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Analysis of a tripartite entanglement distribution switch

Philippe Nain[1] · **Gayane Vardoyan**[2] · **Saikat Guha**[3] · **Don Towsley**[4]

Abstract

We study a quantum switch that distributes *tripartite* entangled states to sets of users. The entanglement switching process requires two steps: First, each user attempts to generate bipartite entanglement between itself and the switch, and second, the switch performs local operations and a measurement to create multipartite entanglement for a set of three users. In this work, we study a simple variant of this system, wherein the switch has infinite memory and the links that connect the users to the switch are identical. This problem formulation is of interest to several distributed quantum applications, while the technical aspects of this work result in new contributions within queueing theory. The state of the system is modeled as continuous-time Markov chain (CTMC), and performance metrics of interest (probability of an empty system, switch capacity, expectation, and variance of the number of qubit-pairs stored) are computed via the solution of a two-dimensional functional equation obtained by reducing it to a boundary value problem on a closed curve. This work is a follow-up of Nain et al. (Proc ACM Measure Anal Comput Syst(POMACS) 4, 2020) where a switch distributing entangled *multipartite* states to sets of users was studied, but only the switch capacity and the expected number of stored qubits were derived.

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1 Introduction

Entanglement is an essential component of quantum computation, information, and communication. Its applications include quantum cryptography (for example, [3,4, 12]), distributed quantum computing (for example, [9,22]), and quantum sensing (for example, multipartite entanglement for quantum metrology [18] and spectroscopy [25]). These applications drive the increasing need for a quantum switching network that can supply end-to-end entanglement to groups of endpoints that request them [32, 35,37]. To realize such quantum systems, several architectures have been proposed to support high entanglement generation rates, high fidelity, and long coherence times [1, 5,19,20,24,26]. In this paper, we focus on the most basic and fundamental component of a quantum network—a single quantum switch that serves k users in a star topology. Each user has a dedicated link connected to the switch as shown in Fig. 1. (A detailed description of this figure is provided below.) In the most general case, the switch serves n-partite entangled states to sets of users according to incoming requests, where $n \leq k$.

To achieve this, link-level entangled states are generated at a constant rate across each link, resulting in two-qubit maximally entangled states (for example, Bell pairs or EPR states). These qubits are stored at local quantum memories: one from each Bell pair at the user and the other one at the switch. We consider the algorithm where the switch performs multi-qubit measurements to provide end-to-end entanglement to user groups of size n when enough of these bipartite states are accrued (at least n of them). When $n = 2$, the switch uses Bell-state measurements (BSMs), and when $n \geq 3$, it uses n-qubit Greenberger–Horne–Zeilinger (GHZ) basis measurements [31]. For additional background on quantum switches, the interested reader is referred to [38, Section 2] and [39, Section II].

Figure 1 depicts a quantum switch serving k users in a star topology. Each user is connected to the switch via a dedicated connection or link (for example, optical fiber). All nodes (including the switch) are assumed to have quantum memories, or buffers. When bipartite entanglement is successfully generated on a link, a user's qubit becomes entangled with a qubit located at the switch (represented with black dashed line in the figure). When entanglement is generated between the switch and a second user, the switch may perform a Bell state measurement (BSM) on its two locally held entangled qubits; as a result, the two user's qubits become entangled up to local Pauli corrections. This process can be generalized to three or more users, with the modification of a GHZ-basis measurement instead of a BSM at the switch. The result is a GHZ state entangling the users. In this paper, we consider the situation when three user's qubits are entangled (*i.e.*, $n = 3$), links are homogeneous, and all buffers are infinite. The situation when $n = 2$ was studied in [39] under a variety of assumptions.

Let us now introduce the problem in abstract, mathematical terms. As we shall see, while the problem considered in this work was initially motivated by its application to entanglement switching, its relevance reaches far beyond this example, and it is of
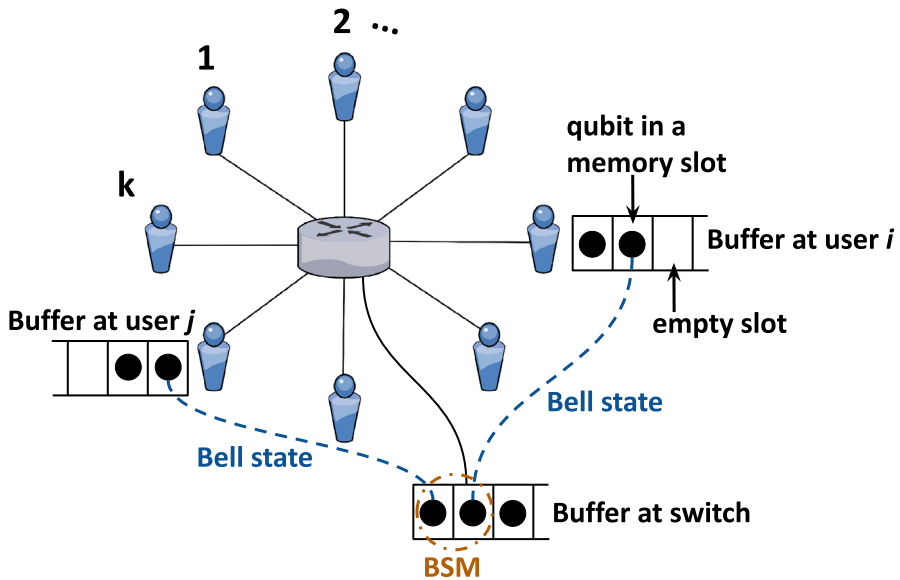
Fig. 1 A quantum switch serving k users in a star topology

interest to queueing theory in general. Consider a server and k clients in a star topology like in Fig. 1. Each client has a dedicated link to the server; one may think of these links either as communication channels or physical paths that may be used for object delivery. For instance, in the former scenario the clients may send data packets to the server for processing, and in the latter, the central node may be an assembly plant that receives components of a product from geographically distant manufacturing plants. (As soon as n components (or parts) from n distinct clients arrive at the central node, they are processed (or assembled) and immediately leave the system.) We assume that the processing/assembly step is instantaneous, but succeeds with probability q. (The fact that the assembly step can fail is important in the context of a quantum switch.) Finally, assume that the arrival process on link l ($l = 1, \ldots, k$) is a Poisson process with a constant rate $\mu > 0$ and that these k arrival processes are mutually independent. Each link is equipped with an infinite buffer and so is the switch.

If we denote by $Q_l(t)$ the number of entangled qubit-pairs stored by link l at time t, the process $\{(Q_1(t), \ldots, Q_k(t))\}_t$ is a continuous-time Markov chain (CTMC). As a consequence of the algorithm described above, when $n = 2$, at most one component of the vector $(Q_1(t), \ldots, Q_k(t))$ is nonzero. Exploiting this property, the state of the switch can be represented by a birth and death process thereby yielding the stationary distribution—and from it the main performance measures of interest—explicitly or in closed form both when the switch has a finite or infinite memory [39]. When the memory is infinite, it is shown in [39] that the system is stable when $k > 2$ and, among other results, that the system capacity—defined as the maximum achievable number of successful assemblies per time unit—is $q \sum_{l=1}^{k} \mu_l / 2$ when the switch has an infinite memory. Results are also obtained in [39] when quantum state decoherence

and an associated cutoff policy are added to the model, which amounts to assuming that stored entanglements have an exponential lifetime with a constant rate.

Getting explicit/closed-form results for $n \geq 3$ when link rates are arbitrary is very challenging. Even when all links are identical, say $\mu := \mu_l$ for $l = 1, \ldots, k$, and the switch has infinite memory, the problem becomes significantly more difficult to analyze when $n \geq 3$. In particular, the stability analysis requires the introduction of a Markov chain embedded in the original Markov chain. Via a careful choice of a Lyapunov function and application of Foster's theorem, the ergodicity condition of the embedded Markov chain, yielding the ergodicity condition of the original one, is found to be $k > n$ in [29]. In [29], the switch capacity and the expected number of stored qubits at the switch are also derived and are given by $\frac{q\mu k}{n}$ and $\frac{k(n-1)}{2(k-n)}$, respectively. The latter results are obtained by setting the drift of appropriate Lyapunov functions to zero. These are all first-order moment properties. The objective of this work is to get higher-order moment properties and, in particular, the variance of the number of stored qubits when $n = 3$. In the setting of entanglement switching, the case $n = 3$ is referred to as *tripartite* entanglement distribution. With a slight abuse of notation, the number of stored qubits will refer to the number of qubits located in the switch memory.

As noted above, this model extends far beyond entanglement switching. In general, one may view the system as a stochastic assembly-like queue, or a "kitting" process, for example, as in [21,23,33,36], since, in a sense, the switch "assembles" multipartite entangled states from bipartite ones, whose "arrival" into the system is driven by a stochastic process. Interestingly, none of these similar problem formulations found in the literature have a direct correspondence to our problem, as in our case the number of links or users being serviced by the central node is allowed to be, in theory, infinite, and our goal is to derive exact results, as opposed to approximate ones, or bounds. Hence, the problem studied here is novel, and the results derived in this work are of independent interest to queueing theory.

The paper is organized as follows: In Sect. 2, we identify a two-dimensional CTMC that models the state of the switch when $n = 3$ and write a two-dimensional functional equation satisfied by its generating function. The main performance metrics are expressed in Sect. 3 either in explicit form (we retrieve the capacity and expected number of stored entangled qubits found in [29] and reported above) or in terms of the generating function (variance). In Sect. 4, we solve the functional equation when the number of users is infinite (*i.e.*, $k = \infty$) and derive the variance of the number of stored qubits. For finite k, the functional equation is solved in Sect. 5 by reducing it to the solution of a boundary value problem (BVP) on a closed curve. From that, we derive in Sect. 6 the variance of the number of stored qubits. Formulas obtained for the probability that the system is empty and for the variance of the number of stored qubits are amenable to numerical evaluation as shown in Sect. 7. Works related to the application of BVP techniques to the solution of certain two-dimensional functional equations are discussed in Sect. 8. The paper is concluded in Sect. 9.

A word on the notation. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the set of nonnegative integers and $\underline{e}_l$ the unit vector in $\mathbb{N}^k$ whose entries all equal zero except for the lth one that equals one. The first, second, and third derivatives at point x, when they exist, of a function φ are denoted by $\varphi'(x)$, $\varphi''(x)$, and $\varphi'''(x)$, respectively.

2 The model

Throughout the paper we assume that the stability condition $k > 3$ holds [29].

Let $Q_l(t)$ denote the number of entangled qubits stored by link $l = 1, \ldots, k$. At time t, the state of the switch can be represented by the vector $(Q_1(t), \ldots, Q_k(t))$, whose state space is $\mathcal{S} = \{i\underline{e}_l + j\underline{e}_{l'} : i, j \in \mathbb{N}, l, l' \in \{1, 2, \ldots, k\}, l \neq l'\}$ since at most two links can store qubit-pairs at the same time. The latter constraint suggests that more compact representations can found. We now introduce one such representation, which we have found particularly convenient to work with.

The state of the switch will be represented by the two-dimensional irreducible continuous-time Markov chain (CTMC) $\{(X(t), Y(t)), t \geq 0\}$ on $\mathbb{N}^2$, whose nonzero transitions are displayed in Fig. 2. In this representation, states (i, j) and (j, i), for $i, j \in \mathbb{N}$, are equivalent. We choose to differentiate between them because it simplifies the analysis of the model, despite increasing the number of total states in the CTMC. Consider state $(0, 0)$ when no qubit-pairs are stored at the links—referred to as the *empty system*. From there, an entanglement is generated on one of the k links at rate $k\mu$; hence, we split this rate in half among the states $(0, 1)$ and $(1, 0)$. The state $(i, 0)$ (resp. $(0, i)$) corresponds to the situation where one link stores i qubit-pairs and the $k - 1$ other links do not store anything. From there, the system may evolve to state $(i + 1, 0)$ at rate μ if an additional entanglement is generated on the busy link, and to state $(i, 1)$ at rate $(k - 1)\mu$ if an entanglement is generated on of the $k - 1$ non-busy links. Finally, state $(i, j) \in \{1, 2, \ldots\}^2$ (resp. $(j, i) \in \{1, 2, \ldots\}^2$) describes the situation where two links are storing qubit-pairs, one is storing i qubit-pairs and the other one is storing j qubit-pairs. From this state, the system may join state $(0, 0)$ if an entanglement appears on one of the $k - 2$ empty links, which occurs at rate $(k - 2)\mu$, or join state $(i + 1, j)$ (resp. $(i, j + 1)$) if an additional entanglement is generated, at rate μ, at one of the two busy links. For convenience, we temporarily introduce the rates $\lambda = \mu$ and $\nu = (k - 2)\mu$. These rates can be interpreted as arrival and departure rates, respectively. The nonzero transition rates of the CTMC $\{(X(t), Y(t)), t \geq 0\}$ (see Fig. 2) are

$$(0, 0) \to (1, 0) \quad \text{with rate } \frac{2\lambda + \nu}{2},$$

$$(0, 0) \to (0, 1) \quad \text{with rate } \frac{2\lambda + \nu}{2},$$

$$\text{For } i \geq 1: \quad (i, 0) \to (i, 1) \quad \text{with rate } \lambda + \nu,$$

$$\text{For } j \geq 1: \quad (0, j) \to (1, j) \quad \text{with rate } \lambda + \nu,$$

$$\text{For } i \geq 1: \quad (i, 0) \to (i + 1, 0) \quad \text{with rate } \lambda,$$

$$\text{For } j \geq 1: \quad (0, j) \to (0, j + 1) \quad \text{with rate } \lambda,$$

$$\text{For } i \geq 1, j \geq 1: (i, j) \to (i, j + 1) \quad \text{with rate } \lambda,$$

$$\text{For } i \geq 1, j \geq 1: (i, j) \to (i + 1, j) \quad \text{with rate } \lambda,$$

$$\text{For } i \geq 1, j \geq 1: (i, j) \to (i - 1, j - 1) \quad \text{with rate } \nu.$$

