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HARDY SPACES AND DILATIONS ON HOMOGENEOUS GROUPS

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Abstract. On a homogeneous group, we characterize the one-parameter groups of dilations whose associated Hardy spaces in the sense of Folland and Stein are the same.

1. Introduction

Let G be a homogeneous group, i.e., a connected, simply connected nilpotent Lie group whose Lie algebra $\mathfrak{g}$ admits automorphic dilations

$$\delta_r^A = \exp(\ln(r)A), \qquad r > 0,$$

for a diagonalizable matrix $A \in \mathrm{GL}(\mathfrak{g})$ with positive eigenvalues. As G is simply connected and nilpotent, its exponential map is a global diffeomorphism, and the automorphisms δ_r^A induce automorphisms of G which we still denote by δ_r^A.

Following Folland and Stein [11], we consider Hardy spaces associated to the dilations $(\delta_r^A)_{r>0}$ on G as follows. Given a Schwartz function $\phi \in \mathcal{S}$ on G, the associated radial maximal function of a tempered distribution $f \in \mathcal{S}'$ is

$$M^0_{\phi,A} f = \sup\nolimits_{r>0} r^{\mathrm{tr}(A)} |f * (\phi \circ \delta_r^A)|,$$

where $*$ denotes the convolution on G. Upon fixing a commutative approximate identity ϕ (see [9, 13]), the Hardy space H^p_A, with $p \in (0, 1]$, is the space

$$H^p_A = \{ f \in \mathcal{S}' : M^0_{\phi,A} f \in L^p \}$$

endowed with the quasi-norm $f \mapsto \|M^0_{\phi,A} f\|_p^p$. It is well known that there are several other equivalent definitions, see, e.g., [8, 11, 14, 17], but we leave further discussions on such choice to a later stage, cf. Remark 3.2 below.

The aim of this paper is to characterize those dilation matrices $A, B \in \mathrm{GL}(\mathfrak{g})$ as above which induce, via the associated dilations $(\delta_r^A)_{r>0}$ and $(\delta_r^B)_{r>0}$ respectively, the same Hardy spaces H^p_A and H^p_B. Our main result is the following theorem.

Theorem 1.1. *$H^p_A = H^p_B$ for some (equivalently, all) $p \in (0, 1]$ if and only if $A = cB$ for some $c > 0$.*

Notice that we do not assume that the spaces have equivalent quasi-norms, but just being the same as sets. We also emphasize that $H^p_A = H^p_B$ is in general *not* equivalent to the homogeneous quasi-norms on G induced by A and B being equivalent; cf. Proposition 2.1 below.

The problem of characterizing the dilations which give rise to the same function spaces looks rather natural. In the case when G is abelian, that is, when G is some Euclidean space $\mathbb{R}^n$, this has already been studied for Hardy spaces associated to

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anisotropic or parabolic dilations [1, 3–6] in [1]. More recently, similar problems have been investigated for Besov and Triebel–Lizorkin spaces in, e.g., [7, 12, 16].

If G is abelian, Theorem 1.1 may be obtained from a combination of results in [1, 3] on Hardy spaces defined by expansive dilation matrices. The novelty of Theorem 1.1 is that it is the first instance of such results on noncommutative groups. Our approach to the problem is strongly influenced by the aforementioned papers, in particular by Bownik's [1]. Nevertheless, the noncommutative setting requires a number of nontrivial modifications which we shall discuss along the way. It finally goes without saying that Folland and Stein's book [11] plays a key role in the paper, too.

The structure of the paper is as follows. In the following Section 2 we characterize those matrices A and B whose induced homogeneous norms on G are equivalent: we show that this happens if and only if $A = B$. In Section 3 we introduce Hardy spaces on G and describe equivalent characterizations of their semi-norms in terms of atomic decompositions and grand maximal functions. In the final Section 4 we prove Theorem 1.1 and discuss an analogous result for BMO spaces.

