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RESEARCH ARTICLE

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Longest cycles in vertex-transitive and highly connected graphs

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Abstract

We present progress on three old conjectures about longest paths and cycles in graphs. The first pair of conjectures, due to Lovász from 1969 and Thomassen from 1978, respectively, states that all connected vertex-transitive graphs contain a Hamiltonian path, and that all sufficiently large such graphs even contain a Hamiltonian cycle. The third conjecture, due to Smith from 1984, states that for $r \geq 2$ in every r -connected graph any two longest cycles intersect in at least r vertices. In this paper, we prove a new lemma about the intersection of longest cycles in a graph, which can be used to improve the best known bounds toward all the aforementioned conjectures: First, we show that every connected vertex-transitive graph on $n \geq 3$ vertices contains a cycle (and hence path) of length at least $\Omega(n^{13/21})$, improving on $\Omega(n^{3/5})$ from DeVos [arXiv:2302:04255, 2023]. Second, we show that in every r -connected graph with $r \geq 2$, any two longest cycles meet in at least $\Omega(r^{5/8})$ vertices, improving on $\Omega(r^{3/5})$ from Chen, Faudree, and Gould [J. Combin. Theory, Ser. B, **72** (1998) no. 1, 143–149]. Our proof combines combinatorial arguments, computer search, and linear programming.

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1 | INTRODUCTION

One of the oldest and most fundamental research areas in graph theory is concerned with *longest cycles*[†] in graphs. In particular, sufficient conditions for the Hamiltonicity of a graph, that is, the existence of a cycle of length n in a graph on n vertices, have been widely studied, going back to the classical result of Dirac [19] stating that every graph with minimum degree at least $n/2$ is Hamiltonian for all $n \geq 3$. The minimum degree condition is tight in Dirac's theorem, so to obtain similar results for sparser graphs one needs to add additional constraints, such as connectivity conditions. For 2-connected regular graphs, the degree condition was improved to $n/3$ by Jackson [33] and later to the tight $n/3 - 1$ by Hilbig [32], answering a question of Szekeres (see [33]). For 3-connected regular graphs, asymptotically confirming a conjecture[‡] independently made by Bollobás [8] and Häggkvist [31], Kühn, Lo, Osthus, and Staden [37] showed that minimum degree $n/4 + o(n)$ is sufficient to guarantee the graph being Hamiltonian. More generally, in the same paper, they also considered some sufficient conditions on the length of the longest cycle of a graph. An old conjecture of Bondy [9], confirmed by Wei [47], says that for all $r \geq 3$, every sufficiently large 2-connected regular graph on n vertices with degree at least n/r has a cycle of length at least $2n/(r - 1)$. Kühn, Lo, Osthus, and Staden [37] considered a generalization of this conjecture for t -connected graphs and showed that degree at least $n/r + o(n)$ guarantees a cycle of length at least $\min\{tn/(r - 1), (n - o(n))\}$.

In contrast to the dense case, sufficient conditions for Hamiltonicity that also address sparse graphs, with, say, a constant average degree, seem to be rather scarce. The positive results that apply to sparse graphs can be roughly clustered into three (mostly orthogonal) research directions: conditions related to pseudo-randomness such as expanders (see, e.g., the recent breakthrough [20]), toughness conditions [7, 16], and symmetry conditions such as vertex-transitivity. Note that for dense vertex-transitive graphs, that is, of degree $d = \Theta(n)$, a lower bound of $(1 - o(1))n$ on the maximum cycle length follows from the aforementioned results: Indeed, every connected vertex-transitive graph of degree d is $2(d + 1)/3$ -connected [26], and thus the results from [37] discussed above imply the existence of a cycle of length $(1 - o(1))n$.

In fact, even stronger results hold: Christofides, Hladký, and Máthé [15] showed that all connected vertex-transitive graphs of degree of order $\Theta(n)$ contain a Hamiltonian cycle. Our first main result concerns longest cycles in general connected vertex-transitive graphs.

There are two main open problems in the study of Hamiltonicity on vertex-transitive graphs that are commonly attributed to Lovász and Thomassen, respectively. The first is a conjecture posed by Lovász [40] in 1969 and asserts that every connected vertex-transitive graph admits a Hamiltonian path. The second is a conjecture by Thomassen from 1978 (see, e.g., [6], Conjecture 1 and the references therein) and states that apart from finitely many exceptional graphs, all connected vertex-transitive graphs even contain a Hamiltonian cycle. In fact, to this date, only four connected vertex-transitive graphs on at least 3 vertices are known, which do not have a Hamiltonian cycle (one of which is the infamous Petersen graph). It is worth mentioning that Lovász originally posed his conjecture in the opposite form, but nowadays the abovementioned positive version has become standard. Curiously, Babai [5] has proposed a conjecture contradicting Lovász

[†] Throughout this paper, a longest cycle in a graph will be defined as a cycle attaining the maximum possible length among all cycles in the graph.

[‡] Their conjecture was made for general t -connected d -regular graphs, $t \geq 3$, and said that $d \geq n/t + 1$ is sufficient to guarantee Hamiltonicity. However, this was disproved for $t \geq 4$ by Jung [35] and independently by Jackson, Li, and Zhu [34].

conjecture, namely that there exists an absolute constant $\varepsilon > 0$ such that for infinitely many n , there exists a connected vertex-transitive graph of order n with no path of length larger than $(1 - \varepsilon)n$. Despite half a century of study, during which the conjectures of Lovász and Thomassen received much attention by the research community, they remain wide open today. In particular, for 44 years the best known general lower bound on the maximum length of a path or cycle in a connected vertex-transitive graph on $n \geq 3$ vertices remained $\Omega(\sqrt{n})$ due to Babai [4], and only recently this bound was improved to $\Omega(n^{3/5})$ by DeVos [18]. There is some further recent evidence of this conjecture being true though: Indeed results of Draganić, Montgomery, Munhá Correia, Pokrovskiy, and Sudakov [20], together with earlier results on expansion properties of random Cayley graphs obtained by Alon and Roichman [1], imply that random Cayley graphs with generator set of size at least $\Omega(\log n)$ are Hamiltonian. For more related work, we refer to the surveys [2, 38] on the partial progress toward Lovász' and Thomassen's conjectures. As our first main result, we obtain the following improved lower bound for the length of longest cycles in vertex-transitive graphs.

Theorem 1.1. *Every connected vertex-transitive graph on $n \geq 3$ vertices contains a cycle of length at least $\Omega(n^{13/21})$.*

The second main result of our paper addresses a related old conjecture about longest cycles in graphs from 1984, which is commonly attributed to Smith.

Conjecture 1.2 (cf. Conjecture 5.2 in [29] and Conjecture 4.15 in [10]). *For all $r \geq 2$, every two longest cycles in an r -connected graph meet in at least r vertices.*

Smith's conjecture has been verified for all values $r \leq 6$ by Grötschel [29] and for $r \in \{7, 8\}$ by Stewart and Thompson [43]. Furthermore, Gutiérrez and Valqui [30] recently proved the conjecture for $r \geq (n + 16)/7$. Regarding general values of r , Burr and Zamfirescu (see [10]) proved that every two longest cycles in an r -connected graph must meet in at least $\sqrt{r} - 1$ vertices, and this bound was improved to $\Omega(r^{3/5})$ by Chen, Faudree, and Gould [13] in 1998. For further background on Smith's conjecture and related problems, we refer to the survey article [42] on intersections of longest cycles and paths in graphs. Our second main result gives the first asymptotic improvement of the lower bound toward Smith's conjecture since the result of Chen, Faudree, and Gould from 26 years ago.

