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DOI
10.23919/ACC50511.2021.9483077

Publication date
2021

Document Version
Accepted author manuscript

Published in
Proceedings of the American Control Conference, ACC 2021

Citation (APA)
Dong, J., Kolarijani, A. S., & Esfahani, P. M. (2021). Multimode Diagnosis for Switched Affine Systems. In *Proceedings of the American Control Conference, ACC 2021* (pp. 870-875). IEEE.
https://doi.org/10.23919/ACC50511.2021.9483077

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Multimode Diagnosis for Switched Affine Systems

Jingwei Dong, Arman Sharifi Kolarijani and Peyman Mohajerin Esfahani

Abstract— **In this paper, we propose an approach to detect mode transitions and to isolate active modes in discrete-time, switched affine systems. The proposed approach is in particular constructed for systems in which the controller is oblivious of the switching signal. The diagnosis approach consists of two main parts: construction of a bank of output filters (that generate desired residuals) and definition of a certain type of residual/threshold-based diagnosis rules. The filters' construction is cast as linear feasibility problems. These feasibility problems enforce desirable diagnosis relationships between each subsystem's affine constant and each residual. The diagnosis rules are inspired by the well-known generalized observer scheme. Moreover, we provide a method to compute each mode's diagnosis time based on the diagnosis rules and properly chosen residual thresholds. A numerical example is presented to show the performance of the proposed approach.**

I. INTRODUCTION

As an important class of hybrid systems, switched systems have been the subject of many studies over the past two decades. Many industrial systems, such as chemical plants [13] and aeronautic systems [17], are difficult to be exactly modeled due to the nonlinearity and complexity of the underlying dynamics. Nonetheless, these systems can be effectively modelled by switched systems, see for example [7] and the reference therein. A general approach to control the switched systems is to employ mode-dependent controllers. Crucially, this approach requires the controller to know the switching signal, see for example [15].

In fault diagnosis scenarios, an unexpected transition from a healthy mode to a faulty one can also be treated as a switching. Thus, the hope is to leverage tools from switched systems in fault diagnosis setups. Unlike the common control-theoretic viewpoint of switched systems, the switching signal representing a fault is usually unknown to the controller here. This implies that one first needs to detect switching instances in diagnosis tasks. This step is then followed by isolating the mode enabled by the switching.

To detect changes caused by a switching and/or a fault, traditional model-based methods utilize approaches such as *observer-based* approaches [4] and *parity space* approaches [8]. By doing so, certain *residual* signals are generated that enable to characterize quantitatively the occurrence of these changes. There is an array of studies in the literature that propose how to construct proper residual signals.

Considering linear systems, Nyberg and Frisk [5] first developed the parity-space-like approach to construct filters. The proposed approach has a polynomial framework and is able to find filters with the lowest possible order. The authors' following work in [12] extended the previous approach to linear differential-algebraic equation (DAE) systems. In particular, they provided a criterion for fault *detectability* in the DAE systems. Inspired by [12], the authors in [10] introduced an optimization scheme to construct residual filters. The scheme provides a tractable procedure to find feasible filters. Notice that all of the aforementioned methods are only applicable to systems with a "single" mode and cannot be directly used to identify an active mode in switched systems. In order to identify the active model in switched systems, a set of residuals are usually required. The authors in [2] extended the classical parity space approach to hybrid systems and obtained a set of structured parity residuals. The isolation method in [2] is similar to the *generalized observer scheme* (GOS) [3] in the following sense. Each residual is made sensitive to all but one mode. Following a GOS mindset, there are several studies that construct a bank of parallel observers, e.g., *unknown input* observers [14] and *sliding mode* observers [16]. However, there is usually an undesirable property in using the parity space and observer-based methods: the order of the constructed residual generators is the same as that of the system dynamics. When dealing with large-scale systems, the residual generators will be complex and computationally demanding to implement. Moreover, none of the above-mentioned studies provide the required time to execute a diagnosis task (which we call the *diagnosis time* later on). Practically speaking, the diagnosis time is important since the (human or machine) operator relies on this quantitative measure to make a proper decision, like activating the matched controller.

Our proposed approach: In this paper, we present a diagnosis scheme for switched affine systems to detect transitions among modes and to isolate an active mode. We also provide an approach to compute the diagnosis time. Inspired by the results in [12] and [10], we first reformulate a switched system to a DAE model. Then, we use a parity-space-like method to design a bank of filters. The design task of each filter is formulated as a linear feasibility problem according to some proper relationships that should exist between each mode's affine term and each residual. The diagnosis rules are set similar to those of the GOS to determine the active mode. This means that the residual corresponding to the active mode is set to be zero while all other residuals are set to be nonzero. Based on the constructed filters, we next derive the analytical expression of the residuals and compute the diagnosis time for some properly chosen thresholds. In summary, the main contributions of this paper are as follows.