Setting and notation. All throughout, G denotes a homogeneous group with identity e and Lie algebra $\mathfrak{g}$. The dimension of $\mathfrak{g}$, whence that of G, will be denoted by n. We shall say that a matrix $A \in \mathrm{GL}(\mathfrak{g})$ is *admissible* if it is diagonalizable, has positive eigenvalues and the matrix exponential $\exp(A \ln(r))$, $r > 0$, is an automorphism of $\mathfrak{g}$. Given such a matrix A and $r > 0$, we denote by δ_r^A both the automorphisms $\exp(A \ln(r))$ of $\mathfrak{g}$ and the corresponding group automorphisms of G given by $\exp_G \circ \delta_r^A \circ \exp_G^{-1}$, where $\exp_G \colon \mathfrak{g} \to G$ is the exponential map of G.

Given two functions $f, g \colon X \to [0, \infty)$ on a set X, we write $f \lesssim g$ if there exists $C > 0$ such that $f(x) \leqslant C g(x)$ for all $x \in X$. The notation $f \asymp g$ will be used whenever $f \lesssim g$ and $g \lesssim f$.

2. Equivalence of homogeneous quasi-norms

Given an admissible matrix A, a homogeneous quasi-norm associated with A (equivalently, with the family of dilations $(\delta_r^A)_{r>0}$) is a continuous function $\rho_A \colon G \to [0, \infty)$ which is smooth in $G \setminus \{e\}$ and satisfies, for $x \in G$,

(1) $\rho_A(x^{-1}) = \rho_A(x)$,
(2) $\rho_A(\delta_r^A x) = r\rho_A(x)$,
(3) $\rho_A(x) = 0$ if and only if $x = e$.

Homogeneous quasi-norms do exist for any given family of dilations $(\delta_r^A)_{r>0}$, and any two such quasi-norms ρ_A, ρ'_A are mutually equivalent, in the sense that $\rho_A \asymp \rho'_A$; see [11, p. 8] and [10, Proposition 3.1.35] for proofs of both facts. In addition, for all semi-norms ρ_A there exists $C > 0$ such that

$$\rho_A(xy) \leqslant C(\rho_A(x) + \rho_A(y)) \tag{2.1}$$

for all $x, y \in G$; see [11, p.11] and [10, Proposition 3.1.38].

Given $x_0 \in G$ and $r > 0$, the ball associated to a homogeneous norm ρ_A centered at x_0 with radius r is

$$\mathcal{B}^A(x_0, r) \coloneqq \{x \in G \colon \rho_A(x_0^{-1} x) < r\}.$$

With such definition, we also have

$$\mathcal{B}^A(x_0, r) = x_0 \mathcal{B}^A(e, r), \qquad \mathcal{B}^A(e, r) = \delta_r^A \mathcal{B}^A(e, 1). \tag{2.2}$$

If λ is the Lebesgue measure on $\mathfrak{g}$, we define the associated Haar measure μ on G by $\mu = \lambda \circ \exp_G^{-1}$. Then, for all measurable subsets E of G,

$$\mu(\delta_r^A E) = r^{\mathrm{tr}(A)} \mu(E). \tag{2.3}$$

In particular, $\mu(\mathcal{B}^A(x_0,r)) = r^{\mathrm{tr}(A)}\mu(\mathcal{B}^A(e,1))$. In view of (2.3), the trace of A is often called the *homogeneous dimension* of G (with respect to A).