Fig. 2 CTMC for a system with k homogeneous links and a switch that has infinite buffer and serves only tripartite entanglement. μ is the entanglement generation rate at the link level
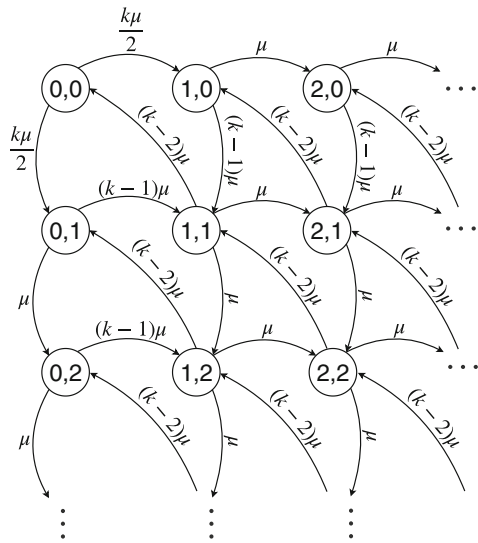
Let $\pi(i, j)$ be the steady-state probability that the chain is in state (i, j). The balance equations read

$$(2\lambda + \nu)\pi(0, 0) = \nu\pi(1, 1), \tag{1}$$

$$(2\lambda + \nu)\pi(1, 0) = \frac{(2\lambda + \nu)}{2}\pi(0, 0) + \nu\pi(2, 1), \tag{2}$$

$$(2\lambda + \nu)\pi(0, 1) = \frac{(2\lambda + \nu)}{2}\pi(0, 0) + \nu\pi(1, 2), \tag{3}$$

$$(2\lambda + \nu)\pi(1, 1) = \nu\pi(2, 2) + (\lambda + \nu)\pi(1, 0) + (\lambda + \nu)\pi(0, 1), \tag{4}$$

$$(2\lambda + \nu)\pi(i, 0) = \nu\pi(i + 1, 1) + \lambda\pi(i - 1, 0), \quad i \geq 2, \tag{5}$$

$$(2\lambda + \nu)\pi(0, j) = \nu\pi(1, j + 1) + \lambda\pi(0, j - 1), \quad j \geq 2, \tag{6}$$

$$(2\lambda + \nu)\pi(i, 1) = \nu\pi(i + 1, 2) + \lambda\pi(i - 1, 1) + (\lambda + \nu)\pi(i, 0), \quad i \geq 2, \tag{7}$$

$$(2\lambda + \nu)\pi(1, j) = \nu\pi(2, j + 1) + (\lambda + \nu)\pi(0, j) + \lambda\pi(1, j - 1), \quad j \geq 2, \tag{8}$$

$$(2\lambda + \nu)\pi(i, j) = \nu\pi(i + 1, j + 1) + \lambda\pi(i - 1, j) + \lambda\pi(i, j - 1), \quad i \geq 2, j \geq 2. \tag{9}$$

Define the generating function

$$F(x, y) = \sum_{i, j \geq 0} \pi(i, j)x^i y^j, \quad |x| \leq 1, |y| \leq 1.$$

It is shown in [30, Appendix A] that, for $|x| \leq 1, |y| \leq 1$,

$$\left(\lambda(1 - x) + \lambda(1 - y) + \nu\left(1 - \frac{1}{xy}\right)\right) F(x, y)$$

$$= v \left(y - \frac{1}{xy} \right) F(x, 0)$$
$$+ v \left(x - \frac{1}{xy} \right) F(0, y) - v \left(\frac{x + y}{2} - \frac{1}{xy} \right) F(0, 0). \tag{10}$$

Replacing λ by μ and v by $(k - 2)\mu$ in (10), and multiplying by $-xy/\mu$ both sides of the resulting equation, yields

$$K(x, y)F(x, y) = (k - 2) \left[a(x, y)F(x, 0) \right.$$
$$\left. + b(x, y)F(0, y) + c(x, y)F(0, 0) \right],$$
$$|x| \le 1, \ |y| \le 1, \tag{11}$$

with

$$K(x, y) := x^2 y + xy^2 - kxy + k - 2,$$
$$a(x, y) := 1 - xy^2,$$
$$b(x, y) := 1 - x^2 y,$$
$$c(x, y) := \frac{x^2 y + xy^2}{2} - 1. \tag{12}$$

Notice that (11) exhibits the following symmetry:

$$F(x, y) = F(y, x), \tag{13}$$

for all $|x| \le 1, |y| \le 1$. When $K(x, x) \ne 0$ and $x = y$, it follows from (11) that

$$F(x, x) = (k - 2) \left(\frac{a(x, x) + b(x, x)}{K(x, x)} F(x, 0) + \frac{c(x, x)}{K(x, x)} F(0, 0) \right), \tag{14}$$

since $F(x, 0) = F(0, x)$. Letting $x \to 1$ in (14), we get by noting that $F(1, 1) = 1$ and by using L'Hôpital's rule in the r.h.s. that

$$1 = (k - 2) \left(\frac{3}{k - 3} F(1, 0) - \frac{3}{2(k - 3)} F(0, 0) \right),$$

yielding

$$2F(1, 0) - F(0, 0) = \frac{2(k - 3)}{3(k - 2)}. \tag{15}$$

3 Main performance metrics

Let N denote the total number of stored entangled qubits at the switch in steady state. Via a Lyapunov function approach, the expectation of N has been computed in [29] in the general case where any combination of n users out of the k users want to

communicate and was found to be $\mathbb{E}[N] = \frac{k(n-1)}{2(k-n)}$. When $n = 3$ as in the present paper, then $\mathbb{E}[N] = \frac{k}{k-3}$. We also obtained in [29] that the capacity C of the switch, defined as the maximum achievable number of successful assemblies (or entanglement swappings) per time unit, is given by $C = \mu q \frac{k}{n}$, where q is the probability that an entanglement swapping succeeds.

For the sake of completeness, we show in Sects. 3.1 and 3.2 how, when $n = 3$, C and $\mathbb{E}[N]$ can be retrieved from the functional equation (11). Interestingly enough, these calculations can be performed without solving this equation. This is in contrast with the variance, which cannot be obtained without solving (11), as discussed in Sect. 3.3.

3.1 Capacity

A GHZ measurement can be generated from any state (i, j) for $i, j \geq 1$: This occurs with rate $\nu = (k - 2)\mu$, whenever a third link (not one of the two links that has a stored entanglement) generates an entanglement. Hence, the system capacity C is

$$C = q\nu \sum_{\substack{i \geq 1 \\ j \geq 1}} \pi(i, j) = q\nu(1 - F(1, 0) - F(0, 1) + F(0, 0))$$

$$= q\mu(k - 2)(1 - 2F(1, 0) + F(0, 0)) = \mu q \frac{k}{3}, \tag{16}$$

by $F(1, 0) = F(0, 1)$ from (13) and (15).

3.2 Expectation

Since $\mathbb{E}[N] = \frac{d}{dx} F(x, x)|_{x=1}$, let $y = x$ in (11), differentiate both sides of the resulting equation with respect to x and then set $x = 1$. This gives, by using (15),

$$\mathbb{E}[N] = \frac{k + 6(k - 2)\frac{d}{dx} F(x, 0)|_{x=1}}{2(k - 3)}. \tag{17}$$

Let us calculate $\frac{d}{dx} F(x, 0)|_{x=1}$.

It is shown in Lemma 2 in Appendix A that, for fixed $|x| = 1$, $K(x, y)$ defined in (12) has a unique zero $y = y(x)$ such that $|y(x)| \leq 1$.

Since $F(x, y)$ is well defined for $|x| \leq 1$, $|y| \leq 1$, the l.h.s. of (11) must vanish when $(x, y) = (x, y(x))$ with $|x| = 1$. This gives, for $|x| = 1$,

$$0 = (xy(x)^2 - 1)F(x, 0) + (x^2 y(x) - 1)F(0, y(x)) - \left(\frac{x^2 y(x) + xy^2(x)}{2} - 1\right)F(0, 0). \tag{18}$$

By twice differentiating (18) with respect to x, we obtain, for $|x| = 1$,

$$0 = \left[4y(x)y'(x) + 2xy'(x)^2 + 2xy(x)y''(x)\right]F(x, 0)$$

$$+2\left[y^2(x) + 2xy(x)y(x)\right]\frac{d}{dx}F(x,0)$$

$$+\left[2y(x) + 4xy'(x) + x^2y''(x)\right]F(0,y(x))$$

$$+2\left[2xy(x) + x^2y'(x)\right]y'(x)\frac{d}{dy}F(0,y(x))$$

$$+\left[x^2y(x) - 1\right]y''(x)F_y'(0,y(x)) + \left[x^2y(x) - 1\right]y'(x)^2\frac{d^2}{dy^2}F(0,y(x))$$

$$-\left[2y(x) + 4xy'(x) + 4y(x)y'(x) + x^2y''(x) + 2xy'(x)^2 + 2xy(x)y''(x)\right]$$

$$\frac{F(0,0)}{2}. \tag{19}$$

By setting $x = 1$ in the above equation and using $y(1) = 1$, $y'(1) = -1$ and $y''(1) = \frac{2(k-2)}{k-3}$ (see Lemma 2), we readily find that (Hint: $\frac{d}{dx}F(x,0)|_{x=1} = \frac{d}{dy}F(0,y)|_{y=1}$ because $F(x,y) = F(y,x)$ for all $|x| \le 1$, $|y| \le 1$ as observed in Sect. 2)

$$\frac{d}{dx}F(x,0)|_{x=1} = \frac{(-4 + 3y''(1))(2F(1,0) - F(0,0))}{8} = \frac{k}{6(k-2)}. \tag{20}$$

by using (15). Hence, by (17),

$$E[N] = \frac{k}{k-3}. \tag{21}$$

3.3 Variance

To obtain the variance of N, let us twice differentiate both sides of (11) with respect to x after setting $y = x$ and then set $x = 1$. By using (20) and (21), we find [30, Appendix C]

$$\frac{d^2}{dx^2}F(x,x)|_{x=1} = \frac{k(2k+3)}{3(k-3)^2} + \frac{3(k-2)}{k-3}\frac{d^2}{dx^2}F(x,0)|_{x=1}. \tag{22}$$

The variance of N is then obtained via the formula

$$\text{var}(N) = \frac{d^2}{dx^2}F(x,x)|_{x=1} + \mathbb{E}[N] - (\mathbb{E}[N])^2 = \frac{2k}{3(k-3)} + \frac{3(k-2)}{k-3}\frac{d^2}{dx^2}F(x,0)|_{x=1}, \tag{23}$$

where the second identity follows from (21) and (22).

It remains to find $\frac{d^2}{dx^2}F(x,0)|_{x=1}$ in (23). Unfortunately this quantity cannot be obtained without solving equation (11). Indeed, the interested reader can check that differentiating (19) w.r.t. x yields the tautology '$0 = 0$'; differentiating (19) twice w.r.t. x gives an equation involving the third and fourth derivative of $F(x,0)$ at $x = 1$. Continuing to differentiate (19) generates an open system of equations (*i.e.*, there are more unknowns than the number of equations) from which we cannot derive $\frac{d^2}{dx^2}F(x,0)|_{x=1}$.

The rest of the paper is devoted to the solution of (11), which will allow us to compute $\frac{d^2}{dx^2}F(x,0)|_{x=1}$ and subsequently var(N).

4 Infinite number of users

Dividing both sides of (11) and letting $k \to \infty$ yields

$$(1 - xy)F(x, y) = (1 - xy^2)F(x, 0) + (1 - x^2y)F(0, y)$$
$$+ \frac{x^2y + xy^2 - 2}{2}F(0, 0), \quad |x| \le 1, |y| \le 1. \quad (24)$$

Setting $y = 1/x$ for $|x| = 1$ (or equivalently $y = \bar{x}$) and noting that $|y| = 1$ yields (Hint: we use that $F(x, 1/x)$ is well defined for $|x| = 1$ so that the l.h.s. of (24) must vanish when $y = 1/x$)

$$0 = \left(1 - \frac{1}{x}\right)F(x, 0) + (1 - x)F(0, 1/x) + \frac{x + 1/x - 2}{2}F(0, 0), \quad |x| = 1,$$

which can be rewritten as

$$F(x, 0) = xF(0, 1/x) + \frac{1 - x}{2}F(0, 0), \quad |x| = 1. \quad (25)$$

The function $F(x, 0)$ is analytic for $|x| < 1$ and continuous for $|x| \le 1$, and the function $F^-(x) := xF(0, 1/x) + \frac{1-x}{2}F(0, 0)$ is analytic in $|x| > 1$ and continuous in $|x| \ge 1$ with a pole at $x = \infty$. Since $F(x, 0)$ and $F^-(x)$ are each other's analytic continuations from (25), Liouville's theorem implies that $F(x, 0) = P(x)$ and $xF(0, 1/x) + \frac{1-x}{2}F(0, 0) = P(x)$ for all $x \in \mathbb{C}$, where $P(x)$ is an arbitrary polynomial of degree one. With $P(x) = a_0 + a_1x$, we obtain, for $x \in \mathbb{C}$,

$$F(x, 0) = a_0 + a_1x, \quad (26)$$
$$xF(0, 1/x) + \frac{1 - x}{2}F(0, 0) = a_0 + a_1x. \quad (27)$$

Letting $x = 0$ in (26) implies that $a_0 = F(0, 0)$. On the other hand, letting $x = 1$ in (26) implies that $a_1 = F(1, 0) - F(0, 0)$. By using (15), we get that $a_1 = \frac{1}{3} - \frac{1}{2}F(0, 0)$. Hence, for $x \in \mathbb{C}$,

$$F(x, 0) = F(0, 0) + \left(\frac{1}{3} - \frac{1}{2}F(0, 0)\right)x, \quad (28)$$
$$xF(0, 1/x) + \frac{1 - x}{2}F(0, 0) = F(0, 0) + \left(\frac{1}{3} - \frac{1}{2}F(0, 0)\right)x. \quad (29)$$

It remains to identify $F(0, 0)$. Dividing (29) by x and letting $x \to \infty$ yields

$$F(0, 0) = \frac{1}{3}, \tag{30}$$

so that

$$F(x, 0) = \frac{1}{3} + \frac{x}{6}, \quad x \in \mathbb{C}. \tag{31}$$

Introducing now (30) and (31) into (24) and using the symmetry property (13) yields

$$F(x, y) = \frac{2 + x + y - xy^2 - x^2y - 2x^2y^2}{6(1 - xy)} = \frac{1}{3} + \frac{x + y}{6} + \frac{xy}{3}, \quad x \in \mathbb{C}, y \in \mathbb{C}. \tag{32}$$

From (31), we find

$$\frac{d}{dx} F(x, 0)|_{x=1} = \frac{1}{6},$$

which agrees with (20) when $k \to \infty$, and $\frac{d^2}{dx^2} F(x, 0)|_{x=1} = 0$, which implies from (23) that

$$\mathrm{var}(N) = \frac{2}{3}, \tag{33}$$

when $k \to \infty$.

