Multilinear operator integrals with second order divided difference symbols in quasi-Banach spaces

Multilinear operator integrals with second order divided difference symbols in quasi-Banach spaces

Master thesis

for the purpose of obtaining the degree of Master of Science
at Delft University of Technology
to be defended publicly on
Thursday April 29th, 2025 at 10:45 am

by

Emiel Huisman

Student number:	5135591
Master programme:	Applied Mathematics
Specialisation:	Mathematics of Partial Differential Equations
Faculty:	Electrical Engineering, Mathematics, and Computer Science
Thesis committee:	Dr. M.P.T. Caspers
	Dr. M.V. Gnann
	Dr. Ir. E. Lorist

For Pleun (of course)

ABSTRACT

The question whether a function f satisfies a Lipschitz estimate in the Schatten-von Neumann space S_p is well understood for $p \in (1, \infty)$. In contrast, the range $p \in (0, 1)$ has only recently been researched, the main challenge being that S_p is a quasi-Banach space in that case.

In 2021, McDonald E. and Sukochev F. showed that Lipschitz functions f belonging to the homogeneous Besov space $\dot{B}^{1/p}_{p/(1-p),p}(\mathbb{R})$ satisfy a Lipschitz estimate when $p \in (0, 1)$. Their result depends on proving the boundedness of Schur multipliers with symbol $f^{[1]}$, the first order divided difference of f, acting on S_p. Additionally, they mention that wavelet analysis in the context of operator Lipschitz functions is a potentially fruitful technique yet to be exploited.

This thesis leverages wavelets, among other techniques, to extend the previous result to multilinear operator integrals with $f^{[2]}$, the second order divided difference of f, as a symbol. It is proven that $f \in \dot{B}^{1/p}_{p/(1-p),p}(\mathbb{R}) \cap \dot{B}^{1/p+1}_{p/(1-p),p}(\mathbb{R})$ being Lipschitz is sufficient for boundedness of the multilinear operator integral with symbol $f^{[2]}$ mapping $S_{p_1} \times S_{p_2} \to S_p$, where $p \in (\frac{1}{2}, 1]$ and $p_1, p_2 \in (1, \infty)$ are such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$.

The proof makes extensive use of properties of multilinear operator integrals, with partitions of unity or Toeplitz functions as symbols, and the relation between homogeneous Besov spaces and wavelet decompositions. Moreover, some boundedness results are given for the full range $p \in (0, 1)$ and a possible strategy for extending the main result to this entire range.

PREFACE

In this segment I'd like to thank several people that made this master thesis possible. Firstly, I'd like to thank Martijn Caspers for suggesting the research topic, his supervision and expertise, and the insightful discussions during our meetings. I fondly look back at the course 'Spectral Theory of Linear Operators' that Martijn taught. From all the courses I've finished, it was the most challenging and *by far* the most rewarding. In addition, I'd like to thank Emiel Lorist and Manuel V. Gnann for being willing to be part of the thesis committee.

Secondly, I'd like to thank my colleagues at the Rijksvastgoedbedrijf. I'm grateful for all my team members that occasionally requested *'Can you explain what your thesis is about? Don't spare any details.'*, before I went of an impromptu pseudo-lecture about non-commutative analysis and its implications in the natural sciences. Marije, Vincent, Lisette, Joost (2x), Duco, Karen, Marc, Jeroen, your enduring enthusiasm made this incredible work-academic life balance possible.

Thirdly, I'd like to thank the following mentors that have supported me thus far: Elwin, Harrie, Rob (2x), Sjon. These archetypes opened new doors for me that I didn't even knew were options at first.

Finally, and most importantly, this thesis would not have been possible without the support of my life partner Pleun. Thank you for always having my back and being there for me.

CONTENTS

1

INTRODUCTION

Multilinear operator integrals are a powerful tool in various problems of applied matrix analysis, mathematical physics, non-commutative geometry, and statistical estimation which allows for the analysis of functions with non-commuting arguments. The concept of a multilinear operator integral arose as an extension of the double operator integral [Ste77; Aza+09; Pel06]. The term 'double operator integral' was first coined in 1956 by Yu. L. Daletskii and S. G. Krein [KD56], who discovered the following identity involving a double operator integral in the study of differentiating operator-valued functions,

$$\frac{d}{dt}f(A+tX)\Big|_{t=0} = \iint_{\mathbb{R}\times\mathbb{R}} \frac{f(x)-f(y)}{x-y} dE_A(x)XdE_A(y) := T^{A,A}_{f^{[1]}}(X). \tag{1.1}$$

where A, X are self-adjoint bounded operators and f is a $C^2(\mathbb{R})$ function mapping $t \mapsto f(A+tX)$. The right-hand side of (1.1) is an iterated Riemann-Stieltjes integral with respect to E_A, the spectral measure of A. Subsequently, the operator norm of $\frac{d}{dt}f(A+tX)|_{t=0}$ could be estimated by analyzing properties of the transformation $T^{A,A}_{f^{[1]}}$.

In the 1960s, the range of the applicability of double operator integrals was significantly expanded by M.S. Birman and M.Z. Solomyak [BS67]. Let H be a seperable Hilbert space. Let A, B be self-adjoint operators densely defined in H and let σ_A, σ_B be their respective spectra and let E_A, E_B be their respective spectral measures. Define the spectral measure E defined on $\sigma_A \times \sigma_B \subset \mathbb{R}^2$ by

$$E(\sigma_A \times \sigma_B)(X) = E_A(\sigma_A)XE_B(\sigma_B)$$

for every X in the Hilbert space S^2 of Hilbert-Schmidt operators on H. M.S. Birman and M.Z. Solomyak defined the double operator integral

1

$T_{\Phi}^{A,B}$ on S^2 as the spectral integral,

$$T_{\Phi}^{A,B}(X) = \iint_{\mathbb{R}\times\mathbb{R}} \Phi(\lambda,\mu)dE(\lambda,\mu)X$$

where $\Phi : \mathbb{R}^2 \mapsto \mathbb{C}$ is a bounded Borel function.

A relatively short proof of (1.1) can be given when f is a monomial acting on a finite d-dimensional Hilbert space H. Let $f(x) = x^m$ for some positive integer m. Further, let $t \mapsto X(t)$ be a C^1-function mapping to self-adjoint bounded operators in H. Let $\{\lambda_i(t)\}_{i=1}^d$ be a complete list of eigenvalues and $\{\xi_i(t)\}_{i=1}^d$ a respective orthonormal basis of eigenvectors, with projections $P_{\xi_i(t)}$, of $X(t)$. Then, applying the product rule and the spectral theorem yields the following equality

$$\begin{aligned}
\frac{df(X(t))}{dt} &= \sum_{i=0}^{m-1} X^i(t)\frac{dX(t)}{dt}X^{m-i-1}(t) \qquad (1.2)\\
&= \sum_{i=0}^{m-1}\sum_{j=1}^{d}\lambda_j^i(t)P_{\xi_j(t)}\frac{dX(t)}{dt}\sum_{k=1}^{d}\lambda_k^{m-i-1}(t)P_{\xi_k(t)}\\
&= \sum_{j=1}^{d}\sum_{k=1}^{d}\left(\sum_{i=0}^{m-1}\lambda_j^i(t)\lambda_k^{m-i-1}(t)\right)P_{\xi_j(t)}\frac{dX(t)}{dt}P_{\xi_k(t)}\\
&= \begin{cases} \sum_{j=1}^d\sum_{k=1}^d \frac{\lambda_j^m(t)-\lambda_k^m(t)}{\lambda_j(t)-\lambda_k(t)}P_{\xi_j(t)}\frac{dX(t)}{dt}P_{\xi_k(t)} & \lambda_j(t)\neq\lambda_k(t)\\ \sum_{j=1}^d\sum_{k=1}^d m\lambda_j^{m-1}P_{\xi_j(t)}\frac{dX(t)}{dt}P_{\xi_k(t)} & \lambda_j(t)=\lambda_k(t)\end{cases}
\end{aligned}$$

The finite-dimensional equivalent of (1.1) follows by setting $X(t) = A + tX$, where A and X are self-adjoint bounded operators in H. Further, the result (1.2) can be extended to general polynomials by linearity and to $C^1(\mathbb{R})$-functions via approximations [ST19, Proposition 5.3.1].

In $T_{f^{[1]}}^{A,A}(X)$, the symbol $f^{[1]}$ denotes the first order divided difference of f, which is defined for $\lambda_0, \lambda_1 \in \mathbb{R}$ as

$$f^{[1]}(\lambda_0,\lambda_1) = \lim_{\lambda\to\lambda_1}\frac{f(\lambda_0)-f(\lambda)}{\lambda_0-\lambda}.$$

For any positive integer n the symbol $f^{[n]}$ denotes the higher order divided difference of f, which is defined recursively acting on distinct points $\lambda_0,\ldots,\lambda_n \in \mathbb{R}$ as [ST19, Section 2.2]

$$f^{[n]}(\lambda_0,\ldots,\lambda_n) = \frac{f^{[n-1]}(\lambda_0,\ldots,\lambda_{n-1})-f^{[n-1]}(\lambda_1,\ldots,\lambda_n)}{\lambda_0-\lambda_n}.$$

To evaluate higher-order operator derivatives there were early attempts to define multilinear operator integrals [Pav71; Ste77]. In these efforts it was shown, under appropriate restrictions on f, that

$$\frac{1}{2}\frac{d^2}{dt^2}f(A+tX)\bigg|_{t=0} = \iiint_{\mathbb{R}\times\mathbb{R}\times\mathbb{R}} f^{[2]}(x,y,z)dE_A(x)XdE_A(y)XdE_A(z). \quad (1.3)$$

However, the class of functions for which (1.3) held was relatively narrow. Later, in 2006, V.V. Peller introduced a new approach to multilinear operator integrals based on integral projective tensor products of L^∞-spaces, which gives a much broader class of integrable functions [Pel06]. (It is noted that a similar approach was also introduced in 2009 by N. A. Azamov, A. L. Carey, P. G. Dodds, and F. A. Sukochev [Aza+09].) In [Pel06] it was shown that f being in the Besov spaces $B^1_{\infty,1}(\mathbb{R})\cap B^n_{\infty,1}(\mathbb{R})$ is sufficient for the following to hold,

$$\frac{1}{n!}\frac{d^n}{dt^n}f(A+tX)\bigg|_{t=0} = \underbrace{\int\dots\int_{\mathbb{R}\times\dots\times\mathbb{R}}}_{n+1} f^{[n]}(\lambda_0,\dots,\lambda_n)dE_A(\lambda_0)X\dots XdE_A(\lambda_n) \quad (1.4)$$

where n is any positive integer, A is a self-adjoint operator and X is a bounded self-adjoint operator. To illustrate, when $n=2$, the right-hand side of (1.4) is called the multilinear operator integral $T^{A,A,A}_{f^{[2]}}(X,X)$. In general, multilinear operator integrals are of the form,

$$T^{A_0,\dots,A_n}_{\Phi}(X_1,\dots,X_n)$$
$$:= \underbrace{\int\dots\int_{\mathbb{R}\times\dots\times\mathbb{R}}}_{n+1} \Phi(\lambda_0,\dots,\lambda_n)dE_{A_0}(\lambda_0)X_1\dots X_n dE_{A_n}(\lambda_n).$$

where Φ is a measurable function, called the symbol, $X_1,\dots,X_n$ are linear operators and $E_{A_0},\dots,E_{A_n}$ are spectral measures of the respective self-adjoint operators $A_0,\dots,A_n$ [Pel15]. The full definition of multilinear operator integrals, based on the approach given in [Pel06], is given in the preliminaries.

Applications of multilinear operator integrals can be found in the natural sciences. For example, quantum mechanics states that observables are represented by self-adjoint operators on a separable complex Hilbert space [GS18, Section 3.2]. If one has a self-adjoint operator A, and wants to retrieve a number depending on the spectrum of A and function f that behaves properly under direct sums, one looks at

1

the spectral action $A \mapsto \text{Tr}(f(A))$ [Con96; Cha98]. The spectral action is used in quantum field theory where A is the Dirac operator and V is the quantum field or quantum mechanics where A is the Hamiltonian, V is the potential and $f(t) = e^{it}$ [CC97]. To understand variational properties of the spectral action when A is perturbed by another self-adjoint operator X, one can apply a Taylor expansion

$$\text{Tr}(f(A+X)) = \text{Tr}(f(A)) + \sum_{n=1}^{\infty} \frac{1}{n!} \text{Tr}\left(\frac{d^n}{dt^n}\bigg|_{t=0} f(A+tX) \right).$$

This expansion leads naturally to the definition of a so-called spectral shift function, which appeared first in the work of I.M. Lifshits in 1952 [Lif52] in the context of solid state theory. Spectral shift functions η_n are an important concept in perturbation theory, mathematically established by M.G. Krein [Kre53] in 1953, for which the following holds

$$\text{Tr}\left(f(A+X) - \sum_{k=0}^{n} \frac{1}{k!} \frac{d^k}{dt^k} f(A+tX)\bigg|_{t=0} \right) = \int_{\mathbb{R}} f^{(n)}(t)\eta_n(t)dt$$

where $n \in \mathbb{N}$. In 1975, a connection was established by M.S. Birman and M.Z. Solomyak [BS75] between spectral shift functions and the theory of double operator integrals. Using the theory of multilinear operator integrals the bound

$$||\eta_n||_{L_1(\mathbb{R})} \leq c_n ||X||_{S_n}^n$$

was found in 2012 by D. Potapov, A. Skripka and F. Sukochev, where c_n is a constant only depending on n and S_n is the n-th Schatten-von Neumann space [PSS12]. An extensive overview of the applications of multilinear operator integrals in perturbation theory can be found in [Pel15; Car+16].

Further, an application of noncommutativity can be found in the generalized uncertainty principle of quantum mechanics

$$\sigma_A \sigma_B \geq |\frac{1}{2i}\overline{[A,B]}|$$

where σ_X (respectively $\overline{X}$) denotes the standard deviation (expectation value) of a physical observable X [GS18, Section 3.5] and $[A,B]$ is the commutator, defined for operators A, B as $[A,B] = AB - BA$. Another example is the time-evolution equation of X in the Heisenberg picture

$$\frac{\partial X}{\partial t} = \frac{i}{\hbar}[H,X]$$

where H is the Hamiltonian [ND13, Section 1.4]. If H is transformed by a function f the resulting commutator may be constructed as [ST19, Theorem 5.1.4]

$$[f(H),X] = T^{H,H}_{f^{[1]}}([H,X]).$$

A different application can be found in the study of Fréchet differentiability. A function f is n times continuously Fréchet S_p-differentiable when $p \in (1,\infty)$ at every bounded self-adjoint operator A if and only if $f \in C^n(\mathbb{R})$. This was proven for $n = 1$ in [Kis+12] using double operator integrals and $n \geq 2$ in [LS20] using multilinear operator integrals. The k-th Fréchet differential is expressed as,

$$D_p^k f(A)(X_1,\dots,X_k) = \sum_{\sigma \in \mathrm{Sym}_k} T_{f^{[k]}}^{A,\dots,A}(X_{\sigma(1)},\dots,X_{\sigma(k)})$$

where Sym_k is the group of all permutations of the set $\{1,\dots,k\}$.

Further, multilinear operator integrals are applied in the study of S_p-operator Lipschitz functions. A function f is called S_p-operator Lipschitz if

$$||f(A) - f(B)||_{S_p} \leq C_{f,p}||A - B||_{S_p}$$

where A,B are arbitrary self-adjoint operators and $C_{f,p}$ is a constant only depending on f and p. The operator Lipschitz property is instructive in determining the change in $f(A)$ when A is perturbed. A recent survey on operator Lipschitz functions can be found in [AP16].

Using the theory of double operator integrals it was established in [PS12], when $p \in (1,\infty)$, that f being Lipschitz is a sufficient and necessary condition for f to be S_p-operator Lipschitz. In contrast, few results exist for the range $p \in (0,1]$. The main challenge is that S_p is a quasi-Banach space when $p \in (0,1)$. This results in the triangle inequality no longer being valid. As a consequence, techniques applied in proofs for $p \in (1,\infty)$ can usually not be applied for $p \in (0,1)$. However, in 2021, E. McDonald and F. Sukochev showed that when f is a Lipschitz function belonging to the homogeneous Besov space $\dot{B}^{1/p}_{p/(1-p),p}(\mathbb{R})$ and $p \in (0,1)$ that

$$||f(A) - f(B)||_{S_p} \leq C_p\left(||f'||_{L_\infty(\mathbb{R})} + ||f||_{\dot{B}^{1/p}_{p/(1-p),p}(\mathbb{R})}\right)||A - B||_{S_p} \tag{1.5}$$

where A, B are arbitrary self-adjoint operators and C_p is a constant only depending on p [MS21]. Besov spaces are efficient in describing the regularity properties of functions and serve as a generalization to elementary function spaces such as Sobolev spaces, a detailed account can be found in [Saw18]. An essential element of the proof of (1.5) is showing that a Schur multiplier with symbol $f^{[1]}$ acting on S_p, when $p \in (0,1)$, is bounded [MS21, Theorem 2.4.3]. A double operator integral can be viewed as a continuous analog of a Schur multiplier [ST19, Section 3.2.2]. In general, boundedness properties of multilinear operator integrals with $f^{[n]}$, where $n \geq 1$, as symbols are used in the study of n-th order Taylor approximations of operator functions [ST19,

1

Section 5.4.2].

This motivates the main problem addressed in this thesis:

Let $p \in (\frac{1}{2}, 1]$ *and* $p_1, p_2 \in (1, \infty)$ *be such that* $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. *Does there exists a class of functions* f *for which a multilinear operator integral with symbol* $f^{[2]}$ *mapping* $S_{p_1} \times S_{p_2}$ *to* S_p *is bounded?*

It is noted that a similar problem for the range $p, p_1, p_2 \in (1, \infty)$ has been addressed in [PSS12; CR25]. However, the main techniques applied there, such as [Con+23, Theorem A] in [CR25], are only applicable when $p \in (1, \infty)$.

The main result obtained in this thesis is that Lipschitz functions in the homogeneous Besov space $\dot{B}^{1/p}_{p/(1-p),p}(\mathbb{R}) \cap \dot{B}^{1/p+1}_{p/(1-p),p}(\mathbb{R})$ are a suitable class of functions for the main problem to be satisfied, as is shown in the main theorems Theorem 1.1 and Theorem 1.2.

Theorem 1.1
Let $p_1, p_2 \in (1, \infty)$ *be such that* $\frac{1}{p_1} + \frac{1}{p_2} = 1$. *Let* $f \in \dot{B}^1_{\infty,1}(\mathbb{R}) \cap \dot{B}^2_{\infty,1}(\mathbb{R})$ *be Lipschitz. Then* $\|T_{f^{[2]}} : S_{p_1} \times S_{p_2} \to S_1\| \lesssim \|f\|_{\dot{B}^1_{\infty,1}} + \|f\|_{\dot{B}^2_{\infty,1}}$.

Theorem 1.2
Let $p \in (\frac{1}{2}, 1)$. *Let* $p_1, p_2 \in (1, \infty)$ *be such that* $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. *Let* $f \in \dot{B}^{1/p}_{p/(1-p),p}(\mathbb{R}) \cap \dot{B}^{1/p+1}_{p/(1-p),p}(\mathbb{R})$ *be Lipschitz. Then* $\|T_{f^{[2]}} : S_{p_1} \times S_{p_2} \to S_p\| \lesssim \|f\|_{\dot{B}^{1/p+1}_{p/(1-p),p}}$.

Essential parts of the method of proof of the main theorems are characteristics of Toeplitz Schur multipliers, as established in [AP02], which are also used extensively in [MS21]. Additionally, a wavelet decomposition of f is applied to show the boundedness of a Schur multiplier with symbol $f^{[1]}$ acting on S_p. Wavelet analysis is a technique already used in a broad range of fields such as physics, biology, mathematics and digital processing [Zay+24; BS98]. E. McDonald and F. Sukochev used wavelet analysis extensively in their proof of (1.5) and note that wavelets are a potentially fruitful technique yet to be exploited in the study of operator Lipschitz functions [MS21].

Another exploited technique is the correspondence between multilinear operator integrals with symbols acting on the real line or torus. The boundedness of multilinear operator integral with symbol $f^{[2]}$ acting on the torus is first proven. Using that result, the boundedness of a multilinear operator integral with symbol $f^{[2]}$ acting on the real line is proven by exploiting the Cayley transform. The relationship between divided differences on the torus and real line in the context of multilinear operator integrals has been researched extensively in [PSS15].

Finally, by decomposing the symbol of a multilinear operator integral as a partition of unity one can bound the multilinear operator integral in terms of each element in the specific partition. This technique has been applied before in the context of Schur multipliers [AP02, Theorem 3.2], but not in the context of multilinear operator integrals to the author's knowledge.

Structure. Section 2 introduces all notation and preliminaries on divided differences, Schatten-Von Neumann spaces, multilinear operator integrals, wavelet analysis and homogeneous Besov spaces. Section 3.1 shows that a compactly supported function f with sufficient regularity defines a bounded multilinear operator integral with $f^{[2]}$ as symbol for quasi-Banach spaces. Section 3.2 proves that a multilinear operator integral with a symbol defined by a partition of unity satisfies a certain bound based on the partition. Section 3.3 defines $\phi_{\alpha,\lambda}$, which serves as an abstraction for a wavelet decomposition. Section 3.4 (respectively 3.5) shows that a Lipschitz function f belonging to the homogeneous Besov space $\dot{B}^1_{\infty,1}(\mathbb{R}) \cap \dot{B}^2_{\infty,1}(\mathbb{R}) \left(\dot{B}^{1/p}_{p/(1-p),p}(\mathbb{R}) \cap \dot{B}^{1/p+1}_{p/(1-p),p}(\mathbb{R})\right)$ defines a bounded multilinear operator integral with $f^{[2]}$ as symbol when $p = 1 \left(p \in (\frac{1}{2}, 1)\right)$. Section 3.6 states boundedness results based on the entire range $p \in (0,1)$ and an outlook for future research directions.

2

PRELIMINARIES

In this chapter, several notational conventions, assumptions and relevant results are introduced that are used throughout this thesis. The reader benefits from having followed a course on spectral theory and functional analysis, such as [Con90].

2.1. NOTATION AND ASSUMPTIONS

- $\mathbb{N}_0$ denotes the natural numbers, including zero. $\mathbb{N} := \mathbb{N}_0 \backslash \{0\}$.
- $\mathbb{K}$ denotes the real line $\mathbb{R}$ or the torus $\mathbb{T}$.
- Let $n \in \mathbb{N}$, denote with $C^n(\mathbb{K})$ the class of functions $f : \mathbb{K} \to \mathbb{C}$ which are n-times differentiable.
- Let $n \in \mathbb{N}$, denote with $C_c^n(\mathbb{R})$ the class of functions $f : \mathbb{R} \to \mathbb{C}$ that have compact support and are n-times differentiable.
- The Fourier coefficients of a function f are denoted by $\{\hat{f}(n)\}_{n\in\mathbb{Z}}$.
- The standard operator norm is denoted by $||\cdot||$.
- The spectrum of an operator is denoted by $\sigma(\cdot)$.
- Let H be a Hilbert space with inner product $<\cdot,\cdot>$ and norm $||\cdot||_H$. Let $B(H)$ denote the algebra of all bounded linear operators on H with the standard operator norm $||\cdot||$.
- All Hilbert spaces are assumed to be separable and complex.
- Let $p \in (0,\infty]$. When $f \in L^p(\mathbb{K},\mu)$, $||f||_p := \left(\int_{\mathbb{K}} |f|^p d\mu\right)^{1/p}$.
- The indicator function of a set S is denoted by χ_S.