Theorem 1.3. *There is a constant $C > 0$ such that for all $r \geq 2$, every two longest cycles in an r -connected graph meet in at least $Cr^{5/8}$ vertices.*

We obtain both our main results, Theorem 1.1 and Theorem 1.3, from the following key lemma, which can be seen as a "local" version of Theorem 1.3. We denote by Q_3^+ the bipartite graph obtained from the 3-dimensional hypercube Q_3 by adding one of the diagonals. Alternatively, Q_3^+ is the graph obtained from $K_{4,4}$ by removing a matching of size 3.

Lemma 1.4. *Let C_1 and C_2 be two longest cycles in a graph G . If $k := |V(C_1) \cap V(C_2)| > 0$ and S is a minimum vertex separator between $A := V(C_1) \setminus V(C_2)$ and $B := V(C_2) \setminus V(C_1)$ in G , then*

$$|S| \leq \text{ex}(2k, \{Q_3^+, K_{3,3}\}) = O(k^{8/5}).$$

The proof of this lemma is the main bulk of work in this paper. It is partly computer-assisted and will be presented in Sections 3 and 4. The upper bound on the extremal function $\text{ex}(n, \{Q_3^+, K_{3,3}\}) = O(n^{8/5})$ follows from the bound $\text{ex}(n, Q_3^+) \leq O(n^{8/5})$ that was first proved by Erdős and Simonovits [21], see also the recent work of Füredi [24] for an independent proof and related results.

Lemma 1.4 strengthens a corresponding result by DeVos (see the proof of Lemma 3 in [18]), who proved that S as in Lemma 1.4 must satisfy $|S| = O(k^2)$ instead of $|S| = O(k^{8/5})$; this is obtained by a simple application of Menger's theorem.

Note that using the upper bound of $\text{ex}(n, \{Q_3^+, K_{3,3}\}) \leq \text{ex}(n, Q_3^+)$ is not necessarily wasteful, as it may seem at first. Namely, a well-known open conjecture by Erdős and Simonovits [22] states that for any finite family of graphs $\mathcal{F}$ the Turán number of the family $\mathcal{F}$ is within a constant factor of the minimum among the Turán numbers of members of $\mathcal{F}$ (see Conjecture 1 in [22]). Since it is known that $\text{ex}(n, K_{3,3}) = (\frac{1}{2} + o(1))n^{5/3}$ (see [11, 23]), this would imply that $\text{ex}(n, \{Q_3^+, K_{3,3}\})$ and $\text{ex}(n, Q_3^+)$ are of the same order.

2 | PROOF OF THEOREM 1.1 AND THEOREM 1.3

In this section, we explain how to deduce both Theorems 1.1 and 1.3 from Lemma 1.4. We start by recording the following folklore observation about intersections of longest paths and cycles in graphs, which we will need repeatedly, see, for example, [51] for a reference.

Observation 2.1. Let G be a connected graph. Then any two longest paths in G meet in some vertex. Additionally, if G is 2-connected, then any two longest cycles in G meet in some vertex.

Let us now continue by first proving Theorem 1.3.

Proof of Theorem 1.3. Let G be an r -connected graph, and let C_1, C_2 be two longest cycles in G . Since $r \geq 2$ by assumption, Observation 2.1 implies that the cycles C_1 and C_2 must overlap in at least one vertex. This implies $k := |V(C_1) \cap V(C_2)| > 0$. So by Lemma 1.4, a smallest vertex-separator S between $V(C_1) \setminus V(C_2)$ and $V(C_2) \setminus V(C_1)$ in G must have size at most $|S| \leq O(k^{8/5})$. On the other hand, since G is r -connected, we must have $|S| \geq r$, implying $k \geq \Omega(r^{5/8})$, as desired. $\square$

Let us now turn to the deduction of Theorem 1.1, which is slightly more elaborate.

Proof of Theorem 1.1. In the following, let us define a *longest cycle transversal* as a set $S \subseteq V(G)$ that intersects every longest cycle of G . For $k \in \mathbb{N}$, let $s(k)$ denote the smallest integer s such that every 2-connected graph G containing two longest cycles C_1 and C_2 such that $|V(C_1) \cap V(C_2)| = k$ has a longest cycle transversal of size at most s . DeVos [18, Lemma 3] showed that $s(k) \leq k^2 + k$, and the remainder of the proof in [18] directly shows that the length of the longest cycle in a connected vertex-transitive graph on at least $n \geq 3$ vertices is at least

$$\min_{k \in \mathbb{N}} \max \left\{ \sqrt{kn}, \frac{n}{s(k)} \right\}. \quad (1)$$

We next show how to obtain the improved upper bound $s(k) = O(k^{8/5})$ from Lemma 1.4. Given two longest cycles C_1 and C_2 in a 2-connected graph G with $|V(C_1) \cap V(C_2)| = k$, by Lemma 1.4,

there exists a vertex-separator S of size at most $O(k^{8/5})$ between $V(C_1) \setminus V(C_2)$ and $V(C_2) \setminus V(C_1)$ in G . We now claim that $S \cup (V(C_1) \cap V(C_2))$ (which is of size $\leq |S| + k = O(k^{8/5})$) intersects all longest cycles in G . Indeed, for any longest cycle C in G , since G is 2-connected and since by Observation 2.1 any two longest cycles in a 2-connected graph must intersect each other, we must have $V(C) \cap V(C_i) \neq \emptyset$ for $i = 1, 2$. Hence, either C contains a vertex in $V(C_1) \cap V(C_2)$ or a segment of C forms a path from $V(C_1) \setminus V(C_2)$ to $V(C_2) \setminus V(C_1)$ in G , in which case, this segment (and thus C) must be hit by S . This shows that indeed, $s(k) \leq O(k^{8/5})$.

Plugging our improved bound $s(k) = O(k^{8/5})$ into (1) yields Theorem 1.1. Indeed, let $C > 0$ be an absolute constant such that $s(k) \leq Ck^{8/5}$ for all $k \geq 1$. The minimum of the expression

$$\max \left\{ \sqrt{kn}, \frac{n}{Ck^{8/5}} \right\}$$

among real values $k > 0$ is achieved for a value of k for which the two bounds $\sqrt{kn}$ and $n/(Ck^{8/5})$ are equal, and setting

$$\sqrt{kn} = \frac{n}{Ck^{8/5}}$$

yields

$$k = n^{(1-1/2)/(1/2+8/5)} / C^{1/(1/2+8/5)} = \Theta(n^{\frac{5}{21}}).$$

Plugging this into the above then gives the desired lower bound of $\sqrt{kn} = \Theta(n^{1/2+5/42}) = \Theta(n^{13/21})$ on the length of a longest cycle. This proves Theorem 1.1. $\square$

3 | REDUCTION TO COMPUTER SEARCH

In this section, we introduce definitions that are required to formalize our computer-based approach and prove Lemma 3.4, which will be used to prove Lemma 1.4. Of course, we cannot simply consider all possible graphs in our computer search, and rather do a brute-force search over the possible “skeletons” that may arise.

