

A CODE VERIFICATION EXERCISE FOR THE UNSTRUCTURED FINITE-VOLUME CFD SOLVER ISIS-CFD

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Abstract. *This paper is devoted to a code verification exercise for the unstructured finite-volume CFD solver ISIS-CFD. This exercise is limited to the verification of the 2D Navier-Stokes solver by prescribing a turbulence eddy-viscosity. The unstructured code is tested using three different types of grid, namely a Cartesian grid, an unstructured triangular grid, and an unstructured quadrilateral grid generated by HEXPRESS. For each type of grid, the order of accuracy of the numerical scheme is determined based on a set of 6 grids and compared to the theoretical order of accuracy. Convergence behaviour of the code on different grids is analysed.*

1 INTRODUCTION

It is not a trivial task to obtain an accurate numerical solution to the Navier-Stokes equation for a turbulent flow. In a finite-volume code, if it is not so difficult to ensure a second order accuracy for the inviscid flux, the same goal is more difficult to achieve for the diffusive flux. It is not unusual to observe a considerable spread in predictions at different workshops. A good way to certify a code is to use the method of manufactured solution to demonstrate that the code is bug free, and the numerical solution converges to the exact solution with the expected order of accuracy. This exercise should be carried out under the conditions for which the code is supposed to be used, which means not only on a idealized Cartesian grid, but also on a less friendly grid corresponding to a realistic situation. The manufactured solution proposed by Eça et al.¹ is employed and the present exercise is limited to the verification of the 2D Navier-Stokes solver by prescribing a turbulence eddy-viscosity. The unstructured code is tested using three different types of grid as mentioned above. While the Cartesian grid makes it easy to analyse the numerical scheme, the triangular unstructured grid is chosen to demonstrate the accuracy of the unstructured

code and, finally, the unstructured quadrilateral grid is selected to verify the convergence behaviour of the code in a non trivial situation since it contains non-conformal elements due to local refinement. The accuracy of the code will be evaluated using a systematic grid refinement.

2 THE MANUFACTURED SOLUTION AND THE BOUNDARY CONDITIONS

In this exercise, we solve only the momentum equations and the continuity constraint to determine the velocity and pressure fields. The turbulence eddy-viscosity is prescribed using the manufactured solution. Different choices for the turbulence eddy-viscosity are possible in¹. We have chosen the solution $\tilde{\nu}$ for the Spalart-Allmaras model designed as MS2 in¹ that has a y^2 asymptotic behaviour near the wall. It should be noticed that it is the solution for the $\tilde{\nu}$ equation that we have chosen rather than the corresponding eddy-viscosity of the Spalart-Allmaras model because the later makes it more difficult to reach the asymptotic convergence range. The kinematic viscosity is 10^{-6} as suggested in¹.

The computational domain is defined by $0.5 \leq x \leq 1$ and $0 \leq y \leq 0.5$. The exact mass flux is imposed on all boundaries. The pressure at the boundary is updated as:

$$P_{bnd}^{num} = P_{internal}^{num} + P_{bnd}^{exact} - P_{internal}^{exact} \quad (1)$$

At the outlet boundary $x=1$, the velocity field is updated in the same way. Those are special boundary conditions for this verification exercise. Dirichlet boundary conditions are applied to the velocity field at the wall $y=0$, at the inlet $x=0.5$ and at the upper boundary $y=0.5$.

3 GRID GENERATION

3.1 Cartesian grid

The manufactured solution is quite smooth and, consequently, there is no large gradient near the wall. The solution can be represented correctly with a uniform grid. However, due to the high Reynolds number, numerical experiments show that it is necessary to use a non-uniform grid in the wall normal direction in order to ensure that the solution is in the asymptotic convergence range. For this reason, we employ a uniform grid distribution in the x direction and a wall-stretched grid distribution in the y direction. The grid aspect ratio h_x/h_y for the first grid cell next to the wall is about 12.5. The dimensions of the six grids used in the present study are 33×65 , 49×97 , 65×129 , 97×192 , 129×257 and 193×385 .

3.2 Unstructured triangular grid

As it is necessary to employ a wall-stretched grid, it is not trivial to generate an unstructured triangular grid using a grid generation software. Therefore, the triangular

grid employed in the present study is obtained by triangulating the previous Cartesian grid.

3.3 Unstructured quadrilateral grid

For high Reynolds number viscous flows, it is not easy to generate a high quality tetrahedral/triangular mesh for the viscous layer. This is the reason why hexahedral/quadrilateral grid is preferred for such applications. Generation of a block-structured hexahedral grid for a complex geometry involving many appendages, for instance, is a not a trivial task. It may take weeks of human time, which explains why hexahedral unstructured grid is a promising alternative. The grid generation software HEXPRESS, developed by NUMECA, allows to generate such a kind of grid. This software is now routinely employed in our CFD group for applications involving very complex geometry like ship or yacht design. HEXPRESS uses a successive refinement technique to capture the geometry, which leads to a grid containing an abrupt change in mesh size and non-conformal elements near the refinement interfaces. It is mandatory to ensure that the numerical solution converges towards the exact solution when such a kind of unstructured grid is employed. Consequently, a set of 6 grids has been generated with a similar number of grid nodes and refinement ratio as the previous Cartesian grid set. However, a perfect grid similarity can not be ensured. Figure 1 displays a zoom of the coarsest grid where the above-mentioned special features can be easily identified. Two refinement regions have been intentionally introduced in order to study the convergence behaviour of the code under this situation.

