p(\mathbb{R}^d)$ we find

$$\begin{aligned}
\|f\|_{B^s_{p,1}(\mathbb{R}^d;X)} &= \sum_{k\geqslant 0} 2^{ks}\|\varphi_k * f\|_{L^p(\mathbb{R}^d;X)} \\
&\lesssim_{d,p} \sum_{k\geqslant 0} 2^{ks}\big\|(1+|2\pi\cdot|^{2L})\varphi_k * f\big\|_{L^\infty(\mathbb{R}^d;X)} \\
&\leqslant \sum_{k\geqslant 0} 2^{ks}\big\|(1+(-\Delta)^L)(\widehat{\varphi}_k\widehat{f})\big\|_{L^1(\mathbb{R}^d;X)},
\end{aligned}$$

where we used the fact that $\mathscr{F}^{-1}$ maps L^1 into L^∞. It remains to estimate $2^{ks}\|\partial^\alpha(\widehat{\varphi}_k\widehat{f})\|_{L^1(\mathbb{R}^d;X)}$ for multi-indices $|\alpha| \leqslant 2L$.

First consider $k \geqslant 1$. Then $\operatorname{supp}\varphi_k \subseteq B_k := \{\xi \in \mathbb{R}^d : 2^{k-1} \leqslant |\xi| \leqslant 3\cdot 2^k\}$ and $|B_k| \lesssim_d 2^{kd}$. By Leibniz' rule and the boundedness on B_k of the functions $\partial^\gamma\widehat{\varphi}_k$ with $|\gamma| \leqslant |\alpha| \leqslant 2L = 2L_{p,d}$,

$$\|\partial^\alpha(\widehat{\varphi}_k\widehat{f})\|_{L^1(\mathbb{R}^d;X)} \lesssim_{d,p} \sum_{|\beta|\leqslant|\alpha|} \|\mathbf{1}_{B_k}\partial^\beta\widehat{f}\|_{L^1(\mathbb{R}^d;X)}.$$

To estimate the terms on the right-hand side, fix an $M \in \mathbb{N}$ which is arbitrary for the moment. Then

$$\begin{aligned}
\|\mathbf{1}_{B_k}\partial^\beta\widehat{f}\|_{L^1(\mathbb{R}^d;X)} &\leqslant \|\mathbf{1}_{B_k}(1+|\cdot|^{2M})^{-1}\|_{L^1(\mathbb{R}^d)}\|(1+|\cdot|^{2M})\partial^\beta\widehat{f}\|_{L^\infty(\mathbb{R}^d;X)} \\
&\leqslant |B_k|(1+2^{2M(k-1)})^{-1}\sum_{|\delta|\leqslant 2M}[\widehat{f}]_{\beta,\delta},
\end{aligned}$$

using the notation (14.2) for the seminorms defining the Schwartz space. Keeping in mind that $|B_k| \lesssim_d 2^{kd}$ we now choose $M = M_{s,p,d} \in \mathbb{N}$ so large that $\sum_{k\geqslant 0} 2^{ks}2^{kd}(1+2^{2M(k-1)})^{-1} < \infty$. With this choice, we obtain the estimate

$$\|f\|_{B^s_{p,1}(\mathbb{R}^d;X)} \lesssim_{d,p,s} \sum_{|\delta|\leqslant 2M}[\widehat{f}]_{\beta,\delta}.$$

A similar estimate in the case $k = 0$ can be obtained in a similar, but simpler, way. Since $\mathscr{F}$ is continuous on $\mathscr{S}(\mathbb{R}^d; X)$ (see Proposition 2.4.22), this proves that we have a continuous embedding $\mathscr{S}(\mathbb{R}^d; X) \hookrightarrow B^s_{p,q}(\mathbb{R}^d; X)$.

Step 2 – Next we prove that $B^s_{p,q}(\mathbb{R}^d; X)$ embeds into $\mathscr{S}'(\mathbb{R}^d; X)$. By (14.22) it is enough to prove that the inclusion mapping $B^s_{p,\infty}(\mathbb{R}^d; X) \subseteq \mathscr{S}'(\mathbb{R}^d; X)$ (by definition $B^s_{p,\infty}(\mathbb{R}^d; X)$ is contained in $\mathscr{S}'(\mathbb{R}^d; X)$) is continuous.

Fix $f \in B^s_{p,\infty}(\mathbb{R}^d; X)$ and $\psi \in \mathscr{S}(\mathbb{R}^d)$, and set $f_k := \varphi_k * f$ and $\psi_k := \varphi_k * \psi$. By Lemma 14.2.10 and (14.10) we have

$$f(\psi) = \sum_{k,\ell \geqslant 0} f_k(\psi_\ell) = \sum_{\ell=-1}^{1} \sum_{k\geqslant 0} f_k(\psi_{k+\ell}).$$

Thus, by (14.13),

$$\begin{aligned}
\|f(\psi)\| &\leqslant \sum_{\ell=-1}^{1} \sum_{k\geqslant 0} \int_{\mathbb{R}^d} \|f_k(x)\| |\psi_{k+\ell}(x)| \, \mathrm{d}x \\
&\leqslant \sum_{\ell=-1}^{1} \big\| (2^{ks} \|f_k(\cdot)\|)_{k\geqslant 0} \big\|_{\ell^\infty(L^p(\mathbb{R}^d;X))} \big\| (2^{-ks} \psi_{k+\ell})_{k\geqslant 0} \big\|_{\ell^1(L^{p'}(\mathbb{R}^d))} \\
&\leqslant 3 \cdot 2^{|s|} \|f\|_{B^s_{p,\infty}(\mathbb{R}^d;X)} \|\psi\|_{B^{-s}_{p',1}(\mathbb{R}^d)}.
\end{aligned}$$

Since $\mathscr{S}(\mathbb{R}^d) \hookrightarrow B^{-s}_{p',1}(\mathbb{R}^d)$ continuously by the previous step, the result follows from this.

Step 3 – To prove density, by Lemma 14.2.1 it suffices to prove the density of $\mathscr{S}(\mathbb{R}^d; X)$ in $B^s_{p,q}(\mathbb{R}^d; X)$.

Fix $f \in B^s_{p,q}(\mathbb{R}^d; X)$ and set $\zeta_n := \sum_{k=0}^n \varphi_k$. By (14.6) we have $\|\zeta_n\|_1 = \|\varphi\|_1$.

We will first show that $\zeta_n * f \to f$ in $B^s_{p,q}(\mathbb{R}^d; X)$. Fix $\varepsilon > 0$ and choose $K \in \mathbb{N}$ such that

$$\sum_{k>K} 2^{ksq} \|\varphi_k * f\|^q_{L^p(\mathbb{R}^d;X)} < \varepsilon^q.$$

By Young's inequality combined with the identity $\|\zeta_n\|_1 = \|\varphi_0\|_1$ we have $\zeta_n * \varphi_k * f \in L^p(\mathbb{R}^d; X)$ and $\|\zeta_n * \varphi_k * f\|_{L^p(\mathbb{R}^d;X)} \leqslant \|\varphi\|_1 \|\varphi_k * f\|_{L^p(\mathbb{R}^d;X)}$. From this we infer that $\zeta_n * f \in B^s_{p,q}(\mathbb{R}^d; X)$ and

$$\sum_{k>K} 2^{ksq} \|\zeta_n * \varphi_k * f\|^q_{L^p(\mathbb{R}^d;X)} < \varepsilon^q \|\varphi\|_1^q.$$

Hence by the triangle inequality in $\ell^q(L^p(\mathbb{R}^d; X))$,

$$\|f - \zeta_n * f\|_{B^s_{p,q}(\mathbb{R}^d;X)}$$

$$= \Big(\sum_{k\geqslant 0} 2^{ksq}\|\varphi_k * (f-\zeta_n * f)\|^q_{L^p(\mathbb{R}^d;X)}\Big)^{1/q}$$
$$\leqslant \Big(\sum_{k=0}^{K} 2^{ksq}\|\varphi_k * (f-\zeta_n * f)\|^q_{L^p(\mathbb{R}^d;X)}\Big)^{1/q} + \varepsilon(1+\|\varphi\|_1).$$

The first term in the last expression tends to zero as $n\to\infty$ by Proposition 1.2.32; here we use that $\zeta_n = 2^{nd}\varphi(2^n\cdot)$ and $\int_{\mathbb{R}^d}\varphi\,\mathrm{d}x = \widehat{\varphi}(0) = 1$. This concludes the proof that $\zeta_n * f \to f$ in $B^s_{p,q}(\mathbb{R}^d;X)$.

It remains to approximate each of the functions $f_n = \zeta_n * f$ by elements in $\mathscr{S}(\mathbb{R}^d;X)$. Observe that $f_n \in L^p(\mathbb{R}^d;X)$ since the functions $\varphi_k * f$ belong to $L^p(\mathbb{R}^d;X)$. Let $\eta\in\mathscr{S}(\mathbb{R}^d)$ be a functions satisfying $\eta(0)=1$ and $\operatorname{supp}(\widehat{\eta})\subseteq\{\xi\in\mathbb{R}^d : |\xi|\leqslant 1\}$. Since $\mathscr{F}(\eta(\delta\cdot)) = \delta^{-d}\widehat{\eta}(\delta^{-1}\cdot)$, for all $\delta\in(0,1)$ the support of $\mathscr{F}(\eta(\delta\cdot)f_n)$ is contained in a ball of radius $3\cdot 2^{n-1}+1\leqslant 2^{n+1}$; here we use the definition of ζ_n and (14.9). Using (14.11), (14.7), and Young's inequality, it follows that

$$\|f_n - \eta(\delta\cdot)f_n\|_{B^s_{p,q}(\mathbb{R}^d;X)} = \Big(\sum_{k=0}^{n+2} 2^{ksq}\|\varphi_k * (f_n-\eta(\delta\cdot)f_n)\|^q_{L^p(\mathbb{R}^d;X)}\Big)^{1/q}$$
$$\leqslant C\|f_n-\eta(\delta\cdot)f_n\|_{L^p(\mathbb{R}^d;X)},$$

where $C = C_{n,s,q} = (\sum_{k=0}^n 2^{ksq})^{1/q}$. For each fixed n, the right-hand side tends to zero as $\delta\downarrow 0$ by the dominated convergence theorem. □

Next we will prove the completeness of the normed space $B^s_{p,q}(\mathbb{R}^d;X)$.

Proposition 14.4.4. *For $p,q\in[1,\infty]$ and $s\in\mathbb{R}$ the space $B^s_{p,q}(\mathbb{R}^d;X)$ is a Banach space.*

The proof requires some preparations. Recall that a sequence $(f_n)_{n\geqslant 1}$ is said to *converge* in $\mathscr{S}'(\mathbb{R}^d;X)$ if there exists an $f\in\mathscr{S}'(\mathbb{R}^d;X)$ such that $f_n(\phi)\to f(\phi)$ in X for all $\phi\in\mathscr{S}(\mathbb{R}^d)$. Likewise, it is said to be *Cauchy* in $\mathscr{S}'(\mathbb{R}^d;X)$ if $(f_n(\phi))_{n\geqslant 1}$ is a Cauchy sequence in X for all $\phi\in\mathscr{S}(\mathbb{R}^d)$.

Lemma 14.4.5. *The space $\mathscr{S}'(\mathbb{R}^d;X)$ is sequentially complete, i.e., every Cauchy sequence in $\mathscr{S}'(\mathbb{R}^d;X)$ is convergent in $\mathscr{S}'(\mathbb{R}^d;X)$.*

Proof. Let $(f_n)_{n\geqslant 1}$ be a Cauchy sequence in $\mathscr{S}'(\mathbb{R}^d;X)$. Since X is complete we may define a linear mapping $f:\mathscr{S}(\mathbb{R}^d)\to X$ by $f(\phi) := \lim_{n\to\infty} f_n(\phi)$. We claim that f is continuous. Indeed, for every $\phi\in\mathscr{S}(\mathbb{R}^d)$ the sequence $(f_n(\phi))_{n\geqslant 1}$ is bounded in X, and therefore the Banach–Steinhaus theorem for topological vector spaces implies that the sequence $(f_n)_{n\geqslant 1}$ is equicontinuous. Hence, given an $\varepsilon>0$, we can find an open neighbourhood V of 0 in $\mathscr{S}(\mathbb{R}^d)$ such that $|f_n(\phi)|\leqslant\varepsilon$ for all $\phi\in V$ and $n\geqslant 1$. Taking limits, it follows that $|f(\phi)|\leqslant\varepsilon$ for all $\phi\in V$. This means that f is continuous at zero and hence continuous. □

A normed space $E \hookrightarrow \mathscr{S}'(\mathbb{R}^d;X)$ is said to have the *Fatou property* if for all sequences $(f_n)_{n\geqslant 1}$ in E such that

$$f_n \to f \text{ in } \mathscr{S}'(\mathbb{R}^d;X) \quad \text{and} \quad \liminf_{n\to\infty} \|f_n\|_E < \infty$$

we have $f \in E$ and $\|f\|_E \leqslant \liminf_{n\to\infty} \|f_n\|_E$.

Lemma 14.4.6. *For all $p, q \in [1,\infty]$ and $s \in \mathbb{R}$ the space $B^s_{p,q}(\mathbb{R}^d;X)$ has the Fatou property.*

Proof. Choose a sequence $(f_n)_{n\geqslant 1}$ of functions from $B^s_{p,q}(\mathbb{R}^d;X)$ with

$$f_n \to f \text{ in } \mathscr{S}'(\mathbb{R}^d;X) \quad \text{and} \quad \liminf_{n\to\infty} \|f_n\|_{B^s_{p,q}(\mathbb{R}^d;X)} < \infty.$$

Then $\lim_{n\to\infty} \varphi_k * f_n = \varphi_k * f$ pointwise. In case $p < \infty$, Fatou's lemma gives

$$\|\varphi_k * f\|_{L^p(\mathbb{R}^d;X)} \leqslant \liminf_{n\to\infty} \|\varphi_k * f_n\|_{L^p(\mathbb{R}^d;X)} < \infty.$$

Multiplying with 2^{ks} and taking ℓ^q-norms, it follows that we have $f \in B^s_{p,q}(\mathbb{R}^d;X)$ and $\|f\|_{B^s_{p,q}(\mathbb{R}^d;X)} \leqslant \liminf_{n\to\infty} \|f_n\|_{B^s_{p,q}(\mathbb{R}^d;X)}$ (by Fatou's lemma if $q < \infty$ and directly if $q = \infty$). For $p = \infty$ the proof is similar. □

Lemma 14.4.7. *Every normed space $E \hookrightarrow \mathscr{S}'(\mathbb{R}^d;X)$ with the Fatou property is complete.*

Proof. Let $(f_n)_{n\geqslant 1}$ be a Cauchy sequence in E. Since $\mathscr{S}'(\mathbb{R}^d;X)$ is sequentially complete by Lemma 14.4.5, and E is continuously embedded in $\mathscr{S}'(\mathbb{R}^d;X)$, it follows that there exists an $f \in \mathscr{S}'(\mathbb{R}^d;X)$ such that $f_n \to f$ in $\mathscr{S}'(\mathbb{R}^d;X)$. Since $(f_n)_{n\geqslant 1}$ is a Cauchy sequence in E it is bounded in E. By the Fatou property of E it follows that $f \in E$. To prove that $f_n \to f$ in E we fix an $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that for all $n, m \geqslant N$ we have $\|f_m - f_n\|_E < \varepsilon$. Using the Fatou property once more, we obtain

$$\|f - f_n\|_E \leqslant \liminf_{m\to\infty} \|f_m - f_n\|_E \leqslant \varepsilon$$

and the result follows. □

Proof of Proposition 14.4.4. Combine Lemmas 14.4.6 and 14.4.7 and Proposition 14.4.3. □

Coming back to the discussion on homogeneous verses inhomogeneous norms (see (14.20) and (14.21)), we have the following remark.

Remark 14.4.8. Let $p, q \in [1,\infty]$ and $s > 0$. For $f \in \mathscr{S}'(\mathbb{R}^d;X)$ one has

$$\big\|(2^{ks}\varphi_k * f)_{k\geqslant 0}\big\|_{\ell^q(L^p(\mathbb{R}^d;X))} \eqsim \big\|(2^{ks}\psi_k * f)_{k\in\mathbb{Z}}\big\|_{\ell^q(L^p(\mathbb{R}^d;X))} + \|f\|_{L^p(\mathbb{R}^d;X)},$$

where both expressions are infinite whenever one of them is. Here the $(\varphi_k)_{k\geqslant 0}$ are as in Definition 14.4.1, and thus the left-hand side of the above identity equals $\|f\|_{B^s_{p,q}(\mathbb{R}^d;X)}$. The $(\psi_k)_{k\in\mathbb{Z}}$ are as in (14.20). The first expression on the right-hand side is equal to the homogeneous Besov norm, which we will not discuss in detail.

To prove the norm equivalence first recall that $\psi_k = \varphi_k$ for $k \geqslant 1$. For "$\lesssim$" it suffices to observe that by Young's inequality

$$\big\|\varphi_0 * f\big\|_{L^p(\mathbb{R}^d;X)} \leqslant \|\varphi_0\|_1 \|f\|_{L^p(\mathbb{R}^d;X)}.$$

Conversely, assume that $f \in B^s_{p,q}(\mathbb{R}^d;X)$. Since $\widehat{\varphi}_0 = 1$ on $\operatorname{supp}(\widehat{\psi}_k)$ for $k \leqslant 0$, we can write

$$\begin{aligned}\|\psi_k * f\|_{L^p(\mathbb{R}^d;X)} &= \|\psi_k * \varphi_0 * f\|_{L^p(\mathbb{R}^d;X)} \\ &\leqslant \|\psi_k\|_1 \|\varphi_0 * f\|_{L^p(\mathbb{R}^d;X)} = \|\psi_0\|_1 \|\varphi_0 * f\|_{L^p(\mathbb{R}^d;X)},\end{aligned}$$

and thus using that $s > 0$ we obtain

$$\begin{aligned}\big\|(2^{ks}\psi_k * f)_{k\leqslant 0}\big\|_{\ell^q(L^p(\mathbb{R}^d;X))} &\leqslant \big\|(2^{ks}\varphi_0 * f)_{k\leqslant 0}\big\|_{\ell^q(L^p(\mathbb{R}^d;X))} \\ &\leqslant C_s \|\varphi_0 * f\|_{L^p(\mathbb{R}^d;X)}.\end{aligned}$$

Moreover, since $s > 0$, from (14.23) $B^s_{p,q}(\mathbb{R}^d;X) \hookrightarrow B^0_{p,1}(\mathbb{R}^d;X)$, and thus by Lemma 14.2.10

$$\begin{aligned}\|f\|_{L^p(\mathbb{R}^d;X)} = \Big\|\sum_{k\geqslant 0} \varphi_k * f\Big\|_{L^p(\mathbb{R}^d;X)} &\leqslant \sum_{k\geqslant 0} \|\varphi_k * f\|_{L^p(\mathbb{R}^d;X)} \\ &= \|f\|_{B^0_{p,1}(\mathbb{R}^d;X)} \leqslant C_{s,q} \|f\|_{B^s_{p,q}(\mathbb{R}^d;X)}\end{aligned}$$

14.4.b Fourier multipliers

The goal of this section is to prove a version of the Mihlin multiplier theorem for operator-valued Fourier multipliers acting on vector-valued Besov spaces. In contrast to the situation in the L^p-setting (cf. Theorems 5.3.18 and 5.5.10), where we had to assume the UMD property, a variant of the Mihlin theorem for Besov spaces holds for arbitrary Banach spaces.

We wish to emphasise that the main result, Theorem 14.4.16 below, is not applicable to multipliers which are non-smooth or even singular near the origin. This is due to the presence of the term φ_0 in the definition of inhomogeneous Littlewood–Paley sequences, whose support contains the origin in its interior. For instance, the Fourier multiplier associated to the Hilbert transform does not satisfy the conditions of the theorem.

Unlike in other chapters, we also include the case $p = \infty$. In order to avoid density issues, we define $\mathfrak{M}L^\infty(\mathbb{R}^d;X,Y)$ as the space of Fourier transforms of operator-valued measures of bounded variation:

Definition 14.4.9. *We define*

$$\mathfrak{M}L^\infty(\mathbb{R}^d; X, Y) := \big\{\widehat{\Phi} : \textit{the operator-valued measure} \\ \Phi : \mathscr{B}(\mathbb{R}^d) \to \mathscr{L}(X, Y) \textit{ is of bounded variation}\big\}.$$

With the norm $\|\widehat{\Phi}\|_{\mathfrak{M}L^\infty(\mathbb{R}^d;X,Y)} = \|\Phi\|(\mathbb{R}^d)$, the space $\mathfrak{M}L^\infty(\mathbb{R}^d; X, Y)$ is a Banach space.

For $m \in \mathfrak{M}L^\infty(\mathbb{R}^d; X, Y)$ and $f \in L^\infty(\mathbb{R}^d; X)$ we define

$$T_m * f := \check{m} * f,$$

recalling that the convolutions with an operator-valued measure of bounded variation has been introduced in Lemma 14.2.4.

Remark 14.4.10. In the scalar case it can be shown that the space $\mathfrak{M}L^\infty(\mathbb{R}^d) = \mathfrak{M}L^\infty(\mathbb{R}^d; \mathbb{C}, \mathbb{C})$ as defined in Definition 14.4.9 coincides with the space of all $m \in L^\infty(\mathbb{R}^d)$ for which the quantity

$$\sup\{\|T_m f\|_\infty : f \in \mathscr{S}(\mathbb{R}^d) \text{ with } \|f\|_\infty \leqslant 1\}$$

is finite, and that this quantity then equals the norm on $\mathfrak{M}L^\infty(\mathbb{R}^d)$ introduced above. This provides further motivation for Definition 14.4.9.

Various properties discussed in Section 5.3.a extend to $p = \infty$. Moreover, from the definition of the Fourier transform one sees that

$$\|\widehat{\Phi}\|_{L^\infty(\mathbb{R}^d;\mathscr{L}(X,Y))} \leqslant \|\Phi\|(\mathbb{R}^d).$$

This induces a contractive embedding

$$\mathfrak{M}L^\infty(\mathbb{R}^d; X, Y) \hookrightarrow L^\infty(\mathbb{R}^d; \mathscr{L}(X, Y)).$$

For $m \in \mathfrak{M}L^\infty(\mathbb{R}^d; X, Y)$ and $f \in \mathscr{S}(\mathbb{R}^d; X)$ one can check that $m\widehat{f} = \mathscr{F}(\check{m} * f)$, and by Lemma 14.2.4 for all $p \in [1, \infty]$ we have

$$\|\check{m} * f\|_{L^p(\mathbb{R}^d;Y)} \leqslant \|\check{m}\|(\mathbb{R}^d)\|f\|_{L^p(\mathbb{R}^d;Y)}.$$

This shows that for all $p \in [1, \infty]$ we have a contractive embedding

$$\mathfrak{M}L^\infty(\mathbb{R}^d; X, Y) \hookrightarrow \mathfrak{M}L^p(\mathbb{R}^d; X, Y). \tag{14.24}$$

In the discussion preceding Lemma 14.2.4 it was observed that for any function $\phi \in L^1(\mathbb{R}^d; \mathscr{L}(X, Y))$, an operator-valued measure $\Phi : \mathscr{B}(\mathbb{R}^d) \to \mathscr{L}(X, Y)$ of bounded variation is obtained by setting

$$\Phi(A) := \int_A \phi \, \mathrm{d}x,$$

and that its total variation satisfies $\|\Phi\|(\mathbb{R}^d) \leqslant \|\phi\|_{L^1(\mathbb{R}^d;\mathscr{L}(X,Y))}$. In this way we obtain contractive embeddings

$$\check{L}^1(\mathbb{R}^d;\mathscr{L}(X,Y)) \hookrightarrow \mathfrak{M}L^\infty(\mathbb{R}^d;X,Y) \hookrightarrow \mathfrak{M}L^p(\mathbb{R}^d;X,Y).$$

In combination with Lemma 14.2.11 we now obtain the following sufficient condition on m for membership of $\mathfrak{M}L^p(\mathbb{R}^d;X,Y)$.

Proposition 14.4.11. *If the multiplier $m \in L^\infty(\mathbb{R}^d;\mathscr{L}(X,Y))$ satisfies $\check{m} \in L^1(\mathbb{R}^d;\mathscr{L}(X,Y))$, then for all $p \in [1,\infty]$ we have $m \in \mathfrak{M}L^p(\mathbb{R}^d;X,Y)$ and*

$$\|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)} \leqslant \|\check{m}\|_{L^1(\mathbb{R}^d;\mathscr{L}(X,Y))}.$$

In particular, if $m \in C^{d+1}(\mathbb{R}^d;\mathscr{L}(X,Y))$ and there exists an $\varepsilon > 0$ such that

$$C_{m,d,\varepsilon} := \max_{|\alpha|\leqslant d+1} \sup_{\xi\in\mathbb{R}^d} (1+|\xi|^{|\alpha|+\varepsilon})\|\partial^\alpha m(\xi)\| < \infty,$$

then $m \in \mathfrak{M}L^p(\mathbb{R}^d;X,Y)$ and $\|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)} \lesssim_{d,\varepsilon} C_{m,d,\varepsilon}$.

Remark 14.4.12. In applications it can be useful to apply Proposition 14.2.11 to a dilated multiplier $m(t\cdot)$ instead of $m(\cdot)$. The $\mathfrak{M}L^p(\mathbb{R}^d;X,Y)$-norm is invariant under dilations, but the expression for $C_{m,d\varepsilon}$ is not. A similar remark applies to Lemma 14.2.11.

Remark 14.4.13. If $m \in C^{d+1}_{\rm c}(\mathbb{R}^d;\mathscr{L}(X,Y))$ is supported in the ball B_R around the origin, one easily checks that $C_{m,d,\varepsilon} \lesssim_R \|m\|_{C^{d+1}_{\rm b}(\mathbb{R}^d;\mathscr{L}(X,Y))}$. As a consequence we obtain that every $m \in C^{d+1}_{\rm c}(\mathbb{R}^d;\mathscr{L}(X,Y))$ belongs to $\mathfrak{M}L^p(\mathbb{R}^d;X,Y)$ and $\|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)} \lesssim_{d,\varepsilon,R} \|m\|_{C^{d+1}_{\rm b}(\mathbb{R}^d;\mathscr{L}(X,Y))}$.

Remark 14.4.14. Multipliers with singularities in the origin, such as the multiplier giving rise to the Hilbert transform, are not covered by Proposition 14.4.11.

Before moving to a Mihlin multiplier theorem for Besov space we present an important result on lifting operators. Recall from Subsection 5.6.a that the *Bessel potential operators* are the continuous operators J_σ, $\sigma \in \mathbb{R}$, acting on $\mathscr{S}'(\mathbb{R}^d;X)$ by

$$J_\sigma u := ((1+4\pi^2|\cdot|^2)^{\sigma/2}\widehat{u})^\vee, \quad u \in \mathscr{S}'(\mathbb{R}^d;X).$$

They satisfy $J_0 = I$ and $J_{\sigma_1+\sigma_2} = J_{\sigma_1} \circ J_{\sigma_2}$.

Proposition 14.4.15 (Lifting). *Let $p,q \in [1,\infty]$ and $s \in \mathbb{R}$. For all $\sigma \in \mathbb{R}$ we have*

$$J_\sigma : B^s_{p,q}(\mathbb{R}^d;X) \simeq B^{s-\sigma}_{p,q}(\mathbb{R}^d;X) \quad \textit{isomorphically.}$$

Proof. Noting that J_σ is a bijection from $\mathscr{S}'(\mathbb{R}^d;X)$ to $\mathscr{S}'(\mathbb{R}^d;X)$, with inverse $J_\sigma^{-1} = J_{-\sigma}$, it suffices to prove that J_σ maps $B^s_{p,q}(\mathbb{R}^d;X)$ into $B^{s-\sigma}_{p,q}(\mathbb{R}^d;X)$ and is bounded for each $\sigma \in \mathbb{R}$.

We claim that there exists a constant $C \geqslant 0$, independent of $k \geqslant 0$, such that for all $f \in \mathscr{S}'(\mathbb{R}^d;X)$,

$$\|\varphi_k * J_\sigma f\|_{L^p(\mathbb{R}^d;X)} \leqslant C 2^{k\sigma} \|\varphi_k * f\|_{L^p(\mathbb{R}^d;X)}.$$

This will imply the result.

To prove the claim we use that $\sum_{\ell=-1}^{1} \widehat{\varphi}_{k+\ell} \equiv 1$ on the support of $\widehat{\varphi}_k$ to write

$$2^{-k\sigma} J_\sigma \varphi_k * f = \sum_{\ell=-1}^{1} \mathscr{F}^{-1}(\widehat{\varphi}_k m \widehat{\varphi}_{k+\ell} \widehat{f}),$$

where $m(\xi) = 2^{-k\sigma}(1+4\pi^2|\xi|^2)^{\sigma/2}$. Using a dilation, Proposition 14.4.11, and the Fourier support property (14.9), for $k \geqslant 1$ we obtain

$$\begin{aligned}
\|\varphi_k m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)} &= \|\varphi_1(2\cdot) m(2^k\cdot)\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)} \\
&\lesssim_d \max_{|\alpha|\leqslant d+1} \sup_{\xi\in\mathbb{R}^d} (1+|\xi|^{|\alpha|+1}) \|\partial^\alpha[\varphi_1(2\cdot) m(2^k\xi)](\xi)\| \\
&\lesssim_d \max_{|\alpha|\leqslant d+1} \sup_{\frac12 \leqslant |\xi| \leqslant \frac32} \|\partial^\alpha[m(2^k\cdot)](\xi)\|,
\end{aligned}$$

where in the last step we applied the Leibniz rule as before and the Fourier support properties of φ_1 given by (14.8) and (14.9). Since $m(2^k\xi) = (2^{-2k} + |\xi|^2)^{\sigma/2}$, it is elementary to check that the latter expression is uniformly bounded in $k \geqslant 1$. A similar argument shows that $\varphi_0 m \in \mathfrak{M}L^p(\mathbb{R}^d;X)$. □

The simple multiplier result of Proposition 14.4.11 is already strong enough to prove the version of Mihlin's multiplier theorem for Besov spaces $B^s_{p,q}(\mathbb{R}^d;X)$ contained in Theorem 14.4.16 below, valid for arbitrary Banach spaces X and integrability exponents $p,q \in [1,\infty]$. In the statement of the theorem the end-points $p=\infty$ and $q=\infty$ create some technical difficulties, since we cannot use the density of the Schwartz functions to define T_m. It is for this reason that in the theorem we assume that the multiplier m is smooth and has derivatives of polynomial growth. Many interesting multipliers satisfy this condition, and to proceed with the development of the theory this version suffices for the time being. A version which avoids this restriction on m will be presented in Theorem 14.5.6.

When $m \in C^\infty(\mathbb{R}^d;\mathscr{L}(X,Y))$ has derivatives of polynomial growth, one can define the Fourier multiplier T_m as an operator from $\mathscr{S}'(\mathbb{R}^d;X)$ into $\mathscr{S}'(\mathbb{R}^d;Y)$ by $T_m f := \mathscr{F}^{-1}(m\widehat{f})$. To see that this is well-defined it suffices to note that $m\widehat{f} \in \mathscr{S}'(\mathbb{R}^d;Y)$ for $f \in \mathscr{S}'(\mathbb{R};X)$. In the next theorem, T_m is understood to be the restriction of this operator to $B^s_{p,q}(\mathbb{R}^d;X)$. The theorem then asserts that, under Mihlin type conditions on m, it maps $B^s_{p,q}(\mathbb{R}^d;X)$ into $B^s_{p,q}(\mathbb{R}^d;Y)$.

Theorem 14.4.16 (Mihlin multiplier theorem for Besov spaces). *Let X and Y be Banach spaces and let $p,q \in [1,\infty]$ and $s \in \mathbb{R}$. Suppose that $m \in C^\infty(\mathbb{R}^d;\mathscr{L}(X,Y))$ has derivatives of polynomial growth, and that*

$$\sup_{|\alpha|\leqslant d+1}\sup_{\xi\in\mathbb{R}^d}(1+|\xi|^{|\alpha|})\|\partial^\alpha m(\xi)\|_{\mathscr{L}(X,Y)} =: K_m < \infty. \tag{14.25}$$

Then the Fourier multiplier $T_m = \mathscr{F}^{-1}m\mathscr{F}$ restricts to a bounded operator from $B^s_{p,q}(\mathbb{R}^d;X)$ to $B^s_{p,q}(\mathbb{R}^d;Y)$ of norm $\|T_m\| \leqslant C_{s,d}K_m$.

The usual Mihlin condition involves a factor $|\xi|^{|\alpha|}$ instead of $1+|\xi|^{|\alpha|}$. A multiplier theorem involving the former can be shown to hold for the scale of homogeneous Besov spaces.

For finite p and q, the condition $m \in C^\infty(\mathbb{R}^d;\mathscr{L}(X,Y))$ can be weakened to $m \in C^{d+1}(\mathbb{R}^d;\mathscr{L}(X,Y))$. This can be seen by taking f in the dense class $\mathscr{S}(\mathbb{R}^d)\otimes X$ in the proof below.

Proof. For $f \in B^s_{p,q}(\mathbb{R}^d;X)$ let $f_n := \varphi_n * f$ for $n \geqslant 0$. Since $\sum_{\ell=-1}^1 \widehat{\varphi}_{k+\ell} \equiv 1$ on the support of $\widehat{\varphi}_k$,

$$\begin{aligned}
\|T_m f\|_{B^s_{p,q}(\mathbb{R}^d;Y)} &= \big\|(2^{ks}\varphi_k * \mathscr{F}^{-1}m\widehat{f})_{k\geqslant 0}\big\|_{\ell^q(L^p(\mathbb{R}^d;Y))}\\
&= \Big\|\Big(2^{ks}\mathscr{F}^{-1}\widehat{\varphi}_k m\sum_{\ell=-1}^{1}\widehat{\varphi}_{k+\ell}\widehat{f}\Big)_{k\geqslant 0}\Big\|_{\ell^q(L^p(\mathbb{R}^d;Y))}\\
&\leqslant \sum_{\ell=-1}^{1}\big\|2^{ks}\mathscr{F}^{-1}(\widehat{\varphi}_k m\widehat{f}_{k+\ell})_{k\geqslant 0}\big\|_{\ell^q(L^p(\mathbb{R}^d;Y))}\\
&\leqslant \sup_{k\geqslant 0}\|\widehat{\varphi}_k m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)}\sum_{\ell=-1}^{1}\big\|(2^{ks}f_{k+\ell})_{n\geqslant 0}\big\|_{\ell^q(L^p(\mathbb{R}^d;Y))}\\
&\leqslant 2^{|s|}\sup_{k\geqslant 0}\|\widehat{\varphi}_k m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)}\|f\|_{B^s_{p,q}(\mathbb{R}^d;X)}.
\end{aligned}$$

To complete the proof we must show that $\sup_{k\geqslant 0}\|\widehat{\varphi}_k m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)} \lesssim_d K_m$.

First consider the case $k \geqslant 1$. Since the multiplier norm is invariant under dilations by Proposition 5.3.8, it suffices to show that

$$\sup_{k\geqslant 1}\|\widehat{\varphi}_1(\cdot)m(2^{k-1}\cdot)\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)} \lesssim_d K_m.$$

By Proposition 14.4.11, it even suffices to show that there exists an $\varepsilon > 0$ such that

$$\max_{|\alpha|\leqslant d+1}\sup_{\xi\in\mathbb{R}^d}(1+|\xi|^{|\alpha|+\varepsilon})\|\partial^\alpha[\widehat{\varphi}_1(\cdot)m(2^{k-1}\cdot)](\xi)\| \lesssim_d K_m.$$

We will verify this bound for $\varepsilon = 1$. By the Fourier support properties of φ_1 implied by (14.8) and (14.9), for $\beta \leqslant \alpha$ with $|\alpha| \leqslant d+1$ we have

$$\sup_{\xi\in\mathbb{R}^d} |(1+|\xi|^{|\alpha|+1})\partial^\beta \widehat{\varphi}_1(\xi)| \leqslant C_{\beta,d}.$$

Hence, by Leibniz's rule the Mihlin condition on m, and the Fourier support property of φ_1 given by (14.9), for all $|\alpha| \leqslant d+1$ we have

$$\begin{aligned}
&\sup_{\xi\in\mathbb{R}^d} (1+|\xi|^{|\alpha|+1})\|\partial^\alpha[\widehat{\varphi}_1(\cdot)m(2^{k-1}\cdot)](\xi)\| \\
&\quad = \sup_{|\xi|\geqslant 1} (1+|\xi|^{|\alpha|+1})\|\partial^\alpha[\widehat{\varphi}_1(\cdot)m(2^{k-1}\cdot)](\xi)\| \\
&\quad \leqslant \sup_{|\xi|\geqslant 1} (1+|\xi|^{|\alpha|+1}) \sum_{\beta\leqslant\alpha} C_{\alpha,\beta}|\partial^\beta \varphi_1(\xi)|\cdot 2^{(k-1)|\alpha-\beta|}|\partial^{\alpha-\beta} m(2^{k-1}\xi)| \\
&\quad \lesssim_d \sup_{|\xi|\geqslant 1} \sum_{\beta\leqslant\alpha} 2^{(k-1)|\alpha-\beta|}|\partial^{\alpha-\beta} m(2^{k-1}\xi)| \\
&\quad \leqslant \sup_{|\xi|\geqslant 1} \sum_{\beta\leqslant\alpha} 2^{(k-1)|\alpha-\beta|}\frac{K_m}{1+|2^{k-1}\xi|^{|\alpha-\beta|}} \\
&\quad \lesssim_d K_m.
\end{aligned} \tag{14.26}$$

The case $k=0$ is proved in similarly, omitting the dilation argument. □

As an application of Theorem 14.4.16, we obtain the following analogue of Theorem 5.6.11.

Proposition 14.4.17. *Let $p\in[1,\infty)$, $q\in[1,\infty]$, and $s\in\mathbb{R}$. For all $k\in\mathbb{N}$,*

$$|||f|||_{B^s_{p,q}(\mathbb{R}^d;X)} := \sum_{|\alpha|\leqslant k} \|\partial^\alpha f\|_{B^{s-k}_{p,q}(\mathbb{R}^d;X)} \tag{14.27}$$

defines an equivalent norm on $B^s_{p,q}(\mathbb{R}^d;X)$

Proof. As a consequence of Proposition 14.4.15 it suffices to prove the equivalence of (14.27) with $\|J_k f\|_{B^{s-k}_{p,q}(\mathbb{R}^d;X)}$. This can be deduced from Theorem 14.4.16 by an argument similar to the one in Theorem 5.6.11. In the present situation it is important to note that the multipliers in the proof of the proposition also satisfy the more restrictive condition (14.25). Below we present a simplification of the argument of Theorem 5.6.11 adapted to the Besov space case. Let $\langle\xi\rangle = (1+|2\pi\xi|^2)^{1/2}$.

First we prove the estimate

$$\|\partial^\alpha f\|_{B^{s-k}_{p,q}(\mathbb{R}^d;X)} \leqslant C\|J_k f\|_{B^{s-k}_{p,q}(\mathbb{R}^d;X)}.$$

Applying the Fourier transform, we have

$$\mathscr{F}[\partial^\alpha f](\xi) = (2\pi i\xi)^\alpha \widehat{f}(\xi) = \frac{(2\pi i\xi)^\alpha}{\langle\xi\rangle^k}\langle\xi\rangle^k \widehat{f}(\xi) =: m_\alpha(\xi)\langle\xi\rangle^k\widehat{f}(\xi).$$

One checks that m_α satisfies the conditions of Theorem 14.4.16, and thus

$$\begin{aligned}\|\partial^\alpha f\|_{B^{s-k}_{p,q}(\mathbb{R}^d;X)} &\leqslant C_\alpha C_{d,p,q}\|\mathscr{F}^{-1}[\langle\cdot\rangle^k \widehat{f}]\|_{B^{s-k}_{p,q}(\mathbb{R}^d;X)} \\ &= C_\alpha C_{d,p,q}\|J_k f\|_{B^{s-k}_{p,q}(\mathbb{R}^d;X)}.\end{aligned}$$

For the reverse estimate it suffices to show

$$\|J_k f\|_{B^{s-k}_{p,q}(\mathbb{R}^d;X)} \leqslant C \sum_{|\alpha|\leqslant k} \|\partial^\alpha f\|_{B^{s-k}_{p,q}(\mathbb{R}^d;X)}.$$

Again we apply Theorem 14.4.16. By induction on k,

$$\langle\xi\rangle^{2k} = (1+|2\pi\xi|^2)^k = \sum_{|\alpha|\leqslant k} c_{\alpha,k}(2\pi i\xi)^\alpha(2\pi i\xi)^\alpha,$$

and therefore

$$\begin{aligned}\langle\xi\rangle^k \widehat{f}(\xi) = \frac{\langle\xi\rangle^{2k}}{\langle\xi\rangle^k}\widehat{f}(\xi) &= \sum_{|\alpha|\leqslant k} c_{\alpha,k} m_\alpha(\xi)(2\pi i\xi)^\alpha \widehat{f}(\xi) \\ &= \sum_{|\alpha|\leqslant k} c_{\alpha,k} m_\alpha(\xi)\widehat{\partial^\alpha f}(\xi),\end{aligned}$$

where $m_\alpha(\xi) = \frac{(2\pi i\xi)^\alpha}{\langle\xi\rangle^k}$. Applying Theorem 14.4.16 to m_α now gives

$$\begin{aligned}\|J_k f\|_{B^{s-k}_{p,q}(\mathbb{R}^d;X)} &= \|\mathscr{F}^{-1}[\langle\cdot\rangle^k \widehat{f}]\|_{B^{s-k}_{p,q}(\mathbb{R}^d;X)} \\ &\leqslant \sum_{|\alpha|\leqslant k} |c_{\alpha,k}|\|T_{m_\alpha}\partial^\alpha f\|_{B^{s-k}_{p,q}(\mathbb{R}^d;X)} \\ &\leqslant C_{d,p,k} \sum_{|\alpha|\leqslant k} \|\partial^\alpha f\|_{B^{s-k}_{p,q}(\mathbb{R}^d;X)}.\end{aligned}$$

□

14.4.c Embedding theorems

We begin by showing that various classes of function spaces lie 'sandwiched' between Besov spaces.

Proposition 14.4.18 (Sandwiching with Besov spaces). *For all $p \in [1,\infty]$, $s \in \mathbb{R}$, and $m \in \mathbb{N}$, we have continuous embeddings*

$$B^s_{p,1}(\mathbb{R}^d;X) \hookrightarrow H^{s,p}(\mathbb{R}^d;X) \hookrightarrow B^s_{p,\infty}(\mathbb{R}^d;X), \tag{14.28}$$

$$B^m_{p,1}(\mathbb{R}^d;X) \hookrightarrow W^{m,p}(\mathbb{R}^d;X) \hookrightarrow B^m_{p,\infty}(\mathbb{R}^d;X), \tag{14.29}$$

$$B^m_{\infty,1}(\mathbb{R}^d;X) \hookrightarrow C^m_{\mathrm{ub}}(\mathbb{R}^d;X) \hookrightarrow B^m_{\infty,\infty}(\mathbb{R}^d;X). \tag{14.30}$$

An improvement for $p \in (1,\infty)$ will be given in Proposition 14.6.13.

Proof. In order to prove (14.28), by Propositions 5.6.3 and 14.4.15 it suffices to consider $s = m = 0$. Similarly, in order to prove (14.29) and (14.30), by Proposition 14.4.17 it suffices to consider $s = m = 0$. Therefore, (14.28) and (14.29) reduce to proving the continuous embeddings

$$B^0_{p,1}(\mathbb{R}^d;X) \hookrightarrow L^p(\mathbb{R}^d;X) \hookrightarrow B^0_{p,\infty}(\mathbb{R}^d;X). \tag{14.31}$$

Fix $f \in B^0_{p,1}(\mathbb{R}^d;X)$. By definition,

$$\|f\|_{B^0_{p,1}(\mathbb{R}^d;X)} = \sum_{k\geqslant 0} \|\varphi_k * f\|_{L^p(\mathbb{R}^d;X)}.$$

In particular, the sum $\sum_{k\geqslant 0} \varphi_k * f$ converges absolutely in $L^p(\mathbb{R}^d;X)$, and the required result follows by Lemma 14.2.10 and the triangle inequality.

To prove the second embedding in (14.31), fix $f \in L^p(\mathbb{R}^d;X)$. By Young's inequality,

$$\begin{aligned}\|f\|_{B^0_{p,\infty}(\mathbb{R}^d;X)} &= \sup_{k\geqslant 0} \|\varphi_k * f\|_{L^p(\mathbb{R}^d;X)} \\ &\leqslant \sup_{k\geqslant 0} \|\varphi_k\|_{L^1(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d;X)} \leqslant 2\|\varphi_0\|_{L^1(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d;X)},\end{aligned}$$

where the last step uses (14.7). This completes the proof of (14.31).

As we already noted, in order to prove the embeddings in (14.30) it suffices to consider the case $m = 0$. Fix $f \in B^0_{\infty,1}(\mathbb{R}^d;X)$. As before we see that the sum $\sum_{k=0}^{\infty} \varphi_k * f$ is absolutely convergent in $L^\infty(\mathbb{R}^d;X)$. By Lemma 14.2.10 its sum equals f and

$$\|f\|_\infty \leqslant \sum_{k=0}^{\infty} \|\varphi_k * f\|_\infty = \|f\|_{B^0_{\infty,1}(\mathbb{R}^d;X)}.$$

To see that f has a uniformly continuous version, we note that by Proposition 2.4.32 we have $\varphi_k * f \in C^\infty(\mathbb{R}^d;X)$ and

$$\|\partial_j(\varphi_k * f)\|_\infty = \|(\partial_j \varphi_k) * f\|_\infty \leqslant \|\partial_j \varphi_k\|_1 \|f\|_\infty \leqslant \|\partial_j \varphi_k\|_1 \|f\|_{B^0_{\infty,1}(\mathbb{R}^d;X)}.$$

In particular, each function $\varphi_k * f$ is Lipschitz continuous and hence uniformly continuous. Therefore $f \in C_{\rm ub}(\mathbb{R}^d;X)$ by uniform convergence.

The second embedding in (14.30) follows by combining the embedding $C^m_{\rm ub}(\mathbb{R}^d;X) \hookrightarrow W^{m,\infty}(\mathbb{R}^d;X)$ and (14.29). □

Theorem 14.4.19 (Sobolev embedding for Besov spaces). *For given $p_0, p_1, q_0, q_1 \in [1,\infty]$, and $s_0, s_1 \in \mathbb{R}$, we have a continuous embedding*

$$B^{s_0}_{p_0,q_0}(\mathbb{R}^d;X) \hookrightarrow B^{s_1}_{p_1,q_1}(\mathbb{R}^d;X)$$

if and only if one of the following three conditions holds:

(i) $p_0 = p_1$ *and* $[s_0 > s_1$ *or* $(s_0 = s_1$ *and* $q_0 \leqslant q_1)]$*;*
(ii) $p_0 < p_1$*,* $q_0 \leqslant q_1$*, and* $s_0 - \frac{d}{p_0} = s_1 - \frac{d}{p_1}$*;*
(iii) $p_0 < p_1$ *and* $s_0 - \frac{d}{p_0} > s_1 - \frac{d}{p_1}$.

The most interesting cases are (ii) and (iii), since they can be used to change the integrability parameter from p_0 into p_1.

For the proof of the sufficiency of the three conditions we need two lemmas. The first provides an L^p-estimate for the derivatives under suitable Fourier support assumptions. Recall from Lemma 14.2.9 that every $f \in \mathscr{S}'(\mathbb{R}^d; X)$ with compact Fourier support belongs to $C^\infty(\mathbb{R}^d; X)$ and has at most polynomial growth.

Lemma 14.4.20 (Bernstein–Nikolskii inequality). *Let* $p_0, p_1 \in [1, \infty]$ *satisfy* $p_0 \leqslant p_1$*. If* $f \in L^{p_0}(\mathbb{R}^d; X)$ *satisfies*

$$\operatorname{supp} \widehat{f} \subseteq \{\xi \in \mathbb{R}^d : |\xi| < t\}$$

for some $t > 0$*, then for any multi-index* $\alpha \in \mathbb{N}^d$ *there is a constant* $C = C_{\alpha,d,p_0,p_1}$ *such that*

$$\|\partial^\alpha f\|_{L^{p_1}(\mathbb{R}^d;X)} \leqslant C t^{|\alpha| + \frac{d}{p_0} - \frac{d}{p_1}} \|f\|_{L^{p_0}(\mathbb{R}^d;X)}.$$

An extension to exponents $0 < p_0 \leqslant p_1 \leqslant \infty$ will be given in Remark 14.6.4.

Proof. By a routine scaling argument it suffices to consider the case $t = 1$.

Let $\psi \in \mathscr{S}(\mathbb{R}^d)$ satisfy $\widehat{\psi} \equiv 1$ on $B_1 := \{x \in \mathbb{R}^d : |x| < 1\}$ and put $\psi_\alpha := \partial^\alpha \psi$. Then $f = \psi * f$, and by Young's inequality with $\frac{1}{p_1} + 1 = \frac{1}{p_0} + \frac{1}{q}$ we obtain

$$\begin{aligned} \|\partial^\alpha f\|_{L^{p_1}(\mathbb{R}^d;X)} &= \|\partial^\alpha(\psi * f)\|_{L^{p_1}(\mathbb{R}^d;X)} \\ &= \|\psi_\alpha * f\|_{L^{p_1}(\mathbb{R}^d;X)} \leqslant \|\psi_\alpha\|_{L^q(\mathbb{R}^d)} \|f\|_{L^{p_0}(\mathbb{R}^d;X)}. \end{aligned}$$

□

The next lemma provides shows how the L^p-norm of $\varphi_k * \varphi_{k+j}$ scales with k.

Lemma 14.4.21. *For all* $j \in \mathbb{Z}$ *there exists a constant* $C_{d,j,p} \geqslant 0$ *such that for all* $k \geqslant 0$ *and* $k + \ell \geqslant 0$ *we have*

$$\|\varphi_{k+\ell} * \varphi_k\|_{L^p(\mathbb{R}^d)} = C_{\ell,p,d} 2^{kd/p'}.$$

Proof. The identity $\widehat{\varphi}_k(\xi) = \widehat{\varphi}_1(2^{-k+1}\xi)$ implies $\varphi_k(x) = 2^{(k-1)d}\varphi_1(2^{k-1}x)$ and therefore, by a change of variables in x and y,

$$\begin{aligned} &\|\varphi_{k+j} * \varphi_k\|^p_{L^p(\mathbb{R}^d)} \\ &\quad = \int_{\mathbb{R}^d} \Big| 2^{(k-1)d} 2^{(k+j-1)d} \int_{\mathbb{R}^d} \varphi_1(2^j 2^{k-1}(x-y))\varphi_1(2^{k-1}y) \, \mathrm{d}y \Big|^p \, \mathrm{d}x \end{aligned}$$

$$= 2^{kd(p-1)} \underbrace{2^{jdp-d(p-1)} \int_{\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} \varphi_1(2^j(x-y))\varphi_1(y)\,\mathrm{d}y\Big|^p \,\mathrm{d}x}_{=:C^p_{j,p,d}}$$

and the result follows. □

Proof of Theorem 14.4.19. For the sufficiency of (i), first consider the case $s_0 \geqslant s_1$ and $q_0 \leqslant q_1$. Then the result follows from the fact that for any scalar sequence $(a_k)_{k\geqslant 0}$,

$$\big\|(2^{ks_1}a_k)_{k\geqslant 0}\big\|_{\ell^{q_1}} \leqslant \big\|(2^{ks_0}a_k)_{k\geqslant 0}\big\|_{\ell^{q_0}}.$$

If $s_0 > s_1$, the result follows from (14.23).

If (ii) holds, then writing $f_k := \varphi_k * f$ for $k \geqslant 0$, from Lemma 14.4.20 we infer that

$$\|f_k\|_{L^{p_1}(\mathbb{R}^d;X)} \leqslant C2^{k(\frac{d}{p_0}-\frac{d}{p_1})}\|f_k\|_{L^{p_0}(\mathbb{R}^d;X)} = C2^{k(s_0-s_1)}\|f_k\|_{L^{p_0}(\mathbb{R}^d;X)}.$$

It follows that

$$\begin{aligned}
\|f\|_{B^{s_1}_{p_1,q_1}(\mathbb{R}^d;X)} &= \big\|(2^{ks_1}f_k)_{k\geqslant 0}\big\|_{\ell^{q_1}(L^{p_1}(\mathbb{R}^d;X))} \\
&\leqslant C\big\|(2^{ks_0}f_k)_{k\geqslant 0}\big\|_{\ell^{q_1}(L^{p_0}(\mathbb{R}^d;X))} \\
&= C\|f\|_{B^{s_0}_{p_0,q_1}(\mathbb{R}^d;X)} \leqslant C\|f\|_{B^{s_0}_{p_0,q_0}(\mathbb{R}^d;X)},
\end{aligned}$$

using (14.22) in the last step.

Suppose now that (iii) holds and let $t := s_0 - \frac{d}{p_0} + \frac{d}{p_1}$. Then $t - \frac{d}{p_1} = s_0 - \frac{d}{p_0}$ and therefore, by the previous step,

$$\|f\|_{B^{s_1}_{p_1,q_1}(\mathbb{R}^d;X)} \leqslant C\|f\|_{B^{t}_{p_0,q_1}(\mathbb{R}^d;X)}.$$

Since $t > s_1$, it follows that the conditions (i) are satisfied, and thus

$$\|f\|_{B^{t}_{p_0,q_1}(\mathbb{R}^d;X)} \leqslant C\|f\|_{B^{s_0}_{p_0,q_0}(\mathbb{R}^d;X)}.$$

Next we move to the necessity of the conditions (i), (ii), and (iii). It suffices to consider the case $X = \mathbb{K}$.

Suppose that we have the continuous embedding stated in the theorem. By the closed graph theorem there is a constant $C = C_{d,p_0,p_1,q_0,q_1,s_0,s_1}$ such that for all $f \in B^{s_0}_{p_0,q_0}(\mathbb{R}^d)$,

$$\|f\|_{B^{s_1}_{p_1,q_1}(\mathbb{R}^d)} \leqslant C\|f\|_{B^{s_0}_{p_0,q_0}(\mathbb{R}^d)}. \tag{14.32}$$

First we will derive

$$s_0 - \frac{d}{p_0} \geqslant s_1 - \frac{d}{p_1} \quad \text{and} \quad p_0 \leqslant p_1. \tag{14.33}$$

By (14.22), (14.32) also holds (with a possibly different constant) for $q_0 = 1$ and $q_1 = \infty$. The Fourier support properties (14.8) and (14.9) of φ_k then imply

$$2^{ks_1}\|\varphi_k * \varphi_k\|_{L^{p_1}(\mathbb{R}^d)} \leqslant \|\varphi_k\|_{B^{s_1}_{p_1,\infty}(\mathbb{R}^d)} \leqslant C\|\varphi_k\|_{B^{s_0}_{p_0,1}(\mathbb{R}^d)} \leqslant C2^{ks_0}\sum_{j=-1}^{1}\|\varphi_k * \varphi_{k+j}\|_{L^{p_0}(\mathbb{R}^d)}.$$

By Lemma 14.4.21 this implies

$$2^{ks_1}2^{kd/p_1'} \leqslant \tilde{C}2^{ks_0}2^{kd/p_0'}$$

for some possibly different constant $\widetilde{C}$ independent of k. Upon letting $k \to \infty$, this gives the inequality $s_1 - \frac{d}{p_1'} \leqslant s_0 - \frac{d}{p_0'}$, or equivalently, $s_1 - \frac{d}{p_1} \leqslant s_0 - \frac{d}{p_0}$.

Define $f_t : \mathbb{R}^d \to \mathbb{C}$ by $\widehat{f_t}(x) := \widehat{\varphi}_0(t^{-1}\cdot)$. Then $\widehat{\varphi}_0 = 1$ and $\varphi_k = 0$ for $k \geqslant 1$ on $\operatorname{supp}(\widehat{f_t})$ for $t > 0$ small enough. Therefore,

$$t^{-\frac{d}{p_j}}\|f_1\|_{L^{p_j}(\mathbb{R}^d)} = \|f_t\|_{L^{p_j}(\mathbb{R}^d)} = \|\varphi_0 * f_t\|_{L^{p_j}(\mathbb{R}^d)} = \|f_t\|_{B^{s_j}_{p_j,q_j}(\mathbb{R}^d)}$$

Combining this with (14.32) gives

$$t^{-\frac{d}{p_1}}\|f_1\|_{L^{p_1}(\mathbb{R}^d)} \leqslant Ct^{-\frac{d}{p_0}}\|f_1\|_{L^{p_0}(\mathbb{R}^d)}.$$

Upon letting $t \downarrow 0$, we find that $p_0 \leqslant p_1$. This completes the proof of (14.33).

Now there are two possibilities: (i) $p_0 < p_1$, or (ii) $p_0 = p_1$. First consider the case (i). If $s_0 - \frac{d}{p_0} > s_1 - \frac{d}{p_1}$, then (iii) follows. Still assuming (i), if $s_0 - \frac{d}{p_0} = s_1 - \frac{d}{p_1}$, then in order to deduce (ii) it suffices to show that $q_0 \leqslant q_1$. We claim that for any finite sequence of scalars $(a_k)_{k=1}^n$,

$$\|(a_k)_{k=1}^n\|_{\ell^{q_1}} \leqslant C\|(a_k)_{k=1}^n\|_{\ell^{q_0}}, \tag{14.34}$$

where C is a constant independent of $n \geqslant 1$ and the sequence $(a_k)_{k=1}^n$. Once established, this claim gives $q_0 \leqslant q_1$.

To prove the claim fix a scalar sequence $(a_k)_{k=1}^n$. Applying (14.32) to the function $f := \sum_{k=1}^n 2^{-3k(s_0+\frac{d}{p_0'})}a_k\varphi_{3k} = \sum_{k=1}^n 2^{-3k(s_1+\frac{d}{p_1'})}a_k\varphi_{3k}$ gives the inequality

$$\begin{aligned}\Big(\sum_{m\geqslant 0} 2^{ms_1q_1}\Big\|\sum_{k=1}^n 2^{-3k(s_1+\frac{d}{p_1'})}a_k\varphi_m * \varphi_{3k}\Big\|^{q_1}_{L^{p_1}(\mathbb{R}^d)}\Big)^{1/q_1} \\ \leqslant C\Big(\sum_{m\geqslant 0} 2^{ms_0q_0}\Big\|\sum_{k=1}^n 2^{-3k(s_0+\frac{d}{p_0'})}a_k\varphi_m * \varphi_{3k}\Big\|^{q_0}_{L^{p_0}(\mathbb{R}^d)}\Big)^{1/q_0}.\end{aligned} \tag{14.35}$$

Let us analyse the expressions on the left-hand and right-hand sides for general values of p, q, and s. We have $\varphi_m * \varphi_{3k} \neq 0$ only for $m = 3k + \ell$ with

$\ell \in \{-1, 0, 1\}$. This suggests splitting the sum over m into the sums over $m = 3j + \ell$ for $\ell \in \{-1, 0, 1\}$. Using the lemma, they evaluate as

$$\begin{aligned}
\Big(\sum_{j\geqslant 0} 2^{(3j+\ell)sq}\Big\| \sum_{k=1}^n 2^{-3k(s+\frac{d}{p'})} a_k \varphi_{3j+\ell} * \varphi_{3k}\Big\|^q_{L^p(\mathbb{R}^d)}\Big)^{1/q} \\
&= \Big(\sum_{j=1}^n 2^{(3j+\ell)sq}\Big\| 2^{-3j(s+\frac{d}{p'})} a_j \varphi_{3j+\ell} * \varphi_{3j}\Big\|^q_{L^p(\mathbb{R}^d)}\Big)^{1/q} \\
&= C_{\ell,p,d}\Big(\sum_{j=1}^n 2^{(3j+\ell)sq} 2^{-3j(sq+\frac{dq}{p'})} \|a_j\|^q 2^{3jdq/p'}\Big)^{1/q} \\
&= 2^{\ell s} C_{\ell,p,d}\Big(\sum_{j=1}^n \|a_j\|^q\Big)^{1/q}.
\end{aligned}$$

We thus find (using the triangle inequality in ℓ^q_3 for the upper estimate)

$$\Big(\sum_{m\geqslant 0} 2^{msq}\Big\| \sum_{k=1}^n 2^{-3k(s+\frac{d}{p'})} a_k \varphi_m * \varphi_{3k}\Big\|^q_{L^p(\mathbb{R}^d)}\Big)^{1/q} \eqsim_{d,p,s} \Big(\sum_{\ell=1}^n \|a_\ell\|^q\Big)^{1/q}.$$

Inserting this norm equivalence into (14.35) (taking $(p, q, s) = (p_0, q_0, s_0)$ on the left and $(p, q, s) = (p_1, q_1, s_1)$ on the right) we obtain (14.34).

Finally suppose that (ii) holds. Then from $s_0 - \frac{d}{p_0} \geqslant s_1 - \frac{d}{p_1}$ we see that $s_0 \geqslant s_1$. If $s_0 = s_1$, then by arguing as above it follows that $q_0 \leqslant q_1$ and (i) follows. □

14.4.d Difference norms

In this section we show that Besov spaces with smoothness parameter $s > 0$ admit a characterisation in terms of difference norms. This characterisation can be often used to effectively check whether a given concrete function belongs to a given Besov space. For example, we check in Corollary 14.4.26 that the Besov spaces $B^s_{\infty,\infty}(\mathbb{R}^d; X)$ coincide with certain spaces of s-Hölder continuous functions.

For functions $f : \mathbb{R}^d \to X$ and vectors $h \in \mathbb{R}^d$, the function $\Delta_h f : \mathbb{R}^d \to X$ is defined by

$$\Delta_h f(x) := f(x + h) - f(x).$$

Clearly, the *difference operator* Δ_h thus defined is bounded as an operator on $L^p(\mathbb{R}^d; X)$ for all $1 \leqslant p \leqslant \infty$, with norm at most 2. We have the following formula for the powers $\Delta^m_h = (\Delta_h)^m$.

Lemma 14.4.22. *For all $f \in L^1(\mathbb{R}^d; X)$ and $h, \xi \in \mathbb{R}^d$ we have*

$$\Delta^m_h f = \sum_{j=0}^m \binom{m}{j} (-1)^j f(\cdot + (m - j)h).$$

Proof. The identity $\mathscr{F}(f(\cdot+h))(\xi) = e^{2\pi i h\cdot\xi}\widehat{f}$ implies $\mathscr{F}(\Delta_h f)(\xi) = (e^{2\pi i h\cdot\xi} - 1)\widehat{f}(\xi)$, from which it follows that

$$\mathscr{F}(\Delta_h^m f)(\xi) = (e^{2\pi i h\cdot\xi} - 1)^m \widehat{f}(\xi) = \sum_{j=0}^{m} \binom{m}{j} (-1)^j e^{2\pi i h\cdot\xi(m-j)} \widehat{f}(\xi).$$

Now apply the inverse Fourier transform. □

Definition 14.4.23 (Difference norm for Besov spaces). *Let $p, q, \tau \in [1, \infty]$, $s \in \mathbb{R}$, and $m \in \mathbb{N} \setminus \{0\}$. For functions $f \in L^p(\mathbb{R}^d; X)$ we define the* difference norm *by setting*

$$[f]^{(m,\tau)}_{B^s_{p,q}(\mathbb{R}^d;X)} := \Big(\int_0^\infty t^{-sq} \Big\| \Big(⨍_{\{|h|\leqslant t\}} \|\Delta_h^m f\|^\tau \,\mathrm{d}h \Big)^{1/\tau} \Big\|^q_{L^p(\mathbb{R}^d)} \frac{\mathrm{d}t}{t} \Big)^{1/q}$$

with obvious modifications for $q = \infty$ and/or $\tau = \infty$ where the integral with respect to $\mathrm{d}t/t$ and the average are replaced by essential suprema, and

$$|||f|||^{(m,\tau)}_{B^s_{p,q}(\mathbb{R}^d;X)} := \|f\|_{L^p(\mathbb{R}^d;X)} + [f]^{(m,\tau)}_{B^s_{p,q}(\mathbb{R}^d;X)}.$$

Here we used the notation $⨍_F := \frac{1}{|F|}\int_F$ to denote the average over the set F. In typical applications one takes $\tau \in \{1, p, \infty\}$.

It is clear that $\tau_0 \leqslant \tau_1$ implies

$$[f]^{(m,\tau_0)}_{B^s_{p,q}(\mathbb{R}^d;X)} \leqslant [f]^{(m,\tau_1)}_{B^s_{p,q}(\mathbb{R}^d;X)}. \tag{14.36}$$

The next theorem implies that if $s > 0$, then each of the norms $|||\cdot|||^{(m,\tau)}_{B^s_{p,q}(\mathbb{R}^d;X)}$ with $m > s$ defines an equivalent norm on $B^s_{p,q}(\mathbb{R}^d; X)$.

Theorem 14.4.24 (Difference norms for Besov spaces). *Let $p, q \in [1, \infty]$, $s > 0$, $\tau \in [1, \infty]$, and let $m > s$ be an integer. A function $f \in L^p(\mathbb{R}^d; X)$ belongs to $B^s_{p,q}(\mathbb{R}^d; X)$ if and only if $[f]^{m,\tau}_{B^s_{p,q}(\mathbb{R}^d;X)} < \infty$, and the following equivalence of norms holds:*

$$\|f\|_{B^s_{p,q}(\mathbb{R}^d;X)} \eqsim_{d,m,s} |||f|||^{(m,\tau)}_{B^s_{p,q}(\mathbb{R}^d;X)}.$$

Before turning to the details of the proof we give some simple applications. The first two identify the Sobolev–Slobodetskii spaces and the Hölder spaces (cf. Section 14.1 for the relevant notation) as Besov spaces.

Corollary 14.4.25 (Sobolev–Slobodetskii spaces). *Let $p \in [1, \infty)$ and $s \in (0, 1)$. Then*

$$B^s_{p,p}(\mathbb{R}^d; X) = W^{s,p}(\mathbb{R}^d; X)$$

with equivalent norms. In fact,

$$[f]^{(1,p)}_{B^s_{p,p}(\mathbb{R}^d;X)} = \frac{1}{(sp+d)^{1/p}|B_1|}[f]_{W^{s,p}(\mathbb{R}^d;X)}. \tag{14.37}$$

Proof. By Theorem 14.4.24 it suffices to prove the identity (14.37) for the seminorms, which follows from Fubini's theorem and a change of variable:

$$\begin{aligned}|B_1|^p\big([f]^{(1,p)}_{B^s_{p,p}(\mathbb{R}^d;X)}\big)^p &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_0^\infty \mathbf{1}_{\{|h|\leqslant t\}} t^{-sp-d-1}\|\Delta_h f(x)\|^p \,\mathrm{d}t\,\mathrm{d}h\,\mathrm{d}x\\ &= (sp+d)^{-1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|h|^{-sp-d}\|\Delta_h f(x)\|^p\,\mathrm{d}h\,\mathrm{d}x\\ &= (sp+d)^{-1}[f]^p_{W^{s,p}(\mathbb{R}^d;X)}.\end{aligned}$$

□

Corollary 14.4.26 (Hölder spaces). *Let X be a Banach space and let $s \in (0,\infty)\setminus\mathbb{N}$. Then*

$$B^s_{\infty,\infty}(\mathbb{R}^d;X) = C^s_{\mathrm{ub}}(\mathbb{R}^d;X)$$

with equivalent norms.

Proof. Let $s = k+\theta$, where $k\in\mathbb{N}$ and $\theta\in(0,1)$. It follows from Proposition 14.4.18 and Theorem 14.4.19 that we have continuous embeddings

$$B^s_{\infty,\infty}(\mathbb{R}^d;X)\hookrightarrow B^k_{\infty,1}(\mathbb{R}^d;X)\hookrightarrow C^k_{\mathrm{ub}}(\mathbb{R}^d;X).$$

Therefore there is no loss of generality in assuming that our functions are k-times continuously differentiable. For functions $f\in C^k_{\mathrm{ub}}(\mathbb{R}^d;X)$ and multi-indices $|\alpha|\leqslant k$, from Theorem 14.4.24 we infer the equivalences

$$\|\partial^\alpha f\|_{B^\theta_{\infty,\infty}(\mathbb{R}^d;X)} \eqsim_{d,\theta} \|\partial^\alpha f\|^{(1,\infty)}_{B^\theta_{\infty,\infty}(\mathbb{R}^d;X)} = \|\partial^\alpha f\|_{C^\theta_{\mathrm{ub}}(\mathbb{R}^d;X)},$$

where we used the continuous version of $\partial^\alpha f$ to replace the essential supremum by a supremum. Now the result follows after summation over all multi-indices $|\alpha|\leqslant k$ and an application of Proposition 14.4.17. □

Corollary 14.4.27 (Embeddings into Hölder spaces). *Let $p_0, q\in[1,\infty]$ and $s_0, s_1\geqslant 0$ satisfy $s_0-\frac{d}{p_0}\geqslant s_1$. Then we have the following continuous embeddings:*

(1) $B^{s_0}_{p_0,q}(\mathbb{R}^d;X)\hookrightarrow C^{s_1}_{\mathrm{ub}}(\mathbb{R}^d;X)$ *if* $s_1\notin\mathbb{N}$;
(2) $B^{s_0}_{p_0,1}(\mathbb{R}^d;X)\hookrightarrow C^{s_1}_{\mathrm{ub}}(\mathbb{R}^d;X)$.

Proof. (1): By Theorem 14.4.19 and Corollary 14.4.26,

$$B^{s_0}_{p_0,q}(\mathbb{R}^d;X)\hookrightarrow B^{s_1}_{\infty,\infty}(\mathbb{R}^d;X) = C^{s_1}_{\mathrm{ub}}(\mathbb{R}^d;X).$$

(2): The case $s_1\notin\mathbb{N}$ follows from the previous case. If $s_1\in\mathbb{N}$, then by Theorem 14.4.19 and Proposition 14.4.18,

$$B^{s_0}_{p_0,1}(\mathbb{R}^d;X)\hookrightarrow B^{s_1}_{\infty,1}(\mathbb{R}^d;X)\hookrightarrow C^{s_1}_{\mathrm{ub}}(\mathbb{R}^d;X).$$

□

The proof of Theorem 14.4.24 makes use the following simple lemma. Recall the Fourier multiplier notation of Subsection 14.4.b.

Lemma 14.4.28. *For non-zero $\xi, h \in \mathbb{R}^d$ let*

$$m_h(\xi) := \frac{e^{2\pi i h\cdot\xi} - 1}{2\pi i h \cdot \xi}.$$

Then for all $p \in [1,\infty]$ we have $m_h \in \mathfrak{M}L^p(\mathbb{R}^d;X)$ and $\|m_h\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)} \leqslant 1$.

Proof. By an elementary computation, the associated Fourier multiplier is given by

$$T_{m_h} f(x) = \int_0^1 f(x - ht)\,\mathrm{d}t = \mu_h * f(x), \quad f \in L^p(\mathbb{R}^d;X),$$

where $\mu_h(A) = \int_0^1 \mathbf{1}_{th\in A}\,\mathrm{d}t$ defines a measure by monotone convergence. Hence the result follows from (14.24). For $p < \infty$, one can also use the direct estimate

$$\|T_{m_h} f\|_{L^p(\mathbb{R}^d;X)} \leqslant \int_0^1 \|f(\cdot - ht)\|_{L^p(\mathbb{R}^d;X)}\,\mathrm{d}t = \|f\|_{L^p(\mathbb{R}^d;X)}.$$

□

Proof of Theorem 14.4.24. Let

$$I_p^{m,\tau}(f,k) := \Big\| \Big(\fint_{\{|h|\leqslant 1\}} \|\Delta^m_{2^{-k}h} f\|^\tau \,\mathrm{d}h \Big)^{1/\tau} \Big\|_{L^p(\mathbb{R}^d)},$$

where the integral average has to be replaced by $\sup_{|h|\leqslant 1}$ if $\tau = \infty$. Discretising the integral over t in the definition of the difference norm (Definition 14.4.23) and noting that

$$\fint_{\{|h|\leqslant t\}} \leqslant \frac{1}{\omega_d 2^{-kd}} \int_{\{|h|\leqslant 2^{-k+1}\}} = 2^d \fint_{\{|h|\leqslant 2^{-k+1}\}},$$

we obtain

$$\begin{aligned}
[f]^{(m,\tau)}_{B^s_{p,q}(\mathbb{R}^d;X)} &= \Big(\sum_{k\in\mathbb{Z}} \int_{2^{-k}}^{2^{-k+1}} t^{-sq-1} \Big\| \Big(\fint_{\{|h|\leqslant t\}} \|\Delta^m_h f\|^\tau \,\mathrm{d}h \Big)^{1/\tau} \Big\|^q_{L^p(\mathbb{R}^d)} \,\mathrm{d}t \Big)^{1/q} \\
&\leqslant 2^{d/\tau} \Big(\sum_{k\in\mathbb{Z}} 2^{ksq} \Big\| \Big(\fint_{\{|h|\leqslant 2^{-k+1}\}} \|\Delta^m_h f\|^\tau \,\mathrm{d}h \Big)^{1/\tau} \Big\|^q_{L^p(\mathbb{R}^d)} \Big)^{1/q} \\
&= 2^{d/\tau} \Big(\sum_{j\in\mathbb{Z}} 2^{(j+1)sq} \Big\| \Big(\fint_{\{|h|\leqslant 1\}} \|\Delta^m_{2^{-j}h} f\|^\tau \,\mathrm{d}h \Big)^{1/\tau} \Big\|^q_{L^p(\mathbb{R}^d)} \Big)^{1/q} \\
&= 2^{s+d/\tau} \Big(\sum_{j\in\mathbb{Z}} 2^{jsq} \Big\| \Big(\fint_{\{|h|\leqslant 1\}} \|\Delta^m_{2^{-j}h} f\|^\tau \,\mathrm{d}h \Big)^{1/\tau} \Big\|^q_{L^p(\mathbb{R}^d)} \Big)^{1/q}.
\end{aligned}$$

Similarly,

$$[f]^{(m,\tau)}_{B^s_{p,q}(\mathbb{R}^d;X)} \geqslant 2^{-s-1-d/\tau}\Big(\sum_{j\in\mathbb{Z}} 2^{jsq}\Big\|\Big(\fint_{\{|h|\leqslant 1\}} \|\Delta^m_{2^{-j}h} f\|^\tau \,\mathrm{d}h\Big)^{1/\tau}\Big\|^q_{L^p(\mathbb{R}^d)}\Big)^{1/q}.$$

Hence,

$$[f]^{(m,\tau)}_{B^s_{p,q}(\mathbb{R}^d;X)} \eqsim_{d,s,\tau} \big\|(2^{ks} I^{m,\tau}_p(f,k))_{k\in\mathbb{Z}}\big\|_{\ell^q(\mathbb{Z})}. \tag{14.38}$$

In view of (14.36) and (14.38) it thus suffices to prove the two estimates

$$\|f\|_{L^p(\mathbb{R}^d;X)} + \big\|(2^{ks} I^{m,\infty}_p(f,k))_{k\in\mathbb{Z}}\big\|_{\ell^q(\mathbb{Z})} \lesssim_{d,m,s} \|f\|_{B^s_{p,q}(\mathbb{R}^d;X)}, \tag{14.39}$$

$$\|f\|_{L^p(\mathbb{R}^d;X)} + \big\|(2^{ks} I^{m,1}_p(f,k))_{k\in\mathbb{Z}}\big\|_{\ell^q(\mathbb{Z})} \gtrsim_{s,m,d} \|f\|_{B^s_{p,q}(\mathbb{R}^d;X)}. \tag{14.40}$$

Throughout the proof of (14.39) and (14.40) we will use the standard algebraic properties of L^p-multipliers discussed in Section 5.3.a.

Put $f_j := \varphi_j * f$ for $j \geqslant 0$. By Hölder's inequality,

$$\|f\|_{L^p(\mathbb{R}^d;X)} \leqslant \sum_{j\geqslant 0} \|f_j\|_{L^p(\mathbb{R}^d;X)} \leqslant \big\|(2^{-js})_{j\geqslant 0}\big\|_{\ell^{q'}} \|f\|_{B^q_{p,q}(\mathbb{R}^d;X)},$$

where the assumption $s > 0$ implies the finiteness of the $\ell^{q'}$-norm. To prove (14.39) and (14.40) it therefore remains to estimate $I^{m,\infty}_p(f,k)$ from above and $I^{m,1}_p(f,k)$ from below.

Step 1 – We begin with the proof of (14.39). By Lemma 14.2.10 and the triangle inequality,

$$I^{m,\infty}_p(f,k) \leqslant \sum_{\ell=-1}^{1}\sum_{j\geqslant 0} I^{m,\infty}_p(\varphi_j * f_{j+\ell}, k),$$

observing the standing convention $\varphi_{-1} \equiv 0$ which implies that $f_{-1} \equiv 0$. Keeping in mind the operator norm inequality $\|\Delta_h\| \leqslant 2$ and (14.7), for $j \geqslant 1$ and arbitrary $g \in L^p(\mathbb{R}^d; X)$ we have

$$\begin{aligned} I^{m,\infty}_p(\varphi_j * g, k) &= \sup_{|h|\leqslant 1|} \big\|\Delta^m_{2^{-k}h}\varphi_j * g\big\|_{L^p(\mathbb{R}^d;X)} \\ &\leqslant 2^m \|\varphi_j * g\|_p \leqslant 2^{m+1}\|\varphi\|_1 \|g\|_p. \end{aligned} \tag{14.41}$$

On the other hand, using that $\widehat{\varphi}_j(\xi) = \widehat{\varphi}_1(2^{-(j-1)}\xi)$, we find that

$$\begin{aligned} I^{m,\infty}_p(\varphi_j * g, k) &\leqslant \sup_{|h|\leqslant 1} \|\mathscr{F}(\Delta^m_{2^{-k}h}\varphi_j)\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)}\|g\|_p \\ &\leqslant \sup_{|h|\leqslant 1} \|\xi \mapsto (e^{2\pi i 2^{-k}h\cdot\xi} - 1)^m \widehat{\varphi}_1(2^{-(j-1)}\cdot)\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)}\|g\|_p. \end{aligned} \tag{14.42}$$

By Lemma 14.4.28 and a dilation

$$\|\xi \mapsto (e^{2\pi i 2^{-k} h\cdot\xi} - 1)^m (h\cdot\xi)^{-m}\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)} \leqslant (2\pi)^m 2^{-km}. \tag{14.43}$$

Moreover, since φ_1 is a Schwartz function, dilation, and $|h| \leqslant 1$,

$$\begin{aligned}\|\xi \mapsto (h\cdot\xi)^m \widehat{\varphi}_1(2^{-(j-1)}\xi)\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)} &= 2^{(j-1)m}\|\xi \mapsto (h\cdot\xi)^m \widehat{\varphi}_1(\xi)\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)} \\ &\leqslant 2^{(j-1)m} \sum_{|\alpha|=m} c_{\alpha,m}\|\xi \mapsto \xi^\alpha \widehat{\varphi}_1(\xi)\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)} \\ &\leqslant C_{m,d} 2^{(j-1)m},\end{aligned} \tag{14.44}$$

where in the last step we used Proposition 14.4.11 with $\partial^\alpha \varphi_1 \in L^1(\mathbb{R}^d)$. Combining (14.41) with (14.42), estimating the latter using (14.43) and (14.44), we obtain the estimate

$$I_p^{m,\infty}(\varphi_j * g, k) \lesssim_{d,m} \min\{1, 2^{(j-k)m}\}\|g\|_p, \quad j \geqslant 1.$$

Similarly one checks that

$$I_p^{m,\infty}(\varphi_0 * g, k) \lesssim_{d,m} \min\{1, 2^{-km}\}\|g\|_p$$

Therefore, with $a_{j,m} = \min\{1, 2^{jm}\}$,

$$\begin{aligned}&\big\|(2^{ks} I_p^{m,\infty}(f,k))_{k\in\mathbb{Z}}\big\|_{\ell^q(\mathbb{Z})} \\ &\leqslant \sum_{\ell=-1}^{1} \Big\|\Big(2^{ks}\sum_{j\geqslant 0} I_p^{m,\infty}(\varphi_j * f_{j+\ell}, k)\Big)_{k\in\mathbb{Z}}\Big\|_{\ell^q(\mathbb{Z})} \\ &\lesssim_{d,m,s} \sum_{\ell=-1}^{1} \Big\|\Big(\sum_{j\geqslant 0} 2^{-(j-k)s} a_{j-k,m} 2^{(j+\ell)s}\|f_{j+\ell}\|_p\Big)_{k\in\mathbb{Z}}\Big\|_{\ell^q(\mathbb{Z})} \\ &\lesssim_s \big\|(2^{-js}a_{j,m})_{j\geqslant 0}\big\|_{\ell^1} \big\|(2^{(j+\ell)s}\|f_j\|_p)_{j\geqslant 0}\big\|_{\ell^q} \\ &\lesssim_s \|f\|_{B^s_{p,q}(\mathbb{R}^d;X)},\end{aligned}$$

where we applied the discrete version of Young's inequality and used the assumption $m > s$ for the finiteness of the ℓ^1 norm.

Step 2 – In this step we prove (14.40). For $k \geqslant 0$ let $T_k f := 2^{kd}\varphi(2^k\cdot) * f$ and $S_k f := \varphi_k * f$. By (14.3), for $k \geqslant 1$ we have $S_k = T_k - T_{k-1} = (I - T_{k-1}) - (I - T_k)$ and therefore

$$\begin{aligned}\|f\|_{B^s_{p,q}(\mathbb{R}^d;X)} &= \big\|(2^{ks}\|S_k f\|_{L^p(\mathbb{R}^d;X)})_{k\geqslant 0}\big\|_{\ell^q} \\ &\leqslant \|S_0 f\|_{L^p(\mathbb{R}^d;X)} + 2\big\|(2^{ks}\|T_k f - f\|_{L^p(\mathbb{R}^d;X)})_{k\geqslant 0}\big\|_{\ell^q}.\end{aligned} \tag{14.45}$$

By Young's inequality,

$$\|S_0 f\|_{L^p(\mathbb{R}^d;X)} \leqslant \|\varphi_0\|_1 \|f\|_{L^p(\mathbb{R}^d;X)}. \tag{14.46}$$

It remains to estimate the terms with $k \geqslant 0$ by the difference norm.

Choose $\psi \in \mathscr{S}(\mathbb{R}^d)$ such that $\widehat{\psi}(\xi) = 1$ if $|\xi| \leqslant 1$ and $\widehat{\psi}(\xi) = 0$ if $|\xi| \geqslant 3/2$. Let $\varphi \in \mathscr{S}(\mathbb{R}^d)$ be given by

$$\widehat{\varphi}(\xi) = (-1)^{m+1} \sum_{j=0}^{m-1} \binom{m}{j} (-1)^j \widehat{\psi}(-(m-j)\xi)$$

and define the sequence $(\varphi_k)_{k\geqslant 0}$ as in (14.3). For $|\xi| \leqslant 1/m$ and $0 \leqslant j \leqslant m-1$ we have $\widehat{\psi}(-(m-j)\xi) = 1$ and therefore

$$\widehat{\varphi}(\xi) = (-1)^{m+1} \sum_{j=0}^{m-1} \binom{m}{j} (-1)^j = (-1)^{m+1} \Big(\sum_{j=0}^{m} \binom{m}{j} (-1)^j - (-1)^m \Big) = 1$$

by the binomial theorem, and for $|\xi| \geqslant 3/2$ we have $\widehat{\varphi}(\xi) = 0$. Furthermore the Fourier supports of φ_j and φ_k are disjoint for $|j-k| \geqslant N_m$, where $N_m \in \mathbb{N}$ only depends on m (rather than for $|j-k| \geqslant 2$ as in (14.10) in the case of an inhomogeneous Littlewood–Paley sequence). Thanks to these properties, the proof of Proposition 14.4.2 may be repeated to see that this system leads to an equivalent norm on $B^s_{p,q}(\mathbb{R}^d;X)$.

Let $f \in L^p(\mathbb{R}^d;X)$. We claim that

$$T_k f(x) - f(x) = (-1)^{m+1} \int_{\mathbb{R}^d} \Delta^m_{2^{-k}y} f(x) \psi(y)\,\mathrm{d}y \tag{14.47}$$

Indeed, taking Fourier transforms in the x-variable and using Lemma 14.4.22 and the fact that $\widehat{\psi}(0) = 1$, we have

$$\begin{aligned}
\widehat{T_k f}(\xi) - \widehat{f}(\xi) &= (\widehat{\varphi}(2^{-k}\xi) - 1)\widehat{f}(\xi) \\
&= \Big((-1)^{m+1} \sum_{j=0}^{m-1} \binom{m}{j} (-1)^j \widehat{\psi}(-(m-j)2^{-k}\xi) - 1 \Big) \widehat{f}(\xi) \\
&= (-1)^{m+1} \sum_{j=0}^{m} \binom{m}{j} (-1)^j \widehat{\psi}(-(m-j)2^{-k}\xi) \\
&= (-1)^{m+1} \sum_{j=0}^{m} \binom{m}{j} (-1)^j \int_{\mathbb{R}^d} e^{2\pi i (m-j) 2^{-k} y\cdot\xi} \psi(y)\,\mathrm{d}y \\
&= (-1)^{m+1} \int_{\mathbb{R}^d} (e^{2\pi i 2^{-k} y\cdot\xi} - 1)^m \widehat{f}(\xi) \psi(y)\,\mathrm{d}y \\
&= (-1)^{m+1} \int_{\mathbb{R}^d} \mathscr{F}(\Delta^m_{2^{-k}y} f)(\xi) \psi(y)\,\mathrm{d}y
\end{aligned}$$

and the claim follows.

Fix a real number $r > 0$, the numerical value of which will be fixed in a moment. Taking norms in (14.47), using that $\sup_{x\in\mathbb{R}^d}(1+|x|^r)|\psi(x)| < \infty$, and writing $B_R := \{\xi \in \mathbb{R}^d : |\xi| \leqslant R\}$, it follows that

$$
\begin{aligned}
&\|f(x) - T_k f(x)\| \\
&\quad \leqslant \int_{\mathbb{R}^d} \|\Delta^m_{2^{-k}y} f(x)\psi(y)\| \, \mathrm{d}y \\
&\quad \lesssim_\psi \int_{B_1} \|\Delta^m_{2^{-k}y} f(x)\| \, \mathrm{d}y + \sum_{j\geqslant 0} 2^{-(j+1)r} \int_{B_{2^{j+1}}\setminus B_{2^j}} \|\Delta^m_{2^{-k}y} f(x)\| \, \mathrm{d}y \\
&\quad = \int_{B_1} \|\Delta^m_{2^{-k}y} f(x)\| \, \mathrm{d}y + \sum_{j\geqslant 0} 2^{-(j+1)(r-d)} \int_{B_1\setminus B_{\frac12}} \|\Delta^m_{2^{j+1-k}h} f(x)\| \, \mathrm{d}h \\
&\quad \leqslant \sum_{j\geqslant 0} 2^{-j(r-d)} \int_{B_1} \|\Delta^m_{2^{j-k}h} f(x)\| \, \mathrm{d}h.
\end{aligned}
$$

Taking L^p-norms with respect to x, we obtain the estimate

$$
\|T_k f - f\|_{L^p(\mathbb{R}^d;X)} \lesssim_{d,\psi} \sum_{j\geqslant 0} 2^{-j(r-d)} I^{m,1}_p(f, k-j).
$$

Taking ℓ^q-norms with respect to $k \geqslant 0$ and choosing $r > d + s$, we obtain

$$
\begin{aligned}
&\big\|(2^{ks}\|T_k f - f\|_{L^p(\mathbb{R}^d;X)})_{k\geqslant 0}\big\|_{\ell^q} \\
&\quad \lesssim_{d,\psi} \Big\|\Big(\sum_{j\geqslant 0} 2^{-j(r-d)} 2^{ks} I^{m,1}_p(f, k-j)\Big)_{k\geqslant 0}\Big\|_{\ell^q} \\
&\quad = \Big\|\Big(\sum_{j\geqslant 0} 2^{-j(r-d-s)} 2^{(k-j)s} I^{m,1}_p(f, k-j)\Big)_{k\geqslant 0}\Big\|_{\ell^q} \\
&\quad \leqslant \sum_{j\geqslant 0} 2^{-j(r-d-s)} \big\|(2^{(k-j)s} I^{m,1}_p(f, k-j))_{k\geqslant 0}\big\|_{\ell^q} \\
&\quad \leqslant \sum_{j\geqslant 0} 2^{-j(r-d-s)} \big\|(2^{ks} I^{m,1}_p(f, k))_{k\in\mathbb{Z}}\big\|_{\ell^q} \\
&\quad = \sum_{j\geqslant 0} 2^{-j(r-d-s)} \big\|(2^{ks} I^{m,1}_p(f, k))_{k\in\mathbb{Z}}\big\|_{\ell^q}.
\end{aligned}
$$

In combination with (14.45) and (14.46) this proves estimate (14.40). □

14.4.e Interpolation

In order to consider interpolation for Besov spaces, we will now introduce the so-called retraction and co-retraction operators, which allow us to reduce questions about the interpolation of Besov spaces to the corresponding questions about the spaces $\ell^q_{w_s}(L^p(\mathbb{R}^d;X))$.

Lemma 14.4.29. *Let $p, q \in [1, \infty]$ and $s \in \mathbb{R}$. For $k \geqslant 0$ set $\psi_k := \varphi_{k-1} + \varphi_k + \varphi_{k+1}$. Define the operators*

$$R : \ell^q_{w_s}(L^p(\mathbb{R}^d; X)) \to B^s_{p,q}(\mathbb{R}^d; X)$$
$$S : B^s_{p,q}(\mathbb{R}^d; X) \to \ell^q_{w_s}(L^p(\mathbb{R}^d; X))$$

by

$$R((f_k)_{k\geqslant 0}) = \sum_{k\geqslant 0} \psi_k * f_k, \quad Sf = (\varphi_k * f)_{k\geqslant 0}.$$

Then R is bounded of norm $\leqslant 60\|\varphi_0\|_1^2 4^{|s|}$, S is an isometry, and $RS = I$.

Proof. It is clear from the definitions of the spaces involved that S is an isometry. Next we turn to the proof that R is well defined and bounded. By (14.7) and Young's inequality, $\|\varphi_{k+\ell} * \psi_k\|_1 \leqslant 12\|\varphi_0\|_1^2$. Therefore, by another application of Young's inequality and (14.11),

$$\begin{aligned}
\Big\|\sum_{k\geqslant 0} \psi_k * f_k\Big\|_{B^s_{p,q}(\mathbb{R}^d;X)} &= \Big\|\Big(\varphi_j * \sum_{k\geqslant 0} \psi_k * f_k\Big)_{j\geqslant 0}\Big\|_{\ell^q_{w_s}(L^p(\mathbb{R}^d;X)} \\
&= \Big\|\Big(\varphi_j * \sum_{|\ell|\leqslant 2} \psi_{j+\ell} * f_{j+\ell}\Big)_{j\geqslant 0}\Big\|_{\ell^q_{w_s}(L^p(\mathbb{R}^d;X)} \\
&\leqslant \sum_{|\ell|\leqslant 2} \big\|(\varphi_j * \psi_{j+\ell} * f_{j+\ell})_{j\geqslant 0}\big\|_{\ell^q_{w_s}(L^p(\mathbb{R}^d;X))} \\
&\leqslant 12\|\varphi_0\|_1^2 \sum_{|\ell|\leqslant 2} \big\|(f_{j+\ell})_{j\geqslant 0}\big\|_{\ell^q_{w_s}(L^p(\mathbb{R}^d;X))} \\
&\leqslant 60\|\varphi_0\|_1^2 4^{|s|} \|(f_j)_{j\geqslant 0}\|_{\ell^q_{w_s}(L^p(\mathbb{R}^d;X))},
\end{aligned}$$

the convergence of the sum $\sum_{k\geqslant 0} \psi_k * f_k$ in $B^s_{p,q}(\mathbb{R}^d; X)$ being a consequence of the convergence of the sum $\sum_{j\geqslant 0} 2^{js} f_j$ in $L^p(\mathbb{R}^d; X)$, for this allows to first perform the same estimates for differences of partial sums.

The identity $RS = I$ follows from Lemma 14.2.10 and the fact that $\widehat{\psi}_k \equiv 1$ on $\mathrm{supp}(\widehat{\varphi}_k)$. ∎

Now we are ready identify the complex interpolation spaces of Besov spaces in a very general setting. In contrast to the complex interpolation results for Sobolev and Bessel potential spaces in Section 5.6, where it was necessary to impose UMD assumptions, no geometric restrictions on the interpolation couple (X_0, X_1) are needed.

Theorem 14.4.30 (Complex interpolation of Besov spaces). *Let (X_0, X_1) be an interpolation couple of Banach spaces, let $p_0, p_1, q_0, q_1 \in [1, \infty]$ satisfy $\min\{p_0, p_1\} < \infty$ and $\min\{q_0, q_1\} < \infty$, and let $s_0, s_1 \in \mathbb{R}$ and $\theta \in (0, 1)$. Furthermore let $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, and $s = (1-\theta)s_0 + \theta s_1$. Then*

$$[B^{s_0}_{p_0,q_0}(\mathbb{R}^d;X_0), B^{s_1}_{p_1,q_1}(\mathbb{R}^d;X_1)]_\theta = B^s_{p,q}(\mathbb{R}^d;[X_0,X_1]_\theta)$$

with equivalent norms.

Proof. Let $R : \ell^q_{w_s}(L^p(\mathbb{R}^d;X)) \to B^s_{p,q}(\mathbb{R}^d;X)$ and $S : B^s_{p,q}(\mathbb{R}^d;X) \to \ell^q_{w_s}(L^p(\mathbb{R}^d;X))$ be the retraction and co-retraction operators of Lemma 14.4.29. Set

$$E_j := \ell^{q_j}_{w_{s_j}}(L^{p_j}(\mathbb{R}^d;X_j)), \quad F_j := B^{s_j}_{p_j,q_j}(\mathbb{R}^d;X_j), \quad j \in \{0,1\},$$

and

$$E_\theta := (E_0,E_1)_\theta, \quad F_\theta := (F_0,F_1)_\theta, \quad X_\theta := [X_0,X_1]_\theta.$$

By Theorem 2.2.6 and Proposition 14.3.3, $E_\theta = \ell^q_{w_s}(L^p(\mathbb{R}^d;X_\theta))$ isometrically. Therefore,

$$[B^{s_0}_{p_0,q_0}(\mathbb{R}^d;X_0), B^{s_1}_{p_0,q_0}(\mathbb{R}^d;X_1)]_\theta = F_\theta = RSF_\theta \subseteq RE_\theta \subseteq B^s_{p,q}(\mathbb{R}^d;X_\theta),$$

and for all $f \in F_\theta$ we have

$$\|f\|_{B^s_{p,q}(\mathbb{R}^d;X_\theta)} = \|Sf\|_{\ell^q_{w_s}(L^p(\mathbb{R}^d;X_\theta))} = \|Sf\|_{E_\theta} \leqslant \|f\|_{F_\theta}$$

using Theorem C.3.3. Conversely, by Theorem C.3.3,

$$B^s_{p,q}(\mathbb{R}^d;X_\theta) = RSB^s_{p,q}(\mathbb{R}^d;X_\theta) \subseteq RE_\theta \subseteq F_\theta,$$

and for all $f \in B^s_{p,q}(\mathbb{R}^d;X_\theta)$ we have

$$\|f\|_{F_\theta} = \|RSf\|_{F_\theta} \leqslant C\|Sf\|_{E_\theta} = C\|Sf\|_{\ell^r_{w_s}(L^p(\mathbb{R}^d;X_\theta))} = C\|f\|_{B^s_{p,q}(\mathbb{R}^d;X_\theta)},$$

where $C = 60\|\varphi_0\|_1^2 4^{|s|}$ is the constant of Lemma 14.4.29. □

In the next result we identify the Besov spaces as the real interpolation spaces of Besov spaces, Bessel potential spaces, and Sobolev spaces, allowing only non-negative integer values of s in the latter case. In contrast to the case of complex interpolation, the integrability exponent p as well as the range space X are fixed.

Theorem 14.4.31 (Real interpolation of Besov spaces). *Let X be a Banach space, let $p,q,q_0,q_1 \in [1,\infty]$, let $s_0,s_1 \in \mathbb{R}$ satisfy $s_0 \neq s_1$, and let $\theta \in (0,1)$ and $s = (1-\theta)s_0 + \theta s_1$. Then*

$$(B^{s_0}_{p,q_0}(\mathbb{R}^d;X), B^{s_1}_{p,q_1}(\mathbb{R}^d;X))_{\theta,q} = B^s_{p,q}(\mathbb{R}^d;X), \tag{14.48}$$

$$(H^{s_0,p}(\mathbb{R}^d;X), H^{s_1,p}(\mathbb{R}^d;X))_{\theta,q} = B^s_{p,q}(\mathbb{R}^d;X), \tag{14.49}$$

with equivalent norms. If we additionally assume that $s_0,s_1 \in \mathbb{N}$, then

$$(W^{s_0,p}(\mathbb{R}^d;X), W^{s_1,p}(\mathbb{R}^d;X))_{\theta,q} = B^s_{p,q}(\mathbb{R}^d;X) \tag{14.50}$$

with equivalent norms. If instead we additionally assume that $p \in [1,\infty)$ *and* $s_0, s_1 \in (0,1)$*, then*

$$(W^{s_0,p}(\mathbb{R}^d;X), W^{s_1,p}(\mathbb{R}^d;X))_{\theta,q} = B^s_{p,q}(\mathbb{R}^d;X) \tag{14.51}$$

with equivalent norms.

Proof. The identification (14.51) follows from (14.48) and Corollary 14.4.25. We will give the proof of the remaining identifications in two steps.

Step 1 – If we can prove that (14.48) holds for $q_0 = q_1 \in \{1,\infty\}$, then all remaining cases can be inferred as follows. Let $\mathcal{A}^{s_j,p}_{q_j} \in \{B^{s_j}_{p,q_j}, H^{s_j,p}, W^{s_j,p}\}$, where we assume that $s_j \in \mathbb{N}$ if $\mathcal{A}^{s_j,p}_{q_j} = W^{s_j,p}$. Then by (14.48), Theorem C.3.3, (14.22), and Theorem 14.4.18, we have continuous embeddings

$$\begin{aligned}
B^s_{p,q}(\mathbb{R}^d;X) &= (B^{s_0}_{p,1}(\mathbb{R}^d;X), B^{s_1}_{p,1}(\mathbb{R}^d;X))_{\theta,q} \\
&\hookrightarrow (\mathcal{A}^{s_0,p}(\mathbb{R}^d;X), \mathcal{A}^{s_1,p}(\mathbb{R}^d;X))_{\theta,q} \\
&\hookrightarrow (B^{s_0}_{p,\infty}(\mathbb{R}^d;X), B^{s_1}_{p,\infty}(\mathbb{R}^d;X))_{\theta,q} = B^s_{p,q}(\mathbb{R}^d;X),
\end{aligned}$$

and (14.48), (14.49), (14.50) follow.

Step 2 – It remains to prove (14.48) for $r := q_0 = q_1 \in \{1,\infty\}$. The argument is similar to that of Theorem 14.4.30.

Let R and S be the retraction and co-retraction operators considered in Lemma 14.4.29. Let

$$E_j := \ell^r_{w_{s_j}}(L^p(\mathbb{R}^d;X)), \quad F_j := B^{s_j}_{p,r}(\mathbb{R}^d;X), \quad j \in \{0,1\},$$

and

$$E_{\theta,q} := (E_0, E_1)_{\theta,q}, \quad F_{\theta,q} := (F_0, F_1)_{\theta,q}.$$

By Proposition 14.3.5, $E_{\theta,q} = \ell^q_{w_s}(L^p(\mathbb{R}^d;X))$ with equivalent norms, say with constants C_1, C_2 (depending on θ, p, q, s_0, s_1), i.e.,

$$C_1^{-1}\|g\|_{E_{\theta,q}} \leqslant \|g\|_{\ell^q_{w_s}(L^p(\mathbb{R}^d;X))} \leqslant C_2\|g\|_{E_{\theta,q}}.$$

From Theorem C.3.3 it follows that

$$(B^{s_0}_{p,r}(\mathbb{R}^d;X), B^{s_1}_{p,r}(\mathbb{R}^d;X))_{\theta,q} = F_{\theta,q} = RSF_{\theta,q} \subseteq RE_{\theta,q} \subseteq B^s_{p,q}(\mathbb{R}^d;X),$$

and for all $f \in F_{\theta,q}$ we have

$$\|f\|_{B^s_{p,q}(\mathbb{R}^d;X)} = \|Sf\|_{\ell^q_{w_s}(L^p(\mathbb{R}^d;X))} \leqslant C_2\|Sf\|_{E_{\theta,q}} = C_2\|f\|_{F_{\theta,q}}.$$

In the converse direction, interpolation R and S by Theorem C.3.3,

$$B^s_{p,q}(\mathbb{R}^d;X) = RSB^s_{p,q}(\mathbb{R}^d;X) \subseteq RE_{\theta,q} \subseteq F_{\theta,q},$$

and for all $f \in B^s_{p,q}(\mathbb{R}^d;X)$ we have

$$\begin{aligned}
\|f\|_{F_{\theta,q}} &= \|RSf\|_{F_{\theta,q}} \\
&\leqslant C\|Sf\|_{E_{\theta,q}} \lesssim C\|Sf\|_{\ell^q_{w_s}(L^p(\mathbb{R}^d;X))} = C_3C_1\|f\|_{B^s_{p,q}(\mathbb{R}^d;X)},
\end{aligned}$$

where $C = 60\|\varphi_0\|_1^2 4^{|s|}$ is the constant of Lemma 14.4.29. □

Corollary 14.4.32. *Let $s_0, s_1 \in [0,\infty)$ satisfy $s_0 \neq s_1$, let $\theta \in [0,1]$, and put $s := (1-\theta)s_0 + \theta s_1$. Then*

$$(C^{s_0}_{\rm ub}(\mathbb{R}^d;X), C^{s_1}_{\rm ub}(\mathbb{R}^d;X))_{\theta,\infty} = B^s_{\infty,\infty}(\mathbb{R}^d;X)$$

with equivalent norms. Moreover, if $s \notin \mathbb{N}$, then $B^s_{\infty,\infty}(\mathbb{R}^d;X) = C^s_{\rm ub}(\mathbb{R}^d;X)$ with equivalent norms and therefore

$$(C^{s_0}_{\rm ub}(\mathbb{R}^d;X), C^{s_1}_{\rm ub}(\mathbb{R}^d;X))_{\theta,\infty} = (C^s_{\rm ub}(\mathbb{R}^d;X).$$

Proof. By Corollary 14.4.26 it suffices to prove the first identity. Since by Proposition 14.4.18 we have continuous embeddings $B^{s_j}_{\infty,1}(\mathbb{R}^d;X) \hookrightarrow C^{s_j}_{\rm ub}(\mathbb{R}^d;X) \hookrightarrow B^{s_j}_{\infty,\infty}(\mathbb{R}^d;X)$ we can straightforwardly adapt the proof of Theorem 14.4.31. □

As a simple application we show that multiplication by a smooth function leads to a bounded operator on Besov spaces.

Example 14.4.33 (Pointwise multiplication by smooth functions – I). Let $p,q \in [1,\infty]$ and $s > 0$, and let $k \in (s,\infty) \cap \mathbb{N}$. If $\zeta \in C^k_{\rm b}(\mathbb{R}^d;\mathscr{L}(X,Y))$, then pointwise multiplication

$$f \mapsto \zeta f$$

defines a bounded operator from $B^s_{p,q}(\mathbb{R}^d;X)$ into $B^s_{p,q}(\mathbb{R}^d;Y)$ of norm

$$\|f \mapsto \zeta f\|_{\mathscr{L}(B^s_{p,q}(\mathbb{R}^d;X),B^s_{p,q}(\mathbb{R}^d;Y))} \lesssim_{k,s} \|\zeta\|_{C^k_{\rm b}(\mathbb{R}^d;\mathscr{L}(X,Y))}.$$

Indeed, $f \mapsto \zeta f$ is bounded as a mapping from $W^{j,p}(\mathbb{R}^d;X)$ into $W^{j,p}(\mathbb{R}^d;Y)$ for each $j \in \{0,\dots,k\}$. Interpolating between the cases $j=0$ and $j=k$ by the real method with parameters $(\frac{s}{k},q)$ and applying Theorems 14.4.31 and C.3.3, the desired result is obtained. Alternatively one can prove the boundedness as a consequence of Theorem 14.4.24.

14.4.f Duality

The main result of this section identifies the duals of Besov spaces $B^s_{p,q}(\mathbb{R}^d;X)$ for $p,q \in [1,\infty)$. It is interesting that no geometric assumptions are needed on X. This contrasts with the situation for vector-valued Bochner spaces: recall that, by Theorem 1.3.10, for σ-finite measures spaces one has $L^p(S;X) = L^{p'}(S;X^*)$ if and only if X^* has the Radon–Nikodým property.

We start with the preliminary observation that elements in the duals of Besov spaces can be naturally identified with tempered distributions. Indeed, if $g \in B^s_{p,q}(\mathbb{R}^d;X)^*$, then for all $\varphi \in \mathscr{S}(\mathbb{R}^d)$ and $x \in X$ we have

$$|\langle \varphi\otimes x, g\rangle| \leqslant \|\varphi\otimes x\|_{B^s_{p,q}(\mathbb{R}^d;X)}\|g\|_{B^s_{p,q}(\mathbb{R}^d;X)^*} = \|\varphi\|_{B^s_{p,q}(\mathbb{R}^d)}\|g\|_{B^s_{p,q}(\mathbb{R}^d;X)^*}\|x\|,$$

where we used Proposition 14.4.3 to identify the Schwartz function φ with an element of $B^s_{p,q}(\mathbb{R}^d)$. Thus the mapping $x \mapsto \langle \varphi\otimes x, g\rangle$ defines an element

$g_\varphi \in X^*$, of norm $\|g_\varphi\| \leqslant \|\varphi\|_{B^s_{p,q}(\mathbb{R}^d)} \|g\|_{B^s_{p,q}(\mathbb{R}^d;X)^*}$. By the continuity of the embedding $\mathscr{S}(\mathbb{R}^d) \hookrightarrow B^s_{p,q}(\mathbb{R}^d)$ (see Proposition 14.4.3), this implies that the mapping $\varphi \to g_\varphi$ defines an element in $\mathscr{S}'(\mathbb{R}^d; X^*)$.

In the converse direction, for $g \in \mathscr{S}'(\mathbb{R}^d; X^*)$ and elements $f = \sum_{n=1}^N \zeta_n \otimes x_n$ in $\mathscr{S}(\mathbb{R}^d) \otimes X$, we can define

$$g(f) := \sum_{n=1}^N \langle x_n, g(\zeta_n)\rangle. \tag{14.52}$$

In order to check whether the mapping $f \mapsto g(f)$ defines an element of $B^s_{p,q}(\mathbb{R}^d; X)^*$, with $p, q \in [1, \infty)$, by the density results contained in Lemma 14.2.1 and Proposition 14.6.8, it suffices to check that there is a constant $C \geqslant 0$ such that

$$|g(f)| \leqslant C\|f\|_{B^s_{p,q}(\mathbb{R}^d;X)}, \quad f \in \mathscr{S}(\mathbb{R}^d) \otimes X. \tag{14.53}$$

Theorem 14.4.34. *Let X be a Banach space and let $p, q \in [1, \infty)$ and $s \in \mathbb{R}$. Then every $g \in B^{-s}_{p',q'}(\mathbb{R}^d; X^*)$, when viewed as an element of $\mathscr{S}'(\mathbb{R}^d; X^*)$, determines a unique element of $B^s_{p,q}(\mathbb{R}^d; X)^*$, and this identification sets up a natural isomorphism of Banach spaces*

$$B^s_{p,q}(\mathbb{R}^d; X)^* \simeq B^{-s}_{p',q'}(\mathbb{R}^d; X^*).$$

Proof. The second assertion follows from the first, combined with Corollary 14.4.25.

As a preliminary observation to the proof of the first assertion, we recall Proposition 2.4.32, which asserts that if $g \in \mathscr{S}'(\mathbb{R}^d; X^*)$ and $\zeta \in \mathscr{S}(\mathbb{R}^d)$, then $\zeta * g$ is in $C^\infty(\mathbb{R}^d; X^*)$ and $\partial^\alpha g$ has polynomial growth for any $\alpha \in \mathbb{N}^d$. Moreover, by Lemma 14.2.10, and the support properties (14.11), (14.12), we have the identity

$$g(\zeta) = \sum_{j\geqslant 0} \int_{\mathbb{R}^d} \langle \zeta(t), g_j(t)\rangle \,\mathrm{d}t = \sum_{\ell=-1}^{1} \sum_{j\geqslant 0} \int_{\mathbb{R}^d} \langle \varphi_{j+\ell} * \zeta(t), g_j(t)\rangle \,\mathrm{d}t, \tag{14.54}$$

where $g_j := \varphi_j * g$.

We split the proof of the theorem into three steps.

Step 1 – First let $g \in B^{-s}_{p',q'}(\mathbb{R}^d; X^*)$. Identifying g with an element of $\mathscr{S}'(\mathbb{R}^d; X^*)$, in order to prove that g defines an element of $B^s_{p,q}(\mathbb{R}^d; X)^*$ we will check that the duality given by (14.52) satisfies the bound (14.53).

By (14.54), if $f \in \mathscr{S}(\mathbb{R}^d) \otimes X$ is as in (14.52), then with $f_j := \varphi_j * f$ we have

$$g(f) = \sum_{\ell=-1}^{1} \sum_{j\geqslant 0} \int_{\mathbb{R}^d} \langle f_{j+\ell}(t), g_j(t)\rangle \,\mathrm{d}t.$$

By Hölder's inequality,

$$
\begin{aligned}
|g(f)| &\leqslant \sum_{\ell=-1}^{1} \sum_{j\geqslant 0} \int_{\mathbb{R}^d} |\langle f_{j+\ell}(t), g_j(t)\rangle| \, \mathrm{d}t \\
&\leqslant \sum_{\ell=-1}^{1} 2^{-\ell s} \big\| (2^{(j+\ell)s} f_{j+\ell})_{j\geqslant 0} \big\|_{\ell^q(L^p(\mathbb{R}^d;X))} \big\| (2^{-js} g_j)_{j\geqslant 0} \big\|_{\ell^{q'}(L^{p'}(\mathbb{R}^d;X^*))} \\
&\leqslant 3 \cdot 2^{|s|} \|f\|_{B^s_{p,q}(\mathbb{R}^d;X)} \|g\|_{B^{-s}_{p',q'}(\mathbb{R}^d;X^*)}.
\end{aligned}
$$

This verifies the bound (14.53).

Step 2 – Suppose next that $g \in B^s_{p,q}(\mathbb{R}^d;X)^*$. As explained above, we can identify g with an element of $\mathscr{S}'(\mathbb{R}^d;X^*)$. Let $(f_j)_{j\geqslant 0}$ be any finitely non-zero sequence in $\mathscr{S}(\mathbb{R}^d)\otimes X$ such that $\|(2^{js}f_j)_{j\geqslant 0}\|_{\ell^q(L^p(\mathbb{R}^d;X))} \leqslant 1$. Put $f := R(f_j)_{j\geqslant 0}$, where $R : \ell^q_{w_s}(L^p(\mathbb{R}^d;X)) \to B^s_{p,q}(\mathbb{R}^d;X)$ is the operator considered in Lemma 14.4.29. Then by (14.54) and the fact that $\widehat{\psi}_j = \widehat{\varphi}_{j-1}+\widehat{\varphi}_j+\widehat{\varphi}_{j+1} = 1$ on $\mathrm{supp}(\widehat{\varphi}_j)$ we see that

$$
g(f) = \sum_{j\geqslant 0} \int_{\mathbb{R}^d} \langle f(t), g_j(t)\rangle \, \mathrm{d}t = \sum_{j\geqslant 0} \int_{\mathbb{R}^d} \langle f_j(t), g_j(t)\rangle \, \mathrm{d}t.
$$

Therefore,

$$
\begin{aligned}
\Big| \sum_{j\geqslant 0} \int_{\mathbb{R}^d} \langle 2^{js} f_j(t), 2^{-js} g_j(t)\rangle \, \mathrm{d}t \Big| = |g(f)| &\leqslant \|f\|_{B^s_{p,q}(\mathbb{R}^d;X)} \|g\|_{B^s_{p,q}(\mathbb{R}^d;X)^*} \\
&\leqslant \|R\| \, \|g\|_{B^s_{p,q}(\mathbb{R}^d;X)^*}.
\end{aligned}
$$

Taking the supremum over all admissible finitely non-zero sequences $(f_j)_{j\geqslant 0}$, Propositions 1.3.1 and 1.3.3 imply that g belongs to $B^{-s}_{p',q'}(\mathbb{R}^d;X^*)$ and

$$
\|g\|_{B^{-s}_{p',q'}(\mathbb{R}^d;X^*)} = \big\| (2^{-js} g_j)_{j\geqslant 0} \big\|_{\ell^q(L^p(\mathbb{R}^d;X^*))} \leqslant \|R\| \, \|g\|_{B^s_{p,q}(\mathbb{R}^d;X)^*}.
$$

Step 3 – Since the identifications in Steps 2 and 3 are inverse to each other, they set up a bijective correspondence, and the estimates in the above proof show that this correspondence is bounded in both directions. □

Theorem 14.4.34 permits an extension of Example 14.4.33 to negative smoothness exponents.

Example 14.4.35 (Pointwise multiplication by smooth functions – II). Let X and Y be Banach spaces, let $p \in (1,\infty)$, $q \in [1,\infty]$, $s \leqslant 0$, and let $k \in (|s|,\infty)\cap\mathbb{N}$. For functions $\zeta \in C^k_{\mathrm{b}}(\mathbb{R}^d;\mathscr{L}(X,Y))$, the pointwise multiplication

$$
f \mapsto \zeta f
$$

defines a bounded operator from $B^s_{p,q}(\mathbb{R}^d;X)$ into $B^s_{p,q}(\mathbb{R}^d;Y)$ of norm

$$\|f \mapsto \zeta f\|_{\mathscr{L}(B^s_{p,q}(\mathbb{R}^d;X),B^s_{p,q}(\mathbb{R}^d;Y))} \lesssim_{k,s} \|\zeta\|_{C^k_{\rm b}(\mathbb{R}^d;\mathscr{L}(X,Y))}. \tag{14.55}$$

To prove this, first assume that $q \in (1,\infty)$ and $s < 0$. From Example 14.4.33 we obtain the boundedness of $g \mapsto \zeta^* g$ from $B^{-s}_{p',q'}(\mathbb{R}^d;Y^*)$ into $B^{-s}_{p',q'}(\mathbb{R}^d;X^*)$. Therefore, by Theorem 14.4.34, the adjoint mapping $f \mapsto \zeta f$ is bounded from $B^s_{p,q}(\mathbb{R}^d;X^{**})$ into $B^s_{p,q}(\mathbb{R}^d;Y^{**})$. Restricting to $\mathscr{S}(\mathbb{R}^d;X)$ and using density (Proposition 14.4.3) we obtain boundedness from $B^s_{p,q}(\mathbb{R}^d;X)$ into $B^s_{p,q}(\mathbb{R}^d;Y)$.

Next let $q \in \{1,\infty\}$ and $s < 0$. Interpolating the inequality (14.55) for the cases $B^{s+\varepsilon}_{p,2}$ and $B^{s-\varepsilon}_{p,2}$ by the real method with parameters $(\frac12,q)$, and using Theorems 14.4.31 to the effect that $(B^{s+\varepsilon}_{p,2},B^{s-\varepsilon}_{p,2})_{\frac12,q} = B^s_{p,q}$ we obtain boundedness in the endpoint cases $q \in \{1,\infty\}$ by Theorem C.3.3.

Finally, if $q \in [1,\infty]$ and $s = 0$, then by interpolating the cases $B^{\varepsilon}_{p,q}$ and $B^{-\varepsilon}_{p,q}$ by the real method with parameters $(\frac12,q)$ we obtain the boundedness also in this case.

As another application of interpolation and duality we present a density result, which at first sight looks a bit technical. It will be used to derive an analogues density result for Triebel–Lizorkin spaces (see Proposition 14.6.17) which will serve to show that several end-point results do not hold (see the text below Theorem 14.6.32 and Example 14.6.33). Moreover, some of these density results will be used to prove results on pointwise multiplication by the non-smooth function $\mathbf{1}_{\mathbb{R}_+}$ (see Sections 14.6.h and 14.7.d).

Let

$$\ddot{\mathbb{R}}^d := (\mathbb{R}\setminus\{0\}) \times \mathbb{R}^{d-1}.$$

Proposition 14.4.36 (Density of compactly supported functions). *Let $p,q \in [1,\infty)$ and $s \in \mathbb{R}$. Then $C^\infty_{\rm c}(\ddot{\mathbb{R}}^d)\otimes X$ is dense in $B^s_{p,q}(\mathbb{R}^d;X)$ in each of the following situations:*

(1) $s < 1/p$;
(2) $p,q \in (1,\infty)$ *and* $s = 1/p$.

Proof. By Proposition 14.4.3 it suffices to show that for every $f \in C^\infty_{\rm c}(\mathbb{R}^d)$ there exist $f_n \in C^\infty_{\rm c}(\ddot{\mathbb{R}}^d)$ such that $f_n \to f$ in $B^s_{p,q}(\mathbb{R}^d)$. Moreover, by the embedding (14.23) and Theorem 14.4.19 it suffices to prove (2).

In order to prove (2) let $f_n := \zeta_n f$, where $\zeta_n(x) = \zeta(nx_1,x_2,\dots,x_n)$ is multiplication by n in the first coordinate, and where $\zeta \in C^\infty(\mathbb{R}^d)$ satisfies $\zeta = 1$ if $|x_1| \geqslant 2$ and $\zeta = 0$ if $|x_1| \leqslant 1$. Then by Theorem 14.4.31 the following interpolation inequality holds:

$$\|f_n\|_{B^{1/p}_{p,q}(\mathbb{R}^d)} \leqslant C\|f_n\|^{1/p'}_{L^p(\mathbb{R}^d)}\|f_n\|^{1/p}_{W^{1,p}(\mathbb{R}^d)}.$$

Since

$$\|f_n\|_{L^p(\mathbb{R}^d)} \leqslant \|f\|_\infty \|\zeta_n\|_{L^p(\mathbb{R}^d)} \lesssim_\zeta n^{-1/p} \|f\|_\infty$$

and similarly

$$\|f_n\|_{W^{1,p}(\mathbb{R}^d)} \lesssim_\zeta n^{1/p'}(\|f\|_\infty + \|\nabla f\|_\infty),$$

the interpolation inequality implies that $(f_n)_{n\geqslant 1}$ is a bounded sequence in $B^{1/p}_{p,q}(\mathbb{R}^d)$. Using the reflexivity of $B^{1/p}_{p,q}(\mathbb{R}^d)$ (which follows Theorem 14.4.34) we find that $(f_n)_{n\geqslant 1}$ has a weakly convergent subsequence, say $f_{n_k} \to g$ weakly in $B^{1/p}_{p,q}(\mathbb{R}^d)$. Since also $f_n \to f$ in $\mathscr{S}'(\mathbb{R}^d)$, we find that $g = f$ and therefore $f_{n_k} \to f$ weakly in $B^{1/p}_{p,q}(\mathbb{R}^d)$. Therefore, $f \in \overline{C_c^\infty(\ddot{\mathbb{R}})}^{w} = \overline{C_c^\infty(\ddot{\mathbb{R}})}^{\|\cdot\|}$, where the closures are taken in the weak and norm topology of $B^{1/p}_{p,q}(\mathbb{R}^d)$, respectively. This completes the proof. □

14.5 Besov spaces, random sums, and multipliers

In the preceding subsections we have proved various results on embedding Besov spaces into other function spaces and vice versa. In the present subsection we take a look at the embeddability of Besov spaces into spaces of γ-radonifying operators. This question turns out to be intimately connected with the type and cotype properties of the space X.

The point of departure is provided by Theorems 9.2.10 and 9.7.3, by which we have the following natural continuous embeddings:

- $L^2(S;X) \hookrightarrow \gamma(L^2(S),X)$ if and only if X has type 2;
- $\gamma(L^2(S),X) \hookrightarrow L^2(S;X)$ if and only if X has cotype 2;
- $W^{\frac{1}{p}-\frac{1}{2},p}(\mathbb{R};X) \hookrightarrow \gamma(L^2(\mathbb{R}),X)$ if and only if X has type p.

In the first two embeddings $(S,\mathscr{A},\mu)$ is an arbitrary measure space.

The main result of this section is the following characterisation of type p and cotype q in terms of embedding properties:

Theorem 14.5.1 (γ-Sobolev embedding – I). *Let X be a Banach space and let $p \in [1,2]$ and $q \in [2,\infty]$.*

(1) *X has type p if and only if the identity mapping on $C_c^\infty(\mathbb{R}^d) \otimes X$ extends to a continuous embedding*

$$B^{(\frac{1}{p}-\frac{1}{2})d}_{p,p}(\mathbb{R}^d;X) \hookrightarrow \gamma(L^2(\mathbb{R}^d),X);$$

(2) *X has cotype q if and only if the identity mapping on $C_c^\infty(\mathbb{R}^d)\otimes X$ extends to a continuous embedding*

$$\gamma(L^2(\mathbb{R}^d),X) \hookrightarrow B^{(\frac{1}{q}-\frac{1}{2})d}_{q,q}(\mathbb{R}^d;X).$$

In particular, for any Banach space X we have continuous embeddings

$$B^{\frac{1}{2}d}_{1,1}(\mathbb{R}^d;X) \hookrightarrow \gamma(L^2(\mathbb{R}^d),X) \hookrightarrow B^{-\frac{1}{2}d}_{\infty,\infty}(\mathbb{R}^d;X).$$

The proof of Theorem 14.5.1 provides quantitative estimates for the norms of these embeddings. It relies on the following Gaussian version of the Bernstein–Nikolskii inequality (Lemma 14.4.20).

Lemma 14.5.2 (γ-Bernstein–Nikolskii inequality). *Let $p \in [1, 2]$ and $q \in [2, \infty]$.*

(1) *Let X have type p. If $f \in \mathscr{S}(\mathbb{R}^d; X)$ satisfies* $\operatorname{supp} \widehat{f} \subseteq \{\xi \in \mathbb{R}^d : |\xi| < t\}$, *then for all multi-indices $\alpha \in \mathbb{N}^d$ we have*

$$\|\partial^\alpha f\|_{\gamma(\mathbb{R}^d,X)} \leqslant \kappa_{2,p}\tau^\gamma_{p,X}\pi^{|\alpha|}t^{|\alpha|+\frac{d}{p}-\frac{d}{2}}\|f\|_{L^p(\mathbb{R}^d;X)}.$$

(2) *Let X have cotype q. If $f \in \mathscr{S}(\mathbb{R}^d; X)$ satisfies* $\operatorname{supp} \widehat{f} \subseteq \{\xi \in \mathbb{R}^d : |\xi| < t\}$, *then for all multi-indices $\alpha \in \mathbb{N}^d$ we have*

$$\|\partial^\alpha f\|_{L^q(\mathbb{R}^d;X)} \leqslant \kappa_{q,2}c^\gamma_{q,X}\pi^{|\alpha|}t^{|\alpha|+\frac{d}{2}-\frac{d}{q}}\|f\|_{\gamma(\mathbb{R}^d;X)}.$$

Here, $\kappa_{2,p}$ and $\kappa_{q,2}$ are the Kahane–Khintchine constants introduced in Section 6.2 and $\tau^\gamma_{q,X}$ and $c^\gamma_{q,X}$ are the Gaussian type and cotype constants of X, respectively, introduced in Section 7.1.d.

Proof. (1): By a scaling argument it suffices to consider the case $t = \frac{1}{2}$. By Example 9.6.5, $\partial^\alpha f \in \gamma(\mathbb{R}^d; X)$ if and only if $\xi \mapsto \xi^\alpha \widehat{f} \in \gamma(\mathbb{R}^d; X)$ and in this case

$$\|\partial^\alpha f\|_{\gamma(\mathbb{R}^d;X)} = (2\pi)^{|\alpha|}\|\xi \mapsto \xi^\alpha \widehat{f}(\xi)\|_{\gamma(\mathbb{R}^d;X)}.$$

In order to show that $\xi \mapsto \xi^\alpha \widehat{f}(\xi) \in \gamma(\mathbb{R}^d; X)$, by Examples 9.1.12 and 9.4.4 it suffices to check $\widehat{f} \in \gamma(Q; X)$, where $Q := [-\frac{1}{2}, \frac{1}{2}]^d$; in that case

$$(2\pi)^{|\alpha|}\|\xi \mapsto \xi^\alpha \widehat{f}\|_{\gamma(\mathbb{R}^d;X)} \leqslant (2\pi)^{|\alpha|}\|\xi \mapsto \xi^\alpha \widehat{f}\|_{\gamma(Q;X)} \leqslant \pi^{|\alpha|}\|\widehat{f}\|_{\gamma(Q;X)}.$$

The assertion $\widehat{f} \in \gamma(Q; X)$ is short-hand for the statement that the Pettis integral operator $\mathbb{I}_{\widehat{f}} : L^2(Q) \to X$ defined by

$$\mathbb{I}_{\widehat{f}}g := \int_Q \widehat{f}(\xi)g(\xi)\,\mathrm{d}\xi, \quad g \in L^2(Q),$$

belongs to $\gamma(L^2(Q), X)$ (see Section 9.2.a). We will prove the latter by testing against an orthonormal bases, making use of Theorem 9.1.17.

Let $e_n(\xi) := e^{2\pi i n\cdot\xi}$ for $n \in \mathbb{Z}^d$ and $\xi \in Q$. These functions define an orthonormal basis for $L^2(Q)$ and we have

$$\mathbb{I}_{\widehat{f}}e_n = \int_Q \widehat{f}(\xi)e^{2\pi i n\cdot\xi}\,\mathrm{d}\xi = f(n).$$

By the Kahane–Khintchine inequalities (Theorem 6.2.6) and the type p condition, for any finite subset $F \subseteq \mathbb{Z}^d$ we have

$$\begin{aligned}\Big\| \sum_{n\in F} \gamma_n \mathbb{I}_{\widehat{f}} e_n \Big\|_{L^2(\Omega;X)} &= \Big\| \sum_{n\in F} \gamma_n f(n) \Big\|_{L^2(\Omega;X)} \\ &\leqslant \kappa_{2,p} \Big\| \sum_{n\in F} \gamma_n f(n) \Big\|_{L^p(\Omega;X)} \\ &\leqslant \kappa_{2,p} \tau_{p,X}^{\gamma} \Big(\sum_{n\in F} \|f(n)\|^p \Big)^{1/p}.\end{aligned}$$

It follows from Theorem 9.1.17 that $\widehat{f} \in \gamma(Q, X)$ and, by the above observations,

$$\|\partial^\alpha f\|_{\gamma(\mathbb{R}^d;X)} \leqslant \pi^{|\alpha|} \|\widehat{f}\|_{\gamma(Q;X)} \leqslant \kappa_{2,p} \tau_{p,X}^{\gamma} \pi^{|\alpha|} \Big(\sum_{n\in\mathbb{Z}^d} \|f(n)\|^p \Big)^{1/p}.$$

To deduce the estimate in the statement of the theorem from it, for $h \in Q$ and $s \in \mathbb{R}^d$ put $f_h(s) := f(s + h)$. Then $\operatorname{supp} \widehat{f_h} \subseteq Q$ and

$$\|\partial^\alpha f\|_{\gamma(\mathbb{R}^d;X)} = \|\partial^\alpha f_h\|_{\gamma(\mathbb{R}^d;X)} \leqslant \kappa_{2,p} \tau_{p,X}^{\gamma} \pi^{|\alpha|} \Big(\sum_{n\in\mathbb{Z}^d} \|f_h(n)\|^p \Big)^{1/p}.$$

Raising both sides to the power p and integrating over $h \in Q$ we obtain

$$\begin{aligned}\|\partial^\alpha f\|_{\gamma(\mathbb{R}^d;X)} &\leqslant \kappa_{2,p} \tau_{p,X}^{\gamma} \Big(\int_Q \sum_{n\in\mathbb{Z}^d} \|f_h(n)\|^p \,\mathrm{d}h \Big)^{1/p} \\ &= \kappa_{2,p} \tau_{p,X}^{\gamma} \Big(\int_{\mathbb{R}^d} \|f(s)\|^p \,\mathrm{d}s \Big)^{1/p}.\end{aligned}$$

(2): This is proved similarly. □

Proof of Theorem 14.5.1. (1): First we prove the 'only if' part and assume that X has type p. Let $f \in \mathscr{S}(\mathbb{R}^d; X)$, put $f_k := \varphi_k * f$, and note that $\operatorname{supp} \widehat{f_0} \subseteq \{\xi \in \mathbb{R}^d : |\xi| \leqslant \frac{3}{2}\}$ and

$$\operatorname{supp} \widehat{f_k} \subseteq S_k := \{\xi \in \mathbb{R}^d : 2^{k-1} \leqslant |\xi| \leqslant 2^{k+1}\}, \quad k \geqslant 1.$$

By Lemma 14.5.2, $f_k \in \gamma(\mathbb{R}^d; X)$ and

$$\|f_k\|_{\gamma(\mathbb{R}^d;X)} \leqslant \kappa_{2,p} \tau_{p,X}^{\gamma} 2^{k(\frac{1}{p}-\frac{1}{2})d} \|f_k\|_{L^p(\mathbb{R}^d;X)}.$$

By Proposition 9.4.13, applied to the decompositions $(S_{2k})_{k\geqslant 0}$ and $(S_{2k+1})_{k\geqslant 0}$ of $\mathbb{R}^d \setminus \{0\}$, for $n \geqslant m \geqslant 0$ we obtain

$$\Big\| \sum_{k=2m}^{2n} f_k \Big\|_{\gamma(\mathbb{R}^d;X)} \leqslant \kappa_{2,p} \tau_{p,X}^{\gamma} \tau_{p,X} \Big(\sum_{j=m}^{n} 2^{2j(\frac{1}{p}-\frac{1}{2})pd} \|f_{2j}\|_{L^p(\mathbb{R}^d;X)}^p \Big)^{1/p}$$

$$+\,\kappa_{2,p}\tau_{p,X}^{\gamma}\tau_{p,X}\Big(\sum_{j=m}^{n-1}2^{(2j+1)(\frac1p-\frac12)pd}\|f_{2j+1}\|_{L^p(\mathbb{R}^d;X)}^p\Big)^{1/p}.$$

Sums of the form $\sum_{k=2m}^{2n+1}$, $\sum_{k=2m+1}^{2n}$, and $\sum_{k=2m+1}^{2n+1}$ can be estimated in a similar way. Since $f=\sum_{k\in\mathbb{Z}}\varphi_k * f=\sum_{k\in\mathbb{Z}}f_k$ in $\mathscr{S}(\mathbb{R}^d;X)$ (by Lemma 14.2.10) and hence in $\gamma(\mathbb{R}^d;X)$ (by the continuous embedding $\mathscr{S}(\mathbb{R}^d;X)\hookrightarrow\gamma(\mathbb{R}^d;X)$), it follows that $f\in\gamma(\mathbb{R}^d;X)$ and

$$\begin{aligned}\|f\|_{\gamma(\mathbb{R}^d;X)}&\leqslant 2\kappa_{2,p}\tau_{p,X}^{\gamma}\tau_{p,X}\Big(\sum_{j\in\mathbb{Z}}2^{j(\frac1p-\frac12)pd}\|f_{2j}\|_{L^p(\mathbb{R}^d;X)}^p\Big)^{1/p}\\&=2\kappa_{2,p}\tau_{p,X}^{\gamma}\tau_{p,X}\|f\|_{B_{p,p}^{(\frac1p-\frac12)d}(\mathbb{R}^d;X)}.\end{aligned}$$

Since $\mathscr{S}(\mathbb{R}^d;X)$ is dense in $B_{p,p}^{(\frac1p-\frac12)d}(\mathbb{R}^d;X)$ by Proposition 14.4.3, the identity mapping on $\mathscr{S}(\mathbb{R}^d;X)$ extends to a bounded operator from $B_{p,p}^{(\frac1p-\frac12)d}(\mathbb{R}^d;X)$ into $\gamma(\mathbb{R}^d;X)$ of norm at most $2\kappa_{2,p}\tau_{p,X}^{\gamma}\tau_{p,X}$. The simple proof that this extension is injective is left to the reader.

Next we prove the ‘if’ part. Since every Banach space has type 1, the ‘if’ part is trivial for $p=1$. In the rest of the proof of (1) we may therefore assume that $p\in(1,2]$. We will prove the stronger statement that if for some $r\in(1,\infty]$ the identity operator on $\mathscr{S}(\mathbb{R}^d;X)$ extends to a bounded operator, say I, from $B_{p,r}^{(\frac1p-\frac12)d}(\mathbb{R}^d;X)$ into $\gamma(L^2(\mathbb{R}^d),X)$, X has type r (and then necessarily $r\in(1,2]$).

Let $\psi\in\mathscr{S}(\mathbb{R}^d)$ be such that $\|\psi\|_{L^2(\mathbb{R}^d)}=1$ and $\operatorname{supp}(\widehat{\psi})\subseteq\{\xi\in\mathbb{R}^d:\widehat{\varphi}_1(\xi)=1\}$. For $n\geqslant 1$, let $\psi_n\in\mathcal{S}(\mathbb{R}^d)$ be defined by

$$\widehat{\psi}_n(\xi):=2^{(-n+1)d/2}\widehat{\psi}(2^{-n+1}\xi).$$

Then $(\psi_n)_{n\geqslant1}$ is an orthonormal system in $L^2(\mathbb{R}^d)$. By Proposition 9.1.3, for any finite sequence $(x_n)_{n=1}^N$ in X we then have, with $f:=\sum_{n=1}^N\psi_n\otimes x_n$,

$$\|f\|_{\gamma(\mathbb{R}^d;X)}^2=\mathbb{E}\Big\|\sum_{n=1}^N\gamma_nx_n\Big\|^2.$$

On the other hand, since $\varphi_k*\psi_n=\delta_{kn}\psi_n$ (this is seen by taking Fourier transforms and using the Fourier support properties of φ_k),

$$\|f\|^q_{B_{p,r}^{(\frac1p-\frac12)d}(\mathbb{R}^d;X)}=\sum_{n=1}^N2^{(\frac1p-\frac12)dr}\|\psi_n\|_p^r\|x_n\|^r=\|\psi\|_p^r\sum_{n=1}^N\|x_n\|^r.$$

By putting things together we see that X has type r, with Gaussian type r constant $\tau_{r,X}^{\gamma}\leqslant\|\psi\|_p\|I\|$.

(2): This is proved similarly. □

14.5.a The Fourier transform on Besov spaces

This section presents some mapping properties of the Fourier transform on spaces of functions taking values in a Banach space with (co)type or Fourier type properties. Recall from Section 2.4.b that a Banach space has *Fourier type* $p \in [1,2]$ if the Fourier transform, initially defined on $L^1(\mathbb{R}^d;X) \cap L^p(\mathbb{R}^d;X)$, extends to a bounded operator from $L^p(\mathbb{R}^d;X)$ into $L^{p'}(\mathbb{R}^d;X)$. If that is the case, the norm of this extension is denoted by $\varphi_{p,X}(\mathbb{R}^d)$.

Proposition 14.5.3 (Integrability of Fourier transforms – II). *Let $p \in [1,2]$, and suppose that one of the following two conditions holds:*

(i) $q \in [p,\infty]$ *and X has Fourier type p;*
(ii) $q \in [2,\infty]$ *and X has type p and cotype* 2.

Let $\mathscr{F}$ denote the Fourier transform on $\mathscr{S}'(\mathbb{R}^d;X)$ and let $s := (\frac{1}{p} - \frac{1}{q})d$.

(1) *$\mathscr{F}$ restricts to a bounded operator from $B^s_{p,q'}(\mathbb{R}^d;X)$ into $L^{q'}(\mathbb{R}^d;X)$;*
(2) *$\mathscr{F}$ restricts to a bounded operator from $W^{\lfloor s\rfloor+1,p}(\mathbb{R}^d;X)$ into $L^{q'}(\mathbb{R}^d;X)$.*

The case $q = \infty$ gives sufficient conditions for the Fourier transform to take values in $L^1(\mathbb{R}^d;X)$. Different conditions guaranteeing this have been discussed in Lemma 14.2.11, where growth assumptions on the functions and their derivatives where imposed.

Proof. We start with case (i). Accordingly, let $q \in [p,\infty]$ and let X have Fourier type p

(1): Let $f \in B^s_{p,q'}(\mathbb{R}^d;X)$. Put $f_k := \varphi_k * f$ for $k \geqslant 0$. Let $I_0 = \{\xi \in \mathbb{R}^d : |\xi| < 1\}$ and

$$I_n := \{\xi \in \mathbb{R}^d : 2^{n-1} \leqslant |\xi| < 2^n\}, \quad n \geqslant 1.$$

The sets I_n thus defined are pairwise disjoint, we have $\bigcup_{n\geqslant 0} I_n = \mathbb{R}^d$, and

$$\|\widehat{f}\|_{q'} = \Big(\sum_{n\geqslant 0} \|\mathbf{1}_{I_n}\widehat{f}\|_{q'}^{q'}\Big)^{1/q'} \leqslant \sum_{\ell=-1}^{1} \Big(\sum_{n\geqslant 0} \|\mathbf{1}_{I_n}\widehat{f}_{n+\ell}\|_{q'}^{q'}\Big)^{1/q'},$$

where we used that $\mathrm{supp}(\widehat{\varphi}_k) \cap I_n = \varnothing$ for $|n-k| \geqslant 2$ and that $\sum_{k\geqslant 0}\widehat{\varphi}_k = 1$. By Hölder's inequality with $\frac{1}{q'} = \frac{s}{d} + \frac{1}{p'}$ and the Fourier type p assumption, for $\ell \in \{-1,0,1\}$ we have

$$\begin{aligned}\|\mathbf{1}_{I_n}\widehat{f}_{n+\ell}\|_{q'} &\leqslant \|\mathbf{1}_{I_n}\|_{\frac{d}{s}}\|\widehat{f}_{n+\ell}\|_{p'} \\ &\leqslant \varphi_{p,X}(\mathbb{R}^d)2^{(n+1)s}\|f_{n+\ell}\|_{p'} \leqslant 2^{2s}\varphi_{p,X}(\mathbb{R}^d)2^{(n+\ell)s}\|f_{n+\ell}\|_{p'}.\end{aligned}$$

Taking $\ell^{q'}$-norms on both sides we obtain $\widehat{f} \in L^{q'}(\mathbb{R}^d;X)$ and

$$\|\widehat{f}\|_{q'} \leqslant 2^{2s}3\varphi_{p,X}(\mathbb{R}^d)\|f\|_{B^s_{p,q'}(\mathbb{R}^d;X)}.$$

(2): This follows from (1) since by Proposition 14.4.18 and Theorem 14.4.19 we have the embeddings $W^{\lfloor s\rfloor+1,p}(\mathbb{R}^d;X) \hookrightarrow B^{\lfloor s\rfloor+1}_{p,\infty}(\mathbb{R}^d;X) \hookrightarrow B^s_{p,1}(\mathbb{R}^d;X)$.

Case (ii): Assume now that $q\in[2,\infty]$ and that X has type p and cotype 2. Using the same notation as in case (i), by Hölder's inequality with $\frac{1}{q'}=\frac{1}{r}+\frac{1}{2}$, Theorem 9.2.10, and Lemma 14.5.2 we have

$$\begin{aligned}
\|\mathbf{1}_{I_n}\widehat{f}_{n+\ell}\|_{q'} &\leqslant \|\mathbf{1}_{I_n}\|_r\|\widehat{f}_{n+\ell}\|_2\\
&\leqslant c^{\gamma}_{2,X}2^{d/r(n+1)}\|f_{n+\ell}\|_{\gamma(\mathbb{R}^d;X)}\\
&\lesssim_{d,p} c_{2,X}\tau_{p,X}2^{(n+1)d/r}2^{(n+1)(\frac{1}{p}-\frac{1}{2})d}\|f_{n+\ell}\|_p\\
&= c_{2,X}\tau_{p,X}2^{(n+1)s}\|f_{n+\ell}\|_p.
\end{aligned}$$

The proof can now be finished as in case (i). □

As an application of Proposition 14.5.3 using the Fourier type of X, we give an improvement of the Mihlin multiplier theorem for vector-valued Besov spaces presented in Theorem 14.4.16. Before we do that we derive an immediate consequence of Propositions 14.4.11.

Corollary 14.5.4 (Fourier multiplier theorem for L^p under Fourier type). *Let $p\in[1,\infty]$ and $s\in\mathbb{R}$, let X and Y be Banach spaces, and suppose that one of the following conditions holds:*

(i) *Y has Fourier type τ;*
(ii) *Y has type τ and cotype 2.*

Then we have a continuous embedding

$$B^{d/\tau}_{\tau,1}(\mathbb{R}^d;\mathscr{L}(X,Y)) \hookrightarrow \mathfrak{M}L^p(\mathbb{R}^d;X,Y),$$

i.e., every $m\in B^{d/\tau}_{\tau,1}(\mathbb{R}^d;\mathscr{L}(X,Y))$ defines a bounded operator T_m from $L^p(\mathbb{R}^d;X)$ to $L^p(\mathbb{R}^d;Y)$.

Proof. The result is immediate from the fact that $\check{m}\in L^1(\mathbb{R}^d;\mathscr{L}(X,Y))$ by Proposition 14.5.3. □

Remark 14.5.5. It is possible to prove a result as in Corollary 14.5.4 under assumptions on m and m^* in the strong operator topology if X (equivalently X^*) has Fourier type τ_1 and Y has Fourier type τ_2. Indeed, assume there is a constant C_m such that

$$\|mx\|_{B^{d/\tau_2}_{\tau_2,1}(\mathbb{R}^d;Y)} \leqslant C_m\|x\|, \quad x\in X, \tag{14.56}$$

$$\|m^*y^*\|_{B^{d/\tau_1}_{\tau_1,1}(\mathbb{R}^d;X^*)} \leqslant C_m\|y^*\|, \quad y^*\in Y^*. \tag{14.57}$$

First observe that by (14.56), (14.57) and Proposition 14.5.3,

$$\begin{aligned} \|\check{m}x\|_{L^1(\mathbb{R}^d;Y)} &\leqslant C_{\tau_2,Y} C_m \|x\| \\ \|\check{m}^* y^*\|_{L^1(\mathbb{R}^d;X^*)} &\leqslant C_{\tau_1,X} C_m \|y^*\|. \end{aligned} \tag{14.58}$$

Here $\check{m}x := \mathscr{F}^{-1}(mx)$ and $\check{m}^* y^* := \mathscr{F}^{-1}(m^* y^*)$. Therefore, for $f \in \mathscr{S}(\mathbb{R}^d) \otimes X$, by Fubini's theorem one can write

$$\begin{aligned} \|\check{m} * f\|_{L^1(\mathbb{R}^d;Y)} &\leqslant \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \|\check{m}(t-s) f(s)\| \, \mathrm{d}s \, \mathrm{d}t \\ &\leqslant \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \|\check{m}(r) f(s)\| \, \mathrm{d}r \, \mathrm{d}s \leqslant C_m \|f\|_{L^1(\mathbb{R}^d;Y)}. \end{aligned}$$

This proves that T_m extends uniquely to $T_m \in \mathscr{L}(L^1(\mathbb{R}^d;X), L^1(\mathbb{R}^d;Y))$. Since the second line of (14.58) trivially implies that the kernel $\check{m}$ satisfies the dual Hörmander's condition, it follows from the Calderón–Zygmund extrapolation theorem (Theorem 11.2.5) that T_m extends uniquely to $T_m \in \mathscr{L}(L^p(\mathbb{R}^d;X), L^p(\mathbb{R}^d;Y))$ for all $p \in [1,\infty)$. By a duality argument a similar result can be derived for $p = \infty$.

It is clear from the above proof that we can replace the Fourier type conditions by the conditions that Y has type τ_2 and cotype 2, and X^* has type τ_1 and cotype 2.

We continue with an improvement of Theorem 14.4.16 using the Fourier type or type and cotype Y.

Theorem 14.5.6 (Mihlin multiplier theorem for $B^s_{p,q}(\mathbb{R}^d;X)$ under type conditions). *Let $p, q \in [1,\infty]$ and $s \in \mathbb{R}$ and X and Y be Banach spaces and suppose that one of the following conditions holds:*

(i) *Y has Fourier type τ;*
(ii) *Y has type τ and cotype 2.*

If $m \in C^{\lfloor \frac{d}{\tau} \rfloor + 1}(\mathbb{R}^d; \mathscr{L}(X,Y))$ satisfies

$$K_m := \sup_{|\alpha| \leqslant \lfloor \frac{d}{\tau} \rfloor + 1} \sup_{\xi \in \mathbb{R}^d} (1 + |\xi|^{|\alpha|}) \|\partial^\alpha m(\xi)\|_{\mathscr{L}(X,Y)} < \infty,$$

then there is a bounded operator $T : B^s_{p,q}(\mathbb{R}^d;X) \to B^s_{p,q}(\mathbb{R}^d;Y)$ with $\|T\| \leqslant C_{d,s,X,Y} K_m$ such that $Tf = \mathscr{F}^{-1}(m\widehat{f})$ for all $f \in \mathscr{S}(\mathbb{R}^d) \otimes X$.

Note that in the case $p, q < \infty$, one has that T is the unique bounded extension of $T_m : \mathscr{S}(\mathbb{R}^d) \otimes X \to \mathscr{S}'(\mathbb{R}^d;Y)$. In the end point case $p = \infty$ or $q = \infty$ this does not make sense since $\mathscr{S}(\mathbb{R}^d) \otimes X$ is not dense in $B^s_{p,q}(\mathbb{R}^d;X)$. This is the main reason for the unusual formulation in Theorem 14.5.6.

By a duality argument one can also formulate the (Fourier) (co)type conditions on X^*, but the end-point cases require some caution.

Proof. For $f \in B^s_{p,q}(\mathbb{R}^d;X)$ let $f_k = \varphi_k * f$ and $m_k = \widehat{\varphi}_k m$. Define

$$Tf = \sum_{\ell=-1}^{1} \sum_{k\geqslant 0} T_{m_{k+\ell}} f_k. \tag{14.59}$$

We will check that the series converges in $\mathscr{S}'(\mathbb{R}^d; Y)$ and defines an element in $B^s_{p,q}(\mathbb{R}^d; Y)$.

The proof follows the lines of Theorem 14.4.16. First we show that m_k bound $\mathfrak{M}L^p(\mathbb{R}^d; X, Y)$ with a uniform bound in $k \geqslant 0$. First let $k \geqslant 1$. By invariance under dilations (see Proposition 5.3.8), Corollary 14.5.4, and the embeddings (14.23) and (14.29), we have

$$\begin{aligned}\|m_k\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)} &= \|m_k(2^{k-1}\cdot)\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)} \\ &\leqslant C_{\tau,Y}\|m_k(2^{k-1}\cdot)\|_{B^{d/\tau}_{\tau,1}(\mathbb{R}^d;\mathscr{L}(X,Y))} \\ &\leqslant C_{\tau,Y}\|m_k(2^{k-1}\cdot)\|_{W^{\lfloor \frac{d}{\tau}\rfloor+1,\tau}(\mathbb{R}^d;\mathscr{L}(X,Y))}\end{aligned}$$

Since $m_k(2^{k-1}\cdot) = \widehat{\varphi}_1(\cdot)m(2^{k-1}\cdot)$, by the support properties of $\widehat{\varphi}_1$ is suffices to bound $\partial^\alpha[\widehat{\varphi}_1(\xi)m(2^{k-1}\xi)]$ for $|\alpha| \leqslant \lfloor\frac{d}{\tau}\rfloor + 1$, uniformly in $k \geqslant 1$ and $1 \leqslant |\xi| \leqslant 3$. This can be done in the same way as in (14.26). The case $k = 0$ can be proved in the same way without the dilation argument. We can conclude that

$$\|T_{m_{k+\ell}} f_k\|_{L^p(\mathbb{R}^d;Y)} \leqslant C_{d,s,X,Y} K_m \|f_k\|_{L^p(\mathbb{R}^d;X)} \tag{14.60}$$

Next we check the convergence of the series in (14.59). For $\zeta \in \mathscr{S}(\mathbb{R}^d)$ one has $T_{m_{k+\ell}} f_k(\zeta) = \sum_{j=-1}^{1} T_{m_{k+\ell}} f_k(\zeta_{k+j})$, where $\zeta_k = \varphi_k * \zeta$, and thus

$$\begin{aligned}\|T_{m_{k+\ell}} f_k(\zeta)\|_Y &\leqslant \|T_{m_{k+\ell}} f_k\|_{L^p(\mathbb{R}^d;Y)} \sum_{j=-1}^{1} \|\zeta_{k+j}\|_{L^{p'}(\mathbb{R}^d)} \\ &\leqslant C_{d,s,X,Y} K_m 2^{sk} \|f_k\|_{L^p(\mathbb{R}^d;Y)} \sum_{j=-1}^{1} 2^{|s|} 2^{-s(k+j)} \|\zeta_{k+j}\|_{L^{p'}(\mathbb{R}^d)}\end{aligned}$$

Summing over k we see that

$$\begin{aligned}\sum_{k\geqslant 0} \|T_{m_{k+\ell}} f_k(\zeta)\|_Y &\leqslant C_{d,s,X,Y} K_m \sum_{k\geqslant 0} 2^{sk} \|f_k\|_{L^p(\mathbb{R}^d;Y)} \sum_{j=-1}^{1} 2^{-sk} \|\zeta_{k+j}\|_{L^{p'}(\mathbb{R}^d)} \\ &\leqslant 3\cdot 2^{|s|} C_{d,s,X,Y} K_m \|f\|_{B^s_{p,q}(\mathbb{R}^d;X)} \|\zeta\|_{B^{-s}_{p',q'}(\mathbb{R}^d)},\end{aligned}$$

which gives the required convergence.

By the properties of $(\varphi_n)_{n\geqslant 0}$ we can write

$$\mathscr{F}(\varphi_j * Tf) = \sum_{k=j-1}^{j+1} \widehat{\varphi}_j \sum_{\ell=-1}^{1} \widehat{\varphi}_{k+\ell} m \widehat{\varphi}_k f = \sum_{k=j-1}^{j+1} \widehat{\varphi}_j m \widehat{\varphi}_k f = \sum_{\ell-1}^{1} m_j f_{j+\ell}.$$

Therefore, the boundedness follows from

$$\begin{aligned}\|Tf\|_{B^s_{p,q}(\mathbb{R}^d;Y)} &\leqslant \sum_{\ell-1}^{1} \|(T_{m_j} f_{j+\ell})_{j\geqslant 0}\|_{\ell^q(L^p(\mathbb{R}^d;Y))} \\ &\leqslant C_{d,s,X,Y} K_m \sum_{\ell-1}^{1} \|(f_{j+\ell})_{j\geqslant 0}\|_{\ell^q(L^p(\mathbb{R}^d;X))} \\ &\leqslant C'_{d,s,X,Y} K_m \|f\|_{B^s_{p,q}(\mathbb{R}^d;X)}.\end{aligned}$$

It remains to observe that for $f \in \mathscr{S}(\mathbb{R}^d) \otimes X$, the following identities hold in $\mathscr{S}'(\mathbb{R}^d;X)$

$$\widehat{Tf} = \sum_{k\geqslant 0} \sum_{\ell=-1}^{1} \widehat{\varphi}_{k+\ell} m \widehat{\varphi}_k \widehat{f} = \sum_{k\geqslant 0} m \widehat{\varphi}_k \widehat{f} = m\widehat{f}.$$

□

A further consequence of Proposition 14.5.3 is a Fourier multiplier theorem of a very different nature, in which the multiplier is non-smooth but the domain and range spaces have different integrability and smoothness exponents.

Proposition 14.5.7. *Let X and Y be Banach spaces with Fourier type $p \in [1,2]$ and let $s := (\frac{1}{p} - \frac{1}{p'})d$. Let $m : \mathbb{R}^d \to \mathscr{L}(X,Y)$ be strongly measurable in the strong operator topology and uniformly bounded. Then the Fourier multiplier $T_m = \mathscr{F}^{-1} m \mathscr{F}$ is bounded as an operator from $B^s_{p,p}(\mathbb{R}^d;X)$ into $L^{p'}(\mathbb{R}^d;Y)$ with norm*

$$\|T_m\|_{\mathscr{L}(B^s_{p,p}(\mathbb{R}^d;X),L^{p'}(\mathbb{R}^d;Y))} \lesssim_p \varphi_{p,X}(\mathbb{R}^d)\varphi_{p,Y}(\mathbb{R}^d) \sup_{\xi\in\mathbb{R}^d} \|m(\xi)\|_{\mathscr{L}(X,Y)}.$$

Proof. By the Fourier type p of Y,

$$\begin{aligned}\|T_m f\|_{L^{p'}(\mathbb{R}^d;Y)} &\leqslant \varphi_{p,Y}(\mathbb{R}^d)\|m\widehat{f}\|_{L^p(\mathbb{R}^d;Y)} \\ &\leqslant \varphi_{p,Y}(\mathbb{R}^d) \sup_{\xi\in\mathbb{R}^d} \|m(\xi)\|_{\mathscr{L}(X,Y)} \|\widehat{f}\|_{L^p(\mathbb{R}^d;X)},\end{aligned}$$

The Fourier type p of X and Proposition 14.5.3, applied with $q = p'$, give

$$\|\widehat{f}\|_{L^p(\mathbb{R}^d;X)} \lesssim_p \varphi_{p,X}(\mathbb{R}^d)\|f\|_{B^s_{p,p}(\mathbb{R}^d;X)},$$

and the result follows. □

14.5.b Smooth functions have *R*-bounded ranges

In Chapter 8 we have seen several instances of the general principle that sufficiently smooth operator-valued functions have R-bounded ranges. The amount of smoothness needed depends on the geometry of the underlying Banach spaces. For instance, it was shown in Theorem 8.5.21 that if X has cotype q and Y has type p, and if $T \in W^{s,r}(\mathbb{R}^d; \mathscr{L}(X,Y))$ with $(\frac{1}{p} - \frac{1}{q})d < \frac{d}{r} < s < 1$, then T has a continuous version whose range is R-bounded.

In the present section we will show that if the Besov scale is used instead of the Sobolev scale, the analogous result holds for the optimal smoothness exponent $s = (\frac{1}{p} - \frac{1}{q})d$ and the restriction $s < 1$ can be omitted. The precise statement reads as follows.

Theorem 14.5.8 (Besov functions with R-bounded range – I). *Let X and Y be Banach spaces, X having cotype $q \in [2,\infty]$ and Y having type $p \in [1,2]$. If $r \in [1,\infty]$ satisfies $\frac{1}{r} \geqslant \frac{1}{p} - \frac{1}{q}$, then every $T \in B^{d/r}_{r,1}(\mathbb{R}^d; \mathscr{L}(X,Y))$ has R-bounded range, with R-bound*

$$\mathscr{R}(T(t) : t \in \mathbb{R}^d) \leqslant C\|T\|_{B^{d/r}_{r,1}(\mathbb{R}^d;\mathscr{L}(X,Y))}, \tag{14.61}$$

where C is a constant depending on d, p, q, r, X, Y.

By Theorem 14.4.19, the spaces $B^{d/r}_{r,1}(\mathbb{R}^d; \mathscr{L}(X,Y))$ increasing in the exponent $r \in [1,\infty]$ and we have continuous embeddings

$$B^{d/r}_{r,1}(\mathbb{R}^d; \mathscr{L}(X,Y)) \hookrightarrow B^0_{\infty,1}(\mathbb{R}^d; \mathscr{L}(X,Y)) \hookrightarrow C_{\rm ub}(\mathbb{R}^d; \mathscr{L}(X,Y)), \tag{14.62}$$

the second being a consequence of Proposition 14.4.18. The continuous version provided by (14.62) is used in the left-hand side of (14.61).

In the proof below, we will use the Lorentz space $L^{r',\sigma}(\mathbb{R}^d)$ with $\sigma = \frac{\min\{\frac{1}{p'},\frac{1}{q}\}}{\frac{1}{p'}+\frac{1}{q}} \in (0,1]$. Referring to Appendix F, we recall that the Lorentz space $L^{r',\sigma}(\mathbb{R}^d)$ is the space of all measurable functions $f : \mathbb{R}^d \to \mathbb{K}$ for which the (quasi-)norm

$$\|f\|_{L^{r',\sigma}(\mathbb{R}^d)} := \big\|\tau \mapsto \tau^{1/r'} f^*(\tau)\big\|_{L^\sigma(\mathbb{R}_+,\frac{d\tau}{\tau})}$$

is finite, where

$$f^*(\tau) := \inf\big\{\lambda > 0 : |\{|f| > \lambda\}| \leqslant \tau\big\}, \quad \tau \in \mathbb{R}_+,$$

is the non-increasing rearrangement of f.

Proof. By the observation before (14.62) it suffices to prove the theorem in the case $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. In the proof we will only consider $r \in (1,\infty]$; in Theorem 14.5.9 a stronger result is proved which covers the case $r = 1$ of the present theorem.

Let us write

$$T = \sum_{k\geqslant 0} T_k = \sum_{\ell=-1}^{1} \sum_{k\geqslant 0} \varphi_{k+\ell} * T_k,$$

where $T_k = \varphi_k * T$, and we used (14.12) in the second identity. Since $T \in B^0_{\infty,1}(\mathbb{R}^d;\mathscr{L}(X,Y))$ (see (14.62)), the series $\sum_{k\geqslant 0} T_k$ converges uniformly on $\mathbb{R}^d$ with respect to the operator norm of $\mathscr{L}(X,Y)$. By Propositions 8.1.19 and 8.1.22,

$$\mathscr{R}(T(t) : t \in \mathbb{R}^d) \leqslant \sum_{\ell=-1}^{1} \sum_{k\geqslant 0} \mathscr{R}(\varphi_{k+\ell} * T_k(t) : t \in \mathbb{R}^d), \tag{14.63}$$

provided of course that the operator families occurring in the sums are R-bounded and their R-bounds are summable. Proving this will occupy us in the remainder of the proof.

Fix an integer $n \geqslant 1$. Starting from the identity $\varphi_n(t) = 2^{(n-1)d}\varphi_1(2^{n-1}t)$ (see (14.4)), it is elementary to check that the non-increasing rearrangements satisfy $\varphi_n^*(\tau) = 2^{(n-1)d}\varphi_1^*(2^{n-1}\tau)$. Therefore,

$$\begin{aligned}\|\varphi_n\|_{L^{r',\sigma}(\mathbb{R}^d)} &= 2^{(n-1)d}\|\tau \mapsto \tau^{1/r'}\varphi_1^*(2^{n-1}\tau)\|_{L^\sigma(\mathbb{R}_+,\frac{\mathrm{d}\tau}{\tau})} \\ &= 2^{(n-1)d/r}\|\tau \mapsto \tau^{1/r'}\varphi_1^*(\tau)\|_{L^\sigma(\mathbb{R}_+,\frac{\mathrm{d}\tau}{\tau})} = 2^{(n-1)d/r}\|\varphi_1\|_{L^{r',\sigma}(\mathbb{R}^d)},\end{aligned}$$

the latter being finite since $\varphi_1 \in \mathscr{S}(\mathbb{R}^d)$. A similar calculation can be done for $n = 0$.

For $t \in \mathbb{R}^d$ define $\varphi_{n,t} \in \mathscr{S}(\mathbb{R}^d)$ by $\varphi_{n,t}(s) := \varphi_n(t-s)$. Then $\varphi_{n,t}$ is identically distributed with φ_n. Letting $T_{k,\varphi_{n,t}} \in \mathscr{L}(X,Y)$ be the integral operator from Proposition 8.5.16, i.e.,

$$T_{k,\varphi_{n,t}}x := \int_{\mathbb{R}^d} \varphi_{n,t}(s)T_k(s)x \,\mathrm{d}s,$$

it follows from Proposition 8.5.16 with $\sigma = r'\min\{\frac{1}{p'},\frac{1}{q}\}$ and $\psi = \varphi_n$ that for all $n \geqslant 0$ and $k \geqslant 0$ the set $\{\varphi_n * T_k(t) : t \in \mathbb{R}^d\}$ is R-bounded, with R-bound

$$\mathscr{R}(\varphi_n * T_k(t) : t \in \mathbb{R}^d) = \mathscr{R}(T_{k,\varphi_{n,t}} : t \in \mathbb{R}^d) \leqslant C2^{nd/r}\|T_k\|_{L^r(\mathbb{R}^d;\mathscr{L}(X,Y))}.$$

With (14.63) we conclude that

$$\begin{aligned}\mathscr{R}(T(t) : t \in \mathbb{R}^d) &\leqslant C\sum_{\ell=-1}^{1}\sum_{k\geqslant 0} 2^{(k+\ell)d/r}\|T_k\|_{L^r(\mathbb{R}^d;\mathscr{L}(X,Y))} \\ &\leqslant 3\cdot 2^{\frac{d}{r}}C\|T\|_{B^{d/r}_{r,1}(\mathbb{R}^d;\mathscr{L}(X,Y))}.\end{aligned}$$

□

We have the following variation of this result for the strong operator topology:

Theorem 14.5.9 (Besov functions with R-bounded range – II). *Let X and Y be Banach spaces and assume that Y has type $p \in [1,2]$. Suppose that $T : \mathbb{R}^d \to \mathscr{L}(X,Y)$ satisfies $Tx \in B^{d/p}_{p,1}(\mathbb{R}^d;Y)$ for all $x \in X$ and*

$$\|Tx\|_{B^{d/p}_{p,1}(\mathbb{R}^d;Y)} \leqslant C_T\|x\|, \quad x \in X.$$

Then the family $\{T(t) \in \mathscr{L}(X,Y) : t \in \mathbb{R}^d\}$ *is R-bounded, with R-bound*

$$\mathscr{R}(T(t) \in \mathscr{L}(X,Y) : t \in \mathbb{R}^d) \leqslant CC_T,$$

where C *is a constant depending on* p *and* Y.

Proof. We begin with the case $p = 1$, which corresponds to the case where Y is an arbitrary Banach space. By Proposition 14.5.3 we have $\widehat{T}x \in L^1(\mathbb{R}^d;Y)$ and

$$\|\widehat{T}x\|_{L^1(\mathbb{R}^d;Y)} \lesssim_d \|Tx\|_{B^{d/p}_{p,1}(\mathbb{R}^d;Y)} \leqslant C_T\|x\|.$$

This implies that we have the integral representation

$$T(t)x = \int_{\mathbb{R}^d} e^{2\pi i\xi\cdot t}\widehat{T}(\xi)x\,\mathrm{d}\xi, \quad t \in \mathbb{R}^d,$$

where the operator-valued kernel is strongly in L^1. Now Theorem 8.5.4 implies that the family $\{T(t) : t \in \mathbb{R}^d\}$ is R-bounded, with R-bound $\mathscr{R}_p(T(t) : t \in \mathbb{R}^d) \lesssim_d C_T$.

Next assume that $p \in (1,2]$. For $k \geqslant 0$ and $x \in X$ set $T_k(t)x := \varphi_k * T(t)x$. By Theorem 14.5.1,

$$\|T_kx\|_{\gamma(L^2(\mathbb{R}^d),Y)} \leqslant C\|T_kx\|_{B^{(\frac{1}{p}-\frac{1}{2})d}_{p,p}(\mathbb{R}^d;Y)} \leqslant C_{d,p,s}2^{kd(\frac{1}{p}-\frac{1}{2})}\|T_kx\|_{L^p(\mathbb{R}^d;Y)}, \tag{14.64}$$

where (setting $s = d(\frac{1}{p}-\frac{1}{2})$ for brevity) the second inequality follows from

$$\begin{aligned}
\|T_kx\|^p_{B^s_{p,p}(\mathbb{R}^d;Y)} &= \sum_{n\geqslant 0} 2^{nsp}\|\varphi_n * \varphi_k * Tx\|^p_{L^p(\mathbb{R}^d;Y)} \\
&= \sum_{\ell=-1}^{1} 2^{(k+\ell)sp}\|\varphi_{k+\ell} * \varphi_k * Tx\|^p_{L^p(\mathbb{R}^d;Y)} \\
&\leqslant \sum_{\ell=-1}^{1} 2^{(k+\ell)sp}\|\varphi_{k+\ell}\|_1^p\|\varphi_k * Tx\|^p_{L^p(\mathbb{R}^d;Y)} \\
&\leqslant 3\cdot 2^{(k+1)sp}\cdot 2^p\|\varphi\|_1^p\|T_kx\|^p_{L^p(\mathbb{R}^d;Y)}
\end{aligned}$$

using (14.11) and (14.7).

Choose arbitrary finite sequences $(t_m)_{m=1}^M$ in $\mathbb{R}^d$ and $(x_m)_{m=1}^M$ in X, and let $(\varepsilon_m)_{m=1}^M$ be a Rademacher sequence on a probability space $(\Omega,\mathbb{P})$. Since Y has type $p > 1$ it follows from Theorem 9.6.14 with constant $L_{p,Y}$ hat

$$\Big\|\sum_{m=1}^{M}\varepsilon_m T(t_m)x_m\Big\|_{L^2(\Omega;Y)}$$

$$
\begin{aligned}
&\leqslant \sum_{k\geqslant 0}\sum_{\ell=-1}^{1}\Big\|\sum_{m=1}^{M}\varepsilon_m\varphi_{k+\ell}*T_k(t_m)x_m\Big\|_{L^2(\Omega;Y)}\\
&=\sum_{k\geqslant 0}\sum_{\ell=-1}^{1}\Big\|\sum_{m=1}^{M}\varepsilon_m\int_{\mathbb{R}^d}T_k(u)x_m\,\varphi_{k+\ell}(t_m-u)\,\mathrm{d}u\Big\|_{L^2(\Omega;Y)}\\
&\leqslant L_{p,Y}\sum_{k\geqslant 0}\sum_{\ell=-1}^{1}\|\varphi_{k+\ell}\|_{L^2(\mathbb{R}^d)}\Big\|\sum_{m=1}^{M}\varepsilon_m T_k x_m\Big\|_{L^2(\Omega;\gamma(L^2(\mathbb{R}^d),Y))}\\
&\leqslant L_{p,Y}C_\varphi\sum_{k\geqslant 0}2^{kd/2}\Big\|T_k\Big(\sum_{m=1}^{M}\varepsilon_m x_m\Big)\Big\|_{L^2(\Omega;\gamma(L^2(\mathbb{R}^d),Y))},
\end{aligned}
$$

where we used that (14.9) implies $\|\varphi_{k+\ell}\|_{L^2(\mathbb{R}^d)} = \|\widehat{\varphi}_{k+\ell}\|_{L^2(\mathbb{R}^d)} \leqslant C_\varphi 2^{kd/2}$. Applying (14.64) pointwise in Ω, setting $C_0 := L_{p,Y}C_\varphi C_{d,p,s}$, and using the Kahane-Khintchine inequalities, we continue estimating

$$
\begin{aligned}
&\leqslant C_0\sum_{k\geqslant 0}2^{kd/2}2^{k(\frac{1}{p}-\frac{1}{2})d}\Big\|T_k\Big(\sum_{m=1}^{M}\varepsilon_m x_m\Big)\Big\|_{L^2(\Omega;L^p(\mathbb{R}^d;Y))}\\
&\leqslant C_0\kappa_{2,1}\int_\Omega\sum_{k\geqslant 0}2^{kd/p}\Big\|T_k\Big(\sum_{m=1}^{M}\varepsilon_m x_m\Big)\Big\|_{L^p(\mathbb{R}^d;Y)}\,\mathrm{d}\mathbb{P}\\
&=C_0\kappa_{2,1}\int_\Omega\Big\|T\Big(\sum_{m=1}^{M}\varepsilon_m x_m\Big)\Big\|_{B^{d/p}_{p,1}(\mathbb{R}^d;Y)}\,\mathrm{d}\mathbb{P}\\
&\leqslant C_0\kappa_{2,1}C_T\int_\Omega\Big\|\sum_{m=1}^{M}\varepsilon_m x_m\Big\|_X\,\mathrm{d}\mathbb{P}\\
&\leqslant C_0\kappa_{2,1}C_T\Big\|\sum_{m=1}^{M}\varepsilon_m x_m\Big\|_{L^2(\Omega;X)}.
\end{aligned}
$$

Putting things together gives the required R-boundedness estimate. □

Remark 14.5.10.

(1) The method of proof for $p = 1$ in Theorem 14.5.9 could be extended to $p \in (1, 2]$ if Y has Fourier type p. We have not done this, because Proposition 7.3.6 shows that having type p is weaker than having Fourier type p.
(2) In the case $p = 1$ and $d = 1$, a variation of the argument in Proposition 8.5.7 actually gives a stronger result than Theorem 14.5.9, namely that if $Tx \in W^{d,1}(\mathbb{R}^d; \mathscr{L}(X, Y))$ for all $x \in X$, then the range of T is R-bounded.

14.6 Triebel–Lizorkin spaces

As we have seen in the preceding sections, the study of Besov spaces is intimately connected with the space $\ell^q(L^p(\mathbb{R}^d;X))$ through the very definition, which features the norm

$$\|f\|_{B^p_{q,s}(\mathbb{R}^d;X)} = \big\|(2^{ks}\varphi_k * f)_{k\geqslant 0}\big\|_{\ell^q(L^p(\mathbb{R}^d;X))}.$$

The class of Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{R}^d;X)$ is obtained upon replacing $\ell^q(L^p(\mathbb{R}^d;X))$ by $L^p(\mathbb{R}^d;\ell^q(X))$, putting

$$\|f\|_{F^s_{p,q}(\mathbb{R}^d;X)} = \big\|(2^{ks}\varphi_k * f)_{k\geqslant 0}\big\|_{L^p(\mathbb{R}^d;\ell^q(X))}.$$

The theory of Triebel–Lizorkin spaces is in many respect analogous to the theory of Besov spaces, but the occurrence of the ℓ^q-norm inside the L^p-norm precludes the use of Young's inequality to estimate the norm of term-wise convolutions, a technique that was critically used in our treatment of Besov spaces. This makes the norm of Triebel–Lizorkin spaces more difficult to deal with.

14.6.a The Peetre maximal function

The obstruction just noted already makes itself felt if one tries to adapt the proof that Besov spaces are independent up to an equivalent norm of the inhomogeneous Littlewood–Paley sequence $(\varphi_k)_{k\geqslant 0}$ to Triebel–Lizorkin spaces. The encountered difficulty will be resolved by a variant on the Fefferman–Stein inequality due to Peetre, to which we turn in the present preliminary subsection.

Throughout this section, unless otherwise stated X is an arbitrary Banach space. For a strongly measurable function $f:\mathbb{R}^d\to X$ and $r\in(0,\infty)$ we let

$$M_r f(x) := (M(\|f\|^r)(x))^{1/r}, \quad x\in\mathbb{R}^d, \tag{14.65}$$

where M is the Hardy–Littlewood maximal operator introduced in Section 2.3,

$$Mf(x) := \sup_{B\ni x}\frac{1}{|B|}\int_B \|f(y)\|\,\mathrm{d}y,$$

the supremum being taken over all Euclidean balls B in $\mathbb{R}^d$ that contain x.

Lemma 14.6.1 (Peetre's maximal inequality). *Fix $r,t\in(0,\infty)$ and a multi-index $\alpha\in\mathbb{N}^d$, and let $f\in\mathscr{S}'(\mathbb{R}^d;X)$ satisfy*

$$\operatorname{supp}\widehat{f}\subseteq B_t := \{\xi\in\mathbb{R}^d : |\xi|\leqslant t\}.$$

Then $f\in C^\infty(\mathbb{R}^d;X)$ and there exist constants C_1 and C_2, depending only on $|\alpha|$, d, r such that for all $x\in\mathbb{R}^d$ we have

$$\sup_{z\in\mathbb{R}^d} t^{-|\alpha|}\frac{\|\partial^\alpha f(x-z)\|}{(1+t|z|)^{d/r}} \leqslant C_1 \sup_{z\in\mathbb{R}^d} \frac{\|f(x-z)\|}{(1+t|z|)^{d/r}} \leqslant C_2 M_r f(x)$$

In particular, taking $z=0$, for all $x\in\mathbb{R}^d$ we have

$$t^{-|\alpha|}\|\partial^\alpha f(x)\| \leqslant \|f(x)\| \leqslant C_2 M_r f(x).$$

Proof. That the tempered distribution f is represented by a function in $C^\infty(\mathbb{R}^d;X)$ has already been observed in Lemma 14.2.9. In the remainder of the proof we assume that this identification has been made.

By an iteration argument it suffices to consider multi-indices satisfying $|\alpha|=1$. The short-hand notation $\|\nabla f(x)\| = \sum_{j=1}^d \|\partial_j f(x)\|$ will be used throughout the proof. We first consider the case $f\in\mathscr{S}(\mathbb{R}^d;X)$. Replacing f by $f(t^{-1}\cdot)$, it suffices to prove the result for $t=1$.

Step 1 – Choose $\psi\in\mathscr{S}(\mathbb{R}^d)$ such that $\widehat{\psi}\equiv 1$ on B_1. Since $\widehat{f}$ is supported on B_1, we have $f=\psi * f$ and $\nabla f = (\nabla\psi)*f$. It follows that for $x,z\in\mathbb{R}^d$ and $\lambda>0$,

$$\begin{aligned}\|\partial_j f(x-z)\| &\leqslant \int_{\mathbb{R}^d} |\partial_j\psi(x-z-y)|\,\|f(y)\|\,\mathrm{d}y\\ &\leqslant c_\lambda \int_{\mathbb{R}^d} (1+|x-z-y|)^{-\lambda}\,\|f(y)\|\,\mathrm{d}y,\end{aligned}$$

where $c_\lambda = \sup_{y\in\mathbb{R}^d}(1+|y|)^\lambda|\partial_j\psi(y)|$. Clearly we have $1+|x-y|\leqslant(1+|x-z-y|)(1+|z|)$, and upon taking $\lambda=d+1+d/r$ we obtain

$$\begin{aligned}\frac{\|\partial_j f(x-z)\|}{(1+|z|)^{d/r}} &\leqslant c_\lambda \int_{\mathbb{R}^d}(1+|x-z-y|)^{-\lambda}(1+|z|)^{-d/r}\,\|f(y)\|\,\mathrm{d}y\\ &\leqslant c_\lambda \int_{\mathbb{R}^d}(1+|x-z-y|)^{-d-1}(1+|x-y|)^{-d/r}\,\|f(y)\|\,\mathrm{d}y\\ &\leqslant C_1 \sup_{y\in\mathbb{R}^d}\frac{\|f(x-y)\|}{(1+|y|)^{d/r}},\end{aligned}$$

where $C_1 = c_\lambda\int_{\mathbb{R}^d}(1+|y|)^{-d-1}\,\mathrm{d}y$. This gives the first inequality in the statement of the lemma.

Step 2 – Fix $\varepsilon>0$ and let Q_ε be the closed cube centred at zero and of side-length ε. We claim that for all $g\in C^1(Q_\varepsilon;X)$,

$$\|g(0)\| \leqslant \frac{\varepsilon}{2}\sup_{y\in Q_\varepsilon}\|\nabla g(y)\| + \Big(\fint_{Q_\varepsilon}\|g(y)\|^r\,\mathrm{d}y\Big)^{1/r}, \qquad (14.66)$$

where we write $\fint_Q = \frac{1}{|Q|}\int_Q$ for averages. By scaling it suffices prove (14.66) for $\varepsilon=1$.

Fix $g\in C^1(Q_1;X)$. For all $y\in Q_1$ we have $\|y\|\leqslant\frac12$ and

$$g(0) = g(y) + \int_0^1 \nabla g(ty) \cdot y \, \mathrm{d}t.$$

Therefore, $\|g(0)\| \leqslant \|g(y)\| + \frac{1}{2}\sup_{y\in Q_1} \|\nabla g(y)\|$. Taking L^r-average over Q_1 gives (14.66) for $\varepsilon = 1$.

Step 3 – By Step 2, applied to the function $f(x - z - \cdot)$,

$$\|f(x-z)\| \leqslant \frac{\varepsilon}{2} \sup_{y\in Q_\varepsilon} \|\nabla f(x-z-y)\| + \Big(\fint_{Q_\varepsilon} \|f(x-z-y)\|^r \, \mathrm{d}y\Big)^{1/r}. \tag{14.67}$$

Now let $\varepsilon \in (0,1]$. It follows from $z - Q_\varepsilon \subseteq Q_{1+|z|}$ that

$$\begin{aligned}\fint_{Q_\varepsilon} \|f(x-z-y)\|^r \, \mathrm{d}y &= \fint_{z-Q_\varepsilon} \|f(x-y)\|^r \, \mathrm{d}y \\ &\leqslant \frac{|Q_{1+|z|}|}{|Q_\varepsilon|} \fint_{Q_{1+|z|}} \|f(x-y)\|^r \, \mathrm{d}y \\ &\leqslant \varepsilon^{-d}(1+|z|)^d M(\|f\|^r)(x).\end{aligned}$$

Substituting this into (14.67) and dividing by $(1+|z|)^{d/r}$, it follows that

$$\begin{aligned}\sup_{z\in\mathbb{R}^d} \frac{\|f(x-z)\|}{(1+|z|)^{d/r}} &\leqslant \frac{\varepsilon}{2} \sup_{z\in\mathbb{R}^d} \sup_{y\in Q_\varepsilon} \frac{\|\nabla f(x-z-y)\|}{(1+|z|)^{d/r}} + \varepsilon^{-d/r} M_r f(x) \\ &\leqslant \varepsilon 2^{d/r-1} \sup_{z\in\mathbb{R}^d} \frac{\|\nabla f(x-z)\|}{(1+|z|)^{d/r}} + \varepsilon^{-d/r} M_r f(x),\end{aligned}$$

where we used that $(1+|z|) \geqslant \frac{1}{2}(1+|y+z|)$ for $|y| \leqslant \varepsilon \leqslant 1$ and performed a change of variables. Combining this estimate with the first inequality in the statement of the lemma, and taking $\varepsilon \in (0,1]$ small enough, the result follows.

Step 4 – Next let $f \in \mathscr{S}'(\mathbb{R}^d; X)$ and $t > 0$. Let $f_\delta = \psi(\delta\cdot)f$, where $\psi \in \mathscr{S}(\mathbb{R}^d)$ satisfies $\psi(0) = 1$, $\operatorname{supp} \widehat{\psi} \subseteq \{\xi \in \mathbb{R}^d : |\xi| \leqslant 1\}$ and $\delta \in (0, \min\{1,t\})$. Recalling that $f \in C^\infty(\mathbb{R}^d; X)$, clearly we have $f_\delta \in \mathscr{S}(\mathbb{R}^d; X)$, $\widehat{f_\delta}$ has support in B_{2t} and therefore, by the previous steps, the second inequality in the statement of the lemma holds if in the two expressions on the left-hand side f is replaced by f_δ and for the right-hand side we note that $M_r f_\delta(x) \leqslant \|\psi\|_\infty M_r f(x)$. It remains to let $\delta \to 0$ on the left-hand side and note that $f_\delta(x-z) \to f(x-z)$ and similarly for its derivatives. □

Using the pointwise estimate of Lemma 14.6.1, we will now deduce a maximal inequality in $L^p(\mathbb{R}^d; \ell^q)$.

Proposition 14.6.2 (Boundedness of Peetre's maximal function). *Let $p \in [1,\infty)$, $q \in [1,\infty]$, and let $r \in (0, \min\{p,q\})$. Let $f = (f_k)_{k\geqslant 0}$ in $L^p(\mathbb{R}^d; \ell^q(X))$ be such that* $\operatorname{supp}(\widehat{f_k}) \subseteq S_k$ *for all $k \geqslant 0$, where $S_k \subseteq \mathbb{R}^d$ is a* *compact set with diameter* $\delta_k > 0$*. There exists a constant* $C \geqslant 0$*, depending only on* d*,* p*,* q*,* r*, such that*

$$\Big\| \Big(\sup_{z\in\mathbb{R}^d} \frac{\|f_k(\cdot - z)\|}{(1+\delta_k|z|)^{d/r}} \Big)_{k\geqslant 0} \Big\|_{L^p(\mathbb{R}^d;\ell^q)} \leqslant C\|f\|_{L^p(\mathbb{R}^d;\ell^q(X))}.$$

Proof. We use the short-hand notation $f = (f_k)_{k\geqslant 0}$ and $f^*_{d/r} = (f^*_{k,d/r})_{k\geqslant 0}$, where

$$f^*_{k,d/r}(x) = \sup_{z\in\mathbb{R}^d} \frac{\|f_k(x-z)\|}{(1+\delta_k|z|)^{d/r}}, \quad x \in \mathbb{R}^d. \tag{14.68}$$

Multiplying $f_k(x)$ with $e^{2\pi i h_k\cdot x}$ for suitable $h_k \in \mathbb{R}^d$, we may assume that each $\widehat{f}_k$ has support in $B_k = \{\xi \in \mathbb{R}^d : |\xi| \leqslant \delta_k\}$ for $k \geqslant 0$.

Let $g_k(x) := f_k(\delta_k^{-1}x)$. Then $\widehat{g}_k$ has support in a ball of radius 1 centred around the origin. Thus by Lemma 14.6.1 there is a constant c, depending only on d and r, such that for all $k \geqslant 0$ and $x \in \mathbb{R}^d$ we have

$$\sup_{z\in\mathbb{R}^d} \frac{\|g_k(x-z)\|}{(1+|z|)^{d/r}} \leqslant cM_r g_k(x).$$

Rewriting this in terms of f_k gives

$$f^*_{k,d/r}(x) = \sup_{z\in\mathbb{R}^d} \frac{\|f_k(x-z)\|}{(1+\delta_k|z|)^{d/r}} \leqslant cM_r f_k(x).$$

Taking $L^p(\mathbb{R}^d;\ell^q)$ norms and applying the Fefferman–Stein maximal Theorem 3.2.28 in the space $L^{p/r}(\mathbb{R}^d;\ell^{q/r})$, we find that

$$\begin{aligned}\|f^*_{d/r}\|_{L^p(\mathbb{R}^d;\ell^q)} \leqslant c\|(M_r f_k)_{k\geqslant 0}\|_{L^p(\mathbb{R}^d;\ell^q)} &= c\big\|(M(\|f_k\|^r))_{k\geqslant 0}\big\|^{1/r}_{L^{p/r}(\mathbb{R}^d;\ell^{q/r})} \\ &\lesssim_{p,q,r} c\big\|(\|f_k\|^r)_{k\geqslant 0}\big\|^{1/r}_{L^{p/r}(\mathbb{R}^d;\ell^{q/r})} = c\|f\|_{L^p(\mathbb{R}^d;\ell^q(X))}.\end{aligned}$$

□

As a first application we derive a Fourier multiplier theorem for certain functions in $L^p(\mathbb{R}^d;\ell^q)$ for $p \in [1,\infty)$ and $q \in [1,\infty]$ which is essential for later considerations about Triebel–Lizorkin spaces. The main difficulty arises if $p = 1$ or $q = 1$ since the maximal function is not bounded in these cases. The case $q = 1$ turns out to be of particular importance in Section 14.7.a.

The statement of the following theorem, which is needed in the proof of the Mihlin multiplier theorem for Triebel–Lizorkin spaces (theorem 14.6.11) is admittedly somewhat technical. We recall from Subsection 2.4.a that $\check{L}^1(\mathbb{R}^d;X)$ denotes the subspace in $L^\infty(\mathbb{R}^d;X)$ of all functions whose inverse Fourier transform belongs to $L^1(\mathbb{R}^d;X)$.

Theorem 14.6.3. *Let X and Y be Banach spaces and let $p \in [1,\infty)$, $q \in [1,\infty]$, and $r \in (0, \min\{p,q\})$. Let $S_k \subseteq \mathbb{R}^d$, $k \geqslant 0$, be compact sets with diameter $\delta_k > 0$. Then for all sequences $m = (m_k)_{k\geqslant 0}$ in $\check{L}^1(\mathbb{R}^d; \mathscr{L}(X,Y))$ and all $f = (f_k)_{k\geqslant 0} \in L^p(\mathbb{R}^d; \ell^q(X))$ with $\operatorname{supp} \widehat{f_k} \subseteq S_k$ for each $k \geqslant 0$ we have $\mathscr{F}^{-1} m \mathscr{F} f \in L^p(\mathbb{R}^d; \ell^q(Y))$ and*

$$
\begin{aligned}
&\|(\mathscr{F}^{-1} m \mathscr{F} f\|_{L^p(\mathbb{R}^d;\ell^q(Y))} \\
&\quad\leqslant C \sup_{k\geqslant 0} \|(1+\delta_k|\cdot|)^{d/r})\mathscr{F}^{-1} m_k(\cdot)\|_{L^1(\mathbb{R}^d;\mathscr{L}(X,Y))} \|f\|_{L^p(\mathbb{R}^d;\ell^q(X))} \\
&\quad= C \sup_{k\geqslant 0} \|(1+|\cdot|)^{d/r})\mathscr{F}^{-1}[m_k(\delta_k \cdot)]\|_{L^1(\mathbb{R}^d;\mathscr{L}(X,Y))} \|f\|_{L^p(\mathbb{R}^d;\ell^q(X))}
\end{aligned}
$$

where the constant $C \geqslant 0$ depends only on d, p, q, r, provided the supremum on the right-hand side is finite.

Proof. The kernels $K_k := \mathscr{F}^{-1} m_k$ are in $L^1(\mathbb{R}^d; \mathscr{L}(X,Y))$ by assumption. Therefore, the functions $\mathscr{F}^{-1}(m_k \widehat{f_k}) = K_k * f_k$ are well defined in $L^p(\mathbb{R}^d; Y)$ by Young's inequality. Let

$$
c_m := \sup_{k\geqslant 0} \|(1+\delta_k|\cdot|)^{d/r} K_k(\cdot)\|_{L^1(\mathbb{R}^d;\mathscr{L}(X,Y))}.
$$

Then, using the notation introduced in (14.68),

$$
\begin{aligned}
\|K_k * f_k(x)\| &\leqslant \int_{\mathbb{R}^d} \|K_k(x-y)\|(1+\delta_k|x-y|)^{d/r} \frac{\|f_k(y)\|}{(1+\delta_k|x-y|)^{d/r}} \,\mathrm{d}y \\
&\leqslant f^*_{n,d/r}(x) \int_{\mathbb{R}^d} \|K_k(x-y)\|(1+\delta_k|x-y|)^{d/r} \,\mathrm{d}y \leqslant c_m f^*_{n,d/r}(x).
\end{aligned}
$$

The required result follows from this by taking $L^p(\mathbb{R}^d; \ell^q)$-norms and applying Proposition 14.6.2.

The final identity of the theorem simply follows by a substitution together with the dilation property $\delta_k^{-1}(\mathscr{F}^{-1} m_k)(\delta_k^{-1} \cdot) = \mathscr{F}^{-1}[m_k(\delta_k \cdot)]$ of the Fourier transform. □

Remark 14.6.4. Lemma 14.6.1 can be used to extend the Bernstein–Nikolskii inequality presented in Lemma 14.4.20 to the full range $0 < p_0 \leqslant p_1 \leqslant \infty$. To this end let ψ be as in the proof of the lemma and note that it suffices to consider the case that $\widehat{f}$ has support in the unit ball.

First consider $0 < p_0 < p_1 \leqslant \infty$ and $\alpha = 0$. If $p_0 \in (0,1)$ and $p_1 = \infty$, then

$$
\begin{aligned}
|f(x)| &\leqslant \int_{\mathbb{R}^d} |\psi(x-y)| \|f(y)\| \,\mathrm{d}y| \\
&\leqslant \|\psi\|_\infty \int_{\mathbb{R}^d} \|f(y)\|^{1-p_0} \|f(y)\|^{p_0} \,\mathrm{d}y| \leqslant \|\psi\|_\infty \|f\|_\infty^{1-p_0} \|f\|_{p_0}^{p_0}
\end{aligned}
$$

and consequently $\|f\|_\infty \lesssim_{p_0,\psi} \|f\|_{p_0}$. Since we already knew the result for $p_0 \geqslant 1$, this inequality holds for $p_0 \in (0,\infty)$. In the remaining case $p_0 < p_1 < \infty$, we similarly find that

$$\|f\|_{p_1} \leqslant \|f\|_\infty^{1-p_0/p_1} \|f\|_{p_0}^{p_0/p_1} \lesssim_{p_0,p_1,\psi} \|f\|_{p_0}.$$

The case $p_0 = p_1$ and $\alpha \neq 0$ follows by taking L^{p_1}-norms in the pointwise estimate $\|\partial^\alpha f(x)\| \leqslant CM_r(f)(x)$ with $r \in (0,p_1)$ (see Lemma 14.6.1) and using the $L^{p_1/r}$-boundedness of the Hardy–Littlewood maximal function, to conclude that $\|\partial^\alpha f\|_{p_1} \leqslant C\|f\|_{p_1}$.

If $p_0 < p_1$ and $\alpha \neq 0$ combining the previous two cases gives

$$\|\partial^\alpha f\|_{p_1} \leqslant C\|f\|_{p_1} \leqslant C'\|f\|_{p_0}.$$

14.6.b Definitions and basic properties

We now introduce our main characters. Recall that we have fixed a inhomogeneous Littlewood–Paley sequence $(\varphi_k)_{k\geqslant 0}$ in Subsection 14.2.c (see Convention 14.2.8).

Definition 14.6.5 (Triebel–Lizorkin spaces). *Let $p \in [1,\infty)$, $q \in [1,\infty]$, and $s \in \mathbb{R}$. The* Triebel–Lizorkin space $F^s_{p,q}(\mathbb{R}^d;X)$ *is the space of all $f \in \mathscr{S}'(\mathbb{R}^d;X)$ for which the quantity*

$$\|f\|_{F^s_{p,q}(\mathbb{R}^d;X)} := \big\|(2^{ks}\varphi_k * f)_{k\geqslant 0}\big\|_{L^p(\mathbb{R}^d;\ell^q(X))}$$

is finite.

We comment on the case $p = \infty$ and $q < \infty$ in the Notes, as this exceptional case behaves differently. Below we will check that the above definition is independent on the choice of the Littlewood–Paley sequence up to an equivalent norm and that the resulting spaces are Banach spaces.

It is immediate from Young's inequality that $\psi * f \in F^s_{p,q}(\mathbb{R}^d;X)$ whenever $\psi \in L^1(\mathbb{R}^d)$ and $f \in F^s_{p,q}(\mathbb{R}^d;X)$, and more generally the analogue of Proposition 14.2.3 in valid.

By Fubini's theorem, for all $p \in [1,\infty)$ we have

$$F^s_{p,p}(\mathbb{R}^d;X) = B^s_{p,p}(\mathbb{R}^d;X).$$

We have continuous embeddings

$$F^s_{p,q_0}(\mathbb{R}^d;X) \hookrightarrow F^s_{p,q_1}(\mathbb{R}^d;X), \quad 1 \leqslant q_0 \leqslant q_1 \leqslant \infty, \tag{14.69}$$

and, by Hölder's inequality for the ℓ^q-norm,

$$F^{s_0}_{p,q_0}(\mathbb{R}^d;X) \hookrightarrow F^{s_1}_{p,q_1}(\mathbb{R}^d;X), \quad q_0,q_1 \in [1,\infty],\ s_0 > s_1. \tag{14.70}$$

Next we prove that, up to equivalence of norm, the Triebel–Lizorkin spaces are independent of the choice of the inhomogeneous Littlewood–Paley sequence $(\varphi_k)_{k\geqslant 0}$. The corresponding result for Besov spaces, Proposition 14.4.2, was rather easy to prove. The case of Triebel–Lizorkin spaces is not so easy and is based on Proposition 14.6.2. For $p > 1$ and $q > 1$ the use of this theorem can be avoided by using instead the estimate $\|\varphi_k * f\| \leqslant cMf$ together with the Fefferman–Stein Theorem 3.2.28.

Proposition 14.6.6. *Let $p \in [1,\infty)$, $q \in [1,\infty]$, and $s \in \mathbb{R}$. Up to an equivalent norm, the space $F^s_{p,q}(\mathbb{R}^d;X)$ is independent of the choice of the inhomogeneous Littlewood–Paley sequence $(\varphi_k)_{k\geqslant 0}$.*

Proof. Fix inhomogeneous Littlewood–Paley sequences $(\varphi_k)_{k\geqslant 0}$ and $(\psi_k)_{k\geqslant 0}$. For all $j,k \geqslant 0$ with $|j-k| \geqslant 2$ we have $\psi_k * \varphi_j = \mathscr{F}^{-1}(\widehat{\psi}_k\widehat{\varphi}_j) = 0$. Therefore, writing $f = \sum_{j\geqslant 0} f_j$ with $f_j = \varphi_j * f$,

$$\big\|(2^{ks}\psi_k * f)_{k\geqslant 0}\big\|_{L^p(\mathbb{R}^d;\ell^q(X))} \leqslant \sum_{\ell=-1}^{1} \big\|(2^{ks}\psi_k * f_{\ell+k})_{k\geqslant 0}\big\|_{L^p(\mathbb{R}^d;\ell^q(X))}.$$

Fix an arbitrary $r \in (0,\min\{p,q\})$, say $r = r_{p,q} = \frac{1}{2}\min\{p,q\}$. Applying Theorem 14.6.3 with $\delta_k = 3\cdot 2^k$ and $m_k = \widehat{\psi}_k$ to $(2^{ks}f_{\ell+k})_{k\geqslant 0}$ we obtain

$$\begin{aligned}\big\|(2^{ks}\psi_k * f)_{k\geqslant 0}\big\|_{L^p(\mathbb{R}^d;\ell^q(X))} &\leqslant C_{\psi,d,p,q,s}\big\|(2^{ks}f_{\ell+k})_{k\geqslant 0}\big\|_{L^p(\mathbb{R}^d;\ell^q(X))}\\ &\leqslant C'_{\psi,d,p,q,s}\big\|(2^{ks}\varphi_k * f)_{k\geqslant 0}\big\|_{L^p(\mathbb{R}^d;\ell^q(X))}.\end{aligned}$$

Since $(\psi_k)_{k\geqslant 0}$ and $(\varphi_k)_{k\geqslant 0}$ were arbitrary, this completes the proof. □

The same argument and (14.5) lead to the following useful estimate.

Lemma 14.6.7. *Let $f \in F^s_{p,q}(\mathbb{R}^d;X)$, let $(\psi_k)_{k\geqslant 0}$ be a Littlewood–Paley sequence, and set*

$$S_n f := \sum_{k=0}^{n} \psi_k * f, \quad n \geqslant 0.$$

Then $S_n f \in F^s_{p,q}(\mathbb{R}^d;X)$ and there exists a constant $C = C(p,q,d,\psi)$ such that

$$\|S_n f\|_{F^s_{p,q}(\mathbb{R}^d;X)} \leqslant C\|f\|_{F^s_{p,q}(\mathbb{R}^d;X)}, \quad n \geqslant 0.$$

We have the following analogue of Proposition 14.4.18 for Triebel–Lizorkin spaces:

Proposition 14.6.8 (Sandwiching with Besov spaces). *For all $p \in [1,\infty)$, $q \in [1,\infty]$, and $s \in \mathbb{R}$, we have the natural continuous embeddings*

$$\mathscr{S}(\mathbb{R}^d;X) \hookrightarrow F^s_{p,q}(\mathbb{R}^d;X) \hookrightarrow \mathscr{S}'(\mathbb{R}^d;X),$$

the first of which is dense if $p,q \in [1,\infty)$, and

$$B^s_{p,p\wedge q}(\mathbb{R}^d;X) \hookrightarrow F^s_{p,q}(\mathbb{R}^d;X) \hookrightarrow B^s_{p,p\vee q}(\mathbb{R}^d;X).$$

By combining the first of these inclusions with Lemma 14.2.1 we see that if $p, q \in [1, \infty)$, then $C_c^\infty(\mathbb{R}^d) \otimes X$ is dense in $F^s_{p,q}(\mathbb{R}^d; X)$.

Proof. First let $p > q$. For $f \in B^s_{p,q}(\mathbb{R}^d; X)$ it follows from the triangle inequality in $L^{p/q}(\mathbb{R}^d)$ that

$$\begin{aligned}\|f\|^q_{F^s_{p,q}(\mathbb{R}^d;X)} &= \Big\| \sum_{k\geqslant 0} 2^{ksq} \|\varphi_k * f\|^q \Big\|_{L^{p/q}(\mathbb{R}^d)} \\ &\leqslant \sum_{k\geqslant 0} 2^{ksq} \|\varphi_k * f\|^q_{L^p(\mathbb{R}^d;X)} = \|f\|^q_{B^s_{p,q}(\mathbb{R}^d;X)}.\end{aligned}$$

This gives the first embedding in the second displayed line of the proposition. The second embedding follows from (14.69), which gives $F^s_{p,q}(\mathbb{R}^d; X) \hookrightarrow F^s_{p,p}(\mathbb{R}^d; X) = B^s_{p,p}(\mathbb{R}^d; X)$ continuously. The case $p \leqslant q$ is handled similarly.