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- **Filter Design:** We propose a multimode diagnosis approach to construct residual filters for switched affine systems in the case of asynchronous switching, employing an optimization approach in the DAE framework (Theorem 1). Compared to the classical observer-based methods [16], the proposed filter here typically has a lower order dynamics, thus a simpler architecture. The extension of the proposed approach to deal with measurement noises is also discussed (Remark 2).
- **Diagnosis time computation:** We provide expression of the matched residual and show that the matched residual is independent of the system matrices (Corollary 1). With the expression of the matched residual and given some user-defined thresholds, we present an approach to compute the diagnosis time for each mode (Theorem 2).

Notations: The set of all positive reals is $\mathbb{R}_{>0}$. The sets $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ represent the space of n dimensional vectors and the space of $m\times n$ dimensional matrices with real entries, respectively. The set $\mathbb{Z}_{>0}$ ($\mathbb{Z}_{\geq 0}$) denotes all positive (non-negative) integers. The set $\{1,\dots,n\}$ is represented by $[n]$. The identity matrix with appropriate dimensions is denoted by I. For a matrix $A\in\mathbb{R}^{m\times n}$, $A^\top$ is the transpose of A. For a row vector $v=[v^1,\dots,v^n]$, the ℓ_∞-norm of v is $\|v\|_\infty := \max_{i\in[n]}|v^i|$, and $|v|=[|v^1|,\dots,|v^n|]$. For $k_1,k_2\in\mathbb{Z}_{>0}$, where $k_1<k_2$, the discrete sequence $[k_1,k_1+1,\dots,k_2]$ is denoted by $[k_1,k_2]$. The operator $\wedge$ is the logical "AND" operator. For $c\in\mathbb{R}_{>0}$, $\lceil c\rceil$ rounds up c to the nearest integer. Given a signal $\{s(k)\}_{k\in\mathbb{Z}_{\geq 0}}$ and an operator $\boldsymbol{O}$, the notation $\boldsymbol{O}[s](k)$ denotes the application of the operator $\boldsymbol{O}$ on the signal s at the time instance k.

II. Model and Problem description

Consider the discrete-time system, that is comprised of n perturbed subsystems,

$$\begin{aligned} x(k+1) &= A_{\sigma(k)}x(k)+B_{\sigma(k)}u(k)+E_{\sigma(k)}d(k)\\ y(k) &= C_{\sigma(k)}x(k), \end{aligned} \tag{1}$$

where $x(k)\in\mathbb{R}^{n_x}$, $u(k)\in\mathbb{R}^{n_u}$, and $y(k)\in\mathbb{R}^{n_y}$ are the state, input and output of the system, respectively. The disturbance is denoted by $d(k)\in\mathbb{R}^{n_d}$. The switching map $\sigma:\mathbb{Z}_{\geq 0}\to[n]$ is a piecewise constant function describing the active mode at each instance k. Matrices $A_{\sigma(k)}$, $B_{\sigma(k)}$, $E_{\sigma(k)}$, and $C_{\sigma(k)}$ are all known and have appropriate dimensions. For all $i\in[n]$, let $\Gamma_i := \{A_i,B_i,E_i,C_i\}$ denote mode i. The system switches from one mode to another mode in $[n]$ at each switching instance t_s in $\mathscr{S} := \{t_s : s\in\mathbb{Z}_{>0}, \sigma(t_s)\neq\sigma(t_s-1)\}$. For each mode $i\in[n]$, we suppose that the control law

$$u(k) := K_i y(k) \tag{2}$$

is available. (One can employ for example the $\mathscr{H}_\infty$-method proposed in [1] to design the feedback gains K_i.)

Assumption 1 (Dwell time [6]): For all $t_s\in\mathscr{S}$, there exists a large enough constant $\tau_d>0$ such that $t_{s+1}-t_s\geq\tau_d$. The quantity τ_d is the so-called *dwell time* constant.

Assumption 2 (Perturbation regularities): We stipulate the following conditions.

(a) (Bounded disturbance) Given two constants $d_{\min}, d_{\max}\in\mathbb{R}_{>0}$, it holds that $d_{\min}\leq|d^j(k)|\leq d_{\max}$ for all $j\in[n_d]$, where $d^j(k)$ is the j-th entry of $d(k)$.

(b) (Switched affine systems) The disturbance $d(k)$ is constant for each mode, that is $d(k)=d_{\sigma(k)}$ for all $k\in\mathbb{R}_{\geq 0}$.

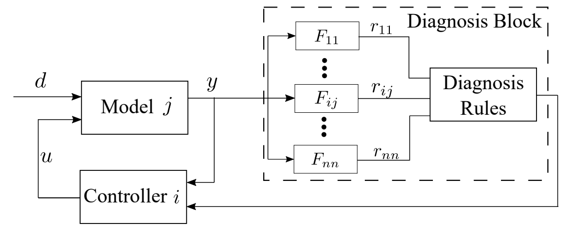

Fig. 1. Structure of the closed-loop dynamics and the diagnosis block.