4 VERIFICATION OF THE TRUNCATION ERROR

4.1 The truncation error

Two types of verification have been performed. The first one is the verification of the truncation error by applying the discrete operator to the exact solution. Let us take the U momentum equation as example. The transport equation is written as

$$\frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x} \left[(\nu + \nu_t) \frac{\partial u}{\partial x} \right] + \frac{\partial}{\partial y} \left[(\nu + \nu_t) \frac{\partial u}{\partial y} \right] + \frac{\partial \nu_t}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial \nu_t}{\partial y} \frac{\partial v}{\partial x} + S_u \quad (2)$$

where S_u is the source term specific to the manufactured solution. Integration of the above equation on a Cartesian grid with a uniform space increment h in both directions leads to the following discrete relation:

$$F_e^x - F_w^x + F_n^y - F_s^y = (S_{\nu_t} + S_u) h^2 \quad (3)$$

with

$$F^x = \left[uu + p - (\nu + \nu_t) \frac{\partial u}{\partial x} \right] h$$

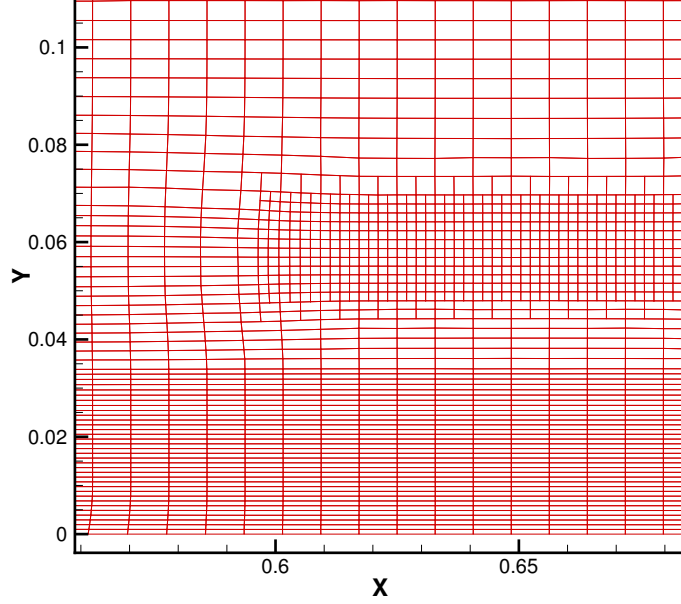


Figure 1: Unstructured quadrilateral grid

$$\begin{aligned}
 F^y &= \left[uu - (\nu + \nu_t) \frac{\partial u}{\partial y} \right] h \\
 S_{\nu_t} &= \frac{\partial \nu_t}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial \nu_t}{\partial y} \frac{\partial v}{\partial x}
 \end{aligned} \tag{4}$$

The turbulence source term S_{ν_t} originally discretized in a conservative form in the ISIS-CFD code is computed as a volumic source term in this verification exercise for a reason that will be explained later in the paper. The ISIS-CFD code uses a linear reconstruction scheme for each control volume and the gradient required for this reconstruction is computed with a least square approach in the present verification exercise. The inviscid flux at the interface takes the value of the upwind side obtained with the linear reconstruction. This ensures a second order accuracy for the inviscid flux. For the diffusive flux, it is reconstructed using the values and gradients of the solution unknowns at both sides of the interface with central difference scheme and distance weighted linear interpolation. Unlike for the inviscid flux, the accuracy of the diffusive flux can be ensured only to first order. However, second order accuracy can be achieved on regular grid due to error cancellation. We call the truncation error the quantity given by

$$Res = \frac{1}{h^2} (F_e^x - F_w^x + F_n^y - F_s^y) - (S_{\nu_t} + S_u) \tag{5}$$

where the manufactured solution is used to reconstruct the flux and to compute the source terms. For a finite-difference method, it is mandatory to ensure that the truncation

error tends towards zero as the mesh is refined. This is the well known consistency condition. For a finite-volume method, this consistency condition is too strict and not always satisfied, which is clearly demonstrated with the use of manufactured solution.

4.2 The momentum equations

The convergence behaviour of internal cells is different from that of boundary cells. Figures 2 and 3 display the L1 norm of truncation error for internal cells and boundary cells, respectively, of the U momentum equation for the three different types of grid. The observed order of accuracy based on two successive grids is also indicated in the figure. As the grid is non-uniform, the L1 norm is computed as

$$|Res|_{L1} = \frac{\sum (|Res_i| Vol_i)}{\sum Vol_i} \quad (6)$$

The truncation error shows a monotonous convergence behaviour for the three different

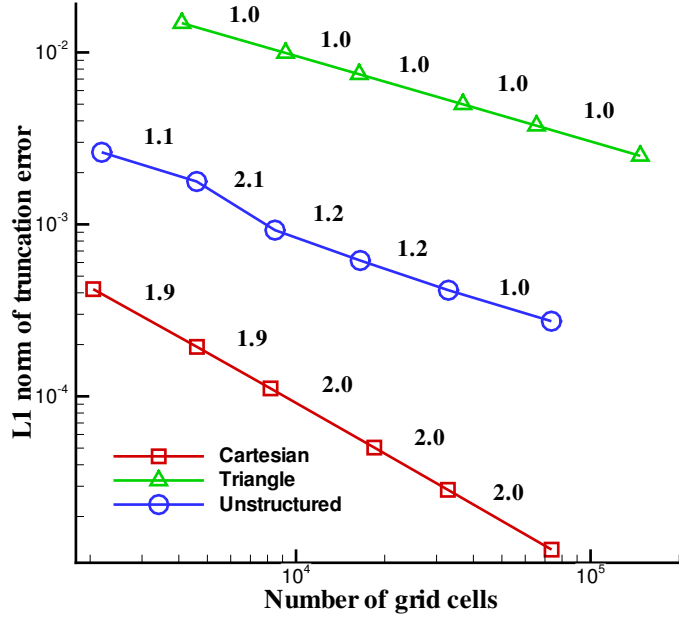


Figure 2: L1 norm of truncation error for U momentum equation for internal cells.