The continuous embeddings in the first line now follow from the corresponding result for Besov spaces contained in Proposition 14.4.3.

Let us finally show that $\mathscr{S}(\mathbb{R}^d; X)$ is dense in $F^s_{p,q}(\mathbb{R}^d; X)$. The proof is similar to Step 3 of the proof of Proposition 14.4.3. Let $f \in F^s_{p,q}(\mathbb{R}^d; X)$ and set $\zeta_n := \sum_{k=0}^n \varphi_k$. By (14.6) we have $\|\zeta_n\|_1 \leqslant \|\varphi_0\|_1$.

We will first show that $\zeta_n * f \to f$ in $F^s_{p,q}(\mathbb{R}^d; X)$. Let $\varepsilon > 0$ and choose $K \geqslant 0$ such that

$$\Big\| \Big(\sum_{k>K} 2^{ksq} \|\varphi_k * f\|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^d)} < \varepsilon.$$

By Young's inequality,

$$\Big\| \zeta_n * (2^{ks} \varphi_k * f)_{k>K} \Big\|_{L^p(\mathbb{R}^d;\ell^q(X))} < \varepsilon \|\varphi_0\|_1.$$

It follows that

$$\begin{aligned}&\|f - \zeta_n * f\|_{F^s_{p,q}(\mathbb{R}^d;X)} \\ &\quad\leqslant \varepsilon(1 + \|\varphi_0\|_1) + \Big\| \Big(\sum_{k=0}^K 2^{ksq} \|\varphi_k * f - \zeta_n * \varphi_k * f\|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^d)} \\ &\quad\leqslant \varepsilon(1 + \|\varphi_0\|_1) + \sum_{k=0}^K 2^{ks} \|\varphi_k * f - \zeta_n * \varphi_k * f\|_{L^p(\mathbb{R}^d;X)}\end{aligned}$$

The last term tends to zero as $n \to \infty$ by Proposition 1.2.32.

It remains to approximate each of the functions $\zeta_n * f$ by elements in $\mathscr{S}(\mathbb{R}^d; X)$. This can be done as in Proposition 14.4.3. □

This result enables us to give a quick proof of the completeness of Triebel–Lizorkin spaces:

Proposition 14.6.9. *For $p \in [1,\infty)$, $q \in [1,\infty]$, and $s \in \mathbb{R}$, the space $F^s_{p,q}(\mathbb{R}^d;X)$ is a Banach space.*

Proof. As in the Besov case one proves that for all $p \in [1,\infty)$, $q \in [1,\infty]$, and $s \in \mathbb{R}$, the space $F^s_{p,q}(\mathbb{R}^d;X)$ has the Fatou property. Since Triebel–Lizorkin spaces embed into $\mathscr{S}'(\mathbb{R}^d;X)$ by Proposition 14.6.8, the completeness of $F^s_{p,q}(\mathbb{R}^d;X)$ follows from Lemma 14.4.7. □

14.6.c Fourier multipliers

The main result of this subsection is a version of the Mihlin multiplier theorem for Triebel–Lizorkin spaces. Before we state it we first prove an important lifting property as we saw in Proposition 14.4.15 for Besov spaces.

Proposition 14.6.10 (Lifting). *Let $p \in [1,\infty)$, $q \in [1,\infty]$, and $s \in \mathbb{R}$. Then for all $\sigma \in \mathbb{R}$,*

$$J_\sigma : F^s_{p,q}(\mathbb{R}^d;X) \simeq F^{s-\sigma}_{p,q}(\mathbb{R}^d;X) \quad \textit{isomorphically.} \tag{14.71}$$

Proof. As in Proposition 14.4.15 it suffices to show that J_σ maps $F^s_{p,q}(\mathbb{R}^d;X)$ into $F^{s-\sigma}_{p,q}(\mathbb{R}^d;X)$ and is bounded for each $\sigma \in \mathbb{R}$ We must show that $(2^{n(s-\sigma)}\varphi_n * J_\sigma f)_{n\geqslant 0}$ belongs to $L^p(\mathbb{R}^d;\ell^q(X))$. This will be done by applying the multiplier Theorem 14.6.3 to a multiplier $m = (m_n)_{n\geqslant 0}$ naturally associated with J_σ.

Write

$$2^{-n\sigma}\varphi_n * J_\sigma f = \sum_{\ell=-1}^{1} \mathscr{F}^{-1} m_n \widehat{\varphi}_{n+\ell}\widehat{f},$$

where

$$m_n(\xi) = 2^{-n\sigma}(1+4\pi^2|\xi|^2)^{\sigma/2}\widehat{\varphi}_n(\xi).$$

We have $m_n \in C^\infty(\mathbb{R}^d)$ and, putting $\delta_n = 3\cdot 2^n$,

$$\operatorname{supp}\widehat{\varphi}_n(\delta_n\cdot) \subseteq \Big\{\xi \in \mathbb{R}^d : \frac{1}{6} \leqslant |\xi| \leqslant \frac{1}{2}\Big\}, \quad (n \geqslant 1)$$
$$\operatorname{supp}\widehat{\varphi}_0(\delta_0\cdot) \subseteq \Big\{\xi \in \mathbb{R}^d : |\xi| \leqslant \frac{1}{2}\Big\}.$$

Lemma 14.2.12, applied with $\lambda = d+1+\lceil d/r\rceil$ with an arbitrary $r = r_{p,q} \in (0,\min\{p,q\})$, gives the estimate

$$\begin{aligned}&\|(1+|\cdot|)^{d/r}\mathscr{F}^{-1}[m_n(\delta_n\cdot)]\|_{L^1(\mathbb{R}^d;\mathscr{L}(X,Y))}\\ &\qquad \leqslant C_d\|m_n(\delta_n\cdot)\|_{C^{d+1+\lceil d/r\rceil}(\mathbb{R}^d;\mathscr{L}(X,Y))} \leqslant C_{m,d,r} = C_{m,d,p,q},\end{aligned}$$

where the last inequality is elementary to verify.

Since for $\ell \in \{-1,0,1\}$ we have $\operatorname{supp}(\widehat{\varphi}_{n+\ell}\widehat{f}) \subseteq \{\xi \in \mathbb{R}^d : |\xi| \leqslant \delta_n\}$ we are now in a position to apply Theorem 14.6.3 and obtain

$$\begin{aligned}
\big\|(2^{n(s-\sigma)}\varphi_n * J_\sigma f)_{n\geqslant 0}\big\|_{L^p(\mathbb{R}^d;\ell^q(X))} & \\
&\leqslant \sum_{\ell=-1}^{1} \|(\mathscr{F}^{-1} m_n 2^{ns}\widehat{\varphi}_{n+\ell}\widehat{f})_{n\geqslant 0}\|_{L^p(\mathbb{R}^d;\ell^q(X))} \\
&\leqslant C_{m,d,p,q} \sum_{\ell=-1}^{1} \|(2^{ns}\varphi_{n+\ell} * f)_{n\geqslant 0}\|_{L^p(\mathbb{R}^d;\ell^q(X))} \\
&\leqslant C'_{m,d,p,q}\|f\|_{F^s_{p,q}(\mathbb{R}^d;X)}.
\end{aligned}$$

□

We continue with the Mihlin multiplier theorem for Triebel–Lizorkin spaces. Note that the Besov space case was considered in Theorems 14.4.16 and 14.5.6.

Theorem 14.6.11 (Mihlin multiplier theorem for Triebel–Lizorkin spaces). *Let $p \in [1,\infty)$, $q \in [1,\infty]$, $s \in \mathbb{R}$, and X and Y be Banach spaces, and set $N := d + 1 + \lceil \max\{\frac{d}{p}, \frac{d}{q}\}\rceil$. If $m \in C^N(\mathbb{R}^d; \mathscr{L}(X,Y))$ satisfies*

$$K_m := \sup_{|\alpha|\leqslant N} \sup_{\xi\in\mathbb{R}^d} (1 + |\xi|^{|\alpha|})\|\partial^\alpha m(\xi)\|_{\mathscr{L}(X,Y)} < \infty,$$

then there is a bounded operator $T : F^s_{p,q}(\mathbb{R}^d; X) \to F^s_{p,q}(\mathbb{R}^d; Y)$ with $\|T\| \leqslant C_{d,p,q,s,X,Y} K_m$ such that $Tf = \mathscr{F}^{-1}(m\widehat{f})$ for all $f \in \mathscr{S}(\mathbb{R}^d) \otimes X$.

Note that in the case $q < \infty$, one has that T is the unique bounded extension of $T_m : \mathscr{S}(\mathbb{R}^d) \otimes X \to \mathscr{S}'(\mathbb{R}^d; Y)$.

Proof. We define T in the same was as in (14.59) of the proof of Theorem 14.5.6:

$$Tf = \sum_{\ell=-1}^{1} \sum_{k\geqslant 0} T_{m_{k+\ell}} f_k,$$

where $f \in F^s_{p,q}(\mathbb{R}^d; X)$, $f_k = \varphi_k * f$ and $m_k = \widehat{\varphi}_k m$. Since $F^s_{p,q}(\mathbb{R}^d; X) \subseteq B^s_{p,\infty}(\mathbb{R}^d; X)$ it follows from the proof of Theorem 14.5.6 that the above series converges in $\mathscr{S}'(\mathbb{R}^d; Y)$, and that $Tg = \mathscr{F}^{-1}(m\widehat{g})$ for all $g \in \mathscr{S}(\mathbb{R}^d) \otimes X$.

To prove the required boundedness, note that

$$\|T_m f\|_{F^s_{p,q}(\mathbb{R}^d;Y)} \leqslant \sum_{\ell=-1}^{1} \big\|2^{ks}\mathscr{F}^{-1}(m\widehat{\varphi}_{k+\ell}\widehat{\varphi}_k\widehat{f})_{k\geqslant 0}\big\|_{L^p(\mathbb{R}^d;\ell^q(Y))}.$$

Fix $\ell \in \{-1, 0, 1\}$. Then $\operatorname{supp} \widehat{f}_{k+\ell} \subseteq \{|\xi| \leqslant \delta_k\}$, where $\delta_k = 3 \cdot 2^k$.

To estimate further it is sufficient to apply Theorem 14.6.3, for which we choose $r = r_{d,p,q} \in (0, \min\{p, q\})$ such that $N = d + 1 + \lceil d/r\rceil$. To check the assumptions of the theorem we have to show that

$$\sup_{k\geqslant 0} \|(1+|\cdot|)^{d/r}\mathscr{F}^{-1}(\widehat{\varphi}_k(\delta_k\cdot)m(\delta_k\cdot))\|_{L^1(\mathbb{R}^d;\mathscr{L}(X,Y))} \leqslant CK_m,$$

where $C \geqslant 0$ is a constant depending only on d and r. Since $\widehat{\varphi}_k(\delta_k\cdot)m(\delta_k\cdot)$ has support in $\{\xi \in \mathbb{R}^d : |\xi| \leqslant 1\}$, the estimate follows from Lemma 14.2.12. □

The following result is proved in the same way as Proposition 14.4.17.

Proposition 14.6.12. *Let $p \in [1,\infty)$, $q \in [1,\infty]$, and $s \in \mathbb{R}$. For all $k \in \mathbb{N}$ the expression*

$$|||f|||_{F^s_{p,q}(\mathbb{R}^d;X)} := \sum_{|\alpha|\leqslant k} \|\partial^\alpha f\|_{F^{s-k}_{p,q}(\mathbb{R}^d;X)}$$

defines an equivalent norm on $F^s_{p,q}(\mathbb{R}^d;X)$.

14.6.d Embedding theorems

We have already noted the continuous inclusions

$$\mathscr{S}(\mathbb{R}^d;X) \hookrightarrow F^s_{p,q}(\mathbb{R}^d;X) \hookrightarrow \mathscr{S}'(\mathbb{R}^d;X)$$

and

$$B^s_{p,p\wedge q}(\mathbb{R}^d;X) \hookrightarrow F^s_{p,q}(\mathbb{R}^d;X) \hookrightarrow B^s_{p,p\vee q}(\mathbb{R}^d;X)$$

for $s \in \mathbb{R}$, $p \in [1,\infty)$ and $q \in [1,\infty]$. Moreover, for any $q \in [1,\infty]$, it is immediate from the definitions that

$$B^s_{p,1}(\mathbb{R}^d;X) \hookrightarrow F^s_{p,q}(\mathbb{R}^d;X) \hookrightarrow B^s_{p,\infty}(\mathbb{R}^d;X). \tag{14.72}$$

The next result compares Triebel–Lizorkin spaces with the Bessel potential and Sobolev spaces. It can be improved if X is UMD and has type and cotype properties (see Proposition 14.7.6 below).

Proposition 14.6.13 (Sandwiching with Triebel–Lizorkin spaces). *For $p \in (1,\infty)$, $s \in \mathbb{R}$, and $m \in \mathbb{N}$, we have the following continuous embeddings:*

$$F^s_{p,1}(\mathbb{R}^d;X) \hookrightarrow H^{s,p}(\mathbb{R}^d;X) \hookrightarrow F^s_{p,\infty}(\mathbb{R}^d;X), \tag{14.73}$$

$$F^m_{p,1}(\mathbb{R}^d;X) \hookrightarrow W^{m,p}(\mathbb{R}^d;X) \hookrightarrow F^m_{p,\infty}(\mathbb{R}^d;X). \tag{14.74}$$

In view of the embeddings $B^s_{p,1}(\mathbb{R}^d;X) \hookrightarrow F^s_{p,1}(\mathbb{R}^d;X)$ and $F^s_{p,\infty}(\mathbb{R}^d;X) \hookrightarrow B^s_{p,\infty}(\mathbb{R}^d;X)$, (14.73) and (14.74) improve the corresponding embeddings in Proposition 14.4.18.

Proof. For (14.73) and (14.74), by Propositions 5.6.3, 14.6.10 and 14.6.12 it suffices to consider the special case $s = m = 0$, for which $H^{0,p}(\mathbb{R}^d;X) = W^{0,p}(\mathbb{R}^d;X) = L^p(\mathbb{R}^d;X)$. It thus remains to show the continuous embeddings

$$F^0_{p,1}(\mathbb{R}^d;X) \hookrightarrow L^p(\mathbb{R}^d;X) \hookrightarrow F^0_{p,\infty}(\mathbb{R}^d;X). \tag{14.75}$$

The first embedding in (14.75) is true for any $p \in [1,\infty)$: writing $f = \sum_{k\geqslant 0} \varphi_k * f$ it follows that

$$\|f\|_{L^p(\mathbb{R}^d;X)} \leqslant \sum_{k\geqslant 0} \|\varphi_k * f\|_{L^p(\mathbb{R}^d;X)} = \|f\|_{F^0_{p,1}(\mathbb{R}^d;X)}.$$

For the second embedding in (14.75) observe that since $\varphi \in \mathscr{S}(\mathbb{R}^d)$, it has a radially decreasing majorant which is integrable. Therefore, by Theorem 2.3.8 there is a constant $C_d \geqslant 0$ such that for all $k \geqslant 0$ and almost all $x \in \mathbb{R}^d$, $\|\varphi_k * f(x)\| \leqslant C_d Mf(x)$. Therefore, by the L^p-boundedness of the Hardy–Littlewood maximal function (Theorem 2.3.2),

$$\|f\|_{F^0_{p,\infty}(\mathbb{R}^d;X)} = \Big\| \sup_{k\geqslant 0} \|\varphi_k * f\| \Big\|_{L^p(\mathbb{R}^d)} \leqslant C_d \|Mf\|_{L^p(\mathbb{R}^d)} \lesssim_p C_d \|f\|_{L^p(\mathbb{R}^d;X)}.$$

This completes the proof. □

We continue with a version of the Sobolev embedding theorem. A surprising feature is that in case of the Triebel–Lizorkin spaces there is an improvement in the microscopic parameter q.

Theorem 14.6.14 (Sobolev embedding for Triebel–Lizorkin spaces). *For given $p_0, p_1 \in [1,\infty)$, $q_0, q_1 \in [1,\infty]$, and $s_0, s_1 \in \mathbb{R}$, we have a continuous embedding*

$$F^{s_0}_{p_0,q_0}(\mathbb{R}^d;X) \hookrightarrow F^{s_1}_{p_1,q_1}(\mathbb{R}^d;X)$$

if and only if one of the following two conditions holds:

(i) $p_0 = p_1$ *and* $[s_0 > s_1$ *or* $(s_0 = s_1$ *and* $q_0 \leqslant q_1)]$*;*
(ii) $p_0 < p_1$ *and* $s_0 - \frac{d}{p_0} \geqslant s_1 - \frac{d}{p_1}$.

The main ingredient is a version of the Gagliardo–Nirenberg inequality with a microscopic improvement.

Proposition 14.6.15 (Gagliardo–Nirenberg inequality for Triebel–Lizorkin spaces). *Let $p, p_0, p_1 \in [1,\infty)$, $q, q_0, q_1 \in [1,\infty]$, let $s_0, s_1 \in \mathbb{R}$ with $s_0 < s_1$, let $\theta \in (0,1)$, and assume that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $s = (1-\theta)s_0 + \theta s_1$. For all $f \in F^{s_0}_{p_0,q_0}(\mathbb{R}^d;X) \cap F^{s_1}_{p_1,q_1}(\mathbb{R}^d;X)$ we have $f \in F^s_{p,q}(\mathbb{R}^d;X)$ and*

$$\|f\|_{F^s_{p,q}(\mathbb{R}^d;X)} \leqslant C \|f\|^{1-\theta}_{F^{s_0}_{p_0,q_0}(\mathbb{R}^d;X)} \|f\|^{\theta}_{F^{s_1}_{p_1,q_1}(\mathbb{R}^d;X)},$$

where the constant $C \geqslant 0$ depends only on θ, s_0, s_1.

Proof. Proposition 14.3.5 (applied with $q_0 = q_1 = \infty$) implies that

$$\big\|(2^{ks}a_k)_{k\geqslant 0}\big\|_{\ell^q} \leqslant C_{s_0,s_1,s}\big\|(2^{ks_0}a_k)_{k\geqslant 0}\big\|_{\ell^\infty}^{1-\theta}\big\|(2^{ks_1}a_k)_{k\geqslant 0}\big\|_{\ell^\infty}^{\theta} \tag{14.76}$$

for all sequences of scalars $(a_k)_{k\geqslant 0}$ for which the expression on the right-hand side is finite.

To prove the desired estimate, by (14.69) it suffices to consider the case $q_0 = q_1 = \infty$. Taking $a_k(x) = \|\varphi_k * f(x)\|$ with $x \in \mathbb{R}^d$ in (14.76), raising to the power p and integrating over $\mathbb{R}^d$, by Hölder's inequality (with exponents $\frac{p_0}{(1-\theta)p}$ and $\frac{p_1}{\theta p}$) we obtain

$$\|f\|_{F^s_{p,q}(\mathbb{R}^d;X)} \leqslant C_{s_0,s_1,s}\|f\|^{1-\theta}_{F^{s_0}_{p_0,\infty}(\mathbb{R}^d;X)}\|f\|^{\theta}_{F^{s_1}_{p_1,\infty}(\mathbb{R}^d;X)}$$

as required. □

In a similar way one can prove the following variant for the end-point $p_1 = \infty$.

Proposition 14.6.16 (Gagliardo–Nirenberg inequality for Triebel–Lizorkin spaces – II). *Let $p, p_0, \in [1,\infty)$, $q, q_0 \in [1,\infty]$, let $s_0, s_1 \in \mathbb{R}$ with $s_0 < s_1$, let $\theta \in (0,1)$, and assume that $\frac{1}{p} = \frac{1-\theta}{p_0}$ and $s = (1-\theta)s_0 + \theta s_1$. For all $f \in F^{s_0}_{p_0,q_0}(\mathbb{R}^d;X) \cap B^{s_1}_{\infty,\infty}(\mathbb{R}^d;X)$ we have $f \in F^s_{p,q}(\mathbb{R}^d;X)$ and*

$$\|f\|_{F^s_{p,q}(\mathbb{R}^d;X)} \leqslant C\|f\|^{1-\theta}_{F^{s_0}_{p_0,q_0}(\mathbb{R}^d;X)}\|f\|^{\theta}_{B^{s_1}_{\infty,\infty}(\mathbb{R}^d;X)},$$

where the constant $C \geqslant 0$ depends only on θ, s_0, s_1.

Proof of sufficiency in Theorem 14.6.14. For the sufficiency of (i) first assume that $p_0 = p_1$, $q_0 \leqslant q_1$, and $s_0 \geqslant s_1$. Under these assumptions the result follows from the fact that

$$\big\|(2^{ks_1}a_k)_{k\geqslant 0}\big\|_{\ell^{q_1}} \leqslant \big\|(2^{ks_0}a_k)_{k\geqslant 0}\big\|_{\ell^{q_0}}.$$

If $p_0 = p_1$, $q_0 > q_1$, and $s_0 > s_1$, the result follows from (14.23) and (14.72):

$$\begin{aligned} F^{s_0}_{p_0,q_0}(\mathbb{R}^d;X) = F^{s_0}_{p_1,q_0}(\mathbb{R}^d;X) &\hookrightarrow B^{s_0}_{p_1,\infty}(\mathbb{R}^d;X) \\ &\hookrightarrow B^{s_1}_{p_1,\infty}(\mathbb{R}^d;X) \hookrightarrow F^{s_1}_{p_1,q_1}(\mathbb{R}^d;X). \end{aligned}$$

This completes the proof of (i).

Let us now assume that (ii) holds. By (14.70) it suffices to consider the case $s_0 - \frac{d}{p_0} = s_1 - \frac{d}{p_1}$. By (14.69) we may furthermore assume that $q_1 = 1$.

First take $f \in \mathscr{S}(\mathbb{R}^d;X)$. Let $\theta_0 \in [0,1)$ be such that $\frac{1}{p_1} - \frac{1-\theta_0}{p_0} = 0$. Choose $\theta \in (\theta_0, 1)$ arbitrary and let r be defined by $\frac{1}{p_1} = \frac{1-\theta}{p_0} + \frac{\theta}{r}$. Note that $p_0 < p_1 < \infty$ implies $r \in (p_1, \infty)$. Let further $t \in \mathbb{R}$ be defined by $t - \frac{d}{r} = s_0 - \frac{d}{p_0}$. Observe that $t < s_0$ and $s_1 = \theta t + (1-\theta)s_0$ (write out the expression for θt and use the formula for θ/r). Therefore, by Proposition 14.6.15,

$$\|f\|_{F^{s_1}_{p_1,1}(\mathbb{R}^d;X)} \leqslant C_{s_0,s_1,\theta}\|f\|^{1-\theta}_{F^{s_0}_{p_0,q_0}(\mathbb{R}^d;X)}\|f\|^{\theta}_{F^{t}_{r,r}(\mathbb{R}^d;X)}. \tag{14.77}$$

By the case (ii) in Theorem 14.4.19 (using that $r > p_1$),

$$\begin{aligned}\|f\|_{F^{t}_{r,r}(\mathbb{R}^d;X)} &= \|f\|_{B^{t}_{r,r}(\mathbb{R}^d;X)} \leqslant C\|f\|_{B^{s_1}_{p_1,p_1}(\mathbb{R}^d;X)} \\ &= C\|f\|_{F^{s_1}_{p_1,p_1}(\mathbb{R}^d;X)} \leqslant C\|f\|_{F^{s_1}_{p_1,1}(\mathbb{R}^d;X)},\end{aligned}$$

where in the last step we used (14.69). Substituting the latter estimate into (14.77), we obtain

$$\|f\|_{F^{s_1}_{p_1,1}(\mathbb{R}^d;X)} \leqslant C^{1/(1-\theta)}_{s_0,s_1,\theta} C^{\theta/(1-\theta)}\|f\|_{F^{s_0}_{p_0,q_0}(\mathbb{R}^d;X)}. \tag{14.78}$$

Now if $q_0 < \infty$, then the result follows from the density of $\mathscr{S}(\mathbb{R}^d;X)$ in $F^{s_0}_{p_0,q_0}(\mathbb{R}^d;X)$.

If $q_0 = \infty$ and $f \in F^{s_0}_{p_0,\infty}(\mathbb{R}^d;X)$, we let $S_n f = \sum_{k=0}^n \varphi_k * f$. Then by Young's inequality and the fact that $\varphi_j * S_n f = 0$ for $j \geqslant n+1$, we have $S_n f \in B^{s_0}_{p_0,1}(\mathbb{R}^d;X)$. Thus Theorem 14.4.19 implies $S_n f \in B^{s_1}_{p_1,1}(\mathbb{R}^d;X)$. Moreover, by Proposition 14.6.8 and (14.69) we also have $S_n f \in F^{s_0}_{p_0,1}(\mathbb{R}^d;X) \hookrightarrow F^{s_0}_{p_0,\infty}(\mathbb{R}^d;X)$ and $S_n f \in F^{s_1}_{p_1,1}(\mathbb{R}^d;X)$. Therefore, by (14.78),

$$\|S_n f\|_{F^{s_1}_{p_1,1}(\mathbb{R}^d;X)} \leqslant C^{1/(1-\theta)}_{s_0,s_1} C^{\theta/(1-\theta)}\|S_n f\|_{F^{s_0}_{p_0,\infty}(\mathbb{R}^d;X)} \leqslant \tilde{C}\|f\|_{F^{s_0}_{p_0,\infty}(\mathbb{R}^d;X)},$$

where the last estimate follows from Lemma 14.6.7. Since $S_n f \to f$ in $\mathscr{S}'(\mathbb{R}^d;X)$ by Lemma 14.2.10, the assertion now follows from the fact that $F^{s}_{p,q}(\mathbb{R}^d;X)$ has the Fatou property. □

Proof of necessity in Theorem 14.6.14. By Proposition 14.6.8,

$$B^{s_0}_{p_0,1}(\mathbb{R}^d;X) \hookrightarrow F^{s_0}_{p_0,q_0}(\mathbb{R}^d;X) \hookrightarrow F^{s_1}_{p_1,q_1}(\mathbb{R}^d;X) \hookrightarrow B^{s_1}_{p_1,\infty}(\mathbb{R}^d;X).$$

Therefore, Theorem 14.4.19 implies that $p_0 \leqslant p_1$. If $p_0 = p_1$, then (i) follows from (i). If $p_0 < p_1$, then (ii) follows from (iii) and (ii). □

Proposition 14.4.36 has the following analogue for Triebel–Lizorkin spaces:

Proposition 14.6.17 (Density of compactly supported functions). *Let*

$$\ddot{\mathbb{R}}^d := \mathbb{R}\setminus\{0\}\times\mathbb{R}^{d-1}.$$

Let $p,q \in [1,\infty)$ and $s \in \mathbb{R}$. Then $C^\infty_{\rm c}(\ddot{\mathbb{R}}^d)\otimes X$ is dense in $F^{s}_{p,q}(\mathbb{R}^d;X)$ and $H^{s,p}(\mathbb{R}^d;X)$ in each of the following situations:

(1) *$s < 1/p$;*
(2) *$p \in (1,\infty)$ and $s = 1/p$.*

Proof. First consider the Triebel–Lizorkin case. As in the proof of Proposition 14.4.36 (using Propositions 14.6.8) we can reduce to the smooth and scalar-valued setting. Thus it suffices to show that an arbitrary $f \in C_c^\infty(\mathbb{R}^d)$ there exist functions $f_n \in C_c^\infty(\dot{\mathbb{R}}^d)$ such that $f_n \to f$ in $F^{1/p}_{p,q}(\mathbb{R}^d)$. By the embedding (14.70) and Theorem 14.6.14, it suffices to prove this for the case (2). However, for this case the result follows from Proposition 14.4.36 and the estimate

$$\|f - f_n\|_{F^{1/p}_{p,q}(\mathbb{R}^d)} \leqslant C\|f - f_n\|_{F^{1/r}_{r,r}(\mathbb{R}^d)} = C\|f - f_n\|_{B^{1/r}_{r,r}(\mathbb{R}^d)}, \quad r \in (1,p),$$

which follows from Theorem 14.6.14.

The same proof for Bessel potential spaces holds, where we note that for the reduction to the scalar situation one can use Proposition 5.6.4, and the embedding $F^{1/r}_{r,r}(\mathbb{R}^d) \hookrightarrow H^{1/p,p}(\mathbb{R}^d)$ follows from Proposition 14.6.13 and Theorem 14.6.14. □

The proof of Theorem 14.5.1 shows that the existence of a continuous embedding

$$B^{(\frac{1}{p}-\frac{1}{2})d}_{p,r}(\mathbb{R}^d;X) \hookrightarrow \gamma(L^2(\mathbb{R}^d),X)$$

implies that X has type r, and that the existence of a continuous embedding $\gamma(L^2(\mathbb{R}^d),X) \hookrightarrow B^{(\frac{1}{q}-\frac{1}{2})d}_{q,r}(\mathbb{R}^d;X)$ implies that X has cotype r. Therefore in the Besov scale the embeddings of Theorem 14.5.1 cannot be improved by using the microscopic parameter r. For the Triebel–Lizorkin spaces the situation is different, as witnessed the following result.

Corollary 14.6.18 (γ-Sobolev embedding – II). *Let $1 \leqslant p_0 \leqslant 2 \leqslant q_0 < \infty$.*

(1) *If X has type p_0, then for all $p \in [1,p_0)$ and all $r \in [1,\infty]$ we have a continuous embedding*

$$F^{(\frac{1}{p}-\frac{1}{2})d}_{p,r}(\mathbb{R}^d;X) \hookrightarrow \gamma(L^2(\mathbb{R}^d),X).$$

(2) *If X has cotype q_0, then for all $q \in (q_0,\infty)$ and all $r \in [1,\infty]$ we have a continuous embedding*

$$\gamma(L^2(\mathbb{R}^d),X) \hookrightarrow F^{(\frac{1}{q}-\frac{1}{2})d}_{q,r}(\mathbb{R}^d;X).$$

Proof. We give the proof of (1), the proof of (2) being similar. Let $1 \leqslant p < p_0$. Let $s_0 = (\frac{1}{p_0}-\frac{1}{2})d$ and $s = (\frac{1}{p}-\frac{1}{2})d$. By Theorem 14.6.14 we have a continuous embedding

$$F^s_{p,r}(\mathbb{R}^d;X) \hookrightarrow F^{s_0}_{p_0,p_0}(\mathbb{R}^d;X) = B^{s_0}_{p_0,p_0}(\mathbb{R}^d;X).$$

Now the result follows from Theorem 14.5.1. □

14.6.e Difference norms

In Section 14.4.d we have discussed a difference norm characterisation for Besov spaces of positive smoothness. We will now prove a similar result for the Triebel–Lizorkin spaces. Recall the notation

$$\Delta_h f(x) = f(x+h) - f(x)$$

and $\Delta_h^m = (\Delta_h)^m$.

Definition 14.6.19 (Difference norm for Triebel–Lizorkin spaces). *Let $p \in [1,\infty)$, $q \in [1,\infty]$, $s > 0$, $m \in \mathbb{N} \setminus \{0\}$ and $\tau \in [1,\infty)$. For $f \in L^p(\mathbb{R}^d; X)$ we define the* difference norm *by setting*

$$[f]^{(m,\tau)}_{F^s_{p,q}(\mathbb{R}^d;X)} := \Big\| \Big(\int_0^\infty t^{-sq} \Big(\fint_{\{|h|\leqslant t\}} \|\Delta_h^m f\|_X^\tau \,\mathrm{d}h \Big)^{q/\tau} \frac{\mathrm{d}t}{t} \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^d)},$$

with obvious modifications if $q = \infty$, and

$$|\!|\!| f |\!|\!|^{(m,\tau)}_{F^s_{p,q}(\mathbb{R}^d;X)} := \|f\|_{L^p(\mathbb{R}^d;X)} + [f]^{(m,\tau)}_{F^s_{p,q}(\mathbb{R}^d;X)}.$$

It will be shown shortly that each of the norms $|\!|\!| \cdot |\!|\!|^{(m,\tau)}_{F^s_{p,q}(\mathbb{R}^d;X)}$ with $m > s$ and $s > \frac{d}{\min\{p,q\}} - \frac{d}{\tau}$ defines an equivalent norm on $F^s_{p,q}(\mathbb{R}^d;X)$.

The expression for the seminorm simplifies for $\tau = q \in [1,\infty)$. Indeed, by Fubini's theorem we have

$$[f]^{(m,q)}_{F^s_{p,q}(\mathbb{R}^d;X)} = \frac{1}{(sq+d)^{1/q}|B_1|} \Big\| \Big(\int_{\mathbb{R}^d} |h|^{-(s+d)q} \|\Delta_h^m f(x)\|^q \,\mathrm{d}h \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^d)}.$$

Theorem 14.6.20 (Difference norms for Triebel–Lizorkin spaces). *Let X be a Banach space and let $p,\tau \in [1,\infty)$, $q \in [1,\infty]$, $s > 0$, let $m > s$ be an integer, and suppose that*

$$s > \frac{d}{\min\{p,q\}} - \frac{d}{\tau}. \tag{14.79}$$

Then for all $f \in \mathscr{S}(\mathbb{R}^d;X)$ the following norm equivalence holds:

$$\|f\|_{F^s_{p,q}(\mathbb{R}^d;X)} \eqsim_{d,m,p,q,s,\tau} |\!|\!| f |\!|\!|^{(m,\tau)}_{F^s_{p,q}(\mathbb{R}^d;X)}, \tag{14.80}$$

whenever one of these expressions is finite.

Note that the condition (14.79) holds trivially holds if $\tau \leqslant \min\{p,q\}$, and in particular if $\tau = 1$. The condition (14.79) is only used in the proof of "$\gtrsim$" of (14.80).

For the proof we will use a discretised version of $|\!|\!| f |\!|\!|^{(m,\tau)}_{F^s_{p,q}(\mathbb{R}^d;X)}$. Put

$$J^{m,\tau}(f,k)(x) := \Big(\fint_{|h|\leqslant 1} \|\Delta^m_{2^{-k}h} f(x)\|^\tau \,\mathrm{d}h\Big)^{1/\tau}.$$

As in (14.38) we have

$$[f]^{(m,\tau)}_{F^s_{p,q}(\mathbb{R}^d;X)} \eqsim_{d,s} \big\|(2^{ks} J^{m,\tau}(f,k))_{k\in\mathbb{Z}}\big\|_{L^p(\mathbb{R}^d;\ell^q(\mathbb{Z}))}.$$

Therefore, to obtain (14.80) it suffices to prove the norm equivalence

$$\|f\|_{F^s_{p,q}(\mathbb{R}^d;X)} \eqsim \|f\|_{L^p(\mathbb{R}^d;X)} + \big\|(2^{ks} J^{m,\tau}(f,k))_{k\in\mathbb{Z}}\big\|_{L^p(\mathbb{R}^d;\ell^q(\mathbb{Z}))}, \tag{14.81}$$

where the implicit constant may depend on d, p, q, m, s, τ. The proof of the estimate $\lesssim$ in (14.81) is similar to Step 2 of the proof of Theorem 14.4.24 except that instead of Proposition 14.4.2 one has to use Proposition 14.6.6 and towards the end of the proof one has to take $L^p(\mathbb{R}^d;\ell^q)$-norms instead of $\ell^q(L^p(\mathbb{R}^d))$-norms.

In the remainder of this subsection we will concentrate on proving the inequality $\gtrsim$ in (14.81). We begin with a lemma involving the maximal function

$$M_r := (M(\|f\|^r)(x))^{1/r}$$

introduced in (14.65).

Lemma 14.6.21. *Let $f \in \mathscr{S}'(\mathbb{R}^d;X)$ satisfy* $\operatorname{supp}(\widehat{f}) \subseteq \{|\xi| \leqslant t\}$. *Then $f \in C^\infty(\mathbb{R}^d;X)$ and for all $r \in (0,\infty)$, $m \in \mathbb{N}$, and all $x, h \in \mathbb{R}^d$ we have*

$$\|\Delta^m_h f(x)\| \lesssim_{d,m,r} (t|h|)^{d/r} M_r(f)(x) \qquad \text{if } |h| > t^{-1}; \tag{14.82}$$

$$\|\Delta^m_h f(x)\| \lesssim_{d,m,r} (t|h|)^m M_r(f)(x) \qquad \text{if } |h| \leqslant t^{-1}. \tag{14.83}$$

Proof. That f belongs to $C^\infty(\mathbb{R}^d;X)$ follows from Lemma 14.2.9.

Recall that by Lemma 14.6.1

$$\|\partial^\alpha f(x+h)\| \lesssim_{|\alpha|,d,r} t^{|\alpha|}(1+t|h|)^{d/r} M_r f(x). \tag{14.84}$$

The estimate (14.82) follows from (14.84) and Lemma 14.4.22, for if $|h| > t^{-1}$, then

$$\begin{aligned}\|\Delta^m_h f(x)\| &\leqslant \sum_{j=0}^m \binom{m}{j} \|f(x+hj)\| \\ &\lesssim_{d,r} 2^m (1+t|h|m)^{d/r} M_r f(x) \lesssim_{d,m,r} (t|h|)^{d/r} M_r f(x).\end{aligned}$$

To prove (14.83) fix $|h| \leqslant t^{-1}$. Set $\phi(s) := f(x+sh)$ for $s \in \mathbb{R}$. Then $\Delta^m_h f(x) = \Delta^m_1 \phi(0)$. Since for any $g \in C^1(\mathbb{R};X)$ we have $\|\Delta_1 g(s)\| \leqslant \sup_{\theta\in[s,s+1]} \|g'(\theta)\|$, an induction argument gives

$$\|\Delta^m_1 \phi(s)\| \leqslant \sup_{\theta\in[s,s+m]} \|\phi^{(m)}(\theta)\|, \quad s \in \mathbb{R}.$$

In particular,

$$\|\Delta_h^m f(x)\| \leqslant \sup_{\theta\in[0,m]} \|\phi^{(m)}(\theta)\| \leqslant |h|^m \sup_{\theta\in[0,m]} \Big(\sum_{|\alpha|=m} \|\partial^\alpha f(x+\theta h)\|^2\Big)^{1/2}.$$

By (14.84) and the fact that $t|h| \leqslant 1$, for $\theta \in [0,m]$ we have

$$\|\partial^\alpha f(x+\theta h)\| \lesssim_{d,m,r} t^m (1+tm|h|)^{d/r} M_r f(x) \lesssim_{d,r,m} t^m M_r f(x).$$

Substituting this into the previous estimate gives the required estimate. □

Proof of Theorem 14.6.20. It remains to prove the inequality $\gtrsim$ in (14.81).

To begin with, from (i) we have inequality

$$\|f\|_{L^p(\mathbb{R}^d;X)} \leqslant \Big\| \sum_{j=0}^{\infty} \|f_j\|_X \Big\|_{L^p(\mathbb{R}^d)} = \|f\|_{F^0_{p,1}(\mathbb{R}^d;X)} \lesssim_{d,p,q,s} \|f\|_{F^s_{p,q}(\mathbb{R}^d;X)},$$

where $f_j = \varphi_j * f$ as always.

To deal with the seminorm, note that from the assumption (14.79) it follows that we can find $r \in (0,\infty)$ and $\lambda \in (0,1]$ such that

$$p, q > \max\{r, \lambda\tau\} \quad \text{and} \quad s > (1-\lambda)d/r. \tag{14.85}$$

Since $f = \sum_{n\in\mathbb{Z}} f_{n+k}$ in $L^p(\mathbb{R}^d;X)$ for any $k \in \mathbb{Z}$ (recall the convention that we set $\varphi_j = 0$ for $j \leqslant -1$, so that we may put $f_j = 0$ for $j \leqslant 1$), we have

$$\big\|(2^{ks} J^{m,\tau}(f,k))_{k\in\mathbb{Z}}\big\|_{L^p(\mathbb{R}^d;\ell^q(\mathbb{Z}))} \leqslant \sum_{n\in\mathbb{Z}} \big\|(2^{ks} J^{m,\tau}(f_{n+k},k))_{k\in\mathbb{Z}}\big\|_{L^p(\mathbb{R}^d;\ell^q(\mathbb{Z}))}.$$

For $n \leqslant 0$, by (14.83) with $t = 2^{k+n+1}$, we have

$$\begin{aligned} J^{m,\tau}(f_{n+k},k)(x) &= \Big(\fint_{|h|\leqslant 1} \|\Delta^m_{2^{-k}h} f_{n+k}(x)\|^\tau \,\mathrm{d}h\Big)^{1/\tau} \\ &\lesssim_{d,m,r} \Big(\fint_{|h|\leqslant 1} (|2^n h|^m M_r(f_{n+k})(x))^\tau \,\mathrm{d}h\Big)^{1/\tau} \\ &\leqslant 2^{nm} M_r(f_{n+k})(x), \end{aligned}$$

and therefore

$$\begin{aligned} &\big\|(2^{ks} J^{m,\tau}(f_{n+k},k)(x))_{k\geqslant 0}\big\|_{\ell^q(\mathbb{Z})} \\ &\qquad\lesssim_{d,m,r} (2^{ks} 2^{nm} M_r f_{n+k}(x))_{k\geqslant 0}\|_{\ell^q(\mathbb{Z})} \\ &\qquad= 2^{n(m-s)} \big\|(2^{s(k+n)} 2^{nm} M_r f_{n+k}(x))_{k\geqslant 0}\big\|_{\ell^q(\mathbb{Z})}. \end{aligned}$$

Since $s < m$ and M_r is bounded on $L^p(\mathbb{R}^d;\ell^q)$ by the Fefferman–Stein maximal Theorem 3.2.28, we obtain

$$\begin{aligned}\sum_{n\leqslant 0} &\big\|(2^{ks}J^{m,\tau}(f_{n+k},k\geqslant 0))_k\big\|_{L^p(\mathbb{R}^d;\ell^q(\mathbb{Z}))}\\ &\lesssim_{d,m,r} \sum_{n\leqslant 0} 2^{n(m-s)}\big\|(2^{(k+n)s}2^{nm}M_r f_{n+k})_{k\geqslant 0}\big\|_{L^p(\mathbb{R}^d;\ell^q(\mathbb{Z}))}\\ &\lesssim_{d,m,s} \|f\|_{F^s_{p,q}(\mathbb{R}^d;X)}.\end{aligned}$$

Next take $n\geqslant 1$. Fixing $\lambda\in(0,1]$ for the moment, we have

$$\begin{aligned}J^{m,\tau}&(f_{n+k},k)(x)\\ &\leqslant \sup_{|h|\leqslant 1}\|\varDelta^m_{2^{-k}h}f_{n+k}(x)\|^{1-\lambda}\Big(\fint_{\{|h|\leqslant 1\}}\|\varDelta^m_{2^{-k}h}f_{n+k}(x)\|^{\tau\lambda}\,\mathrm{d}h\Big)^{1/\tau}\\ &=: T_1(x)\times T_2(x).\end{aligned}$$

From (14.82) we obtain the pointwise bound

$$T_1(x)\leqslant 2^{dn(1-\lambda)/r}M_r(f_{n+k})(x)^{1-\lambda}.$$

To estimate T_2, we use Lemma 14.4.22 and the inequality $(\sum_{j=1}^m|a_j|)^{\lambda\tau}\lesssim_{\lambda,m,\tau}\sum_{j=1}^m|a_j|^{\lambda\tau}$ to obtain

$$\|\varDelta^m_{2^{-k}h}f_{n+k}(x)\|^{\tau\lambda}\lesssim_{\lambda,m,\tau}\|f_{n+k}(x)\|^{\tau\lambda}+\sum_{j=1}^m\binom{m}{j}\|f_{n+k}(x+2^{-k}hj)\|^{\tau\lambda}.$$

Estimating both terms by the maximal function, we obtain the pointwise bound that $T_2(x)$ is less than a constant depending on λ,m,τ times

$$\begin{aligned}&\Big(M_{\tau\lambda}(f_{n+k})(x)^{\tau\lambda}+\sum_{j=1}^m\binom{m}{j}\fint_{\{|h|\leqslant 1\}}\|f_{n+k}(x+2^{-k}hj)\|^{\tau\lambda}\,\mathrm{d}h\Big)^{1/\tau}\\ &=\Big(M_{\tau\lambda}(f_{n+k})(x)^{\tau\lambda}+\sum_{j=1}^m\binom{m}{j}\fint_{|y|\leqslant j2^{-k}}\|f_{n+k}(x+y)\|^{\tau\lambda}\,\mathrm{d}y\Big)^{1/\tau}\\ &\leqslant (2^m+1)^{1/\tau}M_{\tau\lambda}(f_{n+k})(x)^{\lambda}.\end{aligned}$$

Combining the estimates for T_1 and T_2, we conclude that

$$J^{m,\tau}(f_{n+k},2^{-k})\lesssim_{d,\lambda m,r,\tau} 2^{dn(1-\lambda)/r}M_r(f_{n+k})^{1-\lambda}M_{\tau\lambda}(f_{n+k})^{\lambda}.$$

Since $s>\frac{(1-\lambda)d}{r}$ (see (14.85)), by Hölder's inequality (applied twice) we obtain

$$\begin{aligned}\sum_{n\geqslant 1}&\big\|(2^{ks}J^{m,\tau}(f_{n+k},k))_{k\in\mathbb{Z}}\big\|_{L^p(\mathbb{R}^d;\ell^q(\mathbb{Z}))}\\ &\lesssim\sum_{n\geqslant 0}2^{-n(s-\frac{(1-\lambda)d}{r})}\big\|\big(2^{(n+k)s}M_r(f_{n+k})^{1-\lambda}M_{\tau\lambda}(f_{n+k})^{\lambda}\big)_{k\in\mathbb{Z}}\big\|_{L^p(\mathbb{R}^d;\ell^q)}\end{aligned}$$

$$\lesssim_{d,\lambda,r,s} \big\|\big(2^{js}M_r(f_j)\big)_{j\geqslant 0}\big\|^{1-\lambda}_{L^p(\mathbb{R}^d;\ell^q)}\big\|\big(2^{js}M_{\tau\lambda}(f_j)\big)_{j\geqslant 0}\big\|^{\lambda}_{L^p(\mathbb{R}^d;\ell^q)}$$
$$\lesssim_{\lambda,p,q,r,\tau} \big\|\big(2^{js}f_j\big)_{j\geqslant 0}\big\|_{L^p(\mathbb{R}^d;\ell^q)} = \|f\|_{F^s_{p,q}(\mathbb{R}^d;X)},$$

where in the last estimate we used the boundedness of M_r and $M_{\tau\lambda}$ on and $L^p(\mathbb{R}^d;\ell^q)$ thanks to (14.85). □

14.6.f Interpolation

In order to prove interpolation results for the scale of Triebel–Lizorkin spaces we need the following variation of Lemma 14.4.29.

Lemma 14.6.22. *Let $s \in \mathbb{R}$, $p \in (1,\infty)$ and $q \in (1,\infty]$. For $k \geqslant 0$ set $\psi_k = \varphi_{k-1} + \varphi_k + \varphi_{k+1}$. Define the operators*

$$R : L^p(\mathbb{R}^d;\ell^q_{w_s}(X)) \to F^s_{p,q}(\mathbb{R}^d;X)$$
$$S : F^s_{p,q}(\mathbb{R}^d;X) \to L^p(\mathbb{R}^d;\ell^q_{w_s}(X))$$

by

$$R(f_k)_{k\geqslant 0} = \sum_{k\geqslant 0}\psi_k * f_k, \quad Sf = (\varphi_k * f)_{k\geqslant 0}.$$

Then S is an isometry, R is bounded, and $RS = I$.

Proof. All assertions follow in the same way as in Lemma 14.4.29, except for the boundedness of R. To see that $\sum_{k\geqslant 0}\psi_k * f_k$ converges in $\mathscr{S}'(\mathbb{R}^d;X)$ note that $L^p(\mathbb{R}^d;\ell^q_{w_s}(X)) \hookrightarrow L^p(\mathbb{R}^d;\ell^p_{w_t}(X)) = \ell^p_{w_t}(L^p(\mathbb{R}^d;X))$ for any $t < s$ by Hölder's inequality, so the convergence follows from Lemma 14.4.29. To see that R is bounded, note that since $\widehat{\psi}_k \equiv 1$ on $\text{supp}(\widehat{\varphi}_k)$ we have

$$\begin{aligned}\|R(f_k)_{k\geqslant 0}\|_{F^s_{p,q}(\mathbb{R}^d;X)} &\leqslant \sum_{|\ell|\leqslant 2}\big\|(\|\varphi_j * \psi_{j+\ell} * f_{j+\ell}\|_X)_{j\geqslant 0}\big\|_{L^p(\mathbb{R}^d;\ell^q_{w_s})}\\ &\lesssim \sup_{|\ell|\leqslant 2}\big\|(M(\|f_{j+\ell}\|_X))_{j\geqslant 0}\big\|_{L^p(\mathbb{R}^d;\ell^q_{w_s})}\\ &\lesssim_{d,p,q} \sup_{|\ell|\leqslant 2}\big\|(\|f_{j+\ell}\|_X)_{j\geqslant 0}\big\|_{L^p(\mathbb{R}^d;\ell^q_{w_s})}\\ &\leqslant 4^{|s|}\big\|(\|f_j\|_X)_{j\geqslant 0}\big\|_{L^p(\mathbb{R}^d;\ell^q_{w_s})},\end{aligned}$$

where we used Proposition 2.3.9 and the boundedness of the Hardy–Littlewood maximal function M on $L^p(\mathbb{R}^d;\ell^q_{w_s})$, which is an immediate consequence of the Fefferman–Stein theorem (Theorem 3.2.28); here we use the assumptions $p \in (1,\infty)$ and $q \in (1,\infty]$. □

Using the operators R and S from Lemma 14.6.22 in the same way as in Theorem 14.4.30, the following theorem identifies the complex interpolation spaces of Triebel–Lizorkin.

Theorem 14.6.23 (Complex interpolation of Triebel–Lizorkin spaces). *Let (X_0, X_1) be an interpolation couple of Banach spaces and let $p_0, p_1 \in (1, \infty)$, $q_0, q_1 \in (1, \infty]$ with $\min\{q_0, q_1\} < \infty$, $s_0, s_1 \in \mathbb{R}$ and let $\theta \in (0, 1)$. Define $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, and $s = (1-\theta)s_0 + \theta s_1$. Then*

$$[F^{s_0}_{p_0,q_0}(\mathbb{R}^d; X_0), F^{s_1}_{p_1,q_1}(\mathbb{R}^d; X_1)]_\theta = F^s_{p,q}(\mathbb{R}^d; X_\theta),$$

isomorphically, where $X_\theta = [X_0, X_1]_\theta$.

The following result on the real interpolation of Triebel–Lizorkin spaces can be derived from the corresponding result for Besov spaces in the same way as Theorem 14.4.31, but now using the sandwich result of Proposition 14.6.13.

Proposition 14.6.24 (Real interpolation of Triebel–Lizorkin spaces). *Let $p \in [1, \infty)$, $q_0, q_1, q \in [1, \infty]$, and $s_0 \neq s_1 \in \mathbb{R}$. For $\theta \in (0, 1)$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, and $s = (1-\theta)s_0 + \theta s_1$ we have*

$$(F^{s_0}_{p,q_0}(\mathbb{R}^d; X), F^{s_1}_{p,q_1}(\mathbb{R}^d; X))_{\theta,q} = B^s_{p,q}(\mathbb{R}^d; X).$$

Our next aim is an interpolation result which will be used improve the Sobolev embedding result of Theorems 14.4.19 and 14.6.14.

Proposition 14.6.25. *Let $p_0, p_1 \in (1, \infty)$, $q \in (1, \infty]$, and $s \in \mathbb{R}$. For $\theta \in (0, 1)$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ we have*

$$\begin{aligned}
(F^s_{p_0,q}(\mathbb{R}^d; X), F^s_{p_1,q}(\mathbb{R}^d; X))_{\theta,p} &= F^s_{p,q}(\mathbb{R}^d; X), \\
(F^s_{p_0,1}(\mathbb{R}^d; X), F^s_{p_1,1}(\mathbb{R}^d; X))_{\theta,p} &\hookrightarrow F^s_{p,1}(\mathbb{R}^d; X).
\end{aligned}$$

Proof. The first interpolation identity can be proved as in Theorem 14.4.31, using Lemma 14.6.22 and the isomorphic identification

$$(L^{p_0}(\mathbb{R}^d; \ell^q_{w_s}(X)), L^{p_1}(\mathbb{R}^d; \ell^q_{w_s}(X)))_{\theta,p} = L^p(\mathbb{R}^d; \ell^q_{w_s}(X))$$

which follows from Theorem 2.2.10 and Proposition 14.3.5. The case $q = 1$ can be deduced from the proof of Theorem 14.4.31 as well. Indeed, since the operator S of Lemma 14.6.22 is an isometry also for $q = 1$, we find

$$\begin{aligned}
\|f\|_{F^s_{p,1}(\mathbb{R}^d;X)} &= \|Sf\|_{L^p(\mathbb{R}^d;\ell^1_{w_s}(X))} \\
&\eqsim_{p,p_0,p_1,\theta} \|Sf\|_{(L^{p_0}(\mathbb{R}^d;\ell^1_{w_s}(X)),L^{p_1}(\mathbb{R}^d;\ell^1_{w_s}(X)))_{\theta,p}} \\
&\lesssim_{p,p_0,p_1,\theta} \|f\|_{(F^s_{p_0,1}(\mathbb{R}^d;X),F^s_{p_1,1}(\mathbb{R}^d;X))_{\theta,p}}.
\end{aligned}$$

□

As an application we can prove some further embedding results.

Theorem 14.6.26 (Jawerth–Franke). *Let* $p_0, p_1, q \in [1,\infty]$ *and* $s_0, s_1 \in \mathbb{R}$ *satisfy* $1 \leqslant p_0 < p_1 \leqslant \infty$ *and* $s_0 > s_1$. *If* $s_0 - \frac{d}{p_0} \geqslant s_1 - \frac{d}{p_1}$, *then we have continuous embeddings*

$$B^{s_0}_{p_0,p_1}(\mathbb{R}^d;X) \hookrightarrow F^{s_1}_{p_1,q}(\mathbb{R}^d;X) \quad \text{if } p_1 < \infty; \tag{14.86}$$

$$F^{s_0}_{p_0,q}(\mathbb{R}^d;X) \hookrightarrow B^{s_1}_{p_1,p_0}(\mathbb{R}^d;X). \tag{14.87}$$

Since the embedding $F^{s_0}_{p_0,p_0}(\mathbb{R}^d;X) \hookrightarrow B^{s_0}_{p_0,p_1}(\mathbb{R}^d;X)$ holds trivially, (14.86) improves the embedding in Theorem 14.6.14. In a similar way one sees that (14.87) is an improvement of Theorem 14.6.14. Consequently, it follows from Theorem 14.6.14 that, under the assumption $p_0 < p_1$, the condition $s_0 - \frac{d}{p_0} \geqslant s_1 - \frac{d}{p_1}$ is also necessary for both (14.86) and (14.87).

Proof. By the trivial embeddings (14.23) and (14.70), it suffices to consider $s_0 - \frac{d}{p_0} = s_1 - \frac{d}{p_1}$.

To prove (14.86), assume that $p_1 < \infty$. In view of (14.70) it suffices to consider $q = 1$. Fix $p_0 < r_0 < p_1 < r_1$ and $\theta \in (0,1)$ such that $\frac{1}{p_1} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$. Let $t_0, t_1 \in \mathbb{R}$ be such that

$$t_0 - \frac{d}{p_0} = s_1 - \frac{d}{r_0} \quad \text{and} \quad t_1 - \frac{d}{p_0} = s_1 - \frac{d}{r_1}.$$

Then $(1-\theta)t_0 + \theta t_1 = s_0$ and therefore, using Proposition 14.6.24, Theorem 14.6.14, and Proposition 14.6.25,

$$\begin{aligned} B^{s_0}_{p_0,p_1}(\mathbb{R}^d;X) &= (F^{t_0}_{p_0,1}(\mathbb{R}^d;X), F^{t_1}_{p_0,1}(\mathbb{R}^d;X))_{\theta,p_1} \\ &\hookrightarrow (F^{s_1}_{r_0,1}(\mathbb{R}^d;X), F^{s_1}_{r_1,1}(\mathbb{R}^d;X))_{\theta,p_1} \hookrightarrow F^{s_1}_{p_1,1}(\mathbb{R}^d;X), \end{aligned}$$

which implies the embedding (14.86).

To prove (14.87) it suffices to consider $q = \infty$. Moreover, by Theorems 14.4.19 and 14.6.14 we may assume that $1 < p_0 < p_1 < \infty$. Fix $1 < r_0 < p_0 < r_1 < p_1$ and $\theta \in (0,1)$ such that $\frac{1}{p_0} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$. Let $t_0, t_1 \in \mathbb{R}$ be such that

$$t_0 - \frac{d}{p_1} = s_0 - \frac{d}{r_0} \quad \text{and} \quad t_1 - \frac{d}{p_1} = s_0 - \frac{d}{r_1}.$$

Then $(1-\theta)t_0 + \theta t_1 = s_1$. By Proposition 14.6.25, Theorem 14.6.14 and Proposition 14.6.24,

$$\begin{aligned} F^{s_0}_{p_0,\infty}(\mathbb{R}^d;X) &= (F^{s_0}_{r_0,\infty}(\mathbb{R}^d;X), F^{s_0}_{r_1,\infty}(\mathbb{R}^d;X))_{\theta,p_0} \\ &\hookrightarrow (F^{t_0}_{p_1,\infty}(\mathbb{R}^d;X), F^{t_1}_{p_1,\infty}(\mathbb{R}^d;X))_{\theta,p_0} = B^{s_1}_{p_1,p_0}(\mathbb{R}^d;X). \end{aligned}$$

□

As an interesting consequence of Theorem 14.6.26 we have the following improvement of Corollary 14.4.27 (2), extending it to the case $p_0 = 1$. The result is false for integrability exponents $p_0 > 1$. Indeed, if $s - \frac{d}{p_0} \geqslant 0$ and it would hold that $F^s_{p_0,q}(\mathbb{R}^d) \hookrightarrow C_{\rm ub}^{s-\frac{d}{p_0}}(\mathbb{R}^d)$ for $q = \infty$, then it would also hold for all $q \in [1,\infty)$. However, by Proposition 14.6.17 this would imply that every function in $F^s_{p_0,q}(\mathbb{R}^d)$ is zero at $x_1 = 0$, which is of course not true.

Corollary 14.6.27. *If $s \geqslant d$ is an integer, then $F^s_{1,\infty}(\mathbb{R}^d; X) \hookrightarrow C^{s-d}_{\rm ub}(\mathbb{R}^d; X)$ continuously.*

The result also holds in the case where $s > d$ is not integer. However, in this case Corollary 14.4.27 (2) gives a better result.

Proof. By Theorem 14.6.26 and Proposition 14.4.18,

$$F^s_{1,\infty}(\mathbb{R}^d; X) \hookrightarrow B^{s-d}_{\infty,1}(\mathbb{R}^d; X) \hookrightarrow C^{s-d}_{\rm ub}(\mathbb{R}^d; X).$$

□

14.6.g Duality

The next theorem identifies the duals of vector-valued Triebel–Lizorkin spaces.

Theorem 14.6.28. *Let $p, q \in (1,\infty)$ and $s \in \mathbb{R}$. Then*

$$F^s_{p,q}(\mathbb{R}^d; X)^* \simeq F^{-s}_{p',q'}(\mathbb{R}^d; X^*)$$

isomorphically.

The proof is similar to that of Theorem 14.4.34. The restriction $p, q > 1$ comes in through Lemma 14.6.22.

14.6.h Pointwise multiplication by $\mathbf{1}_{\mathbb{R}_+}$ in $B^s_{p,q}$ and $F^s_{p,q}$

In this section we apply the difference norm characterisation of Theorem 14.6.20, as well as the interpolation and duality results proved in this section, to study pointwise multiplication in Triebel–Lizorkin spaces with the non-smooth function $\mathbf{1}_{\mathbb{R}_+}$. The corresponding result for Besov spaces will be derived afterwards by real interpolation.

As a preparation we first deduce several fractional Hardy inequalities.

Proposition 14.6.29 (Hardy–Young inequality). *Let $p \in [1,\infty]$ and $\alpha \in \mathbb{R} \setminus \{0\}$, and let $f : \mathbb{R}_+ \to X$ be strongly measurable and integrable on every finite interval $(0,t)$. Each of the conditions*

(1) $\alpha > 0$ *and* $\lim_{t\to 0} \frac{1}{t}\int_0^t f(\tau)\,\mathrm{d}\tau = 0$
(2) $\alpha < 0$ *and* $\lim_{t\to \infty} \frac{1}{t}\int_0^t f(\tau)\,\mathrm{d}\tau = 0$

implies

$$\begin{aligned}&\|t\mapsto t^{-\alpha}f(t)\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)}\\&\qquad\leqslant(1+|\alpha|^{-1})\Big\|t\mapsto t^{-\alpha}\Big(f(t)-\fint_0^t f(\tau)\,\mathrm{d}\tau\Big)\Big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)}\end{aligned}$$

provided the right-hand side is finite.

Proof. (1): Let $F(t):=f(t)-\fint_0^t f(\tau)\,\mathrm{d}\tau$. Integrating by parts on $[t,\sigma]$ we obtain

$$I:=\int_t^\sigma\frac{1}{s^2}\int_0^s f(r)\,\mathrm{d}r\,\mathrm{d}s=-\frac{1}{\sigma}\int_0^\sigma f(r)\,\mathrm{d}r+\frac{1}{t}\int_0^t f(r)\,\mathrm{d}r+\int_t^\sigma f(s)\,\frac{\mathrm{d}s}{s}.$$

Therefore,

$$\begin{aligned}\int_t^\sigma F(s)\,\frac{\mathrm{d}s}{s}=\int_t^\sigma f(s)\,\frac{\mathrm{d}s}{s}-I&=\fint_0^\sigma f(r)\,\mathrm{d}r-\fint_0^t f(r)\,\mathrm{d}r\\&=f(\sigma)-F(\sigma)-\fint_0^t f(r)\,\mathrm{d}r.\end{aligned}\tag{14.88}$$

Letting $t\downarrow 0$ in (14.88) and taking norms, we obtain the estimate

$$\|f(\sigma)\|\leqslant\|F(\sigma)\|+\int_0^\sigma\|F(s)\|\,\frac{\mathrm{d}s}{s},\quad t>0.$$

Applying Hardy's inequality (see Lemma L.3.2(1)) with $\widetilde{\alpha}:=\alpha-1>-1$ to the function $s\mapsto\|F(s)\|$ we obtain

$$\|\sigma\mapsto\sigma^{-\alpha}f(\sigma)\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}\sigma}{\sigma};X)}\leqslant(1+\alpha^{-1})\|\sigma\mapsto\sigma^{-\alpha}F(\sigma)\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}\sigma}{\sigma};X)}$$

which gives the required estimate.

(2): We argue in the same way, but this time we rewrite the right-hand side of (14.88) as

$$\int_t^\sigma F(s)\,\frac{\mathrm{d}s}{s}=\fint_0^\sigma f(r)\,\mathrm{d}r-f(t)+F(t).$$

Letting $\sigma\to\infty$ and taking norms, we obtain the estimate

$$\|f(t)\|\leqslant\|F(t)\|+\int_t^\infty\|F(s)\|\,\frac{\mathrm{d}s}{s},\quad t>0.$$

Now the proof is finished as before, this time applying Lemma L.3.2(2) with $\widetilde{\alpha}:=\alpha-1<-1$. $\square$

As an immediate consequence we obtain the following result.

Proposition 14.6.30 (Fractional Hardy inequality). *Let $p \in [1,\infty)$ and $\beta \in \mathbb{R}$, and let $f : \mathbb{R}_+ \to X$ is strongly measurable and integrable on every finite sub-interval $(0,t)$. Each of the conditions*

(1) $\beta \in (1/p,\infty)$ *and* $\lim_{t\downarrow 0} \fint_0^t \|f(\tau)\|\,\mathrm{d}\tau = 0$
(2) $\beta \in (-\infty,1/p)$ *and* $\lim_{t\to\infty} \fint_0^t \|f(\tau)\|\,\mathrm{d}\tau = 0$

implies

$$\begin{aligned}\|f\|_{L^p(\mathbb{R}_+,t^{-\beta p}\,\mathrm{d}t;X)} &\leqslant C\Big\|x \mapsto x^{-\beta}\Big(\fint_{(0,x)} \|f(x)-f(x-h)\|\,\mathrm{d}h\Big)\Big\|_{L^p(\mathbb{R}_+)}\\ &\leqslant C\Big\|x \mapsto \sup_{t>0}\ t^{-\beta}\fint_{(0,x\wedge t)} \|f(x)-f(x-h)\|\,\mathrm{d}h\Big\|_{L^p(\mathbb{R}_+)}\end{aligned}$$

with $C := 1 + \frac{1}{|\beta-\frac{1}{p}|}$, provided the right-hand side is finite.

Proof. By Proposition 14.6.29 with $\alpha = \beta - \frac{1}{p}$,

$$\begin{aligned}\|f\|_{L^p(\mathbb{R}_+,t^{-\beta p}\,\mathrm{d}t;X)} &\leqslant C\Big\|x \mapsto x^{-\beta}\Big\|f(x) - \fint_{(0,x)} f(\tau)\,\mathrm{d}\tau\Big\|\Big\|_{L^p(\mathbb{R}_+;X)}\\ &\leqslant C\Big\|x^{-\beta}\Big(\fint_{(0,x)} \|f(x)-f(x-h)\|\,\mathrm{d}h\|_{L^p(\mathbb{R}_+)}\\ &\leqslant C\Big\|\sup_{t>0}\ t^{-\beta}\Big(\fint_{(0,x\wedge t)} \|f(x)-f(x-h)\|\,\mathrm{d}h\Big\|_{L^p(\mathbb{R}_+)}.\end{aligned}$$

This gives the required estimate in both cases. □

For $p \in [1,\infty)$, $q \in [1,\infty]$, and $s \in (1/p,1)$ we define the following closed subspaces of $H^{s,p}(\mathbb{R};X)$ and $F^s_{p,q}(\mathbb{R};X)$, respectively:

$$\begin{aligned}{}_0H^{s,p}(\mathbb{R};X) &:= \{f \in H^{s,p}(\mathbb{R};X) : f(0) = 0\},\\ {}_0F^s_{p,q}(\mathbb{R};X) &:= \{f \in F^s_{p,q}(\mathbb{R};X) : f(0) = 0\}.\end{aligned}$$

Here we use the bounded continuous version for f (which exists by Corollary 14.4.27 combined with Propositions 14.6.8 and 14.6.13) respectively. The continuity of the embeddings in Corollary 14.4.27 gives the closedness of these subspaces.

We can now prove the following fractional Hardy inequality in terms of the spaces $F^s_{p,q}$ and $H^{s,p}$ and their analogues ${}_0F^s_{p,q}$ and ${}_0H^{s,p}$.

Corollary 14.6.31. *Let $p \in [1,\infty)$ and $q \in [1,\infty]$.*

(1) *If $s \in (1/p,1)$, then each of the spaces ${}_0F^s_{p,q}(\mathbb{R};X)$ and ${}_0H^{s,p}(\mathbb{R};X)$ continuously embeds into $L^p(\mathbb{R},|t|^{-sp}\,\mathrm{d}t;X)$.*
(2) *If $s \in (0,1/p)$, then each of the spaces $F^s_{p,q}(\mathbb{R};X)$ and $H^{s,p}(\mathbb{R};X)$ (if $p \neq 1$) continuously embeds into $L^p(\mathbb{R},|t|^{-sp}\,\mathrm{d}t;X)$.*

Since $W^{s,p}(\mathbb{R};X) = F^s_{p,p}(\mathbb{R};X)$ for $s \in (0,1)$, the corollary also covers fractional Sobolev spaces.

Proof. By the embeddings (14.69) and (14.73) it suffices to prove the result for ${}_0F^s_{p,\infty}(\mathbb{R};X)$ and $F^s_{p,\infty}(\mathbb{R};X)$.

By Proposition 14.6.30, using that for bounded continuous functions $f : \mathbb{R} \to X$ we have $⨍_0^t f(\tau)\,\mathrm{d}\tau \to f(0) = 0$ as $t \downarrow 0$ in case (1) and $⨍_0^t f(\tau)\,\mathrm{d}\tau \to 0$ as $t \to \infty$ in case (2), we have

$$\begin{aligned}\|\mathbf{1}_{\mathbb{R}_+} f\|_{L^p(\mathbb{R},|t|^{-sp}\,\mathrm{d}t;X)} &\leqslant C\Big\| x \mapsto x^{-s} ⨍_{(0,x)} \|f(x) - f(x-h)\|\,\mathrm{d}h\Big\|_{L^p(\mathbb{R}_+)} \\ &\leqslant 2C\Big\| x \mapsto \sup_{t>0} t^{-s} ⨍_{(-t,t)} \|\Delta_h f(x)\|\,\mathrm{d}h\Big\|_{L^p(\mathbb{R})} \\ &= 2C[f]^{(1)}_{F^s_{p,\infty}(\mathbb{R};X)} \lesssim_{p,s} \|f\|_{F^s_{p,\infty}(\mathbb{R};X)}\end{aligned}$$

where in the last step we used Theorem 14.6.20 with $m = 1$. A similar estimate holds for f on the negative real axis. □

As a consequence we obtain the following result on pointwise multiplication.

Theorem 14.6.32 (Pointwise multiplication by $\mathbf{1}_{\mathbb{R}_+}$). *Let $p \in [1,\infty)$, $q \in [1,\infty]$, and $s \in (0,1)$. Each of the two conditions*

(1) $s \in (0, 1/p)$ *and* $f \in F^s_{p,q}(\mathbb{R};X)$
(2) $s \in (1/p, 1)$ *and* $f \in {}_0F^s_{p,q}(\mathbb{R};X)$

implies that $\mathbf{1}_{\mathbb{R}_+} f \in F^s_{p,q}(\mathbb{R};X)$ *and*

$$\|\mathbf{1}_{\mathbb{R}_+} f\|_{F^s_{p,q}(\mathbb{R};X)} \leqslant C\|f\|_{F^s_{p,q}(\mathbb{R};X)}.$$

Without the condition $f(0) = 0$, the result is false for $s > 1/p$. Indeed, this is clear from the fact that, by combining Corollary 14.4.27 and Proposition 14.6.13, we have a continuous embedding $F^s_{p,q}(\mathbb{R};X) \hookrightarrow C_{\mathrm{ub}}(\mathbb{R};X)$. A counterexample to the case $s = 1/p$ will be discussed in Example 14.6.33. It shows that Propositions 14.6.29, 14.6.30, and Corollary 14.6.31 do not hold for $\alpha = 0$ and $s = 1/p$.

Proof. Clearly, $\|\mathbf{1}_{\mathbb{R}_+} f\|_{L^p(\mathbb{R}^d;X)} \leqslant \|f\|_{L^p(\mathbb{R}^d;X)}$. Therefore, using the difference norm of Theorem 14.6.20 it remains to estimate $[\mathbf{1}_{\mathbb{R}_+} f]^{(1)}_{F^s_{p,q}(\mathbb{R};X)}$ in terms of $\|f\|_{F^s_{p,q}(\mathbb{R};X)}$ and $[f]^{(1)}_{F^s_{p,q}(\mathbb{R};X)}$. We give the proof for $q \in [1,\infty)$; the case $q = \infty$ requires the usual obvious modifications.

By the triangle inequality,

$$\begin{aligned}&[\mathbf{1}_{\mathbb{R}_+} f]^{(1)}_{F^s_{p,q}(\mathbb{R};X)} \\ &\leqslant \Big(\int_{\mathbb{R}_+} \Big(\int_{\mathbb{R}_+} t^{-sq}\Big(\frac{1}{t}\int_{(-t,t)\cap(-x,\infty)} \|f(x+h) - f(x)\|\,\mathrm{d}h\Big)^q \frac{\mathrm{d}t}{t}\Big)^{p/q}\,\mathrm{d}x\Big)^{1/p}\end{aligned}$$

$$
\begin{aligned}
&+\Big(\int_{\mathbb{R}_+}\Big(\int_{\mathbb{R}_+} t^{-sq}\Big(\frac{1}{t}\int_{(-t,t)\cap(-\infty,-x)}\|f(x)\|\,\mathrm{d}h\Big)^q\,\frac{\mathrm{d}t}{t}\Big)^{p/q}\,\mathrm{d}x\Big)^{1/p}\\
&+\Big(\int_{(-\infty,0)}\Big(\int_{\mathbb{R}_+} t^{-sq}\Big(\frac{1}{t}\int_{(-t,t)\cap(-x,\infty)}\|f(x+h)\|\,\mathrm{d}h\Big)^q\,\frac{\mathrm{d}t}{t}\Big)^{p/q}\,\mathrm{d}x\Big)^{1/p}\\
&=: (I)+(II)+(III).
\end{aligned}
$$

We estimate these three terms separately. Clearly, $(I)\leqslant [f]^{(1)}_{F^s_{p,q}(\mathbb{R};X)}$ and, with $C=1+\frac{p}{|sp-1|}$,

$$
\begin{aligned}
(II) &\leqslant \Big(\int_{\mathbb{R}_+}\Big(\int_x^\infty t^{-sq}\,\frac{\mathrm{d}t}{t}\Big)^{p/q}\|f(x)\|^p\,\mathrm{d}x\Big)^{1/p}\\
&\leqslant (sq)^{-1/q}\Big(\int_0^\infty x^{-sp}\|f(x)\|^p\,\mathrm{d}x\Big)^{1/p}\\
&\lesssim_{s,p,q} \|f\|_{F^s_{p,q}(\mathbb{R};X)},
\end{aligned}
$$

using Corollary 14.6.31 in the last step.

To estimate (III) fix $x\in(-\infty,0)$. By Minkowski's inequality (Theorem 1.2.22),

$$
\begin{aligned}
&\Big(\int_{\mathbb{R}_+} t^{-sq}\Big(\frac{1}{t}\int_{(-t,t)\cap(-x,\infty)}\|f(x+h)\|\,\mathrm{d}h\Big)^q\,\frac{\mathrm{d}t}{t}\Big)^{p/q}\\
&\qquad\leqslant \Big(\int_{\mathbb{R}_+}\Big(\int_{\mathbb{R}_+} t^{-sq-q}\mathbf{1}_{(h,\infty)}(t)\,\frac{\mathrm{d}t}{t}\Big)^{1/q}\mathbf{1}_{(-x,\infty)}(h)\|f(x+h)\|\,\mathrm{d}h\\
&\qquad= K_{q,s}\int_{\mathbb{R}_+} h^{-s-1}\mathbf{1}_{(-x,\infty)}(h)\|f(h+x)\|\,\mathrm{d}h\\
&\qquad= K_{q,s}\int_{\mathbb{R}_+}(y-x)^{-s-1}\|f(y)\|\,\mathrm{d}y,
\end{aligned}
$$

where $K_{q,s}=(sq+q)^{1/q}$. Setting $z=-x$ and $\phi_p(z)=z^{1/p}(1+z)^{-s-1}$, (III) can be estimated using Young's inequality for convolutions for the multiplicative group $\mathbb{R}_+$ with Haar measure $\frac{\mathrm{d}z}{z}$:

$$
\begin{aligned}
(III) &\leqslant K_{q,s}\Big(\int_{\mathbb{R}_+}\Big(\int_{\mathbb{R}_+}(y+z)^{-s-1}\|f(y)\|\,\mathrm{d}y\Big)^p\,\mathrm{d}z\Big)^{1/p}\\
&= K_{q,s}\Big(\int_{\mathbb{R}_+}\Big(\int_{\mathbb{R}_+}\phi_p(z/y)y^{-s+\frac{1}{p}}\|f(y)\|\,\frac{\mathrm{d}y}{y}\Big)^p\frac{\mathrm{d}z}{z}\Big)^{1/p}\\
&\leqslant K_{q,s}\|\phi_p\|_{L^1(\mathbb{R}_+,\frac{\mathrm{d}z}{z})}\Big(\int_{\mathbb{R}_+} y^{-sp}\|f(y)\|^p\,\mathrm{d}y\Big)^{1/p}\\
&\lesssim_{p,q,s} \|f\|_{F^s_{p,q}(\mathbb{R};X)},
\end{aligned}
$$

using Corollary 14.6.31 as in the estimate for (II). □

Example 14.6.33. Theorem 14.6.32 is false for $s = 1/p$ even in the scalar-valued case. Indeed, $f \in C_c^\infty(\mathbb{R})$ is any function satisfying $f \equiv 1$ on $[-1,1]$, then for all $p \in [1,\infty)$ we have $f \in F^{1/p}_{p,q}(\mathbb{R})$. Let us prove that $\mathbf{1}_{\mathbb{R}_+} f \notin F^{1/p}_{p,q}(\mathbb{R})$. To this end it suffices to take $q = \infty$. In case $p \in (1,\infty)$ we can use Theorem 14.6.20 to find

$$\begin{aligned}
\|\mathbf{1}_{\mathbb{R}_+} f\|_{F^{1/p}_{p,\infty}(\mathbb{R}^d;X)} &\eqsim_p \|\mathbf{1}_{\mathbb{R}_+} f\|^{(1)}_{F^{1/p}_{p,\infty}(\mathbb{R}^d;X)} \\
&\geqslant \Big\| x \mapsto \sup_{t>0} t^{-\frac{1}{p}-1} \int_{-t}^{-x} |f(x)| \,\mathrm{d}h \Big\|_{L^p(0,1)} \\
&= \Big\| x \mapsto \sup_{t>x} t^{-\frac{1}{p}-1} (t-x) \Big\|_{L^p(0,1)} \\
&\gtrsim_p \|x \mapsto x^{-\frac{1}{p}}\|_{L^p(0,1)} = \infty.
\end{aligned}$$

For $p = 1$ we note that $F^1_{1,q}(\mathbb{R}) \hookrightarrow F^{1/r}_{r,\infty}(\mathbb{R})$ for all $r \in (p,\infty)$ by Theorem 14.6.14, and therefore $\mathbf{1}_{\mathbb{R}_+} f \notin F^1_{1,q}(\mathbb{R})$.

One could still hope that the boundedness of $f \mapsto \mathbf{1}_{\mathbb{R}_+} f$ for $s = 1/p$ holds on the closure in $F^{1/p}_{p,q}(\mathbb{R})$ of the smooth functions satisfying $f(0) = 0$. This turns out to be false as well. Indeed, in the case $q < \infty$ the latter space coincides with $F^{1/p}_{p,q}(\mathbb{R})$ by Proposition 14.6.17. If $q = \infty$, the boundedness is also fails, as follows from the previous example and the embedding $F^{1/p}_{p,\infty}(\mathbb{R}) \hookrightarrow F^{1/r}_{r,r}(\mathbb{R})$ for all $r \in (p,\infty)$ contained in Theorem 14.6.14.

By duality and interpolation, we now extend Theorem 14.6.32 to smoothness exponents $s \leqslant 0$, which excludes the end-point cases.

Corollary 14.6.34 (Pointwise multiplication by $\mathbf{1}_{\mathbb{R}_+}$). *Let $p \in (1,\infty)$, $q \in (1,\infty)$, and $s \in (-1/p',0]$. For all $f \in F^s_{p,q}(\mathbb{R};X)$ we have $\mathbf{1}_{\mathbb{R}_+} f \in F^s_{p,q}(\mathbb{R};X)$ and*

$$\|\mathbf{1}_{\mathbb{R}_+} f\|_{F^s_{p,q}(\mathbb{R};X)} \leqslant C\|f\|_{F^s_{p,q}(\mathbb{R};X)}.$$

Proof. By density it suffices to consider $f \in C^\infty(\mathbb{R}\setminus\{0\}) \otimes X$. We use duality result. By Theorems 14.6.28 and 14.6.32 for any $g \in \mathscr{S}(\mathbb{R}^d;X^*)$ we have

$$\begin{aligned}
|\langle \mathbf{1}_{\mathbb{R}_+} f, g\rangle| = |\langle f, \mathbf{1}_{\mathbb{R}_+} g\rangle| &\leqslant C\|f\|_{F^s_{p,q}(\mathbb{R};X)} \|\mathbf{1}_{\mathbb{R}_+} g\|_{F^{-s}_{p',q'}(\mathbb{R};X^*)} \\
&\leqslant C'\|f\|_{F^s_{p,q}(\mathbb{R};X)} \|g\|_{F^{-s}_{p',q'}(\mathbb{R};X^*)}.
\end{aligned}$$

Since $\mathscr{S}(\mathbb{R}^d;X^*)$ is dense in $F^{-s}_{p',q'}(\mathbb{R};X^*)$, the result follows by another application of Theorem 14.6.28.

The case $s = 0$ follows by complex interpolation between the cases s and $-s$ for $s > 0$ small enough, using Theorems C.2.6 and 14.6.23. □

Applying the real interpolation method instead, we obtain the following for the Besov scale.

Corollary 14.6.35 (Pointwise multiplication by $\mathbf{1}_{\mathbb{R}_+}$). *Let $p \in (1,\infty)$, $q \in [1,\infty]$, and $s \in (-1/p, 1/p)$. For all $f \in B^s_{p,q}(\mathbb{R};X)$ we have $\mathbf{1}_{\mathbb{R}_+} f \in B^s_{p,q}(\mathbb{R};X)$ and*

$$\|\mathbf{1}_{\mathbb{R}_+} f\|_{B^s_{p,q}(\mathbb{R};X)} \leqslant C\|f\|_{B^s_{p,q}(\mathbb{R};X)}, \quad f \in B^s_{p,q}(\mathbb{R};X).$$

Proof. First let $s > 0$. Since $(F^{s-\varepsilon}_{p,2}, F^{s+\varepsilon}_{p,2})_{1/2,q} = B^s_{p,q}$ by Theorem 14.4.31, the result follows from Theorems 14.6.32 and C.3.3. Here we can allow $p = 1$ as well.

The result for $s < 0$ and $q \in (1,\infty)$ follows from Theorem 14.4.34 in the same way as in Corollary 14.6.34. The cases $q = 1$ and $q = \infty$ can be obtained by another real interpolation argument as we did in Example 14.4.35.

The case $s = 0$ follows by real interpolation between the cases s and $-s$ for $s > 0$ small. □

14.7 Bessel potential spaces

In this section we prove Sobolev embeddings and norm estimates for Bessel potential spaces. Some results will depend on the geometry of X. Real interpolation for $H^{s,p}(\mathbb{R}^d;X)$ has already been considered in Theorem 14.4.31. Duality for $H^{s,p}(\mathbb{R}^d;X)$ has already been considered in Proposition 5.6.7.

14.7.a General embedding theorems

We begin with the following Sobolev embedding theorem.

Theorem 14.7.1 (Sobolev embedding for Bessel potential spaces and Sobolev spaces). *Let $p_0, p_1 \in (1,\infty)$ and $s_0, s_1 \in \mathbb{R}$. We have a continuous embedding*

$$H^{s_0,p_0}(\mathbb{R}^d;X) \hookrightarrow H^{s_1,p_1}(\mathbb{R}^d;X)$$

if and only if one of the following two conditions holds:

$$p_0 = p_1 \quad \text{and} \quad s_0 \geqslant s_1; \tag{14.89}$$

$$p_0 < p_1 \qquad \text{and} \qquad s_0 - \frac{d}{p_0} \geqslant s_1 - \frac{d}{p_1}. \tag{14.90}$$

If $s_0, s_1 \in \mathbb{N}$, then the same necessary and sufficient conditions give the existence of a continuous embedding

$$W^{s_0,p_0}(\mathbb{R}^d;X) \hookrightarrow W^{s_1,p_1}(\mathbb{R}^d;X).$$

Proof. We first prove the result for Bessel potential spaces.

'If': By Proposition 14.6.13, for $p \in (1,\infty)$ and $s \in \mathbb{R}$ we have continuous embeddings

$$F^s_{p,1}(\mathbb{R}^d;X) \hookrightarrow H^{s,p}(\mathbb{R}^d;X) \hookrightarrow F^s_{p,\infty}(\mathbb{R}^d;X). \tag{14.91}$$

From Theorem 14.6.14 we see that if either (14.89) or (14.90) holds, then $F^{s_0}_{p_0,\infty}(\mathbb{R}^d;X) \hookrightarrow F^{s_1}_{p_1,1}(\mathbb{R}^d;X)$. Therefore the required embedding follows from (14.91) with $s = s_0, s_1$ and $p = p_0, p_1$.

'Only if': If the stated embedding holds, then by (14.91) with $s = s_0, s_1$ and $p = p_0, p_1$, we also have a continuous embedding $F^{s_0}_{p_0,1}(\mathbb{R}^d;X) \hookrightarrow F^{s_1}_{p_1,\infty}(\mathbb{R}^d;X)$. Therefore, either (14.89) or (14.90) must hold by Theorem 14.6.14.

The corresponding result for Sobolev spaces with integer smoothness can be proved in the same way, noting that the analogue of (14.91) holds for these spaces. □

Remark 14.7.2. The embedding of Theorem 14.7.1 for Bessel potential spaces can be restated as the boundedness of $J_{-(s_0-s_1)} = (1-\Delta)^{-(s_0-s_1)}$ from $L^{p_0}(\mathbb{R}^d;X)$ into $L^{p_1}(\mathbb{R}^d;X)$. Since $J_{-(s_0-s_1)}$ is a positive operator by Proposition 5.6.6, we infer from Theorem 2.1.3 that the boundedness in the scalar case is actually equivalent to boundedness in the vector-valued situation.

By the same argument as in Theorem 14.7.1, the following result can be deduced from Proposition 14.6.15.

Proposition 14.7.3 (Gagliardo–Nirenberg inequality for Bessel potential spaces). *Let $p_0, p_1 \in (1,\infty)$, $-\infty < s_0 < s_1 < \infty$, and $\theta \in (0,1)$, and let*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad s = (1-\theta)s_0 + \theta s_1.$$

There exists a constant $C = C_{\theta,p_0,p_1,s_0,s_1} \geqslant 0$ such that for all $f \in H^{s_0,p_0}(\mathbb{R}^d;X) \cap H^{s_1,p_1}(\mathbb{R}^d;X)$ we have $f \in H^{s,p}(\mathbb{R}^d;X)$ and

$$\|f\|_{H^{s,p}(\mathbb{R}^d;X)} \leqslant C\|f\|^{1-\theta}_{H^{s_0,p_0}(\mathbb{R}^d;X)}\|f\|^{\theta}_{H^{s_1,p_1}(\mathbb{R}^d;X)}.$$

If, in Proposition 14.7.3, $s_0, s_1 \geqslant 0$ are integers and $p \in (1,\infty)$, the same argument gives that $f \in W^{s_0,p_0}(\mathbb{R}^d;X) \cap W^{s_1,p_1}(\mathbb{R}^d;X)$ implies $f \in W^{s,p}(\mathbb{R}^d;X)$ and

$$\|f\|_{W^{s,p}(\mathbb{R}^d;X)} \leqslant C\|f\|^{1-\theta}_{W^{s_0,p_0}(\mathbb{R}^d;X)}\|f\|^{\theta}_{W^{s_1,p_1}(\mathbb{R}^d;X)}. \tag{14.92}$$

The latter estimate extends to $p_0 \in (1,\infty]$ and $p_1 \in (1,\infty]$. Indeed, if only one of the exponents is infinite, then (14.92) is a consequence of Proposition 14.6.16 and the sandwich results of Propositions 14.4.18 (see (14.29)) and

14.6.13. If $p = p_0 = p_1 \in [1, \infty]$, (14.92) can be deduced from these sandwich results and real interpolation and (L.2):

$$\begin{aligned}(W^{s_0,p}(\mathbb{R}^d;X), W^{s_1,p}(\mathbb{R}^d;X))_{\theta,1} &\hookrightarrow (B^{s_0}_{p,\infty}(\mathbb{R}^d;X), B^{s_1}_{p,\infty}(\mathbb{R}^d;X))_{\theta,1}\\ &= B^s_{p,1}(\mathbb{R}^d;X) \qquad \text{(by (14.48))}\\ &\hookrightarrow W^{s,p}(\mathbb{R}^d;X).\end{aligned}$$

Note that this even gives (14.92) for $p = p_0 = p_1 = 1$.

The estimate (14.92) self-improves to the following Gagliardo–Nirenberg type inequality for $W^{s,p}(\mathbb{R}^d;X)$:

Theorem 14.7.4 (Schmeisser–Sickel). *Let $p_0, p_1, p \in (1, \infty]$, $m \in \mathbb{N}$, and $|\alpha| \leqslant m$ satisfy*

$$\theta = \frac{|\alpha|}{m} \quad \text{and} \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

There exists a constant $C \geqslant 0$ such that for all $f \in L^{p_0}(\mathbb{R}^d;X) \cap W^{m,p_1}(\mathbb{R}^d;X)$ we have

$$\|\partial^\alpha f\|_{L^p(\mathbb{R}^d;X)} \leqslant C \|f\|^{1-\theta}_{L^{p_0}(\mathbb{R}^d;X)} \Big(\sum_{|\beta|=m} \|\partial^\beta f\|_{L^{p_1}(\mathbb{R}^d;X)} \Big)^\theta.$$

Moreover, the same holds if $p = p_0 = p_1 = 1$.

Proof. For $\theta = \frac{|\alpha|}{m} \in \{0, 1\}$ there is nothing to prove, so we may assume that $\theta \in (0, 1)$. Taking $s = |\alpha|$, $s_0 = 0$, and $s_1 = m$ in (14.92), it follows that

$$\|\partial^\alpha f\|_{L^p(\mathbb{R}^d;X)} \leqslant C \|f\|^{1-\theta}_{L^{p_0}(\mathbb{R}^d;X)} \Big(\sum_{|\beta|\leqslant m} \|\partial^\beta f\|_{L^{p_1}(\mathbb{R}^d;X)} \Big)^\theta.$$

Applying this to the function $f(\lambda\cdot)$ for $\lambda > 0$, we obtain

$$\begin{aligned}&\lambda^{|\alpha| - \frac{d}{p}} \|\partial^\alpha f\|_{L^p(\mathbb{R}^d;X)}\\ &\qquad \leqslant C(\lambda^{-\frac{d}{p_0}} \|f\|_{L^{p_0}(\mathbb{R}^d;X)})^{1-\theta} \Big(\sum_{|\beta|\leqslant m} \lambda^{|\beta| - \frac{d}{p_1}} \|\partial^\beta f\|_{L^{p_1}(\mathbb{R}^d;X)} \Big)^\theta.\end{aligned}$$

Now divide both sides by $\lambda^{|\alpha| - \frac{d}{p}}$ and pass to the limit $\lambda \to \infty$. □

14.7.b Embedding theorems under geometric conditions

Littlewood–Paley inequality for Bessel potential spaces

The aim of this paragraph is to prove the following Littlewood–Paley inequality with smooth cut-offs for $H^{s,p}(\mathbb{R}^d;X)$.

Theorem 14.7.5 (Littlewood–Paley theorem for Bessel potential spaces). *Let X be a UMD space, $p \in (1,\infty)$, and $s \in \mathbb{R}$. A tempered distribution $f \in \mathscr{S}'(\mathbb{R}^d;X)$ belongs to $H^{s,p}(\mathbb{R}^d;X)$ if and only if*

$$\|\!|f|\!\|_{H^{s,p}(\mathbb{R}^d;X)} := \sup_{n\geqslant 0}\Big\|\sum_{k=0}^{n}\varepsilon_k 2^{ks}\varphi_k * f\Big\|_{L^p(\Omega\times\mathbb{R}^d;X)} < \infty.$$

In this situation the sum $\sum_{k\geqslant 0}\varepsilon_k 2^{ks}\varphi_k * f$ converges, both in $L^p(\Omega\times\mathbb{R}^d;X)$ and almost surely in $L^p(\mathbb{R}^d;X)$, and we have an equivalence of norms

$$\|\!|f|\!\|_{H^{s,p}(\mathbb{R};X)} \eqsim_{d,p,s,X} \|f\|_{H^{s,p}(\mathbb{R}^d;X)}.$$

For $s=0$ the above estimate yields an equivalent norm on $L^p(\mathbb{R}^d;X)$ which is slightly different from the Littlewood–Paley estimate with smooth cut-offs of Theorem 5.5.22, where the summation was taken over $\mathbb{Z}$ and the functions ψ_k were of the form $2^k\psi(2^k\cdot)$ for a Littlewood–Paley function ψ in the sense of Definition 5.5.20.

Proof. 'Only if': Fix $f \in H^{s,p}(\mathbb{R}^d;X)$. Fix a sequence of signs $\epsilon = (\epsilon_k)_{k\geqslant 0}$ in $\{z\in\mathbb{K} : |z|=1\}$. For integers $n\geqslant 0$, define the function $m_n \in C^\infty(\mathbb{R}^d)$ by

$$m_n(\xi) := \sum_{k=0}^{n}\epsilon_k 2^{ks}(1+|\xi|^2)^{-s/2}\widehat{\varphi}_k(\xi).$$

From the location of the supports of the functions $\widehat{\varphi}_k$ one sees three things: first, that for each $\xi\in\mathbb{R}^d$ at most three terms in this sum are non-zero (the sum therefore converges for trivial reasons); second, that $\|\partial^\beta\widehat{\varphi}_k\|_\infty \leqslant C_\beta 2^{-k|\beta|}$; and third, that

$$c_d = \sup_{\epsilon}\sup_{n\geqslant 0}\sup_{\alpha\in\{0,1\}^d}\sup_{\xi\neq 0}|\xi|^{|\alpha|}|\partial^\alpha m_n(\xi)|$$

is finite, the outer supremum being taken over all sequences of signs $\epsilon = (\epsilon_k)_{k\geqslant 0}$.

By the Mihlin multiplier theorem (Theorem 5.5.10), the Fourier multiplier operators T_{m_n} associated with m_n are bounded on $L^p(\mathbb{R}^d;X)$, with estimates uniform in n and signs ϵ, say $\sup_\epsilon\sup_{n\geqslant 0}\|T_{m_n}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))}\leqslant C_{X,p,d}$. Since

$$\sum_{k=0}^{n}\epsilon_k 2^{ks}\varphi_k * f = T_{m_n}J_s f, \tag{14.93}$$

we obtain

$$\Big\|\sum_{k=0}^{n}\epsilon_k 2^{ks}\varphi_k * f\Big\|_{L^p(\mathbb{R}^d;X)} \leqslant C_{X,p,d}\|J_s f\|_{L^p(\mathbb{R}^d;X)} = C_{X,p,d}\|f\|_{H^{s,p}(\mathbb{R}^d;X)}.$$

Taking $\epsilon_k = \varepsilon_k(\omega)$ and passing to the $L^p(\Omega)$-norms, we obtain the estimate

$$\Big\| \sum_{k=0}^{n} \varepsilon_k 2^{ks} \varphi_k * f \Big\|_{L^p(\Omega\times\mathbb{R}^d;X)} \leqslant C_{X,p,d} \|f\|_{H^{s,p}(\mathbb{R}^d;X)}.$$

'If': Assume now that $f \in \mathscr{S}'(\mathbb{R}^d;X)$ satisfies $|\!|\!| f |\!|\!|_{H^{s,p}(\mathbb{R}^d;X)} < \infty$. We claim that $\sum_{k\geqslant 0} \varepsilon_k 2^{ks}\varphi_k * f$ converges in $L^p(\Omega; L^p(\mathbb{R}^d;X))$ and almost surely in $L^p(\mathbb{R}^d;X)$. Indeed, $L^p(\mathbb{R}^d;X)$ is a UMD space by Proposition 4.2.15, so by Proposition 4.2.19 it does not contain an isomorphic copy of c_0. The convergence of the sum, in $L^p(\Omega\times\mathbb{R}^d;X)$ and almost surely in $L^p(\mathbb{R}^d;X)$, now follows from Corollary 6.4.12. Moreover, by Fatou's lemma and the Kahane contraction principle,

$$|\!|\!| f |\!|\!|_{H^{s,p}(\mathbb{R}^d;X)} = \Big\| \sum_{k\geqslant 0} \varepsilon_k 2^{ks} \varphi_k * f \Big\|_{L^p(\Omega\times\mathbb{R}^d;X)}.$$

For $k \in \{0,1\}$ choose $\psi_k \in C_c^\infty(\mathbb{R})$ such that $0 \leqslant \widehat{\psi}_k \leqslant 1$, $\operatorname{supp}\widehat{\psi}_0 \subseteq \{0 \leqslant |\xi| \leqslant 2\}$ and $\operatorname{supp}\widehat{\psi}_1 \subseteq \{\frac14 \leqslant |\xi| \leqslant 4\}$, and $\widehat{\psi}_k \equiv 1$ on $\operatorname{supp}\widehat{\varphi}_k$. For $k \geqslant 2$ we define $\widehat{\psi}_k := \widehat{\psi}_1(2^{-(k-1)}\cdot)$. For $\omega \in \Omega$ put

$$m_\omega := \sum_{j\geqslant 0} \overline{\varepsilon_j(\omega)} 2^{-js}(1+|\cdot|^2)^{s/2}\widehat{\psi}_j, \qquad g_\omega := \sum_{k\geqslant 0} \varepsilon_k(\omega) 2^{ks}\varphi_k * f.$$

As before,

$$C_m = \sup_{\omega\in\Omega} \sup_{\alpha\in\{0,1\}^d} \sup_{\xi\neq 0} |\xi|^{|\alpha|} |\partial^\alpha m_\omega(\xi)| < \infty.$$

Therefore, by the Mihlin multiplier Theorem 5.5.10,

$$\|T_{m_\omega} g_\omega\|_{L^p(\mathbb{R}^d;X)} \leqslant C \|g_\omega\|_{L^p(\mathbb{R}^d;X)}$$

for almost every $\omega \in \Omega$. Considering finite sums first, one checks that $\omega \mapsto T_{m_\omega} g_\omega$ is strongly measurable. Since $\omega \mapsto g_\omega$ belongs to $L^p(\Omega; L^p(\mathbb{R}^d;X))$, it follows that so does $\omega \mapsto T_{m_\omega} g_\omega$. By the condition $\widehat{\psi}_k \equiv 1$ on $\operatorname{supp}\widehat{\varphi}_k$, as in (14.93) we have

$$\int_\Omega T_{m_\omega} g_\omega \,\mathrm{d}\mathbb{P}(\omega) = J_s f.$$

By Jensen's inequality and Fubini's theorem, $f \in H^{s,p}(\mathbb{R}^d;X)$ and

$$\begin{aligned}
\|f\|^p_{H^{s,p}(\mathbb{R}^d;X)} &= \|J_s f\|^p_{L^p(\mathbb{R}^d;X)} \\
&= \Big\| \int_\Omega T_{m_\omega} g_\omega \,\mathrm{d}\mathbb{P}(\omega) \Big\|^p_{L^p(\mathbb{R}^d;X)} \\
&\leqslant \int_\Omega \|T_{m_\omega} g_\omega\|^p_{L^p(\mathbb{R}^d;X)} \,\mathrm{d}\mathbb{P}(\omega)
\end{aligned}$$

$$\leqslant C \int_{\Omega} \|g_\omega\|^p_{L^p(\mathbb{R}^d;X)} \, \mathrm{d}\mathbb{P}(\omega) = C\|f\|^p_{H^{s,p}(\mathbb{R}^d;X)}.$$

□

We continue with an embedding result under additional geometric assumptions on X. The cases $p_0 = 1$ and $q_0 = \infty$ were proved for general Banach spaces in Propositions 14.4.18 and 14.6.13.

Proposition 14.7.6 (Sandwich theorem under type and cotype). *Let X be a UMD Banach space with type $p_0 \in [1,2]$ and cotype $q_0 \in [2,\infty]$. For all $p \in (1,\infty)$ and $s \in \mathbb{R}$ we have continuous embeddings*

$$F^s_{p,p_0}(\mathbb{R}^d;X) \hookrightarrow H^{s,p}(\mathbb{R}^d;X) \hookrightarrow F^s_{p,q_0}(\mathbb{R}^d;X).$$

Proof. We only prove $F^s_{p,p_0}(\mathbb{R}^d;X) \hookrightarrow H^{s,p}(\mathbb{R}^d;X)$; the other embedding is proved similarly.

Let $f \in F^s_{p,p_0}(\mathbb{R}^d;X)$. By Theorem 14.7.5, the Kahane–Khintchine inequality (Theorem 6.2.4) and the type p_0 property of X, we have

$$\begin{aligned}
\|f\|_{H^{s,p}(\mathbb{R}^d;X)} &\leqslant C \sup_{n\geqslant 1} \Big\| \sum_{k=0}^n \varepsilon_k 2^{ks} \varphi_k * f \Big\|_{L^p(\Omega\times\mathbb{R}^d;X)} \\
&\eqsim_p C \sup_{n\geqslant 1} \Big(\int_{\mathbb{R}^d} \Big\| \sum_{k=0}^n \varepsilon_k 2^{ks} \varphi_k * f \Big\|^p_{L^{p_0}(\Omega;X)} \, \mathrm{d}x \Big)^{1/p} \\
&\leqslant C \sup_{n\geqslant 1} \Big(\int_{\mathbb{R}^d} \Big(\sum_{k=0}^n \|2^{ks}\varphi_k * f\|^{p_0} \Big)^{p/p_0} \mathrm{d}x \Big)^{1/p} \\
&= C\|f\|_{F^s_{p,p_0}(\mathbb{R}^d;X)}.
\end{aligned}$$

□

In combination with Proposition 14.6.13 and Corollary 14.6.18 we obtain:

Corollary 14.7.7 (γ-Sobolev embedding – III). *Let $p_0 \in [1,2]$ and $q_0 \in [2,\infty]$.*

(1) *If X has type p_0, then for all $p \in [1,p_0)$ we have a continuous embedding*

$$H^{(\frac1p-\frac12)d,p}(\mathbb{R}^d;X) \hookrightarrow \gamma(L^2(\mathbb{R}^d),X).$$

(2) *If X has cotype q_0, then for all $q \in (q_0,\infty)$ we have a continuous embedding*

$$\gamma(L^2(\mathbb{R}^d),X) \hookrightarrow H^{(\frac1q-\frac12)d,q}(\mathbb{R}^d;X)$$

By Theorem 9.2.10, for $p_0 = 2$ assertion (1) also holds for $p = 2$, and for $q_0 = 2$ assertion (2) also holds for $q = 2$.

Necessity of the type and cotype assumptions

Proposition 14.7.8. *Let $p \in (1,\infty)$, $q \in [1,\infty]$, $s \in \mathbb{R}$, and $m \in \mathbb{N}$. Then the following assertions hold with $\mathcal{A} \in \{B, F\}$:*

(1) *If $\mathcal{A}^s_{p,q}(\mathbb{R}^d; X) \hookrightarrow H^{s,p}(\mathbb{R}^d; X)$ continuously, then X has type q.*
(2) *If $H^{s,p}(\mathbb{R}^d; X) \hookrightarrow \mathcal{A}^s_{p,q}(\mathbb{R}^d; X)$ continuously, then X has cotype q.*
(3) *If $\mathcal{A}^k_{p,q}(\mathbb{R}^d; X) \hookrightarrow W^{m,p}(\mathbb{R}^d; X)$ continuously, then X has type q.*
(4) *If $W^{m,p}(\mathbb{R}^d; X) \hookrightarrow \mathcal{A}^m_{p,q}(\mathbb{R}^d; X)$ continuously, then X has cotype q.*

Proof. (1): By the lifting properties of Propositions 14.4.15, 14.6.10, and 5.6.3, it suffices to consider $s = 0$. Fix a finitely non-zero sequence $(x_n)_{n\geqslant 1}$ in X. Let $\psi \in \mathscr{S}(\mathbb{R}^d)$ be a non-zero function satisfying $\mathrm{supp}(\widehat{\psi}) \subseteq [-\frac{1}{4}, -\frac{1}{8}]^d$ and put

$$f(t,\omega) := \psi(t) \sum_{n\geqslant 1} \varepsilon_n(\omega) e^{2\pi i 2^n t_1} x_n,$$

where as always $(\varepsilon_n)_{n\geqslant 1}$ is a Rademacher sequence. Since $(\varepsilon_n e^{2\pi i 2^n t_1})_{n\geqslant 1}$ is a Rademacher sequence for each $t \in \mathbb{R}^d$, we have

$$\begin{aligned} \mathbb{E}\|f\|^p_{L^p(\mathbb{R}^d;X)} &= \int_{\mathbb{R}^d} |\psi(t)|^p \mathbb{E}\Big\| \sum_{n\geqslant 1} \varepsilon_n e^{2\pi i 2^n t_1} x_n \Big\|^p \,\mathrm{d}t \\ &= \|\psi\|^p_{L^p(\mathbb{R}^d)} \mathbb{E}\Big\| \sum_{n\geqslant 1} \varepsilon_n x_n \Big\|^p. \end{aligned} \tag{14.94}$$

On the other hand, the Fourier support properties of $\widehat{\psi}(\cdot - 2^n t_1 e_1)$ and $\widehat{\varphi}_n$ (see (14.8) and (14.9)) imply that $\|f(\cdot,\omega) * \varphi_n\|_X = |\psi(t)|\|x_n\|$ and $\|f(\cdot,\omega) * \varphi_0\|_X = 0$. Therefore,

$$\|f(\cdot,\omega)\|_{\mathcal{A}^s_{p,q}(\mathbb{R}^d;X)} = \|\psi\|_{L^p(\mathbb{R}^d)} \|(x_n)_{n\geqslant 1}\|_{\ell^q(X)}. \tag{14.95}$$

Applying the assumption (1) pointwise in Ω, we obtain

$$\begin{aligned} \|\psi\|^p_{L^p(\mathbb{R})} \mathbb{E}\Big\| \sum_{n\geqslant 1} \varepsilon_n x_n \Big\|^p &= \mathbb{E}\|f\|^p_{L^p(\mathbb{R}^d;X)} \\ &\leqslant C^p \mathbb{E}\|f\|^p_{\mathcal{A}^s_{p,q}(\mathbb{R}^d;X)} = C^p \|\psi\|^p_{L^p(\mathbb{R}^d)} \|(x_n)_{n\geqslant 1}\|^p_{\ell^q(X)}. \end{aligned}$$

By the Kahane–Khintchine inequalities, this shows that X has type q.

(2): This follows from the previous proof upon replacing "$\leqslant$" by "$\geqslant$".

(3): The idea of the proof is the same as in (1), but this case is slightly more technical. Let $(x_n)_{n\geqslant 1}$ and ψ be as before and put

$$f(t,\omega) := \psi(t) \sum_{n\geqslant 1} 2^{-mn} \varepsilon_n(\omega) e^{2\pi i 2^n t_1} x_n =: \psi(t) f_m(t,\omega).$$

By Leibniz's rule we obtain

$$\partial^\alpha f(t,\omega) = \sum_{|\beta|+j=|\alpha|} c_{\beta,\gamma}\partial^\beta \psi(t) f_{m-j}(t,\omega),$$

For $j \in \{0,\dots,m-1\}$,

$$\begin{aligned}\|(\partial^\beta\psi) f_{m-j}(\cdot,\omega)\|_{L^p(\mathbb{R}^d;X)} &\leqslant \|\partial^\beta\psi\|_\infty \|f_{m-j}(\cdot,\omega)\|_{L^p(\mathbb{R}^d;X)} \\ &\leqslant \|\partial^\beta\psi\|_\infty \sum_{n\geqslant 1} 2^{-(m-j)n}\|x_n\| \leqslant \|\partial^\beta\psi\|_\infty \sup_{n\geqslant 1}\|x_n\|.\end{aligned}$$

For $j = m$, as in (14.94) we have

$$\mathbb{E}\|f_0\|^p_{L^p(\mathbb{R}^d;X)} = \|\psi\|^p_{L^p(\mathbb{R})}\mathbb{E}\Big\|\sum_{n\geqslant 1}\varepsilon_n x_n\Big\|^p.$$

By the reverse triangle inequality, this shows that there exists a constant $C = C(d,m,p,\psi)$ such that

$$\Big|\|f\|_{L^p(\Omega;W^{m,p}(\mathbb{R}^d;X))} - \|\psi\|_{L^p(\mathbb{R})}\Big\|\sum_{n\geqslant 1}\varepsilon_n x_n\Big\|_{L^p(\Omega;X)}\Big| \leqslant C\sup_{n\geqslant 1}\|x_n\|. \quad (14.96)$$

Stated differently, up a relatively small term the norm $\|f\|_{L^p(\Omega;W^{m,p}(\mathbb{R}^d;X))}$ is equivalent to the norm $\|\sum_{n\geqslant 1}\varepsilon_n x_n\|$ of the random sum. As in (14.95) we see that

$$\|f(\cdot,\omega)\|_{\mathcal{A}^m_{p,q}(\mathbb{R}^d;X)} = \|\psi\|_{L^p(\mathbb{R}^d)}\|(x_n)_{n\geqslant 1}\|_{\ell^q(X)}.$$

Now from (14.96) and the assumptions, we obtain

$$\begin{aligned}\|\psi\|_{L^p(\mathbb{R})}\Big\|\sum_{n\geqslant 1}\varepsilon_n x_n\Big\|_{L^p(\Omega;X)} &\leqslant \|f\|_{L^p(\Omega;W^{m,p}(\mathbb{R}^d;X))} + C\sup_{n\geqslant 1}\|x_n\| \\ &\lesssim \|f\|_{L^p(\Omega;\mathcal{A}^m_{p,q}(\mathbb{R}^d;X))} + \sup_{n\geqslant 1}\|x_n\| \\ &\lesssim \|(x_n)_{n\geqslant 1}\|_{\ell^q(X)}.\end{aligned}$$

(4): This can be proved in the same way as (3). By (14.96) and the Kahane contraction principle, which implies bound $\sup_{n\geqslant 1}\|x_n\|^p \leqslant \mathbb{E}\|\sum_{n\geqslant 1}\varepsilon_n x_n\|^p$, from the assumption (4) we obtain

$$\begin{aligned}\|\psi\|_{L^p(\mathbb{R}^d)}\|(x_n)_{n\geqslant 1}\|_{\ell^q(X)} &= \|f\|_{L^p(\Omega;\mathcal{A}^m_{p,q}(\mathbb{R}^d;X))} \\ &\lesssim \|f\|_{L^p(\Omega;W^{m,p}(\mathbb{R}^d;X))} \lesssim \Big\|\sum_{n\geqslant 1}\varepsilon_n x_n\Big\|_{L^p(\Omega;X)}.\end{aligned}$$

□

A Hilbert space characterisation

The equality $F^s_{p,2}(\mathbb{R}^d;X) = H^{s,p}(\mathbb{R}^d;X)$ with equivalent norms characterises Hilbert spaces:

Theorem 14.7.9 (Han–Meyer). *Let $p \in (1,\infty)$, $s \in \mathbb{R}$, and $m \in \mathbb{N}$. The following assertions are equivalent:*

(1) $F^m_{p,2}(\mathbb{R}^d;X) = W^{m,p}(\mathbb{R}^d;X)$ *with equivalent norms;*
(2) $F^s_{p,2}(\mathbb{R}^d;X) = H^{s,p}(\mathbb{R}^d;X)$ *with equivalent norms;*
(3) X *is isomorphic to a Hilbert space.*

Proof. (1)⇒(3) and (2)⇒(3): By Proposition 14.7.8, X has type 2 and cotype 2. Therefore X is isomorphic to a Hilbert space by Theorem 7.3.1.

(3)⇒(2): This is immediate from Proposition 14.7.6 and the fact that Hilbert spaces are UMD (by Theorem 4.2.14) and have type 2 and cotype 2 (by the result of Example 7.1.2).

(3)⇒(1): This is a special case of the previous implication since Theorem 5.6.11 implies $W^{m,p}(\mathbb{R}^d;X) = H^{m,p}(\mathbb{R}^d;X)$ with equivalent norms. □

14.7.c Interpolation

Real interpolation of vector-valued Bessel potential spaces has already been considered in Theorem 14.4.31. Complex interpolation was considered in Theorem 5.6.9, but only in the case $p_0 = p_1$ and $X_0 = X_1$. In order to treat a more general case we need a variant of the complex interpolation results for $\ell^p_{w_s}(X)$ of Proposition 14.3.3.

Let $(\varepsilon_k)_{k\geqslant 0}$ be a Rademacher sequence on a probability space Ω. Let $p \in (1,\infty)$ and $s \in \mathbb{R}$, and let $\varepsilon^{s,p}(X)$ denote the space of all sequences $(x_k)_{k\geqslant 0}$ in X for which

$$\big\|(x_k)_{k\geqslant 0}\big\|_{\varepsilon^{s,p}(X)} := \sup_{n\geqslant 1}\Big\|\sum_{k=0}^{n}\varepsilon_k 2^{ks}x_k\Big\|_{L^p(\Omega;X)} < \infty.$$

The spaces $\varepsilon^p(X) := \varepsilon^{0,p}(X)$ have been introduced in Section 6.3. Clearly the mapping $(x_k)_{k\geqslant 0} \mapsto (2^{ks}x_k)_{k\geqslant 0}$ defines an isometric isomorphism from $\varepsilon^{s,p}(X)$ onto $\varepsilon^p(X)$. For fixed $s \in \mathbb{R}$ the spaces $\varepsilon^{s,p}(X)$, $1 < p < \infty$, coincide, with pairwise equivalent norms; this follows from the Kahane–Khintchine inequalities as in Proposition 6.3.1. If X does not contain a copy isomorphic to c_0, then Corollary 6.4.12 implies that for any $(x_k)_{k\geqslant 0}$ in $\varepsilon^{s,p}(X)$ the sum $\sum_{k\geqslant 0}\varepsilon_k 2^{ks}x_k$ converges in $L^p(\Omega;X)$ and almost surely in X, and in this case

$$\big\|(x_k)_{k\geqslant 0}\big\|_{\varepsilon^{s,p}(X)} = \Big\|\sum_{k\geqslant 0}\varepsilon_k 2^{ks}x_k\Big\|_{L^p(\Omega;X)}.$$

In particular, the partial sum projections $P_n : (x_k)_{k\geqslant 0} \mapsto (x_k)_{k=0}^n$ are uniformly bounded and strongly convergent to the identity as operators on $\varepsilon^{s,p}(X)$.

The next result extends Theorem 7.4.16, which corresponds to the special case $s = 0$.

Lemma 14.7.10. *For $j \in \{0,1\}$ let X_j be a K-convex space and let $p_j \in (1,\infty)$. For $\theta \in (0,1)$ set $X_\theta := [X_0, X_1]_\theta$. Then*

$$[\varepsilon^{s_0,p_0}(X_0), \varepsilon^{s_1,p_1}(X_1)]_\theta = \varepsilon^{s,p}(X_\theta),$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $s = (1-\theta)s_0 + \theta s_1$.

Proof. By Proposition 7.4.15, X_θ is K-convex. By Proposition 7.4.5 and Lemma 7.4.11, X_θ, does not contain an isomorphic copy of c_0, and hence the partial sum projections P_n on $\varepsilon^{s,p}(X_\theta)$ are strongly convergent to the identity.

To prove the required identity one can repeat the argument in Theorem 14.3.1 to reduce the result to the unweighted setting considered in Theorem 7.4.16. □

As a final preparation for the complex interpolation of Bessel potential spaces, we prove a version of Lemma 14.4.29 for Bessel potential spaces.

Lemma 14.7.11. *Let X be a UMD space and let $p \in [1,\infty]$, $q \in [1,\infty]$, and $s \in \mathbb{R}$. For $k \geqslant 0$ set $\psi_k = \varphi_{k-1} + \varphi_k + \varphi_{k+1}$. The operators*

$$R : \varepsilon^{s,p}(L^p(\mathbb{R}^d; X)) \to H^{s,p}(\mathbb{R}^d; X)$$
$$S : H^{s,p}(\mathbb{R}^d; X) \to \varepsilon^{s,p}(L^p(\mathbb{R}^d; X))$$

defined by

$$R(f_k)_{k\geqslant 0} = \sum_{k\geqslant 0} \psi_k * f_k, \quad Sf = (\varphi_k * f)_{k\geqslant 0},$$

are bounded and satisfy $RS = I$.

Proof. The identity $RS = I$ is proved as in Lemma 14.4.29. The boundedness of S follows from Theorem 14.7.5. It remains to prove that R is bounded. Let $E := L^p(\Omega; L^p(\mathbb{R}^d; X))$. By Theorem 14.7.5 and a density argument it suffices to show that, for all finitely non-zero sequences $(f_\ell)_{\ell\geqslant 0}$ in $L^p(\mathbb{R}^d; X)$,

$$\Big\| \sum_{k=0}^n \varepsilon_k 2^{ks} \varphi_k * \sum_{j\geqslant 0} \psi_j * f_j \Big\|_E \leqslant C \Big\| \sum_{k\geqslant 0} \varepsilon_k 2^{ks} f_k \Big\|_E, \quad n \geqslant 0.$$

From Theorem 14.7.5 (with $s = 0$) and Proposition 8.4.6(i) we see that the sequence $\{\varphi_k * : k \geqslant 0\}$ is R-bounded in $\mathscr{L}(L^p(\mathbb{R}^d; X))$, with R-bound at most by $C_{p,X}$. Hence also the sequence $\{\psi_k * : k \geqslant 0\}$ is R-bounded in this space, with R-bound at most $3C_{p,X}$. Therefore, by the Fourier support properties (14.8) and (14.9) of φ_k,

$$\Big\| \sum_{k=0}^n \varepsilon_k 2^{ks} \varphi_k * \sum_{j\geqslant 0} \psi_j * f_j \Big\|_E \leqslant \sum_{|\ell|\leqslant 2} \Big\| \sum_{k=0}^n \varepsilon_k 2^{ks} \varphi_k * \psi_{k+\ell} * f_{k+\ell} \Big\|_E$$

$$\leqslant 3C_{p,X}^2 \sum_{|\ell \leqslant 2} \Big\| \sum_{k=0}^{n} \varepsilon_k 2^{ks} f_{k+\ell} \Big\|_E$$
$$\leqslant 3C_{p,X}^2 4^{|s|} \Big\| \sum_{k\geqslant 0} \varepsilon_k 2^{ks} f_k \Big\|_E,$$

where in the last step we used Kahane's contraction principle. □

Theorem 14.7.12 (Complex interpolation of Bessel potential spaces). *Let (X_0, X_1) be an interpolation couple of UMD Banach spaces and let $p_0, p_1 \in (1,\infty)$, $s_0, s_1 \in \mathbb{R}$, and $\theta \in (0,1)$. Then*

$$[H^{s_0,p_0}(\mathbb{R}^d; X_0), H^{s_1,p_1}(\mathbb{R}^d; X_1)]_\theta = H^{s,p}(\mathbb{R}^d; X_\theta) \quad \textit{with equivalent norms,}$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $s = (1-\theta)s_0 + \theta s_1$, and $X_\theta = [X_0, X_1]_\theta$.

Proof. Let R and S be the operator of Lemma 14.7.11. Let

$$E_j := \varepsilon^{s_j,p_j}(L^{p_j}(\mathbb{R}^d; X_j)), \quad F_j := H^{s_j,p_j}(\mathbb{R}^d; X_j), \quad j \in \{0,1\},$$

and set $E_\theta := (E_0, E_1)_{\theta,q}$ and $F_\theta := (F_0, F_1)_{\theta,q}$. Then, by Theorem 2.2.6 and Lemma 14.7.10, $E_\theta = \varepsilon^{s,p}(L^p(\mathbb{R}^d; X_\theta))$ isomorphically. Now the proof can be completed in the same way as in Theorem 14.4.30, replacing $\ell^q_{w_s}$ by $\varepsilon^{s,p}$ and $B^s_{p,q}$ by $H^{s,p}$ everywhere. □

Theorem 14.7.12 contains several results of Volume I as special cases. To begin with, it contains Theorem 5.6.9, which asserts that if X is a UMD space, $p \in (1,\infty)$, and $s_0 < s_1$, then

$$[H^{s_0,p}(\mathbb{R}^d; X), H^{s_1,p}(\mathbb{R}^d; X)]_\theta = H^{s_\theta,p}(\mathbb{R}^d; X)$$

and, if in addition $s \geqslant 0$,

$$[L^p(\mathbb{R}^d; X), H^{s,p}(\mathbb{R}^d; X)]_\theta = H^{\theta s,p}(\mathbb{R}^d; X)$$

up to equivalent norms. It also contains Theorem 5.6.1, which asserts that if X is a UMD space, $p \in (1,\infty)$, and $k \geqslant 1$ is an integer, then

$$[L^p(\mathbb{R}^d; X), W^{k,p}(\mathbb{R}^d; X)]_\theta = H^{\theta k,p}(\mathbb{R}^d; X)$$

up to an equivalent norm. This result is obtained by taking $X_0 = X_1 = X$, $p_0 = p_1 = p$, $s_0 = 0$, and $s_1 = k$ in Theorem 14.7.12 and noting that $H^{k,p}(\mathbb{R}^d; X) = W^{k,p}(\mathbb{R}^d; X)$ up to equivalent norm by Theorem 5.6.11.

Upon combining Theorem 14.7.12 with Theorem 5.6.11 we obtain another extension of Theorem 5.6.1:

Corollary 14.7.13 (Complex interpolation for Sobolev spaces). *Let (X_0, X_1) an interpolation couple of UMD Banach spaces and let $p_0, p_1 \in (1,\infty)$, $k_0, k_1 \in \mathbb{N}$, and $\theta \in (0,1)$. Then*

$$[W^{k_0,p_0}(\mathbb{R}^d; X_0), W^{s_1,p_1}(\mathbb{R}^d; X_1)]_\theta = H^{k_\theta,p}(\mathbb{R}^d; X_\theta) \quad \textit{with equivalent norms,}$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $k_\theta = (1-\theta)k_0 + \theta k_1$, and $X_\theta = [X_0, X_1]_\theta$.

As in Examples 14.4.33 and 14.4.35, we can use this corollary to prove boundedness of pointwise multiplication by smooth functions:

Example 14.7.14 (Pointwise multiplication by smooth functions – I). Let X and Y be UMD spaces, let $p \in [1, \infty]$ and $s \in \mathbb{R}$, and let $k \in [s, \infty) \cap \mathbb{N}$ be an integer. If $\zeta \in C_{\rm b}^k(\mathbb{R}^d; \mathscr{L}(X, Y))$, then pointwise multiplication

$$f \mapsto \zeta f$$

defines a bounded mapping from $H^{s,p}(\mathbb{R}^d; X)$ into $H^{s,p}(\mathbb{R}^d; Y)$ of norm $\lesssim_{k,s} \|\zeta\|_{C_{\rm b}^k(\mathbb{R}^d;\mathscr{L}(X,Y))}$.

Indeed, the pointwise multiplier $f \mapsto \zeta f$ is bounded as a mapping from $W^{j,p}(\mathbb{R}^d; X)$ into $W^{j,p}(\mathbb{R}^d; Y)$ for each $j \in \{0, \dots, k\}$. Therefore, for $s \in \mathbb{N}$ the result is immediate from Theorem 5.6.11. If $-s \in \mathbb{N}$, then the result follows by the duality result of Proposition 5.6.7 and Theorem 5.6.11. If $s \in (0, \infty)$, then the result follows by interpolation between the cases $j = 0$ and $j = k$ by the complex method $[\cdot, \cdot]_{\frac{s}{k}}$ and applying Theorem C.2.6 and Corollary 14.7.13. Finally, the case $s \in (-\infty, 0)$ follows by duality again.

14.7.d Pointwise multiplication by $\mathbf{1}_{\mathbb{R}_+}$ in $H^{s,p}$

To conclude this section we present a result on pointwise multiplication by $\mathbf{1}_{\mathbb{R}_+}$ for vector-valued Bessel potential spaces. The cases of vector-valued Besov spaces and Triebel–Lizorkin space have been considered in Section 14.6.h; in both cases, values in general Banach spaces X could be allowed. In the Bessel potential case, the proof below requires the UMD property of the range space X. It seems to be an open problem whether this conditions is actually necessary.

Theorem 14.7.15 (Pointwise multiplication by $\mathbf{1}_{\mathbb{R}_+}$). *Let $p \in (1, \infty)$ and $s \in (-1/p', 1/p)$, and let X be a UMD space. For all $f \in H^{s,p}(\mathbb{R}; X)$ we have $\mathbf{1}_{\mathbb{R}_+} f \in H^{s,p}(\mathbb{R}; X)$ and*

$$\|\mathbf{1}_{\mathbb{R}_+} f\|_{H^{s,p}(\mathbb{R};X)} \leqslant C\|f\|_{H^{s,p}(\mathbb{R};X)}, \quad f \in H^{s,p}(\mathbb{R}; X).$$

The UMD property of X will only be used through the following proposition.

Proposition 14.7.16. *Let $p \in (1, \infty)$ and $s > 0$, and let X be a UMD space.*

(1) *The operator $(-\Delta)^s : \mathscr{S}(\mathbb{R}^d; X) \to \mathscr{S}'(\mathbb{R}^d; X)$ given by*

$$(-\Delta)^s f = |2\pi \cdot|^s \widehat{f}$$

uniquely extends to $(-\Delta)^s \in \mathscr{L}(H^{s,p}(\mathbb{R}^d; X), L^p(\mathbb{R}^d; X))$.

(2) *For all $f \in H^{s,p}(\mathbb{R}^d; X)$ the following norm equivalence holds*

$$\|f\|_{H^{s,p}(\mathbb{R}^d;X)} \eqsim_{p,X} \|f\|_{L^p(\mathbb{R}^d;X)} + \|(-\Delta)^{s/2} f\|_{L^p(\mathbb{R}^d;X)}.$$

Proof. (1): Let $m_1(\xi) = \frac{|2\pi\xi|^s}{(1+|2\pi\xi|^2)^{s/2}}$. Using Mihlin's multiplier Theorem 5.5.10 one can check that $m_1 \in \mathfrak{M}L^p(\mathbb{R}^d; X, Y)$. Therefore,

$$\|(-\Delta)^s f\|_p = \|T_{m_1} J_s f\|_p \leqslant \|m_1\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)} \|J_s f\|_p \leqslant C_{p,X} \|f\|_{H^{s,p}(\mathbb{R}^d;X)}.$$

(2): Note that since $s > 0$, Proposition 5.6.6 gives that $H^{s,p}(\mathbb{R}^d; X) \hookrightarrow L^p(\mathbb{R}^d; X)$ contractively. This combined with (1) gives the estimate "$\gtrsim$".

The estimate $\lesssim$ follows similarly. Let $m_2(\xi) = \frac{(1+|2\pi\xi|^2)^{s/2}}{1+|2\pi\xi|^s}$. Then $m_2 \in \mathfrak{M}L^p(\mathbb{R}^d; X, Y)$ as before. Therefore,

$$\begin{aligned} \|f\|_{H^{s,p}(\mathbb{R}^d;X)} &= \|T_{m_2}(I + (-\Delta)^{s/2}) f\|_p \\ &\leqslant \|m_2\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)} (\|f\|_p + \|(-\Delta)^{2/s} f\|_p. \end{aligned}$$

□

We need two more preparatory results. The first one is a concrete formula for $(-\Delta)^{s/2} f$ as an integral operator.

Lemma 14.7.17. *Let $s \in (0,1)$. For $f \in \mathscr{S}(\mathbb{R}; X)$ we have*

$$(-\Delta)^{s/2} f = c_s \int_{\mathbb{R}} \frac{f(\cdot + h) - f(\cdot)}{|h|^{1+s}} dh, \quad x \in \mathbb{R},$$

where the integral on the right-hand side converges absolutely pointwise $\mathbb{R}$, and as a Bochner integral in $L^p(\mathbb{R}; X)$ for any $p \in [1,\infty)$. Here $c_s \in \mathbb{R} \setminus \{0\}$ is a constant only depending on s.

Proof. The convergence of the integral for $|h| > 1$ is immediate. The convergence for $|h| < 1$ follows by writing $f(x+h) - f(x) = \int_0^1 f'(x+th)h \,\mathrm{d}t$.

To prove the stated identity we take Fourier transforms on the right-hand side and use Fubini's theorem to obtain

$$\mathscr{F} \int_{\mathbb{R}} \frac{f(\cdot + h) - f(\cdot)}{|h|^{1+s}} \,\mathrm{d}h \,\mathrm{d}x = \int_{\mathbb{R}} \frac{e^{2\pi i h\xi} - 1}{|h|^{1+s}} \widehat{f}(\xi) \,\mathrm{d}h = k_s |\xi|^s \widehat{f}(\xi),$$

where from the fact that the odd part of the integral cancels we see that $k_s = 2\int_{\mathbb{R}_+} \frac{\cos(2\pi t)-1}{t^{1+s}} \,\mathrm{d}t$ is in $(-\infty, 0)$. This proves the result with constant $c_s = k_s^{-1}(2\pi)^s$. □

We also need the following inequality.

Lemma 14.7.18 (Hilbert absolute inequality). *Let $p \in (1,\infty)$. For $f \in L^p(\mathbb{R}_+)$ one has*

$$\Big\| x \mapsto \int_{\mathbb{R}_+} \frac{|f(y)|}{x+y} \,\mathrm{d}y \Big\|_{L^p(\mathbb{R}_+)} \leqslant C_p \|f\|_{L^p(\mathbb{R}_+)}.$$

Proof. Letting $\zeta_p(y) = \frac{x^{1/p}}{x+1}$, after rewriting the integral, we can use Young's inequality for the multiplicative group $\mathbb{R}_+$ with Haar measure $\frac{dx}{x}$ to obtain

$$\Big\| x \mapsto \int_{\mathbb{R}_+} \frac{|f(y)|}{x+y}\,dy \Big\|_{L^p(\mathbb{R}_+)} = \int_{\mathbb{R}_+} \Big(\int_{\mathbb{R}_+} \zeta_p(x/y) y^{1/p} f(y) \frac{dy}{y} \Big)^p \frac{dx}{x} \Big)^{1/p}$$
$$\leqslant \|\zeta_p\|_{L^1(\mathbb{R}_+, \frac{dx}{x})} \|f\|_{L^p(\mathbb{R}_+)}.$$

□

Proof of Theorem 14.7.15. By Proposition 14.6.17 it suffices to prove the desired estimate for f in the dense class $C_c^\infty(\mathbb{R} \setminus \{0\}) \otimes X$. In that case one actually has $g := \mathbf{1}_{\mathbb{R}_+} f$ is in the same class and thus is smooth as well.

We claim that

$$\|(-\Delta)^{s/2} g\|_p \leqslant \|(-\Delta)^{s/2} f\|_p + C_{p,s} \|f\|_{H^{s,p}(\mathbb{R};X)}. \tag{14.97}$$

As soon as we proved the claim, then the result follows. Indeed, applying Proposition 14.7.16 twice we obtain

$$\begin{aligned} \|g\|_{H^{s,p}(\mathbb{R};X)} &\eqsim_{p,X} \|g\|_p + \|(-\Delta)^{s/2} g\|_p \\ &\overset{(14.97)}{\leqslant} \|f\|_p + \|(-\Delta)^{s/2} f\|_p + C_{p,s} \|f\|_{H^{s,p}(\mathbb{R};X)}. \\ &\eqsim_{p,X} \|f\|_{H^{s,p}(\mathbb{R};X)}. \end{aligned}$$

To rewrite $(-\Delta)^{s/2} g$ in a suitable way, let

$$S := \{(x,h) \in \mathbb{R}^2 : (x > 0 \text{ and } h < -x) \text{ or } (x < 0 \text{ and } h > -x)\}.$$

Then applying Lemma 14.7.17 twice, by elementary considerations we see that for all $x \in \mathbb{R}$,

$$\begin{aligned} (-\Delta)^{s/2} g(x) &= c_s \int_{\mathbb{R}} \frac{g(x+h) - g(x)}{|h|^{1+s}} dh \\ &= c_s \int_{\mathbb{R}} \frac{f(x+h) - f(x)}{|h|^{1+s}} dh - c_s \operatorname{sgn}(x) \int_{\mathbb{R}} \mathbf{1}_S(x,h) \frac{f(x+h)}{|h|^{1+s}}\,dh \\ &= (-\Delta)^{s/2} f(x) - c_s \operatorname{sgn}(x) \int_{\mathbb{R}} \mathbf{1}_S(x,h) \frac{f(x+h)}{|h|^{1+s}}\,dh. \end{aligned}$$

Taking L^p-norms, we see that (14.97) holds if we can show that

$$\Big\| x \mapsto \int_{\mathbb{R}} \mathbf{1}_S(x,h) \frac{\|f(x+h)\|}{|h|^{1+s}}\,dh \Big\|_{L^p(\mathbb{R})} \lesssim_{p,s} \|f\|_{H^{s,p}(\mathbb{R};X)}. \tag{14.98}$$

To prove (14.98) we only consider the part $L^p(\mathbb{R}_+)$ as the other one is similar. By elementary considerations

$$\int_0^\infty \Big(\int_{\mathbb{R}} \mathbf{1}_S(x,h) \frac{\|f(x+h)\|}{|h|^{1+s}}\,dh \Big)^p dx = \int_0^\infty \Big(\int_{-\infty}^{-x} \frac{\|f(x+h)\|}{|h|^{1+s}}\,dh \Big)^p dx$$

$$
\begin{aligned}
&= \int_0^\infty \Big(\int_0^\infty \frac{\|f(-y)\|}{(y+h)^{1+s}}\, \mathrm{d}h \Big)^p \, \mathrm{d}y \\
&\leqslant \int_0^\infty \Big(\int_0^\infty \frac{y^{-s}\|f(-y)\|}{y+h}\, \mathrm{d}h \Big)^p \, \mathrm{d}y \\
&\overset{(i)}{\leqslant} C_p^p \|y \mapsto |y|^{-s} f(y)\|^p_{L^p(\mathbb{R};X)} \\
&\overset{(ii)}{\leqslant} C_p^p C_{p,s}^p \|f\|^p_{H^{s,p}(\mathbb{R};X)},
\end{aligned}
$$

where in (i) we applied Lemma 14.7.18 to the function $y \mapsto y^{-s}\|f(-y)\|$, and (ii) follows from Corollary 14.6.31(2). This completes the proof of the remaining estimate (14.98). □

14.8 Notes

Early influential monographs on function spaces are those of Adams [1975] (see also Adams and Fournier [2003]), Bergh and Löfström [1976], Peetre [1976], and Triebel [1978]. After these works appeared, a new maximal function argument was discovered by Peetre [1975] which made it possible to study Besov and Triebel–Lizorkin spaces in the full range $p, q \in (0, \infty]$. This theory is presented in detail in the monograph of Triebel [1983] and the more recent works of Triebel [1992, 2006, 2020, 2013, 2014]; further expositions are due to Bahouri, Chemin, and Danchin [2011], Denk and Kaip [2013], Grafakos [2009], Maz'ya [2011], Runst and Sickel [1996], and Sawano [2018].

Standard references for function spaces in the vector-valued setting include the works of Amann [1995, 1997, 2019], Triebel [1997], König [1986], Schmeisser [1987], Schmeisser and Sickel [2001], and Schmeisser and Sickel [2005]. A unified treatment of Besov and Triebel–Lizorkin spaces and related classes of function spaces is given by Lindemulder [2021], where the axiomatic setting of Hedberg and Netrusov [2007] is extended to the vector-valued context. In particular, this covers the weighted and anisotropic settings, and it allows for Banach function space other than the spaces $\ell^q(L^p)$ or $L^p(\ell^q)$ employed in the construction of the Besov and Triebel–Lizorkin spaces.

The theory of function spaces is a vast topic, and by necessity our treatment does not cover a number of important topics such as approximation theory, wavelets, atomic decompositions, weighted spaces, paraproducts, anisotropic spaces, and typical aspects for bounded domains and manifolds such as traces, extension operators, boundary values, and interpolation with boundary conditions (although some of these topics will be briefly visited in these notes). Of the omitted themes, we specifically mention the ϕ-transform of Frazier and Jawerth [1990], which allows the identification of Besov and Triebel–Lizorkin spaces with subspaces of appropriate discrete sequence spaces. In this identification, the question of boundedness of various operators on the original function spaces is transformed into the question of boundedness of infinite matrices on the corresponding sequence spaces, which in turn can be deduced from natural almost diagonality estimates of these matrices, in certain analogy with our proof of the $T(1)$ theorem on $L^p(\mathbb{R}^d;X)$ spaces through estimates of the matrix coefficients of T with respect to the Haar basis. This approach lies behind many of the proofs of $T(1)$ theorems in Besov and Triebel–Lizorkin spaces that we discussed in the Notes of Chapter 12.

The 'classical' Besov and Triebel–Lizorkin spaces considered in this chapter are modelled on the gradient ∇ in the setting of $\mathbb{R}^d$. It is possible to introduce Besov and Triebel–Lizorkin spaces based on different types of sectorial operators and to study them in the setting of manifolds; we refer to Batty and Chen [2020], Haase [2006], Kriegler and Weis [2016], Kunstmann and Ullmann [2014], Taylor [2011a], Taylor [2011b], Taylor [2011c], Taylor [1974], and Voigtlaender [2022].

Section 14.2

Lemma 14.2.1 is taken from Amann [1995]. The other results of this section are standard in the scalar-valued case, and their extensions to the vector-valued setting are straightforward.

Section 14.3

The complex and real interpolation results for vector-valued and weighted L^q-spaces of Theorems 14.3.1 and 14.3.4 extend Theorems 2.2.6 and 2.2.10, where the unweighted case was treated. The scalar-valued case goes back to Stein and Weiss [1958], and the extension to the vector-valued weighted setting is well-known, at least for complex interpolation. The case of real interpolation is included in the work of Kreĭn, Petunīn, and Semënov [1982], and a different approach based on Stein interpolation for the real method is due to Lindemulder and Lorist [2022]. The interpolation results for $q_0 = q_1 = \infty$ are false in general. Indeed, already Triebel [1978, 1.18.1] gave an example where $[\ell^\infty_{w_{s_0}}(X_0), \ell^\infty_{w_{s_1}}(X_1)]_\theta \neq \ell^\infty_{w_s}([X_0, X_1]_\theta)$ with $w_s(n) = 2^{ns}$. Propositions 14.3.3 and 14.3.5 are presented by Triebel [1978], who attributes the real case to Peetre [1967]. More generally, Triebel [1978, Section 1.18] identifies the complex and real interpolation spaces of $\ell^{p_0}((X_j)_{j\geqslant 1})$ and $\ell^{p_1}((Y_j)_{j\geqslant 1})$ for $p_0, p_1 < \infty$ and for sequences of interpolation couples $(X_j, Y_j)_{j\geqslant 1}$; here $\ell^p((Z_j)_{j\geqslant 1})$ is the space of all sequences $(z_j)_{j\geqslant 1}$ with $z_j \in Z_j$ such that $(\|z_j\|_{Z_j})_{j\geqslant 1}$ belongs to ℓ^p, $Z \in \{X, Y\}$. Proposition 14.3.3 then follows by taking $X_j = 2^{js}X$ and $X_j = 2^{js}Y$. It seems that Proposition 14.3.5 can only be stated for a single space X unless further assumptions on q_0 and q_1 are made.

Section 14.4

Our introduction of vector-valued Besov spaces is self-contained up to a modest number of prerequisites from earlier chapters. Part of the section follows the presentation by Schmeisser and Sickel [2001]. For the history of Besov spaces, we refer the reader to Bergh and Löfström [1976] and Triebel [1978, 1983]. Besov spaces appear naturally as real interpolation spaces between L^p and $W^{k,p}$ (see Theorem 14.4.31). As such, they have important applications in the theory of evolution equations (see Chapter 18). Moreover, by choosing the microscopic parameter q suitably, one can often include end-point cases into the considerations.

In contrast to the theory of the spaces $W^{k,p}(\mathbb{R}^d;X)$ and $H^{s,p}(\mathbb{R}^d;X)$, where assumptions on the space X such as the Radon–Nikodým property or the UMD property are often needed, many key results on vector-valued Besov spaces hold for general Banach spaces X.

Lemma 14.4.5 on the sequential completeness of $\mathscr{S}'(\mathbb{R}^d;X)$ is a standard result. It is possible to endow the space $C_c^\infty(U;X)$ with a complete locally convex topology in such a way that sequential convergence in this topology coincides with the *ad hoc* notion of sequential convergence used here. A detailed construction is presented by Rudin [1991].

Fourier multipliers

Fourier multipliers for vector-valued Besov spaces have been discussed by Amann [1997], Weis [1997], Girardi and Weis [2003a], Hytönen [2004], and Hytönen and Weis [2006a]. In Theorem 14.4.16, we only considered smooth m, and this restriction was removed in Theorem 14.5.6. The latter result and related ones can be found in the work of Girardi and Weis [2003a], who showed that the operator T is a continuous extension (with respect to a weaker topology) of T_m also if $\max\{p,q\}=\infty$. Fourier multipliers for vector-valued Besov spaces have been applied by Weis [1997] to obtain sharp exponential stability results of C_0-semigroups in spaces with Fourier type p.

Embedding

The sandwich result of Proposition 14.4.18 is very useful in avoiding additional conditions on the Banach space X. The Sobolev embedding result of Theorem 14.4.19 is standard. Especially the sufficiency is simple to prove via Lemma 14.4.20. For the proof of this lemma and its extension to all $0<p_0<p_1<\infty$ in Remark 14.6.4, we follow Schmeisser and Sickel [2001].

Difference norms

The difference norm characterisation of Besov spaces can be found in many places. It was already used before the Fourier analytic description of Besov spaces was given. We refer the reader to Bergh and Löfström [1976], Triebel [1983], and references therein for historical details. The difference norms have the advantage that in certain cases one can check by hand whether a given function belongs to some given Besov space. By choosing the parameter τ in Theorem 14.4.24 appropriately, the Besov spaces can be identified with other classical spaces, as we have done in Corollaries 14.4.25 and 14.4.26 for $W^{s,p}$ and $C^s_{\rm ub}$.

In Step 1 of the proof of Theorem 14.4.24 we follow the presentation of Bergh and Löfström [1976], where the case $\tau = \infty$ was given. Step 2 of the proof is based on the presentation of Schmeisser and Sickel [2001].

Interpolation

Interpolation of Besov spaces is discussed by Bergh and Löfström [1976], König [1986], and Triebel [1978, 1983]; further references to the literature can be found in these works. The method to reduce the proofs to interpolation of $\ell^q(L^p)$-spaces fits into a more general retraction–co-retraction scheme explained by [Triebel, 1978, Theorem 1.2.4].

The complex interpolation result of Theorem 14.4.30 is folklore, although we are not aware of a reference containing the general form with an interpolation couple (X_0, X_1) presented here. In the special case $X = X_0 = X_1$, the theorem can be proved in the same way as in the scalar-valued case, and some end-point results are valid as well. For instance, we have

$$[B^{s_0}_{p_0,q_0}(\mathbb{R}^d; X), B^{s_1}_{p_1,q_1}(\mathbb{R}^d; X)]_\theta = B^s_{p,q}(\mathbb{R}^d; X), \quad p_j, q_j \in [1, \infty], \; s_j \in \mathbb{R},$$

with equivalent norms, where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \qquad s = (1-\theta)s_0 + \theta s_1.$$

The real interpolation result of Theorem 14.4.31 is well known, and the proof is a simple generalisation of the standard proof for the scalar-valued case. Several other real interpolation results can be proved with the same methods. For instance, if $\min\{p_0, p_1\} < \infty$, $\min\{q_0, q_1\} < \infty$, and $s_0, s_1 \in \mathbb{R}$, then

$$(B^{s_0}_{p_0,q_0}(\mathbb{R}^d; X_0), B^{s_1}_{p_1,q_1}(\mathbb{R}^d; X_1))_{\theta,p} = B^s_{p,p}(\mathbb{R}^d; (X_0, X_1)_{\theta,p}),$$

with equivalent norms, where again $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ and $s = (1-\theta)s_0 + \theta s_1$. This follows Theorem 14.3.4 in a similar way as in Theorem 14.4.30.

Duality

In Theorem 14.4.34, we identified the dual of $B^s_{p,q}(\mathbb{R}^d; X)$ with respect to the duality for $\mathscr{S}(\mathbb{R}^d; X)$ and $\mathscr{S}'(\mathbb{R}^d; X)$. Unlike in the L^p-setting treated in Section 1.3, no conditions on X are needed. A result of this type in a more general abstract setting (including weights and anisotropic function spaces) is presented by Lindemulder [2021]. The proof that we have given follows Agresti, Lindemulder, and Veraar [2023].

Section 14.5

The characterisations in Theorem 14.5.1 of type and cotype in terms of embedding properties of Besov spaces into spaces of γ-radonifying operators are due to Kalton, Van Neerven, Veraar, and Weis [2008]. This paper also contains the γ-Bernstein–Nikolskii inequality of Lemma 14.5.2, as well as optimal embedding results for the smooth spaces $\gamma(H^{-s,2}(\mathbb{R}^d); X)$. The consequences for Bessel potential spaces discussed in Corollary 14.7.7 are taken from Veraar [2013]. This work also contained the following result:

Theorem 14.8.1. *Let X be a Banach lattice, and $1 \leqslant p \leqslant 2 \leqslant q < \infty$. If X is p-convex and q-concave, then*

$$H^{(\frac{1}{p}-\frac{1}{2})d,p}(\mathbb{R}^d; X) \hookrightarrow \gamma(L^2(\mathbb{R}^d), X),$$
$$\gamma(L^2(\mathbb{R}^d), X) \hookrightarrow H^{(\frac{1}{q}-\frac{1}{2})d,q}(\mathbb{R}^d; X).$$

It is an open problem to characterise the Banach spaces for which these embeddings hold (see Problem Q.14).

Mapping properties of the Fourier transform

The mapping properties of the vector-valued Fourier transform $\mathscr{F}$ for Banach spaces X with Fourier type p contained in Proposition 14.5.3 appear in the papers by García-Cuerva, Kazaryan, Kolyada, and Torrea [1998], König [1991], and Girardi and Weis [2003a]. Real interpolation of the end-point cases $q = p$ and $q = \infty$ in Proposition 14.5.3 gives an alternative proof of some of the results in the papers just mentioned:

Theorem 14.8.2. *Suppose that X has Fourier type $p \in (1, 2]$. Let $q \in (p, \infty)$, $r \in [1, \infty]$, and $s = \frac{d}{p} - \frac{d}{q}$. Then $\mathscr{F}$ is bounded from $B^s_{p,r}(\mathbb{R}^d; X)$ into the Lorentz space $L^{q',r}(\mathbb{R}^d; X)$.*

Proposition 14.5.3 contains a parallel result under the assumption that X has type p and cotype 2. Recall from Proposition 13.1.35 that, under these assumptions, X has Fourier type r for any $r \in [1, p)$.

The mapping properties of the Fourier transform on vector-valued L^p-spaces with power weights have been recently studied by Dominguez and Veraar [2021], who show that a version of the classical Pitt inequalities holds if and only if X has non-trivial Fourier type. In particular, the following result was proved:

Theorem 14.8.3. *Let X be of Fourier type $p_0 \in (1, 2]$. Let $1 < p \leqslant q < \infty$ and $\beta, \gamma \geqslant 0$. If*

$$\max\left\{0, d\left(\frac{1}{\min\{p, p_0\}} + \frac{1}{q} - 1\right)\right\} < \gamma < \frac{d}{q} \quad \text{and} \quad \beta - \gamma = d\Big(1 - \frac{1}{p} - \frac{1}{q}\Big),$$

then $\mathscr{F}$ extends boundedly from $L^p(\mathbb{R}^d, |\cdot|^{\beta p}; X)$ into $L^q(\mathbb{R}^d, |\cdot|^{-\gamma q}; X)$.

In the limiting case $\gamma = \max\{0, d(\frac{1}{\min\{p,p_0\}} + \frac{1}{q} - 1)\}$, the above boundedness of $\mathscr{F}$ still holds true under further restrictions on p and q. Surprisingly, if X has non-trivial Fourier type (equivalently, by Theorem 13.1.33, non-trivial type), one can allow $p = q = 2$ by choosing the weights suitably. A similar result holds in the periodic setting, but the problem is open for more general orthogonal systems that have been considered by Stein [1956].

R-boundedness

R-boundedness of smooth operator-valued functions is studied by Girardi and Weis [2003c] under Fourier type conditions, and by Hytönen and Veraar [2009] under (co)type conditions; the latter paper contains Theorems 14.5.8 and 14.5.9.

Section 14.6

In this section, we followed part of the presentation of Schmeisser and Sickel [2001]. For a detailed description of the history of Triebel–Lizorkin spaces, we refer the reader to Bergh and Löfström [1976], and Triebel [1978, 1983]. Below, we only discuss those aspects of Triebel–Lizorkin spaces that are specific for this class of spaces.

Triebel–Lizorkin spaces $F^s_{p,q}$ were originally introduced as a natural variant of Besov spaces, with the roles of L^p and ℓ^q interchanged in the definition. The special case $q = 2$ leads to the equality $F^s_{p,2} = H^{s,p}$ with equivalent norms for $p \in (1,\infty)$, and in the early days of the theory the cases $q \neq 2$ were mostly studied for reasons of mathematical curiosity. The definition of Triebel–Lizorkin spaces given here does not cover the spaces $F^s_{\infty,q}$. The latter are known to be connected to BMO spaces, and require a modification of the definition for which we refer to Triebel [1983]. These spaces are naturally contained, as $F^s_{\infty,q} = F^{s,1/p}_{p,q}$ for any $p \in (0,\infty)$, in the general framework of *Triebel–Lizorkin-type spaces* $F^{s,\tau}_{p,q}$ with a fourth parameter $\tau \in [0,\infty)$, which has been introduced by Yang and Yuan [2008] and studied in several subsequent works.

Genesis of (vector-valued) Triebel–Lizorkin spaces

Vector-valued Triebel–Lizorkin spaces are needed for the treatment of parabolic boundary value problems in the spaces $L^p(0,T;L^q(\mathbb{R}^d_+))$. Such applications first appeared in the works of Weidemaier [2002] for $q \leqslant p$ and scalar second order equations with inhomogeneous Dirichlet boundary conditions, and of Denk, Hieber, and Prüss [2007] for $p, q \in (1,\infty)$ and more general systems and boundary conditions. Kunstmann [2015] introduced a new interpolation method $(\cdot,\cdot)_{\theta,\ell^q}$ and shows that $F^s_{p,q} = (L^p, W^{k,p})_{s/k,\ell^q}$ with equivalent norms. This interpolation method fits into the axiomatic setting of discrete interpolation recently developed by Lindemulder and Lorist [2021].

As in the Besov space case, results for vector-valued Triebel–Lizorkin spaces typically hold without restrictions on the target Banach space X. Thanks to the sandwich result

$$B^s_{p,1} \hookrightarrow F^s_{p,1} \hookrightarrow H^{s,p} \hookrightarrow F^s_{p,\infty} \hookrightarrow B^s_{p,\infty},$$

one can sometimes deduce results about vector-valued Bessel potential spaces as well. Within the Triebel–Lizorkin scale, one can get closer to $H^{s,p}$ than in the Besov scale, which often makes Triebel–Lizorkin spaces more useful. For instance, the sandwich result can be combined with the Sobolev Embedding Theorem 14.6.14, which allows arbitrary microscopic improvement for Triebel–Lizorkin spaces. Further flexibility in sandwiching and embedding theorems can be built in by introducing weights such as $|x|^\gamma$ or $|x_1|^\gamma$ as was done by Meyries and Veraar [2012, 2014a].

The boundedness of the Peetre maximal function proved in Proposition 14.6.2 appears in the book of Triebel [1997]. This proposition extends results of Triebel [1983, Theorem 1.6.3] and Triebel [1997, Formula 15.3(iv)] to the vector-valued setting.

Theorems 14.6.3 and 14.6.11 are presented by Triebel [1997] for scalar-valued multipliers m. An operator-valued extension is due to Bu and Kim [2005].

Gagliardo–Nirenberg inequalities and Sobolev embedding

The Gagliardo–Nirenberg inequalities of Proposition 14.6.15 and 14.6.16 are taken from Brezis and Mironescu [2001]. Our presentation follows Schmeisser and Sickel [2001, 2005]. Proposition 14.6.13 and Theorem 14.6.14 can also be found in these works. Gagliardo–Nirenberg inequalities in the Besov scale can be found in the paper of Brezis and Mironescu [2018]; they do not allow for a microscopic improvement.

Difference norms

Difference norm characterisations of Triebel–Lizorkin spaces appear in the works of Kaljabin [1977, 1979], and Triebel [1983]. Our presentation of Theorem 14.6.20 follows Schmeisser and Sickel [2001], who consider the case $\tau = 1$.

Interpolation and duality

The interpolation and duality results for Triebel–Lizorkin spaces are similar to their Besov space counterparts. In our presentation, the end-point $q = 1$ is excluded, since the Fefferman–Stein inequality for the maximal operator is not valid in $L^p(\mathbb{R}^d; \ell^1)$. This problem can be circumvented by a reduction to interpolation identities for vector-valued Hardy spaces instead of $L^p(\mathbb{R}^d; \ell^q(X))$ (see Triebel [1983]). The embedding (14.87) of Theorem 14.6.26 is due to Jawerth [1977], and the one of (14.86) to Franke [1986].

Fractional Hardy inequalities

The fractional Hardy inequalities of Proposition 14.6.30 and Corollary 14.6.31 are variations of those by Krugljak, Maligranda, and Persson [2000], who proved the results with a fractional Sobolev norm $W^{s,p}$ on the right-hand side. The advantage of our formulation is that both the $H^{s,p}$ and the $W^{s,p}$ cases are consequences of the stronger estimate using the space $F^s_{p,\infty}$. Higher-dimensional versions of fractional Hardy inequalities can be deduced from the work of Meyries and Veraar [2012], where Sobolev embedding with power weights are discussed.

Pointwise multiplication by $\mathbf{1}_{\mathbb{R}_+}$

Pointwise multiplier results such as the one of Theorem 14.6.32 and Corollaries 14.6.34 and 14.6.35 were proved via paraproducts estimates in more generality by Runst and Sickel [1996]. Some of the results from this monograph were extended to the weighted vector-valued setting by Meyries and Veraar [2015]. In particular, some of the end-points can be included, and higher dimensional versions of the results hold. The results of the present section merely serve as an illustration of how the theory can be applied. Since the work of Grisvard [1967] and Seeley [1972], it is known that results on pointwise multipliers stand at the basis of interpolation with boundary conditions. The one-dimensional case is useful for evolution equations, since ${}_0F^1_{p,q}(\mathbb{R}_+;X)$ and ${}_0B^1_{p,q}(\mathbb{R}_+;X)$ can be used as the domain of the time-derivative. As in the work of Lindemulder, Meyries, and Veraar [2018], one can identity the real and complex interpolation spaces between ${}_0F^1_{p,q}(\mathbb{R};X)$ and $F^0_{p,q}(\mathbb{R};X)$ for $p,q \in (1,\infty)$ using the theory of this section, and similarly for Besov spaces for $p \in (1,\infty)$ and $q \in [1,\infty]$.

Section 14.7

The Embedding Theorems 14.7.1, 14.7.3, and 14.7.4 are taken from Schmeisser and Sickel [2001, 2005]. The end-point cases, where $\min\{p_0,p_1\} = 1 < \max\{p_0,p_1\}$, are not completely understood; we refer the reader to Brezis and Mironescu [2018] for a further discussion.

The Littlewood–Paley theorem 14.7.5 is taken from Meyries and Veraar [2015], who also consider a weighted setting.

The improved embeddings for Besov, Triebel–Lizorkin, and Bessel potential spaces under UMD and (co)type assumptions stated in Proposition 14.7.6 are due to Veraar [2013]. The converse result presented in Proposition 14.7.8 seems to be new. In the case $p = q$, Hytönen and Merikoski [2019] have shown the following more precise result.

Theorem 14.8.4. *For $k \in \mathbb{N}$ and $p \in [2,\infty)$, there is a continuous embedding*

$$B^k_{q,q}(\mathbb{R}^d;X) \hookrightarrow W^{k,q}(\mathbb{R}^d;X)$$

if and only if X has martingale cotype q.

In case the embedding constant depend on d in a polynomial way, such results have applications to quantitative affine approximation in infinite dimensions, as discussed by Hytönen, Li, and Naor [2016] and Hytönen and Naor [2019]. The proof of Theorem 14.8.4 is based on ideas from these works and results of Xu [1998] and Martínez, Torrea, and Xu [2006] connecting Littlewood–Paley–Stein inequalities and martingale (co)type. Some of these results have been extended by Xu [2020]. For open problems related to Theorem 14.8.4, we refer the reader to Problem Q.13.

Theorem 14.7.9 is due to Han and Meyer [1996], who obtained it as a consequence of a more general Littlewood–Paley theorem for $L^p(\mathbb{R}^d; X)$. Our approach is more direct.

The interpolation result of Theorem 14.7.12 was discovered independently by Amann [2019], Hummel [2019], and Lindemulder and Veraar [2020]. In the first reference, the anisotropic setting was also covered, and weighted spaces are included in the latter two references.

Pointwise multipliers

Theorem 14.7.15 is due to Meyries and Veraar [2015], where it appears as a special case of a general pointwise multiplier theorem for weighted vector-valued Bessel potential spaces. It is unknown whether the UMD condition is necessary (see Problem Q.12). The proof presented here is simplified from that of Lindemulder, Meyries, and Veraar [2018]. Another proof, based on a difference norm characterisation, is due to Lindemulder [2017]. The scalar case of Theorem 14.7.15 is due to Shamir [1962] and Strichartz [1967]. Their proof extends to the vector-valued setting only when the range space is isomorphic to a Hilbert spaces (see Walker [2003]).

Interpolation with boundary conditions

Applications to complex interpolation with boundary conditions are given by Lindemulder, Meyries, and Veraar [2018]. Among other things, the domains of the fractional powers of the first order derivative with Dirichlet boundary conditions are identified as $\mathsf{D}(\partial_t^s) = {}_0H^{s,p}(\mathbb{R}_+; X)$ for $s \in (0,1)$. This extends a special case of a result of Seeley [1972] to the vector-valued setting. Certain difficulties in obtaining such identities were overlooked in applications to evolution equations for several years. The boundedness of pointwise multiplication by indicator functions was proved recently in the anisotropic setting by Lindemulder [2022]. This solves an open problem of Amann [2019], who used the boundedness to obtain vector-valued and anisotropic extensions of some of the results of Seeley [1972] on interpolation with boundary conditions.

Function spaces on domains and extension operators

Function spaces on domains $\mathcal{O} \subseteq \mathbb{R}^d$ are usually defined by restriction, declaring that $f \in \mathcal{A}^s_{p,q}(\mathcal{O})$ if there exists $g \in \mathcal{A}^s_{p,q}(\mathbb{R}^d)$ such that $f = g|_{\mathcal{O}}$ in the distributional sense; the norm on $\mathcal{A}^s_{p,q}(\mathcal{O})$ is then taken to be the corresponding quotient norm. From this definition, it is often complicated to decide whether a given function belongs to $\mathcal{A}^s_{p,q}(\mathcal{O})$ and to estimate its norm. Extension operators help to get a better grip on this problem. Given a domain $\mathcal{O} \subseteq \mathbb{R}^d$, an extension operator for $\mathcal{O}$ is a bounded linear operator $E_{\mathcal{O}} : \mathcal{A}^s_{p,q}(\mathcal{O}) \to \mathcal{A}^s_{p,q}(\mathbb{R}^d)$ such that

$$(E_{\mathcal{O}} f)|_{\mathcal{O}} = f, \quad f \in \mathcal{A}^s_{p,q}(\mathcal{O}).$$

For Lipschitz domains $\mathcal{O}$, Rychkov [1999] constructed a 'universal' extension operator $E_{\mathcal{O}}$ which enjoys this property for all $s \in \mathbb{R}$, $p, q \in (0, \infty]$, and $\mathcal{A} \in \{B, F\}$. His proof extends to the vector-valued and weighted setting. A crucial ingredient is the work of Bui, Paluszyński, and Taibleson [1996, 1997], where the restriction that the Littlewood–Paley function φ should have compact Fourier support is relaxed to a moment condition on φ and a Tauberian condition on $\widehat{\varphi}$.

Once an extension operator is available, one often tries to obtain an intrinsic characterisation of the functions in $\mathcal{A}^s_{p,q}(\mathcal{O})$, e.g., in terms of differences and moduli of smoothness. As a consequence of the result of Rychkov [1999], a difference characterisation for $B^s_{p,q}(\mathcal{O})$ was obtained in Dispa [2003] for Lipschitz domains $\mathcal{O}$. A difference norm characterisations for $F^s_{p,q}(\mathcal{O})$ was obtained by Prats [2019] for ε-uniform domains (in particular, for Lipschitz domains).

Other ways to construct extension operators can be found in the books of Triebel [1983, 1992]. A classical method is to find an extension operator for $W^{k,p}(\mathcal{O})$, and use real and complex interpolation and duality to obtain an extension operators for $B^s_{p,q}(\mathcal{O})$ and $H^{s,p}(\mathcal{O})$ with $|s| < k$ and $q \in [1, \infty]$. This approach also works for Triebel–Lizorkin spaces if one uses the ℓ^q-interpolation method from Kunstmann [2015] and Lindemulder and Lorist [2021]. These techniques can also be used for vector-valued function spaces.

Another way to define function spaces on domains is by using wavelets; see Triebel [2006].

Weighted function spaces

Bui [1982] defined and studied the spaces $B^s_{p,q}(\mathbb{R}^d, w)$ and $F^s_{p,q}(\mathbb{R}^d, w)$ for all weights w in the class $A_\infty = \bigcup_{p>1} A_p$, where A_p denotes the class of Muckenhoupt weights as defined in Appendix J. Crucial to this approach is the Peetre maximal function and the weighted version of Theorem 3.2.28. The vector-valued setting was introduced and studied by Meyries and Veraar [2012, 2015, 2014b], Lindemulder, Meyries, and Veraar [2018], and, from a more abstract point of view, Lindemulder [2021].

Matrix-weighted Besov spaces have been introduced and investigated by Roudenko [2003, 2004] for $p \in [1, \infty)$, and by Frazier and Roudenko [2004, 2008] for $p \in (0, \infty)$. The special case $F^0_{p,2}(W)$ of matrix-weighted Triebel–Lizorkin spaces, and its identification with $L^p(W)$, was already considered by Nazarov and Treil [1996] and Volberg [1997], and more recently by Isralowitz [2021], but a systematic introduction and study of the full scale of these spaces is only recently due to Frazier and Roudenko [2021]. Matrix-weighted versions $F^{s,\tau}_{p,q}(W)$ of the generalised Triebel–Lizorkin-type spaces of Yang and Yuan [2008] have been subsequently studied by Bu, Hytönen, Yang, and Yuan [2023].

Two-weight Sobolev embedding

Haroske and Skrzypczak [2008] characterised the validity of the continuous embedding

$$B^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0) \hookrightarrow B^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)$$

in terms of the weights $w_0, w_1 \in A_\infty$, the exponents $p_0, p_1, q_0, q_1 \in (0, \infty]$, and the smoothness parameters $s_0 \geqslant s_1$. The compactness of this embedding was characterised as well. A characterisation for Triebel–Lizorkin spaces was obtained by Meyries and Veraar [2014b] under the additional assumption $p_0 \leqslant p_1$; as in Theorem 14.6.14, a microscopic improvement occurs. In the vector-valued setting, the case of power weights is fully understood; see Meyries and Veraar [2012].

L^p–L^q-multipliers

In the scalar-valued case, L^p–L^q Fourier multiplier theorems for $p < q$ first appeared in the pioneering work of Hörmander [1960]. The scalar-valued case has the advantage that one can often factor through an L^2-space and use Plancherel's identity. In the Banach space-valued case, this is no longer possible unless additional conditions on the spaces are imposed. The singularities in L^p–L^q-multiplier theorems for $p < q$ usually behave in a different way from the case $p = q$. Often they are absolutely integrable in some appropriate sense, and then trivially extend to the vector-valued setting by Proposition 2.1.3. A typical example where this happens is the classical Hardy–Littlewood–Sobolev inequality on the L^p–L^q-boundedness of $f \mapsto |\cdot|^{-s} * f$.

For operator-valued L^p–L^q-Fourier multipliers, different phenomena arise. For details and applications to stability of C_0-semigroups we refer the reader to Rozendaal and Veraar [2018a, 2017, 2018c,b] and the survey by Rozendaal [2023]. The homogeneous version of Corollary 14.7.7 implies the following multiplier result of Rozendaal and Veraar [2018a].

Theorem 14.8.5. *Let X be a Banach space with type $p_0 \in (1, 2]$ and let Y be a Banach space with cotype $q_0 \in [2, \infty)$. Let $p \in (1, p_0)$ and $q \in (q_0, \infty)$, where we allow $p = 2$ if $p_0 = 2$ and $q = 2$ if $q_0 = 2$. Let $r \in [1, \infty]$ satisfy $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. If $m : \mathbb{R}^d \setminus \{0\} \to \mathscr{L}(X, Y)$ is a strongly measurable function in the strong operator topology, and such that*

$$\{|\xi|^{d/r} m(\xi) : \xi \in \mathbb{R}^d \setminus \{0\}\}$$

is γ-bounded, then T_m uniquely extends to a bounded operator from $L^p(\mathbb{R}^d; X)$ to $L^q(\mathbb{R}^d; Y)$).

The proof of this theorem is based on factorisation through $\gamma(L^2(\mathbb{R}^d), X)$ and uses the γ-boundedness of the stated operator family. To obtain a homogeneous condition on m, one needs the homogeneous version of the γ-Sobolev embedding. It is not known whether Theorem 14.8.5 holds for $p = p_0$ and $q = q_0$. An exception is the case where X and Y are p-convex and q-concave Banach lattices, respectively; the result then follows from the homogeneous version of Theorem 14.8.1. Theorem 14.8.5 was used by Rozendaal [2019] to obtain boundedness of the H^∞-calculus on fractional domain spaces for strip type operators. Rozendaal and Veraar [2018a] also prove the following multiplier theorem under Fourier type assumptions.

Theorem 14.8.6. *Let X be a Banach space with Fourier type $p_0 \in (1,2]$ and let Y be a Banach space with Fourier type $q_0' \in (1,2]$. Let $p \in (1,p_0)$ and $q \in (q_0,\infty)$, and let $r \in [1,\infty)$ satisfy $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. If $m : \mathbb{R}^d \setminus \{0\} \to \mathscr{L}(X,Y)$ is a strongly measurable functions and $m \in L^{r,\infty}(\mathbb{R}^d; \mathscr{L}(X,Y))$, then T_m uniquely extends to a bounded operator from $L^p(\mathbb{R}^d;X)$ to $L^q(\mathbb{R}^d;Y))$.*

The condition $m \in L^{r,\infty}(\mathbb{R}^d; \mathscr{L}(X,Y))$ allows for singularities of the form $|\cdot|^{-d/r}$. The proof in the case $C_{m,r} := \big\| \|m\|_{\mathscr{L}(X,Y)} \big\|_{L^r(\mathbb{R}^d)} < \infty$ with $\frac{1}{p_0} - \frac{1}{q_0} = \frac{1}{r_0}$ is completely straightforward. Indeed, by Hölder's inequality,

$$
\begin{aligned}
\|T_m f\|_{q_0} \leqslant \varphi_{q_0',Y}(\mathbb{R}^d)\|m\widehat{f}\|_{q_0'} &\leqslant \varphi_{q_0',Y}(\mathbb{R}^d) C_{m,r} \|\widehat{f}\|_{p_0'} \\
&\leqslant \varphi_{q_0',Y}(\mathbb{R}^d)\varphi_{p_0,Y}(\mathbb{R}^d) C_{m,r} \|f\|_{p_0}.
\end{aligned}
$$

Theorem 14.8.6 can be deduced from this estimate by an interpolation argument.

The above Fourier multiplier theorems are stated for one specific value of p and q. However, if the kernel (see Hörmander [1960]) or the multiplier (see Rozendaal and Veraar [2017]) satisfies certain Hörmander conditions, boundedness from L^u into L^v can be shown for all $u, v \in (1,\infty)$ satisfying $\frac{1}{u} - \frac{1}{v} = \frac{1}{p} - \frac{1}{q} =: \frac{1}{r}$. For example, a sufficient condition is

$$
\sup_{\xi \neq 0} |\xi|^{|\alpha| + d/r} \|\partial^\alpha m(\xi)\| < \infty, \quad |\alpha| \leqslant \lfloor \tfrac{d}{r'} \rfloor + 1.
$$

Under Fourier type assumptions on X and Y, the number of derivatives can be further reduced.

Proposition 14.5.7 can be viewed as a mixed Besov–L^q-Fourier multiplier theorem in the same spirit as Theorems 14.8.5 and 14.8.6.

15

Extended calculi and powers of operators

In this chapter we address two strongly interwoven topics: How to verify the boundedness of the H^∞-calculus of an operator and how to represent and estimate its fractional powers. For concrete operators such as the Laplace operator or elliptic partial differential operators, the fractional domain spaces can often be identified with certain function spaces considered in Chapter 14 and the imaginary powers of the operator are related to singular integral and pseudo-differential operators treated in Chapters 11 and 13.

Throughout this chapter, unless otherwise stated, we let A be a sectorial operator on a Banach space X. We work over the complex scalar field.

15.1 Extended calculi

In Chapter 10 we have introduced the Dunford calculus

$$f \mapsto f(A),$$

defined for functions $f \in H^1(\Sigma_\sigma)$, the space of holomorphic functions on Σ_σ that are integrable with respect to the measure $\frac{\mathrm{d}z}{z}$ (in the sense of (15.1) below). We performed a detailed study of the class of operators whose Dunford calculus, when restricted to $H^1(\Sigma_\sigma)\cap H^\infty(\Sigma_\sigma)$ extends to a functional calculus for functions in $H^\infty(\Sigma_\sigma)$.

In the present section we extend the Dunford calculus of a sectorial operator A to holomorphic functions f of polynomial growth on Σ_σ. Although the operators $f(A)$ in this calculus are generally unbounded, the mapping $f \mapsto f(A)$ still shares many properties with bounded functional calculi. This extended calculus includes all functions in $H^\infty(\Sigma_\sigma)$, and it agrees with the $H^\infty(\Sigma_\sigma)$-calculus of A when this operator has a bounded $H^\infty(\Sigma_\sigma)$-calculus. In the next section, it will enable us to define the fractional powers A^α in terms of the holomorphic functions z^α. Sectorial operators A whose imaginary powers A^{it} are bounded are of special interest in view of their close relationship with a variety of topics studied in these volumes.

T. Hytönen et al., *Analysis in Banach Spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics 76,
https://doi.org/10.1007/978-3-031-46598-7_5

We briefly recall some notation and terminology introduced in Volume II that will be used throughout this chapter. For $0 < \sigma < \pi$ we denote by

$$\Sigma_\sigma := \{z \in \mathbb{C} \setminus \{0\} : \ |\arg(z)| < \sigma\}$$

the open sector of angle σ in the complex plane; the argument is taken in the interval $(-\pi, \pi)$. A linear operator $(A, \mathsf{D}(A))$ is *sectorial* if there exists $\sigma \in (0, \pi)$ such that the spectrum $\sigma(A)$ is contained in $\overline{\Sigma_\sigma}$ and

$$M_{\sigma,A} := \sup_{z \in \complement \overline{\Sigma_\sigma}} \|zR(z, A)\| < \infty.$$

Here, for $z \in \varrho(A)$, the resolvent set of A, $R(z, A) := (z - A)^{-1}$ denotes the resolvent of A. In this situation we say that A is *σ-sectorial with constant $M_{\sigma,A}$*. The infimum of all $\sigma \in (0, \pi)$ such that A is σ-sectorial is called the *angle of sectoriality* of A and is denoted by $\omega(A)$.

By $H^1(\Sigma_\sigma)$ we denote the Banach space of all holomorphic functions $f : \Sigma_\sigma \to \mathbb{C}$ for which

$$\|f\|_{H^1(\Sigma_\sigma)} := \sup_{|\nu|<\sigma} \|t \mapsto f(e^{i\nu}t)\|_{L^1(\mathbb{R}_+, \frac{\mathrm{d}t}{t})} < \infty. \tag{15.1}$$

Our objective in this section is to extend the Dunford calculus $f \mapsto f(A)$ to larger classes of functions. This is achieved in two steps: in Subsection 15.1.a we adjoin the constant-one function and the function $(1+z)^{-1}$. Among other things, this allows us to treat bounded rational functions as well as bounded functions such as $\exp(-z)$. This calculus provides the starting point for Subsections 15.1.b and 15.1.c, where we extend the calculus to a class of unbounded functions whose growth at the origin and at infinite is controlled by a regularising function. Among other things this, extended Dunford calculus will allow us to define fractional powers of A.

15.1.a The primary calculus

Our first aim is to extend the Dunford calculus $f \mapsto f(A)$ of a sectorial operator A to a slightly larger class of functions f for which one still obtains bounded operators, while preserving the multiplicativity of the calculus.

Definition 15.1.1. *For $0 < \sigma < \pi$ we define $E(\Sigma_\sigma)$ to be the vector space of holomorphic functions $f : \Sigma_\sigma \to \mathbb{C}$ of the form*

$$f(z) = f_0(z) + \frac{a}{1+z} + b,$$

where $f_0 \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ and $a, b \in \mathbb{C}$.

We could, more generally, allow functions $f_0 \in H^1(\Sigma_\sigma)$ here, but not much is gained by doing so because any such function belongs to $H^\infty(\Sigma_\nu)$ for all

$0 < \nu < \sigma$ (see Proposition H.1.3). This additional generality would in fact cause some inconvenience in the statement of the multiplicativity rule (Proposition 15.1.4), where one would be forced to switch to slightly smaller angles. A further advantage of the present definition is that $E(\Sigma_\sigma)$ is contained in $H^\infty(\Sigma_\sigma)$ as a linear subspace.

Lemma 15.1.2. *A bounded holomorphic function $f : \Sigma_\sigma \to \mathbb{C}$ belongs to $E(\Sigma_\sigma)$ if and only if it has* integrable limits *at 0 and ∞, by which we mean that there exist constants $c_0, c_\infty \in \mathbb{C}$ such that $f - c_0$ and $f - c_\infty$ are integrable with respect to $\frac{\mathrm{d}z}{z}$ near 0 and ∞, respectively, in the sense that*

$$\sup_{|\nu|<\sigma} \|t \mapsto f(e^{i\nu}t) - c_0\|_{L^1((0,1),\frac{\mathrm{d}t}{t})} < \infty$$

and

$$\sup_{|\nu|<\sigma} \|t \mapsto f(e^{i\nu}t) - c_\infty\|_{L^1((1,\infty),\frac{\mathrm{d}t}{t})} < \infty.$$

Proof. If $f = E(\Sigma_\sigma)$ is of the form $f(z) = f_0(z) + \frac{a}{1+z} + b$ one may take $c_0 = a + b$ and $c_\infty = b$. In the converse direction, if the bounded holomorphic function $f : \Sigma_\sigma \to \mathbb{C}$ has integrable limits c_0 and c_∞ at 0 and ∞, respectively, then $f_0(z) := f(z) - \frac{c_0 - c_\infty}{1+z} - c_\infty$ belongs to $H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$. □

The following functions belong to $E(\Sigma_\sigma)$:

$$z \mapsto \frac{z^m}{(1+z)^n} \qquad \text{for } 0 < \sigma < \pi \text{ and integers } n \geqslant m \geqslant 0;$$

$$z \mapsto \exp(-\zeta z) \qquad \text{for } 0 < \sigma < \tfrac{1}{2}\pi \text{ and } \zeta \in \Sigma_{\frac{1}{2}\pi - \sigma}.$$

For the first this follows by multiplicativity (proved in Proposition15.1.4 below) and the fact that $z \mapsto (1+z)^{-1}$ and $z \mapsto z(1+z)^{-1} = 1-(1+z)^{-1}$ belong to $E(\Sigma_\sigma)$. For the second this follows by noting that both $\exp(-\zeta z)-(1+\zeta z)^{-1}$ and $(1 + \zeta z)^{-1} - (1 + z)^{-1}$ are in $H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$. Another example will be encountered in the proof of Theorem 15.2.8.

Definition 15.1.3 (Primary calculus). *Let A be a sectorial operator on a Banach space X and let $\omega(A) < \sigma < \pi$. For functions $f \in E(\Sigma_\sigma)$ the bounded operator $f(A) \in \mathscr{L}(X)$ is defined by*

$$f(A) := f_0(A) + a(I + A)^{-1} + bI,$$

where

$$f(z) = f_0(z) + \frac{a}{1+z} + b$$

with $f_0 \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ and $a, b \in \mathbb{C}$, and with $f_0(A)$ defined through the Dunford calculus.

Since the constants a and b are uniquely determined by f this is well defined. For functions in $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ the primary calculus of a sectorial operator A agrees with the Dunford calculus. If A has a bounded $H^\infty(\Sigma_\sigma)$-calculus and $\mathsf{D}(A) \cap \mathsf{R}(A)$ is dense in X, then for functions $f \in E(\Sigma_\sigma)$ the definitions of $f(A)$ through the primary calculus agrees with that through the H^∞-calculus; this is because in the H^∞-calculus we have $\frac{1}{1+z}(A) = (I+A)^{-1}$ and $\mathbf{1}(A) = I$ by Theorem 10.2.13.

Proposition 15.1.4. *Let A be a sectorial operator on a Banach space X and let $\omega(A) < \sigma < \pi$. For all $f, g \in E(\Sigma_\sigma)$ we have $fg \in E(\Sigma_\sigma)$ and*

$$(fg)(A) = f(A)g(A).$$

Proof. Let $f, g \in E(\Sigma_\sigma)$ be represented as in Definition 15.1.1. It is clear that the product $f_0 g_0$ belongs to $H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ and that the product of $z \mapsto (1+z)^{-1}$ with a function in $H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ is in $H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ again. Finally,

$$\frac{1}{1+z} \cdot \frac{1}{1+z} = \frac{1}{1+z} - \frac{z}{(1+z)^2}$$

and the right-hand side is in $E(\Sigma_\sigma)$. This proves that $fg \in E(\Sigma_\sigma)$.

We have $f_0 g_0 \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$, and the multiplicativity of the Dunford calculus gives

$$f_0(A) g_0(A) = (f_0 g_0)(A).$$

Also, with $\phi(z) = 1/(1+z)$ and $\zeta(z) = z/(1+z^2)$,

$$\phi(A)^2 = (I+A)^{-2} = \phi(A) - \zeta(A) = (\phi - \zeta)(A) = \phi^2(A),$$

where we used Proposition 10.2.3 to see that $\zeta(A) = A(I+A)^{-2}$ in the Dunford calculus and hence in the primary calculus. Thus it remains to check that $\phi(A) f_0(A) = (\phi f_0)(A)$. This follows by applying the resolvent identity and Cauchy's theorem to the contour integral representation of the Dunford calculus:

$$\begin{aligned}
\phi(A) f_0(A) &= \frac{1}{2\pi i} \int_\Gamma f_0(z)(I+A)^{-1} R(z,A)\, \mathrm{d}z \\
&= \frac{1}{2\pi i} \int_\Gamma \frac{f_0(z)}{1+z} [R(z,A) - R(-1,A)]\, \mathrm{d}z \\
&= \frac{1}{2\pi i} \int_\Gamma \frac{f_0(z)}{1+z} R(z,A)\, \mathrm{d}z \\
&= (\phi f_0)(A).
\end{aligned}$$

This completes the proof. □

Example 15.1.5 (Bounded rational functions). As a first application let us prove that if A is sectorial, then for all integers $m \geqslant n \geqslant 0$ we have

$$\frac{z^m}{(1+z)^n}(A) = A^m(I+A)^{-n},$$

noting that $z \mapsto \frac{z^m}{(1+z)^n}$ belongs to $E(\Sigma_\sigma)$ for all $0 < \sigma < \pi$.

By Proposition 15.1.4,

$$\begin{aligned}\frac{z^m}{(1+z)^n}(A) &= \Big(\frac{z}{1+z}(A)\Big)^m \Big(\frac{1}{1+z}(A)\Big)^{n-m} \\ &= (A(I+A)^{-1})^m (I+A)^{m-n} = A^m(I+A)^{-n},\end{aligned}$$

where we used that

$$\frac{z}{1+z}(A) = \mathbf{1}(A) - \frac{1}{1+z}(A) = I - (I+A)^{-1} = A(I+A)^{-1}.$$

Example 15.1.6 (Exponential functions). In this example we assume that A is sectorial with $\omega(A) < \frac{1}{2}\pi$. For $\omega(A) < \sigma < \frac{1}{2}\pi$ and $\zeta \in \Sigma_{\frac{1}{2}\pi-\sigma}$ define

$$\exp(-\zeta A) := \exp(-\zeta z)(A),$$

noting that $z \mapsto \exp(-\zeta z)$ belongs to $E(\Sigma_\sigma)$.

By Proposition 15.1.4,

$$\exp(-\zeta_1 A)\exp(-\zeta_2 A) = \exp(-(\zeta_1+\zeta_2)z)(A).$$

Furthermore, for all $x \in X$ and $n \geqslant 1$ we have $\exp(-\zeta A)x \in \mathsf{D}(A^n)$ and

$$(z^n \exp(-\zeta z))(A)x = A^n \exp(-\zeta A)x.$$

To see this denote the left-hand side by $g(A)$. By Proposition 15.1.4 and Example 15.1.5,

$$\begin{aligned}(I+A)^{-n}g(A) &= \frac{1}{(1+z)^n}(A)g(A) = \Big(\frac{z^n}{(1+z)^n}\exp(-\zeta z)\Big)(A) \\ &= \frac{z^n}{(1+z)^n}(A)\exp(-\zeta z)(A) = A^n(I+A)^{-n}\exp(-\zeta z)(A),\end{aligned}$$

from which the claim follows.

The preceding example connects with semigroup theory through Proposition 10.2.7 in Volume II which can be restated in the present language of primary calculus as follows.

Theorem 15.1.7. *Let A be a densely defined sectorial operator on X with angle $\omega(A) < \frac{1}{2}\pi$, and let $\omega(A) < \sigma < \frac{1}{2}\pi$. Then the bounded holomorphic C_0-semigroup $(S(z))_{z\in\Sigma_{\frac{1}{2}\pi-\sigma}}$ generated by $-A$ is given by the primary calculus through*

$$S(z) = \exp(-zA), \quad z \in \Sigma_{\frac{1}{2}\pi-\sigma},$$

where $\exp(-zA) = \exp(-z\,\cdot)(A)$ *as in the preceding example.*

15.1.b The extended Dunford calculus

Throughout this section, A is a sectorial operator on a Banach space X and we fix $\omega(A) < \sigma < \pi$. We proceed to define an extension of the primary calculus $f \mapsto f(A)$ for suitable unbounded functions f. The idea is to use a regularising function ϱ to "tame" the growth of f near the origin and at infinity. The resulting operators $f(A)$ are unbounded in general, but they nevertheless enjoy various good properties. For functions $f \in H^\infty(\Sigma_\sigma)$ and $\varrho(z) = z/(1+z)^2$ the construction proposed in the definition has already been used in Volume II (see (10.14)).

Definition 15.1.8 (Regularisers, extended Dunford calculus). *Let A be a sectorial operator on a Banach space X and let $\omega(A) < \sigma < \pi$. Let $f : \Sigma_\sigma \to \mathbb{C}$ be holomorphic. A function $\varrho \in E(\Sigma_\sigma)$ is called a* regulariser on *Σ_σ for the pair (f, A) if the following two conditions are met:*

- *$\varrho f \in E(\Sigma_\sigma)$;*
- *the operator $\varrho(A)$ defined by the primary calculus is injective.*

*We say that (f, A) is Σ_σ-*regularisable *if such a regulariser exists, and in that case we define*

$$\mathsf{D}(f(A)) := \big\{x \in X : (\varrho f)(A)x \in \mathsf{R}(\varrho(A))\big\},$$
$$f(A)x := \varrho(A)^{-1}(\varrho f)(A)x, \quad x \in \mathsf{D}(f(A)).$$

The mapping $f \mapsto f(A)$ is referred to as the extended calculus!Dunford *of A.*

If ϱ is a Σ_σ-regulariser for the pair (f, A), then so is $\rho\varrho$ for any $\rho \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ such that $\rho(A)$ is injective. Since $\rho\varrho \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$, this shows that regularisers may be assumed to lie in $H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ whenever this is convenient.

In what follows we omit the prefix 'Σ_σ-' whenever the choice of the angle σ is clear from the context.

A trivial consequence of the first assertion in Proposition 15.1.4 is that if $\varrho \in E(\Sigma_\sigma)$, then for every function $f \in E(\Sigma_\sigma)$ we have $\varrho f \in E(\Sigma_\sigma)$. If $\varrho(A)$ is injective, Proposition 15.1.4 implies that for all $f \in E(\Sigma_\sigma)$ the definitions of $f(A)$ in Definitions 15.1.3 and 15.1.8 agree.

The following proposition shows that the definition of the operator $f(A)$ is independent of the regulariser.

Proposition 15.1.9 (Well-definedness). *Let A be a sectorial operator on a Banach space X and let $\omega(A) < \sigma < \pi$. Let $f : \Sigma_{\sigma\vee\tau} \to \mathbb{C}$ be holomorphic, where $\sigma, \tau \in (\omega(A), \pi)$. If $\varrho \in E(\Sigma_\sigma)$ and $\vartheta \in E(\Sigma_\tau)$ are regularisers for (f, A), then*

$$\big\{x \in X : (\varrho f)(A)x \in \mathsf{R}(\varrho(A))\big\} = \big\{x \in X : (\vartheta f)(A)x \in \mathsf{R}(\vartheta(A))\big\}$$

and, for all $x \in X$ belonging to this common set,

$$\varrho(A)^{-1}(\varrho f)(A)x = \vartheta(A)^{-1}(\vartheta f)(A)x.$$

Proof. Replacing σ and τ by $\sigma \wedge \tau$ we may assume that $\sigma = \tau$. Denote the domains defined in the statement of the lemma by $\mathsf{D}_\varrho(f(A))$ and $\mathsf{D}_\vartheta(f(A))$. If $x \in \mathsf{D}_\varrho(f(A))$, then $(\varrho f)(A)x = \varrho(A)y$ for some $y \in X$. By Proposition 15.1.4 we have $\vartheta\varrho f \in E(\Sigma_\sigma)$ and

$$\varrho(A)(\vartheta f)(A)x = (\vartheta\varrho f)(A)x = \vartheta(A)(\varrho f)(A)x = \vartheta(A)\varrho(A)y = \varrho(A)\vartheta(A)y,$$

and therefore $(\vartheta f)(A)x = \vartheta(A)y$ by the injectivity of $\varrho(A)$. This shows that $(\vartheta f)(A)x \in \mathsf{R}(\vartheta(A))$, so $x \in \mathsf{D}_\vartheta(f(A))$, and

$$\vartheta(A)^{-1}(\vartheta f)(A)x = y = \varrho(A)^{-1}(\varrho f)(A)x.$$

Interchanging the roles of ϱ and ϑ, one also sees that if $x \in \mathsf{D}_\vartheta(f(A))$, then $x \in \mathsf{D}_\varrho(f(A))$. This concludes the proof. □

The following observation is an immediate consequence of Proposition 15.1.4.

Lemma 15.1.10. *Let A be a sectorial operator on a Banach space X and let $\omega(A) < \sigma < \pi$. If $f, g : \Sigma_\sigma \to \mathbb{C}$ are holomorphic functions and $\varrho \in E(\Sigma_\sigma)$ and $\vartheta \in E(\Sigma_\sigma)$ are regularisers for (f, A) and (g, A), respectively, then $\varrho\vartheta$ is a regulariser for both (f, A) and (g, A).*

Proposition 15.1.11. *Let A be a sectorial operator on a Banach space X and let $\omega(A) < \sigma < \pi$. Let $f : \Sigma_\sigma \to \mathbb{C}$ be a holomorphic function such that the pair (f, A) is regularisable.*

(1) *the operator $f(A)$ is closed;*
(2) *if $\varrho \in E(\Sigma_\sigma)$ regularises (f, A), then $\mathsf{R}(\varrho(A)) \subseteq \mathsf{D}(f(A))$ and*

$$f(A)x = (\varrho f)(A)\varrho(A)^{-1}x, \quad x \in \mathsf{R}(\varrho(A)).$$

Proof. (1): Let $x_n \in \mathsf{D}(f(A))$ satisfy $x_n \to x$ and $f(A)x_n \to y$ in X as $n \to \infty$. Then $(\varrho f)(A)x_n \to (\varrho f)(A)x$ since $(\varrho f)(A)$ is bounded, and $\varrho(A)^{-1}[(\varrho f)(A)x_n] = f(A)x_n \to y$ by the definition of $f(A)$. The closedness of $\varrho(A)^{-1}$ implies $(\varrho f)(A)x \in \mathsf{D}(\varrho(A)^{-1}) = \mathsf{R}(\varrho(A))$ and $\varrho(A)^{-1}[(\varrho f)(A)x] = y$. By the definition of $\mathsf{D}(f(A))$, this means that $x \in \mathsf{D}(f(A))$ and $f(A)x = y$. This proves the closedness of $f(A)$.

(2): For $x \in \mathsf{R}(\varrho(A))$, say $x = \varrho(A)y$, we have

$$(\varrho f)(A)x = (\varrho f)(A)\varrho(A)y = \varrho(A)(\varrho f)(A)y \in \mathsf{R}(\varrho(A)).$$

Therefore $x \in \mathsf{D}(f(A))$ and

$$f(A)x = \varrho(A)^{-1}(\varrho f)(A)x = (\varrho f)(A)y = (\varrho f)(A)\varrho(A)^{-1}x.$$

□

Proposition 15.1.12. *Let A be a sectorial operator on a Banach space X and let $\omega(A) < \sigma < \pi$. Let $f, g : \Sigma_\sigma \to \mathbb{C}$ be holomorphic functions such that the pairs (f, A) and (g, A) are regularisable.*

(1) *for all* $a, b \in \mathbb{C}$ *the pair* $(af + bg, A)$ *is regularisable, and for all* $x \in \mathsf{D}(f(A)) \cap \mathsf{D}(g(A))$ *we have* $x \in \mathsf{D}((af + bg)(A))$ *and*

$$(af + bg)(A)x = af(A)x + bg(A)x.$$

(2) *the pair* (fg, A) *is regularisable and*

$$\mathsf{D}(f(A)g(A)) = \mathsf{D}(g(A)) \cap \mathsf{D}((fg)(A)),$$

and for all $x \in X$ *belonging to the common set we have*

$$(fg)(A)x = f(A)g(A)x.$$

In particular, $f(A)g(A)x$ *is closable. If* $g(A)$ *is bounded, then*

$$(fg)(A) = f(A)g(A)$$

with equal domains.

(3) *if* f *is zero-free and the pair* $(1/f, A)$ *is regularisable, then* $f(A)$ *is injective and*

$$\left(\frac{1}{f}\right)(A) = f(A)^{-1}$$

with equality of domains. In particular, if A *is injective and if we set* $\mathrm{inv}(z) := 1/z$*, then* (inv, A) *is regularisable and*

$$\mathrm{inv}(A) = A^{-1}.$$

Proof. By Lemma 15.1.10 we may select a function $\varrho \in E(\Sigma_\sigma)$ that regularises both (f, A) and (g, A) (in parts (1) and (2)), respectively both (f, A) and $(1/f, A)$ (in part (3)).

(1): It is clear that if $\varrho f, \varrho g \in E(\Sigma_\sigma)$, then $\varrho(af + bg) \in E(\Sigma_\sigma)$. The assumption $x \in \mathsf{D}(f(A)) \cap \mathsf{D}(g(A))$ implies that $(\varrho f)(A)x$ and $(\varrho g)(A)x$ belong to $\mathsf{R}(\varrho(A))$ and therefore we have $(\varrho(af + bg))(A)x \in \mathsf{R}(\varrho(A))$. Hence $x \in \mathsf{D}((af + bg)(A))$ and

$$\begin{aligned}(af + bg)(A)x &= \varrho(A)^{-1}(\varrho(af + bg))(A)x \\ &= a\varrho(A)^{-1}(\varrho f)(A)x + b\varrho(A)^{-1}(\varrho g)(A)x = af(A)x + bg(A)x.\end{aligned}$$

(2): By assumption we have $\varrho f, \varrho g \in E(\Sigma_\sigma)$. By Proposition 15.1.4 we also have $\varrho^2 fg \in E(\Sigma_\sigma)$. By multiplicativity we have $\varrho^2(A) = (\varrho(A))^2$, so $\varrho^2(A)$ is injective. It follows that the operator $(fg)(A)$ is well defined in the extended Dunford calculus.

Let $x \in \mathsf{D}(g(A)) \cap \mathsf{D}((fg)(A))$. Then, by the definition of $g(A)x$, multiplicativity, and the definition of $(fg)(A)x$,

$$\begin{aligned}(\varrho f)(A)g(A)x &= (\varrho f)(A)\varrho(A)^{-1}(\varrho g)(A)x \\ &= \varrho(A)^{-1}(\varrho f)(A)(\varrho g)(A)x\end{aligned}$$

$$= \varrho(A)^{-1}(\varrho^2 fg)(A)x$$
$$= \varrho(A)\varrho(A)^{-2}(\varrho^2 fg)(A)x$$
$$= \varrho(A)(fg)(A)x.$$

This shows that $(\varrho f)(A)g(A)x \in \mathsf{R}(\varrho(A))$ and therefore $g(A)x \in \mathsf{D}(f(A))$, i.e., $x \in \mathsf{D}(f(A)g(A))$, and

$$(fg)(A)x = \varrho(A)^{-1}(\varrho f)(A)g(A)x = f(A)g(A)x.$$

In the converse direction, let $x \in \mathsf{D}(f(A)g(A))$. Then $x \in \mathsf{D}(g(A))$ and $g(A)x \in \mathsf{D}(f(A))$, so $(\varrho f)(A)g(A)x \in \mathsf{R}(\varrho(A))$, say $(\varrho f)(A)g(A)x = \varrho(A)y$. Then,

$$(\varrho^2 fg)(A)x = (\varrho f)(A)(\varrho g)(A)x = \varrho(A)(\varrho f)(A)g(A)x = \varrho(A)^2 y = \varrho^2(A)y.$$

This shows that $(\varrho^2 fg)(A)x$ belongs to $\mathsf{R}(\varrho^2(A))$, so $x \in \mathsf{D}((fg)(A))$ by Proposition 15.1.9 and

$$(fg)(A)x = \varrho^2(A)^{-1}(\varrho^2 fg)(A)x = y = \varrho(A)^{-1}(\varrho f)(A)g(A)x = f(A)g(A)x.$$

By part (1) of Proposition 15.1.11 the operator $(fg)(A)$ is closed, and the above argument shows that it extends $f(A)g(A)$, so $f(A)g(A)$ is closable.

(3): Noting that $\mathsf{D}((1/f)f)(A) = \mathsf{D}(\mathbf{1}(A)) = \mathsf{D}(I) = X$, it follows from part (2) that if $x \in \mathsf{D}(f(A))$, then $x \in \mathsf{D}((1/f)(A)f(A))$ and $(1/f)(A)f(A)x = x$. Reversing the roles of f and $1/f$ we also obtain that if $x \in \mathsf{D}((1/f)(A))$, then $(1/f)(A)x \in \mathsf{D}(f(A))$ and $f(A)(1/f)(A)x = x$.

The second assertion follows by considering, e.g., the regulariser $\varrho(z) = z/(1+z)$. □

As a consequence of what has been shown in the course of the proof of part (2), and by applying (2) with f and g interchanged, we find that $f(A)$ and $g(A)$ commute in the following sense: we have

$$f(A)g(A)x = g(A)f(A)x = (fg)(A)x$$

for $x \in \mathsf{D}(f(A)) \cap \mathsf{D}(g(A)) \cap \mathsf{D}(fg(A))$.

We continue with a characterisation of the domain of $f(A)$ which, in view of later applications, we formulate in two versions. For integers $n \geqslant 1$ we write

$$r_n(z) := \frac{n}{n+z}, \qquad \zeta_n(z) := \frac{n}{n+z} - \frac{1}{1+nz}.$$

These functions belong to $E(\Sigma_\sigma)$.