We further suppose that the switching map σ is unknown to the controller (2). Let $i,j\in[n]$ where $i\neq j$, and denote the *transition* from Γ_i to Γ_j by M_{ij}. Thus, the transition M_{ij} is understood as a faulty situation because there is a mismatch between model Γ_j and controller gain K_i. We use S_{ij} to denote the *status* of the closed-loop dynamics that consist of Γ_j and K_i. The diagnosis process for the system (1) breaks down to two steps: detection and isolation. In the detection step, the goal is to identify that a transition has occurred, i.e., the model is not Γ_i anymore. The isolation step determines Γ_j occurred after the transition in order to employ the correct controller gain K_j. Fig. 1 depicts the structure of the fault diagnosis block proposed for the closed-loop dynamics (1)-(2). For each $i,j\in[n]$, the block $\boldsymbol{F}_{ij}$ represents a filter. Mathematically speaking, each filter $\boldsymbol{F}_{ij}$ is an MISO transfer function that maps y to the *residual* r_{ij}, i.e.,

$$r_{ij}(k) := \boldsymbol{F}_{ij}[y](k),$$

that is responsible to identify transition M_{ij} has occurred. Notice that there are in total n^2 filters to be designed to. Let the *diagnosis time* $\Delta\tau$ be the time that the diagnosis block takes to perform the two steps of detection and isolation. Generally, the proposed approach needs to guarantee that $\Delta\tau<\tau_d$. To compute $\Delta\tau$, we introduce the following definitions, given two thresholds $\varepsilon_{\text{det}},\varepsilon_{\text{iso}}\in\mathbb{R}_{>0}$.

Definition 1 (Detection time): A quantity $t_{\text{det}}(\Gamma_i,\varepsilon_{\text{det}})\in\mathbb{Z}_{>0}$ is called the *detection time* of mode Γ_i if

$$t_{\text{det}}(\Gamma_i,\varepsilon_{\text{det}}) := \max\big\{\hat{k} : |r_{ii}(t_s+k)|\leq\varepsilon_{\text{det}},\ \mathsf{M}_{ij}\text{ where } j\neq i,\ \forall k\in[0,\hat{k}]\big\}.$$

Definition 2 (Isolation time): A quantity $t_{\text{iso}}(\varepsilon_{\text{iso}})\in\mathbb{Z}_{>0}$ is called the *isolation time* if

$$t_{\text{iso}}(\varepsilon_{\text{iso}}) := \max_{\mathsf{M}_{ij},i\neq j,d}\min\big\{k' : |r_{ij}(t_s+k)|\leq\varepsilon_{\text{iso}}, \forall k\geq k'-t_s\big\}.$$

Definition 1 indicates that the transition M_{ij} is detected once the residual r_{ii} exceeds the threshold ε_{det}. Also, the isolation time $t_{\text{iso}}(\varepsilon_{\text{iso}})$ is defined for all modes. By means of this definition of $t_{\text{iso}}(\varepsilon_{\text{iso}})$, it is guaranteed that no matter which mode occurs, the matched residual r_{ij} will remain smaller than ε_{iso} after $t_{\text{iso}}(\varepsilon_{\text{iso}})$. Hence, the active model can be isolated.

The problem addressed in this paper is as follows.

Problem 1: Consider the closed-loop dynamics (1)-(2) subject to Assumptions 1 and 2, and the structure of the diagnosis block in Fig. 1.

- For all $i, j \in [n]$, design a bank of filters $\boldsymbol{F}_{ij}$ such that the input-output maps between d and r_{ij} satisfy

$$d \xrightarrow{\mathsf{S}_{ij}} r_{ij} = 0, \tag{3a}$$

$$d \xrightarrow{\mathsf{S}_{ih}} r_{ij} \neq 0, \quad \forall h \in [n]/\{j\}. \tag{3b}$$

- Given a pair of thresholds ε_{det} and ε_{iso}, compute the diagnosis time, i.e.,

$$\begin{aligned}\Delta\tau_i(\varepsilon_{\text{det}}, \varepsilon_{\text{iso}}) :&= \Delta\tau(\Gamma_i, \varepsilon_{\text{det}}, \varepsilon_{\text{iso}}) \\ &= t_{\text{det}}(\Gamma_i, \varepsilon_{\text{det}}) + t_{\text{iso}}(\varepsilon_{\text{iso}}).\end{aligned}$$

Remark 1 (Required number of residual comparisons): In order to conduct a detection/isolation task, only the residuals r_{ih} where $h \in [n]$ are required to be compared (i.e., the residuals that share the same index representing the controller). This has to do with fact that the information of the controller is known.

III. Main results

In this section, we present the main results of this work. First, the approach to design the filters is introduced. We then provide a closed-form expression of the matched residual. Lastly, we focus on the task of determining the diagnosis time, given a pair of thresholds.