2

- When A, B are expressions. $A \lesssim B$ denotes that there exists a finite constant c such that $A \leq c \cdot B$. The subscript $\lesssim_{...}$ indicates whether the constant has an explicit dependency.
- When A, B are expressions. $A \approx B$ denotes equivalence between A and B up to some constants. The subscript $\approx_{...}$ indicates whether these constants have an explicit dependency.
- When $0 < p \leq 1$, then $p^{\#} := \begin{cases} \frac{p}{1-p} & 0 < p < 1 \\ \infty & p = 1 \end{cases}$
- When $0 < p \leq 2$, then $p^{\flat} := \begin{cases} \frac{2p}{2-p} & 0 < p < 2 \\ \infty & p = 2 \end{cases}$.

2.2. DIVIDED DIFFERENCES

Definition 2.2.1 ([ST19, Section 2.2]).
Let $n \in \mathbb{N}$ and $f \in C^n(\mathbb{K})$. Let λ_0, ..., λ_n be points in $\mathbb{K}$. The divided difference of $f^{[0]}$ of order 0 is the function f itself. The divided difference $f^{[n]}$ of order n is defined recursively by

$$f^{[n]}(\lambda_0, \ldots, \lambda_n) = \lim_{\lambda \to \lambda_n} \frac{f^{[n-1]}(\lambda_0, \ldots, \lambda_{n-2}, \lambda_{n-1}) - f^{[n-1]}(\lambda_0, \ldots, \lambda_{n-2}, \lambda)}{\lambda_{n-1} - \lambda}. \tag{2.1}$$

Using this, several examples of divided difference functions follow

$$f^{[1]}(\lambda_0, \lambda_1) = \lim_{\lambda \to \lambda_1} \frac{f(\lambda_0) - f(\lambda)}{\lambda_0 - \lambda}$$

and

$$\begin{aligned} f^{[2]}(\lambda_0, \lambda_1, \lambda_2) &= \lim_{\lambda \to \lambda_2} \frac{f^{[1]}(\lambda_0, \lambda_1) - f^{[1]}(\lambda_1, \lambda)}{\lambda_0 - \lambda} \\ &= \lim_{\lambda \to \lambda_2} \frac{\lim_{\gamma \to \lambda_1} \frac{f(\lambda_0) - f(\gamma)}{\lambda_0 - \gamma} - \lim_{\xi \to \lambda} \frac{f(\lambda_1) - f(\xi)}{\lambda_1 - \xi}}{\lambda_0 - \lambda}. \end{aligned}$$

If λ_0, ..., λ_n are distinct points in $\mathbb{K}$ then the divided difference functions simplify as

$$f^{[1]}(\lambda_0, \lambda_1) = \frac{f(\lambda_0) - f(\lambda_1)}{\lambda_0 - \lambda_1}$$

and

$$f^{[2]}(\lambda_0, \lambda_1, \lambda_2) = \frac{\frac{f(\lambda_0) - f(\lambda_1)}{\lambda_0 - \lambda_1} - \frac{f(\lambda_1) - f(\lambda_2)}{\lambda_1 - \lambda_2}}{\lambda_0 - \lambda_2}.$$

Further, it is a well known result that $f^{[n]}(\lambda_0, \ldots, \lambda_n)$ is invariant under all permutations of λ_0, ..., λ_n [ST19, Section 2.2].

2.3. SCHATTEN-VON NEUMANN SPACES

Definition 2.3.1 ([MS21, Section 2.1]).
Let H be a Hilbert space and $0 < p < \infty$. Define the Schatten-von Neumann $S_p(H)$ space as the space of all compact operators $T \in B(H)$ such that its singular value sequence is in ℓ_p

$$||T||_{S_p} := Tr(|T|^p)^{1/p} = (\sum_{j=0}^{\infty} \mu(j,T)^p)^{(1/p)} = ||\mu(T)||_{\ell_p} < \infty.$$

where $\mu(T)$ is the sequence of eigenvalues of the absolute value $|T|$ arranged in non-increasing order with multiplicities. When H is an arbitrary Hilbert space $S_p(H)$ is denoted by S_p.

For $p \geq 1$, $|| \cdot ||_{S_p}$ defines a Banach norm on S_p. For $0 < p < 1$, this is only a quasi-norm obeying the p-triangle inequality

$$||A+B||_{S_p}^p \leq ||A||_{S_p}^p + ||B||_{S_p}^p, \quad T, S \in S_p. \tag{2.2}$$

2.4. MULTILINEAR OPERATOR INTEGRALS

Let $n \in \mathbb{N}_0$ and $\mathbb{K} \in \{\mathbb{T}, \mathbb{R}\}$, then $\mathcal{A}_n(\mathbb{K})$ denotes the class of functions $\phi : \mathbb{K}^{n+1} \to \mathbb{C}$ admitting the representation

$$\phi(\lambda_0, \dots, \lambda_n) = \int_\Omega \Pi_{i=0}^n a_i(\lambda_i, \omega) d\mu(\omega)$$

for some finite measure space (Ω, μ) and bounded Borel functions $a_j(\cdot, \omega) : \mathbb{K} \to \mathbb{C}$. Denote by $\mathcal{A}_n^c(\mathbb{K})$ the class of functions in $\mathcal{A}_n(\mathbb{K})$ having continuous $a_j(\cdot, \omega)$ [PSS15, Section 2].

Definition 2.4.1 ([Aza+09, Definition 4.1]).
Let $\phi \in \mathcal{A}_n(\mathbb{T})$ (or $\mathcal{A}_n(\mathbb{R})$) and let A_0, ..., A_n be closed densely defined unitary (or self-adjoint) operators. The multilinear operator integral,

$$T_\phi^{A_0,\dots,A_n} : S_{\alpha_1} \times \dots \times S_{\alpha_n} \to S_\alpha$$

where $\alpha_1, \dots, \alpha_n, \alpha \in (0, \infty)$ and $\sum_{i=1}^n \frac{1}{\alpha_i} = \frac{1}{\alpha}$, is defined as the unique element in S_α such that for all $h \in H$

$$T_\phi^{A_0,\dots,A_n}(V_1, \dots, V_n)h$$

$$= \int_\Omega a_0(A_0, \omega) V_1 a_1(A_1, \omega) V_2 a_2(A_2, \omega) \dots V_n a_n(A_n, \omega) h d\mu(\omega) \tag{2.3}$$

where $a_i(A_i, \omega)$ for $i \in \{0, \dots, n\}$ is defined by spectral calculus, and the integral is a Bochner integral [Aza+09, Remark 4.2]. The function ϕ is *called the symbol of* $T_\phi^{A_0,\dots,A_n}(V_1,\dots,V_n)$.

Notation 2.4.2
When $A_0 = \dots = A_n = A$, *the operator* $T_\phi^{A_0,\dots,A_n}$ *is written as* T_ϕ^A. *Further, when* $A_0,\dots,A_n$ *are fixed implicitly the operator* $T_\phi^{A_0,\dots,A_n}$ *is written as* T_ϕ.

2

Additionally, when $\alpha \in (0,1)$, inequality (2.2) implies that for $\phi, \psi \in \mathcal{A}_n(\mathbb{K})$

$$||T_\phi + T_\psi||^\alpha \le ||T_\phi||^\alpha + ||T_\psi||^\alpha. \tag{2.4}$$

Lemma 2.4.3 ([PSS12, Lemma 3.2(iii-iv)]).
Let $p_1,\dots,p_n \in (0,\infty)$. *Let* $x_i \in S_{p_i}$, *where* $i \in \{1,\dots,n\}$. *Let* $\phi_1 : \mathbb{R}^{k+1} \to \mathbb{C}$ *and* $\phi_2 : \mathbb{R}^{n-k+1} \to \mathbb{C}$ *and* $\phi_3 : \mathbb{R}^{n-k+2} \to \mathbb{C}$ *be bounded Borel functions such that* T_{ϕ_1} *and* T_{ϕ_2} *and* T_{ϕ_3} *exist. If* $\psi_1(\lambda_0,\dots,\lambda_n) = \phi_1(\lambda_0,\dots,\lambda_k)\phi_2(\lambda_k,\dots,\lambda_n)$, *then*

$$T_{\psi_1}(x_1,\dots,x_n) = T_{\phi_1}(x_1,\dots,x_k)T_{\phi_2}(x_{k+1},\dots,x_n).$$

If $\psi_2(\lambda_0,\dots,\lambda_n) = \phi_1(\lambda_0,\dots,\lambda_k)\phi_3(\lambda_0,\lambda_k,\dots,\lambda_n)$, *then*

$$T_\psi(x_1,\dots,x_n) = T_{\phi_3}(T_{\phi_1}(x_1,\dots,x_k),x_{k+1},\dots,x_n).$$

2.5. WAVELET ANALYSIS

A wavelet is a function $\phi \in L_2(\mathbb{R})$ such that the family

$$\phi_{j,k}(t) = 2^{\frac{j}{2}}\phi(2^j - k), \quad j,k \in \mathbb{Z}, \quad t \in \mathbb{R}$$

of translations and dilations of ϕ forms an orthonormal basis of $L_2(\mathbb{R})$ [Gra14, Definition 6.6.1]. It is a well known result that, for every $r > 0$, there exists a compactly supported C^r wavelet [Dau88]. Having defined wavelets, the wavelet decomposition of a function f is constructed as

$$f_j(t) = \sum_{k\in\mathbb{Z}} 2^{\frac{j}{2}}\phi(2^j t - k) < f, \phi_{j,k} > . \tag{2.5}$$

The wavelet coefficient $< f, \phi_{j,k} >$ exists when f is locally integrable on $\mathbb{R}$ and ϕ is continuous and compactly supported [MS21, Equation 4.1].

Lemma 2.5.1 ([MS21, Lemma 4.1.2]).
Let f *be a locally integrable function on* $\mathbb{R}$. *For every* $p \in (0,\infty]$ *and wavelet decomposition* f_j, *that is computed with respect to a compactly supported continuous wavelet* ϕ, *it holds that*

$$||f_j||_p \approx_\phi 2^{j(\frac{1}{2}-\frac{1}{p})}\left(\sum_{k\in\mathbb{Z}} |< f, \phi_{j,k} >|^p\right)^{1/p}.$$

2.6. HOMOGENEOUS BESOV SPACES

The following definition of homogeneous Besov spaces is repurposed from [MS21, Section 2.3]. Firstly, for $f \in C^\infty(\mathbb{R})$ and $\alpha, \beta \in \mathbb{N}$ define the seminorms

$$\rho_{\alpha,\beta}(f) := \sup_{x \in \mathbb{R}} |x^\alpha \partial^\beta f(x)|$$

Then, define the Schwartz space as

$$S(\mathbb{R}) := \{f \in C^\infty(\mathbb{R}) : \rho_{\alpha,\beta}(f) < \infty : \forall \alpha, \beta \in \mathbb{N}\}$$

and denote by $S'(\mathbb{R})$ its topological dual [Hyt+16, Definition 2.4.21 and Definition 2.4.24]. Now, let $\Phi \in C^\infty(\mathbb{R})$ be supported in the set $[-2, -1 + \frac{1}{7}) \cup (1 - \frac{1}{7}, 2]$ and identically equal to 1 in the set $[-2 + \frac{2}{7}, -1) \cup (1, 2 - \frac{2}{7}]$. Additionally, assume that

$$\sum_{k \in \mathbb{Z}} \Phi(2^{-k}\xi) = 1, \quad \xi \neq 0.$$

Denote by $\{\Delta_k\}_{k \in \mathbb{Z}}$ the homogeneous Littlewood-Paley decomposition where Δ_k is the operator on $S'(\mathbb{R})$ of Fourier multiplication by the function $\xi \mapsto \Phi(2^{-k}\xi)$.

Definition 2.6.1 ([MS21, Section 2.3]).
Let $s \in \mathbb{R}$ and $p, q \in (0, \infty]$. The homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R})$ is defined as the class of tempered distributions $f \in S'(\mathbb{R})$ where

$$\|f\|_{\dot{B}^s_{p,q}} := \|\{2^{js}\|\Delta_j f\|_p\}_{j \in \mathbb{Z}}\|_{\ell_q(\mathbb{Z})} < \infty.$$

Below, two results from [MS21] are restated which relate wavelet analysis to homogeneous Besov spaces.

Lemma 2.6.2 ([MS21, Lemma 4.1.3]).
Let $p, q \in (0, \infty]$ and $s \in \mathbb{R}$. Let f be locally integrable function, and let ϕ be a compactly supported C^r for $r > |s|$. Then f belongs to the homogeneous Besov class $\dot{B}^s_{p,q}(\mathbb{R})$ if and only if,

$$\|f\|_{\dot{B}^s_{p,q}} \approx_{p,q,s,\phi} \left(\sum_{j \in \mathbb{Z}} 2^{jsq} \|f_j\|_p^q \right)^{1/q} < \infty.$$

where f_j is the wavelet decomposition of a function f with respect to the wavelet ϕ as defined in (2.5).

Lemma 2.6.3 ([MS21, Lemma 4.1.4]).

Let f be a Lipschitz function on $\mathbb{R}$ such that $f \in \dot{B}^{(1/p)}_{p^\#,p}(\mathbb{R})$, where $0 < p \leq 1$. There exists a constant $c \in \mathbb{R}$ such that

$$f(t) = f(0) + ct + \sum_{j\in\mathbb{Z}} f_j(t) - f_j(0), \quad t \in \mathbb{R}$$

and the series $\sum_{j\in\mathbb{Z}} f_j(t) - f_j(0)$ converges uniformly on compact sets.

3

BOUNDEDNESS OF DIVIDED DIFFERENCE OPERATORS

In this chapter, the necessary results will be presented to prove the main theorems: Theorem 1.1 and Theorem 1.2. Firstly, it is proven that a compactly supported function f with sufficient regularity defines a bounded multilinear operator integral with $f^{[2]}$ as a symbol mapping $S_{p_1} \times S_{p_2}$ to S_p. Secondly, it is proven that a multilinear operator integral with a symbol defined by a partition of unity satisfies a certain bound based on the specific partition. Then, $\phi_{\alpha,\lambda}$ is defined, which serves as an abstraction for a wavelet decomposition. Then, by decomposing $\phi_{\alpha,\lambda}$ in a specific way it is shown that the multilinear operator integral with the symbol $\phi_{\alpha,\lambda}$ is bounded by the wavelet coefficients introduced in (2.5). Then, by applying Lemma 2.6.2, which links wavelet decompositions to homogeneous Besov classes, the main theorems, Theorem 1.1 and Theorem 1.2, are proven. In the last section, some boundedness results on the entire range $p \in (0,1)$ are presented.

3.1. BOUNDING $T_{\phi^{[2]}} : S_{p_1} \times S_{p_2} \to S_p, \quad \phi \in C_c^{\beta}(\mathbb{R})$

Lemma 3.1
Let $0 < p \leq 1$ and $p_1, p_2 \in (0,\infty)$ be such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ and $\frac{3}{p} < \beta \in \mathbb{N}$. If $\phi \in C^{\beta}(\mathbb{T})$, then $||T_{\phi^{[2]}} : S_{p_1} \times S_{p_2} \to S_p|| < \infty$.

Proof. Firstly, note that due to [Gra14, Proposition 3.2.5]

$$\phi(z) = \sum_{n \in \mathbb{Z}} \hat{\phi}(n) z^n.$$

3

Thus,

$$\phi^{[2]}(x,y,z) = \sum_{n\in\mathbb{N}_0} \hat{\phi}(n) \frac{\frac{x^n-y^n}{x-y} - \frac{y^n-z^n}{y-z}}{x-z} + \sum_{n\in-\mathbb{N}} \hat{\phi}(n) \frac{\frac{x^n-y^n}{x-y} - \frac{y^n-z^n}{y-z}}{x-z}.$$

Note that on the circle $\mathbb{T}$ for $n \geq 0$,

$$\frac{z^n - w^n}{z-w} = \sum_{k=0}^{n-1} z^k w^{n-k-1}$$

and for $n < 0$

$$\frac{z^n - w^n}{z-w} = -\sum_{k=0}^{-n-1} z^{k+n} w^{-k-1}.$$

Now, observe that

$$\sum_{n\in\mathbb{N}_0} \hat{\phi}(n) \frac{\frac{x^n-y^n}{x-y} - \frac{y^n-z^n}{y-z}}{x-z} = \sum_{n\in\mathbb{N}_0} \hat{\phi}(n) \frac{\sum_{k=0}^{n-1} x^k y^{n-k-1} - \sum_{k=0}^{n-1} z^k y^{n-k-1}}{x-z}$$

$$= \sum_{n\in\mathbb{N}_0} \hat{\phi}(n) \sum_{k=0}^{n-1} y^{n-k-1} \frac{x^k - z^k}{x-z} = \sum_{n\in\mathbb{N}_0} \hat{\phi}(n) \sum_{k=0}^{n-1} \sum_{l=0}^{k-1} y^{n-k-1} x^l z^{k-l-1}$$

and

$$\sum_{n\in-\mathbb{N}} \hat{\phi}(n) \frac{\frac{x^n-y^n}{x-y} - \frac{y^n-z^n}{y-z}}{x-z}$$

$$= \sum_{n\in-\mathbb{N}} \hat{\phi}(n) \frac{\sum_{k=0}^{-n-1} z^{k+n} y^{-k-1} - \sum_{k=0}^{-n-1} x^{k+n} y^{-k-1}}{x-z}$$

$$= -\sum_{n\in-\mathbb{N}} \hat{\phi}(n) \sum_{k=0}^{-n-1} y^{-k-1} \frac{z^{n+k} - x^{n+k}}{z-x} =$$

$$= \sum_{n\in-\mathbb{N}} \hat{\phi}(n) \sum_{k=0}^{-n-1} \sum_{l=0}^{-n-k-1} y^{-k-1} x^{n+k+l} z^{-l-1}.$$

Let $\alpha, \beta, \gamma \in \mathbb{Z}$. Define $\psi_{\alpha,\beta,\gamma}(x,y,z) = x^\alpha y^\beta z^\gamma$. Further, let X, Y, Z be arbitrary unitary operators. Then observe that for all $V \in S_{p_1}$, $W \in S_{p_2}$

$$T^{X,Y,Z}_{\psi_{\alpha,\beta,\gamma}}(V,W) = X^\alpha V Y^\beta W Z^\gamma.$$

from which, by repeatedly applying Hölder's inequality [DR21, Equation 1.8], it follows that,

$$||T^{X,Y,Z}_{\psi_{\alpha,\beta,\gamma}}(V,W)||_{S_p}$$

$$\leq ||X^{\alpha}||||V||_{S_{p_1}}||Y^{\beta}||||W||_{S_{p_2}}||Z^{\gamma}|| = ||V||_{S_{p_1}}||W||_{S_{p_2}}$$

which implies that

$$||T_{\psi_{\alpha,\beta,\gamma}}|| \leq 1$$

Now, using the p-triangle inequality yields

$$||T_{\phi^{[2]}}||^p$$

$$\leq \sum_{n\in\mathbb{N}_0} |\hat{\phi}(n)|^p \sum_{k=0}^{n-1}\sum_{l=0}^{k-1} ||T_{\psi_{l,n-k-1,k-l-1}}||^p$$

$$+ \sum_{n\in-\mathbb{N}} |\hat{\phi}(n)|^p \sum_{k=0}^{-n-1}\sum_{l=0}^{-n-k-1} ||T_{\psi_{n+k+l,-k-1,-l-1}}||^p.$$

$$\leq \sum_{n\in\mathbb{N}_0} |\hat{\phi}(n)|^p \sum_{k=0}^{n-1}\sum_{l=0}^{k-1} 1 + \sum_{n\in-\mathbb{N}} |\hat{\phi}(n)|^p \sum_{k=0}^{-n-1}\sum_{l=0}^{-n-k-1} 1$$

$$= \sum_{n\in\mathbb{N}_0} |\hat{\phi}(n)|^p \frac{n(n-3)}{2} + \sum_{n\in-\mathbb{N}} |\hat{\phi}(n)|^p \frac{n(n+1)}{2}.$$

When $h \in C^{\beta}(\mathbb{T})$, the Fourier coefficients $\{\hat{h}(n)\}_{n\in\mathbb{Z}}$ satisfy $|\hat{h}(n)| \lesssim (1+|n|)^{-\beta}$ [Gra14, Proposition 3.3.12].

$$||T_{\phi^{[2]}}||^p \lesssim \sum_{n\in\mathbb{N}_0} (1+n)^{-\beta p}\frac{n(n-3)}{2} + \sum_{n\in-\mathbb{N}} (1+|n|)^{-\beta p}\frac{n(n+1)}{2}$$

$$= \sum_{n\in\mathbb{N}}(1+n)^{-\beta p}\frac{n(n-3)}{2} + \sum_{n\in\mathbb{N}}(1+n)^{-\beta p}\frac{n(n-1)}{2} \leq 2\sum_{n\in\mathbb{N}} n^{2-\beta p} < \infty.$$

Apply in the last step that $\sum_{n\in\mathbb{N}} n^{2-\beta p}$ converges due to $2-\beta p < -1$. □

The following lemma is primarily based on [PSS15, Theorem 2.7].

Theorem 3.2

Let H be a densely defined self-adjoint operator, let U denote the unitary operator $(H+i)(H-i)^{-1}$, and let $W_1,\ldots,W_n \in S_p$, where $p \in (0,\infty)$. Let $h(z) = i\frac{z+1}{z-1}$ and $g(\lambda) = \frac{\lambda+i}{\lambda-i}$ be functions defined on $\mathbb{T}$ and $\mathbb{R}$ respectively, and let $\phi \in \mathcal{A}_n^c(\mathbb{T})$. Define $\psi(\lambda_0,\ldots,\lambda_n) = \phi(g(\lambda_0),\ldots,g(\lambda_n))$. Then,

$$T_{\phi}^U(W_1,\ldots,W_n) = T_{\psi}^H(W_1,\ldots,W_n).$$

Proof. Firstly, as $\phi \in \mathcal{A}_n^c(\mathbb{T})$,

$$\phi(z_0,z_1,\ldots,z_n) = \int_{\Omega} a_0(z_0,\omega)a_1(z_1,\omega)\ldots a_n(z_n,\omega)d\mu(\omega)$$

3

where each $a_j(\cdot,\omega)$ is continuous. Furthermore, note that $\psi \in \mathcal{A}_n^c(\mathbb{R})$ and

$$\psi(\lambda_0,\lambda_1,\dots,\lambda_n) = \int_\Omega b_0(\lambda_0,\omega)b_1(\lambda_1,\omega)\dots b_n(\lambda_n,\omega)d\mu(\omega)$$

where each $b_j(\cdot,\omega)$ is continuous and given by $b_j(\lambda,\omega) = a_j(g(\lambda),\omega)$.