In addition to homogeneous quasi-norms, we will also make use of a function on G that is homogeneous with respect to dilations by multiples of the identity matrix. For this, endow $\mathfrak{g}$ with an orthonormal basis $\{Y_1,\dots,Y_n\}$ and let $\|\cdot\|$ be the associated Euclidean norm. Then extend it to a function on G by means of the exponential map, i.e., (with a slight abuse) $\|x\| = \|\exp_G^{-1} x\|$ for $x \in G$. Observe that $\|x^{-1}\| = \|x\|$, though $\|\cdot\|$ *is not* a norm nor a quasi-norm on G unless G is abelian. We denote the "ball" of center x_0 and radius r with respect to $\|\cdot\|$ simply by

$$\mathcal{B}(x_0,r) := \{x \in G\colon \|x_0^{-1}x\| < r\} = x_0\mathcal{B}(e,r).$$

The function $\|\cdot\|$ on G is homogeneous with respect to the classical (Euclidean) dilations $\delta_t^I := \exp_G \circ \exp(I\ln(t)) \circ \exp_G^{-1}$, $t > 0$, namely $\|\delta_t^I x\| = t\|x\|$ for $x \in G$. However, we remark that such dilations are automorphisms of G if and only if G is abelian. More generally, we shall write

$$\delta_t^\Lambda := \exp_G \circ \exp(\ln(t)\Lambda) \circ \exp_G^{-1}$$

for $t > 0$ and general $\Lambda \in \mathrm{GL}(\mathfrak{g})$, which do not need to be automorphisms. Since the identity I and its multiples commute with all matrices, the dilations δ_t^I commute with any other dilation δ_r^Λ: for any $x \in G$, then,

$$\delta_t^I \delta_r^\Lambda x = \delta_r^\Lambda \delta_t^I x, \quad r,t > 0. \tag{2.4}$$

Given $\Lambda \in \mathrm{GL}(\mathfrak{g})$, we shall denote by $\|\Lambda\|_{\mathrm{GL}(\mathfrak{g})}$ its operator norm associated to the Euclidean norm $\|\cdot\|$ on $\mathfrak{g}$. Observe that δ_1^Λ is the identity map for all $\Lambda \in \mathrm{GL}(\mathfrak{g})$.

Given two admissible matrices A and B, we say that two associated quasi-norms ρ_A and ρ_B are *equivalent* if $\rho_A \asymp \rho_B$, namely (we recall it for future use) if there exists a constant $C > 0$ such that

$$C^{-1}\rho_B(x) \leqslant \rho_A(x) \leqslant C\rho_B(x) \tag{2.5}$$

for all $x \in G$.

In the following proposition we show that the equivalence of homogeneous quasi-norms is a rather rigid condition; cf. [1, Lemma 10.2].

Proposition 2.1. *Let $A, B \in \mathrm{GL}(\mathfrak{g})$ be admissible matrices and ρ_A, ρ_B associated quasi-norms. Then $\rho_A \asymp \rho_B$ if and only if $A = B$.*

Proof. Since all homogeneous quasi-norms associated to an admissible matrix are equivalent, it follows that $\rho_A \asymp \rho_B$ for any choice of ρ_A and ρ_B whenever $A = B$. As for the converse, since A, B are admissible, their exponentials $\exp(A)$ and $\exp(B)$ have only strictly positive eigenvalues, and thus admit a unique logarithm, see, e.g., [15, Theorem 1.31]. Consequently, $\exp(A) = \exp(B)$ if and only if $A = B$, and it is then enough to prove that if $\rho_A \asymp \rho_B$ then $\exp(A) = \exp(B)$.

If (2.5) holds, then $\mathcal{B}^A(e,r) \subseteq \mathcal{B}^B(e,Cr)$ for all $r > 0$. By (2.2), this amounts to

$$\delta_r^A \mathcal{B}^A(e,1) \subseteq \delta_r^B \mathcal{B}^B(e,C),$$

which implies by (2.3), for $r > 0$,

$$r^{\mathrm{tr}(A)}\mu(\mathcal{B}^A(e,1)) \leqslant r^{\mathrm{tr}(B)}\mu(\mathcal{B}^B(e,C)).$$