A. Filter designs

Let us first reiterate the conditions (3). Consider a transition M_{ij}. To design the filter $\boldsymbol{F}_{ij}$, the basic idea is to force the residual r_{ij} to be zero while the rest of residuals to be non-zero.

$$\begin{cases}(|r_{ii}| > \varepsilon) \equiv \text{a transition occurred,} \\ (|r_{ij}| \leq \varepsilon) \wedge (|r_{ih}| > \varepsilon, \forall h \in [n]/\{j\}) \equiv M_{ij} \text{ occurred.}\end{cases} \tag{4}$$

Suppose now M_{ij} happens at some switching instance t_s. The closed-loop dynamics (1)-(2) become

$$x(k+1) = \begin{cases} A^{\text{cl}}_{ij}x(k) + E_j d(k), \; k \in [t_s, t_s + \Delta\tau_i(\varepsilon)) \\ A^{\text{cl}}_{jj}x(k) + E_j d(k), \; k \in [t_s + \Delta\tau_i(\varepsilon), t_{s+1}) \end{cases} \tag{5}$$
$$y(k) = C_j x(k),$$

where $A^{\text{cl}}_{ij} := A_j + B_j K_i C_j$. To construct the filters, we rewrite the closed-loop dynamics (5) in a DAE format, that is

$$\begin{cases} H_{ij}(p)X(k) + L(p)y(k) = 0, \; k \in [t_s, t_s + \Delta\tau_i(\varepsilon)) \\ H_{jj}(p)X(k) + L(p)y(k) = 0, \; k \in [t_s + \Delta\tau_i(\varepsilon), t_{s+1}), \end{cases} \tag{6}$$

where $X(k) := \begin{bmatrix} x(k)^\top & d(k)^\top \end{bmatrix}^\top$, the operator p is the time-shift operator, and the matrices $H_{ij}(p)$ and $L(p)$ are

$$H_{ij}(p) := p \times H_{ij,1} + H_{ij,0} = \begin{bmatrix} -pI + A^{cl}_{ij} & E_j \\ C_j & 0 \end{bmatrix},$$
$$L(p) := L_0 = \begin{bmatrix} 0 \\ -I \end{bmatrix}.$$

In what follows, we now spell out the involved variables in the construction of each filter $\boldsymbol{F}_{ij}$. For the status S_{ij}, we define the corresponding filter $\boldsymbol{F}_{ij}$ as

$$\boldsymbol{F}_{ij} := a^{-1}(p)N_{ij}(p)L(p), \tag{7}$$

where the polynomial row vector $N_{ij}(p) := \sum_{m=0}^{d_N} N_{ij,m}p^m$ and $N_{ij,m} \in \mathbb{R}^{1\times(n_x+n_y)}$ are constant row vectors, d_N denotes the degree of $N_{ij}(p)$, and finally $a(p)$ is a (d_N+1)-th order polynomial with all roots inside the unit disk. We define

$$a(p) := p^{d_N+1} + a_1 p^{d_N} + \cdots + a_{d_N}p + a_{d_N+1}, \tag{8}$$

where for each $m \in [d_N+1]$, a_m is some constant coefficient. Notice that the role of $a(p)$ is to ensure that the obtained filter $\boldsymbol{F}_{ij}$ is proper and stable. For the sake of simplicity of exposition, we further suppose that all the filters are of the same degree. Observe that both $N_{ij}(p)$ and $a(p)$ are the design parameters. In order to make the design process tractable, we however fix $a(p)$ and find a feasible $N_{ij}(p)$ in the following theorem. To simplify the notation, let $\bar{N}_{ij} := [N_{ij,0} \; N_{ij,1} \; \ldots \; N_{ij,d_N}]$,

$$\bar{H}_{ij} := \begin{bmatrix} H_{ij,0} & H_{ij,1} & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & H_{ij,0} & H_{ij,1} \end{bmatrix}, \; \bar{L} := \begin{bmatrix} L_0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & L_0 \end{bmatrix}.$$

Theorem 1 (Filter design as feasibility problem): Consider the closed-loop dynamics (1)-(2) subject to Assumptions 1 and 2-(a). If the filter (7) satisfies

$$\bar{N}_{ij}\bar{H}_{ij} = 0, \tag{9a}$$

$$\|a^{-1}(1)\bar{N}_{ij}\bar{L}[\overbrace{I \; \ldots \; I}^{d_N+1}]^\top C_h(I - A^{\text{cl}}_{ih})^{-1}E_h\|_\infty \geq 1, \quad \forall h \in [n]/\{j\}, \tag{9b}$$

then, the properties (3) hold.

Proof. We first show that equality (9a) enforces the desired property (3a). According to the rules of multiplication of polynomial matrices [11, Section III-A], we have

$$N_{ij}(p)H_{ij}(p) = \bar{N}_{ij}\bar{H}_{ij}[I \; pI \; \ldots \; p^{d_N+1}I]^\top,$$

and thus, equality (9a) implies $N_{ij}(p)H_{ij}(p) = 0$. Now, multiply from left the first DAE expression in (6) by $N_{ij}(p)$ and observe that $N_{ij}(p)H_{ij}(p)X(k) + N_{ij}(p)L(p)y(k) = N_{ij}(p)L(p)y(k) = 0$. Recall the construction of filter $\boldsymbol{F}_{ij}$ in (7). It holds that

$$r_{ij}(k) = \boldsymbol{F}_{ij}[y](k) = \big(a^{-1}(p)N_{ij}(p)L(p)\big)[y](k) = 0. \tag{10}$$

This concludes the first part of the proof.