Now, fix the finite rank operators $V_1, V_2, \dots, V_n$. Let V be a finite rank operator. Let E_H denote the spectral measure of H. Then, the set function

$$\Omega_0 \times \Omega_1 \times \dots \times \Omega_n \to \mathrm{Tr}(E_H(\Omega_0)V_1E_H(\Omega_1)V_2\dots V_nE_H(\Omega_n)V).$$

defined on the rectangular sets of $\mathbb{R}^{n+1}$ extends to a finite countably additive measure ν_H on the Borel subsets of $\mathbb{R}^{n+1}$ [BS96]. Now, observe that $h(U)$ is well defined due to $1 \notin \sigma_p(U)$ [Con90, Corollary X.3.5]. Similarly, the set function, where E_U is the spectral measure of U,

$$\Omega_0 \times \Omega_1 \times \dots \times \Omega_n \to \mathrm{Tr}(E_U(\Omega_0)V_1E_U(\Omega_1)V_2\dots V_nE_U(\Omega_n)V).$$

extends to a finite countably additive measure ν_U on the Borel subsets of $\mathbb{T}^{n+1}$. Since h and g are inverses of one another, $\nu_U = \nu_H \circ h$. Making a change of variables in the scalar integrals yields

$$\int_{\mathbb{T}^{n+1}} \phi(z_0,\dots,z_n)d\nu_U(z_0,\dots,z_n) = \int_{\mathbb{R}^{n+1}} \psi(\lambda_0,\dots,\lambda_n)d\nu_H(\lambda_0,\dots,\lambda_n). \tag{3.1}$$

Now, apply the definition of multiple operator integral (2.3), [PSS12, Lemma 3.10], Fubini's theorem and (3.1) to observe that

$$\begin{aligned}
&\mathrm{Tr}(T_\phi^U(V_1,\dots,V_n)V)\\
&= \mathrm{Tr}(\int_\Omega\int_{\mathbb{T}} a_0(z,\omega)dE_U(z)V_1\cdot\ldots\cdot V_n\int_{\mathbb{T}} a_n(z,\omega)dE_U(z)Vd\mu(\omega))\\
&= \int_\Omega\int_{\mathbb{T}^{n+1}} a_0(z,\omega)\cdot\ldots\cdot a_n(z,\omega)\mathrm{Tr}(dE_U(z)V_1dE_U(z)V_2\dots V_ndE_U(z)V)d\mu(\omega)\\
&= \int_{\mathbb{T}^{n+1}}\int_\Omega a_0(z,\omega)\cdot\ldots\cdot a_n(z,\omega)d\mu(\omega)\mathrm{Tr}(dE_U(z)V_1dE_U(z)V_2\dots V_ndE_U(z)V)\\
&= \int_{\mathbb{T}^{n+1}} \phi(z_0,\dots,z_n)d\nu_U(z_0,\dots,z_n) = \int_{\mathbb{R}^{n+1}} \psi(\lambda_0,\dots,\lambda_n)d\nu_H(\lambda_0,\dots,\lambda_n)\\
&= \int_{\mathbb{R}^{n+1}}\int_\Omega b_0(z,\omega)\cdot\ldots\cdot b_n(z,\omega)d\mu(\omega)\mathrm{Tr}(dE_H(\lambda)V_1dE_H(\lambda)V_2\dots V_ndE_H(\lambda)V)\\
&= \int_\Omega\int_{\mathbb{R}^{n+1}} b_0(z,\omega)\cdot\ldots\cdot b_n(z,\omega)\mathrm{Tr}(dE_H(\lambda)V_1dE_H(\lambda)V_2\dots V_ndE_H(\lambda)V)d\mu(\omega)
\end{aligned}$$

$$= \mathrm{Tr}(\int_\Omega \int_\mathbb{R} b_0(z,\omega)dE_H(\lambda)V_1 \cdot \ldots \cdot V_n \int_\mathbb{R} b_n(z,\omega)dE_H(\lambda)V d\mu(\omega))$$
$$= \mathrm{Tr}(T_\psi^H(V_1, \ldots, V_n)V).$$

Therefore,

$$T_\phi^U(V_1, \ldots, V_n) = T_\psi^H(V_1, \ldots, V_n).$$

Furthermore, the finite rank operators are dense in S_p. Construct for each W_i, where $i \in \{1, \ldots, n\}$, a sequence $\{V_{i,j} : j \in \mathbb{N}\}$ of finite rank operators such that $V_{i,j} \to W_i$ in the so-topology. Apply [PSS12, Proposition 4.9] to see that

$$T_\phi^U(W_1, \ldots, W_n)$$
$$= \lim_{j\to\infty} T_\phi^U(V_{1,j}, \ldots, V_{n,j}) = \lim_{j\to\infty} T_\psi^H(V_{1,j}, \ldots, V_{n,j})$$
$$= T_\psi^H(W_1, \ldots, W_n).$$

□

Theorem 3.3
Let $0 < p \le 1$ and $p_1, p_2 \in (0,\infty)$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ and $\frac{3}{p} < \beta \in \mathbb{N}$. If $f \in C_c^\beta(\mathbb{R})$, then $||T_{f^{[2]}} : S_{p_1} \times S_{p_2} \to S_p|| < \infty$.

Proof. Consider the image of f under the Cayley transform

$$h(z) := f(i\frac{z+1}{z-1})$$

$$f(z) := h(\frac{z+i}{z-i}),$$

where $z \in \mathbb{T}$.

Since $f \in C_c^\beta(\mathbb{R})$ and the Cayley transform is C^∞, it follows that $h \in C^\beta(\mathbb{T})$. Therefore $||T_{h^{[2]}}|| < \infty$ due to Lemma 3.1. Further, because $\beta > \frac{2}{p}$, note that $||T_{h^{[1]}}|| < \infty$ by [MS21, Theorem 4.3.1].

Define the following class of functions

$$G_m(\mathbb{T}) := \{\phi \in C(\mathbb{T}) : \phi(z) = \sum_{n\in\mathbb{N}_0} a_n z^n : \sum_{n\in\mathbb{N}_0} |a_n| n^m < \infty\}.$$

Additionally, due to $h \in C^4(\mathbb{T})$ the function h can be expressed as,

$$h(z) = \sum_{n\in\mathbb{Z}} \hat{h}(n) z^n = \sum_{n\in\mathbb{N}_0} \hat{h}(n) z^n + \sum_{n\in\mathbb{N}} \hat{h}(-n) z^{-n} := h_1(z) + h_2(z^{-1})$$

3

where

$$h_1(z) = \sum_{n\in\mathbb{N}_0} a_n z^n$$

$$h_2(z) = \sum_{n\in\mathbb{N}_0} b_n z^n$$

and

$$a_n = \hat{h}(n)$$

$$b_n = \begin{cases} \hat{h}(-n) & n > 0 \\ 0 & n = 0 \end{cases}.$$

Observe that $|a_n| \lesssim (1+|n|)^{-4}$ and $|b_n| \lesssim (1+|n|)^{-4}$ due to [Gra14, Proposition 3.3.12]. Thus,

$$\sum_{n\in\mathbb{N}_0} |a_n| n^2 \lesssim \sum_{n\in\mathbb{N}_0} (1+|n|)^{-4} n^2 < \sum_{n\in\mathbb{N}} n^{-2} < \infty$$

$$\sum_{n\in\mathbb{N}_0} |b_n| n^2 \lesssim \sum_{n\in\mathbb{N}_0} (1+|n|)^{-4} n^2 < \sum_{n\in\mathbb{N}} n^{-2} < \infty.$$

Therefore $h_1, h_2 \in G_2(\mathbb{T})$. Thus $h \in G_2(\mathbb{T})$ as well. Due to $G_2(\mathbb{T}) \subset G_1(\mathbb{T})$ it follows that $h^{[1]} \in \mathcal{A}_1^c(\mathbb{T})$ and $h^{[2]} \in \mathcal{A}_2^c(\mathbb{T})$ due to [PSS15, Theorem 2.4(i)].

Further, let $f \in C^2(\mathbb{R})$ and $h \in C^2(\mathbb{T})$ and define $z := g(\lambda) = \frac{\lambda+i}{\lambda-i} \in \mathbb{T}$ and $\lambda := h(z) = i\frac{z+1}{z-1} \in \mathbb{R}$. Then, it follows from [PSS15, Lemma 2.3(ii)] that

$$f^{[2]}(\lambda_0, \lambda_1, \lambda_2)$$

$$= \sum_{k=1}^{2} \sum_{0=i_0<\ldots<i_k=2} \frac{(-1)^{k+1} i^{3-k}}{2^{3-k}} h^{[k]}(z_{i_0}, \ldots, z_{i_k})$$

$$\Pi_{j=1}^{k-1}(z_{i_j}-1)^2 \Pi_{l\in\{0,1,2\}\setminus\{i_1,\ldots,i_{k-1}\}}(z_l-1)$$

$$= -\frac{1}{4} h^{[1]}(z_0, z_2) - \frac{i}{2} h^{[2]}(z_0, z_1, z_2)(z_0-1)(z_1-1)^2(z_2-1).$$

Let H be a bounded densely defined self-adjoint closed operator and let U denote the unitary operator $(H+i)(H-i)^{-1}$. Now define

$$u_{0,2}(\lambda_0, \lambda_1, \lambda_2) := h^{[1]}(g(\lambda_0), g(\lambda_2))$$

$$u_{0,1,2}(\lambda_0, \lambda_1, \lambda_2) := h^{[2]}(g(\lambda_0), g(\lambda_1), g(\lambda_2))(g(\lambda_0)-1)(g(\lambda_1)-1)^2(g(\lambda_2)-1).$$

It then follows from [PSS15, Lemma 2.2(i)] that

$$T_{f^{[2]}}^{H,H,H}(X,Y) = -\frac{1}{4} T_{u_{0,2}}^{H,H,H}(X,Y) - \frac{i}{2} T_{u_{0,1,2}}^{H,H,H}(X,Y).$$

Additionally, set

$$v_{0,2}(z_0,z_1,z_2) := u_{0,2}(h(z_0),h(z_1),h(z_2)) = h^{[1]}(z_0,z_2) \tag{3.2}$$

$$\begin{aligned} v_{0,1,2}(z_0,z_1,z_2) &:= u_{0,1,2}(h(z_0),h(z_1),h(z_2)) \\ &= h^{[2]}(z_0,z_1,z_2)(z_0-1)(z_1-1)^2(z_2-1). \end{aligned} \tag{3.3}$$

Due to $h^{[1]} \in \mathcal{A}_1^c(\mathbb{T})$, $h^{[2]} \in \mathcal{A}_2^c(\mathbb{T})$, observe that $v_{0,2}$, $v_{0,1,2} \in \mathcal{A}_2^c(\mathbb{T})$. Now, apply Theorem 3.2 to obtain

$$T^{H,H,H}_{u_{0,2}}(X,Y) = T^{U,U,U}_{v_{0,2}}(X,Y)$$

$$T^{H,H,H}_{u_{0,1,2}}(X,Y) = T^{U,U,U}_{v_{0,1,2}}(X,Y).$$

Now, recall that $v_{0,2}(z_0,z_1,z_2)$ admits the representation

$$v_{0,2}(z_0,z_1,z_2) = \int_\Omega a_0(z_0,\omega)a_1(z_1,\omega)a_2(z_2,\omega)d\mu(\omega).$$

Additionally, recall that $h^{[1]}(z_0,z_2)$ admits the representation,

$$h^{[1]}(z_0,z_1,z_2) := \int_\Omega b_0(z_0,\omega)b_1(z_1,\omega)b_2(z_2,\omega)d\mu(\omega).$$

where b_1 is the identity. Using (3.2), observe that

$$T^{U,U,U}_{v_{0,2}}(X,Y) = \int_\Omega a_0(U,\omega)Xa_1(U,\omega)Ya_2(U,\omega)d\mu(\omega).$$

$$= \int_\Omega b_0(U,\omega)Xb_1(U,\omega)Yb_2(U,\omega)d\mu(\omega) = T^{U,U}_{h^{[1]}}(XY).$$

Secondly, assume that $v_{0,1,2}(z_0,z_1,z_2)$ admits the representation

$$v_{0,1,2}(z_0,z_1,z_2) = \int_\Omega c_0(z_0,\omega)c_1(z_1,\omega)c_2(z_2,\omega)d\mu(\omega).$$

Additionally, assume that $h^{[2]}(z_0,z_1,z_2)$ admits the representation

$$h^{[2]}(z_0,z_1,z_2) = \int_\Omega d_0(z_0,\omega)d_1(z_1,\omega)d_2(z_2,\omega)d\mu(\omega).$$

Apply [PSS15, Lemma 2.2(iii)] and [PSS16, Lemma 2.3(iii)] and $d_1(U,\omega)(U-1) = (U-1)d_1(U,\omega)$ to observe that

$$T^{U,U,U}_{v_{0,1,2}}(X,Y) = \int_\Omega c_0(U,\omega)Xc_1(U,\omega)Yc_2(U,\omega)d\mu(\omega)$$

3

$$= \int_\Omega d_0(U,\omega)(U-1)X(U-1)d_1(U,\omega)(U-1)Y(U-1)d_2(V,\omega)d\mu(\omega)$$
$$= T^{U,U,U}_{h^{[2]}}((U-1)X(U-1),(U-1)Y(U-1)).$$

Now, apply the p-triangle inequality (2.2), the boundedness of $T_{h^{[1]}}$ and $T_{h^{[2]}}$ and Hölder's inequality [DR21, Equation 1.8] to observe that

$$||T^{H,H,H}_{f^{[2]}}(X,Y)||^p_{S_p}$$
$$= ||-\frac{1}{4}T^{U,U}_{h^{[1]}}(XY)-\frac{i}{2}T^{U,U,U}_{h^{[2]}}((U-1)X(U-1),(U-1)Y(U-1))||^p_{S_p}$$
$$\leq \frac{1}{4^p}||T^{U,U}_{h^{[1]}}(XY)||^p_{S_p}+\frac{1}{2^p}||T^{U,U,U}_{h^{[2]}}((U-1)X(U-1),(U-1)Y(U-1))||^p_{S_p}$$
$$\lesssim ||T^{U,U}_{h^{[1]}}(XY)||^p_{S_p}+||T^{U,U,U}_{h^{[2]}}((U-1)X(U-1),(U-1)Y(U-1))||^p_{S_p}$$
$$\lesssim ||XY||^p_{S_p}+||(U-1)X(U-1)||^p_{S_{p_1}}||(U-1)Y(U-1)||^p_{S_{p_2}}$$
$$\lesssim ||XY||^p_{S_p}+||U-1||^{4p}||X||^p_{S_{p_1}}||Y||^p_{S_{p_2}} \lesssim ||X||^p_{S_{p_1}}||Y||^p_{S_{p_2}}.$$

From which it follows that $||T^{H,H,H}_{f^{[2]}}||<\infty$.

Now, let A,B,C be arbitrary densely defined self-adjoint operators. After careful examination, one observes that the techniques used in the proof of [CSZ21, Corollary 2.4] are also valid for $0<p<1$. Applying [CSZ21, Corollary 2.4] gives that $||T^{A,B,C}_{f^{[2]}}||<\infty$. Thus, $||T_{f^{[2]}}||<\infty$.

□

3.2. BOUNDING PARTITIONS OF UNITY

The following three results are used to prove a result analogous to [MS21, Lemma 2.2.3] in the context of multilinear operator integrals.

Lemma 3.4
Let $A,B\in B(H)$ be arbitrary normal operators such that $AB=BA=0$ and let $\Omega=\sigma(A)\cup\sigma(B)\cup(\sigma(A)+\sigma(B))$. Let $f\in C(\Omega)$ and $f(0)=0$. Then, $f(A+B)=f(A)+f(B)$.

Proof. Define $q(x)=\sum_{m=1}^M a_m x^m$, where $\{a_m\}_{m\in\mathbb{N}}$ is an arbitrary bounded sequence. Apply the binomial expansion to observe that

$$q(A+B)=\sum_{m=1}^{M} a_m(A+B)^m=\sum_{m=1}^{M} a_m\sum_{k=0}^{m}\binom{m}{k}A^{m-k}B^k \qquad (3.4)$$

$$= \sum_{m=1}^{M} a_m A^m + \sum_{m=1}^{M} a_m B^m = q(A) + q(B).$$

By the Stone–Weierstrass theorem there exists a sequence of polynomials $\{p_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} ||f - p_n||_\infty = 0$ on Ω. Define $\widetilde{p_n}(x) = p_n(x) - p_n(0)$. Note that $f(0) = 0$ implies that $\lim_{n\to\infty} ||p_n(0)||_\infty = 0$. Then,

$$||f - \widetilde{p_n}||_\infty \le ||f - p_n||_\infty + ||p_n - \widetilde{p_n}||_\infty = ||f - p_n||_\infty + ||p_n(0)||_\infty \tag{3.5}$$

on Ω, where the right-hand side converges to 0 as n goes to ∞.

Due to (3.4-3.5) and the functional calculus it follows that

$$||f(A+B) - f(A) - f(B)|| \tag{3.6}$$

$$\le ||f(A+B) - \widetilde{p_n}(A+B)|| + ||f(A) - \widetilde{p_n}(A)|| + ||f(B) - \widetilde{p_n}(B)||$$

$$= ||f(A+B) - \widetilde{p_n}(A+B)||_\infty + ||f(A) - \widetilde{p_n}(A)||_\infty + ||f(B) - \widetilde{p_n}(B)||_\infty$$

where the right-hand side converges to 0 as n goes to ∞. □

Lemma 3.5
Let $p \in (1, \infty)$. Additionally, let H_1, H_2 be arbitrary self-adjoint operators. Let $\{s_k\}_{k\in\mathbb{Z}}$ and $\{t_k\}_{k\in\mathbb{Z}}$ be partitions of $\mathbb{R}$ such that $s_k \cap s_{k'} = t_k \cap t_{k'} = \emptyset$ whenever $k \neq k'$. Then the mapping $\Omega : S_p \to S_p : V \mapsto \sum_{k\in\mathbb{Z}} \chi_{s_k}(H_1) V \chi_{t_k}(H_2)$ is a contraction.

Proof. Let $x \in H \oplus H$ be arbitrary. Thus, $x = (h_1, h_2)$ where $h_1, h_2 \in H$. Observe that any $T \in B(H \oplus H)$ can be represented as

$$Tx = (Ah_1 + Bh_2, Ch_1 + Dh_2)$$

where $A, B, C, D \in B(H)$. Thus T can be identified with the 2×2 matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Let $T' \in B(H \oplus H)$ be identified with the 2×2 matrix $\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$. Observe that regular matrix multiplication $T'T = \begin{pmatrix} A'A + B'C & A'B + B'D \\ C'A + D'C & B'C + D'D \end{pmatrix}$ and addition $T + T' = \begin{pmatrix} A + A' & B + B' \\ C + C' & D + D' \end{pmatrix}$ is equivalent with multiplication and addition on $B(H \oplus H)$. Thus, $B(H \oplus H) \cong \begin{pmatrix} B(H) & B(H) \\ B(H) & B(H) \end{pmatrix}$.

Construct the following operator $P_k \in B(H \oplus H)$, where $k \in \mathbb{Z}$,

$$P_k = \begin{pmatrix} \chi_{s_k}(H_1) & 0 \\ 0 & \chi_{t_k}(H_2) \end{pmatrix}.$$

Note that $P_k^* = P_k$ and $P_k P_{k'} = P_{k'} P_k = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ when $k \neq k'$ due to $\chi_{s_k}(H_1)$ and $\chi_{t_k}(H_2)$ being pairwise orthogonal projections.

Define the following map $E : B(H \oplus H) \to B(H \oplus H)$ defined by

$$EX = \sum_{k \in \mathbb{Z}} P_k X P_k.$$

3

Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of operators in $B(H \oplus H)$ such that $\lim_{n\to\infty} ||X_n||_{B(H\oplus H)} = 0$. Additionally, suppose that there exists some $Y \in B(H \oplus H)$ such that $\lim_{n\to\infty} ||EX_n||_{B(H\oplus H)} = ||Y||_{B(H\oplus H)}$. Let $\vartheta \in \mathrm{Ran}(P_\ell)$ for some $\ell \in \mathbb{Z}$. Then $\lim_{n\to\infty} ||EX_n\vartheta||_{H\oplus H} = \lim_{n\to\infty} ||P_\ell X_n P_\ell \vartheta||_{H\oplus H} \leq \lim_{n\to\infty} ||X_n||_{B(H\oplus H)} ||\vartheta||_{H\oplus H} = 0$. From this it follows that $Y = 0$ and due to the Closed Graph Theorem [Con90, Proposition III.12.7] it follows that E is a bounded operator.

Further, observe that E is idempotent due to

$$E^2 X = \sum_{k \in \mathbb{Z}} P_k \left(\sum_{k' \in \mathbb{Z}} P_{k'} X P_{k'} \right) P_k$$

$$= \sum_{k,k' \in \mathbb{Z}} P_k P_{k'} X P_{k'} P_k = \sum_{k \in \mathbb{Z}} P_k X P_k = EX.$$

Additionally, when $A \in B(H \oplus H)$ is positive there exists a positive self-adjoint compact operator $B \in B(H \oplus H)$ such that $A = B^2$ [Con90, Definition II.2.12, Theorem II.7.16]. Let $h \in H \oplus H$ be arbitrary. Then EA is positive due to

$$< EAh, h >=< \sum_{k \in \mathbb{Z}} P_k A P_k h, h >= \sum_{k \in \mathbb{Z}} < P_k A P_k h, h >$$

$$= \sum_{k \in \mathbb{Z}} < P_k B^2 P_k h, h >= \sum_{k \in \mathbb{Z}} < BP_k h, BP_k h > \geq 0.$$

Thus, E is completely positive.

Let $\epsilon > 0$ and $\xi \in H \oplus H$ be arbitrary. Let $\{X_n\}_{n \in J}$ be a net of bounded operators in $B(H \oplus H)$ converging strongly to $X \in B(H \oplus H)$. Now, let $k \in \mathbb{Z}$ and $\epsilon_k > 0$ be arbitrary. Then one can pick $N_k \geq 1$ sufficiently large such that for all $n_k \geq N_k$ it holds that

$$||P_k X_{n_k} P_k h - P_k X P_k h||_{H\oplus H} < \epsilon_k.$$

Denote an orthonormal basis of $H \oplus H$ by $\{q_n\}_{n \in \mathbb{N}}$, where each $q_n \in \mathrm{Ran}(P_{k'})$ for some $k' \in \mathbb{Z}$ depending on n. One can choose $N_A \geq 1$

sufficiently large such that there exists an $\eta \in \operatorname{span}(\{q_i\}_{i=1}^{N_A})$ such that $||\xi - \eta||_{H\oplus H} < \frac{\epsilon}{3}(||E||$
$\max(\sup_{n\geq N_A}||X_n||, ||X||))^{-1}$. It follows that there exists $\tilde{R}^-, \tilde{R}^+ \in \mathbb{Z}$, where $\tilde{R}^- \leq \tilde{R}^+$, such that for all $Y \in B(H\oplus H)$: $EY\eta = \sum_{k=R^-}^{R^+} P_k Y P_k \eta$. Thus, one can choose $N_B \geq \max(N_{R^-}, N_{(R^-+1)}, \dots, N_{R^+})$ sufficiently large such that for all $n \geq N_B$ the following inequality holds

$$||\sum_{k=R^-}^{R^+} P_k X_n P_k \xi - \sum_{k=R^-}^{R^+} P_k X P_k \xi||_{H\oplus H} \leq \sum_{k=R^-}^{R^+} ||P_k X_n P_k \xi - P_k X P_k \xi||_{H\oplus H}$$

$$\leq (R^+ - R^-)\cdot \max(\epsilon_{R^-}, \epsilon_{(R^-+1)}, \dots, \epsilon_{R^+}).$$

Since $\epsilon_{R^-}, \epsilon_{(R^-+1)}, \dots, \epsilon_{R^+}$ are arbitrary one can bound

$$||\sum_{k=R^-}^{R^+} P_k X_n P_k \xi - \sum_{k=R^-}^{R^+} P_k X P_k \xi||_{H\oplus H} \leq \frac{\epsilon}{3}.$$

Choose $N_{\max} = \max(N_A, N_B)$. Then, for all $n \geq N_{\max}$, one observes that

$$||EX_n\xi - EX\xi||_{H\oplus H}$$

$$\leq ||EX_n\xi - EX_n\eta||_{H\oplus H} + ||EX\eta - EX\xi||_{H\oplus H} + ||EX_n\eta - EX\eta||_{H\oplus H}$$

$$\leq ||EX_n||||\xi - \eta||_{H\oplus H} + ||EX||||\xi - \eta||_{H\oplus H} + ||EX_n\eta - EX\eta||_{H\oplus H}$$

$$\leq ||E||||X_n||||\xi - \eta||_{H\oplus H} + ||E||||X||||\xi - \eta||_{H\oplus H} + ||EX_n\eta - EX\eta||_{H\oplus H}$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

Thus, $\{EX_n\}_{n\in J}$ is a net of bounded operators in $B(H\oplus H)$ converging strongly to $EX \in B(H\oplus H)$. Then, from [Tak11, Theorem II.2.6] it follows that E is normal.