Proposition 15.1.13. *Let A be a sectorial operator on a Banach space X and let $\omega(A) < \sigma < \pi$. Let $f : \Sigma_\sigma \to \mathbb{C}$ be a holomorphic function, and fix an integer $k \geqslant 1$.*

(1) *If* $\mathsf{D}(A)$ *is dense in* X *and* $r_n^k f \in E(\Sigma_\sigma)$, *then* $\mathsf{D}(A^k)$ *is densely contained in* $\mathsf{D}(f(A))$, *we have*

$$\mathsf{D}(f(A)) = \Big\{x \in X : \lim_{n\to\infty} (r_n^k f)(A)x \text{ exists in } X\Big\},$$

and, for all $x \in \mathsf{D}(f(A))$,

$$f(A)x = \lim_{n\to\infty} (r_n^k f)(A)x.$$

(2) *If* $\mathsf{D}(A) \cap \mathsf{R}(A)$ *is dense in* X *and* $\zeta_n^k f \in E(\Sigma_\sigma)$, *then* $\mathsf{D}(A^k) \cap \mathsf{R}(A^k)$ *is densely contained in* $\mathsf{D}(f(A))$, *we have*

$$\mathsf{D}(f(A)) = \Big\{x \in X : \lim_{n\to\infty} (\zeta_n^k f)(A)x \text{ exists in } X\Big\},$$

and, for all $x \in \mathsf{D}(f(A))$,

$$f(A)x = \lim_{n\to\infty} (\zeta_n^k f)(A)x.$$

In either case, $f(A)$ *is densely defined.*

Proof. (1): Let $\varrho(z) := r_1(z) = (1+z)^{-1}$. Then $\varrho^k = r_n^k \in E(\Sigma_\sigma)$ and $\mathsf{D}(A^k) = \mathsf{R}(\varrho^k(A))$, so the inclusion $\mathsf{R}(\varrho^k(A)) \subseteq \mathsf{D}(f(A))$ of Proposition 15.1.11 implies that $\mathsf{D}(A^k) \subseteq \mathsf{D}(f(A))$.

Let $x \in \mathsf{D}(f(A))$ and set $x_n := r_n^k(A)x$. Then $x_n \in \mathsf{D}(A^k) \subseteq \mathsf{D}(f(A))$, and by Proposition 10.1.7 we have $\lim_{n\to\infty} x_n = x$ (here we use that $\mathsf{D}(A)$ is dense) and

$$\lim_{n\to\infty} f(A)x_n = \lim_{n\to\infty} f(A)r_n^k(A)x = \lim_{n\to\infty} r_n^k(A)f(A)x = f(A)x,$$

where the middle identity follows from the second part of Proposition 15.1.11, observing that r_n^k is a regulariser for (f, A). This shows that $\mathsf{D}(A^k)$ is dense in $\mathsf{D}(f(A))$.

If $x \in \mathsf{D}(f(A))$, multiplicativity and the fact that $\varrho^k r_n^k f \in E(\Sigma_\sigma)$ imply

$$\begin{aligned} r_n^k(A)f(A)x &= \varrho^k(A)^{-1} r_n^k(A)(\varrho f)(A)x \\ &= \varrho^k(A)^{-1}(\varrho r_n^k f)(A)x = (r_n^k f)(A)x. \end{aligned}$$

Therefore $\lim_{n\to\infty}(r_n^k f)(A)x$ exists and equals $f(A)x$.

Conversely, suppose that $x \in X$ is such that $\lim_{n\to\infty}(r_n^k f)(A)x =: y$ exists. Put $z_n := r_n^k(A)(\varrho^k f)(A)x$. Then $z_n \in \mathsf{D}(A^k)$, so $z_n \in \mathsf{R}(\varrho^k(A))$. Moreover $z_n \to (\varrho^k f)(A)x$, and, by multiplicativity,

$$\begin{aligned} \varrho^k(A)^{-1} z_n &= \varrho^k(A)^{-1} r_n^k(A)(\varrho^k f)(A)x \\ &= \varrho^k(A)^{-1}(\varrho^k r_n^k f)(A)x = (r_n^k f)(A)x \to y. \end{aligned}$$

Since $\varrho^k(A)^{-1}$ is closed it follows that $(\varrho^k f)(A)x$ belongs to $\mathsf{D}(\varrho^k(A)^{-1}) = \mathsf{R}(\varrho^k(A))$, and therefore $x \in \mathsf{D}(f(A))$.

(2): This is proved in the same way as (1), replacing the use of r_n and Proposition 10.1.7 by ζ_n and Proposition 10.2.6. □

The following result improves Proposition 15.1.12(2) under an additional assumption.

Proposition 15.1.14. *Let A be a sectorial operator on a Banach space X and let $\omega(A) < \sigma < \pi$. Let $f, g : \Sigma_\sigma \to \mathbb{C}$ be holomorphic functions such that the pairs (f, A) and (g, A) are regularisable. Then $f(A)g(A)$ is closable and*

$$\overline{f(A)g(A)} = (fg)(A)$$

in each of the following two cases:

(1) $\mathsf{D}(A)$ *is dense in X, and f and g are bounded near 0 and have at most polynomial growth near ∞;*
(2) $\mathsf{D}(A) \cap \mathsf{R}(A)$ *is dense in X, and f and g have at most polynomial growth near 0 and ∞.*

Proof. The closability of $f(A)g(A)$ has already been proved in Proposition 15.1.12. We prove (1); the proof of (2) is entirely similar.

With $\varrho := r_1$ as in the previous proof, the growth assumption implies that for large enough $k \geqslant 1$ the functions $\varrho^k f$, $\varrho^k g$, and $\varrho^{2k} fg$ belong to $E(\Sigma_\sigma)$. Moreover, $\mathsf{D}(A^k) = \mathsf{R}(\varrho^k(A))$. The domain $\mathsf{D}(A^{2k})$ equals $\mathsf{R}(\varrho^{2k}(A))$, which in turn is contained in $\mathsf{D}((fg)(A))$ by Proposition 15.1.11 applied with ϱ^{2k} and fg. We also have $\mathsf{D}(A^{2k}) \subseteq \mathsf{D}(A^k) \subseteq \mathsf{D}(g(A))$, and hence $\mathsf{D}(A^{2k}) \subseteq \mathsf{D}(f(A)g(A))$ by Proposition 15.1.12. Moreover, since $\mathsf{D}(A)$ is dense in X, $\mathsf{D}(A^{2k})$ is dense in $\mathsf{D}((fg)(A))$ by Proposition 15.1.13. It follows that $\mathsf{D}(f(A)g(A))$ is dense in $\mathsf{D}((fg)(A))$. □

Theorem 15.1.15 (Composition). *Let A be a sectorial operator on a Banach space X and let $\omega(A) < \sigma < \pi$. Let $f : \Sigma_\sigma \to \mathbb{C}$ be a holomorphic function such that the pair (f, A) is regularisable, and assume that*

$$f(\Sigma_\sigma) \subseteq \Sigma_\tau$$

for some $0 < \tau < \pi$. Suppose furthermore that $f(A)$ is sectorial with $\omega(f(A)) < \tau$. If $g : \Sigma_\tau \to \mathbb{C}$ is a holomorphic function such that the pairs $(g, f(A))$ and $(g \circ f, A)$ are regularisable, then

$$g(f(A)) = (g \circ f)(A)$$

holds under either one of the following additional assumptions:

(i) $g \in E(\Sigma_\tau)$;
(ii) $(g, f(A))$ *admits a regulariser $\varphi \in E(\Sigma_\tau)$ such that $\varphi \circ f \in E(\Sigma_\sigma)$.*

The proof depends on the following lemma.

Lemma 15.1.16. *Let A be a sectorial operator on a Banach space X and let $\omega(A) < \sigma < \pi$. Let $f : \Sigma_\sigma \to \mathbb{C}$ be a holomorphic function such that*

$$f(\Sigma_\sigma) \subseteq \Sigma_\tau$$

for some $0 < \tau < \pi$. *If* $\varrho \in E(\Sigma_\sigma)$ *be a regulariser for* (f, A) *and* $\lambda \notin \overline{\Sigma_\tau}$, *then it is a regulariser for* $((\lambda - f)^{-1}, A)$ *as well, and*

$$\frac{1}{\lambda - f(z)}(A) = R(\lambda, f(A)).$$

Proof. By assumption, $\varrho f \in E(\Sigma_\sigma)$ and $\varrho(A)$ is injective. By Lemma 15.1.2, ϱ and ϱf have integrable limits at 0 and at ∞, say c_0, c_∞ and d_0, d_∞, respectively. Putting $f_\lambda := 1/(\lambda - f)$, we wish to show that ϱf_λ has integrable limits at 0 and at ∞; another application of Lemma 15.1.2 then implies that this function belongs to $E(\Sigma_\sigma)$, so ϱ regularises (f_λ, A). The identity in the statement of the lemma then follows from Proposition 15.1.12(3).

If $c_\infty = 0$, then $|f(\cdot) - \lambda| \geqslant \delta_1 := \operatorname{dist}(\lambda, \overline{\Sigma_\tau}) > 0$ implies that $\frac{\varrho(\cdot)}{\lambda - f(\cdot)}$ has integrable limit 0 at ∞. Suppose next that $c_\infty \neq 0$. We claim that $d_\infty / c_\infty \in \overline{\Sigma_\tau}$. Indeed, otherwise we had

$$\Big| f(\cdot) - \frac{d_\infty}{c_\infty} \Big| \geqslant \delta_2 := \operatorname{dist}(d_\infty / c_\infty, \overline{\Sigma_\tau}) > 0. \tag{15.2}$$

Since both ϱf and $\frac{d_\infty}{c_\infty}\varrho$ have integrable limit d_∞ at ∞, the identity

$$\varrho f = \varrho(f - \frac{d_\infty}{c_\infty}) + \frac{d_\infty}{c_\infty}\varrho$$

implies that $\varrho(f - \frac{d_\infty}{c_\infty})$ has integrable limit 0 at ∞. But then (15.2) would imply that ϱ has integrable limit 0 at ∞, contradicting the assumption that this integrable limit satisfies $c_\infty \neq 0$. This proves the claim.

With $\delta := \min\{\delta_1, \delta_2\}$ it now follows from

$$\begin{aligned}
&\Big| \frac{\varrho(z)}{\lambda - f(z)} - \frac{c_\infty^2}{c_\infty \lambda - d_\infty} \Big| \\
&\quad \leqslant \Big| \frac{\varrho(z)}{\lambda - f(z)} - \frac{c_\infty \varrho(z)}{c_\infty \lambda - d_\infty} \Big| + \Big| \frac{c_\infty(\varrho(z) - c_\infty)}{c_\infty \lambda - d_\infty} \Big| \\
&\quad = \Big| \frac{c_\infty(\varrho(z) f(z) - d_\infty) - d_\infty(\varrho(z) - c_\infty)}{(\lambda - f(z))(c_\infty \lambda - d_\infty)} \Big| + \Big| \frac{c_\infty(\varrho(z) - c_\infty)}{c_\infty \lambda - d_\infty} \Big| \\
&\quad \leqslant \frac{1}{c_\infty \delta^2}\Big(|c_\infty||\varrho(z) f(z) - d_\infty| + |d_\infty||\varrho(z) - c_\infty| \Big) + \frac{1}{\delta}|\varrho(z) - c_\infty|
\end{aligned}$$

that $\varrho f_\lambda = \frac{\varrho(\cdot)}{\lambda - f(\cdot)}$ has integrable limit $\frac{c_\infty^2}{c_\infty \lambda - d_\infty}$ at ∞.

Replacing c_∞ and d_∞ by c_0 and d_0, in the same way one sees that ϱf_λ has integrable limit 0 at 0 if $c_0 = 0$, and integrable limit $\frac{c_0^2}{c_0 \lambda - d_0}$ at 0 if $c_0 \neq 0$. $\square$

Proof of Theorem 15.1.15. We begin with the proof of the theorem under the additional assumption made in (i), namely, that $g \in E(\Sigma_\tau)$.

Step 1 – For $g = \mathbf{1}$ the theorem is trivial since $\mathbf{1} \circ f = \mathbf{1}$ and $\mathbf{1}(f(A)) = (\mathbf{1} \circ f)(A) = I$. For $g(z) = 1/(1+z)$ we have $g(f(A)) = (I + f(A))^{-1}$ and $(g \circ f)(A) = (1 + f(z))^{-1}(A) = (I + f(A))^{-1}$, the former by Definition 15.1.3 applied to $f(A)$ and the latter by Lemma 15.1.16.

Step 2 – We now consider a general $g \in E(\Sigma_\tau)$, and write $g = g_0 + a/(1+z) + b$ with $a, b \in \mathbb{C}$ and $g_0 \in H^1(\Sigma_\tau) \cap H^\infty(\Sigma_\tau)$. Let $\varrho \in E(\Sigma_\sigma)$ be a regulariser for (f, A). By Lemma 15.1.10 we may assume that ϱ also regularises $(g \circ f, A)$, and by the observation after Definition 15.1.8 we may also assume that $\varrho \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$. As the proof of Lemma 15.1.16 shows, ϱ also regularises $(\frac{1}{\lambda - f(\cdot)}, A)$ for $\lambda \notin \overline{\Sigma_\tau}$.

Fix $\omega(A) < \mu < \sigma$ and $\omega(f(A)) < \nu < \tau$. By the Dunford calculus of $f(A)$, the operator $g_0(f(A))$ is bounded and for all $x \in X$ we have

$$g_0(f(A))x = \frac{1}{2\pi i} \int_{\partial \Sigma_\nu} g_0(z) R(z, f(A))x \,\mathrm{d}z.$$

If $z \in \partial \Sigma_\nu$, then by Lemma 15.1.16 for all $x \in X$ we have

$$R(z, f(A))x = \frac{1}{z - f(\cdot)}(A)x. \tag{15.3}$$

Using (15.3) and multiplicativity of the primary calculus of A, Fubini's theorem, the Cauchy integral theorem, and keeping in mind that $\varrho \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$, we obtain

$$\begin{aligned}
\varrho(A) g_0(f(A))x &= \frac{1}{2\pi i} \int_{\partial \Sigma_\nu} g_0(\lambda) \varrho(A) R(\lambda, f(A))x \,\mathrm{d}\lambda \\
&= \frac{1}{2\pi i} \int_{\partial \Sigma_\nu} g_0(\lambda) \frac{\varrho(\cdot)}{\lambda - f(\cdot)}(A)x \,\mathrm{d}\lambda \\
&= \Big(\frac{1}{2\pi i}\Big)^2 \int_{\partial \Sigma_\nu} g_0(\lambda) \Big(\int_{\partial \Sigma_\mu} \frac{\varrho(z)}{\lambda - f(z)} R(z, A)x \,\mathrm{d}z \Big) \,\mathrm{d}\lambda \\
&= \Big(\frac{1}{2\pi i}\Big)^2 \int_{\partial \Sigma_\mu} \varrho(z) \Big(\int_{\partial \Sigma_\nu} \frac{g_0(\lambda)}{\lambda - f(z)} \,\mathrm{d}\lambda \Big) R(z, A)x \,\mathrm{d}z \\
&= \frac{1}{2\pi i} \int_{\partial \Sigma_\mu} \varrho(z) g_0(f(z)) R(z, A)x \,\mathrm{d}z \\
&= (\varrho \cdot (g_0 \circ f))(A)x.
\end{aligned}$$

Setting $h_0(z) := a/(1+z) + b$, by Step 1 we also have $h_0(f(A)) = (h_0 \circ f)(A)$ and therefore, by Proposition 15.1.4,

$$\varrho(A) h_0(f(A))x = (\varrho \cdot (h_0 \circ f))(A)x, \quad x \in X.$$

Adding up, we obtain

$$\varrho(A) g(f(A))x = (\varrho \cdot (g \circ f))(A)x, \quad x \in X,$$

the operator $g(f(A))$ being defined by the primary functional calculus of $f(A)$. Since ϱ regularises $(g \circ f, A)$, this implies that every $x \in X$ belongs to $\mathsf{D}((g \circ f)(A))$ and

$$g(f(A))x = (g \circ f)(A)x, \quad x \in X.$$

This proves that $g(f(A)) = (g \circ f)(A)$, and both operators are bounded. This concludes the proof under the assumption made (i).

For the proof of the theorem under the assumption made in (i), let $\varphi \in E(\Sigma_\tau)$ be a regulariser for the pair $(g, f(A))$ such that $\varphi \circ f \in E(\Sigma_\tau)$, and let ρ be a regulariser for $(g \circ f, A)$.

We claim that under these circumstances, $\rho \cdot (\varphi \circ f)$ regularises $(g \circ f, A)$. To this end we must show:

- $\rho \cdot (\varphi \circ f) \cdot (g \circ f) \in E(\Sigma_\sigma)$;
- $(\rho \cdot (\varphi \circ f))(A)$ is injective.

The first assertion follows from $\rho \cdot (g \circ f) \in E(\Sigma_\sigma)$ (since ρ be a regulariser for $(g \circ f, A)$) and $\varphi \circ f \in E(\Sigma_\tau)$ (by assumption). For the second assertion we use the multiplicativity rule of Proposition 15.1.12 (noting that $(\varphi \circ f)(A) = \varphi(f(A))$ by the result of Step 2 and the fact that $\varphi \in E(\Sigma_\tau)$) to see that

$$(\rho \cdot (\varphi \circ f))(A) = \rho(A)(\varphi \circ f)(A) = \rho(A)\varphi(f(A)).$$

The right-hand side is the composition of two injective operators; this is because ρ is a regulariser for $(g \circ f, A)$ and φ is a regulariser for $(g, f(A))$. This proves the claim.

In the following computation, in (i) we use the definition of a regulariser, in (ii) we apply the result of Step 2 to $\varphi g \in E(\Sigma_\tau)$, noting that φg satisfies the conditions of the theorem since g does, (iii) follows from Proposition 15.1.12, noting that $((\varphi g) \circ f))(A) = (\varphi g)(f(A))$ is a bounded operator since $\varphi g \in E(\Sigma_\tau)$, (iv) is a simple rewriting, (v) follows from the definition of a regulariser, noting that $\rho \cdot (\varphi \circ f)$ regularises $(g \circ f, A)$, (vi) follows by another application of Proposition 15.1.12, and (vii) uses the result of Step 2 once again:

$$\begin{aligned}
\rho(A)\varphi(f(A))g(f(A)) &\overset{\text{(i)}}{=} \rho(A)(\varphi g)(f(A)) \\
&\overset{\text{(ii)}}{=} \rho(A)((\varphi g) \circ f)(A) \\
&\overset{\text{(iii)}}{=} (\rho \cdot ((\varphi g) \circ f))(A) \\
&\overset{\text{(iv)}}{=} (\rho \cdot (\varphi \circ f) \cdot (g \circ f))(A) \\
&\overset{\text{(v)}}{=} (\rho \cdot (\varphi \circ f))(A)(g \circ f)(A) \\
&\overset{\text{(vi)}}{=} \rho(A)(\varphi \circ f)(A)(g \circ f)(A) \\
&\overset{\text{(vii)}}{=} \rho(A)\varphi(f(A))(g \circ f)(A).
\end{aligned}$$

The identity $g(f(A)) = (g \circ f)(A)$ follows from this since both $\rho(A)$ and $\varphi(f(A))$ are injective. □

Our next aim is to relate the extended Dunford calculus with the H^∞-calculus.

Theorem 15.1.17 (Boundedness of the extended Dunford calculus). *Let A be a sectorial operator on X with $\mathsf{D}(A) \cap \mathsf{R}(A)$ dense in X, and let $\omega(A) < \sigma < \pi$. Then for all functions $f \in H^\infty(\Sigma_\sigma)$ the pair (f, A) is regularisable and the following assertions are equivalent:*

(1) *the operator $f(A)$ defined through the extended Dunford calculus is bounded for all $f \in H^\infty(\Sigma_\sigma)$;*
(2) *the operator A has bounded $H^\infty(\Sigma_\sigma)$-calculus.*

In this situation the operators $f(A)$ defined through the extended Dunford calculus and the H^∞-calculus agree.

Proof. By Proposition 10.1.8, the density of $\mathsf{D}(A) \cap \mathsf{R}(A)$ implies that A is injective. As a consequence, for every $f \in H^\infty(\Sigma_\sigma)$ the function $\zeta(z) = z/(1+z)^2$ is a regulariser for the pair (f, A).

(1)⇒(2): By the boundedness of $f(A)$ and the closedness of $\zeta(A)^{-1}$, the identity $f(A) = \zeta(A)^{-1}(\zeta f)(A)$ (note that $\mathsf{D}(f(A)) = X$) implies that $f \mapsto f(A)$ is closed as a linear map from $H^\infty(\Sigma_\sigma)$ to $\mathscr{L}(X)$, and therefore bounded, by the closed graph theorem. Denoting its norm by M, it follows that

$$\|f(A)\| \leqslant M\|f\|_\infty$$

for all $f \in H^\infty(\Sigma_\sigma)$. In particular, this bound holds for all $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$. For such functions the extended Dunford calculus agrees with the Dunford calculus, and therefore the estimate tells us that A has a bounded $H^\infty(\Sigma_\sigma)$-calculus.

(2)⇒(1): If $x \in \mathsf{D}(A) \cap \mathsf{R}(A)$, then $x \in \mathsf{R}(\zeta(A))$, say $x = \zeta(A)y$. For the operator $f(A)$ defined through the H^∞-calculus we have, by the multiplicativity of the H^∞-calculus,

$$f(A)x = \zeta(A)f(A)y = (\zeta f)(A)y,$$

where the operator on the right-hand side is again defined by the H^∞-calculus. We can also define the operator $(\zeta f)(A)$ through the primary calculus, and these two definitions agree (they agree for functions in $H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ and for the functions $(1 + z)^{-1}$ and $\mathbf{1}$). It follows that

$$f(A)y = \zeta(A)^{-1}(\zeta f)(A)y.$$

Since $\mathsf{D}(A) \cap \mathsf{R}(A)$ is dense in X, this implies that $f(A) = \zeta(A)^{-1}(\zeta f)(A)$. The operator on the right-hand side equals the operator $f(A)$ defined through the extended Dunford calculus, which is therefore bounded. □

We finish this section with a perturbation result that will be useful in connection with bounded imaginary powers (see the proof of Lemma 15.3.8).

Theorem 15.1.18. *Let A be a densely defined sectorial operator on a Banach space X and let $\omega(A) < \sigma < \pi$. Let $f \in H^\infty(\Sigma_\sigma)$ be given. If the operator $f(A)$, defined through the extended Dunford calculus of A is bounded, then also the operator $f(A+I)$, defined through the extended Dunford calculus of $A+I$, is bounded and we have*

$$\|f(A+I)\| \leqslant (1+M_{\sigma,A})^2(\|f(A)\| + C_\sigma\|f\|_{H^\infty(\Sigma_\sigma)}),$$

where C_σ is a constant depending only on σ and $M_{\sigma,A}$ is the sectoriality constant of A at angle σ.

Proof. Note that $\omega(A+I) \leqslant \omega(A)$ and fix $\omega(A) < \nu < \sigma$. The injectivity of $A+I$ implies that the function $\zeta(z) = z/(z+1)^2$ is a regulariser for $(f, A+I)$ for any $f \in H^\infty(\Sigma_\sigma)$. Since $\mathsf{D}(A+I) = \mathsf{D}(A)$ and $\mathsf{R}(A+I) = X$, the second part of Proposition 15.1.13 implies that $f(A+I)$ is densely defined.

By the extended Dunford calculus of $A+I$, for all $x \in \mathsf{D}(f(A+I))$ we have

$$f(A+I)x = (\zeta(A+I))^{-1}\frac{1}{2\pi i}\int_{\partial\Sigma_\nu} \zeta(z)f(z)R(z,A+I)x\,\mathrm{d}z.$$

We have $1/\zeta(z) = z+2+z^{-1}$, and this easily implies $(\zeta(A+I))^{-1} = (A+I)+2I+(A+I)^{-1}$. Now,

$$\begin{aligned}&\Big\|(2I+(A+I)^{-1})\frac{1}{2\pi i}\int_{\partial\Sigma_\nu} \zeta(z)f(z)R(z,A+I)x\,\mathrm{d}z\Big\|\\ &\qquad\leqslant (2+M_{\nu,A})M_{\nu,A+I}\|x\|\Big(\frac{1}{2\pi}\int_{\partial\Sigma_\nu}\frac{1}{|z+1|^2}\,|\mathrm{d}z|\Big)\|f\|_{H^\infty(\Sigma_\sigma)}\end{aligned}$$

and, noting that $R(z,A+I) = R(z,A) + R(z,A+I)R(z,A)$ by the resolvent identity,

$$\begin{aligned}&\Big\|(A+I)\frac{1}{2\pi i}\int_{\partial\Sigma_\nu} \zeta(z)f(z)R(z,A+I)x\,\mathrm{d}z\Big\|\\ &\quad\leqslant \Big\|(A+I)\frac{1}{2\pi i}\int_{\partial\Sigma_\nu} \zeta(z)f(z)R(z,A)x\,\mathrm{d}z\Big\|\\ &\qquad + \Big\|\frac{1}{2\pi i}\int_{\partial\Sigma_\nu} \zeta(z)f(z)(A+I)R(z,A+I)R(z,A)x\,\mathrm{d}z\Big\|\\ &\quad\leqslant \|(A+I)\zeta(A)\|\|f(A)\|\|x\|\\ &\qquad + (1+M_{\nu,A+I})M_{\nu,A}\|x\|\Big(\frac{1}{2\pi}\int_{\partial\Sigma_\nu}\frac{1}{|z+1|^2}\,|\mathrm{d}z|\Big)\|f\|_{H^\infty(\Sigma_\sigma)}.\end{aligned}$$

Since $\|A\zeta(A+I)\| = \|(A+I)A(A+I)^{-2}\| \leqslant M_{\nu,A}$ and $M_{\nu,A+I} \leqslant M_{\nu,A}$, this proves the estimate

$$\|f(A+I)x\| \leqslant (1+M_{\nu,A})^2\|f(A)\| + C_\nu(1+M_{\nu,A})^2\|f\|_{H^\infty(\Sigma_\sigma)}\|x\|$$

for $x \in \mathsf{D}(f(A+I))$, with $C_\nu = \frac{1}{\pi}\int_{\partial\Sigma_\nu} \frac{1}{|z+1|^2}\,|\mathrm{d}z|$. Since $\mathsf{D}(f(A+I))$ is dense, this estimate extends to arbitrary $x \in X$. To conclude the proof we let $\nu \uparrow \sigma$ and note that $M_{\nu,A} \to M_{\sigma,A}$ by an easy estimate based on the resolvent identity. □

15.1.c Extended calculus via compensation

For functions $f \in H^\infty(\Sigma_\sigma)$ and regulariser $\varrho(z) := \zeta(z) = z/(1+z)^2$ there is different approach to the extended Dunford calculus via the Cauchy integral formula, which we outline presently.

Let A be a sectorial operator and let $f \in H^\infty(\Sigma_\sigma)$. For $\omega(A) < \tau < \sigma' < \sigma$, $\mu \in \Sigma_{\sigma'} \setminus \Sigma_\tau$, and $x \in \mathsf{D}(A) \cap \mathsf{R}(A)$ define

$$f(A)x := f(\mu)x + \frac{1}{2\pi i}\int_{\partial\Sigma_{\sigma'}} f(z)\Big(R(z,A) - \frac{1}{z-\mu}\Big)x\,\mathrm{d}z. \tag{15.4}$$

Let us check that the integrand converges absolutely. Since $x \in \mathsf{D}(A) \cap \mathsf{R}(A)$ we may pick $y \in \mathsf{D}(A)$ with $Ay = x$. Then

$$\Big\|R(z,A)x - \frac{x}{z-\mu}\Big\| = \Big\|\frac{(A-\mu)R(z,A)x}{z-\mu}\Big\| \leqslant \frac{\|R(z,A)\|}{|z-\mu|}(\|Ax\| + \mu\|x\|),$$

which is of the order $O(|z|^{-2})$ as $|z| \to \infty$ along $\partial\Sigma_\tau$. Also,

$$\Big\|R(z,A)x - \frac{x}{z-\mu}\Big\| = \Big\|R(z,A)Ay - \frac{x}{z-\mu}\Big\| \leqslant \|R(z,A)Ay\| + \frac{\|x\|}{|z-\mu|}$$

which is of the order $O(1)$ as $|z| \to 0$ along $\partial\Sigma_\tau$, noting that $\|R(z,A)Ay\| = \|R(z,A)[(A-z)+z]y\| \leqslant (1+\|zR(z,A)\|)\|y\|$. This establishes the claim. By an application of Cauchy's theorem, $f(A)$ is independent of $\mu \in \Sigma_\tau \setminus \Sigma_{\tau'}$. Since the integrand is an integrable $\overline{\mathsf{R}(A)}$-valued function, we see that

$$f(A)x \in \overline{\mathsf{R}(A)}, \quad x \in \mathsf{D}(A) \cap \mathsf{R}(A).$$

Note that if $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$, the above definition of $f(A)x$ agrees with (10.7).

We will now check that the definition of $f(A)x$ by (15.4) agrees with the one via Definition 15.1.1 for the regulariser $\varrho(z) = \zeta(z) = z/(1+z)^2$. Suppose that $x \in \mathsf{D}(A) \cap \mathsf{R}(A)$, say $x = \zeta(A)y$. Starting from the latter definition we have

$$f(A)x = (f\zeta)(A)y = \frac{1}{2\pi i}\int_{\partial\Sigma_\tau} \frac{zf(z)}{(1+z)^2}R(z,A)y\,\mathrm{d}z.$$

Fix $\omega(A) < \tau' < \tau$ and $\mu \in \Sigma_\tau \setminus \Sigma_{\tau'}$. To check that (15.4) agrees with Definition 15.1.1 we must show that

$$f(\mu)A(I+A)^{-2}y \\ = \frac{1}{2\pi i}\int_{\partial\Sigma_\tau} \frac{f(z)}{z-\mu}\Big[\frac{z(z-\mu)}{(1+z)^2}R(z,A)y - \big((z-\mu)R(z,A)-I\big)x\Big]\,\mathrm{d}z.$$

By Cauchy's formula, the right-hand side integral evaluates to

$$= \Big[f(z)\Big(\frac{z(z-\mu)}{(1+z)^2}R(z,A)y - \big((z-\mu)R(z,A)-I\big)x\Big)\Big]\Big|_{z=\mu} = f(\mu)x,$$

as was to be proved. We have thus proved:

Proposition 15.1.19. *Let A be a sectorial operator on a Banach space X, let $\omega(A) < \tau' < \tau < \sigma < \pi$. Let $f \in H^\infty(\Sigma_\sigma)$. For all $\mu \in \Sigma_\tau \setminus \Sigma_{\tau'}$ and $x \in \mathsf{D}(A) \cap \mathsf{R}(A)$ the integral*

$$f(A)x := f(\mu)x + \frac{1}{2\pi i}\int_{\partial\Sigma_\tau} f(z)\Big(R(z,A) - \frac{1}{z-\mu}\Big)x\,\mathrm{d}z$$

converges absolutely and we have $f(A)x = (f\zeta)(A)y$, in agreement with the definition of $f(A)x$ through the extended Dunford calculus.

The attentive reader will have noticed that we already used this procedure in Proposition 10.2.7.

15.2 Fractional powers

In this section, we will apply the extended Dunford calculus to introduce the fractional powers A^α of a sectorial operator A. Particular instances of fractional powers such as $(-\Delta)^{1/2}$, the square root of the negative Laplacian, appear all over in Analysis. On a theoretical level, domains of fractional powers encode useful smoothness properties of the elements in their domains, and correspond to (or are closely connected with) interpolation scales between the domain $\mathsf{D}(A)$ and the underlying Banach space X. For example, if the imaginary powers A^{it}, $t \in \mathbb{R}$, are bounded operators, then for all $0 < \alpha < 1$ the fractional domain $\mathsf{D}(A^\alpha)$ equals the complex interpolation space $[X, \mathsf{D}(A)]_\alpha$ as a subspace of X, and as a Banach space up to equivalent norms. As we have seen in Chapter 4, for the negative Laplacian $A = -\Delta$ on $X = L^p(\mathbb{R}^d)$, the latter can be identified as the Bessel potential space $H^{2\alpha,p}(\mathbb{R}^d)$.

After introducing fractional powers, we establish several basic algebraic properties and prove several useful representation formulas. In the next section, we then take a closer look at the class of sectorial operators whose imaginary powers are bounded, and prove a number of non-trivial theorems connecting this property with (R-, γ-)sectoriality and boundedness of the H^∞-calculus.

15.2.a Definition and basic properties

In what follows, unless otherwise stated we let A be a sectorial operator acting in a Banach space X. When additional assumptions are needed, they will always be stated explicitly.

For $\alpha \in \mathbb{C}$ it is natural to try to define the fractional power A^α by applying the extended Dunford calculus to the function

$$f_\alpha(z) := z^\alpha := e^{\alpha \log z},$$

where we use the branch of the logarithm that is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$. Let $0 < |\nu| < \sigma < \pi$. For $z = re^{i\nu}$ with $r \geqslant 0$ we have

$$|f_\alpha(z)| = |r^\alpha|\,|e^{i\nu\alpha}| \leqslant |z|^{\Re\alpha} e^{\sigma|\Im\alpha|}.$$

For all integers $m, n \in \mathbb{N}$, the function

$$\varrho_{m,n}(z) := z^m(1+z)^{-m-n}$$

belongs to $E(\Sigma_\sigma)$, and

- if $\Re\alpha > 0$, then $\varrho_{m,n} f_\alpha \in E(\Sigma_\sigma)$ for all integers $m \geqslant 0$, $n > \Re\alpha$;
- if $\Re\alpha = 0$, then $\varrho_{m,n} f_\alpha \in E(\Sigma_\sigma)$ for all integers $m, n \geqslant 1$;
- if $\Re\alpha < 0$, then $\varrho_{m,n} f_\alpha \in E(\Sigma_\sigma)$ for all integers $m > |\Re\alpha|$, $n \geqslant 0$.

The operator $\varrho_{m,n}(A) = A^m(I+A)^{-m-n}$ (cf. Example 15.1.5) is injective if $m = 0$ or A is injective (or both). This shows:

Proposition 15.2.1. *Let A be a sectorial operator on a Banach space X. The pair (f_α, A) is regularisable in each of the following two cases:*

- $\Re\alpha > 0$
- $\Re\alpha \leqslant 0$ *and A is injective.*

In the first case $\varrho_{0,n}(z) = (1+z)^{-n}$ with $n > \Re\alpha$ is a regulariser; in the second case $\varrho_{n,n}(z) = z^n(1+z)^{-2n}$ with $n > |\Re\alpha|$ is a regulariser.

In view of these considerations the extended Dunford calculus allows us to make the following definition.

Definition 15.2.2 (Fractional powers). *Let A be a sectorial operator on a Banach space X. For $\alpha \in \mathbb{C}$ we define*

$$A^\alpha := f_\alpha(A), \quad \alpha \in \mathbb{C},$$

in each of the following two cases:

- $\Re\alpha > 0$
- $\Re\alpha \leqslant 0$ *and A is injective.*

These operators are closed. Moreover, if $\Re\lambda > 0$ and $\mathsf{D}(A)$ is dense, then A^α is densely defined; if $\Re\alpha \leqslant 0$ and $\mathsf{D}(A)\cap\mathsf{R}(A)$ is dense, then A is injective and A^α is densely defined. Using the results of Section 10.1.b, These assertions follow from Proposition 15.1.11, the domain identifications $\mathsf{D}(A^n) = \mathsf{R}(\varrho_{0,n})$ and $\mathsf{D}(A^n)\cap\mathsf{R}(A^n) = \mathsf{R}(\varrho_{n,n})$, and the fact that $\mathsf{D}(A^n)$ is dense if $\mathsf{D}(A)$ dense, respectively $\mathsf{D}(A^n)\cap\mathsf{R}(A^n)$ is dense in X if $\mathsf{D}(A)\cap\mathsf{R}(A)$ is dense in X.

We begin our study of fractional powers with a consistency check.

Proposition 15.2.3. *Let A be a sectorial operator on a Banach space X. For all $n = 0, 1, 2, \dots$ and $f_n(z) = z^n$ we have*

$$f_n(A) = A^n \quad \textit{with equal domains.} \tag{15.5}$$

If in addition A is injective, this identity extends to all $n \in \mathbb{Z}$.

Proof. For $n = 0$ this reduces to the identity $\mathbf{1}(A) = I$. For $n \geqslant 1$, consider the function $\varrho_n(z) = (1+z)^{-n}$ and let $x \in \mathsf{D}(A^n) = \mathsf{R}(\varrho_n(A))$, say $x = (I+A)^{-n}y$. Then

$$f_n(A)x = \varrho_n(A)^{-1}(\varrho_n f_n)(A)x = A^n x,$$

where we used that $\varrho_n(A) = (I+A)^{-n}$ in the primary calculus, and that $(\varrho_n f_n)(A) = \frac{z^n}{(1+z)^n}(A) = A^n(I+A)^{-n}$ in the primary calculus. This proves that $A^n \subseteq f_n(A)$. In the converse direction, if $x \in \mathsf{D}(f_n(A))$, then

$$A^n(I+A)^{-n}x = (\varrho_n f_n)(A)x \in \mathsf{R}(\varrho_n(A)) = \mathsf{D}(A^n),$$

forcing $x \in \mathsf{D}(A^n)$. This completes the proof of (15.5) for $n \geqslant 1$. For $n = -1, -2, \dots$ the result follows by applying Proposition 15.1.12(3). $\square$

From the definition of the extended Dunford calculus we immediately deduce the following result.

Proposition 15.2.4. *Let A be a sectorial operator on a Banach space X, and fix an integer $k \geqslant 1$.*

(1) *For all $x \in \mathsf{D}(A^k)$ the function $z \mapsto A^z x$ is well defined and holomorphic on $\{0 < \Re z < k\}$.*
(2) *If A is injective, then for all $x \in \mathsf{D}(A^k)\cap\mathsf{R}(A^k)$ the function $z \mapsto A^z x$ is well defined and holomorphic on $\{-k < \Re z < k\}$.*

Theorem 15.2.5. *Let A be a sectorial operator on a Banach space X, and let $\alpha, \alpha_1, \alpha_2 \in \mathbb{C}$.*

(1) *If A is injective and $\alpha \in \mathbb{C}$, then A^α is injective and*

$$A^{-\alpha} = (A^\alpha)^{-1} = (A^{-1})^\alpha \quad \textit{with equality of domains.}$$

(2) *If $\Re\alpha_1 > \Re\alpha_2 > 0$, then*

$$\mathsf{D}(A^{\alpha_1}) \subseteq \mathsf{D}(A^{\alpha_2}) \quad \textit{and} \quad \mathsf{R}(A^{\alpha_2}) \supseteq \mathsf{R}(A^{\alpha_1}),$$

(3) *If A is injective and $\Re\alpha_1 < \Re\alpha_2 < 0$, then*

$$\mathsf{D}(A^{\alpha_1}) \supseteq \mathsf{D}(A^{\alpha_2}) \quad \text{and} \quad \mathsf{R}(A^{\alpha_1}) \subseteq \mathsf{R}(A^{\alpha_2}).$$

(4) *If $\Re\alpha_1 > 0$ and $\Re\alpha_2 > 0$, then*

$$A^{\alpha_1+\alpha_2} = A^{\alpha_1}A^{\alpha_2} \quad \text{with equality of domains.}$$

(5) *If A is injective and $\Re\alpha_1 < 0$ and $\Re\alpha_2 < 0$, then*

$$A^{\alpha_1+\alpha_2} = A^{\alpha_1}A^{\alpha_2} \quad \text{with equality of domains.}$$

Proof. (1): The injectivity of A^α and the identity $A^{-\alpha} = (A^\alpha)^{-1}$ follow from Proposition 15.1.12(3). The identity $A^{-\alpha} = (A^{-1})^\alpha$ follows from Theorem 15.1.15, noting that A^{-1} is sectorial with the same angle as A.

(2): We consider the regulariser $\varrho_k(z) = (1+z)^{-k}$, for which we have $\mathsf{R}(\varrho_k(A)) = \mathsf{D}(A^k)$.

Let $x \in \mathsf{D}(A^{\alpha_1})$ and fix an integer $k > \max\{\Re\alpha_2, \Re\alpha_1 - \Re\alpha_2\}$. In order to prove that $x \in \mathsf{D}(A^{\alpha_2})$ we must show that $((1+z)^{-k}z^{\alpha_2})(A)x \in \mathsf{D}(A^k)$.

Since $2k > \Re\alpha_1$, by the definition of $\mathsf{D}(A^{\alpha_1})$ we have $((1+z)^{-2k}z^{\alpha_1})(A)x \in \mathsf{D}(A^{2k})$. Using the multiplicativity of the Dunford calculus, this implies that

$$\begin{aligned} A^k(I+A)^{-2k}((1+z)^{-k}z^{\alpha_2})(A)x &= \frac{z^{k+\alpha_2}}{(1+z)^{3k}}(A)x \\ &= \frac{z^{k-(\alpha_1-\alpha_2)}}{(1+z)^k}(A)\frac{z^{\alpha_1}}{(1+z)^{2k}}(A)x \end{aligned}$$

belongs to $\mathsf{D}(A^{2k})$. It follows that $(I+A)^{-2k}((1+z)^{-k}z^{\alpha_2})(A)x \in \mathsf{D}(A^{3k})$ and $((1+z)^{-k}z^{\alpha_2})(A)x \in \mathsf{D}(A^k)$ as desired. The opposite inclusion of the ranges follows from part (4) proved below.

(3): If A is injective and $\Re\alpha_1 < \Re\alpha_2 < 0$ we can apply parts (2) and (1) with $\beta_1 = -\alpha_1$ and $\beta_2 = \alpha$, noting that $\mathsf{D}(A^{\alpha_j}) = \mathsf{D}(A^{-\beta_j}) = \mathsf{R}(A^{\beta_j})$ and $\mathsf{R}(A^{\alpha_j}) = \mathsf{R}(A^{-\beta_j}) = \mathsf{D}(A^{\beta_j})$.

(4): Let $\Re\alpha_1 > 0$ and $\Re\alpha_2 > 0$. Proposition 15.1.12 implies that $A^{\alpha_1}A^{\alpha_2}x = A^{\alpha_1+\alpha_2}x$ for all $x \in \mathsf{D}(A^{\alpha_1}A^{\alpha_2}) = \mathsf{D}(A^{\alpha_2}) \cap \mathsf{D}(A^{\alpha_1+\alpha_2})$. It remains to prove that $\mathsf{D}(A^{\alpha_1+\alpha_2}) \subseteq \mathsf{D}(A^{\alpha_2})$. But this follows from part (2).

(5): This follows from (1) and (4) by taking inverses. □

Proposition 15.2.6. *Let A be a sectorial operator on a Banach space X. Let $c \in \mathbb{C} \setminus \{0\}$ satisfy $|\arg c| < \pi - \omega(A)$. Then:*

(1) *the operator cA is sectorial with angle $\omega(cA) \leqslant \omega(A) + |\arg(c)|$, and for all $\omega(A) < \sigma < \pi - |\arg c|$ we have $M_{\sigma+|\arg c|,cA} \leqslant M_{\sigma,A}$;*
(2) *for all $\alpha \in \mathbb{C}$, and assuming A to be injective if $\Re\alpha \leqslant 0$, we have*

$$(cA)^\alpha = c^\alpha A^\alpha \quad \text{with equality of domains.}$$

Proof. Since $(\lambda - cA)^{-1} = c^{-1}(c^{-1}\lambda - A)^{-1}$, the condition $|\arg(c)| < \pi - \omega(A)$ guarantees that cA is sectorial with $\omega(cA) \leqslant \omega(A) + |\arg c|$. Also, for $\omega(A) < \sigma < \pi - |\arg c|$ and $\lambda \in \complement\overline{\Sigma_{\sigma+|\arg c|}}$ we have $c^{-1}\lambda \in \complement\overline{\Sigma_\sigma}$ and

$$\|\lambda R(\lambda, cA)\| = \|c^{-1}\lambda R(c^{-1}\lambda, A)\| \leqslant M_{\sigma,A},$$

which gives the bound $M_{\sigma+|\arg c|,cA} \leqslant M_{\sigma,A}$.

Choose $\omega > \omega(A)$ such that $\omega + |\arg c| < \pi$. Fix $\alpha \in \mathbb{C}$. Then, for $x \in X$ and $k > |\Re\alpha|$,

$$(\rho_k f_\alpha)(cA)x = \frac{1}{2\pi i}\int_{\partial\Sigma_{\omega+|\arg c|}} \rho_k(z)z^\alpha R(z, cA)x \,\mathrm{d}z$$

with $f_\alpha(z) = z^\alpha$, and $\rho_k(z) := \varrho_{0,k}(z) = (1+z)^{-k}$ if $\Re\alpha > 0$ and $\rho_k(z) := \varrho_{k,k}(z) = z^k/(1+z)^{2k}$ if $\Re\alpha \leqslant 0$. By Cauchy's theorem we can deform the path in the above integral to $\Gamma = c\cdot\partial\Sigma_\omega$ and obtain, by a change of variables,

$$\begin{aligned}(\rho_k f_\alpha)(cA)x &= \frac{1}{2\pi i}\int_\Gamma \rho_k(z)z^\alpha c^{-1}R(c^{-1}z, A)x\,\mathrm{d}z \\ &= c^\alpha \frac{1}{2\pi i}\int_{\partial\Sigma_\omega} \rho_k(cz)z^\alpha R(z,A)x\,\mathrm{d}z = c^\alpha(\rho_n^k(c\,\cdot)f_\alpha)(A)x.\end{aligned} \tag{15.6}$$

If $x \in \mathsf{D}(f_\alpha(A))$, then $(\rho_k(c\,\cdot)f_\alpha)(A)x \in \mathsf{R}(A)$ (by the definition of $\mathsf{D}(f_\alpha(A))$, since $\rho_k(c\,\cdot)$ is a regulariser for (f_α, A)), and (15.6) implies that $(\rho_k f_\alpha)(cA)x \in \mathsf{R}(A) = \mathsf{R}(cA)$. But this implies that $x \in \mathsf{D}(f_\alpha(cA))$ (by the definition of $\mathsf{D}(f_\alpha(cA))$, since ρ_k is a regulariser for (f_α, cA)). This gives the inclusion $\mathsf{D}(f_\alpha(A)) \subseteq \mathsf{D}(f_\alpha(cA))$. The same argument in reverse direction gives the inclusion $\mathsf{D}(f_\alpha(cA)) \subseteq \mathsf{D}(f_\alpha(A))$. Moreover, for any x in this common domain,

$$\begin{aligned} f_\alpha(cA)x &= (\rho_k(cA))^{-1}(\rho_k f_\alpha)(cA)x, \\ c^\alpha f_\alpha(A)x &= c^\alpha(\rho_k(c\,\cdot)(A))^{-1}(\rho_k(c\,\cdot)f_\alpha)(A)x = (\rho_k(c\,\cdot)(A))^{-1}(\rho_k f_\alpha)(cA)x,\end{aligned}$$

the last identity being a consequence of (15.6). Since the right-hand sides are obviously equal, this gives the result. □

Theorem 15.2.7. *Let A be a sectorial operator on a Banach space X. If $0 < \alpha < \pi/\omega(A)$, then A^α is sectorial, we have*

$$\omega(A^\alpha) = \alpha\omega(A),$$

and for all $\beta \in \mathbb{C}$ we have

$$(A^\alpha)^\beta = A^{\alpha\beta} \quad \textit{with equality of domains.}$$

If A is R-sectorial and $0 < |\alpha| < \pi/\omega_R(A)$, then A^α is R-sectorial and

$$\omega_R(A^\alpha) = \alpha\omega_R(A).$$

Proof. The proof proceeds in a number of steps.

Step 1 – First consider an arbitrary $\alpha > 0$. In this step we will prove that for all $\mu \notin \overline{\Sigma_{\alpha\omega(A)}}$ we have $\mu \in \varrho(A^\alpha)$ and

$$\mu R(\mu, A^\alpha) = -|\mu|^{1/\alpha} R(-|\mu|^{1/\alpha}, A) + \psi_\tau(|\mu|^{-1/\alpha} A),$$

where $\tau = \arg \mu$ and

$$\psi_\tau(z) = \frac{e^{i\tau} z + z^\alpha}{(e^{i\tau} - z^\alpha)(1+z)}.$$

Note that $\psi_\tau \in H^1(\Sigma_\sigma)$ for all $\sigma < |\tau|/\alpha$.

A straightforward calculation shows

$$\frac{\mu}{\mu - z^\alpha} - \frac{|\mu|^{1/\alpha}}{|\mu|^{1/\alpha} + z} = \frac{\mu z + |\mu|^{1/\alpha} z^\alpha}{(\mu - z^\alpha)(|\mu|^{1/\alpha} + z)} = \psi_\tau(|\mu|^{-1/\alpha} z).$$

Hence

$$\frac{1}{\mu - z^\alpha} = \frac{1}{\mu}\Big(\frac{|\mu|^{1/\alpha}}{|\mu|^{1/\alpha} + z} + \psi_\tau(|\mu|^{-1/\alpha} z)\Big).$$

Proposition 15.1.12 implies that $(\frac{1}{\mu - (\cdot)^\alpha})(A)$ is indeed the inverse of $(\mu - (\cdot)^\alpha)(A) = \mu - A^\alpha$. Thus $\mu \in \varrho(A^\alpha)$ and

$$\begin{aligned} R(\mu, A^\alpha) &= \frac{1}{\mu}\Big(\frac{|\mu|^{1/\alpha}}{|\mu|^{1/\alpha} + z} + \psi_\tau(|\mu|^{-1/\alpha} z)\Big)(A) \\ &= \frac{1}{\mu}\Big(- |\mu|^{1/\alpha} R(-|\mu|^{1/\alpha}, A) + \psi_\tau(|\mu|^{-1/\alpha} A)\Big) \end{aligned} \tag{15.7}$$

using that if $\lambda \in \complement\overline{\Sigma_\sigma}$, then $\frac{1}{\lambda - \cdot}(A)x = R(\lambda, A)x$, and observing that $\psi_\tau(|\mu|^{-1/\alpha} A)$ is well defined and bounded by the Dunford calculus of A.

Step 2 – Now let $0 < \alpha < \pi/\omega(A)$. We will prove that the operator A^α is sectorial, with $\omega(A^\alpha) \leqslant \alpha\omega(A)$.

By Step 1, for $\tau > \alpha\omega(A)$ we have $\mu \in \varrho(A)$ if $|\arg \mu| \geqslant \tau$. Furthermore, for $\sigma \in (\omega(A), \tau/\alpha)$ have

$$\begin{aligned} \psi_\tau(|\mu|^{-1/\alpha} A) &= \frac{1}{2\pi i} \int_{\partial\Sigma_\sigma} \psi_\tau(|\mu|^{-1/\alpha} z) \frac{\mathrm{d}z}{z} \\ &= \frac{1}{2\pi i} \int_{\partial\Sigma_\sigma} \psi_\tau(z) \frac{\mathrm{d}z}{z}. \end{aligned}$$

Hence we may estimate

$$\|\psi_\tau(|\mu|^{-1/\alpha} A)\| \leqslant \frac{M_{\sigma,A}}{2\pi} \int_{\partial\Sigma_\sigma} |\psi_\tau(z)| \frac{|\mathrm{d}z|}{|z|}.$$

Therefore by (15.7) the sectoriality of A implies the sectoriality of A^α with $\omega(A^\alpha) \leqslant \alpha\omega(A)$.

Step 3 – Having proved that A^α is sectorial, the identity $(A^\alpha)^\beta = A^{\alpha\beta}$ follows from the composition rule of Theorem 15.1.15.

Since $\pi/\omega(A) > 1$ we have $0 < 1/\alpha < \pi/(\alpha\omega(A)) \leqslant \pi/\omega(A^\alpha)$. Hence we may apply the inequality of the angles of sectoriality of Step 2 to A^α to obtain $\omega(A) = \omega((A^\alpha)^{1/\alpha}) \leqslant (1/\alpha)\omega(A^\alpha)$, the equality $A = (A^\alpha)^{1/\alpha}$ being a consequence what we just proved. In combination with Step 2, this proves the equality $\omega(A^\alpha) = \alpha\omega(A)$.

Step 4 – Using Proposition 10.3.2, the final assertion is proved in the same way. □

The next theorem shows that $\alpha \mapsto \|A^\alpha x\|$ satisfies a useful log–convexity property.

Theorem 15.2.8 (Interpolation estimate). *Let A be a sectorial operator on a Banach space X. Let $\alpha, \beta, \gamma \in \mathbb{C}$ satisfy*

$$0 < \Re\alpha < \Re\gamma < \Re\beta \quad \text{or} \quad 0 = \alpha < \Re\gamma < \Re\beta$$

and let $\theta \in (0,1)$ be such that $\Re\gamma = (1-\theta)\Re\alpha + \theta\Re\beta$. Then

$$\mathsf{D}(A^\alpha) \cap \mathsf{D}(A^\beta) \subseteq \mathsf{D}(A^\gamma),$$

and for all $x \in \mathsf{D}(A^\alpha) \cap \mathsf{D}(A^\beta)$ and $\omega(A) < \sigma < \pi$ we have

$$\|A^\gamma x\| \leqslant \frac{C}{\theta(1-\theta)} \|A^\alpha x\|^{1-\theta} \|A^\beta x\|^\theta,$$

where C is a constant depending only on $\Re\beta - \Re\alpha$, σ, and A.

Proof. Let m be the smallest integer strictly greater than $\Re\beta - \Re\alpha$. We will use the auxiliary function $\psi(z) = cz^m(1+z)^{-2m}$, where c is chosen so that $\int_0^\infty \psi(s)\,\frac{\mathrm{d}s}{s} = 1$. Then the functions

$$g(z) := \int_0^1 \psi(sz)\,\frac{\mathrm{d}s}{s} \quad \text{and} \qquad h(z) := \int_1^\infty \psi(sz)\,\frac{\mathrm{d}s}{s}$$

are well defined for all $z \in \mathbb{C}$ and satisfy

$$g(z) + h(z) = \int_0^\infty \psi(sz)\,\frac{\mathrm{d}s}{s} = \int_0^\infty \psi(s)\,\frac{\mathrm{d}s}{s} = 1.$$

We claim that $g, h \in E(\Sigma_\sigma)$. Indeed, we have

$$|g(z)| \leqslant \int_0^1 |\psi(sz)|\,\frac{\mathrm{d}s}{s} \leqslant C_{\sigma,m}|z|^m \int_0^1 s^m\,\frac{\mathrm{d}s}{s} = \frac{|z|^m}{m},$$
$$|h(z)| \leqslant \int_1^\infty |\psi(sz)|\,\frac{\mathrm{d}s}{s} \leqslant C_{\sigma,m}|z|^{-m} \int_0^1 s^{-m}\,\frac{\mathrm{d}s}{s} = \frac{|z|^{-m}}{m}.$$

It follows that g an h have integrable limits 0 at 0 and ∞ in the sense of Lemma 15.1.2, respectively. From $g = 1 - h$ and $h = 1 - g$ we see that g an h have integrable limits 1 at ∞ and 0, respectively. Therefore Lemma 15.1.2 implies the claim.

For all $t > 0$, it follows from the claim that

$$g(tA) + h(tA) = I \tag{15.8}$$

in the primary calculus of the sectorial operator tA.

Now let $x \in \mathsf{D}(A^\alpha) \cap \mathsf{D}(A^\beta)$. Then $x \in \mathsf{D}(A^\gamma)$ and (15.8) implies

$$A^\gamma x = g(tA)A^\gamma x + h(tA)A^\gamma x. \tag{15.9}$$

The functions $\widetilde{g}(z) = z^{\gamma-\beta} g(z)$ and $\widetilde{h}(z) = z^{\gamma-\alpha} h(z)$ belong to $E(\Sigma_\sigma)$; this follows from the choice of m and redoing the above computation for $\widetilde{g}$ and $\widetilde{h}$. We have

$$g(tA)A^\gamma x = \widetilde{g}(tA)(tA)^{\beta-\gamma} A^\gamma x = t^{\beta-\gamma}\widetilde{g}(tA)A^\beta x.$$

Here, the first identity can be justified by viewing $\widetilde{g}(t\,\cdot)$ as a regulariser for $(z^{\beta-\gamma}, tA)$ and noting that $A^\gamma x \in \mathsf{D}(A^{\beta-\gamma})$; the second identity follows by first applying Proposition 15.2.6 and then Theorem 15.2.5. Similarly we have

$$h(tA)A^\gamma x = t^{\alpha-\gamma}\widetilde{h}(tA)A^\alpha x.$$

From (15.9) it now follows that

$$A^\gamma x = t^{\beta-\gamma}\widetilde{g}(tA)A^\beta x + t^{\alpha-\gamma}\widetilde{h}(tA)A^\alpha x.$$

Therefore,

$$\begin{aligned}\|A^\gamma x\| &\leqslant t^{\Re\beta-\Re\gamma}\|\widetilde{g}(tA)\|\|A^\beta x\| + t^{\Re\alpha-\Re\gamma}\|\widetilde{h}(tA)\|\|A^\alpha x\| \\ &\leqslant C\big(t^{\Re\beta-\Re\gamma}\|A^\beta x\| + t^{\Re\alpha-\Re\gamma}\|\|A^\alpha x\|\big),\end{aligned}$$

where the constant C only depends on $\Re\beta - \Re\alpha$, σ, and A; we used that from the definition of the primary calculus for it follows that $\sup_{t>0} \|f(tA)\| \leqslant C < \infty$ for $f \in \{\widetilde{g}, \widetilde{h}\}$, using by (10.9) and the sectoriality of A.

Optimising the choice of $t > 0$, we arrive at the estimate

$$\|A^\gamma x\| \leqslant C\bigg[\Big(\frac{\theta}{1-\theta}\Big)^{1-\theta} + \Big(\frac{1-\theta}{\theta}\Big)^{\theta}\bigg]\|A^\alpha x\|^{1-\theta}\|A^\beta x\|^\theta.$$

Since the term in the square brackets is bounded above by $1/(\theta(1-\theta))$, this gives the second estimate. □

Remark 15.2.9. It is tempting to believe that

$$g(A)x = \int_0^1 \psi(sA)x\,\frac{\mathrm{d}s}{s} \quad \text{and} \qquad h(A)x = \int_1^\infty \psi(sA)x\,\frac{\mathrm{d}s}{s},$$

but these integrals may fail to converges at 0 (the first) and ∞ (the second). Calderón's reproducing formula (Proposition 10.2.5) guarantees their convergence (as improper integrals) for elements $x \in \overline{\mathsf{D}(A) \cap \mathsf{R}(A)}$ if $z \mapsto \psi(z)$ belongs to $H^1(\Sigma_\sigma)$, and for $x \in \mathsf{D}(A) \cap \mathsf{R}(A)$ if $z \mapsto \psi(z) \log z$ belongs to $\in H^1(\Sigma_\sigma)$. The above proof does not depend on these matters; all we needed there were bounds on the operators $g(A)$ and $h(A)$ that follow directly from the definitions of these operators through the extended Dunford calculus.

Corollary 15.2.10. *Let A be a sectorial operator on a Banach space X with $0 \in \varrho(A)$. Then for all $\Re\alpha > 0$ the operator $A^{-\alpha}$ is bounded. Moreover, for $0 < \Re\alpha < n$ we have*

$$\|A^{-\alpha}\| \leqslant \frac{CM_{\sigma,A}}{\frac{\Re\alpha}{n}(1-\frac{\Re\alpha}{n})}\|A^{-1}\|^{\Re\alpha},$$

where C is a universal constant.

Proof. Let $0 < \Re\alpha < n$. By Theorem 15.2.5 and 15.2.8, applied with $\theta = 1 - \Re\alpha/n$, for all $x \in X$ we have

$$\begin{aligned}\|A^{-\alpha}x\| = \|A^{n-\alpha}(A^{-n}x)\| &\leqslant \frac{CM_{\sigma,A}}{\frac{\Re\alpha}{n}(1-\frac{\Re\alpha}{n})}\|A^{-n}x\|^{\Re\alpha/n}\|x\|^{1-\Re\alpha/n} \\ &\leqslant \frac{CM_{\sigma,A}}{\frac{\Re\alpha}{n}(1-\frac{\Re\alpha}{n})}\|A^{-1}\|^{\Re\alpha}\|x\|,\end{aligned}$$

where C is a universal constant. It follows that $A^{-\alpha}$ is bounded and satisfies the bound in the statement of the corollary. □

Proposition 15.2.11. *Let A be a sectorial operator on a Banach space X; when considering A^α for $\Re\alpha \leqslant 0$ we assume A to be injective. If A has a bounded H^∞-calculus and $0 < |\alpha| < \pi/\omega_{H^\infty}(A)$, then A^α has a bounded H^∞-calculus and $\omega_{H^\infty}(A^\alpha) = \alpha\,\omega_{H^\infty}(A)$.*

Proof. This follows directly from the identity $f(A)x = g(A^\alpha)x$ for $x \in \mathsf{D}(A) \cap \mathsf{R}(A)$ and $f \in H^\infty(\Sigma_\sigma)$, with $g \in H^\infty(\Sigma_{|\alpha|\sigma})$ given by $f(z) = g(z^\alpha)$. □

If A is sectorial, then $A+\varepsilon$ is sectorial and boundedly invertible. We conclude this section a some useful result that applies in this situation.

Proposition 15.2.12. *Let A be a sectorial operator on a Banach space X. If $\mathsf{D}(A) \cap \mathsf{R}(A)$ is dense in X, then for all $\alpha > 0$ and $\varepsilon > 0$ we have $\mathsf{D}(A^\alpha) = \mathsf{D}((\varepsilon + A)^\alpha)$ with equivalent graph norms.*

Proof. The result is clear for $\alpha = 1, 2, \ldots$. Next let $\alpha \in (0,1)$. The functions

$$f(z) := \frac{(\varepsilon + z)^\alpha}{\varepsilon + z^\alpha} - 1, \quad g(z) = \frac{\varepsilon + z^\alpha}{(\varepsilon + z)^\alpha} - 1$$

belong to $H^1(\Sigma_\sigma)$ for all $0 < \sigma < \pi$. For $x \in \mathsf{D}(A^k) \cap \mathsf{R}(A^k)$ with k large enough, Proposition 15.1.12 gives

$$f(A)x = (\varepsilon + A)^\alpha(\varepsilon + A^\alpha)^{-1}x - x, \quad g(A)x = (\varepsilon + A^\alpha)(\varepsilon + A)^{-\alpha})x - x.$$

Since $f(A)$ and $g(A)$ are bounded, these identities imply $\mathsf{D}(A^\alpha) = \mathsf{D}((\varepsilon+A)^\alpha)$. The equivalence of the norms follows from the open mapping theorem.

If $\beta = \alpha + n$ with $n \in \mathbb{N}$ and $\alpha \in (0,1)$ then $\mathsf{D}((\varepsilon + A)^\beta) \subseteq \mathsf{D}((\varepsilon + A)^n)$ by Theorem 15.2.5. Thus we obtain

$$\begin{aligned}
\mathsf{D}((\varepsilon + A)^\beta) &= \mathsf{D}((\varepsilon + A)^n(\varepsilon + A)^\alpha) \\
&= \{x \in \mathsf{D}((\varepsilon + A)^n) : (\varepsilon + A)^\alpha x \in \mathsf{D}((\varepsilon + A)^n)\} \\
&= \{x \in \mathsf{D}(A^n) : (\varepsilon + A)^\alpha x \in \mathsf{D}(A^n)\} \\
&= \{x \in \mathsf{D}(A^n) : A^\alpha x \in \mathsf{D}(A^n)\} \\
&= \mathsf{D}(A^n A^\alpha) = \mathsf{D}(A^\beta).
\end{aligned}$$

Equivalence of norms now follows easily. □

15.2.b Representation formulas

The aim of this section is to prove various integral representations for the fractional powers of sectorial operators.

Theorem 15.2.13 (Balakrishnan). *Let A be a sectorial operator on a Banach space X and let $\omega(A) < \sigma < \pi$. For all $0 < \Re\alpha < 1$ and $x \in \mathsf{D}(A)$ we have*

$$A^\alpha x = \frac{1}{2\pi i}\int_{\partial\Sigma_\sigma} z^{\alpha-1}R(z,A)Ax \,\mathrm{d}z = \frac{\sin\pi\alpha}{\pi}\int_0^\infty t^{\alpha-1}(t+A)^{-1}Ax\,\mathrm{d}t.$$

If in addition A is densely defined and $\omega(A) < \frac{1}{2}\pi$, then for all $x \in \mathsf{D}(A)$ we have

$$A^\alpha x = \frac{1}{\Gamma(1-\alpha)}\int_0^\infty s^{-\alpha}S(t)Ax\,\mathrm{d}t,$$

where $(S(t))_{t\geqslant 0}$ is the bounded analytic C_0-semigroup generated by $-A$.

Note that $\lim_{z\downarrow 0} R(z,A)Ax = 0$ for $x \in \mathsf{D}(A)$ by Proposition 10.1.7, so the first integral is absolutely convergent. By the same reasoning the second integral is absolutely convergent. The absolute convergence of the third integral follows near $t = 0$ from the fact that $x \in \mathsf{D}(A)$, and near $t = \infty$ from the bound $\|S(t)Ax\| \leqslant Ct^{-1}\|x\|$ (see Theorem G.5.3).

Integrating by parts and using with the identity $-\alpha\Gamma(-\alpha) = \Gamma(1-\alpha)$, the third identity in Balakrishnan's theorem may equivalently be presented as

$$A^\alpha x = \frac{1}{\Gamma(-\alpha)} \int_0^\infty t^{-\alpha-1}(S(t)x - x)\,\mathrm{d}t, \quad x \in \mathsf{D}(A).$$

The absolute convergence of this integral follows from the bound $\|S(t)x - x\| = O(t)$ as $t \downarrow 0$ for $x \in \mathsf{D}(A)$.

Proof. For all $\varepsilon > 0$ the function $z \mapsto \frac{z^\alpha}{z+\varepsilon}$ belongs to $H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ and therefore the operator $(\frac{z^\alpha}{z+\varepsilon})(A)$ can be defined by the Dunford calculus and is bounded. Fix $x \in \mathsf{D}(A)$. Then $x \in \mathsf{D}(A^\alpha)$, and therefore by multiplicativity of the extended Dunford calculus (Proposition 15.1.12),

$$A^\alpha x = \Big(\frac{z^\alpha}{z+\varepsilon}\Big)(A)(\varepsilon + A)x.$$

Similarly,

$$\Big(\frac{z^\alpha}{z+\varepsilon}\Big)(A)x = \Big(\frac{z^\alpha}{(z+\varepsilon)(z+1)}\Big)(A)(I + A)x.$$

Combining these identities, we compute

$$\begin{aligned}
A^\alpha x &= \varepsilon\Big(\frac{z^\alpha}{(z+\varepsilon)(z+1)}\Big)(A)(I+A)x + \Big(\frac{z^\alpha}{z+\varepsilon}\Big)(A)Ax \\
&= \frac{\varepsilon}{2\pi i}\int_{\partial\Sigma_\sigma} \frac{z^\alpha}{(z+\varepsilon)(z+1)} R(z,A)(I+A)x\,\mathrm{d}z \\
&\qquad + \frac{1}{2\pi i}\int_{\partial\Sigma_\sigma} \frac{z^\alpha}{z+\varepsilon} R(z,A)Ax\,\mathrm{d}z \\
&= (I) + (II).
\end{aligned}$$

Noting that $z \mapsto z^{\alpha-1}R(z,A)Ax$ is integrable along $\partial\Sigma_\sigma$, the term (I) tends to 0 as $\varepsilon \downarrow 0$ by dominated convergence. Also,

$$(II) = \frac{1}{2\pi i}\int_{\partial\Sigma_\sigma} \frac{z}{z+\varepsilon} z^{\alpha-1}R(z,A)Ax\,\mathrm{d}z \to \frac{1}{2\pi i}\int_{\partial\Sigma_\sigma} z^{\alpha-1}R(z,A)Ax\,\mathrm{d}z$$

as $\varepsilon \downarrow 0$ by dominated convergence. This proves the first identity.

Turning to the second identity, write $\partial\Sigma_\sigma = \Gamma_\sigma \cup \Gamma_{-\sigma}$ where $\Gamma_{\pm\sigma} = \{re^{\pm i\sigma} \in \mathbb{C} : r > 0\}$. It follows from Cauchy's theorem that

$$\begin{aligned}
A^\alpha x &= \frac{1}{2\pi i}\int_{\partial\Sigma_\sigma} z^{\alpha-1}R(z,A)Ax\,\mathrm{d}z \\
&= \frac{1}{2\pi i}\int_{\Gamma_\sigma} z^{\alpha-1}R(z,A)Ax\,\mathrm{d}z \\
&\qquad + \frac{1}{2\pi i}\int_{\Gamma_{-\sigma}} z^{\alpha-1}R(z,A)Ax\,\mathrm{d}z \\
&= -\frac{1}{2\pi i}\int_0^\infty (re^{i\sigma})^{\alpha-1}R(re^{i\sigma},A)Ax\, e^{i\sigma}\,\mathrm{d}r
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2\pi i}\int_0^\infty (re^{-i\sigma})^{\alpha-1}R(re^{-i\sigma},A)Ax\,e^{-i\sigma}\,\mathrm{d}r\\
&\to \frac{1}{2\pi i}\int_0^\infty r^{\alpha-1}(e^{-i\pi(\alpha-1)}-e^{i\pi(\alpha-1)})R(-r,A)Ax\,\mathrm{d}r \quad (\text{as } \sigma\to\pi)\\
&= \frac{\sin\alpha\pi}{\pi}\int_0^\infty r^{\alpha-1}(r+A)^{-1}Ax\,\mathrm{d}r.
\end{aligned}
$$

The minus sign in the third identity comes from the fact that Γ_σ is downwards oriented. The convergence is a consequence of the dominated convergence theorem.

To prove the third formula we use the identity just proved together with the Laplace transform representation of the resolvent (Proposition G.4.1) to get

$$
\begin{aligned}
A^\alpha x &= \frac{\sin\pi\alpha}{\pi}\int_0^\infty t^{\alpha-1}\int_0^\infty e^{-ts}S(s)Ax\,\mathrm{d}s\,\mathrm{d}t\\
&= \frac{\sin\pi\alpha}{\pi}\int_0^\infty \Big(\int_0^\infty t^{\alpha-1}e^{-ts}\,\mathrm{d}t\Big)S(s)Ax\,\mathrm{d}s\\
&= \frac{1}{\Gamma(1-\alpha)}\int_0^\infty s^{-\alpha}S(s)Ax\,\mathrm{d}s,
\end{aligned}
$$

where we used the identity $\frac{\sin\pi\alpha}{\pi} = \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}$. □

From this theorem it is rather easy to re-deduce a special case of Theorem 15.2.8 as follows. Let $0<\alpha<1$. Let $M\geqslant 0$ be such that $\|(t+A)^{-1}\|\leqslant M/t$ for all $t>0$. By Theorem 15.2.13, for all $x\in\mathsf{D}(A)$ we have

$$
\begin{aligned}
\|A^\alpha x\| &\leqslant \Big|\frac{\sin\pi\alpha}{\pi}\Big|\int_0^\infty t^{\alpha-1}\|(t+A)^{-1}Ax\|\,\mathrm{d}t\\
&\leqslant \Big|\frac{\sin\pi\alpha}{\pi}\Big|\int_0^\rho t^{\alpha-1}\|(t+A)^{-1}A\|\|x\|\,\mathrm{d}t\\
&\qquad + \Big|\frac{\sin\pi\alpha}{\pi}\Big|\int_\rho^\infty t^{\alpha-1}\|(t+A)^{-1}\|\|Ax\|\,\mathrm{d}t\\
&\leqslant \Big|\frac{\sin\pi\alpha}{\pi\alpha}\Big|(1+M)\rho^\alpha\|x\| + \Big|\frac{\sin\pi\alpha}{\pi(1-\alpha)}\Big|M\rho^{\alpha-1}\|Ax\|
\end{aligned}
$$

with absolute convergence of all integrals. Up to this point we have assumed that $x\in\mathsf{D}(A)$. The estimate extends to general $x\in\mathsf{D}(A^\alpha)$ by approximation as in the proof of that theorem. The estimate of Theorem 15.2.8 is obtained by optimising over ρ as in the proof of the theorem.

Corollary 15.2.14. *Let A be a sectorial operator on a Banach space X and let $\omega(A)<\sigma<\pi$. Let $0<\alpha<1$ and $\lambda\in\complement\overline{\Sigma_\sigma}$.*

(1) *The operator $A^\alpha R(\lambda,A)$ is bounded and*

$$\|A^\alpha R(\lambda, A)\| \leqslant C_\alpha M_{\sigma,A}(M_{\sigma,A}+1)|\lambda|^{\alpha-1},$$

where $C_\alpha = \frac{\sin(\pi\alpha)}{\pi\alpha}\frac{1}{1-\alpha}$.

(2) *If, in addition, A is densely defined and $\omega(A) < \frac12\pi$, and $(S(t))_{t\geqslant 0}$ denotes the bounded analytic C_0-semigroup generated by $-A$, then for all $t>0$ the operator $A^\alpha S(t)$ is bounded and*

$$\|A^\alpha S(t)\| \leqslant C_\alpha M_A t^{-\alpha},$$

where $C_\alpha = \frac{1}{\Gamma(1-\alpha)}\int_0^\infty \tau^{-\alpha}(1+\tau)^{-1}\|x\|\,\mathrm{d}\tau$ *and* $M_A = \sup_{t>0} t\|AS(t)\|$.

From Theorem G.5.3 we recall that $\sup_{t>0} t\|AS(t)\| < \infty$.

Proof. For the first assertion, fix $\lambda \in \complement\overline{\Sigma_\sigma}$. The boundedness of $A^\alpha R(\lambda, A)$ is evident from the inclusion $\mathsf{D}(A) \subseteq \mathsf{D}(A^\alpha)$. For all $x \in \mathsf{D}(A)$, by Theorem 15.2.13 we have

$$A^\alpha R(\lambda, A)x = \frac{\sin(\pi\alpha)}{\pi}\int_0^\infty t^{\alpha-1}(t+A)^{-1}R(\lambda,A)Ax\,\mathrm{d}t.$$

We split the integral on the right into two parts and estimate them separately. First, writing $A = (A+t)-t$,

$$\begin{aligned}\Big\|\int_0^{|\lambda|} t^{\alpha-1}(t+A)^{-1}R(\lambda,A)Ax\,\mathrm{d}t\Big\| &\leqslant \int_0^{|\lambda|} t^{\alpha-1}\|[I-t(t+A)^{-1}]R(\lambda,A)x\|\,\mathrm{d}t\\ &\leqslant |\lambda|^{-1}\int_0^{|\lambda|} t^{\alpha-1}(1+M)\|\lambda R(\lambda,A)x\|\,\mathrm{d}t\\ &\leqslant \frac{M(M+1)}{\alpha}|\lambda|^{\alpha-1}\|x\|.\end{aligned}$$

Similarly, but now writing $A = (A-\lambda)+\lambda$,

$$\begin{aligned}\Big\|\int_{|\lambda|}^\infty t^{\alpha-1}(t+A)^{-1}R(\lambda,A)Ax\,\mathrm{d}t\Big\| &\leqslant (1+M)\|x\|\int_{|\lambda|}^\infty t^{\alpha-2}\|t(t+A)^{-1}\|\,\mathrm{d}t\\ &\leqslant \frac{M(M+1)}{1-\alpha}|\lambda|^{\alpha-1}\|x\|.\end{aligned}$$

Turning to the second assertion, by analyticity the operators $S(t)$ map X into $\mathsf{D}(A)$ and $\sup_{t>0} t\|AS(t)\| < \infty$. The boundedness of the operators $A^\alpha S(t)$ follows from the boundedness of $AS(t)$ and the inclusion $\mathsf{D}(A) \subseteq \mathsf{D}(A^\alpha)$. To prove the estimate, note that for all $x \in X$ we have

$$A^\alpha S(t)x = \frac{1}{\Gamma(1-\alpha)}\int_0^\infty s^{-\alpha}AS(t+s)x\,\mathrm{d}s,$$

so, for $t>0$,

$$\begin{aligned}\|A^\alpha S(t)x\| &\leqslant \frac{C}{\Gamma(1-\alpha)}\int_0^\infty s^{-\alpha}(t+s)^{-1}\|x\|\,\mathrm{d}s\\ &= \frac{Ct^{-\alpha}}{\Gamma(1-\alpha)}\int_0^\infty \tau^{-\alpha}(1+\tau)^{-1}\|x\|\,\mathrm{d}\tau.\end{aligned}$$

□

As a corollary to Theorem 15.2.13 we have the following representation formula for the negative fractional powers of A.

Corollary 15.2.15. *Let A be an injective sectorial operator on a Banach space X and let $\omega(A) < \sigma < \pi$. For all $0 < \Re\alpha < 1$ and $x \in \mathsf{R}(A)$ we have*

$$A^{-\alpha}x = \frac{1}{2\pi i}\int_{\partial\Sigma_\sigma} z^{-\alpha}R(z,A)x\,\mathrm{d}z = \frac{\sin\pi\alpha}{\pi}\int_0^\infty t^{-\alpha}(t+A)^{-1}x\,\mathrm{d}t.$$

If, in addition, A is densely defined and $\omega(A) < \frac{1}{2}\pi$, and if $(S(t))_{t\geqslant 0}$ denotes the bounded analytic C_0-semigroup generated by $-A$, then for all $x \in \mathsf{R}(A)$ we have

$$A^{-\alpha}x = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{-\alpha}S(t)x\,\mathrm{d}t.$$

Note that if $x = Ay$ with $y \in \mathsf{D}(A)$, then $R(z,A)x = -y + zR(z,A)y$, so the first integral is absolutely convergent. In the same way it is checked that the second integral is absolutely convergent. From $\|S(t)x\| = \|AS(t)y\| = O(1/t)$ as $t \to \infty$ (by Theorem G.5.3) we see that the third integral is absolutely convergent.

Proof. Writing $x = Ay$ with $y \in \mathsf{D}(A)$ we have $A^{-\alpha}x = A^{1-\alpha}y$, and Theorem 15.2.13 gives

$$A^{-\alpha}x = A^{1-\alpha}y = \frac{1}{2\pi i}\int_{\partial\Sigma_\sigma} z^{-\alpha}R(z,A)Ay\,\mathrm{d}z = \frac{1}{2\pi i}\int_{\partial\Sigma_\sigma} z^{-\alpha}R(z,A)x\,\mathrm{d}z.$$

The second identity is proved in the same way. The third follows from the second by following the lines of the proof of Theorem 15.2.13. □

When A boundedly invertible, the identities in the corollary hold for arbitrary $x \in X$. If in addition A is densely defined, the result extends to arbitrary $\Re\alpha > 0$ as follows:

Theorem 15.2.16. *Let A be a densely defined sectorial operator on a Banach space X with $0 \in \varrho(A)$, and let $\omega(A) < \sigma < \pi$. Then for all $\Re\alpha > 0$ we have*

$$A^{-\alpha}x = \frac{1}{2\pi i}\int_{\partial(\Sigma_\sigma\setminus B_\varepsilon)} z^{-\alpha}R(z,A)x\,\mathrm{d}z, \qquad x \in X,$$

with $B_\varepsilon := \{z \in \mathbb{C} : |z| < \varepsilon\}$, where $\varepsilon > 0$ is so small that $B_\varepsilon \subseteq \varrho(A)$.

Proof. First let $x \in \mathsf{D}(A^k)$ with $k > \Re\alpha$ and set $y = \zeta(A)^{-k}x = (I + A)^{2k}A^{-k}x$, where $\zeta(z) = z/(z+1)^2$. The integral

$$T_\alpha x := \frac{1}{2\pi i}\int_{\partial(\Sigma_\sigma\setminus B_\varepsilon)} z^{-\alpha}R(z,A)x\,\mathrm{d}z$$

is absolutely convergent and defines a bounded operator T_α. We may now repeat the proof of the multiplicativity of the Dunford calculus (Theorem 10.2.2) to obtain, with $\omega(A) < \nu < \sigma$,

$$\begin{aligned}T_\alpha x = (T_\alpha\circ\zeta^k(A))y &= T_\alpha\circ\frac{1}{2\pi i}\int_{\partial(\Sigma_\nu\setminus B_{\varepsilon/2})}\zeta(z)^kR(z,A)y\,\mathrm{d}z\\ &= \frac{1}{2\pi i}\int_{\partial(\Sigma_\nu\setminus B_{\varepsilon/2})} z^{-\alpha}\zeta(z)^kR(z,A)y\,\mathrm{d}z\\ &= \frac{1}{2\pi i}\int_{\partial\Sigma_\nu} z^{-\alpha}\zeta(z)^kR(z,A)y\,\mathrm{d}z.\end{aligned}$$

In the last step, the assumption $k > |\Re\lambda|$ was used to justify the change of contour by Cauchy's theorem. By the definition of $A^{-\alpha}x$ via the extended Dunford calculus, the right hand side equals $A^{-\alpha}x$. This proves the first identity for $x \in \mathsf{D}(A^k)$. Using the second part of Proposition 15.1.13, the general case follows from it by approximation, noting that T_α is a bounded operator on X. □

Theorem 15.2.17. *Let $-A$ be the generator of a bounded C_0-semigroup $(S(t))_{t\geqslant 0}$ on X. Then A is densely defined and sectorial of angle $\omega(A) \leqslant \frac12\pi$, for all $0 < \alpha < 1$ the operator A^α is densely defined and sectorial of angle $\omega(A^\alpha) \leqslant \frac12\pi\alpha$, and the bounded analytic C_0-semigroup generated by $-A^\alpha$ is given by*

$$S_\alpha(t)x = \int_0^\infty f_{\alpha,t}(s)S(s)x\,\mathrm{d}s, \qquad t > 0,\ x \in X,$$

where, for $t > 0$,

$$f_{\alpha,t}(s) := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{sz-tz^\alpha}\,\mathrm{d}z, \qquad s > 0,$$

is a non-negative function which is independent of $c > 0$ and satisfies

$$\int_0^\infty f_{\alpha,t}(s)\,\mathrm{d}s = 1.$$

Proof. By generalities from semigroup theorem (see Section G.2), the assumptions imply that A is densely defined and sectorial with $\omega(A) \leqslant \frac12\pi$. By Proposition 15.2.7, A^α is densely defined and sectorial of angle $\frac12\pi\alpha$ and consequently $-A^\alpha$ generates a bounded analytic C_0-semigroup by Theorem G.5.2.

By Example 15.1.6 we furthermore have $S_\alpha(t) = e_t(A^\alpha)$, where $e_t(z) = e^{-tz}$. Hence by the composition rule of Theorem 15.1.15 we have

$$S_\alpha(t) = g_{\alpha,t}(A),$$

where $g_{\alpha,t}(z) = e^{-tz^\alpha}$.

Let $\frac{1}{2}\pi < \nu < \sigma < \min\{\frac{1}{2}\pi/\alpha, \pi\}$. By the Phillips calculus (Proposition 10.7.2(2)),

$$S_\alpha(t)(A)x = \int_0^\infty f_{\alpha,t}(s)S(s)x\,\mathrm{d}s, \qquad x \in X,$$

where $f_{\alpha,t} \in L^1(\mathbb{R}_+)$ is given (with $B_\varepsilon = \{z \in \mathbb{C} : |z| = \varepsilon\}$) by

$$\begin{aligned} f_{\alpha,t}(s) &= -\frac{1}{2\pi i}\int_{\partial(\Sigma_\nu\setminus B_\varepsilon)} e^{sz-tz^\alpha}\,\mathrm{d}z \\ &= -\frac{1}{2\pi i}\int_{\partial\Sigma_\nu} e^{sz-tz^\alpha}\,\mathrm{d}z = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{sz-tz^\alpha}\,\mathrm{d}z \end{aligned}$$

for $c > 0$. The second and third identity follow from Cauchy's formula, the use of which is justified by noting that for $z = re^{i\sigma u}$ with $u \geqslant 0$ we have

$$\begin{aligned} |e^{sz-tz^\alpha}| &= \exp(sr\cos\sigma - t\Re e^{\alpha(\ln r + i\sigma)}) \\ &= \exp(sr\cos\sigma - tr^\alpha\cos(\alpha\sigma)), \end{aligned}$$

from which it follows that $z \mapsto e^{sz-tz^\alpha}$ is integrable along $\partial\Sigma_\nu$. In its stated form, Proposition 10.7.2(2) requires $g_{\alpha,t} = e^{-tz^\alpha}$ to be in $H^1(\Sigma_\sigma)$, which is not the case. The reader may check, however, that the proof still works in the present situation if we replace integration over $\partial\Sigma_\nu$ by integration over $\partial(\Sigma_\nu \setminus B_\varepsilon)$. For $\lambda > 0$ we have

$$\begin{aligned} \int_0^\infty e^{-\lambda s} f_{\alpha,t}(s)\,\mathrm{d}s &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_0^\infty e^{-\lambda s}e^{sz-tz^\alpha}\,\mathrm{d}s\,\mathrm{d}z \\ &= -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{-tz^\alpha}}{z-\lambda}\,\mathrm{d}z = e^{-t\lambda^\alpha}. \end{aligned} \tag{15.10}$$

Using the non-negativity of $f_{\alpha,t}$, upon passing to the limit $\lambda \downarrow 0$ gives $\int_0^\infty f_{\alpha,t}(s)\,\mathrm{d}s = 1$.

Finally, the fact that $f_{\alpha,t}$ is non-negative follows from (15.10), the fact that $\lambda \mapsto e^{-t\lambda^\alpha}$ is completely monotone and the Post–Widder real inversion theorem for the Laplace transform. We refer the reader to the Notes for further details. □

We finish with two examples.

Example 15.2.18 (Fractional derivatives). For $1 < p < \infty$, the operator $A = \mathrm{d}/\mathrm{d}t$ with domain $\mathsf{D}(A) = \{f \in W^{1,p}(0,T;X) : f(0) = 0\}$ is sectorial on $L^p(0,T;X)$ of angle $\frac{1}{2}\pi$ and for all $\Re\alpha > 0$ and $f \in L^p(0,T;X)$ we have

$$A^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-y)^{\alpha-1} f(y)\,\mathrm{d}y \quad \text{for almost all } x \in \mathbb{R}.$$

The operators $A^{-\alpha}$ are called the (Liouville) *fractional derivatives*. In particular,

$$A^{-1} f(x) = \int_0^x f(y)\,\mathrm{d}y$$

The operator $V := A^{-1}$ is called the *Volterra operator*. These formulas are special cases of Theorem 15.2.16 once we note that $-A$ is the generator of the C_0-semigroup on $L^p(0,T;X)$ given by

$$S(t)f(s) = \begin{cases} f(s-t), & s \in [0,T],\ s > t, \\ 0, & \text{otherwise.} \end{cases}$$

To see that the generator of this semigroup is indeed $-A$, let us denote the generator by B for the moment. It is clear that $Y := \{f \in C^1([0,T];X) : f(0) = 0\}$ is contained in $\mathsf{D}(B)$ and $Bf = -f' = -Af$ for all $f \in Y$. Since Y is also invariant under the semigroup, Y is dense in $\mathsf{D}(B)$ by Lemma G.2.4. But A is a closed operator and Y is also dense in $\mathsf{D}(A)$, and therefore $B = -A$ with equal domains.

Example 15.2.19 (Poisson semigroup). Let A be the Laplace operator on $L^p(\mathbb{R}^d;X)$, where $1 \leqslant p < \infty$ is fixed and X is a Banach space. This operator has been introduced in Section 5.5 by declaring

$$\begin{aligned} \mathsf{D}(A) &:= H^{2,p}(\mathbb{R}^d;X), \\ Af &:= \Delta f, \quad f \in \mathsf{D}(A), \end{aligned}$$

where $H^{2,p}(\mathbb{R}^d;X)$ is the Banach space of all $f \in L^p(\mathbb{R}^d;X)$ admitting a weak Laplacian Δf in $L^p(\mathbb{R}^d;X)$ (see (5.44)). As was noted in Lemma 5.5.5, $C_c^\infty(\mathbb{R}^d;X)$ is dense in $\mathsf{D}(A)$, and consequently A can be equivalently defined as the closure of the operator $f \mapsto \Delta f$ acting in $C_c^\infty(\mathbb{R}^d;X)$, where Δf is now defined in terms of the classical second order derivatives of f. For UMD spaces X and exponents $1 < p < \infty$, Proposition 5.5.4 shows that

$$H^{2,p}(\mathbb{R}^d;X) = W^{2,p}(\mathbb{R}^d;X),$$

and Theorem 5.6.11 establishes a Fourier analytic characterisation of these spaces as the Banach space of all tempered distributions $u \in \mathscr{S}'(\mathbb{R}^d;X)$ such that the tempered distribution

$$((1+4\pi^2|\cdot|^2)\widehat{u})\check{}$$

belongs to $L^p(\mathbb{R}^d;X)$.

Let us now return to the general situation where $1 \leqslant p < \infty$ and X is a general Banach space. From this point on, we will simply write Δ for the

Laplace operator in $L^p(\mathbb{R}^d;X)$. As was shown in Example G.5.6, $-\Delta$ is the generator of a C_0-semigroup of contractions $(H(t))_{t\geqslant 0}$ on $L^p(\mathbb{R}^d;X)$, the *heat semigroup*, given by $H(0)=I$ and

$$H(t)f := k_t * f, \quad t>0,$$

where $k_t(x) = (4\pi t)^{-d/2}e^{-|x|^2/(4t)}$ is the heat kernel. It was shown in the same example that this semigroup extends analytically to $\{z\in\mathbb{C} : \Re z>0\}$ by the formula

$$H(z)f = k_z * f, \quad \Re z>0,$$

and that this extension is uniformly bounded and strongly continuous on every sector Σ_ω with $0<\omega<\frac12\pi$. As a consequence, $-\Delta$ is a densely defined sectorial operator of angle $\omega(\Delta)=0$.

By Theorem 15.2.17, the operator $(-\Delta)^{1/2}$ is densely defined and sectorial of angle 0 and generates a bounded analytic C_0-semigroup $(P(z))_{z\in\Sigma_\omega}$ for every $0<\omega<\frac12\pi$ on $L^p(\mathbb{R}^d;X)$, the so-called *Poisson semigroup*. By Theorem 15.1.7, in the primary calculus of $(-\Delta)^{1/2}$ this semigroup is given by

$$P(z)f = \exp(-z\Delta^{1/2}), \quad z\in\Sigma_\omega,\ f\in L^p(\mathbb{R}^d;X).$$

An explicit representation is obtained from Theorem 15.2.17, from which it follows that

$$P(t)f = \int_0^\infty k_t(s)H(s)x\,\mathrm{d}s, \qquad t>0,\ f\in L^p(\mathbb{R}^d;X),$$

where, for $t>0$,

$$k_t(s) := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{sz-tz^{1/2}}\,\mathrm{d}z, \qquad s>0.$$

is a non-negative function which is independent of $c>0$ and satisfies

$$\int_0^\infty f_{\alpha,t}(s)\,\mathrm{d}s = 1.$$

We wish to prove here that

$$P(t)f = p_t * f, \quad t\geqslant 0,$$

where

$$p_t(x) = \frac{\Gamma(\frac12(d+1))}{\pi^{\frac12(d+1)}}\frac{t}{(t^2+|x|^2)^{\frac12(d+1)}}$$

is the *Poisson kernel*. For $d=1$ it takes the simpler form

$$p_t(x) = \frac{1}{\pi}\frac{t}{t^2+x^2}.$$

By Theorem 15.2.5 we have $((-\Delta)^{1/2})^2 f = -\Delta f$ for $f \in \mathsf{D}(\Delta)$ and therefore, by the composition rule of Theorem 15.1.15,

$$\exp(t(-\Delta)^{1/2})f = \phi_t(\Delta)f, \quad f \in \mathsf{D}(\Delta),$$

with $\phi_t(z) = e^{-tz^{1/2}}$. It follows from Proposition 15.1.13 that, for $f \in \mathsf{D}(\Delta)$,

$$P(t)f = \phi_t(\Delta)f = \lim_{n\to\infty} \phi_t(\Delta)\psi_n(\Delta)f,$$

where

$$\psi_n(z) = \frac{n}{n+z}, \quad n \geqslant 1.$$

The remainder of the proof will be devoted to proving the identity

$$\phi_t(\Delta)\phi_n(\Delta)f = p_t * \psi_n(\Delta)f. \tag{15.11}$$

These functions are regularisers for $(\exp(-t\,\cdot), \Delta)$. Once this has been shown the identity

$$P(t)f = p_t * f, \quad f \in L^p(\mathbb{R}^d; X),$$

follows from Proposition 10.1.7 by passing to the limit $n \to \infty$ in (15.11).

Fixing $f \in \mathsf{D}(\Delta) \cap \mathsf{R}(\Delta)$ and $t > 0$. Below we will show that

$$e^{-tz^{1/2}} = \int_0^\infty \frac{te^{-t^2/4s}}{2\pi^{1/2}s^{3/2}} e^{-zs}\,\mathrm{d}s. \tag{15.12}$$

Assuming this identity for the moment, by Fubini's theorem and Example 15.1.6 we have

$$\begin{aligned}
\phi_t(\Delta)\psi_n(\Delta)f &= (\phi_t\psi_n)(\Delta)f \\
&= \frac{1}{2\pi i}\int_{\partial\Sigma_\sigma} e^{-tz^{1/2}}\psi_n(z)R(z,\Delta)f\,\mathrm{d}z \\
&= \frac{1}{2\pi i}\int_{\partial\Sigma_\sigma}\int_0^\infty \frac{te^{-t^2/4s}}{2\pi^{1/2}s^{3/2}} e^{-zs}\psi_n(z)R(z,\Delta)f\,\mathrm{d}s\,\mathrm{d}z \\
&= \int_0^\infty \frac{te^{-t^2/4s}}{2\pi^{1/2}s^{3/2}} \frac{1}{2\pi i}\int_{\partial\Sigma_\sigma} e^{-zs}\psi_n(z)R(z,\Delta)f\,\mathrm{d}z\,\mathrm{d}s \\
&= \int_0^\infty \frac{te^{-t^2/4s}}{2\pi^{1/2}s^{3/2}} \exp(-s\Delta)\psi_n(\Delta)f\,\mathrm{d}s.
\end{aligned} \tag{15.13}$$

On the other hand,

$$\begin{aligned}
p_t(x) &\overset{(*)}{=} \frac{1}{(4\pi)^{\frac12(d+1)}}\frac{t}{(t^2+|x|^2)^{\frac12(d+1)}}\int_0^\infty s^{-\frac12(d+1)}e^{-1/4s}\,\frac{\mathrm{d}s}{s} \\
&= \frac{t}{(4\pi)^{\frac12(d+1)}}\int_0^\infty s^{-\frac12(d+3)}e^{-(t^2+|x|^2)/4s}\,\mathrm{d}s
\end{aligned}$$

$$= \int_0^\infty \frac{te^{-t^2/4s}}{2\pi^{1/2}s^{3/2}} k_t(x)\,\mathrm{d}s,$$

where $k_s(x) = (4\pi s)^{-d/2}e^{-|x|^2/4s}$ denotes the heat kernel associated with Δ and $(*)$ follows from

$$\int_0^\infty s^{-\frac{1}{2}(d+1)}e^{-1/4s}\,\frac{\mathrm{d}s}{s} = 4^{\frac{1}{2}(d+1)}\int_0^\infty u^{\frac{1}{2}(d+1)}e^{-u}\,\frac{\mathrm{d}u}{u} = 4^{\frac{1}{2}(d+1)}\Gamma(\frac{1}{2}(d+1)).$$

Now Fubini's theorem implies

$$\begin{aligned}
&p_t * \psi_n(\Delta)f \\
&= \int_{-\infty}^\infty \int_0^\infty \frac{te^{-t^2/4s}}{(4\pi)^{1/2}s^{3/2}} k_s(\cdot - y)\frac{1}{2\pi i}\int_{\partial\Sigma_\sigma} \psi_n(z)R(z,\Delta)f(y)\,\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}y \\
&= \int_0^\infty \frac{te^{-t^2/4s}}{2\pi^{1/2}s^{3/2}}\frac{1}{2\pi i}\int_{\partial\Sigma_\sigma} \psi_n(z)\int_{-\infty}^\infty k_s(\cdot - y)R(z,\Delta)f(y)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}s \\
&= \int_0^\infty \frac{te^{-t^2/4s}}{2\pi^{1/2}s^{3/2}}\frac{1}{2\pi i}\int_{\partial\Sigma_\sigma} \psi_n(z)\exp(-s\Delta)R(z,\Delta)f\,\mathrm{d}z\,\mathrm{d}s \\
&= \int_0^\infty \frac{te^{-t^2/4s}}{2\pi^{1/2}s^{3/2}}\exp(-s\Delta)\psi_n(\Delta)f\,\mathrm{d}s.
\end{aligned} \tag{15.14}$$

The identity (15.11) is obtained by combining (15.13) and (15.14).

It remains to prove (15.12). First, the substitution $u = c/t$ gives

$$\int_0^\infty e^{-(\frac{c}{t}-t)^2}\,\mathrm{d}t = \int_0^\infty \frac{c}{u^2}e^{-(\frac{c}{u}-u)^2}\,\mathrm{d}u.$$

Renaming the second integration variable and adding the two formulas, the substitution $s = \frac{c}{u} - u$ gives

$$\int_0^\infty \frac{c}{u^2}e^{-(\frac{c}{u}-u)^2}\,\mathrm{d}u = \frac{1}{2}\int_0^\infty \Big(1 + \frac{c}{u^2}\Big)e^{-(\frac{c}{u}-u)^2}\,\mathrm{d}u = \frac{1}{2}\int_{-\infty}^\infty e^{-s^2}\,\mathrm{d}s = \frac{1}{2}\pi^{1/2}.$$

We will apply this identity with $c = \frac{1}{2}tz^{1/2}$. Completing squares and changing variables twice, we obtain

$$\begin{aligned}
e^{tz^{1/2}}\int_0^\infty \frac{te^{-t^2/4s}}{s^{3/2}}e^{-zs}\,\mathrm{d}s &= \int_0^\infty \frac{t}{s}e^{-(t/2\sqrt{s}-z^{1/2}\sqrt{s})^2}\,\frac{\mathrm{d}s}{2\sqrt{s}} \\
&= \int_0^\infty \frac{t}{u^2}e^{-(t/2u-z^{1/2}u)^2}\,\mathrm{d}u = \pi^{1/2},
\end{aligned}$$

and this is the formula (15.12) we wanted to prove.

15.3 Bounded imaginary powers

A special role is played by sectorial operators whose purely imaginary fractional powers A^{it} are bounded. As their definition requires that $\mathsf{D}(A) \cap \mathsf{R}(A)$ be dense it will be convenient to introduce the following terminology.

Definition 15.3.1 (Standard sectorial operators). *A* standard sectorial operator *is a sectorial operator A with the property that $\mathsf{D}(A) \cap \mathsf{R}(A)$ is dense in X.*

The following proposition recalls some results proved in Proposition 10.1.8.

Proposition 15.3.2. *Let A be a sectorial operator on a Banach space X. Then:*

(1) *if A is standard, then A is injective;*
(2) *A is standard if and only if it is densely defined and has dense range;*
(3) *if X is reflexive, the following assertions are equivalent:*
  (i) *A is standard sectorial;*
  (ii) *A is injective;*
  (iii) *A has dense range.*

In view of (1), the fractional powers A^α of a standard sectorial operator A are well defined for all $\alpha \in \mathbb{C}$, and all results from the previous section are applicable to A.

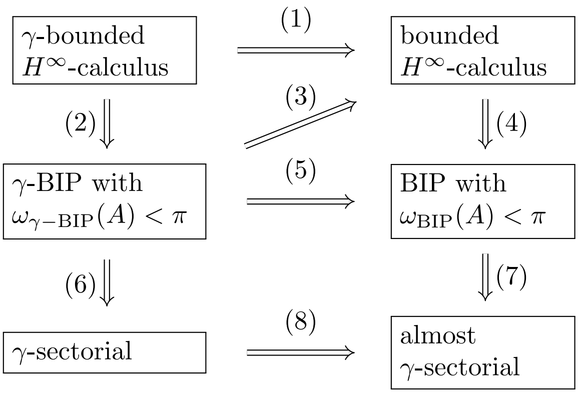
In applications, standardness is hardly a restrictive assumption. In most situations the Banach space will be reflexive and even UMD, and in such spaces for a sectorial operator A we have the direct sum decomposition

$$X = \mathsf{N}(A) \oplus \overline{\mathsf{R}(A)}.$$

By (2), the part of A in $\overline{\mathsf{R}(A)}$ is standard sectorial, and the part of A in $\mathsf{N}(A)$ is identically zero.

Example 15.3.3 (Standardness of the Laplacian on $L^p(\mathbb{R}^d;X)$). Let us consider the Laplace operator Δ on $L^p(\mathbb{R}^d;X)$, where $1 < p < \infty$ and X is a UMD space, with domain $\mathsf{D}(\Delta) = H^{2,p}(\mathbb{R};X)$. It is shown in Example 10.1.5 that $-\Delta$ is sectorial of angle 0 on $L^p(\mathbb{R}^d;X)$ for all $1 \leqslant p < 1$, and standard sectorial if and only if $1 < p < \infty$.

For standard sectorial operators A on a Banach space X, the next diagram summarises the main results of this section.



The implications (1), (2), (4), (5), and (8) are trivial. The implication (3) follows from Theorem 15.3.21, where it is also shown that equivalence holds when X has Pisier's contraction property. The implications (1)–(5) are equivalence when X is a Hilbert space. The implication (6) follows from Theorem 15.3.19, and the implication (7) is Theorem 15.3.16.

15.3.a Definition and basic properties

For $t \in \mathbb{R}$ consider the function

$$f_t(z) := z^{it} := \exp(it \log z),$$

where we use the branch of the logarithm that is holomorphic on $\mathbb{C} \setminus (-\infty, 0]$. From $|f_t(z)| = \exp(-t \arg(z))$ it follows that $f_t \in H^\infty(\Sigma_\sigma)$ for each $0 < \sigma < \pi$ and

$$\|f_t\|_{H^\infty(\Sigma_\sigma)} \leqslant \exp(\sigma |t|).$$

Thus if A is a standard sectorial operator with a bounded $H^\infty(\Sigma_\sigma)$-calculus, the operators

$$A^{it} := f_t(A)$$

are well defined as bounded operators on X. Some examples of operators with bounded imaginary powers will be discussed in Subsection 15.3.h.

When A is merely standard sectorial, we may use the extended Dunford calculus to define the operators A^{it}, $t \in \mathbb{R}$, as closed and densely defined operators in X. This suggests the following definition.

Definition 15.3.4 (BIP). *A linear operator A acting in a Banach space X is said to have* bounded imaginary powers *(briefly, A has* bounded imaginary powers*) if A is standard sectorial and the operators A^{it} are bounded for all $t \in \mathbb{R}$.*

Examples of operators with bounded imaginary powers will be given in Section 15.3.h.

Proposition 15.3.5. *If A has bounded imaginary powers, then the family $(A^{it})_{t\in\mathbb{R}}$ is a C_0-group.*

Proof. It is evident from the definition through the extended Dunford calculus that $t\mapsto A^{it}x$ is strongly measurable for all $x\in X$. We have already seen that $A^{i0}x=\mathbf{1}(A)x=x$ for all $x\in \mathsf{D}(A)\cap\mathsf{R}(A)$, so $A^{i0}=I$. The identity $A^{is}A^{it}x=A^{i(s+t)}x$ follows from Proposition 15.1.12. Proposition G.2.7 implies that $t\mapsto A^{it}x$ is continuous for all $x\in X$. □

When A has bounded imaginary powers, then by the above result and the general theory of C_0-(semi)groups, there exist constants $M\geqslant 1$ and $\omega\in\mathbb{R}$ such that

$$\|A^{it}\|\leqslant Me^{\omega|t|}.$$

This allows us to define the abscissa

$$\omega_{\rm BIP}(A):=\inf\big\{\omega\in\mathbb{R}:\ \sup_{t\in\mathbb{R}} e^{-\omega|t|}\|A^{it}\|<\infty\big\}.$$

We have the following improvement of Corollary 15.2.10 in the presence of bounded imaginary powers. The point of the estimate in part (1) is that boundedness of the imaginary powers permits us to obtain an estimate that is uniform all the way up the imaginary axis.

Proposition 15.3.6. *If A has bounded imaginary powers and $0\in\varrho(A)$, and if $\|A^{it}\|\leqslant Me^{-\omega|t|}$ for all $t\in\mathbb{R}$, then:*

(1) *the operator A^{-z} is bounded for every $\Re z\geqslant 0$, and*

$$\|A^{-z}\|\leqslant C_A M e^{\omega|\Im z|}\|A^{-1}\|^{\Re z},\quad \Re z\geqslant 0,$$

where C_A depends only on $M_A:=\sup_{t>0}(1+t)\|(t+A)^{-1}\|$ and $\|A^{-1}\|$;
(2) *for all $\Re z_1\geqslant 0$ and $\Re z_2\geqslant 0$ we have $A^{-z_1}A^{-z_2}=A^{-(z_1+z_2)}$;*
(3) *for all $x\in X$ the mapping $z\mapsto A^{-z}x$ is continuous on $\{\Re z\geqslant 0\}$ and holomorphic on $\{\Re z>0\}$.*

Proof. (1) and (2): By assumption for all $t\in\mathbb{R}$ the operators A^{it} are bounded, and for $\Re z>0$ the operators A^{-z} are bounded by Corollary 15.2.10. For $\Re z_1\geqslant 0$ and $\Re z_2\geqslant 0$ the identity $A^{-z_1}A^{-z_2}=A^{-(z_1+z_2)}$ follows from Proposition 15.1.12, noting that all operators occurring in this identity are bounded.

We next prove the norm estimate. We begin by noting that

$$C_A':=\sup_{s\in[0,1]}\|A^{-s}\|<\infty$$

by Corollary 15.2.15, with a constant C_A depending only on the constant M_A (which is finite since A is boundedly invertible).

By writing $z=s+it$ with $s\in[0,1]$, it follows from the identity $A^{-z}=A^{-s}A^{-it}$ that

$$\sup_{0\leqslant \Re z\leqslant 1} \|A^{-z}\| \leqslant C'_A \sup_{t\in\mathbb{R}} \|A^{-it}\| \leqslant C'_A M e^{-\omega|t|} \leqslant C_A M e^{-\omega|t|}\|A^{-1}\|^{\Re z},$$

where $C_A = C'_A/\max\{1, \|A\|^{-1}\}$. This gives the desired bound in (1) for $0 \leqslant \Re z \leqslant 1$.

For $z = z' + n$ with $n \geqslant 1$ and $0 \leqslant \Re z' < 1$, the estimate in (1) now follows from

$$\begin{aligned}\|A^{-z}\| = \|A^{-z'-n}\| \leqslant \|A^{-z'}\|\|A^{-n}\| &\leqslant C_A M^{-\omega|t|}\|A^{-1}\|^{\Re z'}\|A^{-1}\|^n \\ &= C_A M^{-\omega|t|}\|A^{-1}\|^{\Re z}.\end{aligned}$$

(3): Fix an arbitrary integer $k \geqslant 1$ and fix an element $x \in \mathsf{D}(A^k) \cap \mathsf{R}(A^k)$. We have already seen that $z \mapsto A^{-z}x$ is holomorphic on $\{|\Re z| < k\}$; in particular $z \mapsto A^{-z}x$ is continuous on $\{0 \leqslant \Re z < k\}$. The holomorphy on $\{|\Re z| < k\}$ and continuity on $\{0 \leqslant \Re z < k\}$ of $z \mapsto A^{-z}x$ for general $x \in X$ follows by approximation $x_n \to x$ with $x_n \in \mathsf{D}(A^k) \cap \mathsf{R}(A^k)$, noting that the above norm estimate implies that the convergence $A^{-z}x_n \to A^{-z}x$ is locally uniform on $\{0 \leqslant \Re z < k\}$. □

15.3.b Identification of fractional domain spaces

An important justification for studying boundedness of imaginary powers comes from Theorem 15.3.9 below, which states that boundedness of the imaginary powers implies the coincidence of the fractional power scale and the complex interpolation scale. For the proof of this result we need some lemmas. The first extends the relation $A^\alpha A^\beta = A^{\alpha+\beta}$, which has been proved so for only for α, β satisfying $\Re\alpha \cdot \Re\beta > 0$.

Lemma 15.3.7. *If A has bounded imaginary powers, then for all $\alpha \in \mathbb{C}$ and $t \in \mathbb{R}$ we have*

$$A^\alpha A^{it} = A^{it} A^\alpha = A^{\alpha+it}$$

with equality of domains.

Proof. Since A^{it} is bounded it is clear that $\mathsf{D}(A^\alpha) = \mathsf{D}(A^{it}A^\alpha)$. From Proposition 15.1.12(2) we already know the inclusion $\mathsf{D}(A^{it}A^\alpha) \subseteq \mathsf{D}(A^{\alpha+it})$ with $A^{it}A^\alpha x = A^{\alpha+it}x$ for all $x \in \mathsf{D}(A^{it}A^\alpha)$, as well as the equality $\mathsf{D}(A^{it}A^\alpha) = \mathsf{D}(A^{\alpha+it}) \cap \mathsf{D}(A^\alpha)$. Combining these results, we obtain $A^{it}A^\alpha = A^\alpha$ with equal domains. □

The second lemma considers bounded imaginary powers for shifted operators:

Lemma 15.3.8. *If A has bounded imaginary powers, then $A + \varepsilon$ has bounded imaginary powers for all $\varepsilon > 0$. If $\|A^{it}\| \leqslant Me^{\omega|t|}$ and $\omega(A) < \sigma < \pi$, then*

$$\|(A+\varepsilon)^{it}\| \leqslant M' e^{(\omega\vee\sigma)|t|},$$

for some constant M' independent of $\varepsilon > 0$.

Proof. It is immediate from Theorem 15.1.18 applied to $\varepsilon^{-1}A$ that $A+\varepsilon$ has bounded imaginary powers. By Proposition 15.2.6 we have $(\varepsilon^{-1}A)^{it} = \varepsilon^{-it}A^{it}$ and $(A+\varepsilon)^{it} = \varepsilon^{it}(\varepsilon^{-1}A+I)^{it}$ with equal domains in both cases. Hence, by the estimates provided by Theorem 15.1.18 and Proposition 15.2.6, for any fixed $\omega(A) < \sigma < \pi$ we have

$$\begin{aligned}\|(A+\varepsilon)^{it}\| &= \|(\varepsilon^{-1}A+I)^{it}\| \\ &\leqslant (1+M_{\sigma,\varepsilon^{-1}A})^2(\|A^{it}\| + C_\sigma\|z\mapsto z^{it}\|_{H^\infty(\Sigma_\sigma)}) \\ &\leqslant (1+M_{\sigma,A})^2(Me^{\omega|t|} + C_\sigma e^{\sigma|t|}).\end{aligned}$$

□

Theorem 15.3.9 (Fractional powers through complex interpolation). *If A has bounded imaginary powers, then for all $\alpha > 0$ and $0 < \theta < 1$,*

$$\mathsf{D}(A^{\alpha\theta}) = [X, \mathsf{D}(A^\alpha)]_\theta \quad \textit{with equivalent norms.}$$

Proof. By Proposition 15.2.12 and Lemma 15.3.8 we may replace A by $A+I$ if necessary, and thereby assume without loss of generality that $0 \in \varrho(A)$. This allows us to use the results of Proposition 15.3.6.

Choose $M \geqslant 1$ and $\omega \in \mathbb{R}$ such that $\|A^{it}\| \leqslant Me^{\omega|t|}$ for all $t \in \mathbb{R}$. We begin by proving the inclusion $\mathsf{D}(A^{\alpha\theta}) \subseteq [X, \mathsf{D}(A^\alpha)]_\theta$. Fix $0 < \theta < 1$ and $x \in \mathsf{D}(A^{\alpha\theta})$, and put

$$f(z) := e^{(z-\theta)^2}A^{-\alpha z}A^{\alpha\theta}x, \quad z \in \overline{\mathbb{S}},$$

where $\mathbb{S} = \{z \in \mathbb{C} : 0 < \Re z < 1\}$ is the unit strip in the complex plane. Then f is holomorphic as an X-valued function on $\mathbb{S}$ and satisfies $f(\theta) = x$. Moreover, by Proposition 15.3.6, f is continuous and uniformly bounded on $\overline{\mathbb{S}}$. Using the notation introduced in Appendix C, to prove that $x \in [X, \mathsf{D}(A^\alpha)]_\theta$ we must check that $f \in \mathscr{H}(X, \mathsf{D}(A^\alpha))$. For this it remains to be checked that $t \mapsto f(it)$ belongs to $C_{\rm b}(\mathbb{R}; X)$ and $t \mapsto f(1+it)$ belongs to $C_{\rm b}(\mathbb{R}; \mathsf{D}(A^\alpha))$. The former follows from what has already been said, and for the latter we write $\|f(1+it)\|_{\mathsf{D}(A^\alpha)} = \|f(1+it)\| + \|A^\alpha f(1+it)\|$. Again by what has already been said, the function $t \mapsto f(1+it)$ belongs to $C_{\rm b}(\mathbb{R}; X)$. The second term can be estimated as follows:

$$\begin{aligned}\|A^\alpha f(1+it)\| &= \|e^{(1+it-\theta)^2}A^\alpha A^{-\alpha(1+it)}A^{\alpha\theta}x\| \\ &= \|e^{(1+it-\theta)^2}A^{-i\alpha t}A^{\alpha\theta}x\| \leqslant e^{(1-\theta)^2-t^2}Me^{\alpha\omega|t|}\|A^{\alpha\theta}x\|,\end{aligned}$$

and this is a bounded function of $t \in \mathbb{R}$. Here we used Lemma 15.3.7, which implies that $\mathsf{D}(A^\alpha) = \mathsf{D}(A^{-\alpha(1+it)})$ and $A^{-\alpha(1+it)}y = A^{-\alpha}A^{it}y$ for $y \in X$.

To prove the reverse inclusion $[X, \mathsf{D}(A^\alpha)]_\theta \subseteq \mathsf{D}(A^{\alpha\theta})$ we will use the results and notation of Appendix C. Fix $x \in [X, \mathsf{D}(A^\alpha)]_\theta$ and let $f \in \mathscr{H}(X, \mathsf{D}(A^\alpha))$ satisfy $f(\theta) = x$. By Corollary C.2.8 there is a sequence of functions $f_n \in \mathscr{H}_0(X, \mathsf{D}(A^\alpha); \mathsf{D}(A^\alpha))$ such that $f_n(\theta) =: x_n \to x$ in $[X, \mathsf{D}(A^\alpha)]_\theta$.

Since $\mathsf{D}(A^\alpha) \subseteq \mathsf{D}(A^{\alpha z})$ for $z \in \overline{\mathbb{S}}$ and f_n takes values in $\mathsf{D}(A^\alpha)$ we may define

$$g_n(z) := e^{(z-\theta)^2} A^{\alpha z} f_n(z), \quad z \in \overline{\mathbb{S}}.$$

With respect to the norm of X, each function g_n is bounded on $\overline{\mathbb{S}}$. By the three lines lemma,

$$\|x_n\| = \|f_n(\theta)\| \leqslant \max\Big\{\sup_{t\in\mathbb{R}} \|f_n(it)\|, \sup_{t\in\mathbb{R}} \|f_n(1+it)\|\Big\},$$

$$\|A^{\alpha\theta} x_n\| = \|g_n(\theta)\| \leqslant \max\Big\{\sup_{t\in\mathbb{R}} \|g_n(it)\|, \sup_{t\in\mathbb{R}} \|g_n(1+it)\|\Big\}.$$

Moreover, for all $t \in \mathbb{R}$,

$$\begin{aligned}\|g_n(it)\| &\leqslant e^{\theta^2-t^2}\|A^{i\alpha t} f_n(it)\| \leqslant e^{\theta^2-t^2} M e^{\alpha\omega|t|}\|f_n(it)\|,\\ \|g_n(1+it)\| &\leqslant e^{(1-\theta)^2-t^2}\|A^{i\alpha t}A^\alpha f_n(1+it)\|\\ &\leqslant e^{\theta^2-t^2} M e^{\alpha\omega|t|}\|f_n(1+it)\|_{\mathsf{D}(A^\alpha)}.\end{aligned}$$

Here we used Lemma 15.3.7, which implies that $\mathsf{D}(A^\alpha) = \mathsf{D}(A^{\alpha+i\alpha t})$ and $A^{\alpha+i\alpha t}y = A^{i\alpha t}A^\alpha y$ for $y \in \mathsf{D}(A^\alpha)$.

It follows from these estimates that $\|x_n\| \lesssim \|f_n\|_{\mathscr{H}(X,X)} \leqslant \|f_n\|_{\mathscr{H}(X,\mathsf{D}(A^\alpha))}$ and $\|A^{\alpha\theta}x_n\| \lesssim \|f_n\|_{\mathscr{H}(X,\mathsf{D}(A^\alpha))}$, and therefore $\|x_n\|_{\mathsf{D}(A^{\alpha\theta})} \lesssim \|f_n\|_{\mathscr{H}(X,\mathsf{D}(A^\alpha))}$. Replacing x_n by $x_n - x_m$ in the above argument, we find that the sequence $(x_n)_{n\geqslant 1}$ is Cauchy in $\mathsf{D}(A^{\alpha\theta})$ and therefore converges to a limit. Since $x_n \to x$ in X, this limit must be x. This proves that $x \in \mathsf{D}(A^{\alpha\theta})$ and that $\|x\|_{\mathsf{D}(A^{\alpha\theta})} \lesssim \|f\|_{\mathscr{H}(X,\mathsf{D}(A^\alpha))}$. Taking the infimum with respect to f it follows that $\|x\|_{\mathsf{D}(A^{\alpha\theta})} \lesssim \|x\|_{[X,\mathsf{D}(A^\alpha)]_\theta}$. □

This theorem self-improves in an obvious manner. Upon replacing X by $\mathsf{D}(A^\beta)$ and using that $\mathsf{D}(A^\gamma) = \mathsf{D}(A^{\gamma+it})$ we arrive at the following more general result.

Corollary 15.3.10. *If A has bounded imaginary powers, then for all $\alpha, \beta \in \mathbb{C}$ with $0 \leqslant \alpha < \beta < \infty$ we have*

$$\mathsf{D}(A^{(1-\theta)\alpha+\theta\beta}) = [\mathsf{D}(A^\alpha), \mathsf{D}(A^\beta)]_\theta$$

with equivalent norms.

Let us revisit the Laplace operator Δ on $L^p(\mathbb{R}^d;X)$, where $1 < p < \infty$ and X is a UMD space, with domain $\mathsf{D}(\Delta) = H^{2,p}(\mathbb{R};X)$. It was already noted above that $-\Delta$ is standard sectorial of angle 0 on $L^p(\mathbb{R}^d;X)$ for all $1 \leqslant p < 1$, and by Theorem 10.2.25 it has a bounded H^∞-calculus of angle 0. As a consequence, $-\Delta$ has bounded imaginary powers. Applying Theorem 15.3.9, for all $0 < \theta < 1$ we obtain

$$\mathsf{D}((-\Delta)^\theta) = [L^p(\mathbb{R}^d;X), H^{2,p}(\mathbb{R}^d;X)]_\theta \ \text{ with equivalent norms.}$$

In Chapter 5 we have proved Seeley's theorem (Theorem 5.6.9), from which it follows that if X is a UMD space and $1 < p < \infty$, then for all $0 < \theta < 1$ we have

$$[L^p(\mathbb{R}^d; X), H^{2,p}(\mathbb{R}^d; X)]_\theta = H^{2\theta,p}(\mathbb{R}^d; X) \text{ with equivalent norms.}$$

Thus we obtain the following result.

Theorem 15.3.11 (Laplacian on $L^p(\mathbb{R}^d; X)$). *Consider the Laplace operator Δ on $L^p(\mathbb{R}^d; X)$, where $1 < p < \infty$ and X is a UMD space, with domain $\mathsf{D}(\Delta) = H^{2,p}(\mathbb{R}; X)$. Then for all $0 < \theta < 1$ we have*

$$\mathsf{D}((-\Delta)^\theta) = H^{2\theta,p}(\mathbb{R}^d; X) \text{ with equivalent norms.}$$

15.3.c Connections with sectoriality

It is part of the definition that an operator with bounded imaginary powers is standard sectorial, but there is no obvious *a priori* relation between the abscissa $\omega_{\mathrm{BIP}}(A)$ and the angle of sectoriality $\omega(A)$. The main result of this section is the following result, which says that $\omega(A) \leqslant \omega_{\mathrm{BIP}}(A)$. Moreover, if X is a UMD space, then A is R-sectorial of angle $\omega_R(A) \leqslant \omega_{\mathrm{BIP}}(A)$.

Theorem 15.3.12 (Clément–Prüss). *Let A be an operator with bounded imaginary powers on a Banach space X, and assume that $\omega_{\mathrm{BIP}}(A) < \pi$.*

(1) *A is sectorial of angle $\omega(A) \leqslant \omega_{\mathrm{BIP}}(A)$.*
(2) *If X is a UMD space, then A is R-sectorial of angle $\omega_R(A) \leqslant \omega_{\mathrm{BIP}}(A)$.*

The key lemma is the following representation formula. It expresses the resolvent of A in terms of the imaginary powers A^{it}, and a such it provides the key insight behind the Clément–Prüss theorem.

Lemma 15.3.13 (Prüss–Sohr). *Let A be an operator with bounded imaginary powers on a Banach space X, and assume that $\omega_{\mathrm{BIP}}(A) < \pi$. Let $\lambda = re^{i\theta}$ with $r > 0$ and $|\theta| < \pi - \omega_{\mathrm{BIP}}(A)$. Then for all $x \in \mathsf{D}(A) \cap \mathsf{R}(A)$ we have*

$$(I + \lambda A)^{-1}x = \frac{1}{2}x + \frac{1}{2\pi i}\text{p.v.}\int_{-\infty}^{\infty} \frac{\pi}{\sinh(\pi t)} \lambda^{-it} A^{-it} x \, \mathrm{d}t,$$

the convergence of the principal value integral on the right-hand side being part of the assertion. Furthermore, for all $0 < s < 1$,

$$\lambda^s A^s (1 + \lambda A)^{-1} x = \frac{1}{2}\int_{\mathbb{R}} \frac{1}{\sin(\pi(s - it))} \lambda^{it} A^{it} x \, \mathrm{d}t. \tag{15.15}$$

Proof. We begin with the proof of the first identity. It proceeds in three steps.

Step 1 – First take $r = 1$ and $\theta = 0$. In this step, for all $x \in X$ we will prove that

$$\frac{1}{2}x + \frac{1}{2\pi i}\text{p.v.}\int_{-\infty}^{\infty} \frac{\pi}{\sinh(\pi s)} A^{-is}x \,\mathrm{d}s$$
$$= \lim_{c\downarrow 0} \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\pi}{\sin(\pi z)} A^{-z}x \,\mathrm{d}z,$$

the convergence of the principal value integral being part of the assertion. Note that the integrals occurring on right-hand side converge absolutely thanks to the estimates

$$|\sinh(\pi(c+it))| = O(e^{\pi|t|}) \quad \text{as} \quad t \to \pm\infty$$

and

$$\|A^{-c-it}x\| \leqslant Me^{\omega|t|}\|A^{-c}x\|, \quad t \in \mathbb{R},$$

for all $\omega_{\mathrm{BIP}}(A) < \omega < \pi$, with $M \geqslant 1$ a constant depending on ω.

By Cauchy's theorem we have

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\pi}{\sin(\pi z)} A^{-z}x \,\mathrm{d}z = \frac{1}{2\pi i}\int_{\Gamma_c} \frac{\pi}{\sin(\pi z)} A^{-z}x \,\mathrm{d}z,$$

where Γ_c is the (upwards oriented) contour consisting of the union of the two half-lines $\Gamma_c^{(1)} = \{is : \ s \leqslant -c\}$ and $\Gamma_c^{(3)} = \{is : \ s \geqslant c\}$ and the semi-circle $\Gamma_c^{(2)} = \{ce^{i\vartheta} : \ \vartheta \in [-\frac{1}{2}\pi, \frac{1}{2}\pi]\}$. As $c \downarrow 0$, the contributions along the two half-lines converge to the principal value integral and the contribution along the semi-circle converges to $\frac{1}{2}x$. The latter follows by noting that $A^{-z}x \to x$ as $z \to 0$ in the closed right-half plane, by the continuity of $z \mapsto A^{-z}x$ on that set (see Proposition 15.3.6). Hence

$$\lim_{c\downarrow 0} \frac{1}{2\pi i}\int_{\Gamma_c^{(2)}} \frac{\pi}{\sin(\pi z)} \,\mathrm{d}z = \lim_{c\downarrow 0} \frac{1}{2\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\pi c e^{i\varphi}}{\sin(\pi c e^{i\varphi})} \,\mathrm{d}\varphi = \frac{1}{2}$$

(since $\sin(\pi c e^{i\varphi}) = \pi c e^{i\varphi} + O(c^3)$ as $c \downarrow 0$.

Step 2 – In this step we will prove the lemma for $r = 1$ and $\theta = 0$ with $x \in \mathsf{D}(A) \cap \mathsf{R}(A)$, i.e., we show that

$$(I + A)^{-1}x = \frac{1}{2}x + \frac{1}{2\pi i}\text{p.v.}\int_{-\infty}^{\infty} \frac{\pi}{\sinh(\pi s)} A^{-is}x \,\mathrm{d}s$$

for all $x \in \mathsf{D}(A) \cap \mathsf{R}(A)$. (Note that $I + A$ is boundedly invertible as part of the definition of bounded imaginary powers, since A is assumed to be standard sectorial).

Let $y := (I + A)x$. Then

$$
\begin{aligned}
&\frac{1}{2}y + \frac{1}{2\pi i}\text{p.v.}\int_{-\infty}^{\infty} \frac{\pi}{\sinh(\pi s)} A^{-is} y \,\mathrm{d}s \\
&\quad = \frac{1}{2}(I+A)x + \frac{1}{2\pi i}\text{p.v.}\int_{-\infty}^{\infty} \frac{\pi}{\sinh(\pi s)} A^{-is} x \,\mathrm{d}s \\
&\qquad + \frac{1}{2\pi i}\text{p.v.}\int_{-\infty}^{\infty} \frac{\pi}{\sinh(\pi s)} A^{1-is} x \,\mathrm{d}s \\
&\quad = \frac{1}{2}(I+A)x + \frac{1}{2\pi i}\text{p.v.}\int_{\Gamma_c} \frac{\pi}{\sin(\pi z)} A^{-z} y \,\mathrm{d}x \\
&\qquad + \frac{1}{2\pi i}\text{p.v.}\int_{\Gamma_c} \frac{\pi}{\sin(\pi z)} A^{1-z} y \,\mathrm{d}x.
\end{aligned}
\tag{15.16}
$$

In view of $\sin(\pi z) = -\sin(\pi(1-z))$, after a change of variable in the last integral the contributions over all four half-lines cancel and we are left with

$$
\frac{1}{2}(I+A)x + \frac{1}{2\pi i}\text{p.v.}\int_{\Gamma_c^{(2)}} \frac{\pi}{\sin(\pi z)} A^{-z} x \,\mathrm{d}z - \frac{1}{2\pi i}\text{p.v.}\int_{\widetilde{\Gamma}_c^{(2)}} \frac{\pi}{\sin(\pi z)} A^{-z} x \,\mathrm{d}z,
$$

where $\widetilde{\Gamma}_c^{(2)} = \{1 - ce^{i\vartheta} : \ \vartheta \in [-\frac{1}{2}\pi, \frac{1}{2}\pi]\}$. As $c \downarrow 0$, the first integral tends to $\frac{1}{2}y$ and the second to $-\frac{1}{2}Ax$. In the limit $c \downarrow 0$ the three terms on the right-hand side of (15.16) therefore add up to x. This proves the identity

$$
x = \frac{1}{2}y + \frac{1}{2\pi i}\text{p.v.}\int_{-\infty}^{\infty} \frac{\pi}{\sinh(\pi s)} A^{-is} y \,\mathrm{d}s.
$$

Multiplying on both sides with $(I+A)^{-1}$ gives the desired result.

Step 3 – The general case follows by applying the result of Step 2 to the operator λA, which by Proposition 15.2.6 has bounded imaginary powers and satisfies $(\lambda A)^{-is} = \lambda^{-is}A^{-is}$. This completes the proof of the first identity. Using it, and fixing $0 < s < 1$, for $x \in \mathsf{D}(A) \cap \mathsf{R}(A)$ we obtain

$$
\begin{aligned}
\lambda^s A^s (I+\lambda A)^{-1} x &= \frac{1}{2}\lambda^s A^s x + \frac{1}{2\pi i}\text{p.v.}\int_{-\infty}^{\infty} \frac{\pi}{\sinh(\pi t)} \lambda^{s-it} A^{s-it} x \,\mathrm{d}t \\
&= \frac{1}{2}\lambda^s A^s x - \frac{1}{2}\int_{-\infty-is}^{\infty-is} \frac{1}{\sin(\pi(s-it))} \lambda^{it} A^{it} x \,\mathrm{d}t \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{\sin(\pi(s-it))} \lambda^{it} A^{it} x \,\mathrm{d}t
\end{aligned}
$$

by the Cauchy theorem, noting that $A^s A^{it} = A^{s+it}$ by Theorem 15.2.5 in the first step. This gives the second identity. □

Remark 15.3.14. A more direct proof of the second identity can be given as follows. Starting from the identity

$$
\int_{-\infty}^{\infty} \frac{e^{2\pi i t\xi}}{\sin(\pi(s-it))} \,\mathrm{d}t = \frac{2e^{2\pi s\xi}}{1+e^{2\pi\xi}}, \qquad 0 < s < 1, \ \xi \in \mathbb{R}, \tag{15.17}
$$

the substitution $z = e^{2\pi\xi}$ gives

$$\int_{-\infty}^{\infty} \frac{z^{it}}{\sin(\pi(s-it))}\,\mathrm{d}t = \frac{2z^s}{1+z}, \qquad 0 < s < 1,\ z \in \mathbb{R}_+. \tag{15.18}$$

By analytic continuation this extends to all $z \in \mathbb{C}$ with $|\arg(z)| < \pi$.

Let $\lambda \in \mathbb{C}\setminus\{0\}$ as in the statement of the lemma. For $x \in \mathsf{D}(A)\cap\mathsf{R}(A)$ it follows from Proposition 15.1.19 that $A^{it}x$ is given by the Bochner integral

$$A^{it}x = \mu^{it}x + \frac{1}{2\pi i}\int_{\Gamma_\nu} z^{it}\Big(R(z,A) - \frac{1}{z-\mu}\Big)x\,\mathrm{d}z,$$

where $\omega(A) < |\arg\mu| < \nu$. Substituting this identity into the right-hand side of (15.15), a short computation involving Fubini's theorem, (15.18), and Cauchy's theorem gives the result. At the expense of some additional computations, instead of invoking Proposition 15.1.19 one may also directly use the definition for $A^{it}x$ as given in Definition 15.1.8.

Proof of Theorem 15.3.12. (1): First let $\lambda = re^{i\theta}$ with $r > 0$ and $|\theta| < \pi - \omega(A)$. By subtraction we obtain the identity

$$(I+\lambda A)^{-1}x = (I+A)^{-1}x + \frac{1}{2\pi i}\mathrm{p.v.}\int_{-\infty}^{\infty}\frac{\pi}{\sinh(\pi s)}(\lambda^{-is}-1)A^{-is}x\,\mathrm{d}s$$

for $x \in \mathsf{D}(A^2)\cap\mathsf{R}(A^2)$. The crux is that the term $\lambda^{-is}-1$ is of the order $O(|s|)$ near $s = 0$ and can therefore be estimated as $|\lambda^{-is}-1| \lesssim |s|\wedge 1$. Similarly, $|\sinh(s)| \lesssim (|s|\wedge 1)e^{\pi|s|}$. Therefore the principal value integral is actually absolutely convergent and bounded in x. As a consequence of this, the identity extends to arbitrary $x \in X$.

The proof is completed by observing that the integral in the right hand side of the identity

$$(I+\lambda A)^{-1}x = (I+A)^{-1}x + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\pi}{\sinh(\pi s)}(\lambda^{-is}-1)A^{-is}x\,\mathrm{d}s$$

is absolutely convergent for any $\lambda = re^{i\theta}$ with $r > 0$ and $|\theta| < \pi - \omega_{\mathrm{BIP}}(A)$. Indeed, recalling the estimates for $\lambda^{-is}-1$ and $\sinh(s)$ mentioned earlier, choosing $\omega_{\mathrm{BIP}}(A) < \omega < \pi$ so that $|\theta| < \pi - \omega$ we estimate

$$\Big\|\int_{-\infty}^{\infty}\frac{\pi}{\sinh(\pi s)}(\lambda^{-is}-1)A^{-is}x\,\mathrm{d}s\Big\| \lesssim \int_{|s|\geqslant 1}\pi e^{-\pi|s|}M_\omega e^{\omega|s|}\|x\|\,\mathrm{d}s$$

with a constant independent of x. The right-hand side defines a holomorphic extension of the function $\lambda \mapsto (I+\lambda A)^{-1}x$ to the open sector $\Sigma_{\pi-\omega_{\mathrm{BIP}}(A)}$. As a consequence the spectrum of A must be contained in the closure of $\Sigma_{\omega_{\mathrm{BIP}}(A)}$. Finally, the sectoriality estimate on the complement of this closure follows from the estimate.

(2): Fix $\omega_{\rm BIP}(A) < \omega < \nu < \pi$ and choose numbers $\lambda_n = r_n e^{i\theta_n}$ with $r_n > 0$ and $|\theta_n| < \pi - \nu$, as well as vectors $x_n \in X$; $n = 1, \dots, N$. We wish to show that there exists a constant C, independent of the choices just made, such that

$$\Big\| \sum_{n=1}^N \varepsilon_n (I + \lambda_n A)^{-1} x_n \Big\|_{L^2(\Omega;X)} \leqslant C \Big\| \sum_{n=1}^N \varepsilon_n x_n \Big\|_{L^2(\Omega;X)},$$

where $(\varepsilon_n)_{n=1}^N$ is a Rademacher sequence defined on a probability space $(\Omega, \mathbb{P})$. By a simple approximation argument, there is no loss of generality in assuming that $x_n \in \mathsf{D}(A) \cap \mathsf{R}(A)$ for all $n = 1 \dots, N$.

Since $\omega(A) \leqslant \omega_{\rm BIP}(A)$ by the Clément–Prüss theorem, Lemma 15.3.13 (with $\lambda = 1$), the representation formulas of Lemma 15.3.13 hold for $\lambda = re^{i\theta}$ with $r > 0$ and $|\theta| < \pi - \nu$, with $x \in \mathsf{D}(A^2) \cap \mathsf{R}(A^2)$, and

$$\begin{aligned}
(I + re^{i\theta} A)^{-1} x &= \frac{1}{2} x + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \psi_\theta(s) r^{-is} A^{-is} x \, \mathrm{d}s \\
&\quad + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \eta(s) r^{-is} A^{-is} x \, \mathrm{d}s, \\
&\quad + \frac{1}{2\pi i} \mathrm{p.v.} \int_{-1}^{1} r^{-is} A^{-is} x \, \frac{\mathrm{d}s}{s} \\
&=: \frac{1}{2} x + T_{r,\theta} x + S_r x + R_r x,
\end{aligned}$$

where

$$\psi_\theta(s) = \frac{\pi}{\sinh(\pi s)} (e^{\theta s} - 1), \quad \eta(s) := \frac{\pi}{\sinh(\pi s)} - \frac{\mathbf{1}_{(-1,1)}(s)}{s}.$$

Applying this to $\lambda = \lambda_n$ we obtain

$$\begin{aligned}
&\Big\| \sum_{n=1}^N \varepsilon_n (I + \lambda_n A)^{-1} x_n \Big\|_{L^2(\Omega;X)} \\
&\quad \leqslant \frac{1}{2} \Big\| \sum_{n=1}^N \varepsilon_n x_n \Big\|_{L^2(\Omega;X)} + \Big\| \sum_{n=1}^N \varepsilon_n T_{r_n,\theta_n} x_n \Big\|_{L^2(\Omega;X)} \\
&\quad\quad + \Big\| \sum_{n=1}^N \varepsilon_n S_{r_n} x_n \Big\|_{L^2(\Omega;X)} + \Big\| \sum_{n=1}^N \varepsilon_n R_{r_n} x_n \Big\|_{L^2(\Omega;X)}.
\end{aligned}$$

We will estimate the last three expressions separately.

To start with the first, we note that $|\psi_{\theta_n}(s)| \lesssim e^{(\theta_n - \pi)|s|} \leqslant e^{-\nu|s|}$. Therefore, by the Kahane contraction principle and the bound $\|A^{is}\| \leqslant M e^{\omega|s|}$,

$$\Big\| \sum_{n=1}^N \varepsilon_n T_{r_n,\theta_n} x_n \Big\|_{L^2(\Omega;X)} \leqslant \frac{1}{2\pi} \int_{-\infty}^{\infty} \Big\| A^{-is} \sum_{n=1}^N \varepsilon_n \psi_{\theta_n}(s) x_n \Big\|_{L^2(\Omega;X)} \mathrm{d}s$$

$$\begin{aligned}
&\lesssim \frac{1}{2\pi}\int_{-\infty}^{\infty} Me^{\omega|s|}\Big\|\sum_{n=1}^N \varepsilon_n\psi_{\theta_n}(s)x_n\Big\|_{L^2(\Omega;X)}\,\mathrm{d}s\\
&\lesssim \frac{1}{2\pi}\int_{-\infty}^{\infty} Me^{(\omega-\nu)|s|}\Big\|\sum_{n=1}^N \varepsilon_n x_n\Big\|_{L^2(\Omega;X)}\,\mathrm{d}s\\
&= C_{A,\nu}\Big\|\sum_{n=1}^N \varepsilon_n x_n\Big\|_{L^2(\Omega;X)}.
\end{aligned}$$

The second term is treated similarly, now using that $|\eta(s)|\lesssim e^{-\pi|s|}$:

$$\begin{aligned}
\Big\|\sum_{n=1}^N \varepsilon_n S_{r_n}x_n\Big\|_{L^2(\Omega;X)} &\leqslant \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big\|A^{-is}\sum_{n=1}^N \varepsilon_n\eta(s)x_n\Big\|_{L^2(\Omega;X)}\,\mathrm{d}s\\
&\lesssim \frac{1}{2\pi}\int_{-\infty}^{\infty} Me^{\omega|s|}\Big\|\sum_{n=1}^N \varepsilon_n\eta(s)x_n\Big\|_{L^2(\Omega;X)}\,\mathrm{d}s\\
&\lesssim \frac{1}{2\pi}\int_{-\infty}^{\infty} Me^{(\omega-\pi)|s|}\Big\|\sum_{n=1}^N \varepsilon_n x_n\Big\|_{L^2(\Omega;X)}\,\mathrm{d}s\\
&= C'_{A,\nu}\Big\|\sum_{n=1}^N \varepsilon_n x_n\Big\|_{L^2(\Omega;X)}.
\end{aligned}$$

For estimating the third term we use the UMD property of X through the boundedness of the Hilbert transform on $L^2(\mathbb{R};X)$.

We begin with a preliminary observation. Let us set $U_n(s)=(r_nA)^{-is}=r_n^{-is}A^{-is}$ for brevity. Then by the Kahane contraction principle, for all $s\in\mathbb{R}$ we have

$$\begin{aligned}
\Big\|\sum_{n=1}^N \varepsilon_n U_n(s)x_n\Big\|_{L^2(\Omega;X)} &\leqslant \Big\|\sum_{n=1}^N \varepsilon_n A^{-is}x_n\Big\|_{L^2(\Omega;X)}\\
&\leqslant Me^{\omega|s|}\Big\|\sum_{n=1}^N \varepsilon_n x_n\Big\|_{L^2(\Omega;X)}.
\end{aligned}\tag{15.19}$$

Fix $0<\delta<1$ and $t\in[-\frac12,\frac12]$. Then

$$\begin{aligned}
\sum_{n=1}^N \varepsilon_n\int_{\delta<|s|<1} U_n(s)x_n\,\frac{\mathrm{d}s}{s} &= \sum_{n=1}^N \varepsilon_n U_n(t)\int_{\delta<|s|<1} U_n(s-t)x_n\,\frac{\mathrm{d}s}{s}\\
&= \sum_{n=1}^N \varepsilon_n U_n(t)\int_{|s|>\delta}\varphi_n(t-s)\,\frac{\mathrm{d}s}{s}\\
&\quad -\sum_{n=1}^N \varepsilon_n\int_1^{1+t} U_n(s)x_n\,\frac{\mathrm{d}s}{s}
\end{aligned}$$

$$+\sum_{n=1}^{N}\varepsilon_n\int_{-1}^{-1+t}U_n(s)x_n\,\frac{\mathrm{d}s}{s},$$

where $\varphi_n(\tau)=\mathbf{1}_{(-1,1)}(\tau)U_n(-\tau)x_n$. Integrating over $t\in(-\frac12,\frac12)$, we obtain

$$\begin{aligned}\sum_{n=1}^{N}\varepsilon_n\int_{\delta<|s|<1}U_n(s)x_n\,\frac{\mathrm{d}s}{s}&=\sum_{n=1}^{N}\varepsilon_n\int_{-\frac12}^{\frac12}U_n(t)\int_{|s|>\delta}\varphi_n(t-s)\,\frac{\mathrm{d}s}{s}\,\mathrm{d}t\\&\quad-\sum_{n=1}^{N}\varepsilon_n\int_{-\frac12}^{\frac12}\int_{1}^{1+t}U_n(s)x_n\,\frac{\mathrm{d}s}{s}\,\mathrm{d}t\\&\quad+\sum_{n=1}^{N}\varepsilon_n\int_{-\frac12}^{\frac12}\int_{-1}^{-1+t}U_n(s)x_n\,\frac{\mathrm{d}s}{s}\,\mathrm{d}t.\end{aligned}$$

Since X is UMD and $\phi_n\in L^2(\mathbb{R};X)$, the limit

$$\lim_{\delta\downarrow0}\int_{|s|>\delta}\varphi_n(\cdot-s)\,\frac{\mathrm{d}s}{s}=\lim_{\substack{\delta\downarrow0\\R\to\infty}}\int_{\delta<|s|<R}\varphi_n(\cdot-s)\,\frac{\mathrm{d}s}{s}$$

exists in $L^2(\mathbb{R};X)$ by Theorem 5.1.1 and equals $\pi H\phi_n$, where H is the Hilbert transform. As a result we obtain

$$\begin{aligned}\sum_{n=1}^{N}\varepsilon_nR_{r_n}x_n&=\sum_{n=1}^{N}\varepsilon_n\mathrm{p.v.}\int_{-1}^{1}U_n(s)x_n\,\frac{\mathrm{d}s}{s}\\&=\sum_{n=1}^{N}\varepsilon_n\lim_{\delta\downarrow0}\int_{\delta<|s|<1}U_n(s)x_n\,\frac{\mathrm{d}s}{s}\\&=\pi\sum_{n=1}^{N}\varepsilon_n\int_{-\frac12}^{\frac12}U_n(t)H\varphi_n(t)\,\mathrm{d}t\\&\qquad-\sum_{n=1}^{N}\varepsilon_n\int_{-\frac12}^{\frac12}\int_{1}^{1+t}U_n(s)x_n\,\frac{\mathrm{d}s}{s}\,\mathrm{d}t\\&\qquad+\sum_{n=1}^{N}\varepsilon_n\int_{-\frac12}^{\frac12}\int_{-1}^{-1+t}U_n(s)x_n\,\frac{\mathrm{d}s}{s}\,\mathrm{d}t\\&=:(I)+(II)+(III).\end{aligned}$$

It remains to estimate the three terms on the right-hand side. For estimating (I) we use that $\|H\|_{\mathscr{L}(L^2(\mathbb{R};X))}\leqslant 2\beta^+_{2,X}\beta^-_{2,X}$ (see Theorem 5.1.13). Applying the Kahane–Khintchine inequality, this gives

$$\Big\|\sum_{n=1}^{N}\varepsilon_n\int_{-1/2}^{1/2}U_n(t)H\varphi_n(t)\,\mathrm{d}t\Big\|_{L^2(\Omega;X)}$$

$$
\begin{aligned}
&\eqsim \Big\| \sum_{n=1}^{N} \varepsilon_n \int_{-1/2}^{1/2} U_n(t) H\varphi_n(t)\,\mathrm{d}t \Big\|_{L^1(\Omega;X)} \\
&= \Big\| \int_{-1/2}^{1/2} \sum_{n=1}^{N} \varepsilon_n U_n(t) H[\mathbf{1}_{(-1,1)}(\cdot) U_n(-\,\cdot\,)x_n](t)\,\mathrm{d}t \Big\|_{L^1(\Omega;X)} \\
&\leqslant \int_{-1/2}^{1/2} \Big\| \sum_{n=1}^{N} \varepsilon_n U_n(t) H[\mathbf{1}_{(-1,1)}(\cdot) U_n(-\,\cdot\,)x_n](t) \Big\|_{L^1(\Omega;X)} \mathrm{d}t \\
&\leqslant M \int_{-1/2}^{1/2} \Big\| \sum_{n=1}^{N} \varepsilon_n H[\mathbf{1}_{(-1,1)}(\cdot) U_n(-\,\cdot\,)x_n](t) \Big\|_{L^1(\Omega;X)} \mathrm{d}t \\
&= M_{1/2}\mathbb{E} \int_{-1/2}^{1/2} \Big\| H\Big[\mathbf{1}_{(-1,1)}(\cdot) \sum_{n=1}^{N} \varepsilon_n U_n(-\,\cdot\,)x_n\Big](t) \Big\|\,\mathrm{d}t \\
&\leqslant M_{1/2}\mathbb{E} \Big\| H\Big[\mathbf{1}_{(-1,1)}(\cdot) \sum_{n=1}^{N} \varepsilon_n U_n(-\,\cdot\,)x_n\Big] \Big\|_{L^2(\mathbb{R};X)} \\
&\leqslant 2\beta_{2,X}^{+}\beta_{2,X}^{-} M_{1/2}\mathbb{E} \Big\| \sum_{n=1}^{N} \varepsilon_n \mathbf{1}_{(-1,1)}(\cdot) U_n(-\,\cdot\,)x_n \Big\|_{L^2(\mathbb{R};X)}
\end{aligned}
$$

and, by (15.19),

$$
\begin{aligned}
\mathbb{E}\Big\| \sum_{n=1}^{N} \varepsilon_n \mathbf{1}_{(-1,1)}(\cdot) U_n(-\,\cdot\,)x_n \Big\|_{L^2(\mathbb{R};X)}. \\
&= \mathbb{E}\Big\| \sum_{n=1}^{N} \varepsilon_n(\cdot) U_n(-\,\cdot\,)x_n \Big\|_{L^2(-1,1;X)} \\
&\leqslant \Big\| \sum_{n=1}^{N} \varepsilon_n(\cdot) U_n(-\,\cdot\,)x_n \Big\|_{L^2(\Omega;L^2(-1,1;X))} \\
&= \Big\| \sum_{n=1}^{N} \varepsilon_n(\cdot) U_n(-\,\cdot\,)x_n \Big\|_{L^2(-1,1;L^2(\Omega;X))} \\
&\leqslant M_{1/2} \Big\| \sum_{n=1}^{N} \varepsilon_n x_n \Big\|_{L^2(-1,1;L^2(\Omega;X))} \\
&= M_{1/2} \Big\| \sum_{n=1}^{N} \varepsilon_n x_n \Big\|_{L^2(\Omega;X))}
\end{aligned}
$$

where $M_{1/2} := \sup_{|t|\leqslant 1/2} \|A^{-it}\|$.

To estimate (II) we use (15.19) again:

$$
\Big\| \sum_{n=1}^{N} \varepsilon_n \int_{-1/2}^{1/2} \int_{1}^{1+t} U_n(s)x_n \,\frac{\mathrm{d}s}{s}\,\mathrm{d}t \Big\|_{L^2(\Omega;X)}
$$

$$\begin{aligned}
&\leqslant \int_{-1/2}^{1/2}\int_1^{1+t} \Big\| \sum_{n=1}^N \varepsilon_n U_n(s)x_n \Big\|_{L^2(\Omega;X)} \frac{\mathrm{d}s}{s}\,\mathrm{d}t \\
&\leqslant M_2 \int_{-1/2}^{1/2}\int_1^{1+t} \Big\| \sum_{n=1}^N \varepsilon_n x_n \Big\|_{L^2(\Omega;X)} \frac{\mathrm{d}s}{s}\,\mathrm{d}t \\
&\leqslant M_2 \Big\| \sum_{n=1}^N \varepsilon_n x_n \Big\|_{L^2(\Omega;X)}
\end{aligned}$$

where $M_2 := \sup_{|t|\leqslant 2} \|A^{-it}\|$.

The estimation of (III) is entirely similar. □

15.3.d Connections with almost γ-sectoriality

We have consistently limited our treatment of the H^∞-calculus and related topics to sectorial operators. It is of some interest to consider the wider class of so-called *almost sectorial* operators, defined as follows.

Definition 15.3.15 (Almost sectorial operators). *Let $\sigma \in (0,\pi)$. A linear operator A acting in a Banach space X is called:*

(i) σ-almost sectorial *if $\sigma(A) \subseteq \overline{\Sigma_\sigma}$ and the set*

$$\big\{\lambda A R(\lambda,A)^2 : \ \lambda \in \mathbb{C}\setminus \overline{\Sigma_\sigma}\big\}$$

is uniformly bounded;

(ii) σ-almost γ-sectorial *if $\sigma(A) \subseteq \overline{\Sigma_\sigma}$ and the set*

$$\big\{\lambda A R(\lambda,A)^2 : \ \lambda \in \mathbb{C}\setminus \overline{\Sigma_\sigma}\big\}$$

is γ-bounded.

The operator A is called almost sectorial*, respectively almost* γ-sectorial *if it is σ-almost sectorial, respectively σ-almost γ-sectorial, for some $\sigma \in (0,\pi)$.*

Almost R-sectorial operators are defined similarly, replacing γ-boundedness by R-boundedness.

For an almost sectorial, respectively an almost γ-sectorial operator A, we define

$$\begin{aligned}
\widetilde{\omega}(A) &:= \inf\big\{\sigma \in (0,\pi) : \ A \text{ is } \sigma\text{-almost sectorial}\big\}, \\
\widetilde{\omega}_\gamma(A) &:= \inf\big\{\sigma \in (0,\pi) : \ A \text{ is } \sigma\text{-almost } \gamma\text{-sectorial}\big\}.
\end{aligned}$$

The identity

$$\lambda A R(\lambda,A)^2 = [\lambda R(\lambda,A)]^2 - \lambda R(\lambda,A)$$

shows that every (γ-)sectorial operator is almost (γ-)sectorial and

$$\widetilde{\omega}(A) \leqslant \omega(A), \quad \text{respectively} \quad \widetilde{\omega}_\gamma(A) \leqslant \omega_\gamma(A).$$

The above definitions may appear somewhat *ad hoc* at first sight, but the motivation to introduce them is as follows. The operators $\lambda R(\lambda, A)$ used in the definition of sectoriality can be represented in the primary calculus of A as

$$\lambda R(\lambda, A) = r_\lambda(A) \quad \text{with} \quad R_\lambda(z) = \frac{\lambda}{\lambda - z}.$$

Indeed, the functions r_λ belong to the class $E(\Sigma_\sigma)$ introduced in Section 15.1.a as long as $0 < \sigma < |\Re\lambda|$. They do not belong to $H^1(\Sigma_\sigma)$, however, and this fact is responsible for some of the technical issues encountered in several proofs. In contrast, the operators $\lambda AR(\lambda, A)^2$ used in the definition of almost sectoriality can be represented in the Dunford calculus of A, for we have

$$\lambda AR(\lambda, A)^2 = \widetilde{r}_\lambda(A) \quad \text{with} \quad \widetilde{r}_\lambda(z) = \frac{\lambda z}{(\lambda - z)^2}.$$

Indeed, the functions $\widetilde{r}_\lambda$ belong to $H^1(\Sigma_\sigma)$ for $0 < \sigma < |\Re\lambda|$. Further motivation will be given in the Notes at the end of the chapter.

The following result gives a version of the (second part of) Clément–Prüss theorem (Theorem 15.3.12) holds without making any assumptions on the Banach space X. The price to pay is that only almost γ-sectoriality is obtained:

Theorem 15.3.16. *Let A be an operator with bounded imaginary powers on a Banach space X. Then A is almost γ-sectorial of angle $\widetilde{\omega}_\gamma(A) \leqslant \omega_{\mathrm{BIP}}(A)$.*

Proof. Fix $\omega_{\mathrm{BIP}}(A) < \theta' < \theta < \pi$ and suppose that $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ are nonzero and satisfy $|\arg(\lambda_k)| \geqslant \theta$. Note that $|\arg(\mu_k)| \leqslant \pi - \theta$. Set $\mu_k := -1/\lambda_k$. Then for all choices $x_1, \dots, x_n \in X$ we have, by Lemma 15.3.13,

$$\begin{aligned}
&\mathbb{E}\Big\|\sum_{k=1}^n \gamma_k \lambda_k^{1/2} A^{1/2} R(\lambda_k, A) x_k\Big\| \\
&= \mathbb{E}\Big\|\sum_{k=1}^n \gamma_k \mu_k^{1/2} A^{1/2} (1 + \mu_k A)^{-1} x_k\Big\| \\
&\leqslant \frac{1}{2}\int_{\mathbb{R}} \frac{1}{|\sin(\pi(\frac{1}{2} - it))|} \mathbb{E}\Big\|\sum_{k=1}^n \gamma_k \mu_k^{it} A^{it} x_k\Big\| \,\mathrm{d}t \\
&\overset{(*)}{\leqslant} \frac{1}{2}\int_{\mathbb{R}} \frac{e^{(\pi-(\theta-\theta'))|t|}}{|\sin(\pi(\frac{1}{2} - it))|} \|e^{-(\pi-(\theta-\theta'))|t|} A^{it}\| \Big(\sup_{1\leqslant k\leqslant n} |\mu_k^{it}|\Big) \mathbb{E}\Big\|\sum_{k=1}^n \gamma_k x_k\Big\| \,\mathrm{d}t \\
&\overset{(**)}{\leqslant} \frac{1}{2}\int_{\mathbb{R}} \frac{e^{(\pi-(\theta-\theta'))|t|}}{|\sin(\pi(\frac{1}{2} - it))|} \,\mathrm{d}t \sup_{t\in\mathbb{R}} \|e^{-\theta'|t|} A^{it}\| \,\mathbb{E}\Big\|\sum_{k=1}^n \gamma_k x_k\Big\| \\
&= C\mathbb{E}\Big\|\sum_{k=1}^n \gamma_k x_k\Big\|,
\end{aligned}$$

where in $(*)$ we used the contraction principle and in $(**)$ the fact that for $|\arg(\mu)| \leqslant \pi - \theta$ and $t \in \mathbb{R}$ we have

$$\big\|e^{-(\pi-(\theta-\theta'))|t|}A^{it}\big\||\mu^{it}| = \big\|e^{-(\pi-(\theta-\theta'))|t|}e^{-\arg(\mu)t}A^{it}\big\| \leqslant \big\|e^{-\theta'|t|}A^{it}\big\| \leqslant C',$$

where $C' := \sup_{t\in\mathbb{R}} \|e^{-\theta'|t|}A^{it}\|$ is finite since $\omega_{\mathrm{BIP}}(A) < \theta'$, and where

$$C := \frac{C'}{2}\int_{\mathbb{R}} \frac{e^{(\pi-(\theta-\theta'))|t|}}{|\sin(\pi(\frac12 - it))|}\,\mathrm{d}t.$$

We have shown that the family

$$\big\{\lambda^{1/2}A^{1/2}R(\lambda, A) : |\arg(\lambda)| \geqslant \theta\big\}$$

is γ-bounded. Taking squares, it follows that the family

$$\big\{\lambda AR(\lambda, A)^2 : |\arg(\lambda)| \geqslant \theta\big\}$$

is γ-bounded as well. Moreover we see that $\widetilde{\omega}_\gamma(A) \leqslant \theta$. This being true for all $\omega_{\mathrm{BIP}}(A) < \theta < \pi$, it follows that $\widetilde{\omega}_\gamma(A) \leqslant \omega_{\mathrm{BIP}}(A)$. □

15.3.e Connections with γ-sectoriality

We start with a definition.

Definition 15.3.17 (γ-bounded imaginary powers). *An operator A is said to have* γ-bounded imaginary powers *(briefly, A has* γ-BIP*) if it has bounded imaginary powers and the family*

$$\{A^{it} : |t| \leqslant 1\}$$

is γ-bounded.

If A has γ-bounded imaginary powers, the group property $A^{is}A^{it} = A^{i(s+t)}$ combined with Proposition 8.1.20 (or rather, the elementary bound in the discussion preceding it) implies that set

$$\{e^{-\omega|t|}A^{it} : t \in \mathbb{R}\}$$

is γ-bounded for large enough $\omega > 0$. Thus it makes sense to define the abscissa

$$\omega_{\gamma\text{-BIP}}(A) := \inf\big\{\omega \geqslant 0 : \ \{e^{-\omega|t|}A^{it} : t \in \mathbb{R}\} \text{ is } \gamma\text{-bounded}\big\}.$$

Replacing γ-boundedness by R-boundedness, we may similarly introduce operators A with *R-BIP* along with their abscissa $\omega_{R\text{-BIP}}(A)$ Since finite cotype implies equivalence of Rademacher sums and Gaussian sums (Corollary 7.2.10), an operator A on a Banach space with finite cotype has R-bounded imaginary powers if and only A has γ-bounded imaginary powers. As the ensuing proofs will make clear, operators with γ-bounded imaginary powers can be effectively studied using the continuous square functions introduced in Section 10.4.b. It is for this reason that our results will be stated for operators with γ-bounded imaginary powers. The analogous results for operators with R-bounded imaginary powers automatically follow if the underlying Banach space has finite cotype.

Proposition 15.3.18. *If A has γ-bounded imaginary powers, then $\omega_{\rm BIP}(A) = \omega_{\gamma\text{-BIP}}(A)$.*

Proof. Let $\omega_{\rm BIP}(A) < \nu < \theta$. For each $n \in \mathbb{Z}$ the singleton $\{A^{in}\}$ is γ-bounded, with γ-bound $\gamma(\{A^{in}\} = \|A^{in}\| \leqslant Me^{\nu|n|}$, where M is a constant independent of $n \in \mathbb{Z}$. By Proposition 8.1.20 (with $p = 1$ and $q = \infty$), the set

$$\{e^{-\theta|n|}A^{in} : \ n \in \mathbb{Z}\}$$

is γ-bounded. Combined with the fact that $\{A^{is} : \ s \in [-1, 1]\}$ is γ-bounded, by Proposition 8.1.19(3) we obtain that $\omega_{\gamma\text{-BIP}}(A) < \theta$. □

We have seen in Theorem 15.3.12 that bounded imaginary powers imply sectoriality with angle $\omega(A) \leqslant \omega_{\rm BIP}(A)$. The next theorem provides the analogue for γ-bounded imaginary powers.

Theorem 15.3.19. *If A has γ-bounded imaginary powers with $\omega_{\gamma\text{-BIP}} < \pi$, then A is γ-sectorial with $\omega_\gamma(A) \leqslant \omega_{\gamma\text{-BIP}}(A)$.*

Proof. The proof proceeds in three steps.

Step 1 – In Steps 2 and 3 we will prove that each of the families of operators

$$\Gamma_s := \{t^s A^s (1 + tA)^{-1} : \ t > 0\}, \qquad \text{where} \ \ 0 < s < \frac{1}{2},$$

is γ-bounded, uniformly with respect to the parameter $s \in (0, \frac{1}{2})$. In the present step we show how the theorem follows from this.

For $x \in \mathsf{D}(A) \cap \mathsf{R}(A)$ we have

$$\lim_{s \downarrow 0} t^s A^s (I + tA)^{-1} x = (I + tA)^{-1} x.$$

Hence by Fatou's lemma, for all finite sequences $x_1, \ldots, x_n \in \mathsf{D}(A) \cap \mathsf{R}(A)$ and $t_1, \ldots, t_n > 0$ we have

$$\begin{aligned} &\mathbb{E}\Big\| \sum_{k=1}^n \gamma_k (I + t_k A)^{-1} x_k \Big\|^2 \\ &\qquad \leqslant \liminf_{s \downarrow 0} \mathbb{E}\Big\| \sum_{k=1}^n \gamma_k t_k^s A^s (I + t_k A)^{-1} x_k \Big\|^2 \leqslant C\mathbb{E}\Big\| \sum_{k=1}^n \gamma_k x_k \Big\|^2, \end{aligned}$$

where C is any finite upper bound for the γ-bounds of the families Γ_s, $s \in (0, \frac{1}{2})$. This proves that the set $\{(I + tA)^{-1} : t > 0\}$ is γ-bounded.

Applying this reasoning to operators $e^{i\theta}$ with $0 < |\theta| < \pi - \omega_{\gamma\text{-BIP}}$ (and noting that the identity $(e^{\pm i\theta}A)^{it} = e^{\mp\theta}A^{it}$ implies that these operators still have γ-bounded imaginary powers) and using Proposition 8.5.8 to extrapolate γ-boundedness from the boundary of a sector to the full sector, it follows that A is γ-sectorial and $\omega_\gamma(A) \leqslant \omega_{\gamma\text{-BIP}}(A)$.

Step 2 – We now turn to the proof of the γ-boundedness of the families Γ_s uniformly with respect to $s \in (0, \frac{1}{2})$. We claim that it suffices to prove that for all $f \in \mathscr{S}(\mathbb{R}; X)$ we have

$$\Big\| t \mapsto \int_{\mathbb{R}} k_s(t-u) A^{i(t-u)} f(u)\,\mathrm{d}u \Big\|_{\gamma(\mathbb{R};X)} \leqslant C\|f\|_{\gamma(\mathbb{R};X)}, \tag{15.20}$$

where the constant C is independent of $0 < s < \frac{1}{2}$ and

$$k_s(t) := \frac{1}{2\sin(\pi(s - it))}, \qquad t \in \mathbb{R}.$$

Indeed, suppose that (15.20) has been proved. By Fubini's theorem and the second identity of Lemma 15.3.13, for all $\xi \in \mathbb{R}$ we have

$$\begin{aligned}\int_{\mathbb{R}} \int_{\mathbb{R}} & k_s(t-u) A^{i(t-u)} f(u)\, e^{-2\pi i t\xi}\,\mathrm{d}u\,\mathrm{d}t \\ &= \int_{\mathbb{R}} k_s(t) A^{it} e^{-2\pi i t\xi}\,\mathrm{d}t \int_{\mathbb{R}} f(u) e^{-2\pi i u\xi}\,\mathrm{d}u = e^{-2\pi\xi s} A^s (1 + e^{-2\pi\xi}A)^{-1} \widehat{f}(\xi).\end{aligned}$$

Hence by (15.20) and the fact, observed in Example 9.6.5, that the Fourier transform extends to an isometry on $\gamma(\mathbb{R}; X)$, we obtain

$$\|\xi \mapsto e^{-2\pi s\xi} A^s (1 + e^{-2\pi\xi}A)^{-1} \widehat{f}(\xi)\|_{\gamma(\mathbb{R};X)} \leqslant C\|f\|_{\gamma(\mathbb{R};X)} = C\|\widehat{f}\|_{\gamma(\mathbb{R};X)}.$$

Since the Fourier transform maps $\mathscr{S}(\mathbb{R}; X)$ onto itself and this space is dense in $\gamma(\mathbb{R}; X)$, this estimate extends to all strongly measurable function $g\colon S \to X$ representing an element of $\gamma(\mathbb{R}; X)$ by density. Then converse to the γ-multiplier theorem (Proposition 9.5.6) implies that Γ_s is γ-bounded, with γ-bound at most C.

Step 3 – To complete the proof of the theorem it remains to prove the bound (15.20) with a uniform constant C independent of $s \in (0, \frac{1}{2})$. We start with the observation that by (15.18) we have

$$\widehat{k_s}(\xi) = \frac{e^{-s\xi}}{1 + e^{-\xi}},$$

which implies that $\widehat{k_s} \in L^\infty(\mathbb{R})$ uniformly in $s \in (0, \frac{1}{2})$.

Fix $s \in (0, \frac{1}{2})$. For $n \in \mathbb{Z}$, set $I_n := [2n - 1, 2n + 1)$ and define, for $\varphi \in C_{\mathrm{c}}(\mathbb{R})$,

$$T_s^{(n)}\varphi(u) := \int_{\mathbb{R}} K_s^{(n)}(u,v)\varphi(v)\,\mathrm{d}v, \qquad u \in \mathbb{R},$$

where

$$K_s^{(n)}(u,v) := \sum_{j\in\mathbb{Z}} k_s(u-v)\mathbf{1}_{I_j}(u)\mathbf{1}_{I_{j+n}}(v).$$

This sum trivially converges pointwise in (t,v), since each such point is contained in at most one rectangle $I_j \times I_{j+n}$. We wish to show that the operator $T_s^{(n)}$ thus defined extends to a bounded operator on $L^2(\mathbb{R})$, uniformly in $s \in (0, \frac{1}{2})$.

For $\varphi \in C_c(\mathbb{R})$ we have, by the disjointness of the intervals I_j, monotone convergence, and a change of variables,

$$\begin{aligned}
\|T_s^{(n)}\varphi\|_2^2 &= \int_{\mathbb{R}} \Big|\sum_{j\in\mathbb{Z}} \mathbf{1}_{I_j}(u) \int_{\mathbb{R}} k_s(u-v)\mathbf{1}_{I_{j+n}}(v)\varphi(v)\,\mathrm{d}v\Big|^2\,\mathrm{d}u \\
&= \int_{\mathbb{R}} \sum_{j\in\mathbb{Z}} \mathbf{1}_{I_j}(u)\Big|\int_{\mathbb{R}} k_s(u-v)\mathbf{1}_{I_{j+n}}(v)\varphi(v)\,\mathrm{d}v\Big|^2\,\mathrm{d}u \\
&\leqslant \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} \Big|\int_{\mathbb{R}} k_s(u-v)\mathbf{1}_{I_{j+n}}(v)\varphi(v)\,\mathrm{d}v\Big|^2\,\mathrm{d}u \\
&= \sum_{j\in\mathbb{Z}} \|\widehat{k}_s\widehat{\mathbf{1}_{I_{j+n}}\varphi}\|_2^2 \leqslant \sum_{j\in\mathbb{Z}} \|\widehat{\mathbf{1}_{I_{j+n}}\varphi}\|_2^2 \\
&= \sum_{j\in\mathbb{Z}} \int_{I_{j+n}} |\varphi(u)|^2\,\mathrm{d}u = \|\varphi\|_2^2.
\end{aligned}$$

This shows that $T_s^{(n)}$ extends to a bounded operator on $L^2(\mathbb{R})$. Moreover, since

$$|K_s^{(n)}(u,v)| \leqslant |k_s(u-v)|\mathbf{1}_{\{|u-v|\geqslant 2(|n|-1)\}}$$

and

$$|k_s(u)| \leqslant \frac{1}{2|\sinh(\pi u)|} \lesssim e^{-\pi|u|}, \qquad |u| \geqslant 1,$$

by Young's inequality we have

$$\|T_s^{(n)}\| \leqslant \|K_s^{(n)}\|_1 \lesssim \int_{\{|u|\geqslant 2(|n|-1)\}} e^{-\pi|u|}\,\mathrm{d}u \lesssim e^{-2\pi|n|}, \qquad |n| \geqslant 2.$$

By the γ-extension theorem (Theorem 9.6.1), the operators $T_s^{(n)}$ extend to bounded operators on $\gamma(\mathbb{R};X)$ and

$$\|T_s^{(n)}\|_{\mathscr{L}(\gamma(\mathbb{R};X))} \leqslant C_0\, e^{-2\pi|n|}.$$

for some absolute constant $C_0 \geqslant 0$.

Define $p : \mathbb{R} \to \mathbb{Z}$ by $p(t) := 2j$ when $t \in I_j$. Then $|p(t) - t| \leqslant 1$ for all $t \in \mathbb{R}$. Let $\omega_{\gamma\text{-BIP}}(A) < \theta < \pi$ and let $C_1, C_2 > 0$ be such that

$$\gamma\{A^{is} : s \in [-1,1]\} \leqslant C_1, \quad \gamma\{A^{is} : s \in \mathbb{R}\} \leqslant C_2\, e^{\theta|s|}.$$

We may of course relate these constants, but that would only complicate the estimate below a bit. Fix a Schwartz function $f \in \mathscr{S}(\mathbb{R};X)$ and an integer $n \in \mathbb{Z}$. If (u,t) belongs to the support of $K_s^{(n)}$, then $u \in I_j$ and $v \in I_{j+n}$ for some $j \in Z$, from which it follows that $p(u) = p(v) - 2n$. Therefore we may estimate

$$\begin{aligned}
&\Big\| u \mapsto \int_{\mathbb{R}} K_s^{(n)}(u,v) A^{i(u-v)} f(v)\,\mathrm{d}v \Big\|_{\gamma(\mathbb{R};X)} \\
&= \Big\| u \mapsto \int_{\mathbb{R}} K_s^{(n)}(u,v) A^{i(u-p(u)+p(v)-v-2n)} f(u)\,\mathrm{d}u \Big\|_{\gamma(\mathbb{R};X)} \\
&\leqslant C_1 \Big\| u \mapsto \int_{\mathbb{R}} K_s^{(n)}(u,v) A^{i(p(v)-v-2n)} f(v)\,\mathrm{d}u \Big\|_{\gamma(\mathbb{R};X)} \\
&\leqslant C_0 C_1\, e^{-2\pi|n|} \big\| u \mapsto A^{i(p(u)-u-2n)} f(u) \big\|_{\gamma(\mathbb{R};X)} \\
&\leqslant C_0 C_1^2 C_2\, e^{-2(\pi-\theta)|n|} \|f\|_{\gamma(\mathbb{R};X)}
\end{aligned}$$

using the γ-multiplier theorem (Theorem 9.5.1) in the second and fourth step. Since

$$k_s(u-v) = \sum_{n\in\mathbb{Z}} K_s^{(n)}(u,v), \qquad u,v \in \mathbb{R},$$

the bound (15.20) now follows from the triangle inequality. □

15.3.f Connections with boundedness of the H^∞-calculus

It has already been observed that standard sectorial operators with a bounded H^∞-calculus have bounded imaginary powers and $\omega_{\rm BIP}(A) \leqslant \omega_{H^\infty}(A)$, the angle of the H^∞-calculus of A (see Definition 10.2.10). The following theorem gives a more precise version of this result.

Theorem 15.3.20 (Cowling–Doust–McIntosh–Yagi). *If A is a standard sectorial operator with a bounded $H^\infty(\Sigma_\sigma)$-calculus for some $\omega(A) < \sigma < \pi$, then A has bounded imaginary powers and*

$$\omega_{\rm BIP}(A) = \omega_{H^\infty}(A).$$

Moreover,

$$\|A^{it}\| \leqslant M_{\sigma,A}^\infty e^{\sigma|t|}, \quad t \in \mathbb{R},$$

where $M_{\sigma,A}^\infty$ is the boundedness constant of the $H^\infty(\Sigma_\sigma)$-calculus of A.

Proof. It remains to prove the inequality $\omega_{H^\infty}(A) \leqslant \omega_{\mathrm{BIP}}(A)$. In view of the Clément–Prüss theorem (Theorem 15.3.12), which asserts that $\omega(A) \leqslant \omega_{\mathrm{BIP}}(A)$, it suffices to prove that if $\omega(A) < \mu < \nu \leqslant \sigma$ with $\|A^{it}\| \leqslant Me^{\mu|t|}$ for all $t \in \mathbb{R}$, then A has a bounded $H^\infty(\Sigma_\nu)$-calculus.

To this end let $f \in H^\infty(\Sigma_\nu)$. We will show that

$$f(z) = \sum_{k\in\mathbb{Z}} z^{ik} f_k(z), \quad z \in \Sigma_\nu, \tag{15.21}$$

for suitable functions $f_k \in H^\infty(\Sigma_\sigma)$ satisfying

$$\sum_{k\in\mathbb{Z}} e^{\mu|k|} \|f_k\|_{H^\infty(\Sigma_\sigma)} \leqslant C\|f\|_{H^\infty(\Sigma_\nu)} \tag{15.22}$$

with constant $C \geqslant 0$ independent of f. Once this has been shown, we may set

$$f(A) := \sum_{k\in\mathbb{Z}} A^{ik} f_k(A),$$

with convergence in the norm of $\mathscr{L}(X)$; here, the operators $f_k(A)$ are defined through the $H^\infty(\Sigma_\sigma)$-calculus of A. The bound (15.22) implies that

$$\|f(A)\| \leqslant CM\|f\|_\infty. \tag{15.23}$$

To complete the proof that A admits a bounded $H^\infty(\Sigma_\nu)$-calculus, we will show that for $f \in H^1(\Sigma_\nu) \cap H^\infty(\Sigma_\nu)$ the operator $f(A)$ thus defined agrees with the Dunford calculus of A.

Step 1 – In this step we prove everything up to and including (15.23). Using the change of variables $z = e^w$ we transform sectors to horizontal strips and must show that every $g \in H^\infty(S_\nu)$ can be expressed as

$$g(w) = \sum_{k\in\mathbb{Z}} e^{ikw} g_k(w), \quad w \in S_\nu,$$

where $S_\theta = \{z \in \mathbb{C} : |\Im(z)| < \theta\}$ and the functions $g_k \in H^\infty(S_\sigma)$ satisfy

$$\sum_{k\in\mathbb{Z}} e^{\mu|k|} \|g_k\|_{H^\infty(S_\sigma)} \leqslant C\|g\|_{H^\infty(S_\nu)}.$$

Let $\phi \in C_{\mathrm{c}}(\mathbb{R})$ satisfy

(i) $0 \leqslant \phi(x) \in \mathbf{1}_{(-1,1)}(\xi)$ for all $\xi \in \mathbb{R}$,
(ii) $\sum_{k\in\mathbb{Z}} \phi(\xi - k) = 1$ for all $\xi \in \mathbb{R}$,

and set

$$g_k(w) := \int_{\mathbb{R}} \check{\phi}(w - t) g(t) e^{-ikt} \,\mathrm{d}t, \quad w \in S_\sigma.$$

By the Paley–Wiener theorem, $\check{\phi}$ is an entire function with sufficient decay to ensure the convergence of the integral for every $w \in \mathbb{C}$. Fixing $w \in S_\sigma$

and $k \in \mathbb{Z}$, and sing Cauchy's theorem to shift the path of integration, for $\epsilon \in \{-1, 1\}$ we may write

$$g_k(w) := \int_{\mathbb{R}} \check{\phi}(w - t - i\epsilon\nu) g(t + i\epsilon\nu) e^{-ik(t+i\epsilon\nu)} \,\mathrm{d}t, \quad w \in S_\sigma.$$

Taking $\epsilon = -\operatorname{sgn}(k)$ gives the bound

$$\|g_k\|_\infty \leqslant C_{\sigma,\nu} e^{-\nu|k|} \|g\|_\infty, \quad k \in \mathbb{Z},$$

with

$$C_{\sigma,\nu} = \sup_{|y|<\sigma+\nu} \int_{\mathbb{R}} |\check{\phi}(x + iy)| \,\mathrm{d}x < \infty.$$

Setting $h_k(\xi) := \phi(\xi - k)\widehat{g}(\xi)$, a simple calculation gives

$$\widecheck{h_k}(w) = \int_{\mathbb{R}} \check{\phi}(w - t) g(t) e^{-ik(w-t)} \,\mathrm{d}t = e^{ikw} g_k(w), \quad w \in S_\sigma.$$

Since $\sum_{k\in\mathbb{Z}} h_k(\xi) = \widehat{g}(\xi)$ for all $\xi \in \mathbb{R}$, the result follows by taking inverse Fourier transforms.

Step 2 – It remains to show that for $f \in H^1(\Sigma_\nu) \cap H^\infty(\Sigma_\nu)$ the operator $f(A)$ defined by (15.21) agrees with the Dunford calculus. For this it suffices to observe that for such functions f, the functions g_k constructed in Step 1 belong to $H^1(S_\sigma) \cap H^\infty(S_\sigma)$ and

$$\|g_k\|_{H^1(S_\sigma)} \leqslant C_{\sigma,\nu} e^{-\nu|k|} \|g\|_{H^1(S_\nu)}, \quad k \in \mathbb{Z},$$

with $C_{\sigma,\nu}$ as before. It follows that the sum defining $f(A)$ also converges in $H^1(\Sigma_\nu)$. The required consistency now follows by interchanging summation and integration, along with the fact that $(z \mapsto z^{ik} f_k(z))(A) = A^{ik} f_k(A)$ in the extended Dunford calculus, hence *a posteriori* also in the Dunford calculus. □

With Theorem 15.3.19 at our disposal we will now investigate the connection between the γ-boundedness of the imaginary powers A^{it} and the boundedness of the H^∞-calculus of A. In preparation of the next result, it is useful to point out that in some of these results in Chapter 10 the finite cotype assumption can be dropped if one defines discrete square functions in terms of Gaussian sums instead of using Rademacher sums. To be explicit, assuming Definition 10.4.1 to have been modified in this way, the finite cotype assumption can be dropped in the following results:

- Proposition 10.4.15(2). Indeed, the proof uses the finite cotype assumption only to pass from Gaussian sums to the Rademacher sums used in the definition of discrete square functions.
- Theorem 10.4.16(1). Indeed, the finite cotype assumption was only used to apply Proposition 10.4.15(2).

- Proposition 10.4.20. Indeed, the finite cotype assumption was only used to apply Proposition 10.4.15(2).

The next theorem establishes the connection between γ-bounded imaginary powers and boundedness of the H^∞-calculus.

Theorem 15.3.21 (Bounded H^∞-calculus $\Leftrightarrow$ γ-BIP). *Let A be standard sectorial on a Banach space X.*

(1) *If A has γ-bounded imaginary powers with $\omega_{\gamma\text{-}BIP}(A) < \pi$, then A has a bounded H^∞-calculus and*

$$\omega_{H^\infty}(A) \leqslant \omega_{\gamma\text{-}BIP}(A).$$

(2) *If A has a bounded H^∞-calculus and X has Pisier's contraction principle, then A has γ-bounded imaginary powers and*

$$\omega_{\gamma\text{-}BIP}(A) \leqslant \omega_{H^\infty}(A).$$

Since Pisier's contraction principle implies finite cotype (Corollary 7.5.13), a sectorial operator A acting in a Banach space with this property has γ-bounded imaginary powers if and only A has R-bounded imaginary powers, and in that case

$$\omega_{\gamma\text{-BIP}}(A) = \omega_{R\text{-BIP}}(A).$$

Before turning to the proof of the theorem, we isolate a lemma which is essentially contained in the proof of Theorem 10.4.16. For the reader's convenience we repeat the argument here.

Lemma 15.3.22. *Let A be standard sectorial, let $\omega(A) < \sigma < \pi$, and suppose that there is a constant $C \geqslant 0$ such that for all $\psi \in H^1(\Sigma_\sigma)$ and $x^* \in \mathsf{D}(A^*) \cap \mathsf{R}(A^*)$ we have*

$$\big\| t \mapsto \psi(tA^*)x^* \big\|_{\gamma(\mathbb{R}_+, \frac{dt}{t}; X^*)} \leqslant C\|\psi\|_{H^1(\Sigma_\sigma)}\|x^*\|.$$

Then for all non-zero $\phi \in H^1(\Sigma_\sigma)$ there is a constant $c_\phi \geqslant 0$ such that for all $x \in \mathsf{D}(A) \cap \mathsf{R}(A)$ we have

$$\|x\| \leqslant 2Cc_\phi M_{\sigma,A}\big\| t \mapsto \phi(tA)x \big\|_{\gamma(\mathbb{R}_+, \frac{dt}{t}; X)}.$$

Note that the assumptions on x, x^*, ϕ, and ψ imply that $t \mapsto \phi(tA)x \in \gamma(\mathbb{R}_+, \frac{dt}{t}; X)$ and $t \mapsto \psi(tA^*)x^* \in \gamma(\mathbb{R}_+, \frac{dt}{t}; X^*)$ by Lemma 10.4.14 (which only assumes sectoriality and can therefore be applied to both A and A^*).

Proof. Fix a non-zero $\phi \in H^1(\Sigma_\sigma)$ and fix an arbitrary $\psi \in H^1(\Sigma_\sigma)$ such that $\int_0^\infty \phi(t)\psi(t)\,\frac{dt}{t} = 1$. For all $x \in \mathsf{D}(A) \cap \mathsf{R}(A)$ and $x^* \in \mathsf{D}(A^*) \cap \mathsf{R}(A^*)$, from the reproducing formula of Proposition 10.2.5, the trace duality inequality of Theorem 9.2.14, and our assumption we obtain

$$\begin{aligned}|\langle x, x^*\rangle| &= \Big|\int_0^\infty \langle \phi(tA)\psi(tA)x, x^*\rangle \frac{\mathrm{d}t}{t}\Big| \\ &\leqslant \big\|t\mapsto \phi(tA)x\big\|_{\gamma(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)}\big\|t\mapsto \psi(tA)^*x^*\big\|_{\gamma(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X^*)} \\ &\leqslant C\|\psi\|_{H^1(\Sigma_\sigma)}\big\|t\mapsto \phi(tA)x\big\|_{\gamma(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)}\|x^*\|,\end{aligned}$$

where we used that $\psi(tA)^* = \psi(tA^*)$. Taking the supremum over all $x^* \in \mathsf{D}(A^*)\cap\mathsf{R}(A^*)$ of norm $\leqslant 1$, the result now follows from Lemma 10.2.19, with $c_\phi = \inf\{\|\psi\|_{H^1(\Sigma_\sigma)} : \int_0^\infty \phi(t)\psi(t)\,\frac{\mathrm{d}t}{t} = 1\}$. □

Proof of Theorem 15.3.21. (1): Fix $\omega_{\gamma\text{-BIP}}(A) < \sigma < \pi$. Then the set $\{e^{-\sigma|t|}A^{it} : t\in\mathbb{R}\}$ is γ-bounded.

Step 1 – In this step we prove that for all $\vartheta > \sigma$ and $x \in X$ the function $t\mapsto e^{-\vartheta|t|}A^{it}x$ belongs to $\gamma(\mathbb{R};X)$.

By the result of Example 9.4.12 (taking $H = \mathbb{C}$), the function $t \mapsto e^{-(\vartheta-\sigma)|t|}\otimes x$ belongs to $\gamma(\mathbb{R};X)$ and

$$\big\|t\mapsto e^{-(\vartheta-\sigma)|t|}\otimes x\big\|_{\gamma(\mathbb{R};X)} = \|t\mapsto e^{-(\vartheta-\sigma)|t|}\|_{L^2(\mathbb{R})}\|x\| \eqsim \frac{1}{(\vartheta-\sigma)^{1/2}}\|x\|.$$

Hence by the γ-multiplier theorem (Theorem 9.5.1), $t\mapsto e^{-\vartheta|t|}A^{it}x$ belongs to $\gamma_\infty(\mathbb{R};X)$ and

$$\big\|t\mapsto e^{-\vartheta|t|}A^{it}x\big\|_{\gamma_\infty(\mathbb{R};X)} \lesssim \frac{1}{(\vartheta-\sigma)^{1/2}}\gamma(\{e^{-\sigma|t|}A^{it} : t\in\mathbb{R}\}). \tag{15.24}$$

We claim that the functions $t\mapsto e^{-\vartheta|t|}A^{it}x$ actually belong to the closed subspace $\gamma(\mathbb{R};X)$ of $\gamma_\infty(\mathbb{R};X)$. To prove this, let B be the generator of the C_0-group $(A^{it})_{t\in\mathbb{R}}$. For all $x\in\mathsf{D}(B)$ and all $0<a<b<\infty$ and $-\infty<a<b<0$ the function $t\mapsto e^{-\vartheta|t|}A^{it}x$ belongs to $C^1([a,b];X)$, and hence to $\gamma(a,b;X)$ by Proposition 9.7.1. Since $\mathsf{D}(B)$ is dense in X, the claim now follows from Corollary 9.5.2.

Step 2 – The formula

$$a^{-\frac12+it} = \frac{\cosh(\pi t)}{\pi}\int_0^\infty u^{-\frac12+it}(u+a)^{-1}\,\mathrm{d}u, \quad a>0,\ t\in\mathbb{R}, \tag{15.25}$$

may be proved by a contour integration argument. Alternatively, it can be obtained from a standard identity for the Mellin transform of the function $(1+t)^{-1}$ and some substitutions.

Set $\theta := \pi-\vartheta$. By analytic continuation, the identity (15.25) extends to complex $a\in\mathbb{C}\setminus(-\infty,0]$. For $z\in\Sigma_{\pi-\theta}$ we may substitute $a = e^{-i\theta}z$ to obtain, after a bit of algebra,

$$e^{\theta t}z^{it} = \frac{\cosh(\pi t)}{\pi}e^{\frac12 i\theta}\int_0^\infty u^{-\frac12+it}z^{1/2}(e^{i\theta}u+z)^{-1}\,\mathrm{d}u.$$

Since $\omega(A) < \pi - \theta$ (this is because $\omega(A) \leqslant \omega_{\mathrm{BIP}}(A) = \omega_{\gamma\text{-}\mathrm{BIP}}(A)$ by the Clément–Prüss theorem and Proposition 15.3.18, and $\omega_{\gamma\text{-}\mathrm{BIP}}(A) < \vartheta = \pi - \theta$ by assumption), we can apply Lemma 10.2.17 (with $p = 1$) to this identity and obtain, for all $x \in X$,

$$\begin{aligned} e^{\theta t} A^{it} x &= \frac{\cosh(\pi t)}{\pi} \int_0^\infty e^{\frac{1}{2} i\theta} u^{-\frac{1}{2}+it} A^{1/2}(e^{i\theta}u + A)^{-1} x \, \mathrm{d}u \\ &= \frac{\cosh(\pi t)}{\pi} \int_{-\infty}^\infty e^{itv} e^{\frac{1}{2}v + \frac{1}{2}i\theta} A^{1/2}(e^{i\theta}e^v + A)^{-1} x \, \mathrm{d}v, \end{aligned} \tag{15.26}$$

where the second identity results from the substitution $u = e^v$.

By Step 1, the function $t \mapsto e^{-\vartheta|t|}A^{it}x$ belongs to $\gamma(\mathbb{R}; X)$. Since $\cosh(\pi t) \sim e^{\pi t}$ and $\pi - \theta = \vartheta$, this implies that the function

$$t \mapsto \frac{e^{-\theta|t|}A^{it}x}{\cosh(\pi t)}$$

belongs to $\gamma(\mathbb{R}; X)$.

By Theorem 9.6.1, the γ-extension of Fourier–Plancherel transform is an isometry from $\gamma(\mathbb{R}; X)$ onto itself. Dividing both sides of (15.26) by $\cosh(\pi t)$ and applying this isometry, it follows that the function

$$v \mapsto e^{\pi v + \frac{1}{2}i\theta} A^{1/2}(e^{i\theta}e^{2\pi v} + A)^{-1}x$$

belongs to $\gamma(\mathbb{R}; X)$ and

$$\begin{aligned} \big\| v \mapsto e^{\pi v + \frac{1}{2}i\theta} A^{1/2}(e^{i\theta}e^{2\pi v} + A)^{-1}x \big\|_{\gamma(\mathbb{R};X)} &\eqsim \Big\| t \mapsto \frac{e^{\theta t}}{\cosh(\pi t)} A^{it}x \Big\|_{\gamma(\mathbb{R};X)} \\ &\eqsim \big\| t \mapsto e^{-\vartheta|t|}A^{it}x \big\|_{\gamma(\mathbb{R};X)} \\ &\lesssim_A \frac{1}{\vartheta - \sigma}\|x\|, \end{aligned}$$

using (15.24) in the last step. Substituting back $e^v = u$ and leaving out terms of modulus one since they do not affect the γ-norms,

$$\big\| u \mapsto u^{1/2}A^{1/2}(e^{i\theta}u + A)^{-1}x \big\|_{\gamma(\mathbb{R}, \frac{\mathrm{d}u}{u}; X)} \lesssim_A \frac{1}{\vartheta - \sigma}\|x\|.$$

The term in the norm on the left-hand side is of the form $\phi(u^{-1}A)$ with $\phi(z) = z^{1/2}(e^{i\theta} + z)^{-1}$. This function belongs to $H^1(\Sigma_{\vartheta'})$ for all $0 < \vartheta' < \vartheta = \pi - \theta$, and the estimate can be interpreted as giving the square function estimate

$$\big\| t \mapsto \phi(tA)x \big\|_{\gamma(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X)} \lesssim \|x\|, \quad x \in X.$$

Note that up to this point we only have used that A is sectorial and has bounded imaginary powers (the γ-sectoriality assumption will only be used towards the end of the proof). Because of this, we can apply the same reasoning to the part $A^{\odot}$ of A^* in $X^{\odot} := \overline{\mathsf{D}(A^*)}$. Indeed, this operator is sectorial and has bounded imaginary powers on $X^{\odot}$ and $(A^{\odot})^{it}x^* = (A^{it})^*x^*$ for $x^* \in X^{\odot}$; we leave the easy verification as an exercise to the reader. Together with the identity $\phi(tA)^*x^* = \phi(tA^{\odot})x^*$, which is equally easy to verify, this gives the dual square function estimate

$$\big\|t \mapsto \phi(tA)^*x^*\big\|_{\gamma(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X^*)} \lesssim \|x^*\|, \quad x^* \in X^{\odot} = \overline{\mathsf{D}(A^*)}.$$

Hence by Lemma 15.3.22,

$$\|x\| \lesssim \big\|t \mapsto \phi(tA)x\big\|_{\gamma(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)}, \quad x \in \mathsf{D}(A) \cap \mathsf{R}(A),$$

with an implied constant independent of x. We may now apply Theorem 10.4.19 (noting that thanks to Theorem 15.3.19 we have $\omega_\gamma(A) \leqslant \omega_{\gamma\text{-BIP}}(A)$) to conclude that A has a bounded $H^\infty(\Sigma_{\vartheta'})$-calculus for all $\omega_{\gamma\text{-BIP}}(A) < \vartheta' < \theta$. This completes the proof.

(2): Let A have a bounded $H^\infty(\Sigma_\sigma)$-calculus for some $\omega(A) < \sigma < \pi$, and let $\vartheta > \sigma$. Recalling the bound $|z^{it}| \leqslant e^{|t||\arg(z)|}$, the R-boundedness (and hence the γ-boundedness, as the Pisier contraction property implies finite cotype) of the set $\{e^{-\theta|t|}A^{it}\}$ follows from Theorem 10.3.4(3). This shows that A has γ-bounded imaginary powers and $\omega_{\gamma\text{-BIP}}(A) \leqslant \vartheta$. □

15.3.g The Hilbert space case

The last main result of this chapter is the following characterisation of sectorial operators on Hilbert spaces with bounded imaginary powers.

Theorem 15.3.23. *For any standard sectorial operator A on a Hilbert space H the following assertions are equivalent:*

(1) *A has a bounded H^∞-calculus;*
(2) *A has bounded imaginary powers.*

In this situation we have

$$\omega_{H^\infty}(A) = \omega_{\mathrm{BIP}}(A).$$

If in addition we have $0 \in \varrho(A)$, then the above conditions are equivalent to

(3) $\mathsf{D}(A^{1/2}) = (H, \mathsf{D}(A))_{\frac{1}{2},2}$ *with equivalent norms.*

In view of the equivalence of uniform boundedness and γ-boundedness for families of Hilbert space operators, the equivalence of (1) and (2) is a special case of the results in the preceding subsection. A version of the equivalence of these conditions with (3) for general Banach spaces will be discussed in the Notes at the end of the chapter.

Proof. It remains to prove the implications (2)⇒(3)⇒(1) under the additional assumption $0 \in \varrho(A)$. As a preliminary observation we point out that this assumption implies that we have equivalences of norms

$$\|x\|_{\mathsf{D}(A^{1/2})} \eqsim \|A^{1/2}x\| \tag{15.27}$$

and

$$\|x\|_{(H,\mathsf{D}(A))_{\frac{1}{2},2}} \eqsim \big\|\lambda \mapsto \lambda^{1/2}A(\lambda+A)^{-1}x\big\|_{L^2(\mathbb{R}_+,\frac{\mathrm{d}t}{t};H)}. \tag{15.28}$$

Indeed, (15.27) follows by writing $x = A^{-1/2}A^{1/2}x$ and using Corollary 15.2.10 to get

$$\|A^{1/2}x\| \leqslant \|x\| + \|A^{1/2}x\| \leqslant (\|A^{-1/2}\|+1)\|A^{1/2}x\|.$$

The equivalence (15.28) follows from Proposition K.4.1.

(2)⇒(3): The equality $\mathsf{D}(A^{1/2}) = (H,\mathsf{D}(A))_{\frac{1}{2},2}$ is an immediate consequence of Peetre's theorem (Theorem C.4.1), which in the present situation implies that for each $\theta \in (0,1)$ we have

$$(H,\mathsf{D}(A))_{\theta,2} = [H,\mathsf{D}(A)]_\theta \ \text{ with equivalent norms,}$$

and Theorem 15.3.9, which identifies $[H,\mathsf{D}(A)]_{1/2}$ as the fractional domain space $\mathsf{D}(A^{1/2})$ up to an equivalent norm.

(3)⇒(1): On H define

$$|\!|\!| x |\!|\!| := \big\|\lambda \mapsto \lambda^{1/2}A^{1/2}(\lambda+A)^{-1}x\big\|_{L^2(\mathbb{R}_+,\frac{\mathrm{d}t}{t};H)}.$$

In view of (15.27) and (15.28) and the assumption in (3), we have the norm equivalences

$$|\!|\!| x |\!|\!| \eqsim \|A^{-1/2}x\|_{(H,\mathsf{D}(A))_{\frac{1}{2},2}} \eqsim \|A^{-1/2}x\|_{\mathsf{D}(A^{1/2})} \eqsim \|x\|.$$

Consequently, $|\!|\!| \cdot |\!|\!|$ defines an equivalent Hilbertian norm on H. Recalling that $\gamma(L^2(\mathbb{R}_+\frac{\mathrm{d}t}{t}),H) = L^2(\mathbb{R}_+,\frac{\mathrm{d}t}{t};H)$ isometrically, the implication now follows from Theorem 10.4.21. □

15.3.h Examples

It has already been noted that every standard sectorial operator A with a bounded $H^\infty(\Sigma_\sigma)$-calculus for some $0<\sigma<\pi$ has bounded imaginary powers. Here we wish to highlight two examples:

Example 15.3.24 (Laplacian). Let $1<p<\infty$ and let X be a Banach space. It was already noted in the discussion preceding Theorem 15.3.11 that if X is a UMD space, then the negative of the Laplace operator Δ on $L^p(\mathbb{R}^d;X)$ with domain $\mathsf{D}(\Delta) = H^{2,p}(\mathbb{R}^d;X)$ has bounded imaginary powers. In the converse direction, it was shown in Section 10.5 that if $-\Delta$ has bounded imaginary powers on $L^p(\mathbb{R}^d;X)$, then X is a UMD space.

Example 15.3.25 (First derivative). Let $1 < p < \infty$ and let X be a UMD space.

(1) The operator $A = \mathrm{d}/\mathrm{d}x$ on $L^p(\mathbb{R}; X)$ with domain $\mathsf{D}(A) = W^{1,p}(\mathbb{R}; X)$ has bounded imaginary powers with angle $\frac{1}{2}\pi$.
(2) The operator $A = \mathrm{d}/\mathrm{d}t$ on $L^p(\mathbb{R}_+; X)$ with domain $\mathsf{D}(A) = \{f \in W^{1,p}(\mathbb{R}_+; X) : f(0) = 0\}$ has bounded imaginary powers with angle $\frac{1}{2}\pi$.
(3) The operator $A = \mathrm{d}/\mathrm{d}t$ on $L^p(0,T; X)$ with domain $\mathsf{D}(A) = \{f \in W^{1,p}(0,T; X) : f(0) = 0\}$ has bounded imaginary powers with angle $\frac{1}{2}\pi$ and, more precisely, we have the estimate

$$\|A^{is}\| \lesssim_T (1+s^2)e^{\frac{1}{2}\pi|s|}, \quad s \in \mathbb{R}.$$

For the proofs of (1), (2), and the first part of (3) one may observe that in each of these three cases A is standard sectorial.

In the case (1), $-A$ generates the translation group on $L^p(\mathbb{R}; X)$, and in the other two cases $-A$ is the generator of the C_0-semigroup on $L^p(I; X)$ (with $I = \mathbb{R}_+$ resp. $(0,T)$) given by

$$S(t)f(s) = \begin{cases} f(s-t), & s \in I,\ s > t, \\ 0, & \text{otherwise.} \end{cases}$$

All three semigroups are contractive and, in the scalar-valued case, positive. It follows that we can apply the Hieber–Prüss theorem (Theorem 10.7.12), which gives that each of these operators has a bounded H^∞-calculus of angle $\frac{1}{2}\pi$. It then follows from Theorem 15.3.20 that each of the operators has bounded imaginary powers.

15.4 Strip type operators

It has already been noted in Volume II that the theories of Hardy spaces over a sector and a strip large rather similar. This similarity can be lifted to the operator level by introducing the 'strip' version of sectorial operators. Such operator admit again a Dunford calculus, a primary calculus, and an extended calculus, and one may ask about the boundedness of their H^∞-calculus. Since this topic is somewhat peripheral to the mainstream of these volumes, we will not embark on a systematic exploration of strip type operator, but rather concentrate on the relationship between sectorial operators and strip type operators. We have already seen several examples, both in Volume II and the present volume, where the relationship between sectorial operators and bisectorial operators (the mediating function being $z \mapsto z^2$) can be exploited in the study of sectorial operators. Likewise the connection with strip type operators (the mediating function being $z \mapsto e^z$) can sometimes be exploited. At the end of this section we demonstrate this by giving a proof of the Dore–Venni theorem by using the properties of strip type operators.