In the second part of the proof, we establish that inequalities (9b) imply the satisfaction of the desired property (3b). In doing so, let us first define the transfer function $\boldsymbol{T}_{dr_{ij}}(p)$ from d to r_{ij}, that is $r_{ij}(k) := \boldsymbol{T}_{dr_{ij}}(p)[d](k)$. Consider now that the status of the closed-loop system is S_{ih} for some $h \in [n]/\{j\}$. According to (5), we have

$$y(k) = \big(C_h(pI - A^{\text{cl}}_{ih})^{-1}E_h\big)[d](k),$$

where we used the identity $x(k+1)=px(k)$. By virtue of (10), we arrive at

$$\boldsymbol{T}_{dr_{ij}}(p)=a^{-1}(p)N_{ij}(p)L(p)\times C_h(pI-A_{ih}^{\mathrm{cl}})^{-1}E_h$$
$$=a^{-1}(p)\bar{N}_{ij}\bar{L}[I\ pI\ \ldots\ p^{d_N}I]^\top\times C_h(pI-A_{ih}^{\mathrm{cl}})^{-1}E_h,$$

where the multiplication of polynomial matrices is used in the second equality. We force the ℓ_∞-norm of the steady-state gain of $\boldsymbol{T}_{dr_{ij}}(p)$ to be greater than or equal to 1, that is

$$\|\boldsymbol{T}_{dr_{ij}}(1)\|_\infty=\|a^{-1}(1)\bar{N}_{ij}\bar{L}[I\ \ldots\ I]^\top C_h(I-A_{ih}^{\mathrm{cl}})^{-1}E_h\|_\infty\geq 1.$$

In light of Assumption 2-(a), it follows that r_{ij} is nonzero when the status is not S_{ij}. This concludes the second part of the proof. ■

Notice that (9) is not a convex feasibility problem because of the non-convex constraint (9b). However, (9) can be viewed as a union of several linear feasibility problems. Thus, it is not a difficult task to find a feasible solution for this problem. Observe that unlike observer-based methods the proposed approach can construct filters that may have lower order than the underlying dynamics. In fact, when the filter degree d_N is the same as the dimension of the original system dynamics n_x, one can show that the coefficients of any observer-based filter is a feasible solution to the program (9) [10]. We finally provide a lower-bound on d_N in order to guarantee the existence of non-trivial solutions to (9).

Proposition 1 (Existence of non-zero solutions): The equality (9a) has non-zero solution if

$$(d_N+1)(n_x+n_y)>Rank(\bar{H}_{ij}).\qquad(11)$$

Proof. The equality (9a) has non-zero solution if the dimension of the left null space of $\bar{H}_{ij}$ is nonzero. Then (11) is a straightforward result of *Rank Plus Nullity Theorem* [9, Chapter 4]. ■

Remark 2 (Measurement noises/Uncertainties): To deal with model uncertainties and process/measurement noises, one can add a proper objective function to the feasibility problem (9), e.g., $\mathscr{H}_2$ gain of the transfer function from noises to residuals. This is done in order to reduce the impacts of noise and uncertainties on the residuals.

B. Transient behavior of matched filer

In the previous section, we showed that for a matched status-residual pairing (S_{ij},r_{ij}), the matched residual r_{ij} is enforced to become zero. Nonetheless, the residual $r_{ij}(t_s)$ is nonzero at a switching instance t_s and this has to do with the fact that the system's states $x(t_s)\neq 0$. In simple words, it takes some time for r_{ij} to reach the preset threshold ε after a transition M_{ij}. In what follows, our goal is to quantitatively capture this transient behavior in terms of the diagnosis time.

We use the Z-transform to incorporate the impact of $x(t_s)$. Let the polynomial row vector $N_{ij}(z):=[\hat{N}_{ij}(z)\ \check{N}_{ij}(z)]$, where $\hat{N}_{ij}(z)$ and $\check{N}_{ij}(z)$ have dimensions n_x and n_y, respectively. The following result captures the relationship between r_{ij} and $x(t_s)$.

Corollary 1 (Matched residual expression): Suppose that the hypotheses in Theorem 1 hold and that the filter $\boldsymbol{F}_{ij}$ is designed using (9). Let t_s be the last switching instance. Then,

$$r_{ij}(z)=-\frac{\hat{N}_{ij}(z)}{a(z)}zx(t_s).\qquad(12)$$

Proof. Consider the closed-loop dynamics (5). Apply the Z-transform on $x(k+1)=A_{ij}^{\mathrm{cl}}x(k)+E_jd(k)$ with the initial condition $x(t_s)$. We have $x(z)=(zI-A_{ij}^{\mathrm{cl}})^{-1}(E_jd(z)+zx(t_s))$. Apply now the Z-transform on $y(k)=C_jx(k)$. It follows that

$$y(z)=C_j(zI-A_{ij}^{\mathrm{cl}})^{-1}(E_jd(z)+zx(t_s)).\qquad(13)$$

Recall now that we showed in the proof of Theorem 1 that $N_{ij}(p)H_{ij}(p)=0$ and $N_{ij}(p)L(p)=0$. Replace the time-shifter p with the Z-transform operator z in these two relations. We get