types of grid with different levels and slopes of convergence. The observed order of convergence is a more relevant indicator. For the Cartesian grid set, the observed orders of accuracy based on the L1 norm for internal cells are 2.0 for the last three grids, indicating that the discretization error is in the asymptotic convergence range for the last four grids. For boundary cells, it also decreases with grid refinement, but becomes less than 1 for the fine grids. This behaviour is due to the discretization of the diffusion operator on

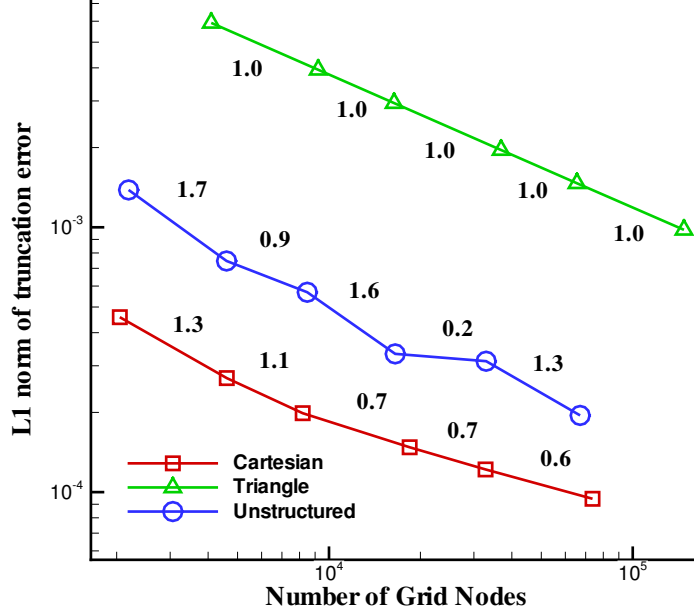


Figure 3: L1 norm of truncation error for U momentum equation only for boundary cells.

the boundary stencil layout employed in the ISIS-CFD code. It can be illustrated with a simple 1D example as follows. The boundary stencil layout employed in the ISIS-CFD code is shown in figure 4. A cell-centered layout is employed. Circles represent grid nodes, and squares the locations of the solution unknown. At boundary cells, the diffusion operator is discretized with

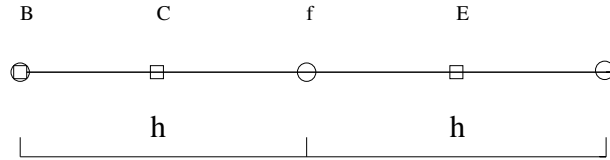


Figure 4: Boundary stencil layout in 1D.

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{h} \left(\frac{\partial \phi}{\partial x} \Big|_f - \frac{\partial \phi}{\partial x} \Big|_B \right) \quad (7)$$

The two diffusive fluxes are approximated by

$$\frac{\partial \phi}{\partial x} \Big|_f = \frac{\phi_E - \phi_C}{h} \quad (8)$$

and

$$\left. \frac{\partial \phi}{\partial x} \right|_B = \frac{\phi_C - \phi_B}{h/2} \quad (9)$$

The first approximation is second order accurate, while the second is only first order. Consequently, the approximation provided by (7) has a zero order accuracy, which can be confirmed by the development in Taylor expansions:

$$\left. \frac{\partial^2 \phi}{\partial x^2} \right|_C = \frac{1}{h^2} (\phi_E - 3\phi_C + 2\phi_B) = \left. \frac{\partial^2 \phi}{\partial x^2} \right|_C - \frac{1}{4} \left. \frac{\partial^2 \phi}{\partial x^2} \right|_C + \frac{1}{8} \frac{\partial^3 \phi}{\partial x^3} h + O(h^2) \quad (10)$$

For this reason, the truncation error for boundary cells has a theoretical order of accuracy of zero. With the present manufactured solution however, this zero order accuracy can not be detected at the wall, since the second derivative of velocity component in the y direction is zero. But because of the zero order accuracy present at other boundaries due to the discretization of the diffusion term, the L1 norm of the truncation error computed only next to the boundaries, becomes less than 1 on fine grids. However, it must be noticed that such a zero order accuracy on the truncation error does not imply that the numerical solution will not converge towards the exact solution when the grid size tends to zero. According to the weak consistency condition for finite-volume method², as long as the flux reconstruction schemes (8) and (9) are consistent, the finite-volume solution converges towards the exact solution when the mesh size tends to zero.

The above example shows that, in a finite-volume method, when a numerical flux is evaluated at one interface of a control volume with a first order accuracy, and at another interface with a second order accuracy, the accuracy of the truncation error may become zero order. Similarly, if the turbulence source term S_{ν_t} in equation (4) is discretized in a conservative form, the order of accuracy of the truncation error will become zero for the first two cells at the boundary. To avoid this issue, this source term is computed as a volumic integration for the present verification exercise. However, it would be easy to maintain a second-order accuracy close to the wall by using a one-sided discretisation of the gradient based on a least square approach using a wider set of unknowns in the domain. This alternate discretization for the wall diffusive fluxes will be evaluated in the future.