Discussion: The interest of the approach used to derive (32) lies in the fact that it allows us to solve the functional equation (11) when $k = \infty$. There is, however, a more direct way to obtain $F(x, y)$ when $k = \infty$, which we now discuss. It is based on the intuitive argument that when k is infinite, only the states $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$ are (positive) recurrent states. To see why this is true, let us consider the discrete-time Markov chain (DTMC) obtained by uniformizing, for finite k, the continuous-time Markov chain whose nonzero transitions are displayed in Fig. 2. Nonzero transitions of this DMTC are obtained from the nonzero transition rates of the CTMC $\{(X(t), Y(t)), t \geq 0\}$ found in Sect. 2 and are given by

$$(0, 0) \to (1, 0) \quad \text{with probability } 1/2,$$
$$(0, 0) \to (0, 1) \quad \text{with probability } 1/2,$$
$$\text{For } i \geq 1: \quad (i, 0) \to (i, 1) \quad \text{with probability } (k - 1)/k,$$
$$\text{For } j \geq 1: \quad (0, j) \to (1, j) \quad \text{with probability } (k - 1)/k,$$
$$\text{For } i \geq 1: \quad (i, 0) \to (i + 1, 0) \quad \text{with probability } 1/k,$$
$$\text{For } j \geq 1: \quad (0, j) \to (0, j + 1) \quad \text{with probability } 1/k,$$
$$\text{For } i \geq 1, j \geq 1: (i, j) \to (i, j + 1) \quad \text{with probability } 1/k,$$
$$\text{For } i \geq 1, j \geq 1: (i, j) \to (i + 1, j) \quad \text{with probability } 1/k,$$
$$\text{For } i \geq 1, j \geq 1: (i, j) \to (i - 1, j - 1) \quad \text{with probability } (k - 2)/k.$$

Letting $k \to \infty$ in the above yields the nonzero transition rates

$$
\begin{aligned}
(0, 0) &\to (1, 0) \quad \text{with probability } 1/2, \\
(0, 0) &\to (0, 1) \quad \text{with probability } 1/2, \\
\text{For } i \geq 1: \quad (i, 0) &\to (i, 1) \quad \text{with probability } 1, \\
\text{For } j \geq 1: \quad (0, j) &\to (1, j) \quad \text{with probability } 1, \\
\text{For } i \geq 1, j \geq 1: (i, j) &\to (i-1, j-1) \quad \text{with probability } 1.
\end{aligned}
$$

It is easy to see that starting from any state (i, j) out of the set $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$ the DTMC will enter that set in a finite number of time steps. (Starting from (i, i) with $i \geq 2$, the DTMC will join state $(1, 1)$ in $i - 1$ time steps, and starting from (i, j) with $i > j \geq 0$, the DTMC will join state $(1, 0)$ in $2(i - 1) - j$ time steps.) Consider state $(0, 0)$. From it the DTMC can either follow the path $(1, 0) \to (1, 1) \to (0, 0)$ with probability $1/2$ or the path $(0, 1) \to (1, 1) \to (0, 0)$ with probability $1/2$. We conclude that the stationary probabilities of states $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$ are $1/3$, $1/6$, $1/6$, and $1/3$, respectively, thereby yielding $F(x, y) = \frac{1}{3} + \frac{x+y}{6} + \frac{xy}{3}$, in agreement with (32). It is worth noting that this argument can be extended to a multipartite (and not only tripartite) entanglement distribution switch.

5 Finite number of users

In this section, we solve the functional equation (11) when k is finite with $k > 3$ (stability condition). Throughout, for any complex number z, $\bar{z}$ denotes its complex conjugate and $\Re(z)$ its real part.

5.1 Preliminary results

Define the set

$$
\mathcal{L} = \{ w \in \mathbb{C}, |w| \leq 1 : w|w|^2 + \overline{w}|w|^2 - k|w|^2 + k - 2 = 0 \}. \tag{34}
$$

Lemma 1 (The contour $\mathcal{L}$) *$\mathcal{L}$ is a smooth closed contour that does not intersect with itself, is symmetric with respect to the real axis, and has the representation*

$$
\mathcal{L} = \left\{ w \in \mathbb{C}, |w| \leq 1 : w = \rho(\theta)e^{i\theta}, -\pi \leq \theta \leq \pi \right\}, \tag{35}
$$

with $\rho(\theta)$ the unique zero in $(0, 1]$ of the polynomial in the variable ρ defined by $2\rho^3 \cos\theta - k\rho^2 + k - 2$.

The function $\rho(\theta)$ is periodic with period 2π and is infinitely differentiable on $\mathbb{R}$, $\rho(\theta) = \rho(-\theta)$ for all θ, and

$$
\rho(0) = 1, \quad \rho'(0) = 0, \quad \rho''(0) = \frac{1}{3 - k}, \quad \rho'''(0) = 0. \tag{36}
$$

The proof is in Appendix B.

Let γ be the conformal mapping of the unit disk $\mathcal{D} := \{z \in C : |z| < 1\}$ onto the interior of the contour $\mathcal{L}$, denoted by $\mathcal{L}^+$. This mapping is uniquely determined by the conditions $\gamma(0) = 0$ and $\frac{d}{dz}\gamma(z) > 0$ at $z = 0$ [27, Theorem 1.2, 1.3]. By Theorem 2.24 in [27], the conformal mapping $\gamma(z)$ is continuous in the region $\{z \in \mathbb{C} : |z| \le 1\}$ and maps the unit circle $\mathcal{C}$ one-to-one onto the contour $\mathcal{L}$. Moreover, the symmetry of $\mathcal{L}$ leads to the following property: for $|z| \le 1$,

$$\gamma(\bar{z}) = \overline{\gamma(z)}. \tag{37}$$

The one-to-one conformal mapping from $\mathcal{C}$ onto $\mathcal{L}$ is given by

$$\gamma(t) = \rho(\phi(\theta))e^{i\phi(\theta)}, \tag{38}$$

for $t = e^{i\theta}$, where $\phi(\theta)$ is the unique continuous and strictly increasing mapping in $[-\pi, \pi]$ solution of the integral equation

$$\phi(\theta) = \theta - \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(\rho(\phi(u))) \cot\left(\frac{u-\theta}{2}\right) du, \quad \theta \in [-\pi, \pi]. \tag{39}$$

Note that (39) can be rewritten as

$$\phi(\theta) = \theta - \frac{1}{2\pi} \int_{-\pi}^{\pi} (\log(\rho(\phi(u))) - \log(\rho(\phi(\theta)))) \cot\left(\frac{u-\theta}{2}\right) du, \quad \theta \in [-\pi, \pi], \tag{40}$$

since $\forall \theta \in [-\pi, \pi]$ the singular integral $\int_{-\pi}^{\pi} \cot\left(\frac{u-\theta}{2}\right) du$ is equal to 0 in the sense of Cauchy principal value [11, Chapter I.1.5.] (Hint: $2\log\left(\sin\left(\frac{u-\theta}{2}\right)\right)$ is a primitive of $\cot\left(\frac{u-\theta}{2}\right)$, yielding $\lim_{\epsilon \to 0}\left(\int_{\pi}^{\theta-\epsilon} \cot\left(\frac{u-\theta}{2}\right) d\theta + \int_{\theta+\epsilon}^{\pi} \cot\left(\frac{u-\theta}{2}\right) d\theta\right) = 0$).

Since $-\phi(-\theta)$ is also a solution of (39), the uniqueness of the solution implies that, for all $\theta \in [-\pi, \pi]$,

$$\phi(\theta) = -\phi(-\theta). \tag{41}$$

In particular, (41), implies that $\phi(0) = 0$, which in turn yields

$$\gamma(1) = 1, \tag{42}$$

from (38) together with $\rho(0) = 1$ (cf. Lemma 1). Also, it is a simple exercise to show that $\phi(-\pi) = -\pi$ and $\phi(\pi) = \pi$ (Hint: $\cot\left(\frac{u\pm\pi}{2}\right) = -\tan(u/2)$ and $\rho(\phi(u)) = \rho(\phi(-u))$ from (41), and $\rho(\theta) = \rho(-\theta)$).

The conformal mapping from $\mathcal{D}$ onto $\mathcal{L}^+$ is given by

$$\gamma(t) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\gamma(u)}{u-t} du = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\rho(\phi(\theta))}{e^{i\theta} - t} e^{i\phi(\theta)+i\theta} d\theta, \quad |t| < 1. \tag{43}$$

The part between (38) and (43) is taken from [16, Chapter II] (see also [6, Chapters I.6 and IV.3]).
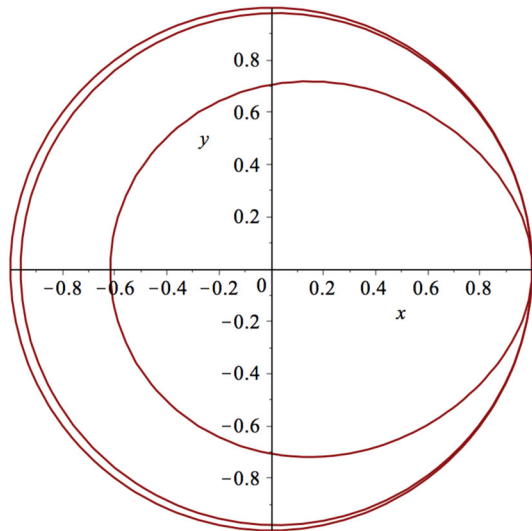
We will also need the inverse of the conformal mapping γ, called γ_0. Fortunately the mapping γ_0 can be obtained in explicit form from [7, Section 7]. To this end, note that the contour $L(1, 0)$ in [7, Eq. (7)] reduces to the contour $\mathcal{L}$ in (35) upon setting $\beta(s) = \frac{k-2}{k-2+s}$ and $\alpha = \frac{1}{2}$ so that the constant a in [7, Eq. (5)]) is given by $a = \frac{2}{k-2}$. Then, by [7, Section 7],

$$\gamma_0(w) = 1 - \frac{2\delta}{w} \frac{(1-w)^2}{(1-\delta)^2} \left(\frac{k-2-w}{k-2-\delta} \right) \left[1 + \frac{1}{\delta} \left(\frac{w-\delta}{1-w} \right) \sqrt{\frac{k-2-w\delta^2}{k-2-w}} \right],$$

$$\text{for } w \in \mathcal{L}, \tag{44}$$

with $\delta := \frac{k-2-\sqrt{(k-2)^2+8(k-2)}}{4}$, and for $w \in \mathcal{L}^+$

$$\gamma_0(w) = \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\gamma_0(u)}{u - w} du. \tag{45}$$

On the other hand, the r.h.s. of (44) can be analytically continued up to the region $\{w \in \mathbb{C} : |w| < k - 2\}$, which shows (in particular) that $\gamma_0(w)$ is analytic for $|w| < 1$ and continuous for $|w| \leq 1$. The mapping γ_0 is given by (45) for $w \in \mathcal{L}^+$ and by (44) for $w \in \{w \in C : |w| \leq 1\} - \mathcal{L}^+$. This property will be used in the proof of Proposition 1.

Last,

$$\gamma_0(0) = 0 \quad \text{and} \quad \gamma_0(1) = 1, \tag{46}$$

which follows from $\gamma(0) = 0$, $\gamma(1) = 1$ and $\gamma_0(\gamma(t)) = t$ for $t \in C^+ \cup C$.

The contour $\mathcal{L}$ is depicted in Fig. 3 for $k = 4$ and $k = 50$. Not surprisingly, we note that as k becomes large, the contour $\mathcal{L}$ becomes close to the unit circle. This result follows from the fact that, for all θ, $\rho(\theta) \to 1$ as $k \to \infty$ (Hint: write the polynomial $Q(\rho) := 2\rho^3 \cos\theta - k\rho^2 + k - 2$ as $Q(\rho) = k(2\rho^3 \cos\theta/k - \rho^2 + 1 - 2/k)$; for any θ, as $k \to \infty$, the polynomial $Q(\rho)$ may vanish only when $\rho \to 1$, which in turn implies from (39) that $\phi(\theta) \to \theta$.

5.2 Reduction of (11) to a boundary value problem

Take $w \in \mathcal{L}$ and notice that $K(w, \overline{w}) = 0$. Since $F(w, \overline{w})$ is well defined for $w \in \mathcal{L}$ (since $|w| \le 1$ and $|\overline{w}| \le 1$ when $w \in \mathcal{L}$), the left-hand side of (11) must vanish when $(x, y) = (w, \overline{w})$. This generates the relation

$$(1-\overline{w}|w|^2)F(w, 0)+(1-w|w|^2)F(0, \overline{w})+(\Re(w)|w|^2-1)F(0, 0) = 0, \quad \text{for } w \in \mathcal{L}. \tag{47}$$

Because $1 - \overline{w}|w|^2 = 0$ if and only if $w = 1$, we can divide both sides of (47) by $1 - \overline{w}|w|^2$ when $w \ne 1$ to obtain

$$F(w, 0) = -\left(\frac{1 - w|w|^2}{1 - \overline{w}|w|^2}\right) F(0, \overline{w})+\left(\frac{1 - \Re(w)|w|^2}{1 - \overline{w}|w|^2}\right) F(0, 0), \quad \text{for } w \in \mathcal{L}-\{1\}. \tag{48}$$

On the other hand,

$$\lim_{\substack{w \to 1 \\ w \in \mathcal{L}}} \frac{1 - w|w|^2}{1 - \overline{w}|w|^2} = \lim_{\theta \to 0} \frac{1 - \rho(\theta)^3(\cos\theta + i\sin\theta)}{1 - \rho(\theta)^3(\cos\theta - i\sin\theta)} = -1,$$

by L'Hôpital's rule (Hint: $\rho(0) = 1$ and $\frac{d}{d\theta}\rho(\theta)|_{\theta=0} = 0$ by Lemma 1), and

$$\lim_{\substack{w \to 1 \\ w \in \mathcal{L}}} \frac{1 - \Re(w)|w|^2}{1 - \overline{w}|w|^2} = \lim_{\theta \to 0} \frac{1 - \rho(\theta)^3 \cos\theta}{1 - \rho(\theta)^3(\cos\theta - i\sin\theta)} = 0,$$

by again using L'Hôpital's rule. This shows that the coefficients of $F(w, 0)$ and $F(0, \overline{w})$ in (48) are well defined when $w = 1$, which implies that (49) can be analytically continued up to $w = 1$, to give

$$F(w, 0) = -\left(\frac{1 - w|w|^2}{1 - \overline{w}|w|^2}\right) F(0, \overline{w}) + \left(\frac{1 - \Re(w)|w|^2}{1 - \overline{w}|w|^2}\right) F(0, 0), \quad \text{for } w \in \mathcal{L}. \tag{49}$$

Take $|t| = 1$. Letting $w = \gamma(t)$ in (49) and using $\overline{w} = \gamma(1/t)$ by (37) gives

$$F(\gamma(t), 0) = -G(t)F(0, \gamma(1/t)) + g(t)F(0, 0), \quad \forall |t| = 1, \tag{50}$$

where

$$G(t) := -\frac{1 - \gamma(t)|\gamma(t)|^2}{1 - \overline{\gamma(t)}|\gamma(t)|^2}, \tag{51}$$

$$g(t) := \frac{1 - \Re(\gamma(t))|\gamma(t)|^2}{1 - \overline{\gamma(t)}|\gamma(t)|^2}. \tag{52}$$

A key observation is that $F(\gamma(t), 0)$ is regular for $|t| < 1$ and $F(0, \gamma(1/t))$ is regular for $|t| > 1$ since $\mathcal{L}^+ \subset \{z \in \mathbb{C} : |z| < 1\}$. As a result, (50) defines a *non-homogeneous Hilbert Boundary Value Problem* (BVP) on $\mathcal{C}$ [11,14,17,28], which can be solved provided the functions G and g fulfill certain conditions (see the proof of Proposition 1).