$$\begin{bmatrix}\hat{N}_{ij}(z) & \check{N}_{ij}(z)\end{bmatrix}\begin{bmatrix}-zI+A_{ij}^{\mathrm{cl}} & E_j\\ C_j & 0\end{bmatrix}=0,$$
$$\begin{bmatrix}\hat{N}_{ij}(z) & \check{N}_{ij}(z)\end{bmatrix}\begin{bmatrix}0\\ -I\end{bmatrix}=0,$$

where we used the expansion $\begin{bmatrix}\hat{N}_{ij}(z) & \check{N}_{ij}(z)\end{bmatrix}$ for $N_{ij}(z)$. As a result, we have

$$\hat{N}_{ij}(z)=\check{N}_{ij}(z)C_j(zI-A_{ij}^{\mathrm{cl}})^{-1},\qquad(14a)$$
$$\hat{N}_{ij}(z)E_j=0,\qquad(14b)$$
$$N_{ij}(z)L(z)=-\check{N}_{ij}(z).\qquad(14c)$$

Consider the filter in (10) and again replace the time-shifter p with z, that is, $r_{ij}(z)=a(z)^{-1}N_{ij}(z)L(z)y(z)$. In light of the above arguments, we have

$$\begin{aligned}r_{ij}(z)&\overset{(13)}{=}\frac{N_{ij}(z)L(z)}{a(z)}\times C_j(zI-A_{ij}^{\mathrm{cl}})^{-1}(E_jd(z)+zx(t_s))\\&\overset{(14c)}{=}\frac{-\check{N}_{ij}(z)}{a(z)}C_j(zI-A_{ij}^{\mathrm{cl}})^{-1}(E_jd(z)+zx(t_s))\\&\overset{(14a)}{=}\frac{-\hat{N}_{ij}(z)}{a(z)}(E_jd(z)+zx(t_s))\\&\overset{(14b)}{=}\frac{-\hat{N}_{ij}(z)}{a(z)}zx(t_s).\end{aligned}$$

The claim of the corollary thus follows. ■

Remark 3 (Guaranteed residual convergence): Notice that Corollary 1 states that the matched residual r_{ij} is independent of the system matrix A_{ij}^{cl} of the status S_{ij}. This implies that even if A_{ij}^{cl} is non-Hurwitz, the residual r_{ij} converges to zero (and the other residuals may diverge).

C. Diagnosis time

We now present an approach to compute the diagnosis time $\Delta\tau_i(\varepsilon_{\mathrm{det}},\varepsilon_{\mathrm{iso}})$ with a preset pair of thresholds $(\varepsilon_{\mathrm{det}},\varepsilon_{\mathrm{iso}})$. Recall that $\Delta\tau_i(\varepsilon_{\mathrm{det}},\varepsilon_{\mathrm{iso}})=t_{\mathrm{det}}(\Gamma_i,\varepsilon_{\mathrm{det}})+t_{\mathrm{iso}}(\varepsilon_{\mathrm{iso}})$. In order to improve the readability of the next result, let us first introduce several notations. Define $\lambda_{\max}:=\max\{|\lambda_1|,\ldots,|\lambda_{d_N+1}|\}$ where for each $m\in[d_N+1]$, λ_m is a root of the polynomial $a(z)$ defined in (8). Moreover, we design $a(z)$ such that all its roots are distinct, i.e., $\lambda_m\neq\lambda_n$ when $m\neq n$. The following theorem shows the computation method of $\Delta\tau_i(\varepsilon_{\mathrm{det}},\varepsilon_{\mathrm{iso}})$.

Theorem 2 (Detection and isolation times): Suppose that the hypotheses in Corollary 1 hold and that also Assumption 2-(b) is satisfied. For 1-dimensional disturbance, there exists a sufficiently small threshold $\varepsilon_{\text{det}} \in (0, \bar{\varepsilon}_{\text{det}})$ such that

$$t_{\text{det}}(\Gamma_i, \varepsilon_{\text{det}}) = 1, \tag{15a}$$

Moreover, for any threshold $\varepsilon_{\text{iso}} \in (0, \bar{\varepsilon}_{\text{iso}})$, it holds that

$$t_{\text{iso}}(\varepsilon_{\text{iso}}) = \lceil \log_{\lambda_{\max}} \frac{\varepsilon_{\text{iso}}}{t_{\max} d_{\max}} \rceil, \tag{16a}$$

where

$$t_{\max} := \max_{i,j\in[n]} \left\{ \left\| \sum_{m=1}^{d_N+1} \left| \frac{-\hat{N}_{ij}(z)(z-\lambda_m)}{a(z)} \right|_{z=\lambda_m} \left| \left(I - A_{ii}^{\text{cl}}\right)^{-1} E_i \right| \right\|_{\infty} \right\},$$

$$\bar{\varepsilon}_{\text{det}} := \min_{i,j\in[n], i\neq j} |a(1)^{-1} N_{ij}(1) L(1) C_i (I - A_{ii}^{\text{cl}})^{-1} E_i| d_{\min},$$

$$\bar{\varepsilon}_{\text{iso}} := \min_{i,j,h\in[n], i\neq j\neq h} |a(1)^{-1} N_{ih}(1) L(1) C_j (I - A_{ij}^{\text{cl}})^{-1} E_j| d_{\min}.$$

Proof. Suppose that a transition M_{ij} occurs at t_s. It follows from Assumption 1 that $x(t_s)$ is the steady state of S_{ii}. Then, by virtue of Assumption 2-(b), $x(t_s) = (I - A_{ii}^{\text{cl}})^{-1} E_i d_i$, recalling matrix A_{ii}^{cl} is Hurwitz.