15.4.a Nollau's theorem

For $\vartheta > 0$ let

$$\mathbb{S}_\vartheta := \{z \in \mathbb{C} : |\Im z| < \vartheta\}$$

be the strip of height ϑ. From Appendix H we recall the definition of the Hardy space $H^p(\mathbb{S}_\vartheta)$, $1 \leqslant p \leqslant \infty$, as the Banach space of all holomorphic functions $f : \mathbb{S}_\vartheta \to \mathbb{C}$ for which the norm

$$\|f\|_{H^p(\mathbb{S}_\vartheta)} := \sup_{|y|<\vartheta} \|t \mapsto f(t+iy)\|_{L^p(\mathbb{R})}$$

is finite.

Definition 15.4.1. *A linear operator A acting in a Banach space X is said to be of* strip type $\omega > 0$ *if $\sigma(A) \subseteq \overline{\mathbb{S}_\omega}$ and*

$$\sup_{z \notin \overline{\mathbb{S}_\omega}} (|\Im z| - \omega)\|R(z, A)\| < \infty.$$

It is said to be of standard strip type $\omega > 0$ *if it is* strip type $\omega > 0$ *and* $\mathsf{D}(A) \cap \mathsf{R}(A)$ *is dense in X.*

The operator A is said to be of *(standard) strip type* if it is of (standard) strip type ω for some $\omega > 0$. The number

$$\omega^{\mathbb{S}}(A) := \inf\{\omega > 0 : \ A \text{ is of strip type } \omega\}$$

is called the *height* of A.

Example 15.4.2. By the easy part of the Hille–Yosida theorem, if iA is the generator of a C_0-group $(U(t))_{t\in\mathbb{R}}$ satisfying $\|U(t)\| \leqslant Me^{\omega|t|}$ for all $t \in \mathbb{R}$ and certain $M \geqslant 1$ and $\omega > 0$, then A is of strip type ω.

Theorem 15.4.3 (Nollau). *If A is standard sectorial, then* $\log(A)$ *is of standard strip type with $\omega^{\mathbb{S}}(A) \leqslant \omega(A)$, and the following Poisson type formula holds:*

$$R(z, \log(A)) = -\int_0^\infty \frac{1}{(z - \log t)^2 + \pi^2}(t + A)^{-1}\,\mathrm{d}t, \quad |\Im z| > \pi.$$

Proof. We proceed in two steps.

Step 1 – First we assume in addition that A is bounded and invertible. Let $\omega(A) < \nu' < \nu < \sigma < \pi$ and fix $\lambda \in \mathbb{C}$ with $|\Im\lambda| > \pi$ and $\mu \in \Sigma_\nu \setminus \Sigma_{\nu'}$. The function $z \mapsto 1/(\lambda - \log z)$ is holomorphic and bounded on Σ_σ. Let $x \in X$. Then by Proposition 15.1.19,

$$\frac{1}{\lambda - \log}(A)x$$

$$
\begin{aligned}
&= \frac{1}{\lambda - \log \mu} x + \frac{1}{2\pi i} \int_{\partial \Sigma_\nu} \frac{1}{\lambda - \log z} \Big(R(z, A) - \frac{1}{z - \mu} \Big) x \, \mathrm{d}z \\
&= \frac{1}{\lambda - \log \mu} x - \frac{1}{2\pi i} \int_0^\infty \frac{e^{i\nu}}{\lambda - i\nu - \log t} \Big(R(te^{i\nu}, A) - \frac{1}{te^{i\nu} - \mu} \Big) x \, \mathrm{d}t \\
&\qquad + \frac{1}{2\pi i} \int_0^\infty \frac{e^{-i\nu}}{\lambda + i\nu - \log t} \Big(R(e^{-i\nu}, A) - \frac{1}{te^{-i\nu} - \mu} \Big) x \, \mathrm{d}t.
\end{aligned}
$$

By dominated convergence we may pass to the limit $\nu \to \pi$ and obtain

$$
\begin{aligned}
&\frac{1}{\lambda - \log}(A)x \\
&\quad = \frac{1}{\lambda - \log \mu} x - \frac{1}{2\pi i} \int_0^\infty \frac{1}{\lambda - i\pi - \log t} \Big((t + A)^{-1} - \frac{1}{t + \mu} \Big) x \, \mathrm{d}t \\
&\qquad\qquad + \frac{1}{2\pi i} \int_0^\infty \frac{1}{\lambda + i\pi - \log t} \Big((t + A)^{-1} - \frac{1}{t + \mu} \Big) x \, \mathrm{d}t \\
&\quad = \frac{1}{\lambda - \log \mu} x - \int_0^\infty \frac{1}{(\lambda - \log t)^2 + \pi^2} \Big((t + A)^{-1} - \frac{1}{t + \mu} \Big) x \, \mathrm{d}t \\
&\quad = - \int_0^\infty \frac{1}{(\lambda - \log t)^2 + \pi^2} (t + A)^{-1} x \, \mathrm{d}t,
\end{aligned}
$$

where we used that

$$
\begin{aligned}
\int_0^\infty \frac{1}{(\lambda - \log t)^2 + \pi^2} \frac{1}{t + \mu} \, \mathrm{d}t &= - \lim_{\nu \to \pi} \frac{1}{2\pi i} \int_{\partial \Sigma_\nu} \frac{1}{\lambda - \log z} \frac{1}{z - \mu} \, \mathrm{d}z \\
&= - \frac{1}{\lambda - \log \mu}.
\end{aligned}
$$

By the multiplicativity of the extended calculus, $\frac{1}{\lambda - \log}(A)$ is inverse to $\lambda - \log(A)$. This gives $\lambda \in \varrho(\log(A))$ as well as the identity for the resolvent. The resolvent estimate follows from the following estimates, where we write $z = x + iy$ and set $M := \sup_{t>0} \|t(t + A)^{-1}\|$:

$$
\begin{aligned}
\|R(z, \log(A))\| &\leqslant M \int_0^\infty \frac{1}{|(z - \log t)^2 + \pi^2|} \frac{\mathrm{d}t}{t} \\
&\leqslant M \int_{-\infty}^\infty \frac{1}{|(z - s)^2 + \pi^2|} \, \mathrm{d}s \\
&= M \int_{-\infty}^\infty \frac{1}{(((x - s)^2 - y^2 + \pi^2)^2 + (2(x - s)y)^2)^{1/2}} \, \mathrm{d}s \\
&= M \int_{-\infty}^\infty \frac{1}{((r^2 - y^2 + \pi^2)^2 + 4r^2 y^2)^{1/2}} \, \mathrm{d}r \\
&\leqslant M \int_{-\infty}^\infty \frac{1}{r^2 + y^2 - \pi^2} \, \mathrm{d}r \\
&= \frac{M\pi}{(y^2 - \pi^2)^{1/2}}
\end{aligned}
$$

$$\leqslant \frac{M\pi}{|y| - \pi},$$

where we used the elementary inequalities

$$\begin{aligned}(r^2 - y^2 + \pi^2)^2 + 4r^2y^2 &= r^4 + y^4 + \pi^4 + 2r^2y^2 + 2\pi^2r^2 - 2\pi^2y^2 \\ &\geqslant r^4 + y^4 + \pi^4 + 2r^2y^2 - 2\pi^2r^2 - 2\pi^2y^2 \\ &= (r^2 + y^2 - \pi^2)^2\end{aligned}$$

and (keeping in mind that $|y| > \pi$, so $2|y| - \pi > \pi$)

$$y^2 - \pi^2 \geqslant y^2 - \pi(2|y| - \pi) = (|y| - \pi)^2.$$

This proves the theorem under the additional assumption that A is bounded and has bounded inverse.

Step 2 – To deduce the general case, for $\varepsilon > 0$ we consider the operators

$$A_\varepsilon = (A + \varepsilon)(I + \varepsilon A)^{-1}.$$

For $\lambda \geqslant 0$ we have

$$\begin{aligned}\lambda + A_\varepsilon &= \lambda(I + \varepsilon A)(I + \varepsilon A)^{-1} + (A + \varepsilon)(I + \varepsilon A)^{-1} \\ &= (\lambda + \varepsilon + (\lambda\varepsilon + 1)A)(I + \varepsilon A)^{-1},\end{aligned}$$

and therefore $\lambda + A_\varepsilon$ is invertible. For $\lambda > 0$ we estimate

$$\begin{aligned}\|(\lambda + A_\varepsilon)^{-1}\| &= \|(I + \varepsilon A)(\lambda + \varepsilon + (\lambda\varepsilon + 1)A)^{-1}\| \\ &= \frac{\varepsilon}{\lambda\varepsilon + 1}\Big\|(\frac{1}{\varepsilon} + A)(\frac{\lambda + \varepsilon}{\lambda\varepsilon + 1} + A)^{-1}\Big\| \\ &= \frac{\varepsilon}{\lambda\varepsilon + 1}\Big\|I + (\frac{1}{\varepsilon} - \frac{\lambda + \varepsilon}{\lambda\varepsilon + 1})(\frac{\lambda + \varepsilon}{\lambda\varepsilon + 1} + A)^{-1}\Big\| \\ &\leqslant \frac{1}{\lambda} + \frac{\varepsilon}{\lambda\varepsilon + 1}\Big(\frac{1}{\varepsilon} - \frac{\lambda + \varepsilon}{\lambda\varepsilon + 1}\Big)\frac{\lambda\varepsilon + 1}{\lambda + \varepsilon}M_A \\ &= \frac{1}{\lambda} + \frac{1 - \varepsilon^2}{(\lambda + \varepsilon)(\lambda\varepsilon + 1)}M_A \\ &\leqslant \frac{1 + M_A}{\lambda},\end{aligned}$$

where $M_A = \sup_{\lambda > 0} \|\lambda(\lambda + A)^{-1}$. It follows that

$$\sup_{\varepsilon > 0}\Big(\sup_{\lambda > 0} \|\lambda(\lambda + A_\varepsilon)^{-1}\|\Big) \leqslant 1 + M_A,$$

and therefore the operators A_ε are uniformly sectorial. In particular the results of Step 1 apply to A_ε, with bounded that are uniform in $\varepsilon > 0$.

Step 3 – Take $0 < \nu < \pi$ close enough to π so that $\partial\Sigma_\delta$ is contained in the resolvent set of each A_ε. Noting that $R(z, A_\varepsilon)x \to R(z, A)x$ uniformly on

$\partial\Sigma_\delta$ (this follows from uniform sectoriality and similar estimates as above), we may pass to the limit $\varepsilon \downarrow 0$ and obtain

$$\begin{aligned}\frac{1}{\lambda - \log}(A)x &= \frac{1}{\lambda - \log\mu}x + \frac{1}{2\pi i}\int_{\partial\Sigma_\nu}\frac{1}{\lambda-\log z}\Big(R(z,A) - \frac{1}{z-\mu}\Big)x\,\mathrm{d}z\\ &= \lim_{\varepsilon\downarrow 0}\frac{1}{\lambda - \log\mu}x + \frac{1}{2\pi i}\int_{\partial\Sigma_\nu}\frac{1}{\lambda-\log z}\Big(R(z,A_\varepsilon) - \frac{1}{z-\mu}\Big)x\,\mathrm{d}z\\ &= \lim_{\varepsilon\downarrow 0} -\int_0^\infty \frac{1}{(\lambda-\log t)^2+\pi^2}(t+A_\varepsilon)^{-1}x\,\mathrm{d}t\\ &= -\int_0^\infty \frac{1}{(\lambda-\log t)^2+\pi^2}(t+A)^{-1}x\,\mathrm{d}t.\end{aligned}$$

Next we show that $\lambda \in \varrho(\log(A))$ and $\frac{1}{\lambda-\log}(A)$ is a two-sided inverse for $\lambda - \log(A)$. For $x \in \mathsf{D}(A^2)\cap\mathsf{R}(A^2)$, say $x = \zeta^2(A)y$, by the general properties of the extended Dunford calculus we have $\frac{1}{\lambda-\log}(A)x \in \mathsf{D}(A)\cap\mathsf{R}(A) \subseteq \mathsf{D}(\log(A))$ and

$$\begin{aligned}(\lambda - \log(A))\frac{1}{\lambda-\log}(A)x &= \zeta(A)(\lambda-\log(A))\frac{\zeta}{\lambda-\log}(A)y\\ &= (\zeta(\lambda-\log))(A)\frac{\zeta}{\lambda-\log}(A)y\\ &= \Big(\zeta(\lambda-\log)\frac{\zeta}{\lambda-\log}\Big)(A)y\\ &= \zeta^2(A)y = x\end{aligned}$$

and similarly $\frac{1}{\lambda-\log}(A)(\lambda-\log(A))x = x$. By density and closedness, these identities extend to general $x \in X$ and $x \in \mathsf{D}(\log(A))$, respectively.

Finally, the strip type estimate for A follows from the corresponding estimate for A_ε proved above, by letting $\varepsilon \downarrow 0$ and using dominated convergence once more. □

One may set up a Dunford calculus and extended Dunford calculus for strip type operators in much the same way as we did for sectorial operators as follows. For an operator A of strip type and $f \in H^1(\mathbb{S}_\sigma)$, where $\sigma > \omega^{\mathbb{S}}(A)$, the Dunford integral

$$f(A)x := \frac{1}{2\pi i}\int_{\partial\mathbb{S}_\nu} f(z)R(z,A)x\,\mathrm{d}x,$$

defines a bounded operator $f(A)$ on X. The defining integral converges absolutely and by Cauchy's theorem it is independent of the choice of ν. Moreover,

$$\|f(A)\| \leqslant \limsup_{\nu\downarrow\omega^{\mathbb{S}}(A)}\frac{1}{2\pi}\frac{C}{\nu-\omega}\int_{|\Im z|=\nu}|f(z)|\,|\mathrm{d}z| \leqslant \frac{1}{\pi}\frac{C}{\sigma-\omega^{\mathbb{S}}(A)}\|f\|_{H^1(\mathbb{S}_\sigma)}.$$

The elementary properties of the extended Dunford calculus extend to the strip case.

15.4.b Monniaux's theorem

We have seen (Proposition 15.3.5) that if an operator B in a Banach space X has bounded imaginary powers, then the bounded operators B^{it} form a C_0-group on X. In this subsection we will show that if X is a UMD space, then conversely every C_0-group on X of growth type less than π with injective generator is of the form $U(t) = B^{it}$ for some operator B in X with bounded imaginary powers:

Theorem 15.4.4 (Monniaux). *Let $(U(t))_{t\in\mathbb{R}}$ be a C_0-group on a UMD space X satisfying $\|U(t)\| \leqslant Me^{\omega|t|}$ for all $t \in \mathbb{R}$ and some $M \geqslant 1$ and $0 \leqslant \omega < \pi$. Assume furthermore that its generator iA is injective. Then there exists an operator B in X with bounded imaginary powers, given by*

$$B^{it} = U(t), \quad t \in \mathbb{R}.$$

Moreover, we have $A = \log(B)$ with equal domains.

Intuitively, one has $B = e^A$; the identity $B^{it} = U(t)$ then corresponds to the intuition that $(e^A)^{it} = e^{itA}$.

The proof of the theorem relies on several ingredients. The first is the following lemma.

Lemma 15.4.5. *Let $(U(t))_{t\in\mathbb{R}}$ be a C_0-group on a UMD space X. If, for some $M \geqslant 1$ and $\omega \in \mathbb{R}$, we have $\|U(t)\| \leqslant Me^{\omega|t|}$ for all $t \in \mathbb{R}$, then for all $x \in X$ the principal value integral*

$$\text{p.v.} \int_{-1}^{1} U(t)x \,\frac{\mathrm{d}t}{t}$$

converges in X and has norm

$$\Big\|\text{p.v.} \int_{-1}^{1} U(t)x \,\frac{\mathrm{d}t}{t}\Big\| \leqslant 6C^2 \hbar_{2,X} \|x\|,$$

where $\hbar_{2,X} := \|H\|_{\mathscr{L}(L^2(\mathbb{R};X))}$, and $C := \sup_{|t|\leqslant 2} \|U(t)\|$.

Proof. All we need to do is stripping the Rademacher sums from the estimates in the last part of the proof of Theorem 15.3.12(2). For the reader's convenience we include the proof that results from this.

Fix $0 < \delta < 1$, $s \in [-\frac{1}{2}, \frac{1}{2}]$, and $x \in X$. Then

$$\begin{aligned}\int_{\delta<|t|<1} U(t)x \,\frac{\mathrm{d}t}{t} &= U(s) \int_{\delta<|t|<1} U(t-s)x \,\frac{\mathrm{d}s}{s} \\ &= U(s) \int_{|t|>\delta} \varphi_x(s-t) \,\frac{\mathrm{d}t}{t} \\ &\qquad - \int_{1}^{1+s} U(t)x \,\frac{\mathrm{d}t}{t} + \int_{-1}^{-1+s} U(t)x \,\frac{\mathrm{d}t}{t},\end{aligned}$$

where $\varphi_x(\tau) = \mathbf{1}_{(-1,1)}(\tau)U(-\tau)x$. Integrating over $[-\frac{1}{2}, \frac{1}{2}]$, we obtain

$$\int_{\delta<|t|<1} U(t)x \frac{\mathrm{d}t}{t} = \int_{-\frac{1}{2}}^{\frac{1}{2}} U(s) \int_{|t|>\delta} \varphi_x(s-t) \frac{\mathrm{d}t}{t} \,\mathrm{d}s - \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{1}^{1+s} U(t)x \frac{\mathrm{d}t}{t} \,\mathrm{d}s + \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-1}^{-1+s} U(t)x \frac{\mathrm{d}t}{t} \,\mathrm{d}s.$$

Since X is UMD and $\phi_x \in L^2(\mathbb{R}; X)$, the limit

$$\lim_{\delta\downarrow 0} \int_{|t|>\delta} \varphi_x(\cdot - t) \frac{\mathrm{d}t}{t} = \lim_{\substack{\delta\downarrow 0\\ R\to\infty}} \int_{\delta<|t|<R} \varphi_x(\cdot - t) \frac{\mathrm{d}t}{t}$$

exists in $L^2(\mathbb{R}; X)$ by Theorem 5.1.1 and equals $\pi H\phi_x$, where H denotes the Hilbert transform. As a result we obtain

$$\begin{aligned} \text{p.v.} \int_{-1}^{1} U(t)x \frac{\mathrm{d}t}{t} &= \lim_{\delta\downarrow 0} \int_{\delta<|t|<1} U(t)x \frac{\mathrm{d}t}{t} \\ &= \pi \int_{-\frac{1}{2}}^{\frac{1}{2}} U(s)H\varphi_x(s) \,\mathrm{d}s \\ &\quad - \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{1}^{1+s} U(t)x \frac{\mathrm{d}t}{t} \,\mathrm{d}s + \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-1}^{-1+s} U(t)x \frac{\mathrm{d}t}{t} \,\mathrm{d}s \\ &=: I + II + III. \end{aligned}$$

With constants $C := \sup_{|t|\leqslant 2} \|U(t)\|$ and $\hbar_{2,X} := \|H\|_{\mathscr{L}(L^2(\mathbb{R};X))}$, we have

$$\|I\| \leqslant \pi C\Big(\int_{-\frac{1}{2}}^{\frac{1}{2}} \|H\varphi_x(s)\|^2 \,\mathrm{d}s\Big)^{\frac{1}{2}} \leqslant \pi C\hbar_{2,X} \|\varphi_x\|_{L^2(\mathbb{R};X)} \leqslant \sqrt{2}\pi C^2 \hbar_{2,X} \|x\|.$$

The other two terms are elementary with

$$\|II\| \leqslant \int_{-\frac{1}{2}}^{\frac{1}{2}} C\Big| \int_{1}^{1+s} \frac{\mathrm{d}t}{t}\Big| \|x\| \,\mathrm{d}s \leqslant C\log 2\|x\|,$$

and III can bounded in exactly the same way. Note that both $C \geqslant \|U(0)\| = 1$ and $\hbar_{2,X} \geqslant 1$, and $\sqrt{2}\pi + 2\log 2 < 6$. □

We will use this lemma for the second ingredient for the proof of Theorem 15.4.4, a primary calculus for strip type operators. We work under the assumptions of Theorem 15.4.4 and let $\omega < \sigma < \pi$. For functions

$$g \in L^1_\omega(\mathbb{R}) = \big\{g \in L^1_{\mathrm{loc}}(\mathbb{R}) : \, t \mapsto e^{\omega|t|}g(t) \in L^1(\mathbb{R})\big\}$$

we define the bounded operator $\widehat{g}(A) := \varPhi_g(A)$ by the Phillips calculus (see Section 10.7.a):

$$\widehat{g}(A)x := \int_0^\infty g(t)U(t)x \,\mathrm{d}t, \quad x \in X.$$

Obviously,

$$\|\widehat{g}(A)\|_{\mathscr{L}(X)} \leqslant M\|g\|_{L^1_\omega(\mathbb{R})},$$

where $\|g\|_{L^1_\omega(\mathbb{R})} := \|t \mapsto e^{\omega|t|}g(t)\|_{L^1(\mathbb{R})}$ and M is in Theorem 15.4.4. The following lemma enables us to enrich this calculus with certain bounded functions in $H^\infty(\mathbb{S}_\sigma)$ which have limits for $\Re z \to \pm\infty$.

Lemma 15.4.6. *The Fourier transform of the distribution*

$$h(t) := \mathrm{p.v.}\, \frac{1}{t}\mathbf{1}_{(-1,1)}(t) \qquad \Big(\langle h, \phi\rangle := \lim_{\varepsilon\downarrow 0} \int_{\varepsilon<|t|<1} \phi(t)\, \frac{\mathrm{d}t}{t} \quad \forall \phi \in \mathscr{S}(\mathbb{R})\Big),$$

equals

$$\widehat{h}(\xi) = \int_{-1}^1 e^{-2\pi i t\xi}\, \frac{\mathrm{d}t}{t} = 2\int_0^1 \sin(2\pi t\xi)\, \frac{\mathrm{d}t}{t}$$

and its analytic continuation to $\mathbb{S}_\sigma$ satisfies

$$\lim_{\substack{|\Im z|<\sigma \\ \Re z\to\infty}} \widehat{h}(z) = \pm\pi.$$

Proof. The first assertion follows by elementary computation and the second from the standard improper integral

$$\int_0^\infty \sin t\, \frac{\mathrm{d}t}{t} = \frac{\pi}{2}$$

and a change of variables. □

For small $\varepsilon > 0$ let $h_\varepsilon(t) := t^{-1}\mathbf{1}_{(-1,-\varepsilon]\cup[\varepsilon,1)}(t)$. Applying the Phillips calculus, we obtain

$$\widehat{h_\varepsilon}(A)x = \int_{-1}^1 h_\varepsilon(t)U(t)x \,\mathrm{d}t$$

and therefore the principal value integral

$$\widehat{h}(A)x := \mathrm{p.v.} \int_{-\pi}^{\pi} U(t)x\, \frac{\mathrm{d}t}{t}$$

exists and satisfies

$$\|\widehat{h}(A)x\| \leqslant 6C^2\hbar_{2,X}\|x\|$$

with constants as in Lemma 15.4.5.

Now we are in a position to define our primary calculus:

Definition 15.4.7 (Primary calculus). *Let A be a strip type operator and let $\omega^{\mathbb{S}}(A) < \omega < \sigma$. For functions $f : \mathbb{S}_\sigma \to \mathbb{C}$ of the form*

$$f = \widehat{g} + a\widehat{h} + b$$

with $a, b \in \mathbb{C}$, $g \in L^1_\omega(\mathbb{R})$, and h as above, we set

$$f(A) := \widehat{g}(A) + a\widehat{h}(A) + bI.$$

The condition on f is satisfied if f' is bounded and there exists an $\alpha > 1$ such that

$$f'(z) = O(|z|^\alpha) \ \text{ as } \ |\Re z| \to \infty. \tag{15.29}$$

The primary calculus enjoys similar properties as the one for sectorial operators; in particular it is multiplicative and consistent with the Dunford calculus. The proof is elementary but a bit tedious and it is therefore left to the reader.

For every $r \in \mathbb{R}$ the primary calculus can be applied to $A + r$ in place of A, noting that $i(A+r)$ generates the C_0-group $(e^{irt}U(t))_{t\in\mathbb{R}}$. It is immediate from the above constructions that the estimates are uniform with respect to r, i.e., for all $f : \mathbb{S}_\sigma \to \mathbb{C}$ of the above form we have

$$\sup_{r\in\mathbb{R}} \|f(A+r)\| < \infty. \tag{15.30}$$

We will now exploit the fact that, for $0 \leqslant |\omega| < \sigma < \pi$, the exponential function $z \mapsto e^z$ maps the line $\Im z = \omega$ bijectively onto the ray $\arg(z) = \omega$. Thus, it maps the strip $\mathbb{S}_\sigma$ bijectively onto the sector Σ_σ. From this, we infer that if $\lambda \in \complement\overline{\Sigma_\sigma}$, then the function

$$r_\lambda(z) := \frac{1}{\lambda - e^z} \tag{15.31}$$

is bounded and holomorphic on $\mathbb{S}_\sigma$. What is more, this function is of the form discussed above and therefore $r_\lambda(A)$ is well defined in the primary calculus (as its derivative satisfies (15.29)).

Remark 15.4.8. In hindsight, one could have introduced the primary calculus using the functions r_λ instead of h. This would restore the symmetry with the definition of the primary calculus for sectorial operator. Hoever, this would require an independent construction of the operators $r_\lambda(A) = R(\lambda, B)$ by different means.

By the algebraic properties of the functions r_λ and the multiplicativity properties of the calculus, the operators R_λ form a *pseudo-resolvent* in the sense of the following proposition.

Lemma 15.4.9 (Pseudo-resolvents). *Let $V \subseteq \mathbb{C}$ be a non-empty connected open set and let $(R_\lambda)_{\lambda\in V}$ be a family of bounded operators on a Banach space X satisfying the resolvent identity*

$$R_\lambda - R_\mu = (\mu - \lambda)R_\lambda R_\mu, \quad \lambda, \mu \in V.$$

If R_{λ_0} is injective for some $\lambda_0 \in V$, then there exists a unique closed operator B on X such that $V \subseteq \varrho(B)$ and $R_\lambda = (\lambda - B)^{-1}$ for all $\lambda \in V$.

Proof. The resolvent identity implies that any two R_λ and R_μ commute.

If $R_\lambda x = 0$, the identity $R_\lambda - R_{\lambda_0} = (\lambda_0 - \lambda)R_{\lambda_0} R_\lambda$ implies that $R_{\lambda_0} x = 0$ and therefore $x = 0$. It follows that R_λ is injective. If $y \in \mathsf{R}(R_{\lambda_0})$, there is a unique $x \in X$ such that $y = R_{\lambda_0} x$. Then $y = R_\lambda(I - (\lambda_0 - \lambda)R_{\lambda_0})x \in \mathsf{R}(R_\lambda)$.

This shows that $\mathsf{N} := \mathsf{N}(R_\lambda) = \{0\}$ and $\mathsf{R} := \mathsf{R}(R_\lambda)$ are independent of $\lambda \in V$. Hence if $y \in \mathsf{R}$, then for all $\lambda \in V$ there is a unique $x_\lambda \in X$ such that $y = R_\lambda x_\lambda$. Then, by the resolvent identity,

$$(\mu - \lambda)R_\lambda R_\mu y = R_\lambda y - R_\mu y = R_\lambda R_\mu x_\mu - R_\mu R_\lambda x_\lambda = R_\lambda R_\mu (x_\mu - x_\lambda).$$

It follows that $(\mu - \lambda)y = (x_\mu - x_\lambda)$. This implies that $\mu y - x_\mu = \lambda y - x_\lambda$ is independent of $\lambda, \mu \in V$. Denoting this element by By, we obtain a linear operator $B : y \mapsto \lambda y - x_\lambda$ with domain $\mathsf{D}(B) = \mathsf{R}(R_{\lambda_0})$. It satisfies

$$R_\lambda(\lambda - B)y = R_\lambda x_\lambda = y,$$

so R_λ is a left inverse to $\lambda - B$. Applying this to $R_\lambda y$ instead of y we also obtain $R_\lambda(\lambda - B)R_\lambda y = R_\lambda y$, and the injectivity of R_λ therefore gives $(\lambda - B)R_\lambda y = y$, so R_λ is a right inverse to $\lambda - B$. This proves that $\lambda \in \varrho(B)$ and $(\lambda - B)^{-1} = R_\lambda$. That B is closed follows from the fact that its resolvent set contains the non-empty set V. □

This construction gives us a rigorous way to construct the operator e^A as the closed operator B given by the lemma.

Proof of Theorem 15.4.4. By Lemma 15.4.9 there exists a unique closed operator B on X such that $R_\lambda = r_\lambda(A) = (\lambda - B)^{-1}$ for all $\lambda \in V := \complement\overline{\Sigma_\sigma}$. We note that

$$\lambda R(\lambda, B) = (I - \lambda^{-1} B)^{-1} = r_1(A - \log \lambda),$$

and therefore (15.30) implies that B is σ-sectorial.

Since sectorial operators on reflexive Banach spaces are densely defined (see Proposition 10.1.9), the operator B is densely defined. We claim that B in injective. If $\lambda, \mu \in \complement\overline{\Sigma_\sigma}$, then

$$(\lambda - B)^{-1}(\mu - B)^{-1}B \subseteq f_{\lambda,\mu}(B) \ \text{ with } \ f_{\lambda,\mu}(z) = \frac{e^z}{(\lambda - e^z)(\mu - e^z)}.$$

The operator $f_{\lambda,\mu}(B)$ is injective in view if the identity

$$f_{\lambda,\mu}(B) = -\frac{1}{\lambda - e^z}(B)\frac{\mu^{-1}}{\mu^{-1} - e^{-z}}(B).$$

This proves the claim. As a consequence of Proposition 15.3.2, we obtain that B is standard sectorial.

The identity $B^{it} = U(t)$ follows by writing out the definition of B^{it} in the extended calculus of B. This results in an expression involving a Dunford integral containing the resolvent of B. This resolvent can be expressed, via the definition of the primary calculus, in terms of the Phillips calculus of the C_0-group U. The details are laborious and are left to the reader. From this, and the general properties of the extended calculus, it follows that $A = \log(B)$. □

15.4.c The Dore–Venni theorem

In this section we apply Monniaux's theorem (Theorem 15.4.4) to prove the celebrated Dore–Venni theorem on the closedness of sums of closed operators. We base our proof on the following lemma. It uses the fact that if iG is the generator of a bounded C_0-group, then G is bisectorial of angle 0 (see Example 10.6.3). In what follows we use that the extended Dunford calculus can be developed also for bisectorial operators. When we cite results from Section 15.1, it is understood that we actually refer to their bisectorial counterparts. We leave it to the reader to verify that these counterparts do indeed hold.

Lemma 15.4.10. *Let $(U(t))_{t\in\mathbb{R}}$ and $(V(t))_{t\in\mathbb{R}}$ be commuting C_0-groups on a Banach space X with generators $-iA$ and $-iB$, respectively, such that*

$$\|U(t)\| \leqslant M_A e^{\omega_A|t|} \quad \text{and} \quad \|V(t)\| \leqslant M_B e^{\omega_B|t|}$$

for all $t \in \mathbb{R}$, where $M_A, M_B \geqslant 1$ and $\omega_A, \omega_B \geqslant 0$ satisfy $\omega_A + \omega_B \leqslant \pi$. Let $-iC$ denote the generator of the C_0-group $W(t) = U(t)V(t)$, $t \in \mathbb{R}$. Then for all $x \in \mathsf{D}(e^A e^B)$ we have $x \in \mathsf{D}(e^C)$ and

$$e^A e^B x = e^C x.$$

Proof. We begin by noting that

$$\|U(t)\| \leqslant M_A M_B e^{(\omega_A+\omega_B)|t|}, \quad t \in \mathbb{R}.$$

In what follows we fix $\omega_A + \omega_B < \sigma < \pi$.

Fix $0 < \vartheta < \frac{1}{2}\pi$ and consider the functions $f, g \in H^1(\Sigma_\vartheta)$ given by

$$f(z) = \begin{cases} e^{2z}(1+e^z)^{-3}, & z \in \Sigma_\vartheta^+, \\ e^{-z}(1+e^{-z})^{-3}, & z \in \Sigma_\vartheta^-, \end{cases}$$

and

$$g(z) = \begin{cases} e^z(1+e^z)^{-3}, & z \in \Sigma_\vartheta^+, \\ e^{-2z}(1+e^{-z})^{-3}, & z \in \Sigma_\vartheta^-. \end{cases}$$

For $G \in \{A, B, C\}$ the operators $f(G)$ and $g(G)$ are well defined and bounded in the bisectorial Dunford calculus. Moreover, by (the bisectorial counterpart of) Proposition 15.1.12 these operators are injective and $e^z = f(z)/g(z)$ implies

$$e^G = g(G)^{-1}f(G).$$

Our aim is to prove that

$$f(A)f(B)g(C) = g(A)g(B)f(C). \tag{15.32}$$

Once we have shown this, from $f(A)g(B)^{-1} \subseteq g(B)^{-1}f(A)$ (this follows by observing that $\mathsf{D}(g(B)^{-1}) = \mathsf{R}(g(B))$ and using that $f(A)$ and $g(B)$ commute, we obtain

$$\begin{aligned} e^A e^B &= g(A)^{-1}f(A)g(B)^{-1}f(B) \\ &\subseteq g(A)^{-1}g(B)^{-1}f(A))f(B) \\ &= (g(A)^{-1}g(B)^{-1}g(C)^{-1})(g(C)f(A))f(B)) \\ &= (g(C)^{-1}g(A)^{-1}g(B)^{-1})(g(A)g(B)f(C)) \\ &= (g(C)^{-1}g(A)^{-1}g(B)^{-1})(g(B)g(A)f(C)) \\ &= g(C)^{-1}f(C) = e^C, \end{aligned}$$

using that $g(A)$ and $g(B)$ commute in the penultimate equality.

The proof of (15.32) relies on the properties of the Phillips calculus (see Section 10.7.a). We recall from Proposition 10.7.2 and Lemma 10.7.3 that if iG generates a bounded C_0-group $(W(t))_{t\in\mathbb{R}}$ on X, then for $0 < \vartheta < \frac{1}{2}\pi$ and $h \in H^1(\Sigma_\vartheta^{\text{bi}})$ one has

$$h(G)x = \int_{-\infty}^{\infty} \phi_h(t)W(t)x \,\mathrm{d}t, \quad x \in X,$$

where $\phi_h \in L^1(\mathbb{R})$ is given by

$$\widehat{\phi_h}(\xi) = h(-2\pi\xi), \quad \xi \in \mathbb{R}.$$

Applying this to $G \in \{-A, -B, -C\}$ and $h \in \{f, g\}$, and keeping in mind that $-iA$, $-iB$, and $-iC$ generate the groups $U(t)$, $V(t)$, and $U(t)V(t)$, respectively, the identity (15.32) takes the form

$$\begin{aligned} &\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \phi_g(r)\phi_f(t)\phi_f(s)U(t+r)V(s+r)\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}t \\ &\qquad = \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \phi_f(r)\phi_g(t)\phi_g(s)U(t+r)V(s+r)\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}t, \end{aligned}$$

or equivalently,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \phi_g(r)F(t,s)U(t)V(s)\,\mathrm{d}s\,\mathrm{d}t = \int_{\mathbb{R}}\int_{\mathbb{R}} \phi_g(r)G(t,s)U(t)V(s)\,\mathrm{d}s\,\mathrm{d}t$$

with

$$F(t,s) = \int_{\mathbb{R}} \phi_g(r)\phi_f(t-r)\phi_f(s-r)\,\mathrm{d}r, \quad G(t,s) = \int_{\mathbb{R}} \phi_f(r)\phi_g(t-r)\phi_g(s-r)\,\mathrm{d}r.$$

Taking Fourier transforms, we obtain

$$\begin{aligned}
\widehat{F}(x,y) &= \int_{\mathbb{R}}\int_{\mathbb{R}} F(t,s)e^{-2\pi i(tx+sy)}\,\mathrm{d}t\,\mathrm{d}s \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \phi_g(r)\phi_f(t-r)\phi_f(s-r)e^{-2\pi i(tx+sy)}\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}s \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \phi_g(r)\phi_f(t-r)\phi_f(s-r)e^{-2\pi itx}e^{-2\pi isy}e^{-2\pi ir(x+y)}\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}s \\
&= \widehat{\phi_g}(x+y)\widehat{\phi_f}(x)\widehat{\phi_f}(y) \\
&= g(-2\pi(x+y))f(-2\pi x)f(-2\pi y)
\end{aligned}$$

and similarly

$$\widehat{G}(x,y) = f(-2\pi(x+y))g(-2\pi x)g(-2\pi y).$$

It is evident from the definitions of f and g that the two right-hand sides are equal. Therefore $F = G$ be the uniqueness of Fourier transforms. This completes the proof of (15.32). □

Theorem 15.4.11 (Dore–Venni). *Suppose that A and B are resolvent commuting standard sectorial operators on a UMD Banach space X. If both A and B have bounded imaginary powers with*

$$\omega_{\mathrm{BIP}}(A) + \omega_{\mathrm{BIP}}(B) < \pi,$$

then there exists a constant $K \geqslant 0$ such that

$$\|Ax\| + \|Bx\| \leqslant K\|(A+B)x\|, \quad x \in \mathsf{D}(A) \cap \mathsf{D}(B).$$

As a consequence $A + B$, with its natural domain $\mathsf{D}(A+B) = \mathsf{D}(A) \cap \mathsf{D}(B)$, is closed.

Proof. Fix $\omega_{\mathrm{BIP}}(A) + \omega_{\mathrm{BIP}}(B) < \omega < \pi$. Since A and B resolvent commute, the operators $U_A(s) = A^{is}$ and $U_B(t) = B^{it}$ commute for all $s, t \in \mathbb{R}$ and $U(t) := A^{it}B^{-it}$ is a C_0-group satisfying $\|U(t)\| \leqslant Ke^{\omega|t|}$ for all $t \in \mathbb{R}$ and some $K \geqslant 1$. Hence by Monniaux's Theorem 15.4.4 we have $U(t) = C^{it}$, $t \in \mathbb{R}$, for some standard sectorial operator C having bounded imaginary powers. The generators of the C_0-groups equal $i\log A$, $-i\log B$, and $i\log C$. Since $I+C$ is invertible there is a constant $M\geqslant 0$ such that have

$$\|x\|\leqslant M\|x+Cx\|,\quad x\in \mathsf{D}(C).$$

By Lemma 15.4.10 applied to $i\log A$ and $-i\log B$, for all $x\in\mathsf{D}(AB^{-1})$ we have $x\in\mathsf{D}(C)$ and $AB^{-1}x=Cx$. Hence for all $x\in\mathsf{D}(A)\cap\mathsf{D}(B)$ we have $Bx\in\mathsf{D}(AB^{-1})\subseteq\mathsf{D}(C)$, and therefore

$$\|Bx\|\leqslant M\|Bx+CBx\|=M\|Bx+Ax\|.$$

The same argument with the roles of A and B interchanged gives the inequality

$$\|Ax\|\leqslant M\|Bx+CBx\|=M\|Bx+Ax\|.$$

Together, these two inequalities imply the inequality in the statement of the theorem. This implies the closedness of $A+B$ by a routine argument. □

15.5 The bisectorial H^∞-calculus revisited

The bisectorial H^∞-calculus has been introduced and studied in Section 10.6. The purpose of the present section is to study in more detail the spectral projections associated with the left and right halves of the bisector. These can be thought of as abstract Riesz projections. The main result is Theorem 15.5.2, which establishes that if A is a standard bisectorial operator with a bounded $H^\infty(\Sigma_\sigma^{\rm bi})$-calculus, then A^2 is a standard 2σ-sectorial operator and

$$\mathsf{D}((A^2)^{1/2})=\mathsf{D}(A),$$

with equivalence of norms

$$\|(A^2)^{1/2}x\|\eqsim\|Ax\|.$$

We use the notation introduced in Section 10.6. Specifically, for $0<\omega<\frac12\pi$ we define $\Sigma_\omega^+:=\Sigma_\omega$ and $\Sigma_\omega^-:=-\Sigma_\omega$, and denote by

$$\Sigma_\omega^{\rm bi}:=\Sigma_\omega^+\cup\Sigma_\omega^-$$

the *bisector* of angle ω. Recall that a linear operator A on a Banach space X is said to be *bisectorial* if there exists an $\omega\in(0,\frac12\pi)$ such that the spectrum $\sigma(A)$ is contained in $\overline{\Sigma_\omega^{\rm bi}}$ and

$$M_{\omega,A}^{\rm bi}:=\sup_{z\in\complement\overline{\Sigma_\omega^{\rm bi}}}\|zR(z,A)\|<\infty.$$

In this situation we say that A is *ω-bisectorial.* The infimum of all $\omega\in(0,\frac12\pi)$ such that A is ω-bisectorial is called the *angle of bisectoriality* of A and is denoted by $\omega^{\rm bi}(A)$.

15.5.a Spectral projections

What distinguishes the theory of bisectorial operators from its sectorial counterpart is the possibility to consider the functions $\mathbf{1}_{\Sigma^+}$ and $\mathbf{1}_{\Sigma^-}$, both of which are bounded and holomorphic as functions on the bisector $\Sigma^{\rm bi} = \Sigma^+ \cup \Sigma^-$. If a bisectorial operator A has a bounded $H^\infty(\Sigma^{\rm bi})$-calculus, the operators $\mathbf{1}_{\Sigma^+}(A)$ and $\mathbf{1}_{\Sigma^-}(A)$ are well defined as bounded operators on $\overline{\mathsf{D}(A) \cap \mathsf{R}(A)}$ and take the role of "spectral projections" associated with the sectors Σ^+ and Σ^-. (The reason for writing quotations is that one has to be a bit careful here since 0 may belong to the spectrum of A.) From the multiplicativity of the H^∞-calculus it follows that the operators $\mathbf{1}_{\Sigma^+}$ and $\mathbf{1}_{\Sigma^-}$ are indeed projections, and that they are mutually orthogonal in the sense that

$$\mathbf{1}_{\Sigma^+}(A)\mathbf{1}_{\Sigma^-}(A) = \mathbf{1}_{\Sigma^-}(A)\mathbf{1}_{\Sigma^+}(A) = 0.$$

The injectivity of A on $\overline{\mathsf{D}(A) \cap \mathsf{R}(A)}$, which follows from Proposition 10.1.8, will be seen to imply the identity

$$\mathbf{1}_{\Sigma^-}(A) + \mathbf{1}_{\Sigma^-}(A) = I.$$

The importance of the operators $\mathbf{1}_{\Sigma^+}(A)$ and $\mathbf{1}_{\Sigma^-}(A)$ stems from their analogy to the Riesz projections; in particular, their difference

$$\mathbf{1}_{\Sigma^+}(A) - \mathbf{1}_{\Sigma^-}(A) =: \operatorname{sgn}(A)$$

may be thought of as an abstract analogue of the Hilbert transform.

Proposition 15.5.1. *If A is a bisectorial operator on a Banach space X with a bounded $H^\infty(\Sigma_\sigma^{\rm bi})$-calculus for some $\omega^{\rm bi}(A) < \sigma < \frac{1}{2}\pi$, then the operators*

$$P^+ := \mathbf{1}_{\Sigma_\sigma^+}(A), \quad P^- := \mathbf{1}_{\Sigma_\sigma^-}(A),$$

are well defined as bounded projections on $\overline{\mathsf{D}(A) \cap \mathsf{R}(A)}$. As such they are mutually orthogonal in the sense that

$$P^+P^- = P^-P^+ = 0,$$

and complementary in the sense that

$$P^+ + P^- = I.$$

Denoting the parts of A in $X^+ := \mathsf{R}(P^+)$ and $X^- := \mathsf{R}(P^-)$ by A^+ and A^- respectively, then both A^+ and $-A^-$ are sectorial on X^+ and X^- and have bounded $H^\infty(\Sigma_\sigma^+)$-calculus on these spaces. We have

$$\sigma(A^\pm) \subseteq \sigma(A) \cap \overline{\Sigma_\sigma^\pm},$$

and if $\mathsf{D}(A) \cap \mathsf{R}(A)$ is dense we also have

$$(\sigma(A) \cap \overline{\Sigma_\sigma^\pm}) \setminus \{0\} \subseteq \sigma(A^\pm) \setminus \{0\}.$$

There is some abuse of notation in writing $X^\pm$ for the range of $P^\pm$, as these operators are projections defined on $\overline{\mathsf{D}(A) \cap \mathsf{R}(A)}$ only, but we do not want to overburden the notation.

Proof. That the operators P^+ and P^- are mutually orthogonal projections on $\overline{\mathsf{D}(A) \cap \mathsf{R}(A)}$ and add up to the identity follows from the general properties of the H^∞-calculus. It is also clear that both projections commute with the resolvent of A. Thus the spaces X^+ and X^- are invariant under the resolvent of A. Denote by A^+ and A^- the parts of A to X^+ and X^-. It suffices to prove that A^+ has the asserted properties; the result for A^- then follows by applying it to the bisectorial operator $-A$.

Step 1a – In this step we prove that $\sigma(A^+) \subseteq \sigma(A) \cap \overline{\Sigma_\sigma^+}$.

Let us write $p^+(z) = \mathbf{1}_{\Sigma_\sigma^+}(z)$ (so $p^+(A)x = P^+x$) and $r_\mu(z) = (\mu - z)^{-1}$ for brevity.

The crux of the argument is the observation that if $\mu \in \complement\overline{\Sigma_\sigma^+}$, then $r_\mu p^+$ is a bounded holomorphic function on the bisector $\Sigma_\sigma^{\rm bi}$ even when $\mu \in \overline{\Sigma_\sigma^-}$. Therefore the operator $(r_\mu p^+)(A)$ is well defined by the $H^\infty(\Sigma_\sigma^{\rm bi})$-calculus of A.

We shall prove next that the restriction of this operator to X^+ is a two-sided inverse of $\mu - A^+$. This will show that inclusion $\sigma(A^+) \subseteq \overline{\Sigma_\sigma^+}$. By general spectral considerations we also have $\sigma(A^+) \subseteq \sigma(A)$, and together these inclusions prove

$$\sigma(A^+) \subseteq \sigma(A) \cap \overline{\Sigma_\sigma^+}.$$

First we prove that the restriction of $(r_\mu p^+)(A)$ to X^+ is a two-sided inverse of $\mu - A^+$ for $\mu \in \complement\overline{\Sigma_\sigma^{\rm bi}}$. Fix $x \in \overline{\mathsf{D}(A) \cap \mathsf{R}(A)}$. We have $r_\mu(A)x = R(\mu, A)x$ by the properties of the $H^\infty(\Sigma_\sigma^{\rm bi})$-calculus. The multiplicativity of this calculus then implies

$$\begin{aligned} (r_\mu p^+)(A)x &= r_\mu(A)p^+(A)x = R(\mu, A)P^+x, \\ (r_\mu p^+)(A)x &= (p^+ r_\mu)(A)x = P^+R(\mu, A)x. \end{aligned} \tag{15.33}$$

It follows that X^+ is invariant under $R(\mu, A)$. Multiplying the first identity on the left and the second on the right by $\mu - A^+$ we see $(r_\mu p^+)(A)$ is a two-sided inverse to $\mu - A^+$. It follows that $\mu \in \varrho(A^+)$ and

$$R(\mu, A^+) = (r_\mu p^+)(A)|_{X^+}. \tag{15.34}$$

We now consider the case of a general $\mu \in \complement\overline{\Sigma_\sigma}$, which will be handled by the resolvent identity. To this end fix an arbitrary $\lambda \in \complement\overline{\Sigma_\sigma^{\rm bi}}$. The scalar identity

$$\frac{1}{\mu - z} = \frac{1}{\lambda - z} + \frac{\lambda - \mu}{(\lambda - z)(\mu - z)}$$

translates, after multiplying with $p^+(z)$ and using the additivity and multiplicativity of the H^∞-calculus, into the identity

$$(r_\mu p^+)(A)x = (r_\lambda p^+)(A)x + (\lambda - \mu)(r_\lambda p^+)(A)(r_\mu p^+)(A)x, \qquad (15.35)$$

still for $x \in \overline{\mathsf{D}(A) \cap \mathsf{R}(A)}$. Among other things it implies that X^+ is invariant under $(r_\mu p^+)(A)$, since we have just proved that $(r_\lambda p^+)(A) = P^+ R(\mu, A)$ maps into X^+. By (15.33) and (15.34) (applied with λ in place of μ), the right-hand side of (15.35) (hence also the left-hand side) belongs to $\mathsf{D}(A)$, and for $x \in X^+$ we obtain

$$\begin{aligned}(\mu - A)(r_\mu p^+)(A)x &= (\mu - \lambda)(r_\mu p^+)(A)x + (\lambda - A)(r_\mu p^+)(A)x \\ &= (\mu - \lambda)(r_\mu p^+)(A)x + \Big[x + (\lambda - \mu)(r_\mu p^+)(A)x\Big] \\ &= x.\end{aligned}$$

It follows that $(r_\mu p^+)(A)x \in \mathsf{D}(A^+)$ and $(r_\mu p^+)(A)$ is a right inverse of $\mu - A^+$ on X^+. Also, using (15.34) (again with λ in place of μ) and the fact that $(r_\lambda p^+)(A)$ and $(r_\mu p^+)(A)$ in (15.35) commute, for $x \in \mathsf{D}(A^+)$ we obtain

$$\begin{aligned}(r_\mu p^+)(A)(\mu - A^+)x &= (r_\mu p^+)(A)(\mu - \lambda)x + (r_\mu p^+)(A)(\lambda - A^+)]x \\ &= (\mu - \lambda)(r_\mu p^+)(A)x + [x + (\lambda - \mu)(r_\mu p^+)(A)x] = x,\end{aligned}$$

and therefore $(r_\mu p^+)(A)$ is also a right inverse.

Step 1b – We now prove that if $\mu \neq 0$ belongs to $\sigma(A) \cap \overline{\Sigma_\sigma^+}$ and $\mathsf{D}(A) \cap \mathsf{R}(A)$ is dense, then $\mu \in \sigma(A^+)$.

To this end let $\mu \in \overline{\Sigma_\sigma^+}$. Since $\mu \neq 0$, we have $\mu \in \complement\overline{\Sigma_\sigma^-}$ and therefore, by the version of Step 1a for A^-, $\mu \in \varrho(A^-)$. Now if we also had $\mu \in \varrho(A^+)$, then along the decomposition $X = \overline{\mathsf{D}(A) \cap \mathsf{R}(A)} = X^+ \oplus X^-$, the operator $R(\mu, A^+) \oplus R(\mu, A^-)$ would be a two-sided inverse for $\mu - A$ and it would follow that $\mu \in \varrho(A)$. Hence for $\mu \neq 0$ this proves the inclusion $\sigma(A) \cap \overline{\Sigma_\sigma^+} \subseteq \sigma(A^+)$.

Step 2 – We next establish the resolvent bound for A^+. The uniform boundedness of $zR(z, A^+)$ on $\complement\overline{\Sigma_\sigma}^{\rm bi}$ follows from the uniform boundedness of $zR(z, A)$ on this set by taking restrictions. In particular, for any $\frac{1}{2}\pi < \vartheta < \pi - \sigma$, $zR(z, A^+)$ is uniformly bounded on the two rays $re^{\pm i\vartheta}$, and then the sectorial version of the three lines lemma implies the uniform boundedness of $zR(z, A^+)$ on Σ_ϑ^-.

Step 3 – The prescription $f(A^+) := (p^+ f)(A)$ defines a bounded linear multiplicative mapping from $H^\infty(\Sigma_\sigma^+)$ into $\mathscr{L}(\overline{\mathsf{D}(A) \cap \mathsf{R}(A)})$ that agrees with the Dunford calculus of A^+ for functions $f \in H^1(\Sigma_\sigma^+) \cap H^\infty(\Sigma_\sigma^+)$. This immediately implies that A^+ has a bounded $H^\infty(\Sigma_\sigma^+)$-calculus. □

15.5.b Sectoriality versus bisectoriality

The next theorem establishes a relationship between a bisectorial operator A and the square root of the sectorial operator A^2. In the proof we shall use the fact, which can be routinely checked by redoing the arguments of Section 15.1, that an extended Dunford calculus can be set up for bisectorial operators and that it enjoys similar properties as in the sectorial case.

Theorem 15.5.2. *If A is a standard bisectorial operator with a bounded $H^\infty(\Sigma_\sigma^{\rm bi})$-calculus on a Banach space X, then A^2 is a standard 2σ-sectorial operator and*

$$\mathsf{D}((A^2)^{1/2}) = \mathsf{D}(A).$$

For all x in this common domain we have

$$\|(A^2)^{1/2}x\| \eqsim \|Ax\|$$

with constants independent of x.

Proof. The function $a(z) := (z^2)^{1/2}$ is holomorphic on $\Sigma_\sigma^{\rm bi}$ and equals z on Σ_σ^+ and $-z$ on Σ_σ^-. Likewise, the function $\mathrm{sgn}(z) := z/(z^2)^{1/2}$ is holomorphic on $\Sigma_\sigma^{\rm bi}$ and equals 1 on Σ_σ^+ and -1 on Σ_σ^-. Thus, $\mathrm{sgn}(z)a(z) = z$ for all $z \in \Sigma_\sigma^{\rm bi}$. By the multiplicativity of the extended functional calculus (cf. Proposition 15.1.12 for the sectorial case), it follows that, if $x \in \mathsf{D}((A^2)^{1/2})$, then $x \in \mathsf{D}(A)$ and $Ax = \mathrm{sgn}(A)(A^2)^{1/2}x$. Taking norms, we find that

$$\|Ax\| \leqslant M\|(A^2)^{1/2}x\|,$$

where M is the boundedness constant of the $H^\infty(\Sigma_\sigma^{\rm bi})$-calculus of A.

In the same way, the identity $a(z) = \mathrm{sgn}(z)z$ implies that, if $x \in \mathsf{D}((A^2)^{1/2})$, then $x \in \mathsf{D}(A)$ and $(A^2)^{1/2}x = \mathrm{sgn}(A)Ax$. Taking norms gives

$$\|(A^2)^{1/2}x\| \leqslant M\|Ax\|.$$

□

It is of some interest to interpret this theorem for the Hodge–Dirac operator D on $L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d;\mathbb{C}^d)$ of Example 10.6.5,

$$D = \begin{pmatrix} 0 & \nabla^* \\ \nabla & 0 \end{pmatrix}.$$

where $\nabla^* = -\mathrm{div}$ is the adjoint of ∇. Its square is of the form

$$D^2 = \begin{pmatrix} -\Delta & 0 \\ 0 & * \end{pmatrix},$$

where $*$ equals (at least formally) $-\nabla \circ \mathrm{div}$. Taking $g(z) = \mathrm{sgn}(z) := z/(z^2)^{1/2}$, then (at least formally)

$$g(D) = D(D^2)^{-1/2} = \begin{pmatrix} 0 & -\mathrm{div} \\ \nabla & 0 \end{pmatrix} \cdot \begin{pmatrix} (-\Delta)^{-1/2} & 0 \\ 0 & * \end{pmatrix} = \begin{pmatrix} 0 & * \\ \nabla/(-\Delta)^{1/2} & 0 \end{pmatrix}.$$

Thus we see the Riesz transform

$$R = \nabla(-\Delta)^{1/2}$$

appear as an entry in the functional calculus of D.

15.6 Notes

Section 15.1

The idea to use regularising functions to extend the functional calculus to suitable classes of unbounded functions goes back to McIntosh [1986]. A comprehensive discussion of extended functional calculi is presented by Haase [2006]; see also Haase [2020]. Our treatment in Sections 15.1 and 15.2 is based on Haase [2006] and Kunstmann and Weis [2004]. The proof of Theorem 15.1.18 is taken from the former reference.

Section 15.2

The first unified account of the theory of fractional powers was undertaken by Komatsu in a series of papers starting with Komatsu [1966]. This paper contains the results presented here and much more. Some earlier works on the subject are due to Balakrishnan [1960], Hille and Phillips [1957], Kato [1960, 1961], Krasnosel′skiĭ and Sobolevskiĭ [1959], Phillips [1952], Watanabe [1961], and Yosida [1960]. Modern accounts include Albrecht, Duong, and McIntosh [1996], Denk, Hieber, and Prüss [2003], Dore [1999, 2001], Haase [2006], and Martínez Carracedo and Sanz Alix [2001]. Our treatment barely scratches the surface of this rich and vast subject, and we have only included the most basic results needed for the treatment of bounded imaginary powers. Our approach based on the extended Dunford calculus has the advantage of keeping the technical details at a minimum, but the price to be paid is that we must make somewhat restrictive assumptions on the operator A.

Theorem 15.2.8 is from Komatsu [1966], but the proof presented here is taken from Haase [2006]. Theorem 15.2.13, 15.2.17, and 15.2.16 are due to Balakrishnan [1960] (see also Yosida [1980]). Some authors take one of the formulas in the first and third theorem as the definition of the fractional powers. For further information on the classical approach to fractional powers via integral representations, we refer the reader to the monographs Butzer and Berens [1967] and Martínez Carracedo and Sanz Alix [2001]. A complete proof of the non-negativity of the function $f_{\alpha,t}$ in Theorem 15.2.17 can be found in Yosida [1980, Proposition IX.11.2].

Section 15.3

A detailed account of the theory of bounded imaginary powers is presented by Haase [2006]; see also Amann [1995] and Prüss and Simonett [2016].

Example 15.3.25 is from Dore and Venni [1987]. Lemma 15.3.8 is from Prüss and Sohr [1990], where a different proof based on properties of the Mellin transform is given. Alternative proofs were obtained subsequently by Monniaux [1997] and Uiterdijk [1998]. The elementary proof presented here, based on the perturbation result of Theorem 15.1.18, is taken from Haase

[2006]. Theorem 15.3.9, identifying domains of fractional powers with complex interpolation spaces in the presence of bounded imaginary powers, is due to Seeley [1971]; see Triebel [1978] for references to earlier results in this direction.

Theorem 15.3.12, connecting the angles (R-)sectoriality and bounded imaginary powers, is due to Prüss and Sohr [1990] ($\omega(A) \leqslant \omega_{\rm BIP}(A)$) and Clément and Prüss [2001] ($\omega_R(A) \leqslant \omega_{\rm BIP}(A)$, in a UMD space). The estimation of the three terms in the last part of the proof are patterned after the proof of Lemma 15.4.5, which is taken from Monniaux [1999] and extends earlier results of Zsidó [1983] and Berkson, Gillespie, and Muhly [1986a]. Lemma 15.3.13 is from Prüss and Sohr [1990]. In Remark 15.3.14, the identity (15.17) can be equivalently stated in terms of the Mellin transform; see, e.g., Titchmarsh [1986]. The two theorems about (almost) γ-sectoriality and (γ-)bounded imaginary powers, Theorem 15.3.16 ($\widetilde{\omega}_\gamma(A) \leqslant \omega_{\rm BIP}(A)$) and its proof, as well as Theorem 15.3.19 ($\omega_\gamma(A) \leqslant \omega_{\gamma\text{-}{\rm BIP}}(A)$), are taken from Kalton, Lorist, and Weis [2023]. Theorem 15.3.20 ($\omega_{\rm BIP}(A) = \omega_{H^\infty}(A)$) is from Cowling, Doust, McIntosh, and Yagi [1996], whose proof we follow. The result proved in this paper shows that $\omega_{H^\infty}(A) = \max\{\omega(A), \omega_{\rm BIP}(A)\}$; as was noted in the main text, this improves to $\omega_{H^\infty}(A) = \omega_{\rm BIP}(A)$ by virtue of the Clément–Prüss Theorem 15.3.12. A different proof of Theorem 15.3.20, based on the theory of Euclidean structures, is presented by Kalton, Lorist, and Weis [2023]. Theorem 15.3.21, on the equivalence of bounded H^∞-calculus and γ-bounded imaginary powers, is taken from Kalton and Weis [2016]. An alternative proof is presented by Kalton, Lorist, and Weis [2023, Theorem 4.5.6 and Corollary 4.5.7]

An example of a sectorial operator on the space c_0 without bounded imaginary powers was was given by Komatsu [1966]. Hilbert space examples were constructed subsequently by McIntosh and Yagi [1990] and Baillon and Clément [1991], where a general way to construct such examples using conditional bases was invented. Venni [1993] showed that, in any Banach space with a Schauder basis, there are densely defined sectorial operators A for which some, but not all, imaginary powers are bounded. More precisely, it can be arranged that $A^{ik\pi} = I$ if k is an even integer and $A^{ik\pi}$ is unbounded if k is an odd integer. Hieber [1996] constructed an example of a pseudo-differential operator acting in $L^p(\mathbb{R})$, $1 < p < \infty$, $p \neq 2$, that is sectorial but does not admit bounded imaginary powers. Examples of operators in $L^p(S)$, $p \neq 2$, with bounded imaginary powers but without a bounded H^∞-calculus can be found in Cowling, Doust, McIntosh, and Yagi [1996]. This reference also contains the proof of the inequality $\omega_{H^\infty}(A) \leqslant \omega_{\rm BIP}(A)$.

Theorem 15.3.23 is due to McIntosh [1986]. In this connection, it is also of interest to mention the result of Yagi [1984, Theorem B] that an invertible sectorial operator A on a Hilbert space has bounded imaginary powers if $\mathsf{D}(A^\alpha) = \mathsf{D}(A^{\star\alpha})$ for all $\alpha \in [0, \epsilon)$.

Around almost sectoriality: further results

The next result, a proof of which is given by Kalton, Lorist, and Weis [2023, Proposition 4.2.4], connects the (almost) γ-sectoriality of A to γ-boundedness of the associated semigroup.

Proposition 15.6.1. *Let A be a sectorial operator on X with $\omega(A) < \frac{1}{2}\pi$ and let $\omega(A) < \sigma < \frac{1}{2}\pi$. Then*

(i) *A is γ-sectorial with $\omega_\gamma(A) \leqslant \sigma$ if and only if the set*
$$\{e^{-zA} :\ z \in \Sigma_\nu\}$$
is γ-bounded for all $0 < \nu < \frac{1}{2}\pi - \sigma$;

(ii) *A is almost γ-sectorial with $\widetilde{\omega}_\gamma(A) \leqslant \sigma$ if and only if the set*
$$\{zAe^{-zA} :\ z \in \Sigma_\nu\}$$
is γ-bounded for all $0 < \nu < \frac{1}{2}\pi - \sigma$.

For operators A that are diagonal with respect to a Schauder basis, the notion of γ-almost sectoriality captures a natural property of the basis. In order to formulate this in the form of a proposition, we first recall that a sequence $(x_n)_{n\in\mathbb{Z}}$ in a Banach space X is called a *Schauder basis* of X if every $x \in X$ has a unique representation of the form $x = \sum_{n\in\mathbb{Z}} c_n x_n$. Associated with a Schauder basis is its sequence of coordinate projections $(P_n)_{n\in\mathbb{Z}}$ on X, defined by
$$P_n \sum_{j\in\mathbb{Z}} c_j x_j := x_n, \qquad n \in \mathbb{Z},$$
and the sequence of partial sum projections $(S_n)_{n\in\mathbb{N}}$, defined by
$$S_n \sum_{j\in\mathbb{Z}} x_j := \sum_{j=-n}^{n} x_j, \qquad n \in \mathbb{N},$$
that is, $S_n = \sum_{k=-n}^{n} P_k$. For any Schauder basis, the set of partial sum projections is uniformly bounded, and by taking differences the same is seen to be true for the set of coordinate projections.

On a Banach space X with a Schauder basis $(x_n)_{n\in\mathbb{Z}}$, we may consider the diagonal operator A defined by
$$Ax_n := 2^n x_n, \qquad x_n \in X_n, \quad n \in \mathbb{Z},$$
with its natural maximal domain. It was shown in Proposition 10.2.28 that A is sectorial of angle $\omega(A) = 0$, and that A has a bounded H^∞-calculus if and only if $(x_n)_{n\in\mathbb{Z}}$ is unconditional. The following result is due to Kalton, Lorist, and Weis [2023, Proposition 6.1.3].

Proposition 15.6.2. *For the operator A just defined, the following is true:*

(1) *A is γ-sectorial if and only if the sequence $(S_n)_{n\in\mathbb{N}}$ is γ-bounded;*
(2) *A is almost γ-sectorial if and only if the sequence $(P_n)_{n\in\mathbb{Z}}$ is γ-bounded.*

We revisit this result in the Notes of Chapter 17 in connection with the problem of finding examples of sectorial operators that are not R-sectorial.

Around the Hilbert space characterisation: the γ-interpolation method

Theorem 15.3.23 asserts that a standard sectorial operator A on a Hilbert space H has a bounded H^∞-calculus if and only if it has bounded imaginary powers. This equivalence is nothing but the specialisation to Hilbert spaces of the equivalence, for any standard sectorial operator A on a Banach space X, of having a bounded H^∞-calculus and having γ-bounded imaginary powers, as stated in Theorem 15.3.21. The aim of this section is to explain that also the third equivalence in Theorem 15.3.23 is the specialisation to Hilbert spaces of a corresponding statement for Banach spaces. Recall that this third equivalence states, under the additional assumptions that $0 \in \varrho(A)$ and $\omega(A) < \frac{1}{2}\pi$, that bounded H^∞-calculus and boundedness of imaginary powers for A are equivalent to the equality

$$\mathsf{D}(A^{1/2}) = (X, \mathsf{D}(A))_{1/2,2} \text{ with equivalent norms.} \tag{15.36}$$

The Banach space version of this equivalence, due to Kalton, Lorist, and Weis [2023, Corollary 5.3.9], replaces (15.36) with the condition

$$\mathsf{D}(A^{1/2}) = (X, \mathsf{D}(A))^{\gamma}_{1/2} \text{ with equivalent norms,}$$

the interpolation space on the right-hand side being obtained via the so-called *γ-interpolation method* which we briefly outline next.

A discrete version of the γ-interpolation method was already considered by Kalton, Kunstmann, and Weis [2006], where Rademacher variables were used instead of Gaussian variables. In that paper, the method was used to study perturbations of the H^∞-calculus for various differential operators. The continuous version of the Gaussian method was introduced by Suárez and Weis [2006, 2009], where Gaussian interpolation of Bochner spaces $L^p(S;X)$ and square function spaces $\gamma(S;X)$, as well as a Gaussian version of abstract Stein interpolation, was studied. An abstract framework covering the γ-interpolation method, as well as the real and complex interpolation methods, has been recently developed by Lindemulder and Lorist [2021]. The present treatment follows the memoir of Kalton, Lorist, and Weis [2023]; theorem numbers in brackets refer to this memoir. As was pointed out in this reference, the results in Kalton, Kunstmann, and Weis [2006] and Suárez and Weis [2006, 2009] were based on a draft version of the memoir.

Let (X_0, X_1) be an interpolation couple of Banach spaces and let $\theta \in (0,1)$. We call an operator

$$T : L^2(\mathbb{R}) + L^2(\mathbb{R}, e^{-2t}\,\mathrm{d}t) \to X_0 + X_1$$

admissible, and write $T \in \mathscr{A}$, if $T \in \gamma(L^2(\mathbb{R}, e^{-2jt}\,\mathrm{d}t), X_j)$ for $j = 0, 1$. For such operators we define

$$\|T\|_{\mathscr{A}} := \max_{j=0,1} \|T_j\|_{\gamma(L^2(\mathbb{R}, e^{-2jt}\,\mathrm{d}t), X_j)},$$

where T_j denotes the operator T from $L^2(\mathbb{R}, e^{-2jt}\,\mathrm{d}t)$ into X_j. It is routine to check that $\mathscr{A}$ is complete with respect to this norm.

Denoting $e_\theta : t \mapsto e^{\theta t}$, we define $(X_0, X_1)^\gamma_\theta$ as the space of all $x \in X_0 + X_1$ for which the quantity

$$\|x\|_{(X_0,X_1)^\gamma_\theta} := \inf\bigl\{\|T\|_{\mathscr{A}} :\ T \in \mathscr{A},\ T(e_\theta) = x\bigr\}$$

is finite. This space is a quotient space of $\mathscr{A}$, and as such it is a Banach space. By [Proposition 3.3.2], the set of finite rank operators T is dense in $\mathscr{A}$; in particular, we have:

Proposition 15.6.3. *$X_0 \cap X_1$ is dense in $(X_0, X_1)^\gamma_\theta$.*

[Proposition 3.3.1] gives the following interpolation result.

Theorem 15.6.4 (γ-interpolation of operators). *Suppose that (X_0, X_1) and (Y_0, Y_1) are interpolation couples of Banach spaces. Let $S\colon X_0 + X_1 \to Y_0 + Y_1$ be a bounded operator such that $S(X_0) \subseteq Y_0$ and $S(X_1) \subseteq Y_1$. Then S maps $(X_0, X_1)^\gamma_\theta$ to $(Y_0, Y_1)^\gamma_\theta$ boundedly, with norm*

$$\|S\|_{\mathscr{L}((X_0,X_1)^\gamma_\theta,(Y_0,Y_1)^\gamma_\theta)} \leqslant \|S\|^{1-\theta}_{\mathscr{L}(X_0,Y_0)} \|S\|^{\theta}_{\mathscr{L}(X_1,Y_1)}.$$

By [Proposition 3.4.1], the norm of $(X_0, X_1)^\gamma_\theta$ can be equivalently expressed as follows.

Proposition 15.6.5. *Let $\mathscr{A}_\bullet$ be the set of all strongly measurable functions $f : \mathbb{R}_+ \to X_0 \cap X_1$ such that $t \mapsto t^j f(t)$ belongs to $\gamma(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X_j)$ for $j = 0, 1$. For $f \in \mathscr{A}_\bullet$, define*

$$\|f\|_{A_\bullet} := \max_{j=0,1} \big\|t \mapsto t^j f(t)\big\|_{\gamma(\mathbb{R}_+,\, \frac{\mathrm{d}t}{t}; X_j)}.$$

Then for all $x \in (X_0, X_1)^\gamma_\theta$ we have

$$\|x\|_{(X_0,X_1)^\gamma_\theta} = \inf\Big\{\|f\|_{\mathscr{A}_\bullet} :\ f \in \mathscr{A}_\bullet,\ \int_0^\infty t^\theta f(t)\,\mathrm{d}t = x\Big\},$$

where the integral converges in the Bochner sense in $X_0 + X_1$.

[Theorem 3.4.4] contains the following relationship of the γ-interpolation method with the real and complex methods.

Theorem 15.6.6 (Relationship with the real and complex method). *Let (X_0, X_1) be an interpolation couple of Banach spaces, and let $0 < \theta < 1$.*

(1) *If X_0 and X_1 have type $p_0, p_1 \in [1,2]$ and cotype $q_0, q_1 \in [2,\infty]$ respectively, then we have the continuous embeddings*

$$(X_0, X_1)_{\theta,p} \hookrightarrow (X_0, X_1)^\gamma_\theta \hookrightarrow (X_0, X_1)_{\theta,q}$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

(2) *If X_0 and X_1 have type 2, then we have the continuous embedding*

$$[X_0, X_1]_\theta \hookrightarrow (X_0, X_1)^\gamma_\theta.$$

If X_0 and X_1 have cotype 2, then we have the continuous embedding

$$(X_0, X_1)^\gamma_\theta \hookrightarrow [X_0, X_1]_\theta.$$

Since a Banach space X is isomorphic to a Hilbert space if and only if X has type 2 and cotype 2 (by Kwapień's Theorem 7.3.1), we obtain the corollary (cf. the Corollary of Peetre's Theorem C.4.1 for the first equivalence):

Corollary 15.6.7. *Let (H_0, H_1) be an interpolation couple of Hilbert spaces, and let $0 < \theta < 1$. Then*

$$(H_0, H_1)_{\theta,2} = [H_0, H_1]_\theta = (H_0, H_1)^\gamma_\theta$$

with equivalent norms.

Section 15.4

Much of the theory developed in the first three sections of this chapter has an analogue for strip type operators. The general theory of this class of operators is developed in the book of Haase [2006], which also treats their connections with logarithms of sectorial operators. Analogues of the results of Sections 10.3 and 10.4 are presented by Kalton and Weis [2016].

Theorem 15.4.3, on the strip type property and integral representation of the logarithm $\log(A)$ of a standard sectorial A, is due to Nollau [1969]. Our proof is a variation of the presentation by Haase [2006]. The converse problem to Theorem 15.4.3, whether the exponent of a striptal operator is sectorial, is subtle; we refer to Haase [2006] for a counterexample. Theorem 15.4.4, on the identification of C_0-groups on a UMD space as bounded imaginary powers of a standard sectorial operator, can be viewed as a partial result in the positive direction. It was first proved by Monniaux [1999] with a very different argument based on the notion of analytic generator. The proof presented here is essentially that of Haase [2009]. Another proof can be found in Haase [2007].

Theorem 15.4.11 about the sum of operators, both of which have bounded imaginary powers, is due to Dore and Venni [1987] under the slightly stronger assumption on A and B that they satisfy the resolvent bounds

$$\|(t+A)^{-1}\|, \|(t+B)^{-1}\| \leqslant \frac{M}{1+t}, \quad t>0.$$

In its present form, where it is only assumed that

$$\|(t+A)^{-1}\|, \|(t+B)^{-1}\| \leqslant \frac{M}{t}, \quad t>0,$$

the result was obtained by Prüss and Sohr [1990]. The original proof of Dore and Venni [1987] is ingenious and relatively short, and has been sketched in the Notes of Chapter 5. It depends on a representation formula for $(A+B)^{-1}$ in terms of fractional powers of A and B. The refinement of these arguments by Prüss and Sohr [1990], to obtain the more general case, depends on subtle approximation arguments for operators A with bounded imaginary powers which, like the proof presented here, use the functional calculus associated with the C_0-group $(A^{it})_{t\in\mathbb{R}}$ and Mellin transform techniques.

The beautiful proof of the Dore–Venni Theorem 15.4.11 presented here is due to Haase [2007] and fits well in the mainstream of ideas developed in this volume. This paper also contains our proof of Theorem 15.4.4, which is originally due to Monniaux [1999] with a different proof based on the notion of an analytic generator. Our presentation uses some ideas of Haase [2006, Section 4.2], where a detailed presentation of the theory of strip type operators if given. With these methods, the operator $B = e^{tA}$ can also be defined using the extended Dunford calculus.

The importance of the Dore–Venni Theorem 15.4.11 is mostly historical, and the more recent sum-of-operator theorems proved in the next chapter have turned out to be more versatile in their usage. It is for this reason that we have contented ourselves with a somewhat sketchy presentation, leaving a few tedious details to the reader.

Section 15.5

The results of this section follow Duelli [2005] and Duelli and Weis [2005], where Theorem 15.5.2 ($\|(A^2)^{1/2}x\| \eqsim \|Ax\|$) is proved. By the Hieber–Prüss Theorem 10.7.10, it covers the case where iA generates a bounded C_0-group. A version of Theorem 15.5.2 (with inhomogeneous estimates) for the case that iA generates a C_0-group of exponential growth type $\omega \geqslant 0$ is due to Haase [2007].

The spectral projections $P^{\pm}$ of Proposition 15.5.1 are studied in more detail by Arendt and Zamboni [2010], Duelli [2005], and Duelli and Weis [2005]. In particular, Arendt and Zamboni [2010] show that, if A is an invertible bisectorial operator and $\delta>0$ is so small that the closure of the ball $B(0,\delta)$ belongs to the resolvent set of A, then for $x \in \mathsf{D}(A)$, these projections are given by

$$P^{\pm}x = \frac{1}{2\pi i}\int_{\partial(\Sigma_\nu^{\pm}\setminus B(0,\delta))} R(z,A)Ax\,\frac{\mathrm{d}z}{z},$$

where $\omega^{\rm bi}(A) < \nu < \sigma$ is arbitrary. The extended Dunford calculus for bisectorial operators, in particular the analogue of Proposition 15.1.12, which was used in the proof of Theorem 15.5.2, has been studied by Duelli [2005].

The Kato square root problem

A long-standing question about fractional powers of operators, and a major motivation for the development of the theory of H^∞-calculus at large, was the *square root problem* of Kato [1961]. To present this problem, we recall that a linear operator A in a Hilbert space H with inner product $(\ |\ )$ is called

- *accretive*, if $\Re(Au|u) \geqslant 0$ for all $u \in \mathsf{D}(A)$;
- *maximal accretive*, if an extension $\widetilde{A} \supseteq A$ is accretive exactly when $\widetilde{A} = A$;
- *regularly accretive*, if A is maximally accretive and there is an associated sesquilinear form $\mathfrak{a}$ in H such that $\Re\mathfrak{a}(v,v) \geqslant 0$ for all $v \in \mathsf{D}(\mathfrak{a})$, and

$$(Au|v) = \mathfrak{a}(u,v) \text{ for all } u \in \mathsf{D}(A) \subseteq \mathsf{D}(\mathfrak{a}) \text{ and all } v \in \mathsf{D}(\mathfrak{a}).$$

For a regularly accretive operator, Kato [1961] defines its real part $\Re A$ as the unique maximal accretive operator associated with the sesquilinear form

$$\Re\mathfrak{a} : (u,v) \mapsto \frac{1}{2}[\mathfrak{a}(u,v) + \overline{\mathfrak{a}(v,u)}]$$

in the above sense. He then proceeds to show that

$$\begin{aligned} \mathsf{D}(A^\alpha) &= \mathsf{D}((A^*)^\alpha), && \text{if } A \text{ is maximal accretive and } \alpha \in [0,\tfrac{1}{2}), \\ &= \mathsf{D}((\Re A)^\alpha), && \text{if } A \text{ is regularly accretive and } \alpha \in [0,\tfrac{1}{2}), \end{aligned}$$

and that these identities can fail for $\alpha > \frac{1}{2}$, "but it is not known whether or not $\alpha = \frac{1}{2}$ can be included". Kato [1961, Remark 1] writes:

> This is perhaps not true in general. But the question is open even when A is regularly accretive. In this case it appears reasonable to suppose that both $\mathsf{D}(A^{\frac{1}{2}})$ and $\mathsf{D}((A^*)^{\frac{1}{2}})$ coincide with $\mathsf{D}((\Re A)^{\frac{1}{2}}) = \mathsf{D}(\mathfrak{a})$, where $\Re A$ is the real part of A and $\mathfrak{a}$ is the regular sesquilinear form which defines A.

As suspected by Kato [1961], a counterexample to the general case of maximal accretive operators was found shortly after by Lions [1962], but the regularly accretive case was only disproved a decade later by McIntosh [1972].

What came to be known as Kato's square root problem was subsequently redefined by McIntosh [1982], making the case that, what Kato [1961] "really had in mind", was differential operators $A = -\operatorname{div} B(x)\nabla$, where $B \in L^\infty(\mathbb{R}^d;\mathbb{C}^d)$ is such that $\Re(B(x)e|e) \geqslant \delta > 0$ for a.e. $x \in \mathbb{R}^d$ and all $e \in \mathbb{C}^d$ of norm one. For such A, the associated sesquilinear form is

$$\mathfrak{a}(u,v) = \int_{\mathbb{R}^d} (B(x)\nabla u(x)|\nabla v(x))\,\mathrm{d}x$$

with domain $\mathsf{D}(\mathfrak{a}) = \mathsf{D}(\nabla) = W^{1,2}(\mathbb{R}^d)$, and the problem thus takes the form

$$\|\sqrt{-\operatorname{div} B\nabla}u\|_{L^2(\mathbb{R}^d)} \overset{?}{\eqsim} \|\nabla u\|_{L^2(\mathbb{R}^d;\mathbb{C}^d)}. \tag{15.37}$$

McIntosh [1982] further suggested that this question was related to Calderón's problem about the L^2-boundedness of the Cauchy integral on a Lipschitz graph (discussed in the Notes of Chapter 12). As pointed out by Alan McIntosh in several oral communications with the authors of this book over the first decade of this century (the quote within the first sentence of this paragraph is also from this oral source), before his making this connection, Kato's question was generally regarded as being a soft one, several levels easier than the problem of Calderón, which everyone agreed to be hard. Nevertheless, the connection suggested by McIntosh [1982] turned out to be a fruitful one, and the combined efforts of Coifman, McIntosh, and Meyer [1982] led to a proof of both the boundedness of the Cauchy integral and, what turned out to be equivalent, McIntosh's reformulation of Kato's square root problem in dimension $d = 1$.

After this, the status of the redefined square root problem increased substantially, and important progress was made by Coifman, Deng, and Meyer [1983], Fabes, Jerison, and Kenig [1985], McIntosh [1985], Alexopoulos [1991], Journé [1991], Auscher and Tchamitchian [1998], and Auscher, Hofmann, Lewis, and Tchamitchian [2001], but it took two decades from the one-dimensional result of Coifman, McIntosh, and Meyer [1982] until a complete solution was achieved by Hofmann, Lacey, and McIntosh [2002] in dimension $d = 2$ and then by Auscher, Hofmann, Lacey, McIntosh, and Tchamitchian [2002] in all dimensions.

While heavily building on ideas and results about functional calculus of the second-order operator $A = -\operatorname{div} B\nabla$, the original solution of the square root problem was not quite a "pure" functional calculus theorem in the sense that the gradient featuring in (15.37) does not have the form $f(A)$ of objects in the functional calculus of A. This "issue" was fixed by a new approach developed by Axelsson, Keith, and McIntosh [2006] which, in contrast to the sectorial calculus of second-order operators employed by Auscher et al. [2002], was based on the bi-sectorial calculus of first-order differential operators, and promoted the relevance of bi-sectorial operators and bi-sectorial H^∞-calculus in subsequent research. Quoting the MathSciNet review of Axelsson et al. [2006] by Ian Doust, this paper provided "a remarkable consolidation of many of the ideas that have arisen in the so-called Calderón program", not only reproving the square root theorem of Auscher et al. [2002] and several other results by a unified approach, but also obtaining new geometric applications to the behaviour of the Hodge–Dirac operator on a Riemannian manifold under measurable perturbations of the Riemannian metric. In fact, the very framework of Axelsson et al. [2006] is based on a general notion of *perturbed Hodge–Dirac operators*; in the application to the Kato square root problem, these take the form

$$D_B := \begin{pmatrix} 0 & -\operatorname{div} B \\ \nabla & 0 \end{pmatrix}, \qquad \mathsf{D}(D_B) = \mathsf{D}(\nabla) \oplus \mathsf{D}(\operatorname{div} B)$$

so that, at least formally,

$$(D_B^2)^{\frac12} = \begin{pmatrix} -\operatorname{div} B\nabla & 0 \\ 0 & -\nabla \operatorname{div} B \end{pmatrix}^{\frac12} = \begin{pmatrix} (-\operatorname{div} B\nabla)^{\frac12} & 0 \\ 0 & (-\nabla \operatorname{div} B)^{\frac12} \end{pmatrix},$$
$$\mathsf{D}((D_B^2)^{\frac12}) = \mathsf{D}((-\operatorname{div} B\nabla)^{\frac12}) \oplus \mathsf{D}((-\nabla \operatorname{div} B)^{\frac12}).$$

Hence, if one can establish bounded bi-sectorial H^∞-calculus of D_B, as Axelsson et al. [2006] do, the Kato conjecture (15.37) will be an immediate consequence of the estimate $\|(D_B^2)^{\frac12} u\| \eqsim \|D_B u\|$ provided by Theorem 15.5.2.

This first-order approach of Axelsson, Keith, and McIntosh [2006] has been influential for several subsequent developments. Of particular interest for the themes of the present volumes is a version of the Kato square root theorem in $L^p(\mathbb{R}^d;X)$. This was obtained by Hytönen, McIntosh, and Portal [2008] by a Banach space extension of the methods of Axelsson et al. [2006]. In the language of the original operator $A = -\operatorname{div} B\nabla$, the result of Hytönen, McIntosh, and Portal [2008] can be stated as follows:

Theorem 15.6.8. *Let X be a UMD space, and suppose that both X and X^* have the RMF property (Definition 3.6.10). Let $B, B^{-1} \in L^\infty(\mathbb{R}^d; \mathscr{L}(\mathbb{C}^d))$, where $B^{-1}(x) := B(x)^{-1}$ is the pointwise inverse of the matrix-valued function B. Let $1 \leqslant p_- < p_+ \leqslant \infty$, and suppose that $A := -\operatorname{div} B\nabla$ is sectorial in $L^p(\mathbb{R}^d;X)$ for all $p \in J := (p_-, p_+)$. Then the following are equivalent:*

(1) *For all $p \in J$, the following four sets are R-bounded in $L^p(\mathbb{R}^d;X)$:*

$$\mathscr{A}_{a,b} := \Big\{(t\sqrt{-\Delta})^a (I + t^2 A)^{-1} (t\sqrt{-\Delta})^b : t > 0\Big\}, \quad a, b \in \{0, 1\}.$$

(2) *For all $p \in J$, A has a bounded H^∞-calculus in $L^p(\mathbb{R}^d;X)$, and*

$$\|\sqrt{A} u\|_{L^p(\mathbb{R}^d;X)} \eqsim \|\nabla u\|_{L^p(\mathbb{R}^d;X)^d}$$

Remark 15.6.9. Several remarks concerning Theorem 15.6.8 are in order:

(a) While (2) contains a full analogue of (15.37) in $L^p(\mathbb{R}^d;X)$, along with the bounded H^∞-calculus of independent interest, the characterising condition (1) is less satisfactory than in the scalar-valued case, as it involves non-trivial R-boundedness properties of operators on $L^p(\mathbb{R}^d;X)$. However, note that the R-boundedness of $\mathscr{A}_{0,0}$ is simply the R-sectoriality of A which, by Theorem 10.3.4(2), is a general necessary condition for the bounded H^∞-calculus contained in (2). When $X = \mathbb{C}$ and $p = 2$, verifying (1) from easy-to-check pointwise conditions on B is straightforward operator theory, and the difficult harmonic analysis enters in passing from (1) to (2). Curiously, in the Banach space valued generality, the easy part becomes unavailable but the difficult part may still be pushed through.

(b) Another shortcoming of Theorem 15.6.8 compared to the scalar-valued L^2 case is that, in order to get (2) for a given p, one needs to verify (1) for an open range of exponents $(p-\varepsilon, p+\varepsilon)$. However, it was subsequently shown by Hytönen and McIntosh [2010], and later with a different argument by Auscher and Stahlhut [2013], that conditions of type (1) self-improve from one exponent p to a small range around it. This allows one to obtain a version of Theorem 15.6.8 for a fixed p in place of the range of p as stated.

(c) The RMF property (Definition 3.6.10) and the related Rademacher maximal function (Definition 3.6.8) were first introduced by Hytönen, McIntosh, and Portal [2008] for the very needs of running the argument to prove Theorem 15.6.8, but these notions (or their extensions) have proven to be useful in other contexts, notably in the study of Banach space valued multilinear operators by Di Plinio and Ou [2018], Di Plinio, Li, Martikainen, and Vuorinen [2020b], and Amenta and Uraltsev [2020].

(d) For L^p-estimates related to Kato's square root problem in the scalar-valued case, there is an alternative approach based on a generalisation of the Calderón–Zygmund theory discussed in Chapter 11, extrapolating from the L^2-results of Auscher, Hofmann, Lacey, McIntosh, and Tchamitchian [2002]. The operators now under consideration are far less regular than those covered in Chapter 11, and the extrapolation yields their boundedness, in general, only in some subinterval $(p_-, p_+) \subseteq (1, \infty)$ determined by the details of the operator in question. Based on an extrapolation theory for "non-integral operators" developed by Blunck and Kunstmann [2003], a systematic investigation of the maximal ranges of p for various L^p estimates related to the Kato square root problem is carried out by Auscher [2007].

(e) Yet another approach to the scalar-valued L^p theory is due to Frey, McIntosh, and Portal [2018]. As in the approach of Hytönen, McIntosh, and Portal [2008] and in contrast to that of Auscher [2007], they work directly in L^p instead of extrapolating from L^2; also their "proof shows that the heart of the harmonic analysis in L^2 extends to L^p for all $p \in (1, \infty)$, while the restrictions in p come from the operator-theoretic part of the L^2 proof". A novelty in their approach is using the theory of tent spaces; on the side of the results, this allows them to dispense with the R-boundedness assumptions required by Hytönen, McIntosh, and Portal [2008].

Given the focus of these volumes on analysis in Banach spaces, we have not covered, in the discussion above, the rich literature of extensions and applications of the machinery of Auscher, Hofmann, Lacey, McIntosh, and Tchamitchian [2002] and Axelsson, Keith, and McIntosh [2006] in the L^2 theory of partial differential operators and equations; for this, we refer the reader to the numerous papers citing these pioneering contributions. The first-order approach to the Kato square root problem of Axelsson, Keith, and McIntosh [2006] has been adapted in Maas and Van Neerven [2009] to the Gaussian setting to prove a nonsymmetric version of the Meyer inequalities in Malliavin calculus. This work belongs to a circle of ideas that will be treated in Volume IV.

16

Perturbations and sums of operators

In this chapter we address a couple of topics in the theory of H^∞-calculus centering around the question what can be said about an operator of the form $A+B$ when A and B have certain "good" properties such as being (R-)sectorial or admitting a bounded H^∞-calculus. The chapter is divided into two sections. The first considers the case where B is "smaller" than A in certain ways, and the second considers the case where A and B are essentially on equal footing. The results of this chapter play an important role in applications as well in the further development of the abstract theory and will be needed in our treatment, in the next to chapter, of the maximal regularity problem.

16.1 Sums of unbounded operators

In general it is a rather delicate problem to give a meaning to the operator sum $A+B$ when A and B are unbounded operators acting in a Banach space X. The simplest approach is to define

$$\begin{aligned} \mathsf{D}(A+B) &:= \mathsf{D}(A) \cap \mathsf{D}(B), \\ (A+B)x &:= Ax + Bx, \quad x \in \mathsf{D}(A+B), \end{aligned} \tag{16.1}$$

but in concrete cases this definition may be vacuous due to the possibility that $\mathsf{D}(A) \cap \mathsf{D}(B)$ could be unreasonably small or even trivial, i.e., equal to $\{0\}$. Various methods to deal with this problem have been developed, such as the method of forms. In the context of evolution equations, the two prime applications one has in mind are cases where either A is the linear operator governing the equation, e.g., a linear differential operator in the space variables, and B is the derivative with respect to time, or both A and B are differential operators in the space variable, typically with B being of lower order than A. In both of these cases, the resolvent operators $R(\lambda, A)$ and $R(\mu, B)$ commute and $\mathsf{D}(A) \cap \mathsf{D}(B)$ is "large", in that it contains all elements of the form $R(\lambda, A)R(\mu, B)x$ with $x \in X$. In fact we have the following result.

T. Hytönen et al., *Analysis in Banach Spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics 76,
https://doi.org/10.1007/978-3-031-46598-7_6

Proposition 16.1.1. *If A and B are sectorial operators acting in X whose resolvents commute, then $\mathsf{D}(A) \cap \mathsf{D}(B)$ is dense in both $\mathsf{D}(A)$ and $\mathsf{D}(B)$*

Proof. As a consequence of the resolvent commutation and Proposition 10.1.7 we have

$$\lim_{\lambda \to -\infty} \lambda R(\lambda, A) R(\mu, B) x = R(\mu, B) x \text{ in } \mathsf{D}(B)$$

for all $\mu \in \varrho(B)$ and

$$\lim_{\mu \to -\infty} \mu R(\lambda, A) R(\mu, B) x = R(\lambda, A) x \text{ in } \mathsf{D}(A)$$

for all $\lambda \in \varrho(A)$ □

It is for this reason that we will stick to the somewhat naive approach embodied in (16.1); the operator sum $A + B$ will always be understood as given in this way.

Let us briefly clarify the meaning of the term 'resolvent commutation' used in the above proposition. If commutation identity

$$R(\lambda, A) R(\mu, B) = R(\mu, B) R(\lambda, A)$$

holds for some $\lambda \in \varrho(A)$ and $\mu \in \varrho(B)$, then it holds for all $\lambda' \in \varrho(A)$ and $\mu' \in \varrho(B)$ in the connected components of $\varrho(A)$ containing λ and μ, respectively. This is an easy consequence of the Taylor series identities

$$R(\lambda', A) = \sum_{n=0}^{\infty} (\lambda - \lambda')^n R(\lambda, A)^{n+1},$$
$$R(\mu', B) = \sum_{n=0}^{\infty} (\mu - \mu')^n R(\mu, B)^{n+1},$$

which follow from repeated application of the resolvent identity (see Section 10.1.b). The following definition then suggests itself naturally:

Definition 16.1.2 (Resolvent commutation). *The sectorial operators A and B are said to* resolvent commute *when*

$$R(\lambda, A) R(\mu, B) = R(\mu, B) R(\lambda, A)$$

holds for some (or equivalently, all) λ, μ in the connected set $\complement\overline{\Sigma_\sigma} \cap \complement\overline{\Sigma_\tau}$ for some (or equivalently, all) $\omega(A) < \sigma < \pi$ and $\omega(B) < \tau < \pi$.

16.2 Perturbation theorems

When it comes to checking the boundedness of the H^∞-calculus of concrete operators, in particular elliptic differential operators, perturbation theorems are often the method of choice. Perturbation arguments compare a "complicated" operator with a more "basic" operator such as the Laplace operator or an elliptic operator with constant coefficients. In order to cover a multitude of concrete situations we phrase these perturbation arguments in the framework of sectorial operators and their scale of fractional domain spaces. The case of lower-order perturbations of the form

$$L = A + B \quad \text{with} \quad B : \mathsf{D}(A^{\alpha}) \to X \quad \text{for some } 0 < \alpha < 1$$

(Theorem 16.2.7) is readily obtained from the corresponding theorem about relatively bounded perturbations of the form

$$L = A + B \text{ with } \ \|Bx\| \leqslant \delta\|Ax\| \ \text{ for small } \delta > 0$$

(Theorem 16.2.3). In contrast to sectoriality, boundedness of the H^{∞}-functional calculus is not preserved under small relatively bounded perturbations, unless additional relative boundedness assumptions are made with respect to the fractional domains (Example 16.2.10 and Theorem 16.2.8). Analogous perturbation theorems for R-sectorial operators are proved as well.

Because of their importance in applications, in particular for the study of non-linear evolution equations, the literature on perturbation theorems is extensive. We can present only a representative selection of such theorems and some model applications serving as illustrations. Variants and extensions of these results, in particular to elliptic operators and pseudo-differential operators, will be discussed in the Notes.

We next introduce some notation which will be used throughout this chapter and the next ones. Recalling from Definition 10.1.1 that an operator A is called *σ-sectorial* if the set $\{\lambda \neq 0, \, |\arg(\lambda)| > \sigma\}$ is contained in the resolvent set $\varrho(A)$ and

$$\sup_{\lambda \neq 0, \, |\arg(\lambda)| \geqslant \sigma} \|\lambda R(\lambda, A)\| < \infty,$$

we define

$$\begin{aligned} M_{\sigma,A} &:= \sup\{\|\lambda R(\lambda, A)\| : \, \lambda \neq 0, \, |\arg(\lambda)| > \sigma\}, \\ \widetilde{M}_{\sigma,A} &:= \sup\{\|AR(\lambda, A)\| : \, \lambda \neq 0, \, |\arg(\lambda)| > \sigma\}. \end{aligned}$$

When A is *σ-R-sectorial* (the definition being similar), for $p \in [1, \infty)$ we set

$$\begin{aligned} \widetilde{M}^{R_p}_{\sigma,A} &:= \mathscr{R}_p(\{\lambda R(\lambda, A) : \, \lambda \neq 0, \, |\arg(\lambda)| > \sigma\}), \\ \widetilde{M}^{R_p}_{\sigma,A} &:= \mathscr{R}_p(\{AR(\lambda, A) : \, \lambda \neq 0, \, |\arg(\lambda)| > \sigma\}), \end{aligned}$$

where $\mathscr{R}_p(\mathscr{T})$ denote the R-bound with exponent p (see Remark 8.1.2).

16.2.a Perturbations of sectorial operators

To set the stage for the results to follow, we begin with an elementary perturbation result for sectorial operators.

Proposition 16.2.1. *If A is an σ-sectorial operator on X and $B \in \mathscr{L}(X)$ is bounded, then for all $\lambda_0 \geqslant M\|B\|$ the operator $\lambda_0 + A + B$ is σ-sectorial.*

Proof. Set $M := M_{\sigma,A}$ for brevity. Fix a non-zero $\lambda \in \mathbb{C}$ with $|\arg(\lambda)| > \sigma$. Then $\lambda \in \varrho(A)$ and $\|R(\lambda, A)\| \leqslant M/|\lambda|$. Because

$$(\lambda - (A + B)) = (I - BR(\lambda, A))(\lambda - A)$$

and $\|BR(\lambda, A)\| \leqslant M\|B\|/|\lambda|$, for $|\lambda| > M\|B\|$ the operator $I - BR(\lambda, A)$ is invertible. For such λ it follows that $\lambda \in \varrho(A + B)$ and

$$R(\lambda, A + B) = R(\lambda, A) \sum_{n=0}^{\infty} [BR(\lambda, A)]^n$$

by the Neumann series. This gives the bound

$$\|R(\lambda, A + B)\| \leqslant \frac{M}{|\lambda|} \frac{1}{1 - M\|B\|/|\lambda|} = \frac{M}{|\lambda| - M\|B\|},$$

valid for non-zero $\lambda \in \mathbb{C}$ satisfying $|\arg(\lambda)| > \sigma$ and $|\lambda| > M\|B\|$. Shifting $A + B$ over $\lambda_0 \geqslant M\|B\|$, the result follows from this. □

The following lemma describes a useful technique that will enable us to deal with lower-order and relatively bounded perturbations.

Lemma 16.2.2 (The method of continuity). *Let E and F be Banach spaces. Let $(L_t)_{t\in[0,1]}$ be a family of bounded linear operators from E into F such that $t \mapsto L_t$ is continuous from $[0, 1]$ into $\mathscr{L}(E, F)$. Suppose furthermore that there exists a constant $C > 0$ such that for all $t \in [0, 1]$ and all $x \in E$ we have*

$$\|x\| \leqslant C\|L_t x\|.$$

Then L_0 is surjective if and only if L_1 is surjective.

Proof. Since $[0, 1]$ is compact, $t \mapsto L_t$ is uniformly continuous. Therefore we can find $\delta > 0$ such that $|t - s| < \delta$ implies $\|L_t - L_s\| \leqslant \frac{\varepsilon}{2C}$.

The assumption of the lemma imply that the operators L_t are injective. Now suppose that L_s is invertible for a given $s \in [0, 1]$. We will show that L_t is invertible for all $t \in [0, 1]$ satisfying $|t - s| < \delta$. Clearly, this implies the required result by an iteration argument.

Fix $f \in F$ and let $T : E \to E$ be the mapping given by $T(x) = y$, where $y \in E$ is the unique solution to $L_s y = f + L_s x - L_t x$. We claim that T is a uniform contraction. Indeed, by the assumed *a priori* estimate,

$$\|T(x_1) - T(x_2)\| = \|y_1 - y_2\| \leqslant C\|L_s y_1 - L_s y_2\|$$

Since $L_s y_1 - L_s y_2 = (L_s - L_t)(x_1 - x_2)$ we obtain

$$\|T(x_1) - T(x_2)\| \leqslant C\|L_s - L_t\| \, \|x_1 - x_2\| \leqslant \frac{1}{2}\|x_1 - x_2\|.$$

This proves the claim. By the Banach fixed point theorem, T has a unique fixed point x. It follows that $L_s x = f + L_s x - L_t x$, and hence $L_t x = f$. □

As a first application of this lemma we prove the following result on relatively bounded perturbations of sectorial operators.

Theorem 16.2.3 (Relatively bounded perturbations of sectorial operators). *Let A be an σ-sectorial operator, and let $B : \mathsf{D}(A) \to X$ be a linear operator that satisfies*

$$\|Bx\| \leqslant \delta\|Ax\| + K\|x\|, \quad x \in \mathsf{D}(A), \tag{16.2}$$

where $K \geqslant 0$ and $\delta \in (0,1)$ satisfies $\delta\widetilde{M}_{\sigma,A} < 1$. Then the operator $A+B$ with domain $\mathsf{D}(A+B) := \mathsf{D}(A)$ is closed, and the following assertions hold:

(1) *For all $\lambda \in \mathbb{R}$ large enough, $\lambda + A + B$ is σ-sectorial.*
(2) *If (16.2) holds with $K = 0$, then $A + B$ is σ-sectorial.*

Proof. Observe that for all $x \in \mathsf{D}(A)$,

$$\|Ax\| \leqslant \|(A+B)x\| + \|Bx\| \leqslant \|(A+B)x\| + \delta\|Ax\| + K\|x\|. \tag{16.3}$$

Therefore, $(1-\delta)\|Ax\| \leqslant \|(A+B)x\| + K\|x\|$. By a routine argument, (16.2) and (16.3) imply that $A+B$ is closed.

We will prove both assertions at the same time by showing that $\lambda_0 + A + B$ is sectorial for any fixed $\lambda_0 \geqslant 0$ large enough, permitting $\lambda_0 = 0$ if (16.2) holds with $K = 0$.

Fix $\lambda \in \lambda_0 + \Sigma_\sigma$. We will apply Lemma 16.2.2 to $E = \mathsf{D}(A)$, $F = X$, and the operators $L_t : \mathsf{D}(A) \to X$ given by

$$L_t x := (\lambda + A + tB)x, \qquad t \in [0,1],$$

where $\mathsf{D}(A)$ will be equipped with the equivalent norm

$$|\!|\!|x|\!|\!| = \|(\lambda - \lambda_0)x\| + \|Ax\|.$$

We first prove the following *a priori* estimate: For all $\lambda_0 \geqslant 0$ large enough there exists a constant $C \geqslant 0$ such that

$$|\!|\!|x|\!|\!| \leqslant C\|L_t x\|, \qquad x \in \mathsf{D}(A),\ t \in [0,1]. \tag{16.4}$$

Let $x \in \mathsf{D}(A)$ and set $y := L_t x$. Then $(\lambda + A)x = y - tBx$. Multiplying this identity with $A(\lambda + A)^{-1}$ on both sides and using (16.2), we obtain

$$\|Ax\| \leqslant \widetilde{M}_{\sigma,A}\|y\| + \widetilde{M}_{\sigma,A}\|Bx\| \leqslant \widetilde{M}_{\sigma,A}\|y\| + \widetilde{M}_{\sigma,A}\delta\|Ax\| + \widetilde{M}_{\sigma,A}K\|x\|.$$

Since $\widetilde{M}_{\sigma,A}\delta < 1$, it follows that

$$\|Ax\| \leqslant C_0\|y\| + C_0 K\|x\|, \tag{16.5}$$

where $C_0 = \widetilde{M}_{\sigma,A}(1 - \widetilde{M}_{\sigma,A}\delta)^{-1}$. To estimate $\|x\|$, writing $\lambda x = y - tBx - Ax$ we find that

$$\begin{aligned}|\lambda|\|x\| &\leqslant \|y\| + \|Bx\| + \|Ax\| \\ &\leqslant \|y\| + (\delta+1)\|Ax\| + K\|x\| \leqslant C_1\|y\| + C_2K\|x\|\end{aligned}$$

where $C_1 := 1 + (\delta+1)C_0$ and $C_2 := (\delta+1)C_0 + 1$, so that

$$\|x\| \leqslant \frac{C_1}{|\lambda| - C_2K}\|y\| =: D\|y\|,$$

provided we take $\lambda_0 \geqslant C_2K$ sufficiently large (in order that $|\lambda| > C_2K$). Such choices of λ_0 imply that $|\lambda - \lambda_0| \leqslant C_\sigma|\lambda|$ and, together with (16.5),

$$\begin{aligned}\|\!|x|\!\| = \|Ax\| + \|(\lambda-\lambda_0)x\| &\leqslant C_0\|y\| + C_0K\|x\| + |\lambda-\lambda_0|\|x\| \\ &\leqslant C_0\|y\| + C_0K\|x\| + C_\sigma(C_1\|y\| + C_2K\|x\|) \\ &\leqslant C\|y\| = C\|L_tx\|\end{aligned}$$

where $C := (C_0 + C_\sigma C_1) + (C_0 + C_\sigma C_2)DK$, which is (16.4). Scrutinising the proof, we see that $\lambda_0 = 0$ can be allowed if (16.2) holds with $K = 0$.

Since $L_0 = \lambda + A$ is surjective, Lemma 16.2.2 gives that $L_1 = \lambda + A + B$ is surjective, and hence boundedly invertible by (16.4). Also by (16.4), for all $y \in X$ and $\lambda \in \lambda_0 + \Sigma_{\pi-\sigma}$ (where we may take $\lambda_0 = 0$ if $K = 0$),

$$\|(\lambda-\lambda_0)(\lambda + A + B)^{-1}y\| \leqslant \|\lambda(\lambda + A + B)^{-1}y\| \leqslant C\|y\|,$$

which proves $\lambda_0 + A + B$ is σ-sectorial. □

Theorem 16.2.4 (Relatively bounded perturbations of R-sectorial operators). *Let A be σ-R-sectorial, and suppose that $B : \mathsf{D}(A) \to X$ is a linear operator which satisfies*

$$\|Bx\| \leqslant \delta\|Ax\| + K\|x\|, \quad x \in \mathsf{D}(A), \tag{16.6}$$

where $K \geqslant 0$ and $\delta \in (0,1]$ satisfies $\delta\widetilde{M}^{R_p}_{\sigma,A} < 1$ for some $p \in [1,\infty)$. Then the operator $A + B$ with domain $\mathsf{D}(A+B) := \mathsf{D}(A)$ is closed, and the following assertions hold:

(1) *For all $\lambda \in \mathbb{R}$ large enough, $\lambda + A + B$ is σ-R-sectorial.*
(2) *If (16.6) holds with $K = 0$, then $A + B$ is σ-R-sectorial.*

Proof. The method of proof is similar to that of Theorem 16.2.3. Again we will prove both assertions at the same time. Let $(\Omega, \mathbb{P})$ be a probability space supporting a Rademacher sequence $(\varepsilon_n)_{n\geqslant 1}$. For notational convenience we write $\|\cdot\|_p = \|\cdot\|_{L^p(\Omega;X)}$. We will show that $\lambda_0 + A + B$ is R-sectorial for all $\lambda_0 \geqslant 0$ large enough, and that we may take $\lambda_0 = 0$ if (16.6) holds with $K = 0$.

The assumptions of the theorem imply those of Theorem 16.2.3, and therefore $A + B$ satisfies its conclusions. It remains to prove the R-boundedness of the set

$$\big\{(\lambda-\lambda_0)(\lambda + A + B)^{-1} : \lambda \neq 0,\, \lambda \in \lambda_0 + \Sigma_{\pi-\sigma}\big\}.$$

To this end let $n \geqslant 1$, non-zero $\lambda_1, \ldots, \lambda_n \in \lambda_0 + \Sigma_{\pi-\sigma}$, and $y_1, \ldots, y_n \in X$ be arbitrary and fixed. Let $x_j \in X$ be the unique solution to $(\lambda_j + A + B)x_j = y_j$ for each $j \in \{1, \ldots, n\}$. It suffices to show that there is a constant $C \geqslant 0$ such that

$$\Big\| \sum_{j=1}^n \varepsilon_j (\lambda_j - \lambda_0) x_j \Big\|_p \leqslant C \Big\| \sum_{j=1}^n \varepsilon_j y_j \Big\|_p.$$

Since $Ax_j = A(\lambda_j + A)^{-1}[y_j - Bx_j]$, the R-sectoriality of A gives

$$\begin{aligned}\Big\| \sum_{j=1}^n \varepsilon_j A x_j \Big\|_p &\leqslant M \Big\| \sum_{j=1}^n \varepsilon_j y_j \Big\|_p + M \Big\| \sum_{j=1}^n \varepsilon_j B x_j \Big\|_p \\ &\leqslant M \Big\| \sum_{j=1}^n \varepsilon_j y_j \Big\|_p + M\delta \Big\| \sum_{j=1}^n \varepsilon_j A x_j \Big\|_p + MK \Big\| \sum_{j=1}^n \varepsilon_j x_j \Big\|_p,\end{aligned}$$

where $M := \widetilde{M}^{R_p}_{\sigma,A}$ for brevity. Therefore,

$$\Big\| \sum_{j=1}^n \varepsilon_j A x_j \Big\|_p \leqslant C_0 \Big\| \sum_{j=1}^n \varepsilon_j y_j \Big\|_p + C_0 K \Big\| \sum_{j=1}^n \varepsilon_j x_j \Big\|_p,$$

where $C_0 = CM(1-\delta M)^{-1}$. Since $\lambda_j x_j = y_j - Bx_j - Ax_j$, we also find

$$\begin{aligned}\Big\| \sum_{j=1}^n \varepsilon_j \lambda_j x_j \Big\|_p &\leqslant \Big\| \sum_{j=1}^n \varepsilon_j y_j \Big\|_p + \Big\| \sum_{j=1}^n \varepsilon_j B x_j \Big\|_p + \Big\| \sum_{j=1}^n \varepsilon_j A x_j \Big\|_p \\ &\leqslant \Big\| \sum_{j=1}^n \varepsilon_j y_j \Big\|_p + K \Big\| \sum_{j=1}^n \varepsilon_j x_j \Big\|_p + (1+\delta) \Big\| \sum_{j=1}^n \varepsilon_j A x_j \Big\|_p \\ &\leqslant C_1 \Big\| \sum_{j=1}^n \varepsilon_j y_j \Big\|_p + C_1 K \Big\| \sum_{j=1}^n \varepsilon_j x_j \Big\|_p,\end{aligned} \tag{16.7}$$

where $C_1 = 1 + (1+\delta)C_0$.

Next we claim that there exist $D \geqslant 0$ and $\lambda_0 \geqslant 0$ such that

$$|\lambda_j - \lambda_0| \leqslant D(|\lambda_j| - 2C_1 K), \tag{16.8}$$

Writing $\lambda_j = \lambda_0 + re^{i\phi}$ with $|\phi| < \pi - \sigma$, (16.8) can be equivalently written as

$$(r + 2DC_1K)^2 \leqslant D^2(\lambda_0^2 + r^2 + 2\lambda_0 r \cos\phi).$$

If $|\phi| \leqslant \frac{1}{2}\pi$, then $\cos\phi \geqslant 0$ and the estimate holds with $D = \sqrt{2}$ and $\lambda_0 = C_1 K$. If $\frac{1}{2}\pi < |\phi| < \pi$, set $\delta := 1 + \cos\phi$ and note that $\delta \in (0,1)$. It then follows that

$$\lambda_0^2 + r^2 + 2\lambda_0 r\cos\phi = \lambda_0^2 + r^2 - 2\lambda_0 r(1-\delta)$$
$$= \delta(\lambda_0^2 + r) + (1-\delta)(\lambda_0 - r)^2 \geqslant \delta(\lambda_0^2 + r^2)$$

and the estimate holds with $D = \sqrt{8/\delta}$ and $\lambda_0 = DC_1K$. This proves the claim.

The claim implies $|(\lambda_j - \lambda_0)/\lambda_j| \leqslant D$ and $2C_1DK \leqslant D|\lambda_j|$, and therefore the Kahane contraction principle (see Theorem 6.1.13) implies

$$\Big\|\sum_{j=1}^n \varepsilon_j(\lambda_j - \lambda_0)x_j\Big\|_p = \Big\|\sum_{j=1}^n \varepsilon_j \frac{\lambda_j - \lambda_0}{\lambda_j}\lambda_j x_j\Big\|_p \leqslant D\Big\|\sum_{j=1}^n \varepsilon_j\lambda_j x_j\Big\|_p$$

and

$$2C_1DK\Big\|\sum_{j=1}^n \varepsilon_j x_j\Big\|_p \leqslant D\Big\|\sum_{j=1}^n \varepsilon_j|\lambda_j|x_j\Big\|_p = D\Big\|\sum_{j=1}^n \varepsilon_j\lambda_j x_j\Big\|_p.$$

Taking the averages of the last two estimates we obtain

$$\frac{1}{2}\Big\|\sum_{j=1}^n \varepsilon_j(\lambda_j - \lambda_0)x_j\Big\|_p \leqslant D\Big\|\sum_{j=1}^n \varepsilon_j\lambda_j x_j\Big\|_p - C_1DK\Big\|\sum_{j=1}^n \varepsilon_j x_j\Big\|_p$$
$$\leqslant C_1D\Big\|\sum_{j=1}^n \varepsilon_j y_j\Big\|_p,$$

where in the last step we applied (16.7). □

As a simple corollary to the above results we show that the smallness conditions on the constants can be lifted in the case of lower order perturbations. The notation is as in Appendix C.

Corollary 16.2.5 (Lower order perturbations of (R-)sectorial operators). *Let A be sectorial (resp. R-sectorial) and let $\theta \in (0,1)$. If*

$$B : \mathsf{D}(A^\theta) \to X$$

is a bounded linear operator, then for all large enough $\lambda \in \mathbb{R}$ the operator $\lambda + A + B$ is sectorial (resp. R-sectorial) with $\omega(\lambda + A + B) \leqslant \omega(A)$ (resp. $\omega_R(\lambda + A + B) \leqslant \omega_R(A)$).

Proof. It suffices to check the conditions of Theorems 16.2.3 and 16.2.4. For $x \in \mathsf{D}(A)$, by the interpolation estimate of Theorem 15.2.8 we obtain

$$\|Bx\| \leqslant \|B\|\,\|x\|_{\mathsf{D}(A^\theta)} \leqslant \|B\|\,\|x\|^{1-\theta}\|x\|_{\mathsf{D}(A)}^\theta.$$

Using the inequality $a^{1-\theta}b^\theta \leqslant (1-\theta)a + \theta b$, for all $\varepsilon > 0$ we obtain

$$\|x\|^{1-\theta}\|x\|_{\mathsf{D}(A)}^\theta \leqslant (1-\theta)\varepsilon^{-\frac{1}{1-\theta}}\|x\| + \varepsilon^{\frac{1}{\theta}}\|x\|_{\mathsf{D}(A)}.$$

The result now follows by combining the estimates and choosing $\varepsilon > 0$ small enough. □

The same proof works if one assumes that $B : (X, \mathsf{D}(A))_{\theta,p} \to X$ is a bounded operator for some $\theta \in (0,1)$ and $p \in [1,\infty]$, or that $B : [X, \mathsf{D}(A)]_\theta \to X$ is a bounded operator for some $\theta \in (0,1)$. A similar remark applies to Theorem 16.2.8 below.

16.2.b Perturbations of the H^∞-calculus

Having studied perturbations of sectorial and R-sectorial operators, we now turn to perturbation of the H^∞-calculus. The first proposition addresses shifts by a positive multiple of the identity. In certain applications it enables one to improve "for sufficiently large $\nu > 0$" to "for all $\nu > 0$".

Proposition 16.2.6 (Perturbation by a multiple of the identity). *Let A be a sectorial operator on X.*

(1) *If A has a bounded $H^\infty(\Sigma_\sigma)$-calculus, then $A+\nu I$ has a bounded $H^\infty(\Sigma_\sigma)$-calculus for all $\nu > 0$, and $M^\infty_{\sigma,A+\nu} \leqslant M^\infty_{\sigma,A}$.*
(2) *If $A + \nu_0 I$ has a bounded $H^\infty(\Sigma_\sigma)$-calculus for some $\nu_0 > 0$, then $A + \nu I$ has a bounded $H^\infty(\Sigma_\sigma)$-calculus for all $\nu > 0$.*

Proof. Assertion (1) is obtained by applying the bounded H^∞-calculus of A to the function $f_\nu(z) = f(z + \nu)$, noting that $f_\nu(A) = f(A + \nu)$; since $\|f(\cdot+\nu)\|_{H^\infty(\Sigma_\sigma)} \leqslant \|f\|_{H^\infty(\Sigma_\sigma)}$, this also gives the bound for the boundedness constants of the H^∞-calculi.

For the proof of assertion (2) we fix $\nu > 0$. Writing $A+\nu = (A+\varepsilon)+(\nu-\varepsilon)$ we see that there is no loss of generality in assuming that A is invertible. We also may assume that $0 < \nu < \delta$, where $\delta > 0$ is to be specified later, for once we have the converse for such ν the general case follows by repeated application of the first part of the proposition.

For $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ consider

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_\Gamma f(\lambda)R(\lambda, A+\nu_0)\,\mathrm{d}\lambda - \frac{1}{2\pi i}\int_\Gamma f(\lambda)R(\lambda, A+\nu)\,\mathrm{d}\lambda \\
&\quad = (\nu_0 - \nu)\frac{1}{2\pi i}\int_\Gamma f(\lambda)R(\lambda, A+\nu)R(\lambda, A+\nu_0)\,\mathrm{d}\lambda \\
&\quad = (\nu_0 - \nu)\frac{1}{2\pi i}\int_\Gamma f(\lambda)R(\lambda, A+\nu)R(0, A+\nu_0)\,\mathrm{d}\lambda \\
&\qquad + (\nu_0 - \nu)\frac{1}{2\pi i}\int_\Gamma f(\lambda)R(\lambda, A+\nu)[R(\lambda, A+\nu_0) - R(0, A+\nu_0)]\,\mathrm{d}\lambda \\
&\quad = (\nu_0 - \nu)R(-\nu_0, A)\frac{1}{2\pi i}\int_\Gamma f(\lambda)R(\lambda, A+\nu)\,\mathrm{d}\lambda \\
&\qquad - (\nu_0 - \nu)\frac{1}{2\pi i}\int_\Gamma \lambda f(\lambda)R(\lambda, A+\nu)R(\lambda-\nu_0, A)R(-\nu_0, A)\,\mathrm{d}\lambda.
\end{aligned}
$$

If we call the last integral $I(f)$, we have

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_\Gamma f(\lambda)R(\lambda, A+\nu_0)\,\mathrm{d}\lambda \\
&\quad = [I+(\nu_0-\nu)R(-\nu_0,A)]\frac{1}{2\pi i}\int_\Gamma f(\lambda)R(\lambda,A+\nu)\,\mathrm{d}\lambda - (\nu_0-\nu)I(f).
\end{aligned}
$$

For invertible A the operator $-AR(-\nu_0,A) = I+\nu_0 R(-\nu_0,A)$ is invertible as well. Since the set of invertible operators is open in $\mathscr{L}(X)$, there exists an $r>0$ so small that $I+(\nu_0-\nu)R(-\nu_0,A)$ is invertible if $\nu\|R(-\nu_0,A)\|<r$, i.e., if $\nu<\delta := r/\|R(-\nu_0,A)\|$. Under this assumption we have the representation

$$
f(A+\nu) = [I+(\nu_0-\nu)R(-\nu_0,A)]^{-1}[f(A+\nu_0)+(\nu_0-\nu)I(f)].
$$

Hence

$$
\|f(A+\nu)\| \leqslant \big\|[I-(\nu_0-\nu)R(-\nu_0,A)]^{-1}\big\|\big(\|f(A+\nu_0)\|+(\nu_0-\nu)\|I(f)\|\big). \tag{16.9}
$$

By the assumptions we have $\|f(A+\nu_0)\|\leqslant C\|f\|_{H^\infty(\Sigma_\sigma)}$. We estimate the integral $I(f)$ by splitting it into $\Gamma_1=\Gamma\cap\{|\lambda|\leqslant 1\}$ and $\Gamma_2=\Gamma\cap\{|\lambda|\geqslant 1\}$ and using

$$
R(\lambda-\nu_0,A) = (\lambda-\nu_0)^{-1}[R(\lambda-\nu_0,A)A+I].
$$

This gives

$$
\begin{aligned}
I(f) = \frac{1}{2\pi i}&\int_{\Gamma_1}\lambda f(\lambda)R(\lambda,A+\nu)R(\lambda-\nu_0,A)\,\mathrm{d}\lambda\;[R(-\nu_0,A)]\\
&+\frac{1}{2\pi i}\int_{\Gamma_2}\lambda f(\lambda)R(\lambda,A+\nu)(\lambda-\nu_0)^{-1}R(\lambda-\nu_0,A)\,\mathrm{d}\lambda\;[AR(-\nu_0,A)]\\
&+\frac{1}{2\pi i}\int_{\Gamma_2}\lambda f(\lambda)R(\lambda,A+\nu)R(-\nu_0,A)(\lambda-\nu_0)^{-1}\,\mathrm{d}\lambda\\
&=(I)+(II)+(III).
\end{aligned}
$$

The integrals (I) and (II) can be estimated by $C\|f\|_{H^\infty(\Sigma_\sigma)}$ with constant C only depending on A, ν_0, σ. The third can be rewritten with the help of the resolvent identity and Cauchy's formula:

$$
\begin{aligned}
(III) &= \frac{1}{2\pi i}\int_{\Gamma_2}\frac{f(\lambda)}{\lambda-\nu_0}R(-\nu_0,A)\,\mathrm{d}\lambda - \frac{1}{2\pi i}\int_{\Gamma_2}\frac{f(\lambda)}{\lambda-\nu_0}R(\lambda,A+\nu)\,\mathrm{d}\lambda\\
&= f(\nu_0)R(-\nu,A) - \frac{1}{2\pi i}\int_{\Gamma_1}\frac{f(\lambda)}{\lambda-\nu_0}R(-\nu_0,A)\,\mathrm{d}\lambda\\
&\qquad - \frac{1}{2\pi i}\int_{\Gamma_2}\frac{f(\lambda)}{\lambda-\nu_0}R(\lambda,A+\nu)\,\mathrm{d}\lambda.
\end{aligned}
$$

The two remaining integrals can again be estimated by $C\|f\|_{H^\infty(\Sigma_\sigma)}$ with constant C only depending on A, ν_0, σ. With (16.9) we arrive at

$$
\|f(A+\nu)\|\leqslant C'\|f\|_{H^\infty(\Sigma_\sigma)},\quad f\in H^1(\Sigma_\sigma)\cap H^\infty(\Sigma_\sigma),
$$

thus completing the proof. ∎

We continue with the following result for lower order perturbations.

Theorem 16.2.7 (Lower order perturbations of the H^∞-calculus). *Let A be a sectorial operator and suppose that B is linear operator in X satisfying*

$$\mathsf{D}(A^\alpha) \subseteq \mathsf{D}(B)$$

and

$$\|Bx\| \leqslant a\|A^\alpha x\| + b\|x\|, \quad x \in \mathsf{D}(A),$$

for suitable real numbers $a, b \geqslant 0$ and $\alpha \in (0,1)$. If A has a bounded $H^\infty(\Sigma_\sigma)$-calculus in X for some $\omega(A) < \sigma < \pi$, then $A+B+\nu$ has a bounded $H^\infty(\Sigma_\sigma)$-calculus in X for all sufficiently large $\nu > 0$.

Proof. By Proposition 16.2.3, for large enough $\nu > 0$ the operator $A + B + \nu$ is sectorial and $\omega(A+B+\nu) \leqslant \omega(A)$. By taking ν larger if necessary, we may assume that $0 \in \varrho(A + B + \nu)$.

We start from the identity

$$\begin{aligned} R(\lambda, A+B+\nu) &= R(\lambda, A+\nu) + R(\lambda, A+B+\nu)BR(\lambda, A+\nu) \\ &= R(\lambda, A+\nu) + M(\lambda), \end{aligned}$$

which may be verified by applying $\lambda - (A + B + \nu)$ on both sides, and where

$$M(\lambda) = R(\lambda, A+B+\nu)[B(A+\nu)^{-\alpha}](A+\nu)^\alpha R(\lambda, A+\nu).$$

For functions $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ this gives the Dunford integral

$$f(A+B) = f(A) + \frac{1}{2\pi i}\int_{\Gamma_\eta} f(\lambda)M(\lambda)\,\mathrm{d}\lambda,$$

where the contour $\Gamma_\eta = \partial\Sigma_\eta$ with $\omega(A) < \eta < \sigma$ is chosen as usual. Near the origin, the integrand is bounded since we assumed that $0 \in \varrho(A+B+\nu)$. For large values of $|\lambda|$ the integrand may be estimated pointwise by

$$\|f(\lambda)M(\lambda)\| \leqslant M_1 M_2 |\lambda|^{-\alpha}\|B(A+\nu)^{-\alpha}\|\|f\|_{H^\infty(\Sigma_\sigma)},$$

since

$$M_1 := \sup\{\|\lambda R(\lambda, A+B+\nu)\| : |\arg\lambda| = \eta\}$$

is finite by sectoriality of $A + B + \nu$ and

$$M_2 := \sup\{\|\lambda^{1-\alpha}(A+\nu)^\alpha R(\lambda, A+\nu)\| : |\arg\lambda| = \eta\}$$

is finite by sectoriality of $A + \nu$ and Corollary 15.2.14. It follows that the integral converges absolutely and its norm is bounded by a constant times $\|f\|_{H^\infty(\Sigma_\sigma)}$. This completes the proof. □

Our main perturbation theorem asserts that the H^∞-calculus of an R-sectorial operator is preserved under relatively bounded perturbations of the H^∞-calculus if we add an additional relative boundedness assumption in the fractional domains scale.

Theorem 16.2.8 (Relatively bounded perturbations of the H^∞-calculus). *Let A be a densely defined sectorial operator with a bounded $H^\infty(\Sigma_\sigma)$-calculus and let B be a densely defined τ-R-sectorial operator on X, with*

$$\mathsf{D}(A) \subseteq \mathsf{D}(B) \quad \textit{and} \quad 0 \in \varrho(A)$$

and satisfying the relative bound

(i) $\|Bx\| \leqslant C_0\|Ax\|$ *for all* $x \in \mathsf{D}(A)$.

Suppose that at least one of the following two additional relative bounds is also satisfied:

(ii) *there exists an* $\alpha \in (0,1)$ *such that* B *maps* $\mathsf{D}(A^{1+\alpha})$ *into* $\mathsf{D}(A^\alpha)$ *and*

$$\|A^\alpha Bx\| \leqslant C_1\|A^{1+\alpha}x\|, \quad x \in \mathsf{D}(A^{1+\alpha});$$

(iii) *there exists an* $\alpha \in (0,1)$ *such that*

$$\|A^{-\alpha}Bx\| \leqslant C_1\|A^{1-\alpha}x\|, \quad x \in \mathsf{D}(A^{1-\alpha}).$$

Then, given the constant C_1 in (ii) or (iii), there is a small enough constant $C \geqslant 0$ so that if (i) holds with $0 \leqslant C_0 \leqslant C$, then $A + B$ has a bounded $H^\infty(\Sigma_{\sigma\vee\tau})$-calculus.

If in (ii) or (iii) we have $C_1 < 1/\widetilde{M}_{\sigma\vee\tau,A}$, then the condition $0 \in \varrho(A)$ may be replaced by the weaker condition that A be injective and B maps $\mathsf{D}(A^{1-\alpha})$ into $\mathsf{D}(A^{-\alpha})$.

In the last line of the statement of the theorem, recall the notation $\widetilde{M}_{\theta,A} = \sup\{\|AR(\lambda,A)\| : \lambda \neq 0, \|\arg(\lambda)| \geqslant \theta\}$.

If X has the triangular contraction property, in particular if X is a UMD space, then by Theorem 10.3.4 we have $\omega_R(A) \leqslant \omega_{H^\infty}(A)$ and therefore the theorem applies.

At the end of the section, an example will be presented which shows that the additional assumptions (ii) and (iii) cannot be omitted.

We will reduce the theorem to the following technical lemma.

Lemma 16.2.9. *Let A be a densely defined sectorial operator with a bounded $H^\infty(\Sigma_\sigma)$-calculus and let B a densely defined R-sectorial operator on X. Let $\omega(A) < \sigma < \pi$ and $\omega_R(B) < \tau < \pi$, and set $\mu := \max\{\sigma,\tau\}$. Suppose there exists a holomorphic function $M : \{|\arg(\lambda)| > \mu\} \to \mathscr{L}(X)$ with R-bounded range and a $\beta \in (0,1)$ such that*

$$R(\lambda,B) = R(\lambda,A) + A^\beta R(\lambda,A)M(\lambda)A^{1-\beta}R(\lambda,A), \quad |\arg(\lambda)| \geqslant \mu. \quad (16.10)$$

Then B has a bounded $H^\infty(\Sigma_\mu)$-calculus.

Proof. Our aim is to prove that there exists a function $\phi \in H^1(\Sigma_\mu)$ and a constant $C \geqslant 0$ such that for all integers $N \geqslant 1$, all scalars $\epsilon_{-N}, \ldots, \epsilon_N$ of modulus one, and all $t > 0$ we have

$$\Big\| \sum_{|n| \leqslant N} \epsilon_n \phi(t2^n B) \Big\| \leqslant C.$$

Once we have this, it follows from Proposition 10.4.11 (and tracking angles in its proof) that B has a bounded $H^\infty(\Sigma_\mu)$-calculus.

Let $\mu < \nu < \pi$ and consider the function $\psi_\nu \in H^1(\Sigma_\mu)$ given by

$$\psi_\nu(z) = \frac{z^{1/2}}{(e^{i\nu} - z)^{1/2}(2e^{i\nu} - z)^{1/2}}, \quad z \in \Sigma_\mu,$$

so that $\phi_\nu := \psi_\nu^2$ satisfies

$$\phi_\nu(z) = \frac{z}{(e^{i\nu} - z)(2e^{i\nu} - z)} = \frac{1}{e^{i\nu} - z} - \frac{2}{2e^{i\nu} - z}.$$

By (16.10),

$$\begin{aligned}
R(\lambda, t2^n B) &= t^{-1}2^{-n} R(t^{-1}2^{-n}\lambda, B) \\
&= t^{-1}2^{-n} R(t^{-1}2^{-n}\lambda, A) \\
&\qquad + t^{-1}2^{-n} A^\beta R(t^{-1}2^{-n}\lambda, A) M(t^{-1}2^{-n}\lambda) A^{1-\beta} R(t^{-1}2^{-n}\lambda, A) \\
&= R(\lambda, t2^n A) \\
&\qquad + t2^n A^\beta R(\lambda, t2^n A) M(t^{-1}2^{-n}\lambda) A^{1-\beta} R(\lambda, t2^n A).
\end{aligned}$$

By Corollary 15.2.14, the right-hand side has decay of order $|\lambda|^{-1}$ as $|\lambda| \to \infty$ in the complement of Σ_μ. Hence, by Cauchy's theorem and taking $\mu < \tau < \nu$,

$$\begin{aligned}
\phi_\nu(t2^n B) &= \frac{1}{2\pi i} \int_{\partial \Sigma_\tau} \phi_\nu(\lambda) R(\lambda, t2^n B) \, \mathrm{d}\lambda \\
&= \phi_\nu(t2^n A) + t2^n A^\beta R(e^{i\nu}, t2^n A) M(t^{-1}2^{-n} e^{i\nu}) A^{1-\beta} R(e^{i\nu}, t2^n A) \\
&\quad - t2^{n+1} A^\beta R(2e^{i\nu}, t2^n A) M(t^{-1}2^{-n} 2e^{i\nu}) A^{1-\beta} R(2e^{i\nu}, t2^n A) \\
&= \phi_\nu(t2^n A) + t2^n A^\beta R(e^{i\nu}, t2^n A) M(t^{-1}2^{-n} e^{i\nu}) A^{1-\beta} R(e^{i\nu}, t2^n A) \\
&\quad - t2^{n-1} A^\beta R(e^{i\nu}, t2^{n-1} A) M(t^{-1}2^{-(n-1)} e^{i\nu}) A^{1-\beta} R(e^{i\nu}, t2^{n-1} A) \\
&= \phi_\nu(t2^n A) + \phi_{\beta,\nu}(t2^n A) M(t^{-1}2^{-n} e^{i\nu}) \phi_{1-\beta,\nu}(t2^n A) \\
&\quad - \phi_{\beta,\nu}(t2^{n-1} A) M(t^{-1}2^{-(n-1)} e^{i\nu}) \phi_{1-\beta,\nu}(t2^{n-1} A) \\
&= (I) + (II) + (III),
\end{aligned} \tag{16.11}$$

where for $\alpha > 0$ we define $\phi_{\alpha,\nu} \in H^1(\Sigma_\mu)$ by

$$\phi_{\alpha,\nu}(z) := \frac{z^\alpha}{e^{i\nu} - z}.$$

In the penultimate identity of (16.11) we used the identity

$$\phi_{\alpha,\nu}(\tau A) = \tau^\alpha A^\alpha R(e^{i\nu}, \tau A),$$

which follows from Propositions 15.1.12 and 15.2.6.

We estimate the terms (I)–(III) separately. We begin with (II). Fixing $x \in X$, by randomisation with a Rademacher sequence $(\varepsilon_n)_{n\in\mathbb{Z}}$,

$$\begin{aligned}
&\Big\| \sum_{|n|\leqslant N} \epsilon_n \phi_{\beta,\nu}(t2^n A) M(t^{-1}2^{-n}e^{i\nu}) \phi_{1-\beta,\nu}(t2^n A)x \Big\| \\
&= \sup_{\|x^*\|\leqslant 1} \Big| \sum_{|n|\leqslant N} \epsilon_n \langle M(t^{-1}2^{-n}e^{i\nu}) \phi_{1-\beta,\nu}(t2^n A)x, \phi_{\beta,\nu}(t2^n A)^* x^* \rangle \Big| \\
&= \sup_{\|x^*\|\leqslant 1} \Big| \mathbb{E} \Big\langle \sum_{|n|\leqslant N} \epsilon_n \varepsilon_n M(t^{-1}2^{-n}e^{i\nu}) \phi_{1-\beta,\nu}(t2^n A)x, \sum_{|n|\leqslant N} \varepsilon_n \phi_{\beta,\nu}(t2^n A^*)x^* \Big\rangle \Big| \\
&\leqslant \mathbb{E} \Big\| \sum_{|n|\leqslant N} \epsilon_n \varepsilon_n M(t^{-1}2^{-n}e^{i\nu}) \phi_{1-\beta,\nu}(t2^n A)x \Big\| \\
&\qquad\qquad \times \sup_{\|x^*\|\leqslant 1} \mathbb{E} \Big\| \sum_{|n|\leqslant N} \varepsilon_n \phi_{\beta,\nu}(t2^n A^*)x^* \Big\| \\
&\lesssim_M \mathbb{E} \Big\| \sum_{|n|\leqslant N} \varepsilon_n \phi_{1-\beta,\nu}(t2^n A)x \Big\| \sup_{\|x^*\|\leqslant 1} \mathbb{E} \Big\| \sum_{|n|\leqslant N} \varepsilon_n \phi_{\beta,\nu}(t2^n A^*)x^* \Big\|,
\end{aligned}$$

where the implicit constant in the last step is the R-boundedness constant of M. Similarly, shifting the index by one and using the contraction principle, we estimate (III) as follows:

$$\begin{aligned}
&\Big\| \sum_{|n|\leqslant N} \epsilon_n \phi_{\beta,\nu}(t2^n A) M(t^{-1}2^{-n}e^{i\nu}) \phi_{1-\beta,\nu}(t2^n A)x \Big\| \\
&\lesssim_M \mathbb{E} \Big\| \sum_{|n|\leqslant N} \varepsilon_n \phi_{1-\beta,\nu}(t2^{n-1} A)x \Big\| \sup_{\|x^*\|\leqslant 1} \Big\| \sum_{|n|\leqslant N} \varepsilon_n \phi_{\beta,\nu}(t2^{n-1} A^*)x^* \Big\| \\
&\leqslant \mathbb{E} \Big\| \sum_{|n|\leqslant N+1} \varepsilon_n \phi_{1-\beta,\nu}(t2^n A)x \Big\| \sup_{\|x^*\|\leqslant 1} \mathbb{E} \Big\| \sum_{|n|\leqslant N+1} \varepsilon_n \phi_{\beta,\nu}(t2^n A^*)x^* \Big\|.
\end{aligned}$$

By the same argument, for (I) we obtain

$$\Big\| \sum_{|n|\leqslant N} \epsilon_n \phi_\nu(t2^n A)x \Big\| \leqslant \mathbb{E} \Big\| \sum_{|n|\leqslant N} \varepsilon_n \psi_\nu(t2^n A)x \Big\|.$$

Taking the supremum over $N \geqslant 1$ and $t > 0$, this proves the square function bound

$$\sup_{|n|\geqslant N}\sup_{t>0}\Big\|\sum_{|n|\leqslant N}\epsilon_n\phi_\nu(t2^nB)x\Big\| \leqslant C\|x\| + 2C'\sup_{t>0}\|x\|_{\phi_{1-\beta},A}\sup_{\|x^*\|\leqslant 1}\|x^*\|_{\phi_\beta,A^*}$$
$$\leqslant C''\|x\|,$$

where the estimate in the last step follows from the boundedness of the $H^\infty(\Sigma_\sigma)$-calculus of A through Theorem 10.4.4. □

Proof of Theorem 16.2.8. By the second part of Theorem 16.2.4, assumption (i) implies that $A + B$ is σ-R-sectorial operator provided the smallness condition on C_0 in (i) holds. Moreover, if we impose $C_0 < 1$, then for all $x \in \mathsf{D}(A + B) = \mathsf{D}(A)$ we have $\|Ax\| \leqslant \|(A + B)x\| + \|Bx\| \leqslant \|(A+B)x\| + C_0\|Ax\|$ and therefore $\|Ax\| \leqslant (1 - C_0)^{-1}\|(A + B)x\|$, while at the same time $\|(A+B)x\| \leqslant \|Ax\| + \|Bx\| \leqslant (1 + C_0)\|Ax$. We conclude that

$$\|Ax\| \eqsim_{C_0} \|(A + B)x\|, \quad x \in \mathsf{D}(A + B) = \mathsf{D}(A).$$

Furthermore, for $\lambda \in \complement\overline{\Sigma_{\sigma\vee\tau}}$ we have $\lambda \in \varrho(A+B)$ and the resolvent operator is represented by the perturbation formula of Proposition 16.2.1,

$$R(\lambda, A + B) = R(\lambda, A)\sum_{n=0}^{\infty}[BR(\lambda, A)]^n, \quad |\arg\lambda| > \sigma \vee \tau, \tag{16.12}$$

again provided C_0 is small enough, for then $\|BR(\lambda, A)\| \leqslant C_0\|AR(\lambda, A)\| \leqslant C_0\|\lambda R(\lambda, A) - I\| \leqslant C_0(1 + M_{\sigma\vee\tau,A}) < 1$ and the series converges absolutely.

First we assume that (i) and (iii) hold. For the time being, we do not assume that $0 \in \varrho(A)$ (in which case A^{-1} is bounded by Corollary 15.2.10), but only assume that A is invertible and B maps $\mathsf{D}(A^{1-\alpha})$ into $\mathsf{D}(A^{-\alpha})$. Then $U := A^{-\alpha}BA^{\alpha-1}$ is bounded on X of norm $\|U\| = C_1$ and we have

$$R(\lambda, A)BR(\lambda, A) = R(\lambda, A)A^\alpha UA^{1-\alpha}R(\lambda, A).$$

If $C_1 < \widetilde{M}^{-1}_{\sigma\vee\tau,A}$, then the sum

$$M(\lambda) := \sum_{k\geqslant 0}[UAR(\lambda, A)]^kU$$

converges in operator norm and defines a holomorphic function for $|\arg\lambda| > \sigma \vee \tau$. We then can rewrite (16.12) in the form

$$R(\lambda, A + B) = R(\lambda, A) + A^\alpha R(\lambda, A)\sum_{k\geqslant 0}[UAR(\lambda, A)]^kUA^{1-\alpha}R(\lambda, A)$$
$$= R(\lambda, A) + A^\alpha R(\lambda, A)M(\lambda)A^{1-\alpha}R(\lambda, A).$$

By the R-sectoriality of A and Proposition 8.1.24, the set $\{M(\lambda) : |\arg(\lambda)| > \sigma\}$ is R-bounded. Thus we derived the representation required in Lemma 16.2.9 and we can conclude that $A+B$ also has a bounded $H^\infty(\Sigma_{\sigma\vee\tau}$-calculus.

It remains to prove that the smallness assumption $C_1 < \widetilde{M}_{\sigma\vee\tau,A}^{-1}$ can be removed if $0 \in \varrho(A)$. Under this assumption, let (i) and (iii) hold, but not necessarily the smallness condition $C_1 < \widetilde{M}_{\sigma\vee\tau,A}^{-1}$.

Using the scale of homogeneous fractional domain spaces $X_\alpha := \mathsf{D}(A^\alpha)$ with norms $\|x\|_{X_\alpha} := \|A^\alpha x\|$ (recall that we are assuming $0 \in \varrho(A)$) we can restate our assumptions as stating that B extends to a bounded operator from X_1 to X and from $X_{1-\alpha}$ to $X_{-\alpha}$, with norm at most C_0 and C_1 respectively. By complex interpolation B acts as a bounded operator $[X_1, X_{1-\alpha}]_\theta$ to $[X, X_{-\alpha}]_\theta$ with norm $\leqslant C_0^{1-\theta} C_1^\theta$, $0 < \theta < 1$.

Since A has a bounded H^∞-calculus and therefore bounded imaginary powers, by Corollary 15.3.10 we have

$$[X_1, X_{1-\alpha}]_\theta = X_{1-\theta\alpha}, \quad [X, X_{-\alpha}]_\theta = X_{-\theta\alpha}$$

with equivalent norms, with equivalence constants which may be chosen independent of $\theta \in (0,1)$. Thus we obtain that B acts as a bounded operator from $X_{1-\theta\alpha}$ to $X_{-\theta\alpha}$ with norm $\lesssim C_0^{1-\theta} C_1^\theta$, $0 < \theta < 1$.

We can choose θ so small that B satisfies (iii) for $\alpha' = \theta\alpha$ with $C_1' < \widetilde{M}_{\sigma\vee\tau,A}^{-1}$ no matter how big C_1 was. This completes the proof of the case (iii).

Finally assume that (ii) holds for some $\alpha \in (0,1)$. By Proposition 15.1.12 we have $A^{\alpha-1} \subseteq A^\alpha A^{-1}$ and $A^{1+\alpha} \subseteq A^\alpha A$ (in fact we have equality in the second case by Theorem 15.2.5), and therefore

$$\|A^{\alpha-1}Bx\| = \|A^\alpha A^{-1}Bx\| \leqslant C_1\|A^{1+\alpha}A^{-1}Bx\| = C_1\|A^\alpha Bx\|$$

implies that (iii) holds for the exponent $1-\alpha \in (0,1)$ and $x \in \mathsf{D}(A^{1+\alpha})$. Since $\mathsf{D}(A^{1+\alpha})$ is dense in $\mathsf{D}(A^\alpha)$ by Proposition 15.1.13, (iii) holds for the exponent $1-\alpha \in (0,1)$ and $x \in \mathsf{D}(A^\alpha)$. □

We conclude this section with an example, due to McIntosh and Yagi, shows that boundedness of the H^∞-calculus is not preserved by small relatively bounded perturbations even when X is a Hilbert space. This shows that the additional assumptions (ii) or (iii) in Theorem 16.2.7 cannot be omitted, no matter how small the constant on (i) is chosen.

Example 16.2.10. We construct a bisectorial operator A on Hilbert space H admitting a bounded bisectorial H^∞-calculus with $\omega_{H^\infty}^{\rm bi}(A) = 0$, such that for any given $\varepsilon > 0$, an operator B_ε on H exists which is relatively bounded with respect to A, with $\|B_\varepsilon x\| \leqslant \varepsilon\|Ax\|$, and such that $A + B_\varepsilon$ fails to have a bounded bisectorial H^∞-calculus. This operator moreover satisfies $(A + B_\varepsilon)^2 = A^2 + C_\varepsilon$, where C is relatively bounded with respect to A^2, with $\|C_\varepsilon x\| \leqslant 2\varepsilon\|A^2 x\|$.

By the first part of Theorem 10.6.7, the operator A^2 has a bounded H^∞-calculus with $\omega_{H^\infty}(A) = 0$. If $A^2 + C_\varepsilon = (A + B_\varepsilon)^2$ had a bounded H^∞-calculus, then by the second part of Theorem 10.6.7 $A + B_\varepsilon$ would have a bounded bisectorial H^∞-calculus, and this is not the case. We conclude that $A^2 + C_\varepsilon$ does not have a bounded H^∞-calculus.

Let us proceed to the construction of the operators A and B_ε. Fix $\varepsilon > 0$. Omitting subscripts ε in what follows, for $n = 1, 2, \dots$ we will construct bounded operators A_n and B_n on a finite-dimensional H_n with the following properties for any $0 < \sigma < \frac{1}{2}\pi$:

- A_n and $A_n + B_n$ are σ-bisectorial with $\|B_n\| \leqslant \varepsilon \|A_n\|$;
- A_n^2 and $(A_n + B_n)^2 = A_n^2 + C_n$ with $\|C_n\| \leqslant 2\varepsilon \|A_n^2\|$;
- the spectra of A_n and $A_n + B_n$ is contained in $(-\infty, 1] \cup [1, \infty)$;
- the resolvents of A_n and $A_n + B_n$ satisfy
$$\|R(\lambda, A_n)\| \leqslant 1/\Im(\lambda), \quad \|R(\lambda, A_n + B_n)\| \leqslant (1+\varepsilon)/\Im(\lambda),$$
for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$;
- A_n and $A_n + B_n$ have contractive, respectively bounded, $H^\infty(\Sigma_\sigma^\pm)$-calculi;
- the spectral projections $\mathbf{1}_{\Sigma_\sigma^\pm}(A_n + B_n)$ have norm $\geqslant n$.

The counterexample with the stated properties is obtained by taking

$$H = \bigoplus_{n\geqslant 1} H_n, \quad A := \bigoplus_{n\geqslant 1} T_n, \quad B := \bigoplus_{n\geqslant 1} B_n, \quad C := \bigoplus_{n\geqslant 1} C_n.$$

The operator A has a contractive $H^\infty(\Sigma_\sigma^\pm)$-calculus. Furthermore, the inequalities $\|B_n\| \leqslant \varepsilon \|A_n\|$ imply that $\mathsf{D}(A) \subseteq \mathsf{D}(B)$ and B is relatively bounded with respect to A, with relative bound $\leqslant \varepsilon$. The operator $A+B$ with domain $\mathsf{D}(A+B) = \mathsf{D}(A)$ doesn't have a bounded $H^\infty(\Sigma_\sigma^{\mathrm{bi}})$-calculus: for if it had, then the associated spectral projections would be bounded; but if they were, then their restrictions to H_n would be uniformly bounded in n; but these restrictions have norm $\geqslant n$.

We now turn to the details of the construction. Choose $N_n \geqslant 1$ so large that $\frac{2\varepsilon}{3\pi}\log(N_n + 1)$. On $\mathbb{C}^{N_n+1}$ consider the matrices $T_n = (t_{jk}^{(n)})_{j,k=0}^{N_n}$ and $S_n = (s_{jk}^{(n)})_{j,k=0}^{N_n}$ given by

$$t_{jk}^{(n)} = 2^j \delta_{jk}, \qquad s_{jk}^{(n)} := \frac{\varepsilon}{\pi(k-j)}\delta_{j\neq k}.$$

Then T_n is self-adjoint and $S_n T_n$ is skew-adjoint. The self-adjoint matrix iS_n is the $N_n \times N_n$ Toeplitz matrix with *generating function* $\varepsilon\theta/\pi$, $\theta \in (-\pi, \pi)$, that is, we have

$$s_{jk} = \widehat{f}_{j-k}, \quad j, k = 0, \dots, N_n.$$

Since the norm of a Toeplitz matrix with bounded real-valued generating function f is bounded by $\|f\|_{L^\infty(\mathbb{T})}$, we see that $\|S_n\| \leqslant \varepsilon$.

The matrix $Z_n = (z_{jk}^{(n)})_{j,k=0}^{N_n}$ given by

$$z_{jk}^{(n)} = \frac{2^k \varepsilon}{\pi(k-j)(2^j + 2^k)}\delta_{j\neq k}$$

has norm

$$
\begin{aligned}
\|Z_n\| \geqslant \|Ze_{N_n}\| &= \frac{2^{N_n}\varepsilon}{\pi} \sum_{j=0}^{N_n-1} \frac{1}{(N_n-j)(2^j+2^{N_n})} \\
&\geqslant \frac{\varepsilon}{\pi} \frac{2^{N_n}}{(2^{N_n-1}+2^{N_n})} \Big(\frac{1}{N_n} + \frac{1}{N_n-1} + \dots + 1\Big) \\
&\geqslant \frac{2\varepsilon}{3\pi} \log(N_n+1) \geqslant n,
\end{aligned} \tag{16.13}
$$

where $(e_n)_{n=0}^{N_n}$ denote the standard unit vectors in $\mathbb{C}^{N_n+1}$, and it satisfies

$$
T_n Z_n + Z_n T_n = S_n T_n. \tag{16.14}
$$

On $H_n := C^{N_n+1} \times C^{N_n+1}$ define the operators

$$
A_n := \begin{bmatrix} T_n & 0 \\ 0 & -T_n \end{bmatrix}, \quad B_n := \begin{bmatrix} 0 & S_n T_n \\ 0 & 0 \end{bmatrix}, \quad P_n^+ := \begin{bmatrix} I & Z_n \\ 0 & 0 \end{bmatrix}, \quad P_n^- := \begin{bmatrix} 0 & -Z_n \\ 0 & I \end{bmatrix}.
$$

One checks that

$$
B_n = \begin{bmatrix} 0 & S_n \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_n & 0 \\ 0 & -T_n \end{bmatrix} = \begin{bmatrix} 0 & S_n \\ 0 & 0 \end{bmatrix} A_n,
$$

so

$$
\|B_n\| \leqslant \|S_n\| \|A_n\| \leqslant \varepsilon \|A_n\|.
$$

Also, using that $S_n T_n = -(T_n S_n)^*$, we have

$$
(A_n+B_n)^2 = \begin{bmatrix} T_n^2 & T_n S_n T_n - S_n T_n^2 \\ 0 & T_n^2 \end{bmatrix} = A_n^2 + \begin{bmatrix} 0 & T_n S_n T_n - S_n T_n^2 \\ 0 & 0 \end{bmatrix} =: A_n^2 + C_n
$$

with

$$
\|C_n\| \leqslant \|T_n S_n T_n - S_n T_n^2\| \leqslant 2\|S_n\| \|T_n\|^2 \leqslant 2\varepsilon \|A_n^2\|,
$$

where we used that $T_n^* = -T_n$, so T_n is normal and therefore $\|T_n\|^2 = \|T^2\|$.

Furthermore, one checks that $\sigma(A_n) = \sigma(A_n + B_n)$ and

$$
R(\lambda, A_n + B_n) = \begin{bmatrix} R(\lambda, T_n) & R(\lambda, T_n) S_n T_n R(\lambda, -T_n) \\ 0 & R(\lambda, -T_n) \end{bmatrix} \tag{16.15}
$$

for all $\lambda \in \varrho(A_n) = \varrho(A_n + B_n)$. In particular,

$$
\begin{aligned}
\sigma(A_n + B_n) &= \sigma(A_n) \\
&= \sigma(T_n) \cup \sigma(-T_n) = \{1, 2, 4, \dots, 2^{N_n}\} \cup \{-1, -2, -4, \dots, -2^{N_n}\}
\end{aligned}
$$

By self-adjointness, for $\lambda \notin \mathbb{R}$ we have $\|A_n\| \leqslant |\Im(\lambda)|^{-1}$, so A_n is σ-bisectorial for all $0 < \sigma < \frac{1}{2}\pi$. By (16.15), for $\lambda \notin \mathbb{R}$ we have $\lambda \in \varrho(A_n + B_n)$, and for $\lambda \notin \overline{\Sigma_\sigma}$

$$\|R(\lambda, A_n + B_n)\| \leqslant |\Im(\lambda)|^{-1} + \|R(\lambda, T_n) S_n T_n R(\lambda, T_n)\| \lesssim_\sigma (1+\varepsilon)|\Im(\lambda)|^{-1}.$$

It follows that $A_n + B_n$ is σ-bisectorial for all $0 < \sigma < \frac{1}{2}\pi$.

The operators P_n^+ and P_n^- are projections,

$$P_n^+ + P_n^- = I, \quad P_n^+ P_n^- = P_n^- P_n^+ = 0,$$

and by (16.13) their norms satisfy

$$\|P_n^+\| \geqslant \|Z_n\| \geqslant n, \quad \|P_n^-\| \geqslant \|Z_n\| \geqslant n.$$

To complete the construction we will show that

$$P_n^\pm = \mathbf{1}_{\Sigma_\sigma^\pm}(A_n + B_n).$$

Indeed, using (16.14) and (16.15), for $0 < \nu < \sigma$ we formally compute

$$\begin{aligned}
&\mathbf{1}_{\Sigma_\sigma^\pm}(A_n + B_n) \\
&\quad = \frac{1}{2\pi i} \int_{\partial \Sigma_\nu^\pm} R(z, A_n + B_n)\,\mathrm{d}z \\
&\quad = \frac{1}{2\pi i} \int_{\partial \Sigma_\nu^\pm} \begin{bmatrix} R(z, T_n) & R(z, T_n) S_n T_n R(z, -T_n) \\ 0 & R(z, -T_n) \end{bmatrix} \mathrm{d}z \\
&\quad = \frac{1}{2\pi i} \int_{\partial \Sigma_\nu^\pm} \begin{bmatrix} R(z, T_n) & R(z, T_n)(T_n Z_n + Z_n T_n) R(z, -T_n) \\ 0 & R(z, -T_n) \end{bmatrix} \mathrm{d}z \\
&\quad = \frac{1}{2\pi i} \int_{\partial \Sigma_\nu^\pm} \begin{bmatrix} R(z, T_n) & R(z, T_n) Z_n + Z_n R(z, -T_n) \\ 0 & R(z, -T_n)x \end{bmatrix} \mathrm{d}z \\
&\quad \overset{(*)}{=} \begin{bmatrix} I & Z_n \\ 0 & 0 \end{bmatrix} = P_n^+,
\end{aligned}$$

where $(*)$ is a consequence of Cauchy's theorem, which gives

$$\frac{1}{2\pi i} \int_{\partial \Sigma_\nu^\pm} R(z, T_n)\,\mathrm{d}z = I, \quad \frac{1}{2\pi i} \int_{\partial \Sigma_\nu^\pm} R(z, -T_n)\,\mathrm{d}z = 0,$$

noting that $\sigma(T_n) = \{1, 2, 4, \dots, 2^{N_n}\}$ is contained in Σ_σ^+. To make the computation rigorous, one brings in additional terms $\zeta_k(T_n)$, where $\zeta_k(z) = \frac{k}{k+z} - \frac{1}{1+kz}$ as in Proposition 10.2.6, to be able to work with the Dunford calculus for functions in $H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ throughout; one passes to the limit $k \to \infty$ at the end. The proof that $\mathbf{1}_{\Sigma_\sigma^\pm}(A_n + B_n) = P_n^-$ is entirely similar.

16.3 Sum-of-operator theorems

The perturbations B studied in Section 16.2 have the property that $\mathsf{D}(B)$ is contained in $\mathsf{D}(A)$, so that the sum the sum $A + B$ may be defined unambiguously by the prescription $(A + B)x := Ax + Bx$. In all these cases, B is "small" in comparison with A. Under a resolvent commutation assumption, in the present section we treat A and B on a more equal footing.

We begin with a general result (Theorem 16.3.2) which says that the sum $A+B$ of two resolvent commuting sectorial operators A and B satisfying $\omega(A)+\omega(B)<\pi$ always has a sectorial extension, and that this extension is the closure of $A+B$ if both A and B are densely defined. In applications to maximal regularity of solution of evolution equations – the topic of the last two chapters of this book – more is needed, namely, that $A+B$ is closed and the following inequality holds:

$$\|Ax\|+\|Bx\|\leqslant C\|(A+B)x\|, \quad x\in \mathsf{D}(A)\cap\mathsf{D}(B) \tag{16.16}$$

with a constant C independent of $x\in\mathsf{D}(A)\cap\mathsf{D}(B)$. For later use we record the simple fact that this inequality in fact implies closedness:

Proposition 16.3.1. *If A and B are closed operators satisfying* (16.16)*, then the operator $A+B$ with its natural domain $\mathsf{D}(A+B)=\mathsf{D}(A)+\mathsf{D}(B)$ is closed.*

Proof. The proof is immediate: if $x_n\to x$ and $(A+B)x_n\to y$, then (16.16) implies that the sequences $(Ax_n)_{n\geqslant 1}$ and $(Bx_n)_{n\geqslant 1}$ are Cauchy. The closedness of A and B implies that $x\in\mathsf{D}(A)\cap\mathsf{D}(B)$ and $y=\lim_{n\to\infty}(A+B)x_n = Ax+Bx=\lim_{n\to\infty}Ax_n+\lim_{n\to\infty}Bx_n = Ax+Bx=(A+B)x$. □

As it turns out, the inequality (16.16) is rather delicate, and it only holds under additional assumptions on A, B, and X. We have already encountered one such situation: the Dore–Venni theorem (Theorem 15.4.11), which assumes that A and B resolvent commute and have bounded imaginary powers, with $\omega_{\rm BIP}(A)+\omega_{\rm BIP}(B)<\pi$, and the underlying Banach space X is a UMD space. In applications, however, one if often confronted with the situation where one of the operator is only (R-)sectorial, whilst the other operator has better properties such as a bounded H^∞-calculus. In the present section, for resolvent commuting sectorial operators A and B acting in a Banach X we will prove the following results:

- If A and B are densely defined, A has a bounded H^∞-calculus and B is R-sectorial, and if $\omega_{H^\infty}(A)+\omega_R(B)<\pi$, then $A+B$ is densely defined and sectorial, with
$$\omega(A+B)\leqslant\max\{\omega_{H^\infty}(A),\omega_R(B)\}$$
and the reverse triangle inequality (16.16) holds. If in addition X has the triangular contraction property, then $A+B$ is R-sectorial and
$$\omega_R(A+B)\leqslant\max\{\omega_{H^\infty}(A),\omega_R(B)\}.$$
(Theorem 16.3.6).

- If A and B are densely defined and have bounded H^∞-calculi with $\omega_{H^\infty}(A) + \omega_{H^\infty}(B) < \pi$ and X has Pisier's contraction property, then $A+B$ has a bounded H^∞-calculus with

$$\omega_{H^\infty}(A+B) \leqslant \max\{\omega_{H^\infty}(A), \omega_{H^\infty}(B)\}$$

and the reverse triangle inequality (16.16) holds (Theorem 16.3.10).
- If A has an absolute calculus with

$$\omega_{\rm abs}(A) + \omega(B) < \pi,$$

then the reverse triangle inequality (16.16) holds (Theorem 16.3.14). The same conclusion holds if X is a Hilbert space, A has bounded imaginary powers and B is densely defined, and $\omega_{\rm BIP}(A) + \omega(B) < \pi$ (Theorem 16.3.15).

To conclude this section we provide an example of the type of applications that will be studied in depth in the next two chapters and which indeed have motivated the development of the abstract approach to sums of operators presented here.

Suppose that $-A$ generates a C_0-semigroup on a Banach space X and consider the inhomogeneous abstract Cauchy problem

$$\begin{cases} u'(t) + Au(t) = f(t), & t \in [0,T], \\ \qquad\qquad u(0) = 0. \end{cases} \tag{ACP}$$

As we will explain in the next chapter, a thorough understanding of this problem is of paramount importance to the study of more general classes of nonlinear, possibly time-dependent, evolution equations. In order to connect (ACP) with operator sums we consider the weak derivative

$$Du := u'$$

viewed as a closed operator on $L^p(0,T;X)$ (with $1 \leqslant p \leqslant \infty$) with domain

$$\mathsf{D}(D) := {}_0W^{1,p}(0,T;X) = \big\{u \in W^{1,p}(0,T;X) : \, u(0) = 0\big\}$$

It will be checked in the next chapter (see Section 17.3.c) that this operator is sectorial of angle $\frac{1}{2}\pi$. Using this operator, we can rewrite (ACP) as the abstract operator equation

$$(D + \widetilde{A})u = f$$

in $L^p(0,T;X)$, where $\widetilde{A}$ is the natural extension of A to a closed operator acting in $\widetilde{X} := L^p(0,T;X)$, defined on $\mathsf{D}(\widetilde{A}) := L^p(0,T;\mathsf{D}(A))$ by

$$(\widetilde{A}f)(t) := A(f(t)), \quad t \in (0,T).$$

In the next chapter (see Propositions 17.3.14 and 17.3.15) we prove that the following assertions are equivalent:

(1) the inverse triangle inequality (16.16) holds, i.e., there is a constant $C \geqslant 0$ such that

$$\|\widetilde{A}u\|_p + \|Du\|_p \leqslant C\|(\widetilde{A}+D)u\|_p, \quad u \in \mathsf{D}(\widetilde{A}) \cap \mathsf{D}(D);$$

(2) $\widetilde{A}+D$ is closed;
(3) $\widetilde{A}+D$ boundedly invertible;
(4) A has maximal L^p-regularity on $(0,T)$.

For the problem (ACP), *maximal L^p-regularity* means that the unique mild solution of the problem, which is given in terms of the semigroup S generated by $-A$ as

$$u(t) = \int_0^t S(t-s)f(s)\,\mathrm{d}s$$

belongs to $L^p(0,T;\mathsf{D}(A)) \cap {}_0W^{1,p}(0,T;X) = \mathsf{D}(\widetilde{A}) \cap \mathsf{D}(D)$. As we will see in the next chapter, the bounded invertibility of $\widetilde{A}+D$ corresponds to the existence and uniqueness of mild solutions for (ACP). Maximal L^p-regularity will be studied in depth in the next chapter, where also a version of the above equivalences with $(0,T)$ replaced by $\mathbb{R}_+$ will be proved.

16.3.a The sum of two sectorial operators

We begin with a general result about sums of resolvent commuting operators. It is not quite as useful as the deeper sums-of-operator theorems proved in the next sections, but its virtue lies in the generality of its assumptions, namely, it is only required that A and B are sectorial with $\omega(A)+\omega(B) < \pi$. The price to be paid is that we do not obtain sectoriality, or even closedness, of $A+B$, but only the weaker result that $A+B$ has a sectorial extension. A second reason to present this result in fair detail is that some techniques that go into the proof will resurface in later proofs.

Theorem 16.3.2 (Sums of sectorial operators). *If A and B are resolvent commuting sectorial operators satisfying*

$$\omega(A)+\omega(B) < \pi$$

then the operator $A+B$ with its natural domain $\mathsf{D}(A+B) = \mathsf{D}(A)+\mathsf{D}(B)$ has a closed extension to a sectorial operator C which satisfies

$$\omega(C) \leqslant \max\{\omega(A), \omega(B)\}.$$

Furthermore,

(1) *If A or B is injective, then C is injective;*
(2) *If A and B are densely defined, then C is densely defined;*
(3) *If A and B are densely defined and A or B is standard sectorial, then C is standard sectorial.*

If (2) *holds (and hence if* (3) *holds), then* C *equals the closure of* $A+B$.

The proof of this theorem will be given shortly. We first pause a brief moment to explain why the condition

$$\omega(A) + \omega(B) < \pi$$

enters naturally in this theorem. Variants of this condition appear in all sum-of-operator theorems we are about to encounter. Arguing naively, one would like to realise the operator sum $A+B$ through a 'bivariate' extended Dunford calculus as $(z+w)(A+B)$, where $z+w$ is short-hand for the function $(z,w) \mapsto z+w$. With this notation, to prove sectoriality of $A+B$ one must estimate the norms of

$$\lambda R(\lambda, A+B) = \frac{\lambda}{\lambda - (z+w)}(A,B)$$

for all $\lambda \in \mathbb{C}$ in the complement of a sector Σ_ω containing all sums $z+w$ with $z \in \Sigma_\sigma$ and $w \in \Sigma_\tau$, where $\omega(A) < \sigma < \pi$ and $\omega(B) < \tau < \pi$ as usual. But the algebraic sum $\Sigma_\sigma + \Sigma_\tau$ is a *sector* only if $\sigma + \tau \leqslant \pi$! Under this condition, $\Sigma_\sigma + \Sigma_\tau = \Sigma_{\max\{\sigma,\tau\}}$. In contrast, when $\sigma+\tau > \pi$ the reader may check that $\Sigma_\sigma + \Sigma_\tau = \mathbb{C}$. Clearly, the condition $\sigma + \tau \leqslant \pi$ forces $\omega(A)+\omega(B) < \pi$, and in that case we may replace σ and τ by slightly smaller values to arrange that $\sigma + \tau < \pi$. Incidentally, this heuristic argument also shows that the inequality $\omega(A+B) \leqslant \max\{\omega(A), \omega(B)\}$ is natural to expect.

Let us now turn to the proof Theorem 16.3.2. Let A and B be resolvent commuting sectorial operators in X satisfying $\omega(A) + \omega(B) < \pi$, and let $\omega(A) < \sigma < \pi$ and $\omega(B) < \tau < \pi$ be such that $\sigma + \tau < \pi$. The construction of the sectorial operator C extending $A+B$ is based on the following observation, which makes use of the primary calculus involving the spaces $E(\Sigma)$ introduced in Section 15.1.a. For a holomorphic function $h \in E(\Sigma_\sigma) \otimes E(\Sigma_\tau)$ of the form

$$h(z,w) = \sum_{n=1}^{N} f_n(z) g_n(w)$$

with all $f_n \in E(\Sigma_\sigma)$ and $g_n \in E(\Sigma_\tau)$, we may define

$$h(A,B) := \sum_{n=1}^{N} f_n(A) g_n(B).$$

It is not difficult that the operator $h(A,B)$ is well defined, in the sense that it does not depend on the particular representation of h. We now observe that

$$h(z,w) := \frac{z+w}{(1+z)(1+w)} = \Big(1 - \frac{1}{1+z}\Big)\frac{1}{1+w} + \frac{1}{1+z}\Big(1 - \frac{1}{1+w}\Big).$$

This identifies the left-hand side as an element of $E(\Sigma_\sigma) \otimes E(\Sigma_\tau)$. Thinking of

$$\rho(z,w) := \frac{1}{(1+z)(1+w)}$$

as a regulariser for the function $(z,w) \mapsto z+w$, we define

$$C := (I+A)(I+B)h(A,B)$$

with domain

$$\mathsf{D}(C) := \big\{x \in X : \, h(A,B)x \in \mathsf{R}((I+B)^{-1}(I+A)^{-1}\big\}.$$

A bit of algebra reveals that

- $x \in \mathsf{D}(C) \iff (A+B)(I+B)^{-1}(I+A)^{-1}x \in \mathsf{R}((I+B)^{-1}(I+A)^{-1})$

  and, for $x \in \mathsf{D}(C)$,

$$Cx = (I+A)(I+B)(A+B)(I+B)^{-1}(I+A)^{-1}x.$$

From this equivalence, by a standard argument one deduces that

- C is closed.

Proof of Theorem 16.3.2. We will prove that C defined by the above procedure has the required properties.

It is immediate from the definition that $\mathsf{D}(A)\cap\mathsf{D}(B)$ is contained in $\mathsf{D}(C)$; this is the same as saying that C is an extension of $A+B$. In fact, a moment's reflection shows that

$$\mathsf{D}(A) \cap \mathsf{D}(C) = \mathsf{D}(A) \cap \mathsf{D}(B) = \mathsf{D}(C) \cap \mathsf{D}(B). \tag{16.17}$$

Choose $\omega(A) < \sigma < \pi$ and $\omega(B) < \tau < \pi$ in such a way that $\sigma + \tau < \pi$. As was already observed above, the condition $\sigma + \tau < \pi$ implies that

$$\Sigma_\sigma + \Sigma_\tau := \{z + w : \, z \in \Sigma_\sigma, \, w \in \Sigma_\tau\} = \Sigma_{\max\{\sigma,\tau\}}.$$

For $z \in \Sigma_\sigma$, $w \in \Sigma_\tau$, and $\lambda \in \mathbb{C}$ with $\max\{\sigma,\tau\} < |\arg(\lambda)| < \pi$, we write

$$\frac{\lambda}{\lambda - (z+w)} = \frac{\lambda^2}{(\lambda - z)(\lambda - w)} + \frac{\lambda z w}{(\lambda - (z+w))(\lambda - z)(\lambda - w)}.$$

These functions are holomorphic on $\Sigma_\sigma \times \Sigma_\tau$ and one may check that

$$\lambda R(\lambda, C) = \lambda^2 R(\lambda, A)R(\lambda, B) + f_\lambda(A,B), \tag{16.18}$$

where $f_\lambda(A,B)$ can be defined in terms of the function

$$f_\lambda(z,w) := \frac{\lambda z w}{(\lambda - (z+w))(\lambda - z)(\lambda - w)}$$

as the absolutely convergent Dunford integral

$$f_\lambda(A,B) := \frac{1}{(2\pi i)^2}\int_{\partial\Sigma_\tau}\int_{\partial\Sigma_\sigma} f_\lambda(z,w)R(z,A)R(w,B)\,\mathrm{d}z\,\mathrm{d}w.$$

With $\mu := \lambda/|\lambda|$ we then have

$$\|f_\lambda(A,B)\| \lesssim_{\sigma,\tau,A,B} \int_{\partial\Sigma_\tau}\int_{\partial\Sigma_\sigma} \frac{1}{|\mu-(z+w)||\mu-z||\mu-w|}\,|\mathrm{d}z|\,|\mathrm{d}w| \quad (16.19)$$

and the sectoriality of C with angle $\omega(C) \leqslant \omega$ now easily follows from the fact that value of the integral on the right-hand side of (16.19) is uniformly bounded with respect to μ on the arc $\{|\mu| = 1, |\arg(\mu)| \geqslant \max\{\sigma,\tau\}$.

Since the choices $\omega(A) < \sigma < \pi$ and $\omega(B) < \tau < \pi$ and $\max\{\sigma,\tau\} < \omega < \pi$ were arbitrary, it follows that $\omega(C) \leqslant \max\{\omega(A),\omega(B)\}$.

It remains to prove the assertions (1)–(3).

(1): Suppose that is injective and let $x \in \mathsf{D}(C)$ be such that $Cx = 0$. By the definition of C, this means that $h(A,B)x = (I+B)^{-1}(I+A)^{-1}y$ for some $y \in X$ and $Cx = (I+A)(I+B)h(A,B)x = y = 0$. Consider the function

$$g(z,w) = \frac{z}{(1+z)^2(z+w)}, \quad z,w \in \Sigma_\sigma.$$

By the primary calculus, for fixed $z \in \Sigma_\sigma$ we have

$$g(z,B) = z(1+z)^{-2}(z+B)^{-1}.$$

Borrowing some terminology from the next subsection, this function belongs to $H^1(\Sigma_\sigma;\mathscr{A})$, where $\mathscr{A}$ is the set of operators in $\mathscr{L}(X)$ commuting with the resolvent of A, and we may define a bounded operator $g(A,B)$ through the Dunford integral

$$g(A,B) := \frac{1}{2\pi i}\int_{\partial\Sigma_\nu} z(1+z)^{-2}(z+B)^{-1}(z-A)^{-1}\,\mathrm{d}z.$$

In view of

$$\begin{aligned}
&(z+B)^{-1}(z-A)^{-1}C \\
&\quad = (z+B)^{-1}(z-A)^{-1}[(I+A)(I+B)h(A,B)] \\
&\quad = [(1+z)(z-A)^{-1} - I][(1-z)(z+B)^{-1} + I]h(A,B) \\
&\quad = [(1+z)(z-A)^{-1} - I][(1-z)(z+B)^{-1} + I](A+B)(I+A)^{-1}(I+B)^{-1} \\
&\quad = [(1+z)(z-A)^{-1} - I][I-(I+A)^{-1}][(1-z)(z+B)^{-1} + I](I+B)^{-1} \\
&\qquad + [(1+z)(z-A)^{-1} - I][(1-z)(z+B)^{-1} + I][I-(I+B)^{-1}](I+A)^{-1} \\
&\quad = (z-A)^{-1} - (z+B)^{-1},
\end{aligned}$$

where the last line follows by the resolvent identity. By Cauchy's theorem,

$$\int_{\partial\Sigma_\nu} \frac{z}{(1+z)^2}\Big((z-A)^{-1} - (z+B)^{-1}\Big)\,\mathrm{d}z = \int_{\partial\Sigma_\nu} \frac{z}{(1+z)^2}(z-A)^{-1}\,\mathrm{d}z$$

$$= A(I+A)^{-2}x$$

It follows that

$$0 = g(A,B)Cx = A(I+A)^{-2}x.$$

Since A and $(I-A)^{-1}$ are injective, this forces $x = 0$.

If B is injective and $Cx = 0$, the same argument (with the roles of A and B reversed) again shows that $x = 0$.

(2): If A and B are densely defined, then so is C by (16.17). If $x \in \mathsf{D}(A)$, then the vectors $x_n := n^2(n+A)^{-1}(n+B)^{-1}x$ belong to $\mathsf{D}(A+B)$ and converge to x in X as $n \to \infty$. Similarly, the vectors $Cx_n = n^2(n+A)^{-1}(n+B)^{-1}Cx$ converge to Cx in X as $n \to \infty$. This shows that $\mathsf{D}(A+B)$ is dense in $\mathsf{D}(C)$ with respect to the graph norm.

(3): Suppose now that A and B are standard sectorial. Then $\mathsf{D}(A)$ and $\mathsf{D}(B)$ are dense, and therefore $\mathsf{D}(C)$ is dense by (2). Furthermore, arguing as in part (1) we see that for all $x \in X$ we have $g(A,B)x \in \mathsf{D}(C)$ and $Cg(A,B)x = A(I+A)^{-2}x$. Since $\mathsf{R}(A(I+A)^{-2}) = \mathsf{D}(A) \cap \mathsf{R}(A)$, it follows that $\mathsf{D}(A) \cap \mathsf{R}(A) \subseteq \mathsf{R}(C)$ and therefore $\mathsf{R}(C)$ is dense. By Proposition 10.1.8, this implies that $\mathsf{D}(C) \cap \mathsf{R}(C)$ is dense, i.e., C is standard sectorial. □

In the next proposition we assume that A and B are sectorial operators in X satisfying $\omega(A) + \omega(B) < \pi$, and choose $\omega(A) < \nu_A < \sigma_A < \pi$, $\omega(B) < \nu_B < \sigma_A < \pi$, and $\max\{\nu_A, \nu_B\} < \nu < \sigma < \pi$. The operator C is as in Theorem 16.3.2.

Proposition 16.3.3. *Every* $\lambda \notin \overline{\Sigma_{\max\{\sigma,\tau\}}}$ *belongs to* $\varrho(C)$ *and*

$$\varrho(A)R(\lambda,C)\varrho(B) = \frac{1}{(2\pi i)^2}\int_{\partial\Sigma_{\nu_A}}\int_{\partial\Sigma_{\nu_B}} \frac{\varrho(z)\varrho(w)}{\lambda-(z+w)} R(z,A)R(w,B)\,\mathrm{d}w\,\mathrm{d}z.$$

In its stated form, the proposition will be useful in the proof of Theorem 16.3.10. It is clear from the proof that the proposition could be stated with $\varrho(A)$, $\varrho(B)$, and $R(\lambda,C)$ replaced by more general operators $\phi(A)$, $\psi(B)$, and $f(A)$ under suitable conditions on the functions ϕ, ψ, and f. We leave the details to the interested reader.

Proof. It has already been observed that every $\lambda \notin \overline{\Sigma_{\max\{\nu_A,\nu_B\}}}$ belongs to the resolvent set of C, and by (16.18) (using the notation introduced there) we have $R(\lambda,C) = \lambda R(\lambda,A)R(\lambda,B) + g_\lambda(A,B)$, where

$$g_\lambda(z,w) := \frac{zw}{(\lambda-(z+w))(\lambda-z)(\lambda-w)} = \frac{1}{(\lambda-(z+w))} - \frac{\lambda}{(\lambda-z)(\lambda-w)}. \tag{16.20}$$

Inserting this into the Dunford integral

$$\begin{aligned}&\varrho(A)R(\lambda, C)\varrho(B)\\&\quad= \frac{1}{(2\pi i)^2}\int_{\partial\Sigma_{\nu_A}}\int_{\partial\Sigma_{\nu_B}} \varrho(z)\varrho(w)R(z,A)R(\lambda,C)R(w,B)\,\mathrm{d}w\,\mathrm{d}z,\end{aligned}$$

we see that this results in the sum of three integrals, where (I) corresponds to the contribution $\lambda R(\lambda, A)R(\lambda, B)$, and (II) and (III) correspond to the splitting of g_λ by (16.20). By a simple computation involving Fubini's theorem and Cauchy's theorem, the integrals (I) and (III) cancel, and the integral (II) equals the one in the statement of the lemma. □

16.3.b Operator-valued H^∞-calculus and closed sums

In this section we extend the Dunford calculus of a sectorial operator A to an operator-valued Dunford calculus and study the question when this calculus is bounded with respect to the H^∞-norm. The idea is to obtain (16.16) from the boundedness of the operator $f(A, B)$ in terms of the function $f(\lambda, B) = B(\lambda + B)^{-1}$ in the operator-valued calculus. Loosely speaking, this gives a way to define an operator "$A(A+B)^{-1}$" even when $A+B$ fails to be bounded invertible. With the operator at hand, it is possible to run a rigorous version of the estimate

$$\|Ax\| = \|A(A+B)^{-1}(A+B)x\| \leqslant C\|(A+B)x\|$$

with $C = \|A(A+B)^{-1}\|$. From this one also obtains the estimate

$$\|Bx\| \leqslant \|(A+B)x\| + \|Ax\| \leqslant (1+C)\|(A+B)x\|,$$

and together these estimates give (16.16), with implied constant $1 + 2C$.

In what follows, A always denotes a sectorial operator on a Banach space X, and we fix $\omega(A) < \sigma < \pi$. Let $\mathscr{A}$ be a closed sub-algebra of $\mathscr{L}(X)$ resolvent commuting with A, i.e.,

$$TR(z, A) = R(z, A)T \quad \text{for all} \quad T \in \mathscr{A} \quad \text{and} \quad z \in \varrho(A).$$

We then denote by $H^1(\Sigma_\sigma; \mathscr{A})$ the space of all holomorphic functions $F : \Sigma_\sigma \to \mathscr{A}$ for which

$$\|F\|_{H^1(\Sigma_\sigma;\mathscr{A})} := \sup_{|\nu|<\sigma}\int_{\mathbb{R}_+} \|F(e^{i\nu}t)\|\,\frac{\mathrm{d}t}{t}$$

is finite. It is easily checked that, which respect to this norm, $H^1(\Sigma_\sigma; \mathscr{A})$ is a Banach space. For functions $F \in H^1(\Sigma_\sigma; \mathscr{A})$ we can define a bounded operator $F(A) \in \mathscr{A}$ by means of the operator-valued Dunford integral

$$F(A) = \frac{1}{2\pi i}\int_{\partial\Sigma_\nu} F(z)R(z, A)\,\mathrm{d}z,$$

where $\omega(A) < \nu < \sigma$. The resulting operator is independent of the particular choice of ν, and it satisfies

$$\|f(A)\| \leqslant \frac{M_{\sigma,A}}{\pi} \|F\|_{H^1(\Sigma_\sigma;\mathscr{A})},$$

where $M_{\sigma,A} = \sup_{\lambda \in \complement \overline{\Sigma_\sigma}} \|\lambda R(\lambda, A)\|$.

As in Proposition 10.2.2, this calculus is multiplicative and satisfies the following convergence property: if $F_n, F \in H^1(\Sigma_\sigma;\mathscr{A})$ are uniformly bounded and satisfy $F_n(z)x \to F(z)x$ for all $z \in \Sigma_\sigma$ and $x \in X$, then for all $g \in H^1(\Sigma_\sigma)$ we have

$$\lim_{n\to\infty} (f_n g)(A)x = (fg)(A)x, \quad x \in X.$$

Denote by $H^\infty(\Sigma_\sigma;\mathscr{A})$ the space of all holomorphic functions $F : \Sigma_\sigma \to \mathscr{A}$ for which the set $\{F(z) : z \in \Sigma_\sigma\}$ is uniformly bounded. Endowed with the norm

$$\|F\|_{H^\infty(\Sigma_\sigma;\mathscr{A})} := \sup\{\|F(z)\| : z \in \Sigma_\sigma\},$$

this space is easily seen to be Banach space. In the same way one defines $RH^\infty(\Sigma_\nu;\mathscr{A})$ as the space of all holomorphic functions $F : \Sigma_\nu \to \mathscr{A}$ for which the set $\{F(z) : z \in \Sigma_\nu\}$ is R-bounded. Endowed with the norm

$$\|F\|_{RH^\infty(\Sigma_\sigma;\mathscr{A})} := \mathscr{R}(\{F(z) : z \in \Sigma_\sigma\})$$

(the R-bound of $\{F(z) : z \in \Sigma_\sigma\}$), this space is a Banach space.

The main result of this section is the following theorem.

Theorem 16.3.4. *Let A be a sectorial operator on a Banach space X, let $\omega(A) < \sigma < \pi$, and suppose that A has a bounded $H^\infty(\Sigma_\sigma)$-calculus. Then there exists a unique bounded linear mapping $F \mapsto F(A)$ from $RH^\infty(\Sigma_\sigma;\mathscr{A})$ into $\mathscr{L}(\overline{\mathsf{D}(A) \cap \mathsf{R}(A)})$ with the following properties:*

(1) *For every function $F \in RH^\infty(\Sigma_\sigma;\mathscr{A}) \cap H^1(\Sigma_\sigma;\mathscr{A})$ the operator $F(A)$ coincides with the one defined by the Dunford integral;*
(2) *For all $F, G \in RH^\infty(\Sigma_\sigma;\mathscr{A})$ we have $FG \in RH^\infty(\Sigma_\sigma;\mathscr{A})$ and*

$$(FG)(A) = F(A)G(A) = G(A)F(A);$$

(3) *Whenever the functions $F_n, F \in RH^\infty(\Sigma_\sigma;\mathscr{A})$ are uniformly bounded and satisfy $F_n \to F$ pointwise on Σ_σ, then $\lim_{n\to\infty} F_n(A)x = F(A)x$ for all $x \in \overline{\mathsf{D}(A) \cap \mathsf{R}(A)}$.*

Furthermore, if X has Pisier's contraction property and $\mathscr{T}$ is an R-bounded subset of $\mathscr{A}$, then for all $0 < \sigma < \nu < \pi$ the family

$$\big\{F(A) : F \in RH^\infty(\Sigma_\nu;\mathscr{A}),\ F(z) \in \mathscr{T} \text{ for all } z \in \Sigma_\nu\big\}$$

is R-bounded.

Parts (2) and (3) are analogues of the corresponding results in Theorem 10.2.13 and the proofs are similar. The proof of (1), which is the non-trivial part of the theorem, is based on an extension of Lemma 10.3.13, which states that if A is a sectorial operator on a Banach space X and $F \in H^1(\Sigma_\sigma; \mathscr{A})$ is given, with $\omega(A) < \sigma < \pi$, then for all $\omega(A) < \nu < \sigma$ we have

$$F(A) = \frac{1}{2\pi i} \int_{\partial \Sigma_\nu} z^{1/2} F(z) \phi_z(A) \frac{\mathrm{d}z}{z}, \tag{16.21}$$

where $\phi_z(\lambda) := \lambda^{1/2}/(z - \lambda)$. The proof is identical to that of Lemma 10.3.13; all one needs to do is to replace $H^1(\Sigma_\sigma)$ by $H^1(\Sigma_\sigma; \mathscr{A})$ throughout, and so is the justification of the well-definedness of the operators $\phi_z(A)$ and the convergence of integral on the right-hand side of (16.21).

We also need the following strengthening of Lemma 10.3.8:

Lemma 16.3.5. *Let A be a sectorial operator on a Banach space X with a bounded H^∞-calculus, and let $\omega_{H^\infty}(A) < \nu < \sigma < \pi$. Suppose $\phi, \psi \in H^1(\Sigma_\sigma)$, and let $\mathscr{T} \subseteq \mathscr{A}$ be R-bounded. Then for all finite subsets $F \subseteq \mathbb{Z}$, all scalars $|a_j| \leqslant 1$ and operators $T_j \in \mathscr{T}$ $(j \in F)$, and all $x \in \overline{\mathsf{D}(A) \cap \mathsf{R}(A)}$,*

$$\sup_{t>0} \Big\| \sum_{j\in F} a_j T_j \phi(2^j tA) \psi(2^j tA) x \Big\| \leqslant C \|\phi\|_{H^1(\Sigma_\sigma)} \|\psi\|_{H^1(\Sigma_\sigma)} \|x\|,$$

where C is a constant depending only on ν, σ, and A.

Proof. Let A_0 denote the part of A in $X_0 := \overline{\mathsf{D}(A) \cap \mathsf{R}(A)}$. This operator is standard sectorial and has a bounded H^∞-calculus, with the same bounds, and the same holds for its adjoint A_0^*. Let $(\varepsilon_j)_{j\in\mathbb{Z}}$ be a Rademacher sequence. For norm one vectors $x \in X_0$ and $x^* \in X_0^*$, and for any fixed $t > 0$ and finite subset $F \subseteq \mathbb{Z}$ we may estimate

$$\begin{aligned}
&\Big| \Big\langle \sum_{j\in F} a_j T_j \phi(2^j tA) \psi(2^j tA) x, x^* \Big\rangle \Big| \\
&\quad = \Big| \sum_{j\in F} a_j \langle T_j \psi(2^j tA_0) x, \phi(2^j tA_0^*) x^* \rangle \Big| \\
&\quad = \Big| \mathbb{E} \Big\langle \sum_{j\in F} \varepsilon_j a_j T_j \psi(2^j tA_0) x, \sum_{k\in F} \varepsilon_k \phi(2^k tA_0^*) x^* \Big\rangle \Big| \\
&\quad \leqslant \Big(\mathbb{E} \Big\| \sum_{j\in F} \varepsilon_j a_j T_j \psi(2^j tA_0) x \Big\|^2 \Big)^{1/2} \Big(\mathbb{E} \Big\| \sum_{k\in F} \varepsilon_k \phi(2^k tA_0^*) x^* \Big\|^2 \Big)^{1/2} \\
&\quad \leqslant K_{\sigma-\nu} (M^\infty_{\nu,A})^2 \mathscr{R}(\mathscr{T}) \|\phi\|_{H^1(\Sigma_\sigma)} \|\psi\|_{H^1(\Sigma_\sigma)} \|x\| \|x^*\|
\end{aligned}$$

using R-boundedness, the Kahane contraction principle, and Theorem 10.4.4 (and its notation) in the last step. The result now follows by taking the supremum over all $x^* \in X_0^*$ of norm at most 1. ∎

Proof of Theorem 16.3.4. Let $\omega(A) < \nu < \sigma < \pi$ and let F be as in (1). As in the proof of Theorem 10.3.4(3), for $x \in \mathsf{D}(A) \cap \mathsf{R}(A)$ and $F \in H^1(\Sigma_\sigma; \mathscr{A}) \cap RH^\infty(\Sigma_\sigma; \mathscr{A})$ we find

$$F(A)x = \sum_{j \in \mathbb{Z}} \sum_{\epsilon = \pm 1} \frac{1}{2\pi i} \epsilon e^{-\epsilon i \nu/2} \int_1^2 F(e^{-\epsilon i\nu} 2^j t) \phi_{e^{-\epsilon i\nu}}(t^{-1} 2^{-j} A) x \, \frac{\mathrm{d}t}{t},$$

where $\phi_z(\lambda) = \lambda^{1/2}/(z - \lambda)$. Then, with $a_j(\epsilon) = \epsilon e^{-\epsilon i\nu/2}$,

$$\|F(A)x\| \leqslant \frac{1}{\pi} \sup_{\epsilon = \pm 1} \sup_{k \geqslant 1} \sup_{t > 0} \Big\| \sum_{|j| \leqslant k} a_j(\epsilon) F(e^{-\epsilon i\nu} 2^j t) \phi_{e^{-\epsilon i\nu}}(t^{-1} 2^{-j} A) x \Big\|.$$

Now we choose $\mathscr{T}$ to be the R-bounded range of F, and we let $\phi = \psi = (\phi_{e^{-i\nu}})^{1/2}$ if $\epsilon = 1$ and $\phi = \psi = (\phi_{e^{i\nu}})^{1/2}$ if $\epsilon = -1$. Applying the lemma twice, we obtain

$$\|F(A)x\| \leqslant \frac{2}{\pi} C \max_{\epsilon = \pm 1} \|\phi_{e^{-\epsilon i\nu}}\|_{H^1(\Sigma_\sigma)} \|x\|,$$

where C is the constant of the lemma.

The proofs of multiplicativity and the convergence property proceed as in Theorem 10.2.13.

Regarding the final assertion, we may adapt the proof of Theorem 10.3.4(3), replacing the scalar functions f_n and f by $\mathscr{A}$-valued functions F_n and F. □

As an application of the operator-valued calculus we prove a useful variant of the Dore–Venni theorem (Theorem 15.4.11). In that theorem, both A and B were assumed to have bounded imaginary powers and act in a UMD Banach space X. In the present theorem, we weaken the assumption on A and strengthen the assumption on B.

Theorem 16.3.6 (The sum of an R-sectorial operator and an operator with bounded H^∞-calculus). *Let A and B be resolvent commuting densely defined (respectively, standard) sectorial operators on a Banach space X. Assume that A has a bounded H^∞-calculus, B is R-sectorial, and*

$$\omega_{H^\infty}(A) + \omega_R(B) < \pi.$$

Then $A+B$ is a densely defined (respectively, standard) sectorial operator and

$$\omega(A+B) \leqslant \max\{\omega_{H^\infty}(A), \omega_R(B)\}.$$

Moreover, there exists a constant $C \geqslant 0$ such that

$$\|Ax\| + \|Bx\| \leqslant C\|(A+B)x\|, \quad x \in \mathsf{D}(A) \cap \mathsf{D}(B). \tag{16.22}$$

If X has the triangular contraction property, then $A+B$ is R-sectorial with

$$\omega_R(A+B) \leqslant \max\{\omega_R(A), \omega_{H^\infty}(B)\}.$$

Proof. The idea is to define $A(A+B)^{-1}$ and $B(A+B)^{-1}$ as bounded operators on $\overline{\mathsf{D}(A)\cap\mathsf{R}(A)}$ using the operator-valued functional calculus for A.

Step 1 – We first assume that A and B are standard sectorial. Let $\mathscr{A}$ denote the closed sub-algebra of $\mathscr{L}(X)$ comprised of all operators resolvent commuting with A. Choose $\omega_{H^\infty}(A) < \sigma < \pi$ and $\omega_R(B) < \tau < \pi$ and such that $\sigma + \tau < \pi$. We wish to apply the operator-valued calculus of A to the function

$$F(z) := B(z+B)^{-1} = I - zR(z,-B).$$

This function belongs to $RH^\infty(\Sigma_\sigma;\mathscr{A})$ since the spectrum of $-B$ is contained in the closure of $-\Sigma_\tau = \{z\in\mathbb{C} : |\arg(z)| > \pi-\tau\}$ and $\sigma < \pi - \tau$. Furthermore, the function

$$G(z) := \zeta_n(z)^2(z+B)\zeta_n(B)^2,$$

with $\zeta_n(z) = \frac{n}{n+z} - \frac{1}{1+nz}$ as in Proposition 10.2.6, is easily seen to belong to $H^1(\Sigma_\sigma;\mathscr{A})\cap RH^\infty(\Sigma_\sigma;\mathscr{A})$ by R-sectoriality. We have

$$(FG)(z) = F(z)G(z) = \zeta_n(z)^2 B\zeta_n(B)^2,$$

and in the operator-valued Dunford calculus the operators $G(A)$ and $(FG)(A)$ are given by

$$\begin{aligned} G(A) &= B\zeta_n(B)^2\zeta_n(A)^2 + A\zeta_n(A)^2\zeta_n(B)^2, \\ (FG)(A) &= B\zeta_n(B)^2\zeta_n(A)^2, \end{aligned}$$

using resolvent commutation to do some rewriting. By the multiplicativity of the operator-valued H^∞-calculus of A we have

$$\begin{aligned} B\zeta_n(B)^2\zeta_n(A)^2 = (FG)(A) &= F(A)G(A) \\ &= F(A)\big(B\zeta_n(B)^2\zeta_n(A)^2 + A\zeta_n(A)^2\zeta_n(B)^2\big). \end{aligned}$$

The boundedness of the operator-valued H^∞-calculus of A then gives, for $x\in\mathsf{D}(A)\cap\mathsf{D}(B)$,

$$\|\zeta_n(A)^2\zeta_n(B)^2Bx\| \lesssim_{\sigma,A} \big\|\big(\zeta_n(A)^2\zeta_n(B)^2Bx + \zeta_n(B)^2\zeta_n(A)^2Ax\big)\big\|$$

Letting $n\to\infty$ and using A and B are standard sectorial, we obtain the inequality

$$\|Bx\| \lesssim_{\sigma,A} \|(A+B)x\|.$$

From this we also obtain

$$\|Ax\| \leqslant \|Bx\| + \|Ax+Bx\| \lesssim_{\sigma,A} \|(A+B)x\|.$$

We have already observed in Proposition 16.3.1 that (16.22) implies the closedness of $A+B$. The standard sectoriality of $A+B$ now follows from Theorem 16.3.2.

Step 2 – We now assume that A and B are densely defined, but not necessarily standard sectorial. Then the operators $A_\varepsilon := A+\varepsilon$ and $B_\varepsilon :+B+\varepsilon$ are standard sectorial, and we may apply the above reasoning with $F_\varepsilon(z) := B_\varepsilon(z+B_\varepsilon)^{-1}$ and $G_\varepsilon(z) := \zeta_n(z)^2(z+B_\varepsilon)\zeta_n(B_\varepsilon)^2$. This results in the estimate

$$\|A_\varepsilon x\| + \|B_\varepsilon x\| \lesssim_{\sigma,A} \|(A_\varepsilon + B_\varepsilon)x\|$$

with an implied constant that is uniform in $\varepsilon > 0$ and independent of x. The estimate

$$\|Ax\| + \|Bx\| \lesssim_{\sigma,A} \|(A+B)x\|$$

follows from this by letting $\varepsilon \downarrow 0$.

Step 3 – Suppose finally that X has the triangular contraction property. From the proof of Theorem 16.3.2 (and keeping in mind that $A+B$ equals the operator C of that theorem by what we have already proved) we recall the identity

$$\begin{aligned}&\lambda R(\lambda, A+B)\\ &\quad = \lambda^2 R(\lambda,A)R(\lambda,B) + \frac{1}{(2\pi i)^2}\int_{\partial\Sigma_\tau}\int_{\partial\Sigma_\sigma} f_\lambda(z,w)R(z,A)R(w,B)\,\mathrm{d}z\,\mathrm{d}w.\end{aligned}$$

Outside the closure of $\Sigma_{\sigma+\tau}$ the operators $\lambda^2 R(\lambda,A)R(\lambda,B)$ are R-bounded, by the R-sectoriality of A (which follows from the second part of Theorem 10.3.4) and B (by assumption). The operators corresponding to the Dunford integral with f_λ are R-bounded by Theorem 8.5.2; the integrability properties required to apply theorem have already been observed in the proof of Theorem 16.3.2 (see (16.19)). □

Since standard sectorial operators with a bounded H^∞-calculus on a Banach space with the triangular contraction property are R-sectorial with $\omega_R(A) = \omega_{H^\infty}(A)$ (see Corollary 10.4.10), we have the following corollary.

Corollary 16.3.7. *Let A and B be resolvent commuting densely defined (respectively, standard) sectorial operators with bounded H^∞-calculi satisfying $\omega_{H^\infty}(A) + \omega_{H^\infty}(B) < \pi$ on a Banach space with the triangular contraction property. Then $A+B$ is a densely defined (respectively, standard) sectorial operator and* (16.22) *holds.*

16.3.c The joint H^∞-calculus

As a first application of the operator-valued functional calculus we construct the *joint functional calculus* of resolvent commuting standard sectorial operators.

Denote by $H^1(\Sigma_{\sigma_1}\times\cdots\times\Sigma_{\sigma_n})$ the space of holomorphic functions on $\Sigma_{\sigma_1}\times\cdots\times\Sigma_{\sigma_n}$ which obey the obvious integrability estimate extending the case $n=1$. For functions $f \in H^1(\Sigma_{\sigma_1}\times\cdots\times\Sigma_{\sigma_n})$ we define the *joint Dunford calculus* by

$$f(A_1,\dots,A_n) = \Big(\frac{1}{2\pi i}\Big)^n \int_{\partial\Sigma_{\nu_n}} \dots \int_{\partial\Sigma_{\nu_1}} f(\lambda_1,\dots,\lambda_n) \prod_{j=1}^n R(\lambda_j, A_j)\, \mathrm{d}\lambda_1 \dots \mathrm{d}\lambda_n \tag{16.23}$$

where $\omega(A_i) < \nu_i < \sigma_i$ for $i = 1,\dots,n$.