Thus, E is a normal conditional expectation. Now, applying [HJX09, Theorem 5.1] and [HJX09, Remark 5.6] yields the result that E is a contraction. This implies that E restricted to $\begin{pmatrix} 0 & S_p(H) \\ 0 & 0 \end{pmatrix} \supseteq S_p(H\oplus H)$ is also a contraction. Further, observe that

$$||\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}||_{S_p(H\oplus H)}^p = \operatorname{Tr}(\left(\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}\right)^{p/2})$$

$$= \operatorname{Tr}(\begin{pmatrix} 0 & 0 \\ 0 & |X|^2 \end{pmatrix}^{p/2}) = \operatorname{Tr}(|X|^p) = ||X||_{S_p(H)}^p$$

3

and
$$P_k \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} P_k = \begin{pmatrix} 0 & \chi_{s_k}(H_1) X \chi_{t_k}(H_2) \\ 0 & 0 \end{pmatrix}.$$
From which it follows that
$$\|\sum_{k\in\mathbb{Z}} \chi_{s_k}(H_1) X \chi_{t_k}(H_2)\|_{S_p(H)} = \|\sum_{k\in\mathbb{Z}} P_k \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} P_k\|_{S_p(H\oplus H)}$$
$$\leq \|\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}\|_{S_p(H\oplus H)} = \|X\|_{S_p(H)}.$$
□

Lemma 3.6
Let $\{\phi_k : \mathbb{R}^3 \to \mathbb{C}\}_{k\in\mathbb{Z}}$ be a sequence of disjointly supported functions in all variables. That is, if $\phi_k(x, y, z) \neq 0$, then for all $k' \neq k \in \mathbb{Z}$ and $s, t, u \in \mathbb{R}$ it holds that $\phi_{k'}(x, t, u) = \phi_{k'}(s, y, u) = \phi_{k'}(s, t, z) = 0$. Choose $\phi : \mathbb{R}^3 \to \mathbb{C}$ such that $\phi(x, y, z) = \sum_{k\in\mathbb{Z}} \phi_k(x, y, z)$ for all $x, y, z \in \mathbb{R}$. Let $p \in (\frac{1}{2}, 1]$ and $p_1, p_2 \in (1, \infty)$ be such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. Then $\|T_\phi : S_{p_1} \times S_{p_2} \to S_p\| \leq \|\{\|T_{\phi_k} : S_{p_1} \times S_{p_2} \to S_p\|\}_{k\in\mathbb{Z}}\|_{\ell_{p^\#}}$.

Proof. Without loss of generality it may be assumed that
$$\phi_k(x, y, z) = \phi_k(x, y, z)\chi_{s_k}(x)\chi_{t_k}(y)\chi_{u_k}(z).$$
where $\{s_k\}_{k\in\mathbb{Z}}$, $\{t_k\}_{k\in\mathbb{Z}}$, $\{u_k\}_{k\in\mathbb{Z}}$ are pairwise disjoint partitions of $\mathbb{R}$ and for all $k' \neq k$ it holds that $s_{k'} \cap s_k = t_{k'} \cap t_k = u_{k'} \cup u_k = \emptyset$. Let H_1, H_2, H_3 be arbitrary self-adjoint operators. Let $V \in S_{p_1}$ and $W \in S_{p_2}$ be arbitrary. Further, define $V_k := \chi_{s_k}(H_1) V \chi_{t_k}(H_2)$ and $W_k := \chi_{t_k}(H_2) W \chi_{u_k}(H_3)$. Note that the mappings $S_{p_1} \to S_{p_1} : V \mapsto \sum_{k\in\mathbb{Z}} V_k$ and $S_{p_2} \to S_{p_2} : W \mapsto \sum_{k\in\mathbb{Z}} W_k$ are contractions due to Lemma 3.5. Combining this with (2.3) and Hölder's inequality for sequence spaces yields
$$\|T_\phi^{H_1,H_2,H_3}(V, W)\|_{S_p} \leq \left(\sum_{k\in\mathbb{Z}} \|T_{\phi_k}^{H_1,H_2,H_3}(V, W)\|_{S_p}^p\right)^{1/p} \tag{3.7}$$
$$= \left(\sum_{k\in\mathbb{Z}} \|T_{\phi_k}^{H_1,H_2,H_3}(V_k, W_k)\|_{S_p}^p\right)^{1/p}$$
$$\leq \left(\sum_{k\in\mathbb{Z}} \|T_{\phi_k}^{H_1,H_2,H_3}\|^p \|V_k\|_{S_{p_1}}^p \|W_k\|_{S_{p_2}}^p\right)^{1/p}$$
$$\leq \|\{\|T_{\phi_k}\|\}_{k\in\mathbb{Z}}\|_{\ell_{p^\#}} \|\{\|V_k\|_{S_{p_1}} \|W_k\|_{S_{p_2}}\}_{k\in\mathbb{Z}}\|_{\ell_1}$$
$$\leq \|\{\|T_{\phi_k}\|\}_{k\in\mathbb{Z}}\|_{\ell_{p^\#}} \|\{\|V_k\|_{S_{p_1}} \|W_k\|_{S_{p_2}}\}_{k\in\mathbb{Z}}\|_{\ell_p}$$
$$\leq \|\{\|T_{\phi_k}\|\}_{k\in\mathbb{Z}}\|_{\ell_{p^\#}} \|\{\|V_k\|_{S_{p_1}}\}_{k\in\mathbb{Z}}\|_{\ell_{p_1}} \|\{\|W_k\|_{S_{p_2}}\}_{k\in\mathbb{Z}}\|_{\ell_{p_2}}.$$

Further,

$$\|\{\|V_k\|_{S_{p_1}}\}_{k\in\mathbb{Z}}\|_{\ell_{p_1}} = \left(\sum_{k\in\mathbb{Z}} \|V_k\|^{p_1}_{S_{p_1}}\right)^{1/p_1} \tag{3.8}$$

$$= \left(\sum_{k\in\mathbb{Z}} \mathrm{Tr}\left((V_k^* V_k)^{p_1/2}\right)\right)^{1/p_1} = \left(\mathrm{Tr}\left(\sum_{k\in\mathbb{Z}} (V_k^* V_k)^{p_1/2}\right)\right)^{1/p_1}.$$

When $k \neq k'$, note that $(V_k^* V_k)(V_{k'}^* V_{k'}) = (V_{k'}^* V_{k'})(V_k^* V_k) = 0$. Thus, Lemma 3.4 can be applied to (3.8) from which it follows that

$$\|\{\|V_k\|_{S_{p_1}}\}_{k\in\mathbb{Z}}\|_{\ell_{p_1}} = \left(\mathrm{Tr}\left(\left(\sum_{k\in\mathbb{Z}} V_k^* V_k\right)^{p_1/2}\right)\right)^{1/p_1} \tag{3.9}$$

$$\left(\mathrm{Tr}\left(\left(\left(\sum_{k\in\mathbb{Z}} V_k\right)^* \left(\sum_{k'\in\mathbb{Z}} V_{k'}\right)\right)^{p_1/2}\right)\right)^{1/p_1} = \mathrm{Tr}\left(|\sum_{k\in\mathbb{Z}} V_k|^{p_1}\right)^{1/p_1}$$

$$= \|\sum_{k\in\mathbb{Z}} V_k\|_{S_{p_1}} \leq \|V\|_{S_{p_1}}.$$

By following an identical procedure it is also valid that

$$\|\{\|W_k\|_{S_{p_2}}\}_{k\in\mathbb{Z}}\|_{\ell_{p_2}} \leq \|W\|_{S_{p_2}}. \tag{3.10}$$

Inserting (3.9-3.10) into (3.7) finishes the proof.

□

3.3. STRUCTURE OF $\phi^{[2]}_{\alpha,\lambda}$

Let $\phi \in C_c^\beta(\mathbb{R})$, where $\beta \in \mathbb{N}$, be a compactly supported C^β function. Let $\alpha = \{\alpha_k\}_{k\in\mathbb{Z}}$ be a bounded sequence of complex numbers and let $\lambda > 0$. Define

$$\phi_{\alpha,\lambda}(x) := \sum_{k\in\mathbb{Z}} \alpha_k \phi(\lambda x - k). \tag{3.11}$$

Comparing (2.5) and (3.11) shows that $\phi_{\alpha,\lambda}(x)$ serves as an abstraction of a wavelet decomposition.

Lemma 3.7
If $\phi \in C^2(\mathbb{R})$, then $\phi^{[2]}_{\alpha,\lambda}(x,y,z) = \sum_{k\in\mathbb{Z}} \alpha_k \lambda^2 \phi^{[2]}(\lambda x - k, \lambda y - k, \lambda z - k)$.

Proof. Observe that

$$\phi^{[2]}_{\alpha,\lambda}(x,y,z) = \frac{\frac{\phi_{\alpha,\lambda}(x)-\phi_{\alpha,\lambda}(y)}{x-y} - \frac{\phi_{\alpha,\lambda}(y)-\phi_{\alpha,\lambda}(z)}{y-z}}{x-z}$$

$$= \sum_{k\in\mathbb{Z}} \alpha_k \frac{\frac{\phi(\lambda x-k)-\phi(\lambda y-k)}{x-y} - \frac{\phi(\lambda y-k)-\phi(\lambda z-k)}{y-z}}{x-z}$$

$$= \sum_{k\in\mathbb{Z}} \alpha_k \lambda \frac{\frac{\phi(\lambda x-k)-\phi(\lambda y-k)}{(\lambda x-k)-(\lambda y-k)} - \frac{\phi(\lambda y-k)-\phi(\lambda z-k)}{(\lambda y-k)-(\lambda z-k)}}{x-z}$$

$$= \sum_{k\in\mathbb{Z}} \alpha_k \lambda \frac{\phi^{[1]}(\lambda x-k,\lambda y-k)-\phi^{[1]}(\lambda y-k,\lambda z-k)}{x-z}$$

$$= \sum_{k\in\mathbb{Z}} \alpha_k \lambda^2 \frac{\phi^{[1]}(\lambda x-k,\lambda y-k)-\phi^{[1]}(\lambda y-k,\lambda z-k)}{(\lambda x-k)-(\lambda z-k)}$$

$$= \sum_{k\in\mathbb{Z}} \alpha_k \lambda^2 \phi^{[2]}(\lambda x-k,\lambda y-k,\lambda z-k).$$

□

3.4. BOUNDING $T_{\phi^{[2]}} : S_{p_1} \times S_{p_2} \to S_p, \quad p = 1$

The next theorem aims to prove a result analogous to [MS21, Theorem 4.2.1], where $p = 1$, for multilinear operator integrals with a second order divided difference function as symbol.

Theorem 3.8
Let $p_1, p_2 \in (1,\infty)$ *such that* $\frac{1}{p_1} + \frac{1}{p_2} = 1$. *If* $\phi \in C_c^4(\mathbb{R})$, *then* $\|T_{\phi^{[2]}_{\alpha,\lambda}} : S_{p_1} \times S_{p_2} \to S_1\| \lesssim (\lambda+\lambda^2)\sup_{k\in\mathbb{Z}}|\alpha_k|$.

Proof. Without loss of generality, assume that the functions $\{\phi(\lambda\cdot -k)\}_{k\in\mathbb{Z}}$ are disjointly supported. Indeed, otherwise one may select $N > 1$ sufficiently large such that $\{\phi(\lambda\cdot -Nk)\}_{k\in\mathbb{Z}}$ are disjointly supported, and write

$$\phi_{\alpha,\lambda} = \sum_{j=0}^{N-1} \phi_{\alpha^{(j)},\lambda}$$

where $\alpha^{(j)}$ is the sequence $\{\alpha_{j+Nk}\}_{k\in\mathbb{Z}}$. Then the assertion may be proven for each $\phi_{\alpha^{(j)},\lambda}$ separately.

Let ρ be a smooth compactly supported function on $\mathbb{R}$ which is equal to 1 in a neighbourhood of zero, define $\mu(t) = \frac{1-\rho(t)}{t}$.

Split the second order divided difference as follows

$$\phi^{[2]}_{\alpha,\lambda}(x,y,z) = \phi^{[2]}_{\alpha,\lambda}(x,y,z)(1-\rho(x-z)) + \phi^{[2]}_{\alpha,\lambda}(x,y,z)\rho(x-z)$$

$$:= A(x,y,z) + B(x,y,z).$$

First examine the $A(x,y,z)$ term

$$A(x,y,z) = \frac{\phi^{[1]}_{\alpha,\lambda}(x,y) - \phi^{[1]}_{\alpha,\lambda}(y,z)}{x-z}(1-\rho(x-z))$$

$$= (\phi^{[1]}_{\alpha,\lambda}(x,y) - \phi^{[1]}_{\alpha,\lambda}(y,z))\mu(x-z)$$

$$= \phi^{[1]}_{\alpha,\lambda}(x,y)\mu(x-z) - \phi^{[1]}_{\alpha,\lambda}(y,z)\mu(x-z).$$

Now, use [MS21, Theorem 4.2.1] to show $\|T_{\phi^{[1]}_{\alpha,\lambda}} : S_{p_1} \times S_{p_2} \to S_1\| \lesssim \lambda \sup_{k\in\mathbb{Z}} |\alpha_k|$. Using [MS21, Lemma 4.2.3 and Proposition 4.2.2.(ii)], observe that $\|T_{(x,z)\mapsto\mu(x-z)} : S_1 \to S_1\| < \infty$. Further, use Lemma 2.4.3 to observe that

$$\|T_A : S_{p_1} \times S_{p_2} \to S_1\|$$

$$\leq \|T_{(x,y,z)\mapsto\phi^{[1]}_{\alpha,\lambda}(x,y)\mu(x-z)} : S_{p_1} \times S_{p_2} \to S_1\|$$

$$+\|T_{(x,y,z)\mapsto\phi^{[1]}_{\alpha,\lambda}(y,z)\mu(x-z)} : S_{p_1} \times S_{p_2} \to S_1\|$$

$$\leq \|T_{\phi^{[1]}_{\alpha,\lambda}} : S_{p_1} \to S_{p_1}\|\|T_{(x,z)\mapsto\mu(x-z)} : S_1 \to S_1\|$$

$$+\|T_{\phi^{[1]}_{\alpha,\lambda}} : S_{p_2} \to S_{p_2}\|\|T_{(x,z)\mapsto\mu(x-z)} : S_1 \to S_1\|$$

$$\lesssim \lambda \sup_{k\in\mathbb{Z}} |\alpha_k|.$$

Now, examine the $B(x,y,z)$ term by splitting it as follows

$$B(x,y,z) = \phi^{[2]}_{\alpha,\lambda}(x,y,z)\rho(x-z)$$

$$= \phi^{[2]}_{\alpha,\lambda}(x,y,z)\rho(x-z)(1-\rho(x-y)) + \phi^{[2]}_{\alpha,\lambda}(x,y,z)\rho(x-z)\rho(x-y)$$

$$:= C(x,y,z) + D(x,y,z).$$

First, examine the $C(x,y,z)$ term and apply that $\phi^{[2]}_{\alpha,\lambda}$ is invariant under permutations [ST19, Section 2.2]

$$C(x,y,z) = \phi^{[2]}_{\alpha,\lambda}(x,y,z)\rho(x-z)(1-\rho(x-y))$$

$$= \phi^{[2]}_{\alpha,\lambda}(x,z,y)\rho(x-z)(1-\rho(x-y))$$

$$= \frac{\phi^{[1]}_{\alpha,\lambda}(x,z) - \phi^{[1]}_{\alpha,\lambda}(z,y)}{x-y}\rho(x-z)(1-\rho(x-y))$$

$$= \phi^{[1]}_{\alpha,\lambda}(x,z)\rho(x-z)\mu(x-y) - \phi^{[1]}_{\alpha,\lambda}(z,y)\rho(x-z)\mu(x-y).$$

3

Now, use [MS21, Theorem 4.2.1] to show $||T_{\phi^{[1]}_{\alpha,\lambda}}|| \lesssim \lambda \sup_{k\in\mathbb{Z}}|\alpha_k|$. Use [MS21, Lemma 4.2.3 and Proposition 4.2.2.(ii)] to observe that $||T_{(x,y)\mapsto\mu(x-y)} : S_{p_1} \to S_{p_1}|| < \infty$. Additionally, due to ρ being Schwartz class it follows that $||T_{(x,z)\mapsto\rho(x-z)} : S_1 \to S_1|| < \infty$ due to [MS21, Proposition 4.2.2(ii)]. Further, use the above and Lemma 2.4.3 to observe that

$$
\begin{aligned}
&||T_C : S_{p_1} \times S_{p_2} \to S_1|| \\
&\le ||T_{(x,y,z)\mapsto\phi^{[1]}_{\alpha,\lambda}(x,y)\rho(x-z)\mu(x-y)} : S_{p_1} \times S_{p_2} \to S_1|| \\
&+ ||T_{(x,y,z)\mapsto\phi^{[1]}_{\alpha,\lambda}(z,y)\rho(x-z)\mu(x-y)} : S_{p_1} \times S_{p_2} \to S_1|| \\
&\le ||T_{\phi^{[1]}_{\alpha,\lambda}} : S_{p_1} \to S_{p_1}|| ||T_{(x,z)\mapsto\rho(x-z)} : S_1 \to S_1|| ||T_{(x,y)\mapsto\mu(x-y)} : S_{p_1} \to S_{p_1}|| \\
&+ ||T_{\phi^{[1]}_{\alpha,\lambda}} : S_{p_2} \to S_{p_2}|| ||T_{(x,z)\mapsto\rho(x-z)} : S_1 \to S_1|| ||T_{(x,y)\mapsto\mu(x-y)} : S_{p_1} \to S_{p_1}|| \\
&\lesssim \lambda \sup_{k\in\mathbb{Z}} |\alpha_k|.
\end{aligned}
$$

Now, observe the $D(x,y,z)$ term. Assume that ρ is bounded in the interval $(-1,1)$. It follows that the function $(x,y,z) \to \phi^{[2]}_{\alpha,\lambda}(x,y,z)\rho(x-z)\rho(x-y)$ is supported in the plane $\{(x,y,z) \in \mathbb{R}^3 : |x-z| < 1, |x-y| < 1\}$. Note that

$$
\begin{aligned}
&\{(x,y,z) \in \mathbb{R}^3 : |x-z| < 1, |x-y| < 1\} \\
&\subset \sum_{i,j\in\{-1,0,1\}} \sum_{k\in\mathbb{Z}} [k,k+1) \times [k+i,k+i+1) \times [k+j,k+j+1).
\end{aligned}
$$

Now, define

$$
\chi_{i,j,k}(x,y,z) = \chi_{[k,k+1)}(x)\chi_{[k+i,k+i+1)}(y)\chi_{[k+j,k+j+1)}(z).
$$

Then

$$
\begin{aligned}
D(x,y,z) &= \sum_{i,j\in\{-1,0,1\}} \sum_{k\in\mathbb{Z}} \phi^{[2]}_{\alpha,\lambda}(x,y,z)\rho(x-z)\rho(x-y)\chi_{i,j,k}(x,y,z) \\
&:= \sum_{i,j\in\{-1,0,1\}} F_{i,j}(x,y,z). \qquad (3.12)
\end{aligned}
$$

Additionally, observe that due to Lemma 3.7

$$
\begin{aligned}
\phi^{[2]}_{\alpha,\lambda}(x,y,z) &= \frac{\phi^{[1]}_{\alpha,\lambda}(x,y) - \phi^{[1]}_{\alpha,\lambda}(y,z)}{x-z} \\
&= \sum_{k\in\mathbb{Z}} \alpha_k \lambda^2 \phi^{[2]}(\lambda x - k, \lambda y - k, \lambda z - k). \qquad (3.13)
\end{aligned}
$$

Inserting (3.13) into (3.12) yields

$$F_{i,j}(x,y,z)$$
$$= \sum_{k' \in \mathbb{Z}} \chi_{i,j,k'}(x,y,z) \sum_{k \in \mathbb{Z}} \alpha_k \lambda^2 \phi^{[2]}(\lambda x - k, \lambda y - k, \lambda z - k)\rho(x-z)\rho(x-y).$$

Due to ϕ being compactly supported for each k' the sum over k has only finitely many terms. In fact, there exists a constant N (depending on ϕ, λ, i and j) such that for $|k - k'| > N$

$$\alpha_k \lambda^2 \phi^{[2]}(\lambda x - k, \lambda y - k, \lambda z - k)$$
$$= \alpha_k \lambda \frac{\phi(\lambda x - k) - \phi(\lambda y - k)}{(x-z)(x-y)} - \alpha_k \lambda \frac{\phi(\lambda y - k) - \phi(\lambda z - k)}{(x-z)(y-z)} = 0.$$

Thus

$$F_{i,j}(x,y,z)$$
$$= \sum_{k' \in \mathbb{Z}} \chi_{i,j,k'}(x,y,z) \sum_{|k-k'| \leq N} \alpha_k \lambda^2 \phi^{[2]}(\lambda x - k, \lambda y - k, \lambda z - k)\rho(x-z)\rho(x-y).$$

Now, fix $i, j \in \{-1, 0, 1\}$. If $\chi_{i,j,n}(x,y,z) \neq 0$, then for all $s, t \in \mathbb{R}$ and $m \neq n$ observe that $\chi_{i,j,m}(s,t,z) = 0$, $\chi_{i,j,m}(x,s,t) = 0$ and $\chi_{i,j,m}(s,y,t) = 0$. Thus $F_{i,j}(x,y,z)$ is disjointly supported in x, y and z.