Section 8 discusses the technique of reducing the solution of certain two-dimensional functional equations (Eq. (11) in our case) to the solution of a boundary value problem (typically a Riemann–Hilbert or Dirichlet problem).

Recall that $\arg[\varphi(z)]_\mathcal{C}$ denotes the variation of the argument of $\varphi(z)$ when z describes the unit circle. Introduce

$$\chi := \frac{1}{2\pi} \arg[G(t)]_\mathcal{C} \in \mathbb{Z}, \tag{53}$$

the index of the BVP defined in (50). The integer number χ gives the increment (positive or negative) of the argument of $G(t)$ when t describes the unit circle once.

Define

$$I_1 := \exp\left(\frac{1}{2\pi i} \int_\mathcal{C} \frac{\log(u^{-1}G(u))}{u - 1} du\right), \tag{54}$$

$$I_2 := \exp\left(\frac{1}{2\pi i} \int_\mathcal{C} \frac{\log(u^{-1}G(u))}{u} du\right), \tag{55}$$

$$I_3 := \frac{1}{2\pi i} \int_\mathcal{C} \frac{g(u)e^{-H(u)}}{u - 1} du, \tag{56}$$

$$I_4 := \frac{1}{2\pi i} \int_\mathcal{C} \frac{g(u)e^{-H(u)}}{u} du. \tag{57}$$

It is shown in Appendix C that $I_2 = 1$.

Proposition 1 gives $F(x, 0)$ for $x \in \mathcal{L}^+ \cup \mathcal{L}$. The function $F(0, x)$ for $x \in \mathcal{L}^+ \cup \mathcal{L}$ is deduced from it since $F(x, 0) = F(0, x)$ for all $|x| \leq 1$.

Proposition 1 ($F(x, 0)$ for $x \in \mathcal{L}^+ \cup \mathcal{L}$) *For $x \in \mathcal{L}^+$,*

$$F(x, 0) = e^{\frac{1}{2\pi i} \int_\mathcal{C} \frac{\log(u^{-1}G(u))}{u - \gamma_0(x)} du} \left(\frac{F(0,0)}{2\pi i} \int_\mathcal{C} \frac{g(u)e^{-H(u)}}{u - \gamma_0(x)} du + a_0 + a_1\gamma_0(x)\right), \tag{58}$$

and for $x_0 \in \mathcal{L}$,

$$
F(x_0, 0) := \lim_{\substack{x \to x_0 \\ x \in \mathcal{L}^+}} F(x, 0)
$$

$$
= e^{H(t_0)} \left(\frac{F(0,0)}{2} g(\gamma_0(x_0)) e^{-H(\gamma_0(x_0))} + \frac{F(0,0)}{2\pi i} \int_{\mathcal{C}} \frac{g(u) e^{-H(u)}}{u - \gamma_0(x_0)} du \right.
$$

$$
\left. + a_0 + a_1 \gamma_0(x) \right), \tag{59}
$$

where $G(t)$ and $g(t)$ are given in (51) and (52), $\gamma_0(x)$ is given in (44) (resp. (45)) for $x \in \mathcal{L}$ (resp. $x \in \mathcal{L}^+$), and

$$
H(t) := \frac{1}{2} \log \left(t^{-1} G(t) \right) + \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\log(u^{-1} G(u))}{u - t} du, \quad t \in \mathcal{C}. \tag{60}
$$

The constants $F(0,0)$, a_0, and a_1 in (58)–(59) are given by

$$
F(0,0) = \Delta^{-1} \frac{2(k-3)}{3(k-2)} I_2, \tag{61}
$$

$$
a_0 := \Delta^{-1} \frac{2(k-3)}{3(k-2)} (1 - I_2 I_4), \tag{62}
$$

$$
a_1 := F(0,0), \tag{63}
$$

with $\Delta := 2I_1 I_2 (I_3 - I_4) + 2I_1(1 + I_2) - I_2$.
  Also,

$$
F(1,0) = \Delta^{-1} \frac{2(k-3)}{3(k-2)} I_1 (I_2 (I_3 - I_4) + I_2 + 1). \tag{64}
$$

Proof It is shown in Lemma 4 in Appendix E that the index χ [defined in (53)] of the BVP defined (50) is equal to 1. For $t \in \mathcal{C}$, define

$$
H(t) = \frac{1}{2} \log \left(t^{-1} G(t) \right) + \frac{1}{2\pi i} \int_{\mathcal{C}} \log(u^{-1} G(u)) \frac{du}{u - t}.
$$

It is shown in Lemma 3 in Appendix D that $G(t) \neq 0$ for $t \in \mathcal{C}$ and that the mappings $\log(t^{-1} G(t))$, $g(t)$ and $g(t) e^{-H(t)}$ satisfy a Hölder condition on $\mathcal{C}$ (as continuous functions on this contour). Hence, the solution of the BVP in (50) is given by [11, Part I.2], [14, Chapter 5], [17, Chapter II], [28, Chapter 5]

$$
F(\gamma(t), 0) = e^{\Gamma(t)} (\Psi(t) + P(t)), \quad |t| < 1, \tag{65}
$$

$$
F(0, \gamma(1/t)) = t^{-1} e^{\Gamma(t)} (\Psi(t) + P(t)), \quad |t| > 1, \tag{66}
$$

$$
\lim_{\substack{t \to t_0 \\ t \in \mathcal{C}^+}} F(\gamma(t), 0) = e^{\Gamma^+(t_0)} (\Psi^+(t_0) + P(t_0)), \quad |t_0| = 1, \tag{67}
$$

$$\lim_{\substack{t \to t_0 \\ t \in \mathcal{C}^-}} F(0, \gamma(1/t)) = t_0^{-1} e^{\Gamma^-(t_0)} \left(\Psi(t_0)^- + P(t_0) \right), \quad |t_0| = 1, \tag{68}$$

with $P(t) = a_0 + a_1 t$ a polynomial of degree one, and

$$\Psi(t) := \frac{F(0,0)}{2\pi i} \int_{\mathcal{C}} g(u) e^{-H(u)} \frac{du}{u-t}, \quad |t| \neq 1,$$

$$\Psi^+(t) := \frac{1}{2} F(0,0) g(t) e^{-H(t)} + \Psi(t), \quad |t| = 1,$$

$$\Psi^-(t) := -\frac{F(0,0)}{2} g(t) e^{-H(t)} + \Psi(t), \quad |t| = 1,$$

$$\Gamma(t) := \frac{1}{2\pi i} \int_{\mathcal{C}} \log(u^{-1} G(u)) \frac{du}{u-t}, \quad |t| \neq 1,$$

$$\Gamma^+(t_0) := \lim_{\substack{t \to t_0 \\ |t| < 1}} \Gamma(t) = \frac{1}{2} \log \left(t_0^{-1} G(t) \right) + \Gamma(t_0) = H(t_0), \quad |t_0| = 1,$$

$$\Gamma^-(t_0) := \lim_{\substack{t \to t_0 \\ |t| > 1}} \Gamma(t) = -\frac{1}{2} \log \left(t_0^{-1} G(t_0) \right) + \Gamma(t_0), \quad |t| = 1.$$

Letting $t = \gamma_0(x)$ in (65) and in (67), we obtain (58) and (59), respectively. Note that the singular integrals in (59) and (60) are defined in the sense of their Cauchy principal value [11, Section I.1.5.].

It remains to find the unknown constants $F(0,0)$, a_0, and a_1. Letting $x = 0$ in (58) and $x = 1$ in (59) and using the identities $\gamma_0(0) = 0$ and $\gamma_0(1) = 1$ (cf. (46)), $G(1) = 1$, and $g(1) = 0$ (cf. Lemma 3) gives the equations

$$(1 - I_2 I_4) F(0,0) - I_2 a_0 = 0, \tag{69}$$

$$F(1,0) - I_1 I_3 F(0,0) - I_1 a_0 - I_1 a_1 = 0. \tag{70}$$

Two additional equations are needed to find the unknowns $a_0, a_1, F(0,0)$, and $F(1,0)$. One is provided by (15), namely

$$2F(1,0) - F(0,0) = \frac{2(k-3)}{3(k-2)}. \tag{71}$$

Another one comes from (66). Letting $t \to \infty$ in (66) and using $\gamma(0) = 0$ generates the equation

$$F(0,0) = a_1. \tag{72}$$

Letting $a_1 = F(0,0)$ in (70) implies that Eqs. (69)–(71) define a linear system of three equations with three unknowns $F(0,0)$, a_0, and $F(1,0)$, whose unique solution is given in (61), (62) and (64). This concludes the proof. $\qquad\square$

The functional equation (11) and Proposition 1 yield the following corollary:

Corollary 1 (Solution of (11) in $\mathcal{L}^+ \cup \mathcal{L} \times \mathcal{L}^+ \cup \mathcal{L}$) *For $x, y \in \mathcal{L}^+ \cup \mathcal{L}$,*

$$F(x, y) = \frac{k-2}{x^2 y + xy^2 - kxy + k - 2}$$
$$\left((1 - xy^2) F(x, 0) + (1 - x^2 y) F(0, y) + \left(\frac{x^2 y + xy^2}{2} - 1 \right) F(0, 0) \right),$$
$$(73)$$

where $F(x, 0)$ is given in (58) (resp. (59)) for $x \in \mathcal{L}^+$ (resp. $x \in \mathcal{L}$) and $F(0, y) = F(y, 0)$ for all $y \in \mathcal{L}^+ \cup \mathcal{L}$.

  In particular, the r.h.s. of (73) gives $F(x, y)$ for all $x, y \in [0, 1]$ as $[0, 1] \subset \mathcal{L}^+ \cup \mathcal{L}$.

Because $F(x, 0)$ is not analytic for $x \in \mathcal{L}$ due to the presence of the terms $\frac{1 - \Re(x)|x|^2}{1 - \bar{x}|x|^2}$ and $\frac{1 - x|x|^2}{1 - \bar{x}|x|^2}$ in (59), the r.h.s. of (73) cannot be analytically extended to the entire region $\{x \in \mathbb{C}, y \in \mathbb{C} : |x| \leq 1, |y| \leq 1\}$, meaning that we do not have an explicit expression for $F(x, y)$ when $(x, y) \in \{x \in \mathbb{C}, y \in \mathbb{C} : |x| \leq 1, |y| \leq 1\} - \{x \in \mathbb{C}, y \in \mathbb{C} : x \in \mathcal{L}^+ \cup \mathcal{L}, y \in \mathcal{L}^+ \cup \mathcal{L}\}$. Fortunately, since the performance metrics of interest can be derived from $F(x, y)$ when $x, y \in [0, 1]$ (see Sect. 7), not knowing $F(x, y)$ in explicit form in the entire region $\{x \in \mathbb{C}, y \in \mathbb{C} : |x| \leq 1, |y| \leq 1\}$ is not problematic.

6 Variance of the number of stored qubits at the switch

This section is devoted to the calculation of the unknown term $\frac{d^2}{dx^2} F(x, 0)|_{x=1}$ in the formula for the variance in (23). To this end, we will use (58) in Proposition 1. However, this calculation requires care as both integrals in the r.h.s. of (58) converge to singular integrals as $x \to 1$.

  For the sake of completeness, we report below known results on Cauchy type integrals that we will use in this calculation. This material can be found in [11, Section I.1.], [17, Chapter 1], and [28, Chapters 1, 2]. All results are stated for the unit circle $\mathcal{C}$ as this is what we need, but they hold for any smooth, non-self-intersecting, and closed contour.

  For any $t_0 \in \mathcal{C}$, the singular integral $\frac{1}{2\pi i} \int_{\mathcal{C}} \frac{du}{u - t_0}$ exists in the sense of its Cauchy principal values[1][11, p. 28, formula (5.4)] and is given by

$$\frac{1}{2\pi i} \int_{\mathcal{C}} \frac{du}{u - t_0} = \frac{1}{2}. \tag{74}$$

Define $\Phi(t) := \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\phi(u)}{u - t} du$ for $t \notin \mathcal{C}$. If $\phi(t)$ satisfies a Hölder condition on $\mathcal{C}$, then the following so-called Plemelj–Sokhotski formulas hold:

---

[1] $\frac{1}{2\pi i} \int_{\mathcal{C}} \frac{du}{u - t_0} = \lim_{r \downarrow 0} \frac{1}{2\pi i} \int_{\mathcal{C}-l} \frac{du}{u - t_0}$, where l is that part of $\mathcal{C}$ cut from $\mathcal{C}$ by a small circle with center at t_0 and radius r.

$$\Phi^+(t_0) := \lim_{\substack{t \to t_0 \\ t \in \mathcal{C}^+}} \Phi(t) = \frac{1}{2}\phi(t_0) + \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\phi(u)}{u - t} du, \tag{75}$$

$$\Phi^-(t_0) := \lim_{\substack{t \to t_0 \\ t \in \mathcal{C}^-}} \Phi(t) = -\frac{1}{2}\phi(t_0) + \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\phi(u)}{u - t} du. \tag{76}$$

With the help of (74), we can rewrite $\Phi^+(t_0)$ as (a similar formula holds for $\Phi^-(t_0)$, which we will not need)

$$\Phi^+(t_0) = \phi(t_0) + \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\phi(u) - \phi(t_0)}{u - t_0} du. \tag{77}$$

If $\phi^{(m)}$, the m-th derivative of ϕ on $\mathcal{C}$, exists, then, for $t \notin \mathcal{C}$,

$$\Phi^{(m)}(t) := \frac{d^m}{dt^m} \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\phi(u)}{u - t} du = \frac{m!}{2\pi i} \int_{\mathcal{C}} \frac{\phi(u)}{(u - t)^{m+1}} du \tag{78}$$

$$= \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\phi^{(m)}(u)}{u - t} du, \tag{79}$$

for $m = 1, 2, \ldots$; if in addition $\phi^{(m)}$ satisfies a Hölder condition on $\mathcal{C}$, the Plemelj–Sokhotski formula (75) applies, to give, for $t_0 \in \mathcal{C}$,

$$[\Phi^{(m)}(t_0)]^+ := \lim_{\substack{t \to t_0 \\ t \in \mathcal{C}^+}} \Phi^{(m)}(t) = \frac{1}{2}\phi^{(m)}(t_0) + \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\phi^{(m)}(u)}{u - t_0} du \tag{80}$$

$$= \phi^{(m)}(t_0) + \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\phi^{(m)}(u) - \phi^{(m)}(t_0)}{u - t_0} du, \tag{81}$$

for $m = 1, 2, \ldots$, where (81) follows from (74). Notice that the integrals in (77) and (81) are well defined as Riemann improper integrals.