The observed orders of accuracy obtained on the unstructured triangular grid are 1.0 for all grids with both norms for both types of cells. It does not provide too much useful information.

The Lmax norm of the truncation error of the U momentum equation for internal cells decreases with grid refinement for the Cartesian grids and for the unstructured triangular grid with an observed order of accuracy at least equal to 1. But as shown in figure 5, it does not decrease for the unstructured quadrilateral grid. But even in this situation, the observed order based on L1 norm for internal cells is still bigger than 1 as shown in figure 2. This problem has been reported by Coirier³ who demonstrated that, with commonly

employed reconstruction schemes, the truncation error for refinement cells has a zero or even -1 order of accuracy with a finite-volume method because of the discretization of the diffusion terms. Coirier qualified such discretizations as inconsistent but the present authors prefer not to employ this term, since the weak consistency condition for finite volume method is still satisfied when we have a zero order accuracy for the truncation error which ensures that the numerical solution will converge towards the exact solution with grid refinement. However, an observed order of accuracy of -1 based on Lmax norm of the truncation error indicates that the reconstruction scheme is inconsistent at some interfaces. In this case, the convergence towards the exact solution with grid refinement can not be guaranteed.

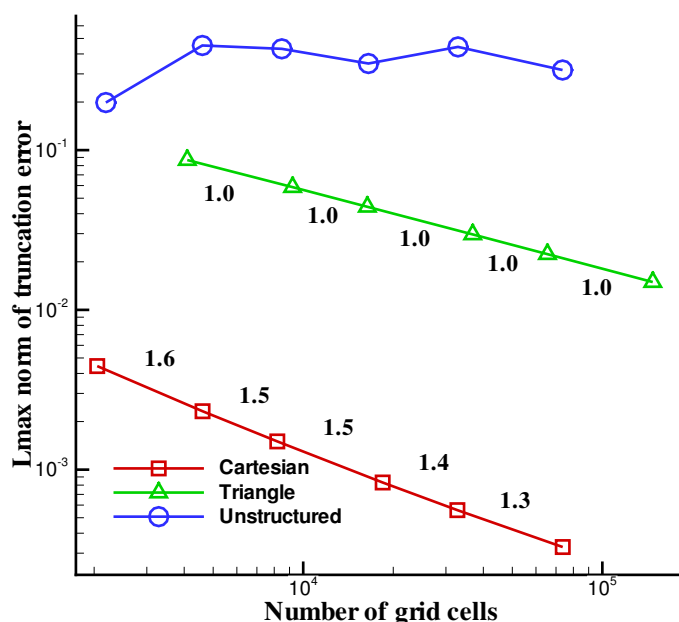


Figure 5: Lmax norm of truncation error of the U momentum equation for internal cells.

4.3 The pressure equation

The L1 norm of the truncation error for the pressure equation for internal cells is displayed in figure 6. The Rhie & Chow interpolation is employed to obtain the pressure equation. An unexpected convergence behaviour is observed: while the theoretical second order accuracy is observed at least for the last four grids for the momentum equations on the Cartesian grid set, the observed order of accuracy for the pressure equation is quite close to first order. One may suspect a coding mistake, but analyses of the Rhie & Chow interpolation implemented in the ISIS-CFD code reveal that this is actually the expected convergence behaviour for this particular implementation of the Rhie and Chow

interpolation.

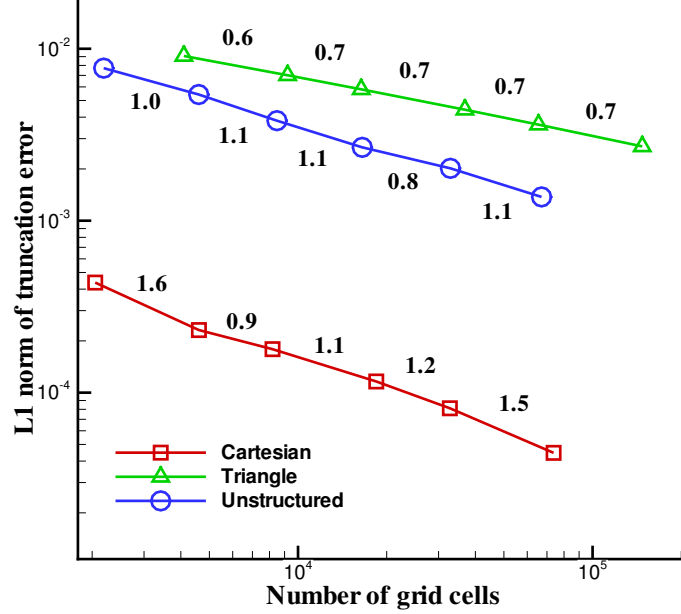


Figure 6: L1 norm of truncation error for the pressure equation for internal cells.