Apply Lemma 3.6 to observe that

$$\|T_{F_{i,j}} : S_{p_1} \times S_{p_2} \to S_1\|$$
$$\leq \sup_{k' \in \mathbb{Z}} \|T_{(x,y,z) \mapsto \chi_{i,j,k'}(x,y,z) \sum_{|k-k'| \leq N} \alpha_k \lambda^2 \phi^{[2]}(\lambda x - k, \lambda y - k, \lambda z - k)\rho(x-z)\rho(x-y)}\|$$
$$= \sup_{k' \in \mathbb{Z}} \sum_{|k-k'| \leq N} |\alpha_k| \lambda^2 \|T_{(x,y,z) \mapsto \chi_{i,j,k'}(x,y,z)\phi^{[2]}(\lambda x - k, \lambda y - k, \lambda z - k)\rho(x-z)\rho(x-y)}\|.$$

Notice that ϕ is a C_c^4 function, thus by Theorem 3.3 it follows that $\|T_{\phi^{[2]}} : S_{p_1} \times S_{p_2} \to S_1\| < \infty$. Similarly, due to ρ being Schwartz class it follows that $\|T_{(x,z) \mapsto \rho(x-z)} : S_1 \to S_1\| < \infty$ and $\|T_{(x,y) \mapsto \rho(x-y)} : S_{p_1} \to S_{p_1}\| < \infty$ due to [MS21, Proposition 4.2.2(ii)]. Combining this once again with Lemma 2.4.3 yields

$$\|T_{F_{i,j}} : S_{p_1} \times S_{p_2} \to S_1\| \lesssim \lambda^2 \sup_{k' \in \mathbb{Z}} \sum_{|k-k'| \leq N} |\alpha_k| \lesssim_N \lambda^2 \sup_{k' \in \mathbb{Z}} |\alpha_{k'}|.$$

From which it follows that

$$\|T_{\phi^{[2]}_{\alpha,\lambda}} : S_{p_1} \times S_{p_2} \to S_1\| \lesssim (\lambda + \lambda^2) \sup_{k \in \mathbb{Z}} |\alpha_k|.$$

3

□

Lemma 3.9
If f is Lipschitz on $\mathbb{R}$ and $f \in \dot{B}^{1/p}_{p^{\#},p}(\mathbb{R})$, where $0 < p \leq 1$, then $f^{[2]}(\lambda_0, \lambda_1, \lambda_2) = \sum_{j\in\mathbb{Z}} f_j^{[2]}(\lambda_0, \lambda_1, \lambda_2)$.

Proof. Apply Lemma 2.6.3 to observe that

$$f(t) - f(0) = ct + \sum_{j\in\mathbb{Z}} f_j(t) - f_j(0), \quad t \in \mathbb{R}.$$

By dividing by t, when $t \neq 0$, it follows that

$$f^{[1]}(\lambda_0, \lambda_1) = c + \sum_{j\in\mathbb{Z}} f_j^{[1]}(\lambda_0, \lambda_1). \tag{3.14}$$

Taking the derivative with respect to t yields

$$f'(t) = c + \sum_{j\in\mathbb{Z}} f_j'(t) \tag{3.15}$$

$$f''(t) = \sum_{j\in\mathbb{Z}} f_j''(t). \tag{3.16}$$

Thus, when $\lambda_0 = \lambda_1 = \lambda_2 = \lambda$ it follows from (3.15) and (3.16) that

$$f^{[2]}(\lambda, \lambda, \lambda) = f''(\lambda) = \sum_{j\in\mathbb{Z}} f_j''(t) = \sum_{j\in\mathbb{Z}} f_j^{[2]}(\lambda, \lambda, \lambda).$$

Additionally, if there exists $i, j \in \{0, 1, 2\}$ such that $i \neq j$ and $\lambda_i \neq \lambda_j$, then it follows from (3.14) that

$$f^{[2]}(\lambda_0, \lambda_1, \lambda_2) = \frac{f^{[1]}(\lambda_0, \lambda_1) - f^{[1]}(\lambda_1, \lambda_2)}{\lambda_0 - \lambda_2}$$

$$= \frac{(c + \sum_{j\in\mathbb{Z}} f_j^{[1]}(\lambda_0, \lambda_1)) - (c + \sum_{j\in\mathbb{Z}} f_j^{[1]}(\lambda_1, \lambda_2))}{\lambda_0 - \lambda_2} = \sum_{j\in\mathbb{Z}} f_j^{[2]}(\lambda_0, \lambda_1, \lambda_2).$$

Thus, the wavelet decomposition of $f^{[2]}$ for arbitrary $\lambda_0, \lambda_1, \lambda_2 \in \mathbb{R}$ is

$$f^{[2]}(\lambda_0, \lambda_1, \lambda_2) = \sum_{j\in\mathbb{Z}} f_j^{[2]}(\lambda_0, \lambda_1, \lambda_2).$$

□

The general strategy in the proof will be to apply the wavelet decomposition to bound $||T_{f^{[2]}} : S_{p_1} \times S_{p_2} \to S_p||^p$ by $\sum_{j\in\mathbb{Z}} ||T_{f_j^{[2]}} : S_{p_1} \times S_{p_2} \to S_p||^p$.

Lemma 3.10
Let f be a locally integrable function on $\mathbb{R}$, and let $j \in \mathbb{Z}$ be such that f_j is bounded where f_j is computed with respect to a compactly supported C^4 wavelet ϕ. Let $p_1, p_2 \in (1, \infty)$ be such that $\frac{1}{p_1} + \frac{1}{p_2} = 1$. Then the map $T_{f_j^{[2]}} : S_{p_1} \times S_{p_2} \to S_1$ is bounded by $||T_{f_j^{[2]}}|| \lesssim (2^j + 2^{2j})||f_j||_\infty$.

Proof. Observe that

$$f_j(t) = \sum_{k \in \mathbb{Z}} 2^{j/2} \phi(2^j t - k) < f, \phi_{j,k} > .$$

Which has the correct form to be able to apply Theorem 3.8 and Lemma 2.5.1

$$||T_{f_j^{[2]}}|| \leq (2^j + 2^{2j}) \sup_{k \in \mathbb{Z}} 2^{j/2} | < f, \phi_{j,k} > | \approx_\phi (2^j + 2^{2j})||f_j||_\infty.$$

□

Collorary 3.11
Let $f \in \dot{B}^1_{\infty,1}(\mathbb{R}) \cap \dot{B}^2_{\infty,1}(\mathbb{R})$ be Lipschitz. Let $p_1, p_2 \in (1, \infty)$ be such that $\frac{1}{p_1} + \frac{1}{p_2} = 1$. Then the map $T_{f^{[2]}} : S_{p_1} \times S_{p_2} \to S_1$ is bounded by $||T_{f^{[2]}}|| \lesssim ||f||_{\dot{B}^1_{\infty,1}} + ||f||_{\dot{B}^2_{\infty,1}}$.

Proof. Apply Lemma 3.9, Lemma 3.10 and Lemma 2.6.2 to observe that

$$||T_{f^{[2]}}|| = || \sum_{j \in \mathbb{Z}} T_{f_j^{[2]}} || \lesssim \sum_{j \in \mathbb{Z}} (2^j + 2^{2j})||f_j||_\infty$$

$$= \sum_{j \in \mathbb{Z}} 2^j ||f_j||_\infty + \sum_{j \in \mathbb{Z}} 2^{2j} ||f_j||_\infty \approx_\phi ||f||_{\dot{B}^1_{\infty,1}} + ||f||_{\dot{B}^2_{\infty,1}}.$$

□

This completes the proof of Theorem 1.1.

3.5. Bounding $T_{\phi^{[2]}} : S_{p_1} \times S_{p_2} \to S_p, \quad p \in (\frac{1}{2}, 1)$

Now the corresponding result of Collorary 3.11 for $p \in (\frac{1}{2}, 1)$ will be proven in the following results. The first step is to prove a result analogous to [MS21, Proposition 4.2.2(ii)] for multilinear operator integrals.

Lemma 3.12
Let $p \in [1, \infty)$. Suppose that $\phi : \mathbb{R}^2 \to \mathbb{C}$ has Toeplitz form. That is, there exists a bounded function μ such that $\phi(t, s) = \mu(t - s)$. Then $||T_\phi : S_p \to S_p|| \leq (2\pi)^{-1} ||\hat{\mu}||_1$.

3

Proof. Firstly, let A, B be arbitrary self-adjoint operators. Further, let $X \in S_p$ be arbitrary. Repeatedly apply Hölder's inequality [DR21, Equation 1.8] to observe that

$$||T^{A,B}_{(t,s)\mapsto e^{i\xi t}e^{-i\xi s}}(X)||_{S_p} = ||e^{i\xi A}Xe^{-i\xi B}||_{S_p} \leq ||e^{i\xi A}||||X||_{S_p}||e^{-i\xi B}|| \leq ||X||_{S_p}.$$

Thus, $||T_{(t,s)\mapsto e^{i\xi t}e^{-i\xi s}} : S_p \to S_p|| \leq 1$. Secondly, apply the fourier transform to observe that $\phi(t,s) = \mu(t-s) = (2\pi)^{-1}\int_{-\infty}^{\infty} e^{i\xi(t-s)}\hat{\mu}(\xi)d\xi = (2\pi)^{-1}\int_{-\infty}^{\infty} e^{i\xi t}e^{-i\xi s}\hat{\mu}(\xi)d\xi$. Then applying the linearity of the multiple operator integral, the triangle inequality, Lemma 2.4.3 yields

$$||T_\phi|| = ||T_{(t,s)\mapsto(2\pi)^{-1}\int_{-\infty}^{\infty} e^{i\xi t}e^{-i\xi s}\hat{\mu}(\xi)d\xi}|| =$$

$$(2\pi)^{-1}||\int_{-\infty}^{\infty} \hat{\mu}(\xi)T_{(t,s)\mapsto e^{i\xi t}e^{-i\xi s}}d\xi||$$

$$\leq (2\pi)^{-1}\int_{-\infty}^{\infty} ||\hat{\mu}(\xi)T_{(t,s)\mapsto e^{i\xi t}e^{-i\xi s}}||d\xi$$

$$= (2\pi)^{-1}\int_{-\infty}^{\infty} |\mu(\hat{\xi})|||T_{(t,s)\mapsto e^{i\xi t}e^{-i\xi s}}||d\xi$$

$$\leq (2\pi)^{-1}\int_{-\infty}^{\infty} |\hat{\mu}(\xi)|d\xi = (2\pi)^{-1}||\hat{\mu}||_1.$$

□

Lemma 3.13
Let $p, R \in (1,\infty)$. Then $||T_{(x,y)\mapsto\chi_{|\lfloor x\rfloor-\lfloor y\rfloor|\leq R}} : S_p \to S_p|| \leq (2\pi)^{-1}(2R+1)^{\frac{1}{2}}$.

Proof. Apply [MS21, Lemma 4.3.3] to observe that,

$$||T_{(x,y)\mapsto\chi_{|\lfloor x\rfloor-\lfloor y\rfloor|\leq R}} : S_p \to S_p|| = ||T_\omega : S_p \to S_p||$$

where $\omega : \mathbb{Z}^2 \to \mathbb{Q}$ is defined as the mapping

$$\omega(n,m) = \chi_{|n-m|\leq R}.$$

Note that ω has Toeplitz form $\omega(n,m) := \mu(n-m)$, where $\mu(l) = \chi_{|l|\leq R}$. When applying the discrete time Fourier transform to μ observe that the Fourier transform $\hat{\mu}$ is supported in the interval $[0,1]$.

One can apply Lemma 3.12, the Hölder inequality for Lebesgue spaces and Plancherel's theorem for the discrete time Fourier transform to show that

$$||T_\omega : S_p \to S_p|| \leq (2\pi)^{-1}||\hat{\mu}||_{L_1([0,1])} \leq (2\pi)^{-1}||\hat{\mu}||_{L_2([0,1])}$$

$$= (2\pi)^{-1}||\mu||_{\ell^2(\mathbb{Z})} = (2\pi)^{-1}\left(\sum_{|j|\leq R} 1\right)^{\frac{1}{2}} = (2\pi)^{-1}(2R+1)^{\frac{1}{2}}.$$

□

Lemma 3.14
Let $p \in (1,\infty)$. Let $R \geq 32$. Then $||T_{(x,y)\mapsto \frac{\chi_{|\lfloor x\rfloor - \lfloor y\rfloor|>R}}{x-y}} : S_p \to S_p|| \leq 2$.

Proof. The first part of this proof is based on the proof of [MS21, Lemma 4.3.5]. The second part of this proof is based on [MS21, Lemma 4.3.4]. Denote by $\{x\}$ and $\{y\}$ the fractional parts of $x, y \in \mathbb{R}$. Then

$$\frac{1}{x-y} = \frac{1}{\lfloor x\rfloor + \{x\} - \lfloor y\rfloor - \{y\}} = \frac{1}{\lfloor x\rfloor - \lfloor y\rfloor} \cdot \frac{1}{1 - \frac{\{y\}-\{x\}}{\lfloor x\rfloor - \lfloor y\rfloor}}.$$

Due to $R > 8$ it follows that $|\frac{\{y\}-\{x\}}{\lfloor x\rfloor - \lfloor y\rfloor}| < 1$. Applying the geometric series yields

$$\frac{\chi_{|\lfloor x\rfloor - \lfloor y\rfloor|>R}}{x-y} = \frac{\chi_{|\lfloor x\rfloor - \lfloor y\rfloor|>R}}{\lfloor x\rfloor - \lfloor y\rfloor} \cdot \frac{1}{1 - \frac{\{y\}-\{x\}}{\lfloor x\rfloor - \lfloor y\rfloor}}$$

$$= \frac{\chi_{|\lfloor x\rfloor - \lfloor y\rfloor|>R}}{\lfloor x\rfloor - \lfloor y\rfloor} \sum_{k=0}^{\infty} \left(\frac{\{y\}-\{x\}}{\lfloor x\rfloor - \lfloor y\rfloor}\right)^k = \sum_{k=0}^{\infty} \frac{\chi_{|\lfloor x\rfloor - \lfloor y\rfloor|>R}}{(\lfloor x\rfloor - \lfloor y\rfloor)^{k+1}}(\{y\}-\{x\})^k.$$

Further, apply the linearity and Lemma 2.4.3 to observe that

$$||T_{(x,y)\mapsto \frac{\chi_{|\lfloor x\rfloor - \lfloor y\rfloor|>R}}{x-y}} : S_p \to S_p|| = ||T_{(x,y)\mapsto \sum_{k=0}^{\infty} \frac{\chi_{|\lfloor x\rfloor - \lfloor y\rfloor|>R}}{(\lfloor x\rfloor - \lfloor y\rfloor)^{k+1}}(\{y\}-\{x\})^k} : S_p \to S_p||$$

$$\leq \sum_{k=0}^{\infty} ||T_{(x,y)\mapsto \frac{\chi_{|\lfloor x\rfloor - \lfloor y\rfloor|>R}}{(\lfloor x\rfloor - \lfloor y\rfloor)^{k+1}}} : S_p \to S_p||||T_{(x,y)\mapsto(\{y\}-\{x\})^k} : S_p \to S_p||$$

$$\leq \sum_{k=0}^{\infty} ||T_{(x,y)\mapsto \frac{\chi_{|\lfloor x\rfloor - \lfloor y\rfloor|>R}}{\lfloor x\rfloor - \lfloor y\rfloor}} : S_p \to S_p||^{(k+1)}||T_{(x,y)\mapsto\{y\}-\{x\}} : S_p \to S_p||^k.$$

Due to $\{x\}$ and $\{y\}$ being bounded above by 1 it follows that

$$||T_{(x,y)\mapsto\{y\}-\{x\}} : S_p \to S_p|| < 2.$$

Inserting this into the previous inequality gives

$$||T_{(x,y)\mapsto \frac{\chi_{|\lfloor x\rfloor - \lfloor y\rfloor|>R}}{x-y}} : S_p \to S_p|| \tag{3.17}$$

$$\leq \sum_{k=0}^{\infty} 2^k ||T_{(x,y)\mapsto \frac{\chi_{|\lfloor x\rfloor - \lfloor y\rfloor|>R}}{\lfloor x\rfloor - \lfloor y\rfloor}} : S_p \to S_p||^{(k+1)}.$$

3

Now, apply [MS21, Lemma 4.3.3] to observe that,

$$\|T_{(x,y)\mapsto \frac{\chi_{|\lfloor x\rfloor-\lfloor y\rfloor|>R}}{\lfloor x\rfloor-\lfloor y\rfloor}} : S_p \to S_p\| = \|T_\omega : S_p \to S_p\|$$

where $\omega : \mathbb{Z}^2 \to \mathbb{Q}$ is defined as the mapping

$$\omega(n,m) = \frac{\chi_{|n-m|>R}}{n-m}.$$

Note that ω has Toeplitz form $\omega(n,m) := \mu(n-m)$, where $\mu(l) = \frac{\chi_{|l|>R}}{l}$. When applying the discrete time Fourier transform to μ observe that the Fourier transform $\hat{\mu}$ is supported in the interval $[0,1]$. Thus, one can apply Lemma 3.12, the Hölder inequality for Lebesgue spaces and Plancherel's theorem for the discrete time Fourier transform to show that

$$\|T_\omega : S_p \to S_p\| \leq (2\pi)^{-1}\|\hat{\mu}\|_{L_1([0,1])} \leq (2\pi)^{-1}\|\hat{\mu}\|_{L_2([0,1])}$$

$$= (2\pi)^{-1}\|\mu\|_{\ell^2(\mathbb{Z})} = (2\pi)^{-1}\left(\sum_{|j|>R}\frac{1}{j^2}\right)^{\frac{1}{2}} \leq \left(\frac{2}{R}\right)^{\frac{1}{2}}$$

Inserting this back into (3.17) and using that $R \geq 32$ and $\frac{2}{R} < 1$ yields

$$\|T_{(x,y)\mapsto \frac{\chi_{|\lfloor x\rfloor-\lfloor y\rfloor|>R}}{x-y}} : S_p \to S_p\| \leq \sum_{k=0}^{\infty} 2^k \left(\frac{2}{R}\right)^{\frac{(k+1)}{2}}$$

$$\leq \sum_{k=0}^{\infty} 2^k \left(\frac{2}{R}\right)^{\frac{k}{2}} \leq \sum_{k=0}^{\infty} 2^{-k} = 2.$$

□

The next step is to prove a result analogous to [MS21, Proposition 4.3.2] for multilinear operator integrals with second order divided difference functions as symbol in the range $p \in (\frac{1}{2}, 1)$.

Theorem 3.15
Let $p \in (\frac{1}{2}, 1)$. Let $p_1, p_2 \in (1,\infty)$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. Let $\phi \in C_c^\beta(\mathbb{R})$, where $\beta \geq \frac{3}{p}$. Then $\|T_{\phi^{[2]}_{\alpha,\lambda}} : S_{p_1} \times S_{p_2} \to S_p\| \lesssim \lambda^2 \|\alpha\|_{\ell_{p^\#}}$.

Proof. Recall that

$$\phi_{\alpha,\lambda} := \sum_{k\in\mathbb{Z}} \alpha_k \phi(\lambda x - k). \tag{3.18}$$

Without loss of generality, it may be assumed that the functions $\{\phi(\lambda\cdot -k)\}_{k\in\mathbb{Z}}$ are disjointly supported. Indeed, otherwise one may select $N > 1$ sufficiently large such that $\{\phi(\lambda \cdot -Nk)\}_{k\in\mathbb{Z}}$ are disjointly supported, and write

$$\phi_{\alpha,\lambda} = \sum_{j=0}^{N-1} \phi_{\alpha^{(j)},\lambda}$$

where $\alpha^{(j)}$ is the sequence $\{\alpha_{j+Nk}\}_{k\in\mathbb{Z}}$. Then the assertion may be proven for each $\phi_{\alpha^{(j)},\lambda}$ separately. Furthermore, due to $\{\phi(\lambda \cdot -k)\}_{k\in\mathbb{Z}}$ being disjointly supported it is assumed that ϕ is supported in $(\xi, \xi+1)$, where $\xi \in \mathbb{R}$. Fix $R \in [32, \infty)$. Now, split $\phi^{[2]}_{\alpha,\lambda}(x,y,z)$ as follows:

3

$$\begin{aligned}
&\phi^{[2]}_{\alpha,\lambda}(x,y,z)\\
&= \phi^{[2]}_{\alpha,\lambda}(x,y,z)\chi_{|\lfloor\lambda x-\xi\rfloor-\lfloor\lambda y-\xi\rfloor|\le R}\chi_{|\lfloor\lambda y-\xi\rfloor-\lfloor\lambda z-\xi\rfloor|\le R}\\
&+\phi^{[2]}_{\alpha,\lambda}(x,y,z)\chi_{|\lfloor\lambda x-\xi\rfloor-\lfloor\lambda y-\xi\rfloor|\le R}\chi_{|\lfloor\lambda y-\xi\rfloor-\lfloor\lambda z-\xi\rfloor|> R}\\
&+\phi^{[2]}_{\alpha,\lambda}(x,y,z)\chi_{|\lfloor\lambda x-\xi\rfloor-\lfloor\lambda y-\xi\rfloor|> R}\\
&:= A_R(x,y,z) + B_R(x,y,z) + C_R(x,y,z)
\end{aligned}$$

First examine the A_R term and observe that

$$\begin{aligned}
A_R(x,y,z) &= \phi^{[2]}_{\alpha,\lambda}(x,y,z)\chi_{|\lfloor\lambda x-\xi\rfloor-\lfloor\lambda y-\xi\rfloor|\le R}\chi_{|\lfloor\lambda y-\xi\rfloor-\lfloor\lambda z-\xi\rfloor|\le R}\\
&= \phi^{[2]}_{\alpha,\lambda}(x,y,z)\chi_{|\lfloor\lambda x-\xi\rfloor-\lfloor\lambda y-\xi\rfloor|\le R}\chi_{|\lfloor\lambda y-\xi\rfloor-\lfloor\lambda z-\xi\rfloor|\le R}\chi_{|\lfloor\lambda x-\xi\rfloor-\lfloor\lambda z-\xi\rfloor|\le 2R}\\
&= \sum_{|a|\le R}\sum_{|b|\le R}\sum_{|c|\le 2R} F_{a,b,c}(x,y,z).
\end{aligned}$$

where

$$\begin{aligned}
&F_{a,b,c}(x,y,z)\\
&= \phi^{[2]}_{\alpha,\lambda}(x,y,z)\chi_{\lfloor\lambda x-\xi\rfloor-\lfloor\lambda y-\xi\rfloor=a}\chi_{\lfloor\lambda y-\xi\rfloor-\lfloor\lambda z-\xi\rfloor=b}\chi_{\lfloor\lambda x-\xi\rfloor-\lfloor\lambda z-\xi\rfloor=c}.
\end{aligned}$$

Note that

$$\chi_{\lfloor v\rfloor-\lfloor w\rfloor=i} = \sum_{k\in\mathbb{Z}} \chi_{[k,k+1)}(v)\chi_{[k+i,k+i+1)}(w).$$