We are now in position to calculate $\frac{d^2}{dx^2} F(x, 0)|_{x=1}$. For $t \in \mathcal{C}$, define

$$\alpha(t) := \log(t^{-1} G(t)), \tag{82}$$
$$\beta(t) := g(t) e^{-H(t)}, \tag{83}$$

where G, g, and H are given in (51), (52), and (60), respectively. Introduce the integrals

$$I_5 := \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\alpha'(u) - \alpha'(1)}{u - 1} du, \tag{84}$$

$$I_6 := \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\beta'(u) - \beta'(1)}{u - 1} du, \tag{85}$$

$$I_7 := \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\alpha''(u) - \alpha''(1)}{u - 1} du, \tag{86}$$

$$I_8 := \frac{1}{2\pi i} \int_C \frac{\beta''(u) - \beta''(1)}{u - 1} du. \tag{87}$$

It is shown in Lemma 6 in Appendix F that the mappings α and β are twice differentiable on C and that their second derivatives are continuous on C, which shows that the integrals $I_5, \ldots, I_8$ are standard (and therefore finite) Riemann integrals.

Differentiating (58) w.r.t x gives, for $x \in C^+$ (Hint: $a_1 = F(0,0)$)

$$\begin{aligned}
\frac{d}{dx} F(x,0) &= \frac{\gamma_0'(x)}{2\pi i} \int_C \frac{\alpha(u)}{(u - \gamma_0(x))^2} du \times F(x,0) \\
&\quad + e^{\frac{1}{2\pi i} \int_C \frac{\alpha(u)}{u - \gamma_0(x)} du} \left(F(0,0) \frac{\gamma_0'(x)}{2\pi i} \int_C \frac{\beta(u)}{(u - \gamma_0(x))^2} du + F(0,0) \gamma_0'(x) \right) \\
&= \gamma_0'(x) \left[\frac{F(x,0)}{2\pi i} \int_C \frac{\alpha(u)}{(u - \gamma_0(x))^2} du \right. \\
&\quad + F(0,0) e^{\frac{1}{2\pi i} \int_C \frac{\alpha(u)}{u - \gamma_0(x)} du} \left(\frac{1}{2\pi i} \int_C \frac{\beta(u)}{(u - \gamma_0(x))^2} du + 1 \right) \right] \\
&= \gamma_0'(x) \left[\frac{F(x,0)}{2\pi i} \int_C \frac{\alpha'(u)}{u - \gamma_0(x)} du \right. \\
&\quad \left. + F(0,0) e^{\frac{1}{2\pi i} \int_C \frac{\alpha(u)}{u - \gamma_0(x)} du} \left(\frac{1}{2\pi i} \int_C \frac{\beta'(u)}{u - \gamma_0(x)} du + 1 \right) \right], \tag{88}
\end{aligned}$$

upon using twice (79) to establish (88). Note that (79) applies to the mappings α and β since they are differentiable on C, as mentioned above and proved in Lemma 6 in Appendix F.

By using (79) and (81), the limit of (88) as $x \to 1$ is

$$\frac{d}{dx} F(x,0)|_{x=1} = \gamma_0'(1) \left[F(1,0) \left(\alpha'(1) + I_5 \right) + F(0,0) I_1 \left(\beta'(1) + I_6 + 1 \right) \right]. \tag{89}$$

We already know that $\frac{d}{dx} F(x,0)|_{x=1} = \frac{k}{6(k-2)}$ [cf. (20)].

Differentiating (88) w.r.t $x \in \mathcal{L}^+$, we obtain after using (79) (the latter result applies to mappings α' and β' as they are differentiable on C—see Lemma 6)

$$\begin{aligned}
\frac{d^2}{dx^2} F(x,0) &= \frac{\gamma_0''(x)}{\gamma_0'(x)} \times \frac{d}{dx} F(x,0) \\
&\quad + \gamma_0'(x) \left[\frac{d}{dx} F(x,0) \times \frac{1}{2\pi i} \int_C \frac{\alpha'(u)}{u - \gamma_0(x)} du \right. \\
&\quad + F(x,0) \frac{\gamma_0'(x)}{2\pi i} \int_C \frac{\alpha''(u)}{u - \gamma_0(x)} du \\
&\quad + F(0,0) \frac{\gamma_0'(x)}{2\pi i} \int_C \frac{\alpha'(u)}{u - \gamma_0(x)} du \times e^{\frac{1}{2\pi i} \int_C \frac{\alpha(u)}{u - \gamma_0(x)} du} \\
&\quad \times \left(\frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\beta'(u)}{u - \gamma_0(x)} du + 1 \right) \\
&\quad \left. + F(0,0) e^{\frac{1}{2\pi i} \int_C \frac{\alpha(u)}{u - \gamma_0(x)} du} \times \frac{\gamma_0'(x)}{2\pi i} \int_{\mathcal{L}} \frac{\beta''(u)}{u - \gamma_0(x)} du \right]. \tag{90}
\end{aligned}$$

Finally, letting $x \rightarrow 1$ in (90) we find by using $\gamma_0(1) = 1$, (79), and (81) with $t_0 = 1$, definitions (84)–(87), and $\frac{d}{dx} F(x, 0)|_{x=1} = \frac{k}{6(k-2)}$ [cf. (20)], that

$$
\begin{aligned}
\frac{d^2}{dx^2} F(x, 0)|_{x=1} = {} & \frac{\gamma_0''(1)}{\gamma_0'(1)} \times \frac{k}{6(k-2)} \\
& + \gamma_0'(1) \Bigg[\frac{k}{6(k-2)} \left(\alpha'(1) + I_5 \right) + F(1, 0)\gamma_0'(1) \left(\alpha''(1) + I_7 \right) \\
& + F(0, 0)\gamma_0'(1) I_1 \left\{ \left(\alpha'(1) + I_5 \right) \left(\beta'(1) + I_6 + 1 \right) + \beta''(1) + I_8 \right\} \Bigg],
\end{aligned}
$$
(91)

where $F(0, 0)$ is given in (61) and $F(1, 0) = \frac{1}{2} F(0, 0) + \frac{2(k-3)}{6(k-2)}$ by (15).

7 Numerical results

In this section, we address the numerical calculation of $F(0, 0)$, the probability that the switch is empty, given in (61), and of the variance of the number of stored qubits at the switch, given in (23) and (91), when $3 < k < \infty$.

7.1 Steps in the numerical evaluation of $F(0, 0)$ and var(N)

Throughout, $\theta \in [-\pi, \pi]$ and $u = e^{i\theta}$. Below are the steps to compute $F(0, 0)$ and var(N).
Step 1: *Computation of $\alpha(u)$.*
  By (82) and (51),
$$
\alpha(u) = \log(e^{-i\theta} \tilde{G}(\phi(\theta))),
$$
(92)

with
$$
\tilde{G}(\theta) := -\frac{1 - \rho(\theta)^3 e^{i\theta}}{1 - \rho(\theta)^3 e^{-i\theta}}.
$$
(93)

In (111), $\rho(\theta)$ is the only zero in $(0, 1]$ of $2\cos(\theta)\rho^3 - k\rho^2 + k - 2$ (see Lemma 1) and in (92) $\phi(\theta)$, defined in (39), is obtained as the output of the following iterative scheme:

$$
\phi_0(\theta) := \theta,
$$
(94)
$$
\phi_{n+1}(\theta) := \theta - \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(\rho(\phi_n(\omega))) \cot\left(\frac{\omega - \theta}{2}\right) d\omega, \quad n = 0, 1, \dots . \text{ (95)}
$$

Step 2: *Computation of $\alpha'(u)$ and $\alpha''(u)$.*
  The mappings $\alpha'(u)$ and $\alpha''(u)$ are given in (113)–(114), with $\tilde{G}'(\theta)$ given in (117) and $\tilde{G}''(\theta)$ given in (120). The first and second derivatives of $\rho(\theta)$ showing up in (117) and (120) are obtained by differentiating the equation $2\cos(\theta)\rho(\theta)^3 - k\rho(\theta)^2 + k = 0$,

which gives

$$\rho'(\theta) = \frac{\rho(\theta)^2 \sin\theta}{3\rho(\theta)\cos\theta - k}, \tag{96}$$

$$\rho''(\theta) = \frac{\rho(\theta)^2}{3\rho(\theta)\cos\theta - k}\left(\cos\theta + \frac{5\rho(\theta)\sin(\theta)^2}{3\rho(\theta)\cos\theta - k} - \frac{3\rho(\theta)^2\sin(\theta)^2\cos\theta}{(3\rho(\theta)\cos\theta - k)^2}\right), \tag{97}$$

where we used (96) to obtain (97). The first and second derivatives of $\phi(\theta)$ in (113)–(114) could be obtained by differentiating the defining equation (39). However, since the mapping γ_0 is known explicitly [see (44)], it is more computationally efficient to differentiate the identity $\gamma_0(\gamma(w)) = w$ with $w = \rho(\phi(\theta))e^{i\phi(\theta)} \in \mathcal{L}$ (see Sect. 5.1), to obtain

$$\phi'(\theta) = \frac{ie^{i(\theta - \phi(\theta))}}{\rho'(\phi(\theta)) + i\rho(\phi(\theta))} \times \frac{1}{\gamma_0'(\rho(\phi(\theta))e^{i\phi(\theta)})}, \tag{98}$$

and

$$\phi''(\theta) = -\frac{e^{i(\theta - \phi(\theta))}}{(\rho'(\phi(\theta)) + i\rho(\phi(\theta)))^2 \gamma_0'\left(\rho(\phi(\theta))e^{i\phi(\theta)}\right)^2}$$

$$\times\left[e^{i\theta}\phi'(\theta)\left(-i\rho(\phi(\theta))^2 - 2\rho'(\phi(\theta))\rho(\phi(\theta)) + i\rho'(\phi(\theta))^2\right)\right.$$

$$\times\gamma_0''\left(\rho(\phi(\theta))e^{i\phi(\theta)}\right)$$

$$+ \left(-2\phi'(\theta)\rho'(\phi(\theta)) + i\phi'(\theta)\rho''(\phi(\theta)) - i\phi'(\theta)\rho(\phi(\theta))\right.$$

$$\left.\left. + i\rho(\phi(\theta)) + \rho'(\phi(\theta))\right)\gamma_0'\left(\rho(\phi(\theta))e^{i\phi(\theta)}\right)\right]. \tag{99}$$

Step 3: *Computation of $\beta(u)$.*

By (83) and (52),

$$\beta(u) = \tilde{g}(\phi(\theta))e^{-H(u)}, \tag{100}$$

where $\tilde{g}$ is given in (112) and H is given in (125).

Step 4: *Computation of $\beta'(u)$ and $\beta''(u)$.*

The mappings $\beta'(u)$ and $\beta''(u)$ are given in (115) and (116), respectively, with the mappings H' and H'' given in (126) and (127), respectively.

Step 5: *Computation of integrals $I_1, I_3, \ldots, I_8$.*

Since the integrands in $I_5, \ldots, I_8$ in (84)–(87) are well defined on $\mathcal{C}$, these integrals can be computed without any problem by using (113)–(116). After easy but lengthy algebra, the constants $\alpha'(1)$ in (84) and $\alpha''(1)$ in (86) have been found to be

$$\alpha'(1) = \phi'(0)\frac{k}{k-3} - 1 \quad \text{and} \quad \alpha''(1) = 1 - \phi'(0)\frac{k}{k-3} + i\phi''(0)\frac{k}{k-3}.$$

The constants $\beta'(1)$ in I_6 and $\beta''(1)$ in I_8 are calculated from (115) and (116), respectively.

Fig. 4 Solid line: Prob. switch memory is empty as a function of number of links k for $k \geq 4$. Dotted line: $P(\text{empty system}) = \frac{1}{3}$ when $k = \infty$—see (30)
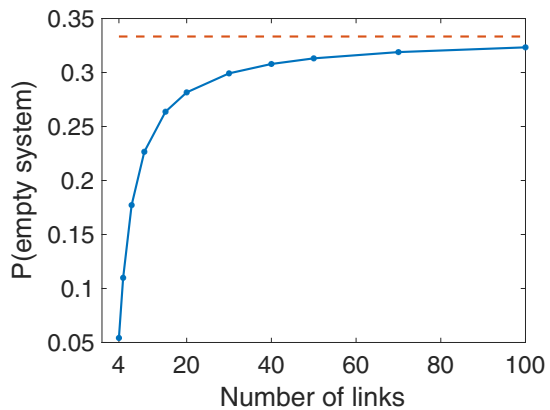
The integrand in I_1 in (54) is well defined on $\mathcal{C}$ since $G(1) = 0$ by Lemma 3 in Appendix D, so that it can be evaluated using (92) since I_1 can be rewritten as $I_1 = \exp\left(\frac{1}{2\pi} \int_{\mathcal{C}} \frac{\alpha(u)}{u-1} du\right)$. The integrand in I_3 in (56) is also well defined as $g(1) = 0$ by Lemma 3 in Appendix D, and can be easily evaluated using (100) since I_3 can be rewritten as $I_3 = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\beta(u)}{u-1} du$. Last, I_4 in (57) is computed by using (100) since I_4 can be rewritten as $I_4 = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\beta(u)}{u} du$.

Steps 1–5 allow us to compute $F(0, 0)$ in (61) and $\text{var}(N)$ in (23). (Recall that $I_2 = 1$, as shown in Appendix C.)