While reading the definitions in this section, the reader may wish to keep the following construction of an auxiliary bipartite graph in mind as motivation. Given a graph G with two longest cycles C_1 and C_2 and a collection of vertex-disjoint paths $\mathcal{P}$ from $C_1 - V(C_2)$ to $C_2 - V(C_1)$, we may define a bipartite graph as follows. The vertices are the connected components of $C_1 - V(C_2)$ and $C_2 - V(C_1)$, and there is an edge between two such components if there is a vertex-disjoint path P between them with $P \in \mathcal{P}$.

An *ordered bipartite graph* is a triple $\mathcal{H} = (H, \sigma_X, \sigma_Y)$, where

- $H = (X, Y, E)$ is a bipartite graph,
- σ_X is a total order on X , and
- σ_Y is a total order on Y .

Given an ordered bipartite graph $\mathcal{H} = (H, \sigma_X, \sigma_Y)$, an $\mathcal{H}$ -*configuration* is a tuple $\tau = (\tau_h)_{h \in V(H)}$, where τ_h is a total order on the neighbors of h for each $h \in V(H)$. Associated with this configuration, we create what we call a *configuration graph*, denoted by G_τ , defined as follows. First we

create a vertex $v_{x,y}$ for each $x \in X$ and $y \in N_H(x)$ and let $C_1^\tau := \{v_{x,y} : x \in X, y \in N_H(x)\}$. We turn this into a cycle using the orders $(\sigma_X, (\tau_x)_{x \in X})$ “lexicographically.” That is, we consider the total order $\leq_{C_1^\tau}$ obtained by

$$\begin{aligned} v_{x_1,y_1} &\leq_{C_1^\tau} v_{x_2,y_2} && \text{if } x_1 <_{\sigma_X} x_2, \\ v_{x_1,y_1} &\leq_{C_1^\tau} v_{x_2,y_2} && \text{if } y_1 \leq_{\tau_{x_1}} y_2. \end{aligned}$$

We place edges to turn C_1^τ into a cycle,[†] adding an edge between the smallest and largest vertex in this order and otherwise adding edges between vertices that are consecutive in the total order. Although C_1^τ is undirected, we also fix a cyclic order on it, which arises naturally by following the order described above. We define C_2^τ analogously by switching the roles of X and Y . Finally, we define G_τ as the union of C_1^τ and C_2^τ together with the edges $\{v_{x,y}, v_{y,x} : \{x,y\} \in E(H)\}$. We will call these “edges corresponding to H ” the *cross edges*. We call C_1^τ and C_2^τ the *main cycles* of G_τ .

Roughly speaking, our goal in this section is to show that when G_τ is a configuration graph for τ an (H, σ_X, σ_Y) -configuration, where $H \in \{Q_3^+, K_{3,3}\}$, then there exists no assignment of “lengths” to its edges such that C_1^τ and C_2^τ are both longest cycles. In fact, provided that C_1^τ and C_2^τ have the same length, we will always be able to find a longer cycle with special properties. The latter will allow us to also find a longer cycle in the setting of Lemma 1.4 in our next section.

Given a configuration graph G_τ , we say an edge $\{v_{x_1,y_1}, v_{x_2,y_2}\}$ of C_1^τ is *dangerous* if $x_1 \neq x_2$, and analogously for the edges of C_2^τ . Edges of C_1^τ and C_2^τ are otherwise said to be *safe*. We say a cycle in G_τ is *good* if it has no dangerous edges from C_1^τ or has no dangerous edges from C_2^τ (or from neither).

We also need an appropriate formal definition for the length of cycles in configuration graphs. Given a configuration graph G_τ , we say a *weight function* on it is a function $w : E(G_\tau) \rightarrow \mathbb{R}_{\geq 0}$ such that $w(e) \geq 1$ for every cross edge e , and such that $w(C_1^\tau) = w(C_2^\tau)$, where we define the weight of any subgraph J of G_τ by $w(J) := \sum_{e \in E(J)} w(e)$. We say a cycle C in a weighted configuration graph (G_τ, w) is *long* if its weight is strictly greater than that of the main cycles of G_τ , that is if $w(C) > w(C_1) = w(C_2)$.

The next lemma will be useful for checking for the existence of good long cycles independent of the weight function.

Lemma 3.1. *Let $\mathcal{H} = (H, \sigma_X, \sigma_Y)$ be an ordered bipartite graph and let τ be an $\mathcal{H}$ -configuration. If there are two good cycles C and C' in G_τ , which together contain all of the edges of its main cycles, as well as at least one cross edge, then for any weight function w , (G_τ, w) contains a good long cycle.*

Proof. As C and C' together contain all edges of the main cycles, and at least one cross edge e (which has nonzero weight),

$$w(C) + w(C') \geq w(C_1) + w(C_2) + w(e) > 2w(C_1),$$

and so $w(C) > w(C_1)$ or $w(C') > w(C_1)$. □

Given an ordered bipartite graph $\mathcal{H} = (H, \sigma_X, \sigma_Y)$ and an $\mathcal{H}$ -configuration τ , we define a linear program $\text{LP}_{\mathcal{H},\tau}$ as follows. We create a variable $w_e \geq 0$ for each edge e of G_τ and put the constraint

[†] If C_1^τ has only one vertex, this yields a loop in the graph, and if C_1^τ has two vertices, this creates parallel edges.

$w_e \geq 1$ for all cross edges e of G_τ . We also have the following constraint that both main cycles must have the same weight:

$$\sum_{e \in E(C_1^\tau)} w_e = \sum_{e \in E(C_2^\tau)} w_e.$$

Finally, for every good cycle C in G_τ , we create the following constraint:

$$\sum_{e \in E(C)} w_e \leq \sum_{e \in E(C_1^\tau)} w_e.$$

We are only interested in the feasibility, so we may set the objective function to 1. This linear program is not feasible (there are no solutions) if and only if for every weight function w , (G_τ, w) contains a good long cycle. We record this fact below.

Observation 3.2. Let $\mathcal{H} = (H, \sigma_X, \sigma_Y)$ be an ordered bipartite graph and let τ be an $\mathcal{H}$ -configuration. Then $\text{LP}_{\mathcal{H}, \tau}$ is infeasible if and only if for all weight functions w , (G_τ, w) contains a good long cycle.

Proof. By definition of $\text{LP}_{\mathcal{H}, \tau}$ and by definition of a weight function, an edge-indexed vector $(w_e)_{e \in E(G_\tau)}$ is feasible for $\text{LP}_{\mathcal{H}, \tau}$ if and only if (1) the corresponding mapping $w : E(G_\tau) \rightarrow \mathbb{R}$, defined as $w(e) := w_e$ for every e , is a weight function; and (2) we have $w(C) = \sum_{e \in E(C)} w_e \leq \sum_{e \in E(C_1^\tau)} w_e = w(C_1^\tau)$ for every good cycle C . The latter is equivalent to saying that there exist no good long cycles. Hence, it follows that $\text{LP}_{\mathcal{H}, \tau}$ is feasible if and only if there exists a weight function w such that (G_τ, w) contains no good long cycle. The claim of the lemma now follows by negating both sides of this equivalence. $\square$

Given that there are many possible configurations to consider, the following lemma will allow us to significantly reduce computation time. A reader who is not interested in the computational efficiency can safely skip over the concepts of substructures that we define below and consider performing a brute-force search instead.