If $n = 2$, by Fubini's theorem we can formally rewrite (16.23) as

$$\begin{aligned} f(A_1, A_2) &= \frac{1}{2\pi i}\int_{\partial\Sigma_{\nu_1}} \Big(\frac{1}{2\pi i}\int_{\partial\Sigma_{\nu_2}} f(\lambda_1,\lambda_2) R(\lambda_2, A_2)\,\mathrm{d}\lambda_2\Big) R(\lambda_1, A_1)\,\mathrm{d}\lambda_1 \\ &= \frac{1}{2\pi i}\int_{\partial\Sigma_{\nu_1}} f(\lambda_1, A_2) R(\lambda_1, A_1)\,\mathrm{d}\lambda_1 \\ &= \Phi_1(f(\cdot, A_2))(A_1), \end{aligned}$$

where $\Phi_1 : g \mapsto g(A_1)$ denotes the operator-valued calculus of A_1, provided of course that all terms are well defined. This indicates the way how to extend (16.23) to $H^\infty(\Sigma_{\sigma_1} \times \dots \times \Sigma_{\sigma_n})$ using induction where each of the operator A_j has a bounded H^∞-calculus. Here, $H^\infty(\Sigma_{\sigma_1} \times \dots \times \Sigma_{\sigma_n})$ denotes the space of bounded holomorphic functions on $\Sigma_{\sigma_1} \times \dots \times \Sigma_{\sigma_n}$.

The following straightforward extension of Lemma 10.2.17 will be useful. As before, by $\mathscr{A}$ we denote the set of bounded operators commuting with the resolvent of A.

Lemma 16.3.8. *Let A have a bounded $H^\infty(\Sigma_\sigma)$-calculus on X. Suppose that $f : [a,b] \times \Sigma_\sigma \to \mathscr{A}$ is a measurable function with the following properties:*

(i) *$z \mapsto f(s,z)$ belongs to $RH^\infty(\Sigma_\sigma; \mathscr{A})$ for all $s \in [a,b]$;*

(ii) $\displaystyle\sup_{|\nu|<\sigma} \int_a^b \int_0^\infty \|f(s, e^{i\nu}t)\| \,\frac{\mathrm{d}t}{t}\,\mathrm{d}s < \infty.$

Then the function $g(z) = \int_a^b f(s,z)\,\mathrm{d}s$ belongs to $H^\infty(\Sigma_\sigma; \mathscr{A})$ and

$$g(A)x = \int_a^b f(s,A)x\,\mathrm{d}s, \qquad x \in X.$$

The straightforward proof is left to the reader.

We will now apply the operator-valued calculus to the sum-of-operators problem next.

Theorem 16.3.9. *Let $A_1,\dots,A_n$ be densely defined resolvent commuting sectorial operators on a Banach space X with the Pisier contraction principle, and assume that A_j has a bounded $H^\infty(\Sigma_{\sigma_j})$-calculus, $j = 1,\dots,n$. Then for $\sigma_j < \nu_j < \pi$, (16.23) extends to an algebra homomorphism $\Phi : H^\infty(\Sigma_{\nu_1} \times \dots \times \Sigma_{\nu_n}) \to \mathscr{L}(\overline{\mathsf{D}(A) \cap \mathsf{R}(A)})$ with the following convergence property:*

If the functions f_m, f are uniformly bounded in $H^\infty(\Sigma_{\nu_1} \times \cdots \times \Sigma_{\nu_n})$ and $\lim_{m\to\infty} f_m = f$ pointwise on $\Sigma_{\nu_1} \times \cdots \times \Sigma_{\nu_n}$, then $\lim_{m\to\infty} \Phi(f_m)x = \Phi(f)x$ for all $x \in \overline{\mathsf{D}(A) \cap \mathsf{R}(A)}$.

Moreover, the set of operators

$$\big\{\Phi(f) : f \in H^\infty(\Sigma_{\sigma_1} \times \cdots \times \Sigma_{\sigma_n}), \|f\|_\infty \leqslant 1\big\}$$

is R-bounded.

Notation. In place of $\Phi(f)$ we shall write $f(A_1, \ldots, A_n)$.

Proof. By $\mathscr{A}_j$ we denote the sub-algebra of all operators in $\mathscr{L}(X)$ that commute with A_j and put $\mathscr{A} := \mathscr{A}_1 \cap \cdots \cap \mathscr{A}_n$. Note that $R(\lambda, A_j) \in \mathscr{A}$ for all $j = 1, \ldots, n$. The case $n = 1$ follows from the general properties of the H^∞-calculus. Assume now that $A_2, \ldots, A_n$ have a joint functional calculus $\Psi : H^\infty(\Sigma_{\sigma_2} \times \cdots \times \Sigma_{\sigma_n}) \to \mathscr{L}(X)$ with the required properties. Since X has Pisier's contraction property, the set

$$\mathscr{T} = \big\{g(A_2, \ldots, A_n) : g \in H^\infty(\Sigma_{\sigma_2} \times \cdots \times \Sigma_{\sigma_n}),\ \|g\|_{H^\infty} \leqslant 1\big\}$$

is an R-bounded subset of $\mathscr{A} \subseteq \mathscr{A}_1$ by Theorem 10.3.4(3). By Φ_1 we denote the operator-valued functional calculus of A_1 defined on $RH^\infty(\Sigma_{\sigma_1}; \mathscr{A}_1)$ as constructed in Theorem 16.3.4.

Given a function $f \in H^1(\Sigma_{\sigma_1} \times \cdots \times \Sigma_{\sigma_n}) \cap H^\infty(\Sigma_{\sigma_1} \times \cdots \times \Sigma_{\sigma_n})$ with $\|f\|_{H^\infty} \leqslant 1$, the set

$$\{f(\lambda_1, \cdot, \ldots, \cdot) : \lambda_1 \in \Sigma_{\sigma_1}\}$$

is uniformly bounded in $H^\infty(\Sigma_{\sigma_2} \times \cdots \times \Sigma_{\sigma_n})$. Hence

$$\Psi\big[f(\lambda_1, \cdot, \ldots, \cdot)\big] = f(\lambda_1, A_2, \ldots, A_n) \in \mathscr{T} \qquad \text{for all } \lambda_1 \in \Sigma_{\sigma_1}.$$

Furthermore, the function

$$\begin{aligned}\lambda_1 &\mapsto f(\lambda_1, A_2, \ldots, A_n) \\ &= \Big(\frac{1}{2\pi i}\Big)^{n-1} \int_{\partial \Sigma_{\nu_2}} \cdots \int_{\partial \Sigma_{\nu_n}} \prod_{j=2}^{n} f(\lambda_1, \ldots, \lambda_n) R(\lambda_j, A_j)\, \mathrm{d}\lambda_2 \ldots \mathrm{d}\lambda_n\end{aligned}$$

is holomorphic on Σ_{σ_1}. Again by Theorem 10.3.4(3), $f(\cdot, A_2, \ldots, A_n) \in RH^\infty(\Sigma_{\sigma_1}; \mathscr{A}_1)$. Consequently we can define

$$\Phi(f) = \Phi_1\big(f(\cdot, A_2, \ldots, A_n)\big)$$

using Theorem 16.3.13. We can extend this definition to arbitrary $f \in H^\infty(\Sigma_{\sigma_1} \times \cdots \times \Sigma_{\sigma_n})$. The required properties of Φ now follow from the corresponding properties of Ψ and Φ_1. For instance, Φ extends (16.23) by Fubini's theorem. To check the multiplicativity, choose $f, g \in H^\infty(\Sigma_{\sigma_1} \times \cdots \times \Sigma_{\sigma_n})$. Then

$$\begin{aligned}\Phi(f\cdot g) &= \Phi_1\big((f\cdot g)(\cdot,A_2,\dots,A_n)\big)\\ &= \Phi_1\big(f(\cdot,A_2,\dots,A_n)\cdot g(\cdot,A_2,\dots,A_n)\big)\\ &= \Phi_1\big(f(\cdot,A_2,\dots,A_n)\big)\Phi_1\big(g(\cdot,A_2,\dots,A_n)\big) = \Phi(f)\Phi(g).\end{aligned}$$

Also, if f_m, f are bounded in $H^\infty(\Sigma_{\sigma_1}\times\dots\times\Sigma_{\sigma_n})$ and $f_m\to f$ pointwise as $m\to\infty$, then by the convergence property of the operator-valued calculus of A_1 we have

$$\lim_{m\to\infty} f_m(\lambda_1,A_2,\dots,A_n)x = f(\lambda_1,A_2,\dots,A_n)x$$

for every fixed $\lambda_1\in\Sigma_{\sigma_1}$ and all $x\in X$. Now apply the convergence property of Φ_1 to $F_m(\lambda) = f_m(\lambda,A_2,\dots,A_n)$ and $F(\lambda) = f(\lambda,A_2,\dots,A_n)$.

The final R-boundedness assertion follows directly from the final assertion of Theorem 16.3.4. □

As an application we have the following variant of Corollary 16.3.7. This is result is actually true for Banach space X with the triangular contraction property; we refer to the Notes for a discussion of this fact.

Theorem 16.3.10. *Let A and B be resolvent commuting standard sectorial operators with bounded H^∞-calculi satisfying $\omega_{H^\infty}(A)+\omega_{H^\infty}(B)<\pi$ on a Banach space X with Pisier's contraction property. Then $A+B$ admits a bounded H^∞-calculus with*

$$\omega_{H^\infty}(A,B)\leqslant \max\{\omega_{H^\infty}(A),\omega_{H^\infty}(B)\}.$$

For the proof we need a technical proposition. For the sake of its formulation, the joint Dunford calculus of two resolvent commuting sectorial operators A and B will be denoted by $\Phi_{A,B}: f\mapsto f(A,B)$, for functions $f\in H^1(\Sigma_\sigma)\times H^1(\Sigma_\tau)$. Likewise, the operator-valued Dunford calculus of A will be denoted by $\Phi_A: F\mapsto F(A)$, for operator-valued functions $F\in H^1(\Sigma_\sigma;\mathscr{A})$ where $\mathscr{A}$ is set of operators resolvent commuting with A.

Proposition 16.3.11. *Let A and B be resolvent commuting sectorial operators acting in a Banach space X satisfying $\omega(A)+\omega(B)<\pi$, and let $\max\{\omega(A),\omega(B)\}<\sigma<\pi$. Let C denote the operator sum of A and B as constructed above. Then for all $f\in H^1(\Sigma_\sigma)$ we have*

$$\begin{aligned}\varrho(A)f(C)\varrho(B) &= \Phi_{A,B}((z,w)\mapsto \varrho(z)f(z+w)\varrho(w))\\ &= \Phi_A(z\mapsto \varrho(z)f(z+B)\varrho(B)),\end{aligned}$$

where $\varrho(z) = z/(1+z)^2$.

Proof. By the definition of the joint Dunford calculus,

$$\Phi_{A,B}((z,w)\mapsto \varrho(z)(\lambda-(z+w))\varrho(w))$$

$$= \frac{1}{(2\pi i)^2} \int_{\partial\Sigma_{\nu_A}} \int_{\partial\Sigma_{\nu_B}} \frac{\varrho(z)\varrho(w)}{\lambda - (z+w)} R(z,A)R(w,B)\,\mathrm{d}w\,\mathrm{d}z.$$

On the other hand, by Proposition 16.3.3 and Cauchy's theorem,

$$\begin{aligned}
&\varrho(A)f(C)\varrho(B)\\
&\quad= \frac{1}{2\pi i}\int_{\partial\Sigma_\nu} f(\lambda)\varrho(A)R(\lambda,C)\varrho(B)\,\mathrm{d}\lambda\\
&\quad= \frac{1}{(2\pi i)^3}\int_{\partial\Sigma_\nu} f(\lambda)\int_{\partial\Sigma_{\nu_A}}\int_{\partial\Sigma_{\nu_B}} \frac{\varrho(z)\varrho(w)}{\lambda-(z+w)} R(z,A)R(w,B)\,\mathrm{d}w\,\mathrm{d}z\,\mathrm{d}\lambda\\
&\quad= \frac{1}{(2\pi i)^2}\int_{\partial\Sigma_{\nu_A}}\int_{\partial\Sigma_{\nu_B}} \varrho(z)\varrho(w)\\
&\qquad\qquad\times\Big(\frac{1}{2\pi i}\int_{\partial\Sigma_\nu}\frac{f(\lambda)}{\lambda-(z+w)}\,\mathrm{d}\lambda\Big)R(z,A)R(w,B)\,\mathrm{d}w\,\mathrm{d}z\\
&\quad= \frac{1}{(2\pi i)^2}\int_{\partial\Sigma_{\nu_A}}\int_{\partial\Sigma_{\nu_B}} \varrho(z)\varrho(w)f(z+w)R(z,A)R(w,B)\,\mathrm{d}w\,\mathrm{d}z
\end{aligned}$$

which equals

$$= \Phi_{A,B}((z,w)\mapsto \varrho(z)f(z+w)\varrho(w))$$

but also

$$\begin{aligned}
&= \frac{1}{2\pi i}\int_{\partial\Sigma_{\nu_A}} \varrho(z)\Big(\frac{1}{2\pi i}\int_{\partial\Sigma_{\nu_B}} \varrho(w)f(z+w)R(w,B)\,\mathrm{d}w\Big)R(z,A)\,\mathrm{d}z\\
&= \Phi_A(z\mapsto \varrho(z)f(z+B)\varrho(B)).
\end{aligned}$$

□

Proof of Theorem 16.3.10. Since X has Pisier's contraction property, A and B admit R-bounded operator-valued H^∞-calculi by Theorem 10.3.4. Choose $\omega(A) < \sigma_A < \pi$ and $\omega(B) < \sigma_B < \pi$ such that $\sigma_A + \sigma_B < \pi$, and let $\max\{\sigma_A, \sigma_B\} < \sigma < \pi$. For $f \in H^\infty(\Sigma_\sigma)$, the function

$$F(z,w) := f(z+w)$$

belongs to $H^\infty(\Sigma_{\sigma_A}) \times H^\infty(\Sigma_{\sigma_B})$. Since B has an R-bounded H^∞-calculus, the set $\{F(z,B) : z \in \Sigma_{\sigma_A}\}$ is an R-bounded subset of the set $\mathscr{A}$ of bounded operators resolvent commuting with A. Applying the operator-valued calculus of A we obtain a bounded operator $F(A,B)$ on $\overline{\mathsf{D}(A)\cap\mathsf{R}(A)}$. By Proposition 16.3.11,

$$\varrho(A)f(A+B)\varrho(B) = \varrho(A)F(A,B)\varrho(B),$$

where $\varrho(z) = z/(1+z)^2$. Since $\varrho(A)$ is injective and $\varrho(B)$ has dense range (we assumed that A and B are standard sectorial), we conclude that $f(A+B) = F(A,B)$ is a bounded operator on $\overline{\mathsf{D}(A)\cap\mathsf{R}(A)}$. The bound $\|f(A,B)\| \lesssim \|f\|_{H^\infty(\Sigma_\sigma)}$ follows by tracing the steps of the proof. □

Inspection of the proof shows that the 'standard' assumption on A may be weakened to 'densely defined and injective'. In reflexive spaces, however, these conditions imply standardness (see Proposition 10.1.9).

16.3.d The absolute calculus and closed sums

The main result of this section (Theorem 16.3.13) provides a version of Theorem 16.3.6 in which no assumption on the Banach space is needed, the assumption on a A is weakened to sectoriality, and the assumption on B is strengthened to having an absolute calculus.

Definition 16.3.12 (Absolute functional calculus). *Let A be a sectorial operator acting in a Banach space X, and let $\omega(A) < \sigma < \pi$. We say that A* admits an absolute calculus on Σ_σ *if there exist a constant $M \geqslant 0$ and $g, h \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$, with $\|h\|_{H^1(\Sigma_\sigma)} = 1$, such that for all $x, y \in \overline{\mathsf{D}(A) \cap \mathsf{R}(A)}$ the validity of the estimate*

$$\|h(tA)g(tA)x\| \leqslant \|g(tA)y\| \quad \textit{for all} \qquad t > 0$$

implies $\|x\| \leqslant M\|y\|$.

We denote

$$\omega_{\text{abs}}(A) := \inf\bigl\{\sigma \in (0, \pi) : A \text{ admits an absolute calculus on } \Sigma_\sigma\bigr\}.$$

Examples of classes of operators with an absolute calculus will be given in the next subsection.

Theorem 16.3.13 (Absolute calculus implies operator-valued H^∞-calculus). *Let A be a sectorial operator in a Banach space X, let $\omega(A) < \sigma < \pi$, and suppose that A admits an absolute calculus on Σ_σ. Then A admits a bounded operator-valued $H^\infty(\Sigma_\sigma)$-calculus. In particular, A admits a bounded $H^\infty(\Sigma_\sigma)$-calculus.*

Proof. Let g, h be as in Definition 16.3.12, and let $F \in H^1(\Sigma_\sigma; \mathscr{A}) \cap H^\infty(\Sigma_\sigma; \mathscr{A})$. Choose $\omega(A) < \nu < \sigma$. For $z \in \mathsf{D}(A) \cap \mathsf{R}(A)$ we estimate

$$\begin{aligned}
\|h(tA)F(A)z\| &\leqslant \frac{1}{2\pi} \int_{\partial\Sigma_\nu} |h(t\lambda)F(\lambda)| \|R(\lambda, A)z\| \, |\mathrm{d}\lambda| \\
&\leqslant \frac{M_{\nu,A}}{2\pi} \|F\|_\infty \int_{\partial\Sigma_\nu} |h(t\lambda)| \frac{|\mathrm{d}\lambda|}{|\lambda|} \|z\| \\
&\leqslant \frac{M_{\nu,A}}{\pi} \|F\|_\infty \|z\|,
\end{aligned}$$

where $M_{\nu,A} = \sup_{\lambda \in \complement\overline{\Sigma_\nu}} \|\lambda R(\lambda, A)\|$ is finite by sectoriality. Now, given $y \in \mathsf{D}(A) \cap \mathsf{R}(A)$, we let $x := F(A)y$, note that $x \in \mathsf{D}(A) \cap \mathsf{R}(A)$, and apply the estimate with $z := g(tA)y$ to obtain

$$\|h(tA)g(tA)x\| \leqslant \frac{M_{\nu,A}}{\pi}\|F\|_\infty\|\|g(tA)y\|.$$

By the absolute calculus, this implies

$$\|F(A)y\| = \frac{M_{\nu,A}}{\pi}\|x\| \leqslant M\frac{M_{\nu,A}}{\pi}\|F\|_\infty\|y\|.$$

By the same argument as in the proof of Theorem 10.2.13, this proves the first assertion.

The second assertion follows by taking $F(\lambda) = f(\lambda)I_X$. □

Theorem 16.3.14 (Closedness from the absolute calculus). *Let A and B be resolvent commuting densely defined sectorial operators. If A has an absolute calculus on Σ_σ and $\sigma + \omega(B) < \pi$, then the operator $A + B$ with domain $\mathsf{D}(A + B) = \mathsf{D}(A) \cap \mathsf{D}(B)$ is closed and*

$$\|Ax\| + \|Bx\| \leqslant C(A + B)x\|, \quad x \in \mathsf{D}(A) \cap \mathsf{R}(A).$$

Proof. By Theorem 16.3.13, A admits a bounded operator-valued $H^\infty(\Sigma_\sigma)$-calculus. Since B is τ-sectorial, the family $f(z, B) = -zR(-z, B)$, $z \in \Sigma_{\pi-\tau}$, is uniformly bounded and commutes with the resolvent of A. Now Theorem 16.3.13 implies that $f(A, B)$ is well defined in the operator-valued calculus as a bounded operator on X. The reverse Hölder inequality

$$\|Ax\| + \|Bx\| \leqslant C\|(A + B)x\|, x \in \mathsf{D}(A) \cap \mathsf{D}(B),$$

is obtained by same argument as in Theorem 16.3.6. It was already observed that the closedness of $A + B$ follows from it. □

We will prove in the next section (see Theorem 16.3.18) that a standard sectorial operator on a Hilbert space has an absolute calculus if and only if it has a bounded imaginary powers. Taking this for granted for now, as a special case of Theorem 16.3.14 we recover the following classical result.

Theorem 16.3.15 (Da Prato–Grisvard). *Let A and B be resolvent commuting sectorial operators in a Hilbert space H. If A has bounded imaginary powers, B is densely defined, and $\omega_{\rm BIP}(A) + \omega(B) < \pi$, then the operator $A + B$ with domain $\mathsf{D}(A + B) = \mathsf{D}(A) \cap \mathsf{D}(B)$ is closed and we have*

$$\|Ax\| + \|Bx\| \leqslant C\|(A + B)x\|, \quad x \in \mathsf{D}(A) \cap \mathsf{D}(B),$$

with C a constant independent of x.

Comparing this result with the Dore–Venni theorem, where both A and B are assumed to have bounded imaginary powers, we observe that here, boundedness of the imaginary powers is imposed only on A.

16.3.e The absolute calculus and real interpolation

In this subsection and the next, we show connect the absolute calculus with the theory of real interpolation. The crucial observation is contained in the following theorem.

Theorem 16.3.16 (L^p-bounds imply absolute calculus). *Let A be a sectorial operator in a Banach space X and let $\omega(A) < \sigma < \pi$. Let $1 \leqslant p \leqslant \infty$, and suppose that there exist $\phi \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ such that*

$$\|x\|_{\phi,p} := \big\|t \mapsto \phi(tA)x\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)}, \quad x \in \mathsf{D}(A) \cap \mathsf{R}(A),$$

induces an equivalent norm on $\overline{\mathsf{D}(A) \cap \mathsf{R}(A)}$, the finiteness of the norms on the right-hand side being part of the assumptions. Then A has an absolute calculus on Σ_σ.

The proof depends on the following lemma.

Lemma 16.3.17. *Let A be a sectorial operator acting in X and let $\omega(A) < \sigma < \pi$. If for some $p \in [1,\infty]$ and some $\psi \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ one has $t \mapsto \psi(tA)x \in L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X)$ for all $x \in \overline{\mathsf{D}(A) \cap \mathsf{R}(A)}$, then for all $\phi \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ one has $t \mapsto \phi(tA)x \in L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X)$ for all $x \in \overline{\mathsf{D}(A) \cap \mathsf{R}(A)}$ and we have the equivalence of norms*

$$\big\|\phi(\cdot A)x\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)} \eqsim_{\phi,\psi,\sigma,A} \big\|\psi(\cdot A)x\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)}$$

with implied constants independent of x.

Proof. Let $\psi \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ have the properties as stated, and let $\phi \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ be arbitrary and fixed. Choose an auxiliary function $g \in H^1(\Sigma_\sigma)$ such that

$$\int_0^\infty g(t)\psi(t)\,\frac{\mathrm{d}t}{t} = 1.$$

First we let $x \in \mathsf{D}(A) \cap \mathsf{R}(A)$. By the Calderón reproducing formula (Proposition 10.2.5) and the multiplicativity of the Dunford calculus,

$$\int_0^\infty g(tA)\psi(tA)x\,\frac{\mathrm{d}t}{t} = x \tag{16.24}$$

with improper convergence of the left-hand side integral. Fix $\omega(A) < \nu < \sigma$.

For all $s > 0$ and $0 < r < R < \infty$, by Fubini's theorem and multiplicativity we have

$$\begin{aligned}\int_r^R \phi(sA)g(tA)\psi(tA)x\,\frac{\mathrm{d}t}{t} &= \int_r^R \Big(\frac{1}{2\pi i}\int_{\partial\Sigma_\nu} \phi(s\lambda)g(t\lambda)R(\lambda,A)\,\mathrm{d}\lambda\Big)\psi(tA)x\,\frac{\mathrm{d}t}{t} \\ &= \frac{1}{2\pi i}\int_{\partial\Sigma_\nu} \phi(s\lambda)\Big(\int_r^R g(t\lambda)R(\lambda,A)\psi(tA)x\,\frac{\mathrm{d}t}{t}\Big)\,\mathrm{d}\lambda.\end{aligned} \tag{16.25}$$

By (16.24) (with x replaced by $\phi(sA)x$), upon passing to the limits $r \downarrow 0$ and $R \to \infty$ in (16.25) (using dominated convergence to deal with the right-hand side) we obtain

$$\begin{aligned}\phi(sA)x &= \int_0^\infty \phi(sA)g(tA)\psi(tA)x \,\frac{\mathrm{d}t}{t} \\ &= \int_0^\infty \Big(\frac{1}{2\pi i}\int_{\partial\Sigma_\nu} \phi(s\lambda)g(t\lambda)R(\lambda,A)\,\mathrm{d}\lambda\Big)\psi(tA)x\,\frac{\mathrm{d}t}{t} \\ &= \frac{1}{2\pi i}\int_{\partial\Sigma_\nu} \phi(s\lambda)G(\lambda)x\,\mathrm{d}\lambda\end{aligned}$$

with

$$G(\lambda) := \int_0^\infty g(t\lambda)R(\lambda,A)\psi(tA)\,\frac{\mathrm{d}t}{t}.$$

Applying Young's inequality for $L^1(\mathbb{R}_+, \frac{\mathrm{d}t}{t})$ twice (after parametrising $\partial\Sigma_\nu$ and substituting $s \mapsto s^{-1}$), we obtain that $\phi(\cdot A)x \in L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X)$ and

$$\big\|\phi(\cdot A)x\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)} \leqslant \frac{M_{\nu,A}}{\pi}\|\phi\|_{H^1(\Sigma_\sigma)}\|g\|_{H^1(\Sigma_\sigma)}\big\|\psi(\cdot A)x\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)}.$$

Now that we know that $t \mapsto \phi(tA)x$ belongs to $L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X)$, the opposite norm estimate is obtained by reversing the roles of ϕ and ψ. This proves the theorem for $x \in \mathsf{D}(A) \cap \mathsf{R}(A)$.

For $x \in \overline{\mathsf{D}(A) \cap \mathsf{R}(A)}$ the result follows by approximation, noting that $\psi(tA)x\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)} \leqslant C\|x$ by a closed graph argument. □

Proof of Theorem 16.3.16. Fix functions $g, h \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$, with $\|h\|_{H^1(\Sigma_\sigma)} = 1$ for some $\omega(A) < \nu < \sigma$, and assume that $x, y \in \mathsf{D}(A) \cap \mathsf{R}(A)$ satisfy

$$\|h(tA)g(tA)x\| \leqslant \|g(tA)y\|, \quad t > 0.$$

Let $1 \leqslant p \leqslant \infty$ and $f \in L^p(\mathbb{R}_+; \frac{\mathrm{d}t}{t}; X)$ be as in the assumptions in the theorem. Then, by Lemma 16.3.17, applied to $\phi = g$ and $\psi = g \cdot h$, the functions $t \mapsto g(tA)x$, $t \mapsto h(tA)g(tA)x = (hg)(tA)x$, and $t \mapsto g(tA)y$ belong to $L^p(\mathbb{R}, \frac{\mathrm{d}t}{t}; X)$ and

$$\begin{aligned}\|x\| &\eqsim \big\|f(\cdot A)x\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)} \eqsim \big\|h(\cdot A)g(\cdot A)x\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)} \\ &\leqslant \big\|g(\cdot A)y\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)} \eqsim \big\|f(\cdot A)y\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)} \eqsim \|y\|\end{aligned}$$

with implied constants independent of x and y. Hence, g and h satisfy the condition in the definition of the absolute calculus. □

An immediate application if the following characterisation of the absolute calculus in Hilbert spaces.

Theorem 16.3.18 (Hilbert space case). *For a standard sectorial operator A acting in a Hilbert space H, the following assertions are equivalent:*

(1) *A has a bounded H^∞-calculus;*
(2) *A has an absolute calculus;*
(3) *A has bounded imaginary powers.*

In this situation we have

$$\omega_{H^\infty}(A) = \omega_{\mathrm{abs}}(A) = \omega_{\mathrm{BIP}}(A).$$

Further equivalences are obtained in Theorems 10.4.21 (square function estimates) and 10.4.22 (generation of a contraction semigroup with respect to some equivalent Hilbertian norm).

Proof. The implication (2)⇒(1) has already been proved in Theorem 16.3.13. The implication (1)⇒(2) follows from the same theorem, because the boundedness of the H^∞-calculus of a sectorial operator on H implies the square function bounds

$$\|x\|_H \eqsim \|t \mapsto g(tA)x\|_{\gamma(L^2(\mathbb{R}_+,\frac{\mathrm{d}t}{t}),H)} = \Big(\int_0^\infty \|g(tA)x\|^2 \frac{\mathrm{d}t}{t}\Big)^{1/2}$$

by Theorem 10.4.16 and Proposition 9.2.9.

For standard sectorial operators A, the equivalence (1)⇔(3) is contained in Theorem 15.3.23. □

The main result of this section is Theorem 16.3.20 which asserts that invertible sectorial operators have a bounded H^∞-calculus on the real interpolation spaces $(X, \mathsf{D}(A))_{\theta,p}$. We begin with a general result which describes these interpolation spaces in terms of the Dunford calculus of A.

Theorem 16.3.19 (Real interpolation spaces between X and $\mathsf{D}(A)$). *Let $0 < \theta < 1$ and $p \in [1,\infty]$, and let A be a sectorial operator on X. Let $\omega(A) < \sigma < \pi$ and suppose that $0 \neq \phi \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ is such that the function $z \mapsto z^{-1}\phi(z)$ belongs to $H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ as well. Then*

$$(X, \mathsf{D}(A))_{\theta,p} = \Big\{x \in X : t \mapsto t^{-\theta}\phi(tA)x \in L^p(\mathbb{R}_+, \tfrac{\mathrm{d}t}{t}; X)\Big\}$$

with equivalence of norms

$$\|x\|_{(X,\mathsf{D}(A))_{\theta,p}} \eqsim \|x\| + \big\|t \mapsto t^{-\theta}\phi(tA)x\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)},$$

where the implied constants only depend on σ, A, and ϕ. If $0 \in \varrho(A)$, we also have equivalence of homogeneous norms

$$\|x\|_{(X,\mathsf{D}(A))_{\theta,p}} \eqsim \big\|t \mapsto t^{-\theta}\phi(tA)x\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)}.$$

The theorem should be compared with the first part of Proposition K.4.1, which asserts that If A is a sectorial operator in X, then

$$(X, \mathsf{D}(A))_{\theta,p} = \Big\{x \in X : \ \lambda \mapsto \lambda^\theta \|GR(\lambda, G)x\| \in L^p(\mathbb{R}_+, \tfrac{\mathrm{d}\lambda}{\lambda})\Big\}$$

with equivalence of norms

$$\|x\|_{(X,\mathsf{D}(G))_{\theta,p}} \eqsim \|x\| + \Big\|\lambda \mapsto \lambda^\theta \|G(\lambda+G)^{-1}x\|\Big\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}\lambda}{\lambda})}.$$

In the $E(\Sigma_\sigma)$-calculus of A we have $G(\lambda+G)^{-1} = \phi(\lambda^{-1}G)$ with $\phi(z) = z/(z+1)$. The case treated in Theorem L.2.4 corresponds to the choice $\phi(z) = ze^{-z}$.

Proof. '$\subseteq$': For $t > 0$ and $x = x_0 + x_1$ with $x_0 \in X$ and $x_1 \in \mathsf{D}(A)$, write

$$\phi(tA)x = \phi(tA)x_0 + \phi(tA)x_1$$

and note that $\phi(tA)x_1 \in \mathsf{D}(A)$ with $A\phi(tA)x_1 = \phi(tA)Ax_1$. Furthermore write $\phi_0(z) := \phi(z)$ and $\phi_1(z) := z^{-1}\phi(z)$. Then

$$\begin{aligned}\|\phi(tA)x\| &\leqslant \|\phi(tA)x\| + t\|(tA)^{-1}\phi(tA)Ax\| \\ &= \|\phi_0(tA)x\| + t\|\phi_1(tA)Ax\| \\ &\leqslant C_{\sigma,A}\big(\|\phi_0\|_{H^1(\Sigma_\sigma)}\|x\| + t\|\phi_1\|_{H^1(\Sigma_\sigma)}\|x\|_{\mathsf{D}(A)}\big),\end{aligned}$$

where $C_{\sigma,A}$ is a constant only depending on σ and A. Taking the infimum over all such decompositions, we obtain

$$\|\phi(tA)x\| \leqslant C_{\sigma,A} \max\{\|\phi_0\|_{H^1(\Sigma_\sigma)}, \|\phi_1\|_{H^1(\Sigma_\sigma)}\} K(t, x; X, \mathsf{D}(A)).$$

It follows that if $x \in (X, \mathsf{D}(A))_{\theta,p}$, then $t \mapsto t^{-\theta}\phi(t, A)x \in L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X)$ and

$$\begin{aligned}&\|t \mapsto t^{-\theta}\phi(t, A)x\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X)} \\ &\qquad \leqslant C_{\sigma,A} \max\{\|\phi_0\|_{H^1(\Sigma_\sigma)}, \|\phi_1\|_{H^1(\Sigma_\sigma)}\}\|x\|_{(X,\mathsf{D}(A))_{\theta,p}}.\end{aligned}$$

'$\supseteq$': Let $x \in X$ be such that $t \mapsto t^{-\theta}\phi(tA)x$ belongs to $L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X)$. Choose $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ in such a way that $f_1(z) := zf(z)$ belongs to $H^1(\Sigma_\sigma)$ and the normalisation condition $\int_0^\infty f(s)\phi(s)\,\frac{\mathrm{d}s}{s} = 1$ is satisfied. Noting that $f\phi \in H^1(\Sigma_\sigma)$, for $z \in \Sigma_\sigma$ we define

$$h(z) := \int_0^1 f(sz)\phi(sz)\,\frac{\mathrm{d}s}{s}, \quad g(z) := \int_1^\infty f(sz)\phi(sz)\,\frac{\mathrm{d}s}{s}.$$

By substitution, $g(z) + h(z) = 1$ for all $z \in \Sigma_\sigma$.

The assumption $t^{-\theta}\phi(tA)x \in L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X)$ implies that $t^{-\theta}(f\phi)(tA)x \in L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X)$ since $\|(f\phi)(tA)x\| = \|f(tA)\phi(tA)x\| \lesssim_{\sigma,A} \|f\|_{H^1(\Sigma_\sigma)}\|\phi(tA)x\|$ by multiplicativity. Hölder's inequality therefore implies that $(f\phi)(tA)x \in$

$L^p((0,1),\frac{dt}{t};X)$ for every $t>0$. Hence, as in the proof of Proposition 10.2.5 we have

$$h(tA)x = \int_0^1 (f\phi)(stA)\,\frac{ds}{s}. \tag{16.26}$$

Next, noting the identities

$$zg(z) = \int_1^\infty zf(sz)\phi(sz)\,\frac{ds}{s} = \int_1^\infty s^{-1}f_1(sz)\phi(sz)\,\frac{ds}{s},$$

reasoning similarly as for h we have $g(A)x \in \mathsf{D}(A)$ and, for $t>0$,

$$Ag(tA)x = \int_1^\infty (st)^{-1}(f_1\phi)(stA)x\,\frac{ds}{s} = \int_t^\infty s^{-1}(f_1\phi)(sA)x\,\frac{ds}{s}.$$

Accordingly, for the decomposition $x = h(tA)x + g(tA)x \in X + \mathsf{D}(A)$ we obtain

$$\begin{aligned} K(t,x;X,\mathsf{D}(A)) &\leqslant \|h(tA)x\| + t\|g(tA)\|_{\mathsf{D}(A)} \\ &= \|h(tA)x\| + t\|g(tA)x\| + t\|Ag(tA)x\| \\ &\leqslant \|h(tA)x\| + t\|g(tA)x\| + t\Big\|\int_t^\infty s^{-1}(f_1\phi)(sA)x\,\frac{ds}{s}\Big\|. \end{aligned} \tag{16.27}$$

We have $t\|g(tA)x\| \lesssim_{\sigma,A,\phi} t\|g\|_{H^1(\sigma_\sigma)}\|x\|$. Trivially, we also have

$$K(t,x;X,\mathsf{D}(A)) \leqslant \|x\|,$$

By taking the minimum of these estimates, it follows that

$$\begin{aligned} K(t,x;X,\mathsf{D}(A)) &\lesssim_{\sigma,A,\phi} \|h(tA)x\| + \min\{1,t\}\|x\| + t\Big\|\int_t^\infty s^{-1}(f_1\phi)(sA)x\,\frac{ds}{s}\Big\| \\ &=: (I) + (II) + (III). \end{aligned}$$

We will estimate $(t^{-\theta}\times)$ (I), (II), (III) separately.

For term (II) it is immediately clear that $t^{-\theta}\min\{1,t\}\|x\|$ belongs to $L^p(\mathbb{R}_+,\frac{dt}{t})$.

By (16.26) we may estimate term (I) by

$$\|h(tA)x\| = \Big\|\int_0^1 (f\phi)(stA)\,\frac{ds}{s}\Big\| \leqslant \int_0^t \|(f\phi)(sA)x\|\,\frac{ds}{s} =: \sigma_x(t).$$

We can now apply the first part of Hardy's inequality (Lemma L.3.2) to obtain that $t\mapsto t^{-\theta}\sigma_x(t) \in L^p(\mathbb{R}_+,\frac{dt}{t};X)$ and

$$\|t\mapsto t^{-\theta}\sigma_x(t)\|_{L^p(\mathbb{R}_+,\frac{dt}{t};X)} \lesssim_{\sigma,A} \frac{1}{\theta}\|t\mapsto t^{-\theta}(f\phi)(tA)x\|_{L^p(\mathbb{R}_+,\frac{dt}{t};X)}$$

$$\lesssim_{\sigma,A} \frac{1}{\theta}\|f\|_{H^1(\Sigma_\sigma)}\|t \mapsto t^{-\theta}\phi(tA)x\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)}.$$

To estimate term (III), we note that

$$t\Big\|\int_t^\infty s^{-1}(f_1\phi)(sA)x\,\frac{\mathrm{d}s}{s}\Big\| \leqslant \|f_1\|_{H^1(\Sigma_\sigma)}\int_t^\infty \|\phi(sA)x\|\,\frac{\mathrm{d}s}{s}$$

by multiplicativity and since $s \geqslant t$ on the domain of integration, Therefore, by the second part of Lemma L.3.2,

$$\Big\|t \mapsto t^{-\theta}\int_t^\infty ts^{-1}(f_1\phi)(sA)x\,\frac{\mathrm{d}s}{s}\Big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)} \\ \lesssim_{\sigma,A} \frac{1}{\theta}\|f_1\|_{H^1(\Sigma_\sigma)}\|t \mapsto t^{-\theta}\phi(tA)x\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)}.$$

Combining things, we have shown that $t^{-\theta}K(t,x;X,\mathsf{D}(A))$ belongs to $L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t})$ and

$$\|t^{-\theta}K(t,x;X,\mathsf{D}(A))\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t})} \lesssim_{\sigma,A} \|x\| + \|t^{-\theta}\phi(tA)x\|_{L^1(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)}.$$

This is the same as saying that $x \in (X,\mathsf{D}(A))_{\theta,p}$ and

$$\|x\|_{(X,\mathsf{D}(A))_{\theta,p}} \lesssim_{\sigma,A} \|cx\| + \|t^{-\theta}\phi(tA)x\|_{L^1(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)}.$$

Finally, if $0 \in \varrho(A)$ we may endow $\mathsf{D}(A)$ with the equivalent norm $x \mapsto \|Ax\|$. In doing so, the term (II) disappears and the first equivalence of homogeneous norms is obtained.

Suppose next that A is invertible, in (16.27) we can estimate

$$\|g(tA)x\| \leqslant \|A^{-1}\|\|g(tA)Ax\|$$

and therefore the second term can be estimated in the same way at the third term appearing in the second line of (16.27). □

Theorem 16.3.20 (Absolute calculus on real interpolation spaces). *Let A be a densely defined sectorial operator with $0 \in \varrho(A)$. Then for all $1 \leqslant p \leqslant \infty$ and $0 < \theta < 1$, the part $A_{\theta,p}$ of A in the real interpolation space $(X,\mathsf{D}(A))_{\theta,p}$ has an absolute calculus.*

The proof of this theorem depends on the following lemma.

Lemma 16.3.21. *Under the assumptions of the theorem, for every $\alpha < 1-\theta$ the norm*

$$\|x\|_\alpha = \Big(\int_0^\infty \big(t^{-\theta}\zeta_\alpha(tA)x\|\big)^p\,\frac{\mathrm{d}t}{t}\Big)^{1/p} \quad \text{with} \quad \zeta_\alpha(z) = \frac{z^\alpha}{(1+z)^{2\alpha}}, \qquad (16.28)$$

is an equivalent norm on $(X,\mathsf{D}(A))_{\theta,p}$.

Proof. This is an immediate consequence of Theorem 16.3.20, as the condition $\alpha > 1-\theta$ ensures that the conditions of the theorem hold for ζ_α. □

Proof of Theorem 16.3.20. We write $Y := (X, \mathsf{D}(A))_{\theta,p}$ for brevity.

Step 1 – We begin by preparing two helpful estimates.

First, for all $s, t > 0$ we have

$$\|\zeta_\alpha(sA)\zeta_\alpha(tA)\| \leqslant C_A\Big(\min\{\frac{s}{t}, \frac{t}{s}\}\Big)^\alpha. \tag{16.29}$$

Indeed, by multiplicativity of the $E(\Sigma_\sigma)$-functional calculus and the identity

$$\begin{aligned} T :&= (tA)^\alpha(1+tA)^{-2\alpha}(sA)^\alpha(1+sA)^{-2\alpha} \\ &= \big(\frac{s}{t}\big)^\alpha\Big(A^{2\alpha}(t^{-1}+A)^{-2\alpha}\Big)\Big((s^{-1})^{2\alpha}(s^{-1}+A)^{-2\alpha}\Big) \end{aligned}$$

we obtain

$$\|T\| \leqslant \big(\frac{s}{t}\big)^\alpha \sup_{t\geqslant 0}\|A(t^{-1}+A)^{-1}\|^{2\alpha}\cdot \sup_{s\geqslant 0}\|s^{-1}(s^{-1}+A)^{-1}\|^{2\alpha} \leqslant C_A\big(\frac{s}{t}\big)^\alpha.$$

Since the same estimate holds with s and t interchanged, this gives (16.29).

Second, for all $s > 0$, $t \in [s, 2s]$, and $x \in X$ we have

$$\|\zeta_\alpha(sA)x\| = \|\zeta_\alpha((s-t/2)A)\zeta((t/2)A)x\| \leqslant C\|\zeta_\alpha((t/2)A)x\|, \tag{16.30}$$

where $C = \sup_{r>0}\|\zeta_\alpha(rA)\|$ is finite by (10.9).

Step 2 – Now we turn to the actual proof of the theorem.

With the notation introduced above, we take $g := \zeta_{2\alpha}$ and $h := \zeta_\delta$, where $\alpha \in \mathbb{N}$ satisfies $\alpha > 1-\theta$ and $\delta > 0$. For $x, y \in Y$ assume that

$$\|h(tA)g(tA)x\|_Y \leqslant \|g(tA)y\|_Y, \quad t > 0. \tag{16.31}$$

Then for all $s > 0$,

$$\begin{aligned} \|\zeta_{3\alpha}(sA)x\| &\lesssim \Big(s^{\theta p}\int_s^{2s} t^{-\theta p}\frac{\mathrm{d}t}{t}\Big)\|\zeta_{3\alpha+\delta}(sA)x\|^p \\ &\lesssim s^{\theta p}\int_s^{2s} t^{-\theta p}\|\zeta_\alpha((t/2)A)\zeta_{2\alpha+\delta}(sA)x\|^p\frac{\mathrm{d}t}{t} && \text{(by (16.30))} \\ &= (2s)^{\theta p}\int_{2s}^{4s} t^{-\theta p}\|\zeta_\alpha(tA)\zeta_{2\alpha+\delta}(sA)x\|^p\frac{\mathrm{d}t}{t} \\ &\lesssim (2s)^{\theta p}\|\zeta_{2\alpha+\delta}(sA)x\|_Y^p && \text{(by (16.28))} \\ &= (2s)^{\theta p}\|h(sA)g(sA)x\|_Y^p \\ &\leqslant (2s)^{\theta p}\|g(sA)y\|_Y^p && \text{(by (16.31))} \\ &\lesssim (2s)^{\theta p}\int_0^\infty t^{-\theta p}\|\zeta_\alpha(tA)\zeta_\alpha(sA)\zeta_\alpha(sA)y\|^p\frac{\mathrm{d}t}{t} && \text{(by (16.28))} \end{aligned}$$

$$\lesssim \|\zeta_\alpha(sA)y\|^p \Big((2s)^{\theta p}\int_0^s \Big(t^{-\theta p}\big(\frac{t}{s}\big)^{\alpha p}\,\frac{\mathrm{d}t}{t} \\ + (2s)^{\theta p}\int_s^\infty t^{-\theta p}\big(\frac{s}{t}\big)^{\alpha p}\,\frac{\mathrm{d}t}{t}\Big) \quad \text{(by (16.29))} \\ \lesssim_{\theta,p} \|\zeta_\alpha(sA)y\|^p$$

with implied constants depending on A, σ, θ, p, and α. Integrating the left- and right-hand sides in this estimate with respect to $s^{-\theta p}\frac{\mathrm{d}s}{s}$ and using (16.28) twice, we see that $\|x\|_Y \lesssim \|y\|_Y$ with implied constant independent of x and y. This proves that $A_{\theta,p}$ has an absolute calculus on Y. □

Corollary 16.3.22 (Dore). *If A is a densely defined sectorial operator on X, with $0 \in \varrho(A)$, then for all $1 \leqslant p \leqslant \infty$ and $0 < \theta < 1$, the part $A_{\theta,p}$ of A in the real interpolation space $(X, \mathsf{D}(A))_{\theta,p}$ has a bounded H^∞-calculus.*

The invertibility assumption cannot be dropped in the corollary, and hence in the theorem. Indeed, let A be a bounded sectorial operator without a bounded H^∞-calculus (such operators exist, even on a separable Hilbert space, by Corollary 10.2.29). Then $\mathsf{D}(A) = X$ and therefore $(X, \mathsf{D}(A))_{\theta,p} = X$ for all $0 < \theta < 1$ and $p \in [1, \infty]$. By assumption, A doesn't have a bounded H^∞-calculus on this space.

16.4 Notes

Section 16.1

The problem of defining the sum of two unbounded operators A and B can be approached from various angles. Besides the direct approach of defining $A+B$ as (the closure of) the operator given on $\mathsf{D}(A)\cap\mathsf{D}(B)$ by $(A+B)x = Ax + Bx$, which works well if A and B have commuting resolvents, various other approaches can be taken. When A and B generate uniformly bounded C_0-semigroups S and T respectively, conditions can be formulated in order that the limit in the *Trotter product formula*

$$V(t)x := \lim_{n\to\infty} (S(t/n)T(t/n))^n x$$

exist for all $x \in X$, and defines a C_0-semigroup whose generator C is the closure of the operator $A+B$ initially defined on $\mathsf{D}(A)\cap\mathsf{D}(B)$ by $(A+B)x = Ax+Bx$ Engel and Nagel [2000]; resolvent commutation is not needed in these results. A different approach is the form method, suitable when A and B are defined on a Hilbert space with inner product $(\cdot|\cdot)$. This method in provides conditions under which the (closure of the) sum $\mathfrak{c} := \mathfrak{a} + \mathfrak{b}$ of the sesquilinear forms

$$\mathfrak{a}(x,y) := (Ax|y), \quad \mathfrak{b}(x,y) = (Bx|y)$$

is associated with a closed operator C satisfying

$$\mathfrak{c}(x,y) = (Cx|y).$$

Like always, subtle domain questions have to be taken care of. A detailed treatment is given in Kato [1995]; for a gentle introduction see, e.g., Van Neerven [2022].

Section 16.2

In this section some classical perturbation theorems for sectorial operators are extended to R-sectorial operators. Theorem 16.2.4 on relatively bounded perturbations of R-sectorial operators is basically from Kunstmann and Weis [2001], with some improvements of constants. More sophisticated perturbation results using real interpolation are contained in Haak, Haase, and Kunstmann [2006].

Proposition 16.2.6 on perturbations of the H^∞ by multiples of the identity and the main theorem of this section, Theorem 16.2.8 on relatively bounded perturbations of the H^∞-calculus, are from Kalton, Kunstmann, and Weis [2006]. This paper contains a number of variants of the relative boundedness conditions, some of them modelled after form perturbations. Part (iii) of Theorem 16.2.8 was proved independently by Denk, Dore, Hieber, Prüss, and Venni [2004]. The perturbation theorem 16.2.7 for lower order perturbations of the H^∞-calculus is due to Amann, Hieber, and Simonett [1994].

Example 16.2.10 is due to McIntosh and Yagi [1990]; a proof of the fact that the norm of a Toeplitz matrix with bounded real-valued generating function f is bounded by $\|f\|_{L^\infty(\mathbb{T})}$ can be found in Garoni and Serra-Capizzano [2017, Theorem 6.1]. A further example can be found in Kalton [2007]; see also the review paper Batty [2009].

The philosophy behind some of these perturbation theorems is that the boundedness of the H^∞-calculus is encoded in the fractional domain spaces of the operator in the following sense (see Kalton, Kunstmann, and Weis [2006]): If A and B are two standard sectorial operators on a reflexive Banach space X, and if for some $0 < \alpha_1 < \alpha_2 < \frac{3}{2}$ and $j = \{0, 1\}$ we have

$$\mathsf{D}(A^{\alpha_j}) = \mathsf{D}(B^{\alpha_j}) \quad \text{and} \quad \|A^{\alpha_j}x\| \eqsim \|B^{\alpha_j}x\| \tag{16.32}$$

for all x in this common domain, then if one of the operators has a bounded H^∞-calculus, then so does the other. Notice that there are no smallness assumptions here.

The basic idea of the proofs of Theorem 16.2.8, the comparison theorem just quoted, and their variants is to use the relative bounded or the equivalence of norms of (16.32) to show the equivalence of the discrete square function norms

$$x \mapsto \Big\| \sum_{j\in\mathbb{Z}} \varepsilon_j \phi(2^j A)x \Big\|$$

with the corresponding ones for B. Here, ϕ is usually of the form $\phi(z) = z^\alpha(1+z)^n$ with $\alpha < n$. The two conditions (ii) and (iii) of Theorem 16.2.8 and (16.32) correspond to the two sides of the square function estimate.

In Kalton, Kunstmann, and Weis [2006] it is also explained how to use these perturbation theorems to establish the boundedness of the H^∞-calculus for rather general classes of elliptic operators on $H^{s,p}(\mathbb{R}^d)$ or $H^{s,p}(D)$ for smooth domains $D \subseteq \mathbb{R}^d$ with Lopatinskii–Shapiro boundary conditions. The idea is to compare them to constant coefficients. A related approach is used in Denk, Hieber, and Prüss [2003]. For more recent results the reader is referred to the Notes of Chapter 17.

Let us mention two further topics related to the H^∞-calculus and its perturbations.

Extrapolation of the H^∞-calculus in the L^p-scale

We have seen in Chapter 11 that for singular integral it is often a successful strategy to first prove a Hilbert space result, then prove a weaker result on L^{p_0} or some endpoint of the L^p scale, and then extend the Hilbert space result to L^p-spaces by interpolation between 2 and p_0. This idea also proves fruitful for perturbation theorems; see Kunstmann and Weis [2017]. As in the case of classical singular integral operators, the Littlewood–Paley theory and so-called *off-diagonal estimates* play a crucial role. Here, the "Littlewood–Paley decomposition" of a standard sectorial operator B on a space $L^p(S)$ with a bounded H^∞-calculus is expressed as the equivalence of norms

$$\|x\|_{L^p(S)} \eqsim \Big\| \Big(\sum_{j\in\mathbb{Z}} |\phi(2^j B)x|^2 \Big)^{1/2} \Big\|_{L^p(S)}$$

for all $x \in L^p(S)$. Given a second standard sectorial operator A on $L^p(S)$, the following R-boundedness condition expresses that A is "close" to B in terms of the " Littlewood–Paley pieces" $\phi(2^j A)$ and $\phi(2^j B)$:

$$\mathscr{R}\Big(\phi(s2^{j+k}A)\psi(t2^k B) : j \in \mathbb{Z}\Big) \lesssim 2^{-\beta k} \tag{16.33}$$

for some $\beta > 0$ and all $k \in \mathbb{Z}$ and $s,t \in [1,2]$. Kunstmann and Weis [2017] contains the following theorem:

Theorem 16.4.1. *Let A and B be standard sectorial operators consistently defined on $L^2(S)$ and $L^{p_0}(S)$, with $p_0 \in (1,\infty)\setminus\{2\}$, and assume that A is R-sectorial on $L^{p_0}(S)$ and B has a bounded H^∞-calculus on both $L^2(S)$ and $L^{p_0}(A)$. If (16.33) holds for $p = 2$, then A has a bounded H^∞-calculus on both $L^2(S)$ and all spaces $L^p(S)$ with p between 2 and p_0.*

This theorem can be extended to the case where A is defined on a complemented subspace of L^p and A is a "retract" of B in a suitable sense. In this way one can, for example, derive the boundedness of the H^∞-calculus of the Stokes operator on the Helmholtz space $L^p_0(D)$ from the boundedness of the H^∞-calculus of the Laplace operator on $L^p(D)$, for bounded Lipschitz domains $D \subseteq \mathbb{R}^d$ with $d \geqslant 3$ and $|\frac{1}{2} - \frac{1}{p}| < \frac{1}{2d}$ (see Kunstmann and Weis [2017]).

Scales of fractional domain spaces and interpolation

Let A be a standard sectorial operator and denote by $\dot{X}_\alpha$ the completion of $\mathsf{D}(A^\alpha)$ with respect to the norm $\|x\|_\alpha := \|A^\alpha x\|$ for $\alpha \in \mathbb{R}$. The methods of Section 15.3.b show that a Hilbert space operator A has a bounded H^∞-calculus if and only if these fractional domain spaces can be identified with the complex interpolation spaces

$$[\dot{X}_\alpha, \dot{X}_\beta]_\theta = \dot{X}_\gamma$$

with $(1-\theta)\alpha + \theta\beta = \gamma$ for $\alpha \neq \beta$ and $\theta \in (0,1)$. This is not true anymore in Banach spaces, where complex interpolation is related to boundedness of the imaginary powers, rather than the boundedness of the H^∞-calculus. However, such an identification is possible with the help of the γ-interpolation method introduced in the Notes to Section 15.3. It is shown in Kalton, Kunstmann, and Weis [2006, Section 5.3] that a standard γ-sectorial operator A on a Banach space X with non-trivial type has a bounded H^∞-calculus if and only if

$$(\dot{X}_\alpha, \dot{X}_\beta)^\gamma_\theta = \dot{X}_\delta$$

with $(1-\theta)\alpha + \theta\beta = \delta$ for $\alpha \neq \beta$ and $\theta \in (0,1)$. Even when A does not have a bounded H^∞-calculus, the spaces $(\dot{X}_\alpha, \dot{X}_\beta)^\gamma_\theta$ can be identified with certain square function spaces $H^\gamma_{s,A}$ which are defined as the completion of $\mathsf{D}(A^m) \cap \mathsf{R}(A^m)$, $m > |s|+1$, with respect to the norm

$$\left\| t \mapsto t^{-s}\phi(tA)x \right\|_{\gamma(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X)}, \quad x \in \mathsf{D}(A^m) \cap \mathsf{R}(A^m)$$

for some $\phi \in H^1(\sigma_\sigma)$ such that $z \mapsto z^{-s}\phi(z)$ still belongs to $H^1(\sigma_\sigma)$. Complete proofs can be found in Kalton, Lorist, and Weis [2023].

Section 16.3

The operator-sum method as a purely functional analytic approach to evolution equations goes back to Da Prato and Grisvard [1975], where already Theorem 16.3.15 is proved. Our proof of Theorem 16.3.2 follows Haase [2006], where further properties of the operator C extending $A+B$ are discussed.

In the setting of Hilbert spaces, the operator-valued H^∞-calculus was introduced in Albrecht, Franks, and McIntosh [1998]. Theorem 16.3.4 is taken from Kalton and Weis [2001]. It is implicit in Lancien and Le Merdy [1998] (see also Lancien, Lancien, and Le Merdy [1998, Remark 6.5] and Albrecht, Franks, and McIntosh [1998]) that any sectorial operator on a Hilbert space with a bounded H^∞-calculus has a bounded operator-valued H^∞-calculus. In these papers the "right" method of proof for Theorem 16.3.4 was already found, but the crucial ingredient of R-boundedness was still missing.

Theorem 16.3.6 is due to Kalton and Weis [2001]. In the next chapter, the connections of Theorem 16.3.6 with maximal L^p-regularity will be discussed in detail. As we will see in Volume IV, the operator-valued functional calculus of Theorem 16.3.4 can be used to give a short proof for stochastic maximal L^p-regularity; see Van Neerven, Veraar, and Weis [2015b]. In Clément and Prüss [2001] it is shown that if A is an injective operator generating a bounded C_0-group on a UMD space X, and B is an invertible closed linear operator in X resolvent commuting with A such that $\pm iB$ is R-sectorial, then the operator $A+B$ with domain $\mathsf{D}(A)\cap\mathsf{D}(B)$ is closed and invertible. If B is also sectorial with angle $\omega(B) < \frac12\pi$, then $A+B$ is sectorial as well, and $\omega(A+B) < \frac12\pi$.

Theorem 16.3.9 is due to Lancien, Lancien, and Le Merdy [1998], Lancien and Le Merdy [1998], who extend an earlier result of Albrecht [1994] on L^p-spaces with $1 < p < \infty$.

That Theorem 16.3.10 holds more generally for Banach spaces with the triangular contraction property was shown by Le Merdy [2003].

The absolute functional calculus was introduced in Kalton and Kucherenko [2010], where Theorems 16.3.13, 16.3.14, and 16.3.20 were proved. The definition of the absolute calculus may be a little off-putting if one is accustomed to thinking in terms of spectral theory, but the benefits of this notion is considerable:

- It implies an operator-valued H^∞-calculus and sum-of-operators theorem without the complexities of R-boundedness, just as in Hilbert spaces (see Theorems 16.3.13 and 16.3.14).
- It leads to a simple sufficient condition for the abstract functional calculus of a sectorial operator A in terms of the equivalence

$$\|x\| \eqsim \Big(\int_0^\infty \|\phi(tA)x\|^p \,\frac{\mathrm{d}t}{t}\Big)^{1/p} \tag{16.34}$$

  (see Theorem 16.3.16, implicit in Kalton and Kucherenko [2010]). The criterion already suffices for the common applications and it shows that in L^1, L^2, and $C(K)$ spaces the absolute functional calculus is equivalent to the H^∞-calculus; see Kalton and Kucherenko [2010]. It also shows that the absolute calculus is mainly a tool for real interpolation spaces and non-UMD spaces. Indeed, for every operator A with a bounded H^∞-calculus on L^p with $1 < p < \infty$ we have (cf. Section 10.4)

$$\|x\|_p \eqsim \big\|t\mapsto \phi(tA)x\big\|_{\gamma(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)} \eqsim \Big\|\Big(\int_0^\infty |\phi(tA)x|^2\,\frac{\mathrm{d}t}{t}\Big)^{1/2}\Big\|_p,$$

which is a norm decidedly different from (16.34) when $p \neq 2$. However, in this setting it provides a unified approach to many results of Dore and Da Prato–Grisvard.

- The absolute functional calculus can be characterised in terms of generalised real interpolation spaces, where the role of $L^p(\mathbb{R}_+, \frac{dt}{t})$ is taken over by more general Banach function spaces E over $\mathbb{R}_+$. Essentially, a standard sectorial operator A, acting on an intermediate spaces X for a couple (X_0, X), where X_0 and X_1 are appropriate fractional domain spaces of X, has an absolute calculus if and only if $X = (X_0, X_1)_{\theta,E}$ for some $\theta \in (0,1)$ and an appropriate choice of such a Banach function space E; see Kalton and Kucherenko [2010], where a precise statement of the result an be found. This characterisation of the absolute calculus compares nicely with the characterisation of the H^∞-calculus through the γ-interpolation method and the close relationship of bounded imaginary powers with the complex interpolation method described in the previous chapter. A necessary condition for the existence of a Banach function space E with $X = (X_0, X_1)_{\theta,E}$ is the monotonicity of the K-functional for the couple (X_0, X_1), in the sense that it has the property that $K(t, x; X_0, X_1) \leqslant K(t, y; X_0, X_1)$ for some $x \in X_0 + X_1$ and $y \in X$, and all $t \geqslant 0$, then $x \in X$ and $\|x\|_X \leqslant c\|y\|_X$. In fact, this is where the definition of the absolute calculus has its origin.

The proof of Theorem 16.3.19 follows Haase [2005], where some additional details have been written out. The theorem also admits a homogeneous version, which is presented in [Haase, 2006, Section 6.4]. A by-product of Theorem 16.3.19 is the equivalence of norms

$$\|x\| + \|t \mapsto t^{-\theta}\phi(tA)x\|_{L^p(\mathbb{R}_+, \frac{dt}{t}; X)} \eqsim \|x\| + \|t \mapsto t^{-\theta}\psi(tA)x\|_{L^p(\mathbb{R}_+, \frac{dt}{t}; X)}$$

for functions $\phi, \psi \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ satisfying the conditions of Theorem 16.3.19. Interestingly, this equivalence of norms remains true under somewhat weaker conditions on ϕ and ψ; see Haase [2006, Theorem 6.4.2]. The proof follows the lines of the equivalence of continuous square functions in Chapter 10, with simplifications due to the fact that various subtleties in the handling of γ-norms can now be avoided. This more general version of the equivalence of norms covers the function $\phi(z) = z/(z+1)$ which is implicit in the first part of Proposition K.4.1.

Corollary 16.3.22 is a classical result due to Dore [1999]. This result was subsequently generalised to standard sectorial operators in Dore [2001], where it was shown that such operators have a bounded H^∞-calculus on $(X, \mathsf{D}(A) \cap \mathsf{R}(A))_{\theta,p}$; see also Kalton and Kucherenko [2010], who establish their absolute functional calculus.

Sums of non-commuting operators

In Section 16.3 we studied the closedness (and further properties) of sums of operator $A + B$ under the assumption that A and B are resolvent commuting. In this paragraph, we briefly comment on the closedness of sums of non-commuting operators, provided suitable condition bounds and, sometimes, domain compatibility assumptions, are imposed on the commutator $[A, B] = AB - BA$. The first such result was obtained by Da Prato and Grisvard [1975], who proved that the closure of $A + B$ is invertible and sectorial under commutator conditions. Closedness of $A + B$ itself was proved under further conditions in the case that X is a Hilbert space. Labbas and Terreni [1987] obtained similar results under a different type of commutator conditions, and Monniaux and Prüss [1997] proved a Dore–Veni type theorem for non-commuting operators under the commutator condition of Labbas–Terreni. In his PhD thesis, Štrkalj [2000] proved a version of Kalton and Weis [2001] for non-commuting operators under the same condition as Labbas–Terreni in the case that X is a B-convex space, and Prüss and Simonett [2007] proved a similar result under either one of the above commutator conditions without any restrictions on the space X. Moreover, under the condition that A and B has a bounded H^∞-calculus, with one of them R-bounded, it was shown in this paper that $A + B$ has a bounded H^∞-calculus. Similar results were proved under a different commutator conditions in Roidos [2018]. Products of non-commuting operators have been considered in Štrkalj [2000], Haller-Dintelmann and Hieber [2005].

Typical applications of these results include parabolic PDE on wedge or cone domains, where an elliptic operator C can be split into two space directions to obtain simpler operators A and B. This type of application was worked out in detail by Prüss and Simonett [2007], Prüss and Simonett [2006] and continued in Nau and Saal [2012], Maier and Saal [2014], Köhne, Saal, and Westermann [2021]. For another typical application to non-autonomous parabolic problems, in which case ∂_t and $A(t)$ are non-commuting on $L^p(0, T; X)$, the reader is referred to Di Giorgio, Lunardi, and Schnaubelt [2005a]. Applications to hyperbolic problems appear in Alouini and Goubet [2014].

An interesting class of operator sums for non-commuting operators arises in connection with (an abstract version of) the Weyl commutation relation for position and momentum operators. The general theory of such operators has an altogether different flavour due to its connections with the Heisenberg group; the reader is referred to Putnam [1967] for a general overview. In connection with the topics treated in this volume, the following result is worth mentioning. Suppose two d-tuples of operators $A = (A_1, \dots, A_d)$ and $B = (B_1, \dots, B_d)$ acting on a Banach space X are given such that $iA_1, \dots, iA_d$ and $iB_1, \dots, iB_d$ generate bounded C_0-groups satisfying the Weyl commutation relations

$$e^{isA_j} e^{itA_k} = e^{itA_k} e^{isA_j}, \quad e^{isB_j} e^{itB_k} = e^{itB_k} e^{isB_j}$$
$$e^{isA_j} e^{itB_k} = e^{-ist\delta_{jk}} e^{itB_k} e^{isA_j}.$$

Here, for clarity of exposition, we use exponential notation for the C_0-groups involved. Under this condition, the operator sum

$$\frac{1}{2}(A^2 + B^2) = \frac{1}{2}\sum_{j=1}^{d} A_j^2 + B_j^2$$

is the abstract counterpart of the quantum harmonic oscillator. Under the assumption that X is a UMD Banach lattice, it is shown in Van Neerven and Portal [2020] (under an additional boundedness assumption of the Weyl calculus associated with the pair (A, B)) and Van Neerven, Portal, and Sharma [2023] that the operator $\frac{1}{2}(A^2 + B^2) - \frac{1}{2}d$ is R-sectoriality and has a bounded of the H^∞-calculus.

17

Maximal regularity

The present chapter is one of the central ones of this book project. Maximal regularity provides a link between the general theory of operator-valued singular integrals and the theory of H^∞-functional calculus with the regularity theory for evolution equations. In some cases progress on these topics was even motivated by applications to maximal regularity.

Maximal L^p-regularity for the Cauchy problem

$$\begin{cases} u'(t) + Au(t) &= f(t), \quad t \in \overline{I}, \\ u(0) &= 0, \end{cases} \tag{17.1}$$

with A an unbounded linear operator A on a Banach space and $I = (0,T)$ or $\mathbb{R}_+$, requires that if the right-hand side f of (17.1) belongs to $L^p(I;X)$, then (17.1) has a unique strong solution u that has the best regularity possible in this situation. Namely, u' and Au should both belong to $L^p(I;X)$ as well, and

$$\|u'\|_{L^p(I;X)} + \|Au\|_{L^p(I;X)} \leqslant C\|u' + Au\|_{L^p(I;X)} = C\|f\|_{L^p(I;X)}. \tag{17.2}$$

This *a priori* estimate is often crucial to fixed point methods for more general classes of non-linear partial differential equations with A as their linear main part. This will be explained in more detail in Chapter 18.

Unfortunately, maximal L^p-regularity for A may fail even if $-A$ is the generator of an analytic semigroup on $L^q(0,1)$ (see Section 17.4.c). Proving maximal L^p-regularity usually requires quite sophisticated methods. In earlier chapters, we have prepared for three approaches to maximal regularity:

- Operator-valued singular integral operators:

Formally using the variation of constants formula for the solution of (17.2), we have

$$Au(t) = \int_0^t Ae^{-(t-s)A} f(s)\,\mathrm{d}s.$$

T. Hytönen et al., *Analysis in Banach Spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics 76,
https://doi.org/10.1007/978-3-031-46598-7_7

Since $\|Ae^{-tA}\| \lesssim \frac{1}{t}$ (and often even $\eqsim$), it is clear that we have to deal with singular integrals to obtain maximal L^p-regularity. The details of this approach will be presented in Section 17.3.a.

- The operator sum method:

This method refers to the inverse triangle inequality stated in (17.2). We show that (17.2) can be obtained by applying the sum-of-operator theorems of Chapter 16 with one operator chosen as $D = \frac{\mathrm{d}}{\mathrm{d}t}$ on $L^p(I;X)$ (see Section 17.3.c).

With both methods one can show that a sectorial operator A on a UMD space X has maximal L^p-regularity if and only if A is R-sectorial of angle $\omega_R(A) < \pi/2$. The notion of R-sectoriality was introduced in Chapter 10.

There is a third method which simplifies the problem somewhat:

- Interpolation theory:

Here, the real interpolation method is of particular interest because the properties of sectorial operators often drastically improve when considered on the real interpolation space $L^p(I;(X,\mathsf{D}(A))_{\theta,q})$ or on a Besov space $B^s_{p,q}(I;X)$ as "regularity" space. In the first case, the part of A on $(X,\mathsf{D}(A))_{\theta,q}$ automatically has a bounded H^∞-calculus if $0 \in \varrho(A)$ (see Corollary 16.3.22). In the second case, since Besov spaces are real interpolation spaces a unified approach can be given via the absolute functional calculus, which was introduced in Section 16.3.d, This approach will be presented in Section 17.3.c.

In Sections 17.1 and 17.2 we discuss the essential properties and variants of maximal L^p-regularity which are useful in applications to evolution equations. For instance, we show that for an additional initial value $u(0) = x$ in (17.1), maximal L^p-regularity with $1 < p < \infty$ is preserved if and only if $x \in (X,\mathsf{D}(A))_{1-\frac{1}{p},p}$ (see Section 17.2.b). In Section 17.2.e we discuss extrapolation of integrability and extrapolation to power weights; this allows the treatment of a wider class of initial values. Perturbation results for maximal L^p-regularity appear in Sections 17.2.g and 17.3.

17.1 The abstract Cauchy problem

Throughout this section we assume that A is a linear operator with domain $\mathsf{D}(A)$ in a Banach space X. We are interested in finding solutions, in an appropriate sense, to the inhomogeneous abstract Cauchy problem

$$\begin{cases} u'(t) + Au(t) &= f(t), \quad t \in I, \\ u(0) &= x, \end{cases} \tag{ACP$_x$}$$

where either $I = \mathbb{R}_+ = (0,\infty)$ or $I = (0,T)$ is a bounded interval, and the initial value x is taken from X. We will use the notation

$$L^1_{\rm loc}(\overline{I};X)$$

for the space of strongly measurable functions on I which are integrable on every bounded interval contained in I. If I is a bounded interval, then of course $L^1_{\rm loc}(\overline{I};X) = L^1(I;X)$.

Definition 17.1.1 (Strong solutions). *A strongly measurable function $u : I \to X$ is called a* strong solution *to the problem* (ACP$_x$) *associated with a given $f \in L^1_{\rm loc}(\overline{I};X)$ if*

(i) *u takes values in $\mathsf{D}(A)$ almost everywhere and Au belongs to $L^1_{\rm loc}(\overline{I};X)$;*
(ii) *u solves the integrated version of* (ACP$_x$)*, that is, for almost all $t \in I$ we have*
$$u(t) + \int_0^t Au(s)\,\mathrm{d}s = x + \int_0^t f(s)\,\mathrm{d}s.$$

It is clear from condition (ii) in the above definition that a strong solution u is equal almost everywhere to a continuous function on $\overline{I}$. In what follows, we will always work with this version, which satisfies the identity of (ii) for all $t \in \overline{I}$ and satisfies
$$u(0) = x.$$

If $(0, T_0)$ is a bounded interval contained in I and u is an strong solution of (ACP$_x$) on I, then by taking the supremum over $[0, T_0]$ in (ii) we obtain

$$\|u\|_{C([0,T_0];X)} \leqslant \|x\| + \|Au\|_{L^1(0,T_0;X)} + \|f\|_{L^1(0,T_0;X)}.$$

By Lemma 2.5.8 and Proposition 2.5.9, indefinite integrals of functions in $L^1(0,T;X)$ belong to $W^{1,1}(0,T;X)$ and are differentiable almost everywhere, with derivative equal to the original function almost everywhere. In the present setting we may apply this to Au and f to obtain that strong solutions u on I are differentiable almost everywhere, and satisfy

$$u' + Au = f \quad \text{almost everywhere on } I \tag{17.3}$$

The definition of a mild solution given in Volume II can be extended to locally bounded strongly measurable semigroups as discussed in Appendix K:

Definition 17.1.2 (Mild solutions). *Suppose that $-A$ generates a locally bounded strongly measurable semigroup S on X. For $f \in L^1_{\rm loc}(\overline{I};X)$ and $x \in \overline{\mathsf{D}(A)}$, the continuous function $u \in L^1_{\rm loc}(\overline{I};X)$ defined by*

$$u(t) := S(t)x + S * f(t) := S(t)x + \int_0^t S(t-s)f(s)\,\mathrm{d}s, \quad t \in \overline{I},$$

is called the mild solution *of the problem* (ACP$_x$)*,*

$$\begin{cases} u'(t) + Au(t) &= f(t), \quad t \in I, \\ u(0) &= x. \end{cases}$$

Since we assume $x \in \overline{\mathsf{D}(A)}$, by Proposition K.1.9 the mild solution u extends continuously to $\overline{I}$ and satisfies $u(0) = x$.

In the case that A is densely defined (by Proposition 10.1.9, this is automatic if X is reflexive), Theorem G.3.2 shows that the mild solution is also the unique weak solution of (ACP$_x$). In the setting considered presently, we do not assume A to be densely defined and weak solutions cannot be defined. Instead, the next proposition states the equivalence of strong and mild solutions. It uses the following terminology.

In line with Definition K.1.2 (and keeping in mind Remark K.1.4), a linear operator $(G, \mathsf{D}(G))$ in X is said to be the *generator* of a locally bounded strongly measurable semigroup S satisfying $\|S(t)\| \leqslant Me^{\omega t}$ for all $t \geqslant 0$, if the following conditions are satisfied:

(i) $\{\lambda \in \mathbb{C} : \Re\lambda > \omega\} \subseteq \varrho(G)$;
(ii) for all $\Re\lambda > \omega$ and $x \in X$ we have

$$R(\lambda, G)x = \int_0^\infty e^{-\lambda t} S(t)x \,\mathrm{d}t.$$

By strong measurability and exponential decay of the integrand, the integral in (ii) converges as a Bochner integral in X. It follows from Proposition K.1.7 that for all $x \in X$ and $t > 0$ we have $\int_0^t S(s)x \,\mathrm{d}s \in \mathsf{D}(G)$ and

$$G \int_0^t S(s)x \,\mathrm{d}s = S(t)x - x;$$

if moreover $x \in \mathsf{D}(G)$, then both expressions are equal to $\int_0^t S(t)Gx \,\mathrm{d}s$. This fact will be used repeatedly below.

Proposition 17.1.3. *Let $-A$ generate a locally bounded strongly measurable semigroup S on a Banach space X. Let $f \in L^p(I; X)$ with $1 \leqslant p \leqslant \infty$, and let $x \in \overline{\mathsf{D}(A)}$. Then for any function $u \in C(\overline{I}; X)$, the following assertions are equivalent:*

(1) *u is a strong solution on I of* (ACP$_x$)*;*
(2) *u is the mild solution of* (ACP$_x$) *on I, u is differentiable almost everywhere, and $u' \in L^1_{\mathrm{loc}}(\overline{I}; X)$;*
(3) *u is the mild solution of* (ACP$_x$)*, u takes values in $\mathsf{D}(A)$ almost everywhere, and $Au \in L^1_{\mathrm{loc}}(\overline{I}; X)$.*

In particular, a strong solution, if it exists, is unique, and it then equals the mild solution.

Proof. Fix $\lambda \in \varrho(-A)$.

(1)⇒(2): We have already seen that u is differentiable almost everywhere and weakly differentiable with $u' \in L^1_{\mathrm{loc}}(I; X)$. It remains to prove that u is the mild solution. Fix $t \in I$ and define $v : [0, t] \to X$ by

$$v(s) := (\lambda + A)^{-1}S(t-s)u(s) = S(t-s)(\lambda + A)^{-1}u(s).$$

Then $v \in C([0,t];X)$ by the strong continuity of S on $\overline{\mathsf{D}(A)}$.

By Proposition K.1.9, for all $x_0 \in \overline{\mathsf{D}(A)}$ the function $v_{x_0}(s) := S(t-s)(\lambda + A)^{-1}x_0$ is continuously differentiable on $(0,t)$. The product rule, which may be applied in the points where u is differentiable, gives that v is differentiable almost everywhere on $(0,t)$ with derivative

$$v'(s) = (\lambda + A)^{-1}S(t-s)u'(s) + A(\lambda + A)^{-1}S(t-s)u(s).$$

Since u is assumed to be a strong solution, it takes values in $\mathsf{D}(A)$ almost everywhere and therefore

$$v'(s) = (\lambda + A)^{-1}S(t-s)(u'(s) + Au(s)) = (\lambda + A)^{-1}S(t-s)f(s)$$

for almost all $s \in (0,t)$. This implies $v \in W^{1,p}(0,t;X)$ and, by Proposition 2.5.9,

$$\begin{aligned} v(t) - (\lambda + A)^{-1}S(t)x = v(t) - v(0) &= \int_0^t v'(s)\,\mathrm{d}s \\ &= (\lambda + A)^{-1}\int_0^t S(t-s)f(s)\,\mathrm{d}s. \end{aligned}$$

This gives $u(t) = (\lambda + A)v(t) = S(t)x + S * f(t)$ as required.

To prove the remaining implications, we first derive a formula in case u is the mild solution. Integrating $A(\lambda + A)^{-1}u$ over $[0,t]$ gives

$$\begin{aligned} \int_0^t A(\lambda + A)^{-1}(u(s) - S(s)x)\,\mathrm{d}s &= \int_0^t \int_0^s A(\lambda + A)^{-1}S(s-r)f(r)\,\mathrm{d}r\,\mathrm{d}s \\ &= \int_0^t \int_r^t A(\lambda + A)^{-1}S(s-r)f(r)\,\mathrm{d}s\,\mathrm{d}r \\ &= \int_0^t A\int_r^t S(s-r)(\lambda + A)^{-1}f(r)\,\mathrm{d}s\,\mathrm{d}r \\ &= -\int_0^t (\lambda + A)^{-1}[S(t-r)f(r) - f(r)]\,\mathrm{d}r \\ &= -(\lambda + A)^{-1}\Big(u(t) - S(t)x - \int_0^t f(r)\,\mathrm{d}r\Big), \end{aligned} \tag{17.4}$$

where we used the definition of a mild solution and Hille's theorem; the identity $A\int_r^t S(s-r)x\,\mathrm{d}s = -[S(t-r)x - x]$ which as used in this computation is justified by Proposition K.1.7.

(2)⇒(3): Since u is almost everywhere differentiable with $u' \in L^1_{\mathrm{loc}}(\overline{I};X)$, and $(\lambda + A)^{-1}S(t)x = S(t)(\lambda + A)^{-1}x$ is differentiable with derivative $A(\lambda + A)^{-1}S(t)x$, (17.4) and Proposition 2.5.9 imply

$$A(\lambda + A)^{-1}u(t) = -(\lambda + A)^{-1}[u'(t) - f(t)] \text{ almost all } t \in I.$$

Combining this with $A(\lambda + A)^{-1}u(t) = u(t) - \lambda(\lambda + A)^{-1}u(t)$ we find that $u \in \mathsf{D}(A)$ almost everywhere and $Au = -u' + f \in L^1_{\rm loc}(\overline{I}; X)$.

(3)⇒(1): Since $u \in \mathsf{D}(A)$ almost everywhere and $Au \in L^1_{\rm loc}(\overline{I}; X)$, (17.4) implies

$$\begin{aligned}(\lambda + A)^{-1} \int_0^t Au(s)\,\mathrm{d}s - A(\lambda + A)^{-1} \int_0^t S(s)x\,\mathrm{d}s \\ = -(\lambda + A)^{-1}\Big[u(t) - S(t)x - \int_0^t f(s)\,\mathrm{d}s\Big].\end{aligned}$$

It follows from this that $A(\lambda+A)^{-1}\int_0^t S(s)x\,\mathrm{d}s \in \mathsf{D}(A)$, which in turn implies that $\int_0^t S(s)x\,\mathrm{d}s \in \mathsf{D}(A)$. Applying $\lambda + A$ on both sides, we obtain

$$\int_0^t Au(s)\,\mathrm{d}s + A\int_0^t S(s)x\,\mathrm{d}s = -u(t) + S(t)x + \int_0^t f(s)\,\mathrm{d}s.$$

Since $A\int_0^t S(s)x\,\mathrm{d}s = S(t)x - x$, this shows that u is a strong solution. □

The following proposition provides a large class of functions $f : [0, T] \to X$ for which the mild solution $u = S * f$ to (ACP_0) is a strong solution.

Proposition 17.1.4. *Let A be sectorial of angle $< \frac{1}{2}\pi$ and let S be the analytic semigroup generated by $-A$. Then for all $f \in C^\alpha([0, T]; X)$ with $\alpha > 0$, the mild solution $u = S * f$ to (ACP_0) satisfies*

$$u \in C([0, T]; X) \cap L^\infty(0, T; \mathsf{D}(A)).$$

In particular, u is a strong solution to (ACP_0).

Proof. In the discussion following Definition 17.1.2 we have already seen that $u \in C([0, T]; X)$. In the remainder of the proof there is no loss of generality in assuming that $0 < \alpha < 1$. For $t \in [0, T]$ we can write

$$u(t) = \int_0^t S(t-r)(f(r) - f(t))\,\mathrm{d}r + \int_0^t S(t-r)f(t)\,\mathrm{d}s =: u_1(t) + u_2(t).$$

Since by (K.4) we have $\sup_{s\in(0,T]} \|sAS(s)\| < \infty$, we find that $u_1(t) \in \mathsf{D}(A)$ and

$$\begin{aligned}\|Au_1(t)\| &\leqslant \int_0^t \|AS(t-r)(f(r) - f(t))\|\,\mathrm{d}r \\ &\leqslant [f]_{C^\alpha([0,T];X)} \int_0^T r^{\alpha-1}\|AS(r)\|\,\mathrm{d}r.\end{aligned}$$

From Proposition K.1.11 we see that $u_2(t) \in \mathsf{D}(A)$ and $Au_2(t) = (I-S(t))f(t)$, and thus

$$\|Au_2(t)\| \leqslant \sup_{t\in[0,T]} \|I - S(t)\| \sup_{t\in[0,T]} \|f(t)\|.$$

□

17.2 Maximal L^p-regularity

After these preparation on the Cauchy problem, we will now turn to the main topic of the chapter.

17.2.a Definition and basic properties

We now discuss solutions, in an appropriate L^p sense, to the inhomogeneous abstract Cauchy problem (ACP$_x$).

Definition 17.2.1 (L^p-solutions). *A strong solution u to* (ACP$_x$) *associated with a function $f \in L^p(I;X)$ is called an* L^p-solution *if $Au \in L^p(I;X)$.*

To connect this to Definition 17.1.1 of strong solutions, note that functions in $L^p(I;X)$ belong to $L^1_{\rm loc}(\overline{I};X)$. A strong solution on I is an L^1-solution on every bounded subinterval contained in I. From this point onwards, in the context of L^p-solutions *it will always be understood that the inhomogeneities f belong to $L^p(I;X)$* even when this is not explicitly mentioned.

We will now concentrate, for a while, on the special initial value $x = 0$, i.e., the Cauchy problem

$$\begin{cases} u'(t) + Au(t) &= f(t), \quad t \in I, \\ u(0) &= 0. \end{cases} \tag{ACP$_0$}$$

If $(0,T)$ is a bounded interval contained in I and u is an L^p-solution of (ACP$_0$) on I, then by using Hölder's inequality and taking the supremum over $t \in [0,T]$ we obtain

$$\|u\|_{C([0,T];X)} \leqslant T^{1/p'}(\|Au\|_{L^p(0,T;X)} + \|f\|_{L^p(0,T;X)}), \tag{17.5}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. By another application of Hölder's inequality it follows that $u \in L^p(0,T;X)$ and

$$\|u\|_{L^p(0,T;X)} \leqslant T(\|Au\|_{L^p(0,T;X)} + \|f\|_{L^p(0,T;X)}). \tag{17.6}$$

In contrast to (17.6), *an L^p-solution on $I = \mathbb{R}_+$ need not belong to $L^p(\mathbb{R}_+;X)$.* In fact, we will see in Proposition 17.2.8 and Corollary 17.2.25 that the following equivalence holds: (ACP$_0$) admits an L^p-solution $u \in L^p(\mathbb{R}_+;X)$ for every $f \in L^p(\mathbb{R}_+;X)$ if and only if $0 \in \varrho(A)$.

Concrete examples of L^p-solutions that do not belong to $L^p(\mathbb{R}_+;X)$ will be presented in Section 17.4. It is true, however, that if u is an L^p-solution on $\mathbb{R}_+$, then for every $\varepsilon > 0$ the rescaled function $t \mapsto e^{-\varepsilon t}u(t)$ belongs to $L^p(\mathbb{R}_+;X)$. Indeed, by (17.5),

$$\|t \mapsto e^{-\varepsilon t}u(t)\|_{L^p(\mathbb{R}_+;X)} \leqslant \sum_{k\geqslant 1} e^{-\varepsilon(k-1)}\|u\|_{L^p(k-1,k;X)}$$

$$\leqslant \sum_{k\geqslant 1} k^{1/p'} e^{-\varepsilon(k-1)} (\|Au\|_{L^p(\mathbb{R}_+;X)} + \|f\|_{L^p(\mathbb{R}_+;X)}).$$

This proves the claim.

By (17.3) we have $u' + Au = f$ almost everywhere on I and therefore

$$\|u'\|_{L^p(I;X)} \leqslant \|Au\|_{L^p(I;X)} + \|f\|_{L^p(I;X)}. \tag{17.7}$$

As a result we have:

Proposition 17.2.2. *If u is an L^p-solution of* (ACP$_0$) *on I, then*

(1) $u \in {}_0\dot{W}^{1,p}_A(I;X) \subseteq {}_0\dot{W}^{1,p}(I;X)$, *where*

$$\begin{aligned}{}_0\dot{W}^{1,p}(I;X) &:= \{v \in W^{1,p}_{\rm loc}(\bar{I};X) : v' \in L^p(I;X), v(0) = 0\},\\ {}_0\dot{W}^{1,p}_A(I;X) &:= \{v \in {}_0\dot{W}^{1,p}(\bar{I};X) : v(\cdot) \in \mathsf{D}(A) \text{ a.e. on } I, Av \in L^p(I;X)\}\end{aligned}$$

are normed spaces with

$$\begin{aligned}\|v\|_{{}_0\dot{W}^{1,p}(I;X)} &:= \|v'\|_{L^p(I;X)},\\ \|v\|_{{}_0\dot{W}^{1,p}_A(I;X)} &:= \max\big\{\|v'\|_{L^p(I;X)}, \|Av\|_{L^p(I;X)}\big\};\end{aligned}$$

(2) *for all bounded subintervals* $(0,T) \subseteq I$ *we have*

$${}_0\dot{W}^{1,p}_A(I;X) \hookrightarrow L^p(0,T;\mathsf{D}(A)) \cap W^{1,p}(0,T;X)$$

with

$$\max\big\{\|v\|_{L^p(0,T;\mathsf{D}(A))}, \|v\|_{W^{1,p}(0,T;X)}\big\} \leqslant (T+1)\|v\|_{{}_0\dot{W}^{1,p}_A(I;X)}. \tag{17.8}$$

In particular, the L^p-solution u belongs to these intersections.

Proof. (1): By Definitions 17.2.1 and 17.1.1, $u(0) = 0$ and u takes values in $\mathsf{D}(A)$ a.e., and $Au \in L^p(I;X)$. By the discussion following Definition 17.1.1, it also follows that $u \in C([0,T];X) \subseteq L^p([0,T];X)$ for every bounded $[0,T] \subseteq \bar{I}$, as well as $u \in W^{1,1}_{\rm loc}(\bar{I};X)$. From (17.7), we get $u' \in L^p(I;X)$, and thus $u \in {}_0\dot{W}^{1,p}_A(I;X)$, as claimed.

The only axiom of a norm that might not be entirely evident is to check that $\|v\|_{{}_0\dot{W}^{1,p}(I;X)} = 0$ only if $v = 0$. However, by Proposition 2.5.9, every $v \in {}_0\dot{W}^{1,p}(I;X)$ satisfies $v(t) = \int_0^t v'(s)\,\mathrm{d}s$ almost everywhere, and hence indeed $v = 0$ if $\|v\|_{{}_0\dot{W}^{1,p}(I;X)} = \|v'\|_{L^p(I;X)} = 0$.

(2): Recall that $\mathsf{D}(A) \subseteq X$ is normed by $\|x\|_{\mathsf{D}(A)} := \|x\|_X + \|Ax\|_X$. Thus

$$\begin{aligned}\|v\|_{L^p(0,T;\mathsf{D}(A))} &\leqslant \|v\|_{L^p(0,T;X)} + \|Av\|_{L^p(0,T;X)}\\ &\leqslant T\|v'\|_{L^p(I;X)} + \|Av\|_{L^p(I;X)} \leqslant (T+1)\|v\|_{{}_0\dot{W}^{1,p}_A(I;X)}.\end{aligned}$$

Similarly,

$$\begin{aligned}\|v\|_{W^{1,p}(0,T;\mathsf{D}(A))} &\leqslant \|v\|_{L^p(0,T;X)} + \|v'\|_{L^p(0,T;X)}\\ &\leqslant T\|v'\|_{L^p(I;X)} + \|v'\|_{L^p(I;X)} \leqslant (T+1)\|v\|_{{}_0\dot{W}^{1,p}_A(I;X)}.\end{aligned}$$

Thus the maximum of the two left-hand sides has this same upper bound. □

Proposition 17.2.3. *If $A : \mathsf{D}(A) \to X$ is a closed operator, then ${}_0\dot{W}^{1,p}_A(I;X)$ is a Banach space.*

Proof. If $u_n \in {}_0\dot{W}^{1,p}(I;X)$ is a Cauchy sequence, then $u_n' \in L^p(I;X)$ have a limit v in this space by the completeness of $L^p(I;X)$. Then $u(t) := \int_0^t v(s)\,\mathrm{d}s$ belongs to ${}_0\dot{W}^{1,p}(I;X)$, and clearly $u_n \to u$ in ${}_0\dot{W}^{1,p}(I;X)$.

Let then $u_n \in {}_0\dot{W}^{1,p}_A(I;X)$ be a Cauchy sequence, and $u \in {}_0\dot{W}^{1,p}(I;X)$ its limit in this larger space. By Proposition 2.5.9, we have

$$\|u_n - u\|_{C([0,T];X)} \leqslant T^{1/p'}\|u_n' - u'\|_{L^p(I;X)} = T^{1/p'}\|u_n - u\|_{{}_0\dot{W}^{1,p}(I;X)}$$

for every bounded interval $[0,T] \subseteq \bar{I}$. Hence $u_n(t) \to u(t)$ for $t \in [0,T]$ and, $[0,T] \subseteq \bar{I}$ being arbitrary, for $t \in I$. (Here we are thinking of continuous versions of the Sobolev functions in question.) On the other hand, $Au_n \in L^p(I;X)$ is a Cauchy sequence, hence convergent to some limit $w \in L^p(I;X)$, and thus a subsequence converges almost everywhere. If $t \in I$ is such a point, then $u_n(t) \to u(t)$ while $Au_n(t) \to w(t)$. Since A is closed, it follows that $u(t) \in \mathsf{D}(A)$ and $Au = w \in L^p(I;X)$. This shows that the limit $u \in {}_0\dot{W}^{1,p}(I;X)$ in fact belongs to ${}_0\dot{W}^{1,p}_A(I;X)$. Since $u_n \to u$ in ${}_0\dot{W}^{1,p}(I;X)$ and $Au_n \to w = Au$ in $L^p(I;X)$, it follows that $u_n \to u \in {}_0\dot{W}^{1,p}_A(I;X)$. □

We now come to the principal notion to be studied in this chapter. *Here, and in the remainder of this subsection, we assume that $1 \leqslant p \leqslant \infty$ unless explicitly stated otherwise.*

Definition 17.2.4 (Maximal L^p-regularity). *A linear operator A has* maximal L^p-regularity *on I if there exists a constant $C \geqslant 0$ such that for all $f \in L^p(I;X)$ problem (ACP$_0$) admits a unique L^p-solution u_f on I and*

$$\|Au_f\|_{L^p(I;X)} \leqslant C\|f\|_{L^p(I;X)}. \tag{17.9}$$

The least admissible constant in this definition will be called the *maximal L^p-regularity constant* of A on I and will be denoted by $M^{\mathrm{reg}}_{p,A}(I)$.

If A has maximal L^p-regularity on I, then for bounded subintervals $(0,T) \subseteq I$, the estimate (17.5) implies that if u_f is an L^p-solution of (ACP$_0$) with $f \in L^p(I;X)$, then its restriction to $(0,T)$ satisfies

$$\|u_f\|_{L^p(0,T;X)} \leqslant T^{1/p}\|u_f\|_{C([0,T];X)} \leqslant T(M^{\mathrm{reg}}_{p,A}(I) + 1)\|f\|_{L^p(I;X)}. \tag{17.10}$$

Moreover, (17.7) implies that if u is a L^p-solution to (ACP$_0$) Unfortunately, we cannot yet conclude that A has maximal L^p-regularity on $(0,T) \subseteq I$, because uniqueness of the solution on $(0,T)$ is unclear. We come back to this issue in Lemma 17.2.16.

Let us immediately observe that maximal L^p-regularity implies closedness:

Proposition 17.2.5 (Closedness). *If a linear operator A has maximal L^p-regularity on I, then A is a closed operator.*

Proof. Let $x_n \to x$ in X, with $x_n \in \mathsf{D}(A)$, and $Ax_n \to y$ in X. Choose a non-zero non-negative function $\phi \in C^1_{\rm c}(I)$ and put $u_n := \phi \otimes x_n$ and $u := \phi \otimes x$. Set $f_n := u_n' + Au_n = \phi' \otimes x_n + \phi \otimes Ax_n$ and $f := \phi' \otimes x + \phi \otimes y$. Then u_n is the unique L^p-solution associated with f_n. Let v be the unique L^p-solution associated with f. Since $f_n \to f$ in $L^p(I;X)$, it follows from (17.10) (applied with $f_n - f$) that $u_n \to v$ in $C([0,T];X)$ for all $[0,T] \subseteq \overline{I}$. Since $u_n \to u$ in $C_{\rm b}(\overline{I};X)$ it follows that $u = v$. Therefore u is an L^p-solution associated with f. This forces $x \in \mathsf{D}(A)$, which implies that $\phi \otimes Ax = Au = f - u' = \phi \otimes y$, so $Ax = y$. This proves that A is closed. □

Corollary 17.2.6. *Let A be a linear operator with maximal L^p-regularity on I. For $f \in L^p(I;X)$ let u_f denote the corresponding L^p-solution in $L^p(I;\mathsf{D}(A)) \cap W^{1,p}(I;X)$. Then the mapping*

$$\mathscr{M} : L^p(I;X) \to {}_0\dot{W}^{1,p}_A(I;X), \quad \mathscr{M}f = u_f,$$

is an isomorphism with

$$\frac{1}{2}\|f\|_{L^p(I;X)} \leqslant \|\mathscr{M}f\|_{{}_0\dot{W}^{1,p}_A(I;X)} \leqslant (M^{\rm reg}_{p,A}(I) + 1)\|f\|_{L^p(I;X)}$$

Proof. If A has maximal L^p-regularity on I, then (17.7) implies that

$$\|u_f'\|_{L^p(I;X)} \leqslant (M^{\rm reg}_{p,A}(I) + 1)\|f\|_{L^p(I;X)}, \tag{17.11}$$

and hence, in combination with (17.9), that

$$\|u_f\|_{{}_0\dot{W}^{1,p}_A(I;X)} \leqslant (M^{\rm reg}_{p,A}(I) + 1)\|f\|_{L^p(I;X)}. \tag{17.12}$$

The existence and uniqueness of the L^p-solution u_f implies that the mapping $f \mapsto u_f$ is linear from $L^p(I;X)$ into ${}_0\dot{W}^{1,p}_A(I;X)$.

Conversely, given $u \in {}_0\dot{W}^{1,p}_A(I;X)$, we have $f := u' + Au \in L^p(I;X)$ with

$$\|f\|_{L^p(I;X)} \leqslant \|u'\|_{L^p(I;X)} + \|Au\|_{L^p(I;X)} \leqslant 2\|u\|_{{}_0\dot{W}^{1,p}_A(I;X)},$$

and it is evident that u is an L^p-solution of (ACP$_0$) with datum f. Thus, by uniqueness, $u = u_f$ is in the range of $\mathscr{M}$, so this operator is onto. □

We recall from Section C that for a Banach couple (X,Y) the intersection $X \cap Y$ is a Banach space with respect to the norm defined by

$$\|x\|_{X\cap Y} := \max\{\|x\|_X, \|x\|_Y\}.$$

Combining the estimate in the definition of maximal L^p-regularity with (17.11) and (17.10) we obtain:

Proposition 17.2.7. *Suppose that A has maximal L^p-regularity on I. If u is the unique L^p-solution on I of (ACP$_0$), then for all bounded subintervals $(0,T) \subseteq I$ we have $u \in L^p(0,T;\mathsf{D}(A)) \cap W^{1,p}(0,T;X)$ and*

$$\|u\|_{L^p(0,T;\mathsf{D}(A))\cap W^{1,p}(0,T;X)} \leqslant (1+T)(M_{p,A}^{\mathrm{reg}}(I)+1)\|f\|_{L^p(I;X)}.$$

Moreover, if $I=(0,T)$, *then*

$$_0\dot{W}_A^{1,p}(I;X) \simeq L^p(0,T;\mathsf{D}(A)) \cap W^{1,p}(0,T;X)$$

and for all v *in this space, we have*

$$\|v\|_{_0\dot{W}_A^{1,p}(I;X)} \leqslant \|v\|_{L^p(0,T;\mathsf{D}(A))\cap W^{1,p}(0,T;X)} \leqslant (T+1)\|v\|_{_0\dot{W}_A^{1,p}(I;X)}.$$

Proof. This first bound is immediate by combining (17.8) and (17.12). For the comparison of the spaces, the first estimate is clear, and the second one is (17.8). □

The next result gives another sufficient condition, also allowing infinite intervals I, that L^p-solutions u do in fact belong to $L^p(I;X)$.

Proposition 17.2.8. *Let* A *be a linear operator with maximal* L^p*-regularity on* I. *If* $0 \in \varrho(A)$, *then*

$$_0\dot{W}_A^{1,p}(I;X) \simeq L^p(I;\mathsf{D}(A)) \cap {}_0W^{1,p}(I;X)$$

and, for all u *in this space,*

$$\|u\|_{_0\dot{W}_A^{1,p}(I;X)} \leqslant \|u\|_{L^p(I;\mathsf{D}(A))\cap {}_0W^{1,p}(I;X)} \leqslant (\|A^{-1}\|+1)\|u\|_{_0\dot{W}_A^{1,p}(I;X)}.$$

In this situation, for all $f \in L^p(I;X)$, *the unique* L^p*-solution* u_f *to* (ACP$_0$) *belongs to this space and satisfies*

$$\|u_f\|_{L^p(I;\mathsf{D}(A))\cap W^{1,p}(I;X)} \leqslant (\|A^{-1}\|+1)(M_{p,A}^{\mathrm{reg}}(I)+1)\|f\|_{L^p(I;X)}.$$

Proof. The boundary value $v(0)=0$ is assumed in both spaces, and the first norm estimate is evident. The second estimate follows by taking the maximum of

$$\begin{aligned}
\|u\|_{W^{1,p}(\mathbb{R}_+;X)} &\leqslant \|u\|_{L^p(\mathbb{R}_+;X)} + \|u'\|_{L^p(\mathbb{R}_+;X)} \\
&\leqslant \|A^{-1}\|\,\|Au\|_{L^p(\mathbb{R}_+;X)} + \|u'\|_{L^p(\mathbb{R}_+;X)} \\
&\leqslant (\|A^{-1}\|+1)\|u\|_{_0\dot{W}_A^{1,p}(\mathbb{R}_+;X)}
\end{aligned}$$

and

$$\begin{aligned}
\|u\|_{L^p(\mathbb{R}_+;\mathsf{D}(A))} &\leqslant \|u\|_{L^p(\mathbb{R}_+;X)} + \|Au\|_{L^p(\mathbb{R}_+;X)} \\
&\leqslant (\|A^{-1}\|+1)\|Au\|_{L^p(\mathbb{R}_+;X)} \leqslant (\|A^{-1}\|+1)\|u\|_{_0\dot{W}_A^{1,p}(\mathbb{R}_+;X)}.
\end{aligned}$$

From these bounds and (17.12), the final claim concerning u_f follows. □

Corollary 17.2.9. *Let* A *be a linear operator with maximal* L^p*-regularity on* I, *where we assume that at least one of the following conditions holds:*

- *$I = (0,T)$ is bounded;*
- *$0 \in \varrho(A)$.*

For $f \in L^p(I;X)$ let u_f denote the corresponding L^p-solution in $L^p(I;\mathsf{D}(A)) \cap W^{1,p}(I;X)$. Then

$$ {}_0\dot{W}^{1,p}_A(I;X) \simeq L^p(I;\mathsf{D}(A)) \cap {}_0W^{1,p}(I;X); $$

for any v in this space, we have

$$ \|v\|_{{}_0\dot{W}^{1,p}_A(I;X)} \leqslant \|v\|_{L^p(I;\mathsf{D}(A))\cap W^{1,p}(I;X)} \leqslant (\min\{\|A^{-1}\|, T\} + 1)\|v\|_{{}_0\dot{W}^{1,p}_A(I;X)}; $$

and $\mathscr{M} : f \mapsto u_f$ is an isomorphism from $L^p(I;X)$ into this space.

Proof. This follows by combining Corollary 17.2.6 with either Proposition 17.2.7 for $I = (0,T)$, or Proposition 17.2.8 for $0 \in \varrho(A)$. (If $I = (0,T)$ *and* $0 \in \varrho(A)$, we could apply either Proposition 17.2.7 or Proposition 17.2.8 for the same qualitative conclusion, but one or the other may give a better quantitative bound, depending on the relative size of T and $\|A^{-1}\|$.) □

By Proposition 17.2.5, maximal L^p-regularity implies closedness. If the operator A is assumed closed to begin with, the next two results provide two different ways of apparently relaxing Definition 17.2.4, yet actually defining the same property in the end. Namely, we may either require

- the existence of a unique L^p-solution with estimate (17.9) only on a dense subspaces of functions $f \in L^p(I;X)$ (Proposition 17.2.10), or
- the existence of a unique L^p-solution for all $f \in L^p(I;X)$, but without postulating any estimate on their size (Proposition 17.2.11).

Proposition 17.2.10. *Let A be a closed linear operator, and let F be a dense subspace of $L^p(I;X)$. Suppose that for all $f \in F$ there exists a unique L^p-solution u_f to* (ACP$_0$)*, and that this solution satisfies* (17.9)*, that is,*

$$ \|Au_f\|_{L^p(I;X)} \leqslant C\|f\|_{L^p(I;X)}, $$

with a constant C independent of $f \in F$. Then A has maximal L^p-regularity on I with $M^{\mathrm{reg}}_{p,A}(I) \leqslant C$.

Proof. Let $f \in L^p(I;X)$ be arbitrary, and suppose that u_1 and u_2 are L^p-solutions to (ACP$_0$) on I. Then $u = u_1 - u_2$ is an L^p-solution to the homogeneous problem $u' + Au = 0$ on I and $u(0) = 0$. Therefore, the uniqueness assumption gives that $u = 0$, and therefore $u_1 = u_2$.

Let again $f \in L^p(I;X)$ be arbitrary, choose functions $f_n \in F$ converging to f in $L^p(I;X)$, and denote the unique L^p-solutions associated with f_n by u_n. By the assumed estimate, the functions Au_n form a Cauchy sequence in $L^p(I;X)$ and therefore converge to a limit v in $L^p(I;X)$. By (17.10) the functions u_n converge uniformly on every bounded interval $[0,T]$ contained in $\overline{I}$ to a continuous function u on $[0,T]$. As a result, by taking limits in the definition of a strong solution we obtain

$$u(t) + \int_0^t v(s)\,\mathrm{d}s = \int_0^t f(s)\,\mathrm{d}s.$$

Since A is closed, a standard subsequence argument furthermore gives that v takes values in $\mathsf{D}(A)$ almost surely and $v = Au$ in $L^p(I;X)$. It follows that u is an L^p-solution of (ACP$_0$). Moreover, taking limits we also see that $\|Au\|_{L^p(I;X)} \leqslant C\|f\|_{L^p(I;X)}$. □

Proposition 17.2.11. *If A is closed and for all $f \in L^p(I;X)$ there exists a unique L^p-solution u_f of (ACP$_0$), then for all $f \in L^p(I;X)$ the estimate (17.9) holds, that is,*

$$\|Au_f\|_{L^p(I;X)} \leqslant C\|f\|_{L^p(I;X)},$$

In particular, A has maximal regularity on I.

Proof. By Proposition 17.2.2, we have $u_f \in {}_0\dot{W}^{1,p}_A(I;X)$. The existence and uniqueness of u_f for every $f \in L^p(I;X)$ guarantees that $\mathscr{M} : f \mapsto u_f$ is a linear mapping from $L^p(I;X)$ to ${}_0\dot{W}^{1,p}_A(I;X)$. Since A is closed, ${}_0\dot{W}^{1,p}_A(I;X)$ is a Banach space by Proposition 17.2.3.

To prove that $\mathscr{M}$ is bounded, by the closed graph theorem, it suffices to check that it is closed. To this end, let $f_n \to f$ in $L^p(I;X)$ and $u_n := \mathscr{M} f_n \to v$ in ${}_0\dot{W}^{1,p}_A(I;X)$. Since $u_n' + Au_n = f_n$, taking limits gives $v' + Av = f$. It follows that v is an L^p-solution corresponding to f. The uniqueness of L^p-solutions therefore gives $v = \mathscr{M} f$.

Thus $\mathscr{M}$ is closed, hence bounded, and therefore

$$\|Au_f\|_{L^p(I;X)} \leqslant \|u_f\|_{{}_0\dot{W}^{1,p}_A(I;X)} \leqslant C\|f\|_{L^p(I;X)},$$

which is the claimed maximal regularity estimate. □

We end this section by showing a commutation relation between A and the solution operator. It will only be presented in the case $I = \mathbb{R}_+$, because that is the case we need in the proof of Theorem 17.2.15 below. It also holds for bounded intervals; this can easily be deduced from Theorem 17.2.19.

Lemma 17.2.12. *Suppose that A has maximal L^p-regularity on $\mathbb{R}_+$, and denote by $\mathscr{M} : f \mapsto u_f$ the solution operator that assigns to a function $f \in L^p(\mathbb{R}_+;X)$ the corresponding L^p-solution u_f of (ACP$_0$). Then for every $f \in L^p(\mathbb{R}_+;\mathsf{D}(A))$ one has $\mathscr{M} f \in L^p(\mathbb{R}_+;\mathsf{D}(A))$ and*

$$A\mathscr{M} f = \mathscr{M} A f$$

as functions in $L^p(\mathbb{R}_+;X)$.

Here, for functions $g \in L^p(\mathbb{R}_+; \mathsf{D}(A))$ we define $Ag \in L^p(\mathbb{R}_+; X)$ pointwise almost everywhere by $(Ag)(t) := A(g(t))$.

Proof. Let $D : {}_0\dot{W}^{1,p}(I;X) \to L^p(I;X)$ be given by $Df := f'$ and $C : L^p(I;X) \to {}_0\dot{W}^{1,p}(I;X)$ by

$$Cf(t) := \int_0^t e^{-s} f(t-s)\,\mathrm{d}s = \int_0^t e^{-(t-s)} f(s)\,\mathrm{d}s.$$

This is well defined, since by Young's inequality (see Lemma 14.2.3)

$$\begin{aligned}\|Cf\|_{L^p(I;X)} &\leqslant \Big\| t \mapsto \int_0^t e^{-s}(\mathbf{1}_I f)(t-s)\,\mathrm{d}s \Big\|_{L^p(\mathbb{R}_+;X)} \\ &\leqslant \|s \mapsto e^{-s}\|_{L^1(\mathbb{R}_+)} \|f\|_{L^p(I;X)} = \|f\|_{L^p(I;X)}.\end{aligned}$$

Moreover, for all $f \in L^p(I;x)$ we have

$$(I+D)Cf = f.$$

Indeed, for $f \in L^p(I;X)$ the product rule and an integration by parts give

$$\begin{aligned}DCf(t) &= \partial_t \Big[e^{-t} \int_0^t e^s f(s)\,\mathrm{d}s \Big] \\ &= -e^{-t} \int_0^t e^s f(s)\,\mathrm{d}s + f(t) = -Cf(t) + f(t).\end{aligned}$$

Similarly, $C(I+D)f = f$ for $f \in {}_0\dot{W}^{1,p}(X)$ by integration by parts. Therefore, $C = (I+D)^{-1}$.

Recall that $\mathscr{M}$ is bounded as an operator from $L^p(I;X)$ to ${}_0\dot{W}^{1,p}_A(I;X)$. Before we give the final argument we state some identities which will be needed. A key identity is

$$(A+D)\mathscr{M} g = g = \mathscr{M}(A+D)g, \quad g \in {}_0\dot{W}^{1,p}_A(I;X),$$

where the first identity (valid more generally for all $g \in L^p(I;X)$) is a restatement of the equation that a solution $\mathscr{M} g$ with datum g must satisfy, and the second one follows by noting that $g \in {}_0\dot{W}^{1,p}_A(I;X)$ has the correct boundary value $g(0) = 0$ by definition, and evidently solves the equation with datum $(A+D)g$, so it must be the L^p-solution by uniqueness. Therefore, if we can prove that $\mathscr{M}$ and D commute in a suitable sense, the desired result will follow.

First observe that, by Hille's theorem (Theorem 1.2.4),

$$A(I+D)^{-1} f = (I+D)^{-1} Af, \quad f \in L^1_{\mathrm{loc}}(\bar{I}; \mathsf{D}(A)).$$

Next we check that $B\mathscr{M} f = \mathscr{M} Bf$ for $f \in L^p(\mathbb{R}_+;X) \subseteq E(X)$ and for $B \in \{(I+D)^{-1}, D(I+D)^{-1}\}$. Clearly, it suffices to consider the case $B = (I+D)^{-1}$. Let $u := \mathscr{M} f$ and $v := (I+D)^{-1}u$. Then,

$$\begin{aligned} v'(t) + Av(t) &= \int_0^t e^{-s} u'(t-s)\,\mathrm{d}s + \int_0^t e^{-s} Au(t-s)\,\mathrm{d}s \\ &= \int_0^t e^{-s} f(t-s)\,\mathrm{d}s = (I+D)^{-1} f(t), \end{aligned}$$

from which it follows that $v = \mathscr{M}(I+D)^{-1} f$. This gives the required result.

With these preparations out of the way, we can turn to the proof of the lemma. It remains to show that for $f \in L^p(\mathbb{R}_+; \mathsf{D}(A))$ one has $\mathscr{M} A f = A \mathscr{M} f$. By the previous observations,

$$\begin{aligned} (1+D)^{-1} \mathscr{M} A f &= \mathscr{M}(1+D)^{-1} A f \\ &= \mathscr{M} A (1+D)^{-1} f \\ &= \mathscr{M}(A+D)(1+D)^{-1} f - \mathscr{M} D (1+D)^{-1} f \\ &= (1+D)^{-1} f - D(1+D)^{-1} \mathscr{M} f \\ &= (1+D)^{-1}(f - D\mathscr{M} f) \\ &= (1+D)^{-1}(A\mathscr{M} f). \end{aligned}$$

The required identity is obtained by applying $(1+D)$ on both sides. □

17.2.b The initial value problem

Up to this point we have considered existence and uniqueness of L^p-solutions for the inhomogeneous problem (ACP_0) with zero initial condition. As we will show presently, by using maximal L^p-regularity it is possible to obtain necessary and sufficient conditions for the existence and uniqueness of L^p-solutions for the problem (ACP_x) with non-zero initial conditions,

$$\begin{cases} u'(t) + Au(t) &= f(t), \quad t \in I, \\ u(0) &= x, \end{cases}$$

where $f \in L^1_{\mathrm{loc}}(\overline{I}; X)$ and $x \in X$; as before, I is either a bounded interval $(0,T)$ or $\mathbb{R}_+ = (0,\infty)$. An obvious necessary condition for the existence of an L^p-solution u is that f should be in $L^p(I;X)$; this is immediate from the definition of an L^p-solution, which imply that u' and Au belong to $L^p(I;X)$, and we then have the trivial bound

$$\|f\|_{L^p(I;X)} \leqslant \|u'\|_{L^p(I;X)} + \|Au\|_{L^p(I;X)}.$$

In view of this observation, *in what follows it will always be assumed that* $f \in L^p(I;X)$ *even when this is not stated explicitly. We also maintain the standing assumption* $1 \leqslant p \leqslant \infty$ *which is in force unless explicitly stated otherwise.*

We begin with a necessary condition on the initial condition x for the existence of an L^p-solution. As a preliminary observation we note that as in Proposition 17.2.2 one shows that if u is an L^p-solution to (ACP$_x$), then $u \in L^p(0,T;\mathsf{D}(A)) \cap W^{1,p}(0,T;X)$ for all bounded intervals $(0,T) \subseteq I$.

In the next two results we interpret $(X,\mathsf{D}(A))_{0,\infty}$ and $(X,\mathsf{D}(A))_{1,\infty}$ as X and $\mathsf{D}(A)$, respectively. It will be necessary to use the equivalent norms

$$\|\cdot\|_{(X,\mathsf{D}(A))_{1-\frac{1}{p},p}} \eqsim \|\cdot\|^{\mathrm{Tr}}_{(X,\mathsf{D}(A))_{1-\frac{1}{p},p}}$$

of Theorem L.2.3.

Proposition 17.2.13 (Necessary condition on the initial condition). *Let $1 \leqslant p < \infty$. If for some $f \in L^p(I;X)$, the problem (ACP$_x$) admits an L^p-solution on I, then $x \in (X,\mathsf{D}(A))_{1-\frac{1}{p},p}$. If $(0,T) \subseteq I$ is a bounded interval, then*

$$\|x\|^{\mathrm{Tr}}_{(X,\mathsf{D}(A))_{1-\frac{1}{p},p}} \leqslant (1+1/T)\max\big\{\|u\|_{L^p(0,T;\mathsf{D}(A))}, \|u\|_{W^{1,p}(0,T;X)}\big\};$$

if $I = \mathbb{R}_+$ and $u \in L^p(\mathbb{R}_+;X)$, then

$$\|x\|^{\mathrm{Tr}}_{(X,\mathsf{D}(A))_{1-\frac{1}{p},p}} \leqslant \max\big\{\|u\|_{L^p(\mathbb{R}_+;\mathsf{D}(A))},\ \|u'\|_{L^p(\mathbb{R}_+;X)}\big\}.$$

Proof. We will prove that $x \in (X,\mathsf{D}(A))^{\mathrm{Tr}}_{1-\frac{1}{p},p}$, and make the estimates related to the corresponding norm. That $x \in (X,\mathsf{D}(A))_{1-\frac{1}{p},p}$ is then immediate using the equivalence of the spaces from Theorem L.2.3.

By Definition L.2.1 (and the discussion after it), this involves identifying x with the initial value $x = v(0)$ of some $v \in W^{1,1}_{\mathrm{loc}}([0,\infty);X)$ (now that $X_0 = X$ and $X_1 = \mathsf{D}(A)$, the sum space is simple $X_0 + X_1 = X$) such that

$$\|t \mapsto t^{1-\frac{1}{p}} v'(t)\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)} = \|v'\|_{L^p(\mathbb{R}_+;X)}$$

and

$$\|t \mapsto t^{1-\frac{1}{p}} v(t)\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};\mathsf{D}(A))} = \|v\|_{L^p(\mathbb{R}_+;\mathsf{D}(A))}$$

are finite. By assumption, we know that $x = u(0)$, where $u = u_f \in W^{1,p}_{\mathrm{loc}}(\bar{I};X)$; however, it might be that I is only a finite interval, and even if $I = \mathbb{R}_+$, the norm $\|u\|_{L^p(\mathbb{R}_+;X)} \leqslant \|u\|_{L^p(\mathbb{R}_+;\mathsf{D}(A))}$ need not be finite.

To fix both problems at once, let $(0,T) \subseteq I$ be finite, and let $\phi(t) := (1-t/T)_+$. Then $v(t) := \phi(t)u(t)$ is defined on all $\mathbb{R}_+$, and it still satisfies $v(0) = u(0) = x$. Since $v' = \phi' u + \phi u'$, where $|\phi'| \leqslant \frac{1}{T}\mathbf{1}_{(0,T)}$ and $|\phi| \leqslant \mathbf{1}_{(0,T)}$, we obtain

$$\begin{aligned}\|v\|_{L^p(\mathbb{R}_+;\mathsf{D}(A))} &\leqslant \|u\|_{L^p(0,T;\mathsf{D}(A))}\\ &\leqslant \|u\|_{L^p(0,T;X)} + \|Au\|_{L^p(0,T;X)}\end{aligned}$$

and

$$\|v'\|_{L^p(\mathbb{R}_+;X)} \leqslant \frac{1}{T}\|u\|_{L^p(0,T;X)} + \|u'\|_{L^p(0,T;X)} \leqslant \Big(\frac{1}{T}+1\Big)\|u\|_{W^{1,p}(0,T;X)}$$

The L^p-norms of Au and u' above are clearly finite for an L^p-solution to (ACP$_x$), and so is the L^p-norm of u on a finite $(0,T) \subseteq I$ by

$$\begin{aligned}\|u\|_{L^p(0,T;X)} &= \Big\|t \mapsto x + \int_0^t u'(s)\,\mathrm{d}s\Big\|_{L^p(0,T;X)} \\ &\leqslant T^{1/p}\|x\|_X + T\|u'\|_{L^p(0,T;X)}.\end{aligned}$$

Thus we have checked that $x \in (X, \mathsf{D}(A))^{\mathrm{Tr}}_{1-\frac{1}{p},p}$. Referring to Definition L.2.1 again, we also have the bound

$$\|x\|_{(X,\mathsf{D}(A))^{\mathrm{Tr}}_{1-\frac{1}{p},p}} \leqslant \max\{\|v'\|_{L^p(\mathbb{R}_+;X)}, \|v\|_{L^p(\mathbb{R}_+;\mathsf{D}(A))}\}.$$

The claimed bound dealing with a bounded $(0,T) \subseteq I$ follows from this at once. The case of $I = \mathbb{R}_+$ follows by taking the limit $T \to \infty$, noting that the term $\frac{1}{T}\|u\|_{L^p(0,T;X)} \to 0$ in this case. □

Proposition 17.2.14 (Sufficient condition on the initial condition). *If A has maximal L^p-regularity on I, then for all $f \in L^p(I;X)$ and $x \in (X, \mathsf{D}(A))_{1-\frac{1}{p},p}$ the problem (ACP$_x$) admits a unique L^p-solution u. Moreover, for $1 < p \leqslant \infty$,*

$$\|Au\|_{L^p(I;X)} \leqslant 3M^{\mathrm{reg}}_{p,A}(I)\|x\|^{\mathrm{Tr}}_{(X,\mathsf{D}(A))_{1-\frac{1}{p},p}} + M^{\mathrm{reg}}_{p,A}(I)\|f\|_{L^p(I;X)}.$$

For $p = 1$, the constant 3 needs to be replaced by a constant depending on A.

We first prove this for $1 < p \leqslant \infty$; the case $p = 1$ will be established below Lemma 17.2.22; there, also the precise estimate for $\|Au\|_{L^1(I;X)}$ is stated (as it turns out, the constant 3 gets replaced by the supremum over I of the norms of the semigroup generated by $-A$).

Proof for $1 < p \leqslant \infty$. Uniqueness of L^p-solutions follows from maximal L^p-regularity. For the proof of existence we first consider the case $1 < p < \infty$. By Theorem L.2.3, for every $\varepsilon > 0$ we can find a function $g \in L^p(\mathbb{R}_+; \mathsf{D}(A)) \cap W^{1,p}(\mathbb{R}_+; X)$ such that $g(0) = x$ and

$$\max\{\|g\|_{L^p(\mathbb{R}_+;\mathsf{D}(A))}, \|g\|_{W^{1,p}(\mathbb{R}_+;X)}\} \leqslant (1+\varepsilon)\|x\|^{\mathrm{Tr}}_{(X,\mathsf{D}(A))_{1-\frac{1}{p},p}}.$$

Let v be the unique L^p-solution of the problem

$$v' + Av = f + g' + Ag, \qquad v(0) = 0.$$

The idea of this construction is that we use g to reduce matters to a problem with zero initial condition, which has been studied in the previous section. By maximal L^p-regularity, the above problem has a unique L^p-solution v, and it satisfies

$$\begin{aligned}\|Av\|_{L^p(I;X)} &\leqslant M^{\rm reg}_{p,A}(I)\|f+g'+Ag\|_{L^p(I;X)}\\ &\leqslant M^{\rm reg}_{p,A}(I)\|f\|_{L^p(I;X)} + 2(1+\varepsilon)M^{\rm reg}_{p,A}(I)\|x\|^{\rm Tr}_{(X,\mathsf{D}(A))_{1-\frac1p,p}}.\end{aligned}$$

Now $u := v+g$ is an L^p-solution to (ACP$_x$), and it satisfies

$$\begin{aligned}\|Au\|_{L^p(I;X)} &\leqslant \|Av\|_{L^p(I;X)} + \|Ag\|_{L^p(I;X)}\\ &\leqslant M^{\rm reg}_{p,A}(I)\|f\|_{L^p(I;X)} + 3(1+\varepsilon)M^{\rm reg}_{p,A}(I)\|x\|^{\rm Tr}_{(X,\mathsf{D}(A))_{1-\frac1p,p}}.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this proves the result.

If $p=\infty$, the above argument can be repeated with $g \equiv x$. □

By combining the necessary and sufficient conditions on the data x and f, for bounded intervals I we obtain that maximal L^p-regularity implies the following norm isomorphism between the data (x,f) on the on hand, and the solution $u = u_{x,f}$ on the other hand, embodied by the mapping

$$X_{1-\frac1p,\frac1p} \times L^p(I;X) \to L^p(I;\mathsf{D}(A)) \cap W^{1,p}(I;X)$$

which maps the data (x,f) to the solution u; this remains correct for $I=\mathbb{R}_+$ if $0 \in \varrho(A)$. This extends Corollary 17.2.9 to the case of non-zero initial data.

17.2.c The role of semigroups

The aim of this section is to prove that if A has maximal L^p-regularity on a bounded interval $(0,T)$, then $-A$ generates an analytic semigroup on X, and that this semigroup is bounded analytic if A has maximal L^p-regularity on $\mathbb{R}_+$. This will enable us to characterise maximal regularity in terms of the boundedness of a convolution operator related to mild solutions.

Maximal regularity implies sectoriality

The main result of this paragraph reads as follows.

Theorem 17.2.15 (Dore). *Let A be a linear operator on a Banach space X, and let $1 \leqslant p \leqslant \infty$ be fixed. Then:*

(1) *if A has maximal L^p-regularity on a bounded interval $(0,T)$, then $-A$ generates an analytic semigroup on X, and $\lambda + A$ is sectorial of angle $< \pi/2$ for all $\lambda \in \mathbb{R}$ large enough. Moreover, for $\Re\lambda$ large enough,*

$$\|A(\lambda+A)^{-1}\| \leqslant 2M^{\rm reg}_{p,A}(\mathbb{R}_+). \tag{17.13}$$

(2) *if A has maximal L^p-regularity on $\mathbb{R}_+$, then $-A$ generates a bounded analytic semigroup on X, and A is sectorial of angle $< \pi/2$. Moreover, for all $\Re\lambda > 0$,*

$$\|A(\lambda + A)^{-1}\| \leqslant M^{\mathrm{reg}}_{p,A}(\mathbb{R}_+). \tag{17.14}$$

In Theorem 17.3.1 we will prove an R-bounded version of (17.14) for exponents $p \in [1, \infty)$.

Proof of Theorem 17.2.15(1). Proposition 17.2.5 shows that A is closed. Let $I = (0, T)$ and denote by

$$\mathscr{M}_I : f \mapsto u_f$$

the operator that assigns to a function $f \in L^p(I; X)$ the corresponding L^p-solution $u_f \in L^p(I; X)$ to the problem (ACP$_0$) with zero initial condition.

Step 1 – Our first aim is to prove the injectivity of $\lambda + A$ for $\Re\lambda > 0$ large enough. Fix $x \in \mathsf{D}(A)$ and $\Re\lambda > 0$ and set

$$\begin{aligned} u(t) &:= e^{\lambda t}x - x, \\ f(t) &:= u'(t) + Au(t) = (\lambda + A)e^{\lambda t}x - Ax. \end{aligned}$$

Since $f \in L^p(I; X)$ and $u = u_f$, by maximal L^p-regularity we have

$$\|Au\|_{L^p(I;X)} \leqslant C\|f\|_{L^p(I;X)},$$

where $C = M^{\mathrm{reg}}_{p,A}(I)$. Substituting the definitions of f and u, this implies

$$\|e^{\lambda(\cdot)} - 1\|_{L^p(I)}\|Ax\| \leqslant C\big[\|e^{\lambda(\cdot)}\|_{L^p(I)}\|(\lambda + A)x\| + T^{1/p}\|Ax\|\big]. \tag{17.15}$$

By scaling and using $\|e^{\lambda(\cdot)} - 1\|_{L^p(I)} \geqslant \|e^{\lambda(\cdot)}\|_{L^p(I)} - \|1\|_{L^p(I)}$, we find that

$$(\|e^{\lambda T(\cdot)}\|_{L^p(0,1)} - 1)\|Ax\| \leqslant C\big[\|e^{\lambda T(\cdot)}\|_{L^p(0,1)}\|(\lambda + A)x\| + \|Ax\|\big]. \tag{17.16}$$

Let $r_0 > 0$ be such that $\|e^{r_0(\cdot)}\|_{L^p(0,1)} - 1 \geqslant 2C$. For $\Re\lambda \geqslant r_0/T$, we obtain

$$\|Ax\| \leqslant \|e^{\lambda T(\cdot)}\|_{L^p(0,1)}\|(\lambda + A)x\|.$$

Now if $(\lambda + A)x = 0$, it follows that $Ax = 0$, $\lambda x = 0$, and thus $x = 0$, which proves the injectivity of $\lambda + A$. This proves the asserted injectivity of $\lambda + A$ for $\Re\lambda \geqslant r_0/T$. To arrive at this conclusion, the detour via (17.16) is not necessary, as we could have directly worked with (17.15), but (17.16) will be useful later.

Step 2 – Next we construct a right-inverse of $\lambda + A$ for $\Re\lambda > 0$ sufficiently large. Fix $\lambda \in \mathbb{C}$ with $\Re\lambda > 0$ and define $f_\lambda \in L^p(I)$ by $f_\lambda(t) := e^{-\overline{\lambda}t}$. For all $x \in X$ we have

$$\|f_\lambda \otimes x\|_{L^p(I;X)} \leqslant \frac{1}{(\Re\lambda)^{1/p}}\|x\| \tag{17.17}$$

Put

$$R_\lambda x := 2\Re\lambda \int_I e^{-\lambda t} \mathscr{M}_I(f_\lambda \otimes x)(t)\,\mathrm{d}t. \tag{17.18}$$

By maximal L^p-regularity, the function $\mathscr{M}_I(f_\lambda \otimes x)$ belongs to $L^p(I;\mathsf{D}(A))$. Interpreting the integral in (17.18) as a Bochner integral in $\mathsf{D}(A)$ and viewing A as a bounded operator from $\mathsf{D}(A)$ to X, we find that $R_\lambda x \in \mathsf{D}(A)$ and

$$AR_\lambda x = 2\Re\lambda \int_I e^{-\lambda t} A\mathscr{M}_I(f_\lambda \otimes x)(t)\,\mathrm{d}t. \tag{17.19}$$

By Hölder's inequality, (17.9), and (17.17),

$$\begin{aligned}\|AR_\lambda x\| &\leqslant 2|\Re\lambda| \|e^{-\lambda(\cdot)}\|_{L^{p'}(I)} \|A\mathscr{M}_I(f_\lambda \otimes x)\|_{L^p(I;X)} \\ &\leqslant 2C|\Re\lambda| \|e^{-\lambda(\cdot)}\|_{L^{p'}(I)} \|f_\lambda \otimes x\|_{L^p(I;X)} \leqslant 2C\|x\|.\end{aligned} \tag{17.20}$$

with $C = M^{\rm reg}_{p,A}(I)$. Once the remainder of the proof has been finished, (17.13) follows from this estimate.

By (17.3) and an integration by parts (which is justified by the fact that L^p-solutions are in $W^{1,p}(I;X)$),

$$\begin{aligned}AR_\lambda x &= -2\Re\lambda \int_I e^{-\lambda t}[(\mathscr{M}_I(f_\lambda \otimes x))'(t) - (f_\lambda \otimes x)(t)]\,\mathrm{d}t \\ &= -2\Re\lambda\, e^{-\lambda T}(\mathscr{M}_I(f_\lambda \otimes x))(T) - \lambda R_\lambda x + 2\Re\lambda \int_0^T e^{-2\Re\lambda t} x\,\mathrm{d}t \\ &= -2\Re\lambda[e^{-\lambda T}(\mathscr{M}_I(f_\lambda \otimes x))(T)] - \lambda R_\lambda x + (1 - e^{-2\Re\lambda T})x.\end{aligned} \tag{17.21}$$

If we could let $T \to \infty$ then this would give $AR_\lambda x = -\lambda R_\lambda x + x$. This is not possible, however, since $I = (0,T)$ is fixed. Instead, we will take $\Re\lambda$ so large that the remainder term has norm $\leqslant 1/2$. To this end let

$$Q_\lambda := 2\Re\lambda[e^{-\lambda T}(\mathscr{M}_I(f_\lambda \otimes x))(T)] + e^{-2\Re\lambda T}x. \tag{17.22}$$

Suppose that $\Re\lambda > 1/T$. By (17.10), (17.17), and the fact that for any $0 < \alpha \leqslant 1$ the function $t \mapsto t^\alpha e^{-t/2}$ is bounded above on $\mathbb{R}_+$ by $(2\alpha/e)^\alpha \leqslant 1$,

$$\begin{aligned}\|Q_\lambda x\| &\leqslant 2\Re\lambda\, e^{-\Re\lambda T} T^{1/p'}(C+1)\|f_\lambda \otimes x\|_p + e^{-2\Re\lambda T}\|x\| \\ &\leqslant 2(\Re\lambda T)^{1/p'} e^{-\Re\lambda\, T}(C+1)\|x\| + e^{-2}\|x\| \\ &\leqslant 2e^{-\frac12 \Re\lambda\, T}(C+1)\|x\| + e^{-2}\|x\|.\end{aligned} \tag{17.23}$$

Therefore, if $\Re\lambda > r_1/T$ with $r_1 := 2\log(4(C+1))$, then

$$\|Q_\lambda\| \leqslant \frac{1}{2} \tag{17.24}$$

and $I - Q_\lambda$ is invertible with $\|(I - Q_\lambda)^{-1}\| \leqslant 2$. Since by (17.21) and (17.22) imply $(\lambda + A)R_\lambda x = (I - Q_\lambda)x$, it follows that $R_\lambda(I - Q_\lambda)^{-1}$ is a right inverse to $\lambda + A$.

Step 3 – Next we will show that the above right-inverse is actually an inverse for $\Re\lambda$ large enough and obtain the required resolvent estimate. Let $r_A = \max\{r_0, r_1\}$. By Steps 1 and 2, $\lambda + A$ is injective and surjective, hence invertible, for $\Re\lambda > r_A/T$. Therefore, for each $\Re\lambda > r_A/T$, $\lambda \in \varrho(-A)$ and $(\lambda + A)^{-1} = R_\lambda(I - Q_\lambda)^{-1}$ coincides with the right-inverse constructed in Step 2. By (17.20), (17.24), and the Neumann series expansion,

$$\begin{aligned}\|\lambda(\lambda + A)^{-1}\| &\leqslant 1 + \|A(\lambda + A)^{-1}\| \leqslant 1 + \|AR_\lambda(I - Q_\lambda)^{-1}\| \\ &\leqslant 1 + 2C\|(I - Q_\lambda)^{-1}\| \leqslant 1 + 4C.\end{aligned}$$

In combination with Lemma G.1.4, this proves that large enough translates of $-A$ are sectorial of angle $< \frac{1}{2}\pi$. The results of Appendix K imply that these translates, and hence $-A$ itself, generate analytic semigroups on X in the sense of Definition K.1.2. □

Proof of Theorem 17.2.15(2) with non-sharp constant. The above proof for $I = (0, T)$ can be repeated for $I = \mathbb{R}_+$, with $\mathscr{M}_I$ replaced by $\mathscr{M}_{\mathbb{R}_+}$, up to equation (17.22) which now reads

$$Q_\lambda := 2\Re\lambda[e^{-\lambda T}(\mathscr{M}_{\mathbb{R}_+}(f_\lambda \otimes x))(T)] + e^{-2\Re\lambda T}x.$$

We can still apply (17.10) to this equation, and in combination with (17.17) we arrive at the bound given in (17.23). This time we can pass to the limit $T \to \infty$ and arrive at the identity $AR_\lambda x = -\lambda R_\lambda x + x$. Therefore, R_λ is a right-inverse of $\lambda + A$. To see that it is a left-inverse it suffices to check that $AR_\lambda = R_\lambda A$. The latter follows from (17.19) and Lemma 17.2.12.

Sectoriality follows from (17.20) with a slightly worse bound than stated in (17.14). The bound will be further improved below. □

Having analytic semigroups available, we can now deduce maximal regularity on sub-intervals by using the equivalence result for strong and mild solutions described in Proposition 17.1.3. Recall from Definition 17.1.2 that if $S = (S(t))_{t>0}$ is a locally bounded strongly measurable semigroup S on X, then for $x \in \overline{\mathsf{D}(A)}$ the continuous function $u : [0, \infty) \to X$ defined by

$$u(t) := S(t)x + \int_0^t S(t - s)f(s)\,\mathrm{d}s, \quad t \in \overline{I},$$

is called the *mild solution* of the problem (ACP$_x$). Continuity of $t \mapsto S(t)x$ for elements for $x \in \overline{\mathsf{D}(A)}$ follows from Proposition K.1.5, but in the present situation there is no need to refer to this result since we consider the initial value $x = 0$.

Lemma 17.2.16 (Maximal L^p-regularity on subintervals). *Let $1 \leqslant p \leqslant \infty$ and let A be a linear operator on a Banach space X with maximal L^p-regularity on I. Then A has maximal L^p-regularity on every subinterval $(0,T) \subseteq I$ and*

$$M^{\rm reg}_{p,A}(0,T) \leqslant M^{\rm reg}_{p,A}(I). \tag{17.25}$$

Proof. The first issue to deal with is the uniqueness of strong solutions on $(0,T)$. Let $u \in L^p(0,T;\mathsf{D}(A))$ be an L^p-solution to (ACP$_0$) on $(0,T)$ with inhomogeneity $f = 0$. Since A has maximal L^p-regularity on I, it generates an analytic semigroup by Dore's Theorem 17.2.15. This allows us to apply Proposition 17.1.3, which guarantees that u equals the mild solution on $(0,T)$, which is obviously identically zero. It follows that $u = 0$.

Now let $f \in L^p(0,T;X)$ be arbitrary, and let u be the L^p-solution to (ACP$_0$) on I with inhomogeneity $\overline{f}$, where $\overline{f}$ is the zero extension of f to I. Then $u|_{(0,T)}$ is a strong solution of (ACP$_0$) on $(0,T)$, and

$$\begin{aligned}\|Au|_{(0,T)}\|_{L^p(0,T;X)} &\leqslant \|Au\|_{L^p(I;X)} \\ &\leqslant M^{\rm reg}_{p,A}(I)\|\overline{f}\|_{L^p(I;X)} = M^{\rm reg}_{p,A}(I)\|f\|_{L^p(0,T;X)}.\end{aligned}$$

By uniqueness, $u|_{(0,T)}$ is the only possible strong solution, we conclude that A has maximal L^p-regularity on $(0,T)$ and that (17.25) holds. □

Remark 17.2.17. For $1 < p < \infty$ the following alternative argument can be used to obtain uniqueness on subintervals. Suppose that A has maximal L^p-regularity on I, and $(0,T) \subseteq I$ be a bounded subinterval. Let u be an L^p-solution to the problem $u' + Au = 0$ on $(0,T)$ with initial condition $u(0) = 0$. We claim that $u = 0$. By Proposition 17.2.2 we have $u \in L^p(0,T;\mathsf{D}(A)) \cap W^{1,p}(0,T;X)$ and thus, by repeating the argument of the proof of Proposition 17.2.13 for $\tilde{u} := u(T-\cdot) \in L^p(0,T;\mathsf{D}(A)) \cap W^{1,p}(0,T;X)$, we obtain $u(T) = \tilde{u}(0) \in (X,\mathsf{D}(A))_{1-\frac{1}{p},p}$. By Proposition 17.2.14, there exists a unique L^p-solution v of $v' + Av = 0$ on I and $v(0) = u(T)$. Let $\overline{u}(t) := u$ for $t \in [0,T]$ and $\overline{u}(t) := v(t-T)$ for $t \in (T,\infty) \cap I$. Then $\overline{u}$ is an L^p-solution to the problem $\overline{u}' + A\overline{u} = 0$ on I with initial condition $\overline{u}(0) = 0$. Therefore, $\overline{u} = 0$ by maximal L^p-regularity, and thus $u = 0$ on $(0,T)$.

We are now in a position complete the proof of Theorem 17.2.15 by showing the estimate (17.14) for the sectoriality constant.

Proof of the sectoriality bound (17.14). We have already proved that A is sectorial of angle $< \pi/2$. It remains to prove the bound (17.14). By Lemma 17.2.16 we know that maximal L^p-regularity holds on $(0,T)$ with estimate (17.25). Replacing the constant C in Step 1 by the larger constant $M^{\rm reg}_{p,A}(\mathbb{R}_+)$, and passing to the limit $T \to \infty$ using that $\|e^{\lambda T\cdot}\|_{L^p(0,1)} \to \infty$ in (17.16), it follows that

$$\|Ax\| \leqslant C\|(\lambda + A)x\|.$$

Since $\lambda + A$ is invertible, this gives (17.14). □

The next result gives a sufficient condition for maximal L^p-regularity on $\mathbb{R}_+$ by establishing uniform estimates on bounded intervals.

Proposition 17.2.18. *Let $1 \leqslant p \leqslant \infty$ and let A be a linear operator. The following are equivalent:*

(1) *For each $T \in (0, \infty)$, the operator A has maximal L^p-regularity on $(0, T)$ and $\sup_{T>0} M^{\rm reg}_{p,A}(0, T) < \infty$;*
(2) *The operator A has maximal L^p-regularity on $\mathbb{R}_+$.*

In this case we have $M^{\rm reg}_{p,A}(\mathbb{R}_+) = \sup_{T>0} M^{\rm reg}_{p,A}(0, T)$.

Proof. (2) ⇒ (1) is clear from Lemma 17.2.16, and the estimate "⩾" follows from (17.25).

(1) ⇒ (2): Let $f \in L^p(\mathbb{R}_+; X)$. For every $0 < T < \infty$ there exists a unique L^p-solution u_T to (ACP$_0$) on $(0, T)$. By uniqueness, $u_T|_{[0,S]} = u_S$ for every $S \leqslant T$. Therefore, we can construct a function $u : [0, \infty) \to X$ such that $u|_{[0,T]} = u_T$ for every $0 < T < \infty$. Then u is a strong solution to (ACP$_0$) on $\mathbb{R}_+$. Moreover, uniqueness follows by taking restrictions to $[0, T]$. Since

$$\|Au\|_{L^p(0,T;X)} \leqslant C\|f\|_{L^p(0,T;X)} \leqslant C\|f\|_{L^p(\mathbb{R}_+;X)},$$

where $C = \sup_{T>0} M^{\rm reg}_{p,A}(0, T)$, the desired result follows from the monotone convergence theorem by passing to the limit $T \to \infty$. □

Now that we know that maximal L^p-regularity of A implies that $-A$ generates an analytic semigroup it is of interest to relate L^p-solutions to mild solutions.

Theorem 17.2.19. *Let $-A$ generate an analytic semigroup on a Banach space X and let $1 \leqslant p \leqslant \infty$. Let F be any dense subspace of $L^p(I; X)$. Then A has maximal L^p-regularity on I if and only if the mapping $f \to Vf$, defined for functions $f \in F$ by*

$$Vf(t) := A \int_0^t S(t-s)f(s)\,\mathrm{d}s, \quad t \in I,$$

is well defined, maps F into $L^p(I; X)$, and there is a constant $C \geqslant 0$ such that

$$\|Vf\|_{L^p(I;X)} \leqslant C\|f\|_{L^p(I;X)}, \quad f \in F.$$

In this situation, V uniquely extends to a bounded operator on $L^p(I; X)$ such that

$$Vf = Au_f,$$

where u_f is the mild and L^p-solution to (ACP$_0$) associated with f, and the least admissible constant C in the above inequality coincides with $M^{\rm reg}_{p,A}(I)$.

Remark 17.2.20 (On the choice of the space F). In the above formulation it is implicit that one should have

$$\int_0^t S(t-s)f(s)\,\mathrm{d}s \in \mathsf{D}(A)$$

for almost all $t \in I$. By Proposition 17.1.4 this holds for instance for $F = C^\alpha(\overline{I};X)$ with $\alpha > 0$ (which is dense if $p \in [1,\infty)$). In the important case when $\overline{\mathsf{D}(A)} = X$, it can be useful to take $F = L^p(I;\mathsf{D}(A))$ (which is dense if $p \in [1,\infty)$); in this case the operator A can even be pulled through the integral. This choice doesn't work for $p = \infty$ even when $\overline{\mathsf{D}(A)} = X$, because the space $L^\infty(I;\mathsf{D}(A))$ in general fails to be dense in $L^\infty(I;X)$. As we will see in Section 17.2.f, this problem can be circumvented by considering the equivalent notion of maximal C-regularity.

Remark 17.2.21 (Connection with singular integrals). For analytic semigroups S with generator $-A$, the bound

$$\|AS(t)\| = \mathcal{O}(1/t) \quad \text{as} \quad t \downarrow 0$$

(apply (K.4) to $\lambda + A$ for λ large enough) shows that in those cases where it is possible to pull A through the integral in the definition of V, a singular convolution integral is obtained. This aspect of the theory will be further studied at later stage in this chapter (see, for example, Theorems 17.2.31, 17.2.39, and Section 17.3).

Proof of Theorem 17.2.19. 'Only if': Every $f \in L^p(I;X)$ gives rise to a unique L^p-solution u_f which, by Proposition 17.1.3, equals the mild solution. Therefore, the mapping $f \mapsto Au_f = Vf$ extends to a bounded operator on $L^p(I;X)$ by maximal L^p-regularity.

'If': Let u_f be the mild solution associated with $f \in F$. The assumptions on F imply that u_f takes values in $\mathsf{D}(A)$ and that $Au \in L^p(I;X)$, so u_f is an L^p-solution by Proposition 17.1.3. Since $Vf = Au_f$ for all $f \in F$, maximal L^p-regularity follows from the boundedness of V, closedness of A, and Proposition 17.2.10. □

We continue with a useful special property of maximal L^1-regularity which will be needed several times below: it leads to a characterisation of maximal L^1-regularity in Theorem 17.3.11, and it allows us to give a proof of Proposition 17.2.14 for $p = 1$, and it will be used in Proposition 17.2.32.

Lemma 17.2.22. *Let A have maximal L^1-regularity on I, and let S be the analytic semigroup generated by $-A$. Then*

$$\int_I \|AS(t)x\|\,\mathrm{d}t \leqslant M^{\mathrm{reg}}_{1,A}(I)C_{I,S}\|x\|, \quad x \in X, \tag{17.26}$$

where $C_{I,S} = \sup_{t\in I}\|S(t)\|$.

Proof. Recall that $-A$ generates an analytic semigroup S by Theorem 17.2.15. Let $C_{I,S} := \sup_{t\in I}\|S(t)\|$. This number is finite: if $I = \mathbb{R}_+$, this follows from the fact that A is sectorial in that case and the semigroup generated by A is uniformly bounded; if $I = (0,T)$ is a bounded interval, this follows from the fact that $\lambda + A$ is sectorial for $\Re\lambda$ large enough, and then $t \mapsto e^{-\lambda t}S(t)$ is uniformly bounded.

Let $\varepsilon \in (0,1)$ be arbitrary and set $I_\varepsilon := I \cap (\varepsilon, 1/\varepsilon)$ and $\phi(t) := \min\{t/\varepsilon, 1\}$. Then ϕ is weakly differentiable and $\|\phi'\|_{L^1(I)} = 1$. For $x \in X$ set

$$u(t) := \phi(t)S(t)x \quad \text{and} \quad f(t) := u'(t) + Au(t) = \phi'(t)S(t)x.$$

Since A has maximal L^1-regularity we obtain that

$$\int_{I_\varepsilon} \|AS(t)x\| \,\mathrm{d}t = \int_{I_\varepsilon} \|Au(t)\| \,\mathrm{d}t \leqslant M^{\mathrm{reg}}_{1,A}(I)\|f\|_{L^1(I;X)} \leqslant M^{\mathrm{reg}}_{1,A}(I)C_{I,S}\|x\|.$$

Letting $\varepsilon \downarrow 0$ we obtain (17.26). □

Remark 17.2.23. A little more can be said. For $t \geqslant s > 0$ with $t \in I$ we have

$$\|S(s)x - S(t)x\| = \Big\|A\int_s^t S(r)x \,\mathrm{d}r\Big\| = \Big\|\int_s^t AS(r)x \,\mathrm{d}r\Big\| \leqslant \int_s^t \|AS(r)x\| \,\mathrm{d}r$$

By this estimate and the Bochner integrability of $AS(\cdot)$, the limit $S(0+)x := \lim_{t\downarrow 0} S(t)x$ exists and it satisfies

$$\|S(0+)x - S(t)x\| \leqslant \int_0^t \|AS(r)x\| \,\mathrm{d}r.$$

In the special case that X is a dual space and A is the adjoint of a C_0-semigroup on the predual of X, the weak*-continuity of S allows us to identify the limit as $S(0+)x = x$. Thus we obtain the following result: *Let G generate a C_0-semigroup on Banach space Y. If G^* has maximal L^1-regularity, then the adjoint semigroup is strongly continuous on Y^*.*

We are now able to prove Proposition 17.2.14 for $p = 1$:

Proof of Proposition 17.2.14 for $p = 1$. Since $x \in (X, \mathsf{D}(A))_{1-\frac{1}{p},p}$ by assumption, Corollary C.3.15 implies that $x \in \overline{\mathsf{D}(A)}$, and therefore $t \mapsto S(t)x$ is strongly continuous by Proposition K.1.5. The function $u = Sx + S * f$ is an L^1-solution of (ACP$_x$), and Theorem 17.2.19 and Lemma 17.2.22 imply that

$$\begin{aligned}\|Au\|_{L^1(I;X)} &\leqslant \|ASx\|_{L^1(I;X)} + \|AS * f\|_{L^1(I;X)} \\ &\leqslant M^{\mathrm{reg}}_{1,A}(I)C_{I,S}\|x\| + M^{\mathrm{reg}}_{1,A}(I)\|f\|_{L^1(I;X)}.\end{aligned}$$

□

In order to prepare for some results in the next sections, we conclude the present section by introducing a constant related to the maximal regularity constant with an additional parameter λ; in applications, this additional flexibility can sometimes be exploited.

From Theorem 17.2.19 we recall that if A is a linear operator in X such that $-A$ generates an analytic semigroup S on a Banach space X, then A has maximal L^p-regularity on some bounded interval $(0,T)$ if and only if

$$Vf(t) := A\int_0^t S(t-s)f(s)\,\mathrm{d}s, \quad t \in [0,T],$$

is well defined for all functions f in some dense subspace of $L^p(I;X)$ and extends to a bounded operator V on $L^p(I;X)$. When $I = (0,T)$ is a *bounded* interval, in these circumstances for all $\lambda \in \mathbb{C}$ the operator V_λ defined by

$$V_\lambda f(t) := A\int_0^t S_\lambda(t-s)f(s)\,\mathrm{d}s, \quad t \in [0,T],$$

has the same properties, where $S_\lambda(t) := e^{-\lambda t}S(t)$ is the rescaled semigroup. Indeed, in this case we have

$$\|V_\lambda f\|_{L^p(I;X)} \leqslant \max\{1, e^{-\Re\lambda T}\}\|V\|_{\mathscr{L}(L^p(I;X))}\|s \mapsto e^{-\lambda s}f(s)\|_{L^p(I;X)}$$

where $I = (0,T)$. We will come back to this in Proposition 17.2.27.

Suppose now that A is a linear operator in X such that $-A$ generates an analytic semigroup S on a Banach space X; we make no maximal regularity assumptions. If $f \in L^p(I;X)$ is a function such that $\int_0^t S_\lambda(t-s)f(s)\,\mathrm{d}s \in \mathsf{D}(A)$ for almost all $t \in I$, we may define the function $V_\lambda f$ as above, that is,

$$V_\lambda f(t) := A\int_0^t S_\lambda(t-s)f(s)\,\mathrm{d}s, \quad t \in \overline{I}. \tag{17.27}$$

Note that $V_\lambda f = Au_\lambda$ almost everywhere on I, where u_λ is the mild solution to the problem

$$\begin{cases} u'(t) + (\lambda + A)u(t) &= f(t), \quad t \in I, \\ u(0) &= 0. \end{cases} \tag{17.28}$$

If the mild solution u_λ exists and takes values in $\mathsf{D}(A)$ for all for all $f \in F$, where F is some dense subspace of $L^p(I;X)$, then the mapping $f \mapsto V_\lambda f$ defined by (17.27) uniquely extends to a bounded linear mapping on $L^p(I;X)$ and we may define

$$M^{\mathrm{reg}}_{p,A,\lambda}(I) := \|V_\lambda\|_{\mathscr{L}(L^p(I;X))}.$$

Thus if A has maximal L^p-regularity on I, then $M^{\mathrm{reg}}_{p,A,0}(I) = M^{\mathrm{reg}}_{p,A}(I)$. For other values of λ, we will derive various bounds in Theorem 17.2.24 and Proposition 17.2.27. We should emphasise that, in general, $M^{\mathrm{reg}}_{p,A,\lambda}(I) \neq M^{\mathrm{reg}}_{p,A+\lambda}(I)$;

the latter would be the norm the operator $\tilde V_\lambda f(t) := (A+\lambda)\int_0^t S_\lambda(t-s)f(s)\,\mathrm{d}s$, while in (17.27), only the semigroup is rescaled, but not the operator A.

If $\alpha, \beta \in \mathbb{R}$ and V_α extends to a bounded operator on $L^p(I;X)$, then so does $V_{\alpha+i\beta}$ and

$$\|V_{\alpha+i\beta}f\|_{L^p(I;X)} = \|V_\alpha e^{i\beta\cdot} f\|_{L^p(I;X)}, \qquad f \in L^p(I;X). \tag{17.29}$$

Clearly, this implies $M^{\mathrm{reg}}_{p,A,\alpha+i\beta}(I) = M^{\mathrm{reg}}_{p,A,\alpha}(I)$ for any $\alpha, \beta \in \mathbb{R}$.

17.2.d Uniformly exponentially stable semigroups

In the previous subsection we have seen that maximal L^p-regularity on $\mathbb{R}_+$ implies maximal L^p-regularity on every bounded interval $(0,T)$. To state a result in the converse direction we need some preparations.

Recall from Appendix K that for a locally bounded semigroup $(S(t))_{t>0}$ there exist $M \geqslant 1$ and $\mu \in \mathbb{R}$ such that $\|S(t)\| \leqslant Me^{\mu t}$ for all $t \geqslant 0$. The semigroup is said to be *uniformly exponentially stable* if μ can be taken strictly negative.

If S is an analytic semigroup, then there exist constants $M \geqslant 1$ and $\mu \in \mathbb{R}$ such that

$$\|S(t)\| \leqslant Me^{\mu t} \quad\text{and}\quad \|tAS(t)\| \leqslant Me^{\mu t}, \quad t \geqslant 0, \tag{17.30}$$

and if such a semigroup S is uniformly exponentially stable we can take $\mu < 0$ in (17.30) (this follows from the proof of the second of these inequalities).

The main result of this section states, among other things, that if A has maximal L^p-regularity on a bounded interval and the analytic semigroup generated by $-A$ is uniformly exponentially stable, then A has maximal L^p-regularity on $\mathbb{R}_+$.

Recall that the constants $M^{\mathrm{reg}}_{p,A,\lambda}(\mathbb{R}_+)$ have been introduced at the end of the previous section.

Theorem 17.2.24 (Dore–Kato). *Let $1 \leqslant p \leqslant \infty$ be fixed, and suppose that the linear operator A has maximal L^p-regularity on some bounded interval $I = (0,T)$. If the analytic semigroup S generated by $-A$ satisfies (17.30) with constants $M \geqslant 1$ and $\mu \in \mathbb{R}$, then for all $\Re\lambda > \mu$ we have*

$$M^{\mathrm{reg}}_{p,A,\lambda}(\mathbb{R}_+) \leqslant 2\max\{e^{\Re\lambda T/2}, e^{-\Re\lambda T}\}M^{\mathrm{reg}}_{p,A}(I) + \frac{2M}{T(\Re\lambda-\mu)}. \tag{17.31}$$

In particular, if S is uniformly exponentially stable with $\omega := -\mu > 0$, then A has maximal L^p-regularity on $\mathbb{R}_+$ and

$$M^{\mathrm{reg}}_{p,A}(\mathbb{R}_+) \leqslant 2M^{\mathrm{reg}}_{p,A}(I) + \frac{2M}{T\omega}. \tag{17.32}$$

Two remarks are in order. First of all, in (17.31) no claim is made with regard to maximal L^p-regularity on $\mathbb{R}_+$; this is done only in (17.32). Secondly, in situations where A has maximal L^p-regularity on every bounded interval $(0,T)$ one can optimise the choice of T in (17.31) by letting it depend on the parameter λ (see, for instance, the proof of Proposition 17.2.27).

Proof. Let $S_\lambda(t) = e^{-\lambda t}S(t)$, where $\lambda = \alpha + i\beta \in \mathbb{C}$ with $\alpha > \mu$. For $f \in L^p(\mathbb{R}_+;X)$ the expression for the mild solution

$$u_{f,\lambda} := S_\lambda * f$$

defines a continuous function $u_{f,\lambda} : [0,\infty) \to X$. We will show that $u_{f,\lambda}$ takes values in $\mathsf{D}(A)$ almost everywhere and that

$$\|Au_{f,\lambda}\|_{L^p(\mathbb{R}_+;X)} \leqslant K\|f\|_{L^p(\mathbb{R}_+;X)} \tag{17.33}$$

where $K = 2e^{\Re\lambda T/2}M^{\rm reg}_{p,A}(I) + 2M/T(\alpha-\mu)$. Once we have shown this, Proposition 17.1.3 implies that $u_{f,\lambda}$ is an L^p-solution to (ACP$_0$) with A replaced by $\lambda + A$, and (17.31) follows from (17.33). The second assertion of the theorem follows by taking $\lambda = 0$.

In view of (17.29), in order to prove (17.33) it suffices to consider $\beta = 0$, and thus $\lambda \in \mathbb{R}$. For $j = 0,1,2,\dots$ put $t_j := \frac{1}{2}jT$. For $t \in (t_j, t_{j+1})$ with $j \geqslant 1$ we write

$$\begin{aligned} u_{f,\lambda}(t) &= \int_{t_{j-1}}^{t} S_\lambda(t-s)f(s)\,\mathrm{d}s + \int_0^{t_{j-1}} S_\lambda(t-s)f(s)\,\mathrm{d}s \\ &= \int_0^{t-t_{j-1}} S_\lambda(t-t_{j-1}-s)f(s+t_{j-1})\,\mathrm{d}s + \int_0^{t_{j-1}} S_\lambda(t-s)f(s)\,\mathrm{d}s \\ &=: u^{(j)}_{f,\lambda}(t) + v^{(j)}_{f,\lambda}(t). \end{aligned}$$

By maximal L^p-regularity and the discussion below (17.29), $u^{(j)}_{f,\lambda} \in \mathsf{D}(A)$ almost everywhere on (t_{j-1}, t_{j+1}), and

$$\begin{aligned} &\|Au^{(j)}_{f,\lambda}\|_{L^p(t_j,t_{j+1};X)} \\ &\quad= \Big\|t \mapsto A\int_0^{t-t_{j-1}} S_\lambda(t-t_{j-1}-s)f(s+t_{j-1})\,\mathrm{d}s\Big\|_{L^p(t_j,t_{j+1};X)} \\ &\quad= \Big\|\tau \mapsto e^{-\lambda\tau}A\int_0^{\tau} S(\tau-s)e^{\lambda s}f(s+t_{j-1})\,\mathrm{d}s\Big\|_{L^p(T/2,T;X)} \\ &\quad\leqslant \max\{e^{-\Re\lambda T/2}, e^{-\Re\lambda T}\}M^{\rm reg}_{p,A}(I)\max\{1, e^{\Re\lambda T}\}\|f(\cdot + t_{j-1})\|_{L^p(I;X)} \\ &\quad= \max\{e^{\Re\lambda T/2}, e^{-\Re\lambda T}\}M^{\rm reg}_{p,A}(I)\|f\|_{L^p(t_{j-1},t_{j+1};X)} \\ &\quad\leqslant b_j + b_{j+1}, \end{aligned}$$

where $b_j := \max\{e^{\lambda T/2}, e^{-\lambda T}\}M^{\rm reg}_{p,A}(I)\|f\|_{L^p(t_{j-1},t_j;X)}$. A similar computation with $t_{-1} := 0$ shows that

$$\|Au_{f,\lambda}\|_{L^p(0,T/2;X)} \leqslant b_1.$$

To estimate the norm of $Av^{(j)}_{f,\lambda}$ we recall from (17.30) that $\|tAS_\lambda(t)\| \leqslant Me^{-(\lambda-\mu)t}$. Since $t - s \geqslant T/2$ for $t \in (t_j, t_{j+1})$ and $s \in (0, t_{j-1})$, we can estimate

$$\begin{aligned}\|Av^{(j)}_{f,\lambda}(t)\| &\leqslant \int_0^{t_{j-1}} \|AS_\lambda(t-s)f(s)\|\,\mathrm{d}s \\ &\leqslant 2MT^{-1}\int_0^t e^{-(\lambda-\mu)(t-s)}\|f(s)\|\,\mathrm{d}s = 2MT^{-1}(\phi_\mu * \|f(\cdot)\|)(t),\end{aligned}$$

where $\phi_\mu(s) = e^{-(\lambda-\mu)s}\mathbf{1}_{\mathbb{R}_+}(s)$. Therefore,

$$\|Av^{(j)}_{f,\lambda}\|_{L^p(t_j,t_{j+1};X)} \leqslant 2MT^{-1}\Big\|\phi_\mu * \|f(\cdot)\|\Big\|_{L^p(t_j,t_{j+1})} =: c_j.$$

This proves that for all $j \geqslant 1$,

$$\|Au_{f,\lambda}\|_{L^p(t_j,t_{j+1};X)} \leqslant b_j + b_{j+1} + c_j. \tag{17.34}$$

As observed above, the estimate (17.34) also holds for $j = 0$ if we set $b_0 = 0$.

Taking ℓ^p-norms over $j \geqslant 0$ in (17.34), we obtain

$$\begin{aligned}&\|Au_{f,\lambda}\|_{L^p(\mathbb{R}_+;X)} \\ &\quad= \Big\|(\|Au_{f,\lambda}\|_{L^p(t_j,t_{j+1};X)})_{j\geqslant 0}\Big\|_{\ell^p} \\ &\quad\leqslant 2\|(b_j)_{j\geqslant 0}\|_{\ell^p} + \|(c_j)_{j\geqslant 0}\|_{\ell^p} \\ &\quad= 2\max\{e^{\Re\lambda T/2}, e^{-\Re\lambda T}\}M^{\mathrm{reg}}_{p,A}(I)\|f\|_{L^p(\mathbb{R}_+;X)} + 2MT^{-1}\Big\|\phi_\mu * \|f(\cdot)\|\Big\|_{L^p(\mathbb{R}_+)} \\ &\quad\leqslant \Big(2\max\{e^{\Re\lambda T/2}, e^{-\Re\lambda T}\}M^{\mathrm{reg}}_{p,A}(I) + 2M(T(\lambda-\mu))^{-1}\Big)\|f\|_{L^p(\mathbb{R}_+;X)},\end{aligned}$$

where in the last step we applied Young's inequality. This proves (17.33). □

We conclude this section by showing the necessity of the invertibility assumption in Proposition 17.2.8.

Corollary 17.2.25. *Suppose that the linear operator A has maximal L^p-regularity on $\mathbb{R}_+$ for a given $1 \leqslant p \leqslant \infty$. If there exists a constant $C \geqslant 0$ such that for all $f \in L^p(\mathbb{R}_+;X)$ the L^p-solution u of* (ACP_0) *satisfies*

$$\|u\|_{L^p(\mathbb{R}_+;\mathsf{D}(A))} \leqslant C\|f\|_{L^p(\mathbb{R}_+;X)},$$

then $-A$ generates an analytic semigroup that is uniformly exponentially stable and, in particular, $0 \in \varrho(A)$.

Proof. From Theorem 17.2.15, we see that $-A$ generates an analytic semigroup. Finally, the uniformly exponential stability follows from Proposition K.2.2. □

17.2.e Permanence properties

We have already encountered several permanence properties for maximal L^p-regularity: if A has maximal L^p-regularity on $\mathbb{R}_+$, then it has maximal L^p-regularity on every bounded interval $(0,T)$, and the converse holds if the semigroup generated by $-A$ is uniformly exponentially stable. Further permanence properties are summarised in the following qualitative theorem. The proofs of the corresponding quantitative versions will be spread out over the remainder of this section.

As before, I denotes a bounded interval $(0,T)$ or $\mathbb{R}_+ = (0,\infty)$.

Theorem 17.2.26. *Let A be a linear operator on a Banach space X and let $1 \leqslant p \leqslant \infty$. If A has maximal L^p-regularity on I, the following assertions hold:*

(1) (Translation) $\lambda + A$ *has maximal L^p-regularity on I in each of the following two situations:*
 (i) $I = \mathbb{R}_+$ *and* $\Re\lambda > 0$;
 (ii) $I = (0,T)$ *is a bounded interval and* $\lambda \in \mathbb{C}$.
(2) (Change of interval) A *has maximal L^p-regularity on every bounded interval* $(0,T')$.
(3) (Scalar multiples) λA *has maximal L^p-regularity on I for all* $\lambda > 0$.
(4) (Extrapolation of exponent) A *has maximal L^q-regularity on I for all* $q \in (1,\infty)$.
(5) (Duality) *If* $\overline{\mathsf{D}(A)} = X$, *then A^* has maximal $L^{p'}$-regularity on I.*

We start preparations for proving (1) by estimating the constants $M^{\rm reg}_{p,A,\lambda}(I)$ introduced after (17.28). Recalling that $M^{\rm reg}_{p,A,\lambda}(I) \neq M^{\rm reg}_{p,A+\lambda}(I)$, note that this will not yet constitute a proof of (1) itself; we only turn to this task after establishing the next proposition.

Proposition 17.2.27 (Translation I). *Let $1 \leqslant p \leqslant \infty$ and let A be a linear operator on a Banach space X with maximal L^p-regularity on I, and consider the problem*

$$\begin{cases} u'(t) + (\lambda + A)u(t) &= f(t), \quad t \in I, \\ u(0) &= 0. \end{cases}$$

(1) *For all $\Re\lambda \geqslant 0$ and $f \in L^p(I;X)$ the above problem admits a unique L^p-solution u, and it satisfies the estimate*

$$\|Au\|_{L^p(I;X)} \leqslant M^{\rm reg}_{p,A}(I)\|f\|_{L^p(I;X)}.$$

In particular, $M^{\rm reg}_{p,A,\lambda}(I) \leqslant M^{\rm reg}_{p,A}(I)$.

(2) *If $I = (0,T)$ is a bounded interval, then for all $\lambda \in \mathbb{C}$ and $f \in L^p(I;X)$ the above problem admits a unique L^p-solution u, and it satisfies the estimate*

$$\|Au\|_{L^p(I;X)} \leqslant M^{\rm reg}_{p,A}(I)e^{|\Re\lambda|T}\|f\|_{L^p(I;X)}.$$

In particular, $M^{\rm reg}_{p,A,\lambda}(I) \leqslant e^{|\Re\lambda|T} M^{\rm reg}_{p,A}(I)$.

If $I = (0,T)$ is bounded *and* $\Re\lambda \geqslant 0$, we can apply either (1) or (2), but the first estimate, which is independent of λ, will be sharper. Both cases can be combined into

$$\|Au\|_{L^p(I;X)} \leqslant M^{\mathrm{reg}}_{p,A}(I)e^{(\Re\lambda)_- T}\|f\|_{L^p(I;X)},$$

where $x_- := \max(-x, 0)$ is the negative part of number.

Proof. We first prove that (2) holds for arbitrary $\lambda \in \mathbb{C}$. Accordingly let $I = (0,T)$ and fix $\lambda \in \mathbb{C}$ and $f \in L^p(0,T;X)$, and consider the function $f_\lambda \in L^p(0,T;X)$ defined by $f_\lambda(t) := e^{-\lambda t}f(t)$. If u denotes the L^p-solution of the problem $u' + Au = f$, then $v(t) := e^{-\lambda t}u(t)$ defines an L^p-solution of the problem $v' + (\lambda + A)v = f_\lambda$. Put $L_\pm = \max\{e^{\pm\Re\lambda T}, 1\}$, so that $L_+L_- = e^{|\Re\lambda|T}$. Using (17.6) and (17.9) we have

$$\begin{aligned}\|Av\|_{L^p(0,T;X)} &= \|Ae^{-\lambda(\cdot)}u\|_{L^p(0,T;X)}\\ &\leqslant L_-\|Au\|_{L^p(0,T;X)}\\ &\leqslant M^{\mathrm{reg}}_{p,A}(I)L_-\|f\|_{L^p(0,T;X)}\\ &\leqslant M^{\mathrm{reg}}_{p,A}(I)L_+L_-\|f_\lambda\|_{L^p(0,T;X)}.\end{aligned}$$

This gives the result with $M^{\mathrm{reg}}_{p,A,\lambda}(I) \leqslant e^{|\Re\lambda|T}M^{\mathrm{reg}}_{p,A}(I)$ for any $\lambda \in \mathbb{C}$.

(1): *Step 1* – First we prove a non-optimal estimate in the case $I = \mathbb{R}_+$. By Theorem 17.2.15, $-A$ generates a bounded analytic semigroup $(S(t))_{t>0}$, and by Proposition K.1.11 and (K.4) we can find a constant $M \geqslant 1$ such that $\|S(t)\| \leqslant M$ and $\|tAS(t)\| \leqslant M$ for all $t > 0$.

Fix an arbitrary $T \in (0,\infty)$. Recall that A has maximal L^p-regularity on $I_T = (0,T)$ with $M^{\mathrm{reg}}_{p,A}(I_T) \leqslant M^{\mathrm{reg}}_{p,A}(\mathbb{R}_+)$ (see (17.25)). By Theorem 17.2.24, for all $\Re\lambda > 0$ one has

$$M^{\mathrm{reg}}_{p,A,\lambda}(\mathbb{R}_+) \leqslant 2e^{\Re\lambda T/2}M^{\mathrm{reg}}_{p,A}(\mathbb{R}_+) + 2M(T\Re\lambda)^{-1}.$$

If $\Re\lambda > 0$, the choice $T = \frac{2}{\Re\lambda}$ gives the bound

$$M^{\mathrm{reg}}_{p,A,\lambda}(\mathbb{R}_+) \leqslant 2eM^{\mathrm{reg}}_{p,A}(\mathbb{R}_+) + M =: K, \quad \Re\lambda > 0. \tag{17.35}$$

Now the idea is to improve this bound by a maximum principle applied to the operator V_λ defined after (17.28). We first consider the case of bounded intervals, and take limits afterwards.

Step 2 – First consider the case where $I = (0,T)$ is a bounded interval. By Theorem 17.2.15, $-A$ generates an analytic semigroup $(S(t))_{t>0}$. Therefore, as before we can find $\mu, M \geqslant 0$ such that $\|S(t)\| \leqslant Me^{\mu t}$ for all $t \geqslant 0$. By (K.2), this implies

$$\|(\mu + 1 + A)^{-1}\| \leqslant \int_0^\infty e^{-(\mu+1)}\|S(t)\|\,\mathrm{d}t \leqslant M,$$

and hence

$$\|A(\mu+1+A)^{-1}\| \leqslant 1 + \|(\mu+1)(\mu+1+A)^{-1}\| \leqslant 1 + M(\mu+1) =: C_\mu,$$

and hence $\|Ax\| \leqslant C_\mu \|(\mu+1+A)x\|$ for $x \in \mathsf{D}(A)$. Therefore, using the λ-analogue of (17.25) and (17.35), for all $\lambda \in \mathbb{C}$ with $\Re\lambda > \mu+1$ we obtain

$$\begin{aligned} M^{\rm reg}_{p,A,\lambda}(I) &\leqslant M^{\rm reg}_{p,A,\lambda}(\mathbb{R}_+) \\ &\leqslant C_\mu M^{\rm reg}_{p,A+\mu+1,\lambda-\mu-1}(\mathbb{R}_+) \\ &\leqslant C_\mu(2eM^{\rm reg}_{p,A+\mu+1}(\mathbb{R}_+) + M) =: K_\mu. \end{aligned}$$

If $0 \leqslant \Re\lambda < \mu+1$, then by the proof of (2) we find that

$$M^{\rm reg}_{p,A,\lambda}(I) \leqslant e^{\Re\lambda T} M^{\rm reg}_{p,A}(I) \leqslant e^{(\mu+1)T} M^{\rm reg}_{p,A}(I).$$

Combining both cases we see that V_λ is uniformly bounded on $L^p(I;X)$ on $\{\Re\lambda \geqslant 0\}$. One can check that V_λ is holomorphic on $\mathbb{C}_+$.

We claim that $\lambda \mapsto V_\lambda$ is continuous on $\overline{\mathbb{C}_+}$. Indeed, let $\lambda_1, \lambda_2 \in \overline{\mathbb{C}_+}$ and $f \in L^p(I;X)$. Then $V_{\lambda_j} f = Au_j$, where u_j is the L^p-solution to $u_j' + (\lambda_j + A)u_j = f$ and $u_j(0) = 0$. By Proposition 17.1.3 we can write $u_j = S_{\lambda_j} * f$, and thus by Young's inequality

$$\|u_j\|_{L^p(I;X)} \leqslant \|S_{\lambda_j}\|_{L^1(0,T;\mathscr{L}(X))} \|f\|_{L^p(I;X)}, \quad j \in \{1,2\}.$$

Subtracting the equations for u_1 and u_2 one sees that $u = u_1 - u_2$ satisfies $u' + (\lambda_1 + A)u = (\lambda_2 - \lambda_1)u_2$. Therefore, by maximal L^p-regularity,

$$\begin{aligned} \|(V_{\lambda_1} - V_{\lambda_2})f\|_{L^p(I;X)} &= \|Au\|_{L^p(I;X)} \\ &\leqslant M^{\rm reg}_{p,A,\lambda_1}(I)|\lambda_2 - \lambda_1| \|u_2\|_{L^p(I;X)}. \\ &\leqslant M^{\rm reg}_{p,A,\lambda_1}(I)|\lambda_2 - \lambda_1| \|S_{\lambda_2}\|_{L^1(0,T;\mathscr{L}(X))} \|f\|_{L^p(I;X)}. \end{aligned}$$

This gives the required continuity of $\lambda \mapsto V_\lambda$.

Since $\lambda \mapsto V_\lambda$ is holomorphic and uniformly bounded on the open half plane, and continuous on the closed half plane, the Phragmén–Lindelöf principle implies that for all $\Re\lambda > 0$,

$$M^{\rm reg}_{p,A,\lambda}(I) = \|V_\lambda\| \leqslant \sup_{\beta\in\mathbb{R}} \|V_{i\beta}\| = M^{\rm reg}_{p,A}(I),$$

where the last identity follows from (17.29).

Step 3 – In case $I = \mathbb{R}_+$, we can just let $T \to \infty$ in the previous estimate, and apply the analogue of Proposition 17.2.18 for the problem with the additional λ. □

Proof of Theorem 17.2.26 (1)–(3). We begin with the proof of (1). First let $I = (0,T)$ be a bounded interval and fix $\lambda \in \mathbb{C}$. By Propositions 17.1.3 and

17.2.27 the mild and L^p-solution to $u' + (\lambda + A)u = f$ with $u(0) = 0$ is given by $u = L_\lambda f := S_\lambda * f$, where $S_\lambda(t) = e^{-\lambda t}S(t)$. It is clear that L_λ is bounded on $L^p(I;X)$, and

$$\begin{aligned}\|(\lambda + A)u\|_{L^p(I;X)} &\leqslant \|\lambda u\|_{L^p(I;X)} + \|Au\|_{L^p(I;X)} \\ &\leqslant \big(\|\lambda L_\lambda\| + M^{\mathrm{reg}}_{p,A,\lambda}(I)\big)\|f\|_{L^p(I;X)}.\end{aligned}$$

Using (the comments right after)Proposition 17.2.27, it follows that

$$M^{\mathrm{reg}}_{p,\lambda+A}(I) \leqslant \|\lambda L_\lambda\| + e^{(\Re\lambda)_- T} M^{\mathrm{reg}}_{p,A}(I).$$

Next let $I = \mathbb{R}_+$ and $\Re\lambda > 0$. The previous considerations apply with $(\Re\lambda)_- = 0$; the boundedness of L_λ now follows from Young's inequality, which gives the estimate

$$\|\lambda L_\lambda\| \leqslant \|t \mapsto \lambda M e^{-t\lambda}\|_{L^1(\mathbb{R}_+)} = \frac{M|\lambda|}{\Re\lambda}$$

where $\|S(t)\| \leqslant M$ for all $t \geqslant 0$. This gives the estimate

$$M^{\mathrm{reg}}_{p,\lambda+A}(\mathbb{R}_+) \leqslant \frac{M|\lambda|}{\Re\lambda} + M^{\mathrm{reg}}_{p,A}(\mathbb{R}_+). \tag{17.36}$$

(2): If $0 < T' < |I|$, the result follows from (17.25) with $M^{\mathrm{reg}}_{p,A}(0,T') \leqslant M^{\mathrm{reg}}_{p,A}(I)$. It remains to consider the case $0 < |I| < T' < \infty$. By Theorem 17.2.15, the operator $-A$ generates an analytic semigroup $(S(t))_{t>0}$. Let $M \geqslant 1$ and $\omega \geqslant 0$ be such that $\|S(t)\| \leqslant Me^{\omega t}$ for all $t > 0$ and fix a number $\lambda > \omega$. By case (1) of the theorem, which we already proved, the translated operator $\lambda + A$ has maximal L^p-regularity on I. Its generated semigroup $S_\lambda(t) = e^{-\lambda t}S(t)$ is uniformly exponentially stable with $\|S_\lambda(t)\| \leqslant Me^{-(\lambda-\omega)t}$, where $\lambda - \omega > 0$ Therefore, by the Dore–Kato theorem 17.2.24, $A + \lambda$ has maximal L^p-regularity on $\mathbb{R}_+$, and then by restriction also on $(0, T')$. Another application of the translation property (case (1) of the theorem) shows that A has maximal L^p-regularity on $(0, T')$.

(3): Let $I_\lambda = \frac{1}{\lambda}I$ and fix an arbitrary $f \in L^p(I_\lambda;X)$. Note that v is the unique L^p-solution to $v' + Av = \lambda^{-1}f(\lambda^{-1}\cdot)$ on I if and only if $u := v(\lambda\cdot)$ is the unique L^p-solution to $u' + \lambda Au = f$ on I_λ, where we use zero initial conditions for both problems. Moreover,

$$\begin{aligned}\|\lambda Au\|_{L^p(I_\lambda;X)} &= \lambda^{1-\frac{1}{p}}\|Av\|_{L^p(I;X)} \\ &\leqslant \lambda^{-\frac{1}{p}} M^{\mathrm{reg}}_{p,A}(I)\|f(\lambda^{-1}\cdot)\|_{L^p(I;X)} = M^{\mathrm{reg}}_{p,A}(I)\|f\|_{L^p(I_\lambda;X)}.\end{aligned}$$

It follows that λA has maximal L^p-regularity on I_λ with $M^{\mathrm{reg}}_{p,\lambda A}(I_\lambda) \leqslant M^{\mathrm{reg}}_{p,A}(I)$. If $I = \mathbb{R}_+$ we are done; if I is bounded we apply (2). □

Remark 17.2.28. The estimate (17.36) shows that the constant $M^{\mathrm{reg}}_{p,\lambda+A}(\mathbb{R}_+)$ is uniformly bounded in $\lambda \in \Sigma_\phi$ for any $\phi \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$.

Next we present a converse to the translation result of Proposition 17.2.27, which can be used to reduce the question whether A has maximal L^p-regularity on $\mathbb{R}_+$ to the case where A is replaced by $\lambda + A$ with $\lambda > 0$.

Proposition 17.2.29 (Translation II). *Let $1 \leqslant p \leqslant \infty$ and let A be a linear operator on a Banach space X. Suppose that there is a constant $C \geqslant 0$ such that for all $0 < T < \infty$ and $0 < \lambda < \infty$ the operator $\lambda + A$ has maximal L^p-regularity on $(0,T)$, and that for all $f \in L^p(0,T;X)$ the L^p-solution u to*

$$\begin{cases} u' + (\lambda + A)u &= f \quad \text{on } (0,T), \\ u(0) &= 0, \end{cases} \tag{17.37}$$

satisfies

$$\|Au\|_{L^p(0,T;X)} \leqslant C\|f\|_{L^p(0,T;X)}. \tag{17.38}$$

Then A has maximal L^p-regularity on $\mathbb{R}_+$ and $M^{\rm reg}_{p,A}(\mathbb{R}_+) = C_A$, where C_A is the infimum of all admissible constants C in (17.38).

Proof. Fix $f \in L^p(\mathbb{R}_+;X)$. By Theorem 17.2.26(1ii), applied to $A+\lambda$ in place of A and $-\lambda$ in place of λ, the operator $A = (A+\lambda) - \lambda$ has maximal L^p-regularity on bounded intervals $(0,T)$. Therefore, as in Proposition 17.2.18 it follows that there exists a function $u : \mathbb{R}_+ \to X$ such that for any $T > 0$ its restriction to $(0,T)$ is an L^p-solution to (17.37) with $\lambda = 0$. Fix $\lambda > 0$. Since $u' + (\lambda + A)u = f + \lambda u$ on $(0,T)$, we obtain

$$\|Au\|_{L^p(0,T,X)} \leqslant C_A\|f + \lambda u\|_{L^p(0,T;X)}.$$

Passing to the limit $\lambda \downarrow 0$ we obtain $M^{\rm reg}_{p,A}(0,T) \leqslant C_A$. As T was arbitrary, Proposition 17.2.18 implies that A has maximal L^p-regularity on $\mathbb{R}_+$ with $M^{\rm reg}_{p,A}(\mathbb{R}_+) \leqslant C_A$. The converse estimate $C_A \leqslant M^{\rm reg}_{p,A}(\mathbb{R}_+)$ follows from Proposition 17.2.27(1). ☐

Extrapolation of integrability

We now set out to prove that maximal L^p-regularity of a linear operator A is independent of the integrability exponent p. This depends on a circle of ideas of a different flavour and will be deduced from the Calderón–Zygmund Theorem 11.2.5. The key point is to interpret maximal L^p-regularity as the L^p-boundedness of the singular integral operator V with kernel $t \mapsto \mathbf{1}_I(t)AS(t)$, where S is the analytic semigroup generated by $-A$ (see Theorem 17.2.19). It is assumed throughout that $I = (0,T)$ is a bounded interval or $I = \mathbb{R}_+$.

First we present a lemma in which we check that the kernel $t \mapsto \mathbf{1}_I(t)AS(t)$ and the operator V satisfy the conditions introduced in Definition 11.2.1. For later purposes we also check the conditions of Definition 11.3.1. We use the terminology and notation introduced in these two definitions.

Lemma 17.2.30. *Let A be a linear operator with maximal L^{p_0}-regularity on I for some $p_0 \in [1,\infty]$. Let S denote the analytic semigroup generated by $-A$, and set $C := \sup_{t\in I} \|tAS(t)\|$. Let $k : \mathbb{R} \to \mathscr{L}(X)$ be given by*

$$k(t) := \begin{cases} AS(t), & t > 0; \\ 0, & t \leqslant 0, \end{cases}$$

and define $K : I \times I \to \mathscr{L}(X)$ by

$$K(t,s) := k(t-s), \quad s,t \in I.$$

Then the following assertions hold:

(1) *the operator V defined in Theorem 17.2.19 has kernel K in the sense of Definition 11.2.1;*
(2) *the kernel K is C^1-Calderón–Zygmund kernel in the sense of Definition 11.3.1 with $c_K = C$ and $c_K^1 = c_K^2 = 4C^2$; a Dini kernel in the sense of the same definition with*

$$\|\omega_K\|_{\mathrm{Dini}} \leqslant 4C(1 + \log_+ 4C);$$

and an operator-Hörmander kernel in the sense of Definition 11.2.1 with

$$\|K\|_{\mathrm{H\ddot{o}r_{op}}(I)} = \|K\|_{\mathrm{H\ddot{o}r^*_{op}}(I)} \leqslant 8C(1 + \log_+ 4C).$$

(3) *the kernel K is an operator-Hörmander kernel in the sense of Definition 11.2.1 with $\|K\|_{\mathrm{H\ddot{o}r_{op}}(I)} = \|K\|_{\mathrm{H\ddot{o}r^*_{op}}(I)} \leqslant 4C^2$;*
(4) *the kernel K is a Dini kernel in the sense of Definition 11.3.1 with $c_K = C$ and $\|\omega\|_{\mathrm{Dini}} = 4C^2$.*

Note that we here specifically apply the Definition 11.2.1(4) with $I = (0,T)$ or $I = \mathbb{R}_+$; these are a one-dimensional cube and a quadrant, respectively.

By Theorem 17.2.15, $-A$ generates an analytic semigroup $(S(t))_{t>0}$, which is bounded if $I = \mathbb{R}_+$. By (K.4), $C := \sup_{t\in I} \|tAS(t)\|$ is a finite quantity. By Proposition K.3.1, $C \geqslant 1/e$ unless A is bounded, hence $4C > 1$, and $\log_+$ may be replaced by log in the Dini and Hörmander bounds of the lemma.

Proof. (1): By Theorem 17.2.19 the operator V extends to a bounded operator on $L^{p_0}(I;X)$. To show that V has kernel K, let $f \in L^{p_0}(I;X)$ be compactly supported and let $F = \mathrm{supp}(f)$ be its support. Fix $t \in I$ such that $\delta := \mathrm{dist}(t,F) > 0$. Then

$$\int_I \|K(t,s)f(s)\| \,\mathrm{d}s \leqslant \sup_{s\in[0,t-\delta]} \|AS(t-s)\| \int_F \|f(s)\| \,\mathrm{d}s < \infty.$$

Moreover, by Hille's Theorem 1.2.4,

$$Vf(t) = A\int_F S(t-s)f(s)\,\mathrm{d}s = \int_F AS(t-s)f(s)\,\mathrm{d}s = \int_I K(t,s)f(s)\,\mathrm{d}s.$$

(2): It is clear that for $s,t \in I$, $\|K(s,t)\| \leqslant C/|s-t|$, thus we can take $c_K = C$ in Definition 11.3.1. Moreover, for all $s,t \in I$ and $r \in \{s,t\}$, we have

$$\begin{aligned}\|(s-t)^2\partial_r K(s,t)\| &= \mathbf{1}_{\mathbb{R}_+}(s-t)\|(s-t)^2A^2S(s-t)\| \\ &\leqslant \mathbf{1}_{\mathbb{R}_+}(s-t)4\Big\|\tfrac{s-t}{2}AS(\tfrac{s-t}{2})\Big\|^2 \leqslant 4C^2.\end{aligned}$$

This verifies the claimed estimates of a C^1-Calderón–Zygmund kernel. (3) and (4): These are immediate from Lemma 11.3.4 in dimension $d = 1$; hence $2^{d+1} = 4$, and $\sigma_{d-1} = 2$ (the unit sphere is now $\{-1,1\}$ with $(d-1) = 0$-dimensional measure 2). □

As an application of Theorem 11.2.5 we obtain the following extrapolation result for maximal L^p-regularity. It gives the assertion (4) on extrapolation of Theorem 17.2.26.

Theorem 17.2.31 (Extrapolation of the exponent). *Let A have maximal L^{p_0}-regularity on I for some $p_0 \in [1,\infty]$. Then A has maximal L^p-regularity on I for all $p \in (1,\infty)$, with*

$$M^{\rm reg}_{p,A}(I) \leqslant c\cdot pp'\Big(M^{\rm reg}_{p_0,A}(I) + C(1+\log_+ C)\Big), \tag{17.39}$$

where c is an absolute constant, and $C := \sup_{t\in I} t\|AS(t)\|$, where S is the analytic semigroup generated by A under the assumptions.

Proof. By Lemma 17.2.30 the operator V has kernel K, where K is as in the lemma, and this kernel satisfies the operator-Hörmander conditions with the bounds stated in the said lemma. By Theorem 11.2.5, the operator V extends to a bounded operator on $L^p(I;X)$ with norm bounded by the right-hand side of (17.39). Therefore, by Theorem 17.2.19, A has maximal L^p-regularity on I for all $p \in (1,\infty)$, with the same bound. □

Duality

The following duality result for maximal L^p-regularity implies assertion (5) of Theorem 17.2.26. As before, we assume that $I = (0,T)$ is a bounded interval or $I = \mathbb{R}_+$.

Proposition 17.2.32 (Duality). *Let A be a densely defined closed operator acting in a Banach space X, and let $1 \leqslant p \leqslant \infty$. The following assertions are equivalent:*

(1) *A has maximal L^p-regularity on I;*
(2) *A^* has maximal $L^{p'}$-regularity on I.*

In this case we have $M^{\rm reg}_{p,A}(I) = M^{\rm reg}_{p',A^*}(I)$.

For the proof of this theorem we need the following lemma, which is a variant of Corollary 1.3.2 except for the fact that we interchange the roles of X and X^* for the sake of the application we have in mind; the corresponding version without this interchange is true as well, with the same proof (or as an application, by passing through the bidual).

Lemma 17.2.33. *Let $I \subseteq \mathbb{R}$ be a (bounded or unbounded) interval. If the subspace $Y \subseteq X$ is norming for X^*, then*

$$\{f \in C_{\mathrm{b}}(\overline{I}; X) : f \text{ takes values in } Y\} \tag{17.40}$$

is norming for $L^1(I; X^*)$ with respect to the duality

$$\langle f, g\rangle = \int_I \langle f(t), g(t)\rangle \,\mathrm{d}t, \quad f \in C_{\mathrm{b}}(\overline{I}; Y),\ g \in L^1(I; X^*).$$

In (17.40), we deliberately avoid denoting the space by $C_{\mathrm{b}}(\overline{I}; Y)$ for the following reason: if Y happens to have a norm $\|\ \|_Y$ of its own, the notation $C_{\mathrm{b}}(\overline{I}; Y)$ refers to functions $f : \overline{I} \to Y$ that are continuous with respect to $\|\ \|_Y$, while in (17.40), we think of continuity relative to the norm $\|\ \|_X$ of the ambient space X.

Proof. We will show that $\|g\|_{L^1(I;X^*)} = \sup_f |\langle f, g\rangle|$, where the supremum is taken over all f in (17.40) of norm $\leqslant 1$. The estimate "$\geqslant$" is clear. To prove the converse estimate, the density of simple functions (Lemma 1.2.19) and the regularity of the Lebesgue measure imply that it suffices to consider simple functions of the form $g = \sum_{j=1}^n \mathbf{1}_{I_j} x_j^*$, where $I_1, \dots, I_n$ are disjoint bounded intervals contained in I, and $x_1^*, \dots, x_n^*$ are elements of X^*. Let $\varepsilon \in (0,1)$ be arbitrary and choose functions $\varphi_j \in C_{\mathrm{c}}(I)$ such that $0 \leqslant \varphi_j \leqslant \mathbf{1}_{I_j}$ and $\int_{I_j} \varphi_j(t)\,\mathrm{d}t \geqslant (1-\varepsilon)[I_J|$ for each j. Choose $y_j \in Y$ of norm $\|y_j\| \leqslant 1$ such that $\|x_j^*\| \leqslant (1+\varepsilon)\langle y_j, x_j^*\rangle$. Then $f := \sum_{j=1}^n \varphi_j y_n$ has norm $\leqslant 1$ and

$$\begin{aligned}\langle f, g\rangle &= \sum_{j=1}^n \langle y_j, x_j^*\rangle \int_I \varphi_j(t)\,\mathrm{d}t \\ &\geqslant \frac{1-\varepsilon}{1+\varepsilon} \sum_{j=1}^n |I_j|\|x_j^*\| = \frac{1-\varepsilon}{1+\varepsilon}\|g\|_{L^1(I;X^*)}.\end{aligned}$$

□

Proof of Proposition 17.2.32. (1)⇒(2) for bounded intervals $I = (0,T)$: Let A have maximal L^p-regularity on $I = (0,T)$. By Theorem 17.2.15 A is sectorial of angle $< \pi/2$ and $-A$ generates an analytic semigroup $(S(t))_{t>0}$. As a consequence (see Remark K.1.12), $-A^*$ generates an analytic semigroup as well, and it is given by $(S^*(t))_{t>0}$, where $S^*(t) := (S(t))^*$. To prove that A^* has maximal $L^{p'}$-regularity on I we will use Theorem 17.2.19.

We begin with the case $p \in (1,\infty)$. Fix $f \in L^p(0,T;\mathsf{D}(A))$ (where, to be sure, we think of $\mathsf{D}(A)$ as a Banach space with respect to the graph norm $\|x\|_{\mathsf{D}(A)} := \|x\|_X + \|Ax\|_X$) and $g \in C^1([0,T];X^*)$, and write $\widetilde{f}(t) := f(T-t)$ and $\widetilde{g}(s) = g(T-s)$ for $s,t \in I$. Then the function $S * \widetilde{f}$ takes values in $\mathsf{D}(A)$, and by Proposition 17.1.4 we have $S^* * g \in C([0,T];X) \cap L^\infty(0,T;\mathsf{D}(A))$. By Fubini's theorem and suitable substitutions,

$$\begin{aligned}\langle f, A^*S^* * g\rangle &= \int_0^T \Big\langle Af(t), \int_0^t S^*(t-s)g(s)\,\mathrm{d}s\Big\rangle\,\mathrm{d}t \\ &= \int_0^T \Big\langle A\int_0^s S(s-t)\widetilde{f}(t)\,\mathrm{d}t, \widetilde{g}(s)\Big\rangle\,\mathrm{d}s = \langle AS * \widetilde{f}, \widetilde{g}\rangle,\end{aligned}$$

where the brackets in the first and last term refer to the duality between $L^p(I;X)$ and $L^{p'}(I;X^*)$, the latter viewed as a closed subspace of the dual of $L^p(I;X)$ (see Proposition 1.3.1). Therefore,

$$\begin{aligned}|\langle f, A^*S^* * g\rangle| = |\langle AS * \widetilde{f}, \widetilde{g}\rangle| &\leqslant \|AS * \widetilde{f}\|_{L^p(I;X)}\|\widetilde{g}\|_{L^{p'}(I;X^*)} \\ &\leqslant M^{\mathrm{reg}}_{p,A}(I)\|f\|_{L^p(I;X)}\|g\|_{L^{p'}(I;X^*)}.\end{aligned}$$

Since the embedding $L^p(I;\mathsf{D}(A)) \hookrightarrow L^p(I;X)$ is dense (recall that $\mathsf{D}(A)$ was assumed to be densely defined in Proposition 17.2.32 that we are proving), this estimate extends to all $f \in L^p(I;X)$ by density. Thus by Proposition 1.3.1

$$\|A^*S^* * g\|_{L^{p'}(I;X^*)} \leqslant M^{\mathrm{reg}}_{p,A}(I)\|g\|_{L^{p'}(I;X^*)}.$$

Therefore, A^* has maximal $L^{p'}$-regularity, and $M^{\mathrm{reg}}_{p',A^*}(I) \leqslant M^{\mathrm{reg}}_{p,A}(I)$ by Theorem 17.2.19 and the density of $C^1([0,T];X^*)$ in $L^{p'}([0,T];X^*)$.

In the case $p = \infty$, the above proof can be modified by using function from the subspace $\{f \in C([0,T];X) : f \text{ takes values in } \mathsf{D}(A)\}$, which is norming for $L^{p'}(0,T;X^*) = L^1(0,T;X^*)$ by Lemma 17.2.33.

Finally, let $p = 1$. Let $g \in L^\infty(I;X^*)$ and $f \in L^1(I;\mathsf{D}(A))$. By Theorem 17.2.31, A has maximal L^q-regularity on I for all $q \in (1,\infty)$. Therefore, by the previous proof, A^* has maximal L^q-regularity for all $q \in (1,\infty)$. In particular, Theorem 17.2.19 implies that $S^* * g$ takes values in $\mathsf{D}(A^*)$ almost everywhere on I. As before one can check that $\langle f, A^*S^* * g\rangle = \langle AS * \widetilde{f}, \widetilde{g}\rangle$, and thus

$$|\langle f, A^*S^* * g\rangle| \leqslant M^{\mathrm{reg}}_{1,A}(I)\|f\|_{L^1(I;X)}\|g\|_{L^\infty(I;X^*)}.$$

Therefore, as before we obtain that A^* has maximal L^∞-regularity, with $M^{\mathrm{reg}}_{\infty,A^*}(I) \leqslant M^{\mathrm{reg}}_{1,A}(I)$. This completes the proof of the "only if" part in the case $I = (0,T)$ is a bounded interval.

(2)⇒(1) for bounded intervals $I = (0,T)$: Let A^* have maximal $L^{p'}$-regularity on $I = (0,T)$. By Theorem 17.2.15, A^* is sectorial of angle $< \pi/2$, and hence A is sectorial of angle $< \pi/2$ as well, and therefore the densely defined (by assumption) A generates an analytic C_0-semigroup $(S(t))_{t>0}$. To complete the proof, first let $p \in [1,\infty)$. Fix functions $f \in L^p(I;\mathsf{D}(A))$ and $g \in L^{p'}(I;X^*)$, and define $\widetilde{f}$ and $\widetilde{g}$ as before. By the maximal $L^{p'}$-regularity of A^* on I, the convolution $S^* * \widetilde{g}$ takes values in $\mathsf{D}(A^*)$ almost everywhere on $(0,T)$. Using Fubini's theorem and the same substitutions as before, we obtain

$$\begin{aligned}\langle AS * f, g\rangle &= \int_0^T \int_0^t \langle S(t-s)Af(s), g(t)\rangle \,\mathrm{d}s\,\mathrm{d}t \\ &= \int_0^T \int_0^s \langle A\widetilde{f}(s), S^*(s-t)\widetilde{g}(t)\rangle \,\mathrm{d}t\,\mathrm{d}s = \langle f, A^*S^* * \widetilde{g}\rangle.\end{aligned}$$

Therefore, the maximal L^p-regularity of A and the bound $M^{\mathrm{reg}}_{p,A}(I) \leqslant M^{\mathrm{reg}}_{p',A^*}(I)$ follow as before.

If $p=\infty$, one can take $f \in L^\infty(I;X)$ and $g \in L^1(I;\mathsf{D}(A^*))$. The density of $\mathsf{D}(A^*)$ in X^* follows from Lemma 17.2.22 and Remark 17.2.23. Therefore, we can argue similarly as in the $p=1$ case of the implication (1)⇒(2) by reversing the roles of (A,p,f) and (A^*,p',g).

(2)⇔(1) for $I=\mathbb{R}_+$: The case $I=\mathbb{R}_+$ follows from the previous cases, and the following identities from Proposition 17.2.18:

$$M^{\mathrm{reg}}_{p',A}(\mathbb{R}_+) = \sup_{T>0} M^{\mathrm{reg}}_{p',A}(0,T), \qquad M^{\mathrm{reg}}_{p,A^*}(\mathbb{R}_+) = \sup_{T>0} M^{\mathrm{reg}}_{p,A^*}(0,T).$$

□

Extrapolation to weighted spaces

Earlier in this section, we have seen that maximal L^{p_0}-regularity for one exponent $p_0 \in [1,\infty]$ extrapolates to maximal L^p-regularity for all exponents $p \in (1,\infty)$. We shall now consider several extrapolation results to weighted spaces. Recall that a *weight* is a measurable function $w:(0,\infty)\to(0,\infty)$ such that $w \in L^1_{\mathrm{loc}}[0,\infty)$. Here we use the notation $L^1_{\mathrm{loc}}(\overline{I};X)$ introduced at the beginning of this chapter, taking $I=(0,\infty)$. To be explicit, the local integrability condition asks for w to be integrable on every bounded subinterval of $(0,\infty)$.