7.2 Plots $k \to \{F(0, 0), \mathbb{E}[N], \text{var}(N)\}$

We recall that N is the (stationary) number of qubits stored at the switch. The mappings $k \to F(0, 0) = P(\text{empty system})$, $k \to \mathbb{E}[N] = \frac{k}{k-3}$ [see (21)], and $k \to \text{var}(N)$ are displayed in Figs. 4, 5 and 6, respectively. On each figure, the dotted line is the corresponding asymptotic result when $k = \infty$ ($F(0, 0) = 1/3$ and $\text{var}(N) = 2/3$—see Sect. 4, and $\mathbb{E}[N] = 1$).

We observe that beyond a certain threshold for the number of users (say $k = 40$ in each figure) the performance metrics do not evolve much. In contrast, below this threshold there are sharp variations in the performance metrics.

The program used to generate Figs. 4 and 6 is available at https://github.com/gvardoyan/TripartiteSwitch.

8 Related works and specificities of equation (11)

The technique of reducing the solution of certain two-dimensional functional equations (Eq. (11) in our case) to the solution of a boundary value problem (typically a Riemann–Hilbert or Dirichlet problem)—whose solution is known in closed form—is due to Fayolle and Iasnogorodski [15] for polynomial kernels (in (11) the kernel is the coefficient of $F(x, y)$, namely, $K(x, y)$), and to Cohen and Boxma [10,11] and Blanc

[6] for Poisson kernels (*i.e.*, kernels of the type $xy - r\beta(\lambda(1 - r_1 x - r_2 y))$, $|r| \leq 1$, $\lambda > 0$, $r_1 \geq 0$, $r_2 \geq 0$, $r_1 + r_2 \leq 1$, where $\beta(s)$ is the Laplace–Stieltjes transform of some probability distribution on $[0, \infty)$) via the use of uniformization techniques.

Subsequent works, involving either polynomial or Poisson kernels, include [2,7,8, 13,14] (non-exhaustive list).

Let us now discuss some specificities of the functional equation (11). Its kernel, $K(x, y)$, is a polynomial of degree two in each variable. The standard approach, invented by Fayolle and Iasnogorodski [15], would consist in finding the algebraic solution $y = Y(x)$ (resp. $x = X(y)$) of the equation $K(x, Y(x)) = 0$ (resp. $K(X(y), y) = 0$). Doing so, we find

$$Y(x) = \frac{x(k - x) \pm \sqrt{x(x(x - k)^2 - 4(k - 2))}}{2x}, \quad x \in \mathbb{C}.$$

$Y(x)$ has four branch real points inside the unit circle given by the four zeros of the discriminant $x(x(x - k)^2 - 4(k - 2))$, two zeros in $[0, 1)$, and two zeros in $(1, \infty)$. The branch points in $[0, 1)$ are $a_1 = 0$ and $a_2 = (k + 2 - \sqrt{(k - 2)(k + 6)})/2$. Similar results hold for $X(y)$ since $K(x, y) = K(y, x)$. Now, this would be the second step

of the standard approach for polynomial kernels, when x swipes twice the segment $[0, a_2]$, and $Y(x)$ describes a contour on which a boundary value problem can be set to calculate $F(x, y)$. Here, however, this contour goes to infinity since $Y(x) \to \infty$ as $x \to 0$, which yields a number of complications. To the best of our knowledge, this is in contrast with all related works involving polynomial kernels for which the contour is always closed.

To avoid the burden of working with an infinite contour, we have used the uniformization technique for Poisson kernels devised by Cohen and Boxma [10,11] and Blanc [6]. This approach has allowed us to set a boundary value problem on the closed contour $\mathcal{L}$ defined in (34). However, this is yet another specificity of Eq. (11), since the point $z = 1$ belongs to the contour $\mathcal{L}$, and the calculation (in particular) of the second-order derivative of $F(x, 0)$ at $x = 1$, involved in the formula for the variance N in (23), requires care (see Sect. 6). This is so as, on $\mathcal{L}$, the function $F(x, 0)$ is defined as a principal value integral [cf. (59)]. Again, this is in contrast with related works we are aware of, as in the latter calculating any order derivative of $F(x, 0)$ at $x = 1$ (resp. $F(0, y)$ at $y = 1$) does not offer any difficulty since this function is in general analytic (or can be extended as an analytic function) at the vicinity of $x = 1$ (resp. $y = 1$). In summary, despite its apparent simplicity, Eq. (11) exhibits non-standard and interesting features.

9 Conclusion

In this work, we analyzed a quantum entanglement distribution switch in which each client has a dedicated link connected to the switch. The switch serves tripartite entanglement to $k \geq 3$ users according to incoming requests. At the link level, entangled states are generated across each link according to independent Poisson processes with a constant and identical rate (μ), resulting in two-qubit maximally entangled states. These qubits are stored at local quantum memories, one at the user and the other at the switch. When three link-level entangled states are accrued, the switch performs three-qubit measurement (successful with probability q) to provide end-to-end entanglement to users of groups of size three. This system can also be seen as an assembly-like stochastic queueing system with one central node (or plant) receiving parts or components of a product from geographically distant manufacturing plants. As soon as three parts from three distinct plants are available at the central plant, they are assembled instantaneously and immediately leave the system. Under simplifying assumptions (identical links, infinite storage capabilities at the users and at the switch, no cutoff times at quantum storage), we have obtained the switch capacity, the probability that no there are no entangled qubits in the switch memory, the expected number of stored qubits at the switch, and the variance of the number of stored qubits at the switch. These quantities are obtained by solving a two-dimensional equation satisfied by the generating function of a two-dimensional Markov chain modeling the state of the system. The solution of the functional equation is obtained by reducing it to a boundary value problem on a closed contour, which we solve. This analysis is of independent interest due to the form of the kernel of the functional equation (cf. Sect. 8) along with the fact that point 1 lies on the contour (see comment at the beginning of

Sect. 6) which generates some difficulties when computing some performance metrics. In terms of future work, it would be interesting to relax the assumption that quantum states are stored indefinitely, as cutoff times have been found to play an important role in controlling quantum state fidelity (quality) [34]. This could be done by assuming that these quantum states have an exponential lifetime with a constant rate.

A Zero of $K(x, y)$ for $|x| = 1$

Lemma 2 *For $|x| = 1$, the equation $K(x, y) = 0$ has a unique solution $y = y(x)$ such that $|y(x)| \leq 1$. Moreover, $y(1) := 1$, $y'(1) = -1$ and $y''(1) = \frac{2(k-2)}{k-3}$.*

Proof Recall that $k > 3$.

Assume first that $x = 1$. Since $K(1, y) = -(y - 1)(y - (k - 2))$, we see that $K(1, y)$ has a unique zero in the unit disk $\{|y| \leq 1\}$, located at $y = 1$. Assume now that x is fixed with $|x| = 1$ and $x \neq 1$. Define $h(x, y) = x(k - x)y$. For $|y| = 1$, we have

$$|K(x, y) - h(x, y)| = |xy^2 + k - 2| \leq k - 1 < |k - x| = |x(k - x)y| = |h(x, y)|,$$

where the second inequality holds since[2] $|x| = 1$ with $x \neq 1$. By Rouché's theorem, we conclude that $K(x, y)$ and $h(x, y)$ have the same number of zeros in the unit disk $\{|y| < 1\}$. Since $h(x, y)$ has the only zero $y = 0$ in $\{|y| < 1\}$, $K(x, y)$ has a unique zero in $\{|y| < 1\}$. This proves the first part of the lemma.

The fact that $y(x)$ has first and second derivatives at $x = 1$ follows[3] from the implicit function theorem [27] that applies here since $K(x, y)$ is analytic in $\mathbb{C} \times \mathbb{C}$, $K(1, 1) = 1$, and $\frac{\partial}{\partial y} K(x, y)|_{(x,y)=(1,1)} = k - 3 \neq 0$.

Differentiating $K(x, y(x))$ with respect to x and letting $x = 1$ gives $y^{(1)}(1) = -1$ by using $y(1) = 1$. Differentiating twice $K(x, y(x))$ with respect to x and letting $x = 1$ gives $y''(1) = \frac{2(k-2)}{k-3}$ upon using $y(1) = 1$ and $y'(1) = -1$. This concludes the proof. □

---

[2] To see that, take $x = e^{i\theta}$ with $x \in (0, 2\pi)$. Then, $|k - x|^2 = k^2 + 1 - 2k \cos(\theta)$ which is strictly larger than $|k - 1^2| = (k - 1)^2 = k^2 - 2k + 1$ when $\cos(\theta) < 1$.

[3] Since $K(x, y)$ is analytic in $\mathbb{C} \times \mathbb{C}$, $K(1, 1) = 1$ and $\frac{\partial}{\partial y} K(x, y)|_{(x,y)=(1,1)} = k - 3 \neq 0$, there exists a neighborhood $\mathcal{V}$ of $x = 1$ in the complex plane and an analytic function $w(x)$ defined on $\mathcal{V}$ such that $K(x, w(x)) = 0$ for all $x \in \mathcal{V}$ and $w(1) = 1$. This necessarily implies that $w(x) = y(x)$ for all $x \in \mathcal{V} \cap \{|x| = 1\}$, and therefore, that $y(x)$ is indefinitely differentiable at $x = 1$ since $1 \in \mathcal{V} \cap \{|x| = 1\}$.

B Proof of Lemma 1

We may rewrite the set $\mathcal{L}$ in (34) as

$$\mathcal{L} := \left\{ 0 \leq \rho \leq 1, \, -\pi \leq \theta \leq \pi : 2\rho^3 \cos\theta - k\rho^2 + k - 2 = 0 \right\}.$$

Define $f(\theta, \rho) := 2\rho^3 \cos\theta - k\rho^2 + k - 2$. Let us show that for any $\theta \in \mathbb{R}$, $f(\theta, \rho)$ has a unique zero in $(0, 1]$, denoted by $\rho = \rho(\theta)$.

Assume first that $\theta = 2l\pi$ for $l \in \mathbb{Z}$. From

$$f(2l\pi, \rho) = (x - 1)\left(x - \frac{k - 2 - \sqrt{k^2 + 4k - 12}}{2} \right)\left(x - \frac{k - 2 + \sqrt{k^2 + 4k - 12}}{2} \right),$$

we deduce that $f(2l\pi, \rho)$ has a single zero in $(0, 1]$ given by $\rho(2l\pi) = 1$.

Assume now that $\theta \in \mathbb{R} - \{2l\pi, l \in \mathbb{Z}\}$. From

$$f(\theta, -1) = -2(\cos\theta + 1) \leq 0, \quad f(\theta, 0) = k - 2 > 0, \quad f(\theta, 1) = 2(\cos\theta - 1) < 0$$

and

$$\lim_{\rho \to -\infty} f(\theta, \rho) = \begin{cases} -\infty, & \text{if } \theta \in (2l\pi, (2l+1)\pi), \\ +\infty, & \text{if } \theta \in ((2l+1)\pi, 2(l+1)\pi), \end{cases}$$

$$\lim_{\rho \to \infty} f(\theta, \rho) = \begin{cases} +\infty, & \text{if } \theta \in (2l\pi, (2l+1)\pi), \\ -\infty, & \text{if } \theta \in (2l+1, 2(l+1)), \end{cases}$$

for $l \in \mathbb{Z}$, we readily deduce that the polynomial $f(\theta, \rho)$ of degree 3 in the variable ρ has a single zero $\rho = \rho(\theta)$ in $(0, 1)$. In summary, for any $\theta \in \mathbb{R}$, $f(\theta, \rho)$ has a unique zero $\rho = \rho(\theta)$ in $(0, 1]$.

Since the partial derivatives $\frac{d^m}{d\theta^m} f(\theta, \rho)$ and $\frac{d^m}{d\rho^m} f(\theta, \rho)$ exist for all $m = 0, 1, \ldots$ and $\frac{d}{d\rho} f(\theta, \rho) = 2\rho(3\rho \cos\theta - k)$ does not vanish for any zero (θ, ρ) of $f(\theta, \rho)$ in $\mathbb{R} \times (0, 1]$ as $k > 3$, we conclude from the implicit function theorem [27] that $\rho(\theta)$ is infinitely differentiable in $\mathbb{R}$.

Because $\rho(\theta) = \rho(\theta + 2\pi)$ for any $\theta \in \mathbb{R}$, the function $\rho(\theta)$ is periodic with period 2π and is infinitely differentiable in (in particular) $[-\pi, \pi]$. The continuity of $\rho(\theta)$ in $[-\pi, \pi]$ together with $\rho(-\pi) = \rho(\pi)$ implies that the contour $\mathcal{L}$ defined by (35) is closed and the differentiability of $\rho(\theta)$ in $[-\pi, \pi]$ implies that it is smooth. The contour $\mathcal{L}$ is symmetric with respect to the real axis since $f(\theta, \rho) = f(-\theta, \rho)$ for all $\theta \in R$, $\rho \geq 0$.

Differentiating the equation $f(\theta, \rho(\theta)) = 0$ once gives $\rho'(0) = 0$ by using $\rho(0) = 1$, differentiating it twice gives $\rho''(0) = \frac{1}{3-k}$ by using $\rho(0) = 1$ and $\rho'(0) = 0$, and differentiating it three times gives $\rho'''(0) = 0$ by using $\rho(0) = 1$, $\rho'(0) = 0$ and $\rho''(0) = \frac{1}{3-k}$.