For simplicity, let's consider an one-dimensional model problem. The starting point for the Rhie & Chow interpolation is the discrete momentum equation in differential form written as

$$C_D u + S(u) + \frac{\partial p}{\partial x} = O(h^\alpha) \quad (11)$$

Here C_D is the diagonal coefficient of the discrete convection-diffusion operator which can be noted as

$$C_D = \frac{c}{h} + \frac{d}{h^2} \quad (12)$$

c and d are functions that change in space and time. In addition, both are positive quantities, due to the use of upwind scheme that ensures the positivity of c and the nature of the diffusion coefficient d . $S(u)$ contains all terms in the discrete momentum equation except the diagonal term and the pressure gradient. The term $O(h^\alpha)$ representing the truncation error of the momentum equation is retained in equation (11) so that its effect on the accuracy of the Rhie & Chow interpolation can be traced. As shown previously, the truncation error may be of zero order. It needs be ensured that even in this case, the Rhie & Chow interpolation leads to a consistent interpolation scheme. Time discretization is

not considered here, since it does not affect the accuracy of the interpolation. Applying the average operator to the equation (11) gives

$$\overline{C_D u} + \overline{S(u)} + \frac{\overline{\partial p}}{\partial x} = \overline{O(h^\alpha)} \quad (13)$$

Introducing the following approximations

$$\overline{C_D u} \approx \overline{C_D} u_f$$

and

$$\frac{\overline{\partial p}}{\partial x} \approx \frac{\partial p}{\partial x} \Big|_f$$

one obtains the so-called Rhie & Chow interpolation

$$u_f = -\frac{1}{\overline{C_D}} \left(\overline{S(u)} + \frac{\partial p}{\partial x} \Big|_f \right) \quad (14)$$

Tracing from equation (13) where no approximation is introduced to equation (14), we obtain the following expression representing the error of the Rhie & Chow interpolation:

$$Err_{RnC} = \frac{1}{\overline{C_D}} \left\{ \left[\overline{O(h^\alpha)} \right] - \left[\frac{\overline{\partial p}}{\partial x} - \frac{\partial p}{\partial x} \Big|_f \right] - \left[\overline{C_D u} - \overline{C_D} u_f \right] \right\} \quad (15)$$

The order of the term $1/\overline{C_D}$ can be estimated with

$$\frac{1}{\overline{C_D}} \approx \frac{h^2}{ch + d} \quad (16)$$

In the present verification exercise, the magnitude of d , proportional to the turbulence eddy-viscosity, has an order of 10^{-3} . With about 100 grid points stretched in the y direction and c of the order of unity, the magnitude of the term ch is about the same as d . Hence, the term $1/\overline{C_D}$ has an undefined order ranging from 1 to 2. The situation is similar in a real application. The term $1/\overline{C_D}$ can only guarantee a practical order of 1 rather than 2. Based on this estimation, the first term in the expression (15) introduces a cubic order of error for regular cells where the truncation error of the discrete momentum equation is expected to be second order. For irregular cells where the truncation error of the discrete momentum equation is of zero order as shown above, this term will introduce an error with an undefined order ranging from 1 to 2.

The second term in the expression (15) introduces an error of a classical linear interpolation. Multiplied by $1/\overline{C_D}$, the order of this term is at least cubic order. Error analysis

of the last term in the expression (15) requires special attention. Developed in Taylor series, we obtain the leading order terms as follow:

$$\begin{aligned}
& 2 \left(\overline{C_D u} - \overline{C_D} u|_f \right) \\
&= \left(\frac{c|_L}{h} + \frac{d|_L}{h^2} \right) u_L + \left(\frac{c|_R}{h} + \frac{d|_R}{h^2} \right) u_R - \left(\frac{c|_L}{h} + \frac{d|_L}{h^2} \right) u_f - \left(\frac{c|_R}{h} + \frac{d|_R}{h^2} \right) u_f \\
&\approx \frac{1}{2} h \frac{\partial u}{\partial x} \left(\frac{c|_R}{h} + \frac{d|_R}{h^2} - \frac{c|_L}{h} - \frac{d|_L}{h^2} \right) \\
&\approx \frac{1}{2} \frac{\partial u}{\partial x} \left(\frac{\partial d}{\partial x} + h \frac{\partial c}{\partial x} \right) \quad (17)
\end{aligned}$$

Hence, the error to the Rhie & Chow interpolation introduced by the last term in the expression (15) has the following form:

$$\frac{1}{4} \frac{\partial u}{\partial x} \frac{h^2}{ch + d} \left(\frac{\partial d}{\partial x} + h \frac{\partial c}{\partial x} \right) \quad (18)$$

Grid space is considered as constant in the above analysis. For laminar flow or flow dominated by convection, the Rhie & Chow interpolation is second order accurate. But for a turbulent flow, it has an undefined order of accuracy ranging from 1 to 2, which confirms the previous numerical results. It should be mentioned that if the equation (11) is first diagonalized with the coefficient C_D before applying the average operator, then, the resulting Rhie & Chow interpolation given by:

$$u_f = \bar{\hat{u}} - \overline{C_p} \left. \frac{\partial p}{\partial x} \right|_f \quad (19)$$

with

$$\hat{u} = - \frac{S(u)}{C_D} \quad (20)$$

and

$$C_p = \frac{1}{C_D} \quad (21)$$

has a second order accuracy without the requirement $ch \ll d$ as for the approach given by equation (14). Results obtained with the interpolation (19) will be presented in future publication.

The Lmax norm of the truncation error for the pressure equation for internal cells is shown in figure 7. The fact that observed order of accuracy on Cartesian grid is lower than 1 is certainly due to the convergence behaviour of the Rhie & Chow interpolation explained above rather than the influence of boundary cells, since the first two layers from the boundary where the truncation error is of zero order are excluded in the evaluation of this Lmax norm. On the unstructured grid, the Lmax norm does not decrease with grid refinement as it was observed for the momentum equation.