Thus,

$$\begin{aligned}
&F_{a,b,c}(x,y,z)\\
&= \phi^{[2]}_{\alpha,\lambda}(x,y,z)\\
&\cdot\left(\sum_{i\in\mathbb{Z}}\chi_{[i,i+1)}(\lambda x-\xi)\chi_{[i+a,i+a+1)}(\lambda y-\xi)\right)\\
&\cdot\left(\sum_{j\in\mathbb{Z}}\chi_{[j,j+1)}(\lambda y-\xi)\chi_{[j+b,j+b+1)}(\lambda z-\xi)\right)
\end{aligned}$$

$$\cdot\left(\sum_{k\in\mathbb{Z}}\chi_{[k,k+1)}(\lambda x-\xi)\chi_{[k+c,k+c+1)}(\lambda z-\xi)\right)$$

$$=\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}G_{a,b,c,i,j,k}(x,y,z).$$

where

$$G_{a,b,c,i,j,k}(x,y,z)$$
$$=\phi^{[2]}_{\alpha,\lambda}(x,y,z)\chi_{[i,i+1)}(\lambda x-\xi)\chi_{[i+a,i+a+1)}(\lambda y-\xi)\chi_{[j,j+1)}(\lambda y-\xi)$$
$$\cdot\chi_{[j+b,j+b+1)}(\lambda z-\xi)\chi_{[k,k+1)}(\lambda x-\xi)\chi_{[k+c,k+c+1)}(\lambda z-\xi)$$
$$=\left(\sum_{\ell\in\mathbb{Z}}\alpha_\ell\lambda^2\frac{\frac{\phi(\lambda x-\ell)-\phi(\lambda y-\ell)}{(\lambda x-\ell)-(\lambda y-\ell)}-\frac{\phi(\lambda y-\ell)-\phi(\lambda z-\ell)}{(\lambda y-\ell)-(\lambda z-\ell)}}{(\lambda x-\ell)-(\lambda z-\ell)}\right)$$
$$\cdot\chi_{[i,i+1)}(\lambda x-\xi)\chi_{[i+a,i+a+1)}(\lambda y-\xi)\chi_{[j,j+1)}(\lambda y-\xi)$$
$$\cdot\chi_{[j+b,j+b+1)}(\lambda z-\xi)\chi_{[k,k+1)}(\lambda x-\xi)\chi_{[k+c,k+c+1)}(\lambda z-\xi).$$

Observe that for $G_{a,b,c,i,j,k}(x,y,z)\neq 0$ it is required that $i=k$, $i+a=j$, $j+b=k+c$. For a,b,c,i,j,k satisfying these constraints,

$$G_{a,b,c,i,j,k}(x,y,z)$$
$$=\left(\sum_{\ell\in\mathbb{Z}}\alpha_\ell\lambda^2\frac{\frac{\phi(\lambda x-\ell)-\phi(\lambda y-\ell)}{(\lambda x-\ell)-(\lambda y-\ell)}-\frac{\phi(\lambda y-\ell)-\phi(\lambda z-\ell)}{(\lambda y-\ell)-(\lambda z-\ell)}}{(\lambda x-\ell)-(\lambda z-\ell)}\right)$$
$$\cdot\chi_{[i,i+1)}(\lambda x-\xi)\chi_{[i+a,i+a+1)}(\lambda y-\xi)\chi_{[i+c,i+c+1)}(\lambda z-\xi).$$

and

$$F_{a,b,c}(x,y,z)=\sum_{i\in\mathbb{Z}}G_{a,b,c,i,i+a,i}(x,y,z).$$

Additionally, it follows from Lemma 3.6 that

$$\|T_{F_{a,b,c}}:S_{p_1}\times S_{p_2}\to S_p\|\leq\|\{\|T_{G_{a,b,c,i,i+a,i}}:S_{p_1}\times S_{p_2}\to S_p\|\}_{i\in\mathbb{Z}}\|_{\ell_{p^\#}}.$$

Fix (x,y,z) such that $G_{a,b,c,i,i+a,i}(x,y,z)$ is nonzero. Then one of the following conditions is satisfied

#1	$\phi(\lambda x-\ell)\neq 0$	$\phi(\lambda y-\ell)\neq 0$	$\phi(\lambda z-\ell)\neq 0$
#2	$\phi(\lambda x-\ell)\neq 0$	$\phi(\lambda y-\ell)\neq 0$	$\phi(\lambda z-\ell)=0$
#3	$\phi(\lambda x-\ell)=0$	$\phi(\lambda y-\ell)\neq 0$	$\phi(\lambda z-\ell)\neq 0$
#4	$\phi(\lambda x-\ell)\neq 0$	$\phi(\lambda y-\ell)=0$	$\phi(\lambda z-\ell)\neq 0$
#5	$\phi(\lambda x-\ell)\neq 0$	$\phi(\lambda y-\ell)=0$	$\phi(\lambda z-\ell)=0$
#6	$\phi(\lambda x-\ell)=0$	$\phi(\lambda y-\ell)\neq 0$	$\phi(\lambda z-\ell)=0$
#7	$\phi(\lambda x-\ell)=0$	$\phi(\lambda y-\ell)=0$	$\phi(\lambda z-\ell)\neq 0.$

Each condition respectively implies one of the following options,

#1	$(\lambda x - \ell) \in (\xi, \xi+1)$	$(\lambda y - \ell) \in (\xi, \xi+1)$	$(\lambda z - \ell) \in (\xi, \xi+1)$
#2	$(\lambda x - \ell) \in (\xi, \xi+1)$	$(\lambda y - \ell) \in (\xi, \xi+1)$	$(\lambda z - \ell) \notin (\xi, \xi+1)$
#3	$(\lambda x - \ell) \notin (\xi, \xi+1)$	$(\lambda y - \ell) \in (\xi, \xi+1)$	$(\lambda z - \ell) \in (\xi, \xi+1)$
#4	$(\lambda x - \ell) \in (\xi, \xi+1)$	$(\lambda y - \ell) \notin (\xi, \xi+1)$	$(\lambda z - \ell) \in (\xi, \xi+1)$
#5	$(\lambda x - \ell) \in (\xi, \xi+1)$	$(\lambda y - \ell) \notin (\xi, \xi+1)$	$(\lambda z - \ell) \notin (\xi, \xi+1)$
#6	$(\lambda x - \ell) \notin (\xi, \xi+1)$	$(\lambda y - \ell) \in (\xi, \xi+1)$	$(\lambda z - \ell) \notin (\xi, \xi+1)$
#7	$(\lambda x - \ell) \notin (\xi, \xi+1)$	$(\lambda y - \ell) \notin (\xi, \xi+1)$	$(\lambda z - \ell) \in (\xi, \xi+1)$.

Firstly, when $a \neq c$ and $a \neq 0$ and $c \neq 0$ observe that there exists no $\ell \in \mathbb{Z}$ such that option 1, 2, 3 or 4 is valid, due to the indicator functions present in $G_{a,b,c,i,i+a,i}$. This leaves options 5, 6 and 7. For option 5 note that the first indicator function implies that $(\lambda x - \xi) \in [i, i+1)$. Combining this with the requirement that $(\lambda x - \ell) \in (\xi, \xi+1)$ implies that $\ell \in [i, i+1) \Rightarrow \ell = i$. Repeating this argumentation yields that for option 6 to be valid it is required that $\ell = i + a$, and for option 7 to be valid it is required that $\ell = i + c$. Thus observe that when $a \neq c$ and $a \neq 0$ and $c \neq 0$ that

$$G_{a,b,c,i,i+a,i}(x,y,z)$$

$$= \alpha_i \lambda^2 \phi^{[2]}(\lambda x - i, \lambda y - i, \lambda z - i)\chi_{[i+a,i+a+1)}(\lambda y - \xi)\chi_{[i+c,i+c+1)}(\lambda z - \xi)$$

$$+\alpha_{i+a}\lambda^2\phi^{[2]}(\lambda x - i - a, \lambda y - i - a, \lambda z - i - a)\chi_{[i,i+1)}(\lambda x - \xi)\chi_{[i+c,i+c+1)}(\lambda z - \xi)$$

$$+\alpha_{i+c}\lambda^2\phi^{[2]}(\lambda x - i - c, \lambda y - i - c, \lambda z - i - c)\chi_{[i,i+1)}(\lambda x - \xi)\chi_{[i+a,i+a+1)}(\lambda y - \xi)$$

and

$$||T_{G_{a,b,c,i,i+a,i}}||^p \leq |\alpha_i|^p \lambda^{2p} ||T_{\phi^{[2]}}||^p + |\alpha_{i+a}|^p \lambda^{2p} ||T_{\phi^{[2]}}||^p + |\alpha_{i+c}|^p \lambda^{2p} ||T_{\phi^{[2]}}||^p.$$

Secondly, when $a = 0$ and $c \neq 0$ observe that there exists no $\ell \in \mathbb{Z}$ such that option 1, 3, 4, 5 or 6 is valid, due to the indicator functions present in $G_{a,b,c,i,i+a,i}$. This leaves options 2 and 7. For option 2 note that the first and second indicator function imply that $(\lambda x - \xi) \in [i, i+1)$ and $(\lambda y - \xi) \in [i, i+1)$. Combining this with the requirement that $(\lambda x - \ell) \in (\xi, \xi+1)$ and $(\lambda y - \ell) \in (\xi, \xi+1)$ implies that $\ell \in [i, i+1) \Rightarrow \ell = i$. For option 7 note that the third indicator function implies that $(\lambda z - \xi) \in [i+c, i+c+1)$. Combining this with the requirement that $(\lambda z - \ell) \in (\xi, \xi+1)$ implies that $\ell \in [i+c, i+c+1) \Rightarrow \ell = i+c$. Thus observe that when $a = 0$ and $c \neq 0$ that

$$G_{a,b,c,i,i+a,i}(x,y,z)$$

$$= \alpha_i \lambda^2 \phi^{[2]}(\lambda x - i, \lambda y - i, \lambda z - i)\chi_{[i+c,i+c+1)}(\lambda z - \xi)$$

$$+\alpha_{i+c}\lambda^2\phi^{[2]}(\lambda x - i - c, \lambda y - i - c, \lambda z - i - c)\chi_{[i,i+1)}(\lambda x - \xi)\chi_{[i,i+1)}(\lambda y - \xi)$$

3

and

$$||T_{G_{a,b,c,i,i+a,i}}||^p \leq |\alpha_i|^p \lambda^{2p} ||T_{\phi^{[2]}}||^p + |\alpha_{i+c}|^p \lambda^{2p} ||T_{\phi^{[2]}}||^p.$$

Thirdly, when $a = c$ and $c \neq 0$ and $a \neq 0$ observe that there exists no $\ell \in \mathbb{Z}$ such that option 1, 2, 4, 6 or 7 is valid, due to the indicator functions present in $G_{a,b,c,i,i+a,i}$. This leaves options 3 and 5. For option 3 note that the second and third indicator function imply that $(\lambda y - \xi) \in [i+a, i+a+1)$ and $(\lambda z - \xi) \in [i+a, i+a+1)$. Combining this with the requirement that $(\lambda y - \ell) \in (\xi, \xi+1)$ and $(\lambda z - \ell) \in (\xi, \xi+1)$ implies that $\ell \in [i+a, i+a+1) \Rightarrow \ell = i+a$. For option 5 note that the third indicator function implies that $(\lambda x - \xi) \in [i, i+1)$. Combining this with the requirement that $(\lambda x - \ell) \in (\xi, \xi+1)$ implies that $\ell \in [i, i+1) \Rightarrow \ell = i$. Thus observe that when $a = c$ and $c \neq 0$ and $a \neq 0$ that

$$G_{a,b,c,i,i+a,i}(x,y,z)$$

$$= \alpha_{i+a}\lambda^2 \phi^{[2]}(\lambda x - i - a, \lambda y - i - a, \lambda z - i - a)\chi_{[i,i+1)}(\lambda x - \xi)$$

$$+\alpha_i \lambda^2 \phi^{[2]}(\lambda x - i, \lambda y - i, \lambda z - i)\chi_{[i+a,i+a+1)}(\lambda y - \xi)\chi_{[i+a,i+a+1)}(\lambda z - \xi)$$

and

$$||T_{G_{a,b,c,i,i+a,i}}||^p \leq |\alpha_i|^p \lambda^{2p} ||T_{\phi^{[2]}}||^p + |\alpha_{i+a}|^p \lambda^{2p} ||T_{\phi^{[2]}}||^p.$$

Fourthly, when $0 = c$ and $a \neq 0$ observe that there exists no $\ell \in \mathbb{Z}$ such that option 1, 2, 3, 5 or 7 is valid, due to the indicator functions present in $G_{a,b,c,i,i+a,i}$. This leaves options 4 and 6. For option 4 note that the first and third indicator function imply that $(\lambda x - \xi) \in [i, i+1)$ and $(\lambda z - \xi) \in [i, i+1)$. Combining this with the requirement that $(\lambda x - \ell) \in (\xi, \xi+1)$ and $(\lambda z - \ell) \in (\xi, \xi+1)$ implies that $\ell \in [i, i+1) \Rightarrow \ell = i$. For option 6 note that the second indicator function implies that $(\lambda y - \xi) \in [i+a, i+a+1)$. Combining this with the requirement that $(\lambda y - \ell) \in (\xi, \xi+1)$ implies that $\ell \in [i+a, i+a+1) \Rightarrow \ell = i+a$. Thus observe that when $0 = c$ and $a \neq 0$ that

$$G_{a,b,c,i,i+a,i}(x,y,z)$$

$$= \alpha_i \lambda^2 \phi^{[2]}(\lambda x - i, \lambda y - i, \lambda z - i)\chi_{[i+a,i+a+1)}(\lambda y - \xi)$$

$$+\alpha_{i+a}\lambda^2 \phi^{[2]}(\lambda x - i - a, \lambda y - i - a, \lambda z - i - a)\chi_{[i,i+1)}(\lambda x - \xi)\chi_{[i,i+1)}(\lambda z - \xi)$$

and

$$||T_{G_{a,b,c,i,i+a,i}}||^p \leq |\alpha_i|^p \lambda^{2p} ||T_{\phi^{[2]}}||^p + |\alpha_{i+a}|^p \lambda^{2p} ||T_{\phi^{[2]}}||^p.$$

Finally, when $0 = a = c$ observe that there exists no $\ell \in \mathbb{Z}$ such that option 2, 3, 4, 5, 6 or 7 is valid, due to the indicator functions present in $G_{a,b,c,i,i+a,i}$. This leaves option 1. For option 1 note that the first, second and third indicator function imply that $(\lambda x - \xi) \in [i, i+1)$ and $(\lambda y - \xi) \in [i, i+1)$ and $(\lambda z - \xi) \in [i, i+1)$. Combining this with the requirement that $(\lambda x - \ell) \in (\xi, \xi+1)$ and $(\lambda y - \ell) \in (\xi, \xi+1)$ and

$(\lambda z - \ell) \in (\xi, \xi + 1)$ implies that $\ell \in [i, i+1) \Rightarrow \ell = i$. Thus observe that when $0 = a = c$ that

$$G_{a,b,c,i,i+a,i}(x,y,z)$$
$$= \alpha_i \lambda^2 \phi^{[2]}(\lambda x - i, \lambda y - i, \lambda z - i)$$

and

$$||T_{G_{a,b,c,i,i+a,i}}||^p \leq |\alpha_i|^p \lambda^{2p} ||T_{\phi^{[2]}}||^p.$$

Combining all the previously found inequalities yields

$$||T_{G_{a,b,c,i,i+a,i}}|| \lesssim (|\alpha_i|^p + |\alpha_{i+a}|^p + |\alpha_{i+c}|^p)^{1/p} \lambda^2 ||T_{\phi^{[2]}}||$$

and it follows from Lemma 3.6 that

$$||T_{F_{a,b,c}}|| \leq \lambda^2 ||T_{\phi^{[2]}}|| ||\{(|\alpha_i|^p + |\alpha_{i+a}|^p + |\alpha_{i+c}|^p)^{1/p}\}_{i \in \mathbb{Z}}||_{\ell_{p^\#}}$$

$$= \lambda^2 ||T_{\phi^{[2]}}|| (\sum_{i \in \mathbb{Z}} (|\alpha_i|^p + |\alpha_{i+a}|^p + |\alpha_{i+c}|^p)^{p^\#/p} \Big)^{1/p^\#}$$

$$= \lambda^2 ||T_{\phi^{[2]}}|| (\sum_{i \in \mathbb{Z}} (|\alpha_i|^p + |\alpha_{i+a}|^p + |\alpha_{i+c}|^p)^{\frac{1}{1-p}} \Big)^{\frac{1-p}{p}}$$

$$= \lambda^2 ||T_{\phi^{[2]}}|| (\left(\left(\sum_{i \in \mathbb{Z}} (|\alpha_i|^p + |\alpha_{i+a}|^p + |\alpha_{i+c}|^p)^{\frac{1}{1-p}} \right)^{1-p} \right)^{1/p}.$$

Now apply Minkowski's inequality for sequence spaces to observe that

$$||T_{F_{a,b,c}}|| \leq \lambda^2 ||T_{\phi^{[2]}}||$$

$$\cdot \left(\left(\sum_{i \in \mathbb{Z}} (|\alpha_i|^p)^{\frac{1}{1-p}} \right)^{1-p} + \left(\sum_{i \in \mathbb{Z}} (|\alpha_{i+a}|^p)^{\frac{1}{1-p}} \right)^{1-p} + \left(\sum_{i \in \mathbb{Z}} (|\alpha_{i+c}|^p)^{\frac{1}{1-p}} \right)^{1-p} \right)^{1/p}$$

$$= \lambda^2 ||T_{\phi^{[2]}}|| (\left(\left(\sum_{i \in \mathbb{Z}} |\alpha_i|^{p^\#} \right)^{1-p} + \left(\sum_{i \in \mathbb{Z}} |\alpha_i|^{p^\#} \right)^{1-p} + \left(\sum_{i \in \mathbb{Z}} |\alpha_i|^{p^\#} \right)^{1-p} \right)^{1/p}$$

$$\approx \lambda^2 ||T_{\phi^{[2]}}|| (\sum_{i \in \mathbb{Z}} |\alpha_i|^{p^\#})^{1/p^\#} = \lambda^2 ||T_{\phi^{[2]}}|| ||\{\alpha_i\}_{i \in \mathbb{Z}}||_{\ell_{p^\#}}.$$

From which it follows that

$$||T_{(x,y,z) \mapsto A_R(x,y,z)} : S_{p_1} \times S_{p_2} \to S_p|| \lesssim \lambda^2 ||T_{\phi^{[2]}}|| ||\alpha||_{\ell_{p^\#}}. \tag{3.19}$$

3

Note that $||T_{\phi^{[2]}}||$ is bounded due to Theorem 3.3.

Now, examine the B_R term

$$B_R(x,y,z) = \phi^{[2]}_{\alpha,\lambda}(x,y,z)\chi_{|\lfloor \lambda x-\xi\rfloor - \lfloor \lambda y-\xi\rfloor| \le R}\chi_{|\lfloor \lambda y-\xi\rfloor - \lfloor \lambda z-\xi\rfloor| > R}.$$

Use that $\phi^{[2]}_{\alpha,\lambda}$ is invariant under permutation of its input variables to observe that

$$B_R(x,y,z) = \phi^{[2]}_{\alpha,\lambda}(y,x,z)\chi_{|\lfloor \lambda x-\xi\rfloor - \lfloor \lambda y-\xi\rfloor| \le R}\chi_{|\lfloor \lambda y-\xi\rfloor - \lfloor \lambda z-\xi\rfloor| > R}$$
$$= \lambda\phi^{[1]}_{\alpha,\lambda}(y,x)\chi_{|\lfloor \lambda x-\xi\rfloor - \lfloor \lambda y-\xi\rfloor| \le R}\frac{\chi_{|\lfloor \lambda y-\xi\rfloor - \lfloor \lambda z-\xi\rfloor| > R}}{(\lambda y-\xi)-(\lambda z-\xi)}$$
$$-\lambda\phi^{[1]}_{\alpha,\lambda}(x,z)\chi_{|\lfloor \lambda x-\xi\rfloor - \lfloor \lambda y-\xi\rfloor| \le R}\frac{\chi_{|\lfloor \lambda y-\xi\rfloor - \lfloor \lambda z-\xi\rfloor| > R}}{(\lambda y-\xi)-(\lambda z-\xi)}.$$

Note that due to Lemma 2.4.3, [MS21, Theorem 4.3.2], Lemma 3.13 and Lemma 3.14 that

$$||T_{(x,y,z)\mapsto B(x,y,z)} : S_{p_1}\times S_{p_2}\to S_p||^p \tag{3.20}$$
$$\le \lambda^p ||T_{(y,x)\mapsto \phi^{[1]}_{\alpha,\lambda}(x,y)} : S_{p_1}\to S_{p_1}||^p$$
$$||T_{(x,y)\mapsto \chi_{|\lfloor \lambda x-\xi\rfloor - \lfloor \lambda y-\xi\rfloor| \le R}} : S_{p_1}\to S_{p_1}||^p ||T_{(y,z)\mapsto \frac{\chi_{|\lfloor \lambda y-\xi\rfloor - \lfloor \lambda z-\xi\rfloor| > R}}{(\lambda y-\xi)-(\lambda z-\xi)}} : S_{p_2}\to S_{p_2}||^p$$
$$+\lambda^p ||T_{(x,z)\mapsto \phi^{[1]}_{\alpha,\lambda}(x,z)} : S_p\to S_p||^p$$
$$||T_{(x,y)\mapsto \chi_{|\lfloor \lambda x-\xi\rfloor - \lfloor \lambda y-\xi\rfloor| \le R}} : S_{p_1}\to S_{p_1}||^p ||T_{(y,z)\mapsto \frac{\chi_{|\lfloor \lambda y-\xi\rfloor - \lfloor \lambda z-\xi\rfloor| > R}}{(\lambda y-\xi)-(\lambda z-\xi)}} : S_{p_2}\to S_{p_2}||^p$$
$$\lesssim \lambda^{2p}||\alpha||^p_{\ell_{p^\#}}.$$

Now, examine the C_R term

$$C_R(x,y,z) = \phi^{[2]}_{\alpha,\lambda}(x,y,z)\chi_{|\lfloor \lambda x-\xi\rfloor - \lfloor \lambda y-\xi\rfloor| > R}.$$

Use that $\phi^{[2]}_{\alpha,\lambda}$ is invariant under permutation of its input variables to observe that

$$C_R(x,y,z) = \phi^{[2]}_{\alpha,\lambda}(x,z,y)\chi_{|\lfloor \lambda x-\xi\rfloor - \lfloor \lambda y-\xi\rfloor| > R}$$
$$= \lambda\phi^{[1]}_{\alpha,\lambda}(x,z)\frac{\chi_{|\lfloor \lambda x-\xi\rfloor - \lfloor \lambda y-\xi\rfloor| > R}}{(\lambda x-\xi)-(\lambda y-\xi)}$$
$$-\lambda\phi^{[1]}_{\alpha,\lambda}(z,y)\frac{\chi_{|\lfloor \lambda x-\xi\rfloor - \lfloor \lambda y-\xi\rfloor| > R}}{(\lambda x-\xi)-(\lambda y-\xi)}.$$

Note that due to Lemma 2.4.3, [MS21, Theorem 4.3.2], Lemma 3.13 and Lemma 3.14 that

$$||T_{(x,y,z)\mapsto C(x,y,z)} : S_{p_1} \times S_{p_2} \to S_p||^p \tag{3.21}$$
$$\leq \lambda^p ||T_{(x,z)\mapsto \phi^{[1]}_{\alpha,\lambda}(x,z)} : S_p \to S_p||^p ||T_{(x,y)\mapsto \frac{\chi_{|\lfloor \lambda x - \xi \rfloor - \lfloor \lambda y - \xi \rfloor| > R}}{(\lambda x - \xi) - (\lambda y - \xi)}} : S_{p_1} \to S_{p_1}||^p$$
$$+ \lambda^p ||T_{(y,z)\mapsto \phi^{[1]}_{\alpha,\lambda}(x,z)} : S_{p_2} \to S_{p_2}||^p ||T_{(x,y)\mapsto \frac{\chi_{|\lfloor \lambda x - \xi \rfloor - \lfloor \lambda y - \xi \rfloor| > R}}{(\lambda x - \xi) - (\lambda y - \xi)}} : S_{p_1} \to S_{p_1}||^p$$
$$\lesssim \lambda^{2p} ||\alpha||^p_{\ell_{p^\#}}.$$

Combining the previously found upper bounds (3.19-3.21) yields

$$||T_{\phi^{[2]}_{\alpha,\lambda}} : S_{p_1} \times S_{p_2} \to S_p|| \lesssim \lambda^2 ||\alpha||_{\ell_{p^\#}}.$$

□

Lemma 3.16
Let $p \in (\frac{1}{2}, 1)$. Let $p_1, p_2 \in (1, \infty)$ be such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. Let f be a locally integrable function on $\mathbb{R}$, and let $j \in \mathbb{Z}$ be such that f_j is bounded where f_j is computed with respect to a compactly supported C^β wavelet ϕ, where $\beta \geq \frac{3}{p}$. Then $||T_{f_j^{[2]}} : S_{p_1} \times S_{p_2} \to S_p|| \lesssim 2^{j(\frac{1}{p}+1)} ||f_j||_{p^\#}$.