Last, if $\mathcal{L}$ intersects with itself, this means that there exist $\theta_1, \theta_2 \in (0, \pi)$, $\theta_1 \neq \theta_2$, such that $\rho(\theta_1) = \rho(\theta_2)$. (Because $\mathcal{L}$ is symmetric with respect to the real axis, we do not need to consider the case where $\theta_1, \theta_2 \in (-\pi, 0)$.) Assume that $\rho(\theta_1) = \rho(\theta_2) =$

ρ_0 for some $\theta_1, \theta_2 \in (0, \pi)$. Note that $\rho_0 \neq 0$ as $\rho = 0$ is not a zero of $f_\theta(\rho)$. The identities $f(\theta_i, \rho_0) = 0$ for $i = 1, 2$ imply that $2\rho_0^3(\cos(\theta_1) - \cos(\theta_2)) = 0$, which in turn implies that $\cos(\theta_1) = \cos(\theta_2)$ and that $\theta_1 = \theta_2$ since $\theta_1, \theta_2 \in (0, \pi)$. Therefore, the contour $\mathcal{L}$ does not intersect with itself. $\square$

C Proof that $I_2 = 1$

We have, by (55), (38), and (51),

$$
\begin{aligned}
\int_{\mathcal{C}} \frac{\log(u^{-1}G(u))}{u} du &= i \int_{-\pi}^{\pi} \log\left(e^{-i\theta} \frac{\rho(\phi(\theta))^3 e^{i\phi(\theta)}}{1 - \rho(\phi(\theta))^3 e^{-i\phi(\theta)}}\right) d\theta \\
&= i \int_{-\pi}^{0} \log\left(e^{-i\theta} \frac{\rho(\phi(\theta))^3 e^{i\phi(\theta)} - 1}{1 - \rho(\phi(\theta))^3 e^{-i\phi(\theta)}}\right) d\theta \\
&\quad + i \int_{-\pi}^{0} \log\left(e^{i\theta} \frac{\rho(\phi(-\theta))^3 e^{i\phi(-\theta)} - 1}{1 - \rho(\phi(-\theta))^3 e^{-i\phi(-\theta)}}\right) d\theta \\
&= i \int_{-\pi}^{0} \log\left(e^{-i\theta} \frac{\rho(\phi(\theta))^3 e^{i\phi(\theta)} - 1}{1 - \rho(\phi(\theta))^3 e^{-i\phi(\theta)}}\right) d\theta \\
&\quad + i \int_{-\pi}^{0} \log\left(e^{i\theta} \frac{\rho(\phi(\theta))^3 e^{-i\phi(\theta)} - 1}{1 - \rho(\phi(\theta))^3 e^{i\phi(\theta)}}\right) d\theta \\
&= i \int_{-\pi}^{0} \log\left(e^{-i\theta} \frac{\rho(\phi(\theta))^3 e^{i\phi(\theta)} - 1}{1 - \rho(\phi(\theta))^3 e^{-i\phi(\theta)}}\right) d\theta \\
&\quad - i \int_{-\pi}^{0} \log\left(e^{-i\theta} \frac{\rho(\phi(\theta))^3 e^{i\phi(\theta)} - 1}{1 - \rho(\phi(\theta))^3 e^{-i\phi(\theta)}}\right) d\theta \\
&= 0, & (101)
\end{aligned}
$$

where we have used (41) and the property that $\rho(\theta) = \rho(-\theta)$ for all θ (cf. Lemma 1) to establish (101). This concludes the proof from the definition of I_2 in (55).

D Properties of the mappings G and g

The following results will be used in the proofs of Lemmas 3 and 4. Since γ_0 is a one-to-one mapping from $\mathcal{L}$ onto $\mathcal{C}$, for any $t = e^{i\theta}$ there exists a unique $w \in \mathcal{L}$ such that $\gamma_0(w) = t$. Hence, cf. (51)–(52),

$$
G(\gamma_0(w)) = -\frac{1 - \rho(\theta)^3 e^{i\theta}}{1 - \rho(\theta)^3 e^{-i\theta}}, \tag{102}
$$

$$
g(\gamma_0(w)) = \frac{1 - \rho(\theta)^3 \cos\theta}{1 - \rho(\theta)^3 e^{-i\theta}}, \tag{103}
$$

for all $w \in \mathcal{L}$ by using $\gamma(\gamma_0(w)) = w$ (in particular) for all $w \in \mathcal{L}$.

Introduce the mappings

$$\tilde{G}(\theta) := \frac{\rho(\theta)^3 e^{i\theta} - 1}{1 - \rho(\theta)^3 e^{-i\theta}}, \tag{104}$$

$$\tilde{g}(\theta) := \frac{1 - \rho(\theta)^3 \cos\theta}{1 - \rho(\theta)^3 e^{-i\theta}}. \tag{105}$$

Observe from (51), (52) and (38) that, for $t = e^{i\theta}$,

$$G(t) = \tilde{G}(\phi(\theta)), \quad g(t) = \tilde{g}(\phi(\theta)). \tag{106}$$

Lemma 3 (Properties of G and g) *The function $G(t)$ does not vanish on $\mathcal{C}$, and $G(t)$, $g(t)$, $\log(t^{-1}G(t))$ and $g(t)e^{-H(t)}$ are continuous on $\mathcal{C}$.*

Also, $G(1) = 1$ and $g(1) = 0$.

Proof Let $t_0 = e^{i\theta_0} \in \mathcal{C}$. Assume first that $\theta_0 \in [-\pi, \pi] - \{0\}$. Since $\phi(\theta)$ is continuous in $[-\pi, \pi]$, the functions $\tilde{G}(\phi(\theta))$ and $\tilde{g}(\phi(\theta))$ in (104)–(105) are continuous at θ_0 if their common denominator $1 - \rho(\phi(\theta))^3 e^{-i\phi(\theta)}$ does not vanish at this point. This is true as $\rho(\theta) < 1$ for all $\theta \in [-\pi, \pi] - \{0\}$ and that $\phi(\theta) = 0$ iff $\theta = 0$. The same argument shows that the numerator of $\tilde{G}(\phi(\theta))$ does not vanish at θ_0. Assume now that $\theta_0 = 0$. Applications of L'Hôpital's rule show that $\tilde{G}(0) = 1$ (or equivalently $G(1) = 1$) and $\tilde{g}(0) = 1$ (or equivalently $g(1) = 0$), thereby proving all statements of the lemma (but the continuity of $g(t)e^{-H(t)}$) thanks to (106).

It remains to show that $g(t)e^{-H(t)}$ with $H(t)$ defined in (60) is continuous on $\mathcal{C}$. We have just shown that the first term $\log(t^{-1}G(t))$ in (60) is continuous on $\mathcal{C}$. Rewriting the integral in (60) as $\int_{-\pi}^{\pi} \frac{\log(e^{-i\theta}G(e^{i\theta})) - \log(t^{-1}G(t))}{e^{i\theta} - t} e^{-i\theta} d\theta$, we observe that it is a continuous function of $t \in \mathcal{C}$ since, for each $\theta \in [-\pi, \pi]$, the mapping $t \to \frac{\log(e^{-i\theta}G(e^{i\theta})) - \log(t^{-1}G(t))}{e^{i\theta} - t} e^{-i\theta}$ is continuous on $\mathcal{C}$ and the range of integration (*i.e.*, $[-\pi, \pi]$) is finite. Therefore, $H(t)$ is continuous on $\mathcal{C}$ as the sum of two continuous functions on $\mathcal{C}$, and so is $g(t)e^{-H(t)}$ as the composition of continuous functions on $\mathcal{C}$. This ends the proof. □

E Calculation of the index χ

Lemma 4 (Index) *The index χ [defined in (53)] is equal to one.*

Proof Thanks to (102), the index χ in (53), is also given by

$$\chi = \frac{1}{2\pi} \arg[G(\gamma_0(w))]_{\mathcal{L}}. \tag{107}$$

Take $w = \rho(\theta)e^{i\theta} \in \mathcal{L}$. When $\theta = \pi$ (resp. $\theta = -\pi$), $w = -\rho(\pi)$ ($w = \rho(-\pi) = \rho(\pi)$) and $G(\gamma_0(w)) = -\frac{1+\rho(\pi)^3}{1+\rho(\pi)^3} = -1$. When $\theta = 0$, $w = \rho(0) = 1$ and

$$G(\gamma_0(1)) = -\lim_{\theta \to 0} \frac{-3\frac{d}{d\theta}\rho(\theta)\rho(\theta)^2 e^{i\theta} - i\rho(\theta)^3 e^{i\theta}}{-3\frac{d}{d\theta}\rho(\theta)\rho(\theta)^2 e^{-i\theta} + i\rho(\theta)^3 e^{-i\theta}} = 1, \qquad (108)$$

since $\frac{d}{d\theta}\rho(\theta)|_{\theta=0} = 0$ and $\rho(\theta) = 1$ by (36), where the second equality holds by L'Hôpital's rule.

The function $G(\gamma_0(w)) = -\frac{1-\rho(\theta)^3 e^{i\theta}}{1-\rho(\theta)^3 e^{-i\theta}}$ in (102) is continuous in $[-\pi, \pi]$ since $\rho(\theta)$ is continuous in $[-\pi, \pi]$ (cf. Lemma 1), since its denominator does not vanish for $\theta \neq 0$ as $\rho(\theta) < 1$ for $\theta \neq 0$ and since $\lim_{\theta \to 0} -\frac{1-\rho(\theta)^3 e^{i\theta}}{1-\rho(\theta)^3 e^{-i\theta}} = 1$ as shown above.

Let us now show that $G(\gamma_0(w))$ crosses the real axis only at $\theta = 0$ and at $\theta = \pi$ (resp. $\theta = -\pi$) when w describes once the contour $\mathcal{L}$ or, equivalently, that $\Im(G(\gamma_0(w))) \neq 0$ when $\theta \in (-\pi, \pi)$ with $\theta \neq 0$. By (102),

$$\Im(G(\gamma_0(w))) = \frac{2(1-\rho(\theta)^3 \cos\theta)\rho(\theta)^3 \sin\theta}{1-2\rho(\theta)^3 \cos\theta + \rho(\theta)^6}. \qquad (109)$$

Fix $\theta_0 \in (-\pi, \pi)$ with $\theta_0 \neq 0$. The numerator of $\Im(G(\gamma_0(w)))$ vanishes if and only if $1 = \rho(\theta_0)^3 \cos(\theta_0)$. Since $\rho(\theta) < 1$ for $\theta \neq 0$, we see that $1 > \rho(\theta_0)^3 \cos(\theta_0)$. On the other hand, the denominator of $\Im(G(\gamma_0(w)))$ does not vanish since $1 - 2\rho(\theta_0)^3 \cos(\theta_0) + \rho(\theta_0)^6 > 1 - 2\rho(\theta_0)^3 + \rho(\theta_0)^6 = (1-\rho(\theta_0)^3)^2 > 0$ by using $\cos(\theta_0) < 1$ and $\rho(\theta_0) < 1$. Therefore, $\Im(G(\gamma_0(w))) > 0$ for $\theta_0 \in (0, \pi)$ and $\Im(G(\gamma_0(w))) < 0$ for $\theta_0 \in (-\pi, 0)$.

In summary, we have shown that as w describes once the contour $\mathcal{L}$ the mapping $G(\gamma_0(w))$ describes once a circuit around zero in the counter-clockwise direction, thereby proving that $\chi = 1$ by (107). □

F Hölder condition for first and second derivatives of α and β

Lemma 5 *The mapping ϕ defined in (39) is twice differentiable in $[-\pi, \pi]$ and its second derivative is continuous in $[-\pi, \pi]$.*

Proof We recall that the stability condition $k \geq 4$ is enforced. By Lemma 3 in [7, p. 876], the conformal mapping $\gamma(z)$ from $\mathcal{C}$ onto $\mathcal{L}$ (see Section 5.1) is differentiable on $\mathcal{C}$. By (98) and the differentiability of γ on $\mathcal{C}$, $\phi'(\theta)$ exists if (a) $\rho'(\phi(\theta)) + i\rho(\phi(\theta)) \neq 0$ for $\theta \in [-\pi, \pi]$. The latter is always true since $\rho'(\theta)$ is a real number (cf. (96)) and $\rho(\theta) \neq 0$ for all $\theta \in [-\pi, \pi]$. This proves the differentiability of ϕ on $[-\pi, \pi]$.

The proof that ϕ is twice differentiable is more tedious. Differentiating the equation $\gamma(\gamma_0(w)) = w$ for $w \in \mathcal{L}$ (see Section 5.1) gives $\gamma_0'(w) = \frac{1}{\gamma'(\gamma_0(w))}$. But Lemma 3 in [7, p. 876] also tells us that $\gamma'(z) \neq 0$ on $\mathcal{C} - \{1\}$ for all $k \geq 4$, which allows us to conclude that

$$\gamma_0'(w) \neq 0, \quad \forall w \in \mathcal{L} - \{1\}. \qquad (110)$$

By (110), the denominator in (99) does not vanish for $\theta \in [-\pi, \pi] - \{0\}$ and it does not vanish either for $\theta = 0$ since $\gamma_0'(\rho(\phi(0))e^{i\phi(0)}) = \gamma_0'(1) \neq 0$ by (136). (Here, we are using that $\rho'(\phi(\theta)) + i\rho(\phi(\theta)) \neq 0$ for $\theta \in [-\pi, \pi]$ as already observed.) On the other hand, a glance at the r.h.s. of (99) indicates that the term within square brackets is well defined on $[-\pi, \pi]$ if (c) $\gamma_0''(w)$ is well defined on $\mathcal{L}$ which, by (131) with $i = 2$, (133) and (135), amounts to showing that (c) $K - w\delta^2 \neq 0$ for $w \in \mathcal{L}$ in the denominator of (135) (Hint: $w \notin \mathcal{L}$ and $K - w$ does not vanish on $\mathcal{L}$ since $K = k - 2 \geq 2$ and $|w| \leq 1$—see Figure 3). It can be checked that property (c) holds (a more direct argument is to notice that (c) holds as otherwise γ_0 in (44) would not be well defined on $\mathcal{L}$). This proves that ϕ is twice differentiable on $\theta \in [-\pi, \pi]$. Last, we observe from (99) that $\phi''(\theta)$ is continuous on $[-\pi, \pi]$ as it is only composed of continuous mappings, which completes the proof.   □

Lemma 6 *The mapping α defined in (82) is twice differentiable on $\mathcal{C}$, and its second derivative is continuous on $\mathcal{C}$. The same result holds for the mapping β defined in (83).*

Proof Define

$$\tilde{G}(\theta) = -\frac{1 - \rho(\theta)^3 e^{i\theta}}{1 - \rho(\theta)^3 e^{-i\theta}}, \tag{111}$$

$$\tilde{g}(\theta) = \frac{1 - \rho(\theta)^3 \cos\theta}{1 - \rho(\theta)^3 e^{-i\theta}}. \tag{112}$$

Since $\rho(\theta) \in [0, 1)$ for $\theta \in (0, 1)$ and $\rho(0) = 1$ (see Sect. 5.1), the denominator in the r.h.s. of (111) and (112) has a single zero of multiplicity one in $[-\pi, \pi]$ at $\theta = 0$. Since both numerators in the r.h.s. of (111) and (112) vanish at $\theta = 0$, this shows that the mappings $\tilde{G}(\theta)$ and $\tilde{g}(\theta)$ are continuous in $[-\pi, \pi]$.