Given two ordered bipartite graphs $\mathcal{H} = (H, \sigma_X, \sigma_Y)$ and $\mathcal{H}' = (H', \sigma'_X, \sigma'_Y)$, we say that $\mathcal{H}'$ is a *subgraph* of $\mathcal{H}$ if H' is a subgraph of H and we have the following conditions on the restrictions of the orders: $\sigma_X|_{X'} = \sigma'_X$ and $\sigma_Y|_{Y'} = \sigma'_Y$.

Furthermore, if $\mathcal{H}'$ is a subgraph of $\mathcal{H}$, τ is an $\mathcal{H}$ -configuration, and τ' is an $\mathcal{H}'$ -configuration, then we say that τ' is a *subconfiguration* of τ if $\tau'_h = \tau_h|_{V(H')}$ for every $h \in V(H')$.

Finally, let τ' be a subconfiguration of τ , w be a weight function on G_τ , and w' be a weight function on $G_{\tau'}$. We say that w *extends* w' if these weight functions are compatible with this relation, that is if they respect the following two conditions.

- If e is a cross edge of $G_{\tau'}$ (which is necessarily also a cross edge of G_τ), then $w'(e) = w(e)$.
- Let $i \in \{1, 2\}$ and suppose $v, w \in V(C_i^{\tau'}) \subseteq V(C_i^\tau)$ satisfy $vw \in E(C_i^{\tau'})$. If P is the unique path[†] between these vertices in C_i^τ that contains no other vertices of $C_i^{\tau'}$, then $w(P) = w'(vw)$.

We are now ready to show that it is sufficient to find good long cycles in subconfigurations.

[†] When $C_1^{\tau'}$ has only two vertices x and x' , we distinguish “the edge from x to x' ” and “the edge from x' to x ” to obtain a unique path here. When $C_1^{\tau'}$ contains only one vertex x , this path refers to the loop on x .

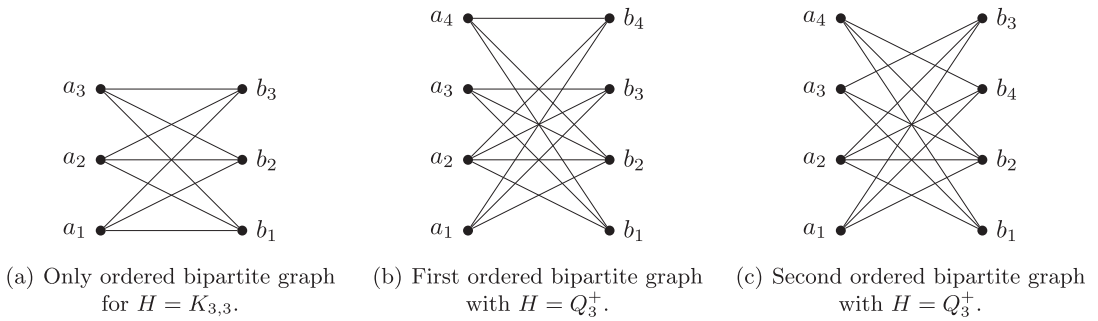


FIGURE 1 The three ordered bipartite graphs (H, σ_X, σ_Y) that need to be considered up to equivalence, as defined in Lemma 3.4. The permutations σ_X, σ_Y are implicit in the drawings.

Lemma 3.3. Let $\mathcal{H}'$ and $\mathcal{H}$ denote ordered subgraphs, τ' an $\mathcal{H}'$ -configuration and τ an $\mathcal{H}$ -configuration. Suppose that τ' is a subconfiguration of τ and w is a weight function on the configuration graph G_τ which extends a weight function w' on the configuration graph $G_{\tau'}$. If $(G_{\tau'}, w')$ contains a good long cycle, then (G_τ, w) also contains a good long cycle.

Proof. Any vertex of $G_{\tau'}$ has a corresponding vertex in G_τ . When there is an edge between two vertices of $G_{\tau'}$ for which there is no edge in G_τ , instead there will be a path following one of the main cycles in G_τ . This gives a corresponding cycle in G_τ for each cycle in $G_{\tau'}$. By the definition of the extension of weight functions, these cycles always have the same weight. Furthermore, if a cycle is good in $G_{\tau'}$, then its corresponding cycle is good in G_τ . This is because whether edges (that are not cross edges) are safe, depends on which vertices in the cycles they connect, and safe edges in $G_{\tau'}$ correspond to (paths consisting of) safe edges in G_τ . This proves the lemma. $\square$

We note that there is another issue that we will need to address to reduce the computation time of our computer search. For an arbitrary bipartite graph H with bipartition (X, Y) , there could be up to $|X|!|Y|!$ possible ordered bipartite graphs. Luckily, symmetry can be used for the graphs of interest to us in order to reduce this number. We will detail the more general criterion in the proof of the main result of this section, which we now proceed with.

Lemma 3.4. Let $\mathcal{H} = (H, \sigma_X, \sigma_Y)$ be an ordered bipartite graph such that either Q_3^+ or $K_{3,3}$ is a subgraph of H , and let τ be an $\mathcal{H}$ -configuration. For any weight function w on G_τ , (G_τ, w) contains a good long cycle.

Proof. By applying Lemma 3.3, we may assume $H \in \{Q_3^+, K_{3,3}\}$.

In our code, we have first implemented checking for equivalence between ordered bipartite graphs. We generate all possible ordered bipartite graphs with $H \in \{Q_3^+, K_{3,3}\}$, and then test for equivalence: There are up to equivalence only two ordered bipartite graphs for Q_3^+ , and of course only one for $K_{3,3}$, as depicted in Figure 1. The equivalence is checked by verifying whether the ordered bipartite graphs admit the same collection of configuration graphs (up to graph isomorphism). For an ordered bipartite graph $\mathcal{H} = (H, \sigma_X, \sigma_Y)$, we place a cycle on the vertices in X and a cycle on the vertices in Y in the cyclic orders given by σ_X and σ_Y , respectively (adding also the edge between the last and first vertex). If, between two such auxiliary graphs, there exists an isomorphism such that X gets mapped to X or to Y , then the set of configuration graphs remains the same (up to relabeling). This follows directly from the definition of the configuration graphs.

We then run a computer search to check that for every configuration τ on these three ordered bipartite graphs, every weighted configuration graph resulting from τ contains a good long cycle. As there are many configurations even on these three ordered bipartite graphs, we proceed by adding edges iteratively. Indeed, it is a direct consequence of Lemma 3.3 that it is sufficient to check the condition for at least one subconfiguration of every possible configuration.

- (1) We fix an order on the edges of H . Let $\emptyset = H_0 \subset H_1 \subset \dots \subset H_{|E(H)|} = H$ be a sequence of subgraphs in which one edge is added at every step.[†] We will write e_i for the unique edge in $E(H_i) \setminus E(H_{i-1})$.
- (2) At step $i = 0$, let S_0 be the set containing only the empty configuration.
- (3) At step $i \in \{1, \dots, |E(H)|\}$, we construct S_i as follows. For each $(H_{i-1}, \sigma_X, \sigma_Y)$ -configuration $\tau' \in S_{i-1}$, consider every $(H_i, \sigma_X, \sigma_Y)$ -configuration τ such that τ' is a subconfiguration of τ . Let R_i be the set of these configurations. Add τ to S_i if the criterion of Lemma 3.1 cannot guarantee that every weighted configuration graph on τ would contain a good long cycle.
- (4) Finally, we create a set S as follows. For every H -configuration τ in $S_{|E(H)|}$, we add τ to S if the more powerful linear program criterion from Observation 3.2 cannot guarantee that every weighted configuration graph on τ would contain a good long cycle.