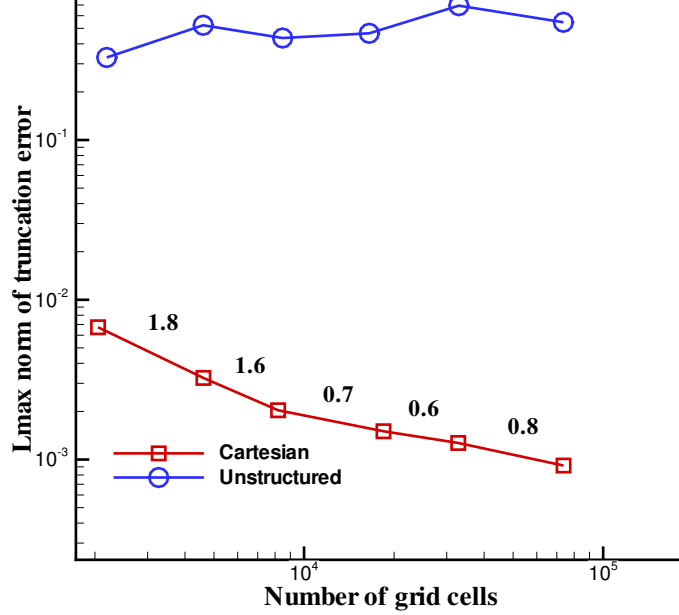


Figure 7: Lmax norm of the truncation error for the pressure equation for internal cells.

5 VERIFICATION OF THE ERROR

The verification of the truncation error in the above section shows that the observed order of accuracy based on the Lmax norm is of zero order for boundary cells on all types of grid, and also for internal cells on the unstructured quadrilateral grids. However, the weak consistency condition is still satisfied. It is expected that the numerical solution converges towards the exact solution when the grid is refined. This will be verified in this section. Unlike for the truncation error, there is no need to distinguish boundary cells from internal cells since all sources of error are dispersed into the whole domain.

The L1 norm of error for the U velocity component is shown in figure 8. Although the L1 norm of the truncation errors on different types of grid are quite different, the L1 norm of errors are similar. It is interesting to note that the solutions obtained on the triangular unstructured grid are almost as accurate as that obtained on the Cartesian grid. The error level in Lmax norm on the unstructured quadrilateral grid is higher as shown in figure 9. Even based on the Lmax norm, the numerical solutions on all types of grid converge towards the exact solution. Deterioration in order of convergence observed on fine grid for Cartesian mesh and triangular mesh is due to the exit boundary rather than to the discretization. These results confirm that the numerical solution obtained by a finite-volume method converges towards the exact solution when the reconstruction scheme is consistent, regardless of the order of accuracy of the truncation error. They also suggest that the truncation error is not a relevant convergence indicator for the

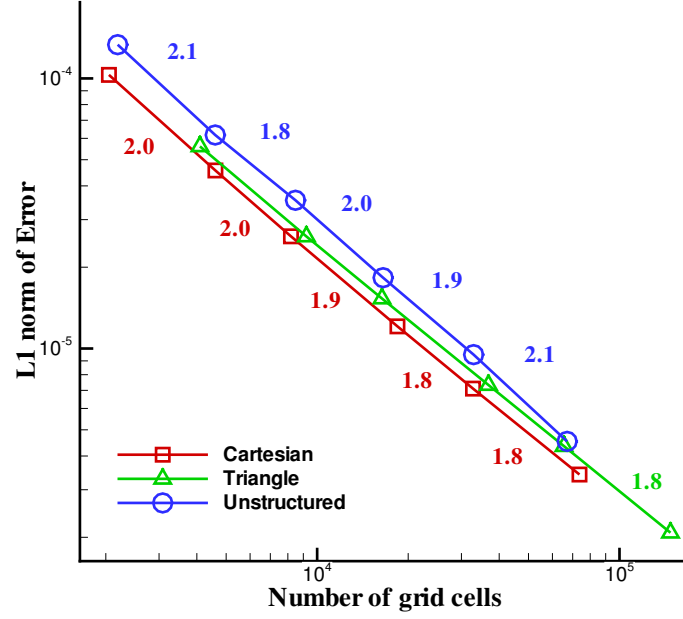


Figure 8: L1 norm of error for the U velocity component.

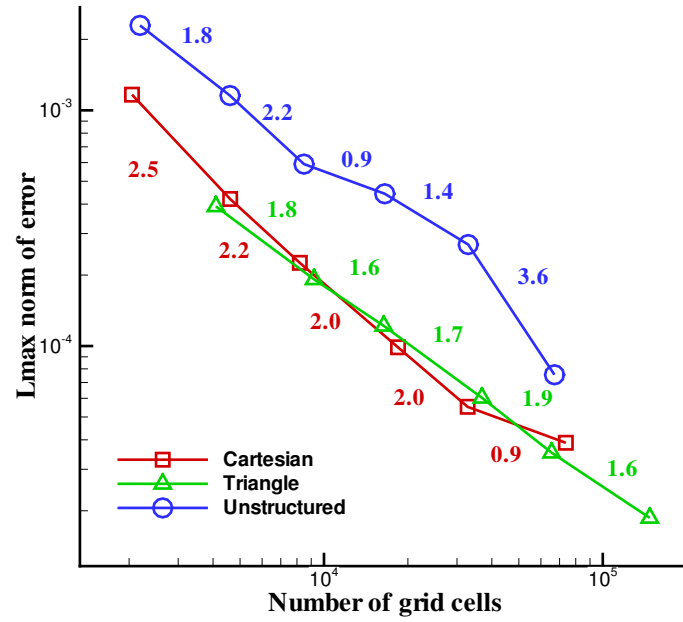


Figure 9: Lmax norm of error for the U velocity component.