Proof. Observe that

$$f_j(t) = \sum_{k\in\mathbb{Z}} 2^{j/2} \phi(2^j t - k) < f, \phi_{j,k} >.$$

Which has the correct form to be able to apply Theorem 3.15 and Lemma 2.5.1 to conclude that

$$||T_{f_j^{[2]}}|| \leq 2^{\frac{5j}{2}} \left(\sum_{k\in\mathbb{Z}} (| < f, \phi_{j,k} > |)^{p^\#} \right)^{\frac{1}{p^\#}} \approx_\phi 2^{\frac{5j}{2}} 2^{j(\frac{1}{p^\#} - \frac{1}{2})} ||f_j||_{p^\#} = 2^{j(\frac{1}{p}+1)} ||f_j||_{p^\#}$$

□

Collorary 3.17
Let $p \in (\frac{1}{2}, 1)$. Let $p_1, p_2 \in (1, \infty)$ be such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. Let $f \in \dot{B}^{1/p}_{p^\#,p}(\mathbb{R}) \cap \dot{B}^{1/p+1}_{p^\#,p}(\mathbb{R})$ be Lipschitz. Then $||T_{f^{[2]}} : S_{p_1} \times S_{p_2} \to S_p|| \lesssim ||f||_{\dot{B}^{1/p+1}_{p^\#,p}}$.

Proof. Apply Lemma 3.9, Lemma 3.16 and Lemma 2.6.2 to conclude that

$$||T_{f^{[2]}}|| \leq \left(\sum_{j\in\mathbb{Z}} ||T_{f_j^{[2]}}||^p \right)^{\frac{1}{p}} \lesssim \left(\sum_{j\in\mathbb{Z}} 2^{j(\frac{1}{p}+1)p} ||f_j||^p_{p^\#} \right)^{\frac{1}{p}} \approx_\phi ||f||_{\dot{B}^{1/p+1}_{p^\#,p}}.$$

□

This completes the proof of Theorem 1.2.

3.6. RESULTS AND OUTLOOK ON $p \in (0,1)$

The next two results indicates what types of bounds may be found in the entire range $p \in (0,1)$. Observe that these results appear mostly similar to previous results but with $p^\#$ replaced by $p^\flat$.

Lemma 3.18
Let $\{\phi_k : \mathbb{R}^3 \to \mathbb{C}\}_{k\in\mathbb{Z}}$ be a sequence of disjointly supported functions in all variables. That is, if $\phi_k(x,y,z) \neq 0$, then for all $k' \neq k \in \mathbb{Z}$ and $s,t,u \in \mathbb{R}$ it holds that $\phi_{k'}(x,t,u) = \phi_{k'}(s,y,u) = \phi_{k'}(s,t,z) = 0$. Choose $\phi : \mathbb{R}^3 \to \mathbb{C}$ such that $\phi(x,y,z) = \sum_{k\in\mathbb{Z}} \phi_k(x,y,z)$ for all $x,y,z \in \mathbb{R}$. Let $p \in (0,1]$. Let $p_1, p_2 \in (0,\infty)$ be such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. Then $||T_\phi : S_{p_1} \times S_{p_2} \to S_p|| \leq ||\{||T_{\phi_k} : S_{p_1} \times S_{p_2} \to S_p||\}_{k\in\mathbb{Z}}||_{\ell_{p^\flat}}$.

Proof. Without loss of generality it may be assumed that

$$\phi_k(x,y,z) = \phi_k(x,y,z)\chi_{s_k}(x)\chi_{t_k}(y)\chi_{u_k}(z).$$

where $\{s_k\}_{k\in\mathbb{Z}}, \{t_k\}_{k\in\mathbb{Z}}, \{u_k\}_{k\in\mathbb{Z}}$ are pairwise disjoint partitions of $\mathbb{R}$ and for all $k' \neq k$ it holds that $s_{k'} \cap s_k = t_{k'} \cap t_k = u_{k'} \cup u_k = \emptyset$. Let H_1, H_2, H_3 be arbitrary self-adjoint operators. Let $V \in S_{p_1}$ and $W \in S_{p_2}$ be arbitrary. Further, define $V_k := \chi_{s_k}(H_1)V\chi_{t_k}(H_2)$ and $W_k := \chi_{t_k}(H_2)W\chi_{u_k}(H_3)$. Applying (2.3) and Hölder's inequality for sequence spaces yields

$$||T_\phi^{H_1,H_2,H_3}(V,W)||_{S_p} \leq \left(\sum_{k\in\mathbb{Z}} ||T_{\phi_k}^{H_1,H_2,H_3}(V,W)||_{S_p}^p\right)^{1/p} \tag{3.22}$$

$$= \left(\sum_{k\in\mathbb{Z}} ||T_{\phi_k}^{H_1,H_2,H_3}(V_k,W_k)||_{S_p}^p\right)^{1/p}$$

$$\leq \left(\sum_{k\in\mathbb{Z}} ||T_{\phi_k}^{H_1,H_2,H_3}||^p ||V_k||_{S_{p_1}}^p ||W_k||_{S_{p_2}}^p\right)^{1/p}$$

$$= ||\{||T_{\phi_k}^{H_1,H_2,H_3}|| ||V_k||_{S_{p_1}} ||W_k||_{S_{p_2}}\}_{k\in\mathbb{Z}}||_{\ell_p}$$

$$\leq ||\{||T_{\phi_k}^{H_1,H_2,H_3}||\}_{k\in\mathbb{Z}}||_{\ell_{p^\flat}} ||\{||V_k||_{S_{p_1}} ||W_k||_{S_{p_2}}\}_{k\in\mathbb{Z}}||_{\ell_2}$$

By using the property that when $q \in (0,2)$ it holds that $\lim_{m\to\infty} ||P_m A||_{S_q}^q \leq \lim_{m\to\infty} ||P_m||^q ||A||_{S_q}^q = ||A||_{S_q}^q$ instead of [FK14, Lemma 2.1(ii)] in the proof of [FK14, Theorem 2.4(i)], it follows by careful observation of the proof of [FK14, Theorem 4.2(i)] that for all $A \in S_q$

$$\left(\sum_{k,l\in\mathbb{Z}} ||\chi_{s_k}(H_1)A\chi_{t_l}(H_2)||_{S_2}^2\right)^{\frac{1}{2}} \leq ||A||_{S_2}. \tag{3.23}$$

Let $r \in [2,\infty)$. Then apply [FK14, Theorem 4.2(ii)] to note that

$$\|\{\|V_k\|_{S_r}\}_{k\in\mathbb{Z}}\|_{\ell_r} = \left(\sum_{k\in\mathbb{Z}} \|\chi_{s_k}(H_1)V\chi_{t_k}(H_2)\|_{S_r}^r\right)^{\frac{1}{r}} \tag{3.24}$$

$$\leq \left(\sum_{k,l\in\mathbb{Z}} \|\chi_{s_k}(H_1)V\chi_{t_l}(H_2)\|_{S_r}^r\right)^{\frac{1}{r}} \leq \|V\|_{S_r}$$

and

$$\|\{\|W_k\|_{S_r}\}_{k\in\mathbb{Z}}\|_{\ell_r} = \left(\sum_{k\in\mathbb{Z}} \|\chi_{t_k}(H_2)W\chi_{u_k}(H_3)\|_{S_r}^r\right)^{\frac{1}{r}} \tag{3.25}$$

$$\leq \left(\sum_{k,l\in\mathbb{Z}} \|\chi_{t_k}(H_2)W\chi_{u_l}(H_3)\|_{S_r}^r\right)^{\frac{1}{r}} \leq \|W\|_{S_r}.$$

Now, analyse all possible combinations of p_1 and p_2. Firstly, examine the case where $p_1 \in (0,2)$ and $p_2 \in (0,2)$. Note that

$$\|\{\|V_k\|_{S_{p_1}}\|W_k\|_{S_{p_2}}\}_{k\in\mathbb{Z}}\|_{\ell_2} \leq \|\{\|V_k\|_{S_{p_1}}\|W_k\|_{S_{p_2}}\}_{k\in\mathbb{Z}}\|_{\ell_1}$$

$$\leq \|\{\|V_k\|_{S_{p_1}}\}_{k\in\mathbb{Z}}\|_{\ell_2}\|\{\|W_k\|_{S_{p_2}}\}_{k\in\mathbb{Z}}\|_{\ell_2}.$$

Then, one can apply (3.23) to show that

$$\|\{\|V_k\|_{S_{p_1}}\}_{k\in\mathbb{Z}}\|_{\ell_2} = \left(\sum_{k\in\mathbb{Z}} \|\chi_{s_k}(H_1)V\chi_{t_k}(H_2)\|_{S_{p_1}}^2\right)^{\frac{1}{2}} \tag{3.26}$$

$$\leq \left(\sum_{k,l\in\mathbb{Z}} \|\chi_{s_k}(H_1)V\chi_{t_l}(H_2)\|_{S_{p_1}}^2\right)^{\frac{1}{2}} \leq \|V\|_{S_{p_1}}$$

and

$$\|\{\|W_k\|_{S_{p_2}}\}_{k\in\mathbb{Z}}\|_{\ell_2} = \left(\sum_{k\in\mathbb{Z}} \|\chi_{t_k}(H_2)W\chi_{u_k}(H_3)\|_{S_{p_2}}^2\right)^{\frac{1}{2}} \tag{3.27}$$

$$\leq \left(\sum_{k,l\in\mathbb{Z}} \|\chi_{t_k}(H_2)W\chi_{u_l}(H_3)\|_{S_{p_2}}^2\right)^{\frac{1}{2}} \leq \|W\|_{S_{p_2}}.$$

Secondly, examine the case where $p_1 \in (0,2)$ and $p_2 \in [2,\infty)$. Note that $\frac{2p_2}{p_2+2} < 2$ and $(\frac{2p_2}{p_2+2})^{-1} = 2^{-1} + p_2^{-1}$. Thus, one can apply (3.26) and (3.25) to observe that

$$\|\{\|V_k\|_{S_{p_1}}\|W_k\|_{S_{p_2}}\}_{k\in\mathbb{Z}}\|_{\ell_2} \leq \|\{\|V_k\|_{S_{p_1}}\|W_k\|_{S_{p_2}}\}_{k\in\mathbb{Z}}\|_{\ell_{\frac{2p_2}{p_2+2}}}$$

$$\leq \|\{\|V_k\|_{S_{p_1}}\}_{k\in\mathbb{Z}}\|_{\ell_2}\|\{\|W_k\|_{S_{p_2}}\}_{k\in\mathbb{Z}}\|_{\ell_{p_2}} \leq \|V\|_{S_{p_1}}\|W\|_{S_{p_2}}$$

Finally, examine the case where $p_1 \in [2,\infty)$ and $p_2 \in (0,2)$. Note that $\frac{2p_1}{p_1+2} < 2$ and $(\frac{2p_1}{p_1+2})^{-1} = 2^{-1} + p_1^{-1}$. Thus, one can apply (3.27) and (3.24) to observe that

$$\|\{\|V_k\|_{S_{p_1}}\|W_k\|_{S_{p_2}}\}_{k\in\mathbb{Z}}\|_{\ell_2} \leq \|\{\|V_k\|_{S_{p_1}}\|W_k\|_{S_{p_2}}\}_{k\in\mathbb{Z}}\|_{\ell_{\frac{2p_1}{p_1+2}}}$$

3

$$\leq \|\{\|V_k\|_{S_{p_1}}\}_{k\in\mathbb{Z}}\|_{\ell_{p_1}}\|\{\|W_k\|_{S_{p_2}}\}_{k\in\mathbb{Z}}\|_{\ell_2} \leq \|V\|_{S_{p_1}}\|W\|_{S_{p_2}}.$$

Notice that $p_1, p_2 \in (2,\infty)$ implies that $p > 1$, which is excluded. Thus, it can be concluded that when $p_1, p_2 \in (0,\infty)$, such that $p_1^{-1} + p_2^{-1} = p^{-1}$, it holds that

$$\|\{\|V_k\|_{S_{p_1}}\|W_k\|_{S_{p_2}}\}_{k\in\mathbb{Z}}\|_{\ell_2} \leq \|V\|_{S_{p_1}}\|W\|_{S_{p_2}}. \tag{3.28}$$

Inserting (3.28) into (3.22) finishes the proof.

□

Theorem 3.19
Let $p \in (0,1)$. Let $p_1, p_2 \in (0,\infty)$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. Let $\phi \in C_c^\beta(\mathbb{R})$, where $\beta \geq \frac{3}{p}$, with supp(ϕ) $\subseteq (\xi, \xi+1)$. Let $R \in (1,\infty)$. Then
$\|T_{\phi^{[2]}_{\alpha,\lambda}\chi_{|\lfloor\lambda x-\xi\rfloor-\lfloor\lambda y-\xi\rfloor|\leq R}\chi_{|\lfloor\lambda y-\xi\rfloor-\lfloor\lambda z-\xi\rfloor|\leq R}} : S_{p_1}\times S_{p_2} \to S_p\| \lesssim \lambda^2\|\alpha\|_{\ell_{p^\flat}}$.

Proof. Recall that

$$\phi_{\alpha,\lambda} := \sum_{k\in\mathbb{Z}} \alpha_k\phi(\lambda x - k). \tag{3.29}$$

Without loss of generality, it may be assumed that the functions $\{\phi(\lambda\cdot - k)\}_{k\in\mathbb{Z}}$ are disjointly supported. Indeed, otherwise one may select $N > 1$ sufficiently large such that $\{\phi(\lambda\cdot - Nk)\}_{k\in\mathbb{Z}}$ are disjointly supported, and write

$$\phi_{\alpha,\lambda} = \sum_{j=0}^{N-1}\phi_{\alpha^{(j)},\lambda}$$

where $\alpha^{(j)}$ is the sequence $\{\alpha_{j+Nk}\}_{k\in\mathbb{Z}}$. Then the assertion may be proven for each $\phi_{\alpha^{(j)},\lambda}$ separately. Furthermore, due to $\{\phi(\lambda\cdot - k)\}_{k\in\mathbb{Z}}$ being disjointly supported it is assumed that ϕ is supported in $(\xi, \xi+1)$, where $\xi \in \mathbb{R}$. Now, observe that

$$\phi^{[2]}_{\alpha,\lambda}(x,y,z)\chi_{|\lfloor\lambda x-\xi\rfloor-\lfloor\lambda y-\xi\rfloor|\leq R}\chi_{|\lfloor\lambda y-\xi\rfloor-\lfloor\lambda z-\xi\rfloor|\leq R}$$

$$= \phi^{[2]}_{\alpha,\lambda}(x,y,z)\chi_{|\lfloor\lambda x-\xi\rfloor-\lfloor\lambda y-\xi\rfloor|\leq R}\chi_{|\lfloor\lambda y-\xi\rfloor-\lfloor\lambda z-\xi\rfloor|\leq R}\chi_{|\lfloor\lambda x-\xi\rfloor-\lfloor\lambda z-\xi\rfloor|\leq 2R}$$

$$= \sum_{|a|\leq R}\sum_{|b|\leq R}\sum_{|c|\leq 2R} F_{a,b,c}(x,y,z).$$

where

$$F_{a,b,c}(x,y,z)$$

$$= \phi^{[2]}_{\alpha,\lambda}(x,y,z)\chi_{\lfloor \lambda x-\xi\rfloor-\lfloor \lambda y-\xi\rfloor=a}\chi_{\lfloor \lambda y-\xi\rfloor-\lfloor \lambda z-\xi\rfloor=b}\chi_{\lfloor \lambda x-\xi\rfloor-\lfloor \lambda z-\xi\rfloor=c}.$$

Note that

$$\chi_{\lfloor v\rfloor-\lfloor w\rfloor=i} = \sum_{k\in\mathbb{Z}} \chi_{[k,k+1)}(v)\chi_{[k+i,k+i+1)}(w).$$

Thus, Observe that for $G_{a,b,c,i,j,k}(x,y,z) \neq 0$ it is required that $i=k$, $i+a=j$, $j+b=k+c$. For a,b,c,i,j,k satisfying these constraints,

3

$$G_{a,b,c,i,j,k}(x,y,z)$$

$$= \left(\sum_{\ell\in\mathbb{Z}} \alpha_\ell \lambda^2 \frac{\frac{\phi(\lambda x-\ell)-\phi(\lambda y-\ell)}{(\lambda x-\ell)-(\lambda y-\ell)} - \frac{\phi(\lambda y-\ell)-\phi(\lambda z-\ell)}{(\lambda y-\ell)-(\lambda z-\ell)}}{(\lambda x-\ell)-(\lambda z-\ell)}\right)$$

$$\cdot\chi_{[i,i+1)}(\lambda x-\xi)\chi_{[i+a,i+a+1)}(\lambda y-\xi)\chi_{[i+c,i+c+1)}(\lambda z-\xi).$$

and

$$F_{a,b,c}(x,y,z) = \sum_{i\in\mathbb{Z}} G_{a,b,c,i,i+a,i}(x,y,z).$$

Additionally, it follows from Lemma 3.6 that

$$\|T_{F_{a,b,c}} : S_{p_1}\times S_{p_2} \to S_p\| \le \|\{\|T_{G_{a,b,c,i,i+a,i}} : S_{p_1}\times S_{p_2} \to S_p\|\}_{i\in\mathbb{Z}}\|_{\ell_{p^\flat}}.$$

Fix (x,y,z) such that $G_{a,b,c,i,i+a,i}(x,y,z)$ is nonzero. Then one of the following conditions is satisfied

#1	$\phi(\lambda x-\ell)\neq 0$	$\phi(\lambda y-\ell)\neq 0$	$\phi(\lambda z-\ell)\neq 0$
#2	$\phi(\lambda x-\ell)\neq 0$	$\phi(\lambda y-\ell)\neq 0$	$\phi(\lambda z-\ell)= 0$
#3	$\phi(\lambda x-\ell)= 0$	$\phi(\lambda y-\ell)\neq 0$	$\phi(\lambda z-\ell)\neq 0$
#4	$\phi(\lambda x-\ell)\neq 0$	$\phi(\lambda y-\ell)= 0$	$\phi(\lambda z-\ell)\neq 0$
#5	$\phi(\lambda x-\ell)\neq 0$	$\phi(\lambda y-\ell)= 0$	$\phi(\lambda z-\ell)= 0$
#6	$\phi(\lambda x-\ell)= 0$	$\phi(\lambda y-\ell)\neq 0$	$\phi(\lambda z-\ell)= 0$
#7	$\phi(\lambda x-\ell)= 0$	$\phi(\lambda y-\ell)= 0$	$\phi(\lambda z-\ell)\neq 0.$

Each condition respectively implies one of the following options,

#1	$(\lambda x-\ell)\in(\xi,\xi+1)$	$(\lambda y-\ell)\in(\xi,\xi+1)$	$(\lambda z-\ell)\in(\xi,\xi+1)$
#2	$(\lambda x-\ell)\in(\xi,\xi+1)$	$(\lambda y-\ell)\in(\xi,\xi+1)$	$(\lambda z-\ell)\notin(\xi,\xi+1)$
#3	$(\lambda x-\ell)\notin(\xi,\xi+1)$	$(\lambda y-\ell)\in(\xi,\xi+1)$	$(\lambda z-\ell)\in(\xi,\xi+1)$
#4	$(\lambda x-\ell)\in(\xi,\xi+1)$	$(\lambda y-\ell)\notin(\xi,\xi+1)$	$(\lambda z-\ell)\in(\xi,\xi+1)$
#5	$(\lambda x-\ell)\in(\xi,\xi+1)$	$(\lambda y-\ell)\notin(\xi,\xi+1)$	$(\lambda z-\ell)\notin(\xi,\xi+1)$
#6	$(\lambda x-\ell)\notin(\xi,\xi+1)$	$(\lambda y-\ell)\in(\xi,\xi+1)$	$(\lambda z-\ell)\notin(\xi,\xi+1)$
#7	$(\lambda x-\ell)\notin(\xi,\xi+1)$	$(\lambda y-\ell)\notin(\xi,\xi+1)$	$(\lambda z-\ell)\in(\xi,\xi+1).$

Firstly, when $a \neq c$ and $a \neq 0$ and $c \neq 0$ observe that there exists no $\ell \in \mathbb{Z}$ such that option 1, 2, 3 or 4 is valid, due to the indicator functions present in $G_{a,b,c,i,i+a,i}$. This leaves options 5, 6 and 7. For option 5 note that the first indicator function implies that $(\lambda x - \xi) \in [i, i+1)$. Combining this with the requirement that $(\lambda x - \ell) \in (\xi, \xi + 1)$ implies that $\ell \in [i, i+1) \Rightarrow \ell = i$. Repeating this argumentation yields that for option 6 to be valid it is required that $\ell = i + a$, and for option 7 to be valid it is required that $\ell = i + c$. Thus observe that when $a \neq c$ and $a \neq 0$ and $c \neq 0$ that

$$G_{a,b,c,i,i+a,i}(x,y,z)$$
$$= \alpha_i \lambda^2 \phi^{[2]}(\lambda x - i, \lambda y - i, \lambda z - i)\chi_{[i+a,i+a+1)}(\lambda y - \xi)\chi_{[i+c,i+c+1)}(\lambda z - \xi)$$
$$+ \alpha_{i+a}\lambda^2\phi^{[2]}(\lambda x - i - a, \lambda y - i - a, \lambda z - i - a)\chi_{[i,i+1)}(\lambda x - \xi)\chi_{[i+c,i+c+1)}(\lambda z - \xi)$$
$$+ \alpha_{i+c}\lambda^2\phi^{[2]}(\lambda x - i - c, \lambda y - i - c, \lambda z - i - c)\chi_{[i,i+1)}(\lambda x - \xi)\chi_{[i+a,i+a+1)}(\lambda y - \xi)$$

and

$$||T_{G_{a,b,c,i,i+a,i}}||^p \leq |\alpha_i|^p \lambda^{2p} ||T_{\phi^{[2]}}||^p + |\alpha_{i+a}|^p \lambda^{2p} ||T_{\phi^{[2]}}||^p + |\alpha_{i+c}|^p \lambda^{2p} ||T_{\phi^{[2]}}||^p.$$