If S is empty, the theorem is a direct consequence of Lemmas 3.1 and 3.3 and Observation 3.2.

This is indeed the case. We have implemented this algorithm in Mathematica [48]. The code is available in [28], and is thoroughly commented. A summary of results per step (number of configurations in R_i , S_i) is presented in Table 1. The computation took under 1 h 49 min[‡] on a 2020 MacBook Air with M1 chip and 16 GB ram running Mathematica 13.0.0.0. $\square$

4 | PROOF OF LEMMA 1.4

We will now deduce Lemma 1.4 from Lemma 3.4.

Proof of Lemma 1.4. Let C_1 and C_2 be two longest cycles in a graph G such that $|V(C_1) \cap V(C_2)| = k > 0$. Let S be a minimum vertex separator between $A = V(C_1) \setminus V(C_2)$ and $B = V(C_2) \setminus V(C_1)$ in G . Suppose toward a contradiction that $|S| \geq \text{ex}(2k, \{Q_3^+, K_{3,3}\}) + 1$.

By Menger's Theorem, there are at least $\text{ex}(2k, \{Q_3^+, K_{3,3}\}) + 1$ vertex-disjoint paths with one endpoint in A and one endpoint in B . Fix such a collection $\mathcal{P}$. We also fix total orderings on C_1 and C_2 by choosing a smallest element and a direction.

We define an ordered bipartite graph as follows. Let X be the set of connected components of $C_1 - V(C_2)$ and let Y be the set of connected components of $C_2 - V(C_1)$. The sets X, Y inherit a total order from C_1, C_2 , respectively.

We add an edge between $x \in X$ and $y \in Y$ if there is some $P \in \mathcal{P}$ with one endpoint of P in the connected component x and the other endpoint in y .

[†] Although our implementation does not optimize this order, it appears one can make the computation quicker by selecting an order on the edges of H , which prunes the most configurations earliest in the process. In particular, as is intuitive, maximizing the number of cycles appearing in H_i in the first steps generally reduces the number of configurations that appear in later steps, with some caveats.

[‡] Of course, the computation time could be greatly reduced through various optimizations, for instance, by implementing the code in a lower level language or by using better pruning heuristics.

TABLE 1 Summary of results per step for the computer-search algorithm presented in the proof of Lemma 3.4. Refer to Figure 1 for the vertex labels.

i	e_i	$ R_i $	$ S_i $
1	$\{a_1, b_1\}$	1	1
2	$\{a_1, b_2\}$	2	2
3	$\{a_1, b_3\}$	6	6
4	$\{a_2, b_1\}$	12	12
5	$\{a_2, b_2\}$	48	36
6	$\{a_2, b_3\}$	216	120
7	$\{a_3, b_1\}$	360	360
8	$\{a_3, b_2\}$	2160	684
9	$\{a_3, b_3\}$	6156	72
		$ S $	0

(a) Results for the ordered bipartite graph with $H = K_{3,3}$.

i	e_i	$ R_i $	$ S_i $
1	$\{a_1, b_2\}$	1	1
2	$\{a_1, b_3\}$	2	2
3	$\{a_2, b_2\}$	4	4
4	$\{a_2, b_3\}$	16	12
5	$\{a_3, b_2\}$	36	36
6	$\{a_3, b_3\}$	216	120
7	$\{a_1, b_4\}$	360	360
8	$\{a_2, b_4\}$	2160	684
9	$\{a_4, b_4\}$	2052	2052
10	$\{a_2, b_1\}$	8208	8208
11	$\{a_3, b_1\}$	49248	8838
12	$\{a_4, b_1\}$	53028	7492
13	$\{a_4, b_2\}$	89904	12
		$ S $	0

(b) Results for the first ordered bipartite graph with $H = Q_3^+$.

i	e_i	$ R_i $	$ S_i $
1	$\{a_1, b_2\}$	1	1
2	$\{a_1, b_3\}$	2	2
3	$\{a_2, b_2\}$	4	4
4	$\{a_2, b_3\}$	16	12
5	$\{a_3, b_2\}$	36	36
6	$\{a_3, b_3\}$	216	120
7	$\{a_1, b_4\}$	360	360
8	$\{a_2, b_4\}$	2160	684
9	$\{a_4, b_4\}$	2052	2052
10	$\{a_2, b_1\}$	8208	8208
11	$\{a_3, b_1\}$	49248	11586
12	$\{a_4, b_1\}$	69516	508
13	$\{a_4, b_2\}$	6096	0
		$ S $	0

(c) Results for the second ordered bipartite graph with $H = Q_3^+$.

For each edge xy with $x \in X$ and $y \in Y$, there is a unique such path that we will denote by P_{xy} . Indeed, suppose on the contrary there were two such paths P_{xy} and P'_{xy} . Let v and v' be the endpoints of P_{xy} and P'_{xy} , respectively, in x and let u, u' be the endpoints of P_{xy}, P'_{xy} , respectively, in y . Let Q be the path between v and v' in x and Q' be the path between u and u' in y . This gives two cycles

$$Q \rightarrow P_{xy} \rightarrow (C_2 \setminus Q') \rightarrow P'_{xy} \rightarrow \text{and}$$

$$Q' \rightarrow P_{xy} \rightarrow (C_1 \setminus Q) \rightarrow P'_{xy} \rightarrow,$$

which together cover all edges of $C_1 \cup C_2$, plus at least two more. Hence one of the two cycles above is longer than C_1 and C_2 , contradicting that they are longest cycles. Thus, there is a unique such path.

This concludes the description of the ordered bipartite graph $\mathcal{H} = (X, Y, E)$. By what we have shown in the previous paragraph, we have $|E| = |\mathcal{P}|$.

We next define an $\mathcal{H}$ -configuration τ . Let $x \in X$. In the configuration graph G_τ of τ , there will be $|N_H(x)|$ copies of x in C_1^τ , one per neighbor $y \in Y$ of x . We order the neighbors of x according to the order in which these “edges” (paths from $\mathcal{P}$) intersect the original cycle C_1 . To be precise, we consider the paths $P_{u_1, v_1}, \dots, P_{u_d, v_d} \in \mathcal{P}$ incident with the connected component of $C_1 \setminus C_2$ corresponding to $x \in X$, with $u_1, \dots, u_d \in V(C_1)$ numbered in order of the total ordering we fixed on C_1 . As we argued above, $v_1, \dots, v_d$ are vertices in distinct components $y_1, \dots, y_d \in Y$, and this is the order we place on the neighbors of x .