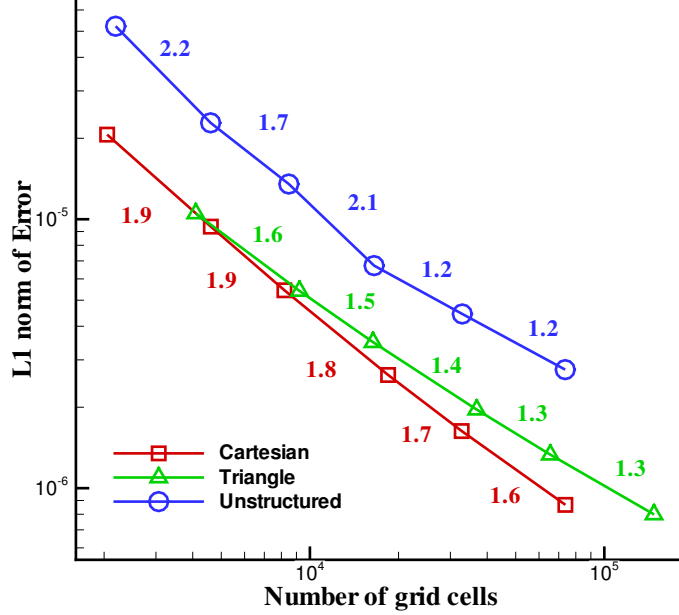


Figure 10: L1 norm of error for the pressure field.

finite-volume method.

Similar results concerning the pressure are shown in figures 10 and 11. As the level of the pressure is not fixed in the computation, the error between the exact solution P^{exa} and the numerical solution P^{num} is evaluated as:

$$error_i = P_i^{exa} - P_i^{num} - \frac{\sum_j (P_j^{exa} - P_j^{num}) Vol_j}{\sum_j Vol_j} \quad (22)$$

Based on the L1 norm, the numerical solutions obtained on all types of grid converge towards the exact solution. Again, more accurate results both in terms of level of error and order of convergence are obtained on the triangular grid rather than on the Cartesian grid, even with the same number of grid cells. Based on the Lmax norm however, error does not decrease with grid refinement if the finest grid is excluded for the unstructured quadrilateral grid. Figure 12 displays the pressure contours near the refinement interface. The kink observed in the figure near the interface is responsible for the local stagnation of the error measured by the Lmax norm. Moreover, unlike for the velocity field, the observed order of accuracy based on the Lmax norm becomes smaller than one on the fine grids both for the Cartesian and triangular grids, leaving therefore a suspicion on the local consistency of the numerical solution. These are unexplained results which require further investigations. It should be noted that the stagnation in Lmax norm is not in contradictory with the convergence in L1 norm, since the number of irregular cells where the kink is observed is proportional to number of grid cell per direction n rather than to

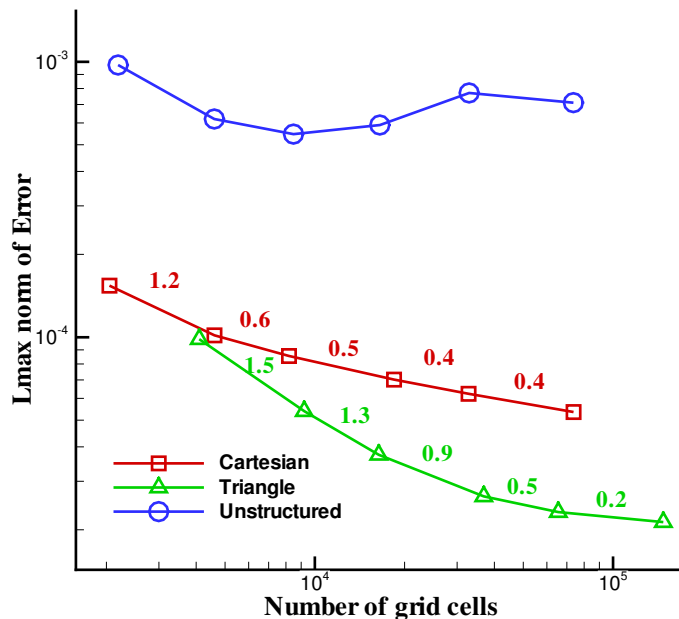


Figure 11: Lmax norm of error for the pressure field.

the number of grid cell n^2 .

Resistance prediction is an important output of a numerical computation. The relative errors for the friction resistance is shown in figure 13. The exact value of the friction resistance is $0.31285313531e-5$. The convergence behaviour is similar to that of the error of the flow field in L1 norm. Predictions obtained on the triangular grids are more accurate. All results converge towards the exact solution without exception. It should be mentioned that the wall friction is computed with a one-sided first order difference scheme. It is interesting to verify if this first order scheme can provide a second order accurate result if the solution is second order accurate. In the present exercise, second order results are obtained. But this may due to the fact that the first order scheme can give a second order accurate result when the second derivative of the velocity component in the direction normal to the wall is zero as it is the case with the present manufactured solution. It would be useful to select another manufactured solution to investigate this issue.