Let $I=(0,T)$ be a bounded interval or $I=\mathbb{R}_+=(0,\infty)$.

Given a weight w on $(0,\infty)$ and an exponent $p\in[1,\infty)$, the space $L^p(I,w;X)$ can be defined as the Banach space of strongly measurable functions $f:I\to X$ such that $t\mapsto w^{1/p}(t)f(t)$ belongs to $L^p(I;X)$. With respect to the natural norm on this space, for functions belonging to this space we have the identity

$$\|f\|_{L^p(I,w;X)} = \|w^{1/p}f\|_{L^p(I;X)}.$$

In order to also cover the exponent $p=\infty$ it is useful to proceed slightly differently.

Definition 17.2.34. *For weights $w \in L^p_{\rm loc}([0,\infty))$ we define $L^p_w(I;X)$ as the Banach space of strongly measurable functions $f : I \to X$ such that $t \mapsto w(t)f(t)$ belongs to $L^p(I;X)$. With respect to the natural norm on this space, we then have the identity*

$$\|f\|_{L^p_w(I;X)} = \|wf\|_{L^p(I;X)}.$$

In what follows we will always make the assumption that *the weight w is chosen in such a way that, as sets, we have the inclusion*

$$L^p_w(I;X) \subseteq L^1_{\rm loc}(\overline{I};X), \tag{17.41}$$

using again the notation introduced at the beginning of the chapter. For $p \in [1,\infty]$, the inclusion (17.41) holds if and only $w^{-1} \in L^{p'}_{\rm loc}(\overline{I})$.

With the spaces $L^p_w(I;X)$ at hand, we may define the notion of *maximal L^p_w-regularity* in the obvious manner, replacing all occurrences of L^p in Definition 17.2.4 by L^p_w. We leave it to the reader to check that several basic results on maximal L^p-regularity extend *mutatis mutandis* to maximal L^p_w-regularity; this includes Propositions 17.2.5, 17.2.10, and Theorem 17.2.19.

The present section will be concerned with identifying situations in which maximal L^p_w-regularity can be inferred from maximal L^q-regularity (where the exponents p and q are possibly different). We start with two elementary results in this direction for power weights t^α and exponential weights $e^{-\lambda t}$. From the point of view of applications to evolution equations, these suffice in most cases; some of them are also valid in some of the end-point cases $p = 1$ and $p = \infty$. After that, we consider the more complicated case of A_p-weights, restricting ourselves to $p \in (1,\infty)$ in that case.

In order to treat extrapolation to power weights, we need the following lemma. At the expense of additional technicalities, the implicit assumption of strong measurability with respect to the uniform operator norm can be relaxed somewhat, but in the applications we have in mind, it is always fulfilled.

Lemma 17.2.35. *Let X and Y be a Banach spaces, and let $p \in (1,\infty]$ and $\alpha \in \mathbb{R}$ be fixed. Let $k : \mathbb{R}_+ \times \mathbb{R}_+ \to [0,\infty)$ be given by*

$$k(t,s) = \mathbf{1}_{t\neq s} \frac{|1-(t/s)^\alpha|}{|t-s|} K(t,s),$$

where $K \in L^\infty(\mathbb{R}^2_+;\mathscr{L}(X,Y))$ is a given function, and let $D \subseteq \mathbb{R}_+ \times \mathbb{R}_+$ be a measurable set. Then the operator $T_k : L^p(\mathbb{R}_+;X) \to L^p(\mathbb{R}_+;Y)$, defined by

$$T_k f(t) = \int_{\mathbb{R}_+} \mathbf{1}_D(t,s)k(t,s)f(s)\,\mathrm{d}s, \quad t \in \mathbb{R}_+,$$

is a bounded of norm at most $C_\alpha\|K\|_\infty$, where $C_{\alpha,p}$ is a constant depending only on α and p, in each of the following cases (with $\frac{1}{p}+\frac{1}{p'}=1$):

(1) $D = \{(t,s) : t > s\}$ *and* $\alpha \in (-\infty, 1/p')$.

(2) $D = \mathbb{R}_+ \times \mathbb{R}_+$ *and* $\alpha \in (-1/p, 1/p')$.

Proof. Clearly, it suffices to consider $\alpha \neq 0$. Let $c : \mathbb{R}_+ \to \mathbb{R}_+$ be given by

$$c(t) = \mathbf{1}_J(t) t^{1/p} \frac{|1 - t^\alpha|}{|t - 1|},$$

where $J = (1, \infty)$ in case (1) and $J = (0, \infty) \setminus \{1\}$ in case (2). Then

$$\begin{aligned}
\|T_k f(t)\| &\leqslant \int_{\mathbb{R}_+} \mathbf{1}_D(t,s) \|k(t,s)\| \|f(s)\| \, \mathrm{d}s \\
&\leqslant \|K\|_\infty t^{-1/p} \int_{\mathbb{R}_+} \mathbf{1}_J(t/s)(t/s)^{1/p} \frac{|1-(t/s)^\alpha|}{|(t/s)-1|} s^{1/p} \|f(s)\| \, \frac{\mathrm{d}s}{s}. \\
&= \|K\|_\infty t^{-1/p} c \star g(t),
\end{aligned}$$

where $g(s) = s^{1/p}\|f(s)\|$ and we used the convolution product $\star$ of the group $(\mathbb{R}_+, \cdot)$ with Haar measure $\mathrm{d}s/s$. Therefore, Young's inequality implies

$$\begin{aligned}
\|T_k f\|_{L^p(\mathbb{R}_+;X)} &\leqslant \|K\|_\infty \|c \star g\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t})} \\
&\leqslant \|K\|_\infty \|c\|_{L^1(\mathbb{R}_+, \frac{\mathrm{d}t}{t})} \|g\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t})} \\
&= \|K\|_\infty \|c\|_{L^1(\mathbb{R}_+, \frac{\mathrm{d}t}{t})} \|f\|_{L^p(\mathbb{R}_+;X)}.
\end{aligned}$$

It remains to check that $\|c\|_{L^1(\mathbb{R}_+, \frac{\mathrm{d}t}{t})} < \infty$. We do this by splitting $\mathbb{R}_+$ into the three parts $(0, 1/2) \cup (1/2, 3/2) \cup (3/2, \infty)$. Using that $\alpha < 1/p'$ and $p > 1$, we find

$$\int_{3/2}^\infty c(t) \, \frac{\mathrm{d}t}{t} \leqslant 3 \int_{3/2}^\infty t^{\frac{1}{p} - 2} \max\{t^\alpha, 1\} \, \mathrm{d}t < \infty.$$

On the interval $(1/2, 3/2)$, c is uniformly bounded by the mean value theorem and therefore $\int_{1/2}^{3/2} c(t) \, \frac{\mathrm{d}t}{t} < \infty$. On the interval $(0, 1/2)$, c is only non-zero in case (2), and in this situation $\alpha > -1/p$ implies

$$\int_0^{1/2} c(t) \, \frac{\mathrm{d}t}{t} \leqslant 2 \int_0^{1/2} t^{\frac{1}{p} - 1} \max\{1, t^\alpha\} \, \mathrm{d}t < \infty.$$

□

Using Lemma 17.2.35 we can prove the equivalence of weighted and unweighted maximal L^p-regularity in case of power weights. The reader may check that the proof below can be extended to various other classes of integral operators. Actually, a similar argument has already been used in Proposition 10.2.31 for proving weighted L^p-boundedness of the Hilbert transform.

Proposition 17.2.36 (Extrapolation with power weights). *Let A be a linear operator on a Banach space X, let $I = (0,T)$ or $I = \mathbb{R}_+$, let $\frac{1}{p} + \frac{1}{p'} = 1$, and consider the weight*

$$w_\alpha(t) = t^\alpha \quad \textit{with} \quad \alpha \in (-1/p, 1/p').$$

Then the following assertions hold:

(1) *If $p \in (1,\infty]$, then A has maximal L^p-regularity on I if and only if A has maximal $L^p_{w_\alpha}$-regularity on I.*
(2) *If A has maximal L^1-regularity on I, then A has maximal $L^1_{w_\alpha}$-regularity on I.*

We do not know about a possible converse of (2), but it does not seem to be particularly relevant for applications.

Proof. (1): Using the scaling properties of the weights w_α, one can check that Theorem 17.2.15 extends to the weighted setting. Therefore, $-A$ generates an analytic semigroup, which is bounded in case $I = \mathbb{R}_+$. In view of Theorem 17.2.19 and its weighted extension, it suffices to prove that the operator V introduced in this theorem is bounded on $L^p(I;X)$ if and only if it is bounded on $L^p_{w_\alpha}(I;X)$.

The boundedness of V on $L^p_{w_\alpha}(I;X)$ is equivalent to the boundedness of the operator V_α on $L^p(I;X)$, where

$$V_\alpha f(t) := A\int_I \mathbf{1}_{t>s} t^\alpha S(t-s)s^{-\alpha} f(s)\,\mathrm{d}s.$$

For $\lambda \in \varrho(A)$,

$$\begin{aligned}(\lambda - A)^{-1}V_\alpha f(t) - &(\lambda - A)^{-1}Vf(t) \\ &= \int_I \mathbf{1}_{t>s} A(\lambda - A)^{-1}S(t-s)((t/s)^\alpha - 1)f(s)\,\mathrm{d}s \\ &= (\lambda - A)^{-1}\int_I k(t,s)f(s)\,\mathrm{d}s,\end{aligned}$$

where $k(t,s)$ is as in Lemma 17.2.35(1) with $K(t,s) = \mathbf{1}_{t>s}\cdot(t-s)AS(t-s)$. Since $K \in L^\infty(\mathbb{R}^2_+;\mathscr{L}(X))$, the lemma gives that $f \mapsto \int_I k(\cdot,s)f(s)\,\mathrm{d}s$ is bounded on $L^p(I;X)$. Applying $\lambda - A$ on both sides of the above identity, we conclude that the boundedness of V and V_α are equivalent.

(2): By Lemma 17.2.22 there is a constant C such that

$$\int_I \|AS(t)x\|\,\mathrm{d}t \leqslant C\|x\|, \quad x \in X.$$

Let $f \in L^1_{w_\alpha}(I;X)$ and set $T = \infty$ if $I = \mathbb{R}_+$. Then, since $\alpha \leqslant 0$,

$$\begin{aligned}\int_I\int_0^t \|AS(t-s)f(s)\|\,\mathrm{d}s\,w_\alpha(t)\,\mathrm{d}t &= \int_I\int_s^T \|AS(t-s)f(s)\|t^\alpha\,\mathrm{d}t\,\mathrm{d}s\\ &\leqslant \int_I\int_I \|AS(t)f(s)\|s^\alpha\,\mathrm{d}t\,\mathrm{d}s\\ &\leqslant C\|f\|_{L^1_{w_\alpha}(I;X)}.\end{aligned}$$

This implies that V is bounded on $L^1_{w_\alpha}(I;X)$ with norm at most C. □

The proof actually shows that, in case $-A$ generates a bounded analytic semigroup, we could also allow any $\alpha \leqslant -1/p$ in Proposition 17.2.36. Since this is unimportant for applications to evolution equations and since such weights are not locally integrable near zero (which would lead to problems in Theorem 17.2.15), we do not elaborate on this any further.

We will apply the above result to the inhomogeneous problem

$$\begin{cases} u'(t) + Au(t) &= f, \quad t \in I,\\ u(0) &= x,\end{cases} \tag{17.42}$$

where $I = (0,T)$ or $I = \mathbb{R}_+$. The aim is to extend Proposition 17.2.14 to the weighted setting. In the absence of weights, this proposition was proved using the trace method. The proof presented here is different and uses the semigroup generated by $-A$.

We fix $p, p' \in [1,\infty]$ satisfying $\frac{1}{p} + \frac{1}{p'} = 1$, and consider the weight

$$w_\alpha(t) = t^\alpha \quad \text{for} \quad \alpha \in (-1/p, 1/p') \cup \{0\}.$$

We define

$$X_{1-\frac{1}{p}-\alpha,p} := (X, \mathsf{D}(A))_{1-\frac{1}{p}-\alpha,p} \ \text{ for } \ \alpha \in (-1/p, 1/p')$$

and set $X_{0,1} := X$ and $X_{1,\infty} := \mathsf{D}(A)$.

Corollary 17.2.37. *Let A be a linear operator acting in a Banach space X. If A has maximal L^p-regularity on I and $f \in L^p_{w_\alpha}(I;X)$, then for all $x \in X_{1-\frac{1}{p}-\alpha,p}$ there is a unique strong solution u of (17.42) such that $Au, u' \in L^p_{w_\alpha}(I;X)$, and there is a constant $C \geqslant 0$, independent of f and x, such that*

$$\|Au\|_{L^p_{w_\alpha}(I;X)} + \|u'\|_{L^p_{w_\alpha}(I;X)} \leqslant C(\|f\|_{L^p_{w_\alpha}(I;X)} + \|x\|_{X_{1-\frac{1}{p}-\alpha,p}}). \tag{17.43}$$

As in Proposition 17.2.13 one checks that the conditions on the data (f,x) are also necessary if $\alpha + \frac{1}{p} \in (0,1)$.

By choosing α appropriately, we can allow any $u(0) = x \in (X, \mathsf{D}(A))_{\theta,p}$ for $\theta \in (0,1)$. On the other hand, if $p \in (1,\infty)$, we can apply Theorem L.4.1 to see that $u \in C(I; X_{1-\frac{1}{p},p})$ (i.e, the solution *regularises instantaneously* from its initial value $u(0)$ into $X_{1-\frac{1}{p},p}$. This agrees with the behaviour of L^p-solutions without weights. Note that an estimate for $\|u\|_{L^p_{w_\alpha}(I;X)}$ is not included in (17.43), but can be obtained in the case of a bounded interval $I = (0,T)$ by the same method as in (17.5).

Proof. By Dore's Theorem 17.2.15, $-A$ generates an analytic semigroup S, and this semigroup is uniformly bounded if $I = \mathbb{R}_+$. Proposition 17.1.3 then gives the uniqueness of strong solutions.

By Proposition 17.2.36, A has maximal $L^p_{w_\alpha}$-regularity on I. To derive existence and the estimate (17.42) we write $u = u_f + u_x$, where $u_f \in L^p_{w_\alpha}(I; \mathsf{D}(A))$ is the strong solution to (17.42) with $x = 0$, and $u_x = S(\cdot)x$. By maximal $L^p_{w_\alpha}$-regularity we obtain

$$\|Au_f\|_{L^p_{w_\alpha}(I;X)} \leqslant C\|f\|_{L^p_{w_\alpha}(I;X)}.$$

Also, by Theorem L.2.4,

$$\|Au_x\|_{L^p_{w_\alpha}(I;X)} \leqslant C\|x\|_{X_{1-\frac{1}{p}-\alpha,p}}.$$

Note that the extremal case $\alpha = 0$ and $p = \infty$ is trivial. For the extremal case $\alpha = 0$ and $p = 1$ we use Lemma 17.2.22.

Clearly, $u = u_f + u_x$ is a strong solution to (17.42), and combining the estimates gives (17.43). □

For later use we observe that, by restriction and uniqueness, the optimal constant C in the estimate is monotone with respect to this interval I.

Next we consider the case of exponential weights. Here, only the case $I = \mathbb{R}_+$ is of interest, as $t \mapsto e^{\lambda t}$ is uniformly bounded from below and above on bounded subsets of $\mathbb{R}_+$.

Proposition 17.2.38 (Extrapolation with exponential weights). *Suppose that A is a linear operator in X that has maximal L^p-regularity on $\mathbb{R}_+$ for some $p \in [1, \infty)$. Then for all $\lambda \geqslant 0$, the operator A has maximal $L^p_{e^{-\lambda(\cdot)}}$-regularity on $\mathbb{R}_+$ with constant at most $M^{\mathrm{reg}}_{p,A}(\mathbb{R}_+)$.*

Proof. Since A has maximal L^p-regularity on $\mathbb{R}_+$, there exists a unique strong solution to the problem (17.42) with initial value $x = 0$. Therefore, it suffices to prove $L^p_{e^{-\lambda(\cdot)}}(\mathbb{R}_+; X)$-estimates for the solution.

For $f \in L^p_{e^{-\lambda(\cdot)}}(\mathbb{R}_+; X)$ we write $f_\lambda = e^{-t\lambda} f$. By Proposition 17.2.27 there exists a unique L^p-solution v of $v' + (A + \lambda)v = f_\lambda$ with initial condition $v(0) = 0$, and

$$\|Av\|_{L^p(\mathbb{R}_+;X)} \leqslant M^{\mathrm{reg}}_{p,A}(\mathbb{R}_+)\|f_\lambda\|_{L^p(\mathbb{R}_+;X)} = M^{\mathrm{reg}}_{p,A}(\mathbb{R}_+)\|f\|_{L^p_{e^{-\lambda(\cdot)}}(\mathbb{R}_+;X)}.$$

It follows that the function u defined by $u(t) = e^{\lambda t} v(t)$ is the unique strong solution to (17.42) with initial condition $x = 0$ and by the previous estimate we obtain

$$\|Au\|_{L^p_{e^{-\lambda(\cdot)}}(\mathbb{R}_+;X)} = \|Av\|_{L^p(\mathbb{R}_+;X)} \leqslant M^{\mathrm{reg}}_{p,A}(\mathbb{R}_+)\|f\|_{L^p_{e^{-\lambda(\cdot)}}(\mathbb{R}_+;X)}.$$

□

We continue with a result that allows us to extrapolate to weighted spaces $L^p(I, w; X)$, with weights w taken from the Muckenhoupt A_p class (see Appendix J). Here we will rely on the A_2 Theorem 11.3.26. In order to make a connection with Propositions 17.2.36 and 17.2.38 we record two simple observations: (i) $|\cdot|^{\alpha p} \in A_p$ if and only if $\alpha \in (-1/p, 1/p')$, and (ii) the exponential weights $e^{\lambda(\cdot)}$ belong to A_p only in the trivial case $\lambda = 0$. Both assertions follow from elementary calculations.

Theorem 17.2.39 (Extrapolation with A_p-weights). *Suppose that A is a linear operator in X that has maximal L^{p_0}-regularity on an interval I for some $p_0 \in [1, \infty]$. Then for all $p \in (1, \infty)$ and weights $w \in A_p(I)$ the operator A has* maximal $L^p(w)$-regularity on I *in the sense that for all $f \in L^p(I, w; X)$ there exists a unique strong solution u of* (ACP$_0$)*, which satisfies $Au \in L^p(I, w; X)$ and*

$$\|Au\|_{L^p(I,w;X)} \leqslant c_p\Big(M^{\rm reg}_{p_0,A}(I) + C(1+\log_+ C)\Big)[w]_{A_p(I)}^{\max(1,\frac{1}{p-1})}\|f\|_{L^p(I,w;X)},$$

where c_p depends only on p, and $C := \sup_{t\in I} t\|AS(t)\|$, where S is the analytic semigroup generated by A under the assumptions.

Proof. Fix $p \in (1, \infty)$ and $w \in A_p(I)$. Theorem 17.2.19 extends to the weighted setting. Accordingly, it suffices to check that the operator V defined in that theorem extends to a bounded operator on $L^p(I, w; X)$. By Lemma 17.2.30, the operator V has kernel K, which is a Dini kernel with $c_K \leqslant C$ and $\|\omega_K\|_{\rm Dini} \leqslant 4C(1 + \log_+(4C))$. Therefore, by Theorem 11.3.26, applied in the case of a one-dimensional cube $I = (0, T)$ or quadrant $I = \mathbb{R}_+$, the operator V extends to a bounded operator on $L^p(I, w; X)$ of norm

$$\begin{aligned}\|V\|_{\mathscr{L}(L^p(I,w;X))} &\leqslant c_p\Big(\|V\|_{\mathscr{L}(L^{p_0}(I;X))} + c_K + \|\omega_K\|_{\rm Dini}\Big)[w]_{A_p(I)}^{\max(1,\frac{1}{p-1})} \\ &\leqslant c_p'\Big(M^{\rm reg}_{p_0,A}(I) + C(1+\log_+ C)\Big)[w]_{A_p(I)}^{\max(1,\frac{1}{p-1})}.\end{aligned}$$

As in Theorem 17.2.19 we see that for all $f \in L^p(I, w; X)$ there exists a unique $L^p(w)$-solution u of (ACP$_0$), which satisfies $Au = Vf$, and hence the claimed estimate for solutions follows by combining the previous operator norm bound with

$$\|Au\|_{L^p(I,w;X)} \leqslant \|V\|_{\mathscr{L}(L^p(I,w;X))}\|f\|_{L^p(I,w;X)}.$$

□

17.2.f Maximal continuous regularity

In this section, we introduce the notion of *maximal continuous regularity*, or, more briefly, *maximal C-regularity*, and show that for densely defined operators it is equivalent to maximal L^∞-regularity. Once this has been proved, it is no longer necessary to distinguish between these notions.

Throughout this section we let $I = (0,T)$ be a bounded interval or $I = \mathbb{R}_+$. As before we consider the inhomogeneous abstract Cauchy problem

$$\begin{cases} u'(t) + Au(t) &= f(t), \quad t \in I, \\ u(0) &= 0, \end{cases} \tag{ACP$_0$}$$

Definition 17.2.40. *A linear operator A acting in a Banach space X has* maximal C-regularity *on I if there exists a constant $C \geqslant 0$ such that for all $f \in C_{\rm b}(\overline{I};X)$ the problem (ACP$_0$) admits a unique strong solution u_f on I, the almost everywhere defined function Au_f is equal almost everywhere to a function in $C_{\rm b}(\overline{I};X)$, and this version satisfies*

$$\|Au_f\|_{C_{\rm b}(\overline{I};X)} \leqslant C\|f\|_{C_{\rm b}(\overline{I};X)}. \tag{17.44}$$

The least admissible constant in this definition will be called the *maximal C-regularity constant* of A on I and will be denoted by $M^{\rm reg}_{{\rm cont},A}(I)$.

Proposition 17.2.41. *If A has maximal C-regularity on I, then:*

(1) *A is closed;*
(2) *for all $f \in C_{\rm b}(\overline{I};X)$, the strong solution u_f takes values in $\mathsf{D}(A)$ and defines a continuous function from I to $\mathsf{D}(A)$.*

Remark 17.2.42. As a consequence of this proposition, an equivalent definition of maximal C-regularity is obtained if these properties are built into the definition. It is with this equivalent version of the definition that we will always work.

Proof. (1): This follows by repeating the proof of the corresponding result for maximal L^p-regularity (Proposition 17.2.5) *verbatim.*

(2): Fix $f \in C_{\rm b}(\overline{I};X)$. It follows from the definition of a strong solution that $u_f(t) \in \mathsf{D}(A)$ for almost all $t \in I$ and $u_f \in C_{\rm b}(\overline{I};X)$. By the definition of maximal C-regularity, the almost everywhere defined function Au_f is equal almost everywhere, say outside the null set N_f, to a function $F \in C_{\rm b}(\overline{I};X)$. Now let $t \in \overline{I}$ be arbitrary and choose a sequence $t_n \to t$, with every t_n in $I \setminus N_f$. Then $u_f(t_n) \to u_f(t)$ in X by continuity, and $Au_f(t_n) \to F(t)$ in X. Hence by closedness, $u_f(t) \in \mathsf{D}(A)$ and similarly $Au_f(t) = F(t)$. This proves the first assertion. The second is now evident from the fact that both u_f and Au_f are well defined pointwise and continuous as X-valued functions. □

As in the case of maximal L^p-regularity, in order to check maximal C-regularity for a given *closed* operator A it suffices to prove existence, uniqueness, and the estimate (17.44) for strong solutions corresponding to inhomogeneities f in a dense subspace F on $C_{\rm b}(\overline{I};X)$. The reader may check that all results in Section 17.2.a have an analogue for maximal C-regularity, with similar proofs. In particular we record the following analogue of Proposition 17.2.7:

Proposition 17.2.43. *Suppose that A has maximal C-regularity on I. If u_f is the unique strong solution on I of* (ACP$_0$)*, then for all bounded subintervals $(0,T) \subseteq I$ we have $u_f \in C([0,T];\mathsf{D}(A)) \cap C^1([0,T];X)$ and*

$$\|u_f\|_{C([0,T];\mathsf{D}(A))\cap C^1([0,T];X)} \leqslant (1+T)(M^{\mathrm{reg}}_{\mathrm{cont},A}(I)+1)\|f\|_{C_{\mathrm{b}}(\overline{I};X)}.$$

The following is the analogue of Dore's Theorem 17.2.15 and can be proved in exactly the same way.

Theorem 17.2.44. *Let A be a linear operator on a Banach space X. Then:*

(1) *if A has maximal C-regularity on a bounded interval $(0,T)$, then $-A$ generates an analytic semigroup on X, and $\lambda + A$ is sectorial of angle $< \pi/2$ for all $\lambda \in \mathbb{R}$ large enough;*
(2) *if A has maximal C-regularity on $\mathbb{R}_+$, then $-A$ generates a bounded analytic semigroup on X, and A is sectorial of angle $< \pi/2$. Moreover,*

$$\|A(\lambda + A)^{-1}\| \leqslant M^{\mathrm{reg}}_{\mathrm{cont},A}(\mathbb{R}_+), \quad \lambda \in \mathbb{C}_+. \tag{17.45}$$

As a consequence, we obtain analogues for maximal C-regularity of Lemma 17.2.16, Proposition 17.2.18, Theorem 17.2.19, Theorem 17.2.24, Corollary 17.2.25, and Theorem 17.2.26(1)–(3).

By Proposition 17.1.3, for every $f \in C(\overline{I};X)$ the strong solution u provided by Definition 17.2.40 coincides with the mild solution:

$$u(t) = \int_0^t S(t-s)f(s)\,\mathrm{d}s, \quad t \in \overline{I}.$$

This will be used in the proof of Theorem 17.2.46.

It is important to observe that $C_{\mathrm{b}}(I;\mathsf{D}(A))$ does not need to be dense in $C_{\mathrm{b}}(I;X)$ if I is unbounded, even when $\mathsf{D}(A)$ is dense in X. (While it is easy to approximate $f \in C_{\mathrm{b}}(I;X)$ by functions with values in $\mathsf{D}(A)$, the difficulty is ensuring that the approximating function would also be bounded with respect to the graph norm $\|x\|_{\mathsf{D}(A)} = \|x\|_X + \|Ax\|_X$ of $\mathsf{D}(A)$.) However, we do have the following result.

Lemma 17.2.45 (Density). *Suppose that A is densely defined and $-A$ generates a C_0-semigroup S. Let $I = \mathbb{R}_+$ or $I = (0,T)$. If there exists a constant $C \geqslant 0$ such that for all $f \in C_{\mathrm{b}}(\overline{I};\mathsf{D}(A))$ we have*

$$\|AS * f(t)\| \leqslant C\|f\|_{C_{\mathrm{b}}(\overline{I};\mathsf{D}(A))}, \quad t \in \overline{I},$$

then A has maximal C-regularity on I and $M^{\mathrm{reg}}_{\mathrm{cont},A}(I) \leqslant C$.

Under the assumptions stated, $S * f$ is continuous with values in $\mathsf{D}(A)$, and therefore $AS * f(t)$ is well defined for all $t \in \overline{I}$.

Proof. If $I = (0,T)$, the result is clear from the density of $C_{\rm b}(\overline{I};\mathsf{D}(A))$ in $C_{\rm b}(\overline{I};X)$. Next let $I = \mathbb{R}_+$. We already observed that $u := S * f$ is continuous with values in $\mathsf{D}(A)$. Let $f \in C_{\rm b}(\overline{I};X)$ and fix $T > 0$. Choose $f_n \in C_{\rm b}([0,T];\mathsf{D}(A))$ such that $f_n \to f|_{[0,T]}$ in $C_{\rm b}([0,T];X)$. Let $\overline{f}(t) = f(T \wedge t)$ and $\overline{f}_n(t) = f_n(T \wedge t)$. Then $\overline{f}_n \to \overline{f}$ in $C_{\rm b}([0,\infty);X)$. It follows that $u_n := S * \overline{f}_n$ is a Cauchy sequence and hence convergent in $C_{\rm b}([0,T];\mathsf{D}(A))$. Since $u_n \to u := S * f$ in X pointwise on $[0,T]$, it follows that $u := S * f \in C_{\rm b}([0,T];\mathsf{D}(A))$ and

$$
\begin{aligned}
\|Au\|_{C_{\rm b}([0,T];X)} &= \lim_{n\to\infty} \|Au_n\|_{C_{\rm b}([0,T];X)} \\
&\leqslant \lim_{n\to\infty} C\|\overline{f}_n\|_{C_{\rm b}([0,T];X)} \\
&\leqslant \lim_{n\to\infty} C\|\overline{f}_n\|_{C_{\rm b}([0,\infty);X)} \leqslant C\|f\|_{C_{\rm b}([0,\infty);X)}
\end{aligned}
$$

Since T was arbitrary this proves that $u = S * f$ satisfies $Au \in C_{\rm b}([0,\infty);X)$ and $\|Au\|_{C_{\rm b}([0,\infty);X)} \leqslant C\|f\|_{C_{\rm b}([0,\infty);X)}$. □

Our next aim is to show that maximal C-regularity coincides with maximal L^∞-regularity.

Theorem 17.2.46 (Continuous versus L^∞-type maximal regularity). *A densely defined linear operator A has maximal C-regularity if and only if it has maximal L^∞-regularity, and in that case $M^{\rm reg}_{{\rm cont},A}(I) = M^{\rm reg}_{\infty,A}(I)$.*

Proof. Each of the two conditions implies that $-A$ generates a C_0-semigroup S on the Banach space X in which A acts.

Suppose first that A has maximal L^∞-regularity. Let $f \in C_{\rm b}(\overline{I};\mathsf{D}(A))$. Then $u = S * f$ is the unique L^∞-solution to (ACP$_0$). Since u is continuous with values in $\mathsf{D}(A)$ it follows that

$$
\sup_{t\in\overline{I}} \|Au(t)\| = \|Au\|_{L^\infty(I;X)} \leqslant M^{\rm reg}_{\infty,A}(I)\|f\|_{L^\infty(I;X)}.
$$

This implies that A has maximal C-regularity. This part of the proof does not use the density of $\mathsf{D}(A)$.

Next suppose that A is densely defined and has maximal C-regularity. As in the proof of Proposition 17.2.32 one shows that A^* has maximal L^1-regularity with constant at most $M^{\rm reg}_{{\rm cont},A}(I)$. Now Proposition 17.2.32 implies that A has maximal L^∞-regularity with $M^{\rm reg}_{\infty,A}(I) = M^{\rm reg}_{1,A^*}(I) \leqslant M^{\rm reg}_{{\rm cont},A}(I)$. □

Conditions for maximal L^∞ and C-regularity will be given in Section 17.3.c.

For later applications we also state a version of Proposition 17.2.36 for maximal C-regularity with power weights. For this purpose, for $\alpha \in \mathbb{R}$ we let $w_\alpha(t) := t^\alpha$ and consider the Banach space

$$
C_{w_\alpha}(I;X) := \Big\{ f \in C(\overline{I}\setminus\{0\};X) : \sup_{t\in I} \|t^\alpha f(t)\| < \infty \Big\}
$$

with norm

$$\|f\|_{C_{w_\alpha}(I;X)} := \sup_{t\in I} \|t^\alpha f(t)\|.$$

Replacing $C_{\rm b}(\overline{I};X)$ by $C_{w_\alpha}(I;X)$ in the definition of maximal C-regularity, we obtain the definition of *maximal C_{w_α}-regularity.*

Proposition 17.2.47 (Extrapolation with power weights). *Let A be a linear operator A in X. For any $\alpha \in [0,1)$, the following assertions are equivalent:*

(1) *A has maximal C-regularity on I;*
(2) *A has maximal $C_{{\rm b},\alpha}$-regularity on I.*

Proof. Fixing $\alpha \in [0,1)$, we can repeat the argument of Proposition 17.2.36, using the additional fact that in Lemma 17.2.35(1) the operator T_k maps $L^\infty(I;X)$ into $C_{\rm b}(\overline{I};Y)$ if K is continuous (using notation introduced in the lemma). To check the latter, let

$$c(t,s) := \mathbf{1}_{t>s}\frac{|1-(t/s)^\alpha|}{|t-s|}$$

and recall from the proof of Lemma 17.2.35 that $c(t,\cdot) \in L^1(I)$. We have

$$\begin{aligned}\|T_k f(t+h) - T_k f(t)\|_Y &\leqslant \|f\|_\infty \int_I c(t,s)\|K(t+h,s) - K(t,s)\|\,{\rm d}s \\ &\quad + \|f\|_\infty\|K\|_\infty \int_I |c(t+h,s) - c(t,s)|\,{\rm d}s.\end{aligned}$$

The first term tends to zero as $h \to 0$ by dominated convergence. For the second term, note that, if $h > 0$, one can write

$$\int_I |c(t+h,s)-c(t,s)|\,{\rm d}s = \int_0^t |c(t+h,s)-c(t,s)|\,{\rm d}s + \int_t^{t+h} c(t+h,s)\,{\rm d}s.$$

To estimate the second term here, the mean value theorem applied to $x^\alpha - 1$ with $x \geqslant 1$, gives

$$\int_t^{t+h} c(t+h,s)\,{\rm d}s \leqslant \alpha \int_t^{t+h} \frac{\frac{t+h}{s}-1}{t+h-s}\,{\rm d}s = \int_t^{t+h} \frac{\alpha}{s}\,{\rm d}s \to 0$$

as $h \downarrow 0$. For the first term, the triangle inequality and the mean value theorem lead to

$$\begin{aligned}|c(t+h,s)-c(t,s)| &\leqslant \frac{(t+h)^\alpha - t^\alpha}{s^\alpha(t+h-s)} + ((t/s)^\alpha - 1)\Big|\frac{1}{t+h-s} - \frac{1}{t-s}\Big| \\ &\leqslant \alpha\frac{t^{\alpha-1}h}{s^\alpha(t+h-s)} + ((t/s)^\alpha - 1)\frac{h}{(t+h-s)(t-s)}\end{aligned}$$

$$=: I + II.$$

Since $\frac{t^{\alpha-1}h}{s^\alpha(t+h-s)} \leqslant \frac{t^{\alpha-1}}{s^\alpha}$ is integrable, part I can be handled via dominated convergence. For part II, we estimate

$$\int_0^{t/2} II \, \mathrm{d}s \leqslant \frac{4h}{t^2} \int_0^{t/2} ((t/s)^\alpha - 1) \, \mathrm{d}s \to 0,$$

and, the mean value theorem and dominated convergence theorem,

$$\int_{t/2}^t II \, \mathrm{d}s \leqslant \int_{t/2}^t \alpha\Big(\frac{t}{s} - 1\Big) \frac{h}{(t+h-s)(t-s)} \, \mathrm{d}s = \int_{t/2}^t \frac{\alpha h}{s(t+h-s)} \, \mathrm{d}s \to 0.$$

The case $h < 0$ can be treated in a similar way. □

Next we present a version of Corollary 17.2.37 for maximal C-regularity. In Chapter 18 we also need the variants of the on subspaces of C_{w_α}, which we first introduce. For $\alpha \in [0, 1)$, $I = (0, T]$ or $I = (0, \infty)$ let

$$C_{w_\alpha,0}(I; X) := \{u \in C_{w_\alpha}(I; X) : \lim_{t \downarrow 0} t^\alpha \|u(t)\| = 0\},$$

$$C^1_{w_\alpha}(I; X) := \{u \in C_{w_\alpha}(I; X) : u' \in C_{w_\alpha}(I; X)\}.$$

$$\|u\|_{C^1_{w_\alpha}(I;X)} = \sup_{t\in I} t^\alpha \|u(t)\| + \sup_{t\in I} t^\alpha \|u'(t)\|$$

for $u \in C^1_{w_\alpha}(I; X)$. Then one can check that both $C_{w_\alpha}(I; X)$ and $C^1_{w_\alpha}(I; X)$ are Banach spaces, and $C_{w_\alpha,0}(I; X)$ is a closed subspace of C_{w_α}.

Corollary 17.2.48. *Let A be a linear operator acting in a Banach space X, and suppose that A has maximal C-regularity on $I = (0, T]$ or $I = \mathbb{R}_+$. For $\alpha \in (0, 1)$ let $X_{1-\alpha,\infty}$ denote the closure of $\mathsf{D}(A)$ in the real interpolation space $(X, \mathsf{D}(A))_{1-\alpha,\infty}$. Then for all $f \in C_{w_\alpha}(I; X)$ and all $x \in X_{1-\alpha,\infty}$ there exists a unique strong solution u of (17.42) such that $Au, u' \in C_{w_\alpha}(I; X)$, and there is a constant $C \geqslant 0$, independent of f and x, such that*

$$\|Au\|_{C_{w_\alpha}(I;X)} + \|u'\|_{C_{w_\alpha}(I;X)} \leqslant C(\|f\|_{C_{w_\alpha}(I;X)} + \|x\|_{X_{1-\alpha,p}}).$$

If additionally $f \in C_{w_\alpha,0}(I; X)$, then $Au, u' \in C_{w_\alpha,0}(I; X)$.

Proof. We argue analogously as in Corollary 17.2.37, where the $L^p_{w_\alpha}$-setting was discussed. For the initial value part, a different argument is needed. Let $u_x = S(\cdot)x$. We claim that $Au_x \in C_{w_\alpha,0}(I; X)$. For $x \in \mathsf{D}(A)$, the claim is clear, and moreover by Corollary 17.2.37 and continuity

$$\|Au_x\|_{C_{w_\alpha}(I;X)} + \|u_x'\|_{C_{w_\alpha}(I;X)} \leqslant C\|x\|_{X_{1-\alpha,p}}.$$

Therefore, by a density argument the latter extends to all $x \in X_{1-\alpha,p}$.

It remains to check the final assertion for $f \in C_{w_\alpha,0}(I; X)$. By density it suffices to consider $f : I \to X$ such that $f = 0$ in a neighbourhood $(0, \delta)$ of zero. By the equivalence with mild solutions (see Proposition 17.1.3), one sees that $u = 0$ on $(0, \delta)$. Therefore, $Au, u' \in C_{w_\alpha,0}(I; X)$ as required. □

17.2.g Perturbation and time-dependent problems

In this section we will prove several perturbation results for maximal L^p-regularity with $p \in [1, \infty]$. Later on, in Corollary 17.3.10, we will improve these results for UMD spaces X and $p \in (1, \infty)$.

We will use the method of continuity introduced in the preceding chapter (see Lemma 16.2.2). For the reader's convenience, we recall its formulation: *Let $(L_t)_{t\in[0,1]}$ be a family of bounded linear operators from a Banach space E into another Banach space F such that $t \mapsto L_t$ is continuous from $[0, 1]$ into $\mathscr{L}(E, F)$. Suppose furthermore that there exists a constant $C \geqslant 0$ such that for all $t \in [0, 1]$ and all $x \in E$ we have*

$$\|x\| \leqslant C\|L_t x\|.$$

Then L_0 is surjective if and only if L_1 is surjective.

We begin by proving an additive perturbation result for *relatively A-bounded* perturbations B in the sense of (17.46) below. The proof uses a variation of the argument in Theorem 16.2.3, as well as the flexibility in the estimates with the additional parameter λ as discussed in Proposition 17.2.27.

Proposition 17.2.49 (Perturbation). *Let A be a linear operator with maximal L^p-regularity on I for some $p \in [1, \infty]$. Suppose that $B : \mathsf{D}(A) \to X$ is a linear operator that satisfies*

$$\|Bx\| \leqslant \delta\|Ax\| + K\|x\|, \quad x \in \mathsf{D}(A), \tag{17.46}$$

where $\delta \in (0, 1)$ satisfies $\delta M^{\rm reg}_{p,A}(I) < 1$ and $K \geqslant 0$. Then the operator $A + B$ with domain $\mathsf{D}(A + B) := \mathsf{D}(A)$ is closed, and the following assertions hold:

(1) *For all $\lambda \in \mathbb{R}$ large enough, $\lambda + A + B$ has maximal L^p-regularity on I.*
(2) *If $I = (0, T)$ is bounded, then $A + B$ has maximal L^p-regularity on I.*
(3) *If $I = \mathbb{R}_+$ and $K = 0$, then $A + B$ has maximal L^p-regularity on I.*

By the p-independence of maximal L^p-regularity for $p \in (1, \infty)$, in applying this proposition, one can choose the value of p for which $M^{\rm reg}_{p,A}(I)$ is minimal, in order to allow the maximal range of $\delta \in (0, 1)$ in the condition $\delta M^{\rm reg}_{p,A}(I) < 1$.

Proof. Closedness of $A + B$ can be proved by repeating the corresponding argument in the proof of Theorem 16.2.3. For notational convenience, in the remainder of the proof we will use the short-hand notation $C := M^{\rm reg}_{p,A}(I)$ and $\|u\|_p = \|u\|_{L^p(I;X)}$.

(1): By Dore's Theorem 17.2.15, $-A$ generates an analytic semigroup $(S(t))_{t>0}$ on X. Let $M \geqslant 0$ and $\omega \in \mathbb{R}$ be such that $\|S(t)\| \leqslant Me^{\omega t}$ for all $t \geqslant 0$, and that $0 \in \varrho(\lambda + A)$ for all $\lambda > \omega$, where the latter is possible by Definition K.1.2(i).

We will first show that $\lambda + A + B$ has maximal L^p-regularity for $\lambda > \omega$ large enough. In order to prove this result, we will apply Lemma 16.2.2 with

$$E = \{u \in W^{1,p}(I;X) \cap L^p(I;\mathsf{D}(A)) : u(0) = 0\},$$

and $F = L^p(I;X)$ with $L_\theta u = u' + (\lambda + A + \theta B)u$ for $\theta \in [0,1]$. The space E is a Banach space with respect to the norm

$$\|u\|_E = (\lambda - \omega)\|u\|_p + \|Au\|_p + \|u'\|_p.$$

We first prove the following *a priori* estimate: for all $\lambda > \omega$ large enough there exists a constant $D \geqslant 0$ such that

$$\|u\|_E \leqslant D\|L_\theta u\|_p, \quad u \in E. \tag{17.47}$$

Let $u \in E$ and set $f := L_\theta u$. Then $u' + (\lambda + A)u = f - \theta Bu$, and therefore, by Proposition 17.2.27,

$$\|Au\|_p \leqslant C\|f\|_p + C\|Bu\|_p \leqslant C\|f\|_p + C\delta\|Au\|_p + CK\|u\|_p.$$

Since $C\delta < 1$, we obtain

$$\|Au\|_p \leqslant C_0\|f\|_p + C_0K\|u\|_p, \tag{17.48}$$

where $C_0 = C(1 - C\delta)^{-1}$. To estimate $(\lambda - \omega)\|u\|_p$, we let $\widetilde{f}$ and $\widetilde{u}$ denote the zero extensions of and f and u to all of $\mathbb{R}_+$ (to deal with the case $I = (0,T)$) and use the assumption $u(0) = 0$ along with Young's inequality to estimate

$$\begin{aligned}\|u\|_{L^p(I)} = \|\widetilde{u}\|_{L^p(\mathbb{R}_+)} &\leqslant \big\|S_\lambda * (\widetilde{f} - \theta B\widetilde{u})\big\|_{L^p(\mathbb{R}_+)} \\ &\leqslant \int_0^\infty \|S_\lambda(t)\|\,\mathrm{d}t \cdot \|(f - \theta Bu)\|_{L^p(I)},\end{aligned}$$

where $S_\lambda(t) = e^{-\lambda t}S(t)$ is the semigroup generated by $-A - \lambda$. Hence, for $\lambda > \omega$,

$$\begin{aligned}(\lambda - \omega)\|u\|_p &\leqslant M\int_0^\infty (\lambda - \omega)e^{-(\lambda-\omega)t}\,\mathrm{d}t \cdot (\|f\|_p + \|Bu\|_p) \\ &\leqslant M(\|f\|_p + \delta\|Au\|_p + K\|u\|_p) \leqslant C_1\|f\|_p + C_1K\|u\|_p,\end{aligned}$$

where in the last step we used (17.48) and took $C_1 = M(\delta C_0 + 1)$. For $\lambda > \omega + C_1K$ this gives

$$\|u\|_p \leqslant C_1(\lambda - \omega - C_1K)^{-1}\|f\|_p. \tag{17.49}$$

Substituting the latter in (17.48) gives

$$\|Au\|_p \leqslant C_2\|f\|_p, \tag{17.50}$$

where $C_2 = C_0 + C_0KC_1(\lambda - \omega - C_1K)^{-1}$. Combining the estimates (17.49) and (17.50) with the equation for u we obtain

$$\|u'\|_p \leqslant \lambda\|u\|_p + \|Au\|_p + \|Bu\|_p + \|f\|_p \leqslant C_3\|f\|_p, \tag{17.51}$$

where C_3 is a constant depending on C, δ, K, M, and λ. Combining the estimates (17.49), (17.50), and (17.51), and recalling that $f = L_\theta u$, we obtain (17.47).

By Theorem 17.2.26(1), $\lambda + A$ has maximal L^p-regularity. Hence the operator $L_0 : u \mapsto u' + (\lambda + A)u$ from ${}_0\dot{W}^{1,p}_{\lambda+A}(I;X)$ to $L^p(I;X)$ is surjective. For the "large enough" λ that we consider, we also have $0 \in \varrho(\lambda + A)$ by Definition K.1.2(i). Under this condition, Proposition 17.2.8 applies to $\lambda + A$ in place of A, and guarantees that

$$\begin{aligned}{}_0\dot{W}^{1,p}_{\lambda+A}(I;X) &\simeq L^p(I;\mathsf{D}(\lambda + A)) \cap {}_0W^{1,p}(I;X)\\ &\simeq L^p(I;\mathsf{D}(A)) \cap {}_0W^{1,p}(I;X) \simeq E,\end{aligned}$$

where E is the space introduced earlier in this proof; we used the easy observation that the domain of $\lambda + A$ and A coincide as sets, and the associated graph norms are equivalent. Thus $L_0 : E \to F = L^p(I;X)$ is surjective.

From Lemma 16.2.2 we deduce that L_1 is surjective, and hence invertible by (17.47). Therefore, for each $f \in L^p(I;X)$ there exists a unique $u \in E$ with $u' + (\lambda + A + B)u = f$ and (17.47) we obtain

$$\begin{aligned}\|(\lambda + A + B)u\|_p &\leqslant \lambda\|u\|_p + \|Au\|_p + \|Bu\|_p\\ &\leqslant (\lambda + K)\|u\|_p + (1+\delta)\|Au\|_p \leqslant \widetilde{D}\|f\|,\end{aligned}$$

where $\widetilde{D} = (\lambda + K)(\lambda - \omega)^{-1}D + (1+\delta)D$. This proves that $\lambda + A + B$ has maximal L^p-regularity.

(2): By (1) and Theorem 17.2.26(1), $A + B$ has maximal L^p-regularity.

(3): Let $f \in L^p(\mathbb{R}_+;X)$. It follows from (2) that $u' + (A+B)u = f$ with $u(0) = 0$ has a unique L^p-solution on every interval $(0,T')$ with $T' < \infty$, and therefore by uniqueness, we can construct a strong solution u on $\mathbb{R}_+$. It remains to estimate $\|(A+B)u\|_p$. Since A has maximal L^p-regularity, writing $u' + Au = f - Bu$ it follows that

$$\|Au\|_p \leqslant M\|f\|_p + M\|Bu\|_p \leqslant M\|f\|_p + M\delta\|Au\|_p.$$

This implies $\|Au\|_p \leqslant M(1 - M\delta)^{-1}\|f\|_p$. Moreover, $\|Bu\|_p \leqslant \delta\|Au\|_p \leqslant \delta M(1 - M\delta)^{-1}\|f\|_p$. Therefore $A + B$ has maximal L^p-regularity on $\mathbb{R}_+$ with constant $M^{\rm reg}_{p,A+B}(\mathbb{R}_+) \leqslant (1+\delta)M(1 - M\delta)^{-1}$. □

As an immediate consequence, maximal L^p-regularity is stable under lower order perturbations:

Corollary 17.2.50. *Let A be a linear operator with maximal L^p-regularity on I for some $p \in [1,\infty]$ and $T \in (0,\infty]$. Suppose that $B : (X, \mathsf{D}(A))_{\theta,1} \to X$ is a bounded linear operator for some $\theta \in (0,1)$.*

(1) *For all* $\lambda \in \mathbb{R}$ *large enough,* $\lambda + A + B$ *has maximal* L^p*-regularity on* I.
(2) *If* $I = (0,T)$ *is bounded, then* $A + B$ *has maximal* L^p*-regularity on* I.

Proof. Under the stated assumptions, the relative smallness condition (17.46) on B was checked in the proof of Corollary 16.2.5. Thus the claims are immediate from Proposition 17.2.49. □

As an application of the previous perturbation results, we prove well-posedness for a non-autonomous Cauchy problem. This result can be seen as an extension of Corollary 17.2.37 to the time-dependent setting. Let X_0 and X_1 be Banach spaces such that $X_1 \hookrightarrow X_0$ with a continuous embedding, and let $A : [0,T] \to \mathscr{L}(X_1, X_0)$ be strongly measurable in the strong operator topology. Consider the time-dependent inhomogeneous problem

$$\begin{cases} u'(t) + A(t)u(t) &= f(t), \quad t \in (0,T) \\ u(0) &= x. \end{cases} \tag{17.52}$$

Mutatis mutandis, the notion of a strong solution can be extended to (17.52).

In the next result we interpret $X_{0,1} = (X_0, X_1)_{0,1}$ as X_0; this case arises if $p = 1$ and $\alpha = 0$. A version for maximal C-regularity instead can be proved in a similar way.

Theorem 17.2.51 (Maximal L^p-regularity for time-dependent A). *Let* X_0 *and* X_1 *be Banach spaces with continuous embedding* $X_1 \hookrightarrow X_0$. *For a given* $p \in [1,\infty)$ *and*

$$\alpha \in [0, 1/p') \cup \{0\} = \begin{cases} [0, 1/p'), & \text{if } p \in (1,\infty), \\ \{0\}, & \text{if } p = 1, \end{cases}$$

consider the space

$$X_{1-\frac{1}{p}-\alpha,p} := (X_0, X_1)_{1-\frac{1}{p}-\alpha,p}.$$

Let $A \in C([0,T]; \mathscr{L}(X_1, X_0))$ *be a mapping with the following two properties:*

(i) *there exists a constant* $L > 0$ *such that for all* $t \in [0,T]$,

$$L^{-1}\|x\|_{X_1} \leqslant \|A(t)x\|_{X_0} + \|x\|_{X_0} \leqslant L\|x\|_{X_1}, \quad x \in X_1; \tag{17.53}$$

(ii) *for all* $t \in [0,T]$ *the operator* $A(t)$, *viewed as an operator acting in* X_0 *with domain* $\mathsf{D}(A(t)) = X_1$, *has maximal* L^p*-regularity on* $(0,T)$ *with*

$$M := \sup_{t\in[0,T]} M^{\mathrm{reg}}_{p,A(t)}(0,T) < \infty.$$

Set $w_\alpha(t) := t^\alpha$. *Under these assumptions, for all* $x \in (X_0, X_1)_{1-\frac{1}{p}-\alpha,p}$ *and* $f \in L^p_{w_\alpha}(0,T;X_0)$ *there exists a unique strong solution*

$$u \in L^p_{w_\alpha}(0,T;X_1) \cap W^{1,p}_{w_\alpha}(0,T;X_0) \cap C([0,T];X_{1-\frac{1}{p}-\alpha,p}) \cap C((0,T];X_{1-\frac{1}{p},p}),$$

of (17.52)*, and there exists a constant* $C \geqslant 0$*, only depending on* A*,* p*,* α*, and* T*, such that*

$$\begin{aligned} &\|u\|_{L^p_{w_\alpha}(0,T;X_1)} + \|u\|_{W^{1,p}_{w_\alpha}(0,T;X_0)} \\ &+ \|u\|_{C([0,T];X_{1-\frac{1}{p}-\alpha,p})} + \sup_{t\in(0,T]} t^\alpha \|u(t)\|_{X_{1-\frac{1}{p},p}} \\ &\qquad\qquad \leqslant C\|f\|_{L^p_{w_\alpha}(0,T;X_1)} + C\|x\|_{X_{1-\frac{1}{p}-\alpha,p}}. \end{aligned} \tag{17.54}$$

Assumption (i) says that the graph norms of $A(t)$, viewed as an operator acting in X_0 with domain $\mathsf{D}(A(t)) = X_1$, are equivalent to the norm of X_1, uniformly with respect to $t \in [0,T]$; Assumption (ii) says that these operators have maximal L^p-regularity on $(0,T)$, uniformly with respect to $t \in [0,T]$.

The fact that $u(t)$, for $t > 0$, takes values in $X_{1-\frac{1}{p},p}$, which is a smaller space than the initial value space $X_{1-\frac{1}{p}-\alpha,p}$ if $\alpha > 0$, is referred to by saying that the solution *instantaneously regularises* if $\alpha > 0$.

The proof of the theorem uses an iteration argument, in which the interval $[0,T]$ is subdivided into smaller intervals $[t_n, t_{n+1}]$. On each of these, the problem (17.52) is solved by a fixed point argument, taking $u(t_n)$ as the initial value for solving (17.52) on $[t_n, t_{n+1}]$. For this to work, it is crucial that $u(t_n)$ belong to the correct trace space; the bookkeeping needed to achieve this is made possible by maximal L^p-regularity and is effected through a judicious choice of the fixed point space. In the process, we ascertain that upon completion of the $(n+1)$th step, a suitable *a priori* inequality that encodes all this information is being extended from $[0,t_n]$ to $[0,t_{n+1}]$.

Implicit in this reasoning are two observations, the easy proofs of which we leave to the reader:

- Gluing strong solutions on subintervals gives a strong solution on the full interval. This follows by iterating the definition of a strong solution.
- Gluing Sobolev functions on subintervals gives a Sobolev function on the full interval. This can be seen, e.g., through the characterisation of $W^{1,p}$-functions as indefinite integrals of L^p-functions (see Section 2.5.c).

Proof. For each $\tau \in (0,T]$ let $\mathscr{M}_\tau : L^p(0,T;X_0) \to L^p(0,T;X_1), f \mapsto u_f$ be the solution map defined in Corollary 17.2.9, associated with $A(\tau)$ in place of A. (Note that the subscript τ of $\mathscr{M}_\tau$ refers only to the operator $A(\tau)$, whereas the time interval under consideration is the same $(0,T)$ for all τ.)

Since $A(\tau)$ has maximal L^p-regularity on $(0,T)$ with constant M, it follows from (17.53) and (17.10) that

$$\begin{aligned} \|u_f\|_{L^p(0,T;X_1)} &\leqslant L\Big(\|A(\tau)u_f\|_{L^p(0,T;X_0)} + \|u_f\|_{L^p(0,T;X_0)}\Big) \\ &\leqslant L\Big(M\|f\|_{L^p(0,T;X_0)} + T(M+1)\|f\|_{L^p(0,T;X_0)}\Big) \\ &\leqslant L(1+T)(M+1)\|f\|_{L^p(0,T;X_0)}, \end{aligned}$$

and hence

$$\|\mathscr{M}_\tau\| \leqslant (1+T)(M+1)L =: \widetilde{M}_1.$$

By the extrapolation of maximal regularity with power weights (Proposition 17.2.36), $A(0)$ has maximal $L^p_{w_\alpha}$-regularity on $(0,T)$ with a certain constant M_0 and therefore in a similar way as before we see that the associated mapping $\mathscr{M}_0 : L^p_{w_\alpha}(0,T;X_0) \to L^p_{w_\alpha}(0,T;X_1)$ is bounded by with constant $\widetilde{M}_0$.

Let $\widetilde{M} = \max\{\widetilde{M}_0, \widetilde{M}_1\}$. By uniform continuity, we can choose $\delta > 0$ such that, for all $s,t \in [0,T]$,

$$|t-s| < \delta \quad \Rightarrow \quad \|A(t) - A(s)\|_{\mathscr{L}(X_1,X_0)} < \frac{1}{2\widetilde{M}}.$$

Let $N \geqslant 1$ be an integer such that $T/N < \delta$. Set $t_n := \frac{nT}{N}$ for $n \in \{0,\dots,N\}$, and set $t_{-1} := 0$.

Let $n \in \{0,\dots,N-1\}$ be fixed. Suppose that a unique strong solution u on $(0,t_n)$ exists, and (17.54) holds on $[0,t_n]$ instead of $[0,T]$ with constant C_n instead of C. (Note that this assumption is vacuous for $n=0$.) We will show how to obtain a unique strong solution and the estimate (17.54) on $[0,t_{n+1}]$. Let $\beta := \alpha$ if $n=0$, and $\beta := 0$ if $n \geqslant 1$.

If $p \in (1,\infty)$, we can apply Theorem L.4.1, to find that there is a constant $C \geqslant 0$ such that

$$\begin{aligned} \|u\|_{C([0,t_{n+1}];X_{1-\frac{1}{p}-\beta,p})} &+ \sup_{t\in(0,t_{n+1}]} t^\beta \|u(t)\|_{X_{1-\frac{1}{p},p}} \\ &\leqslant C\|u\|_{L^p_{w_\beta}(0,t_{n+1};X_1)} + C\|u\|_{W^{1,p}_{w_\beta}(0,t_{n+1};X_0)}. \end{aligned} \tag{17.55}$$

The case $p=1$ is not covered by Theorem L.4.1, and we need to argue differently. Then $\alpha = 0$ by assumption, and thus $\beta = 0$ in all cases. Hence the right-hand side of (17.55) involves unweighted norms only, and the two terms on the left of (17.55) coincide in this case. Moreover, $X_{0,1} = X_0$ by the convention that we made right before the statement of Theorem 17.2.51 that we are proving. Thus (17.55) for $p=1$ follows simply from the continuity of the embedding $W^{1,1}(0,t_{n+1};X_0) \hookrightarrow C([0,t_{n+1}];X_0)$; the first term on the right of (17.55) is not even needed in this case. Summarising, we have established (17.55) for all $p \in [1,\infty)$.

By the validity of the estimate (17.54) on $[0,t_n]$, and the fact that $w_\alpha(t) = t^\alpha$ is bounded away from zero for $t \in [t_1,T]$ (allowing us to pass from the weight w_α used on $[0,t_1]$ to the trivial weight $w_0 = \mathbf{1}$ on the intervals $[t_n,t_{n+1}]$ for $n \geqslant 1$), it is therefore enough to prove the estimate

$$\begin{aligned} \|u\|_{L^p_{w_\beta}(t_n,t_{n+1};X_1)} &+ \|u\|_{W^{1,p}_{w_\beta}(t_n,t_{+1};X_0)} \\ &\leqslant C\|f\|_{L^p_{w_\beta}(t_n,t_{n+1};X_1)} + C\|u(t_n)\|_{X_{1-\frac{1}{p}-\beta,p}}. \end{aligned}$$

Finally, since $u' = -Au + f$ it is enough to estimate the first term on the left with the right-hand side.

To shorten the notation we write

$$Z_j := L^p_{w_\beta}(t_n, t_{n+1}; X_j), \quad j \in \{0, 1\}.$$

By the extrapolation of maximal regularity with power weights (Proposition 17.2.36), the operator $A(t_n)$ has maximal $L^p_{w_\beta}$-regularity. Thus, we can define a new operator $\Phi : Z_1 \to Z_1$ by $\Phi(v) := y$, where $y \in Z_1$ is the unique strong solution to

$$\begin{cases} y'(t) + A(t_n)y(t) &= (A(t_n) - A(t))v(t) + f(t), \quad t \in (t_n, t_{n+1}) \\ y(t_n) &= u(t_n). \end{cases}$$

Note that existence and uniqueness of the strong solution follows by maximal $L^p_{w_\beta}$-regularity, Corollary 17.2.37 concerning the related initial value problem, the fact that the initial value satisfies $u(t_n) \in X_{1-\frac{1}{p}-\beta,p}$, and an easy time shift between the interval (t_n, t_{n+1}) above and the interval $(0, t_{n+1} - t_n)$ of the form considered elsewhere in this chapter.

Since $y = \Phi(v_1) - \Phi(v_2)$ is a strong solution of

$$\begin{cases} y'(t) + A(t_n)y(t) &= (A(t_n) - A(t))(v_1(t) - v_2(t)), \quad t \in (t_n, t_{n+1}) \\ y(t_n) &= 0, \end{cases}$$

it follows that $\Phi(v_1) - \Phi(v_2) = \mathscr{M}_{t_n}(\mathbf{1}_{(t_n,t_{n+1})}(A(t_n) - A(\cdot))(v_1 - v_2))$ on $[t_n, t_{n+1}]$, and thus

$$\|\Phi(v_1) - \Phi(v_2)\|_{Z_1} \leqslant \widetilde{M}\|(A(t_n) - A(\cdot))(v_1 - v_2)\|_{Z_0} \leqslant \frac{1}{2}\|v_1 - v_2\|_{Z_1}.$$

Therefore, the Banach fix point theorem implies that Φ has a unique fix point $y \in Z_1$. Extending u to $[0, t_{n+1}]$ as $u|_{(t_n,t_{n+1}]} = y$, one can check that u turns into a strong solution on $(0, t_{n+1})$.

Finally, we prove the required *a priori* estimate. It suffices to estimate u on $[t_n, t_{n+1}]$. By the above we obtain

$$\|u\|_{Z_1} \leqslant \|\Phi(u) - \Phi(0)\|_{Z_1} + \|\Phi(0)\|_{Z_1} \leqslant \frac{1}{2}\|u\|_{Z_1} + \|\Phi(0)\|_{Z_1}.$$

Therefore, $\|u\|_{Z_1} \leqslant 2\|\Phi(0)\|_{Z_1}$. To estimate the latter, we note that $\Phi(0)$ is the solution z of the initial value problem

$$\begin{cases} z'(t) + A(t_n)z(t) &= f(t), \qquad t \in (t_n, t_{n+1}), \\ z(t_n) &= u(t_n). \end{cases}$$

Thus, we again apply Corollary 17.2.37 concerning such initial value problems, together with (17.54) on $[0, t_n]$, to obtain

$$\begin{aligned} \|\Phi(0)\|_{Z_1} &\leqslant K\|f\|_{Z_0} + K\|u(t_n)\|_{X_{1-\frac{1}{p}-\beta,p}} \\ &\leqslant K\|f\|_{Z_0} + C_n K\|f\|_{L^p_{w_\alpha}(0,t_n;X_0)} \\ &\leqslant (Kt_n^{\beta-\alpha} + C_n K)\|f\|_{L^p_{w_\alpha}(0,t_{n+1};X_0)}, \end{aligned}$$

where of course $0^0 = 1$, as usual. □

17.3 Characterisations of maximal L^p-regularity

So far, we were mostly dealing with situations where maximal L^p-regularity in some form was assumed, with the goal of deriving various consequences of it. we now turn to the problem of finding conditions for maximal L^p-regularity in terms of certain properties of the operator A. We already have seen a necessary condition for maximal L^p-regularity on $\mathbb{R}_+$ in Theorem 17.2.15, namely, that $-A$ generate a bounded analytic semigroup on X, or equivalently, that A be sectorial of angle strictly less than $\frac{1}{2}\pi$.

17.3.a Fourier multiplier approach

The main theorem of this section, Theorem 17.3.1, provides a necessary and sufficient condition for maximal L^p-regularity on $\mathbb{R}_+$ in the setting of UMD Banach spaces.

Theorem 17.3.1 (Maximal L^p-regularity and R-sectoriality). *Suppose that A is a linear operator on a Banach space X and $p \in [1, \infty]$.*

(1) *If A has maximal L^p-regularity on $\mathbb{R}_+$ with constant $M^{\rm reg}_{p,A}(\mathbb{R}_+)$, then A is R-sectorial with angle $\omega_R(A) < \frac{1}{2}\pi$, and if $p \in [1, \infty)$ the following holds:*

$$\mathscr{R}_p(\{AR(\lambda, A) : \ \lambda \in \overline{\mathbb{C}_+} \setminus \{0\}\}) \leqslant M^{\rm reg}_{p,A}(\mathbb{R}_+).$$

(2) *If X is a UMD space, $p \in (1, \infty)$ and A is R-sectorial with angle $\omega_R(A) < \frac{1}{2}\pi$, then A has maximal L^p-regularity on $\mathbb{R}_+$ with*

$$M^{\rm reg}_{p,A}(\mathbb{R}_+) \leqslant 400\hbar_{p,X}\beta^2_{p,X}(M + 1)^2,$$

where $M = \mathscr{R}_p(\{AR(it, A) : \ t \in \mathbb{R} \setminus \{0\}\})$.

(3) *If X is a UMD space, $p \in (1, \infty)$, and $-A$ generates an analytic semigroup $(S(t))_{t\geqslant 0}$ such that the sets $\{S(t) : t \geqslant 0\}$ and $\{tAS(t) : t \geqslant 0\}$ are R-bounded, then A has maximal L^p-regularity on $\mathbb{R}_+$.*

Part (2) of this theorem extends to maximal L^p-regularity on bounded intervals if one considers R-sectoriality of $A + \lambda$ for λ large enough. Similarly, part (3) extends to bounded intervals $[0, T]$ if one considers R-boundedness of $\{S(t) : t \in (0, T]\}$ and $\{tAS(t) : t \in (0, T]\}$. Both assertions follow by combining Theorem 17.3.1 with Theorem 17.2.26(1).

Remark 17.3.2. If $A \neq 0$, then $\liminf_{t\to 0} \|A(it+A)^{-1}\| \geqslant 1$, and in particular the constant in Theorem 17.3.1(2) satisfies $M \geqslant 1$. To see this, let $x \in \mathsf{D}(A)$ be so that both $Ax \neq 0$. Thus $y_t := (it + A)x$ is non-zero when $|t|$ is small enough. Now

$$A(it + A)^{-1}y_t = Ax = y_t - itx,$$

hence $\|A(it+A)^{-1}y_t\| \geqslant \|y_t\| - |t|\|x\|$, and therefore

$$\|A(it+A)^{-1}\| \geqslant 1 - \frac{|t|\|x\|}{\|y_t\|}.$$

Here $\|y_t\| \to \|Ax\| \neq 0$ as $t \to 0$, and thus the right-hand side tends to 1.

The assumption that $A \neq 0$ is obviously necessary, for otherwise $A(it + A)^{-1} = 0$.

To prove Theorem 17.3.1 we use the reformulation of maximal L^p-regularity of Theorem 17.2.19 and the following lemma, where we extend the convolution operator V introduced in Theorem 17.2.19 to an operator on $L^p(\mathbb{R}_+;X)$ or $L^p(\mathbb{R};X)$. After that, we can reformulate the boundedness of the convolution operator in terms of Fourier multipliers.

Lemma 17.3.3. *Let $-A$ be a linear operator that generates an analytic semigroup $(S(t))_{t>0}$ on a Banach space X. Let $p \in [1,\infty)$ and let $F(\mathbb{R}) = C_c^\infty(\mathbb{R}) \times X$, and let $F(\mathbb{R}_+) = \{f|_{\mathbb{R}_+} : f \in F(\mathbb{R})\}$. The following assertions are equivalent:*

(1) *the mapping $f \to Vf$, defined for functions $f \in F(\mathbb{R}_+)$ by*

$$Vf(t) := A\int_0^t S(t-s)f(s)\,\mathrm{d}s,$$

is well defined, takes values in $L^p(\mathbb{R}_+;X)$, and extends to a bounded operator on $L^p(\mathbb{R}_+;X)$;

(2) *the mapping $f \to \overline{V}f$, defined for functions $f \in F(\mathbb{R})$ by*

$$\overline{V}f(t) := A\int_{-\infty}^\infty S(t-s)f(s)\,\mathrm{d}s,$$

where we set $S(-t) = 0$ for $t > 0$, is well defined, takes values in $L^p(\mathbb{R};X)$, and extends to a bounded operator on $L^p(\mathbb{R};X)$.

In this case we have $\|V\| = \|\overline{V}\| = M^{\rm reg}_{p,A}(\mathbb{R}_+)$. Furthermore, the equivalence of (1) and (2) and norm identity hold for $p = \infty$ if we additionally assume that S be uniformly exponentially stable and set $F(I) = L^\infty(I;X)$.

In the above result, it is part of the assumptions that the convolutions take values in $\mathsf{D}(A)$ almost everywhere. If f is sufficiently regular, one can pull the operator A inside the integral by Hille's theorem (Theorem 1.2.4). By Theorem 17.2.19, (1) holds if and only if A has maximal L^p-regularity on $\mathbb{R}_+$.

Proof. (2)⇒(1): This implication hold trivially, and so does the bound $\|V\| \leqslant \|\overline{V}\|$.

(1)⇒(2) for $p \in [1,\infty)$: Without loss of generality we can assume that $F(I)$ are the functions in $L^p(I;X)$ with support in a bounded set.

Fix $f \in F(\mathbb{R})$ and pick $r > 0$ so that the support of f is in $(-r, \infty)$. Let $f_r : \mathbb{R}_+ \to X$ be defined by $f_r(s) := f(s - r)$. Then for all $t \geqslant -r$ we have

$$\overline{V} f(t) = \int_{-\infty}^{\infty} AS(t+r-s) f(s-r)\,\mathrm{d}s = \int_0^{t+r} AS(t+r-s) f_r(s)\,\mathrm{d}s = V f_r(t+r).$$

whereas $\overline{V} f(t) = 0$ for $t < -r$ since the support of f is in $(-r, \infty)$ and $S(t) = 0$ for $t < 0$. Hence

$$\begin{aligned}\|\overline{V} f\|^p_{L^p(\mathbb{R};X)} &= \int_{-r}^{\infty} \|V f_r(t + r)\|^p \,\mathrm{d}t = \int_0^{\infty} \|V f_r(t)\|^p \,\mathrm{d}t \\ &= \|V f_r\|^p_{L^p(\mathbb{R}_+;X)} \leqslant \|V\|^p \|f_r\|^p_{L^p(\mathbb{R}_+;X)} = \|V\|^p \|f\|^p_{L^p(\mathbb{R};X)}.\end{aligned}$$

It follows that $\overline{V}$ is bounded on $L^p(\mathbb{R}; X)$ and $\|\overline{V}\| \leqslant \|V\|$.

(1)⇒(2) for $p = \infty$, under the additional assumption that $(S(t))_{t>0}$ is an uniformly exponentially stable analytic semigroup: By uniform exponential stability there exist $M \geqslant 1$ and $\omega > 0$ such that $\|tAS(t)\| \leqslant Me^{-\omega t}$ for all $t > 0$ (see Theorems 17.2.15 and G.5.3). It suffices to prove that for any fixed $f \in L^\infty(\mathbb{R}; X)$, and $R \leqslant -1$

$$\|\overline{V} f\|_{L^\infty(R/2,\infty;X)} \leqslant \Big(M^{\mathrm{reg}}_{\infty,A}(\mathbb{R}_+) + \frac{Ce^{\omega R/2}}{-R}\Big)\|f\|_{L^\infty(\mathbb{R};X)}.$$

Indeed, by letting $R \to -\infty$, the desired estimate then follows. We split the convolution into integrals over $(-\infty, R)$ and (R, ∞). On the first interval, bringing the norms inside, for $t \in (R/2, \infty)$ we can estimate

$$\begin{aligned}\int_{-\infty}^{R} \|AS(t - s)f(s)\|\,\mathrm{d}s &\leqslant \int_{-\infty}^{R} \frac{M}{t - s} e^{-\omega(t-s)}\,\mathrm{d}s \|f\|_{L^\infty(\mathbb{R};X)} \\ &\leqslant \frac{Me^{-\omega(t-R)}}{(t - R)\omega} \|f\|_{L^\infty(\mathbb{R};X)}. \\ &\leqslant \frac{2Me^{\omega R/2}}{-R\omega} \|f\|_{L^\infty(\mathbb{R};X)}.\end{aligned}$$

For the integral over (R, ∞) let $g(s) = f(s + R)$. Then, by the assumption that $\int_R^\infty S(t - s)f(s)\,\mathrm{d}s = \int_0^\infty S(t - R - s)g(s)\,\mathrm{d}s$ takes vales in $\mathsf{D}(A)$ for almost all $t \in (R/2, \infty)$ and

$$\begin{aligned}\Big\|A \int_R^{\infty} S(t - s)f(s)\,\mathrm{d}s\Big\| &= \Big\|A \int_0^{\infty} S(t - R - s)g(s)\,\mathrm{d}s\Big\| \\ &\leqslant M^{\mathrm{reg}}_{\infty,A}(\mathbb{R}_+)\|g\|_{L^\infty(\mathbb{R}_+;X)} \\ &\leqslant M^{\mathrm{reg}}_{\infty,A}(\mathbb{R}_+)\|f\|_{L^\infty(\mathbb{R};X)}.\end{aligned}$$

The result follows by combining both estimates. □

The space $\mathscr{S}'(\mathbb{R}^d;X)$ of tempered distributions has been introduced in Section 2.4.d. In the next lemma, we deal with the case of $\mathscr{L}(X)$ in place of X.

Lemma 17.3.4. *Let A be sectorial of angle $< \frac{1}{2}\pi$, and let S be the bounded analytic semigroup generated by $-A$. In $\mathscr{S}'(\mathbb{R};\mathscr{L}(X))$ the following identity holds:*

$$\mathscr{F}(A(1+A)^{-1}\mathbf{1}_{\mathbb{R}_+}S) = A(1+A)^{-1}(2\pi i\cdot +A)^{-1}.$$

Somewhat informally the lemma states that $\mathscr{F}(\mathbf{1}_{\mathbb{R}_+}S) = (2\pi i\cdot +A)^{-1}$. The terms $A(1+A)^{-1}$ cannot be left out, however, since AS and $(2\pi i\cdot +A)^{-1}$ need not be locally integrable.

Proof. For $\delta > 0$ and $t\in\mathbb{R}$ let $k_\delta(t) := \mathbf{1}_{(0,\infty)}(t)e^{-\delta t}S(t)$. The function k_δ belongs to $L^\infty(\mathbb{R};\mathscr{L}(X))$, which we view as continuously embedded in $\mathscr{S}'(\mathbb{R};\mathscr{L}(X))$. Then, for all Schwartz functions $\phi\in\mathscr{S}(\mathbb{R};X)$,

$$\begin{aligned}
A(1+A)^{-1}\widehat{k_0}(\phi) &= \int_{\mathbb{R}} A(1+A)^{-1}k_0(t)\widehat{\phi}(t)\,\mathrm{d}t \\
&\overset{\text{(i)}}{=} \lim_{\delta\downarrow 0}\int_{\mathbb{R}} A(1+A)^{-1}k_\delta(t)\widehat{\phi}(t)\,\mathrm{d}t \\
&= \lim_{\delta\downarrow 0} A(1+A)^{-1}\widehat{k_\delta}(\phi) \\
&\overset{\text{(ii)}}{=} \lim_{\delta\downarrow 0}\int_{\mathbb{R}} A(1+A)^{-1}(2\pi i\xi+\delta+A)^{-1}\phi(\xi)\,\mathrm{d}\xi \\
&\overset{\text{(iii)}}{=} \int_{\mathbb{R}} A(1+A)^{-1}(2\pi i\xi+A)^{-1}\phi(\xi)\,\mathrm{d}\xi,
\end{aligned}$$

where (i) and (iii) follow by dominated convergence and (ii) by the definition of a generator (Definition G.2.1). □

We recall from Section 2.4.a that $\check{L}^1(\mathbb{R};X)$ denotes the space of functions that are the inverse Fourier transform of a function in $L^1(\mathbb{R};X)$.

Proposition 17.3.5. *Let A be a sectorial operator of angle $\omega(A) < \frac{1}{2}\pi$, and let $1\leqslant p<\infty$. The following assertions are equivalent:*

(1) *A has maximal L^p-regularity on $\mathbb{R}_+$;*
(2) *the Fourier multiplier operator*

$$T_m : \check{L}^1(\mathbb{R};X)\to\check{L}^1(\mathbb{R};X),\ f\mapsto (m\widehat{f})\check{}$$

associated with the function

$$m(\xi) = A(2\pi i\xi + A)^{-1}$$

extends to a bounded operator on $L^p(\mathbb{R};X)$.

In this situation, we have $\overline{V} = T_m$ *and* $M^{\rm reg}_{p,A}(\mathbb{R}_+) = \|T_m\|_{\mathscr{L}(L^p(\mathbb{R};X))}$, *where* $\overline{V}$ *is the operator defined in Lemma 17.3.3.*

Proof. By the sectoriality assumption, $-A$ generates a bounded analytic semigroup $(S(t))_{t>0}$. Moreover, m belongs to $L^\infty(\mathbb{R};\mathscr{L}(X))$ and hence defines an element of $\mathscr{S}'(\mathbb{R};\mathscr{L}(X))$. By Theorem 17.2.19 and Lemma 17.3.3, condition (1) is equivalent to the boundedness of $\overline{V}$ and we have $M^{\rm reg}_{p,A}(\mathbb{R}_+) = \|\overline{V}\|$. Therefore, to prove the proposition it is enough to show $T_m = \overline{V}$. Clearly, it suffices to show that for all $f \in C_c^\infty(\mathbb{R}) \otimes X$

$$(1 + A)^{-1}T_m f = (1 + A)^{-1}\overline{V} f \tag{17.56}$$

as elements of $\mathscr{S}'(\mathbb{R};X)$. Indeed, after applying $1 + A$ it remains to use the density of $C_c^\infty(\mathbb{R}) \otimes X$ in $L^p(\mathbb{R};X)$.

Fix $f \in C_c^\infty(\mathbb{R})\otimes X$ and consider the function Let $t \mapsto k(t) = \mathbf{1}_{(0,\infty)}(t)S(t)$ in $L^\infty(\mathbb{R};\mathscr{L}(X))$, viewed as an element of $\mathscr{S}'(\mathbb{R};\mathscr{L}(X))$. Then, in $\mathscr{S}'(\mathbb{R};X)$,

$$(1 + A)^{-1}\overline{V} f = A(1 + A)^{-1}k * f.$$

Taking Fourier transforms in $\mathscr{S}'(\mathbb{R};X)$, we obtain

$$\begin{aligned}
\mathscr{F}((1 + A)^{-1}\overline{V} f) &= A(1 + A)^{-1}\mathscr{F}(k * f)\\
&= A(1 + A)^{-1}\widehat{k}\widehat{f}\\
&\overset{(*)}{=} (1 + A)^{-1} m\widehat{f}\\
&= \mathscr{F}(T_{(1+A)^{-1}m} f) = \mathscr{F}((1 + A)^{-1}T_m f),
\end{aligned}$$

where in $(*)$ we used Lemma 17.3.4. Hence (17.56) follows. □

Now that we have connected the maximal L^p-regularity of an operator A to the theory of operator-valued Fourier multipliers, we can use the theory of Chapter 5 to prove Theorem 17.3.1.

Proof of Theorem 17.3.1. (1): First consider $p \in [1,\infty)$. If A has maximal L^p-regularity on $\mathbb{R}_+$, then by Theorem 17.2.15, A is sectorial of angle $< \frac{1}{2}\pi$. By Proposition 17.3.5, the Fourier multiplier T_m, where $m(\xi) = A(i2\pi\xi + A)^{-1}$, is bounded on $L^p(\mathbb{R};X)$. The Clément–Prüss Theorem 5.3.15 then gives the R-boundedness of the set $\{m(\xi) : \xi \in \mathbb{R} \setminus \{0\}\}$, with bound

$$\mathscr{R}_p(\{m(\xi) :\ \mathbb{R} \setminus \{0\}\}) \leqslant \|T_m\|_{\mathscr{L}(L^p(\mathbb{R};X))}.$$

By the observed sectoriality of A, the function $\lambda \mapsto A(\lambda+A)^{-1}$ is bounded and holomorphic in a sector Σ_σ with some $\sigma \in (\frac{1}{2}\pi, \pi)$, and we just noted that it is R-bounded on the boundary (minus the origin) of the smaller sector $\Sigma_{\pi/2} = \mathbb{C}_+$. Hence, by Proposition 8.5.8, this function is R-bounded on $\overline{\mathbb{C}_+} \setminus \{0\}$, with the same R-bound. This in turn implies the R-bound

$$\mathscr{R}_p(\{\lambda(\lambda+A)^{-1} : \lambda \in \overline{\mathbb{C}_+} \setminus \{0\}\}) \leqslant 1 + \|T_m\|_{\mathscr{L}(L^p(\mathbb{R};X))}.$$

By Proposition 10.3.3 (of which the proof extends to the case of non-dense $\mathsf{D}(A)$), the R-boundedness of $\{\lambda(\lambda+A)^{-1} : \lambda \in \overline{\mathbb{C}_+} \setminus \{0\}\}$ implies the R-sectoriality of A with angle $\omega_R(A) < \frac{1}{2}\pi$.

If $p = \infty$, then by Theorem 17.2.31, A has maximal L^q-regularity on $\mathbb{R}_+$ for any $q \in (1,\infty)$, and R-sectoriality follows from the previous case.

(2): If A is R-sectorial of angle $\omega_R(A) < \frac{1}{2}\pi$, then the function $m(\xi) = A(i2\pi\xi + A)^{-1}$ has R-bounded range with R-bound M as in the assumptions. Moreover, $m \in C^1(\mathbb{R} \setminus \{0\}; \mathscr{L}(X))$, and for all $\xi \in \mathbb{R} \setminus \{0\}$ we have

$$\xi m'(\xi) = -2\pi i\xi A(2\pi i\xi + A)^{-2} = -m(\xi)(I - m(\xi)) = m^2(\xi) - m(\xi).$$

This function also has R-bounded range, with R-bound at most $M^2 + M$. Thus m belongs to Mihlin's class (Definition 5.3.17) with norm

$$\begin{aligned}\|m\|_{\mathfrak{M}_p(\mathbb{R};X,Y)} &\leqslant \mathscr{R}_p(\{m(\xi) : \xi \in \mathbb{R} \setminus \{0\}\}) + \mathscr{R}_p(\{\xi m'(\xi) : \xi \in \mathbb{R} \setminus \{0\}\}) \\ &\leqslant M + (M^2 + M) \leqslant (M+1)^2.\end{aligned}$$

The boundedness of T_m on $L^p(\mathbb{R};X)$ now follows from the operator-valued Mihlin Multiplier Theorem 5.3.18, with norm estimate

$$\|T_m\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} \leqslant 400\hbar_{p,X}\beta_{p,X}^2 \|m\|_{\mathfrak{M}_p(\mathbb{R};X,Y)} \leqslant 400\hbar_{p,X}\beta_{p,X}^2(M+1)^2.$$

Now the required result follows from Proposition 17.3.5.

(3): Again this follows from the extension of Proposition 10.3.3 to the non-densely defined setting. Explicit bounds can be obtained from that proof. □

Corollary 17.3.6. *If A has a bounded H^∞-calculus of angle $\omega_{H^\infty}(A) < \frac{1}{2}\pi$ on a UMD space X, then A has maximal L^p-regularity on $\mathbb{R}_+$ for all $1 < p < \infty$.*

Proof. Since X is a UMD space it also satisfies the triangular contraction property by Theorem 7.5.9. Therefore, by Theorem 10.3.4(2), A is R-sectorial of angle $\omega_R(A) \leqslant \omega_{H^\infty}(A) < \frac{1}{2}\pi$. Applying Theorem 17.3.1 gives the result. □

The above results, when combined with various results proved in Volume II, subsume several classical maximal regularity criteria.

Recall from Theorem 15.3.20 that every standard sectorial (Definition 15.3.1) operator A with a bounded H^∞-calculus has bounded imaginary powers and $\omega_{\rm BIP}(A) = \omega_{H^\infty}(A)$; thus, at least for standard sectorial operators, the existence of bounded imaginary powers is less restrictive than having a bounded H^∞-calculus. Nevertheless, it turns out that bounded imaginary powers (with a condition on their angle) already suffice to imply maximal regularity:

Corollary 17.3.7 (Dore–Venni, Prüss–Sohr). *If A has bounded imaginary powers with $\omega_{\rm BIP} < \frac{1}{2}\pi$ on a UMD space X, then A has maximal L^p-regularity on $\mathbb{R}_+$ for all $1 < p < \infty$.*

Proof. By Theorem 15.3.12, such an A is R-sectorial of angle $\omega_R(A) \leqslant \omega_{\rm BIP}(A) < \frac{1}{2}\pi$. Now apply Theorem 17.3.1. □

An alternative approach to this result via sums of operators will be presented in the next section.

In Theorem 17.2.15, we have seen that maximal L^p-regularity of a linear operator A implies that $-A$ generates an analytic semigroup. Moreover, in the case that we have maximal L^p-regularity on the full half-line $\mathbb{R}_+$, this semigroup will have to be bounded. In Hilbert spaces the following converse holds:

Corollary 17.3.8 (De Simon). *Let A be a linear operator on a Hilbert space.*

(1) *If $-A$ generates a bounded analytic C_0-semigroup, then A has maximal L^p-regularity on $\mathbb{R}_+$ for all $p \in (1,\infty)$.*
(2) *If $-A$ generates an analytic C_0-semigroup, then A has maximal L^p-regularity on $(0,T)$ for all $p \in (1,\infty)$ and $T \in (0,\infty)$.*

Proof. We give two proofs of (1).

(i): By Proposition G.2.3(4), the generator $-A$ of a C_0-semigroup is a closed and densely defined operator. For such an operator, Theorem G.5.2 guarantees that generating an analytic semigroup implies sectoriality of angle $< \frac{1}{2}\pi$. In a Hilbert space, every uniformly bounded family of operators is R-bounded by Theorem 8.1.3(1). Thus the sectoriality and R-sectoriality angles coincide, and A is even R-sectorial of angle $\omega_R(A) < \frac{1}{2}\pi$. Hence Theorem 17.3.1(2), together with the fact that Hilbert spaces are UMD (Proposition 4.2.14), implies that A has maximal L^p-regularity on $\mathbb{R}_+$ for all $1 < p < \infty$.

(ii): By Plancherel's theorem, the operator T_m of Proposition 17.3.5 on $L^2(\mathbb{R};X)$ satisfies

$$\|T_m\|_{\mathscr{L}(L^2(\mathbb{R}^d;X))} \leqslant \|m\|_{L^\infty(\mathbb{R};\mathscr{L}(X))} = \sup_{s\in\mathbb{R}} \|A(is+A)^{-1}\| =: M,$$

and A has maximal L^2-regularity on $\mathbb{R}_+$ with constant $M^{\rm reg}_{2,A}(\mathbb{R}_+) \leqslant M$ by the same proposition. Therefore, by Theorem 17.2.31, A has maximal L^p-regularity for all $p \in (1,\infty)$.

(2): Since $-(\lambda + A)$ generates a bounded analytic C_0-semigroup, it has maximal L^p-regularity on $\mathbb{R}_+$ by (1). Therefore, the result follows from the first and second permanence property of Theorem 17.2.26. □

Corollary 17.3.9. *Let $-A$ be the generator of a positive C_0-contraction semigroup $S = (S(t))_{t\geqslant 0}$ on a space $L^q(T)$, where $(T,\mathscr{B},\nu)$ is a measure space and $1 < q < \infty$. If S extends to a bounded analytic C_0-semigroup on $L^q(T)$, then A has maximal L^p-regularity for all $1 < p < \infty$.*

Proof. By Theorem 10.7.13, A is R-sectorial of angle $\omega_R(A) < \frac{1}{2}\pi$, and therefore the result follows from Theorem 17.3.1. □

Consequences for perturbation of maximal L^p-regularity

Another consequence of Theorem 17.3.1 is an improvement of the perturbation result of Proposition 17.2.49. For R-sectorial operators A and for $\omega_R(A) < \omega < \pi$ and $p \in [1,\infty)$, we have defined (see Section 16.2)

$$M^{R_p}_{\omega,A} := \mathscr{R}_p\big(\{AR(z,A) :\ z \in \complement\overline{\Sigma_\omega}\}\big).$$

Corollary 17.3.10 (Perturbation). *Let A be a linear operator, acting in a UMD space X, with maximal L^p-regularity on $\mathbb{R}_+$ for some $p \in (1,\infty)$. Suppose that $B : \mathsf{D}(A) \to X$ is a linear operator satisfying*

$$\|Bx\| \leqslant \delta\|Ax\| + K\|x\|, \quad x \in \mathsf{D}(A),$$

where $\delta \in (0,1)$ is such that $\delta M^{R_p}_{\frac{1}{2}\pi,A}(\mathbb{R}_+) < 1$, and where $K \geqslant 0$. Then the operator $A + B$ with domain $\mathsf{D}(A + B) := \mathsf{D}(A)$ is closed, and the following assertions hold:

(1) *for all $\lambda \in \mathbb{R}$ large enough, $\lambda + A + B$ has maximal L^p-regularity on $\mathbb{R}_+$;*
(2) *if $I = (0,T)$ is bounded, then $A + B$ has maximal L^p-regularity on I;*
(3) *if $K = 0$, then $A + B$ has maximal L^p-regularity on $\mathbb{R}_+$.*

By Theorem 17.3.1, we have $M^{R_p}_{\frac{1}{2}\pi,A}(\mathbb{R}_+) \leqslant M^{\rm reg}_{p,A}(\mathbb{R}_+)$. Therefore, in the case of strict inequality, the above result improves the perturbation result of Proposition 17.2.49 for UMD spaces X.

Proof. (1): By Theorem 17.3.1 it is enough to check that $\lambda + A + B$ is R-sectorial of angle $< \frac{1}{2}\pi$, and this is immediate from the stability of R-sectoriality under relatively small perturbations (Theorem 16.2.4).

(2): This follows from (1) by subtracting λ (see Theorem 17.2.26(1)).

(3): This follows from the proof of (1) by taking $\lambda = 0$. □

17.3.b The end-point cases $p = 1$ and $p = \infty$

In Section 17.3.a we characterised maximal L^p-regularity for $p \in (1,\infty)$ in terms of R-sectoriality of the resolvent along the imaginary axis in case the underlying Banach space has the UMD property. In the present section we provide a characterisation of maximal L^p-regularity for the end-point cases $p = 1$ and $p = \infty$.

Theorem 17.3.11 (Kalton–Portal, maximal L^1-regularity). *Let $I = (0,T)$ or $I = \mathbb{R}_+$. Let $-A$ be the generator of an analytic semigroup $(S(t))_{t>0}$ on a Banach space X, and suppose that $C_{I,S} := \sup_{t\in I}\|S(t)\| < \infty$. Then the following are equivalent:*

(1) *A has maximal L^1-regularity on I;*
(2) *there exists a constant $C > 0$ such that*

$$\int_I \|AS(t)x\| \,\mathrm{d}t \leqslant C\|x\|, \quad x \in X.$$

In this case $C_{I,S}^{-1} C_A \leqslant M_{1,A}^{\mathrm{reg}}(I) \leqslant C_A$, where C_A is the infimum of all admissible C in the above estimate. Moreover, for all $f \in L^1(I;X)$, the function $s \in (0,t) \mapsto S(t-s)f(s) \in \mathsf{D}(A)$ is Bochner integrable and

$$\int_I \int_0^t \|AS(t-s)f(s)\| \,\mathrm{d}s \,\mathrm{d}t \leqslant C_A C_{I,S} \|f\|_{L^1(I;X)}.$$

Proof. The implication (1)⇒(2), with estimate $C_A \leqslant M_{1,A}^{\mathrm{reg}}(I) C_{I,S}$, has already been established Lemma 17.2.22.

(2)⇒(1): Let $f \in L^1(I;X)$ and set $T = \infty$ if $I = \mathbb{R}_+$. Then

$$\begin{aligned}\int_I \int_0^t \|AS(t-s)f(s)\| \,\mathrm{d}s \,\mathrm{d}t &= \int_I \int_s^T \|AS(t-s)f(s)\| \,\mathrm{d}t \,\mathrm{d}s \\ &\leqslant \int_I \int_I \|AS(t)f(s)\| \,\mathrm{d}t \,\mathrm{d}s \leqslant C\|f\|_{L^1(I;X)}.\end{aligned}$$

This proves the convolution estimate $\|AS * f\|_{L^1(I;X)} \leqslant C\|f\|_{L^1(I;X)}$, which completes the proof of (1) and provides the Bochner integrability of $s \mapsto AS(t-s)f(s)$ on $(0,t)$. □

Turning to case $p = \infty$, we point out that we now impose a stronger assumption of the analytic semigroup, involving the prefix 'C_0-', compared to Theorem 17.3.11.

Theorem 17.3.12 (Kalton–Portal, maximal L^∞-regularity). *Let $I = (0,T)$ or $I = \mathbb{R}_+$. Let $-A$ be the generator of an analytic C_0-semigroup $(S(t))_{t>0}$ on a Banach space X, and suppose that $\sup_{t \in I} \|S(t)\| < \infty$. Then the following assertions are equivalent:*

(1) *A has maximal L^∞-regularity on I;*
(2) *A has maximal C-regularity on I;*
(3) *there exists a constant $C > 0$ such that*

$$\|x\| \leqslant \limsup_{t \to T} \|S(t)x\| + C \sup_{t \in I} \|tAS(t)x\|, \quad x \in X,$$

where we set $T = \infty$ if $I = \mathbb{R}_+$.

If these conditions are satisfied, and if we denote

$$M_{I,k} := \sup_{t \in I} \|(tA)^k S(t)\|, \quad k = 0, 1, 2,$$

then (3) holds with $C = (1 + 4M_{\infty,A}^{\mathrm{reg}}(I) M_{I,0})$, and

$$M_{\infty,A}^{\mathrm{reg}}(I) = M_{\mathrm{cont},A}^{\mathrm{reg}}(I) \leqslant M_{I,1} \log(2) + C M_{I,2}.$$

Remark 17.3.13.

(1) If $T < \infty$, we have $\limsup_{t\to T} \|S(t)x\| = \|S(T)x\|$, while if $T = \infty$, then for all $x \in \overline{\mathsf{R}(A)}$ we have $\limsup_{t\to\infty} \|S(t)x\| = 0$.

(2) The quantities $M_{I,k}$ are finite under the assumptions of Theorem 17.3.12, although only case $k = 0$ is explicitly postulated:
For $k = 1$, let first $I = \mathbb{R}_+$. Then we are assuming that $(S(t))_{t>0}$ is a bounded analytic C_0-semigroup, and the finiteness of $M_{I,1}$ is an application of Theorem G.5.3. If $I = (0,T)$ with $T < \infty$, then $(S(t))_{t>0}$ is only assumed to be an analytic C_0-semigroup, but not necessarily bounded. (The finiteness of $M_{I,0}$ is automatic for any semigroup.) However, $S_\lambda(t) = e^{-\lambda t}S(t)$ will be a bounded analytic C_0-semigroup, if $\lambda > 0$ is sufficiently large. Then Theorem G.5.3 applies to show that

$$M_\lambda := \sup_{t>0} \|t(\lambda + A)S_\lambda(t)\| < \infty,$$

and then

$$M_{I,1} \leqslant \sup_{t\in I} \Big(e^{\lambda t}\|t(\lambda + A)S_\lambda(t)\| + \|t\lambda S(t)\|\Big) \leqslant e^{\lambda T} M_\lambda + T\lambda M_{I,0} < \infty.$$

Finally, for $k = 2$, we can use the semigroup property to deduce that

$$M_{I,2} = \sup_{t\in I} \|(tA)^2 S(\tfrac{1}{2}t)^2\| \leqslant 4 \sup_{t\in I} \|(\tfrac{1}{2}tA)S(\tfrac{1}{2}t)\|^2 \leqslant 4M_{I,1}^2$$

Proof of Theorem 17.3.12. The equivalence of (1) and (2) was already obtained in Theorem 17.2.46. Note that $M_{I,k} < \infty$ for each $k \geqslant 0$. This is clear if $T < \infty$ and follows from the uniform boundedness of the semigroup if $k \geqslant 1$.

(3)⇒(2): Let $f \in C_{\rm b}(I; \mathsf{D}(A))$ and fix $t \in \overline{I}$. Since $u := S * f$ takes values in $\mathsf{D}(A)$, we can use the assumption to obtain

$$\|Au(t)\| \leqslant \limsup_{r\to T} \|S(r)Au(t)\| + C \sup_{r\in I} \|rA^2S(r)u(t)\|. \tag{17.57}$$

For the first part of (17.57), the bound $\|AS(\sigma)\| \leqslant M_{I,1}/\sigma$ gives

$$\begin{aligned}\|S(r)Au(t)\| &\leqslant \int_0^t \|AS(t - s + r)f(s)\| \,\mathrm{d}s \\ &\leqslant M_{I,1} \int_0^t \frac{1}{t - s + r} \,\mathrm{d}s \|f\|_{C_{\rm b}(\overline{I};X)} \\ &= M_{I,1} \log\big(1 + \frac{t}{r}\big)\|f\|_{C_{\rm b}(\overline{I};X)}.\end{aligned}$$

As $r \uparrow T$, the logarithm tends to $\log(1 + \frac{t}{T})$, which vanishes if $T = \infty$. For the second part of (17.57), the bound $\|A^2S(\sigma)\| \leqslant M_{I,2}/\sigma^2$ gives

$$\|rA^2S(r)u(t)\| \leqslant \int_0^t \|rA^2S(t - s + r)f(s)\| \,\mathrm{d}s$$

$$\leqslant M_{I,2} \int_0^t \frac{r}{(t-s+r)^2}\,\mathrm{d}s \|f\|_{C_{\mathrm{b}}(\overline{I};X)}$$
$$= M_{I,2} \frac{t}{t+r} \|f\|_{C_{\mathrm{b}}(\overline{I};X)}$$

Therefore, (17.57) becomes

$$\|Au(t)\| \leqslant \Big(M_{I,1} \log\big(1 + \frac{t}{T}\big) + CM_{I,2}\Big) \|f\|_{C_{\mathrm{b}}(\overline{I};X)}.$$

Now Lemma 17.2.45 implies that A has maximal C-regularity.

(1)⇒(3): To obtain the desired estimate, it suffices to consider $I = (0,T)$ with $T < \infty$. The case $I = \mathbb{R}_+$ can then be deduced from the relation of maximal regularity on finite intervals and $\mathbb{R}_+$ established in Proposition 17.2.18. Moreover, the generator $-A$ of a C_0-semigroup has a dense domain by Proposition G.2.3(4), and hence it suffices to consider $x \in \mathsf{D}(A)$.

Proposition 17.2.32 implies that A^* has maximal L^1-regularity with constant at most $M^{\mathrm{reg}}_{\infty,A}(I)$. Therefore, Theorem 17.3.11 gives

$$\int_I \|A^* S(t)^* x^*\|\,\mathrm{d}t \leqslant M^{\mathrm{reg}}_{\infty,A}(I) M_{I,1} \|x^*\|, \quad x^* \in X. \tag{17.58}$$

Since $x \in \mathsf{D}(A)$, an integration by parts implies

$$\int_0^T tA^2 S(t)x\,\mathrm{d}t = -TAS(T)x + \int_0^T AS(t)x\,\mathrm{d}t = -TAS(T)x + S(T)x - x.$$

Therefore, for all $x^* \in X^*$,

$$\langle x, x^*\rangle = \langle S(T)x, x^*\rangle - \langle TAS(T)x, x^*\rangle - \int_0^T \langle tAS(t/2)x, A^*S(t/2)^*x^*\rangle\,\mathrm{d}t.$$

By (17.58),

$$\Big|\int_0^T \langle tAS(t/2)x, A^*S(t/2)^*x^*\rangle\,\mathrm{d}t\Big| \leqslant 4M^{\mathrm{reg}}_{\infty,A}(I) M_{I,1} \sup_{t\in(0,T)} \|tAS(t)x\|\,\|x^*\|,$$

and hence

$$|\langle x, x^*\rangle| \leqslant \|S(T)x\|\,\|x^*\| + (1 + 4M^{\mathrm{reg}}_{\infty,A}(I) M_{I,1}) \sup_{t\in(0,T)} \|tAS(t)x\|\,\|x^*\|.$$

This implies the required result. □

17.3.c Sum-of-operators approach

It follows from the discussion in Section 17.2 that a closed operator A on a Banach space X has maximal L^p-regularity on I if and only if for every $f \in L^p(I;X)$ the inhomogeneous problem

$$\begin{cases} u'(t) + Au(t) = f(t), & t \in I, \\ \qquad u(0) = 0, \end{cases} \tag{17.59}$$

admits a unique L^p-solution u of (17.59) such that $Au, u' \in L^p(I;X)$. Here, $I = (0,T)$ or $I = \mathbb{R}_+$, and u' is the weak derivative of u on I. Let

$$ {}_0W^{1,p}(I;X) := \big\{u \in W^{1,p}(I;X) : u(0) = 0\big\},$$

recalling that functions in $W^{1,p}(I;X)$ admit a unique version that is continuous on $\overline{I}$.

Let D denote the weak derivative, viewed as a closed operator on $\widetilde{X} = L^p(I;X)$ with domain

$$\mathsf{D}(D) := {}_0W^{1,p}(IX).$$

We will also consider D as an operator on $\widetilde{X} = C_{\rm b}(\overline{I};X)$ with domain

$$\mathsf{D}(D) := \big\{u \in C^1_{\rm b}(\overline{I};X) : u(0) = 0\big\}.$$

In Proposition 17.3.16 we investigate various properties of the operator D.

By $\widetilde{A}$ we denote the closed operator $(\widetilde{A}y)(t) = A(y(t))$ on $L^p(I;X)$ with domain $\mathsf{D}(\widetilde{A}) := L^p(I;\mathsf{D}(A))$.

Maximal L^p-regularity can be characterised in terms of the operator sum $\widetilde{A} + D$ as follows.

Proposition 17.3.14. *Let $-A$ be the generator of a C_0-semigroup $(S(t))_{t\geqslant 0}$ on a Banach space X, let $1 \leqslant p \leqslant \infty$, and assume that at least one of the following conditions holds:*

- *$I = (0,T)$ is bounded;*
- *$(S(t))_{t\geqslant 0}$ is uniformly exponentially stable.*

The following assertions are equivalent:

(1) *A has maximal L^p-regularity on I;*
(2) *$\widetilde{A} + D$ boundedly invertible;*
(3) *there is a constant $M \geqslant 0$ such that*

$$\|\widetilde{A}u\|_p + \|Du\|_p \leqslant M\|(\widetilde{A} + D)u\|_p, \quad u \in \mathsf{D}(\widetilde{A}) \cap \mathsf{D}(D);$$

(4) *$\widetilde{A} + D$ is closed.*

The same equivalences holds if in (1) we consider maximal C-regularity and in (2)–(4) we take $\mathsf{D}(\widetilde{A}) = C_{\rm b}(\overline{I};\mathsf{D}(A))$ and $\mathsf{D}(D) = \{u \in C^1_{\rm b}(\overline{I};X) : u(0) = 0\}$.

Proof. Let $\widetilde{X} := L^p(I;X)$ when considering maximal L^p-regularity with $p \in [1,\infty]$, and let $\widetilde{X} = C_{\rm b}(\overline{I};X)$ when considering maximal C-regularity.

The equivalence of maximal L^∞-regularity with (3)–(4) will be considered at the end of the proof. We first deal with maximal L^p-regularity for $1 \leqslant p < \infty$ and with maximal C-regularity.

(1)⇒(2): This follows from the definition of maximal L^p (resp. C)-regularity and Corollary 17.2.9.

(2)⇒(3): This is trivial.

(3)⇒(4): This was already observed in Proposition 16.3.1.

(4)⇒(1): First consider the L^p-setting with $p < \infty$. The closedness assumption implies that $\mathsf{D}(\widetilde{A}) \cap \mathsf{D}(D)$ is a Banach space with respect to the graph norm of $\widetilde{A} + D$. For $f \in \mathsf{D}(\widetilde{A}))$ set $u_f(t) = \int_0^t S(t-s)f(s)\,\mathrm{d}s$. Then $u_f \in \mathsf{D}(\widetilde{A}) \cap \mathsf{D}(D)$ and $\widetilde{A}u_f + Du_f = f$, and therefore

$$\|u_f\|_{\mathsf{D}(\widetilde{A})\cap\mathsf{D}(D)} = \|u_f\|_p + \|(\widetilde{A}+D)u_f\|_p = \|u_f\|_p + \|f\|_p \leqslant (M+1)\|f\|_p,$$

where M is the norm of the operator $f \mapsto u_f$ as a bounded operator on $\widetilde{X}$. The density of $\mathsf{D}(\widetilde{A})$ in $\widetilde{X}$ implies that $f \mapsto u_f$ extends uniquely to a bounded operator from $\widetilde{X}$ into $\mathsf{D}(\widetilde{A}) \cap \mathsf{D}(D)$. Therefore, (1) holds by Theorem 17.2.19.

In the continuous setting, the above proof can be repeated and instead of the density argument we use Lemma 17.2.45.

In the L^∞-setting, (4) implies the C-version of (4), and the C-version of (1) holds. Therefore, A has maximal L^∞-regularity by Theorem 17.2.46. □

Next we present a version of Proposition 17.3.14 in the case where $I = \mathbb{R}_+$ and $-A$ generates a bounded C_0-semigroup, but without the condition $0 \in \varrho(A)$. For $I = \mathbb{R}_+$ the equivalence with (2) no longer holds (see Corollary 17.2.25). Also, the equivalence with (4) does not hold. Indeed, in Section 17.4.c we will see that there exist *bounded* operators A on L^q such that $-A$ generates a bounded analytic C_0-semigroup, but A does not have maximal L^p-regularity on $\mathbb{R}_+$ unless $q = 2$. The boundedness of A clearly implies that $\widetilde{A} + D$ is closed.

In the proposition we also allow $I = (0,T)$, in which case some information on the uniformity of the constants with respect to T can be deduced.

Proposition 17.3.15. *Let $-A$ be the generator of a C_0-semigroup on a Banach space X, let $1 \leqslant p \leqslant \infty$, and $I = (0,T)$ or $I = \mathbb{R}_+$, and suppose that $M_S(I) := \sup_{t\in I} \|S(t)\| < \infty$. The following assertions are equivalent:*

(1) *A has maximal L^p-regularity on I;*
(2) *there is a constant $M \geqslant 0$ such that*

$$\|\widetilde{A}u\|_p + \|Du\|_p \leqslant M\|(\widetilde{A}+D)u\|_p, \quad u \in \mathsf{D}(\widetilde{A}) \cap \mathsf{D}(D);$$

Moreover, letting $M_0(I)$ denote the least admissible constant in (2), one has

$$M_0(I) \leqslant 2M^{\mathrm{reg}}_{p,A}(I) + 1,$$
$$M^{\mathrm{reg}}_{p,A}(0,T) \leqslant M_0(0,T), \text{ and } M^{\mathrm{reg}}_{p,A}(\mathbb{R}_+) \leqslant M_0(\mathbb{R}_+) + 2M_S(I).$$

The same equivalence and estimates hold if in (1) *we consider maximal C-regularity and in* (2) *we take* $\mathsf{D}(\widetilde{A}) = C_{\mathrm{b}}([0,\infty);\mathsf{D}(A))$ *and* $\mathsf{D}(D) = \{u \in C^1_{\mathrm{b}}([0,\infty);X) : u(0) = 0\}$.

Furthermore, in the above situation $\widetilde{A} + D$ *is closed and injective.*

Proof. (1)⇒(2): For every $f \in L^p(\mathbb{R}_+;X)$ there exists a unique L^p-solution (resp C-solution) u_f satisfying $\|\widetilde{A}u_f\|_p \leqslant M^{\mathrm{reg}}_{p,A}(I)\|f\|_p$, where $M^{\mathrm{reg}}_{p,A}(I)$ is the constant of Definition 17.2.4. Then $\|Du_f\|_p \leqslant \|(\widetilde{A}+D)u_f\|_p + \|\widetilde{A}u_f\|_p \leqslant (1 + M^{\mathrm{reg}}_{p,A}(I))\|f\|_p$, and therefore

$$\|\widetilde{A}u_f\|_p + \|Du_f\|_p \leqslant (1+2M)\|f\|_p = (1+2M^{\mathrm{reg}}_{p,A}(I))\|(\widetilde{A}+D)u_f\|_p.$$

On the other hand, given $u \in \mathsf{D}(\widetilde{A}) \cap \mathsf{D}(D)$ we may take $f := (\widetilde{A}+D)u$. Then $u_f = u$, and substituting this in the above inequality gives the desired result.

(2)⇒(1): First consider the L^p-setting with $1 \leqslant p < \infty$. For $f \in \mathsf{D}(\widetilde{A}))$ set $u_f(t) = \int_0^t S(t-s)f(s)\,\mathrm{d}s$. Note that $Au_f + Du_f = f$. In the case $I = (0,T)$, one has $u_f \in \mathsf{D}(\widetilde{A}) \cap \mathsf{D}(D)$ and (2) implies

$$\|Au_f\|_p = \|\widetilde{A}u_f\|_p \leqslant M\|(\widetilde{A}+D)u_f\|_p = \|f\|_p,$$

and conclude the required result by density of $\mathsf{D}(\widetilde{A}))$ in $L^p(0,T;X)$. In the case $I = \mathbb{R}_+$ it need not be true that $u_f \in L^p(\mathbb{R}_+;X)$, and thus we cannot apply the estimate of (2). In order to get around this problem, fix $T > 0$ and define $v_T : [0,\infty) \to X$ by $v_T := \phi_T u_f$, where $\phi_T : [0,\infty) \to [0,1]$ is the piecewise linear function satisfying connecting the points $(0,1)$, $(T,1)$, and $(2T,0)$, and which is zero on the interval $(2T,\infty)$.

We have $v_T \in \mathsf{D}(\widetilde{A}) \cap \mathsf{D}(D)$ and

$$(\widetilde{A}+D)v_T = \phi_T(\widetilde{A}+D)u_f + \phi_T' u_f,$$

and therefore

$$\begin{aligned}\|Au_f\|_{L^p(0,T;X)} &= \|Av_T\|_{L^p(0,T;X)}\\ &\leqslant \|Av_T\|_{L^p(\mathbb{R}_+;X)}\\ &\leqslant M\|(\widetilde{A}+D)v_T\|_{L^p(0,\infty;X)}\\ &\leqslant M\|f\|_{L^p(\mathbb{R}_+;X)} + MT^{-1}\|u_f\|_{L^p(T,2T;X)}.\end{aligned}$$

Writing $M_S := \sup_{t\geqslant 0}\|S(t)\|$, Hölder's inequality gives

$$\frac{\|u_f\|_{L^p(T,2T;X)}}{T} \leqslant \frac{M_S}{T}\Big(\int_T^{2T} t^{p-1}\,\mathrm{d}t\Big)^{1/p}\|f\|_{L^p(\mathbb{R}_+;X)} = M_S c_p\|f\|_{L^p(\mathbb{R}_+;X)},$$

where $c_p^p = \frac{2^p-1}{p} \leqslant 2$. Combining the estimates and letting $T \to \infty$, we obtain

$$\|Au_f\|_{L^p(\mathbb{R}_+;X)} \leqslant M\|f\|_{L^p(\mathbb{R}_+;X)} + M_S c_p \|f\|_{L^p(\mathbb{R}_+;X)}.$$

The density of $\mathsf{D}(\widetilde{A})$ in $L^p(\mathbb{R}_+;X)$ implies that $f \mapsto Au_f$ extends uniquely to a bounded operator from $L^p(\mathbb{R}_+;X)$ into $\mathsf{D}(\widetilde{A}) \cap \mathsf{D}(D)$. Therefore, (1) holds by Theorem 17.2.19.

In the continuous setting, the above proof can be repeated with a C^1-adjustment of the cut-off function ϕ_T. Instead of the density argument we can use Lemma 17.2.45.

In the L^∞-setting, if (2) holds, then the estimate holds for all $u \in C_{\rm b}(\overline{I};\mathsf{D}(A)) \cap C^1_{\rm b}(\overline{I};X)$. Therefore, A has maximal C-regularity on I by the preceding argument. Now Theorem 17.2.46 implies that A has maximal L^∞-regularity on I.

The final assertion on closedness has already been observed in Proposition 16.3.1. To prove injectivity, suppose that $(\widetilde{A}+D)u = 0$. Now (2) gives $Du = 0$, so u is constant. Since $u(0) = 0$, we see that $u = 0$. □

The operator sum $\widetilde{A} + D$ of Propositions 17.3.14 and 17.3.15 actually falls in the setting of Theorem 16.3.2. In the case $I = (0,T)$, Theorem 17.2.15 implies that there exists a $\lambda \in \mathbb{R}$ such that $A+\lambda$ (and thus $\widetilde{A}+\lambda$) is sectorial of angle $< \pi/2$. Moreover, $D - \lambda$ is sectorial of angle $\leqslant \pi/2$ (see Proposition 17.3.16 below). Therefore, $\widetilde{A} + D = (\widetilde{A} + \lambda) + (D - \lambda)$ is as in Theorem 16.3.2. For $I = \mathbb{R}_+$, the assumptions in Proposition 17.3.15 imply that A is sectorial of angle $< \pi/2$. Hence also in this case the sum falls in the setting of Theorem 16.3.2.

Theorem 16.3.6 will now be applied to derive sufficient conditions for maximal L^p-regularity in terms of the operator A and the space X (see Theorem 17.3.19). This will lead to a different proof of Theorem 17.3.1(2). We first collect some properties of the operator D on $L^p(I;X)$ introduced before Proposition 17.3.14. We will only state the next proposition for $p \in [1,\infty)$, since D is not densely defined if $p = \infty$.

Proposition 17.3.16. *Let $p \in [1,\infty)$ and let X be a Banach space. Consider the derivative operator D on $L^p(0,T;X)$ subject to the one-sided Dirichlet boundary condition $u(0) = 0$, as introduced before Proposition 17.3.14. Then the following hold:*

(1) *$\sigma(D) = \varnothing$;*
(2) *for all $\lambda \in \mathbb{C}$ the operator $D + \lambda$ is densely defined, sectorial of angle $\leqslant \pi/2$, and invertible on $L^p(0,T;X)$;*
(3) *if X is a UMD space and $p \in (1,\infty)$, then D has a bounded H^∞-calculus on $L^p(0,T;X)$ of angle $\omega_{H^\infty}(D) \leqslant \pi/2$, and for all $\sigma \in (\pi/2,\pi)$ and $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ we have*

$$\|f(D)\| \leqslant C_\sigma C_{p,X} \|f\|_{H^\infty(\Sigma_\sigma)};$$

(4) *if X is a UMD space and $p \in (1,\infty)$, then for all $\lambda \in \mathbb{R}$ the operator $D+\lambda$ has a bounded H^∞-calculus on $L^p(0,T;X)$ of angle $\omega_{H^\infty}(D+\lambda) \leqslant \pi/2$.*

In (3) and (4), the angles are in fact equal to $\pi/2$. This will not be needed later on.

Proof. We begin by observing that the operator $-D$ generates the C_0-semigroup S of right-shifts on $L^p(0,T;X)$,

$$S(t)f(s) = \begin{cases} f(s-t), & \text{if } s-t \in I; \\ 0, & \text{if } s-t \notin I. \end{cases}$$

This semigroup is the tensor extension of the positive contraction semigroup corresponding to the scalar-valued case.

(1): The semigroup S is nilpotent in the sense that $S(t) = 0$ for all $t \geqslant T$. In particular, for all $\omega \in \mathbb{R}$ there exists a constant $M \geqslant 1$ such that $\|S(t)\| \leqslant Me^{\omega t}$ for all $t \geqslant 0$. Hence by Proposition G.4.1, for all $\omega \in \mathbb{R}$ the half-plane $\{\Re\lambda > \omega\}$ is contained in the resolvent set of $-D$.

(2): It is clear that D is densely defined. We claim that D is sectorial of angle $\leqslant \pi/2$. Indeed, since S is nilpotent we have $\sigma(D) = \varnothing$, and for fixed $\lambda \in \mathbb{C}$ and all $\Re(\mu) > 0$ we have

$$\|R(\mu, -\lambda - D)f\|_p \leqslant \int_0^T e^{-(\Re(\mu+\lambda)t}\|f\|_p \,\mathrm{d}t \leqslant \frac{\max\{e^{-\Re(\lambda)}, 1\}}{\Re(\mu)}\|f\|_p.$$

This implies the asserted sectoriality. The surjectivity of $\lambda + D$ follows from the fact that every $\lambda \in \mathbb{C}$ belongs to the resolvent set $\varrho(D)$ by part (1).

(3): This is immediate from Theorem 10.7.12, with constant $C_{p,X} = \beta_{p,X}^2 \hbar_{p,X}$ and $C_\sigma = 200\frac{1}{\cos\sigma}$.

(4): For $\lambda > 0$ this follows from Proposition 16.2.6(1) applied to the operator D. For $\lambda < 0$, this follows from Proposition 16.2.6(2) applied to the sectorial operator $D + \lambda - 1$. □

For $I = \mathbb{R}_+$, arguing as in the above proof one obtains the following.

Proposition 17.3.17. *Let $p \in [1,\infty)$ and let X be a Banach space. Consider the derivative operator D on $L^p(\mathbb{R}_+;X)$ subject to the one-sided Dirichlet boundary condition $u(0) = 0$, as introduced before Proposition 17.3.14. Then the following hold:*

(1) *the operator $D + \lambda$ is densely defined, sectorial of angle $\leqslant \pi/2$ on $L^p(\mathbb{R}_+;X)$;*
(2) *if X is a UMD space and $p \in (1,\infty)$, then D has a bounded H^∞-calculus on $L^p(\mathbb{R}+;X)$ of angle $\omega_{H^\infty}(D) \leqslant \pi/2$, and for all $\sigma \in (\pi/2,\pi)$ and $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ we have*

$$\|f(D)\| \leqslant C_\sigma C_{p,X}\|f\|_{H^\infty(\Sigma_\sigma)};$$

In the following we give a different proof of Theorem 17.3.1(2) based on H^∞-calculus techniques.

Theorem 17.3.18 (Maximal L^p-regularity through the H^∞-calculus). *Suppose that A is a linear operator on a UMD space X and $p \in (1,\infty)$. Suppose at least one of the following conditions holds:*

(1) *$I = (0,T)$ and there exists a $\lambda \in \mathbb{R}$ such that $A+\lambda$ is R-sectorial of angle $\omega_R(A+\lambda) < \pi/2$;*
(2) *$I = \mathbb{R}_+$, A is R-sectorial of angle $\omega_R(A+\lambda) < \pi/2$.*

Then A has maximal L^p-regularity on I.

Proof. It is straightforward to check that A and D are resolvent commuting. Moreover, since X is reflexive (see Theorem 4.3.3), it follows from Proposition 10.1.9 that $\mathsf{D}(A)$ is dense in X.

(1): By the assumption we can find a (possibly different) $\lambda \in \mathbb{R}$ such that $A+\lambda$ (and thus $\widetilde{A}+\lambda$) is invertible and standard R-sectorial. By Proposition 17.3.16, D has a bounded H^∞-calculus and is standard sectorial. From Theorem 16.3.6, applied with $\widetilde{A}+\lambda$ and D, we see that

$$\|(\lambda+\widetilde{A})u\|_p + \|Du\|_p \leqslant C_{\lambda,T}\|(\lambda+\widetilde{A}+D)u\|_p, \quad u \in \mathsf{D}(\widetilde{A}) \cap \mathsf{D}(D).$$

Therefore, Proposition 17.3.14 implies that $\lambda + A$ has maximal L^p-regularity on $(0,T)$, and by Theorem 17.2.26(1) A has maximal L^p-regularity as well.

(2): By Proposition 17.3.17(1) and (2), D is densely defined and sectorial and has a bounded H^∞-calculus on $L^p(\mathbb{R}_+;X)$. Therefore, Theorem 16.3.6 implies

$$\|\widetilde{A}u\|_p + \|Du\|_p \leqslant C\|(\widetilde{A}+D)u\|_p, \quad u \in \mathsf{D}(\widetilde{A}) \cap \mathsf{D}(D).$$

Therefore, the result follows from Proposition 17.3.15. □

The following theorem is valid for arbitrary Banach spaces X and exponents $1 \leqslant p \leqslant \infty$. We let $I = (0,T)$ or $I = \mathbb{R}_+$, and on $L^p(I;X)$ we define the operator $\widetilde{A}$ by

$$(\widetilde{A}u)(t) := Au(t), \quad t \in I,$$

with domain $\mathsf{D}(\widetilde{A}) := L^p(I;\mathsf{D}(A))$.

Theorem 17.3.19 (Maximal L^p-regularity through the absolute H^∞-calculus). *Suppose that A is a densely defined linear operator on a Banach space X, let $1 \leqslant p < \infty$, and suppose that at least one of the following conditions holds:*

(1) *$I = (0,T)$ and there exists a $\lambda \in \mathbb{C}$ such that $\widetilde{A}+\lambda$ has an absolute $H^\infty(\Sigma_\sigma)$-calculus on $L^p(I;X)$ with $\sigma < \pi/2$;*
(2) *$I = \mathbb{R}_+$ and $\widetilde{A}$ has an absolute $H^\infty(\Sigma_\sigma)$-calculus on $L^p(I;X)$ with $\sigma < \pi/2$.*

Then A has maximal L^q-regularity on I for all $q \in [1,\infty) \cup \{p\}$.

Proof. (1): We use a variation of the argument in Theorem 17.3.18(1). By the assumption we can find a (possibly different) $\lambda \in \mathbb{C}$ such that $\widetilde{A} + \lambda$ has an operator-valued $H^\infty(\Sigma_\sigma)$-functional calculus with $\sigma < \pi/2$, is invertible, and since $p < \infty$ it is standard sectorial. By Proposition 17.3.16, D is sectorial of angle $\leqslant \pi/2$. From Theorem 16.3.14 we see that

$$\|(\lambda + \widetilde{A})u\|_p + \|Du\|_p \leqslant C_{\lambda,T}\|(\lambda + \widetilde{A} + D)u\|_p, \quad u \in \mathsf{D}(\widetilde{A}) \cap \mathsf{D}(D).$$

Therefore, Proposition 17.3.14 implies that $\lambda+A$ has maximal L^q-regularity on $(0,T)$, and thus Theorem 17.2.26(1) implies that A has maximal L^q-regularity, for all $q \in [1,\infty) \cup \{p\}$.

(2): One can repeat the proof of Theorem 17.3.18(2) with a similar variation as in the above proof. □

As a consequence of Theorem 16.3.20 we obtain the following result.

Corollary 17.3.20 (Da Prato–Grisvard for $1 \leqslant p < \infty$). *Let $-A$ generate an analytic C_0-semigroup $(S(t))_{t\geqslant 0}$ on a Banach space X, and suppose that at least one of the following conditions holds:*

(1) $I = (0,T)$*;*
(2) $I = \mathbb{R}_+$ *and S is uniformly exponentially stable.*

Then for all $1 \leqslant p < \infty$ and $0 < \theta < 1$, the part $A_{\theta,p}$ of A in the real interpolation space $(X,\mathsf{D}(A))_{\theta,p}$ has maximal L^q-regularity for all $q \in (1,\infty) \cup \{p\}$.

Proof. By the extrapolation result of Theorem 17.2.26(5), it suffices to consider $q = p$ below.

(1): We first prove the result for $1 \leqslant p < \infty$; the case $p = \infty$ is considered at the end of the proof. By the Stein–Weis theorem (Theorem 14.3.4),

$$(\widetilde{X}, \mathsf{D}(\widetilde{A}))_{\theta,p} = (L^p(I;X), L^p(I;\mathsf{D}(A)))_{\theta,p} = L^p(I;(X,\mathsf{D}(A))_{\theta,p}).$$

Therefore, Theorem 16.3.20 implies that for $\lambda \in \mathbb{R}$ large enough $\lambda + \widetilde{A}_{\theta,p}$ has an absolute functional calculus on $L^p(I;(X,\mathsf{D}(A))_{\theta,p})$ of angle $< \pi/2$. Since Proposition 17.3.16 gives that D is sectorial of angle $\frac{\pi}{2}$, we can apply Theorem 16.3.14 and Proposition 17.3.14 to conclude the required result.

(2): For $1 \leqslant p < \infty$ the result follows from (1) and Theorem 17.2.24. Now let $p = \infty$. We use the equivalent norm of Theorem L.2.4 for $(X,\mathsf{D}(A))_{\theta,\infty}$. For all $t \geqslant 0$,

$$\begin{aligned}\|AS * f(t)\|_X &\leqslant \int_0^t \|AS(t-s)f(s)\|_X \,\mathrm{d}s \\ &\lesssim_{A,\theta} \int_0^t \|(t-s)^{\theta-1}\,\mathrm{d}s\|f\|_{L^\infty(0,T;(X,\mathsf{D}(A))_{\theta,\infty})}\end{aligned}$$

$$\leqslant C_\theta \|f\|_{L^\infty(0,T;(X,\mathsf{D}(A))_{\theta,\infty})}.$$

Moreover,

$$\begin{aligned}
&\|AS * f(t)\|_{(X,\mathsf{D}(A))_{\theta,\infty}} \\
&\eqsim_{\theta,A} \sup_{r>0} r^{1-\theta} \|AS(r)AS * f(t)\| \\
&\leqslant r^{1-\theta} \|AS(r)AS * f(t)\| \\
&\leqslant \sup_{r>0} r^{1-\theta} \int_0^t \|AS((t+r-s)/2)\| \, \|AS((t+r-s)/2)f(s)\| \, \mathrm{d}s \\
&\lesssim \sup_{r>0} r^{1-\theta} \int_0^t (t+r-s)^{\theta-2} \, \mathrm{d}s \|f\|_{L^\infty(0,T;(X,\mathsf{D}(A))_{\theta,\infty})} \\
&\lesssim C_\theta \|f\|_{L^\infty(0,T;(X,\mathsf{D}(A))_{\theta,\infty})}.
\end{aligned}$$

Taking the supremum over $t \geqslant 0$, we infer that $A_{\theta,\infty}$ has maximal L^∞-regularity.

(1) for $p = \infty$: By Theorem 17.2.26(1), the result follows from the case $p = \infty$ of (2). □

Remark 17.3.21. In the proof of the case $p = \infty$ we did not use that $(S(t))_{t\geqslant 0}$ is strongly continuous. Also for $p < \infty$ this assumption can be avoided by using a direct proof (see the notes).

As a consequence of the case $p = \infty$, by a density argument we obtain a similar result for maximal C-regularity using the spaces $C(\overline{I}; (X, \mathsf{D}(A))^0_{\theta,\infty})$, where $(X, \mathsf{D}(A))^0_{\theta,\infty}$ is defined as the closure of $\mathsf{D}(A)$ in $(X, \mathsf{D}(A))_{\theta,\infty}$.

17.3.d Maximal L^p-regularity on the real line

Up to this point we have been concerned with maximal regularity on the half-line and on bounded intervals. In the present section we discuss maximal L^p-regularity on the real line. We have already briefly touched upon it in Section 17.3.a, which in hindsight states that, if $-A$ generates an analytic semigroup, then A has maximal L^p-regularity on the half-line if and only if it has maximal L^p-regularity on the real line.

It is worth to considering the case of the real line separately for two reasons:

(1) there are operators A which have maximal L^p-regularity on $\mathbb{R}$ for which $-A$ does not generate a semigroup (see below Definition 17.3.23);
(2) in certain situations, it is possible to derive maximal regularity on the half-line by checking maximal L^p-regularity on the real line (see Theorem 17.3.32).

For a given function $f \in L^1_{\mathrm{loc}}(\mathbb{R}; X)$, consider the problem

$$u'(t) + Au(t) = f(t), \quad t \in \mathbb{R}. \tag{17.60}$$

Notice that there is no initial value as time runs over all of $\mathbb{R}$.

Definition 17.3.22 (Strong solutions and L^p-solutions). *A strongly measurable function $u : \mathbb{R} \to X$ is called a* strong solution *of* (17.60) *associated with a given $f \in L^1_{\rm loc}(\mathbb{R}; X)$ if*

(i) *u takes values in $\mathsf{D}(A)$ almost everywhere and Au belongs to $L^1_{\rm loc}(\mathbb{R}; X)$;*
(ii) *for almost all $s, t \in \mathbb{R}$ we have*

$$u(t) - u(s) + \int_s^t Au(r)\,\mathrm{d}r = \int_s^t f(r)\,\mathrm{d}r.$$

A strong solution u associated with a function $f \in L^p(\mathbb{R}; X)$ is called an L^p-solution *if $Au \in L^p(\mathbb{R}; X)$.*

As before one shows that if $u : \mathbb{R} \to X$ is a strong solution of (17.60), then u is differentiable almost everywhere, u takes values in $\mathsf{D}(A)$ almost everywhere, both u' and Au belong to $L^1_{\rm loc}(\mathbb{R}; X)$, and (17.60) holds for almost all $t \in \mathbb{R}$. Moreover, strong solutions have a continuous version. If u is an L^p-solution of (17.60), then for all real numbers $a < b$ we have

$$\|u(b) - u(a)\| \leqslant (b-a)^{1/p'}(\|Au\|_{L^p(a,b;X)} + \|f\|_{L^p(a,b;X)})$$

as in (17.5), and as in (17.6) it follows that $u' \in L^p(\mathbb{R}; X)$ and

$$\|u'\|_{L^p(\mathbb{R};X)} \leqslant \|Au\|_{L^p(\mathbb{R};X)} + \|f\|_{L^p(\mathbb{R};X)}.$$

The definition of *maximal L^p-regularity on* $\mathbb{R}$ is as in Definition 17.2.4, except that now we take $I = \mathbb{R}$. However, to avoid technical problems involving uniqueness and approximation we furthermore ask for an *a priori* estimate in $L^p(\mathbb{R}; \mathsf{D}(A))$ instead of just the norm of Au.

Definition 17.3.23 (Maximal L^p-regularity). *A linear operator A has* maximal L^p-regularity on $\mathbb{R}$ *if for every $f \in L^p(\mathbb{R}; X)$ there exists a unique L^p-solution u of* (17.60) *on $\mathbb{R}$, and there is a constant $C \geqslant 0$ independent of f such that*

$$\|u\|_{L^p(\mathbb{R};\mathsf{D}(A))} \leqslant C\|f\|_{L^p(\mathbb{R};X)}. \tag{17.61}$$

The least admissible constant C will be called the maximal L^p-regularity constant *and will be denoted by $M^{\rm reg}_{p,A}(\mathbb{R})$.*

Unlike in the definition of $M^{\rm reg}_{p,A}(I)$ for bounded intervals $I = (0,T)$ or $\mathbb{R}_+$, an *a priori* bound on $\|u\|_{L^p(\mathbb{R};X)}$ is also included in the estimate (17.61) through the use of the graph norm. Also, unlike in the case $I = (0,T)$ or $I = \mathbb{R}_+$, in the above definition the uniqueness is assumed *a priori*. In fact, when uniqueness holds, then A is injective. Indeed, suppose that $Ax = 0$. Then, setting $u = x$, one has $u' + Au = 0$, and therefore $u = 0$ by uniqueness. Conversely, if A is injective, then uniqueness is immediate from the estimate (17.61).

Example 17.3.24. The operator $A = 0$ does not have maximal L^p-regularity on $\mathbb{R}$, but of course it has maximal L^p-regularity on $\mathbb{R}_+$.

The analogue of Lemma 17.2.16 for the real line does not hold: it is *not* true that maximal L^p-regularity on $\mathbb{R}$ implies maximal L^p-regularity on all subintervals of $\mathbb{R}$. This can be seen as follows. By reflection, the following observation is immediate:

Proposition 17.3.25. *Let $1 \leqslant p \leqslant \infty$ be fixed. A linear operator A has maximal L^p-regularity on $\mathbb{R}$ if and only if $-A$ has maximal L^p-regularity on $\mathbb{R}$, and in this case we have $M^{\rm reg}_{p,A}(\mathbb{R}) = M^{\rm reg}_{p,-A}(\mathbb{R})$.*

Now, if maximal L^p-regularity on $\mathbb{R}$ would always imply maximal L^p-regularity on $\mathbb{R}_+$, we could apply this fact to both A and $-A$ and obtain from Theorem 17.2.15 that A generates an analytic group. It is easy to see that this implies that A is bounded. Indeed, denoting the group by $(S(t))_{t\in\mathbb{R}}$, the identity $A = S(-1)(AS(1))$ exhibits A as the product of two bounded operators (the latter being bounded by Theorem G.5.3).

Example 17.3.26. Let A be an unbounded linear operator on X with maximal L^p-regularity on $\mathbb{R}$. Then the operator on $X \times X$ given by $\mathcal{A}(x, y) = (Ax, -Ay)$ has maximal L^p-regularity on $\mathbb{R}$. If $\mathcal{A}$ had maximal L^p-regularity on $(0, T)$, then $\mathcal{A}$ would generate an analytic semigroup by Theorem 17.2.15, and thus both A and $-A$ generate analytic semigroups. This contradicts the unboundedness of A.

Remark 17.3.27. Examples 17.3.24 and 17.3.26 show that the two properties of maximal L^p-regularity on $\mathbb{R}$ and on $\mathbb{R}_+$ are incomparable in both directions; neither implies the other one.

Remark 17.3.28. The reader may check that the following results extend to maximal L^p-regularity on $\mathbb{R}$:

(1) maximal L^p-regularity of A implies closedness of A (Proposition 17.2.5);
(2) it is enough to verify maximal L^p-regularity on a dense subspace (Proposition 17.2.10);
(3) under closedness of A and unique solvability for all f, maximal L^p-regularity follows (Proposition 17.2.11).

Indeed, by using the fact that maximal L^p-regularity on $\mathbb{R}$ is formulated with the more restrictive estimate (17.61), it is straightforward to extend the previous proofs.

Maximal regularity on $\mathbb{R}$ and bisectoriality

In the next paragraph we will discuss the relationship between maximal L^p-regularity on $\mathbb{R}_+$ and $\mathbb{R}$. Prerequisite to this discussion is the following result. It is a version for the real line of Dore's Theorem 17.2.15), which gave the sectoriality of operators with maximal L^p-regularity on $\mathbb{R}_+$.

Theorem 17.3.29 (Mielke). *Let X be a Banach space and let $1 \leqslant p \leqslant \infty$. Let $\lambda \in \mathbb{C}$ and set $\lambda_0 = |\lambda|$ if $\lambda \neq 0$ and $\lambda_0 = 1$ if $\lambda = 0$. Let A be a linear operator in X such that $\lambda + A$ has maximal L^p-regularity on $\mathbb{R}$, and suppose that there exists a constant C such that for all $f \in L^p(\mathbb{R}; X)$ the L^p-solution u of*

$$u' + (\lambda + A)u = f \tag{17.62}$$

satisfies

$$\Big\| \lambda_0 \|u\|_X + \|Au\|_X \Big\|_{L^p(\mathbb{R})} \leqslant C\|f\|_{L^p(\mathbb{R};X)}. \tag{17.63}$$

Then $\lambda + A$ is bisectorial, $0 \in \varrho(\lambda + A)$, and for all $\xi \in \mathbb{R}$ we have

$$\lambda_0 \|(\lambda + i\xi + A)^{-1}\| + \|A(\lambda + i\xi + A)^{-1}\| \leqslant C.$$

In the special case $\lambda = 0$, the left-hand side of (17.63) coincides with $\|u\|_{L^p(\mathbb{R};\mathsf{D}(A))}$ and therefore (17.63) holds with $C = M^{\rm reg}_{p,A}(\mathbb{R})$.

To prove the theorem, we first derive a maximal regularity result with exponential weights, related to Proposition 17.2.38. Given a non-negative locally integrable function w on $\mathbb{R}$, upon replacing the L^p-spaces in Definition 17.3.23 by the spaces L^p_w of function u such that $wu \in L^p$ (not to be confused with $L^p(w)$, where the condition is $w^{1/p}u \in L^p$), we obtain the definition of *maximal L^p_w-regularity on* $\mathbb{R}$. Propositions 17.2.5, 17.2.10, and 17.2.11 extend to this setting.

Proposition 17.3.30. *Suppose that the conditions of Theorem 17.3.29 hold, and let $\alpha \in \mathbb{R}$ satisfy $|\alpha| < \lambda_0/C$. Then the operator $\lambda + A$ has maximal $L^p_{e^{-\alpha|\cdot|}}$-regularity on $\mathbb{R}$. Moreover, for all $f : \mathbb{R} \to X$ such that $e^{-\alpha|\cdot|}f \in L^p(\mathbb{R})$ the $L^p_{e^{-\alpha|\cdot|}}$-solution u to (17.62) satisfies*

$$\Big\| \lambda_0 e^{-\alpha|\cdot|}\|u\|_X + e^{-\alpha|\cdot|}\|Au\|_X \Big\|_{L^p(\mathbb{R})} \leqslant \frac{C}{1 - |\alpha|\lambda_0^{-1}C}\|e^{-\alpha|\cdot|}f\|_{L^p(\mathbb{R};X)}.$$

Proof. As in the proof of extrapolation with exponential weights on $\mathbb{R}_+$ (Proposition 17.2.38), one checks that if $e^{-\alpha|\cdot|}f \in L^p(\mathbb{R}; X)$, then a locally integrable function $u : \mathbb{R} \to X$ is an $L^p_{e^{-\alpha|\cdot|}}$-solution to (17.60) if and only if $v = e^{-\alpha|\cdot|}u$ is an L^p-solution to

$$v' + Av + \alpha\,\mathrm{sgn}(\cdot)v = f_\alpha,$$

where $f_\alpha = e^{-\alpha|\cdot|}f$. Therefore it will be enough to analyse the equation for v, which will be done by a fixed point argument.

Let $Y := L^p(\mathbb{R}; \mathsf{D}(A))$ with norm

$$\|v\|_Y = \Big\| \lambda_0 \|v\|_X + \|Av\|_X \Big\|_{L^p(\mathbb{R})}. \tag{17.64}$$

For $g \in Y$ let $T(g) := v$, where v is the unique L^p-solution to

$$v' + (\lambda + A)v = f_\alpha - \alpha \operatorname{sgn}(\cdot) g.$$

The assumptions imply that T maps Y into itself and

$$\|T(g_1) - T(g_2)\|_Y \leqslant |\alpha| C \|g_1 - g_2\|_{L^p(\mathbb{R};X)} \leqslant \frac{|\alpha| C}{\lambda_0} \|g_1 - g_2\|_Y, \qquad (17.65)$$

where the constant on the right is < 1 by our assumptions. Therefore, T has a unique fixed point $v \in Y$, and this is our required L^p-solution. Moreover, by (17.65) and the assumed estimate of Theorem 17.3.29,

$$\|v\|_Y \leqslant \|T(v) - T(0)\|_Y + \|T(0)\|_Y \leqslant \frac{|\alpha| C}{\lambda_0} \|v\|_Y + C\|f_\alpha\|_{L^p(\mathbb{R};X)}.$$

It follows that $\|v\|_Y \leqslant K\|f_\alpha\|_{L^p(\mathbb{R};X)}$ with $K = \frac{C}{1-|\alpha|C\lambda_0^{-1}}$, and therefore

$$\begin{aligned}\Big\| \lambda_0 e^{-\alpha|\cdot|}\|u\|_X + e^{-\alpha|\cdot|}\|Au\|_X \Big\|_{L^p(\mathbb{R})} = \|v\|_Y &\leqslant K\|f_\alpha\|_{L^p(\mathbb{R};X)} \\ &= K\|e^{-\alpha|\cdot|} f\|_{L^p(\mathbb{R};X)},\end{aligned}$$

as claimed. □

Proof of Theorem 17.3.29. Fix $\alpha \in (0, \lambda_0/C)$, and let $Y = L^p(\mathbb{R}; \mathsf{D}(A))$ be normed as in (17.64).

Let $x \in X$, $s \in \mathbb{R}$, and $\xi \in \mathbb{R}$ be fixed, and for $t \in \mathbb{R}$ set $f_s(t) := e^{i\xi(t+s)}x$. Then $e^{-\alpha|\cdot|} f_s \in L^p(\mathbb{R}; X)$, and by Proposition 17.3.30 there exists a unique $L^p_{e^{-\alpha|\cdot|}}$-solution u_s to

$$u_s' + (\lambda + A)u_s = f_s,$$

and we have

$$\|e^{-\alpha|\cdot|} u_s\|_Y \leqslant \frac{C}{1 - \alpha C\lambda_0^{-1}} \|e^{-\alpha|\cdot|} f_s\|_{L^p(\mathbb{R};X)}. \qquad (17.66)$$

Since $f_s(t) = f_0(s+t) = e^{i\xi s} f_0(t)$, it follows that $u_s(t) = u_0(s+t) = e^{i\xi s} u_0(t)$. Therefore $u_0(s) = e^{i\xi s} y$ with $y := u_0(0)$. Since $e^{-\alpha|\cdot|} u_0 \in Y$, we must have $y \in \mathsf{D}(A)$, and consequently

$$e^{i\xi s}(i\xi + \lambda + A)y = u_0'(s) + (\lambda + A)u_0(s) = f_0(s) = e^{i\xi s} x.$$

This shows that $(\lambda + i\xi + A)y = x$. Moreover, by (17.66) with $s = 0$, upon letting $\alpha \downarrow 0$ we find that $\lambda_0\|y\| + \|Ay\| \leqslant C\|x\|$.

It remains to prove that $\lambda + i\xi + A$ is injective. Let $y \in \mathsf{D}(A)$ be such that $(\lambda + i\xi + A)y = 0$. Then $u(t) := e^{it\xi} y$ satisfies $u' + (\lambda + A)u = 0$, and therefore $u = 0$ by uniqueness of the $L^p_{e^{-\alpha|\cdot|}}$-solution. □

Connections between maximal regularity on $\mathbb{R}$ and $\mathbb{R}_+$

The following two results connect maximal L^p-regularity on $\mathbb{R}_+$ and $\mathbb{R}$ under suitable additional assumptions on A. This provides a way to check maximal L^p-regularity on $\mathbb{R}_+$ via the case $\mathbb{R}$; in fact, this is one of our main motivations to consider the case $\mathbb{R}$.

When A has maximal L^p-regularity on $\mathbb{R}$ we define the operator $\mathscr{M}_{\mathbb{R}} : L^p(\mathbb{R}; X) \to L^p(\mathbb{R}; \mathsf{D}(A))$ by

$$\mathscr{M}_{\mathbb{R}} f := u_f, \tag{17.67}$$

where u_f is the unique L^p-solution of (17.60). The norm of this operator is given by $\|\mathscr{M}_{\mathbb{R}}\| = M^{\rm reg}_{p,A}(\mathbb{R})$.

Proposition 17.3.31 (Maximal regularity: $\mathbb{R}_+$ versus $\mathbb{R}$ – I). *Let $p \in [1, \infty]$. For a linear operator A, the following assertions are equivalent:*

(1) *A has maximal L^p-regularity on $\mathbb{R}_+$ and $0 \in \varrho(A)$;*
(2) *A has maximal L^p-regularity on $\mathbb{R}$ and $-A$ generates a bounded analytic semigroup.*

In this case, one has

$$M^{\rm reg}_{p,A}(\mathbb{R}_+) = \|\overline{V}\|_{\mathscr{L}(L^p(\mathbb{R};X))} = \|A\mathscr{M}_{\mathbb{R}}\|_{\mathscr{L}(L^p(\mathbb{R};X))},$$

and the L^p-solution u to (17.60) satisfies $Au = \overline{V} f = A\mathscr{M}_{\mathbb{R}} f$, where $\overline{V}$ is as in Lemma 17.3.3.

Proof. We start with some general observations. Let $(S(t))_{t>0}$ be a bounded analytic semigroup with generator $-A$, and suppose that $0 \in \varrho(A)$. By Proposition K.2.3, $(S(t))_{t>0}$ is a uniformly exponentially stable analytic semigroup. Define the operator T on $L^p(\mathbb{R}; X)$ by

$$Tf := \int_{-\infty}^t S(t-s) f(s)\,\mathrm{d}s.$$

This operator is well defined and bounded by Young's inequality. For $f \in L^p(\mathbb{R}; X)$ let $u_f := Tf$. As in the proof of Proposition 17.1.3 one can check that for all $s < t$ one has

$$\int_s^t A(1+A)^{-1} u_f(r)\,\mathrm{d}r = -(1+A)^{-1}\Big(u_f(t) - u_f(s) + \int_s^t f(r)\,\mathrm{d}r\Big). \tag{17.68}$$

Therefore, $(1+A)^{-1}u_f$ is an L^p-solution to (17.60) with right-hand side $(1+A)^{-1}f$.

(1)⇒(2): From Theorem 17.2.15 it follows that $-A$ generates a bounded analytic semigroup $(S(t))_{t>0}$ and, since $0 \in \varrho(A)$, it is uniformly exponentially stable by Proposition K.2.3. Moreover, by Theorem 17.2.19 and Lemma

17.3.3, $\overline{V}$ extends to a bounded operator on $L^p(\mathbb{R};X)$ and $\|\overline{V}f\|_{L^p(\mathbb{R};X)} \leqslant M^{\rm reg}_{p,A}(\mathbb{R}_+)\|f\|_{L^p(\mathbb{R};X)}$ for all $f \in L^p(\mathbb{R};X)$. Let $f \in L^p(\mathbb{R};X)$ and set $u_f := A^{-1}\overline{V}f = Tf$. Clearly, $\|Au_f\|_{L^p(\mathbb{R};X)} \leqslant M^{\rm reg}_{p,A}(\mathbb{R}_+)\|f\|_{L^p(\mathbb{R};X)}$ by the boundedness of $\overline{V}$. It remains to check that u_f is an L^p-solution to (17.60).

From (17.68) and the fact that $Au_f \in L^p(\mathbb{R};X)$ we find that

$$(1+A)^{-1}\int_s^t Au_f(r)\,{\rm d}r = -(1+A)^{-1}\Big[u_f(t) - u_f(s) + \int_s^t f(r)\,{\rm d}r\Big].$$

Applying $1+A$ to both sides, it follows that u_f is an L^p-solution to (17.60).

(2)⇒(1): By Theorem 17.3.29, we have $0 \in \varrho(A)$, and thus $(S(t))_{t>0}$ is uniformly exponentially stable by Proposition K.2.3. For $f \in L^p(\mathbb{R};X)$, let $u_f := Tf$. By the observation below (17.68) and by uniqueness, we find that

$$(1+A)^{-1}Tf = (1+A)^{-1}u_f = u_{(1+A)^{-1}f} = (1+A)^{-1}u_f = (1+A)^{-1}\mathscr{M}_{\mathbb{R}}f,$$

where $\mathscr{M}_{\mathbb{R}}$ is as in (17.67). Applying $1+A$ on both sides, it follows that $Tf = \mathscr{M}_{\mathbb{R}}f$. Therefore, $\overline{V} = AT = A\mathscr{M}_{\mathbb{R}}$ are all bounded on $L^p(\mathbb{R};X)$. Now Theorem 17.2.19 and Lemma 17.3.3 imply that A has maximal L^p-regularity on $\mathbb{R}_+$. □

In situations where one does not know *a priori* whether $-A$ generates an analytic semigroup, the following result can often be used. In particular, this is useful in proving maximal L^p-regularity in the case that A is a differential operator and $\mathsf{D}(A)$ includes a boundary condition.

Theorem 17.3.32 (Maximal regularity: $\mathbb{R}_+$ versus $\mathbb{R}$ – part II). *Let A be a linear operator and $1 \leqslant p \leqslant \infty$. Suppose that, for all $\lambda > 0$, the operator $\lambda + A$ has maximal L^p-regularity on $\mathbb{R}$, and suppose that there exist a constant $C \geqslant 0$ and an integer $n \in \mathbb{N}$ such that, for all $\lambda > 0$ and all $f \in L^p(\mathbb{R};X)$, the L^p-solution u_λ of*

$$u' + (\lambda + A)u = f \tag{17.69}$$

satisfies

$$\Big\| \lambda\|u_\lambda\|_X + \|Au_\lambda\|_X \Big\|_{L^p(\mathbb{R})} \leqslant C(1+\lambda)^n\|f\|_{L^p(\mathbb{R};X)}. \tag{17.70}$$

Under these assumptions,

(1) *the estimate (17.70) holds with $n = 0$;*
(2) *$-A$ generates a bounded analytic semigroup;*
(3) *for all $\lambda \geqslant 0$, the operator $\lambda + A$ has maximal L^p-regularity on $\mathbb{R}_+$;*
(4) *for all $\lambda \geqslant 0$ and $f \in L^p(\mathbb{R}_+;X)$, there exists a unique L^p-solution v_λ to*

$$\begin{cases} v'(t) + (\lambda + A)v(t) &= f(t), \quad t \in \mathbb{R}_+, \\ v(0) &= 0; \end{cases} \tag{17.71}$$

(5) *this v_λ satisfies*

$$\Big\| \lambda \|v_\lambda\|_X + \|Av_\lambda\|_X \Big\|_{L^p(\mathbb{R}_+)} \leqslant C\|f\|_{L^p(\mathbb{R}_+;X)}. \qquad (17.72)$$

In particular, $M^{\rm reg}_{p,A}(\mathbb{R}_+) \leqslant C$.

Proof. Fix $\lambda > 0$. By assumption (17.70) and Mielke's Theorem 17.3.29, we have $i\xi \in \varrho(\lambda + A)$ for all $\xi \in \mathbb{R}$ and

$$\lambda\|(\lambda + i\xi + A)^{-1}\| + \|A(\lambda + i\xi + A)^{-1}\| \leqslant C(1 + \lambda)^n, \qquad \xi \in \mathbb{R}. \quad (17.73)$$

As a consequence also $\|(\lambda + i\xi)(\lambda + i\xi + A)^{-1}\| \leqslant C(1 + \lambda)^n + 1$. As the right-hand side remains bounded for $\lambda \downarrow 0$, it follows by a power series expansion that $\overline{\mathbb{C}_+} \setminus \{0\} \subseteq \varrho(-A)$ and (17.73) extends to all $\lambda + i\xi \in \overline{\mathbb{C}_+} \setminus \{0\}$. Fix $\varepsilon > 0$. Setting $\zeta_\varepsilon(z) := \frac{z}{z+\varepsilon} A(z + A)^{-1}$ for $z \in \overline{\mathbb{C}_+} \setminus \{0\}$ and $\zeta_\varepsilon(0) = 0$, upon rewriting (17.73) with $z = \lambda + i\xi$, we see that

$$\|\zeta_\varepsilon(z)\| \leqslant C(1 + \Re z)^n, \quad z \in \overline{\mathbb{C}_+}.$$

Hence, we can apply the Phragmén–Lindelöf principle to obtain

$$\sup_{z\in\overline{\mathbb{C}_+}} \|\zeta_\varepsilon(z)\| \leqslant \sup_{z\in\partial\mathbb{C}_+} \|\zeta_\varepsilon(z)\| \leqslant \sup_{\xi\in\mathbb{R}\setminus\{0\}} \|A(i\xi + A)^{-1}\| \leqslant C,$$

where the last estimate follows by taking $\lambda = 0$ in (17.73). Letting $\varepsilon \downarrow 0$, it follows that A is sectorial of angle $< \frac{1}{2}\pi$, and accordingly $-A$ generates a bounded analytic semigroup.

By Proposition 17.3.31, $\lambda + A$ has maximal L^p-regularity on $\mathbb{R}_+$ for all $\lambda > 0$. Moreover, this result and the fact that $0 \in \varrho(\lambda + A)$ also imply that, for all $\lambda > 0$, the solution u_λ to (17.69) satisfies

$$u_\lambda(t) = \int_{-\infty}^t e^{-\lambda(t-s)} S(t - s) f(s)\, \mathrm{d}s,$$

for $f \in L^p(\mathbb{R}; X)$. Thus $f \mapsto u_\lambda$ defines a bounded mapping $\mathcal{M}^\lambda_{\mathbb{R}} : L^p(\mathbb{R}; X) \to L^p(\mathbb{R}; \mathsf{D}(A))$ and by (17.70) we have

$$\Big\| \|\lambda \mathcal{M}^\lambda_{\mathbb{R}} f\|_X + \|A\mathcal{M}^\lambda_{\mathbb{R}} f\|_X \Big\|_{L^p(\mathbb{R})} \leqslant C(1 + \lambda)^n \|f\|_{L^p(\mathbb{R};X)}.$$

Reasoning as after (17.28), for all $\lambda > 0$ and $\xi \in \mathbb{R}$ we have

$$\|\mathscr{M}^{\lambda+i\xi}_{\mathbb{R}} f\|_{L^p(\mathbb{R};X)} = \|\mathscr{M}^\lambda_{\mathbb{R}} f\|_{L^p(\mathbb{R};X)}.$$

Therefore, in a similar way as before, the Phragmén–Lindelöf principle applied to the mapping $z \in \lambda + \mathbb{C}_+ \mapsto (z\mathcal{M}^z_{\mathbb{R}} f, A\mathcal{M}^z_{\mathbb{R}} f)$ gives that, for all $z \in \lambda + \mathbb{C}_+$,

$$\Big\| \|z\mathcal{M}^z_{\mathbb{R}} f\|_X + \|A\mathcal{M}^z_{\mathbb{R}} f\|_X \Big\|_{L^p(\mathbb{R})} \leqslant C\|f\|_{L^p(\mathbb{R};X)}.$$

Letting $\lambda \downarrow 0$, shows that we can take $n = 0$ in (17.70).

Now fix $\lambda > 0$. By Theorem 17.2.19, the solution v_λ to (17.71) satisfies

$$v_\lambda = \int_0^t e^{-\lambda(t-s)} S(t-s) f(s) \,\mathrm{d}s$$

whenever $f \in L^p(\mathbb{R}_+; X)$. Therefore, we find that $v_\lambda = u_\lambda$ on $\mathbb{R}_+$, where $u_\lambda = \mathscr{M}_{\mathbb{R}}^\lambda(\mathbf{1}_{\mathbb{R}_+} f)$, and thus

$$\begin{aligned}\Big\| \lambda \|v_\lambda\|_X + \|A v_\lambda\|_X \Big\|_{L^p(\mathbb{R}_+)} &\leqslant \Big\| \lambda \|u_\lambda\|_X + \|A u_\lambda\|_X \Big\|_{L^p(\mathbb{R})} \\ &\leqslant C \|f\|_{L^p(\mathbb{R}_+;X)}.\end{aligned}$$

This proves (17.72) for $\lambda > 0$. The extension to $\lambda = 0$ is a consequence of Proposition 17.2.29. □

Theorem 17.3.32 admits the following converse.

Proposition 17.3.33 (Maximal regularity: $\mathbb{R}_+$ versus $\mathbb{R}$ – III). *Let X be a Banach space, $1 \leqslant p \leqslant \infty$, and let A be a linear operator with maximal L^p-regularity on $\mathbb{R}_+$. Then for all $\lambda > 0$, the operator $\lambda + A$ has maximal L^p-regularity on $\mathbb{R}$, and for all $f \in L^p(\mathbb{R}; X)$, the L^p-solution u_λ of*

$$u' + (\lambda + A)u = f \quad \text{on } \mathbb{R} \tag{17.74}$$

satisfies

$$\lambda \|u_\lambda\|_{L^p(\mathbb{R};X)} + \|A u_\lambda\|_{L^p(\mathbb{R};X)} \leqslant (M + M_{p,A}^{\mathrm{reg}}(\mathbb{R}_+)) \|f\|_{L^p(\mathbb{R};X)}, \tag{17.75}$$

where M is such that $\|S(t)\| \leqslant M$ for all $t \geqslant 0$, with $(S(t))_{t>0}$ the bounded analytic semigroup generated by A.

Proof. Fix $\lambda > 0$. By Dore's Theorem 17.2.15, $-A$ generates a bounded analytic semigroup $S = (S(t))_{t>0}$. Thus one can take $\omega = 0$ in the growth bound (K.1), and hence $\lambda + A$ is invertible for $\lambda > 0$ by Definition K.1.2 of a generator. It follows from Theorem 17.2.26(2) that $\lambda + A$ has maximal L^p-regularity on $\mathbb{R}_+$. Since we also have $0 \in \varrho(\lambda + A)$, Proposition 17.3.31 guarantees that $\lambda + A$ has maximal L^p-regularity on $\mathbb{R}$ as well.

Next we prove the estimate (17.75). Define the operator $\mathscr{M}_{\mathbb{R}}^\lambda : L^p(\mathbb{R}; X) \to L^p(\mathbb{R}; \mathsf{D}(A))$ by

$$\mathscr{M}_{\mathbb{R}}^\lambda f = u_\lambda,$$

where u_λ is the unique solution of (17.74). By Proposition 17.3.31,

$$\mathscr{M}_{\mathbb{R}}^\lambda f(t) = \int_{-\infty}^t e^{-\lambda(t-s)} S(t-s) f(s) \,\mathrm{d}s, \quad t \geqslant 0.$$

Therefore, by Young's inequality

$$\lambda\|\mathscr{M}_{\mathbb{R}}^{\lambda} f\|_{L^p(\mathbb{R};X)} \leqslant M\|f\|_{L^p(\mathbb{R};X)}.$$

It remains to prove the bound $\|A\mathscr{M}_{\mathbb{R}}^{\lambda}\|_{\mathscr{L}(L^p(\mathbb{R};X))} \leqslant M_{p,A}^{\mathrm{reg}}(\mathbb{R}_+)$. By Theorem 17.2.19, the L^p-solution v_λ to (17.71) with $f \in L^p(\mathbb{R}_+;X)$ satisfies

$$v_\lambda(t) = \mathscr{M}_{\mathbb{R}_+}^{\lambda} f(t) := \int_0^t e^{-\lambda(t-s)} S(t-s) f(s)\,\mathrm{d}s, \quad t \geqslant 0.$$

Therefore, by Proposition 17.2.27(1)

$$\|A\mathscr{M}_{\mathbb{R}_+}^{\lambda}\|_{\mathscr{L}(L^p(\mathbb{R}_+;X))} \leqslant M_{p,A}^{\mathrm{reg}}(\mathbb{R}_+).$$

The required bound for $A\mathscr{M}_{\mathbb{R}}^{\lambda}$ can be deduced from this by the same argument as in Lemma 17.3.3. □

From Theorem 17.3.32 and Proposition 17.3.33, it follows that, in order to establish maximal L^p-regularity of A on $\mathbb{R}_+$, it suffices to prove maximal L^p-regularity of $\lambda + A$ on $\mathbb{R}$ for all $\lambda > 0$, along with the estimate (17.70) for some $n \in \mathbb{N}$ (or equivalently, for $n = 0$). The advantage of considering problems on $\mathbb{R}$ is that one can apply the Fourier transform in the time variable, and initial value conditions do not play any role. In particular, this leads to simplifications in studying evolution equation with inhomogeneous boundary values and initial values. This will be further explained in the Notes at the end of the chapter.

Fourier multipliers related to maximal L^p-regularity on $\mathbb{R}$

The remainder of this section is devoted to extending some of the Fourier multiplier connections of maximal L^p-regularity on $\mathbb{R}_+$ to the real line. A variation of Proposition 17.3.5 is needed for this, the proof of which has to be adapted since the operators $\overline{V}$ were defined via the bounded analytic semigroup generated by $-A$ (see Lemma 17.3.3). In the present situation, where we assume that A has maximal L^p-regularity on $\mathbb{R}$, Mielke's Theorem 17.3.29 implies that A is bisectorial rather than sectorial, as was the case with Dore's Theorem 17.2.15 on $\mathbb{R}_+$. Below we will only consider the case $0 \in \varrho(A)$ (which is a necessary assumption by Theorem 17.3.29). The multiplier below will be $(2\pi i\xi + A)^{-1}$, instead of the multiplier $A(2\pi i\xi + A)^{-1}$ used in Proposition 17.3.5.

Proposition 17.3.34. *Let A be a bisectorial operator in X with $0 \in \varrho(A)$, and let $1 \leqslant p < \infty$. The following assertions are equivalent:*

(1) *A has maximal L^p-regularity on $\mathbb{R}$;*
(2) *the Fourier multiplier operator*

$$T_m : \check{L}^1(\mathbb{R};X) \to \check{L}^1(\mathbb{R};\mathsf{D}(A)), \ f \mapsto (m\widehat{f})\check{}$$

associated with the function

$$m(\xi) = (2\pi i\xi + A)^{-1}$$

extends to a bounded operator from $L^p(\mathbb{R};X)$ *to* $L^p(\mathbb{R};\mathsf{D}(A))$.

(3) *the Fourier multiplier operators*

$$T_{m_k} : \check{L}^1(\mathbb{R};X) \to \check{L}^1(\mathbb{R};X),\ f \mapsto (m_k\widehat{f})\check{}$$

associated with the functions $m_0(\xi) := m(\xi)$ *and* $m_1(\xi) := Am(\xi)$ *extend to a bounded operators from* $L^p(\mathbb{R};X)$ *to* $L^p(\mathbb{R};X)$.

In this situation, the L^p*-solution* u *to* (17.60) *satisfies* $u = T_m f$, *and we have*

$$\max\{N_0, N_1\} \leqslant M^{\mathrm{reg}}_{p,A}(\mathbb{R}) = \|T_m\|_{\mathscr{L}(L^p(\mathbb{R};X),L^p(\mathbb{R};\mathsf{D}(A)))} \leqslant N_0 + N_1,$$

where $N_k := \|T_{m_k}\|_{\mathscr{L}(L^p(\mathbb{R};X))}$ *for* $k = 0, 1$.

Proof. We start by making some observations that will be needed in the proofs of both implications. Let $f \in \mathscr{S}(\mathbb{R};X)$ be fixed. Then the function $\xi \mapsto (2\pi i\xi+A)^{-1}\widehat{f}(\xi)$ belongs to $\mathscr{S}(\mathbb{R};\mathsf{D}(A))$. Thus we may define $u \in \mathscr{S}(\mathbb{R};\mathsf{D}(A))$ by

$$\widehat{u} := (2\pi i \cdot +A)^{-1}\widehat{f}. \tag{17.76}$$

This gives the identity $T_m f = u$. Moreover, since $\mathscr{F}(u') = (2\pi i\cdot)\widehat{u}$, one has

$$\mathscr{F}(u' + Au) = \mathscr{F}(u') + A\mathscr{F}(u) = (2\pi i \cdot +A)\widehat{u} = \widehat{f},$$

and thus $u' + Au = f$ in $\mathscr{S}(\mathbb{R};X)$.

Next we show that any L^p-solution u to (17.60) satisfies $u = T_m f$. Since u is an L^p-solution we have $u, Au, u' \in L^p(\mathbb{R};X)$ and

$$\mathscr{F}(u') = (2\pi i\cdot)\widehat{u} \quad\text{and}\quad \mathscr{F}(Au) = A\widehat{u}$$

in $\mathscr{S}'(\mathbb{R};X)$. It follows that

$$((2\pi i\cdot) + A)\widehat{u} = \mathscr{F}(u' + Au) = \widehat{f}.$$

We conclude that $\widehat{u} = ((2\pi i\cdot) + A)^{-1}\widehat{f} = m\widehat{f}$, and thus $u = T_m f$.

(1)⇒(2): Let $f \in \mathscr{S}(\mathbb{R};X)$ be given, and let u be the L^p-solution to (17.60). By the preliminary observations we have $u = T_m f$, and by maximal L^p-regularity we have the estimate

$$\|T_m f\|_{L^p(\mathbb{R};\mathsf{D}(A))} = \|u\|_{L^p(\mathbb{R};\mathsf{D}(A))} \leqslant M^{\mathrm{reg}}_{p,A}(\mathbb{R})\|f\|_{L^p(\mathbb{R};X)}.$$

Now the bounded extension is obtained by the density of $\mathscr{S}(\mathbb{R};X)$ in $L^p(\mathbb{R};X)$.

(2)⇒(1): By the preliminary observations, for any given $f \in \mathscr{S}(\mathbb{R};X)$ the function u defined by (17.76) satisfies $u' + Au = f$ in $\mathscr{S}(\mathbb{R};X)$, and $u = T_m f$ is in $\mathscr{S}(\mathbb{R};X)$. Therefore, u is an L^p-solution to (17.60) and it satisfies

$$\|u\|_{L^p(\mathbb{R};\mathsf{D}(A))} = \|T_m f\|_{L^p(\mathbb{R};\mathsf{D}(A))} \leqslant \|T_m\|_{\mathscr{L}(L^p(\mathbb{R};X),L^p(\mathbb{R};\mathsf{D}(A)))} \|f\|_{L^p(\mathbb{R};X)}.$$

If v is another L^p-solution to (17.60), these observations also imply that $v = T_m f = u$. Therefore, any L^p-solution satisfies the required *a priori* estimate.

The equality $M^{\mathrm{reg}}_{p,A}(\mathbb{R}) = \|T_m\|_{\mathscr{L}(L^p(\mathbb{R};X),L^p(\mathbb{R};\mathsf{D}(A)))}$ follows from a combination of the bounds established in the course of proving (1)⇒(2) and (2)⇒(1).

(2)⇔(3): Since $\|x\|_{\mathsf{D}(A)} := \|x\|_X + \|Ax\|_X$, it follows that

$$\begin{aligned}
\|T_{m_0} f\|_{L^p(\mathbb{R};X)} &= \|T_m f\|_{L^p(\mathbb{R};X)} \leqslant \|T_m f\|_{L^p(\mathbb{R};\mathsf{D}(A))},\\
\|T_{m_1} f\|_{L^p(\mathbb{R};X)} &= \|AT_m f\|_{L^p(\mathbb{R};X)} \leqslant \|T_m f\|_{L^p(\mathbb{R};\mathsf{D}(A))},\\
\|T_m f\|_{L^p(\mathbb{R};\mathsf{D}(A))} &\leqslant \|T_m f\|_{L^p(\mathbb{R};X)} + \|AT_m f\|_{L^p(\mathbb{R};X)},
\end{aligned}$$

which readily implies both implications and the related estimates. □

The next theorem presents an extrapolation result for maximal L^p-regularity on $\mathbb{R}$.

Theorem 17.3.35 (Extrapolation). *Let A be a densely defined bisectorial operator that has maximal L^{p_0}-regularity on $\mathbb{R}$ for some $p_0 \in [1,\infty)$. Then for all $p \in (1,\infty)$ and Muckenhoupt weights $w \in A_p$, the operator A has maximal $L^p(w)$-regularity on $\mathbb{R}$. Moreover, the strong solution of (17.60) satisfies*

$$\|u\|_{L^p(\mathbb{R},w;\mathsf{D}(A))} \leqslant C_{p_0,p}(M^2 + M^{\mathrm{reg}}_{p_0,A}(\mathbb{R}))[w]_{A_p}^{\max(1,\frac{1}{p-1})} \|f\|_{L^p(\mathbb{R},w;X)},$$

where $M = \sup_{\xi\in\mathbb{R}\setminus\{0\}} \|(i\xi + A)^{-1}\|_{\mathscr{L}(X,\mathsf{D}(A))}$.

Remark 17.3.36. Note that

$$\begin{aligned}
1 = \|I\|_{\mathscr{L}(X)} &= \|(i\xi + A)(i\xi + A)^{-1}\|_{\mathscr{L}(X)}\\
&\leqslant |\xi|\|(i\xi + A)^{-1}\|_{\mathscr{L}(X)} + \|A(i\xi + A)^{-1}\|_{\mathscr{L}(X)}\\
&\leqslant (|\xi| + 1)\|(i\xi + A)^{-1}\|_{\mathscr{L}(X,\mathsf{D}(A))} \leqslant (|\xi| + 1)M;
\end{aligned}$$

hence, with $\xi \to 0$, we find that $M \geqslant 1$.

Proof of Theorem 17.3.35. By Mielke's Theorem 17.3.29 (applied to $\lambda = 0$), the assumed maximal L^{p_0}-regularity of A on $\mathbb{R}$ implies that $0 \in \varrho(A)$. Thus Proposition 17.3.34 applies to show that the Fourier multiplier operator T_m associated with the function $m(\xi) = (2\pi i\xi + A)^{-1}$, $\xi \in \mathbb{R}$, is bounded from $L^{p_0}(\mathbb{R};X)$ to $L^p(\mathbb{R};\mathsf{D}(A))$. Clearly, $\|m(\xi)\|_{\mathscr{L}(X,\mathsf{D}(A))} \leqslant M$ and $\|\xi m'(\xi)\|_{\mathscr{L}(X,\mathsf{D}(A))} \leqslant M^2 + M$ for all $\xi \in \mathbb{R} \setminus \{0\}$. Therefore, Corollary 13.2.8 implies that T_m is bounded from $L^p(\mathbb{R},w;X)$ to $L^p(\mathbb{R},w;\mathsf{D}(A))$ for all $p \in (1,\infty)$ and all $w \in A_p$, with norm estimate

$$\|T_m\|_{\mathscr{L}(L^{p_0}(\mathbb{R};X),L^{p_0}(\mathbb{R};\mathsf{D}(A)))} \leqslant c_p(M^2 + M^{\mathrm{reg}}_{p_0,A}(\mathbb{R}))[w]_{A_p}^{\max(1,\frac{1}{p-1})},$$

where we absorbed the lower order terms to M^2 by Remark 17.3.36. By Proposition 17.3.34, $T_m f$ is the L^p-solution to (17.60) for all $f \in \mathscr{S}(\mathbb{R};X)$. Since this space is dense in $L^p(\mathbb{R},w;X)$, the result follows. □

Next we present a duality result for maximal regularity on $\mathbb{R}$. Note that unlike in Proposition 17.2.32 we do not consider the end-points $p = 1$ and $p = \infty$.

Proposition 17.3.37 (Duality). *Let X be a Banach space, and let $1 < p < \infty$. Suppose that A is a closed and densely defined linear operator in X. Then A has maximal L^p-regularity on $\mathbb{R}$ if and only if its adjoint A^* has maximal $L^{p'}$-regularity on $\mathbb{R}$, and in this case*

$$\frac{1}{2}M^{\rm reg}_{p,A}(\mathbb{R}) \leqslant M^{\rm reg}_{p',A^*}(\mathbb{R}) \leqslant 2M^{\rm reg}_{p,A}(\mathbb{R}).$$

Proof. By Proposition 17.3.34, maximal L^p-regularity of A on $\mathbb{R}$ is equivalent to the boundedness of the two multipliers $m_k(\xi) = A^k(i2\pi\xi + A)^{-1}$ $(k = 0, 1)$, on $L^p(\mathbb{R}; X)$. Similarly, maximal $L^{p'}$-regularity of A^* on $\mathbb{R}$ is equivalent to the boundedness of the two multipliers $(A^*)^k(i2\pi\xi + A^*)^{-1}$ $(k = 0, 1)$ on $L^{p'}(\mathbb{R}; X^*)$. It is evident that these are the pointwise adjoints $m_k(\xi)^*$ of the $m_k(\xi)$. By Proposition 5.3.7, if $m \in \mathfrak{M}L^p(\mathbb{R}; X, X)$, then its pointwise adjoint satisfies $m^* \in \mathfrak{M}L^{p'}(\mathbb{R}; X^*, X^*)$ with

$$\|T_{m^*}\|_{\mathscr{L}(L^{p'}(\mathbb{R};X^*))} \leqslant \|T_m\|_{\mathscr{L}(L^p(\mathbb{R};X))},$$

and one can extract from the short proof that the converse implication and estimate are also true. A combination of these results proves the proposition at hand, and it is also easy to get the quantitative statement

$$\begin{aligned} M^{\rm reg}_{p',A^*}(\mathbb{R}) &\leqslant \sum_{k=0}^{1} \|T_{m_k^*}\|_{\mathscr{L}(L^{p'}(\mathbb{R};X^*))} = \sum_{k=0}^{1} \|T_{m_k}\|_{\mathscr{L}(L^p(\mathbb{R};X))} \\ &\leqslant 2 \max_{k=0,1} \|T_{m_k}\|_{\mathscr{L}(L^p(\mathbb{R};X))} \leqslant 2M^{\rm reg}_{p,A}(\mathbb{R}). \end{aligned}$$

The converse is proved similarly. □

As in Theorem 17.3.1, we can characterise maximal L^p-regularity on $\mathbb{R}$ by using R-bisectoriality.

Theorem 17.3.38 (Maximal L^p-regularity and R-bisectoriality). *Let A be a densely defined bisectorial operator on a Banach space X with $0 \in \varrho(A)$.*

(1) *If A has maximal L^p-regularity on $\mathbb{R}$ for some $p \in (1, \infty)$ with constant $M^{\rm reg}_{p,A}(\mathbb{R})$, then A is R-bisectorial with angle $\omega_R(A) < \frac{1}{2}\pi$, and*

$$M := \mathscr{R}_p(\{(i\xi + A)^{-1} \in \mathscr{L}(X, \mathsf{D}(A)) : \ \xi \in \mathbb{R} \setminus \{0\}\}) \leqslant M^{\rm reg}_{p,A}(\mathbb{R}).$$

(2) *If X is a UMD space, $p \in (1, \infty)$, and A is R-bisectorial, then A has maximal L^p-regularity on $\mathbb{R}$ with*

$$M^{\rm reg}_{p,A}(\mathbb{R}) \leqslant 400\hbar_{p,X}\beta^2_{p,X}(M + 1)^2.$$

The proof of Theorem 17.3.38 is completely analogous to Theorem 17.3.1 and we therefore omit the details.

17.4 Examples and counterexamples

In this section we will first show that the Laplace and Poisson operator on $L^q(\mathbb{R}^d; X)$ have maximal L^p-regularity if and only if X is a UMD space. Moreover, several end points cases are studied. In the final Subsection 17.4.c we will construct sectorial operators on L^q-spaces with $q \in (1, \infty) \setminus \{2\}$, which fail maximal L^p-regularity on finite time intervals.

17.4.a The heat semigroup and the Poisson semigroup

By Example 10.1.5, the Laplacian $-\Delta$ is sectorial of angle 0 on $L^q(\mathbb{R}^d; X)$, $1 \leqslant q < \infty$, and therefore by Theorem 15.2.7 its fractional powers $(-\Delta)^{\alpha/2}$ are sectorial of angle 0 for all $\alpha > 0$. In particular, every operator $(-\Delta)^{\alpha/2}$ generates an analytic C_0-semigroup on $L^q(\mathbb{R}^d; X)$. Of special interest are the cases $\alpha = 1$ and $\alpha = 2$, which correspond to the Poisson semigroup and the heat semigroup, respectively.

The aim of this section is to characterise when $(-\Delta)^{\alpha/2}$ has maximal L^p-regularity on $L^q(\mathbb{R}^d; X)$. We start with exponents $q \in (1, \infty)$; the case $q = 1$ will be considered in Example 17.4.2.

Theorem 17.4.1. *Let X be a Banach space and let $1 < p, q < \infty$ and $\alpha > 0$, let $\lambda \geqslant 0$, and consider the operator $A = \lambda + (-\Delta)^{\alpha/2}$ as a closed and densely defined operator on $L^q(\mathbb{R}^d; X)$. Let $I = \mathbb{R}_+$ or $I = (0, T)$ with $T \in (0, \infty)$. The following assertions are equivalent:*

(1) *A has maximal L^p-regularity on I;*
(2) *X is a UMD space.*

In this situation we have $\beta_{p,X}^{\mathbb{R}} \leqslant 2M_{p,A}^{\mathrm{reg}}(\mathbb{R}_+)$.

By inspection of the proof, the reader can also track a quantitative estimate in the other direction. However, the present method is sub-optimal for this purpose, and we therefore refrain from being more explicit about the result.

Proof. (2)⇒(1): By Theorem 10.2.25 (which, in turn, was proved with the help of the Mihlin Multiplier Theorem 5.5.10), $-\Delta$ has a bounded H^∞-calculus of angle 0. Moreover, the proof of Theorem 10.2.25 shows that this functional calculus is explicitly given by Fourier multiplier through the formula

$$f(-\Delta) = T_{f(4\pi^2|\cdot|^2)}.$$

By Proposition 15.2.11, the bounded H^∞-calculus is inherited by the powers $(-\Delta)^{\alpha/2}$, and the proof of Proposition 15.2.11 provides the representation

$$f((-\Delta)^{\alpha/2}) = f((\cdot)^{\alpha/2})(-\Delta) = T_{f(2^\alpha \pi^\alpha |\cdot|^\alpha)}.$$

Finally, the translates $(-\Delta)^{\alpha/2} + \lambda$ also inherit the bounded H^∞-calculus, and again we have a representation

$$f((-\Delta)^{\alpha/2}+\lambda) = f(\cdot+\lambda)((-\Delta)^{\alpha/2}) = T_{f(2^\alpha\pi^\alpha|\cdot|^\alpha+\lambda)}.$$

This bounded H^∞-calculus implies maximal L^p-regularity of $(\Delta)^{\alpha/2}$ on $\mathbb{R}_+$ by Corollary 17.3.6. The permanence properties in Theorem 17.2.26 show that it extends to bounded subintervals $(0,T)$ of $\mathbb{R}_+$.

(1)⇒(2): By Theorem 17.2.26, we may assume that $p=q$ and $\lambda>0$. Since the semigroup $S(t):=e^{-tA}=e^{-t(-\Delta)^{\alpha/2}}e^{-\lambda t}$ is uniformly exponentially stable, A has maximal L^p-regularity on $I=\mathbb{R}_+$ by Theorem 17.2.24.

Recall from Theorem 4.2.5 that the UMD property of X is equivalent (with the same constant) to its dyadic version, in which the defining inequality involves only finitely many vectors of X at a time. From this, it is immediate that the UMD property of X is equivalent to the property that all finite-dimensional subspaces of X have the UMD property, with a uniform upper bound for their UMD constants. On the other hand, the operator $A=(-\Delta)^{\alpha/2}+\lambda$ on $L^p(\mathbb{R}^d;X)$ is the tensor extension of a scalar-valued operator on $L^p(\mathbb{R}^d)$. Thus, if $Y\subseteq X$ is a subspace, then $L^p(\mathbb{R}^d;Y)\subseteq L^p(\mathbb{R}^d;X)$ is an invariant subspace for A, and the maximal L^p-regularity of A on $L^p(\mathbb{R}^d;X)$ implies its maximal L^p-regularity on $L^p(\mathbb{R}^d;Y)$, with at most the same constant. Combining these observations, it suffices to shows that $\beta_{p,Y}\leqslant M^{\rm reg}_{p,A|_{L^p(\mathbb{R}^d;Y)}}(\mathbb{R}_+)$ for all finite-dimensional subspaces $Y\subseteq X$. The advantage of this reduction is that we then already know that Y is a UMD space, *qualitatively*, so that the results of the "(2)⇒(1)" part of the proof are available to us, but of course it still remains to prove the good quantitative estimate for $\beta_{p,Y}$.

By Proposition 17.3.5 (with $L^p(\mathbb{R}^d;Y)$ in place of X), the function $m(\eta):=A(2\pi i\eta+A)^{-1}$ defines a bounded Fourier multiplier T_m on $L^p(\mathbb{R};L^p(\mathbb{R}^d;Y))$ of norm $\|T_m\|_{\mathscr{L}(L^p(\mathbb{R};L^p(\mathbb{R}^d;Y)))}=M^{\rm reg}_{p,A}(\mathbb{R}_+)$. For each fixed $\eta\in\mathbb{R}\setminus\{0\}$, it is evident that $A(2\pi i\eta+A)^{-1}=f_\eta(A)$, where $f_\eta(z)=z(2\pi i\eta+z)^{-1}$. Thus, by part "(2)⇒(1)", we have

$$m(\eta)=f_\eta(A)=T_{f_\eta(2^\alpha\pi^\alpha|\cdot|^\alpha+\lambda)}=:T_{M(\eta,\cdot)}.$$

For a function of the form $\phi\otimes\psi$ with $\phi\in\mathscr{S}(\mathbb{R}),\psi\in\mathscr{S}(\mathbb{R}^d;Y)$, we have

$$\begin{aligned}
T_m(\phi\otimes\psi)(t,s) &= \int_{\mathbb{R}} m(\eta)(\widehat{\phi}(\eta)\psi)(s)e^{i2\pi\eta t}\,\mathrm{d}\eta\\
&= \int_{\mathbb{R}} T_{M(\eta,\cdot)}(\widehat{\phi}(\eta)\psi)(s)e^{i2\pi\eta t}\,\mathrm{d}\eta\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^d} M(\eta,\xi)\widehat{\phi}(\eta)\widehat{\psi}(\xi)e^{i2\pi\xi\cdot s}\,\mathrm{d}\xi e^{i2\pi\eta t}\,\mathrm{d}\eta\\
&= \int_{\mathbb{R}\times\mathbb{R}^d} M(\eta,\xi)\widehat{\phi\otimes\psi}(\eta,\xi)e^{i2\pi(\eta,\xi)\cdot(t,s)}\,\mathrm{d}(\eta,\xi)\\
&= T_M(\phi\otimes\psi)(t,s),
\end{aligned}$$

where T_m on the left is the Fourier multiplier on $L^p(\mathbb{R}; L^p(\mathbb{R}^d; Y))$ with operator-valued symbol $m : \mathbb{R} \to \mathscr{L}(L^p(\mathbb{R}^d; Y))$, while T_M on the right is the Fourier multiplier on $L^p(\mathbb{R}\times\mathbb{R}^d; Y)$ with scalar-valued symbol $M : \mathbb{R}^{d+1} \to \mathbb{C}$. Since linear combinations of $\phi \otimes \psi$ as above are dense in $L^p(\mathbb{R} \times \mathbb{R}^d; Y) \simeq L^p(\mathbb{R}; L^p(\mathbb{R}^d; Y))$, we obtain

$$\|M\|_{\mathfrak{M}L^p(\mathbb{R}\times\mathbb{R}^d;Y)} = \|T_m\|_{\mathscr{L}(L^p(\mathbb{R};L^p(\mathbb{R}^d;Y))} \leqslant M^{\rm reg}_{p,A}(\mathbb{R}_+),$$

where

$$M(\eta, \xi) = \frac{\lambda + |2\pi\xi|^\alpha}{2\pi i\eta + \lambda + |2\pi\xi|^\alpha}.$$

Invoking the same scaling argument as in Proposition 5.5.2, we find that the multiplier M_2 defined by

$$M_2(\eta, \xi) = \frac{|\xi|^\alpha}{i\eta + |\xi|^\alpha}$$

satisfies $\|M_2\|_{\mathfrak{M}L^p(\mathbb{R}\times\mathbb{R}^d;X)} \leqslant M^{\rm reg}_{p,A}(\mathbb{R}_+)$. Clearly M_2 satisfies $M_2(\eta, 0) = 0$ and $M(0, \xi) = 1$ for every $\eta \neq 0$ and $\xi \neq 0$. Thus Theorem 13.3.5 applies to show that

$$\beta^{\mathbb{R}}_{p,Y} \leqslant 2\|M_2\|_{\mathfrak{M}L^p(\mathbb{R}\times\mathbb{R}^d;X)} \leqslant 2M^{\rm reg}_{p,A}(\mathbb{R}_+),$$

and this concludes the proof. □

The next example deals with the endpoint $q = 1$.

Example 17.4.2. The aim of this example is to prove that the operator $-\Delta$, viewed as a closed and densely defined operator on $L^1(\mathbb{R}^d)$, fails to have maximal L^p-regularity on any bounded interval $(0, T)$ for any $p \in [1, \infty]$.

Suppose, for a contradiction, that $-\Delta$ has maximal L^p-regularity on $(0, T)$. By Theorem 17.2.24, for large enough $\lambda > 0$, the operator $\lambda - \Delta$ has maximal L^p-regularity on $\mathbb{R}_+$. Then, by Theorem 17.3.1(1), $\lambda - \Delta$ is R-sectorial of angle $< \pi/2$. By Proposition 10.3.3, this implies that the family $\{e^{-(\lambda-\Delta)t} : t \geqslant 0\}$ is R-bounded, and by the contraction principle this implies that the family $\{e^{t\Delta} : t \in [0, 1]\}$ is R-bounded. We will now show that the latter is not the case.

Set $f_\sigma(y) := \sigma^d e^{-\frac{\sigma^2|y|^2}{2}}$. Completing squares, one obtains

$$\begin{aligned} e^{t\Delta} f_\sigma(x) &= \frac{\sigma^d}{(2\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{2t}} e^{-\frac{\sigma^2|y|^2}{2}} \,\mathrm{d}y \\ &= \frac{\sigma^d}{(2\pi t)^{d/2}} e^{-\frac{|\sigma x|^2}{2(\sigma^2 t+1)}} \int_{\mathbb{R}^d} e^{-\frac{\sigma^2 t+1}{2t}|y-\frac{x}{\sigma^2 t+1}|^2} \,\mathrm{d}y \\ &= \frac{\sigma^d}{(\sigma^2 t + 1)^{d/2}} e^{-\frac{|\sigma x|^2}{2(\sigma^2 t+1)}}. \end{aligned}$$

By Fubini's theorem and Khintchine's inequality,

$$\Big\| \sum_{n=0}^{N} \varepsilon_n e^{t_n \Delta} f_\sigma \Big\|_{L^1(\Omega;L^1(\mathbb{R}^d))} \geqslant \kappa_{1,2} \Big\| \Big(\sum_{n=0}^{N} |e^{t_n\Delta} f_\sigma|^2 \Big)^{1/2} \Big\|_{L^1(\mathbb{R}^d)} =: \|F\|_{L^1(\mathbb{R}^d)}$$

Setting $\sigma = 2^N$ and $t_n = 2^{2n-2N}$, we obtain

$$\begin{aligned} \|F\|_{L^1(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \Big(\sum_{n=0}^{N} \frac{\sigma^{2d}}{(2^{2n}+1)^d} e^{-\frac{|\sigma x|^2}{2^{2n}+1}} \Big)^{1/2} \,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} \Big(\sum_{n=0}^{N} \frac{1}{(2^{2n}+1)^d} e^{-\frac{|x|^2}{2^{2n}+1}} \Big)^{1/2} \,\mathrm{d}x \end{aligned}$$

For $m \geqslant 0$ consider the disjoint annuli $A_m = \{x : 2^m < |x| < 2^{m+1}\}$. Splitting the integral and estimating, we obtain

$$\begin{aligned} \|F\|_{L^1(\mathbb{R}^d)} &\geqslant \sum_{m=0}^{N} \int_{A_m} \frac{1}{(2^{2m}+1)^{d/2}} e^{-\frac{|x|^2}{2(2^{2m}+1)}} \,\mathrm{d}x \\ &= \sum_{m=0}^{N} \int_{A_0} \frac{2^{md}}{(2^{2m}+1)^{d/2}} e^{-\frac{|2^m x|^2}{2(2^{2m}+1)}} \,\mathrm{d}x \geqslant (N+1)c_d, \end{aligned}$$

where $c_d > 0$ depends only on the dimension d. On the other hand, by Fubini's theorem and Hölder's inequality in Ω, we find

$$\Big\| \sum_{n=0}^{N} \varepsilon_n f_\sigma \Big\|_{L^1(\Omega;L^1(\mathbb{R}^d))} \leqslant \sqrt{N} \|f_\sigma\|_{L^1(\mathbb{R}^d)} = \sqrt{N}(2\pi)^{d/2}.$$

Therefore, the set $\{e^{t\Delta} : t \in [0,1]\}$ fails to be R-bounded as claimed.

Example 17.4.3. One can check that $-\Delta$ is sectorial of angle 0 on $C_0(\mathbb{R}^d)$. However, the operator $-\Delta$ on $C_0(\mathbb{R}^d)$ does not have maximal L^p-regularity on $(0,T)$ for any $p \in [1,\infty]$. Indeed, since $L^1(\mathbb{R}^d)$ is isometrically embedded in the dual of $C_0(\mathbb{R}^d)$, this easily follows by combining the duality result of Proposition 17.2.32, with Example 17.4.2.

As a consequence $-\Delta$ also does not have maximal L^p-regularity on any space which contains $C_0(\mathbb{R}^d)$ isomorphically (e.g. $C_{\mathrm{ub}}(\mathbb{R}^d)$, $C_{\mathrm{b}}(\mathbb{R}^d)$, and $L^\infty(\mathbb{R}^d)$).

17.4.b End-point maximal regularity versus containment of c_0

In Section 17.3.b we have characterised maximal L^1- and L^∞-regularity, and in Corollary 17.3.20 we have seen that these properties hold on the real interpolation spaces $(X, \mathsf{D}(A))_{\theta,1}$ and $(X, \mathsf{D}(A))_{\theta,\infty}$, respectively. Below we will obtain several results which show that positive results for $p = 1$ and $p = \infty$ are rare, and geometric restrictions on the underlying spaces are required.

From Theorem 17.2.46 we recall that maximal L^∞-regularity coincides with maximal C-regularity in the case that A is densely defined, and therefore all results in this section pertaining to maximal L^∞-regularity also hold for maximal C-regularity.

We begin with a theorem due to Baillon, which implies that in Banach spaces without an isomorphic copy of c_0, only bounded operators A can have maximal L^∞-regularity. Examples of Banach spaces without an isomorphic copy of c_0 are reflexive Banach spaces and L^1-spaces. To see the latter, one may for instance note that L^1-spaces have cotype 2 (Proposition 7.1.4), but c_0 does not (Corollary 7.1.10).

Theorem 17.4.4 (Baillon). *Let A be a linear operator acting in a Banach space X not containing a closed subspace isomorphic to c_0. If A has maximal L^∞-regularity on I, then A is bounded.*

Proof. It suffices to consider the case that $I = (0, T)$ with $T \in (0, \infty)$. Reasoning by contradiction, we assume that A is an unbounded linear operator with maximal L^∞-regularity and construct an isomorphic copy of c_0 inside X. This will be done by means of the Bessaga–Pełczyński theorem (Theorem 1.2.40), which asserts that if $(x_n)_{n\geqslant 1}$ is a sequence in a Banach space X satisfying $\inf_{n\geqslant 1} \|x_n\| > 0$ and

$$\Big\|\sum_{j=1}^{k} \epsilon_j x_j\Big\| \leqslant C$$

for all $k \geqslant 1$ and all signs $\epsilon_1, \dots, \epsilon_k \in \{-1, 1\}$, with a uniform constant C, then the closed linear span of $(x_n)_{n\geqslant 1}$ contains a subspace isomorphic to c_0.

By Theorem 17.2.15, $-A$ generates an exponentially bounded analytic semigroup $(S(t))_{t>0}$. By Proposition K.3.1,

$$\limsup_{t\downarrow 0} t\|AS(t)\| \geqslant \frac{1}{e},$$

and hence there exists a sequence $T \geqslant t_1 > t_2 > \ldots \downarrow 0$ and a sequence $(y_j)_{j\geqslant 1}$ in X of norm one vectors such that

$$t_j\|AS(t_j)y_j\| \geqslant \frac{1}{2e}, \quad j \geqslant 1.$$

By restriction to $[0, t_1]$ and passing to a subsequence, we may assume that

$$t_1 = T \quad\text{and}\quad t_{j+1} \leqslant \frac{t_j}{2^j} \text{ for all } j \geqslant 1.$$

Setting $M := \sup_{t\in[0,T]} \|S(t)\|$ and $f_j(s) := S(s)y_j$, we have

$$t_j\|AS(t_j)y_j\| = \Big\|A\int_0^{t_j} S(t_j-s)f_j(s)\,\mathrm{d}s\Big\| \leqslant C\sup_{t\in[0,T]}\|f_j(t)\| \leqslant CM,$$

where $C = M^{\mathrm{reg}}_{\infty,A}(I)$.

Set $x_j := t_jAS(t_j)y_j$ for $j \geqslant 0$. By the preceding estimates,

$$\frac{1}{2e} \leqslant \inf_{j\geqslant 0}\|x_j\| \leqslant \sup_{j\geqslant 0}\|x_j\| \leqslant CM.$$

To check the second condition of the Bessaga–Pełczyński theorem, fix $k \geqslant 1$ and signs $\epsilon_1,\dots,\epsilon_k \in \{-1,1\}$. Define $f : [0,T] \to X$ by

$$f(t) := \begin{cases} \epsilon_j S(t-T+t_j)y_j, & t \in I_j := [T-t_j, T-t_{j+1}), \quad j = 1,\dots,k,\\ 0, & t \in [T-t_{k+1}, T].\end{cases}$$

Then for all $s \in [T-t_{k+1}, T]$,

$$\begin{aligned}(S*f)(s) &= \sum_{j=1}^k \int_{I_j} S(s-t)f(t)\,\mathrm{d}t\\ &= \sum_{j=1}^k \epsilon_j\int_{I_j} S(s-T+t_j)y_j\,\mathrm{d}t = \sum_{j=1}^k (t_j-t_{j+1})\epsilon_j S(s-T+t_j)y_j.\end{aligned}$$

The function $s \mapsto AS*f(s)$ is continuous on $[T-t_{k+1}, T]$, so we can evaluate it at $s = T$ to obtain

$$(AS*f)(T) = \sum_{j=0}^k (1-\frac{t_{j+1}}{t_j})\epsilon_j x_j,$$

where we used the definition of the vectors x_j. Therefore,

$$\begin{aligned}\Big\|\sum_{j=1}^k \epsilon_j x_j\Big\| &\leqslant \Big\|\sum_{j=1}^k \epsilon_j x_j - (AS*f)(T)\Big\| + \|(AS*f)(T)\|\\ &\leqslant \Big\|\sum_{j=1}^k \epsilon_j \frac{t_{j+1}}{t_j}x_j\Big\| + C\|f\|_{L^\infty(0,T;X)} \leqslant \sum_{j=1}^k \frac{1}{2^j}CM + CM \leqslant 2CM,\end{aligned}$$

where we used that $t_{j+1}/t_j \leqslant 1/2^j$.

Having checked the conditions of the Bessaga–Pełczyński theorem, it follows that X contains an isomorphic copy of c_0. □

We continue with dual version of Baillon's theorem. This time, the condition is that X^* does not contain a closed subspace isomorphic to c_0. Examples of such spaces are reflexive space, as well as the space $L^\infty(S)$, $C_0(K)$, $C_{\mathrm{ub}}(K)$ and $C_{\mathrm{b}}(K)$, where S is a measure space and K is a locally compact Hausdorff space. This follows from the fact that their duals have cotype 2; see the Notes.

Corollary 17.4.5 (Guerre-Delabriere). *Let A be a densely defined linear operator acting in a Banach space X whose dual does not contain a closed subspace isomorphic to c_0. If A has maximal L^1-regularity on I, then A is bounded.*

Proof. It suffices to consider $I = (0, T)$. By Proposition 17.2.32 A^* has maximal L^∞-regularity on $(0, T)$, and the result follows by applying Baillon's theorem to this operator. □

The condition that X or X^* contain no isomorphic copy of c_0 cannot be omitted in Theorem 17.4.4 and Corollary 17.4.5, respectively:

Example 17.4.6. On $X = c_0$ consider the unbounded operator $A : (y_n)_{n\geqslant 1} \mapsto (ny_n)_{n\geqslant 1}$ with maximal domain

$$\mathsf{D}(A) = \{(x_n)_{n\geqslant 1} \in c_0 : \quad (nx_n)_{n\geqslant 1} \in c_0\}.$$

Then A is closed and densely defined. The adjoint operator A^* on ℓ^1 is given by the same multiplication operator and has maximal domain

$$\mathsf{D}(A^*) = \{(x_n)_{n\geqslant 1} \in \ell^1 : \quad (nx_n)_{n\geqslant 1} \in \ell^1\},$$

and is again closed and densely defined. Let $I = \mathbb{R}_+$ or $I = (0, T)$ with $T \in (0, \infty)$ and $p \in [1, \infty]$. Then:

(1) A has maximal L^p-regularity on I if and only if $p \in (1, \infty]$.
(2) A^* has maximal L^p-regularity on I if and only if $p \in [1, \infty)$.

Assertion (1) follows from (2) and the duality result of Proposition 17.2.32. To prove (2), let F be the dense subspace of all $f \in L^1(\mathbb{R}_+; \ell^1)$ for which there exists an $N \geqslant 1$ such that for all $n \geqslant N$ for all $t \in I$ we have $f_n(t) = 0$. Referring to Theorem 17.2.19, it suffices to show that for every $f \in F$ the mild solution u_f of the problem $u' + Au = f$, $u(0) = 0$, takes values in $\mathsf{D}(A^*)$ and

$$\|A^* u_f\|_{L^1(\mathbb{R}_+;\ell^1)} \leqslant \|f\|_{L^1(\mathbb{R}_+;\ell^1)}.$$

We begin by noting that u_f is given by

$$(u_f(t))_n = \int_0^t e^{-n(t-s)} f_n(s)\,\mathrm{d}s.$$

Since f_n vanishes for $n \geqslant N$ it is clear that $A^* u_f(t) \in \mathsf{D}(A^*)$, and by Young's inequality

$$\begin{aligned}
\|A^* u_f\|_{L^1(\mathbb{R}_+;\ell^1)} &= \int_0^\infty \sum_{n\geqslant 1} \Big| \int_0^t n e^{-n(t-s)} f_n(s)\,\mathrm{d}s \,\Big| \,\mathrm{d}t \\
&= \sum_{n\geqslant 1} \int_0^\infty \Big| \int_0^t n e^{-n(t-s)} f_n(s)\,\mathrm{d}s \,\Big| \,\mathrm{d}t
\end{aligned}$$

$$\leqslant \sum_{n\geqslant 1} \|ne^{-n\cdot}\|_{L^1(\mathbb{R}_+)}\|f_n\|_{L^1(\mathbb{R}_+)}$$
$$= \sum_{n\geqslant 1} \|f_n\|_{L^1(\mathbb{R}_+)} \leqslant \|f\|_{L^1(\mathbb{R}_+;\ell^1)},$$

interchanging sum and integral in the last step. From the extrapolation result of Theorem 17.2.31 we find that A^* has maximal L^p-regularity for all $p \in [1,\infty)$. Finally, we note that by Theorem 17.4.4, A^* cannot have maximal L^∞-regularity, for ℓ^1 does not contain a closed subspace isomorphic to c_0.