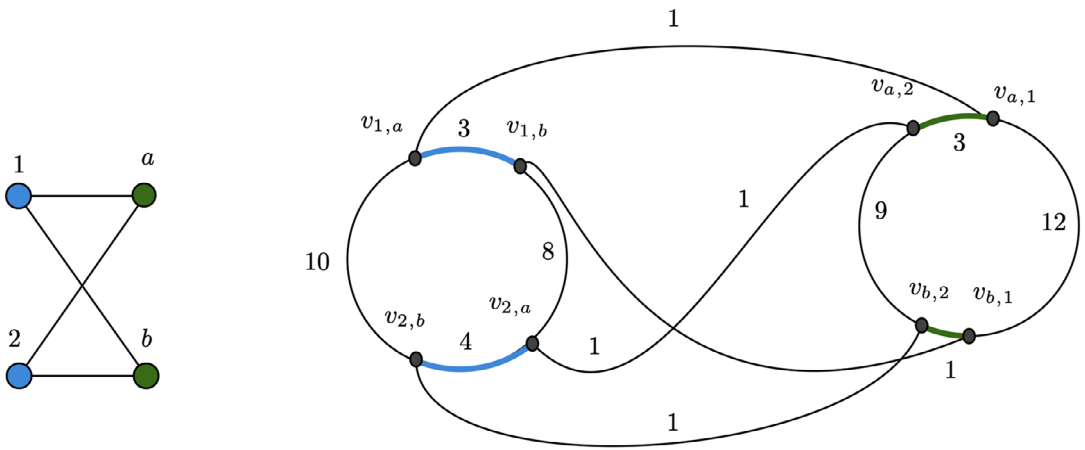


FIGURE 2 The above depicts a bipartite graph H (left) together with a possible configuration graph G_τ (right). The colored edges are safe.

We obtain a configuration graph G_τ with main cycles C_1^τ and C_2^τ as described in Section 3. We included an example in Figure 2. Recall that C_1^τ contains vertices $v_{x,y}$ for $x \in X$ and $y \in N_H(x)$.

We define a weight function $w : E(G_\tau) \rightarrow \mathbb{R}_{\geq 0}$, considering the two types of edges separately.

- If $e = v_{x,y}v_{y,x}$ is a cross edge with $x \in X$ and $y \in Y$, then we have defined a path $P_{xy} \in \mathcal{P}$. We set $w(e)$ as the length of P_{xy} .
- Otherwise, the edge is contained in one of the main cycles. Since both cases are analogous, we describe the case for an edge $e = v_{x,y}v_{x',y'}$ of C_1^τ with $x, x' \in X$ and $y, y' \in Y$. Let $u_{xy}, u_{x'y'} \in V(C_1)$ denote the endpoints in C_1 of $P_{xy}, P_{x'y'} \in \mathcal{P}$, respectively. We set $w(e)$ as the length of the arc of C_1 from u_{xy} to $u_{x'y'}$, in the direction we chose for the total ordering on C_1 .

Since $|E| = |\mathcal{P}| \geq \text{ex}(2k, \{Q_3^+, K_{3,3}\}) + 1$, it follows that the bipartite graph $H = (X, Y, E)$ contains either Q_3^+ or $K_{3,3}$. Thus, by Lemma 3.4, (G_τ, W) contains a good long cycle C^τ . The vertices $v_1^\tau, \dots, v_\ell^\tau$ of C^τ correspond to vertices $v_1, \dots, v_\ell$ of G . The path that was used to define the edge $e = v_i^\tau v_{i+1}^\tau$ of C^τ has length $w(e)$ and is between v_i and v_{i+1} . Therefore, we obtain a closed walk C in G of length $w(C^\tau)$. Since C^τ is long, $w(C^\tau) > w(C_1^\tau)$ and the length of C is strictly larger than the length of C_1 . What remains to show is that C is in fact a cycle. The vertices $v_1, \dots, v_\ell$ are distinct because the vertices $v_1^\tau, \dots, v_\ell^\tau$ are distinct, but we need to show the paths between them do not intersect internally. The paths corresponding to cross edges are chosen to be internally vertex-disjoint and only intersect $V(C_1) \cup V(C_2)$ in their endpoints. For $i \in \{1, 2\}$, the paths corresponding to edges from C_i^τ correspond to internally disjoint arcs of C_i . It remains to rule out intersections between the arcs of C on C_1 and C_2 : By the definition of good, there exists some $i \in \{1, 2\}$ such that all arcs of C_i that are used by C are part of a connected component of $C_i - V(C_{3-i})$ and thus do not intersect any of the arcs that C uses from C_{3-i} . This shows C is indeed a cycle, contradicting C_1 being the longest cycle.

We conclude that $|S| \leq \text{ex}(2k, \{Q_3^+, K_{3,3}\})$ as desired. $\square$



5 | FURTHER REMARKS

One direction toward improving the bounds in Lemma 1.4 is to prove a result analogous to Lemma 3.4 for a graph H with a better extremal number than Q_3^+ . In fact, Chen, Faudree, and Gould [13] already took a similar approach back in 1998 and proved a result analogous to Lemma 3.4 for $H = K_{3,257}$ without the aid of a computer, via a result analogous to our Lemma 3.1, the Erdős-Szekeres theorem and the pigeonhole principle. The perhaps most natural attempt is to consider the complete bipartite graphs $K_{2,t}$, since their extremal functions satisfy $\text{ex}(n, K_{2,t}) = \Theta(n^{3/2}) \ll n^{8/5}$. However, one can show that for every fixed integer $t \in \mathbb{N}$, there exist (weighted) configuration graphs on an ordered bipartite graph of $K_{2,t}$ without good long cycles. Similarly, by computer search, we found that Lemma 3.4 is false when Q_3^+ is replaced by Q_3 : There are (weighted) configuration graphs on an ordered bipartite graph of Q_3 without good long cycles. Indeed, these can be obtained with only slight modifications to the code in [28], see Figure 3 for an example.

We first need some notation. Given a graph G , a *longest path transversal* in G is defined as a set of vertices S such that S intersects every longest path in G . Denote by $\text{lpt}(G)$ and $\text{lct}(G)$ the minimum size of a longest path and cycle transversal in G , respectively. Furthermore, let us denote by $\text{mipc}(G)$ and $\text{micc}(G)$, respectively, the largest number $k \geq 1$ such that for every collection of k longest paths (cycles, respectively) in the graph G , there exists a vertex that is common to all of these k paths (cycles). If all longest paths (cycles) in G meet in a single vertex, then we set $\text{mipc}(G) = \infty$ or $\text{micc}(G) = \infty$, respectively. We observe the following lower bounds on longest

cycles and paths in vertex-transitive graphs in terms of these parameters. The two lower bounds in terms of $\text{lpt}(G)$ and $\text{lct}(G)$ were already stated in a slightly different language in the paper of DeVos [18].

Proposition 5.1. *If G is a connected vertex-transitive graph on $n \geq 3$ vertices, then G contains a path of length at least $\frac{n}{\text{lpt}(G)} - 1$ and a cycle of length at least $\frac{n}{\text{lct}(G)}$. Furthermore, G contains a path of length at least $n^{1-1/\text{mipc}(G)} - 1$ and a cycle of length at least $n^{1-1/\text{micc}(G)}$.*

Proof. It is easy to observe that G must in fact be 2-connected: Removing a leaf of a spanning tree of G leaves a connected graph, and since G is vertex-transitive, this means that removing *any* vertex of G leaves a connected graph. Hence, by Observation 2.1, any two longest paths and any two longest cycles in G intersect each other. A key observation is that if a mapping $\phi : V(G) \rightarrow V(G)$ is drawn uniformly at random from the set of all automorphisms of G , then, since G is vertex-transitive, for every fixed vertex $u \in V(G)$, its image $\phi(u)$ is uniformly distributed among all vertices of G .