6 CONCLUSIONS

The ISIS-CFD code has been verified with the method of manufactured solution for turbulent flow computations using a prescribed manufactured solution for the turbulence field. An accuracy close to second-order has been obtained on the L1 norms of the velocity fields for all types of grid. A slight reduction of accuracy has been observed on

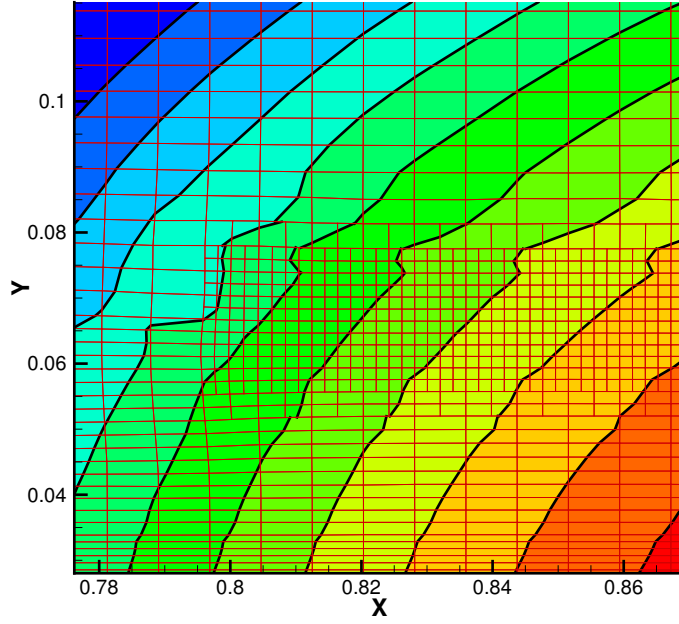


Figure 12: The pressure field near the refinement interface.

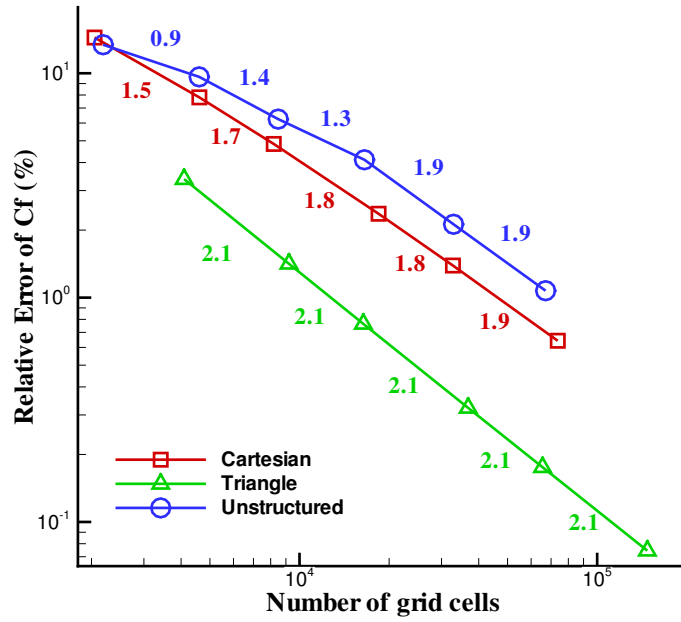


Figure 13: Relative error of the friction resistance.

the pressure due to the Rhie & Chow interpolation. This exercise has also revealed that a saturation is observed on locally refined unstructured grids. It has been established that the peculiar implementation of the Rhie & Chow interpolation can not directly achieve a second order accuracy even when the grid is fine enough to represent the solution. This observation is confirmed by an error analysis which also shows that second order accuracy can fortunately be obtained with an alternative formulation which will be evaluated in a near future. Another outcome of this verification study is that the numerical solution obtained on a triangular mesh has been found to be as accurate as that obtained on a Cartesian grid.

As expected, it has been observed that the truncation error is not a relevant convergence indicator for the finite-volume method since numerical solution obtained with a finite-volume method is expected to converge towards the exact solution as long as the flux reconstruction scheme is consistent. However, this property is not fully confirmed by the present numerical study. On the unstructured quadrilateral grid set, the accuracy of the flux reconstruction schemes at the refinement interfaces reduces to first order both for the viscous flux in the momentum equation and for the mass flux because of the use of Rhie & Chow interpolation in the pressure equation, which proves the consistency of the flux evaluation and leads us to expect that the numerical solution converges towards the exact solution. However, for locally-refined unstructured grids, convergence is not observed on the Lmax norm of the error on the pressure, although the L1 norm converges to zero. Even if the source of this local error is clearly related to a lack of regularity of the grid, one should try to understand the deep origins of this local inconsistency to improve the overall quality of the solution on locally refined unstructured grids since we consider that local grid adaptivity is a key characteristic of future CFD methodologies. Future progresses will likely come from an improvement of the Rhie & Chow mass-flux interpolation and diffusive fluxes reconstruction on faces characterised by strong misalignment.

Finally, the authors suggest two useful modifications for future manufactured solutions. The first one is to choose a solution for the velocity whose second derivative in the normal direction to the wall does not vanish in order to study the effect of the first order one-sided difference scheme both on the evaluation of viscous flux at the wall and on the evaluation of the skin friction coefficient. The second one concerns the term $S_{\nu t}$ in equation (4). It would be better if this term is not zero so that an omission in a CFD code can be detected.

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