Secondly, when $a = 0$ and $c \neq 0$ observe that there exists no $\ell \in \mathbb{Z}$ such that option 1, 3, 4, 5 or 6 is valid, due to the indicator functions present in $G_{a,b,c,i,i+a,i}$. This leaves options 2 and 7. For option 2 note that the first and second indicator function imply that $(\lambda x - \xi) \in [i, i+1)$ and $(\lambda y - \xi) \in [i, i+1)$. Combining this with the requirement that $(\lambda x - \ell) \in (\xi, \xi+1)$ and $(\lambda y - \ell) \in (\xi, \xi+1)$ implies that $\ell \in [i, i+1) \Rightarrow \ell = i$. For option 7 note that the third indicator function implies that $(\lambda z - \xi) \in [i + c, i + c + 1)$. Combining this with the requirement that $(\lambda z - \ell) \in (\xi, \xi + 1)$ implies that $\ell \in [i + c, i + c + 1) \Rightarrow \ell = i + c$. Thus observe that when $a = 0$ and $c \neq 0$ that

$$G_{a,b,c,i,i+a,i}(x,y,z)$$
$$= \alpha_i \lambda^2 \phi^{[2]}(\lambda x - i, \lambda y - i, \lambda z - i)\chi_{[i+c,i+c+1)}(\lambda z - \xi)$$
$$+ \alpha_{i+c}\lambda^2\phi^{[2]}(\lambda x - i - c, \lambda y - i - c, \lambda z - i - c)\chi_{[i,i+1)}(\lambda x - \xi)\chi_{[i,i+1)}(\lambda y - \xi)$$

and

$$||T_{G_{a,b,c,i,i+a,i}}||^p \leq |\alpha_i|^p \lambda^{2p} ||T_{\phi^{[2]}}||^p + |\alpha_{i+c}|^p \lambda^{2p} ||T_{\phi^{[2]}}||^p.$$

Thirdly, when $a = c$ and $c \neq 0$ and $a \neq 0$ observe that there exists no $\ell \in \mathbb{Z}$ such that option 1, 2, 4, 6 or 7 is valid, due to the indicator functions present in $G_{a,b,c,i,i+a,i}$. This leaves options 3 and 5. For option 3 note that the second and third indicator function imply that $(\lambda y - \xi) \in [i + a, i + a + 1)$ and $(\lambda z - \xi) \in [i + a, i + a + 1)$. Combining this with the requirement that $(\lambda y - \ell) \in (\xi, \xi + 1)$ and $(\lambda z - \ell) \in (\xi, \xi + 1)$ implies that $\ell \in [i + a, i + a + 1) \Rightarrow \ell = i + a$. For option 5 note that the third indicator function implies that $(\lambda x - \xi) \in [i, i + 1)$. Combining this with the requirement that $(\lambda x - \ell) \in (\xi, \xi + 1)$ implies that $\ell \in [i, i+1) \Rightarrow \ell = i$. Thus observe that when $a = c$ and $c \neq 0$ and $a \neq 0$ that

$$G_{a,b,c,i,i+a,i}(x,y,z)$$
$$= \alpha_{i+a}\lambda^2\phi^{[2]}(\lambda x - i - a, \lambda y - i - a, \lambda z - i - a)\chi_{[i,i+1)}(\lambda x - \xi)$$
$$+\alpha_i\lambda^2\phi^{[2]}(\lambda x - i, \lambda y - i, \lambda z - i)\chi_{[i+a,i+a+1)}(\lambda y - \xi)\chi_{[i+a,i+a+1)}(\lambda z - \xi)$$

and

$$||T_{G_{a,b,c,i,i+a,i}}||^p \leq |\alpha_i|^p\lambda^{2p}||T_{\phi^{[2]}}||^p + |\alpha_{i+a}|^p\lambda^{2p}||T_{\phi^{[2]}}||^p.$$

Fourthly, when $0 = c$ and $a \neq 0$ observe that there exists no $\ell \in \mathbb{Z}$ such that option 1, 2, 3, 5 or 7 is valid, due to the indicator functions present in $G_{a,b,c,i,i+a,i}$. This leaves options 4 and 6. For option 4 note that the first and third indicator function imply that $(\lambda x - \xi) \in [i, i+1)$ and $(\lambda z - \xi) \in [i, i+1)$. Combining this with the requirement that $(\lambda x - \ell) \in (\xi, \xi+1)$ and $(\lambda z - \ell) \in (\xi, \xi+1)$ implies that $\ell \in [i, i+1) \Rightarrow \ell = i$. For option 6 note that the second indicator function implies that $(\lambda y - \xi) \in [i+a, i+a+1)$. Combining this with the requirement that $(\lambda y - \ell) \in (\xi, \xi+1)$ implies that $\ell \in [i+a, i+a+1) \Rightarrow \ell = i+a$. Thus observe that when $0 = c$ and $a \neq 0$ that

$$G_{a,b,c,i,i+a,i}(x,y,z)$$
$$= \alpha_i\lambda^2\phi^{[2]}(\lambda x - i, \lambda y - i, \lambda z - i)\chi_{[i+a,i+a+1)}(\lambda y - \xi)$$
$$+\alpha_{i+a}\lambda^2\phi^{[2]}(\lambda x - i - a, \lambda y - i - a, \lambda z - i - a)\chi_{[i,i+1)}(\lambda x - \xi)\chi_{[i,i+1)}(\lambda z - \xi)$$

and

$$||T_{G_{a,b,c,i,i+a,i}}||^p \leq |\alpha_i|^p\lambda^{2p}||T_{\phi^{[2]}}||^p + |\alpha_{i+a}|^p\lambda^{2p}||T_{\phi^{[2]}}||^p.$$

Finally, when $0 = a = c$ observe that there exists no $\ell \in \mathbb{Z}$ such that option 2, 3, 4, 5, 6 or 7 is valid, due to the indicator functions present in $G_{a,b,c,i,i+a,i}$. This leaves option 1. For option 1 note that the first, second and third indicator function imply that $(\lambda x - \xi) \in [i, i+1)$ and $(\lambda y - \xi) \in [i, i+1)$ and $(\lambda z - \xi) \in [i, i+1)$. Combining this with the requirement that $(\lambda x - \ell) \in (\xi, \xi+1)$ and $(\lambda y - \ell) \in (\xi, \xi+1)$ and $(\lambda z - \ell) \in (\xi, \xi+1)$ implies that $\ell \in [i, i+1) \Rightarrow \ell = i$. Thus observe that when $0 = a = c$ that

$$G_{a,b,c,i,i+a,i}(x,y,z)$$
$$= \alpha_i\lambda^2\phi^{[2]}(\lambda x - i, \lambda y - i, \lambda z - i)$$

and

$$||T_{G_{a,b,c,i,i+a,i}}||^p \leq |\alpha_i|^p\lambda^{2p}||T_{\phi^{[2]}}||^p.$$

Combining all the previously found inequalities yields

$$||T_{G_{a,b,c,i,i+a,i}}|| \lesssim (|\alpha_i|^p + |\alpha_{i+a}|^p + |\alpha_{i+c}|^p)^{1/p}\lambda^2||T_{\phi^{[2]}}||$$

3

and it follows from Lemma 3.18 that

$$||T_{F_{a,b,c}}|| \leq \lambda^2 ||T_{\phi^{[2]}}|| ||\{(|\alpha_i|^p + |\alpha_{i+a}|^p + |\alpha_{i+c}|^p)^{1/p}\}_{i\in\mathbb{Z}}||_{\ell_{p^\flat}}$$

$$= \lambda^2 ||T_{\phi^{[2]}}|| (\sum_{i\in\mathbb{Z}} (|\alpha_i|^p + |\alpha_{i+a}|^p + |\alpha_{i+c}|^p)^{p^\flat/p}\Big)^{1/p^\flat}$$

$$= \lambda^2 ||T_{\phi^{[2]}}|| (\sum_{i\in\mathbb{Z}} (|\alpha_i|^p + |\alpha_{i+a}|^p + |\alpha_{i+c}|^p)^{\frac{2}{2-p}}\Big)^{\frac{2-p}{2p}}$$

$$= \lambda^2 ||T_{\phi^{[2]}}|| (\left(\left(\sum_{i\in\mathbb{Z}} (|\alpha_i|^p + |\alpha_{i+a}|^p + |\alpha_{i+c}|^p)^{\frac{2}{2-p}}\right)^{\frac{2-p}{2}}\right)^{1/p} .$$

Now apply Minkowski's inequality for sequence spaces to observe that

$$||T_{F_{a,b,c}}|| \leq \lambda^2 ||T_{\phi^{[2]}}||$$

$$\cdot \left(\left(\sum_{i\in\mathbb{Z}} (|\alpha_i|^p)^{\frac{2}{2-p}}\right)^{\frac{2-p}{2}} + \left(\sum_{i\in\mathbb{Z}} (|\alpha_{i+a}|^p)^{\frac{2}{2-p}}\right)^{\frac{2-p}{2}} + \left(\sum_{i\in\mathbb{Z}} (|\alpha_{i+c}|^p)^{\frac{2}{2-p}}\right)^{\frac{2-p}{2}}\right)^{1/p}$$

$$= \lambda^2 ||T_{\phi^{[2]}}|| (\left(\sum_{i\in\mathbb{Z}} |\alpha_i|^{p^\flat}\right)^{\frac{2-p}{2}} + \left(\sum_{i\in\mathbb{Z}} |\alpha_i|^{p^\flat}\right)^{\frac{2-p}{2}} + \left(\sum_{i\in\mathbb{Z}} |\alpha_i|^{p^\flat}\right)^{\frac{2-p}{2}}\Big)^{1/p}$$

$$\approx \lambda^2 ||T_{\phi^{[2]}}|| (\sum_{i\in\mathbb{Z}} |\alpha_i|^{p^\flat})^{1/p^\flat} = \lambda^2 ||T_{\phi^{[2]}}|| ||\{\alpha_i\}_{i\in\mathbb{Z}}||_{\ell_{p^\flat}} .$$

Further, note that $||T_{\phi^{[2]}} : S_{p_1} \times S_{p_2} \to S_p||$ is bounded due to Theorem 3.3. From which it follows that

$$||T_{\phi^{[2]}_{\alpha,\lambda}\chi_{|\lfloor\lambda x-\xi\rfloor-\lfloor\lambda y-\xi\rfloor|\leq R}\chi_{|\lfloor\lambda y-\xi\rfloor-\lfloor\lambda z-\xi\rfloor|\leq R}} : S_{p_1} \times S_{p_2} \to S_p||$$

$$\lesssim \lambda^2 ||T_{\phi^{[2]}} : S_{p_1} \times S_{p_2} \to S_p|| ||\alpha||_{\ell_{p^\flat}} .$$

□

Remark. A possible direction for future research would be to attempt repurpose the proof of Theorem 3.15 for the entire range $p \in (0,1)$. The main question is whether there exists a bound for $||T_{B_R} : S_p \to S_p||$ and $||T_{C_R} : S_p \to S_p||$ when $p \in (0,1)$. A similar question, corresponding to first divided difference functions, was answered by introducing a bound of the form [MS21, Equation 4.8]

$$||T_{\alpha_{\lfloor x\rfloor}\phi(x,y)} : S_p \to S_p|| \leq ||\alpha||_{\ell_{p^\#}} ||T_{\phi(x,y)} : S_1 \to S_1||. \tag{3.30}$$

McDonald E. and Sukochev F. express $\phi_{\alpha,\lambda}(x)$ as $\alpha_{\lfloor \lambda x - \xi \rfloor}\phi_{1,\lambda}(x)$ and apply (3.30) and Lemma 3.12, among other steps, to be able to bound $||T_{f^{[1]}} : S_{p_1} \times S_{p_2} \to S_p||$ for the full range $p \in (0,1)$. If one aims to reuse this strategy to bound $||T_{f^{[2]}} : S_{p_1} \times S_{p_2} \to S_p||$ it could be beneficial to attempt to express $\phi^{[1]}_{\alpha,\lambda}(x,y)$ in a form such that a bound of type (3.30) can be applied. A potentially fruitful expression of $\phi^{[1]}_{\alpha,\lambda}(x,y)$ for this purpose is $\sum_{j\in\mathbb{Z}} \alpha_j f^{[1]}_j(x,y)$ where $f_j(x) := \chi_{[j,j+1)}(\lambda x - \xi)\phi_{1,\lambda}(x)$.

REFERENCES

[AP02] **A. Aleksandrov and V. Peller**. "Hankel and Toeplitz–Schur multipliers". In: *Mathematische Annalen* 324 (Oct. 2002), pp. 277–327. doi: https://doi.org/10.1007/s00208-002-0339-z.

[AP16] **A. B. Aleksandrov and V. V. Peller**. "Operator Lipschitz functions". In: *Russian Mathematical Surveys* 71.4 (Aug. 2016), p. 605. doi: 10.1070/RM9729.

[Aza+09] **N. Azamov, A. Carey, P. Dodds, and F. Sukochev**. "Operator Integrals, Spectral Shift, and Spectral Flow". In: *Canadian Journal of Mathematics* 61 (Apr. 2009), pp. 241–263. doi: 10.4153/CJM-2009-012-0.

[BS75] **M. Birman and M. Solomyak**. "Remarks on the spectral shift function". In: *Journal of Soviet Mathematics* 3 (Apr. 1975), pp. 408–419. doi: https://doi.org/10.1007/BF01084680.

[BS67] **M. Birman and M. Solomyak**. "Double Stieltjes Operator Integrals". In: *Topics in Mathematical Physics* 1 (1967), pp. 25–54.

[BS96] **M. Birman and M. Solomyak**. "Tensor product of a finite number of spectral measures is always a spectral measure". In: *Integral Equations and Operator Theory* 24 (Feb. 1996), pp. 179–187. doi: 10.1007/BF01193459.

[BS98] **M. Bračič and A. Stefanovska**. "Wavelet-based Analysis of Human Blood-flow Dynamics". In: *Bulletin of Mathematical Biology* 60.5 (1998), pp. 919–935. issn: 0092-8240. doi: https://doi.org/10.1006/bulm.1998.0047.

[Car+16] **A. Carey, F. Gesztesy, G. Levitina, R. Nichols, D. Potapov, and F. Sukochev**. "Double Operator Integral Methods Applied to Continuity of Spectral Shift Functions". In: *Journal of Spectral Theory* 6.4 (2016), pp. 747–779. doi: 10.4171/JST/140. url: https://ems.press/journals/jst/articles/14410.

[CR25] **M. Caspers and J. Reimann**. "On the best constants of Schur multipliers of second order divided difference functions". In: *Mathematische Annalen* (Mar. 2025). doi: https://doi.org/10.1007/s00208-025-03111-y.

3

[CSZ21] **M. Caspers, F. Sukochev, and D. Zanin**. "Weak (1,1) estimates for multiple operator integrals and generalized absolute value functions". In: *SpringerLink* (Aug. 2021), pp. 245–271. doi: https://doi.org/10.1007/s11856-021-2179-0.

[CC97] **A. Chamseddine and A. Connes**. "The Spectral Action Principle". In: *Communications in Mathematical Physics* 186 (July 1997), pp. 731–750. doi: https://doi.org/10.1007/s002200050126.

[Cha98] **A. H. Chamseddine**. "Remarks on the spectral action principle". In: *Physics Letters B* 436.1–2 (Sept. 1998), pp. 84–90. issn: 0370-2693. doi: 10.1016/s0370-2693(98)00845-4. url: http://dx.doi.org/10.1016/S0370-2693(98)00845-4.

[Con+23] **J. M. Conde-Alonso, A. M. González-Pérez, J. Parcet, and E. Tablate**. "Schur multipliers in Schatten-von Neumann classes". In: *Annals of Mathematics* 198.3 (2023), pp. 1229–1260. doi: 10.4007/annals.2023.198.3.5. url: https://doi.org/10.4007/annals.2023.198.3.5.

[Con96] **A. Connes**. "Gravity coupled with matter and the foundation of non-commutative geometry". In: *Communications in Mathematical Physics* 182.1 (Dec. 1996), pp. 155–176. issn: 1432-0916. doi: 10.1007/bf02506388. url: http://dx.doi.org/10.1007/BF02506388.

[Con90] **J. B. Conway**. *A Course in Functional Analysis*. Nov. 1990. isbn: 978-0-387-97245-9. doi: https://doi.org/10.1007/978-1-4757-4383-8.

[Dau88] **I. Daubechies**. "Orthonormal bases of compactly supported wavelets". In: *Communications on Pure and Applied Mathematics* 41.7 (Oct. 1988), pp. 909–996. doi: https://doi.org/10.1002/cpa.3160410705.

[DR21] **J. Delgado and M. Ruzhansky**. "Schatten-von Neumann classes of integral operators". In: *Journal de Mathématiques Pures et Appliqués* 154 (May 2021), pp. 1–29. doi: 10.1016/j.matpur.2021.08.006.

[FK14] **T. Formisano and E. Kissin**. "Clarkson-McCarthy Inequalities for l (p) -Spaces of Operators in Schatten Ideals". In: *Integral Equations and Operator Theory* 2 (June 2014). doi: 10.1007/s00020-014-2145-x.

[Gra14] **L. Grafakos**. *Classical Fourier Analysis*. Jan. 2014. isbn: 978-1-4939-1193-6. doi: 10.1007/978-1-4939-1194-3.

[GS18] **D. J. Griffiths and D. F. Schroeter**. *Introduction to quantum mechanics*. Third edition. Cambridge ; New York, NY: Cambridge University Press, 2018. isbn: 978-1-107-18963-8.

[HJX09] **U. Haagerup, M. Junge, and Q. Xu**. "A reduction method for noncommutative Lp-spaces and applications". In: *Transactions of the American mathematical society* 362.4 (Oct. 2009), pp. 2125–2165. doi: http://dx.doi.org/10.1090/S0002-9947-09-04935-6.

[Hyt+16] **T. Hytonen, J. van Neerven, M. Veraar, and L. Weis**. *Analysis in Banach Spaces I - Martingales and Littlewood-Paley Theory*. Dec. 2016. isbn: 978-3-319-48519-5. doi: https://doi.org/10.1007/978-3-319-48520-1.

[Kis+12] **E. Kissin, D. Potapov, V. Shulman, and F. Sukochev**. "Operator smoothness in Schatten norms for functions of several variables: Lipschitz conditions, differentiability and unbounded derivations". In: *Proceedings of the London Mathematical Society* 105.4 (2012), pp. 661–702. doi: https://doi.org/10.1112/plms/pds014.

[Kre53] **M. Krein**. "On the trace formula in perturbation theory". In: *Mat. Sbornik N.S.* 33(75) (1953), pp. 597–626.

[KD56] **S. Krein and Y. L. Daletskii**. "Integration and differentiation of functions of Hermitian operators and applications to the theory of perturbations". In: *Trudy Sem. Functsion. Anal. Voronezh. Gos. Univ* 1 (1956), pp. 81–105.

[LS20] **C. Le Merdy and A. Skripka**. "Higher order differentiability of operator functions in Schatten norms". In: *Journal of the Institute of Mathematics of Jussieu* 19.6 (2020), pp. 1993–2016. doi: 10.1017/S1474748019000033.

[Lif52] **I. Lifshits**. "On a problem of the theory of perturbations connected with quantum statistics". In: *Russian Math. Surveys* 7 (1952).

[MS21] **E. McDonald and F. Sukochev**. "Lipschitz estimates in quasi-Banach Schatten ideals". In: *Mathematische Annalen* 383 (July 2021), pp. 571–619. doi: 10.1007/s00208-021-02247-x.

[ND13] **Y. V. Nazarov and J. Danon**. *Advanced Quantum Mechanics: A Practical Guide*. Cambridge University Press, Feb. 2013. isbn: 9780511980428. doi: https://doi.org/10.1017/CBO9780511980428.

3

[Pav71] **B. S. Pavlov**. "On Multidimensional Integral Operators". In: *Linear Operators and Operator Equations*. Ed. by V. I. Smirnov. Boston, MA: Springer US, 1971, pp. 81–97. isbn: 978-1-4757-0013-8. doi: 10.1007/978-1-4757-0013-8_4. url: https://doi.org/10.1007/978-1-4757-0013-8_4.

[Pel06] **V. Peller**. "Multiple operator integrals and higher operator derivatives". In: *Journal of Functional Analysis* 233.2 (2006), pp. 515–544. issn: 0022-1236. doi: https://doi.org/10.1016/j.jfa.2005.09.003. url: https://www.sciencedirect.com/science/article/pii/S0022123605003307.

[Pel15] **V. Peller**. "Multiple operator integrals in perturbation theory". In: *Bulletin of Mathematical Sciences* 6 (Oct. 2015), pp. 15–88. doi: https://doi.org/10.1007/s13373-015-0073-y.

[PS12] **D. Potapov and F. Sukochev**. "Operator-Lipschitz functions in Schatten–von Neumann classes". In: *Acta Mathematica* 207 (Feb. 2012), pp. 375–389. doi: https://doi.org/10.1007/s11511-012-0072-8.

[PSS12] **D. Potapov, A. Skripka, and F. Sukochev**. "Spectral shift function of higher order". In: *Inventiones mathematicae* 193 (Dec. 2012), pp. 501–538. doi: 10.1007/s00222-012-0431-2.

[PSS15] **D. Potapov, A. Skripka, and F. Sukochev**. "Trace formulas for resolvent comparable operators". In: *Advances in Mathematics* 272 (Feb. 2015), pp. 630–651. doi: 10.1016/j.aim.2014.12.016.

[PSS16] **D. Potapov, A. Skripka, and F. Sukochev**. "Functions of unitary operators: Derivatives and trace formulas". In: *Journal of Functional Analysis* 270 (Jan. 2016), pp. 2048–2072. doi: 10.1016/j.jfa.2016.01.001.

[Saw18] **Y. Sawano**. *Theory of Besov Spaces*. Springer Singapore, Nov. 2018. isbn: 978-981-13-0835-2. doi: https://doi.org/10.1007/978-981-13-0836-9.

[ST19] **A. Skripka and A. Tomskova**. *Multilinear Operator Integrals: Theory and Applications*. Jan. 2019. isbn: 978-3-030-32405-6. doi: 10.1007/978-3-030-32406-3.

[Ste77] **V. Sten'kin**. "Multiple operator integrals". In: *Izv. Vysš. Ucebn. Zaved. Mat.* 179 (1977), pp. 102–115.

[Tak11] **M. Takesaki**. *Theory of Operator Algebras I*. Nov. 2011. isbn: 978-1-4612-6190-2. doi: https://doi.org/10.1007/978-1-4612-6188-9.

[Zay+24] **H. Zaynidinov, U. Juraev, S. Tishlikov, and J. Modullayev**. “Application of Daubechies Wavelets in Digital Processing of Biomedical Signals and Images”. In: *Intelligent Human Computer Interaction*. Ed. by B. J. Choi, D. Singh, U. S. Tiwary, and W.-Y. Chung. Cham: Springer Nature Switzerland, 2024, pp. 194–206. isbn: 978-3-031-53827-8.

3