The proofs of the two statements for paths and cycles are completely analogous, which is why in the following, we only give the arguments for the two lower bounds on the maximum length of a path. Fix some longest path P in G , as well as minimum-size longest path transversal S . Let ϕ be a random automorphism of G . Then clearly, the image $\phi(P)$ is also longest path in G and thus must be hit by S . This implies

$$1 \leq \mathbb{E}[|V(\phi(P)) \cap S|] = \sum_{v \in S} \mathbb{P}(v \in \phi(P)) = \sum_{v \in S} \sum_{u \in V(P)} \mathbb{P}(\phi(u) = v) = \frac{|S||V(P)|}{n}.$$

Rearranging now yields the desired lower bound $|P| = |V(P)| - 1 \geq \frac{n}{\text{lpt}(G)} - 1$.

Let us now move on to the second lower bound. Let $k = \text{mipc}(G)$. Then every collection of k longest paths in G meet in a common vertex. Let $\phi_1, \dots, \phi_k$ be k independently and uniformly drawn random automorphisms of G . Then the images $\phi_1(P), \dots, \phi_k(P)$ form a collection of k longest paths in G and thus must meet in a common vertex. This implies

$$\begin{aligned} 1 &\leq \mathbb{E}\left[\left|\bigcap_{i=1}^k V(\phi_i(P))\right|\right] = \sum_{v \in V(G)} \mathbb{P}\left[\bigwedge_{i=1}^k \{v \in V(\phi_i(P))\}\right] = \sum_{v \in V(G)} \prod_{i=1}^k \mathbb{P}[v \in V(\phi_i(P))] \\ &= \sum_{v \in V(G)} \prod_{i=1}^k \sum_{u \in V(P)} \mathbb{P}[\phi_i(u) = v] = n \cdot \left(\frac{|V(P)|}{n}\right)^k = \frac{|V(P)|^k}{n^{k-1}}. \end{aligned}$$

Rearranging now yields $|V(P)| \geq n^{1-1/k}$ and thus $|P| \geq n^{1-1/\text{mipc}(G)} - 1$, as claimed. This concludes the proof. $\square$

Given the lower bounds on the length of longest cycles from Proposition 5.1, it is natural to ask for the state of the art on bounds for the parameters $\text{lpt}(G)$, $\text{lct}(G)$, $\text{mipc}(G)$, $\text{micc}(G)$, and we conclude with a discussion of these.

A famous question asked by Gallai [25] in 1966 is whether $\text{lpt}(G) = 1$ for every connected graph G , that is, whether all longest paths meet in one vertex. This was answered in the negative by Walther [45], who constructed a connected graph G on 25 vertices with $\text{lpt}(G) = 2$. A smaller

counterexample on 12 vertices was found independently by Walther and Zamfirescu [46, 50]. Later, Grünbaum [27] constructed a connected graph G on 324 vertices with $\text{lpt}(G) = 3$, and amazingly, this remains the best known lower bound for connected graphs to this date. In particular, it is still an open question, raised by both Walther and Zamfirescu [49], whether there exists an absolute constant $c > 0$ such that $\text{lpt}(G) \leq c$ for every connected graph G . Note that this would immediately yield a lower bound of $\Omega(n)$ on the maximum length of paths in vertex-transitive graphs. The question of Walther and Zamfirescu is currently out of reach, and in fact, all the known general upper bounds for longest path and cycle transversals in connected and 2-connected graphs grow polynomially in n , see the recent sequence of papers [36, 39, 41] on this problem. The current state-of-the-art bounds are $\text{lpt}(G) \leq 5n^{2/3}$ for all connected n -vertex graphs and $\text{lct}(G) \leq 5n^{2/3}$ for all 2-connected n -vertex graphs by Kierstead and Ren [36]. Unfortunately, these only provide lower bounds of order $\Omega(n^{1/3})$ on longest paths and cycles in vertex-transitive graphs, but improving the upper bounds on $\text{lpt}(G)$ and $\text{lct}(G)$ to $n^{o(1)}$, for instance, would be very interesting.

Let us now turn to the parameters $\text{mipc}(G)$ and $\text{micc}(G)$. A well-known and longstanding conjecture, popularized by Zamfirescu since the 80s, asks whether every three longest paths in a connected graph must intersect in a common vertex, that is, whether $\text{mipc}(G) \geq 3$ for every connected graph, see, for example, [3, 12, 14, 17, 42, 44, 51]. By Proposition 5.1, the truth of this conjecture would directly yield an improved lower bound of $\Omega(n^{2/3})$ for longest paths in vertex-transitive graphs. Similarly, it remains open whether $\text{micc}(G) \geq 3$ for every 2-connected graph G [42], which would give the same improved lower bound for longest cycles in vertex-transitive graphs. In fact, it is even possible that $\text{mipc}(G) \geq 6$ for every connected G and $\text{micc}(G) \geq 6$ for every 2-connected G , which would yield much improved lower bounds of $\Omega(n^{5/6})$ for longest paths and cycles in vertex-transitive graphs.

We believe that because of this, the problems of bounding the parameters $\text{lpt}(G)$, $\text{lct}(G)$, $\text{mipc}(G)$, $\text{micc}(G)$ on vertex-transitive graphs deserve further attention.