We now show the validity of (15a). When M_{ij} occurs at the instance t_s, the residual r_{ii} satisfies

$$r_{ii}(z) = a(z)^{-1} N_{ii}(z) L(z) C_j (zI - A_{ij}^{\text{cl}})^{-1} (E_j d(z) + z x(t_s)).$$

The absolute value $|r_{ii}(t_s+1)|$ will exceed ε_{det} because of the transient behavior of the above expression for $r_{ii}(z)$. Thus, the transition M_{ij} will be detected in one step.

To compute the isolation time in (16a), we transform the matched residual $r_{ij}(z) = -a(z)^{-1} \hat{N}_{ij}(z) z x(t_s)$ to its time series format. Observe that the partial expansion of $-\hat{N}_{ij}(z)z/a(z)$ is

$$\frac{-\hat{N}_{ij}(z)z}{a(z)} = \sum_{m=1}^{d_N+1} \frac{b_{ij,m} z}{z - \lambda_m},$$

where $b_{ij,m} = \frac{-\hat{N}_{ij}(z)(z-\lambda_m)}{a(z)}\Big|_{z=\lambda_m}$ and the denominator polynomial $a(z) = \prod_{m=1}^{d_N+1}(z - \lambda_m)$. Then, We apply the inverse Z-transform to $r_{ii}(z)$ and arrive at

$$r_{ij}(t_s + k) = Z^{-1}\left[\frac{-\hat{N}_{ij}(z)z}{a(z)} x(t_s)\right] = \sum_{m=1}^{d_N+1} \lambda_m^k b_{ij,m} x(t_s).$$

Recalling that $x(t_s) = (I - A_{ii}^{\text{cl}})^{-1} E_i d_i$, we have

$$\begin{aligned} r_{ij}(t_s+k) &= \sum_{m=1}^{d_N+1} \lambda_m^k b_{ij,m} (I - A_{ii}^{\text{cl}})^{-1} E_i d_i \\ &\leq \lambda_{\max}^k \left\| \sum_{m=1}^{d_N+1} |b_{ij,m}| |(I - A_{ii}^{\text{cl}})^{-1} E_i| \right\|_{\infty} d_{\max} \\ &\leq \lambda_{\max}^k t_{\max} d_{\max}, \end{aligned}$$

where in the last inequality we used the definition of $t_{\max}$ given in the theorem. Setting $\lambda_{\max}^k t_{\max} d_{\max} \leq \varepsilon_{\text{iso}}$, we have

$$t_{\text{iso}}(\varepsilon_{\text{iso}}) = \lceil \log_{\lambda_{\max}} \frac{\varepsilon_{\text{iso}}}{t_{\max} d_{\max}} \rceil.$$

To guarantee that the unmatched residuals are above the thresholds ε_{det} and ε_{iso}, we compute the corresponding minimum steady-state responses $\bar{\varepsilon}_{\text{det}}$ and $\bar{\varepsilon}_{\text{iso}}$ of the unmatched residuals. We finally set $\bar{\varepsilon}_{\text{det}}$ and $\bar{\varepsilon}_{\text{iso}}$ as the upper bounds for ε_{det} and ε_{iso}, respectively. This Concludes the proof of this theorem. ■

Remark 4 (Multi-dimensional disturbance): When the disturbance is multi-dimensional, the steady-state response of the residuals is the result of multiplication of two vectors. As a result, it is difficult to compute $\bar{\varepsilon}_{\text{det}}$ and $\bar{\varepsilon}_{\text{det}}$. To avoid false diagnosis, one should choose sufficiently small values for the two thresholds.

Remark 5 (Disturbance regularities): Recall that Assumption 2-(b) implies that underlying system (1) is in fact a switched "affine" system. However, we would like to emphasize the following fact: The only result of this study that requires Assumption 2-(b) is the one related to the computation of the isolation time $t_{\text{iso}}(\varepsilon_{\text{iso}})$ in (16a).

Remark 6 (Threshold setting): We choose a sufficiently small value for ε_{det}. By doing so, the diagnosis block becomes very sensitive to small residual variations, and a "one step" detection phase is achieved. Moreover, observe that there is a fundamental trade-off in selecting the threshold ε_{iso}. A smaller value for ε_{iso} enables guaranteed diagnosis results and can avoid false isolation. On the other hand, by decreasing the value of ε_{iso}, the isolation phase takes more time. This scenario is in particular undesirable in systems which are unstable during the diagnosis phase. We leave finding a balance to properly select ε_{iso} for our future studies.