It is a classical result due to Sobczyk that if a closed subspace of a *separable* Banach space X is isomorphic to c_0, then this subspace is complemented (see the Notes). Therefore, Example 17.4.6(1) can be extended to any separable Banach space containing an isomorphic copy of c_0. In this sense, the example shows that for separable Banach spaces Theorem 17.4.4 is optimal. By another classical result, due to Bessaga and Pelczyński, the dual a Banach space X contains an isomorphic copy of c_0 if and only if X contains a complemented copy of ℓ^1 (see the Notes). In this sense, the Example 17.4.6(2) shows that Corollary 17.4.5 is optimal.

While the previous results show that maximal L^∞- and L^1-regularity of unbounded operators

- cannot occur in the absence of c_0 (Theorem 17.4.4, Corollary 17.4.5),
- can occur in the presence of c_0 (Example 17.4.6),

it is worth noting that, even in the latter case, such end-point regularity is not something to be typically expected. In particular, Examples 17.4.2 and 17.4.3 show that the Laplacian $-\Delta$ fails both maximal L^1-regularity and L^∞-regularity on $X = L^1(\mathbb{R}^d)$ and $X = C_0(\mathbb{R}^d)$, respectively. Moreover, by Theorem 17.2.46, $-\Delta$ also does not have maximal C-regularity.

We complete our discussion of Example 17.4.6 by also considering the same operator as in this example on the space ℓ^q with $q \in (1,\infty)$.

Example 17.4.7. On ℓ^q with $q \in (1,\infty)$, we consider the closed densely defined operator $A : (y_n)_{n\geqslant 1} \mapsto (ny_n)_{n\geqslant 1}$ with maximal domain

$$\mathsf{D}(A) = \{(x_n)_{n\geqslant 1} \in \ell^q : \quad (nx_n)_{n\geqslant 1} \in \ell^q\}.$$

Let $I = \mathbb{R}_+$ or $I = (0,T)$ with $T \in (0,\infty)$ and let $p \in [1,\infty]$. We will show that A has maximal L^p-regularity if and only if $p \in (1,\infty)$. Indeed, the fact that maximal L^p-regularity fails for $p = \infty$ and $p = 1$ follows from Theorem 17.4.4 and Corollary 17.4.5. For $p = q$ we argue as in Example 17.4.6. For all $f \in L^q(\mathbb{R}_+;\ell^q)$, by Young's convolution inequality we have

$$\int_0^\infty \sum_{n\geqslant 1}\Big|\int_0^t ne^{-n(t-s)}f_n(s)\,\mathrm{d}s\,\Big|^q\,\mathrm{d}t = \sum_{n\geqslant 1}\int_0^\infty \Big|\int_0^t ne^{-n(t-s)}f_n(s)\,\mathrm{d}s\,\Big|^q\,\mathrm{d}t$$

$$\leqslant \sum_{n\geqslant 1} \|ne^{-n(\cdot)}\|_{L^1(\mathbb{R}_+)}^q \|f_n\|_{L^q(\mathbb{R}_+)}^q$$
$$\leqslant \|f\|_{L^q(\mathbb{R}_+;\ell^q)}^q.$$

The case $p \in (1,\infty) \setminus \{q\}$ follows by extrapolation, using Theorem 17.2.31.

17.4.c Analytic semigroups may fail maximal regularity

The aim of the present section is to prove the following theorem.

Theorem 17.4.8 (Kalton–Lancien, Fackler). *For each $1 < q < \infty$ the space $L^q(0,1)$ admits a densely defined invertible sectorial operator A_q, with $\omega(A_q) = 0$, such that the following assertions hold:*

(1) *the operator A_q is R-sectorial if and only if $q = 2$;*
(2) *we have $S_q(t) = S_r(t)$ for all $1 < q, r < \infty$ and $t \geqslant 0$, where S_q and S_r denote the bounded analytic C_0-semigroups generated by $-A_q$ and $-A_r$, respectively.*

Theorem 17.4.8 has a number of interesting consequences:

- In Theorem 17.3.1, one cannot replace 'R-sectorial' by 'sectorial'.
- Combining the theorem with Theorem 17.2.24 and the first part of Theorem 17.3.1, it follows that maximal L^p-regularity on $\mathbb{R}_+$ and $(0,T)$ may fail for densely defined invertible sectorial operators A of zero angle on $L^q(0,1)$ unless $q = 2$.
- It follows that maximal L^p-regularity on $\mathbb{R}_+$ may fail for bounded sectorial operators A of zero angle on $L^q(0,1)$ unless $q = 2$. Indeed, consider A^{-1} from the previous bullet. It is clearly bounded, and sectorial of angle zero since $\lambda R(\lambda, A) = A^{-1}R(1/\lambda, A^{-1})$. However, A^{-1} cannot be R-sectorial, since this identity would then imply that A is R-sectorial.
- Since A_2 has maximal L^p-regularity, the second assertion of the theorem shows that maximal L^p-regularity does not extrapolate from $L^2(0,1)$ to $L^q(0,1)$ in the case of consistent bounded analytic C_0-semigroups on these spaces.
- Theorem 10.7.13 (where the second occurrence of the word 'contraction' should be deleted from the formulation of the theorem as printed), 'contractive for $t \geqslant 0$ and positive' cannot be replaced by 'bounded for $t \geqslant 0$'.

The operator A_q constructed in the proof of the theorem is a diagonal operator with respect to Schauder basis for a complemented subspace of $L^q(0,1)$ that is unconditional and non-homogeneous in the sense of the following definition; on the complement of this subspace we take $A = 0$.

Definition 17.4.9. *Let X be a Banach space.*

(1) *A sequence $(x_n)_{n\geqslant 1}$ in X is said to be a* basic sequence *if it is a Schauder basis for its own closed linear span.*

(2) *Two basic sequences* $(x_n)_{n\geqslant 1}$ *and* $(y_n)_{n\geqslant 1}$ *in* X *are said to be* equivalent *if for any scalar sequence* $(c_n)_{n\geqslant 1}$ *it is true that*

$$\sum_{n\geqslant 1} c_n x_n \text{ converges if and only if } \sum_{n\geqslant 1} c_n y_n \text{ converges.}$$

(3) *A Schauder basis* $(x_n)_{n\geqslant 1}$ *in* X *is said to be* homogeneous *if every two disjoint subsequences of* $(x_n)_{n\geqslant 1}$ *are equivalent as basic sequences.*

If two basic sequences $(x_n)_{n\geqslant 1}$ and $(y_n)_{n\geqslant 1}$ are equivalent, their closed linear spans $[(x_n)_{n\geqslant 1}]$ and $[(y_n)_{n\geqslant 1}]$ are isomorphic as Banach spaces. Indeed, the correspondence

$$\sum_{n\geqslant 1} b_n x_n \mapsto \sum_{n\geqslant 1} b_n y_n$$

gives a bijection between $[(x_n)_{n\geqslant 1}]$ and $[(y_n)_{n\geqslant 1}]$. Since the coordinate projections of a Schauder basis are bounded, this bijection is closed, and therefore bounded by the closed graph theorem. Applying the same argument to its inverse, it follows that this bijection is an isomorphism.

The following lemma isolates the crucial property of $L^q(0,1)$ that will be needed in the proof of Theorem 17.4.8.

Lemma 17.4.10. *The Haar basis of* $L^q(0,1)$ *is unconditional for every* $1 < q < \infty$*, and non-homogeneous for* $q \neq 2$*.*

Using the notation of Section 9.1.h, the (L^2-normalised) Haar basis is given as $(h_n)_{n\geqslant 1}$, where $h_1 \equiv \mathbf{1}$ and $h_n := \phi_{j,k}$ for $n \geqslant 2$, where $n = 2^j + k$ with $j = 0, 1, 2, \dots$ and $k = 1, \dots, 2^j$, and

$$\phi_{j,k} := 2^{j/2}\mathbf{1}_{(\frac{k-1}{2^j}, \frac{k-1/2}{2^j})} - 2^{j/2}\mathbf{1}_{(\frac{k-1/2}{2^j}, \frac{k}{2^j})}.$$

Note that $\phi_{j,k}$ is supported on the interval $(\frac{k-1}{2^j}, \frac{k}{2^j})$.

Proof of Lemma 17.4.10. Unconditionality for $1 < q < \infty$ of the Haar basis is proved in Corollaries 4.5.8 and 4.5.16.

Consider the subsequence $(\phi_{n,2})_{n\geqslant 2}$. The supports of the functions g_n are disjoint, contained in $(0, \frac12)$, and therefore they span a closed subspace isomorphic to ℓ^q. On the other hand, the functions in the sequence $(\phi_{j,k})_{j=1,2,3,\dots;\, k=2^{j-1}+1,\dots 2^j}$ have support in $(\frac12, 1)$ and their linear span contains the '$(\frac12, 1)$-Rademacher' functions $r_j = \sum_{k=2^{j-1}+1}^{2^j} \phi_{j,k}$, $j \geqslant 1$, whose closed span is isomorphic to ℓ^2 by the Khintchine inequalities. Since it can be shown (a proof is included in the final paragraph of this section) that ℓ^q does not contain a closed subspace isomorphic to ℓ^2 unless $q = 2$, it follows that for $1 < q < \infty$ with $q \neq 2$, the Haar basis of $L^q(0,1)$ is unconditional and non-homogeneous. □

With this lemma at hand, Theorem 17.4.8 is seen to be a special case of the following more general result:

Theorem 17.4.11. *Every Banach space with a non-homogeneous unconditional basis admits a densely defined invertible sectorial operator* A*, with* $\omega(A) = 0$*, which is not* R*-sectorial.*

By invoking additional results from the theory of Schauder bases, Kalton and Lancien proved an even stronger result, namely, that if X is a Banach space with an unconditional basis $(x_n)_{n\geqslant 1}$ with the property every sectorial operator of angle zero on X is R-sectorial, then X is isomorphic to ℓ^2. We will comment on this result Notes at the end of the chapter.

Proof of Theorem 17.4.11. Let $(e_n)_{n\geqslant 1}$ be a non-homogeneous unconditional basis for X. By renorming, this basis can be made into an 1-*unconditional* basis, i.e., an unconditional basis satisfying

$$\Big\| \sum_{n\geqslant 1} \lambda_n c_n e_n \Big\| \leqslant \|\lambda\|_{\ell^\infty} \Big\| \sum_{n\geqslant 1} c_n e_n \Big\|$$

for all $x = \sum_{m\geqslant 1} c_n e_n \in X$ and $\lambda = (\lambda_n)_{n\geqslant 1} \in \ell^\infty$. Indeed, the equivalent norm

$$\Big|\!\Big|\!\Big| \sum_{n\geqslant 1} c_n e_n \Big|\!\Big|\!\Big| := \sup_{\|\lambda\|_\infty \leqslant 1} \Big\| \sum_{n\geqslant 1} \lambda_n c_n e_n \Big\|$$

will do. This renorming preserves non-symmetry of the basis, and will be used in the proof below to eliminate inessential unconditionality constants from several estimates. As a side-remark, a Banach space with a 1-unconditional basis is a Banach lattice with respect to the coordinate-wise ordering.

Choose disjoint subsequences $(e_{m_j})_{j\geqslant 1}$ and $(e_{n_j})_{j\geqslant 1}$, as well as a scalar sequence $(a_j)_{j\geqslant 1}$, such that $\sum_{j\geqslant 1} a_j e_{m_j}$ converges and $\sum_{j\geqslant 1} a_j e_{n_j}$ does not converge.

The construction below will produce a sectorial operator A on the closed linear span Y of the two sequences $(e_{m_j})_{j\geqslant 1}$ and $(e_{n_j})_{j\geqslant 1}$ that has all the stated properties. The desired example on the whole space X is then obtained by taking the direct sum with the identity operator on the closed linear span Z of the remaining basis vectors (note that $X = Y \oplus Z$ with contractive projections thanks to the 1-unconditionality).

Since the subsequences $(e_{m_j})_{j\geqslant 1}$ and $(e_{n_j})_{j\geqslant 1}$ do not overlap, the first be relabelled as $(e_{2j})_{j\geqslant 1}$ and the second as $(e_{2j-1})_{j\geqslant 1}$. Define the sequence $(f_j)_{j\geqslant 1}$ in X by

$$f_j = \begin{cases} e_j & (j \text{ odd}) \\ e_j + e_{j-1} & (j \text{ even}) \end{cases}$$

so that $e_{2j} = f_{2j} - f_{2j-1}$. It is elementary to check that $(f_j)_{j\geqslant 1}$ is a Schauder basis for its closed linear span. Consider the diagonal operator

$$A : \sum_{j\geqslant 1} c_j f_j \mapsto \sum_{j\geqslant 1} 2^j c_j f_j$$

with its natural domain. By Lemma 10.2.27 (restricting to the positive integers as index set) this operator is sectorial of zero angle, and moreover $0 \in \varrho(A)$. The analytic C_0-semigroup $(S(t))_{t\geqslant 0}$ generated by $-A$ is bounded and strongly continuous on every sector of angle strictly less than $\frac{1}{2}\pi$.

Assume, for a contradiction, that A is R-sectorial. Let $\lambda = (\lambda_j)_{j\geqslant 1}$ be a sequence of positive real numbers to be chosen shortly. Since $e_{2j} = f_{2j} - f_{2j-1}$, we obtain

$$\begin{aligned}\sum_{j=k}^{\ell} &\varepsilon_j a_j \lambda_j (\lambda_j + A)^{-1}(f_{2j} - f_{2j-1}) \\ &= \sum_{j=k}^{\ell} \varepsilon_j \Big(\frac{\lambda_j a_j}{\lambda_j + 2^{2j}} f_{2j} - \frac{\lambda_j a_j}{\lambda_j + 2^{2j-1}} f_{2j-1}\Big) \\ &= \sum_{j=k}^{\ell} \varepsilon_j \Big(\frac{\lambda_j a_j}{\lambda_j + 2^{2j}} (e_{2j} + e_{2j-1}) - \frac{\lambda_j a_j}{\lambda_j + 2^{2j-1}} e_{2j-1}\Big) \\ &= \sum_{j=k}^{\ell} \varepsilon_j \Big(\frac{\lambda_j a_j}{\lambda_j + 2^{2j}} e_{2j} + a_j \Big(\frac{\lambda_j}{\lambda_j + 2^{2j}} - \frac{\lambda_j}{\lambda_j + 2^{2j-1}}\Big) e_{2j-1}\Big).\end{aligned}$$

Now take $\lambda_j := 2^{2j}$. The above identity then takes the form

$$\sum_{j=k}^{\ell} \varepsilon_j a_j 2^{2j} (2^{2j} + A)^{-1}(f_{2j} - f_{2j-1}) = \sum_{j=k}^{\ell} \varepsilon_j \Big(\frac{1}{2} a_j e_{2j} + \frac{1}{6} a_j e_{2j-1}\Big).$$

The 1-conditionality of $(e_j)_{j\geqslant 1}$ and the assumed R-boundedness imply that

$$\begin{aligned}\Big\| \sum_{j=k}^{\ell} \Big(\frac{1}{2} a_j e_{2j} - \frac{1}{6} a_j e_{2j-1}\Big)\Big\| &= \Big\| \sum_{j=k}^{\ell} \varepsilon_j \Big(\frac{1}{2} a_j e_{2j} + \frac{1}{6} a_j e_{2j-1}\Big)\Big\|_{L^2(\Omega;X)} \\ &\leqslant \mathscr{R}(\{\lambda(\lambda + A)^{-1} \colon \lambda > 0\}) \Big\| \sum_{j=k}^{\ell} \varepsilon_j a_j e_{2j}\Big\|_{L^2(\Omega;X)} \\ &= \mathscr{R}(\{\lambda(\lambda + A)^{-1} \colon \lambda > 0\}) \Big\| \sum_{j=k}^{\ell} a_j e_{2j}\Big\|,\end{aligned}$$

where $\mathscr{R}(\{\lambda(\lambda + A)^{-1} \colon \lambda > 0\})$ denotes the R-bound of the set $\{\lambda(\lambda + A)^{-1} \colon \lambda > 0\}$. Since the right-hand side tends to 0 as $k, \ell \to \infty$, so does the left-hand side. By the convergence of the sum $\sum_{j\geqslant 1} a_j e_{2j}$, this in turn implies

$$\lim_{k,\ell\to\infty} \lim_{k,\ell\to\infty} \Big\| \sum_{j=k}^{\ell} a_j e_{2j-1}\Big\| = 0.$$

This contradicts the fact that $\sum_{j=k}^{\ell} a_j e_{2j-1}$ fails to converge. □

On closed subspaces of ℓ^q

In this paragraph we complete the proof that the Haar basis of $L^q(0,1)$ is non-homogeneous for $q \in [1,\infty)$ unless $q = 2$ (Lemma 17.4.10). The missing piece of information was the following result.

Proposition 17.4.12. *Let $1 \leqslant q < \infty$. The space ℓ^q contains a closed subspace isomorphic to ℓ^2 (if and) only if $q = 2$.*

Since ℓ^q and ℓ^2 are isomorphic (if and) only if $q = 2$ (e.g., by looking at their type and cotype), this proposition is an immediate consequence of the following result.

Proposition 17.4.13. *Every infinite-dimensional closed subspace of ℓ^q contains a closed subspace isomorphic to ℓ^q.*

The proof of this proposition is based on two lemmas, for which we need to introduce some terminology. Recall that a sequence in a Banach space X is said to be a *basic sequence* if it is a Schauder basis for its closed linear span, and that two basic sequences $(x_n)_{n\geqslant 1}$ and $(y_n)_{n\geqslant 1}$ are said to be *equivalent* if for any scalar sequence $(c_n)_{n\geqslant 1}$ it is true that

$$\sum_{n\geqslant 1} c_n x_n \text{ converges if and only if } \sum_{n\geqslant 1} c_n y_n \text{ converges.}$$

The *basis constant* of a basic sequence $(x_n)_{n\geqslant 1}$ is defined as the (finite) number $\sup_{N\geqslant 1} \|P_N\|$, where P_N is the projection in the closed span $[(x_n)_{n\geqslant 1}]$ of $(x_n)_{n\geqslant 1}$ defined by

$$P_N \sum_{n\geqslant 1} c_n x_n := \sum_{n=1}^{N} c_n x_n.$$

A sequence $(y_n)_{n\geqslant 1}$ in X is said to be a *block sequence* of the basic sequence $(x_n)_{n\geqslant 1}$ in X if there exist a strictly increasing sequence of positive integers $(N_n)_{n\geqslant 1}$ and a scalar sequence $(c_n)_{n\geqslant 1}$ such that for all $m \geqslant 1$ we have

$$y_m = \sum_{m=N_{m-1}}^{N_m - 1} c_n x_n,$$

where $N_0 := 1$. Clearly, a block sequence of a basic sequence is a basic sequence.

Lemma 17.4.14. *Let $(x_n)_{n\geqslant 1}$ be a basic sequence in a Banach space X and let $(y_n)_{n\geqslant 1}$ be a sequence in X which satisfies*

$$\sum_{n\geqslant 1} \|x_n - y_n\| \leqslant \frac{1}{3K},$$

where K is the basis constant of $(x_n)_{n\geqslant 1}$. Then $(y_n)_{n\geqslant 1}$ is a basic sequence equivalent to $(x_n)_{n\geqslant 1}$, and the mapping

$$\sum_{n\geqslant 1} c_n x_n \mapsto \sum_{n\geqslant 1} c_n y_n$$

sets up an isomorphism of their closed linear spans.

Proof. The mapping $T : \sum_{n\geqslant 1} c_n x_n \to \sum_{n\geqslant 1} c_n y_n$ is well defined from the closed linear span $[(x_n)_{n\geqslant 1}]$ to the closed linear span $[(y_n)_{n\geqslant 1}]$, because

$$\begin{aligned}\Big\|\sum_{n=m}^{M} c_n y_n\Big\| &\leqslant \Big\|\sum_{n=m}^{M} c_n x_n\Big\| + \sum_{n=m}^{M} |c_n|\|y_n - x_n\| \\ &\leqslant \Big\|\sum_{n=m}^{M} c_n x_n\Big\| + \sum_{n=m}^{M} 2K\|x\|\cdot\|y_n - x_n\|\end{aligned}$$

implies that the partial sums of $\sum_{n\geqslant 1} c_n y_n$ form a Cauchy sequence, and therefore the sum $\sum_{n\geqslant 1} c_n y_n$ converges. Moreover,

$$\|x - Tx\| \leqslant \sum_{n\geqslant 1} |c_n|\|x_n - y_n\| \leqslant 2K\|x\| \cdot \frac{1}{2K} = \frac{2}{3}\|x\|.$$

This implies that $\|Tx\| \geqslant \frac{1}{3}\|x\|$ for all $x \in [(x_n)_{n\geqslant 1}]$, and therefore T is an isomorphism from $[(x_n)_{n\geqslant 1}]$ onto its range in $[(y_n)_{n\geqslant 1}]$, which is dense since it contains all finite linear combinations of the y_n. It follows that T is an isomorphism from $[(x_n)_{n\geqslant 1}]$ onto $[(y_n)_{n\geqslant 1}]$.

Finally, let $y \in [(y_n)_{n\geqslant 1}]$ and let $x := T^{-1}y$. Since $(x_n)_{n\geqslant 1}$ is a Schauder basis for $[(x_n)_{n\geqslant 1}]$ we have a unique representation $x = \sum_{n\geqslant 1} c_n x_n$. Then $y = \sum_{n\geqslant 1} c_n y_n$ and this representation is unique. This proves that $(y_n)_{n\geqslant 1}$ is a Schauder basis for $[(y_n)_{n\geqslant 1}]$, i.e., $(y_n)_{n\geqslant 1}$ is a basic sequence. □

Lemma 17.4.15. *Every infinite-dimensional closed subspace Y of a Banach space X with a Schauder basis $(x_n)_{n\geqslant 1}$ has a closed subspace Z with a normalised Schauder basis $(y_n)_{n\geqslant 1}$, equivalent to a block sequence $(u_n)_{n\geqslant 1}$ of $(x_n)_{n\geqslant 1}$, and with $[(y_n)_{n\geqslant 1}] \simeq [(u_n)_{n\geqslant 1}$ isomorphically.*

Proof. Since Y is infinite-dimensional, for every $k \geqslant 1$ it is possible to find elements in Y of the form $\sum_{n\geqslant k} a_n x_n$.

Let $y_1 \in Y$ be an arbitrary norm one vector in Y, say $y_1 = \sum_{n\geqslant 1} a_n x_n \in Y$. Choose $N_1 \geqslant 1$ so large that $\|y_1 - u_1\| < 1/(4K)$, where $u_1 = \sum_{n=1}^{N_1-1} a_n x_n$ and K is the basis constant of $(x_n)_{n\geqslant 1}$. Next let $y_2 \in Y$ be a norm one vector of the form $y_2 = \sum_{n\geqslant N_1} a_n x_n$ and choose $N_2 \geqslant 1$ so large that $\|y_2 - u_2\| < 1/(4^2K)$, where $u_2 = \sum_{n=N_1}^{N_2-1} a_n x_n$. Continuing in this way, we obtain a block sequence $(u_n)_{n\geqslant 1}$ of $(x_n)_{n\geqslant 1}$ for which we have

$$\sum_{n\geqslant 1}\|y_n-u_n\|\leqslant \frac{1}{K}\sum_{n\geqslant 1}\frac{1}{4^n}=\frac{1}{3K}.$$

By Lemma 17.4.14, $(y_n)_{n\geqslant 1}$ is a normalised basic sequence equivalent to the block sequence $(u_n)_{n\geqslant 1}$ and $[(y_n)_{n\geqslant 1}]\simeq[(u_n)_{n\geqslant 1}$ isomorphically. The closed linear span Z of $(y_n)_{n\geqslant 1}$ has the desired properties. □

Proof of Proposition 17.4.12. Let Y be an infinite-dimensional closed subspace of ℓ^q. By Lemma 17.4.15, Y has a closed subspace Z with a normalised Schauder basis $(y_n)_{n\geqslant 1}$ equivalent to a block sequence $(u_n)_{n\geqslant 1}$ of $(x_n)_{n\geqslant 1}$ and with $[(y_n)_{n\geqslant 1}]\simeq[(u_n)_{n\geqslant 1}$ isomorphically.

If $(c_n)_{n\geqslant 1}$ is any scalar sequence, then

$$\begin{aligned}\Big\|\sum_{n=j}^{k}c_nu_n\Big\|_{\ell^q}^q&=\Big\|\sum_{n=j}^{k}c_j\sum_{n=N_{j-1}}^{N_j-1}a_ne_n\Big\|_{\ell^q}^q\\&=\sum_{n=j}^{k}\sum_{n=N_{j-1}}^{N_j-1}|c_ja_n|^q\overset{(*)}{\eqsim}\sum_{n=j}^{k}|c_j|^q=\Big\|\sum_{n=j}^{k}c_ne_n\Big\|_{\ell^q}^q,\end{aligned}$$

where $(*)$ follows from the fact that $\frac34\leqslant\|u_n\|\leqslant\frac54$ for all $n\geqslant 1$. This computation shows that $[(u_n)_{n\geqslant 1}]\simeq[(e_n)_{n\geqslant 1}]=\ell^q$ isomorphically, and since also $[(y_n)_{n\geqslant 1}]\simeq[(u_n)_{n\geqslant 1}]$ isomorphically we conclude that $Z:=[(y_n)_{n\geqslant 1}]\simeq\ell^q$ isomorphically. □

17.5 Notes

Parabolic maximal regularity estimates of the form $\|u_f'\|_p+\|Au_f\|_p\leqslant C\|f\|_p$ can be traced to back to the Ladyženskaja, Solonnikov, and Ural′ceva [1968]. Early contributions to the abstract operator-theoretic framework include Sobolevskiĭ [1964], De [1964], Grisvard [1969], Da Prato and Grisvard [1975]. More recent expositions can be found in Amann [1995], Kunstmann and Weis [2004], Denk, Hieber, and Prüss [2003], Prüss and Simonett [2016].

There is an extensive literature of applications of maximal L^p-regularity to non-linear parabolic problems; we refer to Prüss and Simonett [2016] and references therein. An abstract approach to quasi-linear and semi-linear evolution equations based on maximal L^p-regularity will be presented in Chapter 18. A small sample of the applications in the literature, which counts several hundreds of papers, is given in the notes of that chapter.

Section 17.1

Most of the semigroups in this chapter are analytic semigroups which are not necessarily strongly continuous. These are widely used in the literature, and an extensive treatment can be found in Lunardi [1995].

Our treatment of the initial value problem

$$\begin{aligned} u'(t) + Au(t) &= f(t), \quad t \in [0,T], \\ u(0) &= x, \end{aligned}$$

for non-densely defined operators A in a Banach space X is focussed on the existence, uniqueness, and regularity of L^p-solutions. For continuous functions $f : [0,T] \to X$, the problem is studied by Da Prato and Sinestrari [1987] where the following result concerning classical solutions is established: If (i) A satisfies the resolvent estimates of the Hille–Yosida theorem, (ii) $f(t) = f(0) + \int_0^t g(s)\,\mathrm{d}s$ for some Bochner integrable function $g : (0,T) \to X$, and (iii) $x \in \mathsf{D}(A)$ satisfies

$$-Ax + f(0) \in \overline{\mathsf{D}(A)},$$

then there exists a unique $u \in C^1([0,T];X) \cap C([0,T];\mathsf{D}(A))$ satisfying the initial value problem pointwise in $t \in [0,T]$. A proof using the theory of locally Lipschitz integrated semigroups was given by Kellerman and Hieber [1989].

Section 17.2

In our definition of maximal L^p-regularity, we only impose that the L^p-solution should satisfy $Au \in L^p(I;X)$. In the early literature it was often assumed that in addition to this one also has $u \in L^p(I;X)$. This imposes a genuine restriction if $I = \mathbb{R}_+$ and makes proofs often simpler. The present definition was proposed by Weis [2001b].

Several results in Section 17.2 are based on the important paper by Dore [2000]. In particular this applies to Propositions 17.2.5, 17.2.7 17.2.10, 17.2.11, Lemma 17.2.12, Theorems 17.2.15 and 17.2.19, the second part of Theorem 17.2.24, and Theorem 17.2.26(1)–(4).

The results of Section 17.2.b are standard applications of the trace method for real interpolation as discussed in Appendix L. An exception is the case $p = 1$, which is treated by an *ad hoc* method here.

The proof of Theorem 17.2.15 follows Dore [2000], but injectivity is shown by an argument from Prüss and Simonett [2016]. Under the *a priori* assumption that $-A$ generates a C_0-semigroup, the result was stated in Sobolevskiĭ [1964]. The sectoriality estimate $\|A(\lambda + A)^{-1}\| \leqslant M^{\mathrm{reg}}_{p,A}(\mathbb{R}_+)$ of (17.14) appears to be new. Propositions 17.2.18 and 17.2.29 seem to be new as well, and turn out to be quite useful in approximation arguments for proving maximal L^p-regularity on $\mathbb{R}_+$.

In Dore [2000], the proof of the second part of Theorem 17.2.24 is attributed to Kato. The technical estimate (17.31) for the operator $\lambda + A$ seems to be new, and is crucial in the proof of a new type of maximum principle for maximal regularity presented in Proposition 17.2.27(1).

Theorems 17.2.19, 17.2.26(1)–(4) are due to Dore [2000], although the proof of the extrapolation of the integrability parameter of part (4) is credited to Kato, who reduced the proof to the extrapolation result Benedek, Calderón, and Panzone [1962, Theorem 2]. This result is not immediately applicable however, since it is phrased for kernels which are locally integrable on $\mathbb{R}^d$. In the present context one needs a corresponding result for kernels which are locally integrable on $\mathbb{R}^d \setminus \{0\}$. The approach taken in the main text circumvents this problem by using the more general extrapolation result of Chapter 11. Under the assumption that $-A$ generate an analytic C_0-semigroup, the extrapolation result of part (4) of Theorem 17.2.31 is due to Cannarsa and Vespri [1986] and Coulhon and Lamberton [1986]; see also Sobolevskiĭ [1964]; the quantitative bound appears to be new. An extrapolation result for maximal L^p-regularity to the vector-valued Hardy space $H^1(I;X)$ is presented in Hytönen [2005].

The duality result of Theorem 17.2.265 is due to Kalton and Portal [2008], who proved it for maximal L^p-regularity on $\mathbb{R}_+$ via discrete maximal regularity. The precise bounds for arbitrary intervals of the duality result obtained in Proposition 17.2.32 seem to be new.

Weighted extrapolation

Extrapolation to weighted estimates is a standard method in harmonic analysis, and we presented it from a modern viewpoint in Chapter 11. Extrapolation of L^p-boundedness of singular integral operators to L^p-spaces with power weights goes back a long way. It was already considered by Hardy and Littlewood [1936] and Babenko [1948] in the one-dimension setting. The d-dimensional case was considered in Stein [1957]. These results do not require any smoothness of the kernel.

Part of Proposition 17.2.47 on extrapolation of maximal C-regularity for weights $w_\alpha(t) = t^\alpha$ appeared in Clément and Simonett [2001]. In Prüss and Simonett [2004], the extrapolation result of Proposition 17.2.36 for the weights w_α was proved for $p \in (1, \infty)$. A proof based on Schur's lemma appears in Auscher and Axelsson [2011]. Our proof is based on Young's inequality for the multiplicative group $\mathbb{R}_+$. Some of the results in the end-points and maximal C-regularity (see Proposition 17.2.47) may be new. The extrapolation result for exponential weights presented in Proposition 17.2.38 is folklore.

The non-quantitative version of Theorem 17.2.39 follows from the weighted Calderón–Zygmund theory in Rubio de Francia, Ruiz, and Torrea [1986]. Extensions to weighted Banach function spaces and applications to non-autonomous maximal L^p-regularity have been given in Chill and Fiorenza [2014]. The class of admissible weights for singular integral operators on the half-line was analysed by Chill and Król [2018].

Maximal C-regularity

Maximal continuous regularity was first shown on real interpolation spaces by Da Prato and Grisvard [1975, 1979], where it was used to study quasilinear evolution equations. This paper also contains one implication of Theorem 17.2.46, as well as the result that positive self-adjoint operators on Hilbert spaces with compact resolvent fail maximal C-regularity. Weighted analogues and simpler proofs of the results in Da Prato and Grisvard [1975, 1979] were presented by Angenent [1990a]. This paper also contains far-reaching applications to fully non-linear equations. Further results, and applications to non-linear equations and their stability, can be found in Amann [1995], Clément and Simonett [2001], Da Prato and Lunardi [1988], LeCrone and Simonett [2020], Lunardi [1995], Shao and Simonett [2014], Travis [1981] and many other papers. Some applications of maximal C-regularity to quasi-linear evolution equations will be presented in Chapter 18. A variant of the results by Dore [2000] for maximal C-regularity can be found in LeCrone and Simonett [2011]. Versions of Corollary 17.3.20 for $p = \infty$ also hold for (weighted) maximal C-regularity, and can be proved via the same method or derived afterwards (see Remark 17.3.21 and Proposition 17.2.47). The equivalence of maximal L^∞ and C-regularity may be known, but we are not aware of a reference stating this result. We also do not know whether the condition that $\mathsf{D}(A)$ be dense in X can be removed.

A different version of the perturbation result in Proposition 17.2.49 appears in Dore [2000]. Our proof uses a less restrictive relative bound.

The application to maximal L^p-regularity of continuous time-dependent operators $A : I \to \mathscr{L}(X_1, X_0)$ as in Theorem 17.2.51 is standard. We will discuss time-dependent problems in more detail below.

Section 17.3

According to Coulhon and Lamberton [1986], the following question was asked by Brezis:

> "à quelles conditions sur l'espace de Banach E a-t-on la régularité L^p pour tout A générateur d'un semi-groupe analytique borné sur E ?"

[Under which conditions on the Banach space E does every bounded analytic semigroup generator have maximal L^p-regularity?]

Corollary 17.3.8, which is due to De [1964], asserts maximal L^p-regularity for negative generators of bounded analytic C_0-semigroups on Hilbert spaces. In Coulhon and Lamberton [1986] it was shown that the Poisson semigroup has maximal L^p-regularity on $L^q(\mathbb{R}^d; X)$ if and only if X is a UMD space (see Theorem 17.4.1 for the precise formulation). Lamberton [1987] subsequently showed that if $-A$ generates a bounded analytic C_0-semigroup S on a space $L^2(\Omega)$ such that the operators $S(t)$ extend to contractions on $L^q(\Omega)$ for every $q \in [1, \infty]$, then A has maximal L^p-regularity for all $p \in (1, \infty)$.

A breakthrough was made by Dore and Venni [1987], where it was shown that every operator A in a UMD Banach space X with bounded imaginary powers of angle $\omega_{\mathrm{BIP}}(A) < \pi/2$ has maximal L^p-regularity for $p \in (1, \infty)$. The approach was based on their preliminary version of (what is nowadays called) the Dore–Venni theorem (Theorem 15.4.11), in which the additional condition that the operators be invertible was made. This invertibility condition was removed in Prüss and Sohr [1990], and Corollary 17.3.7 follows from this paper.

It took approximately ten years Brezis's question was settled definitively:

- Kalton and Lancien [2000] showed that Brezis's question, in the form stated, has a negative answer. For every $q \in (1,\infty) \setminus \{2\}$, they gave an example of an operator $-A$ generating a bounded analytic C_0-semigroup on $L^q(0,1)$ such that A does not have maximal L^p-regularity on finite time intervals (see Theorem 17.4.8).
- It was shown in Weis [2001a,b] that a sectorial operator A on a UMD Banach space X has maximal L^p-regularity if and only if A is R-sectorial (see Theorem 17.3.1).

The negative general answer of Kalton and Lancien [2000] will be further discussed below. The characterisation of maximal L^p-regularity in UMD spaces of Weis [2001a,b] is based on the operator-valued Fourier multiplier theorem for UMD-valued multipliers (Theorem 5.3.18), which was proved in the same paper. This work led to many follow-up studies in which maximal L^p-regularity was proved by checking R-sectoriality, some of which will be discussed below. In particular, Weis's result implies the sufficient conditions for maximal L^p-regularity stated in Corollaries 17.3.7 and 17.3.9. Corollary 17.3.9 resembles the previously mentioned result of Lamberton [1987], which instead of positivity, assumes contractivity on L^q for all $q \in [1,\infty]$. One can check that under the latter condition the argument in the proof of Corollary 17.3.9 can be repeated: the R-sectoriality with $\omega_R(A) < \pi/2$ proved in Theorem 10.7.13 was based on Akcoglu's maximal ergodic Theorem 10.7.14. The latter was stated and proved for semigroups which are positive and contractive for some $q \in (1,\infty)$, but is also valid if contractivity is assumed for all $q \in [1,\infty]$ (see Dunford and Schwartz [1958, Theorem VIII.7.7]).

The perturbation result of Corollary 17.3.10 improves Proposition 17.2.49 for UMD spaces, since the condition of the relative bound is weaker. A similar result under a different condition can be found in Kunstmann and Weis [2001].

The characterisations of maximal L^p-regularity for the endpoints $p = 1$ and $p = \infty$ of Theorems 17.3.11 and 17.3.12 are to due to Kalton and Portal [2008], who proved these results via discrete versions of maximal L^p-regularity. It seems that their proof of Theorem 17.3.12 also assumes that the operator A be densely defined.

The alternative proof of Theorem 17.3.1 presented in Theorem 17.3.18 is due to Kalton and Weis [2001], and is based on their sum-of-operator method explained in Theorem 16.3.6. Proposition 17.3.14 seems to be folklore. The fact that only the weaker result of Proposition 17.3.15 holds in the case $I = \mathbb{R}_+$ seems to be less known. The properties of the derivative operator $Du = u'$ on $L^p(I;X)$ with a Dirichlet condition at the left end-point collected in Proposition 17.3.16 are standard. Extensions to weighted spaces $L^p(\mathbb{R}_+, t^\gamma\, \mathrm{d}, t; X)$ can be found in Prüss and Simonett [2004] and Meyries and Schnaubelt [2012b]. A direct proof based on the Mihlin multiplier theorem and the duality between

$-D$ and D can be found in Lindemulder, Meyries, and Veraar [2018], where also the complex interpolation spaces of $\mathsf{D}(D)$ are identified.

Differential operators with maximal L^p-regularity

The characterisation of Weis [2001a,b] presented in Theorem 17.3.1, led to many follow-up works in which R-sectoriality of angle $\omega_R A < \pi/2$ was checked for concrete differential operators. Moreover, due to the estimate $\omega_{H^\infty}(A) \leqslant \omega_R(A)$ (see Theorem 10.4.9), these R-sectoriality estimate also led to new examples of operators with a bounded H^∞-calculus of angle $\omega_{H^\infty}(A) < \pi/2$. Before the characterisation was known usually BIP-conditions on A were checked which for elliptic differential operators often required α-Hölder regularity of the coefficients and Poisson estimates on the kernels. Details can be found in Duong and Robinson [1996], Hieber and Prüss [1997], Coulhon and Duong [2000] and references therein.

First of all Corollary 17.3.9 and the mentioned result of Lamberton [1987] provides a wide class of examples with maximal L^p-regularity on L^q-spaces. The contractivity condition can however be a serious constraint when dealing with systems or differential operators of order four or higher (see Langer and Maz′ya [1999]).

In their influential memoir, Denk, Hieber, and Prüss [2003] proved maximal L^p-regularity for a large class of uniformly elliptic systems under the following conditions

- domain equals $\mathbb{R}^d$, $\mathbb{R}^d_+$, or $\Omega \subseteq \mathbb{R}^d$ which is open, bounded, and smooth;
- non-divergence form with bounded and uniformly continuous coefficients;
- boundary conditions of Lopatinskii-Shapiro type.

Piasecki, Shibata, and Zatorska [2020] proved maximal L^p-regularity without the need of uniform Lopatinskii-Shapiro conditions. Moreover, on $\mathbb{R}^d$ Haller-Dintelmann, Heck, and Hieber [2006], Heck and Hieber [2003] managed to prove maximal L^p-regularity under VMO (vanishing mean oscillation) conditions on the coefficients, but do not consider any boundary conditions.

In Clément and Prüss [2001] and Weis [2001a] it is explained that often R-sectoriality can be checked by a simple domination by maximal functions, leading to an effective approach to prove maximal L^p-regularity. These ideas emerged into a very flexible framework in a series of papers by Blunck and Kunstmann [2002, 2003, 2004, 2005] (see also Kunstmann and Weis [2004]), who proved R-sectoriality under off-diagonal bounds on the semigroup. They apply their results to elliptic operators in divergence form. Here no symmetry of the coefficients is required and thus contractivity cannot be expected on L^q with $q \neq 2$, and thus Corollary 17.3.9 is not applicable.

Kunstmann [2008] proved maximal L^p-regularity a class of elliptic differential operators A in non-divergence form only assuming that $-A$ generates an analytic semigroup in $L^q(\mathbb{R}^d)$. This can be seen as a positive answer to Brezis's question for a special class of elliptic differential operators on $L^q(\mathbb{R}^d)$.

The following theorem is immediate from Theorem 17.3.1, and the A_p-weighted characterisation of R-boundedness given in Theorem 8.2.6, which trivially extends to open subsets of $\mathbb{R}^d$, if one uses restrictions of A_p-weight on $\mathbb{R}^d$. This method was first used by Fröhlich [2001], Fröhlich [2007] and Haller, Heck, and Hieber [2003] to prove maximal L^p-regularity.

Theorem 17.5.1. *Let $\Omega \subseteq \mathbb{R}^d$ be an open set. Let $r \in (1,\infty)$ and let $\phi_r : [1,\infty) \to [0,\infty)$ be a non-decreasing function. Let A be a θ-sectorial operator on $L^r(\Omega)$ with $\theta \in (0,\pi/2)$, and suppose that for all $w \in A_r$ and $\lambda \in \complement\overline{\Sigma_\theta}$, the resolvent $R(\lambda, A)$ extends to an element in $\mathscr{L}(L^r(\Omega,w))$ and satisfies*

$$\|\lambda R(\lambda, A)\|_{\mathscr{L}(L^r(\Omega,w))} \leqslant \phi([w]_{A_r}).$$

Then A has maximal L^p-regularity on $L^q(\Omega, w)$ for any $p, q \in (1,\infty)$ and any $w \in A_q$.

The above result can be seen as another way to answer to Brezis question on L^r-spaces: if the semigroup is bounded analytic on all $L^r(\Omega,w)$ with $w \in A_r$, then one has maximal L^p-regularity. Theorem 17.5.1 was used by Haller, Heck, and Hieber [2003] to prove maximal L^p-regularity for a large class of elliptic systems on $\mathbb{R}^d$.

Maximal L^p-regularity for linear operators arising in fluid dynamics

Fröhlich [2007] used Theorem 17.5.1 to prove maximal L^p-regularity for the Stokes operator on $C^{1,1}$-domains, and extends the papers Farwig and Sohr [1997], Fröhlich [2003] where $\mathbb{R}^d$ and the half-space were considered, respectively. Currently, such results under the weakest conditions on the domain are due to Kunstmann and Weis [2017], and state that for any $\theta \in (0,\pi/2)$ the Stokes operator has a bounded H^∞-calculus of angle $< \pi/2$ (and thus maximal L^p-regularity) on bounded Lipschitz domains for all $q \in (1,\infty)$ satisfying $|\frac{1}{q} - \frac{1}{2}| < \frac{1}{2d} + \varepsilon$ Here $\varepsilon > 0$ only depends on the Lipschitz domain, and the angle θ. The condition on q comes from the sectoriality result proved by Shen [2012].

There is a large number of maximal L^p-regularity (or even H^∞-calculus) results for operators of Stokes type arising in fluid dynamics. Some of the recent ones include Choudhury, Hussein, and Tolksdorf [2018], Giga, Gries, Hieber, Hussein, and Kashiwabara [2017], Hieber and Prüss [2020], Prüss [2018], Shibata [2020], Shibata and Shimizu [2008], Simonett and Wilke [2022a], Tolksdorf [2018, 2020], Tolksdorf and Watanabe [2020], Watanabe [2022, 2023]. Applications of maximal L^p-regularity to equations of Navier–Stokes type are too numerous to list here, but some are mentioned in Chapter 18.

Inhomogeneous boundary conditions

A Fourier analytic approach to maximal L^p-regularity for inhomogeneous boundary value problems in L^q was developed by Weidemaier [2002], Denk, Hieber, and Prüss [2007]. Extensions to boundary conditions of relaxation type were obtained in Denk, Prüss, and Zacher [2008a] in the case $p = q$. Weight in time were added in Meyries and Schnaubelt [2012a], and the spatial weights $\operatorname{dist}(\cdot, \partial\Omega)^\gamma$ were added in Lindemulder [2020]. A further generalisation was obtained in Hummel and Lindemulder [2022], where the solution space is taken as $L^p(0, T; \mathcal{A}^{s,q})$ and $\mathcal{A}^{s,q}$ is a weighted Bessel potential, Besov, or Triebel–Lizorkin space and the smoothness $s \geqslant 2$ is restricted by the order of the boundary operators, and the integrability parameter q is in $(1, \infty)$. Weights in space have influence on which regularity is needed on the boundary data in a similar way as weights in time do. A maximal L^p-regularity theory for the heat equation with inhomogeneous boundary conditions and power weights outside the A_p-range was obtained by Lindemulder and Veraar [2020]. It is not well-understood which weights one can take for more general equations. Another type of generalisation was obtained by Dong and Gallarati [2018, 2020] who consider the setting of Denk, Hieber, and Prüss [2007] but only require VMO conditions in time and space.

Maximal L^p-regularity on $\mathbb{R}$ vs. $\mathbb{R}_+$

In the study of parabolic PDE the equation $u' + Au = f$ is often considered on the full real line $\mathbb{R}$ instead of $\mathbb{R}_+$; see for instance the monograph Krylov [2008] and the papers Dong and Kim [2011, 2018].

For problems in which A depends on time it is often easier to first deal with the full real line, since then Fourier analytic techniques are available. Such techniques also plays a role in Auscher and Egert [2016, 2023], Dier and Zacher [2017] in the variational setting of Lions where the operator A is given by a time-dependent bilinear form.

For evolution equation with inhomogeneous boundary conditions, it is often simpler to first deal with the (inhomogeneous) boundary conditions in space, without having to deal with compatibility conditions coming from the initial conditions by working on the time line $\mathbb{R}$. The initial conditions are then introduced in the last step. This approach is for instance followed in Lindemulder and Veraar [2020] for the heat equation, and in Hummel and Lindemulder [2022] for elliptic systems under Lopatinskii-Shapiro conditions.

Theorem 17.3.29 is due to Mielke [1987]. The equivalence between maximal L^p-regularity on $\mathbb{R}$ and $\mathbb{R}_+$ is due to Hummel and Lindemulder [2022]. Proposition 17.3.31, Theorem 17.3.32, and Proposition 17.3.33 are extensions of their results. It would be interesting to find versions of these results for time-dependent operators A. Proposition 17.3.34 and Theorem 17.3.38 appear in Arendt and Duelli [2006], where also applications to quasilinear equations are given. Proposition 17.3.37 and Theorem 17.3.35 are standard variants of their corresponding results on $\mathbb{R}_+$.

Maximal regularity on interpolation spaces

Theorem 17.3.19 is based on the ideas in Kalton and Kucherenko [2010]. With a different proof, Corollary 17.3.20 is due to Da Prato and Grisvard [1975], where the result is stated for analytic semigroup which are not necessarily continuous. A direct proof for $1 \leqslant p < \infty$ can be found in Prüss and Simonett [2016]. The proof for $p = \infty$ is taken from Lunardi [1995]. Examples of operators with maximal L^1-regularity on real interpolation spaces of the form $(X, \mathsf{D}(A))_{\theta,1}$ can be obtained from Corollary 17.3.20 or by more direct arguments. This often leads to maximal regularity results for operators acting the Besov spaces $B^s_{q,1}$ or their homogeneous counterparts $\dot{B}^s_{q,1}$. These spaces appear naturally in fluid dynamics; see, e.g., Danchin, Hieber, Mucha, and Tolksdorf [2020], Ogawa and Shimizu [2016, 2021], Xu [2022], and references therein.

Our proof of Corollary 17.3.20 uses the fact that $L^p(I; (X, \mathsf{D}(A))_{\theta,p})$ equals the real interpolation space $(L^p(I; X), L^p(I; \mathsf{D}(A)))_{\theta,p}$ up to an equivalent norm. Alternative results can be obtained if one replaces $L^p(I; X)$ by a real interpolation space such as the Besov space $B^s_{p,q}(I; X)$, or even by other spaces such as the Triebel–Lizorkin spaces $F^s_{p,q}(I; X)$. To prove maximal regularity results in such spaces at least two methods are available which will be discussed briefly. We will only provide the details in the case $I = \mathbb{R}$ in order to avoid difficulties with extension operators and initial value conditions.

The first method is due to Kalton and Kucherenko [2010], and is similar as in Corollary 17.3.20. However, this time the absolute calculus is used for $Du = u'$ instead of A. It follows from Proposition 14.4.17 that D is a closed operator on $B^s_{p,q}(\mathbb{R}; X)$ with domain $B^{s+1}_{p,q}(\mathbb{R}; X)$ for any $s \in \mathbb{R}$, $p, q \in [1, \infty]$. An application of the Mihlin multiplier theorem (Theorem 14.4.16) for Besov spaces shows that for all $\mu > 0$ the operator $\mu + D$ is an invertible sectorial operator of angle $\leqslant \pi/2$. Moreover, by Theorem 14.4.31, $B^s_{p,q}(\mathbb{R}; X) = (B^{s_0}_{p,q}(\mathbb{R}; X), B^{s_1}_{p,q}(\mathbb{R}; X))_{\theta,q}$ if $(1-\theta)s_0 + \theta s_1 = s$. Therefore, Theorem 16.3.14 implies that D has an absolute function calculus of angle $\leqslant \pi/2$. From Theorem 16.3.20 we see that, at least for $p, q < \infty$, for any densely defined sectorial operator A such that $A - \mu$ is also sectorial, the operator $A + D = (A - \mu) + (D + \mu)$ is invertible as an operator from $B^s_{p,q}(\mathbb{R}; \mathsf{D}(A)) \cap B^{s+1}_{p,q}(\mathbb{R}; X)$. This implies a maximal regularity result for A in $B^s_{p,q}(\mathbb{R}; X)$.

The second method is due to Amann [1997], Girardi and Weis [2003a], and Bu and Kim [2005], and uses a Fourier multiplier Mihlin's multiplier theorem for Besov and Triebel Lizorkin spaces (Theorems 14.4.16 and 14.6.11 respectively). Indeed, one can repeat the argument of Theorem 17.3.1(2), except that one needs $0 \in \varrho(A)$ to avoid problems with the Mihlin condition at zero in Theorems 14.4.16 and 14.6.11. An extension of these results to Besov–Orlicz spaces was recently given in Ondreját and Veraar [2020], where it was used to study temporal regularity of stochastic convolutions.

Section 17.4

The special case of $\lambda = 0$ and $\alpha = 1$ of Theorem 17.4.1 is due to Coulhon and Lamberton [1986], where maximal L^p-regularity of $(-\Delta)^{1/2}$ on $L^p(\mathbb{R}; X)$ was characterised in terms of the UMD property for X. A quantitative version appears in Hytönen [2015], where it was shown that

$$\frac{1}{2}\max\{\beta^{\mathbb{R}}_{p,X}, \hbar_{p,X}\} \leqslant M^{\rm reg}_{p,(-\Delta)^{1/2}}(\mathbb{R}_+) \leqslant \beta^{\mathbb{R}}_{p,X} + \hbar_{p,X}. \tag{17.77}$$

Here, $\beta^{\mathbb{R}}_{p,X}$ denotes the (real) UMD constant of X and $\hbar_{p,X}$ denotes the norm of the Hilbert transform. In Theorem 17.4.1 we obtained the lower bound $\frac{1}{2}\beta^{\mathbb{R}}_{p,X} \leqslant M^{\rm reg}_{p,\lambda+(-\Delta)^{\alpha/2}}(\mathbb{R}_+)$ for all dimensions and all $\lambda \geqslant 0$ and $\alpha > 0$ as in the statement of the theorem. It is unknown whether the second estimate of (17.77) extends to this setting. For this one would need an extension of the results in Geiss, Montgomery-Smith, and Saksman [2010] to multipliers $m : \mathbb{R}^{d+1} \to \mathbb{C}$ which satisfy the anisotropic homogeneity condition $m(t\eta, t^{1/\alpha}\xi) = m(\eta, \xi)$ for $\eta \in \mathbb{R}$, $\xi \in \mathbb{R}^d$ and $t > 0$. Dimension free upper estimates for $M^{\rm reg}_{p,-\Delta}(\mathbb{R}_+)$ can be obtained from Krylov and Priola [2017] in the scalar case, which provides another potential method to extend (17.77) to other values of α.

For non-UMD spaces X, Theorem 17.4.1 provides examples of sectorial operators without maximal L^p-regularity. For Banach function spaces the following improvement holds:

Theorem 17.5.2 (Kalton, Lorist, and Weis [2023]). *For an order continuous Banach function space X the following assertions are equivalent:*

(1) $\{\lambda(\lambda - \Delta)^{-1} : \lambda > 0\}$ *is R-bounded on* $L^p(\mathbb{R}^d; X)$*;*
(2) X *is a UMD space.*

Combining their proof with the dilation argument in Theorem 17.4.1, one can show that the same equivalence holds when the negative Laplacian $-\Delta$ is replaced by $A = \lambda + (-\Delta)^{\alpha/2}$. Notice that condition (1) is equivalent to R-sectoriality. It is unknown whether Theorem 17.5.2 holds for general Banach spaces X.

Baillon's result on maximal C-regularity

Theorem 17.4.4 is the main result of Baillon [1980]. Our presentation combines ideas of Baillon [1980] and Eberhardt and Greiner [1992] (in the latter paper we do not understand how the closed graph theorem is applied, for the space of piecewise continuous functions is not complete). Example 17.4.6(1), which shows that the conclusion of Baillon's theorem fails for c_0, is taken from Dore [2000] and seems to be an example due to Kato (See the MathSciNet review of the paper Baillon [1980]). Corollary 17.4.5 and Examples 17.4.6(2) and 17.4.7 are due to Guerre-Delabrière [1995].

Before the statement of Corollary 17.4.5, we mentioned that the duals of $L^\infty(S)$ and certain $C(K)$-spaces have cotype 2. This follows from the fact that these duals are abstract L^1-spaces and thus isometric to an L^1-space, see

Lindenstrauss and Tzafriri [1979, Theorem 1.b.2 and the proof of Theorem 1.b.6].

A simple proof of the result, due to Sobczyk [1941], that isomorphic copies of c_0 in separable Banach spaces are always complemented can be found in Lindenstrauss and Tzafriri [1977, Theorem 2.f.5]. That X contains a complemented copy of ℓ^1 if X^* contains a copy of c_0 is due to Bessaga and Pełczyński [1958]; see also Lindenstrauss and Tzafriri [1977, Proposition 2.e.8]. Variations of Baillon's theorem and connections to control theory can be found in Jacob, Schwenninger, and Wintermayr [2022], where it is shown that if $-A$ generates an analytic C_0-semigroup S, then the boundedness and well-definedness of the operator $\Phi_T : L^\infty(0,T;X) \to X$ given by $\Phi_T(f) = AS * f(T)$ characterises boundedness of A. This shows that one cannot replace the essential supremum by a supremum in the definition of maximal L^∞-regularity. From the proof of Corollary 17.3.20 for $p = \infty$ one sees that in certain situations Φ_T is bounded when S is not strongly continuous on $(X, \mathsf{D}(A))_{\theta,\infty}$.

Counterexamples to maximal L^p-regularity

After partial results by several authors, Brezis's question was finally solved to the negative in Kalton and Lancien [2000], who constructed counterexamples in separable Banach lattices not isomorphic to a Hilbert space, and in particular in L^q-spaces with $1 < q < \infty$, $q \neq 2$. This paper implies part (1) of Theorem 17.4.8. More general counterexamples were constructed in Kalton and Lancien [2002]. The construction presented here is due to Fackler [2014, 2016] and has the additional merits of being explicit and solving to the negative the extrapolation problem for maximal L^p-regularity; in particular, this work implies part (2) of Theorem 17.4.8. Our proof of Theorem 17.4.11 is a simplified version of the proof in Fackler [2016]. The result in this paper is stated with 'non-homogeneous' replaced by 'non-symmetric', and then invokes a result from Singer [1970] to reduce matters to the non-homogeneous setting up to a permutation of one of the two subsequences. To deduce from this the isomorphic characterisation of ℓ^2 mentioned after the statement of Theorem 17.4.11, one uses a result of Lindenstrauss and Zippin [1969] (see also McArthur [1972, Theorem 7.6], Fackler [2016, Proposition 5.5]) which states that if X is a Banach space with an unconditional basis, and if X is not isomorphic to c_0, ℓ^1, or ℓ^2, then X has a normalised unconditional, non-symmetric basis. The spaces c_0 and ℓ^1 can be excluded, because on these spaces it is possible to give simple direct constructions of sectorial operators that are not R-sectorial (Kalton and Lancien [2000], Fackler [2016, Propositions 4.2 and 4.3]). For more on the theory of unconditional bases in Banach spaces the reader may consult Lindenstrauss and Tzafriri [1977], Singer [1970]. Our presentation of Proposition 17.4.12 follows Lindenstrauss and Tzafriri [1977].

In connection with Theorem 17.4.11 it is of interest to observe that the diagonal operator A featuring in the proof is R-sectorial (respectively, almost R-sectorial) if and only if the partial sum projections (respectively, the coordinate projections) of the basis on which it acts are R-bounded; see Kalton, Lorist, and Weis [2023, Proposition 6.1.3].

Several counterexamples to maximal L^p-regularity were presented by Le Merdy [1999] in case $X = L^1(\mathbb{T})$ and $X = C(\mathbb{T})$. These are connected to Examples 17.4.2 and 17.4.3 on $L^1(\mathbb{R}^d)$ and $C_0(\mathbb{R}^d)$ which are motivated by this paper. The mentioned paper also contains a counterexample in $X = \mathscr{K}(\ell^2)$, the space of compact operators on ℓ^2, of a densely defined invertible sectorial operator A of angle $\omega(A) < \frac{1}{2}\pi$ admitting bounded imaginary powers with $\|A^{it}\| = 1$ for all $t \in \mathbb{R}$ but without maximal L^p-regularity on bounded intervals for any $1 < p < \infty$. Some further counterexamples and standard constructions for sectorial and Ritt operators can be found in Arnold and Le Merdy [2019].

Abstract results showing that a large class of weakly compact C_0-semigroups on $C(K)$ and $L^1(\Omega)$ fail to be R-bounded (and thus do not have maximal L^p-regularity) can be found in Hoffmann, Kalton, and Kucherenko [2004]. Further extension are given in Kalton and Kucherenko [2008], Kucherenko and Weis [2005].

Maximal L^p-regularity for non-autonomous equations

Linear non-autonomous evolution equations are of the form

$$\begin{cases} u'(t) + A(t)u(t) &= f(t), \quad t \in I, \\ u(0) &= x, \end{cases}$$

where $(A(t))_{t\in I}$ is a family of unbounded operators on X. One can define maximal L^p-regularity in a similar way as in the autonomous case. The theory distinguishes between the setting where the domains $\mathsf{D}(A(t))$ are constant in time and the setting in which they varies in time. Constant domains typically appear for equations in non-divergence form without boundary, or with Dirichlet boundary conditions. The varying domain case arises in the case of time-dependent boundary conditions and is much harder to treat. We will discuss these cases separately.

Constant domains

Let X_0 and X_1 be Banach spaces with a continuous and dense embedding $X_1 \hookrightarrow X_0$, and suppose that $A : I \to \mathscr{L}(X_1, X_0)$ is a mapping with the property that the operators $A(t)$ are sectorial of angle $\omega(A(t)) < \pi/2$, with uniform estimates with respect to $t \in I$. There are several approaches to evolution equations governed by time-dependent operators of this form. Standard references are Amann [1995], Lunardi [1995], Tanabe [1979], where it is assumed that $A \in C^\alpha(\overline{I}; \mathscr{L}(X_1, X_0))$ for some $\alpha > 0$.

Maximal L^p-regularity for continuous time-dependent operators $A : I \to \mathscr{L}(X_1, X_0)$ was studied in details by Prüss and Schnaubelt [2001], where also the corresponding evolution family generated by $(A(t))_{t\in I}$ is studied. Theorem 17.2.51 can be seen as the weighted analogue of their result. on maximal L^p-regularity. They also show that the maximal L^p-regularity of $A(t_0)$ for each fixed $t_0 \in \overline{I}$ is necessary for maximal L^p-regularity for the time-dependent case. The continuity condition on A was further weakened in Amann [2004] and Arendt, Chill, Fornaro, and Poupaud [2007].

It is an open problem to find a characterisation (or a sharp sufficient condition) on A of maximal L^p-regularity in the case A is only measurable in time (in the uniform operator topology). There are counterexamples available in the literature, which show the limits of possible general theory. One counterexample due to Pierre and Schmitt [1997] goes as follows:

$$\begin{cases} u'(t,x) &= a(t,x)\Delta u(t,x) + f(t,x), \quad t \in [0,T], x \in B, \\ u(t,x) &= 0 \qquad\qquad\qquad\qquad\qquad\quad t \in [0,T], x \in \partial B, \\ u(0) &= 0, \end{cases} \tag{17.78}$$

where B is the unit ball in $\mathbb{R}^d$. It follows from their results that there exist constants $m, M > 0$ and a measurable $a : [0,T] \times \Omega \to \mathbb{R}$ with $m \leqslant a \leqslant M$ on $[0,T] \times B$, such that for $q > 1$ close to 1 there is no maximal L^p regularity for (17.78) on $X_0 = L^q(B)$. Another counterexample for a similar equations but on $\Omega = \mathbb{R}$, can be obtained from Krylov [2016], where also the range of q is analysed for which maximal L^q-regularity in $L^q(\mathbb{R})$ does hold.

For differential operators with coefficients that are measurable in time and VMO in space (with uniform estimates in time), Krylov [2008, Theorem 4.3.7 and Chapter 7] shows that the usual conditions such as uniform ellipticity are sufficient for maximal L^p-regularity in $L^q(\mathbb{R}^d)$ with $p \geqslant q$. The proof is based on several sophisticated maximal function techniques. By a duality argument, the condition $p \geqslant q$ can be removed in case of constant coefficients in space. Generalisations to higher order elliptic systems (and certain boundary conditions) have been obtained in Dong and Kim [2011] for $p = q$.

A more abstract operator-theoretic approach was taken by Gallarati and Veraar [2017a,b], where arbitrary $p, q \in (1,\infty)$ were considered and certain R-boundedness conditions on the evolution family and commutator conditions were assumed. In the special case of elliptic operators with coefficients that are measurable in time, they use their abstract results to prove maximal L^p-regularity with A_p-weights in time (and possibly also in space) in $L^q(\mathbb{R}^d)$. Afterwards they use the method of "freezing the coefficients" to extend to coefficients which are measurable in time and continuous in space (uniformly in time). Using Rubio de Francia's extrapolation theory (see Theorem J.2.1) they then obtain maximal L^p-regularity on $L^q(\mathbb{R}^d)$ for all $p, q \in (1,\infty)$. Dong and Kim [2018] unified the above results and proved estimates involving parabolic weights and coefficients which are measurable in time and VMO in space. In Dong and Krylov [2019] and Krylov [2020], Rubio de Francia's extrapolation theory was used to obtain regularity estimates in mixed L^p-norms for fully non-linear elliptic and parabolic equations with measurable dependence in time and VMO in space.

Non-constant domains and the Acquistapace–Terreni conditions

Standard references for parabolic equations with time-dependent domains are Acquistapace and Terreni [1987, 1992], Yagi [1991, 1990] and the monographs Amann [1995], Tanabe [1979, 1997], Yagi [2010]. An extensive theory of maximal C^α-regularity (weighted and unweighted) has been developed under the conditions introduced in Acquistapace and Terreni [1987]. A special case of their assumptions read as follows:

- there exists a $\omega \in (0, \pi/2)$ such that every $A(t)$ is ω-sectorial, with uniform estimates in $t \in I$, and $0 \in \varrho(A)$;
- there exist $\omega \in (0, \pi/2)$, $L \geqslant 0$, and $\mu, \nu \in (0, 1]$ with $\mu + \nu > 1$, such that for all $\omega \leqslant |\arg(\lambda)| \leqslant \pi$ one has

$$|\lambda|^\nu \|A(t)R(\lambda, A(t))(A(t)^{-1} - A(s)^{-1})\| \leqslant L|t - s|^\mu.$$

The second condition can be understood as a μ-Hölder continuity condition.

Under the above conditions, Hieber and Monniaux [2000a,b] derive maximal L^p-regularity via kernel bounds, and Štrkalj [2000], Portal and Štrkalj [2006] derive maximal L^p-regularity via R-sectoriality. Extensions to maximal L^p-regularity on $\mathbb{R}$ are discussed in Di Giorgio, Lunardi, and Schnaubelt [2005b].

Fackler [2018] proved maximal L^p-regularity in case under fractional Sobolev conditions on A, assuming a further technical condition on certain intermediate spaces between X and $\mathsf{D}(A(t))$. When specialised to the Hilbert space setting, this paper provides sufficient conditions for maximal L^2-regularity, which is related to Lions's problem for non-autonomous evolution equations (see Lions [1961, p. 68]). The reader is referred to the survey Arendt, Dier, and Fackler [2017] for details on recent progress on this problem, and for remaining open questions.

Maximal L^p-regularity results for elliptic operators in divergence form under mixed smoothness conditions in space and time can be found Dier and Zacher [2017] for $X = L^2$, and more recently in Bechtel and Gabel [2022] for $X = L^q$.

Maximal γ-regularity

In this paragraph we briefly discuss the Gaussian counterpart of maximal L^p-regularity, namely, *maximal γ-regularity*, which was introduced and studied in Van Neerven, Veraar, and Weis [2015a]. It turns out that, in any Banach space, a densely defined sectorial operator A has maximal γ-regularity if and only it is γ-sectorial. Combining this result with Theorem 17.5.4, as an immediate corollary one obtains that, in UMD Banach spaces, the notions of maximal

L^p-regularity and maximal γ-regularity are equivalent. This points the way to using maximal regularity techniques beyond the UMD setting.

In what follows, the reader is assumed to be familiar with the basic theory of γ-radonifying operators as presented in Section 9.1, whose notation and terminology we follow here. As in the definition of maximal L^p-regularity, our starting point is the inhomogeneous abstract Cauchy problem

$$\begin{cases} u'(t) + Au(t) &= f(t), \quad t \in \overline{I}, \\ u(0) &= 0, \end{cases} \tag{ACP$_0$}$$

where either $I = (0,T)$ or $I = (0,\infty) = \mathbb{R}_+$.

Definition 17.5.3 (Maximal γ-regularity). *A linear operator A has* maximal γ-regularity on *I if for every $f \in C_c^1(I) \otimes \mathsf{D}(A)$ there exists a unique strongly measurable function $u : I \to X$ with the following properties:*

(i) *u takes values in $\mathsf{D}(A)$ almost everywhere and Au belongs to $\gamma(L^2(I);X)$;*
(ii) *u solves the integrated version of (ACP$_0$), i.e., for almost all $t \in I$ we have*

$$u(t) + \int_0^t Au(s)\,\mathrm{d}s = \int_0^t f(s)\,\mathrm{d}s.$$

(iii) *we have the estimate*

$$\|Au\|_{\gamma(L^2(I),X)} \leqslant C\|f\|_{\gamma(L^2(I);X)},$$

A strongly measurable function $u : I \to X$ is called an γ-solution *of* (ACP$_0$) *associated with a given $f \in \gamma(L^2(I),X)$ if it satisfies conditions* (i) *and* (ii).

In (ii), for an operator $f \in \gamma(L^2(I);X)$ and a measurable subset $F \subseteq I$ of finite measure we define

$$\int_F f\,\mathrm{d}\mu := f(\mathbf{1}_F).$$

Note that

$$\Big\|\int_s^t f(r)\,\mathrm{d}r\Big\| \leqslant (t-s)^{1/2}\|f\|_{\gamma(a,b;X)}, \quad a \leqslant s \leqslant t \leqslant b.$$

It follows that $t \mapsto \int_a^t f(s)\,\mathrm{d}s \in C([a,b];X)$ and

$$\Big\|t \mapsto \int_a^t f(s)\,\mathrm{d}s\Big\|_{C([a,b];X)} \leqslant (b-a)^{1/2}\|f\|_{\gamma(a,b;X)}.$$

It follows from Theorem 9.6.1 that $t \mapsto \int_a^t f(s)\,\mathrm{d}s$ also belongs to $\gamma(a,b;X)$. Indeed, it is trivial to check that for functions $f \in L^2(a,b) \otimes X$ this mapping coincides with the γ-extension of the indefinite integral, viewed as a bounded operator from $L^2(a,b)$ into itself. Since $L^2(a,b)\otimes X$ is dense in $\gamma(a,b;X)$ this proves the claim. The theorem also gives the norm estimate

$$\Big\| t\mapsto \int_a^t f(s)\,\mathrm{d}s\Big\|_{\gamma(a,b;X)} \leqslant (b-a)^{1/2}\|f\|_{\gamma(a,b;X)},$$

observing that the indefinite integral, as an operator on $L^2(a,b)$, has norm $(b-a)^{1/2}$.

In analogy with Theorem 17.2.15, whose proof can be repeated almost *verbatim*, we have:

Theorem 17.5.4. *If A is a closed linear operator with maximal γ-regularity on $(0,T)$, then $-A$ generates an analytic C_0-semigroup. If A has maximal γ-regularity on $\mathbb{R}_+$, this semigroup is bounded.*

Let A be a densely defined γ-sectorial operator, and let $(S(t))_{t\geqslant 0}$ be the bounded analytic C_0-semigroup generated by $-A$. By the preceding remarks, for any $f\in\gamma(L^2(\mathbb{R}_+);X)$ we may define the *mild solution* to (ACP$_0$) by

$$u_f(t) := \int_0^t S(t-s)f(s)\,\mathrm{d}s, \quad t\geqslant 0, \quad t\in\overline{I}.$$

Theorem 17.5.5 (Maximal γ-regularity for the problem (ACP$_0$)). *Let A be densely defined γ-sectorial on a Banach space X with $\omega_\gamma(A)<\frac{1}{2}\pi$, and let $(S(t))_{t\geqslant 0}$ be the semigroup on X generated by $-A$. Then A has maximal γ-regularity on I if and only if for every $f\in\gamma(L^2(I),X)$ the mild solution u_f takes values in $\mathsf{D}(A)$ almost everywhere and $Au\in\gamma(L^2(I);X)$. In this situation, mild solutions and γ-solutions agree.*

The following theorem shows that the UMD assumption of Theorem 17.3.1 can be lifted if maximal L^p-regularity is replaced by maximal γ-regularity.

Theorem 17.5.6 (Characterising maximal γ-regularity). *Let A be a densely defined sectorial operator on a Banach space X. Then:*

(1) *if A is γ-sectorial and $\omega_\gamma(A)<\frac{1}{2}\pi$, then A has maximal γ-regularity on $\mathbb{R}_+$;*
(2) *if A has maximal γ-regularity on $\mathbb{R}_+$, then A is γ-sectorial;*
(3) *if X is a UMD Banach space and $0\in\varrho(A)$, then A has maximal γ-regularity on $\mathbb{R}_+$ if and only if A has maximal L^p-regularity for some/all $p\in(1,\infty)$.*

The final result gives a sufficient condition for pointwise regularity of γ-solutions with a uniform bound in time.

Theorem 17.5.7. *Suppose that A is a densely defined γ-sectorial operator with a bounded H^∞-calculus of angle $\omega_{H^\infty}(A)<\frac{1}{2}\pi$ on a Banach space X with finite cotype. If $0\in\varrho(A)$, then for all $f\in\gamma(L^2(\mathbb{R}_+),X)$ the associated*

γ-solution $u(t)$ takes values in $\mathsf{D}(A^{1/2})$ for all $t \geqslant 0$, the resulting function $u_f : \overline{\mathbb{R}}_+ \to \mathsf{D}(A^{1/2})$ is uniformly continuous and satisfies

$$\sup_{t\in\mathbb{R}_+} \|u(t)\|_{\mathsf{D}(A^{1/2}))} \leqslant C\|f\|_{\gamma(L^2(\mathbb{R}_+),X)},$$

for some constant C independent of f.

Theorem 17.5.7 indicates an important difference with the theory of maximal L^p-regularity consists in terms of the trace space involved. Whereas maximal L^p-regularity allows for the treatment of non-linear problems with initial values in the space real interpolation space $(X, \mathsf{D}(A))_{1-\frac{1}{p},p}$, in the presence of maximal γ-regularity initial values in the complex interpolation space $[X, \mathsf{D}(A)]_{\frac{1}{2}}$ can be allowed. For a fuller discussion of this point the reader is referred to Van Neerven, Veraar, and Weis [2015a].

Reversing the integrability in time and space

In the special case that $X = L^q(S)$, maximal γ-regularity is equivalent to maximal regularity estimates in the spaces $L^q(S; L^2(I))$. Thus compared to the results of this chapter the order of time and space are interchanged. In the special setting of $X = L^q(S)$, one can even study maximal regularity estimates in the spaces $L^q(S; L^p(I))$. A detailed study of the latter is given in Antoni [2017]. In some cases one can also obtain such bounds via $L^p(S; L^p(I))$-estimates if one can add A_p-weights in S, and afterwards apply Rubio de Francia's extrapolation theory (see Theorem J.2.1).

Another subtle way to interchange time and space is to formulate maximal regularity in so-called tent spaces. The definition and some historical details on these spaces can be found in the notes of Chapter 10. For details on maximal regularity in tent spaces the reader is referred to Auscher, Monniaux, and Portal [2012b], Auscher, Kriegler, Monniaux, and Portal [2012a], Auscher, Monniaux, and Portal [2019].

Miscellaneous topics

The discrete-time version of maximal L^p-regularity, usually referred to as maximal ℓ^p-regularity, was studied in detail in Blunck [2001a,b]. In numerical analysis, it can be used to show stability and convergence of numerical schemes. There is a large body of subsequent work on this topic; the reader is referred to Ashyralyev, Piskarev, and Weis [2002], Akrivis, Li, and Lubich [2017], Blunck and Kunstmann [2002], Geissert [2006], Kalton and Portal [2008], Kemmochi [2016], Kovács, Li, and Lubich [2016], Lizama [2015], Portal [2003, 2005], and references therein. In some of these works, connections with the theory of Ritt operators play a role. A discussion on these operators is contained in the Notes of Chapter 10.

Maximal L^p-regularity for evolution equations which are of second order equations in time, are introduced from a modern point of view in Chill and

Srivastava [2005]. In this paper, among other things, characterisations of maximal L^p-regularity in terms of Fourier multipliers are given and applications to quasi-linear evolution equations are presented. Subsequent work on second order equations includes Arendt, Chill, Fornaro, and Poupaud [2007], Batty, Chill, and Srivastava [2008], Chill and Srivastava [2008], Denk and Schnaubelt [2015], Poblete [2009].

Parabolic mixed order systems do not fall within the setting of analytic semigroup theory. Maximal L^p-regularity for such types of systems can still be developed via operator-valued Fourier multiplier theory; this was done by Denk, Saal, and Seiler [2008b], Denk and Volevich [2008], Denk and Kaip [2013].

In Zacher [2005], R-sectoriality was used to prove maximal L^p-regularity for Volterra equations. The weighted setting in time was subsequently introduced in Prüss [2019].

18

Nonlinear parabolic evolution equations in critical spaces

As we have seen in the preceding sections, in the context of inhomogeneous linear evolution equations, maximal regularity enables one to set up an isomorphism between the space of data (initial value and inhomogeneity) and the solution space. In the present section we will show how this idea can be used to study to non-linear evolution equations. Specifically, we consider a class of quasi-linear evolution equations of the form

$$\begin{cases} u'(t) + A(u(t))u(t) &= F(u(t)), \quad t \in (0,T), \\ u(0) &= u_0. \end{cases}$$

The setting is as follows. We are given a pair of Banach spaces (X_0, X_1) along with a continuous embedding $X_1 \hookrightarrow X_0$. The initial value u_0 is taken in a suitable interpolation space of X_0 and X_1, and for each v_0 in some neighbourhood Y of u_0 the operator $A(v_0)$ is a linear operator in X_0 with domain $\mathsf{D}(A(v_0)) = X_1$. As such, we interpret $A(v_0)$ as a bounded linear operator in $\mathscr{L}(X_1, X_0)$. The mapping F is defined on an interpolation space of X_0 and X_1, takes values in X_0, and is assumed to satisfy suitable local Lipschitz conditions; the precise assumptions will be formulated later. Our aim is to present several local well-posedness results, and to discuss a blow-up criterion which can be used to derive global well-posedness.

Before we start with this programme, we first explain the difference between semi-linear and quasi-linear evolution equations. In the *quasi-linear case*, the typical situation is that

$$A : Y \to \mathscr{L}(X_1, X_0) \quad \text{and} \quad F : Z \to X_0$$

are Lipschitz continuous on bounded subsets of Y and Z, where Y and Z are (subsets of) suitable interpolation spaces between X_0 and X_1. In the *semi-linear case* one typically has that

$$A \in \mathscr{L}(X_1, X_0) \quad \text{and} \quad F : Z \to X_0,$$

T. Hytönen et al., *Analysis in Banach Spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics 76,
https://doi.org/10.1007/978-3-031-46598-7_8

where A is a fixed operator in $\mathscr{L}(X_1, X_0)$ and F is a locally Lipschitz continuous mapping on bounded subsets of Z, where Z is as before. Clearly, every semi-linear equation is quasi-linear, but the converse is not true. In principle one can allow A and F to be also time-dependent, but in order to keep the presentation within reasonable bound we will not consider this additional level of generality.

When analysing evolution equations with maximal L^p-regularity methods, one usually takes Y equal to (or a subset of) the real interpolation space $(X_0, X_1)_{1-\frac{1}{p},p}$, at least in the absence of weights. The reason for taking Y of this form is that one has a continuous embedding (see Corollary L.4.6)

$$L^p(0,T;X_1) \cap W^{1,p}(0,T;X_0) \hookrightarrow C([0,T];(X_0,X_1)_{1-\frac{1}{p},p}). \tag{18.1}$$

The space on the left-hand side is the usual space in which solutions lie when maximal L^p-regularity techniques are applicable. For the space Z one can take either take Y, or more generally $(X_0, X_1)_{\beta,1}$ with $\beta \in [1-\frac{1}{p}, 1)$, the latter requiring polynomial growth restrictions on F. In practice, we often split F into two parts $F = F_{\mathrm{Tr}} + F_{\mathrm{c}}$, where

$$F_{\mathrm{Tr}} : Y \to X_0, \quad F_{\mathrm{c}} : Z \to X_0 \tag{18.2}$$

with Y and Z as before. Here the subscript Tr stands for *trace space* and c stands for *critical*. The word *critical* is also used in the title of the chapter. In Section 18.2 we will give a definition of a criticality using only evolution equation terminology. Surprisingly, this often coincides with criticality from a PDE perspective.

The following simple example explains why the additional flexibility in choosing Z may be expected to be useful.

Example 18.0.1. On $\mathbb{R}^d$ consider the equation

$$\begin{cases} \partial_t u - a(u)\Delta u &= -u^3, \\ u(0) &= u_0, \end{cases} \tag{18.3}$$

where $a : \mathbb{R} \to [0,\infty)$ is a given locally Lipschitz continuous function. If a is non-constant, then (18.3) leads to a quasi-linear evolution equation, and if a is constant it leads to a semi-linear evolution equation. In both cases, the spaces X_0 and X_1 need to be chosen as function spaces relative to which the definitions $A(u)v := a(u)\Delta v$ and $F(u) := -u^3$ admit meaningful interpretations. A possible choice is to take

$$X_0 := L^q(\mathbb{R}^d), \quad X_1 := W^{2,q}(\mathbb{R}^d).$$

With these choices, $Y := (X_0, X_1)_{1-\frac{1}{p},p}$ equals the Besov space $B^{2-\frac{2}{p}}_{q,p}(\mathbb{R}^d)$ (see Theorem 14.4.31). If we assume that $2 - \frac{2}{p} - \frac{d}{q} > 0$, then we have the continuous embedding (see Corollary 14.4.27)

$$B^{2-\frac{2}{p}}_{q,p}(\mathbb{R}^d) \hookrightarrow C_{\rm b}(\mathbb{R}^d).$$

The space Y then consists of bounded continuous functions, and consequently for $u \in Y$ we can interpret $a(u)$ as a bounded continuous function. The operator $A(u)$ is then well defined as an element of $\mathscr{L}(X_1, X_0)$. On the other hand, since $F(u) = -u^3$ belongs to X_0 if and only if $u \in L^{3q}(\mathbb{R}^d)$, for $u \in Y$ we can interpret $F(u)$ as an element of X_0 as soon as we have a continuous embedding

$$B^{2-\frac{2}{p}}_{q,p}(\mathbb{R}^d) \hookrightarrow L^{3q}(\mathbb{R}^d).$$

This embedding holds under the (much weaker) condition $2 - \frac{2}{p} - \frac{d}{q} > -\frac{d}{3q}$, where even equality is allowed if $p \leqslant 3q$ (see (14.22), Proposition 14.6.13, and Theorem 14.6.26). To optimally exploit this fact in situations where the more stringent condition $2 - \frac{2}{p} - \frac{d}{q} > 0$ mentioned earlier is not needed, e.g., in the semi-linear case arising when $a \equiv 1$, we may admit functions F defined on a space Z that is larger than Y. Even more flexibility is created if we take time integrability into account as by maximal L^p-regularity methods we actually only need

$$W^{1,p}(0,T;X_0) \cap L^p(0,T;X_1) \hookrightarrow L^{3p}(0,T;L^{3q}(\mathbb{R}^d))$$

in order to define $F(u)$. Conditions for this are given by Corollary L.4.7 which in the current situation $\alpha = 0$, $h = 3$ and thus $\theta = 1 - \frac{2}{3p}$ lead to the requirement $H^{2\theta,q} \hookrightarrow L^{3q}$, which holds if and only if $\frac{d}{q} + \frac{2}{p} \leqslant 3$, which is even weaker than what we saw before. We will come back to this point in Examples 18.1.3 and 18.3.1.

In Section 18.1 we start with the study of local existence and uniqueness for semi-linear equations, where the function F is defined on the trace space $Y = (X_0, X_1)_{1-\frac{1}{p}}$ with $p \in (1,\infty)$. Here we can admit initial values u_0 which belong to the space Y. We present this setting separately, as it allows us to introduce some important techniques in the simplest possible setting.

In Sections 18.2 we turn to the study of local well-posedness in the technically more demanding quasi-linear setting. At the same time, we improve on the assumptions needed to make things work: it is possible to allow exponents $p \in [1,\infty]$ and functions F of the form $F_{\rm Tr} + F_{\rm c}$ as in (18.2), with $F_{\rm Tr}$ defined on Y as before and $F_{\rm c}$ on a larger space Z. Furthermore, we work in a weighted setting in time. This has at least three important advantages:

(i) it allows initial data u_0 belonging to the space $(X_0, X_1)_{1-\alpha-\frac{1}{p}}$, where $\alpha > 0$ is a parameter associated with the weight;
(ii) global existence of solutions can be proved under milder blow-up criteria;
(iii) it allows the inclusion of the endpoint $p = \infty$ (inclusion of the endpoint $p = 1$ is possible for different reasons).

Blow-up criteria will be discussed in Section 18.2.d. After presenting an illustrating example in Section 18.3, the final Section 18.4 presents long term and even global well-posedness results for small initial data in the case $F = F_{\mathrm{c}}$ (i.e., $F_{\mathrm{Tr}} = 0$).

18.1 Semi-linear evolution equations with $F = F_{\mathrm{Tr}}$

In this section we study local well-posedness of semi-linear evolution equations of the form

$$\begin{cases} u'(t) + Au(t) &= F(u(t)), \quad t \in (0,T), \\ u(0) &= u_0, \end{cases} \tag{18.4}$$

where $T > 0$ is fixed. Parabolic partial differential equations of evolution type can often be cast into this form. Some examples will be encountered below.

Our standing assumptions are as follows. We let X_0 and X_1 be Banach spaces, with X_0 continuously embedded into X_0, we fix $p \in (1,\infty)$, and make the following assumptions:

(1) $A : X_1 \to X_0$ is a bounded linear operator;
(2) $F : (X_0, X_1)_{1-\frac{1}{p},p} \to X_0$ is a locally Lipschitz function;
(3) u_0 belongs to $(X_0, X_1)_{1-\frac{1}{p},p}$.

For the sake of brevity, in what follows we will write

$$X_{1-\frac{1}{p},p} := (X_0, X_1)_{1-\frac{1}{p},p}$$

and refer to this space as the *trace space* associated with the problem (18.4).

The following definition extends the notion of L^p-solutions to the present setting.

Definition 18.1.1. *A function $u \in L^p(0,T;X_1) \cap W^{1,p}(0,T;X_0)$ is called an* L^p-solution *to* (18.4) *if for all $t \in [0,T]$ we have*

$$u(t) - u_0 + \int_0^t Au(s)\,\mathrm{d}s = \int_0^t F(u(s))\,\mathrm{d}s.$$

The assumptions imply that Au belongs to $L^p(0,T;X_0)$, and therefore the first integral is well defined as a Bochner integral in X_0. To prove the Bochner integrability of $F(u) : s \mapsto F(u(s))$ in X_0, we note that the assumptions and (18.1) imply that $u \in C([0,T];X_{1-\frac{1}{p},p})$. Consequently, $F(u)$ is well defined as a function in $C([0,T];X_0)$.

In order to get acquainted with the type of arguments involved, we begin by proving a preliminary local existence and uniqueness result. Later on, in Section 18.2, this result will be further improved in several ways, and continuous dependence on the initial data and conditions for global well-posedness will be discussed.

Theorem 18.1.2 (Local well-posedness for semi-linear problems). *Let $X_1 \hookrightarrow X_0$ as stated, let $p \in (1, \infty)$, and assume the conditions (1), (2), (3) to be satisfied. Suppose that*

(1) *The operator A, viewed as a linear operator in X_0 with domain $\mathsf{D}(A) = X_1$, has maximal L^p-regularity on bounded time intervals;*
(2) *There exists an non-decreasing function $\psi : (0, \infty) \to (0, \infty)$ such that for all $r > 0$ and all $x, y \in X_{1-\frac{1}{p},p}$ satisfying*

$$\|x\|_{X_{1-\frac{1}{p},p}} \leqslant r \quad \textit{and} \quad \|y\|_{X_{1-\frac{1}{p},p}} \leqslant r$$

one has

$$\|F(x) - F(y)\|_{X_0} \leqslant \psi(r)\|x - y\|_{X_{1-\frac{1}{p},p}}. \tag{18.5}$$

Then for all $R > 0$ there exists a $T > 0$ such that for all $u_0 \in X_{1-\frac{1}{p},p}$ satisfying $\|u_0\|_{X_{1-\frac{1}{p},p}} \leqslant R$ the problem (18.4) has a unique L^p-solution u. Moreover,

$$u \in L^p(0, T; X_1) \cap W^{1,p}(0, T; X_0).$$

The bound (18.5) is a quantified local Lipschitz assumption, where "local" refers to bounded subsets of $X_{1-\frac{1}{p},p}$. We note that by (18.1) the L^p-solution u satisfies

$$u \in C([0, T]; X_{1-\frac{1}{p},p}).$$

As a preparation for the proof, we first explain how the maximal L^p-regularity of A will be used to prove the theorem. By maximal L^p-regularity, for all $f \in L^p(0, T; X_0)$ the problem

$$\begin{cases} u' + Au &= f \quad \text{on } (0, T) \\ u(0) &= u_0 \end{cases}$$

admits a unique L^p-solution. Moreover, there exists a constant $C_{p,A,T}$, independent of f and u_0, such that

$$\|u\|_{L^p(0,T;X_1) \cap W^{1,p}(0,T;X_0)} \leqslant C_{p,A,T}(\|f\|_{L^p(0,T;X_0)} + \|u_0\|_{X_{1-\frac{1}{p},p}}). \tag{18.6}$$

This follows from Proposition 17.2.14 and a repetition of the argument in (17.10) and (17.11). For the optimal choice of these constants, using (17.25) one can check that $C_{p,A,T} \leqslant C_{p,A,T'}$ whenever $T \leqslant T' < \infty$.

Proof of Theorem 18.1.2. The theorem will be established by applying the Banach fixed point theorem on a suitable bounded closed subset of the *maximal regularity space*

$$\mathrm{MR}^p(0,T) := L^p(0,T;X_1) \cap W^{1,p}(0,T;X_0).$$

The boundedness of the embedding $\mathrm{MR}^p(0,T) \hookrightarrow C([0,T];(X_0,X_1)_{1-\frac{1}{p},p})$ (see (18.1)) enables us to use the local Lipschitz assumption on F.

Let $R > 0$ and fix an $u_0 \in X_{1-\frac{1}{p},p}$ satisfying $\|u_0\|_{X_{1-\frac{1}{p},p}} \leqslant R$. In order to define a suitable subset of $\mathrm{MR}^p(0,T)$ on which the fixed point argument can be performed, we introduce the *reference solution* z_{u_0} as the unique L^p-solution to

$$\begin{cases} z'(t) + Az(t) &= 0, \quad t \geqslant 0, \\ z(0) &= u_0. \end{cases}$$

Note that $z_{u_0} \in \mathrm{MR}^p(0,T)$ for every $T < \infty$. By (18.1) (and its proof) and (18.6) (with $f = 0$) we have

$$\begin{aligned} \sup_{t\in[0,1]} \|z_{u_0}(t)\|_{X_{1-\frac{1}{p},p}} &\leqslant C_{p,T}\|z_{u_0}\|_{\mathrm{MR}^p(0,1)} \\ &\leqslant C_{p,T}C_{p,A,T}\|u_0\|_{X_{1-\frac{1}{p},p}} \leqslant C_{p,T}C_{p,A,T}R =: M_R. \end{aligned}$$

Fix an arbitrary $T \in (0,1]$, and consider the closed ball

$$B_1^T(u_0) := \{u \in \mathrm{MR}^p(0,T) : u(0) = u_0, \ \|u - z_{u_0}\|_{\mathrm{MR}^p(0,T)} \leqslant 1\}.$$

Let $\varPhi : B_1^T(u_0) \to \mathrm{MR}^p(0,T)$ be defined by $\varPhi(v) := u$, where u is the unique L^p-solution to the problem

$$\begin{cases} u' + Au &= F(v), \\ u(0) &= u_0. \end{cases}$$

This unique solution exists by the discussion preceding the proof; note that $F(v) \in C([0,T];X_0)$ by the continuity of F and (18.1).

For later purpose we observe that for all $v_1, v_2 \in B_1^T(u_0)$, Corollary L.4.6 (using $v_1(0) - v_2(0) = 0$ to get T-independent constants) implies that

$$\|v_1 - v_2\|_{C([0,T];X_{1-\frac{1}{p},p})} \leqslant C_p\|v_1 - v_2\|_{\mathrm{MR}^p(0,T)} \leqslant 2C_p. \tag{18.7}$$

In particular, since $T \leqslant 1$, upon taking $v_1 = v \in B_1^T(u_0)$ and $v_2 := z_{u_0} \in B_1^T(u_0)$, we find that

$$\begin{aligned} \|v\|_{C([0,T];X_{1-\frac{1}{p},p})} &\leqslant \|v - z_{u_0}\|_{C([0,T];X_{1-\frac{1}{p},p})} + \|z_{u_0}\|_{C([0,T];X_{1-\frac{1}{p},p})} \\ &\leqslant 2C_p + M_R =: N_R. \end{aligned} \tag{18.8}$$

To be able to apply the Banach fixed point theorem to $\varPhi$, we need to check that $\varPhi$ maps the closed ball $B_1^T(u_0)$ into itself and is uniformly contractive on it. For both assertions it will be necessary to choose $T \in (0,1]$ small enough.

First we check that Φ maps $B_1^T(u_0)$ into itself. For all $v \in B_1^T(u_0)$ one has

$$\begin{aligned}\|F(v(t))\|_{X_0} &\leqslant \|F(v(t)) - F(z_{u_0}(t))\|_{X_0} + \|F(z_{u_0}(t)) - F(0)\|_{X_0} + \|F(0)\|_{X_0} \\ &\leqslant \psi(N_R)(\|v(t) - z_{u_0}(t)\|_{X_{1-\frac{1}{p},p}} + \|z_{u_0}(t)\|_{X_{1-\frac{1}{p},p}}) + \|F(0)\|_{X_0} \\ &\leqslant \psi(N_R)(2C_p + M_R) + \|F(0)\|_{X_0} =: C_{R,F},\end{aligned}$$

where we used (18.5) and (18.8). Thus

$$\|F(v)\|_{L^p(0,T;X_0)} \leqslant T^{1/p} C_{R,F}.$$

Therefore, letting $u = \Phi(v)$ and using the maximal L^p-regularity estimate (18.6) for the equation which $u - z_{u_0}$ satisfies, we find that

$$\|u - z_{u_0}\|_{\mathrm{MR}^p(0,T)} \leqslant C_{p,A,T}\|F(v)\|_{L^p(0,T;X_0)} \leqslant C_{p,A,1}T^{1/p}C_{R,F}.$$

Therefore, for $0 < T \leqslant (C_{p,A,1}C_{R,F})^{-p} \wedge 1$ we find that $u \in B_1^T(u_0)$.

To check that Φ is a uniform contraction, let $v_i \in B_1^T(u_0)$ for $i \in \{1, 2\}$. Using the maximal L^p-regularity estimate (18.6) for the equation which $u_1 - u_2$ satisfies, and (18.5), we find that

$$\begin{aligned}\|\Phi(v_1) - \Phi(v_2)\|_{\mathrm{MR}^p(0,T)} &\leqslant C_{p,A,T}\|F(v_1) - F(v_2)\|_{L^p(0,T;X_0)} \\ &\leqslant C_{p,A,1}T^{1/p}\psi(N_R)\|v_1 - v_2\|_{C([0,T];X_{1-\frac{1}{p},p})} \\ &\leqslant C_{p,A,1}T^{1/p}\psi(N_R)C_p\|v_1 - v_2\|_{\mathrm{MR}^p(0,T)},\end{aligned}$$

where in the last step we used (18.7). Therefore, combining both conditions on T it follows that for $T = \frac{1}{2}((C_{p,A,1}\psi(N_R)C_p)^{-p} \wedge (C_{p,A,1}C_{R,F})^{-p} \wedge 1)$ the mapping Φ is a uniform contraction on $B_1^T(u_0)$ with

$$\|\Phi(v_1) - \Phi(v_2)\|_{\mathrm{MR}^p(0,T)} \leqslant \frac{1}{2}\|v_1 - v_2\|_{\mathrm{MR}^p(0,T)}.$$

By the Banach fixed point theorem, the restriction of Φ to $B_1^T(u_0)$ has a unique fixed point $u \in B_1^T(u_0)$. From the definition of Φ, it is immediate that u is an L^p-solution to (18.4).

It remains to prove the uniqueness. Uniqueness is clear on $B_1^T(u_0)$, but we still need to prove uniqueness in the larger set $\mathrm{MR}^p(0,T)$. Let $u_1, u_2 \in \mathrm{MR}^p(0,T)$ be L^p-solutions to (18.4). Then for every $t \in [0,T]$, by Corollary L.4.6 (with t-independent constant), (18.6), and the remarks below it, and (18.5),

$$\begin{aligned}\|u_1(t) - u_2(t)\|_{X_{1-\frac{1}{p},p}} &\leqslant C_p\|u_1 - u_2\|_{\mathrm{MR}^p(0,t)} \\ &\leqslant C_pC_{p,A,T}\|F(u_1) - F(u_2)\|_{L^p(0,t;X_0)} \\ &\leqslant C_pC_{p,A,T}\psi(N)\|u_1 - u_2\|_{L^p(0,t;X_{1-\frac{1}{p},p})},\end{aligned}$$

where N is such that $\|u_i\|_{C([0,T];X_{1-\frac{1}{p},p})} \leqslant N$ for $i \in \{1,2\}$. Therefore, applying Gronwall's inequality to $\|u_1(t) - u_2(t)\|^p_{X_{1-\frac{1}{p},p}}$, we find that $u_1 \equiv u_2$ on $[0,T]$. □

Here is a simple example to which Theorem 18.1.2 can be applied. Further examples will be given in Section 18.3.

Example 18.1.3. Let $A \in \mathscr{L}(X_1, X_0)$, where

$$X_0 = H^{s,q}(\mathbb{R}^d) \quad \text{and} \quad X_1 = H^{s+2,q}(\mathbb{R}^d)$$

with $s \in (-2,0]$ and $q \in (1,\infty)$. In the present situation, Theorem 14.4.31 shows that $X_{1-\frac{1}{p},p} = B^{s+2-\frac{2}{p}}_{q,p}(\mathbb{R}^d)$. In order to have a concrete equation in mind note that one for instance could take A to be a second order differential operator such as $-\Delta$, and we could consider the PDE

$$\begin{cases} \partial_t u - \Delta u &= f(u), \\ u(0) &= u_0, \end{cases}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a given locally Lipschitz function satisfying $f(0) = 0$.

Suppose now that $s + 2 - \frac{2}{p} - \frac{d}{q} > 0$. Then, by the Sobolev embedding in Corollary 14.4.27, we have a continuous embedding

$$X_{1-\frac{1}{p},p} \hookrightarrow C_{\rm b}(\mathbb{R}^d). \tag{18.9}$$

We claim that the so-called *Nemitskii map*

$$F(u)(x) = f(u(x)), \quad x \in \mathbb{R}^d,$$

is well defined and locally Lipschitz as a mapping from $X_{1-\frac{1}{p},p}$ into X_0. To prove this, fix $N > 0$ and elements $u, v \in X_{1-\frac{1}{p},p}$ satisfying $\|u\|_{X_{1-\frac{1}{p},p}} \leqslant N$ and $\|v\|_{X_{1-\frac{1}{p},p}} \leqslant N$. Then, we obtain that for some constant L depending on f, N, and the embedding constant of (18.9),

$$\begin{aligned} \|F(u) - F(v)\|_{X_0} &\leqslant \Big(\int_{\mathbb{R}^d} |f(u(x)) - f(v(x))|^q \,\mathrm{d}x \Big)^{1/q} \\ &\leqslant L\Big(\int_{\mathbb{R}^d} |u(x) - v(x)|^q \,\mathrm{d}x \Big)^{1/q} \\ &\leqslant LC\|u - v\|_{X_{1-\frac{1}{p},p}}, \end{aligned}$$

where in the last step we used (14.22) and Proposition 14.4.18. Taking $v = 0$ and using the assumption $f(0) = 0$, one also obtains

$$\|F(u)\|_{X_0} \leqslant LC\|u\|_{X_{1-\frac{1}{p},p}}.$$

These two estimates prove the claim.

The above estimates on F are not optimal and the condition on the exponents, namely, $s+2-\frac{2}{p}-\frac{d}{q}>0$ turns out to be far from sharp. We also notice that in the example we can only treat rather smooth initial values $u_0 \in B^{s+2-\frac{2}{p}}_{q,p}(\mathbb{R}^d)$ (in particular they need to be Hölder continuous). This turns out to be far from sharp. Both these sharpness issues will be addressed in the next section.

18.2 Local well-posedness for quasi-linear evolution equations

In the present section and the next, we will study local well-posedness for quasi-linear evolution equations of the form introduced at the beginning of Chapter 18,

$$\begin{cases} u'(t) + A(u(t))u(t) &= F(u(t)), \quad t \in (0,T), \\ u(0) &= u_0. \end{cases}$$

We will make several changes to the simple setting considered in Section 18.1. Besides the fact that the operator A now depends on the solution u, the changes are as follows:

- The non-linearity is of the form

  $$F = F_{\mathrm{Tr}} + F_{\mathrm{c}},$$

  where F_{Tr} plays a similar role as in Section 18.1, and F_{c} is the so-called *critical part* of F. We assume that both F_{Tr} and F_{c} are defined on a suitable subset of $X_{\sigma,p}$ (see (18.10) below) with $\sigma \in [0, 1-\frac{1}{p}]$, and that F_{c} satisfies a suitable polynomial growth condition.
- Weights in time are added (see Corollaries 17.2.37 and 17.2.48). This will enable us to reduce the smoothness conditions on the initial data. At the same time, this makes it possible to formulate flexible conditions for global existence.
- The full range $p \in [1,\infty]$ will be considered.

In Example 18.3.1 we will see that the new setting takes care of the issues raised in the discussion after Example 18.1.3.

18.2.a Setting

Turning to the details, as before we make the standing assumption that we have a continuous embedding of Banach spaces

$$X_1 \hookrightarrow X_0.$$

Without loss of generality we will always assume that the constant in the embedding is $\geqslant 1$.

We further fix

$$p \in [1, \infty]$$

and

$$\alpha \in [0, \tfrac{1}{p'}) \cup \{0\},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$; we take $\alpha > 0$ if $p = \infty$. The exponent α enters into the weight $w_\alpha(t) := t^\alpha$ that will be used later. In applications, the choice of α determines which initial condition u_0 can be allowed; larger values of α permit initial conditions with less smoothness. The exponent

$$\sigma := 1 - \alpha - \frac{1}{p}$$

has already been encountered in Corollary 17.2.37, and will occur frequently in what follows.

For the sake of notational brevity, we will use the conventions that

$$\begin{aligned} (X_0, X_1)_{0,r} &:= X_0 && \text{for } r \in [1, \infty], \\ X_{\theta,r} &:= \overline{X_1}^{(X_0,X_1)_{\theta,r}} && \text{for } \theta \in (0,1) \text{ and } r \in [1, \infty], \\ X_\theta &:= X_{\theta,1} && \text{for } \theta \in (0,1). \end{aligned} \tag{18.10}$$

Note that $X_{\theta,r} = (X_0, X_1)_{\theta,r}$ if $\theta \in (0,1)$ and $r \in [1, \infty)$, because in these ranges $X_0 \cap X_1 = X_1$ is dense in $(X_0, X_1)_{\theta,r}$ by Corollary C.3.15. For $\theta = 0$, $X_{\theta,r} = (X_0, X_1)_{\theta,r}$ holds for all $r \in [1, \infty]$ by definition.

Remark 18.2.1. There is some flexibility with regard to the choice of the spaces X_θ in (18.10). These spaces will appear only in the assumptions on the non-linearity $F_{\rm c}$ through (18.11) below. The only requirement needed is that $X_{\theta,1}$ continuously embeds into this space. In the above definition one could for instance take X_θ to be $X_{\theta,r}$, $[X_0, X_1]_\theta$, or $\mathsf{D}((\omega + A(u_0))^\theta)$ for $\omega \in \mathbb{R}$ large enough.

In addition to the above-stated assumptions on the spaces X_0, X_1 and the parameters p, α, σ, we make the following structural assumptions on the operator A and the non-linearity F.

Assumption 18.2.2. *We fix an open set $O_{\sigma,p} \subseteq X_{\sigma,p}$ and assume:*

(1) *the initial condition u_0 belongs to $O_{\sigma,p}$;*
(2) *there exists a constant $L \geqslant 0$ such that the mapping $A : O_{\sigma,p} \to \mathscr{L}(X_1, X_0)$ satisfies*

$$\|A(u) - A(v)\|_{\mathscr{L}(X_1,X_0)} \leqslant L\|u - v\|_{X_{\sigma,p}}, \quad u, v \in O_{\sigma,p};$$

(3) *the mapping $F : X_1 \cap O_{\sigma,p} \to X_0$ admits a decomposition $F = F_{\rm Tr} + F_{\rm c}$, where*

(i) $F_{\mathrm{Tr}} : X_1 \cap O_{\sigma,p} \to X_0$ *and there exists an* $L_{\mathrm{Tr}} \geqslant 0$ *such that*

$$\|F_{\mathrm{Tr}}(u) - F_{\mathrm{Tr}}(v)\|_{X_0} \leqslant L_{\mathrm{Tr}} \|u - v\|_{X_{\sigma,p}}, \quad u, v \in X_1 \cap O_{\sigma,p};$$

(ii) $F_{\mathrm{c}} : X_1 \cap O_{\sigma,p} \to X_0$ *and there exist* $m \geqslant 1$, $\beta_j \in (\sigma, 1)$, $\rho_j > 0$ *for* $j \in \{1, \ldots, m\}$, *and* $L_{\mathrm{c}} \geqslant 0$ *such that*

$$\|F_{\mathrm{c}}(u) - F_{\mathrm{c}}(v)\|_{X_0} \leqslant L_{\mathrm{c}} \sum_{j=1}^{m} (1 + \|u\|_{X_{\beta_j}}^{\rho_j} + \|v\|_{X_{\beta_j}}^{\rho_j}) \|u - v\|_{X_{\beta_j}} \tag{18.11}$$

for all $u, v \in X_1 \cap O_{\sigma,p}$, *and where*

$$\beta_j \leqslant \frac{1 + \rho_j \sigma}{1 + \rho_j}, \qquad j \in \{1, \ldots, m\}. \tag{18.12}$$

Several clarifying comments are in order.

In typical applications, the set $O_{\sigma,p}$ is a bounded subset of $X_{\sigma,p}$. In situations where A, F_{Tr}, and F_{c} are defined on all of $X_{\sigma,p}$ (in case of A and F_{Tr}) and X_1 (in case of F_{c}), the constants L, L_{Tr} and L_{c} will increase with $O_{\sigma,p}$. Thus, although some of the above Lipschitz estimates are formulated as global Lipschitz conditions on $O_{\sigma,p}$, they should actually be thought of as local Lipschitz conditions on $X_{\sigma,p}$.

The quasi-linear operator A is Lipschitz on the same space as F_{Tr}. In the semi-linear case the operator A can be taken constant on $O_{\sigma,p}$.

The assumptions on the non-linearity F_{Tr} are very similar to the ones in Theorem 18.1.2 in case $\alpha = 0$ and $p \in (1, \infty)$, but for simplicity we chose to let F_{Tr} be defined on the full space $(X_0, X_1)_{1-\frac{1}{p},p}$ in that result. Taking larger values of α leads to more restrictive conditions on F_{Tr}. However, at the same time it will lead to less conditions on the initial data. The mapping F_{Tr} uniquely extends to a continuous mapping on $O_{\sigma,p}$.

A central role is played by the non-linear mapping F_{c}, where c stands for "critical". Let $\beta = \max_{j \in \{1,\ldots,m\}} \beta_j$. By (18.11) and the density of X_1 in X_β we find that F_{c} uniquely extends to a locally Lipschitz function $F_{\mathrm{c}} : \overline{O_{\sigma,p} \cap X_1}^{X_\beta} \to X_0$.

The restriction (18.12) should be seen as a balance between the polynomial growth rate ρ_j of the local Lipschitz constant and the regularity exponent β_j. The larger ρ_j is, the smaller β_j needs to be. The case of equality plays a special role:

Definition 18.2.3 (Criticality). *Let Assumption 18.2.2 hold. The space* $X_{\sigma,p}$ *and the parameter* σ *are called* critical *if equality holds in* (18.12) *for some* $j \in \{1, \ldots, m\}$. *In case of strict inequality in* (18.12) *for all* $j \in \{1, \ldots, m\}$, *the space* $X_{\sigma,p}$ *and the parameter* σ *are called* sub-critical.

In applications to concrete non-linear PDEs, the parameters β_j and ρ_j are determined by spatial Sobolev embedding and the growth order of the polynomial non-linearity. Often, one can choose a minimal σ for which at least one of the inequalities becomes an equality. After σ has been determined, one can choose α and p such that $\sigma = 1 - \alpha - \frac{1}{p}$ holds. Here, p is usually chosen large (and thus $\alpha \in [0, 1/p') \cup \{0\}$ is close to $1 - \sigma$), as this leads to the best time regularity results. Quite often, the critical space (for the initial values) $X_{\sigma,p}$ has some scaling behaviour which fits well to the scaling behaviour of solutions to the corresponding PDE. If $p = \infty$, the unweighted case $\alpha = 0$ cannot be considered due to a technical reason: in the proofs below we need $\alpha + \frac{1}{p} > 0$.

In the case $p = 1$, which is allowed in our setting, the other assumptions enforce $\alpha = 0$ and $\sigma = 0$. In particular, there is not much flexibility for the function $F_{\rm Tr}$ and it needs to be locally Lipschitz on X_0. On the other hand, (18.11) is still quite flexible: for instance if $m = 1$ one can allow $F_{\rm c} : X_{1/(1+\rho)} \to X_0$ where growth of power ρ is allowed for the Lipschitz constant.

Remark 18.2.4 (Time-dependent and inhomogeneous settings). It is possible to extend the above setting to time-dependent mappings $A : [0, T] \times O_{\sigma,p} \to \mathscr{L}(X_1, X_0)$ and $F : [0, T] \times X_1 \cap O_{\sigma,p} \to X_0$. This does not lead to any major changes as long as the mapping properties of A and F and estimates are uniform in $t \in [0, T]$ (or the constants in the estimates satisfy a suitable integrability condition). Usually, A is assumed to be continuous in time, so that maximal regularity of $A(0, u_0)$ can be used in local well-posedness results in a similar way as we did in Theorem 17.2.51. Continuity in time can be avoided by introducing a suitable notion of maximal regularity for the case of time-dependent A.

One can also allow a further inhomogeneity by allowing non-linearities of the form $F = F_{\rm Tr} + F_{\rm c} + f$, where $f : (0, T) \to X_0$ satisfies appropriate integrability assumptions.

We will now proceed to the main theorems on local well-posedness for the quasi-linear problem

$$\begin{cases} u' + A(u)u &= F(u), \quad \text{on } (0, T), \\ u(0) &= v_0, \end{cases} \tag{18.13}$$

where $v_0 \in O_{\sigma,p}$ can be taken as the given u_0 or close to u_0 in $X_{\sigma,p}$-norm. Allowing v_0 to be taken from a neighbourhood of u_0 will be important as we will also give prove continuous dependence on the initial data. Moreover, it will be used to obtain criteria for global well-posedness.

Define

$$\mathrm{MR}^p_\alpha(0, T) := \begin{cases} L^p_{w_\alpha}(0, T; X_1) \cap W^{1,p}_{w_\alpha}(0, T; X_0) & \text{if } p < \infty; \\ C_{w_\alpha,0}((0, T]; X_1) \cap C^1_{w_\alpha,0}((0, T]; X_0) & \text{if } p = \infty, \end{cases}$$

where we recall that the Banach space $C_{w_\alpha,0}((0,T];X_1)$ was defined before Corollary 17.2.48, and is a closed subspace of $L^\infty_{w_\alpha}(0,T;X_1)$. Similar assertions hold for its C^1-variant.

Definition 18.2.5. *Let Assumption 18.2.2 hold. A function $u \in \mathrm{MR}^p_\alpha(0,T)$ is called a $L^p_{w_\alpha}$-solution to (18.13) on $(0,T)$ if u takes values in $O_{\sigma,p}$, $A(u)u, F(u) \in L^1(0,T;X_0)$, and for all $t \in [0,T]$ we have*

$$u(t) - v_0 + \int_0^t A(u(s))u(s)\,\mathrm{d}s = \int_0^t F(u(s))\,\mathrm{d}s.$$

Later on in Lemma 18.2.8, we will see that the integrability assumptions on $A(u)u$ and $F(u)$ are actually redundant, and that one even has $A(u)u, F(u) \in L^p_{w_\alpha}(0,T;X_0)$.

18.2.b Main local well-posedness result

The main result of this section is the following local well-posedness for quasi-linear equations.

Theorem 18.2.6 (Local well-posedness for quasi-linear problems). *Let Assumption 18.2.2 hold. If, for some $u_0 \in O_{\sigma,p}$, the operator $A(u_0)$ has maximal L^p-regularity (maximal C-regularity if $p = \infty$) on finite time intervals, then there exist $T > 0$ and $\varepsilon > 0$ such that for all*

$$v_0 \in B_{X_{\sigma,p}}(u_0,\varepsilon) \subseteq O_{\sigma,p}$$

the problem (18.13) has a unique $L^p_{w_\alpha}$-solution $u_{v_0} \in \mathrm{MR}^p_\alpha(0,T)$. Moreover, there exists a constant $C \geqslant 0$ such that for all $v_0, v_1 \in B_{X_{\sigma,p}}(u_0,\varepsilon)$ we have

$$\|u_{v_0} - u_{v_1}\|_{\mathrm{MR}^p_\alpha(0,T)} \leqslant C\|v_0 - v_1\|_{X_{\sigma,p}}. \tag{18.14}$$

From Corollary L.4.6 we additionally see that

$$u_{v_0} \in C([0,T];X_{1-\alpha-\frac{1}{p},p}) \cap C((0,T];X_{1-\frac{1}{p},p}). \tag{18.15}$$

This shows that for $\alpha > 0$, the solution u instantaneously (that is, for $t \in (0,T]$) regularises from $X_{1-\alpha-\frac{1}{p},p}$ to $X_{1-\frac{1}{p},p}$. By similar arguments, an analogous continuous dependence as in (18.14) holds in $C([0,T];X_{1-\alpha-\frac{1}{p},p})$ and in the weighted space $C_{w_\alpha}((0,T];X_{1-\frac{1}{p},p})$.

The parameters T, ε, and C in Theorem 18.2.6 depend on the choice of u_0 in general. The parameters T and ε need to be small enough for the conclusions of the theorem to hold. This has several reasons. First of all, ε must be small because we need $B_{X_{\sigma,p}}(u_0,\varepsilon)$ to be contained in $O_{\sigma,p}$. More importantly, the proof uses the maximal regularity of $A(u_0)$ to obtain local well-posedness of (18.13) with initial value v_0, via a perturbation argument involving the smallness of $\|u_0 - v_0\|_{X_{\sigma,p}}$.

The time T must be small for two reasons. First of all, we need to assure that u maps $[0,T]$ to $O_{\sigma,p}$. Secondly, in the proof of the theorem we also need T to be small in order to be able to use fixed point arguments. This is hardly surprising: already in the familiar setting of ordinary differential equations, blow-up can occur in the presence of locally Lipschitz continuous non-linearities F. Theorem 18.2.15 will provide conditions under which one can extend the time interval of existence and uniqueness to the full interval $[0,\infty)$. For a special class of semi-linear equations, Theorem 18.2.17 will give large-time well-posedness for small initial data.

18.2.c Proof of the main result

The proof Theorem 18.2.6 uses a fixed point argument similar to the one of Theorem 18.1.2. However, the proof is technically more demanding due to the quasi-linear structure of the problem, the presence of the additional term $F_{\rm c}$, the use of weights, and the admission of the full range $p \in [1,\infty]$; some new ideas are needed to deal with these difficulties.

We will use the following abbreviations to keep the formulas at a reasonable length. For $k \in \{0,1\}$ and $j \in \{1,\dots,m\}$ we let, with notation introduced earlier,

$$E_k := \begin{cases} L^p_{w_\alpha}(0,T;X_k) & \text{if } p<\infty \\ C_{w_\alpha,0}((0,T];X_k) & \text{if } p=\infty \end{cases}$$
$$Y_j := \begin{cases} L^{(\rho_j+1)p}_{w_{\alpha/(\rho_j+1)}}(0,T;X_{\beta_j^*}) & \text{if } p<\infty \\ C_{w_{\alpha/(\rho_j+1)},0}((0,T];X_{\beta_j^*}) & \text{if } p=\infty, \end{cases}$$

where $\beta_j^* := \frac{1+\rho_j\sigma}{1+\rho_j}$. Assumption 18.2.2 implies that $\beta_j \leqslant \beta_j^*$.

Lemma 18.2.7. *Let Assumption 18.2.2 hold. Then for all $T>0$ we have continuous embeddings*

$$\begin{aligned} &\mathrm{MR}^p_\alpha(0,T) \hookrightarrow C([0,T];X_{\sigma,p}), \\ &\mathrm{MR}^p_\alpha(0,T) \hookrightarrow Y_j, \quad j\in\{1,\dots,m\}, \end{aligned}$$

and there exists a constant $M_{1,T} \geqslant 0$ such that for all $u \in \mathrm{MR}^p_\alpha(0,T)$ and $j \in \{1,\dots,m\}$ we have

$$\|u\|_{C([0,T];X_{\sigma,p})} + \|u\|_{Y_j} \leqslant M_{1,T}\|u\|_{\mathrm{MR}^p_\alpha(0,T)}. \tag{18.16}$$

These constants may be chosen so that $\sup_{T\geqslant 1} M_{1,T} < \infty$. For functions $u \in \mathrm{MR}^p_\alpha(0,T)$ satisfying $u(0)=0$, the constants $M_{1,T}$ can be replaced by a constant M_1 independent of $T>0$.

Proof. For $p\in[1,\infty)$, the embeddings and estimates follow from Corollaries L.4.6 and L.4.7, where for $p=1$ we additionally use Remark L.4.2 and Proposition L.4.5.

For $p = \infty$, the same can be done if Y_j is replaced by $C_{w_{\alpha/(\rho_j+1)}}((0,T];X_{\beta_j^*})$. To get the embedding into its closed subspace Y_j, recall $u \in \mathrm{MR}_\alpha^\infty(0,T) = C_{w_{\alpha,0}}((0,T];X_1) \cap C^1_{w_{\alpha,0}}((0,T];X_0)$. Then, for all $j \in \{1,\dots,m\}$, by (L.19),

$$\|u(t)\|_{X_{\beta_j^*}} \leqslant (1-\sigma)^{-1}\|u(t)\|^{\lambda}_{X_{\sigma,\infty}}\|u(t)\|^{1-\lambda}_{X_1},$$

where $\lambda = \frac{1-\beta_j^*}{\alpha} = \frac{\rho_j}{1+\rho_j}$. Hence

$$t^{\alpha/(\rho_j+1)}\|u(t)\|_{X_{\beta_j^*}} \leqslant (1-\sigma)^{-1}\|u\|^{\lambda}_{C([0,T];X_{\sigma,\infty})}(t^{\alpha}\|u(t)\|_{X_1})^{1-\lambda},$$

and the latter tends to zero as $t \downarrow 0$ since $u \in E_1$. This shows that $u \in Y_j$. □

Lemma 18.2.8. *Let Assumption 18.2.2 hold. Let $u, v, z \in \mathrm{MR}_\alpha^p(0,T)$ be given, and assume that u and v take values in $O_{\sigma,p}$. Then we have $A(u)z \in E_0$, $F_{\mathrm{c}}(u) \in E_0$, and $F_{\mathrm{Tr}}(u) \in C([0,T];X_0)$. Moreover,*

$$\|A(u)z - A(v)z\|_{E_0} \leqslant L\|u-v\|_{C([0,T];X_{\sigma,p})}\|z\|_{E_1}$$

and

$$\begin{aligned}\|F_{\mathrm{Tr}}(u) - F_{\mathrm{Tr}}(v)\|_{C([0,T];X_0)} &\leqslant L_{\mathrm{Tr}}\|u-v\|_{C([0,T];X_{\sigma,p})},\\ \|F_{\mathrm{c}}(u) - F_{\mathrm{c}}(v)\|_{E_0} &\leqslant \sum_{j=1}^m C^{\rho_j}_{\beta_j,X} L_{\mathrm{c}}\big[T^{\delta_j} + \|u\|^{\rho_j}_{Y_j} + \|v\|^{\rho_j}_{Y_j}\big]\|u-v\|_{Y_j},\end{aligned} \tag{18.17}$$

where $\delta_j = \frac{\alpha\rho_j}{1+\rho_j} + \frac{\rho_j}{(1+\rho_j)p}$.

This lemma asserts in particular that the integrability assumptions on $A(u)u$ and $F(u)$ in Definition 18.2.5 are automatically satisfied for functions $u \in \mathrm{MR}_\alpha^p(0,T)$.

Proof. First consider the case $p < \infty$. By Assumption 18.2.2(2),

$$\|A(u(t))z(t) - A(v(t))z(t)\|_{X_0} \leqslant L\|u(t)-v(t)\|_{X_{\sigma,p}}\|z(t)\|_{X_1}. \tag{18.18}$$

This gives the required estimate for A. Taking $v \equiv u_0 \in X_{\sigma,p}$ fixed, one also sees that the function $t \mapsto A(u(t))z(t)$ belongs to E_0.

By Assumption 18.2.2(3),

$$\|F_{\mathrm{Tr}}(u(t)) - F_{\mathrm{Tr}}(v(t))\|_{X_0} \leqslant L_{\mathrm{Tr}}\|u(t)-v(t)\|_{X_{\sigma,p}}.$$

This implies the estimate for F_{Tr} in (18.17); the assumptions on F_{Tr} and the continuity of $u : [0,T] \to X_{\sigma,p}$ (see Lemma 18.2.7) imply that $t \mapsto F_{\mathrm{Tr}}(u(t))$ belongs to $C([0,T];X_0)$.

Next, we have $u, v \in Y_j$ by Lemma 18.2.7. Moreover, for all $j \in \{1,\dots,m\}$,

$$
\begin{aligned}
&\Big\|(1+\|u\|_{X_{\beta_j^*}}^{\rho_j} + \|v\|_{X_{\beta_j^*}}^{\rho_j})\|u-v\|_{X_{\beta_j^*}}\Big\|_{L^p_{w_\alpha}(0,T)} \\
&\quad \overset{\text{(i)}}{\leqslant} \Big\|(1 + \|u\|_{X_{\beta_j^*}}^{\rho_j} + \|v\|_{X_{\beta_j^*}}^{\rho_j})\|_{L^{(\rho_j+1)p/\rho_j}_{w_{\alpha\rho_j/(\rho_j+1)}}(0,T)} \|u-v\|_{L^{(\rho_j+1)p}_{w_{\alpha/(\rho_j+1)}}(0,T;X_{\beta_j^*})} \\
&\quad \overset{\text{(ii)}}{\leqslant} \big[T^{\delta_j} + \|u\|_{Y_j}^{\rho_j} + \|v\|_{Y_j}^{\rho_j}\big]\|u-v\|_{Y_j}
\end{aligned}
$$

where in (i) we applied Hölder's inequality with $\frac{1}{(1+\rho_j)} + \frac{\rho_j}{(1+\rho_j)} = 1$ and in (ii) the definition of Y_j and the triangle inequality. The estimate for $F_{\rm c}$ in (18.17) now follows from Assumption 18.2.2 and the inequality $\|x\|_{X_{\beta_j}} \leqslant C_{\beta_j,X}\|x\|_{X_{\beta_j^*}}$ (see Proposition L.1.1(2)), where we used that the embedding constant in $X_1 \hookrightarrow X_0$ is $\geqslant 1$, and thus $C_{\beta_j,X} \geqslant 1$.

The estimate for $F_{\rm Tr}$ immediately extends to $p = \infty$. The estimates for A and $F_{\rm c}$ also extend to $p = \infty$ if we replace $E_0 = C_{w_\alpha,0}((0,T];X_0)$ by $L^\infty_{w_\alpha}(0,T;X_0)$. In order to obtain the estimates in the E_0-norm, it remains to prove that $t \mapsto A(u(t))z(t)$ and $t \mapsto F_{\rm c}(u(t))$ are continuous on $(0,T]$ and $t^\alpha\|A(u(t))z(t)\|_{X_0}$ and $t^\alpha\|F_{\rm c}(u(t))\|_{X_0}$ are bounded and tend to zero as $t \downarrow 0$.

To prove continuity for A, we observe that for $s,t \in (0,T]$

$$
\begin{aligned}
&\|A(u(t))z(t) - A(u(s))z(s)\|_{X_0} \\
&\leqslant \|(A(u(t)) - A(u(s))z(t)\|_{X_0} + \|A(u(s))(z(t)-z(s))\|_{X_0} \\
&\leqslant L\|u(t)-u(s)\|_{X_{\sigma,\infty}}\|z(t)\|_{X_1} + \|A(u(s))\|_{\mathscr{L}(X_1,X_0)}\|(z(t)-z(s))\|_{X_1} \\
&\leqslant \frac{L}{t^\alpha}\|u(t)-u(s)\|_{X_{\sigma,\infty}}\|z\|_{E_1} + \|A(u(s))\|_{\mathscr{L}(X_1,X_0)}\|(z(t)-z(s))\|_{X_1}.
\end{aligned}
$$

The latter tends to zero if $t \to s$, and the desired continuity follows. To prove the bound and convergence of $t^\alpha\|A(u(t))z(t)\|_{X_0}$, we observe that by (18.18), applied with $v \equiv x \in X_1 \cap O_{\sigma,p}$,

$$
\begin{aligned}
\|A(u(t))z(t)\|_{X_0} &\leqslant \|A(x)z(t)\|_{X_0} + \|A(u(t))z(t) - A(x)z(t)\|_{X_0} \\
&\leqslant \|A(x)\|_{\mathscr{L}(X_1,X_0)}\|z(t)\|_{X_1} + L\|u-x\|_{C([0,T];X_{\sigma,p})}\|z(t)\|_{X_1}.
\end{aligned}
$$

Since $z \in C_{w_\alpha,0}((0,T];X_1)$, this implies the desired boundedness and convergence.

To prove continuity for $F_{\rm c}$, note that by Assumption 18.2.2, for $s,t \in (0,T]$ we have

$$
\|F_{\rm c}(u(t)) - F_{\rm c}(u(s))\|_{X_0} \leqslant L_{\rm c}\sum_{j=1}^m (1 + \|u(t)\|_{X_{\beta_j}}^{\rho_j} + \|u(s)\|_{X_{\beta_j}}^{\rho_j})\|u(t)-u(s)\|_{X_{\beta_j}}.
$$

The latter tends to zero as $t \to s$. Indeed, since $u \in Y_j$, $u : (0,T] \to X_{\beta_j^*} \hookrightarrow X_{\beta_j}$ is continuous for each $j \in \{1,\dots,m\}$. To prove the bound and the convergence for $F_{\rm c}$, note that as already mentioned, we have

$$
\|F_{\rm c}(u) - F_{\rm c}(v)\|_{C_{w_\alpha}((0,T];X_0)} \leqslant L_{\rm c}\sum_{j=1}^m C_{\beta_j,X}^{\rho_j}(T^{\delta_j} + \|u\|_{Y_j}^{\rho_j} + \|v\|_{Y_j}^{\rho_j})\|u-v\|_{Y_j}.
$$

From the definitions of δ_j and Y_j it follows that $T^{\delta_j}, \|u\|_{Y_j}, \|v\|_{Y_j} \to 0$ as $T \downarrow 0$. In particular, the estimate implies that $t^\alpha \|F_c(u(t)) - F_c(v(t))\|_{X_0} \to 0$ as $t \downarrow 0$. For $v \equiv x$ with $x \in X_1 \cap O_{\sigma,p}$ it is clear that $t^\alpha \|F_c(v)\|_{X_0} \to 0$ as $t \downarrow 0$. Therefore, $t^\alpha \|F(u(t))\|_{X_0} \to 0$ as $t \downarrow 0$, and hence $F(u) \in E_0$. □

For each $v_0 \in X_{\sigma,p}$ and $T > 0$, we define the *reference solution* $z_{v_0} \in \mathrm{MR}^p_\alpha(0,T)$ as the $L^p_{w_\alpha}$-solution to the following linear problem (see Corollaries 17.2.37 and 17.2.48):

$$\begin{cases} u' + A(u_0)u &= 0, \quad \text{on } \mathbb{R}_+, \\ u(0) &= v_0. \end{cases}$$

Clearly, the mapping $v_0 \mapsto z_{v_0}$ is linear.

Let $\varepsilon, r, T > 0$ be fixed for the moment; these parameters will be chosen small enough shortly. For $v_0 \in B_{X_{\sigma,p}}(u_0, \varepsilon) \subseteq O_{\sigma,p}$ consider the following subset of $\mathrm{MR}^p_\alpha(0,T)$:

$$B^T_r(v_0) = \{v \in \mathrm{MR}^p_\alpha(0,T) : v(0) = v_0, \ \|v - z_{u_0}\|_{\mathrm{MR}^p_\alpha(0,T)} \leqslant r\}. \tag{18.19}$$

Note that $B^T_r(v_0)$ is a closed subset of $\mathrm{MR}^p_\alpha(0,T)$ by the continuity of the trace at zero (see Lemma 18.2.7).

To prove local well-posedness for (18.13), we will apply the Banach fixed point theorem to the mapping $\varPhi_{v_0} : B^T_r(v_0) \to B^T_r(v_0)$ defined by $\varPhi_{v_0}(v) = u$, where u is the $L^p_{w_\alpha}$-solution to

$$\begin{cases} u' + A(u_0)u &= (A(u_0) - A(v))v + F(v), \quad \text{on } (0,T), \\ u(0) &= v_0. \end{cases} \tag{18.20}$$

Below we will first ensure that $B^T_r(v_0) \subseteq O_{\sigma,p}$ for $\varepsilon, r > 0$ small enough, so that $A(v)$ and $F(v)$ are well-defined. Then from its definition, it is clear that $\varPhi_{v_0}$ maps $B^T_r(v_0)$ to $\mathrm{MR}^p_\alpha(0,T)$. Below we will check that for $\varepsilon, r, T > 0$ small enough, $\varPhi_{v_0}$ is well defined as a mapping from $B^T_r(v_0)$ to itself by using the maximal regularity assumption on $A(u_0)$ and the mapping properties of A and F. Note that a function u is an $L^p_{w_\alpha}$-solution to (18.13) if and only if u is an $L^p_{w_\alpha}$-solution (C_{w_α}-solution if $p = \infty$) to (18.20) with $u = v$. Before we turn to the fixed point argument we need several preparatory lemmas.

Choose $\varepsilon_0 > 0$ such that $\overline{B_{X_{\sigma,p}}(u_0, \varepsilon_0)} \subseteq O_{\sigma,p}$. Fix $T_1 > 0$ such that

$$\|z_{u_0} - u_0\|_{C([0,T_1];X_{\sigma,p})} < \varepsilon_0/3. \tag{18.21}$$

By Corollaries 17.2.37 and 17.2.48, there is a constant C_{T_1} such that for every $v_0 \in X_{\sigma,p}$ we have

$$\|z_{v_0}\|_{\mathrm{MR}^p_\alpha(0,T_1)} \leqslant C_{T_1} \|v_0\|_{X_{\sigma,p}}. \tag{18.22}$$

The constant C_{T_1} will depend on T_1 in general, but this will not create problems since T_1 is fixed.

In order to show that $A(v)$ and $F(v)$ in (18.20) are well defined, we need to check that $v(t) \in O_{\sigma,p}$ for all $t \in (0,T)$ when $\varepsilon \in (0,\varepsilon_0)$, $r \in (0,1]$, and $T \in (0,T_1]$ are small enough. This is taken care of in the next lemma.

Lemma 18.2.9. *Let Assumption 18.2.2 hold, and let $\varepsilon_0 > 0$ be chosen as before (18.21). For small enough $r \in (0,1]$ and $\varepsilon \in (0,\varepsilon_0)$ the following holds: For all $v_0 \in B_{X_{\sigma,p}}(u_0,\varepsilon)$, all $T \in (0,T_1]$, and all $v \in B_r^T(v_0)$, one has $\|v - u_0\|_{C([0,T];X_{\sigma,p})} < \varepsilon_0$, and thus $v(t) \in O_{\sigma,p}$ for all $t \in [0,T]$.*

Proof. For notational convenience we write $\|\cdot\|_{\infty,T} = \|\cdot\|_{C([0,T];X_{\sigma,p})}$. For all $v \in B_r^T(v_0)$,

$$\begin{aligned}
\|v - z_{v_0}\|_{\infty,T} &\leqslant M_1\|v - z_{v_0}\|_{\mathrm{MR}^p_\alpha(0,T)} && \text{(by (18.16))}\\
&\leqslant M_1\|v - z_{u_0}\|_{\mathrm{MR}^p_\alpha(0,T)} + M_1\|z_{u_0} - z_{v_0}\|_{\mathrm{MR}^p_\alpha(0,T_1)}\\
&\leqslant M_1 r + M_1 C_{T_1}\|u_0 - v_0\|_{X_{\sigma,p}} && \text{(by (18.22))}.
\end{aligned}$$

Therefore, by (18.16), (18.21), and (18.22)

$$\begin{aligned}
\|v - u_0\|_{\infty,T} &\leqslant \|v - z_{v_0}\|_{\infty,T} + \|z_{v_0} - z_{u_0}\|_{\infty,T_1} + \|z_{u_0} - u_0\|_{\infty,T_1}\\
&\leqslant M_1 r + C_{T_1}(M_1 + M_{1,T_1})\|u_0 - v_0\|_{X_{\sigma,p}} + \|z_{u_0} - u_0\|_{\infty,T_1}\\
&\leqslant M_1 r + C_{T_1}(M_1 + M_{1,T_1})\varepsilon + \varepsilon_0/3.
\end{aligned} \tag{18.23}$$

This implies the required result for all $r, \varepsilon > 0$ small enough. ☐

In the next lemma we collect some estimates for A, F_{Tr}, and F_{c}, which will be used to ensure that Φ_{v_0} maps $B_r^T(v_0)$ to itself.

Lemma 18.2.10 (Smallness). *Let Assumption 18.2.2 hold. Fix $T \in (0,T_1]$ and let $\varepsilon \in (0,\varepsilon_0)$ and $r \in (0,1]$ be as in Lemma 18.2.9. Then for all $v_0 \in B_{X_{\sigma,p}}(u_0,\varepsilon)$ and $v \in B_r^T(v_0)$ we have*

$$\begin{aligned}
&\|(A(v) - A(u_0))v\|_{E_0}\\
&\qquad\leqslant \big(M_1 r + M_{2,T_1}\varepsilon + \|z_{u_0} - u_0\|_{C([0,T];X_{\sigma,p})}\big)(r + \|z_{u_0}\|_{E_1}),\\
&\|F_{\mathrm{Tr}}(v)\|_{E_0} \leqslant T^{\alpha+\frac{1}{p}}\big(L_{\mathrm{Tr}}\varepsilon_0 + \|F_{\mathrm{Tr}}(u_0)\|_{X_0}\big),\\
&\|F_{\mathrm{c}}(v)\|_{E_0} \leqslant C_{\varepsilon,r,T}(u_0)r + C_{\varepsilon,T}(u_0),
\end{aligned}$$

where $C_{\varepsilon,r,T}(u_0)$ and $C_{\varepsilon,T}(u_0)$ are independent of v_0 and v, $C_{\varepsilon,r,T}(u_0)$ and $C_{\varepsilon,T}(u_0)$ are non-decreasing in each of the variables ε, r, and T, and satisfy $C_{\varepsilon,r,T}(u_0), C_{\varepsilon,T}(u_0) \to 0$ as $\varepsilon, r, T \downarrow 0$.

Proof. We use the short-hand notation $\|\cdot\|_{\infty,T} := \|\cdot\|_{C([0,T];X_{\sigma,p})}$.

As in (18.23), one sees that

$$\|v - u_0\|_{\infty,T} \leqslant M_1 r + C_{T_1}(M_1 + M_{1,T_1})\varepsilon + \|z_{u_0} - u_0\|_{\infty,T}.$$

Therefore, by Lemma 18.2.8,

$$\begin{aligned}
&\|(A(v) - A(u_0))v\|_{E_0}\\
&\qquad \leqslant L\|v - u_0\|_{\infty,T}\|v\|_{E_1}\\
&\qquad \leqslant \big(M_1 r + C_{T_1}(M_1 + M_{1,T_1})\varepsilon + \|z_{u_0} - u_0\|_{\infty,T}\big)\|v\|_{E_1}.\\
&\qquad \leqslant \big(M_1 r + C_{T_1}(M_1 + M_{1,T_1})\varepsilon + \|z_{u_0} - u_0\|_{\infty,T}\big)(r + \|z_{u_0}\|_{E_1}).
\end{aligned}$$

For F_{Tr} we have pointwise estimate

$$\begin{aligned}
\|F_{\mathrm{Tr}}(v)\|_{X_0} &\leqslant \|F_{\mathrm{Tr}}(v) - F_{\mathrm{Tr}}(u_0)\|_{X_0} + \|F_{\mathrm{Tr}}(u_0)\|_{X_0}\\
&\leqslant L_{\mathrm{Tr}}\|v - u_0\|_{X_{\sigma,p}} + \|F_{\mathrm{Tr}}(u_0)\|_{X_0}\\
&\leqslant L_{\mathrm{Tr}}\varepsilon_0 + \|F_{\mathrm{Tr}}(u_0)\|_{X_0},
\end{aligned}$$

where in the last step we used Lemma 18.2.9. Taking $L^p_{w_\alpha}$-norms, we obtain

$$\|F_{\mathrm{Tr}}(v)\|_{E_0} \leqslant T^{\alpha+\frac{1}{p}}(L_{\mathrm{Tr}}\varepsilon_0 + \|F_{\mathrm{Tr}}(u_0)\|_{X_0}).$$

The estimate for F_{c} is more difficult to obtain. By the second estimate in (18.17),

$$\|F_{\mathrm{c}}(v) - F_{\mathrm{c}}(z_{u_0})\|_{E_0} \leqslant \sum_{j=1}^{m} C^{\rho_j}_{\beta_j,X} L_{\mathrm{c}}\big(T^{\delta_j} + \|v\|^{\rho_j}_{Y_j} + \|z_{u_0}\|^{\rho_j}_{Y_j}\big)\|v - z_{u_0}\|_{Y_j}.$$

It remains to estimate $\|v\|_{Y_j}$ and $\|v - z_{u_0}\|_{Y_j}$. By (18.16),

$$\begin{aligned}
\|v - z_{u_0}\|_{Y_j} &\leqslant \|v - z_{v_0}\|_{Y_j} + \|z_{v_0} - z_{u_0}\|_{Y_j}\\
&\leqslant M_1\|v - z_{v_0}\|_{\mathrm{MR}^p_\alpha(0,T)} + M_{1,T_1}\|z_{v_0} - z_{u_0}\|_{\mathrm{MR}^p_\alpha(0,T_1)}\\
&\leqslant M_1\|v - z_{u_0}\|_{\mathrm{MR}^p_\alpha(0,T)} + (M_1 + M_{1,T_1})\|z_{v_0} - z_{u_0}\|_{\mathrm{MR}^p_\alpha(0,T_1)}\\
&\leqslant M_1 r + (M_1 + M_{1,T_1})C_{T_1}\|v_0 - u_0\|_{X_{\sigma,p}}\\
&\leqslant M_1 r + (M_1 + M_{1,T_1})C_{T_1}\varepsilon,
\end{aligned}$$

applying (18.22) in the penultimate estimate. Similarly,

$$\|v\|_{Y_j} \leqslant \|v - z_{u_0}\|_{Y_j} + \|z_{u_0}\|_{Y_j} \leqslant M_1 r + (M_1 + M_{1,T_1})C_{T_1}\varepsilon + \|z_{u_0}\|_{Y_j}.$$

Combining things, we obtain the estimate

$$\begin{aligned}
&\|F_{\mathrm{c}}(v)\|_{E_0}\\
&\leqslant \|F_{\mathrm{c}}(v) - F_{\mathrm{c}}(z_{u_0})\|_{E_0} + \|F_{\mathrm{c}}(z_{u_0})\|_{E_0}\\
&\leqslant \sum_{j=1}^{m} C^{\rho_j}_{\beta_j,X} L_{\mathrm{c}}\big(T^{\delta_j} + 2(M_1 r + \widetilde{C}_{T_1}\varepsilon + k_{j,T}(u_0))^{\rho_j}\big)(M_1 r + \widetilde{C}_{T_1}\varepsilon) + k_{c,T}(u_0),
\end{aligned}$$

where have set $\widetilde{C}_{T_1} = (M_1 + M_{1,T_1})C_{T_1}$, $k_{j,T}(u_0) = \|z_{u_0}\|_{Y_j}$, and $k_{c,T}(u_0) = \|F_{\mathrm{c}}(z_{u_0})\|_{E_0}$. Note that $k_{j,T}(u_0) \to 0$ and $k_{c,T}(u_0) \to 0$ as $T \downarrow 0$ since $z_{u_0} \in \mathrm{MR}^p_\alpha(0,T) \subseteq Y_j$ and since $F_{\mathrm{c}}(z_{u_0}) \in E_0$ by Lemma 18.2.8.

The estimate $\|F_{\rm c}(v)\|_{E_0} \leqslant C_{\varepsilon,r,T}(u_0)r + C_{\varepsilon,T}(u_0)$ in the statement of the lemma now follows, with constants

$$C_{\varepsilon,r,T}(u_0) = \sum_{j=1}^{m} C^{\rho_j}_{\beta_j,X} L_{\rm c}\big(T^{\delta_j} + 2(M_1 r + \widetilde{C}_{T_1}\varepsilon + k_{j,T}(u_0))^{\rho_j}\big)M_1$$
$$C_{\varepsilon,T}(u_0) = \widetilde{C}_{T_1}\varepsilon M_1^{-1} C_{\varepsilon,1,T}(u_0) + k_{c,T}(u_0),$$

where we used that $r \in (0, 1]$. □

Remark 18.2.11. In the last part of the proof one does not have $k_{j,T}(u_0) \to 0$ and $k_{c,T}(u_0) \to 0$ as $T \downarrow 0$ if one were to use maximal $L^\infty_{w_\alpha}$-regularity or data u_0 in $(X_0, X_1)_{\sigma,\infty}$ rather than in the closed subspace $X_{\sigma,\infty}$. This is one of the reasons for working with maximal C_{w_α}-regularity and data in $X_{\sigma,\infty}$. It is also clear from the above proof that $\alpha = 0$ leads to difficulties if $p = \infty$. For example, the estimate for $F_{\rm Tr}(v)$ in Lemma 18.2.10 contains a factor $T^{\alpha+\frac{1}{p}}$ which does not vanish in the limit $T \downarrow 0$ if $\alpha = 0$ and $p = \infty$.

The final lemma contains Lipschitz variations of the above estimates, which will be used to show that Φ_{v_0} is a uniform contraction.

Lemma 18.2.12 (Lipschitz estimates). *Let Assumption 18.2.2 hold. Fix $T \in (0, T_1]$ and let $\varepsilon \in (0, \varepsilon_0)$ and $r \in (0, 1]$ be as in Lemma 18.2.9. Then for all $v_{1,0}, v_{2,0} \in B_{X_{\sigma,p}}(u_0, \varepsilon)$, and all $v_1 \in B^T_r(v_{1,0})$ and $v_2 \in B^T_r(v_{2,0})$,*

$$\|(A(v_1) - A(v_2))v_1\|_{E_0} + \|(A(u_0) - A(v_2))(v_1 - v_2)\|_{E_0}$$
$$+ \|F_{\rm Tr}(v_1) - F_{\rm Tr}(v_2)\|_{E_0} + \|F_{\rm c}(v_1) - F_{\rm c}(v_2)\|_{E_0}$$

can be estimated from above by

$$L_{\varepsilon,r,T}(u_0)(\|v_1 - v_2\|_{{\rm MR}^p_\alpha(0,T)} + \|v_{1,0} - v_{2,0}\|_{X_{\sigma,p}}),$$

where $L_{\varepsilon,r,T}(u_0)$ is a constant independent of $v_{1,0}, v_{2,0}, v_1, v_2$, non-decreasing in each of the variables ε, r, T, and satisfying $L_{\varepsilon,r,T}(u_0) \to 0$ as $\varepsilon, r, T \downarrow 0$.

Proof. We use the short-hand notation $\|\cdot\|_{\infty,T} := \|\cdot\|_{C([0,T];X_{\sigma,p})}$.

First we provide an estimate for $\|v\|_{\infty,T}$ and $\|v - u_0\|_{\infty,T}$ for $v \in B^T_r(v_0)$ and $v_0 \in B_{X_{\sigma,p}}(u_0, \varepsilon)$. By (18.22) and (18.16),

$$\begin{aligned}\|v\|_{\infty,T} &\leqslant \|z_{v_0}\|_{\infty,T} + \|v - z_{v_0}\|_{\infty,T}\\ &\leqslant M_{1,T_1} C_{T_1}\|v_0\|_{X_{\sigma,p}} + M_1\|v - z_{v_0}\|_{{\rm MR}^p_\alpha(0,T)}\\ &\leqslant (M_{1,T_1} + M_1)C_{T_1}\|v_0\|_{X_{\sigma,p}} + M_1\|v\|_{{\rm MR}^p_\alpha(0,T)}.\end{aligned} \tag{18.24}$$

Similarly, setting $k_T(u_0) := \|z_{u_0} - u_0\|_{\infty,T}$,

$$\begin{aligned}
&\|v-u_0\|_{\infty,T}\\
&\quad\leqslant \|v-z_{v_0}\|_{\infty,T}+\|z_{v_0}-z_{u_0}\|_{\infty,T}+k_T(u_0)\\
&\quad\leqslant M_1\|v-z_{v_0}\|_{\mathrm{MR}^p_\alpha(0,T)}+M_{1,T_1}\|z_{v_0}-z_{u_0}\|_{\mathrm{MR}^p_\alpha(0,T_1)}+k_T(u_0)\\
&\quad\leqslant M_1\|v-z_{u_0}\|_{\mathrm{MR}^p_\alpha(0,T)}+(M_1+M_{1,T_1})\|z_{v_0}-z_{u_0}\|_{\mathrm{MR}^p_\alpha(0,T_1)}+k_T(u_0)\\
&\leqslant M_1 r+(M_1+M_{1,T_1})C_{T_1}\varepsilon+k_T(u_0).
\end{aligned} \tag{18.25}$$

To estimate the first A-term, by Lemma 18.2.8 we obtain

$$\begin{aligned}
\|(A(v_1)-A(v_2))v_1\|_{E_0} &\leqslant L\|v_1-v_2\|_{\infty,T}\|v_1\|_{E_1}\\
&\leqslant L\|v_1-v_2\|_{\infty,T}(r+\|z_{u_0}\|_{E_1}).
\end{aligned}$$

Therefore, the required estimate follows from (18.24) with $v_0=0$.

For the second A-term, we again use Lemma 18.2.8 and obtain

$$\|(A(u_0)-A(v_2))(v_1-v_2)\|_{E_0}\leqslant L\|u_0-v_2\|_{\infty,T}\|v_1-v_2\|_{E_1}.$$

Therefore, the required estimate follows from (18.25).

For the F_{Tr}-term, we use Lemma 18.2.8 to obtain

$$\|F_{\mathrm{Tr}}(v_1)-F_{\mathrm{Tr}}(v_2)\|_{E_0}\leqslant T^{\alpha+\frac{1}{p}}L_{\mathrm{Tr}}\|v_1-v_2\|_{\infty,T}.$$

Therefore, the estimate follows from (18.24) again.

The F_{c}-term is more difficult to estimate. In the same way as in (18.24) and (18.25) one shows that

$$\|v\|_{Y_j}\leqslant (M_{1,T_1}+M_1)C_{T_1}\|v_0\|_{X_{\sigma,p}}+M_1\|v\|_{\mathrm{MR}^p_\alpha(0,T)} \tag{18.26}$$

and

$$\|v\|_{Y_j}\leqslant M_1 r+(M_1+M_{1,T_1})C_{T_1}\varepsilon+\|z_{u_0}\|_{Y_j}. \tag{18.27}$$

By the second estimate in (18.17),

$$\|F_{\mathrm{c}}(v_1)-F_{\mathrm{c}}(v_2)\|_{E_0}\leqslant \sum_{j=1}^m C^{\rho_j}_{\beta_j,X}L_{\mathrm{c}}\big(T^{\delta_j}+\|v_1\|^{\rho_j}_{Y_j}+\|v_2\|^{\rho_j}_{Y_j}\big)\|v_1-v_2\|_{Y_j}.$$

Using (18.26), we find

$$\|v_1-v_2\|_{Y_j}\leqslant (M_{1,T_1}+M_1)C_{T_1}\|v_{1,0}-v_{2,0}\|_{X_{\sigma,p}}+M_1\|v_1-v_2\|_{\mathrm{MR}^p_\alpha(0,T)}.$$

The required estimate for F_{c} now follows by applying (18.27) to estimate $\|v_1\|^{\rho_j}_{Y_j}$ and $\|v_2\|^{\rho_j}_{Y_j}$. □

After these preparations we are ready to turn to the proof of Theorem 18.2.6. It will be useful to recall the maximal regularity estimate which follows from Corollaries 17.2.37 and 17.2.48: for all $f \in E_0$ and $v_0 \in X_{\sigma,p}$ there exists a unique $L^p_{w_\alpha}$-solution (C_{w_α}-solution if $p = \infty$) to the problem

$$\begin{cases} u' + A(u_0)u &= f \quad \text{on } (0,T), \\ u(0) &= v_0, \end{cases}$$

and there exists a constant $C_T \geqslant 0$, independent of f and v_0, such that

$$\|u\|_{\mathrm{MR}^p_\alpha(0,T)} \leqslant C_T \|f\|_{E_0} + C_T \|v_0\|_{X_{\sigma,p}}. \tag{18.28}$$

This constant C_T also depends on $A(u_0)$ and p, but we can choose it in such a way that $C_T \leqslant C_{T_1}$ whenever $T < T_1$; this follows from a weighted version of (17.25).

Proof of Theorem 18.2.6. Fix $\varepsilon \in (0, \varepsilon_0)$ and $r \in (0,1]$ be as in Lemma 18.2.9, and let $T \in (0, T_1]$. Let $B^T_r(v_0)$ be as in (18.19). Let $\Phi_{v_0} : B^T_r(v_0) \to \mathrm{MR}^p_\alpha(0,T)$ be defined by $\Phi_{v_0}(v) := u$, where u is the $L^p_{w_\alpha}$-solution (C_{w_α}-solution if $p = \infty$) to the problem

$$\begin{cases} u' + A(u_0)u &= (A(u_0) - A(v))v + F(v), \\ u(0) &= v_0. \end{cases} \tag{18.29}$$

Then v takes values in $O_{\sigma,p}$ by Lemma 18.2.9, and we have $(A(v) - A(u_0))v \in E_0$ and $F(v) \in E_0$ by Lemma 18.2.10. Below Theorem 18.2.6 we have already observed that local existence and uniqueness follow if we can show that Φ_{v_0} has a unique fixed point.

Since $u - z_{u_0}$ satisfies (18.29) with v_0 replaced by $v_0 - u_0$, by the maximal regularity estimate (18.28) applied on $(0, T_1)$ (see (18.21) for the definition of T_1) we have

$$\begin{aligned} \|u - z_{u_0}\|_{\mathrm{MR}^p_\alpha(0,T)} &\leqslant C_{A,T_1}\big(\|u_0 - v_0\|_{X_{\sigma,p}} + \|(A(u_0) - A(v))v + F(v)\|_{E_0}\big) \\ &\leqslant C_{A,T_1}\varepsilon + \widetilde{C}_{\varepsilon,r,T}\, r + \widetilde{C}_{\varepsilon,T}, \end{aligned}$$

applying Lemma 18.2.10 in the last step, and where $\widetilde{C}_{\varepsilon,r,T}$ and $\widetilde{C}_{\varepsilon,T}$ are constants such that $\widetilde{C}_{\varepsilon,r,T} \to 0$ as $\varepsilon, r, T \downarrow 0$. Therefore, for $r, \varepsilon, T > 0$ small enough we obtain $\|u - z_{u_0}\|_{\mathrm{MR}^p_\alpha(0,T)} \leqslant r$, and thus $u \in B^T_r(v_0)$.

Next, fix $v_{j,0} \in B_{X_{\sigma,p}}(u_0, \varepsilon)$ and $v_j \in B^T_r(v_{j,0})$ for $j \in \{1,2\}$. Then $u = \Phi_{v_{1,0}}(v_1) - \Phi_{v_{2,0}}(v_2)$ solves the problem

$$\begin{cases} u' + A(u_0)u &= (A(u_0) - A(v_1))v_1 - (A(u_0) - A(v_2))v_2 + F(v_1) - F(v_2), \\ u(0) &= v_{1,0} - v_{2,0}. \end{cases}$$

Therefore, by the maximal regularity estimate (18.28),

$$\|u\|_{\mathrm{MR}^p_\alpha(0,T)} \leqslant C_{A,T_1}(R_A + R_F) + C_{A,T_1}\|v_{1,0} - v_{2,0}\|_{X_{\sigma,p}},$$

where

$$\begin{aligned} R_A &:= \|(A(u_0) - A(v_1))v_1 - (A(u_0) - A(v_2))v_2\|_{E_0} \\ &\leqslant \|(A(v_1) - A(v_2))v_1\|_{E_0} + \|(A(u_0) - A(v_2))(v_1 - v_2)\|_{E_0} \end{aligned}$$

and

$$\begin{aligned} R_F &:= \|F(v_1) - F(v_2)\|_{E_0} \\ &\leqslant \|F_{\mathrm{Tr}}(v_1) - F_{\mathrm{Tr}}(v_2)\|_{E_0} + \|F_{\mathrm{c}}(v_1) - F_{\mathrm{c}}(v_2)\|_{E_0}. \end{aligned}$$

From Lemma 18.2.12 we deduce that

$$\begin{aligned} \|u\|_{\mathrm{MR}^p_\alpha(0,T)} \leqslant C_{A,T_1} L_{\varepsilon,r,T}(u_0)\|v_1 - v_2\|_{\mathrm{MR}^p_\alpha(0,T)} \\ + C_{A,T_1}(L_{\varepsilon,r,T}(u_0) + 1)\|v_{1,0} - v_{2,0}\|_{X_{\sigma,p}}. \end{aligned}$$

Choosing $\varepsilon > 0$, $r > 0$, and $T > 0$ so small that $C_{A,T_1} L_{\varepsilon,r,T}(u_0) \leqslant 1/2$, we obtain

$$\begin{aligned} \|\Phi_{v_{1,0}}(v_1) - \Phi_{v_{2,0}}(v_2)\|_{\mathrm{MR}^p_\alpha(0,T)} \leqslant \frac{1}{2}\|v_1 - v_2\|_{\mathrm{MR}^p_\alpha(0,T)} \\ + (C_{A,T_1} + 1)\|v_{1,0} - v_{2,0}\|_{X_{\sigma,p}}. \end{aligned} \tag{18.30}$$

The estimate (18.30) allows us to finish the proof of local well-posedness. By (18.30), $\Phi_{v_0} : B^T_r(v_0) \to B^T_r(v_0)$ is a uniform contraction, and thus it has a unique fixed point $u_{v_0} \in B^T_r(v_0)$. This is the required solution to (18.13). Moreover, (18.30) implies that for all $v_{1,0}, v_{2,0} \in B_{X_{\sigma,p}}(u_0, \varepsilon)$,

$$\begin{aligned} \|u_{v_{1,0}} - u_{v_{2,0}}\|_{\mathrm{MR}^p_\alpha(0,T)} \leqslant \frac{1}{2}\|u_{v_{1,0}} - u_{v_{2,0}}\|_{\mathrm{MR}^p_\alpha(0,T)} \\ + (C_{A,T_1} + 1)\|v_{1,0} - v_{2,0}\|_{X_{\sigma,p}} \end{aligned}$$

which implies

$$\|u_{v_{1,0}} - u_{v_{2,0}}\|_{\mathrm{MR}^p_\alpha(0,T)} \leqslant 2(C_{A,T_1} + 1)\|v_{1,0} - v_{2,0}\|_{X_{\sigma,p}}.$$

This gives (18.14).

It remains to prove uniqueness. Uniqueness does hold if we only consider solutions in $B^T_r(v_0)$. In order to derive uniqueness for the larger set $\mathrm{MR}^p_\alpha(0,T)$, we will replace ε and T by suitable smaller values $\widetilde{\varepsilon}$ and $\widetilde{T}$. The above estimates then show that $\Phi_{v_0} : B^{\widetilde{T}}_r(v_0) \to B^{\widetilde{T}}_r(v_0)$ and (18.30) holds with T replaced by $\widetilde{T}$.

Let $\widetilde{\varepsilon} := \min\big\{\varepsilon, \frac{r}{8(C_{A,T_1}+1)}\big\}$ and set

$$\widetilde{T} := \inf\Big\{t \in [0,T] : \|u_{u_0} - z_{u_0}\|_{\mathrm{MR}^p_\alpha(0,t)} \geqslant \frac{r}{2}\Big\},$$

where $\inf \varnothing := T$. Then, for all $v_0 \in B_{X_{\sigma,p}}(u_0, \widetilde{\varepsilon})$,

$$\|u_{v_0} - u_{u_0}\|_{\mathrm{MR}^p_\alpha(0,T)} \leqslant 2(C_{A,T_1} + 1)\|v_0 - u_0\|_{X_{\sigma,p}} \leqslant \frac{r}{4}.$$

In particular,

$$\|u_{v_0} - z_{u_0}\|_{\mathrm{MR}^p_\alpha(0,\widetilde{T})} \leqslant \|u_{v_0} - u_{u_0}\|_{\mathrm{MR}^p_\alpha(0,\widetilde{T})} + \|u_{u_0} - z_{u_0}\|_{\mathrm{MR}^p_\alpha(0,\widetilde{T})} \leqslant \frac{3r}{4}.$$

We claim that for every $v_0 \in B_{X_{\sigma,p}}(u_0, \widetilde{\varepsilon})$, the element $u_{v_0} \in \mathrm{MR}^p_\alpha(0,\widetilde{T})$ is the unique $L^p_{w_\alpha}$-solution to (18.13). To show this, we will prove the slightly stronger result (which will play a key role in the construction of the maximal solution in Section 18.2.d) that, for an $\tau > 0$, if $v \in \mathrm{MR}^p_\alpha(0,\tau)$ is an $L^p_{w_\alpha}$-solution to (18.13), then $v \equiv u_{v_0}$ on $[0, \widetilde{T} \wedge \tau]$. This will give the theorem for $\widetilde{T}$ instead of T.

Let

$$\tau_v := \inf\{t \in [0, \widetilde{T} \wedge \tau] : \|v - z_{u_0}\|_{\mathrm{MR}^p_\alpha(0,t)} \geqslant r\},$$

setting $\inf \varnothing := \widetilde{T} \wedge \tau$. Then $v|_{[0,\tau_v]}$ belongs to $B^{\tau_v}_r(v_0)$, and since $\tau_v \leqslant T$ it follows that $v|_{[0,\tau_v]} = u_{v_0}|_{[0,\tau_v]}$ by uniqueness of the fixed point in $B^{\tau_v}_r(v_0)$. Thus we obtain

$$\|v - z_{u_0}\|_{\mathrm{MR}^p_\alpha(0,\tau_v)} = \|u_{v_0} - z_{u_0}\|_{\mathrm{MR}^p_\alpha(0,\tau_v)} \leqslant \|u_{v_0} - z_{u_0}\|_{\mathrm{MR}^p_\alpha(0,\widetilde{T})} < r,$$

and therefore $\tau_v = \widetilde{T} \wedge \tau$. This gives the claimed result. □

18.2.d Maximal solutions

Having established local well-posedness in Theorem 18.2.6, we will now extend the time interval on which the solution exists to a maximal time interval $[0, T_{\max}(v_0))$.

Definition 18.2.13. *Let Assumption 18.2.2 hold and assume that* $v_0 \in O_{\sigma,p}$. *A pair* $(v, T_{\max}(v_0))$ *is called a* maximal $L^p_{w_\alpha}$-solution to (18.13) *if* $T_{\max}(v_0) \in (0,\infty]$ *and* $v : [0, T_{\max}(v_0)) \to X_0$ *are such that*

- *for all* $T \in (0, T_{\max}(v_0))$, $v|_{(0,T)}$ *belongs to* $\mathrm{MR}^p_\alpha(0,T)$ *and is an* $L^p_{w_\alpha}$*-solution to* (18.13) *on* $(0,T)$*;*
- *whenever* $u \in \mathrm{MR}^p_\alpha(0,T)$ *is a unique* $L^p_{w_\alpha}$*-solution to* (18.13) *for some* $T > 0$*, one has* $T \leqslant T_{\max}(v_0)$ *and* $u \equiv v$ *on* $(0,T)$*.*

Note that maximal $L^p_{w_\alpha}$-solutions are unique. An even stronger uniqueness assertion will be derived in Remark 18.2.16 under further restrictions. We will now show that the solution to (18.13) provided by Theorem 18.2.6 can be extended to a maximal $L^p_{w_\alpha}$-solution.

Theorem 18.2.14 (Maximal solutions). *Let Assumption 18.2.2 hold, let $u_0 \in O_{\sigma,p}$, and suppose that $A(u_0)$ has maximal L^p-regularity (C-regularity if $p = \infty$) on finite time intervals. Let $\varepsilon > 0$ be as in Theorem 18.2.6, and let $v_0 \in O_{\sigma,p}$ be such that $\|u_0 - v_0\|_{X_{\sigma,p}} < \varepsilon$. Then there exists a maximal $L^p_{w_\alpha}$-solution $(u, T_{\max}(v_0))$ to (18.13).*

Proof. Let us say that an $L^p_{w_\alpha}$-solution v to (18.13) on $(0,T)$ *has the uniqueness property* if for any $\tau > 0$ and any $L^p_{w_\alpha}$-solution u to (18.13) on $(0,\tau)$, we have $v \equiv u$ on $[0, T \wedge \tau]$. Let $T_{\max}(v_0)$ be the supremum of all $T > 0$ such that there exists an $L^p_{w_\alpha}$-solution to (18.13) on $(0,T)$ with the uniqueness property. Then $T_{\max}(v_0) > 0$ by Theorem 18.2.6. Note that the uniqueness property was established as part of the uniqueness proof. It follows that there exists a maximal $L^p_{w_\alpha}$-solution $u : [0, T_{\max}(v_0)) \to X_0$ to (18.13). □

Theorem 18.2.15 (Global well-posedness for quasi-linear equations). *Let Assumption 18.2.2 hold, and suppose that for all $u_0 \in O_{\sigma,p}$ the operator $A(u_0)$ has maximal L^p-regularity (C-regularity if $p = \infty$) on finite time intervals. Let $v_0 \in O_{\sigma,p}$ and let $v : [0, T_{\max}(v_0)) \to X_0$ be the maximal solution provided by Theorem 18.2.14. If $T_{\max}(v_0) < \infty$, then either*

- $\lim_{t\uparrow T_{\max}(v_0)} v(t)$ *does not exist in $X_{\sigma,p}$, or*
- $v_* := \lim_{t\uparrow T_{\max}(v_0)} v(t)$ *exist in $X_{\sigma,p}$, but $v_* \notin O_{\sigma,p}$.*

The final assertion in the theorem is called a *blow-up criterion*. Blow-up criteria can be used to prove global well-posedness. In typical applications, assuming $T_{\max}(v_0) < \infty$, energy estimates can be used to show that $v_* := \lim_{t\uparrow T_{\max}(v_0)} v(t)$ exists in $O_{\sigma,p}$. This contradicts Theorem 18.2.15 and thus leads to $T_{\max}(v_0) = \infty$, i.e., global existence. Further blow-up criteria are discussed in the Notes.

Proof. Assuming that $T_0 := T_{\max}(v_0) < \infty$ and that $v_* := \lim_{t\uparrow T_0} v(t)$ exists in $X_{\sigma,p}$ with $v_* \in O_{\sigma,p}$, a contradiction will be derived.

The idea is to restart the problem at time T_0 with initial value v_* and apply Theorem 18.2.6 to extend v to a larger time interval $[0, T_0 + \delta]$. However, it is not self-evident that $v \in \mathrm{MR}^p_\alpha(0, T_0 + \delta)$. This problem will be overcome by using a compactness argument.

From the continuity of v and the assumption that the limit v_* at $t = T_0$ exist, it follows that the set

$$K := \{v(t) : t \in [0, T_0)\} \cup \{v_*\}$$

is compact in $X_{\sigma,p}$. By Theorem 18.2.6, for all $x \in K$ there exists an open ball $B(x, \varepsilon_x) \subseteq O_{\sigma,p}$ such that for initial values from $B(x, \varepsilon_x)$ we can find an $L^p_{w_\alpha}$-solution in $\mathrm{MR}^p_\alpha(0, t_x)$ for some $t_x > 0$. Since K is compact, the open cover $\{B(x, \varepsilon_x) : x \in K\}$ has a finite sub-cover $\{B(x_n, \varepsilon_{x_n}) : n = 1, \dots N\}$. Let $\delta := \min_{n=1,\dots,N} t_{x_n}$. Then for all $x \in K$ there exists a unique $L^p_{w_\alpha}$-solution $u_x \in \mathrm{MR}^p_\alpha(0, \delta)$ to the problem

$$\begin{cases} u' + A(u)u &= F(u), \\ u(0) &= x. \end{cases} \tag{18.31}$$

Now we are ready to define a suitable extension of v. Let $x := v(T_0 - \frac{1}{2}\delta)$, and let $u_x \in \mathrm{MR}^p_\alpha(0,\delta)$ be as above. Then $t \mapsto v(T_0 - \frac{1}{2}\delta + t)$ belongs to $\mathrm{MR}^p_\alpha(0,\gamma)$ for all $\gamma \in (0,\delta)$ and is an $L^p_{w_\alpha}$-solution to (18.31). Therefore, uniqueness gives that $v(T_0 - \frac{1}{2}\delta + t) = u_x(t)$ for all $t \in [0,\delta/2)$. Now one can check that the function $v_{\mathrm{ext}} : [0, T_0 + \delta/2] \to X_{\sigma,p}$ defined by

$$v_{\mathrm{ext}}(t) = \begin{cases} v(t), & t \in [0, T_0); \\ u_x(t - T_0 + \frac{1}{2}\delta), & t \in [T_0 - \frac{1}{2}\delta, T_0 + \frac{1}{2}\delta]. \end{cases}$$

is well defined, belongs to $\mathrm{MR}^p_\alpha(0, T + \frac{1}{2}\delta)$, and is an $L^p_{w_\alpha}$-solution to (18.13) on $(0, T_0 + \frac{1}{2}\delta)$. This contradicts the maximality of T_0. □

Under the conditions of Theorem 18.2.15, one can leave out the uniqueness from the second bullet in Definition 18.2.13. This excludes the existence of an (non-unique) $L^p_{w_\alpha}$-solution $u \in \mathrm{MR}^p_\alpha(0,T)$ which extends v.

Remark 18.2.16. Let Assumption 18.2.2 hold, and suppose that for all $u_0 \in O_{\sigma,p}$ the operator $A(u_0)$ has maximal L^p-regularity (C-regularity if $p = \infty$) on finite time intervals. Let $v_0 \in O_{\sigma,p}$ and let $v : [0, T_{\max}(v_0)) \to X_0$ be the maximal solution provided by Theorem 18.2.14. Now suppose that $u \in \mathrm{MR}^p_\alpha(0,T)$ is an $L^p_{w_\alpha}$-solution to (18.13) for some $T > 0$. We claim that $T \leqslant T_{\max}(v_0)$ and $u \equiv v$ on $(0,T)$. To see this, first note that by the uniqueness property of the proof of Theorem 18.2.14 one has $u = v$ on $[0, T \wedge T_{\max}(v_0))$. Thus it remains to show $T \leqslant T_{\max}(v_0)$. Suppose that $T > T_{\max}(v_0)$. Since $u \in \mathrm{MR}^p_\alpha(0,T)$, it follows from Lemma 18.2.7 that

$$v_* := \lim_{t \uparrow T_{\max}(v_0)} v(t) = \lim_{t \uparrow T_{\max}(v_0)} u(t) = u(T_{\max}(v_0)) \;\text{ exists in } X_{\sigma,p},$$

and $v_* \in O_{\sigma,p}$. This contradicts Theorem 18.2.15 and thus the claim follows.

As a consequence of Theorem 18.2.15 we obtain the following criteria for global well-posedness for (18.13) in the semi-linear case.

Theorem 18.2.17 (Global well-posedness for semi-linear equations). *Let Assumption 18.2.2 hold for any bounded open set $O_{\sigma,p}$, and that $A \in \mathscr{L}(X_1, X_0)$ has maximal L^p-regularity (maximal C-regularity if $p = \infty$) on finite time intervals. Then for every $v_0 \in X_{\sigma,p}$ there exists a maximal $L^p_{w_\alpha}$-solution $(v, T_{\max}(v_0))$ to (18.13) with $T_{\max}(v_0) > 0$. Moreover, if either one of the following holds:*

(1) $p < \infty$, $\quad \sup_{t \in [0, T_{\max}(v_0))} \|v(t)\|_{X_{\sigma,p}} + \|v\|_{L^p_{w_\alpha}(0, T_{\max}(v_0); X_1)} < \infty$;

(2) $p = \infty$, $\quad \sup_{t \in [0, T_{\max}(v_0))} \|v(t)\|_{X_{\sigma,\infty}} + t^\alpha \|v(t)\|_{X_1} < \infty$;

(3) $\sup_{t\in[0,T_{\max}(v_0))} \|v(t)\|_{X_{\sigma,p}} < \infty$ *and Assumption 18.2.2 holds in the subcritical case,*

then $T_{\max}(v_0) = \infty$*, and thus the* $L^p_{w_\alpha}$*-solution* v *exists globally.*

Proof. The existence of the maximal solution has already been observed in Theorem 18.2.14.

We start with a preliminary observation. Fix $\rho > 0$ and $T \in (0,\infty)$, and set $\beta^* := 1 - (\alpha + \frac{1}{p})(1 - \frac{1}{\rho+1})$. We claim that for all $\beta \in (\sigma, \beta^*]$ and $u \in L^\infty(0,T;X_{\sigma,p}) \cap L^p_{w_\alpha}(0,T;X_1)$ we have

$$\|u\|_{L^{hp}_{w_{\alpha/h}}(0,T;(X_0,X_1)_{\beta,1})} \leqslant C_T \|u\|^{\lambda}_{L^\infty(0,T;X_{\sigma,p})} \|u\|^{1-\lambda}_{L^p_{w_\alpha}(0,T;X_1)}, \tag{18.32}$$

where $h = \rho + 1$, $\lambda \in (0,1)$ is given by $\lambda = \frac{1-\beta}{\alpha+\frac{1}{p}}$, and where C_T also depends on α, h, p and is non-decreasing in T. From the assumption on β it follows that $\lambda \in [1 - \frac{1}{1+\rho}, 1)$. Moreover, if $\beta < \beta^*$, one even has $\lambda > 1 - \frac{1}{\rho+1}$. To prove (18.32), note that by (C.6), Theorem L.3.1, and (L.2),

$$\begin{aligned}\|u(t)\|_{\beta,1} &\leqslant C\|u(t)\|_{((X_0,X_1)_{\sigma,p},X_1)_{1-\lambda,1}} \\ &\leqslant C\|u(t)\|^{\lambda}_{\sigma,p}\|u(t)\|^{1-\lambda}_{X_1},\end{aligned}$$

with the understanding that $\|u(t)\|_{\sigma,p}$ needs to be replaced by $\|u(t)\|_{X_0}$ in the case $p = 1$. Taking $L^{hp}_{w_{\alpha/h}}(0,T)$-norms on both sides gives

$$\begin{aligned}\|u\|_{L^{hp}_{w_{\alpha/h}}(0,T;(X_0,X_1)_{\beta,1})} &\leqslant C\|u\|^{\lambda}_{L^\infty(0,T;X_{\sigma,p})}\|u\|^{1-\lambda}_{L^{hp(1-\lambda)}_{w_{\alpha/(hp(1-\lambda))}}(0,T;X_1)} \\ &\leqslant C_T\|u\|^{\lambda}_{L^\infty(0,T;X_{\sigma,p})}\|u\|^{1-\lambda}_{L^p_{w_\alpha}(0,T;X_1)},\end{aligned}$$

where we used $h(1-\lambda) = (1+\rho)(1-\lambda) \leqslant 1$.

(1): Suppose, for a contradiction, that $T_{\max}(v_0) < \infty$. Let $O_{\sigma,p} \subseteq X_{\sigma,p}$ be a bounded open set such that $\overline{v([0,T_{\max}(v_0)))} \subseteq O_{\sigma,p}$. Taking $\beta = \beta^*_j$ and $h = \rho_j + 1$ in (18.32), we obtain $u \in Y_j$ for every j, and thus $F_{\rm c}(v) \in L^p_{w_\alpha}(0,T_{\max}(v_0);X_0)$ by Lemma 18.2.8. Since $F_{\rm Tr} : O_{\sigma,p} \to X_0$ has linear growth, it is straightforward to check that

$$F_{\rm Tr}(v) \in L^\infty(0,T_{\max}(v_0);X_0) \subseteq L^p_{w_\alpha}(0,T_{\max}(v_0);X_0).$$

Therefore, maximal L^p-regularity of A implies that $v \in \mathrm{MR}^p_\alpha(0,T_{\max}(v_0))$. In particular, $\lim_{t\uparrow T_{\max}(v_0)} v(t)$ exists in $X_{\sigma,p}$ (see Lemma 18.2.7). This contradicts Theorem 18.2.14. It follows that $T_{\max}(v_0) = \infty$.

(2): This can be proved similarly, this time using maximal C-regularity.

(3): Suppose, for a contradiction, that $T_{\max}(v_0) < \infty$. Let $O_{\sigma,p}$ be as in the proof of (1), and let $T \in (0,T_{\max}(v_0))$. As before, it suffices to prove $v \in \mathrm{MR}^p(0,T_{\max}(v_0))$.

By maximal regularity (see Corollaries 17.2.37 and 17.2.48) we can estimate

$$\|v\|_{\mathrm{MR}^p(0,T)} \leqslant C(\|v_0\|_{X_{\sigma,p}} + \|F_{\mathrm{c}}(v)\|_{L^p_{w_\alpha}(0,T;X_0)} + \|F_{\mathrm{Tr}}(v)\|_{L^p_{w_\alpha}(0,T;X_0)}), \tag{18.33}$$

where the constant C depends on $T_{\max}(v_0)$, but not on T. As before, $\|F_{\mathrm{Tr}}(v)\|_{L^p_{w_\alpha}(0,T;X_0)}$ can be estimated above by $K(1+\|v\|_{L^\infty(0,T_{\max}(v_0);X_{\sigma,p})})$. The F_{c}-term is more complicated to handle; this is where the subcriticality enters. Set

$$\overline{Y}_j := \begin{cases} L^{(\rho_j+1)p}_{w_{\alpha/(\rho_j+1)}}(0,T;X_{\beta_j}) & \text{if } p<\infty; \\ C_{w_{\alpha/(\rho_j+1)},0}((0,T];X_{\beta_j}) & \text{if } p=\infty. \end{cases}$$

Fix $x \in O_{\sigma,p} \cap X_1$. Repeating the proof of the second estimate in (18.17) with β_j^* replaced by β_j, we obtain

$$\begin{aligned} \|F_{\mathrm{c}}(v)\|_{L^p_{w_\alpha}(0,T;X_0)} &\leqslant \|F_{\mathrm{c}}(v)-F_{\mathrm{c}}(x)\|_{L^p_{w_\alpha}(0,T;X_0)} + \|F_{\mathrm{c}}(x)\|_{L^p_{w_\alpha}(0,T;X_0)} \\ &\leqslant L_{\mathrm{c}} \sum_{j=1}^m \big(T^{\delta_j} + \|v\|^{\rho_j}_{\widetilde{Y}_j} + \|x\|^{\rho_j}_{\widetilde{Y}_j}\big)\|v-x\|^{\rho_j}_{\widetilde{Y}_j} \\ &\leqslant L_{\mathrm{c}} \sum_{j=1}^m C_{j,x} + \|v\|^{\rho_j+1}_{\widetilde{Y}_j}, \end{aligned}$$

where in the last step we used Young's inequality in the form $a^\rho b \leqslant a^{\rho+1} + b^{\rho+1}$, and the constant $C_{j,x}$ depends on $T_{\max}(v_0)$ but not on T. Let

$$M := \sup_{t\in[0,T_{\max}(v_0))} \|v(t)\|_{X_{\sigma,p}}.$$

By (18.32) with $h=\rho_j+1$, $\beta=\beta_j$, and $\lambda_j = \frac{1-\beta_j}{\alpha+\frac{1}{p}}$, we find that

$$\begin{aligned} \|v\|^{\rho_j+1}_{\widetilde{Y}_j} &\leqslant C_T^{\rho_j+1} M^{\lambda_j(\rho_j+1)} \|v\|^{(1-\lambda_j)(\rho_j+1)}_{L^p_{w_\alpha}(0,T;X_1)} \\ &\leqslant C_T^{\rho_j+1} C_{j,\varepsilon} M^{(\lambda_j(\rho_j+1))/(1-\beta_j)} + \varepsilon\|v\|_{L^p_{w_\alpha}(0,T;X_1)}, \end{aligned}$$

where we used $\beta_j = (1-\lambda_j)(\rho_j+1) \in (0,1)$ by subcriticality, and we used Young's inequality in the form $ab^{\beta_j} \leqslant \varepsilon^{-\beta_j/(1-\beta_j)} a^{1/(1-\beta_j)} + \varepsilon b$ for arbitrary $\varepsilon>0$. Taking $\sum_{j=1}^m$ this results in the estimate

$$\|F_{\mathrm{c}}(v)\|_{L^p_{w_\alpha}(0,T;X_0)} \leqslant C_{M,\varepsilon} + L_{\mathrm{c}} m\varepsilon \|v\|_{L^p_{w_\alpha}(0,T;X_1)}.$$

Combining this estimate with (18.33), we obtain

$$(1 - C\varepsilon L_{\mathrm{c}} m)\|v\|_{\mathrm{MR}^p(0,T)} \leqslant C(\|v_0\|_{X_{\sigma,p}} + \|F_{\mathrm{Tr}}(v)\|_{L^p_{w_\alpha}(0,T;X_0)}).$$

Setting $\varepsilon = (2CL_{\mathrm{c}} m)^{-1}$ and letting T tend to $T_{\max}(v_0)$, it follows that $v \in \mathrm{MR}^p(0,T_{\max}(v_0))$. ∎

18.3 Examples and comparison

In order to understand the assumptions on the non-linearity F_c in Assumption 18.2.2, we will now discuss in detail a standard situation, and make a comparison with Example 18.1.3 which involved only the non-linearity F_{Tr}.

Example 18.3.1 (Critical spaces and non-linearities). Let

$$X_0 := H^{s,q}(\mathbb{R}^d), \quad X_1 := H^{s+2,q}(\mathbb{R}^d)$$

with $s \in (-2, 0]$ and $q \in (1, \infty)$. Since $s + 2 > 0$, powers of functions in X_1 are well defined. Notice that X_1 features two more derivatives than X_0; this is the typical situation encountered in applications to PDEs with a leading term of second order. Note that (see Theorem 5.6.9)

$$[X_0, X_1]_\beta = H^{s+2\beta,q}(\mathbb{R}^d)$$

and

$$X_{\sigma,p} = B^{s+2\sigma}_{q,p}(\mathbb{R}^d),$$

where $p \in (1, \infty)$ (extensions to the end-points are possible, but not considered here for simplicity) and $\sigma \in (0, 1/p']$ are arbitrary but fixed for the moment.

Suppose now that $f \in C^1(\mathbb{R})$ satisfies

$$f(0) = 0 \quad \text{and} \quad |f'(t)| \leqslant \ell |t|^\rho, \quad t \in \mathbb{R}, \tag{18.34}$$

for a suitable exponent $\rho > 0$ and constant $\ell \geqslant 0$. Let $F_c : X_1 \to X_0$ be given by

$$(F_c(u))(x) := f(u(x)), \quad x \in \mathbb{R}^d.$$

Then F_c is well-defined and Lipschitz on bounded subsets of X_β under suitable conditions. Indeed, for all $u, v \in X_1$,

$$\begin{aligned}
\|F_c(u) - F_c(v)\|_{X_0} \\
&= \|f(u) - f(v)\|_{H^{s,q}} \\
&\leqslant C\|f(u) - f(v)\|_r && \text{(Sobolev embedding)} \\
&\leqslant C\ell\|(|u|^\rho + |v|^\rho)(u - v)\|_r && \text{(mean value theorem)} \\
&\leqslant C\ell(\|u\|^\rho_{(\rho+1)r} + \|v\|^\rho_{(\rho+1)r})\|u - v\|_{(\rho+1)r} && \text{(Hölder inequality)} \\
&\leqslant C\ell(\|u\|^\rho_{X_\beta} + \|v\|^\rho_{X_\beta})\|u - v\|_{X_\beta} && \text{(Sobolev embedding),}
\end{aligned}$$

provided we impose some further restrictions in order to justify the application of the Sobolev embeddings. Specifically, the first Sobolev embedding can be applied if $-\frac{d}{r} = s - \frac{d}{q}$ and $1 < r \leqslant q$, which leads to the condition

$$s > -\frac{d}{q'}. \tag{18.35}$$

The second Sobolev embedding can be applied if

$$s + 2\beta - \frac{d}{q} = -\frac{d}{(\rho+1)r}, \quad \text{and} \quad q \leqslant (\rho+1)r,$$

which after substitution of the identity $-\frac{d}{r} = s - \frac{d}{q}$ leads to the condition

$$s + 2\beta - \frac{d}{q} = \frac{1}{\rho+1}\Big(s - \frac{d}{q}\Big) \quad \text{and} \quad q \leqslant \frac{dq(\rho+1)}{d-qs}.$$

Thus we arrive at the conditions

$$\beta = \frac{\rho}{2(\rho+1)}\Big(\frac{d}{q} - s\Big) \quad \text{and} \quad s \geqslant -\frac{d\rho}{q}. \tag{18.36}$$

Sobolev embeddings can also be applied in sub-optimal cases, but here we wish to demonstrate certain optimality and scaling behaviour which is present only if all Sobolev embeddings are sharp.

Combining (18.36) with the (sub)criticality condition (18.12), we obtain

$$\rho\Big(\frac{d}{q} - s\Big) \leqslant 2 + 2\rho\sigma,$$

and criticality holds if

$$\sigma = \frac{1}{2}\Big(\frac{d}{q} - s\Big) - \frac{1}{\rho}.$$

Since $\sigma \in (0, 1/p']$ we arrive the following condition on (q, s) to obtain a critical setting:

$$0 < \frac{1}{2}\Big(\frac{d}{q} - s\Big) - \frac{1}{\rho} \leqslant \frac{1}{p'}. \tag{18.37}$$

If (18.37) holds for some p, then it also holds for all larger values of p, and one can take the limit $p \to \infty$. Thus (18.35), (18.36), (18.37), and the assumption $s \in (-2, 0]$ imply

$$\max\Big\{-2 + \frac{d}{q} - \frac{2}{\rho}, -2, -\frac{d}{q'}\Big\} < s < \frac{d}{q} - \frac{2}{\rho}, \text{ and } -\frac{d\rho}{q} \leqslant s \leqslant 0. \tag{18.38}$$

In the converse direction, if (18.38) holds, then (18.37) holds for large enough p, so the existence of a triple (p, q, s) satisfying the aforementioned conditions is equivalent to (18.38).

Elementary computations show that we can find pairs (s, q) satisfying these conditions holds if and only if

$$\rho > \frac{2}{d} \quad \text{and} \quad \frac{2}{\rho(\rho+1)} < \frac{d}{q}. \tag{18.39}$$

In this case, the corresponding critical space for the initial data is given by

$$X_{\sigma,p} = B^{s+2\sigma}_{q,p}(\mathbb{R}^d) = B^{\frac{d}{q}-\frac{2}{\rho}}_{q,p}(\mathbb{R}^d). \tag{18.40}$$

An interesting feature of (18.40) is that the parameter s does not appear in the critical space $X_{\sigma,p}$ and the smoothness parameter is independent of p.

Remark 18.3.2. The homogeneous variant of $B^{\frac{d}{q}-\frac{2}{\rho}}_{q,p}(\mathbb{R}^d)$ scales as $\|u(\lambda\cdot)\| \eqsim \lambda^{\frac{2}{\rho}}\|u\|$. It follows from this that if u is a solution to a PDE with leading second order differential operator in the space variables, with non-linearity $f(u) = k|u|^{\rho+1}$ (or similar scaling behaviour), and with initial data u_0, then $(t,x) \mapsto \lambda^{\frac{2}{\rho}} u(\lambda^2 t, \lambda x)$ is a solution to the same equation with initial data $\lambda^{\frac{2}{\rho}} u_0(\lambda\cdot)$. This shows that the scaling of the space we encountered in (18.40) is the correct one (up to being a inhomogeneous Besov space).

Specialising to the case $\frac{d}{q} - \frac{2}{\rho} = 0$ and taking p large enough, we also see that one can consider initial data from $L^q(\mathbb{R}^d)$, as this spaces embeds into $B^0_{q,p}(\mathbb{R}^d)$. This space has the same scaling behaviour as just discussed.

In (18.40) the limiting case where $q = \frac{1}{2}d\rho(\rho+1)$ shows that we can 'almost' treat initial data from the space $B^{-2/(\rho+1)}_{q,p}(\mathbb{R}^d)$. The less important so-called microscopical tuning parameter p in (18.40) needs to be so large that (18.37) holds.

Unlike in Example 18.1.3, it now becomes possible to take the special structure of f into account. The space of initial data which we could consider in the example was $B^{s+2-\frac{2}{p}}_{q,p}(\mathbb{R}^d)$ with $s \in (-2, 0]$ and $s+2-\frac{2}{p}-\frac{d}{q} > 0$. Under these restrictions, the smoothness parameter satisfies $s+2-\frac{2}{p} > \frac{d}{q}$, which leads to a much smaller class of initial data than considered in (18.40). Introducing weights in the set-up of Example 18.1.3, does not change anything.

Remark 18.3.3. When $\mathbb{R}^d$ is replaced by a bounded domain, the condition (18.34) on f in Example 18.3.1 can be weakened to

$$|f'(t)| \leqslant \ell(1 + |t|^\rho), \quad t \in \mathbb{R}.$$

Indeed, the step where Hölder's inequality is used can then be replaced by

$$\|(1 + |u|^\rho + |v|^\rho)(u - v)\|_r \leqslant C\ell(1 + \|u\|^\rho_{(\rho+1)r} + \|v\|^\rho_{(\rho+1)r})\|u - v\|_{(\rho+1)r}.$$

Similarly, one can check that $f(0)$ is allowed to be non-zero.

We finish this section with an example illustrating how Theorems 18.2.6 and 18.2.15 can be applied to obtain local and global well-posedness for certain concrete PDEs.

Example 18.3.4 (Local well-posedness for the Allen-Cahn equation). On $\mathbb{R}^d$ with $d \geqslant 2$ (the case $d = 1$ can be included by making subcritical choices) we consider the so-called *Allen-Cahn equation*

$$\begin{cases} \partial_t u - \Delta u &= -u^3 + u, \\ u(0) &= u_0. \end{cases} \tag{18.41}$$

This equation fits into the setting discussed in Example 18.3.1 with $X_0 = H^{s,q}(\mathbb{R}^d)$ and $X_1 = H^{s+2,q}(\mathbb{R}^d)$ for suitable (q,s). Indeed, taking $\rho = 2$, one checks that (18.39) holds if $1 < q < 3d$. Let $s \in (-2,0]$ be such that (18.38) holds with $\rho = 2$, and set $\sigma := \frac{1}{2}(\frac{d}{q} - s) - \frac{1}{2}$. Choose $p \in (1,\infty)$ so large that (18.37) holds. Then, by Example 18.3.1, $F(u) = -u^3$ satisfies the Assumption 18.2.2. We choose to include the linear part of $-u^3 + u$ into the operator A. Another possibility would be to put it into F as well, and consider $\rho_1 = 2$ and $\rho_2 > 0$ arbitrary small.

From Example G.5.6 it follows that for $s = 0$ the operator $Au = -\Delta u - u$ on X_0, with domain X_1, is sectorial of angle zero. Moreover, by Theorems 17.4.1 and 17.2.26, A has maximal L^p-regularity on finite time intervals for all $p \in (1,\infty)$. Since the Bessel potentials $(1-\Delta)^{t/2}$ commute with Δ, the maximal L^p-regularity extends to the full range $s \in \mathbb{R}$.

From now on we view (18.41) as an abstract problem of the form (18.13). In particular, we say that (18.41) admits a *(maximal)* (p,q,s,σ)*-solution* if (18.13) has a (maximal) $L^p_{w_\alpha}$-solution. Applying Theorems 18.2.6 and 18.2.14, it follows that for every $u_0 \in O_{\sigma,p} = X_{\sigma,p} = B^{\frac{d}{q}-1}_{q,p}(\mathbb{R}^d)$ (see (18.40)), the problem (18.41) admits a maximal (p,q,s,σ)-solution $(u, T_{\max}(u_0))$. Moreover,

$$\begin{aligned} u \in W^{1,p}_{w_\alpha}(0,T;H^{s,q}(\mathbb{R}^d)) \cap L^p_{w_\alpha}(0,T;H^{s+2,q}(\mathbb{R}^d)) \\ \cap C([0,T];B^{\frac{d}{q}-1}_{q,p}(\mathbb{R}^d)) \cap C([\tau,T];B^{s+2-\frac{2}{p}}_{q,p}(\mathbb{R}^d)) \end{aligned} \tag{18.42}$$

for all $0 < \tau < T < T_{\max}(u_0)$, where we used the instantaneous regularisation stated in (18.15).

Global well-posedness can often be obtained via Theorem 18.2.17, but to apply it to the rough initial data considered in the above example requires first performing a (weighted) bootstrap argument to obtain enough regularity in space and time. After that, suitable energy estimate can be applied. Bootstrapping regularity will not be discussed here (a concise discussion of this technique is included in the Notes). Instead, we will only prove global well-posedness for sufficiently smooth initial data. This is done in the next example. In particular, all initial data $u_0 \in L^q(\mathbb{R}^d)$ for $q \in (d,2d)$ are covered if $d \in \{2,3,4,5,6\}$.

Example 18.3.5 (Global well-posedness for the Allen-Cahn equation). Consider again the problem (18.41) in dimension $d \geqslant 2$. In order to obtain that u takes values in $H^{1,q}(\mathbb{R}^d)$, the smallest value of s which we can allow (without bootstrapping) is $s = -1$. Let $q \in (\frac{d}{2}, 2d)$ and $p \in (2,\infty)$ are such that $\frac{d}{q} + \frac{2}{p} \leqslant 2$ (see (18.37)), and set $\rho := 2$, $\sigma := \frac{d}{2q}$, and $\alpha := 1 - \frac{1}{p} - \sigma$. These choices form a special case of Example 18.3.4, and in particular they lead to a critical setting.

Let $u_0 \in B^{\frac{d}{q}-1}_{q,p}(\mathbb{R}^d)$; note that this space contains $L^q(\mathbb{R}^d)$ if $q > d$. By the result of Example 18.3.4, the problem (18.41) admits a (unique) maximal (p, q, s, σ)-solution, and for all $0 < \tau < T < T_{\max}(u_0)$ we have

$$u \in L^p(\tau, T; H^{1,q}(\mathbb{R}^d)) \cap C([\tau, T]; B^{1-\frac{2}{p}}_{q,p}(\mathbb{R}^d)).$$

We will show global existence, i.e., $T_{\max}(u_0) = \infty$, under the more restrictive conditions

$$\max\{d, 2d-6\} < q < 2d \quad \text{and} \quad 2 < p \leqslant \frac{2q}{2d-q}. \tag{18.43}$$

For $d = 2$ we can take $q \in (2, 4)$ and $p \in (2, 2q/(4-q)]$. For $d = 3$ we can take $q \in (3, 6)$ and $p \in (2, 2q/(6-q)]$. We do not claim this is optimal, and we expect that by further bootstrapping some of these conditions can be omitted.

Step 1 – Assuming that $T_{\max}(u_0) < \infty$, we will derive a contradiction with Theorem 18.2.17(1). For the latter it suffices to use Step 2 below. However, we prefer to show the techniques to check Theorem 18.2.17(1) since this can be useful for other situations. This boils down to showing that $u \in L^p_{w_\alpha}(0, T; H^{1,q}(\mathbb{R}^d))$ and

$$\sup_{t\in[0,T_{\max}(v_0))} \|u(t)\|_{B^{\frac{d}{q}-1}_{q,p}(\mathbb{R}^d)} < \infty.$$

By (18.42), both assertions are clear on $[0, \tau]$ for any $\tau < T_{\max}(v_0)$. Thus it suffices to show that, for some $\tau > 0$,

$$u \in L^p(\tau, T_{\max}(u_0); H^{1,q}(\mathbb{R}^d)) \quad \text{and} \quad \sup_{t\in[\tau,T_{\max}(u_0))} \|u(t)\|_{B^{\frac{d}{q}-1}_{q,p}(\mathbb{R}^d)} < \infty. \tag{18.44}$$

Step 2 – We show the second part of (18.44). Since $\frac{d}{q} - 1 < 0$, by the easy embeddings of (14.23) and Proposition 14.4.18, it is enough to show that

$$\sup_{t\in[\tau,T_{\max}(u_0))} \|u(t)\|_{L^q(\mathbb{R}^d)} < \infty.$$

The idea will be to apply the chain rule of Lemma 18.3.6 below. For this we need that $u^3 \in L^1(\tau, T; L^q)$ for $0 < \tau < T < T_{\max}(u_0)$. To see this, note that by Sobolev embedding with $\theta - \frac{d}{q} = -\frac{d}{3q}$ and interpolation,

$$\|u^3\|_{L^q} = \|u\|^3_{L^{3q}} \leqslant C\|u\|^3_{H^{\theta,q}} \leqslant C'\|u\|^{3(1-\theta)}_{L^q}\|u\|^{3\theta}_{H^{1,q}}.$$

As observed before, the L^q-norm of u is uniformly bounded on $[\tau, T]$. Thus for the integrability of $\|u^3\|_{L^q}$ in time it remains to note that $u \in L^p(\tau, T; H^{1,q}) \hookrightarrow L^{3\theta}(\tau, T; H^{1,q})$ since $p > 2 > \frac{2d}{q} = 3\theta$.

Applying the chain rule to the identity

$$u(t) - u(\tau) = \int_\tau^t \Delta u(r)dr + \int_\tau^t -u^3(r) + u(r)\,\mathrm{d}r, \quad t \in [\tau, T],$$

we see that

$$\begin{aligned}\|u(t)\|^q_{L^q(\mathbb{R}^d)} = \|u(\tau)\|^q_{L^q(\mathbb{R}^d)} - q(q-1)\int_\tau^t \int_{\mathbb{R}^d} |u|^{q-2}|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}r \\ + q\int_\tau^t \int_{\mathbb{R}^d} |u|^{q-2}(-u^4 + u^2)\,\mathrm{d}x\,\mathrm{d}r \\ \leqslant \|u(\tau)\|^q_{L^q(\mathbb{R}^d)} + q\int_\tau^t \|u(r)\|^q_{L^q(\mathbb{R}^d)}\,\mathrm{d}r.\end{aligned} \tag{18.45}$$

Therefore, by Gronwall's lemma applied to $t \mapsto \|u(t)\|^q_{L^q(\mathbb{R}^d)}$,

$$\|u(t)\|^q_{L^q(\mathbb{R}^d)} \leqslant \|u(\tau)\|^q_{L^q(\mathbb{R}^d)} e^{q(t-\tau)}.$$

Since we assumed $T_{\max}(u_0) < \infty$, this implies the desired bound

$$N := \sup_{t\in[\tau, T_{\max}(u_0))} \|u(t)\|^q_{L^q(\mathbb{R}^d)} \leqslant \|u(\tau)\|^q_{L^q(\mathbb{R}^d)} e^{qT_{\max}(u_0)} < \infty. \tag{18.46}$$

As a consequence of (18.45), we also find that

$$\int_\tau^{T_{\max}(u_0)} \int_{\mathbb{R}^d} |u|^{q-2}|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}r \leqslant C_{q,T_{\max}(u_0)} \|u(\tau)\|^q_{L^q(\mathbb{R}^d)}, \tag{18.47}$$

where $C_{q,T_{\max}(u_0)} = \frac{(1+qT_{\max}(u_0))}{q(q+1)} e^{q(T_{\max}(u_0))}$.

Step 3 – By (18.42) we have $u(\tau) \in B^{1-\frac{2}{p}}_{q,p} = (X_0, X_1)_{1-\frac{1}{p},p}$. Therefore, if we can show that $-u^3 + u$ belongs to $L^p(\tau, T_{\max}(u_0); H^{-1,q}(\mathbb{R}^d))$, the first part of (18.44) follows from maximal L^p-regularity applied on the interval $(\tau, T_{\max}(u_0))$ with inhomogeneity $u - u^3$.