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REFERENCES

1. N. Alon and Y. Roichman, *Random Cayley graphs and expanders*, Random Structures Algorithms **5** (1994), no. 2, 271–284.
2. B. Alspach, *The search for long paths and cycles in vertex-transitive graphs and digraphs* (K. L. McAvaney, ed.), Combin. Math. VIII, Springer, Berlin, Heidelberg, pp. 14–22, <https://doi.org/10.1007/BFb0091804>.
3. M. Axenovich, *When do three longest paths have a common vertex?*, Discrete Math. Algorithms Appl. **1** (2009), no. 01, 115–120.
4. L. Babai, *Long cycles in vertex-transitive graphs*, J. Graph Theory **3** (1979), no. 3, 301–304.
5. L. Babai, *Automorphism groups, isomorphism, reconstruction*, Handbook of Combin., Elsevier Sci. B. V., Amsterdam, 1995, pp. 1447–1540.
6. J. Barát and M. Kriesell, *What is on his mind?*, Discrete Math. **310** (2010), no. 20, 2573–2583.
7. D. Bauer, H. Broersma, and E. Schmeichel, *Toughness in graphs-a survey*, Graphs Combin. **22** (2006), no. 1, 1–35, <https://doi.org/10.1007/s00373-006-0649-0>.
8. B. Bollobás, *Extremal graph theory*, Dover Publications, Inc., Mineola, NY, 2004. xx+488 pp. ISBN:0-486-43596-2.
9. J. A. Bondy, *Hamilton cycles in graphs and digraphs*, Dept. Combin. Optim., Univ. Waterloo, 1978.
10. J. A. Bondy, *Basic graph theory: paths and circuits*, Handbook of Combinatorics. Vol. 1, 2, 3–110. Elsevier Science B.V., Amsterdam, 1995, ISBN:0-444-88002-X.
11. W. G. Brown, *On graphs that do not contain a Thomsen graph*, Canad. Math. Bull. **9** (1966), no. 3, 281–285.
12. M. R. Cerioli and P. T. Lima, *Intersection of longest paths in graph classes*, Discrete Appl. Math. **281** (2020), 96–105.
13. G. Chen, R. J. Faudree, and R. J. Gould, *Intersections of longest cycles in k -connected graphs*, J. Combin. Theory Ser. B **72** (1998), no. 1, 143–149.
14. G. Chen, R. J. Faudree, and R. J. Gould, *Nonempty intersection of longest paths in series-parallel graphs*, Discrete Math. **340** (2017), no. 3, 287–304.
15. D. Christofides, J. Hladký, and A. Máthé, *Hamilton cycles in dense vertex-transitive graphs*, J. Combin. Theory Ser. B **109** (2014), 34–72.
16. V. Chvátal, *Tough graphs and hamiltonian circuits*, Discrete Math. **5** (1973), no. 3, 215–228.
17. S. F. De Rezende, M. M. Martins, and C. S. de Souza, *Intersecting longest paths*, Discrete Math. **313** (2013), no. 12, 1401–1408.
18. M. DeVos, *Longer cycles in vertex transitive graphs*, arXiv preprint arXiv:2302.04255 (2023).
19. G. A. Dirac, *Some theorems on abstract graphs*, Proc. Lond. Math. Soc. **3** (1952), no. 1, 69–81.
20. N. Draganić, R. Montgomery, D. Munhá Correia, A. Pokrovskiy, and B. Sudakov, *Hamiltonicity of expanders: optimal bounds and applications*, 2024. arXiv:2402.06603.
21. P. Erdős and M. Simonovits, *Some extremal problems in graph theory*, Combinatorial theory and its applications, I–III (Proc. Colloq., Balatonfüred, 1969), pp. 377–390. Colloq. Math. Soc. János Bolyai, 4, North-Holland Publishing Co., Amsterdam-London, 1970.
22. P. Erdős and M. Simonovits, *Compactness results in extremal graph theory*, Combinatorica **2** (1982), 275–288.
23. Z. Füredi, *An upper bound on Zarankiewicz’ problem*, Combin. Probab. Comput. **5** (1996), no. 1, 29–33.
24. Z. Füredi, *On a theorem of Erdős and Simonovits on graphs not containing the cube*, Number theory, analysis, and combinatorics, De Gruyter Proc. Math., De Gruyter, Berlin/Boston, 2013, pp. 113–125.
25. T. Gallai, *Problem 4* (P. Erdős and G. Katona, eds.), Theory graphs, Proc. Tihany, Academic Press, New York, 1968, p. 362.
26. C. Godsil and G. F. Royle, *Algebraic graph theory*, vol. 207, Springer, New York, 2001, <https://doi.org/10.1007/978-1-4613-0163-9>.
27. B. Grünbaum, *Vertices missed by longest paths or circuits*, J. Combin. Theory Ser. A **17** (1974), no. 1, 31–38.
28. C. Groenland, T. P. van der Zanden, J. Skokan, and Y. Zhao, *Code used in the paper “Longest cycles in vertex-transitive and highly connected graphs,”* 2024, <https://github.com/tjeremie/Long-cycles>.

29. M. Grötschel, *On intersections of longest cycles* (B. Bollobás, ed.), Graph Theory Combin.: Proc. Cambridge Combin. Conf. Hon. Paul Erdos, Academic Press, London, pp. 171–189.
30. J. Gutiérrez and C. Valqui, *On two conjectures about the intersection of longest paths and cycles*, Discrete Math. **347** (2024), no. 11, 114148.
31. R. Häggkvist, *Unsolved problem*, Proc. Hunger. Collog. 5th Keszthely, North Holland Publishing Company, Amsterdam, 1976, pp. 1203–1204.
32. F. Hilbig, *Kantenstrukturen in nichthamiltonschen Graphen*, Ph.D. thesis, Technical University, Berlin, Institute of Mathematics, 1986.
33. B. Jackson, *Hamilton cycles in regular 2-connected graphs*, J. Combin. Theory Ser. B **29** (1980), no. 1, 27–46.
34. B. Jackson, H. Li, and Y. Zhu, *Dominating cycles in regular 3-connected graphs*, Discrete Math. **102** (1992), no. 2, 163–176.
35. H. Jung, *Longest circuits in 3-connected graphs*, Finite and Infinite Sets, North-Holland Publishing Company, Amsterdam, 1984, pp. 403–438.
36. H. Kierstead and E. Ren, *Improved upper bounds on longest-path and maximal-subdivision transversals*, Discrete Math. **346** (2023), no. 9, 113514.
37. D. Kühn, A. Lo, D. Osthus, and K. Staden, *The robust component structure of dense regular graphs and applications*, Proc. Lond. Math. Soc. **110** (2015), no. 1, 19–56.
38. K. Kutnar and D. Marušic, *Hamilton cycles and paths in vertex-transitive graphs-current directions*, Discrete Math. **309** (2009), no. 17, 5491–5500.
39. J. A. Long, K. G. Milans, and A. Munaro, *Sublinear longest path transversals*, SIAM J. Discrete Math. **35** (2021), no. 3, 1673–1677.
40. L. Lovász, *Problem 11 in combinatorial structures and their applications*, Proc. Calgary Internat. Conf., Gordon and Breach, New York, 1970, pp. 243–246.
41. D. Rautenbach and J.-S. Sereni, *Transversals of longest paths and cycles*, SIAM J. Discrete Math. **28** (2014), no. 1, 335–341.
42. A. Shabbir, C. T. Zamfirescu, and T. I. Zamfirescu, *Intersecting longest paths and longest cycles: a survey*, Electron. J. Graph Theory Appl. **1** (2013), no. 1, 56–76.
43. I. A. Stewart and B. Thompson, *On the intersections of longest cycles in a graph*, Exp. Math. **4** (1995), no. 1, 41–48.
44. H.-J. Voss, *Cycles and bridges in graphs*, Kluwer Acad. Publ., Dordrecht, 1991.
45. H. Walther, *Über die Nichtexistenz eines Knotenpunktes, durch den alle längsten Wege eines Graphen gehen*, J. Combin. Theory **6** (1969), no. 1, 1–6.
46. H. Walther, *Anwendungen der Graphentheorie*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1978, <https://doi.org/10.1007/978-3-322-84013-4>.
47. B. Wei, *On the circumferences of regular 2-connected graphs*, J. Combin. Theory Ser. B **75** (1999), no. 1, 88–99.
48. Wolfram Research Inc., *Mathematica*, <https://www.wolfram.com/mathematica/>.
49. T. Zamfirescu, *A two-connected planar graph without concurrent longest paths*, J. Combin. Theory Ser. B **13** (1972), 116–121.
50. T. Zamfirescu, *On longest paths and circuits in graphs*, Math. Scand. **38** (1976), no. 2, 211–239.
51. T. Zamfirescu, *Intersecting longest paths or cycles: a short survey*, An. Univ. Craiova Ser. Mat. Inform **28** (2001), no. 1-9, 27, <http://tzamfirescu.tricube.de/TZamfirescu-151.pdf>.