Notice that (cf. (51), (52), (38))

$$G(e^{i\theta}) = \tilde{G}(\phi(\theta)) \quad \text{and} \quad g(e^{i\theta}) = \tilde{g}(\phi(\theta)).$$

We have, for $u = e^{i\theta}$ (cf. (82))

$$\alpha'(u) = \frac{d}{de^{i\theta}} \log(e^{-i\theta} G(e^{i\theta})) = -ie^{-i\theta} \frac{d}{d\theta} \log(e^{-i\theta} \tilde{G}(\phi(\theta)))$$

$$= -ie^{-i\theta} \left(-i + \frac{\phi'(\theta)\tilde{G}'(\phi(\theta))}{\tilde{G}(\phi(\theta))} \right), \tag{113}$$

and

$$\alpha''(u) = \frac{d}{du}\alpha'(u) = \frac{1}{ie^{i\theta}} \frac{d}{d\theta}\alpha'(u)$$

$$= -ie^{-i\theta} \frac{d}{d\theta} \left\{ -ie^{-i\theta} \left(-i + \frac{\phi'(\theta)\tilde{G}'(\phi(\theta))}{\tilde{G}(\phi(\theta))} \right) \right\}, \quad \text{by using (113)},$$

$$= -ie^{-i\theta} \left(-e^{-i\theta} \left(-i + \frac{\phi'(\theta)\tilde{G}'(\phi(\theta))}{\tilde{G}(\phi(\theta))} \right) - ie^{-i\theta} \frac{d}{d\theta} \frac{\phi'(\theta)\tilde{G}'(\phi(\theta))}{\tilde{G}(\phi(\theta))} \right)$$

$$= -e^{-i\theta}\alpha'(u) - e^{-2i\theta}\frac{d}{d\theta}\frac{\phi'(\theta)\tilde{G}'(\phi(\theta))}{\tilde{G}(\phi(\theta))}, \quad \text{by using again (113)}$$

$$= -e^{-i\theta}\alpha'(u) - e^{-2i\theta}\left(\phi''(\theta)\frac{\tilde{G}'(\phi(\theta))}{\tilde{G}(\phi(\theta))} + \phi'(\theta)^2\frac{\tilde{G}''(\phi(\theta))}{\tilde{G}(\phi(\theta))} - \phi'(\theta)^2\left(\frac{\tilde{G}'(\phi(\theta))}{\tilde{G}(\phi(\theta))}\right)^2\right),$$

$$\tag{114}$$

after easy algebra. Similarly, for $u = e^{i\theta}$ [cf. (83)]

$$\beta'(u) = \frac{d}{de^{i\theta}}\left\{g(e^{i\theta})e^{-H(e^{i\theta})}\right\}$$

$$= -ie^{-i\theta}\frac{d}{d\theta}\left\{\tilde{g}(\phi(\theta))e^{-H(e^{i\theta})}\right\}$$

$$= -ie^{-i\theta}\left(\phi'(\theta)\tilde{g}'(\phi(\theta)) - ie^{i\theta}\tilde{g}(\phi(\theta))H'(e^{i\theta})\right)e^{-H(e^{i\theta})}, \quad \tag{115}$$

and

$$\beta''(u) = -e^{-2i\theta}\left[-i\phi'(\theta)\tilde{g}'(\phi(\theta)) + \phi''(\theta)\tilde{g}'(\phi(\theta)) + \phi'(\theta)^2\tilde{g}''(\phi(\theta))\right.$$

$$-ie^{i\theta}\phi'(\theta)\tilde{g}'(\phi(\theta))H'(e^{i\theta})$$

$$+e^{2i\theta}\tilde{g}(\phi(\theta))H''(e^{i\theta}) - e^{i\theta}\left(i\phi'(\theta)\tilde{g}'(\phi(\theta)) + e^{i\theta}\tilde{g}(\phi(\theta))H'(e^{i\theta})\right)H'(e^{i\theta})\right]$$

$$e^{-H(e^{i\theta})}, \tag{116}$$

after lengthy but easy algebra. On the other hand,

$$\tilde{G}'(\theta) = -\frac{2i\rho(\theta)^2}{(1 - \rho(\theta)^3 e^{-i\theta})^2}\left(\rho(\theta)^4 - \rho(\theta)\cos\theta - 3\rho'(\theta)\sin\theta\right), \tag{117}$$

$$\tilde{g}'(\theta) = \frac{i\rho(\theta)^2}{(1 - \rho(\theta)^3 e^{-i\theta})^2}\left(\rho(\theta)^4 - \rho(\theta)\cos\theta - 3\rho'(\theta)\sin\theta\right), \tag{118}$$

so that

$$\tilde{G}'(\theta) = -2\tilde{g}'(\theta), \tag{119}$$

and

$$\tilde{G}''(\theta) = \frac{\rho(\theta)}{(1 - \rho(\theta)^3 e^{-i\theta})^3}\left(-4\rho(\theta)^8 e^{-i\theta} + \rho(\theta)^5(3 + e^{-2i\theta})\right.$$

$$+3\rho(\theta)^4\left(2i\rho'(\theta)(e^{-2i\theta} - 3) - \rho''(\theta)(1 - e^{-2i\theta})\right)$$

$$+12\rho(\theta)^3\rho'(\theta)^2(1 - e^{-2i\theta})$$

$$\left.-2i\rho(\theta)^2\sin\theta + 6i\rho(\theta)\left(2\rho'(\theta)\cos\theta + \rho''(\theta)\sin\theta\right) + 12i\rho'(\theta)^2\sin\theta\right),$$

$$\tag{120}$$

$$\tilde{g}''(\theta) = -\frac{1}{2}\tilde{G}''(\theta) \quad \text{from (119).} \tag{121}$$

In particular,

$$\tilde{G}'(0) = i\frac{k}{k-3}, \quad \tilde{G}''(0) = -\left(\frac{k}{k-3}\right)^2, \quad \tilde{g}'(0) = -\frac{i}{2}\cdot\frac{k}{k-3},$$

$$\tilde{g}''(0) = \frac{1}{2}\left(\frac{k}{k-3}\right)^2.$$

We are now in position to prove the lemma. We have shown in Lemma 5 above that $\phi(\theta)$ is twice differentiable in $[-\pi, \pi]$ and that its second derivative is continuous in $[-\pi, \pi]$. Also, recall that $\rho(\theta)$ is infinitely differentiable in $[-\pi, \pi]$ (cf. Lemma 1).

We start with α. The first derivative of $\alpha(u)$ in (113) is continuous on $\mathcal{C}$ if the numerator in the ratio[4]

$$\frac{\tilde{G}'(\theta)}{\tilde{G}(\theta)} = \frac{2i\rho(\theta)^2}{(1-\rho(\theta)^3e^{-i\theta})(1-\rho(\theta)^3e^{i\theta})}\left(\rho(\theta)^4 - \rho(\theta)\cos\theta - 3\rho'(\theta)\sin\theta\right) \tag{122}$$

vanishes when the denominator vanishes. Since $\rho(\theta) \in [0, 1)$ for $\theta \neq 0$ and $\rho(0) = 1$ (see Lemma 1), the denominator in (122) has a single zero in $[-\pi, \pi]$ at $\theta = 0$ and this zero has multiplicity two. It is easily checked that $\rho(\theta)^4 - \rho(\theta)\cos\theta - 3\rho'(\theta)\sin\theta$ has a zero of order two at $\theta = 0$. This shows that α' is well defined (actually continuous) on $\mathcal{C}$.

Now, since α' is continuous on $\mathcal{C}$, we see from (114) and from the fact that the ratio in (117) is continuous for $\theta \in [-\pi, \pi]$, that α'' is continuous on $\mathcal{C}$ if the numerator in the ratio

$$\frac{\tilde{G}''(\theta)}{\tilde{G}(\theta)} = -\frac{\rho(\theta)}{(1-\rho(\theta)^3e^{-i\theta})^2(1-\rho(\theta)^3e^{i\theta})}$$
$$\left(-4\rho(\theta)^8e^{-i\theta} + \rho(\theta)^5(3+e^{-2i\theta})\right.$$
$$+3\rho(\theta)^4\left(2i\rho'(\theta)(e^{-2i\theta}-3)-\rho''(\theta)(1-e^{-2i\theta})\right)$$
$$+12\rho(\theta)^3\rho'(\theta)^2(1-e^{-2i\theta}) - 2i\rho(\theta)^2\sin\theta$$
$$\left.+6i\rho(\theta)\left(2\rho'(\theta)\cos\theta + \rho''(\theta)\sin\theta\right) + 12i\rho'(\theta)^2\sin\theta\right) \tag{123}$$

vanishes when the denominator vanishes. As observed above, the denominator in (123) has a single zero in $[-\pi, \pi]$ at $\theta = 0$ and this zero has multiplicity three. One can check that the term within the parentheses in (123) has a zero of order three at $\theta = 0$

---

[4] Since the mapping $\phi(\theta)$ is strictly increasing in $[-\pi, \pi]$ with $\phi(-\pi) = -\pi$ and $\phi(\pi) = \pi$ (see Section 5.1), showing that the mapping $\varphi(\theta)$ is continuous in $[-\pi, \pi]$ is the same, thanks to the continuity of $\phi(\theta)$, as showing that $\varphi(\phi(\theta))$ is continuous in $[-\pi, \pi]$. This property is used repeatedly hereafter.

(Hint: $\rho'(0) = \rho'''(0) = 0$ and $\rho''(0) = 1/(3-k)$ from Lemma 1), which proves that α is twice differentiable on $\mathcal{C}$ and that α'' is continuous on $\mathcal{C}$.

We now turn our attention to β. It is easy to show that $\tilde{g}$ is twice differentiable in $[-\pi, \pi]$ and that its second derivative is continuous (Hint: check that the numerator in (118) [resp. (121)] vanishes at $\theta = 0$ as many times as the denominator does). We are therefore left with proving that the mapping H is twice differentiable on $\mathcal{C}$ and that its second derivative is continuous on $\mathcal{C}$. With (82), $H(t)$ in (60) may be written

$$H(t) = \frac{1}{2}\alpha(t) + \frac{1}{2\pi i} \int_\mathcal{C} \frac{\alpha(u)}{u - t} du. \tag{124}$$

Thanks to (74), $H(t)$ in (124) is also given by

$$H(t) = \alpha(t) + \frac{1}{2\pi i} \int_\mathcal{C} \frac{\alpha(u) - \alpha(t)}{u - t} du, \quad t \in \mathcal{C}. \tag{125}$$

For every $t \in \mathcal{C}$, notice that the integrand in (125) is continuous on $\mathcal{C}$. This is clearly true for $u \neq t$. When $u = t$, it is equal to $\alpha'(t)$ by L'Hôpital's rule. Differentiating (125) gives, for $t \in \mathcal{C}$,

$$H'(t) = \alpha'(t) + \frac{1}{2\pi i} \int_\mathcal{C} \frac{\alpha(u) - \alpha(t) - \alpha'(t)(u - t)}{(u - t)^2} du, \tag{126}$$

$$H''(t) = \alpha''(t) + \frac{1}{2\pi i} \int_\mathcal{C} \frac{2(\alpha(u) - \alpha(t)) - 2\alpha'(t)(u - t) - \alpha''(t)(u - t)^2}{(u - t)^3} du. \tag{127}$$

For every $t \in \mathcal{C}$, notice that the integrand in (126) [resp. (127)] is continuous on $\mathcal{C}$. This is clearly true for $u \neq t$. When $u = t$, it is equal to $\frac{1}{2}\alpha''(t)$ (resp. $\frac{1}{3}\alpha'''(t)$) by L'Hôpital's rule. The third derivative $\alpha'''(t)$ can be calculated similarly to the calculation of α'' in (114); this calculation is omitted. (For the sake of simplicity, we recommend discarding the point $u = t$ when evaluating the integral in (127).) Since we have shown that α, α', and α'' are all continuous on $\mathcal{C}$, the mappings H, H' and H'' are all continuous on $\mathcal{C}$ as the range of integration in (125)–(127) is bounded. This concludes the proof. $\square$

G First and second derivatives of γ_0

Set $K = k - 2$. First, rewrite $\gamma_0(w)$ in (44) as

$$\gamma_0(w) = 1 - \frac{2}{(1 - \delta)^2(K - \delta)} \left(\delta \gamma_{0,1}(w) + \gamma_{0,2}(w) \right), \tag{128}$$

with

$$\gamma_{0,1}(w) = \frac{(1 - w)^2(K - w)}{w}, \tag{129}$$

$$\gamma_{0,2}(w) = \frac{(w-\delta)(1-w)}{w}\sqrt{(K-w\delta^2)(K-w)}. \tag{130}$$

For $i = 1, 2$, we find

$$\frac{d^i}{dw^i}\gamma_0(w) = -\frac{2}{(1-\delta)^2(K-\delta)}\left(\delta\frac{d^i}{dw^i}\gamma_{0,1}(w) + \frac{d^i}{dw^i}\gamma_{0,2}(w)\right), \tag{131}$$

with

$$\frac{d}{dw}\gamma_{0,1}(w) = \frac{(w-1)(Kw-2w^2+K)}{w^2}, \tag{132}$$

$$\frac{d^2}{dw^2}\gamma_{0,1}(w) = \frac{2(K-w^3)}{w^3}, \tag{133}$$

and

$$\frac{d}{dw}\gamma_{0,2}(w) = \frac{1}{2w^2\sqrt{(K-w\delta^2)(K-w)}}$$
$$\left(-K\delta^3 w^2 + 3K\delta^2 w^3 + 2\delta^3 w^3 - 4\delta^2 w^4 - K\delta^3 w - K\delta^2 w^2\right.$$
$$\left.+2\delta^2 w^3 - 2K^2 w^2 - K\delta w^2 + 3Kw^3 + 2K^2\delta - K\delta w - Kw^2\right), \tag{134}$$

$$\frac{d^2}{dw^2}\gamma_{0,2}(w) = -\frac{1}{4w^3\left((K-w\delta^2)(K-w)\right)^{3/2}}\left(K^2\delta^5 w^3 + 3K^2\delta^4 w^4\right.$$
$$-12K\delta^4 w^5 + 8\delta^4 w^6 + 3K^2\delta^5 w^2$$
$$+K^2\delta^4 w^3 - 4K\delta^5 w^3 - 4K^3\delta^2 w^3 - 2K^2\delta^3 w^3$$
$$+18K^2\delta^2 w^4 - 12K\delta^2 w^5 - 12K^3\delta^3 w$$
$$+18K^2\delta^3 w^2 - 2K^2\delta^2 w^3 - 4K\delta^3 w^3 - 4K^3 w^3 + K^2\delta w^3$$
$$\left.+3K^2 w^4 + 8K^4\delta - 12K^3\delta w + 3K^2\delta w^2 + K^2 w^3\right). \tag{135}$$

It is easy to see that

$$\frac{d}{dw}\gamma_0(w)\big|_{w=1} = \frac{8\sqrt{2(k-3)}}{\sqrt{6-k+\sqrt{(k+6)(k-2)}}\cdot(3\sqrt{k-2}+\sqrt{k+6}))}. \tag{136}$$

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