It is clear from Step 2 that u has the required regularity, so it remains to consider the term u^3. By Sobolev embedding,

$$\begin{aligned}\|u^3\|_{L^p(\tau,T_{\max}(u_0);H^{-1,q}(\mathbb{R}^d))} &\leqslant C\|u^3\|_{L^p(\tau,T_{\max}(u_0);L^{\frac{qd}{q+d}}(\mathbb{R}^d))} \\ &= C\|u\|^3_{L^{3p}(\tau,T_{\max}(u_0);L^{q_0}(\mathbb{R}^d))},\end{aligned}$$

where $q_0 = \frac{3qd}{q+d}$. To prove that the latter is finite, note that by Sobolev embedding with $\theta - \frac{d}{2} = -\frac{dq}{2q_0}$ (then $\theta \in (0,1]$ by (18.43) and $2q_0/q > 2$ since $q < 2d$),

$$\begin{aligned}\|u\|^{q/2}_{L^{q_0}(\mathbb{R}^d)} &= \||u|^{q/2}\|_{L^{2q_0/q}} \\ &\leqslant C_0\||u|^{q/2}\|_{H^{\theta,2}}\end{aligned}$$

$$
\begin{aligned}
&\leqslant C_1 \| |u|^{q/2} \|_{L^2(\mathbb{R}^d)}^{1-\theta} \| |u|^{q/2} \|_{W^{1,2}(\mathbb{R}^d)}^{\theta} \\
&\leqslant C_2 \big[\| |u|^{q/2} \|_{L^2(\mathbb{R}^d)} + \| |u|^{q/2} \|_{L^2(\mathbb{R}^d)}^{1-\theta} \| \nabla |u|^{q/2} \|_{L^2(\mathbb{R}^d)}^{\theta} \big] \\
&= C_2 \Big[\|u\|_{L^q(\mathbb{R}^d)}^{q/2} + \|u\|_{L^q(\mathbb{R}^d)}^{q(1-\theta)/2} \frac{q^\theta}{2^\theta} \Big(\int_{\mathbb{R}^d} |u|^{q-2} |\nabla u|^2 \, \mathrm{d}x \Big)^{\theta/2} \Big]. \\
&\leqslant C_2 \Big[N^{1/2} + N^{(1-\theta)/2} \frac{q^\theta}{2^\theta} \Big(\int_{\mathbb{R}^d} |u|^{q-2} |\nabla u|^2 \, \mathrm{d}x \Big)^{\theta/2} \Big],
\end{aligned}
$$

where we used (18.46). Therefore, $u \in L^{3p}(\tau, T_{\max}(u_0); L^{q_0}(\mathbb{R}^d))$ follows if we can check that

$$\int_\tau^{T_{\max}(u_0)} \Big(\int_{\mathbb{R}^d} |u|^{q-2} |\nabla u|^2 \, \mathrm{d}x \Big)^{3p\theta/q} \mathrm{d}t < \infty.$$

The latter follows from (18.47) since our choice of θ satisfies $\theta \leqslant \frac{q}{3p}$, which follows from (18.43).

The following chain rule was used in Example 18.3.5.

Lemma 18.3.6 (Chain rule in the weak setting). *Let $q \in [2, \infty)$ and $p \in (1, \infty)$. Suppose that $u \in C([\tau, T]; L^q(\mathbb{R}^d)) \cap L^p(\tau, T; H^{1,q}(\mathbb{R}^d))$, $G \in L^{p'}(\tau, T; L^q(\mathbb{R}^d; \mathbb{R}^d))$, and $g \in L^1(\tau, T; L^q(\mathbb{R}^d))$ are such that for all $t \in [\tau, T]$*

$$u(t) = u(\tau) + \int_\tau^t \nabla \cdot G(s) \, \mathrm{d}s + \int_\tau^t g(s) \, \mathrm{d}s, \tag{18.48}$$

where the equality is meant in the space $H^{-1,q}(\mathbb{R}^d)$. Then

$$
\begin{aligned}
\|u(t)\|_{L^q(\mathbb{R}^d)}^q = \|u(\tau)\|_{L^q(\mathbb{R}^d)}^q &- q(q-1) \int_\tau^t \langle G(s), |u(s)|^{q-2} \nabla u(s) \rangle \, \mathrm{d}s \\
&+ q \int_\tau^t \langle g(s), |u(s)|^{q-2} u(s) \rangle \, \mathrm{d}s,
\end{aligned}
\tag{18.49}
$$

where the duality pairing is in $(L^q, L^{q'})$ in both cases.

In view of the Mihlin multiplier theorem (see Theorem 5.5.10),

$$\|\nabla \cdot G\|_{H^{-1,q}(\mathbb{R}^d)} = \Big\| \mathscr{F}^{-1} [\xi \mapsto \frac{2\pi i \xi}{(1+|\xi|^2)^{1/2}} \cdot \widehat{G}(\xi)] \Big\|_{L^q(\mathbb{R}^d)} \leqslant C_{p,d} \|G\|_{L^q(\mathbb{R}^d;\mathbb{R}^d)},$$

and therefore the integral of $\nabla \cdot G$ exists as a Bochner integral in $H^{-1,q}(\mathbb{R}^d)$.

Proof. Without loss of generality we may assume that $\tau = 0$. First we establish some boundedness properties which also show the well-definedness of the integrals appearing in (18.49). For all $v \in L^\infty(0, T; L^q(\mathbb{R}^d))$ and $w \in L^p(0, T; L^q(\mathbb{R}^d))$, by Hölder's inequality in the space variables with $\frac{1}{q} + \frac{q-2}{q} + \frac{1}{q} = 1$, and subsequently in the time variable with $\frac{1}{p} + \frac{1}{p'} = 1$,

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^d} |G|\,|v|^{q-2}|w| \,\mathrm{d}x \,\mathrm{d}s \\ &\leqslant \int_0^T \|G\|_{L^q(\mathbb{R}^d;\mathbb{R}^d)} \|v\|^{q-2}_{L^q(\mathbb{R}^d)} \|w\|_{L^q(\mathbb{R}^d)} \,\mathrm{d}s \\ &\leqslant \|v\|^{q-2}_{L^\infty(0,T;L^q(\mathbb{R}^d))} \|G\|_{L^{p'}(0,T;L^q(\mathbb{R}^d;\mathbb{R}^d))} \|w\|_{L^p(0,T;L^q(\mathbb{R}^d))}.\end{aligned} \tag{18.50}$$

In a similar way one proves that

$$\int_0^T \int_{\mathbb{R}^d} |g|\,|v|^{q-1} \,\mathrm{d}x \,\mathrm{d}s \leqslant \|g\|_{L^1(0,T;L^q(\mathbb{R}^d))} \|v\|^{q-1}_{L^\infty(0,T;L^q(\mathbb{R}^d))}.$$

Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ be such that $\int_{\mathbb{R}^d} \varphi \,\mathrm{d}x = 1$, and let $\varphi_n := n^d \varphi(n\cdot)$. By Theorem 2.3.8, for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ we have

$$\varphi_n * f \to f \text{ and } |\varphi_n * f| \leqslant Mf \;\text{ almost everywhere,} \tag{18.51}$$

where M denotes the Hardy–Littlewood maximal operator.

Taking convolutions in (18.48), we obtain

$$u_n(t) - u_n(0) = \int_0^t \nabla \cdot G_n(s) \,\mathrm{d}s + \int_0^t g_n(s) \,\mathrm{d}s,$$

where $u_n = \varphi_n * u$, $\nabla G_n = \nabla \cdot (\varphi_n * G) = \varphi_n * (\nabla \cdot G)$, and $g_n = \varphi_n * g$. Fix $x \in \mathbb{R}^d$ and let $R > 0$ be so large that $|u(s,x)| \leqslant R$ for all $s \in [0,T]$. Let $\zeta \in C_c^2(\mathbb{R})$ be such that $\zeta(y) = |y|^q$ for $|y| \leqslant R$. Note that $\zeta'(y) = |y|^{q-2}y$ and $\zeta''(y) = |y|^{q-2}$ for $|y| \leqslant R$. Applying the chain rule for weak derivatives in time to the function $t \mapsto \zeta(u(t,x))$, we obtain

$$\begin{aligned}|u_n(t,x)|^q = |u_n(0,x)|^q + q \int_0^t |u_n(s,x)|^{q-2} u_n(s,x) \nabla \cdot G_n(s,x) \,\mathrm{d}s \\ + q \int_0^t q |u_n(s,x)|^{q-2} u_n(s,x) g_n(s,x) \,\mathrm{d}s.\end{aligned}$$

Integrating over $\mathbb{R}^d$ and using Fubini's theorem and integrating by parts, we obtain

$$\begin{aligned}\|u_n(t)\|^q_{L^q(\mathbb{R}^d)} = \|u_n(0)\|^q_{L^q(\mathbb{R}^d)} - q(q-1) \int_0^t \langle G_n(s), |u_n(s)|^{q-2} \nabla u_n(s) \rangle \,\mathrm{d}s \\ + \int_0^t q \langle g_n(s), |u_n(s)|^{q-2} u_n(s) \rangle \,\mathrm{d}s.\end{aligned}$$

From the observation (18.51) we deduce that $u_n \to u$ in $L^q(\mathbb{R}^d)$ pointwise in $[0,T]$, $u_n \to u$ in $L^p(0,T;H^{1,q}(\mathbb{R}^d))$, $G_n \to G$ in $L^{p'}(0,T;L^q(\mathbb{R}^d))$, and $g_n \to g$ in $L^1(0,T;L^q(\mathbb{R}^d))$. Thus it remains to let $n \to \infty$ in the above identity and use the boundedness/continuity properties from the beginning of the proof to obtain convergence. Indeed, after extracting almost everywhere convergent subsequences and relabelling, convergence follows by dominated convergence. For instance, for the first term,

$$\int_0^T \int_{\mathbb{R}^d} |u_n|^{q-2} G_n \cdot \nabla u_n \, \mathrm{d}x \, \mathrm{d}s \to \int_0^T \int_{\mathbb{R}^d} |u|^{q-2} G \cdot \nabla u \, \mathrm{d}x \, \mathrm{d}s$$

follows since $|u_n|^{q-2} G_n \cdot \nabla u_n$ is dominated by $|Mu|^{q-2} MG\, M|\nabla u|$, which is an integrable function by (18.50) and the boundedness of M on $L^q(\mathbb{R}^d)$. □

18.4 Long-time existence for small initial data and $F = F_{\mathrm{c}}$

Short-time existence and uniqueness has been proved in Theorems 18.1.2 and 18.2.6 In this section we prove that, under suitable conditions, for initial values with small norm in $X_{\sigma,p}$ one can obtain well-posedness on *arbitrary long time intervals* $[0,T]$. This result is typical for the semi-linear setting. The assumptions on F will be similar to the ones of Section 18.2. However, we will assume that $F = F_{\mathrm{c}}$, $F(0) = 0$, and replace (18.11) by the slightly more restrictive condition (18.52) below. Moreover, we assume $A \in \mathscr{L}(X_1, X_0)$, and thus we only consider the semi-linear setting.

Theorem 18.4.1 (Semi-linear equations with small initial data). *Let $p \in [1,\infty]$ and $\alpha \in [0, \frac{1}{p'}) \cup \{0\}$, where we take $\alpha > 0$ if $p = \infty$. Let $\sigma = 1 - \alpha - \frac{1}{p} \in [0, 1/p'] \cap [0,1)$. Let X_0 and X_1 be Banach spaces such that $X_1 \hookrightarrow X_0$ with embedding constant $C_X \geqslant 1$. Let $O_{\sigma,p} \subseteq X_{\sigma,p}$ be an open set and suppose that $0 \in O_{\sigma,p}$. Let $A \in \mathscr{L}(X_1, X_0)$ and suppose that A has maximal L^p-regularity (C-regularity if $p = \infty$) on finite time intervals. Suppose that $F_{\mathrm{c}} : X_1 \cap O_{\sigma,p} \to X_0$ is such that $F_{\mathrm{c}}(0) = 0$ and*

$$\|F_{\mathrm{c}}(u) - F_{\mathrm{c}}(v)\|_{X_0} \leqslant L_{\mathrm{c}} \sum_{j=1}^m (\|u\|_{X_{\beta_j}}^{\rho_j} + \|v\|_{X_{\beta_j}}^{\rho_j}) \|u - v\|_{X_{\beta_j}} \tag{18.52}$$

for all $u, v \in X_1 \cap O_{\sigma,p}$, where $\beta_j \in (\sigma, 1)$, $\rho_j > 0$ are such that $\beta_j \leqslant \frac{1+\rho_j \sigma}{1+\rho_j}$ for $j \in \{1, \ldots, m\}$. Then for every $T \in (0,\infty)$ there exist $\varepsilon > 0$ such that for each $\|v_0\|_{X_{\sigma,p}} \leqslant \varepsilon$, the problem

$$\begin{cases} u' + Au &= F(u), \quad \text{on } (0,T), \\ u(0) &= v_0, \end{cases} \tag{18.53}$$

has a unique $L^p_{w_\alpha}$-solution $u_{v_0} \in \mathrm{MR}^p_\alpha(0,T)$. Moreover, there is a $C \geqslant 0$ such that for all $\|v_0\|_{X_{\sigma,p}}, \|v_1\|_{X_{\sigma,p}} \leqslant \varepsilon$,

$$\|u_{v_0} - u_{v_1}\|_{\mathrm{MR}^p_\alpha(0,T)} \leqslant C \|v_0 - v_1\|_{X_{\sigma,p}}. \tag{18.54}$$

If additionally, A has maximal L^p-regularity (C-regularity if $p = \infty$) on $\mathbb{R}_+$ and $0 \in \varrho(A)$, then the above holds with $(0,T)$ replaced by $\mathbb{R}_+$.

Proof. In the proof we use the notation $E_j = L^p_{w_\alpha}(0,T;X_j)$. Let $u_0 = 0$ and set $T_1 = T$. Without loss of generality we may assume $T \geqslant 1$ and $r \leqslant 1$. Let $\Phi_{v_0} : B^T_r(v_0) \to \mathrm{MR}^p_\alpha(0,T)$ be defined by $\Phi_{v_0}(v) := u$, where u is the unique $L^p_{w_\alpha}$-solution to

$$\begin{cases} u' + Au &= F(v), \\ u(0) &= v_0. \end{cases}$$

Note that for $r \in (0,1]$ and $\varepsilon > 0$ small enough, v takes values in $O_{\sigma,p}$ by Lemma 18.2.9, and by Lemma 18.2.10 we have $F(v) \in E_0$. Below Theorem 18.2.6, it has already been observed that local existence and uniqueness follow if we can show that Φ_{v_0} has a unique fixed point.

By the maximal regularity estimate (18.28) we have $u \in \mathrm{MR}^p_\alpha(0,T)$, $u(0) = v_0$, and

$$\begin{aligned} \|u\|_{\mathrm{MR}^p_\alpha(0,T)} &\leqslant C_{A,T}\|v_0\|_{X_{\sigma,p}} + C_{A,T}\|F(v)\|_{E_0} \\ &\leqslant C_{A,T}\varepsilon + C_{A,T}CL_{\mathrm{c}}\sum_{j=1}^m \|v\|^{\rho_j+1}_{\mathrm{MR}^p_\alpha(0,T)} \\ &\leqslant C_{A,T}\varepsilon + C_{A,T}CL_{\mathrm{c}}\sum_{j=1}^m r^{\rho_j+1}, \end{aligned}$$

where the estimate for $F(v)$ follows from Lemmas 18.2.7 and 18.2.8, the constant C can be taken T-independent since $T \geqslant 1$, and we used (18.19) with $u_0 = 0$ and $z_{u_0} = 0$. Note that the terms T^{δ_j} can be avoided due to the more restrictive condition (18.52). The above estimate shows that for $r, \varepsilon > 0$ small enough, $\|u\|_{\mathrm{MR}^p_\alpha(0,T)} \leqslant r$, and thus $u \in B^T_r(v_0)$.

Next, fix $v_{j,0} \in B_{X_{\sigma,p}}(u_0,\varepsilon)$ and $v_j \in B^T_r(v_{j,0})$ for $j \in \{1,2\}$. Then $u = \Phi_{v_{1,0}}(v_1) - \Phi_{v_{2,0}}(v_2)$ solves the problem

$$\begin{cases} u' + Au &= F(v_1) - F(v_2), \\ u(0) &= v_{1,0} - v_{2,0}. \end{cases}$$

Therefore, by the maximal regularity estimate (18.28),

$$\|u\|_{\mathrm{MR}^p_\alpha(0,T)} \leqslant C_{A,T}\|F(v_1) - F(v_2)\|_{E_0} + C_{A,T}\|v_{1,0} - v_{2,0}\|_{X_{\sigma,p}},$$

From Lemmas 18.2.7 and 18.2.8 we obtain that

$$\begin{aligned} &\|F(v_1) - F(v_2)\|_{E_0} \\ &\qquad \leqslant CL_{\mathrm{c}}\sum_{j=1}^m \big[\|v_1\|^{\rho_j}_{\mathrm{MR}^p_\alpha(0,T)} + \|v_2\|^{\rho_j}_{\mathrm{MR}^p_\alpha(0,T)}\big]\|v_1 - v_2\|_{\mathrm{MR}^p_\alpha(0,T)} \\ &\qquad \leqslant 2CL_{\mathrm{c}}\sum_{j=1}^m r^{\rho_j}\|v_1 - v_2\|_{\mathrm{MR}^p_\alpha(0,T)}. \end{aligned}$$

Therefore, by choosing $r > 0$ small enough,

$$\begin{aligned}\|\Phi_{v_{1,0}}(v_1) - \Phi_{v_{2,0}}(v_2)\|_{\mathrm{MR}^p_\alpha(0,T)} \leqslant \frac{1}{2}\|v_1 - v_2\|_{\mathrm{MR}^p_\alpha(0,T)} \\ + C_{A,T}\|v_{1,0} - v_{2,0}\|_{X_{\sigma,p}}.\end{aligned} \tag{18.55}$$

By (18.55), $\Phi_{v_0} : B_r^T(v_0) \to B_r^T(v_0)$ is a strict contraction, and thus it has a unique fixed point $u_{v_0} \in B_r^T(v_0)$. This is the required solution to (18.53). Moreover, (18.55) implies that for $v_{1,0}, v_{2,0} \in B_{X_{\sigma,p}}(u_0, \varepsilon)$

$$\begin{aligned}\|u_{v_{1,0}} - u_{v_{2,0}}\|_{\mathrm{MR}^p_\alpha(0,T)} \leqslant \frac{1}{2}\|u_{v_{1,0}} - u_{v_{2,0}}\|_{\mathrm{MR}^p_\alpha(0,T)} \\ + C_{A,T}\|v_{1,0} - v_{2,0}\|_{X_{\sigma,p}},\end{aligned}$$

and thus

$$\|u_{v_{1,0}} - u_{v_{2,0}}\|_{\mathrm{MR}^p_\alpha(0,T)} \leqslant 2C_{A,T}\|v_{1,0} - v_{2,0}\|_{X_{\sigma,p}}.$$

which gives (18.54).

In case A has maximal regularity on $\mathbb{R}_+$ and $0 \in \varrho(A)$, then (18.28) holds with $(0,T)$ replaced by $\mathbb{R}_+$. Moreover, one can check that Lemma 18.2.9 holds with $(0,T_1)$ replaced by $\mathbb{R}_+$. Therefore, one can repeat the above argument on the half line. □

18.5 Notes

The theory of abstract non-linear parabolic evolution equations has a long history going back to the work of the Japanese school in the 1960s, with contributions of Fujita, Kato, Tanabe, and others. Excellent monographs on the subject are available, including Amann [1995], Friedman [1969], Henry [1981], Lunardi [1995], Lions [1969], Pazy [1983], Prüss and Simonett [2016], Tanabe [1979], Yagi [2010]. For the purpose of this chapter we chose to limit ourselves to the maximal L^p-regularity approach to quasi-linear evolution equations, mostly focussing on local well-posedness. Other approaches, including maximal Hölder regularity, the so-called Kato approach, and the theory monotone operators, are treated in some of the references just mentioned. Maximal regularity techniques have important applications to a number of topics not covered in this volume, such as linearised stability, semi-flows, higher order regularity, sharp conditions for global well-posedness, numerical analysis, and applications to concrete PDEs. Maximal L^p-regularity for stochastic evolution equations will be covered in Volume IV.

The maximal L^p-regularity approach to quasi-linear evolution equations was initiated by the influential paper Clément and Li [1993/94], and further investigated and extended in Prüss [2002] and Amann [2005]. The semi-linear setting of Theorem 18.1.2 is a special case of the results in these works, and is presented here as a warm-up to the later results.

A local well-posedness result under the assumption of maximal C-regularity was obtained by Clément and Simonett [2001]. The use of weights in time seems essential in the latter (see Remark 18.2.11). Based on the weighted maximal L^p-regularity result of Prüss and Simonett [2004], the L^p-setting was extended to a weighted setting in time by Köhne, Prüss, and Wilke [2010]. The use of weights has several advantages:

- well-posedness in case of rough initial data
- instantaneous regularisation
- compactness properties of orbits

and has become standard in the theory of evolution equations. All of the above works have found applications to concrete quasi- and semi-linear PDEs, many of which are collected and mentioned in the influential monograph Prüss and Simonett [2016]. Since the number of applications is too large to discuss here, we will mainly focus on applications of the theory of critical spaces in these notes.

For parabolic equations it is often possible to bootstrap regularity in time and space. Sometimes one can even derive real analyticity via use the so-called parameter trick of Angenent [1990b,a]; see also Prüss and Simonett [2016, Section 5.2] for a presentation in the setting of abstract quasi-linear evolution equations. Applications of maximal L^p-regularity techniques to the study of linearised stability for non-linear parabolic evolution equation can be found in Lunardi [1995], Prüss [2002], Prüss, Simonett, and Zacher [2009], Matioc and Walker [2020], and references therein.

Critical spaces

In the present abstract evolution equations framework, the splitting $F = F_{\rm Tr} + F_{\rm c}$ was first introduced in LeCrone et al. [2014]. In this paper, local well-posedness in the subcritical case was proved using maximal L^p-regularity for $1 < p < \infty$. Shortly afterwards, it was realised in Prüss and Wilke [2017] that under additional conditions on A and (X_0, X_1), local well-posedness can even be obtained in the critical case. Consequences for the Navier–Stokes equations were discussed in Prüss and Wilke [2018]. Further results and applications to concrete and abstract problems were given in Prüss, Simonett, and Wilke [2018]. In particular, this paper discusses the relationship between scaling invariance and criticality for several concrete PDEs. It is remarkable that an abstract definition for criticality can be given which leads to new insights for many concrete PDEs. In the same paper, by way of an example it is shown that the sub-criticality condition (18.12) cannot be improved. The L^p-framework was extended to maximal C-regularity in LeCrone and Simonett [2020].

Theorem 18.2.6 unifies and extends several of the results mentioned in the preceding discussion. For simplicity, here we only considered the case where A and F are time-independent, but this restriction can be avoided easily (see Remark 18.2.4). The unification lies in the fact that one proof is presented which works for all $p \in [1, \infty]$ and all admissible weights, with $p = \infty$ corresponding to maximal C-regularity. Moreover, we do not need geometric conditions on X_0 such as the UMD property, or further conditions on $A(u_0)$ besides maximal L^p- or C-regularity. In part of the existing literature, the spaces X_{β_j} appearing in (18.11) are taken as the complex interpolation spaces $[X_0, X_1]_{\beta_j}$. Taking the real interpolation spaces $(X_0, X_1)_{\beta_j,1}$ leads to a less restrictive condition on F_c and is easier to work with in the proofs.

The case $p = 1$ of Theorem 18.2.6 seems to be new. It is important to observe that for $p = 1$ one is forced to take $\sigma = \alpha = 0$, which in turn forces the X_0-valued trace part $F_{\rm Tr}$ to be defined on an open subset $O_{\sigma,p}$ of the same space X_0. For non-linearities of the form $F = F_{\rm Tr}$, this requirement rules out many interesting examples of non-linearities. However, by allowing non-linearities with a critical part, i.e., non-linearities of the form $F = F_{\rm Tr} + F_c$, many interesting examples can be covered even when $p = 1$, the point being that it suffices to have F_c locally Lipschitz with respect to the norms of the smaller spaces $X_{1/(1+\rho_j)}$ (with the ρ_j's as in Assumption 18.2.2). On the other hand, according to Theorem 17.4.5, operators with maximal L^1-regularity are rare. An exception is the case where X_0 itself is a real interpolation space in which case the Da Prato–Grisvard theorem applies (see Corollary 17.3.20).

It should be observed that a more flexible condition on F_c could be used in (18.11), namely

$$\|F_c(u) - F_c(v)\|_{X_0} \leqslant L_c \sum_{j=1}^{m} (1 + \|u\|_{X_{\varphi_j}}^{\rho_j} + \|v\|_{X_{\varphi_j}}^{\rho_j}) \|u - v\|_{X_{\beta_j}}, \tag{18.56}$$

with $\varphi_j \in (\sigma, 1)$, $\beta_j \in (\sigma, \varphi_j]$, along with the subcriticality condition

$$\rho_j(\varphi_j - \sigma) + \beta_j \leqslant 1, \qquad j \in \{1, \dots, m\}. \tag{18.57}$$

The formulation (18.56) allows for different space regularity for u, v, and $u - v$ on the right-hand side (see Agresti and Veraar [2022a] and Prüss, Simonett, and Wilke [2018]). However, in all known examples, it suffices to take $\varphi_j = \beta_j$ (as we do in the main text) in order to obtain the sharpest results. Note that by taking $\varphi_j = \beta_j$, (18.57) reduces to the sub-criticality condition (18.12).

Global well-posedness and blow-up criteria

The existence of a maximal time interval in Theorem 18.2.14 is a standard result. Often it is only stated and proved under the more restrictive assumption that $A(v_0)$ have maximal L^p- or C-regularity for all $v_0 \in O_{\sigma,p}$. The present formulation only uses maximal regularity of $A(u_0)$. In a slightly different set-up it appears in Agresti and Veraar [2022a].

The global well-posedness result of Theorem 18.2.15 is also standard. The statement and proof closely follow Prüss and Simonett [2016, Corollary 5.1.2]. The weight t^α can be helpful in proving global well-posedness, as estimates in the space $X_{\sigma,p}$ are easier to obtain for smaller values of σ (i.e., for higher values of α). In the semi-linear case, the blow-up criteria can be further weakened as was done in Theorem 18.2.17. In case of semi-linear functions F of quadratic type, blow-up criteria appear in Prüss, Simonett, and Wilke [2018, Section 2.1]. Some of these were extended, for a more general class of semi-linearities F, in to a stochastic setting in Agresti and Veraar [2022b, Theorem 4.11]. Simplifying this to the deterministic setting, one arrives at the following result:

Theorem 18.5.1 (Serrin-type blow-up criteria). *Let $p \in (1, \infty)$, suppose that Assumption 18.2.2 holds, and let $A \in \mathscr{L}(X_1, X_0)$ have maximal L^p-regularity on finite time intervals. Let $(u, T_{\max}(u_0))$ denote the maximal $L^p_{w_\alpha}$-solution to*

$$\begin{cases} u' + Au &= F(u), \quad \text{on } (0,T), \\ u(0) &= u_0. \end{cases}$$

Suppose that for each $j \in \{1, \dots, m\}$ we have

$$\rho_j < 1 + \alpha p \quad \text{or} \quad (\alpha = 0 \text{ and } \rho_j \leqslant 1).$$

Then the following assertions hold:

- *for all $T < T_{\max}(u_0)$ one has $\|u\|_{L^p(0,T;X_{1-\alpha})} < \infty$;*
- *if $T_{\max}(u_0) < \infty$, then $\|u\|_{L^p(0,T_{\max}(u_0);X_{1-\alpha})} = \infty$.*

In the case of (sub-)quadratic semi-linearity F, one has $\rho_j \leqslant 1$ and the above condition always holds.

Applications

The theory of quasi-linear evolution equations in critical spaces as presented in this chapter has been applied to models in several scientific areas which include fluid dynamics, chemistry, neuroscience, free boundary problems, and differential geometry. For details we refer to the founding papers and books LeCrone, Prüss, and Wilke [2014], Prüss and Simonett [2016], Prüss and Wilke [2018], Prüss, Simonett, and Wilke [2018], LeCrone and Simonett [2020], and for further applications to the more recent papers Hieber and Prüss [2018], Mazzone, Prüss, and Simonett [2019a,b], Simonett and Prüss [2019], Binz, Hieber, Hussein, and Saal [2020], Giga, Gries, Hieber, Hussein, and Kashiwabara [2020], Hieber, Hussein, and Saal [2023], Hieber, Kress, and Stinner [2021], Mazzone [2021], Prüss, Simonett, and Wilke [2021], Court and Kunisch [2022], Simonett and Wilke [2022b]. This list is likely to expand in the near future, as the splitting $F = F_{\mathrm{Tr}} + F_{\mathrm{c}}$ of Theorem 18.2.6 has proved to be very powerful in applications to concrete non-linear parabolic equations of semi- and quasi-linear type. It leads to new insights for many PDEs to which the original framework of Clément and Li [1993/94] was applicable. Moreover, some of the new blow-up criteria can make it possible to obtain global well-posedness results.

The examples considered in Section 18.3 are very basic, and local/global well-posedness is well known for a broad class of initial values. The examples are merely chosen to demonstrate the abstract theorems of Section 18.2 in a simple setting. The method to check the blow criteria in Example 18.3.5 is taken from Agresti and Veraar [2023a], where these techniques are used in several examples.

An extension of the results of Section 18.2 to stochastic quasi-linear evolution equations in critical spaces was recently obtained in Agresti and Veraar [2022a,b], where completely new proofs where required. Applications to stochastic PDE can be found in these works, as well as in Agresti and Veraar [2021, 2022c, 2023b,a], Agresti [2022], Agresti, Hieber, Hussein, and Saal [2022a,b].

Q

Questions

Calderón–Zygmund operators

The extrapolation theory for the boundedness of Calderón–Zygmund operators developed in Chapter 11 is in many ways analogous and parallel to the extrapolation of L^p inequalities for martingale transforms that we discussed in Section 3.5. Specifically, the quantitative statement of Calderón–Zygmund Theorem 11.2.5(3) is analogous to the estimate (3.48) of Martingale Extrapolation Theorem 3.5.4; in both cases, the L^p norm of an operator is controlled by the L^q norm multiplied by the factor $pp' = p+p'$, which exhibits the correct blow up of these norms as $p \to 1$ or $p \to \infty$. However, in the case of martingale transforms with respect to a Paley–Walsh filtration, (3.49) gives a more precise estimate with the factor $\frac{p}{q} + \frac{p'}{q'}$. While this has the same rate of blow up as $p \to 1$ or $p \to \infty$, it gives a better estimate if the "starting point" q is either large or close to 1. (For instance, think of the case that q is large and $p = 2q$.) Given the well-behaved nature of the Lebesgue measure with respect to which the Calderón–Zygmund singular integrals are integrated, it seems reasonable to expect that the behaviour of these operators should be as good as that of martingale transforms with respect to a Paley–Walsh filtration; however, this is not reflected in the quantitative estimates of Calderón–Zygmund Theorem 11.2.5. We therefore pose the question:

Problem Q.1. Under the assumptions of Calderón–Zygmund Theorem 11.2.5 (or even just for more regular operators with a standard kernel), is there an estimate

$$\|T\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \overset{?}{\leqslant} c_d\Big(\frac{p}{q} + \frac{p'}{q'}\Big)\Big(\|T\|_{\mathscr{L}(L^q(\mathbb{R}^d;X),L^q(\mathbb{R}^d;Y))} + \mathfrak{C}_K\Big),$$

for all $p, q \in (1, \infty)$, where c_d depends only on the dimension and $\mathfrak{C}_K$ only on the kernel K of the operator T, either via the quantities $\|K\|_{\text{Hör}}$ and $\|K\|_{\text{Hör}^*}$ from Definition 11.2.1 of a Hörmander kernel, or c_K and ω_K from Definition 11.3.1 of a Calderón–Zygmund kernel? In particular, does the Hilbert transform satisfy an estimate

T. Hytönen et al., *Analysis in Banach Spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics 76,
https://doi.org/10.1007/978-3-031-46598-7

$$\hbar_{p,X} \overset{?}{\leqslant} C\Big(\frac{p}{q}+\frac{p'}{q'}\Big)\hbar_{q,X}, \qquad \hbar_{p,X} := \|H\|_{\mathscr{L}(L^p(\mathbb{R};X))} \tag{Q.1}$$

with some universal constant C?

The prospective estimate (Q.1) would be an analogue of Theorem 4.2.7, which gives a similar bound for the UMD constants $\beta_{p,X}$ and $\beta_{q,X}$ in place of $\hbar_{p,X}$ and $\hbar_{q,X}$.

Recall that Problem O.6 asks about a possible linear dependence between $\beta_{p,X}$ and $\hbar_{p,X}$ for $p = 2$, which remains wide open. Note that *if* this linear relation was true for all $p \in (1,\infty)$, and moreover with constants independent of p, then (Q.1) would immediately follow via this linear relation from Theorem 4.2.7 for the UMD constants. Thus, (Q.1) could be thought of as a simpler model problem related to the presumably difficult Problem O.6.

Sparse domination and weighted inequalities

As we have seen in Chapter 11, and many more examples can be found in the literature, *sparse domination* of operators is very efficient in capturing essential information about their boundedness properties on various function spaces, particularly in view of sharp weighted norm inequalities. As discussed in the Notes of that Chapter, *convex body domination* provides a useful elaboration in view of applications like matrix-weighted inequalities and commutator estimates. The proofs of existing convex body domination results follow the same broad outline as their sparse domination counterparts, but require elaborations at critical points of the argument. Rather than reworking each sparse domination proof for the convex body improvement, it would be useful to have a general statement guaranteeing that one implies the other—or, to know that such a statement is impossible, justifying the need of case-by-case study. This raises the question:

Problem Q.2. Does sparse domination imply convex body domination?

More precisely, let $1 \leqslant p < r < q \leqslant \infty$ and

$$T \in \mathscr{L}(L^r(\mathbb{R}^d;X), L^r(\mathbb{R}^d;Y)).$$

For each $N \in \mathbb{Z}_+$, consider the following property:

For some constants $\varepsilon \in (0,1)$ and $\alpha, C \in [1,\infty)$, for all $f = (f_n)_{n=1}^N \in L_c^\infty(\mathbb{R}^d;X)^N$ and $g = (g_n)_{n=1}^N \in L_c^\infty(\mathbb{R}^d;Y^*)^N$ (or some other suitable test function spaces), there is an ε-sparse collection $\mathscr{S} \subseteq \mathscr{D}$ such that

$$|\langle Tf, g\rangle| \leqslant \sum_{Q\in\mathscr{S}} \frac{C}{|Q|} \sup_{\phi,\psi} \iint_{\alpha Q\times\alpha Q} \Big|\sum_{n=1}^N \langle f_n(s),\phi(s)\rangle\langle\psi(t),g_n(t)\rangle\Big| \,\mathrm{d}s\,\mathrm{d}t, \tag{Q.2}$$

where the supremum is over all $\phi \in L^{p'}(\alpha Q;X^*)$ and $\psi \in L^q(\alpha Q;Y)$ such that

$$\fint_{\alpha Q} \|\phi(s)\|_{X^*}^{p'} \,\mathrm{d}s \leqslant 1, \qquad \fint_{\alpha Q} \|\psi(t)\|_Y^q \,\mathrm{d}t \leqslant 1.$$

If this property holds for $N = 1$, does it follow (a) for $N = 2$, or even (b) for all $N \in \mathbb{Z}_+$? If not, what is a counterexample? Does the implication hold

- under additional assumptions on the operator T?
- with a relaxed conclusion with $p + \varepsilon$ and $q - \varepsilon$ in place of p and q?

For $N = 1$, the estimate (Q.2) reduces, after elementary duality results, to sparse domination in the form

$$|\langle Tf, g\rangle| \leqslant C' \sum_{Q\in\mathscr{S}} |Q| \Big(\fint_{\alpha Q} \|f_n(s)\|_X^p \,\mathrm{d}t\Big)^{1/p} \Big(\fint_{\alpha Q} \|g_n(t)\|_{Y^*}^{q'} \,\mathrm{d}t\Big)^{1/q'}.$$

For $N > 1$, the estimate (Q.2) corresponds to convex body domination of Banach space valued functions in the sense of Hytönen [2023]. While we have stated Problem Q.2 in this generality, it is open even for $X = Y = \mathbb{R}$.

Recall that Rubio de Francia's extrapolation Theorem J.2.1 gives a way to extend the boundedness of T on $L^p(\mathbb{R}^d, w)$ for all $w \in A_p$ with $p \in (1, \infty)$ fixed, to boundedness on $L^q(\mathbb{R}^d, w)$ for all $w \in A_q$ and all $q \in (1, \infty)$.

The following problem has been wide open for many years.

Problem Q.3 (Rubio de Francia extrapolation in UMD spaces). Let X be a UMD Banach space and $p \in (1, \infty)$. Suppose $T \in \mathscr{L}(L^p(w))$ for any $w \in A_p$ and there exists an increasing function $\phi : [1, \infty) \to [1, \infty)$ such that $\|T\|_{\mathscr{L}(L^p(w))} \leqslant \phi([w]_{A_p})$. Does the tensor extension $T \otimes I_X$ extend to an element in $\mathscr{L}(L^p(\mathbb{R}^d; X))$.

The tensor extension is indeed bounded if X is a UMD Banach function spaces, and this was proved by Bourgain [1984] and Rubio de Francia [1986] by using a variation of the Rubio de Francia algorithm. Weighted and rescaled extensions have been obtained in Amenta, Lorist, and Veraar [2019]. An extension to the multilinear limited range setting was obtain in Lorist and Nieraeth [2019]. A solution to the stated problem needs different techniques, because the known proofs in UMD Banach function spaces eventually boil down to a scalar estimate.

If more structure is assumed on a single operator T, then often the UMD property of X is enough to deduce the boundedness of $T \otimes I_X$ from the boundedness of T. Typical examples are:

- $T = T_m$, where $m : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ satisfies Mihlin's conditions;
- $T = \mathfrak{t}$ is an element in $\mathscr{L}(L^s(\mathbb{R}^d))$ in Theorem 12.4.21 with $s \in (1, \infty)$ (and say $t = 1$ and $q = \infty$).

Recently, Lorist and Nieraeth [2022] showed that for UMD Banach function spaces X, sparse domination of T implies, sparse domination of $T \otimes I_X$.

Since sparse domination implies (weighted) boundedness, this leads to the following natural question whether their results remains true for general UMD spaces. In particular this would provide a subclass of operators for which we can answer Problem Q.3.

Problem Q.4. Let X be a UMD space and let $p \in (1,\infty)$. Let $\varepsilon > 0$, and $T \in \mathscr{L}(L^p(\mathbb{R}^d))$ be an operator such that for any $f,g \in L^\infty_c(\mathbb{R}^d)$ there exists a ε-sparse collection of cubes $\mathscr{S}$ such that

$$\int_{\mathbb{R}^d} |Tf| \cdot |g| \,\mathrm{d}x \leqslant \sum_{S\in\mathscr{S}} \fint_S |f| \,\mathrm{d}x \fint_S |g| \,\mathrm{d}x \,|S|.$$

Does the tensor extension $T \otimes I_X$ extend to an element of $\mathscr{L}(L^p(\mathbb{R}^d;X))$? Moreover, does sparse domination hold for $T \otimes I_X$? More precisely, does there exist a $\delta > 0$ and constant C_X such that for all simple functions $f \in L^\infty_c(\mathbb{R}^d, X)$ and $g \in L^\infty_c(\mathbb{R}^d)$ there exists a δ-sparse collection of cubes $\mathscr{S}$ such that

$$\int_{\mathbb{R}^d} \|T \otimes I_X f\|_X \cdot |g| \,\mathrm{d}x \leqslant C_X \sum_{S\in\mathscr{S}} \fint_S \|f\| \,\mathrm{d}x \fint_S |g| \,\mathrm{d}x \,|S|?$$

Paraproducts

In Corollary 12.2.19, we have proved in particular (we now consider dimension $d = 1$ for simplicity) that the paraproduct

$$\Pi_b f := \sum_{I\in\mathscr{D}} \langle b, h_I\rangle \langle f\rangle_I h_I,$$

with a scalar-valued symbol $b \in \mathrm{BMO}(\mathbb{R})$, extends to a bounded operator on $L^p(\mathbb{R};X)$ for all $p \in (1,\infty)$, if X is a UMD space. (As discussed in the Notes of Chapter 12, Figiel [1990] attributes this result to unpublished work of J. Bourgain.) Although we have not treated it in the text, one can show by adapting the same argument (which we leave as an exercise to the reader) that the same conclusion is also valid if X has martingale type 2 (see Section 3.5.d). Since this class is incomparable with UMD, this raises the natural question:

Problem Q.5. What is the largest class of Banach spaces X such that

$$\|\Pi_b f\|_{L^2(\mathbb{R};X)} \leqslant C \|b\|_{\mathrm{BMO}(\mathbb{R})} \|f\|_{L^2(\mathbb{R};X)}$$

for all $b \in \mathrm{BMO}(\mathbb{R})$ and $f \in L^2(\mathbb{R};X)$, where C is independent of b and f?

By what we just discussed, this class should at least include all UMD spaces and all spaces of martingale type 2.

$T(1)$ theorems

In the various $T(1)$ theorems proved in Sections 12.3 and 12.4 and discussed in the related Notes, we have seen the following:

(1) Under assumptions on the Haar coefficient of a bilinear form with respect to a *fixed* dyadic system, the induced operator satisfies $L^p(\mathbb{R}^d;X)$ bounds that are *cubic* in the UMD constant $\beta_{p,X}$ (see Theorem 12.3.26).
(2) Under assumptions on the Haar coefficient of a bilinear form with respect to an ensemble of dyadic system, the induced operator satisfies $L^p(\mathbb{R}^d;X)$ bounds that are *quadratic* in the UMD constant $\beta_{p,X}$ (see Theorem 12.3.35). Such assumptions are verified by weakly defined Calderón–Zygmund operators with standard kernels (see Theorem 12.4.12), and in particular by extensions of scalar-valued Calderón–Zygmund operators on $L^p(\mathbb{R}^d)$ (see Theorem 12.4.21).
(3) Under additional symmetry and smoothness assumptions, the induced operator satisfies $L^p(\mathbb{R}^d;X)$ bounds that are *linear* in the UMD constant $\beta_{p,X}$. These results were not covered in the present volume, but can be found in the works of Geiss et al. [2010] for a class of Fourier multipliers, and of Pott and Stoica [2014] for a class of Calderón–Zygmund operators in dimension $d = 1$.

A major problem is the following:

Problem Q.6. Is there an upper bound that is linear in the UMD constant for all operators in the scope of Theorems 12.3.26, 12.3.35, 12.4.12, and 12.4.21? If not, what is a counterexample?

Answering Problem Q.6 in the positive is expected to be very hard, given that it remains open even for the prototype case of the Hilbert transform (see Problem O.6). On the other hand, Problem Q.6 allows more flexibility for a possible counterexample than Problem O.6. Such a counterexample should still be of interest in indicating the limits of how much one can ever hope to prove with linear dependence on the UMD constant.

We also formulate a more open-ended related task:

Problem Q.7. Can any of the bounds in (1) through (3) be improved? In particular, does $T(1)$ Theorem 12.3.26 with a single dyadic system allow bounds that are quadratic in $\beta_{p,X}$? Do Figiel's elementary operators from Theorems 12.1.25 and 12.1.28 admit such bounds? If not, what is a counterexample?

Given the relatively wide scope of estimates that one can prove with bounds quadratic in the UMD constant (in contrast to the somewhat restricted class of linear estimates currently available), Problem Q.7 would appear to be more approachable than the presumably very hard Problem Q.6. In view of the currently different quantitative bounds in (1) and (2) above, Problem Q.7 has a "philosophical" dimension concerning the role of random dyadic systems in the estimation of singular integral operators: Are the methods based on random cubes genuinely stronger, or could one also recover the same results without resorting to random cubes?

While Problems Q.6 and Q.7 deal with the role of the UMD condition in the boundedness of singular integral operators, one can ask related questions about the roles of type and cotype:

Problem Q.8. Is the interdependence between the required decay of the Haar coefficients (resp. modulus of smoothness of the kernel) and type and cotype of the spaces in $T(1)$ Theorems 12.3.26 and 12.3.35 (resp. 12.4.12 and 12.4.21) sharp? If not, what are the sharp forms of these theorems? What are examples of bilinear forms or operators satisfying weaker forms of the assumptions and failing the conclusions of these theorems concerning boundedness of the induced operator?

This problem is relevant and open even in the scalar-valued case with type and cotype 2, in which case the existing $T(1)$ theorems require Figiel or Dini norms of order $\frac{1}{2}$. Nevertheless, the fact that the Banach space valued theory "interprets" the seemingly arbitrary number $\frac{1}{2}$ as $\max(\frac{1}{t}, \frac{1}{q'})$ with the type and cotype exponents t and q could suggest the construction of possible counterexamples through extremal situations for the type and cotype inequalities even in the scalar-valued case.

On the side of the open-ended Problem Q.8 we pose the more provocative:

Problem Q.9. Is any of the $T(1)$ theorems valid for all classical Dini kernels? If not, what is an example of an operator with a Dini kernel, satisfying all assumptions of a $T(1)$ theorem yet failing to be bounded on $L^2(\mathbb{R}^d)$ or on $L^2(\mathbb{R}^d; X)$ for some UMD space X?

In the scalar-valued $T(1)$ theory, this is related to the investigation of the minimal assumptions in the line of research pursued by Yabuta [1985], Meyer [1986], Han and Hofmann [1993], Yang, Yan, and Deng [1997], and Grau de la Herrán and Hytönen [2018].

Fourier type

According to Proposition 13.1.35, a Banach space of type p and cotype q "close enough", in that $\frac{1}{p} - \frac{1}{q} < \frac{1}{2}$, has Fourier type t for every $t \in (1, r)$, where $\frac{1}{r} = \frac{1}{p} - \frac{1}{q} + \frac{1}{2} < 1$.

By adapting an example of Bourgain [1988a], it has been shown by García-Cuerva, Torrea, and Kazarian [1996] that the conclusion is false, in general, for every $t > r$. This is based on the following construction, for every $r \in (1, 2)$ and $\theta \in (0, 1)$. Let $X_{r,\theta} := [L^{r'}(\mathbb{R}), \breve{L}^r(\mathbb{R})]_\theta$, where

$$\breve{L}^r(\mathbb{R}) := \{\breve{g} : g \in L^r(\mathbb{R})\} \subseteq L^{r'}(\mathbb{R})$$

is equipped with the norm $\|\check{g}\|_{\check{L}^r(\mathbb{R})} := \|g\|_{L^r(\mathbb{R})}$, and the inclusion above follows from the classical Hausdorff–Young inequality.

García-Cuerva et al. [1996] check that $X_{r,\theta}$ has Fourier-type r as well as type p and cotype q determined by

$$\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{r}, \qquad \frac{1}{q} = \frac{1-\theta}{r'} + \frac{\theta}{2},$$

and that none of these numbers can be improved. Here the exponents satisfy

$$\frac{1}{p} - \frac{1}{q} + \frac{1}{2} = (1-\theta)\Big(\frac{1}{2} - \frac{1}{r'}\Big) + \theta\Big(\frac{1}{r} - \frac{1}{2}\Big) + \frac{1}{2} = \Big(\frac{1}{r} - \frac{1}{2}\Big) + \frac{1}{2} = \frac{1}{r},$$

which is exactly the relation in Proposition 13.1.35. The fact that none of these can be improved shows in particular that $X_{r,\theta}$ does not have any Fourier type $t > r$. On the other hand, the situation is slightly better than in the conclusions of Proposition 13.1.35 in that $X_{r,\theta}$ also has Fourier type r, not only Fourier type $t \in (1, r)$. This motivates the question about the end-point case of Proposition 13.1.35 (see Pietsch [2007, 6.1.8.6]):

Problem Q.10. If a Banach space X has type p and cotype q with $\frac{1}{p} - \frac{1}{q} < \frac{1}{2}$, does it follow that X has Fourier type r determined by $\frac{1}{r} = \frac{1}{p} - \frac{1}{q} + \frac{1}{2}$? If not, what is a counterexample?

A particular case of a positive answer to Problem Q.10 would be that every space of type p and cotype 2 would have Fourier type p. If this was the case, then case (i) of Proposition 14.5.3 (and its consequences) would already contain case (ii), making the latter redundant. The fact that one can derive the same conclusions under the seemingly alternative sets of assumptions in Proposition 14.5.3 may be taken as an indication that one of these conditions is actually contained in the other one, supporting a positive answer to Problem Q.10.

Another general open problem is how Fourier type p and Walsh type p are related. The definition of Walsh (co)type 2 can be found before Theorem 7.6.12, and extends to p in the obvious way. As for Fourier type, spaces X have Walsh type p if and only if they have Walsh cotype p' (see García-Cuerva, Kazaryan, Kolyada, and Torrea [1998]). Moreover, Walsh type p implies type p and cotype p'. In particular, spaces with Walsh type 2 are isomorphic to a Hilbert space. The result of Bourgain [1988a] shows that nontrivial type implies Fourier and Walsh type p for some $p \in (1, 2]$. Another result for the corresponding operator version of the relation between Fourier and Walsh type for $p = 2$ can be found in Hinrichs [2001].

Problem Q.11. Let X be a Banach space and $p \in (1, 2)$. Prove or disprove X has Fourier type p if and only if X has Walsh type p.

Function spaces and geometry of Banach spaces

In Theorem 14.7.15 we have seen that for UMD spaces $\mathbf{1}_{\mathbb{R}^d_+}$ acts as pointwise multiplier on $H^{s,p}(\mathbb{R};X)$ for $p \in (1,\infty)$ and $-1/p' < s < 1/p$. The same result holds for general dimension d, and there exist at least three different proofs of this fact (see Meyries and Veraar [2015], Lindemulder [2017], Lindemulder, Meyries, and Veraar [2018]). However, it is not known whether the UMD property or any other condition on X is necessary.

Problem Q.12. Let $p \in (1,\infty)$ and $s \in (0,1/p)$. Characterise those Banach spaces X for which $\mathbf{1}_{\mathbb{R}^d_+}$ acts as pointwise multiplier on $H^{s,p}(\mathbb{R}^d;X)$.

Theorem 14.8.4, due to Hytönen and Merikoski [2019], states that $B^k_{p,p}(\mathbb{R}^d;X)$ embeds continuously into $W^{k,p}(\mathbb{R}^d;X)$ if and only if X has martingale cotype p (see Section 3.5.d).

Problem Q.13. Let $p \in (1,2)$, $k \in \mathbb{N}$ and $s \in \mathbb{R}$. Characterise those Banach spaces X for which one has continuous embeddings

$$W^{k,p}(\mathbb{R}^d;X) \hookrightarrow B^k_{p,p}(\mathbb{R}^d;X) \tag{Q.3}$$

$$H^{s,p}(\mathbb{R}^d;X) \hookrightarrow B^s_{p,p}(\mathbb{R}^d;X). \tag{Q.4}$$

A similar question can be asked with $B^s_{p,p}(\mathbb{R}^d;X)$ replaced by $F^s_{p,q}(\mathbb{R}^d;X)$ or $B^s_{p,q}(\mathbb{R}^d;X)$. Moreover, by Proposition 14.7.8, each of the embeddings (Q.3) and (Q.4) implies that X has type p. In case of UMD Banach spaces X, type p is also sufficient for these embeddings. Therefore, it is natural to conjecture that (Q.3) and (Q.4) are both equivalent to martingale type p.

Theorem 14.5.1, due to Kalton, Van Neerven, Veraar, and Weis [2008], characterises the Banach spaces X for which the Sobolev embedding

$$B^{(\frac{1}{p}-\frac{1}{2})d}_{p,p}(\mathbb{R}^d;X) \hookrightarrow \gamma(L^2(\mathbb{R}^d),X)$$

holds as the Banach space that have type p. A corresponding result for the converse embedding, with 'p' replaced by 'q', characterises Banach spaces with cotype q. It is natural to ask for similar embeddings for Bessel potential spaces and Triebel-Lizorkin spaces.

Problem Q.14. Let $p \in (1,2)$ and $r \in (p,\infty)$. Characterise those Banach spaces X for which one has continuous embeddings

$$H^{(\frac{1}{p}-\frac{1}{2})d,p}(\mathbb{R}^d;X) \hookrightarrow \gamma(L^2(\mathbb{R}^d),X) \tag{Q.5}$$

$$F^{(\frac{1}{p}-\frac{1}{2})d}_{p,r}(\mathbb{R}^d;X) \hookrightarrow \gamma(L^2(\mathbb{R}^d),X). \tag{Q.6}$$

In Corollaries 14.6.18 and 14.7.7 it was shown that having type $r > p$ is sufficient. Moreover, it is known that each of the embeddings implies type p, that and the embedding (Q.5) holds for p-convex Banach lattices; see Veraar

[2013]. It is also known that the embedding (Q.6) for $r = \infty$ characterises stable type p; see Van Neerven, Veraar, and Weis [2007] and Veraar [2013]. The validity of the embedding (Q.5) for Banach spaces X with type p would have consequences for the L^p-L^q-multiplier theory developed in Rozendaal and Veraar [2018a], which in turn has applications to stability of semigroups of operators.

Functional calculus

Several open problems related to the H^∞-functional calculus have been stated in Volume II. Theorem 15.3.9 is a classical result of Seeley [1971] and states that for a sectorial operator A with bounded imaginary powers, the domain of fractional powers $\mathsf{D}(A^\theta)$ coincides with the complex interpolation space $[X, \mathsf{D}(A)]_\theta$. A converse holds for Hilbert spaces X, where boundedness of imaginary powers characterises the boundedness of the H^∞-calculus. For Banach spaces, boundedness of imaginary powers does not imply the boundedness of the H^∞-calculus (see Example 10.2.32).

Problem Q.15. Let A be a sectorial operator on a Banach space X, and let $\theta \in (0, 1)$. Under what conditions on X does the validity of the domain identification $\mathsf{D}(A^\theta) = [X, \mathsf{D}(A)]_\theta$ imply that A has bounded imaginary powers?

Maximal regularity

The characterisation, in Theorem 17.3.1, of maximal L^p-regularity in terms of R-sectoriality was stated for the class of UMD spaces. The necessity of R-sectoriality holds without conditions on the Banach space. Therefore it seems natural to pose the following problem:

Problem Q.16. Characterise those Banach spaces X with the property that all R-sectorial operators A in X have maximal L^p-regularity.

Theorem 17.4.1 shows that maximal L^p-regularity of $-\Delta$ implies that

$$\beta_{p,X}^{\mathbb{R}} \leqslant 2M_{p,-\Delta}^{\mathrm{reg}}(\mathbb{R}_+), \tag{Q.7}$$

where $-\Delta$ is considered on $L^p(\mathbb{R}^d; X)$. A converse bound was obtained as well, but since we applied the Mihlin multiplier theorem in the proof, the bound seems far from optimal. Krylov and Priola [2017] showed that in the scalar case dimension independent bounds hold for $M_{p,-\Delta}^{\mathrm{reg}}(\mathbb{R}_+)$, and their proof seems to extend to the vector-valued setting to improve the bound we obtained. Still, powers of the UMD constant will be needed if one applies the Mihlin multiplier theorem for $d = 1$. In case of the Poisson operator $(-\Delta)^{1/2}$ the following two-sides bound was proved in Hytönen [2015]:

$$\frac{1}{2}\max\{\beta_{p,X}^{\mathbb{R}}, \hbar_{p,X}\} \leqslant M_{p,(-\Delta)^{1/2}}^{\mathrm{reg}}(\mathbb{R}_+) \leqslant \beta_{p,X}^{\mathbb{R}} + \hbar_{p,X} \tag{Q.8}$$

via the results in Geiss, Montgomery-Smith, and Saksman [2010]. Our lower estimate (Q.7) was obtained via an anisotropic extension of part of their result. It seems an interesting problem to try to extend the techniques in the latter paper to find a full analogue of the estimates (Q.8) for the maximal L^p-regularity constant of the Laplace operator:

Problem Q.17. Let $M^{\rm reg}_{p,-\Delta}(\mathbb{R}_+)$ be the maximal L^p-regularity constant of $-\Delta$ on $L^p(\mathbb{R}^d;X)$ on $\mathbb{R}_+$, where X is a UMD space. We ask if exist universal constants $0 < c \leqslant C < \infty$ such that

$$c \max\{\beta_{p,X}, \hbar_{p,X}\} \leqslant M^{\rm reg}_{p,-\Delta}(\mathbb{R}_+) \leqslant C(\beta_{p,X} + \hbar_{p,X}).$$

In Theorem 17.4.8 we have seen that there are operators A such that $-A$ generates an analytic semigroup on L^q, but A does not have maximal L^p-regularity on finite time intervals. This provided a negative answer to Brezis's question as explained in the notes of Chapter 17. The question still remains open for differential operators.

Problem Q.18. Let $\mathcal{O} \subseteq \mathbb{R}^d$ be open, and let A be a differential operator on $L^p(\mathcal{O})$ with $p \in (1,\infty)$ such that $-A$ generates an analytic semigroup on $L^p(\mathcal{O})$. Does A have maximal L^p-regularity?

Some evidence in favor of having maximal L^p-regularity can be found in Blunck and Kunstmann [2002], Kunstmann [2008] for operators in divergence and non-divergence form respectively.

In Theorem 17.2.15 we have seen that maximal L^p-regularity of A implies that $-A$ generates an analytic semigroup. Such a result seems unavailable for time-dependent operators A without any further conditions. Because of the time-dependence, generation of a semigroup has to be replaced by generation of an evolution family. A family of bounded operators $(S(t,s))_{0\leqslant s\leqslant t\leqslant T}$ on a Banach space X is called an *evolution family* if

(1) $S(t,t) = I$ for all $t \in [0,T]$;
(2) $S(t,s)S(s,r) = S(t,r)$ for all $0 \leqslant r \leqslant s \leqslant t \leqslant T$;

It is called *strongly continuous* if $S : \{(t,s) \in [0,T]^2 : s \leqslant t\} \to \mathscr{L}(X)$ is strongly continuous. References for the theory of evolution families include Amann [1995], Engel and Nagel [2000], Lunardi [1995], Pazy [1983], Tanabe [1979] and Yagi [2010].

Problem Q.19. Let X_0 and X_1 be Banach spaces, with X_1 continuously and densely embedded in X_0, and let $p \in [1,\infty]$. Suppose that $A : [0,T] \to \mathscr{L}(X_1, X_0)$ is strongly measurable in the uniform operator topology, and that there exist a constant $0 < C < \infty$ such that

$$C^{-1}\|x\|_{X_1} \leqslant \|A(t)x\|_{X_0} + \|x\|_{X_0} \leqslant C\|x\|_{X_1}, \quad x \in X_1, \ t \in [0,T].$$

Suppose further that A has maximal L^p-regularity in the sense that for all $f \in L^p(0,T;X_0)$ there exists a unique strong solution $u \in W^{1,p}(0,T;X_0) \cap$

$L^p(0,T;X_1)$ with $u(0)=0$ to the problem $u'(t)+A(t)u=f(t)$ for $t\in[0,T]$. Does there exists a strongly continuous evolution family $(S(t,s))_{0\leqslant s\leqslant t\leqslant T}$ on X_0 such that for all $f\in L^p(0,T;X_0)$ the strong solution u is equals

$$u(t)=\int_0^t S(t,s)f(s)\,\mathrm{d}s,\quad t\in[0,T]?$$

The problem is also open if additionally one assumes that for each for fixed $t_0\in[0,T]$ the operator $-A(t_0)$ generates an analytic semigroup on X_0. By a result of Prüss and Schnaubelt [2001], for $A\in C([0,T];\mathscr{L}(X_1,X_0))$ with the property that $-A(t_0)$ generates an analytic semigroup group in X_0 for each fixed $t_0\in[0,T]$, an associated evolution family $(S(t,s))_{0\leqslant s\leqslant t\leqslant T}$.

The following converse to Problem Q.19 is also open.

Problem Q.20. Let X_0 and X_1 be UMD spaces, with X_1 continuously and densely embedded in X_0. Suppose that $A:[0,T]\to\mathscr{L}(X_1,X_0)$ is strongly measurable in the uniform operator topology, and that there exist a constant $0<C<\infty$ such that

$$C^{-1}\|x\|_{X_1}\leqslant\|A(t)x\|_{X_0}+\|x\|_{X_0}\leqslant C\|x\|_{X_1},\quad x\in X_1, t\in[0,T].$$

Let $(S(t,s))_{0\leqslant s\leqslant t\leqslant T}$ be a strongly continuous evolution family on X_0, and suppose that for each $x\in X_1$, $u=S(\cdot,s)x$ is the unique strong solution to the problem

$$\begin{cases} u'(t)+A(t)u(t) &= 0,\quad t\in[s,T],\\ u(s) &= x.\end{cases}$$

Suppose that for all $0\leqslant s<t\leqslant T$ we have $S(t,s)\in\mathscr{L}(X_0,X_1)$ and the families

$$\{S(t,s):\,0\leqslant s\leqslant t\leqslant T\}$$
$$\{(t-s)A(r)S(t,s):0\leqslant s\leqslant t\leqslant T, r\in[0,T]\}$$

are R-bounded as subsets of $\mathscr{L}(X_0)$. Does it follows that A has maximal L^p-regularity?

The problem is also open if additionally one assumes that there exists a $\theta\in(0,\pi/2)$ such that for all $t\in[0,T]$, $A(t)$ is θ-sectorial and the family

$$\{\lambda R(\lambda,A(t)):\lambda\in\complement\Sigma_\theta, t\in[0,T]\}$$

is R-bounded. In Gallarati et al. [2016], Gallarati and Veraar [2017a,b], a maximal L^p-regularity result is proved under hypotheses related to those in Problem Q.20, but with additional commutativity conditions. In applications to concrete PDE, this means that the coefficients of elliptic operators need to be space-independent.

Sufficient conditions for maximal L^2-regularity for elliptic second order operators in divergence form with measurable coefficients in time and space on $X_0 = H^{-1,2}$ can be obtained via the theory of Lions [1961, 1969] (also see Auscher and Egert [2016], Dier and Zacher [2017]). Disser, ter Elst, and Rehberg [2017] extended some of these results to maximal L^p-regularity on $X_0 = H^{-1,q}$ for q and p in an small interval around 2 via Sneiberg extrapolation. Interesting counterexamples to maximal L^p-regularity for time-dependent operators A appear in Pierre and Schmitt [1997].

It would be interesting to see if Problem Q.20 can be understood from the point of singular integrals on $T(1)$ or $T(b)$-theorems (see Chapter 12). For this one could use the kernel $K(t,s) = \mathbf{1}_{s<t} A(0)S(t,s) \in \mathscr{L}(X_0)$.

K

Measurable semigroups

This appendix extends some results of Appendix G to the setting of semigroups that are not necessarily strongly continuous. In applications, such semigroups arise in several natural ways. For instance, the adjoint semigroup on X^* associated with a C_0-semigroup on a Banach space X is weak*-continuous but not necessarily strongly continuous, and if A is a sectorial operator of angle less than $\frac{1}{2}\pi$, then $-A$ generates a semigroup of operators which is a C_0-semigroup if and only if A is densely defined. The framework introduced here treats all such examples in a unified way.

K.1 Measurable semigroups

In what follows we will always assume that X is a Banach space.

Definition K.1.1 (Measurable semigroups). *A family $S = \{S(t)\}_{t>0}$ of bounded linear operators acting on X is called a* semigroup *if $S(t)S(s) = S(t+s)$ for all $t, s > 0$. A semigroup S is said to be:*

- weakly Y-measurable*, where Y is a given subspace of X^*, if for all $x \in X$ and $x^* \in Y$ the mapping $t \mapsto \langle S(t)x, x^*\rangle$ is measurable;*
- weakly measurable*, if it is weakly X^*-measurable;*
- strongly measurable*, if for all $x \in X$ the mapping $t \mapsto S(t)x$ is strongly measurable.*

A semigroup S is locally bounded *if the mapping $t \mapsto S(t)$ is bounded on every bounded interval.*

It has been shown in Proposition G.2.7 that every strongly measurable semigroup is strongly continuous for $t > 0$, that is, for all $x \in X$ the mapping $t \mapsto S(t)x$ is continuous for $t > 0$. The Pettis measurability theorem (Theorem 1.1.20) implies that if Y is a weak*-dense subspace of the dual of a separable Banach space X, then every weakly Y-measurable semigroup on X is strongly measurable, and hence strongly continuous for $t > 0$.

T. Hytönen et al., *Analysis in Banach Spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics 76,
https://doi.org/10.1007/978-3-031-46598-7

Every locally bounded semigroup S satisfies an exponential bound

$$\|S(t)\| \leqslant Me^{\omega t}, \quad t > 0, \tag{K.1}$$

for suitable $M \geqslant 1$ and $\omega \in \mathbb{R}$; this is proved as in Proposition G.2.2.

Definition K.1.2 (Generators). *Let Y be a weak*-dense subspace of X^*, and let S be a locally bounded weakly Y-measurable semigroup on X satisfying (K.1). A linear operator $(G, \mathsf{D}(G))$ in X is said to be the* Y-generator *of S if the following conditions are satisfied:*

(i) $\{\lambda \in \mathbb{C} : \Re\lambda > \omega\} \subseteq \varrho(G)$;
(ii) *for all $\Re\lambda > \omega$ and $t > 0$ we have $S(t)R(\lambda, G) = R(\lambda, G)S(t)$;*
(iii) *for all $\Re\lambda > \omega$, $x \in X$, and $x^* \in Y$ we have*

$$\langle R(\lambda, G)x, x^*\rangle = \int_0^\infty e^{-\lambda t}\langle S(t)x, x^*\rangle \,\mathrm{d}t. \tag{K.2}$$

If a Y-generator exists, it is unique. Also, by the injectivity of the Fourier transform, two locally bounded strongly measurable semigroups agree if they have the same Y-generator. If two locally bounded weakly measurable semigroups S and $\widetilde{S}$ have the same Y-generator, then for all $x \in X$ and $x^* \in Y$ we have $\langle S(t)x, x^*\rangle = \langle \widetilde{S}(t)x, x^*\rangle$ for almost all $t > 0$. In the latter case, more cannot be asserted in general: the identically zero semigroup and the left and right translation semigroups on $\ell^2(\mathbb{R})$ have generator 0.

Remark K.1.3. Suppose that the semigroup S is strongly measurable. Then the integrals $\int_0^\infty e^{-\lambda t}S(t)x\,\mathrm{d}t$ as in (K.2) exist in the strong sense as Bochner integrals in X and define an X-valued holomorphic function for $\Re\lambda > \omega$. By analytic continuation, and using the fact that the resolvents diverge in norm near the spectrum, it follows that $s(G) \leqslant \omega_0(S)$, where

$$s(G) = \inf\big\{\omega \in \mathbb{R} : \ \{\Re\lambda > \omega\} \subseteq \varrho(G)\big\}$$

is the spectral abscissa of G, and

$$\omega_0(S) = \inf\big\{\omega \in \mathbb{R} : \ \exists M \geqslant 1 : \|S(t)\| \leqslant Me^{\omega t} \text{ for all } t > 0\big\}$$

is the exponential growth bound of S.

In what follows, whenever there is no risk of confusion, we understand the weak*-dense subspace Y to be given and drop the prefix 'Y-'.

Remark K.1.4. Assumption (ii) is implied by (i) and (iii) in each of the following two cases:

- S is strongly measurable: for then one has

$$R(\lambda, G)x = \int_0^\infty e^{-\lambda s}S(s)x\,\mathrm{d}s$$

for every $x \in X$ as a Bochner integral, and one may use that bounded operators can be pulled through such integrals;

- $S^*(t)Y \subseteq Y$ for all $t > 0$: for then one has, for all $x^* \in Y$,

$$\begin{aligned}\langle S(t)R(\lambda, A)x, x^*\rangle &= \langle R(\lambda, A)x, S^*(t)x^*\rangle \\ &= \int_0^\infty e^{-\lambda t}\langle S(s)x, S^*(t)x^*\rangle \,\mathrm{d}s \\ &= \int_0^\infty e^{-\lambda t}\langle S(s+t)x, x^*\rangle \,\mathrm{d}s = \langle R(\lambda, A)S(t)x, x^*\rangle.\end{aligned}$$

Proposition K.1.5. *Let G generate the locally bounded weakly Y-measurable semigroup S on X. Then:*

(1) *for all $x \in \mathsf{D}(G)$ we have $S(t)x \in \mathsf{D}(G)$ and $GS(t)x = S(t)Gx$;*
(2) *for all $x \in \mathsf{D}(G)$ and $x^* \in Y$ the function $\langle S(\cdot)x, x^*\rangle$ is differentiable on $(0, \infty)$ and $\frac{\mathrm{d}}{\mathrm{d}t}\langle S(t)x, x^*\rangle = \langle S(t)Gx, x^*\rangle$;*
(3) *if Y is norming, then*

$$\begin{aligned}\overline{\mathsf{D}(G)} &= \Big\{x \in X : \lim_{t \downarrow 0} \|S(t)x - x\| = 0\Big\} \\ &= \Big\{x \in X : \lim_{\lambda \to \infty} \|\lambda R(\lambda, G)x - x\| = 0\Big\}\end{aligned}$$

and S restricts to a C_0-semigroup on $\overline{\mathsf{D}(G)}$ whose generator equals the part of G in $\overline{\mathsf{D}(G)}$.

Proof. (1): Applying the resolvent commutation assumption to $y = (\lambda - G)x$ with $x \in \mathsf{D}(G)$ gives $S(t)x = R(\lambda, G)S(t)(\lambda - G)x$. Since the right-hand side is in $\mathsf{D}(G)$, so is the left-hand side. Applying $\lambda - G$ on both sides gives $(\lambda - G)S(t)x = S(t)(\lambda - G)x$ and the result follows.

(2): If $x \in \mathsf{D}(G)$ and $\lambda > \omega$, we may write $x = R(\lambda, G)y$ for some $y \in X$. Then, for all $s, t > 0$ and $x^* \in Y$,

$$\begin{aligned}&\langle S(t)x - S(s)x, x^*\rangle \\ &\quad = \langle R(\lambda, G)S(t)y - R(\lambda, G)S(s)y, x^*\rangle \\ &\quad = \int_0^\infty e^{-\lambda r}\langle S(t+r)y, x^*\rangle \,\mathrm{d}r - \int_0^\infty e^{-\lambda r}\langle S(s+r)y, x^*\rangle \,\mathrm{d}r \\ &\quad = e^{\lambda t}\int_t^\infty e^{-\lambda r}\langle S(r)y, x^*\rangle \,\mathrm{d}r - e^{\lambda s}\int_s^\infty e^{-\lambda r}\langle S(r)y, x^*\rangle \,\mathrm{d}r \\ &\quad = (e^{\lambda t} - e^{\lambda s})\int_t^\infty e^{-\lambda r}\langle S(r)y, x^*\rangle \,\mathrm{d}r - e^{\lambda s}\int_s^t e^{-\lambda r}\langle S(r)y, x^*\rangle \,\mathrm{d}r.\end{aligned}$$

Fox fixed $t > 0$, the right-hand side is a continuous function of $s > 0$. Hence this is also true for the left-hand side, from which we infer that $s \mapsto \langle S(s)x, x^*\rangle$ is continuous. Dividing by $t - s$ and letting $s \to t$, by the continuity just observed we obtain

$$\lim_{s \to t}\Big\langle \frac{S(t)x - S(s)x}{t - s}, x^*\Big\rangle = \lambda e^{\lambda t}\int_t^\infty e^{-\lambda r}\langle S(r)y, x^*\rangle \,\mathrm{d}r - \langle S(t)y, x^*\rangle$$

$$
\begin{aligned}
&= \lambda\langle S(t)R(\lambda,G)y, x^*\rangle - \langle S(t)y, x^*\rangle \\
&= \langle S(t)GR(\lambda,G)y, x^*\rangle \\
&= \langle S(t)Gx, x^*\rangle.
\end{aligned}
$$

(3): Repeating the argument of (2) with $S(s)x$ replaced by x, and using that Y is norming, the estimate (K.1) implies, for $\lambda > \omega$ and $t > 0$,

$$
\begin{aligned}
\|S(t)x - x\| &= \sup_{\substack{x^*\in Y\\ \|x^*\|\leqslant 1}} |\langle S(t)x - x, x^*\rangle| \\
&\leqslant M\|y\|\Big(|e^{\lambda t} - 1|\int_t^\infty e^{-(\lambda-\omega)r}\,\mathrm{d}r + \int_0^t e^{-(\lambda-\omega)r}\,\mathrm{d}r\Big),
\end{aligned}
$$

with $x = R(\lambda,G)y$ as before. As $t \downarrow 0$, the right-hand side tends to 0. This proves strong continuity for $x \in \mathsf{D}(G)$. Strong continuity for $x \in \overline{\mathsf{D}(G)}$ follows from this by local boundedness. It is clear that the restriction of S to $\overline{\mathsf{D}(G)}$ is a C_0-semigroup on this subspace.

In the converse direction, if $x \in X$ is such that $\lim_{t\downarrow 0} S(t)x = x$ strongly in X, the semigroup property implies that $t \mapsto S(t)x$ is continuous in $[0,\infty)$ (with the convention $S(0)x = x$), and for $\Re\lambda > \omega$ we obtain

$$
\lambda R(\lambda,G)x = \int_0^\infty \lambda e^{-\lambda t} S(t)x\,\mathrm{d}t,
$$

where the integral converges as a Bochner integral in X. By strong continuity, passing to the limit $\lambda \to \infty$ gives $\lim_{\lambda\to\infty} \lambda R(\lambda,G)x = x$. This shows that $x \in \overline{\mathsf{D}(G)}$. This proves the first identity as well as the inclusion '$\subseteq$' for the second identity. Conversely, if $\lim_{\lambda\to\infty} \lambda R(\lambda,G)x = x$, then $x \in \overline{\mathsf{D}(G)}$. This gives the inclusion '$\supseteq$' for the second identity.

Let us denote the generator of $S|_{\overline{\mathsf{D}(G)}}$ by G_0 and the part of G in $\overline{\mathsf{D}(G)}$ by G_1. If $x \in \mathsf{D}(G_0)$, then

$$
R(\lambda,G)(\lambda - G_0)x = \int_0^\infty e^{-\lambda t} S(t)(\lambda - G_0)x\,\mathrm{d}t = R(\lambda,G_0)(\lambda - G_0)x = x,
$$

where the first identity follows from the fact that G generates S and the second from the fact that G_0 generates $S_0 := S|_{\overline{\mathsf{D}(G)}}$, both in the sense of Definition K.1.2; by strong continuity, there is no need to evaluate against functionals in Y. Since the left-hand side belongs to $\mathsf{D}(G)$, so does the right-hand side x. Applying $\lambda - G$ to both sides, we obtain the identity $(\lambda - G_0)x = (\lambda - G)x$. Since the former belongs to $\overline{\mathsf{D}(G)}$, so does the latter. This proves that $x \in \mathsf{D}(G_1)$ and $G_0x = G_1x$.

In the converse direction, if $x \in \mathsf{D}(G_1)$, then writing $x = R(\lambda,G)y$ with $y \in X$ gives, for all $x^* \in Y$,

$$
\langle x, x^*\rangle = \int_0^\infty e^{-\lambda t}\langle S(t)y, x^*\rangle\,\mathrm{d}t
$$

$$= \int_0^\infty e^{-\lambda t} \langle S(t) \underbrace{(\lambda - G)x}_{\in \overline{\mathsf{D}(G)}}, x^* \rangle \, \mathrm{d}t$$

$$= \int_0^\infty e^{-\lambda t} \langle S_0(t)(\lambda - G)x, x^* \rangle \, \mathrm{d}t = \langle R(\lambda, G_0)(\lambda - G)x, x^* \rangle,$$

where the last step follows from the fact that G_0 generates $S_0 = S|_{\overline{\mathsf{D}(G)}}$ in the sense of Definition K.1.2. It follows that $x = R(\lambda, G_0)(\lambda - G)x \in \mathsf{D}(G_0)$. □

Proposition K.1.6 (Rescaling). *Let G be generate the locally bounded weakly Y-measurable semigroup $S = (S(t))_{t>0}$ on X. Then for any $\mu \in \mathbb{C}$ the operator $G_\mu := G - \mu$ generates the locally bounded scalarly measurable semigroup $S_\mu = (e^{-\mu t} S(t))_{t>0}$.*

Proof. We have $\|S_\mu(t)\| \leqslant M e^{(\omega - \Re\mu)t}$, $\{\lambda \in \mathbb{C} : \Re\lambda > \omega - \Re\mu\} \subseteq \varrho(G_\mu)$, and if $\Re\lambda > \omega - \Re\mu$, then $\Re(\lambda + \mu) > \omega$, and therefore for all $x \in X$ and $x^* \in Y$ we have

$$\begin{aligned} \int_0^\infty e^{-\lambda t} \langle S_\mu(t)x, x^* \rangle \, \mathrm{d}t &= \int_0^\infty e^{-(\lambda+\mu)t} \langle S(t)x, x^* \rangle \, \mathrm{d}t \\ &= \langle R(\lambda + \mu, G)x, x^* \rangle = \langle R(\lambda, G_\mu)x, x^* \rangle. \end{aligned}$$

□

The following proposition provides an analogue of Proposition G.2.3(3), which states that if S is a C_0-semigroup on X with generator A, then for all $x \in X$ and $t \geqslant 0$ one has $\int_0^t S(s)x \, \mathrm{d}s \in \mathsf{D}(A)$ and

$$A \int_0^t S(s)x \, \mathrm{d}s = S(s)x - x,$$

and that if $x \in \mathsf{D}(A)$, then also

$$A \int_0^t S(s)x \, \mathrm{d}s = \int_0^t S(s)Ax \, \mathrm{d}s.$$

The difficulty in the present set-up is that the integrals of the semigroup orbits make no *a priori* sense. Establishing that the integrals do indeed exist in X as "weak Y-integrals" is part of our task in proving the proposition. In the strongly measurable case, all this poses no problems and in the proposition below one can simply take

$$x_t' = \int_0^t S(s)x \, \mathrm{d}s$$

as a Bochner integral in X and redo the proof of Proposition G.2.3.

Proposition K.1.7 (Main theorem of calculus). *Let G be generate the locally bounded weakly Y-measurable semigroup S on X, and assume that Y is norming. For all $t > 0$ and $x \in X$ there exists a unique $x'_t \in \mathsf{D}(G)$ such that for all $x^* \in Y$ we have*

$$\langle x'_t, x^* \rangle = \int_0^t \langle S(s)x, x^* \rangle \, \mathrm{d}s, \quad x^* \in Y,$$

and

$$Gx'_t = S(t)x - x.$$

If in addition $x \in \mathsf{D}(G)$, we furthermore have

$$\langle Gx'_t, x^* \rangle = \int_0^t \langle S(s)Gx, x^* \rangle \, \mathrm{d}s, \quad x^* \in Y.$$

Proof. Let $x \in X$ and $t > 0$ be fixed.

Step 1 – We begin with uniqueness. If

$$S(t)x - x = 0 \quad \text{and} \quad \int_0^t \langle S(s)x, x^* \rangle \, \mathrm{d}s = 0, \quad x^* \in Y,$$

then integrating the first identity over $[0, t]$ with $t > 0$, applying x^*, and subtracting the second identity, we are left with the identity $t\langle x, x^* \rangle = 0$, valid for all $x^* \in Y$, and therefore $x = 0$. But for $x = 0$, uniqueness is clear.

Step 2 – For the proof of existence, fix an arbitrary $\lambda > \omega$ and consider the rescaled semigroup S_λ generated by G_λ as in Proposition K.1.6. This semigroup is uniformly exponentially stable, by which we mean that it satisfies (K.1) with a negative exponent. In particular, G_λ is boundedly invertible. The element $x'_{t,\lambda} := G_\lambda^{-1}(S_\lambda(t)x - x)$ belongs to $\mathsf{D}(G_\lambda) = \mathsf{D}(G)$ and satisfies

$$G_\lambda x'_{t,\lambda} = S_\lambda(t)x - x.$$

Applying the definition of a generator to G_λ and performing a change of variables, for all $x^* \in Y$ we obtain

$$\begin{aligned}\langle x'_{t,\lambda}, x^* \rangle &= \langle G_\lambda^{-1}(S_\lambda(t)x - x), x^* \rangle \\ &= -\int_0^\infty \langle S_\lambda(s)(S_\lambda(t)x - x), x^* \rangle \, \mathrm{d}s = \int_0^t \langle S_\lambda(s)x, x^* \rangle \, \mathrm{d}s.\end{aligned}$$

This proves the proposition for the semigroup S_λ.

Step 3 – We wish to use the results of Steps 1 and 2 to derive the proposition for the original semigroup S. To guess the formula for x'_t, we formally write $x'_t = \int_0^t S(s)x \, \mathrm{d}s$ (not worrying about measurability and integrability issues) and integrate by parts to arrive at

$$\begin{aligned} x'_t = \int_0^t S(s)x\,\mathrm{d}s &= \int_0^t e^{\lambda s} S_\lambda(s)x\,\mathrm{d}s \\ &= e^{\lambda t}\int_0^t S_\lambda(r)x\,\mathrm{d}r - \lambda\int_0^t e^{\lambda s}\int_0^s S_\lambda(r)x\,\mathrm{d}r\,\mathrm{d}s \qquad \text{(K.3)} \\ &= e^{\lambda t}x'_{t,\lambda} - \lambda\int_0^t e^{\lambda s}x'_{s,\lambda}\,\mathrm{d}s, \end{aligned}$$

with $x'_{s,\lambda} = G_\lambda^{-1}[S_\lambda(s)x - x] = \int_0^s S_\lambda(r)x\,\mathrm{d}r$ as in Step 2, the second identity being again formal. The last integral appearing in (K.3) is well defined as a Bochner integral in X because $s \mapsto x'_{s,\lambda} = S(s)G_\lambda^{-1}x - G_\lambda^{-1}x$ is continuous as an X-valued function, S being strongly continuous on $\overline{\mathsf{D}(G)} = \overline{\mathsf{D}(G_\lambda)}$.

We now take the right-hand side of (K.3) as the *definition* of x'_t. The strong continuity of S on $\overline{\mathsf{D}(G)} = \overline{\mathsf{D}(G_\lambda)}$ implies that

$$\int_0^t e^{\lambda s}x'_{s,\lambda}\,\mathrm{d}s = \int_0^t e^{\lambda s}(S(s)G_\lambda^{-1}x - G_\lambda^{-1}x)\,\mathrm{d}s$$

belongs to $\mathsf{D}(G)$. Indeed, using the notation used in the proof of Proposition K.1.5 it belongs to $\mathsf{D}(G_0)$, and we have $\mathsf{D}(G_0) \subseteq \mathsf{D}(G)$. Since also $x'_{t,\lambda} \in \mathsf{D}(G)$ (by Step 2), we conclude from the representation (K.3) that x'_t belongs to $\mathsf{D}(G)$, and we have

$$\begin{aligned} Gx'_t &= e^{\lambda t}Gx'_{t,\lambda} - \lambda G\int_0^t e^{\lambda s}x'_{s,\lambda}\,\mathrm{d}s \\ &= GG_\lambda^{-1}(S(t)x - e^{\lambda t}x) - \lambda G\int_0^t e^{\lambda s}(S_\lambda(s)G_\lambda^{-1}x - G_\lambda^{-1}x)\,\mathrm{d}s \\ &= (I + \lambda G_\lambda^{-1})(S(t)x - e^{\lambda t}x) - \lambda G\int_0^t S(s)G_\lambda^{-1}x\,\mathrm{d}s \\ &\qquad + (e^{\lambda t} - 1)GG_\lambda^{-1}x \\ &\overset{(*)}{=} (I + \lambda G_\lambda^{-1})(S(t)x - e^{\lambda t}x) - \lambda(S(s)G_\lambda^{-1}x - G_\lambda^{-1}x) \\ &\qquad + (e^{\lambda t} - 1)(I + \lambda G_\lambda^{-1})x \\ &= S(t)x - x, \end{aligned}$$

where $(*)$ uses the reasoning involving strong continuity of S on $\overline{\mathsf{D}(G)} = \overline{\mathsf{D}(G_\lambda)}$ once more. This proves that x'_t has the required properties.

If $x \in \mathsf{D}(G)$, then by Proposition K.1.5(2), for all $x^* \in Y$ we have

$$\langle Gx'_t, x^*\rangle = \langle S(t)x - x, x^*\rangle\,\mathrm{d}s = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\langle S(s)x, x^*\rangle\,\mathrm{d}s = \int_0^t \langle S(r)Gx, x^*\rangle\,\mathrm{d}r;$$

to justify the second identity we note that $s \mapsto \langle S(s)x, x^*\rangle$ is differentiable on $(0,t]$, with bounded derivative.

Finally, applying this with $G^{-1}x$ in place of x and noting that $(G^{-1}x)'_t = G^{-1}x'_t$, we obtain

$$\langle x'_t, x^* \rangle = \int_0^t \langle S(s)x, x^* \rangle \, \mathrm{d}s.$$

This completes the proof. □

To finish the abstract treatment, let us write

$$x' =: \int_0^t S(s)x \, \mathrm{d}s$$

for the unique element $x' \in \mathsf{D}(G)$ satisfying the conclusions of the proposition. With this notation, we have the following result.

Proposition K.1.8. *Under the assumptions of Proposition K.1.7, for all* $s, t > 0$ *and* $x \in X$ *we have*

$$S(s) \int_0^t S(r)x \, \mathrm{d}r = \int_0^t S(r+s)x \, \mathrm{d}r,$$

where the right-hand side is shorthand for $\int_0^{t+s} S(r)x \, \mathrm{d}r - \int_0^s S(r)x \, \mathrm{d}r$.

Proof. As in the proof of Proposition K.1.7, this is easy for the rescaled semigroup S_λ. Indeed, for all $\in X$ and $x^* \in Y$ we have

$$\begin{aligned} G_\lambda S_\lambda(s) x'_{t,\lambda} &= S_\lambda(s) G_\lambda x'_{t,\lambda} \\ &= S_\lambda(t)[S_\lambda(s)x] - [S_\lambda(s)x] = G_\lambda x'_{t+s,\lambda} - G_\lambda x'_{s,\lambda}. \end{aligned}$$

Applying G_λ^{-1} we obtain the identity

$$S_\lambda(s) x'_{t,\lambda} = x'_{t+s,\lambda} - x'_{s,\lambda}.$$

Applying $x^* \in Y$ to both sides, this can be rewritten as

$$\begin{aligned} \langle S_\lambda(s) x'_{t,\lambda}, x^* \rangle &= \int_0^{t+s} \langle S_\lambda(r)x, x^* \rangle \, \mathrm{d}r - \int_0^s \langle S_\lambda(r)x, x^* \rangle \, \mathrm{d}r \\ &= \int_0^t \langle S_\lambda(r)[S_\lambda(s)x], x^* \rangle \, \mathrm{d}r. \end{aligned}$$

These identities imply that $S(s)x'_{t,\lambda}$ satisfies the two properties of Proposition K.1.7 with x replaced by $S(s)x$. By uniqueness, in the notation introduced above this gives the desired identity

$$S_\lambda(s) \int_0^t S_\lambda(r)x \, \mathrm{d}r = \int_0^t S_\lambda(r+s)x \, \mathrm{d}r.$$

The general case can again be deduced from the rescaled case, by similar arguments as before. We leave the details to the reader. □

We proceed with some important examples. The first demonstrates the consistency of Definition K.1.2 with the corresponding Definition G.2.1 for C_0-semigroups.

Proposition K.1.9 (C_0-semigroups). *If G generates a C_0-semigroup S in the sense of Definition K.1.2, then $G = \widetilde{G}$, where $\widetilde{G}$ is the generator of S in the sense of Definition G.2.1.*

Proof. This is immediate from the fact that both G and $\widetilde{G}$ satisfy (K.2), which implies that their resolvents coincide. □

Proposition K.1.10 (Adjoints of C_0-semigroups). *If G generates the adjoint S^* of a C_0-semigroup S on X in the sense of Definition K.1.2, then $G = A^*$, where A is the generator of S in the sense of Definition G.2.1.*

This is proved in the same way (take $Y = X$ this time).

Proposition K.1.11 (Sectorial operators of angle less than $\frac{1}{2}\pi$). *Let A be a sectorial operator in X of angle $\omega(A) < \frac{1}{2}\pi$, and for $t > 0$ define the bounded operators $S(t)$ on X by*

$$S(t)x := \frac{1}{2\pi i}\int_{\Gamma} e^{-zt}R(z,A)x\,\mathrm{d}z, \quad x \in X,$$

where Γ is the downwards oriented boundary of $\Sigma_\sigma \cup B$, for any $\omega(A) < \sigma < \frac{1}{2}\pi$ and any closed ball B centred at the origin. Then $S = (S(t))_{t>0}$ is a bounded and strongly measurable semigroup on X, and the operator $-A$ generates S in the sense of Definition K.1.2.

Proof. Boundedness of S is proved by repeating the argument of Theorem G.5.2, and strong measurability is evident. The semigroup property is proved in the same way as the multiplicativity of the Dunford calculus (Theorem 10.2.2). For the reader's convenience we present the details. Fix $\zeta, \zeta' \in \Sigma_\eta$ and choose contours Γ and Γ' as above, with Γ to the right of Γ'. Then, by the resolvent identity, Cauchy's theorem, Fubini's theorem, and the Cauchy integral formula,

$$\begin{aligned}
S(z')S(z)x &= \frac{1}{(2\pi i)^2}\int_{\Gamma}\int_{\Gamma'} e^{\lambda z+\mu z'}R(\lambda,A)R(\mu,A)x\,\mathrm{d}\lambda\,\mathrm{d}\mu \\
&= \frac{1}{(2\pi i)^2}\int_{\Gamma}\int_{\Gamma'} e^{\lambda z+\mu z'}\,\frac{R(\lambda,A)x - R(\mu,A)x}{\mu-\lambda}\,\mathrm{d}\lambda\,\mathrm{d}\mu \\
&= \frac{1}{(2\pi i)^2}\int_{\Gamma}\int_{\Gamma'} e^{\lambda z+\mu z'}\,\frac{R(\lambda,A)x}{\mu-\lambda}\,\mathrm{d}\mu\,\mathrm{d}\lambda \\
&= \frac{1}{2\pi i}\int_{\Gamma} e^{\lambda z+\lambda z'}R(\lambda,A)x\,\mathrm{d}\lambda = S(z+z')x.
\end{aligned}$$

Next we check that $-A$ is the generator of S in the sense of Definition K.1.2. Resolvent commutations is clear. For all $x^* \in X^*$, by Fubini's theorem and Cauchy's theorem we have

$$\begin{aligned}\int_0^\infty e^{-\lambda r}\langle S(r)x, x^*\rangle \,\mathrm{d}r &= \frac{1}{2\pi i}\int_\Gamma \Big(\int_0^\infty e^{-(\lambda+z)r}\,\mathrm{d}r\Big)\langle R(z,A)x, x^*\rangle \,\mathrm{d}z \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{1}{\lambda+z}\langle R(z,A)x, x^*\rangle \,\mathrm{d}z \\ &= \langle R(-\lambda, A)x, x^*\rangle = \langle R(\lambda, -A)x, x^*\rangle.\end{aligned}$$

□

Remark K.1.12. Suppose that A is a densely defined sectorial operator of angle $\omega(A) < \frac{1}{2}\pi$. By Theorem G.5.2, $-A$ generates a bounded analytic C_0-semigroup S in the sense of Definition G.2.1. Since the adjoint operator A^* is sectorial and has the same angle, combination of the above two results shows that $-A^*$ generates the bounded analytic (but not necessarily C_0) semigroup S^* in the sense of Definition K.1.2.

K.2 Uniform exponential stability

In this section we state two propositions providing elementary characterisations of uniform exponential stability. A semigroup S on a Banach space X is said to be *uniformly exponentially stable* if there exist $M \geqslant 1$ and $\omega > 0$ such that $\|S(t)\| \leqslant Me^{-\omega t}$ for all $t > 0$. A locally bounded semigroup S is uniformly exponentially stable if and only if

$$\|S(t_0)\| < 1 \ \text{ for some } \ t_0 > 0.$$

The 'only if' part is obvious. To prove the 'if' part, we show that $\|S(t)\| \leqslant Me^{-\omega t}$ with

$$M = e^{\omega t_0}\sup_{s\in(0,t_0]}\|S(s)\|, \quad \omega = \log(\|S(t_0)\|^{-1/t_0}).$$

Indeed, let $t > 0$ be fixed and write $t = (n+\theta)t_0$ with $\theta \in [0,1)$ and $n \in \mathbb{N}$. Then, with the above choices of ω and M,

$$\|S(t)\| \leqslant \|S(\theta t_0)\|\,\|S(t_0)\|^n \leqslant Me^{-\omega t_0}e^{-\omega t_0 n} = Me^{-\omega t_0(n+1)} \leqslant Me^{-\omega t}.$$

Proposition K.2.1 (Datko–Pazy). *Let S be a strongly measurable semigroup on a Banach space X and let $1 \leqslant p < \infty$. The following assertions are equivalent:*

1. *the semigroup S is uniformly exponentially stable;*
2. *the semigroup S is locally bounded and for all $x \in X$ the orbit $t \mapsto S(t)x$ belongs to $L^p(\mathbb{R}_+; X)$.*

Proof. The implication (1)⇒(2) part is clear. To prove the converse implication (2)⇒(1), by the closed graph theorem there exists a constant $C \geqslant 0$ such that $\|S(\cdot)x\|_{L^p(\mathbb{R}_+;X)} \leqslant C\|x\|$ for all $x \in X$. Let $M \geqslant 1$ and $\omega > 0$ be such that $\|S(t)\| \leqslant Me^{\omega t}$ for all $t > 0$. Then

$$\frac{1 - e^{-\omega pt}}{\omega p}\|S(t)x\|^p = \int_0^t e^{-wp(t-s)}\|S(t-s)S(s)x\|^p \,\mathrm{d}s \leqslant M^p C^p \|x\|^p.$$

It follows that $\sup_{t\geqslant 0}\|S(t)\| =: K < \infty$ and, for $t > 0$,

$$t\|S(t)x\|^p = \int_0^t \|S(t-s)S(s)x\|^p \,\mathrm{d}s \leqslant K^p C^p \|x\|^p.$$

Therefore $\|S(t)\| \leqslant CKt^{-1/p}$ and the result follows from the preliminary observation. □

Proposition K.2.2. *Let S be a strongly measurable semigroup and let $1 \leqslant p \leqslant \infty$. The following assertions are equivalent:*

(1) *the semigroup S is uniformly exponentially stable;*
(2) *the semigroup S is locally bounded and for all $f \in L^p(\mathbb{R}_+;X)$ we have $S * f \in L^p(\mathbb{R}_+;X)$.*

Below we will see that $S * f$ can often be interpreted as the so-called mild solution to an abstract Cauchy problem.

Proof. (1)⇒(2): If $\|S(t)\| \leqslant Me^{-\omega t}$ for all $t > 0$, with $\omega > 0$, then $\|S*f(t)\| \leqslant \phi * f(t)$ for all $t > 0$, where $\phi(s) = Me^{-\omega s}\mathbf{1}_{\mathbb{R}_+}(s)$. Taking L^p-norms, Young's inequality gives $S * f \in L^p(\mathbb{R}_+;X)$ and

$$\|S * f\|_{L^p(\mathbb{R}_+;X)} \leqslant \|\phi\|_{L^1(\mathbb{R}_+)}\|f\|_{L^p(\mathbb{R}_+;X)} \leqslant M\omega^{-1}\|f\|_{L^p(\mathbb{R}_+;X)}.$$

(2)⇒(1): By the closed graph theorem there exists a constant $C \geqslant 0$ such that

$$\|S * f\|_{L^p(\mathbb{R}_+;X)} \leqslant C\|f\|_{L^p(\mathbb{R}_+;X)}, \quad f \in L^p(\mathbb{R}_+;X).$$

First consider $p \in [1,\infty)$. Let $M \geqslant 1$ and $\mu \geqslant 0$ be such that $\|S(t)\| \leqslant Me^{\mu t}$ for all $t \geqslant 0$. Fix $\varepsilon > 0$ and $x \in X$, and set $f(t) := e^{-(\mu+\varepsilon)t}S(t)x$ for $t \geqslant 0$. Then

$$\|f\|_{L^p(\mathbb{R}_+;X)} \leqslant \frac{M}{(\varepsilon p)^{1/p}}\|x\|$$

and

$$S * f(t) = \int_0^t S(t-s)f(s)\,\mathrm{d}s = \frac{1 - e^{-(\mu+\varepsilon)t}}{\mu+\varepsilon}S(t)x.$$

Fixing any $\tau > 0$ we obtain

$$\|t \mapsto S(t)x\|_{L^p(\mathbb{R}_+;X)} \leqslant \|t \mapsto S(t)x\|_{L^p(0,\tau;X)} + \|t \mapsto S(t)x\|_{L^p(\tau,\infty;X)}$$

$$
\begin{aligned}
&\leqslant M e^{\mu\tau}\tau^{1/p}\|x\| + \frac{\mu+\varepsilon}{1-e^{-(\mu+\varepsilon)\tau}}\|S*f\|_{L^p(\tau,\infty;X)} \\
&\leqslant M e^{\mu\tau}\tau^{1/p}\|x\| + \frac{\mu+\varepsilon}{1-e^{-(\mu+\varepsilon)\tau}}C\|f\|_{L^p(\mathbb{R}_+;X)} \\
&\leqslant M e^{\mu\tau}\tau^{1/p}\|x\| + \frac{\mu+\varepsilon}{1-e^{-(\mu+\varepsilon)\tau}}\frac{CM}{(\varepsilon p)^{1/p}}\|x\|,
\end{aligned}
$$

using that $\|f(t)\| \leqslant Me^{-\varepsilon t}\|x\|$ in the last inequality. Proposition K.2.1 now gives the required uniformly exponential stability.

For $p = \infty$ the above argument can be repeated to give the bound

$$
\begin{aligned}
\sup_{t\geqslant 0}\|S(t)x\| &\leqslant \|t\mapsto S(t)x\|_{L^\infty(0,\tau;X)} + \|t\mapsto S(t)x\|_{L^\infty(\tau,\infty;X)} \\
&\leqslant M e^{\mu\tau}\|x\| + \frac{\mu+\varepsilon}{1-e^{-(\mu+\varepsilon)\tau}}\|S*f\|_{L^\infty(\mathbb{R}_+;X)} \\
&\leqslant M e^{\mu\tau}\|x\| + \frac{\mu+\varepsilon}{1-e^{-(\mu+\varepsilon)\tau}}C\|f\|_{L^\infty(\mathbb{R}_+;X)} \\
&\leqslant M e^{\mu\tau}\|x\| + \frac{\mu+\varepsilon}{1-e^{-(\mu+\varepsilon)\tau}}CM\|x\| =: K\|x\|.
\end{aligned}
$$

This shows that for every $x \in X$ the function $t \mapsto f_x(t) := S(t)x$ belongs to $L^\infty(\mathbb{R}_+;X)$. Since $S * f_x(t) = tS(t)x$ for all $t \geqslant 0$, we find that

$$
\|S(t)x\| = t^{-1}\|S*f_x(t)\| \leqslant t^{-1}C\|f_x\|_{L^\infty(\mathbb{R}_+;X)} \leqslant t^{-1}K\|x\|.
$$

We now apply the preliminary observation preceding Proposition K.2.1. □

For analytic semigroups we have the following simple necessary and sufficient condition for uniform exponential stability:

Proposition K.2.3. *Let A be a sectorial operator of angle $\omega(A) < \frac{1}{2}\pi$. The analytic semigroup generated by $-A$ is uniformly exponentially stable if and only if $0 \in \varrho(A)$.*

Proof. If the semigroup $(S(t))_{t\geqslant 0}$ generated by $-A$ is uniformly exponentially stable, the observation of Remark K.1.3 implies that $\varrho(-A)$ contains the right-half plane $\Re\lambda > -\delta\}$ for some $\delta > 0$. In particular, $0 \in \varrho(A)$.

In the converse direction, if $0 \in \varrho(A)$, then $A-\delta$ is sectorial with $\omega(A-\delta) < \frac{1}{2}\pi$ for any $\delta > 0$ so small that the disc $\{|z| < 2\delta\}$ is contained in $\varrho(A)$. But that means that the semigroup $(e^{\delta t}S(t))_{t\geqslant 0}$ is bounded, so $(S(t))_{t\geqslant 0}$ is uniformly exponentially stable. □

K.3 Analytic semigroups

Let A be a sectorial operator of angle $\omega(A) < \pi/2$. As we have seen in Proposition K.1.11, $-A$ generates the bounded strongly measurable semigroup given by the formula

$$S(t)x = \frac{1}{2\pi i}\int_\Gamma e^{-zt}R(z,A)x\,\mathrm{d}z, \quad t>0,\ x\in X,$$

where Γ is as described in the proposition. By the same formula, this semigroup extends to complex times $z\in\Sigma_{\frac12\pi-\omega(A)}$. For any $\omega(A)<\sigma<\frac12\pi$, the resulting family $S=S(z))_{z\in\Sigma_{\frac12\pi-\sigma}}$ is bounded and holomorphic.

As in the case of analytic C_0-semigroups, one proves that if A be sectorial of angle $<\frac12\pi$ and S is the analytic semigroup generated by $-A$, then for all $x\in X$ we have $S(t)x\in\mathsf{D}(A)$ and

$$M_A := \sup_{t>0}\|tAS(t)\| < \infty. \tag{K.4}$$

As a consequence, for any integer $n\geqslant 1$, $S(t)$ maps X into $\mathsf{D}(A^n)$ and

$$\sup_{t>0}\|t^nA^nS(t)\| \leqslant \sup_{t>0}\|tAS(t/n)\|^n \leqslant (nM_A)^n.$$

The following result shows that the quantity (K.4) cannot be arbitrarily small unless the operator A is bounded. This fact plays a role in a construction of a counterexample in Chapter 17 (see Theorem 17.4.4).

Proposition K.3.1. *Let A be sectorial of angle $<\frac12\pi$ and let S be the locally bounded strongly measurable semigroup generated by $-A$. If*

$$\limsup_{t\downarrow 0} t\|AS(t)\| < \frac{1}{e},$$

then A is bounded.

Proof. For all $t>0$ and $x\in X$ we have $-AS(t)x=S'(t)x$. To see this, we first use Hille's theorem to move A into the integral, then we write $A=(A-z)+z$ and use that

$$\frac{1}{2\pi i}\int_\Gamma e^{-zt}x\,\mathrm{d}z = 0$$

by Cauchy's theorem, and note that

$$\frac{1}{2\pi i}\int_\Gamma ze^{-zt}R(z,A)x\,\mathrm{d}z = S'(t)x$$

by differentiation under the integral sign. As a consequence, for all fixed $t>0$ the assumption of the lemma implies

$$\limsup_{n\to\infty}\frac{t}{n}\Big\|S'\big(\frac{t}{n}\big)\Big\| < \frac{1}{e}.$$

By induction, $S^{(n)}(t)=(-A)^nS(t)=(-AS(\frac{t}{n}))^nx=(S'(\frac{t}{n}))^n$. The inequality $n^n/n!\leqslant e^n$ implies that $\limsup_{t\to\infty}\frac{n^n}{n!}\|(S'(\frac{t}{n}))^n\|<1$, and therefore there exists a $\delta>0$ such that for every $t>0$ the series

$$\widetilde{S}(z) := \sum_{n=0}^{\infty} \frac{1}{n!}(z-t)^n S^{(n)}(t) = \sum_{n=0}^{\infty} \frac{(z-t)^n}{t^n} \frac{n^n}{n!} \Big(\frac{t}{n} S'\big(\frac{t}{n}\big)\Big)^n$$

converges absolutely and uniformly on the set $B_t := \{z \in \mathbb{C} : |z-t| < (1+\delta)t\}$ and defines an analytic function, and we have $\widetilde{S} = S$ on $B_t \cap (0,\infty)$. Since every B_t contains the origin and $\widetilde{S}(0) = I$, it follows that

$$\lim_{s\downarrow 0} \|S(s) - I\| = \lim_{s\downarrow 0} \|\widetilde{S}(s) - I\| = 0. \tag{K.5}$$

The remainder of the proof is devoted to showing that if G generates an locally bounded strongly measurable semigroup S such that (K.5) holds, then G is bounded. We follow the standard proof of the corresponding result for C_0-semigroups, while keeping an eye on the fact that in the present situation we use a different generator concept.

First of all, we note that (K.5) implies that S extends to a continuous mapping from $[0,\infty)$ to $\mathscr{L}(X)$. By Proposition K.1.9, the generator $-A$ coincides with the generator of S as a C_0-semigroup in the sense of Definition G.2.1. In particular, A is densely defined. Choose $t > 0$ so small that $\|I - \frac{1}{t}\int_0^t S(s)\,\mathrm{d}s\| < 1$, using the shorthand notation $\int_0^t S(s)\,\mathrm{d}s$ for the bounded operator $x \mapsto \int_0^t S(s)x\,\mathrm{d}s$. Then the operator $\frac{1}{t}\int_0^t S(s)\,\mathrm{d}s$ is invertible and therefore the identity

$$\frac{1}{h}(S(h)-I)\int_0^t S(s)\,\mathrm{d}s = \frac{1}{h}\Big(\int_t^{t+h} S(s)\,\mathrm{d}s - \int_0^h S(s)\,\mathrm{d}s\Big)$$

implies that

$$\frac{1}{h}(S(h)-I) = \frac{1}{h}\Big(\int_t^{t+h} S(s)\,\mathrm{d}s - \int_0^h S(s)\,\mathrm{d}s\Big)\Big(\frac{1}{t}\int_0^t S(s)\,\mathrm{d}s\Big)^{-1}.$$

We may now let $h \downarrow 0$ in operator norm and conclude that

$$-A = (S(t)-I)\Big(\frac{1}{t}\int_0^t S(s)\,\mathrm{d}s\Big)^{-1}.$$

This shows that on its dense domain, A coincides with a bounded operator. Since A is also closed, this forces $\mathsf{D}(A) = X$ and consequently A is bounded. □

K.4 An interpolation result

For the proof of Theorem 15.3.23 we need a characterisation of the real interpolation space $(X, \mathsf{D}(A))_{\frac{1}{2},2}$ for sectorial operators A, which will be given presently. A further characterisation, valid in the case where $-A$ is the generator of an analytic semigroup, is given in the next appendix (see Theorem L.2.4).

Proposition K.4.1. *Let $\theta \in (0,1)$ and $1 \leqslant p \leqslant \infty$.*

(1) *If A is a sectorial operator in X, then*

$$(X, \mathsf{D}(A))_{\theta,p} = \Big\{x \in X : \ \lambda \mapsto \lambda^\theta A(\lambda + A)^{-1} x \in L^p(\mathbb{R}_+, \tfrac{\mathrm{d}\lambda}{\lambda}; X)\Big\}$$

with equivalence of norms

$$\|x\|_{(X,\mathsf{D}(A))_{\theta,p}} \eqsim \|x\| + \big\|\lambda \mapsto \lambda^\theta A(\lambda + A)^{-1}x\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}\lambda}{\lambda};X)}.$$

If $0 \in \varrho(A)$, we also have equivalence of homogeneous norms

$$\|x\|_{(X,\mathsf{D}(A))_{\theta,p}} \eqsim \big\|\lambda \mapsto \lambda^\theta A(\lambda + A)^{-1}x\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}\lambda}{\lambda};X)}.$$

(2) *If $-A$ generates a C_0-semigroup $(S(t))_{t>0}$ on X, then*

$$(X, \mathsf{D}(A))_{\theta,p} = \Big\{x \in X : \ t \mapsto t^{-\theta}(S(t)x - x) \in L^p(\mathbb{R}_+, \tfrac{\mathrm{d}t}{t}; X)\Big\}$$

with equivalence of norms

$$\|x\|_{(X,\mathsf{D}(A))_{\theta,p}} \eqsim \|x\| + \big\|t \mapsto t^{-\theta}(S(t)x - x)\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)}.$$

If the semigroup is uniformly exponentially stable, we also have equivalence of homogeneous norms

$$\|x\|_{(X,\mathsf{D}(A))_{\theta,p}} \eqsim \big\|t \mapsto t^{-\theta}(S(t)x - x)\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)}.$$

Proof. Denote the two sets on the right-hand side in (1) and (2) by $X^{(1)}_{\theta,p}$ and $X^{(2)}_{\theta,p}$, respectively. We will write $K(t,x) := K(t,x;X,\mathsf{D}(A))$ for the K-functional of the real interpolation method (see Appendix C).

For (1) we will prove continuous inclusions

$$(X, \mathsf{D}(A))_{\theta,p} \subseteq X^{(1)}_{\theta,p} \subseteq (X, \mathsf{D}(A))_{\theta,p},$$

and for (2) it will then suffice to prove continuous inclusions

$$(X, \mathsf{D}(A))_{\theta,p} \subseteq X^{(2)}_{\theta,p} \subseteq X^{(1)}_{\theta,p}.$$

The norm equivalences are proved along the way.

(1): We start with the inclusion $(X, \mathsf{D}(A))_{\theta,p} \subseteq X^{(1)}_{\theta,p}$. Let $x \in (X, \mathsf{D}(A))_{\theta,p}$ be given. If $x = x_0 + x_1$ with $x_0 \in X$ and $x_1 \in \mathsf{D}(A)$, then for $\lambda > 0$ we have, with $M := \sup_{\mu>0} \|\mu(\mu + A)^{-1}\|$,

$$\begin{aligned}\lambda^\theta\|A(\lambda + A)^{-1}x\| &\leqslant \lambda^\theta\|A(\lambda + A)^{-1}x_0\| + \lambda^\theta\|A(\lambda + A)^{-1}x_1\| \\ &\leqslant (M + 1)\lambda^\theta\|x_0\| + M\lambda^{\theta-1}\|Ax_1\|\end{aligned}$$

$$\leqslant (M+1)\lambda^\theta(\|x_0\| + \lambda^{-1}\|Ax_1\|).$$

It follows that

$$\lambda^\theta \|A(\lambda+A)^{-1}x\| \leqslant (M+1)\lambda^\theta K(\lambda^{-1},x).$$

By a change of variables the right-hand side is seen to belong to $L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t})$, with norm equal to $(M+1)\|x\|_{(X,\mathsf{D}(A))_{\theta,p}}$. It follows that $\lambda^\theta A(\lambda+A)^{-1}x$ belongs to $L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X)$, hence $x \in X^{(1)}_{\theta,p}$, and

$$\big\|\lambda^\theta A(\lambda+A)^{-1}x\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)} \leqslant (M+1)\|x\|_{(X,\mathsf{D}(A))_{\theta,p}}.$$

We next prove the inclusion $X^{(1)}_{\theta,p} \subseteq (X,\mathsf{D}(A))_{\theta,p}$. Suppose $x \in X$ is such that $\lambda \mapsto \lambda^\theta A(\lambda+A)^{-1}x$ belongs to $L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X)$. Then, using the decomposition $x = A(\lambda+A)^{-1}x + \lambda(\lambda+A)^{-1}x \in X + \mathsf{D}(A)$, we obtain

$$\begin{aligned}\lambda^\theta K(\lambda^{-1},x) &\leqslant \lambda^\theta(\|A(\lambda+A)^{-1}x\| + \lambda^{-1}\|\lambda(\lambda+A)^{-1}x\|_{\mathsf{D}(A)})\\ &= \lambda^\theta(2\|A(\lambda+A)^{-1}x\| + \|(\lambda+A)^{-1}x\|).\end{aligned}$$

Since $\|\lambda(\lambda+A)^{-1}\| \leqslant M$ for $\lambda > 0$, the right-hand is in $L^p((1,\infty), \frac{\mathrm{d}\lambda}{\lambda})$, with norm at most

$$\begin{aligned}&2\|x\|^*_{\theta,p} + M\Big(\frac{1}{(1-\theta)p}\Big)^{1/p}\|x\| && \text{if } 1 \leqslant p < \infty;\\ &2\|x\|^*_{\theta,p} + M\|x\| && \text{if } p = \infty,\end{aligned}$$

where $\|x\|^*_{\theta,p} := \|\lambda \mapsto \lambda^\theta(\lambda+A)^{-1}x\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)}$. It follows that $t \mapsto t^{-\theta}K(t,x)$ belongs to $L^p((1,\infty), \frac{\mathrm{d}t}{t})$. Also, since $\mathsf{D}(A) \subseteq X$, we have $K(t,x) \leqslant \|x\|$ and therefore $t \mapsto t^{-\theta}K(t,x)$ is in $L^p((0,1), \frac{\mathrm{d}t}{t})$. Together, these facts imply that $t \mapsto t^{-\theta}K(t,x)$ is in $L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t})$ and therefore $x \in (X,\mathsf{D}(A))_{\theta,p}$. Our estimates moreover show that

$$\|x\|_{(X,\mathsf{D}(A))_{\theta,p}} = \big\|t \mapsto t^{-\theta}K(t,x)\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t})} \lesssim_{\theta,p} \|x\|^*_{\theta,p} + \|x\|.$$

Assume next that $0 \in \varrho(A)$. Since $\lim_{\lambda\downarrow 0}(\lambda+A)^{-1} = A^{-1}$ in $\mathscr{L}(X)$ we may choose $r = r_A > 0$ so small that $\|(\lambda+A)^{-1} - A^{-1}\| \leqslant \frac12\|A^{-1}\|$ for all $0 < \lambda < r$. It follows that for all $x \in X$ we have $\|(\lambda+A)^{-1}x\| \geqslant \frac12\|A^{-1}x\|$, and therefore, for all $x \in \mathsf{D}(A)$, $\|A(\lambda+A)^{-1}x\| \geqslant \frac12\|x\|$. It follows that for all $x \in \mathsf{D}(A)$,

$$\begin{aligned}\|\lambda^\theta A(\lambda+A)^{-1}x\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)} &\geqslant \|\lambda \mapsto \lambda^\theta A(\lambda+A)^{-1}x\|_{L^p((0,r),\frac{\mathrm{d}t}{t};X)}\\ &\geqslant \frac12\|x\|\|\lambda \mapsto \lambda^\theta\|_{L^p((0,r),\frac{\mathrm{d}t}{t})} =: C_{\theta,p,A}\|x\|.\end{aligned}$$

This estimate extends to general $x \in (X, \mathsf{D}(A))_{\theta,p}$ by applying it to $\mu R(\mu, A)x$ and letting $\mu \to \infty$ on both sides. This gives the equivalence with the homogeneous norm.

(2): We begin with the inclusion $(X, \mathsf{D}(A))_{\theta,p} \subseteq X^{(2)}_{\theta,p}$. Fix an arbitrary element $x \in (X, \mathsf{D}(A))_{\theta,p}$. If $x = x_0 + x_1$ with $x_0 \in X$ and $x_1 \in \mathsf{D}(A)$, then for all $t > 0$ we have

$$\begin{aligned} t^{-\theta}\|S(t)x - x\| &\leqslant t^{-\theta}(\|S(t)x_0 - x_0\| + \|S(t)x_1 - x_1\|) \\ &\leqslant t^{-\theta}((M+1)\|x_0\| + tM\|Ax_1\|) \leqslant (M+1)t^{-\theta}K(t,x), \end{aligned}$$

where $M := \sup_{t\geqslant 0}\|S(t)\|$; to obtain the estimate for x_1 we used that $S(t)x_1 - x_1 = \int_0^t S(s)Ax_1 \,\mathrm{d}s$. It follows that $t \mapsto t^{-\theta}(S(t)x - x)$ belongs to $L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X)$ and

$$\Big\| t \mapsto t^{-\theta}\|S(t)x - x\| \Big\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t})} \leqslant (M+1)\|x\|_{(X,\mathsf{D}(A))_{\theta,p}}.$$

To prove the inclusion $X^{(2)}_{\theta,p} \subseteq X^{(2)}_{\theta,p}$, suppose that $x \in X$ is such that $t \mapsto t^{-\theta}(S(t)x - x) \in L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X)$. Using the identity $A(\lambda + A)^{-1} = \lambda(\lambda + A)^{-1}x - x$ and the Laplace transform representation of the resolvent (Proposition G.4.1), for $\lambda > 0$ we have

$$A(\lambda + A)^{-1} = \int_0^\infty \lambda e^{-\lambda t}(S(t)x - x)\,\mathrm{d}t.$$

First let $1 \leqslant p < \infty$. By Jensen's inequality we estimate

$$\begin{aligned} \big\|\lambda \mapsto \lambda^\theta A(\lambda + A)^{-1}x\big\|^p_{L^p(\mathbb{R}_+, \frac{\mathrm{d}\lambda}{\lambda}; X)} &= \int_0^\infty \lambda^{\theta p}\Big\|\int_0^\infty \lambda e^{-\lambda t}(S(t)x - x)\,\mathrm{d}t\Big\|^p \frac{\mathrm{d}\lambda}{\lambda} \\ &\leqslant \int_0^\infty \lambda^{\theta p}\int_0^\infty \lambda e^{-\lambda t}\|S(t)x - x\|^p \,\mathrm{d}t\frac{\mathrm{d}\lambda}{\lambda} \\ &= \int_0^\infty \Big(\int_0^\infty \lambda^{\theta p+1}e^{-\lambda t}\frac{\mathrm{d}\lambda}{\lambda}\Big)\|S(t)x - x\|^p\,\mathrm{d}t \\ &= C_{\theta,p}\int_0^\infty t^{-\theta p-1}\|S(t)x - x\|^p\,\mathrm{d}t \\ &= C_{\theta,p}\big\|t \mapsto t^{-\theta}(S(t)x - x)\big\|^p_{L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X)}, \end{aligned}$$

where $C_{\theta,p} = \int_0^\infty \mu^{\theta p+1}e^{-\mu}\frac{\mathrm{d}\mu}{\mu} = \Gamma(\theta p+1)$. This gives the desired inclusion for $1 \leqslant p < \infty$. For $p = \infty$ we note that for all $\lambda > 0$ we have $\int_0^\infty \lambda^{\theta+1}e^{-\lambda t}t^\theta\,\mathrm{d}t = \int_0^\infty s^\theta e^{-s}\,\mathrm{d}s = \Gamma(1+\theta)$, and therefore

$$\begin{aligned} &\big\|\lambda \mapsto \lambda^\theta A(\lambda + A)^{-1}x\big\|_{L^\infty(\mathbb{R}_+, \frac{\mathrm{d}\lambda}{\lambda}; X)} \\ &\quad = \sup_{\lambda>0}\Big\|\int_0^\infty \lambda^{\theta+1}e^{-\lambda t}(S(t)x - x)\,\mathrm{d}t\Big\| \end{aligned}$$

$$\begin{aligned}&\leqslant \sup_{\lambda>0}\int_0^\infty \lambda^{\theta+1}e^{-\lambda t}t^\theta\,\mathrm{d}t\cdot\sup_{t>0}t^{-\theta}\|S(t)x-x\|\\ &\leqslant C_\theta\big\|t\mapsto t^{-\theta}(S(t)x-x)\big\|_{L^\infty(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)},\end{aligned}$$

with $C_\theta = \Gamma(1+\theta)$.

Assume finally that $\|S(t)\| \leqslant Me^{-\omega t}$ for all $t \geqslant 0$, with $M \geqslant 1$ and $\omega > 0$. Choose $R = R_{M,\omega} > 0$ so large that $\|S(t)\| \leqslant \frac{1}{2}$ for $t \geqslant R$. Then $\|S(t)x - x\| \geqslant \frac{1}{2}\|x\|$ for $t \geqslant R$. If $1 \leqslant p < \infty$, it follows that

$$\big\|t\mapsto t^{-\theta}(S(t)x-x)\big\|^p_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)} \geqslant \int_R^\infty t^{-\theta p}(\|x\|/2)^p\,\frac{\mathrm{d}t}{t} =: C_{R,\theta,p}\|x\|^p.$$

This gives the equivalence with the homogeneous norm for $1 \leqslant p < \infty$. For $p = \infty$ we simply note that

$$\begin{aligned}\big\|t\mapsto t^{-\theta}(S(t)x-x)\big\|_{L^\infty(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)} &\geqslant \big\|t\mapsto t^{-\theta}(S(t)x-x)\big\|_{L^\infty((R,\infty),\frac{\mathrm{d}t}{t};X)}\\ &\geqslant \frac{1}{2}R^{-\theta}\|x\|.\end{aligned}$$

□

K.5 Notes

Definition K.1.2 is in the spirit of the corresponding definition in the theory of integrated semigroups; see, e.g., Arendt [1987], Kellerman and Hieber [1989]. The proof of Proposition K.2.1 is taken from Engel and Nagel [2000, Theorem V.1.8]; a systematic treatment is given in Van Neerven [1996]. Analytic semigroups with non-densely defined generators have been studied in detail in Lunardi [1995]. The proofs of Propositions K.3.1 and K.4.1 are taken from Pazy [1983] and Lunardi [2009], respectively.

L

The trace method for real interpolation

This appendix is a continuation of Appendix C in Volume I, where the real and complex methods were introduced. After recalling some basic definitions (Section L.1), we treat several additional topics including the trace method (Section L.2), reiteration theorems (Section L.3), and Sobolev embedding theorems and mixed derivatives (Section L.4).

L.1 Preliminaries

We begin by recalling some definitions from Appendix C.

Throughout this appendix, (X_0, X_1) is an *interpolation couple*, that is, an ordered pair of Banach spaces, both of which are continuously embedded in a Hausdorff topological vector space $\mathscr{X}$. For $t > 0$ and $x \in X_0 + X_1 := \{x \in \mathscr{X} : x = x_0 + x_1 \text{ with } x_0 \in X_0 \text{ and } x_1 \in X_1\}$, the *K-functional* is defined by

$$K(t, x; X_0, X_1) := \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : \ x_0 \in X_0, \ x_1 \in X_1, \ x = x_0 + x_1\}.$$

The function $t \mapsto K(t, x; X_0, X_1)$ is non-decreasing and continuous, and the function $x \mapsto K(t, x; X_0, X_1)$ is sub-additive. Moreover,

$$tK(t^{-1}, x; X_0, X_1) = K(t, x; X_1, X_0).$$

From now on, when no confusion is likely to occur we abbreviate

$$K(t, x) = K(t, x; X_0, X_1).$$

For $0 < \theta < 1$ and $1 \leqslant p \leqslant \infty$, the real interpolation space $(X_0, X_1)_{\theta,p}$ is the Banach space defined by

$$(X_0, X_1)_{\theta,p} := \big\{x \in X_0 + X_1 : \|x\|_{\theta,p} < \infty\big\},$$

with norm

T. Hytönen et al., *Analysis in Banach Spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics 76,
https://doi.org/10.1007/978-3-031-46598-7

$$\|x\|_{\theta,p} := \begin{cases} \Big(\displaystyle\int_0^\infty [t^{-\theta}K(t,x)]^p \,\frac{\mathrm{d}t}{t}\Big)^{1/p} & \text{if } 1 \leqslant p < \infty; \\ \operatorname*{ess\,sup}_{t>0} t^{-\theta}K(t,x) & \text{if } p = \infty. \end{cases}$$

One has $(X,X)_{\theta,p} = X$ and $(X_0,X_1)_{\theta,p} = (X_1,X_0)_{1-\theta,p}$ with identical norms. By Lemma C.3.12, $X_0 \cap X_1$ is dense in $(X_0,X_1)_{\theta,p}$ whenever $0 < \theta < 1$ and $1 \leqslant p < \infty$. In what follows we let

$$X_{p,\infty} := \overline{X_0 \cap X_1}^{(X_0,X_1)_{\theta,\infty}} \tag{L.1}$$

denote the closure of $X_0 \cap X_1$ in $(X_0,X_1)_{\theta,\infty}$.

Fix an $x \in X_0 \cap X_1$ and consider the mapping $c \mapsto cx$ as operators from $\mathbb{K}$ into X_0 and into X_1, respectively. Real interpolation of these operators (Theorem C.3.3) with $\theta \in (0,1)$ and $p \in [1,\infty]$ gives the estimate

$$\|x\|_{\theta,p} \leqslant \|x\|_{X_0}^{1-\theta}\|x\|_{X_1}^{\theta}, \qquad x \in X_0 \cap X_1. \tag{L.2}$$

In (C.6) we have seen that for all $0 < \theta < 1$ and $1 \leqslant p_0 \leqslant p_1 \leqslant \infty$ we have the continuous inclusion

$$(X_0,X_1)_{\theta,p_0} \hookrightarrow (X_0,X_1)_{\theta,p_1}. \tag{L.3}$$

The next result shows that more can be said in the special case when $X_1 \subseteq X_0$ with continuous inclusion mapping (we write $X_1 \hookrightarrow X_0$ in this situation).

Proposition L.1.1. *If we have a continuous embedding $X_1 \hookrightarrow X_0$ with norm C, the following assertions hold:*

(1) *For all $0 < \theta < 1$ and $1 \leqslant p \leqslant \infty$ we have continuous embeddings*

$$X_1 \hookrightarrow (X_0,X_1)_{\theta,p} \hookrightarrow X_0$$

with

$$\|x\|_{X_0} \leqslant C^\theta \|x\|_{\theta,p} \leqslant C\|x\|_{X_1}, \quad x \in X_1.$$

(2) *For all $0 < \theta_0 < \theta_1 < 1$, and all $1 \leqslant p_0, p_1, p \leqslant \infty$ we have a continuous embedding*

$$(X_0,X_1)_{\theta_1,p_1} \hookrightarrow (X_0,X_1)_{\theta_0,p_0}$$

with

$$\|x\|_{\theta_0,p_0} \leqslant C^{\theta_1-\theta_0}\big((\theta_1-\theta_0)^{-1} + \theta_0^{-1}\big)\|x\|_{\theta_1,p_1}, \quad x \in (X_0,X_1)_{\theta_1,p_1}.$$

Proof. (1): Let $x = x_0 + x_1$ with $x_0 \in X_0$ and $x_1 \in X_1$. Then

$$\min\{1,t\}\|x\|_{X_0} \leqslant \|x_0\|_{X_0} + t\|x_1\|_{X_0} \leqslant \|x_0\|_{X_0} + Ct\|x_1\|_{X_1}.$$

Therefore, $\min\{1,t\}\|x\|_{X_0} \leqslant K(Ct,x)$, and we conclude that

$$\|x\|_{X_0} \leqslant \sup_{t>0} t^{-\theta} K(Ct, x) = C^{\theta} \|x\|_{\theta,\infty} \leqslant C^{\theta} \|x\|_{\theta,p}$$

using the contractivity of the inclusion (L.3). Also, by (L.2) and the inequality just proved,

$$\|x\|_{\theta,p} \leqslant \|x\|_{X_0}^{1-\theta} \|x\|_{X_1}^{\theta} \leqslant C^{\theta(1-\theta)} \|x\|_{\theta,p}^{1-\theta} \|x\|_{X_1}^{\theta}.$$

This implies the remaining estimate $\|x\|_{\theta,p} \leqslant C^{1-\theta} \|x\|_{X_1}$.

(2): By the continuity of the inclusion (L.3), it suffices to consider the case $p_0 = 1$ and $p_1 = \infty$. Fix $x \in X_1$. Since $K(t,x) \leqslant \|x\|_{X_0}$ for all $t > 0$,

$$\begin{aligned}
\|x\|_{\theta_0,1} &= \int_0^C t^{-\theta_0} K(t,x) \frac{\mathrm{d}t}{t} + \int_C^\infty t^{-\theta_0} K(t,x) \frac{\mathrm{d}t}{t} \\
&\leqslant \sup_{t>0} t^{-\theta_1} K(t,x) \int_0^C t^{\theta_1-\theta_0} \frac{\mathrm{d}t}{t} + \|x\|_{X_0} \int_C^\infty t^{-\theta_0} \frac{\mathrm{d}t}{t} \\
&= \frac{C^{\theta_1-\theta_0}}{\theta_1-\theta_0} \|x\|_{\theta_1,\infty} + \frac{C^{-\theta_0}}{\theta_0} \|x\|_{X_0} \\
&\leqslant \frac{C^{\theta_1-\theta_0}}{\theta_1-\theta_0} \|x\|_{\theta_1,\infty} + \frac{C^{\theta_1-\theta_0}}{\theta_0} \|x\|_{X_{\theta_1,\infty}},
\end{aligned}$$

where in the last step we applied the estimate of (1). This gives the first estimate. □

L.2 The trace method

In this section we introduce another interpolation method, the so-called *trace method*, and prove its equivalence with the real interpolation method.

Definition L.2.1 (Trace method). *For $p \in [1,\infty]$ and $\theta \in (0,1)$, the space*

$$(X_0, X_1)^{\mathrm{Tr}}_{\theta,p}$$

is defined as the set of all $x \in X_0 + X_1$ for which there exists a strongly measurable function $u : (0,\infty) \to X_0 + X_1$ which is weakly differentiable, and satisfies the following three properties:

(i) *$t \mapsto t^{1-\theta} u'(t)$ belongs to $L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X_0)$;*
(ii) *$t \mapsto t^{1-\theta} u(t)$ belongs to $L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X_1)$;*
(iii) *$u(0) = x$.*

On $(X_0, X_1)^{\mathrm{Tr}}_{\theta,p}$ we define a norm by

$$\begin{aligned}
&\|x\|_{(X_0,X_1)^{\mathrm{Tr}}_{\theta,p}} \\
&\quad := \max\Big\{ \|t \mapsto t^{1-\theta} u'(t)\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X_0)}, \|t \mapsto t^{1-\theta} u(t)\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X_1)} \Big\},
\end{aligned} \tag{L.4}$$

where the infimum over u extends over all strongly measurable functions $u : (0, \infty) \to X_0 + X_1$ with the above three properties. Note that if $u : (0, \infty) \to X_0 + X_1$ is strongly measurable and satisfies (i) and (ii), then $u \in W^{1,1}((0, T); X_0 + X_1)$ for all $0 < T < \infty$; in particular, u is equal almost everywhere to a (uniquely defined) continuous function from $[0, \infty)$ to $X_0 + X_1$. In condition (iii), we refer to this version when imposing $u(0) = x$.

We continue with a technical result which shows that, in the definition of the trace method, we may restrict ourselves to functions $u \in C^1((0, \infty); X_0 \cap X_1)$ without changing the norm defined by (L.4). Indeed, this is due to the fact that the constant $\varepsilon > 0$ can be taken arbitrarily small in both inequalities in (L.6).

Proposition L.2.2. *For $\varepsilon > 0$ let $g_\varepsilon : [0, \infty) \to \mathbb{R}$ be the 'tent shaped' piecewise linear function which is identically zero on $[0, 1]$ and $[1 + 2\varepsilon, \infty)$ and whose graph connects the points $(1, 0)$, $(1+\varepsilon, \varepsilon^{-1})$ and $(1+2\varepsilon, 0)$ linearly. Let $\varphi_\varepsilon(t) := tg_\varepsilon(t)$. Let $u : (0, \infty) \to X_0 + X_1$ be strongly measurable and satisfy conditions (i) and (ii) of Definition L.2.1, and define $u_\varepsilon : (0, \infty) \to X_0 + X_1$ by*

$$u_\varepsilon(t) := \int_0^\infty \varphi_\varepsilon(t/\tau)u(\tau)\,\frac{\mathrm{d}\tau}{\tau} = \int_0^\infty \varphi_\varepsilon(\tau)u(t/\tau)\,\frac{\mathrm{d}\tau}{\tau}. \tag{L.5}$$

Then $u_\varepsilon \in C([0, \infty); X_0 + X_1) \cap C^1((0, \infty); X_0 \cap X_1)$, we have $u_\varepsilon(0) = u(0)$, and for all $0 < \theta < 1$,

$$\begin{aligned} \|t \mapsto t^{1-\theta}u_\varepsilon'(t)\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X_0)} &\leqslant \|t \mapsto t^{1-\theta}u'(t)\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X_0)}, \\ \|t \mapsto t^{1-\theta}u_\varepsilon(t)\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X_1)} &\leqslant (1 + 2\epsilon)\|t \mapsto t^{1-\theta}u(t)\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X_1)}, \end{aligned} \tag{L.6}$$

and

$$\|t \mapsto t^{2-\theta}u_\varepsilon'(t)\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X_1)} \leqslant \frac{2(1 + 2\varepsilon)^2}{\varepsilon}\|t \mapsto t^{1-\theta}u(t)\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X_1)}.$$

Proof. Observe that

$$\int_0^\infty \varphi_\varepsilon(t)\,\frac{\mathrm{d}t}{t} = \int_0^\infty g_\varepsilon(t)\,\mathrm{d}t = 1, \tag{L.7}$$

and that the function u_ε is defined by convolution on the multiplicative group $(0, \infty)$ with Haar measure $\mathrm{d}\tau/\tau$. From this it is easy to see that $u \in C([0, \infty); X_0 + X_1)$ implies $u_\varepsilon \in C([0, \infty); X_0 + X_1)$ and $u_\varepsilon(0) = x$ (use the last expression in (L.5)). It also implies that $u_\varepsilon \in C^1((0, \infty); X_0 \cap X_1)$ and

$$u_\varepsilon'(t) = \int_0^\infty \varphi(\tau)u'(t/\tau)\,\frac{\mathrm{d}\tau}{\tau^2}.$$

Writing

$$t^{1-\theta}u_\varepsilon'(t) = \int_0^\infty \tau^{-\theta}\varphi(\tau) \cdot (t/\tau)^{1-\theta}u'(t/\tau)\,\frac{\mathrm{d}\tau}{\tau},$$

Young's inequality implies

$$\begin{aligned}
&\|t \mapsto t^{1-\theta} u_\varepsilon'(t)\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X_0)} \\
&\quad\leqslant \|t \mapsto t^{-\theta} \varphi_\varepsilon(t)\|_{L^1(\mathbb{R}_+, \frac{\mathrm{d}t}{t})} \|t \mapsto t^{1-\theta} u'(t)\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X_0)} \\
&\quad\leqslant \|t \mapsto t^{1-\theta} u'(t)\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X_0)},
\end{aligned}$$

where we used (L.7) and the fact that $t^{-\theta} \leqslant 1$ on the support of φ_ε. Similarly,

$$\begin{aligned}
&\|t \mapsto t^{1-\theta} u_\varepsilon(t)\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X_1)} \\
&\quad\leqslant \|t \mapsto t^{1-\theta} \varphi_\varepsilon\|_{L^1(\mathbb{R}_+, \frac{\mathrm{d}t}{t})} \|t \mapsto t^{1-\theta} u(t)\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X_1)} \\
&\quad\leqslant (1+2\varepsilon) \|t \mapsto t^{1-\theta} u(t)\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X_1)}.
\end{aligned}$$

Finally, in view of

$$t^{2-\theta} u_\varepsilon'(t) = \int_0^\infty (t/\tau)^{2-\theta} \varphi_\varepsilon'(t/\tau) \cdot \tau^{1-\theta} u(\tau) \,\frac{\mathrm{d}\tau}{\tau},$$

another application of Young's inequality gives

$$\begin{aligned}
&\|t \mapsto t^{2-\theta} u_\varepsilon'(t)\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X_1)} \\
&\quad\leqslant \|t \mapsto t^{2-\theta} \varphi_\varepsilon'(t)\|_{L^1(\mathbb{R}_+, \frac{\mathrm{d}t}{t})} \|t \mapsto t^{1-\theta} u(t)\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X_1)} \\
&\quad\leqslant (1+2\varepsilon) \|t \mapsto g_\varepsilon(t) + t g_\varepsilon'(t)\|_{L^1(\mathbb{R}_+, \mathrm{d}t)} \|t \mapsto t^{1-\theta} u(t)\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X_1)} \\
&\quad= (1+2\varepsilon) \frac{2 + \frac{7}{2}\varepsilon}{\varepsilon} \|t \mapsto t^{1-\theta} u(t)\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X_1)},
\end{aligned}$$

where the last equality follows by exact computation of the L^1-norm in the preceding line. □

Theorem L.2.3 (Lions). *For all* $0 < \theta < 1$ *and* $1 \leqslant p \leqslant \infty$ *we have*

$$(X_0, X_1)_{\theta,p}^{\mathrm{Tr}} = (X_0, X_1)_{\theta,p}$$

with equivalent norms satisfying

$$\frac{1}{64 C_\theta} \|x\|_{\theta,p} \leqslant \|x\|_{(X_0,X_1)_{\theta,p}^{\mathrm{Tr}}} \leqslant \frac{12}{1-\theta} \|x\|_{\theta,p},$$

where $C_\theta = \max\{\theta^{-1}, (1-\theta)^{-1}\}$.

Proof. Let $x \in (X_0, X_1)_{\theta,p}^{\mathrm{Tr}}$, choose u such that Definition L.2.1(i)-(iii) hold, and let u_ε with $0 < \varepsilon < 1$ be as in Proposition L.2.2.

Setting $v(t) = t u_\varepsilon'(t)$, for $j \in \{0, 1\}$ we find

$$\|t \mapsto t^{j-\theta} v(t)\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X_j)} = \|t \mapsto t^{1+j-\theta} u_\varepsilon'(t)\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t}; X_j)}$$

$$\leqslant C_\varepsilon \|t \mapsto t^{1-\theta} u^{(j)}(t)\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X_j)}$$

with $C_\varepsilon = 2(1+2\varepsilon)^2/\varepsilon$. Therefore,

$$\begin{aligned}
&\|v\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X_0+X_1)} \\
&\qquad \leqslant \|v\|_{L^p(0,1,\frac{\mathrm{d}t}{t};X_0)} + \|v\|_{L^p(1,\infty,\frac{\mathrm{d}t}{t};X_1)} \\
&\qquad \leqslant \|t \mapsto t^{-\theta} v(t)\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X_0)} + \|t \mapsto t^{1-\theta} v(t)\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X_1)} \\
&\qquad \leqslant C_\varepsilon \big(\|t \mapsto t^{1-\theta} u(t)\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X_0)} + \|t \mapsto t^{1-\theta} u'(t)\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X_1)}\big).
\end{aligned}$$

The right-hand side is finite because u satisfies conditions (i) and (ii) of Definition L.2.1. As a consequence, the integral $\int_0^\infty v(t)\,\frac{\mathrm{d}t}{t}$ converges as a Bochner integral in $X_0 + X_1$ (see (C.17)), and

$$\int_0^\infty v(t)\,\frac{\mathrm{d}t}{t} = \int_0^\infty u_\varepsilon'(t)\,\mathrm{d}t = \lim_{t\to\infty} u_\varepsilon(t) - \lim_{t\to 0} u_\varepsilon(t). \tag{L.8}$$

The existence of the limits follows from the convergence of the integral. From Proposition L.2.2 we see that $\lim_{t\to\infty} u_\varepsilon(t) = 0$ in X_1. From the definition of u_ε we obtain $\lim_{t\to 0} u_\varepsilon(t) = u_\varepsilon(0) = u(0) = x$.

From Theorem C.3.14 it follows that $x \in (X_0, X_1)_{\theta,p}$ and, using the notation of the theorem,

$$\begin{aligned}
\|x\|_{\theta,p} &\leqslant 4C_\theta \|x\|_{\theta,p,p} \\
&\leqslant 4C_\theta \max_{j\in\{0,1\}} \|t \mapsto t^{j-\theta} v(t)\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X_j)} \\
&\leqslant 8C_\theta \frac{(1+2\varepsilon)^2}{\varepsilon} \max_{j\in\{0,1\}} \|t \mapsto t^{1-\theta} u^{(j)}(t)\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X_j)}.
\end{aligned}$$

With $\varepsilon = 1/2$, taking the infimum over all admissible functions u gives the bound

$$\|x\|_{\theta,p} \leqslant 64 C_\theta \|x\|_{(X_0,X_1)^{\mathrm{Tr}}_{\theta,p}}.$$

In the converse direction, suppose that $x \in (X_0, X_1)_{\theta,p}$ and fix $\varepsilon > 0$. By Theorem C.3.14 there exists a strongly measurable $v : (0,\infty) \to X_0 \cap X_1$ such that $\int_0^\infty v(t)\,\frac{\mathrm{d}t}{t} = x$ and for $j \in \{0,1\}$,

$$\|t \mapsto t^{j-\theta} v(t)\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X_j)} \leqslant (1+\varepsilon)\|x\|_{\theta,p,p} \leqslant 12(1+\varepsilon)\|x\|_{\theta,p}.$$

Let $u : (0,\infty) \to X_0 + X_1$ be given by

$$u(t) = \int_t^\infty v(\tau)\,\frac{\mathrm{d}\tau}{\tau} = \int_0^1 v(t/\tau)\,\frac{\mathrm{d}\tau}{\tau}.$$

It follows that

$$\|t \mapsto t^{1-\theta} u(t)\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X_1)} \leqslant \int_0^1 \|t \mapsto t^{1-\theta} v(t/\tau)\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X_1)}\,\frac{\mathrm{d}\tau}{\tau}$$

$$= \int_0^1 \tau^{-\theta} \|t \mapsto t^{1-\theta} v(t)\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X_1)} \,\mathrm{d}\tau$$
$$\leqslant 12(1+\varepsilon)(1-\theta)^{-1}\|x\|_{\theta,p}.$$

Similarly, since $u' = -v(t)/t$ we obtain

$$\|t \mapsto t^{1-\theta} u'(t)\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X_0)} \leqslant \|t \mapsto t^{-\theta} v(t)\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X_0)} \leqslant 12(1+\varepsilon)\|x\|_{\theta,p}.$$

Since also $u(0) = x$, we conclude that

$$\begin{aligned}\|x\|^{\mathrm{Tr}}_{\theta,p} &\leqslant \max\Big\{\|t \mapsto t^{\theta} u(t)\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X_0)}, \|t \mapsto t^{\theta} u'(t)\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X_1)}\Big\}\\ &\leqslant 12(1+\varepsilon)(1-\theta)^{-1}\|x\|_{\theta,p}.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this proves the result. □

As an application we obtain the following characterisation of real interpolation spaces using analytic semigroups. We note that for $I = \mathbb{R}_+$, the theorem is a special case of Theorem 16.3.19 in the main text.

Theorem L.2.4. *Let $-A$ generate an analytic semigroup $(S(t))_{t\geqslant 0}$ on a Banach space X. Let either $I = \mathbb{R}_+$ and assume that S is uniformly bounded, or let $I = (0,T)$. Let $p \in [1,\infty]$ and $\theta \in (0,1)$. Then*

$$(X, \mathsf{D}(A))_{\theta,p} = \Big\{x \in X : \|t \mapsto t^{1-\theta} AS(t)x\|_{L^p(I,\frac{\mathrm{d}t}{t};X)}\Big\}$$

with equivalence of norms

$$\|x\|_{(X,\mathsf{D}(A))_{\theta,p}} \eqsim \|x\| + \|t \mapsto t^{1-\theta} AS(t)x\|_{L^p(I,\frac{\mathrm{d}t}{t};X)},$$

with implied constants only depending on θ and A. If $0 \in \varrho(A)$ and $I = \mathbb{R}_+$, we also have equivalence of homogeneous norms

$$\|x\|_{(X,\mathsf{D}(A))_{\theta,p}} \eqsim \|t \mapsto t^{1-\theta} AS(t)x\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)}.$$

Proof. In the proof below we use the shorthand notation

$$|\!|\!|x|\!|\!|_I := \|t \mapsto t^{1-\theta} AS(t)x\|_{L^p(I,\frac{\mathrm{d}t}{t};X)}.$$

First we consider the case $I = \mathbb{R}_+$ under the additional assumption that S is uniformly bounded. Let $M_1, M_2 \geqslant 0$ be such that $\|S(t)\| \leqslant M_1$ and $\|tAS(t)\| \leqslant M_2$ for all $t \geqslant 0$.

Suppose first that $x \in X$ satisfies $|\!|\!|x|\!|\!|_{\mathbb{R}_+} < \infty$. We will prove that $x \in (X, \mathsf{D}(A))_{\theta,p}$. Let $u(t) = e^{-t}S(t)x$. Then $u'(t) = -e^{-t}S(t)x - e^{-t}AS(t)x$, and

$$\|t \mapsto t^{1-\theta} u'(t)\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};X)} \leqslant K_{\theta,p} M_1\|x\| + |\!|\!|x|\!|\!|_{\mathbb{R}_+},$$

$$\|t \mapsto t^{1-\theta}u(t)\|_{L^p(\mathbb{R}_+,\frac{dt}{t};\mathsf{D}(A))} \leqslant K_{\theta,p}M_1\|x\| + \|\!|x|\!\|_{\mathbb{R}_+},$$

where $K_{\theta,p} = \|t \mapsto t^{1-\theta}e^{-t}\|_{L^p(\mathbb{R}_+,\frac{dt}{t})}$. Therefore, by Theorem L.2.3, $x \in (X, \mathsf{D}(A))_{\theta,p}$ and

$$\|x\|_{(X,\mathsf{D}(A))_{\theta,p}} \leqslant 64C_\theta(K_{\theta,p}M_1\|x\| + \|\!|x|\!\|_{\mathbb{R}_+}).$$

If, in addition to the assumptions already made, we also assume that $0 \in \varrho(A)$, then S is uniformly exponentially stable (because $A - \varepsilon$ is sectorial in that case and hence generates a bounded analytic semigroup), and we can use $u(t) = S(t)x$ in the above proof instead. This gives

$$\|t \mapsto t^{1-\theta}u'(t)\|_{L^p(\mathbb{R}_+,\frac{dt}{t};X)} = \|\!|x|\!\|_{\mathbb{R}_+},$$
$$\|t \mapsto t^{1-\theta}u(t)\|_{L^p(\mathbb{R}_+,\frac{dt}{t};\mathsf{D}(A))} \leqslant \max\{1, \|A^{-1}\|\}\|\!|x|\!\|_{\mathbb{R}_+},$$

and consequently

$$\|x\|_{(X,\mathsf{D}(A))_{\theta,p}} \leqslant 64C_\theta \max\{1, \|A^{-1}\|\}\|\!|x|\!\|_{\mathbb{R}_+}.$$

In the conversely direction, suppose that $x \in (X, \mathsf{D}(A))_{\theta,p}$. By the definition of the K-method $t \mapsto t^{-\theta}K(t,x)$ belongs to $L^p(\mathbb{R}_+, \frac{dt}{t})$, where

$$K(t,x) = \inf\big\{\|x_0\|_X + t\|x_1\|_{\mathsf{D}(A)} :\ x_0 \in X,\ x_1 \in \mathsf{D}(A),\ x_0 + x_1 = x\big\}.$$

Let $x_0 \in X$ and $x_1 \in \mathsf{D}(A)$ satisfy $x_0 + x_1 = x$. Then

$$\|AS(t)x\| \leqslant \|AS(t)x_0\| + \|AS(t)x_1\| \leqslant t^{-1}(M_2\|x_0\| + tM_1\|x_1\|_{\mathsf{D}(A)}).$$

Therefore, $\|AS(t)x\| \leqslant t^{-1}\max\{M_1, M_2\}K(t,x)$, and

$$\begin{aligned}\|\!|x|\!\|_{\mathbb{R}_+} &\leqslant \max\{M_1, M_2\}\|t \mapsto t^{-\theta}K(t,x)\|_{L^p(\mathbb{R}_+,\frac{dt}{t};X)} \\ &= \max\{M_1, M_2\}\|x\|_{(X,\mathsf{D}(A))_{\theta,p}}.\end{aligned}$$

Since also $\|x\| \leqslant C\|x\|_{(X,\mathsf{D}(A))_{\theta,p}}$, this concludes the proof for the case $I = \mathbb{R}_+$.

In case $I = (0,T)$, we use a simple scaling argument. For this let $M \geqslant 0$ and $\omega \geqslant 0$ be such that $e^{-\omega t}\|S(t)\| \leqslant Me^{-t}$ and $e^{-\omega t}\|tAS(t)\| \leqslant Me^{-t}$ for all $t \geqslant 0$, and set $S_\omega(t) = e^{-\omega t}S(t)$. In both of the implications below we will use that $(X, \mathsf{D}(\omega - A))_{1-\frac{1}{p},p} = (X, \mathsf{D}(A))_{1-\frac{1}{p},p}$ with equivalent norms.

First suppose that $\|\!|x|\!\|_{(0,T)} := \|t^{1-\theta}AS(\cdot)x\|_{L^p((0,T),\frac{dt}{t};X)} < \infty$. In order to show that $x \in (X, \mathsf{D}(A))_{\theta,p}$ with the desired norm estimate, note

$$\begin{aligned}\|t \mapsto t^{1-\theta}AS_\omega(t)x\|_{L^p(\mathbb{R}_+,\frac{dt}{t};X)} &\leqslant \|\!|x|\!\|_{(0,T)} + M\|t \mapsto t^{-\theta}e^{-t}\|_{L^p((T,\infty),\frac{dt}{t})}\|x\| \\ &\leqslant \|\!|x|\!\|_{(0,T)} + MT^{-\theta-\frac{1}{p}}\|x\|.\end{aligned}$$

Similarly, we obtain

$$\|t \mapsto t^{1-\theta} S_\omega(t)x\|_{L^p(\mathbb{R}_+, \frac{dt}{t}; X)} \leqslant M \|t \mapsto t^{1-\theta} e^{-t}\|_{L^p(\mathbb{R}_+, \frac{dt}{t})} \|x\| =: M L_{\theta,p} \|x\|.$$

Therefore, from the case $I = \mathbb{R}_+$ and the above observation we see that $x \in (X, \mathsf{D}(A))_{\theta,p}$ and

$$\begin{aligned} c\|x\|_{(X,\mathsf{D}(\omega-A))_{\theta,p}} &\leqslant \|t \mapsto t^{1-\theta}(\omega - A)S_\omega(t)x\|_{L^p(\mathbb{R}_+, \frac{dt}{t}; X)} + \|x\| \\ &\leqslant \|x\|_{(0,T)} + (MT^{-\theta-\frac{1}{p}} + ML_{\theta,p} + 1)\|x\|. \end{aligned}$$

In the converse direction, let $x \in (X, \mathsf{D}(A))_{\theta,p}$. Then by the case $I = \mathbb{R}_+$ and the above estimates,

$$\begin{aligned} e^{-\omega T}\|x\|_{(0,T)} &\leqslant \|t \mapsto t^{1-\theta}(\omega - A)S_\omega(t)x\|_{L^p(\mathbb{R}_+, \frac{dt}{t}; X)} \\ &\quad + \omega\|t \mapsto t^{1-\theta} S_\omega(t)x\|_{L^p(\mathbb{R}_+, \frac{dt}{t}; X)} \\ &\leqslant C\|x\|_{(X,\mathsf{D}(\omega-A))_{\theta,p}} + \omega M L_{\theta,p}\|x\|, \end{aligned}$$

which implies the required result. □

L.3 Reiteration

Next we will prove the reiteration theorem.

Theorem L.3.1 (Reiteration). *Let $p \in [1, \infty]$, and fix $0 \leqslant \theta_0 < \theta_1 \leqslant 1$ and $\lambda \in (0,1)$. Suppose that Y_0, Y_1 are Banach spaces with continuous embeddings*

$$(X_0, X_1)_{\theta_j,1} \hookrightarrow Y_j \hookrightarrow (X_0, X_1)_{\theta_j,\infty}, \quad j \in \{0,1\},$$

with norm estimates

$$A^{-1}\|x\|_{\theta_j,\infty} \leqslant \|x\|_{Y_j} \leqslant B\|x\|_{\theta_j,1}, \quad x \in X_{\theta_j,1}, \quad j \in \{0,1\},$$

where we set $Y_0 = X_0$ if $\theta_0 = 0$ and $Y_1 = X_1$ if $\theta_1 = 1$. Then

$$(Y_0, Y_1)_{\lambda,p} = (X_0, X_1)_{\theta,p} \ \text{ with } \ \theta = (1-\lambda)\theta_0 + \lambda\theta_1,$$

with equivalent norms satisfying

$$\frac{\theta_1 - \theta_0}{A}\|x\|_{\theta,p} \leqslant \|x\|_{(Y_0,Y_1)_{\lambda,p}} \leqslant \frac{1024B}{1-\lambda} \max\Big\{\frac{1}{\lambda}, \frac{1}{1-\lambda}\Big\}\|x\|_{\theta,p}.$$

As a special case we have

$$\big((X_0, X_1)_{\theta_0,p_0}, (X_0, X_1)_{\theta_1,p_1}\big)_{\lambda,p} = (X_0, X_1)_{\theta,p} \ \text{ with } \ \theta = (1-\lambda)\theta_0 + \lambda\theta_1 \tag{L.9}$$

with equivalent norms, for all choices $p_0, p_1, p \in [1, \infty]$.

Proof. First let $x \in (Y_0, Y_1)_{\lambda,p}$. Suppose that $x_0 \in Y_0$ and $x_1 \in Y_1$ are such that $x = x_0 + x_1$. Then

$$\begin{aligned} K(t,x;X_0,X_1) &\leqslant K(t,x_0;X_0,X_1) + K(t,x_1;X_0,X_1) \\ &\leqslant t^{\theta_0}\|x_0\|_{\theta_0,\infty} + t^{\theta_1}\|x_1\|_{\theta_1,\infty} \\ &\leqslant t^{\theta_0}A(\|x_0\|_{Y_0} + t^{\theta_1-\theta_0}\|x_1\|_{Y_1}), \end{aligned}$$

Therefore, $K(t,x;X_0,X_1) \leqslant t^{\theta_0}AK(t^{\theta_1-\theta_0},x;Y_0,Y_1)$, and consequently

$$\begin{aligned} \|x\|_{\theta,p} &= \|t \mapsto t^{-\theta}K(t,x;X_0,X_1)\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t})} \\ &\leqslant A\|t \mapsto t^{\theta_0-\theta}K(t^{\theta_1-\theta_0},x;Y_0,Y_1)\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t})} \\ &= A\|t \mapsto t^{-\lambda(\theta_1-\theta_0)}K(t^{\theta_1-\theta_0},x;Y_0,Y_1)\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t})} \\ &\leqslant A(\theta_1-\theta_0)^{-1}\|s \mapsto s^{-\lambda}K(s,x;Y_0,Y_1)\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}s}{s})} \\ &= A(\theta_1-\theta_0)^{-1}\|x\|_{(Y_0,Y_1)_{\lambda,p}}. \end{aligned}$$

In the converse direction, suppose that $x \in (X_0, X_1)_{\theta,p}$ and fix $\varepsilon > 0$. Let u and u_ε be as in Definition L.2.1 and Proposition L.2.2, and set $v_\varepsilon(t) := u_\varepsilon(t^{1/(\theta_1-\theta_0)})$. By the substitution $s = t^{1/(\theta_1-\theta_0)}$, the estimate

$$\|y\|_{Y_i} \leqslant B\|y\|_{\theta_i,1} \leqslant B\|y\|_{X_0}^{1-\theta_i}\|y\|_{X_1}^{\theta_i}, \quad y \in X_0 \cap X_1, \tag{L.10}$$

Hölder's inequality (with exponents $1/(1-\theta_0)$ and $1/\theta_0$), we obtain

$$\begin{aligned} R_0 :&= \big\|t \mapsto t^{1-\lambda}v_\varepsilon'(t)\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};Y_0)} \\ &= \frac{1}{\theta_1-\theta_0}\big\|t \mapsto t^{-\lambda+1/(\theta_1-\theta_0)}u_\varepsilon'(t^{1/(\theta_1-\theta_0)})\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};Y_0)} \\ &= \big\|s \mapsto s^{1-\theta+\theta_0}u_\varepsilon'(s)\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}s}{s};Y_0)} \\ &\leqslant B\Big\|s \mapsto (s^{1-\theta}\|u_\varepsilon'(s)\|)_{X_0}^{1-\theta_0}(s^{2-\theta}\|u_\varepsilon'(s)\|_{X_1})^{\theta_0}\Big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}s}{s})} \\ &\leqslant B\big\|s \mapsto s^{1-\theta}u_\varepsilon'(s)\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}s}{s};X_0)}^{1-\theta_0}\big\|s \mapsto s^{2-\theta}u_\varepsilon'(s)\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}s}{s};X_1)}^{\theta_0} \\ &\leqslant BC_\varepsilon\big\|s \mapsto s^{1-\theta}u'(s)\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}s}{s};X_0)}^{1-\theta_0}\big\|s \mapsto s^{1-\theta}u(s)\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}s}{s};X_1)}^{\theta_0} \\ &\leqslant BC_\varepsilon\max\Big\{\big\|s \mapsto s^{1-\theta}u'(s)\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}s}{s};X_0)}^{1-\theta_0}, \big\|s \mapsto s^{1-\theta}u(s)\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}s}{s};X_1)}^{\theta_0}\Big\} \end{aligned}$$

applying Proposition L.2.2 in the penultimate step, with $C_\varepsilon := \frac{2(1+2\varepsilon)^2}{\varepsilon}$. By same substitution and writing $u_\varepsilon(s) = \int_s^\infty u_\varepsilon'(r)\,\mathrm{d}r$ (see (L.8)), it follows from Hardy's inequality (see Lemma L.3.2(2) below) and (L.10) that

$$\begin{aligned} R_1 :&= \big\|t \mapsto t^{1-\lambda}v_\varepsilon(s)\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};Y_1)} \\ &= \big\|t \mapsto t^{1-\lambda}u_\varepsilon(t^{1/(\theta_1-\theta_0)})\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}t}{t};Y_1)} \end{aligned}$$

$$
\begin{aligned}
&= (\theta_1 - \theta_0)\big\|s \mapsto s^{\theta_1-\theta} u_\varepsilon(s)\big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}s}{s};Y_1)} \\
&\leqslant \frac{\theta_1-\theta_0}{\theta_1-\theta}\|s \mapsto s^{1-\theta+\theta_1} u'_\varepsilon(s)\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}s}{s};Y_1)} \\
&\leqslant \frac{B}{1-\lambda}\|s \mapsto (s^{1-\theta}\|u'_\varepsilon(s)\|_{X_0})^{1-\theta_1}(s^{2-\theta}\|u'_\varepsilon(s)\|_{X_1})^{\theta_1}\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}s}{s})} \\
&\leqslant \frac{B}{1-\lambda}\|s \mapsto s^{1-\theta}u'_\varepsilon(s)\|^{1-\theta_1}_{L^p(\mathbb{R}_+,\frac{\mathrm{d}s}{s};X_0)}\|s \mapsto s^{2-\theta}u'_\varepsilon(s)\|^{\theta_1}_{L^p(\mathbb{R}_+,\frac{\mathrm{d}s}{s};X_1)} \\
&\leqslant \frac{BC_\varepsilon}{1-\lambda}\|s \mapsto s^{1-\theta}u'(s)\|^{1-\theta_1}_{L^p(\mathbb{R}_+,\frac{\mathrm{d}s}{s};X_0)}\|s \mapsto s^{1-\theta}u(s)\|^{\theta_1}_{L^p(\mathbb{R}_+,\frac{\mathrm{d}s}{s};X_1)} \\
&\leqslant \frac{BC_\varepsilon}{1-\lambda}\max\Big\{\big\|s\mapsto s^{1-\theta}u'(s)\big\|^{1-\theta_0}_{L^p(\mathbb{R}_+,\frac{\mathrm{d}s}{s};X_0)}, \big\|s\mapsto s^{1-\theta}u(s)\big\|^{\theta_0}_{L^p(\mathbb{R}_+,\frac{\mathrm{d}s}{s};X_1)}\Big\}
\end{aligned}
$$

with $C_\varepsilon := \frac{2(1+2\varepsilon)^2}{\varepsilon}$ as before.

Combining these estimates and taking the infimum over all admissible functions u, we obtain the bound

$$\|x\|_{(Y_0,Y_1)^{\mathrm{Tr}}_{\lambda,p}} \leqslant \frac{BC_\varepsilon}{1-\lambda}\|x\|_{(X_0,X_1)^{\mathrm{Tr}}_{\theta,p}}.$$

Setting $\varepsilon = \frac{1}{2}$ gives $C_\varepsilon = 16$, and using the estimate from Theorem L.2.3, we obtain

$$
\begin{aligned}
\|x\|_{(Y_0,Y_1)_{\lambda,p}} &\leqslant 64\max\Big\{\frac{1}{\lambda}, \frac{1}{1-\lambda}\Big\}\|x\|_{(Y_0,Y_1)^{\mathrm{Tr}}_{\lambda,p}} \\
&\leqslant \frac{1024B}{1-\lambda}\max\Big\{\frac{1}{\lambda}, \frac{1}{1-\lambda}\Big\}\|x\|_{(X_0,X_1)^{\mathrm{Tr}}_{\theta,p}}.
\end{aligned}
$$

□

In the above proof we used Hardy's inequality.

Lemma L.3.2 (Hardy's inequality). *Let $p \in [1,\infty]$.*

(1) *If $\alpha > -1$ and $f : \mathbb{R}_+ \to [0,\infty]$ is measurable, then*

$$\Big\|s \mapsto s^{-\alpha}\frac{1}{s}\int_0^s f(t)\,\mathrm{d}t\Big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}s}{s})} \leqslant \frac{1}{|\alpha+1|}\|f\|_{L^p(\mathbb{R}_+,t^{-\alpha p}\,\frac{\mathrm{d}t}{t})}$$

(2) *If $\alpha < -1$ and $f : \mathbb{R}_+ \to [0,\infty]$ is measurable, then*

$$\Big\|s \mapsto s^{-\alpha}\frac{1}{s}\int_s^\infty f(t)\,\mathrm{d}t\Big\|_{L^p(\mathbb{R}_+,\frac{\mathrm{d}s}{s})} \leqslant \frac{1}{|\alpha+1|}\|f\|_{L^p(\mathbb{R}_+,t^{-\alpha p}\,\frac{\mathrm{d}t}{t})}$$

Proof. (1): Without loss of generality we may assume that f is integrable on finite intervals. Set $u(s) := \frac{1}{s}\int_0^s f(t)\,\mathrm{d}t$ and note that

$$s^{-\alpha}u(s) = \int_0^1 s^{-\alpha}f(\theta s)\,\mathrm{d}\theta.$$

Taking L^p-norms and applying Minkowski's inequality, we obtain

$$\begin{aligned}\|s \mapsto s^{-\alpha} u(s)\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}s}{s})} &\leqslant \int_0^1 \|s \mapsto s^{-\alpha} f(\theta s)\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}s}{s})} \,\mathrm{d}\theta \\ &= \int_0^1 \theta^\alpha \|t \mapsto t^{-\alpha} f(t)\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t})} \,\mathrm{d}\theta \\ &= \frac{1}{\alpha+1} \|t \mapsto t^{-\alpha} f(t)\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t})}\end{aligned}$$

(2): This is proved similarly, this time setting $v(s) := \frac{1}{s} \int_s^\infty f(t) \,\mathrm{d}t$ and noting that

$$s^{-\alpha} v(s) = \int_1^\infty s^{-\alpha} f(\theta s) \,\mathrm{d}\theta.$$

□

L.4 Mixed derivatives and Sobolev embedding

In this section we prove various weighted embedding results relating to the trace method. This is done first for functions spaces over $\mathbb{R}_+$; the case of bounded intervals is treated afterwards by using extension techniques.

We begin by introducing some terminology. For $\alpha \in [0,1)$ we consider the weight

$$w_\alpha(t) = t^\alpha.$$

Let $I \subseteq \mathbb{R}_+$ be an interval. By definition, a strongly measurable function $u : I \to X$ belongs to $L^p_{w_\alpha}(I;X)$ if $t \mapsto t^\alpha u(t)$ belongs to $L^p(I;X)$, and in this case we set

$$\|u\|_{L^p_{w_\alpha}(I;X)} := \|w_\alpha u\|_{L^p(I;X)}.$$

We furthermore define $W^{1,p}_{w_\alpha}(I;X)$ to be the space of all $f \in L^p_{w_\alpha}(I;X)$ admitting a weak derivative f' belonging to $L^p_{w_\alpha}(I;X)$. As in the unweighted case, this space is a Banach space with respect to the norm

$$\|f\|_{W^{1,p}_{w_\alpha}(I;X)} := \|f\|_{L^p_{w_\alpha}(I;X)} + \|f'\|_{L^p_{w_\alpha}(I;X)}.$$

For $\alpha \in [0, 1/p')$, functions in $W^{1,p}_{w_\alpha}(I;X)$ always have a version that is continuous on $\overline{I}$, and we will always use this version.

As an application of the trace method, for interpolation couples (X_0, X_1) we obtain some mixed derivative result for the spaces

$$W^{1,p}_{w_\alpha}(I;X_0) \cap L^p_{w_\alpha}(I;X_1).$$

L.4.a Results for the half-line

Theorem L.4.1 (Mixed derivatives and traces). *Let $p \in (1, \infty]$ and $\alpha \in [0, 1/p')$, where we take $\alpha \neq 0$ if $p = \infty$, and let $\theta := 1 - \alpha - \frac{1}{p}$. Then for all $u \in W^{1,p}_{w_\alpha}(\mathbb{R}_+; X_0) \cap L^p_{w_\alpha}(\mathbb{R}_+; X_1)$ the estimate*

$$\sup_{t \geqslant 0} \|u(t)\|_{(X_0,X_1)_{\theta,p}} \leqslant K_\theta \|u'\|^{1-\theta}_{L^p_{w_\alpha}(\mathbb{R}_+;X_0)} \|u\|^{\theta}_{L^p_{w_\alpha}(\mathbb{R}_+;X_1)} \tag{L.11}$$

holds with $K_\theta = 64 \max\{\theta^{-1}, (1-\theta)^{-1}\}$. If $p \in (1, \infty)$, we moreover have a continuous embedding

$$W^{1,p}_{w_\alpha}(\mathbb{R}_+; X_0) \cap L^p_{w_\alpha}(\mathbb{R}_+; X_1) \hookrightarrow C_{\rm b}([0, \infty); (X_0, X_1)_{\theta,p}) \tag{L.12}$$

and

$$\sup_{t>0} t^\alpha \|u(t)\|_{(X_0,X_1)_{1-\frac{1}{p},p}} \leqslant K_{1-\frac{1}{p}} \|u'\|^{1/p}_{L^p_{w_\alpha}(\mathbb{R}_+;X_0)} \|u\|^{1-1/p}_{L^p_{w_\alpha}(\mathbb{R}_+;X_1)}. \tag{L.13}$$

The above result is often applied in the setting where we have a continuous embedding $X_1 \hookrightarrow X_0$. In that case, (L.13) shows instantaneous regularisation in case $\alpha > 0$.

Remark L.4.2. The estimate (L.11) extends to the case $p = 1$ and $\alpha = 0$ (in which case we have $\theta = 0$, $X_{\theta,1} = X_0$, and $w_0 \equiv \mathbf{1}$). Indeed, writing $u(t) = -\int_t^\infty u'(s)\,\mathrm{d}s$ and using density of compactly supported functions, we see that $W^{1,1}(\mathbb{R}_; X_0) \hookrightarrow C([0, \infty); X_0)$ continuously, and

$$\sup_{t \geqslant 0} \|u(t)\|_{X_0} \leqslant \|u'\|_{L^p(\mathbb{R}_+;X_0)}.$$

Proof of Theorem L.4.1. Let $u \in W^{1,p}_{w_\alpha}(\mathbb{R}_+; X_0) \cap L^p_{w_\alpha}(\mathbb{R}_+; X_1)$ be arbitrary and fixed.

Step 1 – The assumptions imply that for $j \in \{0, 1\}$ we have $u^{(1-j)} \in L^p_{w_\alpha}(\mathbb{R}_+; X_j)$. Using that $\alpha = 1 - \theta - \frac{1}{p}$, we have

$$\begin{aligned} \|u^{(1-j)}\|_{L^p_{w_\alpha}(\mathbb{R}_+;X_j)} &= \|t \mapsto t^{1-\theta-\frac{1}{p}} u^{(1-j)}(t)\|_{L^p(\mathbb{R}_+;X_j)} \\ &= \|t \mapsto t^{1-\theta} u^{(1-j)}(t)\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t};X_j)}. \end{aligned}$$

By the argument after Definition L.2.1, this implies that $u \in W^{1.1}(0, T); X_0 + X_1)$ for all $T > 0$, and therefore u has a version belonging to $C([0, \infty; X_0 + X_1)$. We will denote this version by u again. By Definition L.2.1, applied to $x := u(0)$, we obtain that $u(0) \in (X_0, X_1)_{\theta,p}$ and, reversing the above identities,

$$\begin{aligned} \|u(0)\|_{(X_0,X_1)_{\theta,p}} &\leqslant K_\theta \|u(0)\|_{(X_0,X_1)^{\rm Tr}_{\theta,p}} \\ &= K_\theta \max_{j \in \{0,1\}} \|t \mapsto t^{1-\theta} u^{(1-j)}(t)\|_{L^p(\mathbb{R}_+, \frac{\mathrm{d}t}{t};X_j)} \\ &= K_\theta \max_{j \in \{0,1\}} \|u^{(1-j)}\|_{L^p_{w_\alpha}(\mathbb{R}_+;X_j)}, \end{aligned} \tag{L.14}$$

Step 2 – In this step we prove the bound (L.11). Applying (L.14) to $u_\lambda := u(\lambda\cdot)$, we obtain

$$\|u(0)\|_{(X_0,X_1)_{\theta,p}} \leqslant K_\theta \max_{j\in\{0,1\}} \lambda^{\theta-j}\|u^{(1-j)}\|_{L^p_{w_\alpha}(\mathbb{R}_+;X_j)}.$$

Optimising over $\lambda > 0$, we obtain

$$\|u(0)\|_{(X_0,X_1)_{\theta,p}} \leqslant K_\theta \|u'\|^{1-\theta}_{L^p_{w_\alpha}(\mathbb{R}_+;X_0)}\|u\|^{\theta}_{L^p_{w_\alpha}(\mathbb{R}_+;X_1)}. \tag{L.15}$$

Applying (L.15) to $u(\cdot+t)$ for fixed $t \geqslant 0$, we obtain

$$\begin{aligned}\|u(t)\|_{(X_0,X_1)_{\theta,p}} &\leqslant K_\theta \|u'(\cdot+t)\|^{1-\theta}_{L^p_{w_\alpha}(\mathbb{R}_+;X_0)}\|u(\cdot+t)\|^{\theta}_{L^p_{w_\alpha}(\mathbb{R}_+;X_1)} \\ &\leqslant K_\theta \|u'\|^{1-\theta}_{L^p_{w_\alpha}(\mathbb{R}_+;X_0)}\|u\|^{\theta}_{L^p_{w_\alpha}(\mathbb{R}_+;X_1)},\end{aligned}$$

which is (L.11).

Step 3 – In the remainder of the proof we assume that $p \in (1,\infty)$. In the present step we prove the continuous embedding (L.12). Applying (L.15) to $u(\cdot+t) - u(\cdot+s)$ with $0 \leqslant s \leqslant t$, we obtain

$$\begin{aligned}&\|u(t) - u(s)\|_{(X_0,X_1)_{\theta,p}} \\ &\quad\leqslant K_\theta \|u'(t+\cdot) - u'(s+\cdot)\|^{1-\theta}_{L^p_{w_\alpha}(\mathbb{R}_+;X_0)}\|u(t+\cdot) - u(s+\cdot)\|^{\theta}_{L^p_{w_\alpha}(\mathbb{R}_+;X_1)} \\ &\quad\leqslant K_\theta \|u'(t-s+\cdot) - u'\|^{1-\theta}_{L^p_{w_\alpha}(\mathbb{R}_+;X_0)}\|u(t-s+\cdot) - u\|^{\theta}_{L^p_{w_\alpha}(\mathbb{R}_+;X_1)}.\end{aligned} \tag{L.16}$$

The latter tends to zero as $t - s \to 0$ by the continuity of translations in $L^p_{w_\alpha}(\mathbb{R}_+;X_i)$ (here we use that $p \in (1,\infty)$).

Step 4 – Still under the assumption that $p \in (1,\infty)$, we will now prove (L.13). Fix an arbitrary $t > 0$. For $j \in \{0,1\}$,

$$t^\alpha \|u^{(1-j)}(\cdot+t)\|_{L^p(\mathbb{R}_+;X_j)} = t^\alpha \|u^{(1-j)}\|_{L^p(t,\infty;X_j)} \leqslant \|u^{(1-j)}\|_{L^p_{w_\alpha}(\mathbb{R}_+;X_j)}. \tag{L.17}$$

Applying (L.15) with $\alpha = 0$ and $\theta = 1 - \frac{1}{p}$ to the function $u(\cdot+t)$, we obtain the estimate

$$\|u(t)\|_{(X_0,X_1)_{1-\frac{1}{p},p}} \leqslant K_{1-\frac{1}{p}} \|u'(\cdot+t)\|^{1/p}_{L^p(\mathbb{R}_+;X_0)}\|u(\cdot+t)\|^{1-1/p}_{L^p(\mathbb{R}_+;X_1)}.$$

Multiplying on both sides with t^α (where now α can be allowed to take any value in $[0, 1/p')$) and using (L.17), we obtain

$$\begin{aligned}t^\alpha\|u(t)\|_{(X_0,X_1)_{1-\frac{1}{p},p}} &\leqslant K_{1-\frac{1}{p}} t^\alpha \|u'(\cdot+t)\|^{1/p}_{L^p(\mathbb{R}_+;X_0)}\|u(\cdot+t)\|^{1-1/p}_{L^p(\mathbb{R}_+;X_1)} \\ &\leqslant K_{1-\frac{1}{p}} \|u'\|^{1/p}_{L^p_{w_\alpha}(\mathbb{R}_+;X_0)}\|u\|^{1-1/p}_{L^p_{w_\alpha}(\mathbb{R}_+;X_1)},\end{aligned}$$

and (L.13) follows. □

We also need a variant of Theorem L.4.1 in the special case of weighted spaces of continuous functions. For $\alpha \in [0,1)$, and $I = (0,T]$ or $I = (0,\infty)$, define

$$\begin{aligned} C_{w_\alpha}(I;X) &:= \{u \in C(I;X) : \sup_{t\in I} t^\alpha \|u(t)\| < \infty\}, \\ C_{w_\alpha,0}(I;X) &:= \{u \in C_{w_\alpha}(I;X) : \lim_{t\downarrow 0} t^\alpha \|u(t)\| = 0\}, \\ C^1_{w_\alpha}(I;X) &:= \{u \in C_{w_\alpha}(I;X) : u' \in C_{w_\alpha}(I;X)\}. \end{aligned}$$

These are Banach spaces with respect to the norms

$$\begin{aligned} \|u\|_{C_{w_\alpha}(I;X)} &:= \sup_{t\in I} t^\alpha \|u(t)\|, \\ \|u\|_{C^1_{w_\alpha}(I;X)} &:= \sup_{t\in I} t^\alpha \|u(t)\| + \sup_{t\in I} t^\alpha \|u'(t)\|, \end{aligned}$$

and $C_{w_\alpha,0}(I;X)$ is a closed subspace of $C_{w_\alpha}(I;X)$.

Corollary L.4.3 (Mixed derivatives and traces). *Let $\alpha \in (0,1)$, set $\theta := 1-\alpha$. Then we have a continuous embedding*

$$C^1_{w_\alpha}((0,\infty);X_0) \cap C_{w_\alpha,0}((0,\infty);X_1) \hookrightarrow C_{\mathrm{b}}([0,\infty))$$

Proof. Fix an arbitrary $C^1_{w_\alpha}((0,\infty);X_0) \cap C_{w_\alpha,0}((0,\infty);X_1)$. It is clear that $u : (0,\infty) \to X_0 \cap X_1$ is continuous, and we have already seen in Theorem L.4.1 that $u : [0,\infty) \to (X_0,X_1)_{\theta,p}$ is continuous. It follows that on $(0,\infty)$, u takes values in $X_{p,\infty}$. Also, by (L.11) (applied with $p=\infty$),

$$\sup_{t\geqslant 0} \|u(t)\|_{(X_0,X_1)_{\theta,p}} \leqslant K_\theta \|u'\|^{1-\theta}_{C_{w_\alpha}(\mathbb{R}_+;X_0)} \|u\|^{\theta}_{C_{w_\alpha}(\mathbb{R}_+;X_1)}. \tag{L.18}$$

It thus remains to show that $u(\tau) \to u(0)$ in $(X_0,X_1)_{\theta,p}$ as $\tau \downarrow 0$; this in particular implies that $u(0) \in X_{\theta,p}$. (Note that this cannot be extracted from (L.12), as the case $p=\infty$ was excluded there).

Replacing u by ϕu, where $\phi \in C^1([0,\infty))$ is such that $\phi = 1$ on $[0,1/2]$ and $\phi = 0$ on $[1,\infty)$, we may assume that $u \equiv 0$ on $[1,\infty)$. Let $\tau > 0$. Applying (L.18) to $u(\cdot+\tau) - u(\cdot)$ and using the estimate

$$\|u' - u'(\cdot+\tau)\|_{C_{w_\alpha}(\mathbb{R}_+;X_0)} \leqslant 2\|u'\|_{C_{w_\alpha}(\mathbb{R}_+;X_0)},$$

from (L.16) we find

$$\|u(\tau) - u(0)\|_{(X_0,X_1)_{\theta,p}} \leqslant 2^{1-\theta} \|u'\|^{1-\theta}_{C_{w_\alpha}(\mathbb{R}_+;X_0)} \|u(\cdot+\tau) - u\|^{\theta}_{C_{w_\alpha}(\mathbb{R}_+;X_1)}.$$

To complete the proof, it thus suffices to show that $\|u(\cdot+\tau)-u\|_{C_{w_\alpha}(\mathbb{R}_+;X_1)} \to 0$ as $\tau \downarrow 0$. To this end let $\varepsilon > 0$, and choose $\delta_0 \in (0,1)$ such that for all $t \in (0,2\delta_0]$ we have $t^\alpha \|u(t)\|_{X_1} < \varepsilon$. By the continuity of u on $(0,\infty)$ as an $X_0 \cap X_1$-valued function and the support condition, u is uniformly continuous on $[\delta_0,\infty)$ as an $X_0 \cap X_1$-valued function, hence also as an X_1-valued function.

Therefore, we can find $\delta_1 > 0$ such that for all $\tau \in (0, \delta_1)$ and $t \geqslant \delta_0$ we have $\|u(t+\tau) - u(t)\|_{X_1} < \varepsilon$. Setting $\delta = \min\{\delta_0, \delta_1\}$, we find that for all $\tau \in (0, \delta)$,

$$\sup_{t\in(0,\delta_0)} t^\alpha \|u(t+\tau) - u(t)\|_{X_1} \leqslant \sup_{t\in(0,2\delta_0)} 2t^\alpha \|u(t)\|_{X_1} < \varepsilon,$$
$$\sup_{t\in[\delta_0,\infty)} t^\alpha \|u(t+\tau) - u(t)\|_{X_1} \leqslant \sup_{t\in[\delta_0,1]} \|u(t+\tau) - u(t)\|_{X_1} < \varepsilon.$$

This gives the claim and completes the proof. □

As a consequence we obtain the following result. It reduces to Theorem L.4.1 upon letting $h \to \infty$.

Theorem L.4.4 (Mixed derivatives and Sobolev embedding). *Let $p \in [1, \infty]$ and $\alpha \in [0, 1/p') \cup \{0\}$ be such that $\alpha + \frac{1}{p} > 0$. Let $h \in (1, \infty)$, and set $r = hp$ and $\theta = 1 - (\alpha + \frac{1}{p})(1 - \frac{1}{h})$. Then we have a continuous embedding*

$$W^{1,p}_{w_\alpha}(\mathbb{R}_+; X_0) \cap L^p_{w_\alpha}(\mathbb{R}_+; X_1) \hookrightarrow L^r_{w_{\alpha/h}}(\mathbb{R}_+; (X_0, X_1)_{\theta,1}),$$

and there exists a constant $C_{p,\alpha,h}$, depending only on (p, α, h), such that

$$\|u\|_{L^r_{w_\alpha}(\mathbb{R}_+;(X_0,X_1)_{\theta,1})} \leqslant C_{p,\alpha,h} \|u'\|^{1-\theta}_{L^p_{w_\alpha}(\mathbb{R}_+;X_0)} \|u\|^\theta_{L^p_{w_\alpha}(\mathbb{R}_+;X_1)}.$$

Proof. First assume that $p > 1$. Let $\sigma = 1 - \alpha - \frac{1}{p}$. (L.9) (taking $\theta_0 = \sigma$ and $\theta_1 = 1$, and A the norm constant in the embedding $(X_0, X_1)_{\sigma,p} \hookrightarrow (X_0, X_1)_{\sigma,\infty}$ in (C.6)) and (L.2),

$$\|x\|_{\theta,1} \leqslant C\|x\|_{((X_0,X_1)_{\sigma,p},X_1)_{1-\lambda,1}} \leqslant C\|x\|^\lambda_{\sigma,p} \|x\|^{1-\lambda}_{X_1} \tag{L.19}$$

if $\lambda \in (0, 1)$ satisfies $\sigma\lambda + (1 - \lambda) = \theta$, that is, if $\lambda = \frac{1-\theta}{\alpha+\frac{1}{p}} = 1 - \frac{1}{h}$. In particular, $r(1 - \lambda) = p$, and therefore,

$$\begin{aligned}\|u\|_{L^r_{w_{\alpha/h}}(\mathbb{R}_+;(X_0,X_1)_{\theta,1})} &\leqslant C\|u\|^\lambda_{L^\infty(\mathbb{R}_+;X_{\sigma,p})} \|u\|^{1-\lambda}_{L^{r(1-\lambda)}_{w_\alpha}(\mathbb{R}_+;X_1)} \\ &= C\|u\|^\lambda_{L^\infty(\mathbb{R}_+;X_{\sigma,p})} \|u\|^{1-\lambda}_{L^p_{w_\alpha}(\mathbb{R}_+;X_1)},\end{aligned} \tag{L.20}$$

where C depends on (p, α, h). On the other hand, by Theorem L.4.1,

$$\|u\|_{L^\infty(\mathbb{R}_+;X_{\sigma,p})} \leqslant K_\sigma \|u'\|^{1-\sigma}_{L^p_{w_\alpha}(\mathbb{R}_+;X_0)} \|u\|^\sigma_{L^p_{w_\alpha}(\mathbb{R}_+;X_1)}.$$

We conclude that

$$\begin{aligned}\|u\|_{L^r_{w_{\alpha/h}}(\mathbb{R}_+;(X_0,X_1)_{\theta,1})} &\leqslant C'K^\lambda_\sigma \|u'\|^{\lambda(1-\sigma)}_{L^p_{w_\alpha}(\mathbb{R}_+;X_0)} \|u\|^{1-\lambda+\lambda\sigma}_{L^p_{w_\alpha}(\mathbb{R}_+;X_1)} \\ &= C'K^{1-\frac{1}{h}}_\sigma \|u'\|^{1-\theta}_{L^p_{w_\alpha}(\mathbb{R}_+;X_0)} \|u\|^\theta_{L^p_{w_\alpha}(\mathbb{R}_+;X_1)},\end{aligned}$$

where C' depends on (p, α, h).

If $p = 1$, then $\alpha = \sigma = 0$, and (L.20) still holds if we replace $X_{\sigma,p}$ by X_0. Combining this with Remark L.4.2, this gives the desired embedding. □

L.4.b Extension operators

By using extension operators, we can obtain versions of the previous results on bounded intervals.

Proposition L.4.5 (Extension operators). *Let $p \in [1,\infty]$, and let $\alpha \in [0,1/p') \cup \{0\}$. There exists a linear mapping $E_T : L^0(0,T;X) \to L^0(\mathbb{R}_+;X)$ with the following properties:*

(1) *For all $u \in L^0(0,T;X)$ we have $E_T u|_{(0,T)} = u$;*
(2) *For all $u \in L^p_{w_\alpha}(0,T;X)$,*

$$\|E_T u\|_{L^p_{w_\alpha}(\mathbb{R}_+;X)} \leqslant 2\|u\|_{L^p_{w_\alpha}(0,T;X)};$$

(3) *For all $u \in W^{1,p}_{w_\alpha}(0,T;X)$,*

$$\|E_T u\|_{W^{1,p}_{w_\alpha}(\mathbb{R}_+;X)} \leqslant (2+3T^{-1})\|u\|_{W^{1,p}_{w_\alpha}(0,T;X)};$$

(4) *For all $u \in W^{1,p}_{w_\alpha}(0,T;X)$ with $u(0)=0$,*

$$\|E_T u\|_{W^{1,p}_{w_\alpha}(\mathbb{R}_+;X)} \leqslant C_{\alpha,p}\|u\|_{W^{1,p}_{w_\alpha}(0,T;X)},$$

where $C_{\alpha,p} = 2+3(1-\alpha p')^{1/p'}$ if $p>1$ and $C_{\alpha,p}=5$ if $p=1$.

The assertions (2), (3), and (4) also hold with $L^p_{w_\alpha}(0,T;X)$ and $W^{1,p}_{w_\alpha}(0,T;X)$ replaced by $C_{w_\alpha}((0,T];X)$ and $C^1_{w_\alpha}((0,T];X)$, respectively.

Often we will need (4) in order to ensure that the bounds are T-independent (note that the bound in (3) involves th term T^{-1}). This is not possible without the condition $u(0)=0$ if $p<\infty$. Indeed, let $u \in W^{1,p}_{w_\alpha}(0,T;X)$ be such that $\|u(0)\|=1$. Then by Sobolev embedding

$$1 = \|u(0)\|_X \leqslant C\|E_T u\|_{W^{1,p}_{w_\alpha}(\mathbb{R}_+;X)} \leqslant C\|E_T\|\,\|u\|_{W^{1,p}_{w_\alpha}(0,T;X)}.$$

Now since $\|u\|_{W^{1,p}_{w_\alpha}(0,T;X)} \to 0$ as $T \downarrow 0$, we see that $\|E_T\|_{W^{1,p}_{w_\alpha}(0,T;X)} \to \infty$ as $T \downarrow 0$.

Proof of Proposition L.4.5. (1), (2), and (3): First let $T=1$. Let $\phi : [1,\infty) \to [0,1]$ be given by $\phi(t) = \max\{3-2t, 0\}$. Set

$$E_1 u(t) := \begin{cases} u(t) & \text{on } (0,1), \\ \phi(t)u(2-t) & \text{on } (1,\frac{3}{2}), \\ 0 & \text{on } [\frac{3}{2},\infty). \end{cases}$$

If $u \in L^p_{w_\alpha}(0,1;X)$, then $E_1 u \in L^p_{w_\alpha}(\mathbb{R}_+;X)$, and if $u \in W^{1,p}_{w_\alpha}(0,1;X)$, then $E_1 u \in W^{1,p}_{w_\alpha}(\mathbb{R}_+;X)$, and since $t^\alpha\phi(t) \leqslant 1$ and $t^\alpha\phi'(t) \leqslant 3$ we obtain

$$\|E_1 u\|_{L^p_{w_\alpha}(\mathbb{R}_+;X)} \leqslant 2\|u\|_{L^p_{w_\alpha}(0,1;X)}, \qquad \text{if } u \in u \in L^p_{w_\alpha}(0,1;X),$$

$$\|(E_1u)'\|_{L^p_{w_\alpha}(\mathbb{R}_+;X)} \leqslant 2\|u'\|_{L^p_{w_\alpha}(0,1;X)} + 3\|u\|_{L^p_{w_\alpha}(0,1;X)} \quad \text{if } u \in W^{1,p}_{w_\alpha}(0,1;X).$$

This implies the result for $T = 1$. In the general case let $u \in L^p_{w_\alpha}(0,T;X)$ or $u \in W^{1,p}_{w_\alpha}(0,T;X)$, and set $u_T(t) := u(tT)$. Define the extension operator by

$$E_T u(t) := E_1(u_T)(t/T).$$

For $u \in L^p_{w_\alpha}(0,T;X)$ this leads to

$$\begin{aligned}\|E_T u\|_{L^p_{w_\alpha}(\mathbb{R}_+;X)} &= T^{\alpha+\frac{1}{p}}\|E_1 u_T\|_{L^p_{w_\alpha}(\mathbb{R}_+;X)} \\ &\leqslant 2T^{\alpha+\frac{1}{p}}\|u_T\|_{L^p_{w_\alpha}(0,1;X)} = 2\|u\|_{L^p_{w_\alpha}(0,T;X)}.\end{aligned}$$

Similarly, for $u \in W^{1,p}_{w_\alpha}(0,T;X)$,

$$\begin{aligned}\|(E_T u)'\|_{L^p_{w_\alpha}(\mathbb{R}_+;X)} &= T^{\alpha+\frac{1}{p}-1}\|(E_1 u_T)'\|_{L^p_{w_\alpha}(\mathbb{R}_+;X)} \\ &\leqslant 2T^{\alpha+\frac{1}{p}-1}\|u_T'\|_{L^p_{w_\alpha}(0,1;X)} + 3T^{\alpha+\frac{1}{p}-1}\|u_T\|_{L^p_{w_\alpha}(0,1;X)} \\ &\leqslant 2\|u'\|_{L^p_{w_\alpha}(0,T;X)} + 3T^{-1}\|u\|_{L^p_{w_\alpha}(0,T;X)}.\end{aligned}$$

(4): We claim that

$$T^{-1}\|u\|_{L^p_{w_\alpha}(0,T;X)} \leqslant c_{\alpha,p}\|u'\|_{L^p_{w_\alpha}(0,T;X)},$$

where $c_{\alpha,p} = (1-\alpha p')^{1/p'}$ if $p > 1$ and $c_{\alpha,p} = 1$ if $p = 1$. Combined with the proof of (3) the claim gives the desired result.

To prove the claim, we restrict ourselves to $p \in (1,\infty)$. the simpler cases $p = 1$ and $p = \infty$ are left to the reader. By Hölder's inequality,

$$\begin{aligned}\|u(t)\| \leqslant \int_0^t \|u'(s)\|\,\mathrm{d}s &\leqslant \|u'\|_{L^p_{w_\alpha}(0,T;X)}\Big(\int_0^t s^{-\alpha p'}\,\mathrm{d}s\Big)^{1/p'} \\ &= c_{\alpha,p}\|u'\|_{L^p_{w_\alpha}(0,T;X)} t^{1-\frac{1}{p}-\alpha},\end{aligned}$$

Taking $L^p_{w_\alpha}$-norms, we obtain

$$\|u\|_{L^p_{w_\alpha}(0,T;X)} \leqslant c_{\alpha,p}\|u'\|_{L^p_{w_\alpha}(0,T;X)}\Big(\int_0^T t^{p-1}\,\mathrm{d}t\Big)^{1/p} \leqslant c_{\alpha,p}T\|u'\|_{L^p_{w_\alpha}(0,T;X)}.$$

To prove the final assertion, let $\varepsilon > 0$. We can repeat the proof of (3) and (4) with a slight deformation of ϕ, worsening the constant by at most an additive term ε. It then remains to let $\varepsilon \downarrow 0$. □

L.4.c Results for bounded intervals

As an application of the results of the foregoing sections we obtain the following mixed derivative type result on bounded intervals. Recall the definition of $X_{p,\infty}$ in (L.1).

Corollary L.4.6 (Mixed derivatives and traces). *Let $p \in (1,\infty]$ and $\alpha \in [0, 1/p')$, where we take $\alpha \neq 0$ if $p = \infty$. Set $\theta := 1 - \alpha - \frac{1}{p}$. Then for all $0 < T < \infty$ we have the (continuous) inclusions*

$$C^1_{w_\alpha}((0,T];X_0) \cap C_{0,w_\alpha}((0,T];X_1) \hookrightarrow C([0,T];X_{\theta,\infty});$$
$$C^1_{w_\alpha}((0,T];X_0) \cap C_{0,w_\alpha}((0,T];X_1) \subseteq C((0,T];X_0 \cap X_1);$$

and, if $p < \infty$,

$$W^{1,p}_{w_\alpha}(0,T;X_0) \cap L^p_{w_\alpha}(0,T;X_1) \hookrightarrow C([0,T];(X_0,X_1)_{\theta,p}),$$
$$W^{1,p}_{w_\alpha}(0,T;X_0) \cap L^p_{w_\alpha}(0,T;X_1) \subseteq C((0,T);(X_0,X_1)_{1-\frac{1}{p},p}).$$

Moreover, for all $u \in W^{1,p}_{w_\alpha}(0,T;X_0) \cap L^p_{w_\alpha}(0,T;X_1)$ the following estimates hold:

$$\sup_{t\in[0,T]} \|u(t)\|_{(X_0,X_1)_{\theta,p}} \leqslant K_\theta C_T \|u\|^{1-\theta}_{W^{1,p}_{w_\alpha}(0,T;X_0)} \|u\|^{\theta}_{L^p_{w_\alpha}(\mathbb{R}_+;X_1)},$$
$$\sup_{t\in(0,T)} t^\alpha \|u(t)\|_{(X_0,X_1)_{1-\frac{1}{p},p}} \leqslant K_{1-\frac{1}{p}} C_T \|u\|^{1-\theta}_{W^{1,p}_{w_\alpha}(0,T;X_0)} \|u\|^{\theta}_{L^p_{w_\alpha}(0,T;X_1)},$$

where $K_\theta = 64 \max\{\theta^{-1}, (1-\theta)^{-1}\}$ and $C_T = 2 + 3T^{-1}$. For functions satisfying $u(0) = 0$, the above constants C_T can be replaced by the T-independent constant $C_{\alpha,p}$ defined in Proposition L.4.5.

By an argument similar to that after the statement of Proposition L.4.5, without the condition $u(0)$ it is not possible to obtain estimates with T-independent constants for small T.

Proof. Let $u \in W^{1,p}_{w_\alpha}(0,T;X_0) \cap L^p_{w_\alpha}(0,T;X_1)$. Combining Theorem L.4.1 and Proposition L.4.5, we obtain that $u \in C([0,T];(X_0,X_1)_{\theta,p})$ and

$$\begin{aligned}
\|u\|_{C([0,T];(X_0,X_1)_{\theta,p})} &\leqslant \|E_T u\|_{C([0,\infty);(X_0,X_1)_{\theta,p})} \\
&\leqslant K_\theta \|E_T u\|^{1-\theta}_{W^{1,p}_{w_\alpha}(\mathbb{R}_+;X_0)} \|E_T u\|^{\theta}_{L^p_{w_\alpha}(\mathbb{R}_+;X_1)} \\
&\leqslant K_\theta (2 + 3T^{-1}) \|u\|^{1-\theta}_{W^{1,p}_{w_\alpha}(0,T;X_0)} \|u\|^{\theta}_{L^p_{w_\alpha}(0,T;X_1)}.
\end{aligned}$$

From this estimate it is clear that if addition $u(0) = 0$ holds, then the constant $2 + 3T^{-1}$ can be replaced by the constant $C_{\alpha,p}$ of Proposition L.4.5.

The remaining cases are proved in the same way, using in addition Corollary L.4.3 and Theorem L.4.4. □

In the same way we obtain the following result from Theorem L.4.4.

Corollary L.4.7 (Mixed derivatives and Sobolev embedding). *Let $p \in [1,\infty]$ and $\alpha \in [0, 1/p') \cup \{0\}$ satisfy $\alpha + \frac{1}{p} > 0$. Let $h \in (1,\infty)$, and set $r = hp$ and $\theta = 1 - (\alpha + \frac{1}{p})(1 - \frac{1}{h})$. Then we have a continuous embedding*

$$W^{1,p}_{w_\alpha}(0,T;X_0) \cap L^p_{w_\alpha}(0,T;X_1) \hookrightarrow L^r_{w_{\alpha/h}}(0,T;(X_0,X_1)_{\theta,1}),$$

and there is a constant $C_{p,\alpha,h}$ only depending on (p,α,h) such that

$$\|u\|_{L^r_{w_\alpha}(0,T;(X_0,X_1)_{\theta,1})} \leqslant C_{p,\alpha,h} C_T \|u\|^{1-\theta}_{W^{1,p}_{w_\alpha}(0,T;X_0)} \|u\|^{\theta}_{L^p_{w_\alpha}(0,T;X_1)},$$

where $C_{p,\alpha,h}$ is as in Theorem L.4.4 and $C_T = 2 + 3T^{-1}$. For functions satisfying $u(0) = 0$, the above constants C_T can be replaced by the T-independent constant $C_{\alpha,p}$ defined in Proposition L.4.5.

L.5 Notes

The trace method of Section L.2 is due to J. L. Lions in a classical series of papers. We follow the presentation in Lunardi [2009] and Triebel [1978], where a detailed historical account is given. A version of the trace method with fractional smoothness was recently obtained in Agresti, Lindemulder, and Veraar [2023]. The reiteration Theorem L.3.1 for real interpolation is due to Lions and Peetre [1964]. A unified presentation of the reiteration method, which covers many interpolation methods (including the real and complex method), can be found in Lindemulder and Lorist [2021].

The mixed derivative result of Theorem L.4.1 is a standard consequence of the trace method. The end-point case of Corollary L.4.3 is less standard, but important in evolution equations, and can be found in Lunardi [1995]. The mixed derivative result of Theorem L.4.4 is also a simple consequence of the trace method, although the simple proof presented here may be new. Fractional versions have been proved in Agresti and Veraar [2022a].

The presentation of Proposition L.4.5, which is a standard result on extension operators for bounded intervals $(0,T)$, follows this reference. Other constructions can be found in Meyries and Schnaubelt [2012b]. A discussion on extension operators for more general domains in $\mathbb{R}^d$ can be found in the notes of Chapter 14. In Corollaries L.4.6 and L.4.7 the extension operators are used to obtain versions of the mixed derivative results on bounded intervals. In Chapter 18 it is important that the embedding constants can be taken independent of the size of the interval if one works with functions vanishing at zero.

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https://doi.org/10.1007/978-3-031-46598-7