Based on the designed filters and the considered diagnosis rules, we summarize the diagnosis approach in Algorithm 1.

Algorithm 1 Diagnosis algorithm for the system (1)-(2)

Set thresholds ε_{det} and ε_{iso} according to Remark 6
for $k = t_s, t_s+1, \ldots$ **do**
  if $|r_{ii}(k)| < \varepsilon_{\text{det}}$ **then**
    No transition enabled
  else
    A transition enabled
    Record $t_{\text{det}} = k - t_s$ and exit loop
  end if
end for
for $k = t_s + t_{\text{det}} + 1, \ldots, t_s + t_{\text{det}} + t_{\text{iso}}$ **do**
  if $k < t_s + t_{\text{det}} + t_{\text{iso}}$ **then**
    Cannot determine the mode
  else
    Check $r_{ih}(k)$, $h \in [n]/\{i\}$
    Determine the mode according to (4)
  end if
end for

IV. Numerical Simulation

In this subsection, the following numerical example is presented to show the effectiveness of the proposed method.

Consider a switched system with three linear subsystems,

$$A_1 = \begin{bmatrix} 0.88 & -0.05 \\ 0.4 & -0.72 \end{bmatrix}, A_2 = \begin{bmatrix} 0.51 & -0.24 \\ 0.8 & 0.32 \end{bmatrix}, A_3 = \begin{bmatrix} 0.3 & 0.16 \\ 0.8 & 0.6 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} -0.3 \\ -0.5 \end{bmatrix}, B_2 = \begin{bmatrix} -1.4 \\ 0.3 \end{bmatrix}, B_3 = \begin{bmatrix} -1.5 \\ 0.1 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} 0.7 \\ 1.3 \end{bmatrix}, E_2 = \begin{bmatrix} 0.2 \\ -1.4 \end{bmatrix}, E_3 = \begin{bmatrix} -1.1 \\ 0.9 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.2 & 0.1 \\ -0.1 & 0.15 \end{bmatrix}, C_2 = \begin{bmatrix} 0.3 & 0.4 \\ 0.1 & -0.2 \end{bmatrix}, C_3 = \begin{bmatrix} -0.1 & 0.2 \\ 0.4 & 0.1 \end{bmatrix},$$

The corresponding controller gains are

$$K_1 = \begin{bmatrix} 0.1295 & -0.2431 \end{bmatrix}, K_2 = \begin{bmatrix} -0.0292 & 0.0977 \end{bmatrix},$$
$$K_3 = \begin{bmatrix} -0.0237 & 0.0729 \end{bmatrix}.$$

To cover all the scenarios, the switching sequence is set as follows: $1 \to 2 \to 3 \to 1 \to 3 \to 2 \to 1$. The thresholds are chosen as $\varepsilon_{\text{det}} = \varepsilon_{\text{iso}} = 0.05$, the disturbance $d = 0.5$, and the initial state $x_0 = [0\ 0\ 0]^\top$. We use Theorem 2 and compute the isolation time $t_{\text{iso}} = 9$.

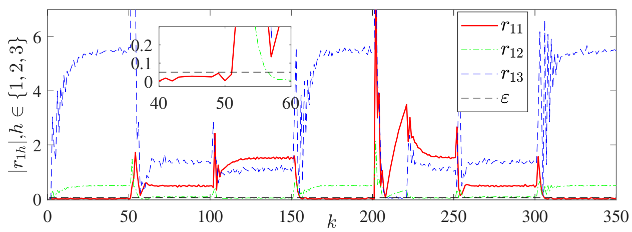

Fig. 2. Residuals r_{1h}, $h \in \{1,2,3\}$ in the presence of noise.

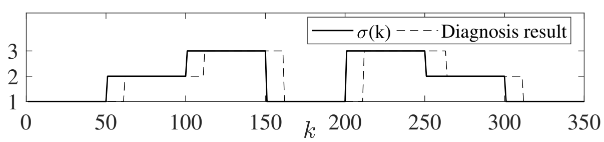

Fig. 3. Diagnosis result of the whole process.

The residuals r_{ih} with the same index i are put together for comparison. Here, only r_{1h} and the transition M_{12} at $k = 51$ are analyzed, and the others are similar. The initial status of the system is S_{11}. Fig. 2 depicts the residuals r_{1h}, for all $h \in \{1,2,3\}$, in the presence of noise. The residual r_{11} remains below the threshold until the system switches to other models at $k = 51$, while the other two residuals oscillate around zero to the their corresponding steady values. As shown in the small figure in Fig. 2, r_{11} crosses over the threshold immediately after the transition happened, such that the switching is detected in one step. According to the proposed diagnosis approach, the diagnosis block waits for $t_{\text{iso}} = 9$ steps. Afterwards, the diagnosis block can determine the active model 2 and then activate the matched controller 2. Notice that the residual r_{12} reaches the threshold ε_{iso} at $k = 57$. The matched residual r_{11} fluctuates around 0. A false diagnosis is possible when the noise level is large. Hence, we will focus on the impact of the measurement noise in our future work. The diagnosis result of the whole process is shown in Fig. 3.

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