Thus, the function $r \mapsto r^{\mathrm{tr}(A)-\mathrm{tr}(B)}$ is bounded on $(0,\infty)$, and hence $\mathrm{tr}(A) = \mathrm{tr}(B)$. In particular, this shows $\det(\exp(A)) = \det(\exp(B))$, so that $\exp(A) = \exp(B)$ follows from [7, Theorem 7.9], provided

$$\sup_{k\in\mathbb{Z}} \|\exp(A)^{-k}\exp(B)^k\|_{\mathrm{GL}(\mathfrak{g})} < \infty. \tag{2.6}$$

Since

$$\sup_{k\in\mathbb{Z}} \|\exp(A)^k \exp(B)^{-k}\|_{\mathrm{GL}(\mathfrak{g})} \leqslant \sup_{r>0} \|\exp(\ln(r)A)\exp(\ln(1/r)B)\|_{\mathrm{GL}(\mathfrak{g})} = \sup_{r>0} \sup_{x\in G\setminus\{e\}} \frac{\|\delta_r^A x\|}{\|\delta_r^B x\|}, \tag{2.7}$$

the desired conclusion will follow once we show that the quantity in (2.7) is finite.

In order to do this, note first that if $x \in G$ is such that $\|x\| = 1$, then

$$\rho_A(\delta^A_{1/r}\delta^B_r x) = r^{-1}\rho_A(\delta^B_r x) \leqslant Cr^{-1}\rho_B(\delta^B_r x) = C\rho_B(x) \leqslant C\sup\{\rho_B(z)\colon \|z\| = 1\} \leqslant D,$$

where the last supremum is finite because $\{z \in G\colon \|z\| = 1\}$ is compact and ρ_B is continuous. In other words,

$$\delta^A_{1/r}\delta^B_r\{x \in G\colon \|x\| = 1\} \subseteq \mathcal{B}^A(e, D).$$

Since $\overline{\mathcal{B}^A(e, D)}$ is compact by [11, Lemma 1.4], there is $R > 0$ such that

$$\delta^A_{1/r}\delta^B_r\{x \in G\colon \|x\| = 1\} \subseteq \mathcal{B}^A(e, D) \subseteq \mathcal{B}(e, R),$$

namely $\|\delta^A_{1/r}\delta^B_r x\| \leqslant R$ for all $r > 0$ and $x \in G$ such that $\|x\| = 1$. If now $x \in G$ is arbitrary, then $\delta^I_{\|x\|^{-1}} x \in \{x \in G\colon \|x\| = 1\}$, and by (2.4),

$$\|\delta^A_{1/r}\delta^B_r x\| \leqslant R\|x\|$$

for all $x \in G$ and $r > 0$. This last inequality is equivalent to

$$\|\delta^A_{1/r} x\| \leqslant R\|\delta^B_{1/r} x\|$$

for all $x \in G$ and $r > 0$, yielding

$$\sup_{x\in G\setminus\{e\}} \sup_{r>0} \frac{\|\delta_r^A x\|}{\|\delta_r^B x\|} = \sup_{x\in G\setminus\{e\}} \sup_{r>0} \frac{\|\delta_{1/r}^A x\|}{\|\delta_{1/r}^B x\|} \leqslant R,$$

which completes the proof. □

3. Hardy spaces on G

For $p \in (0, 1]$ and $\Lambda \in \mathrm{GL}(\mathfrak{g})$, we consider the dilation of a function f on G

$$D_t^{\Lambda,p} f(x) := t^{\mathrm{tr}(\Lambda)/p} f(\delta_t^\Lambda(x)), \qquad x \in G,\ t > 0.$$

We shall equivalently write $f_t^{\Lambda,p}$ for $D_t^{\Lambda,p} f$ and f_t^Λ for $f_t^{\Lambda,1}$. Let us observe that since $\delta_t^\Lambda = \delta^{c\Lambda}_{t^{1/c}}$ for all $t > 0$, one has

$$f_t^\Lambda(x) = t^{\mathrm{tr}(\Lambda)} f(\delta_t^\Lambda(x)) = (t^{1/c})^{\mathrm{tr}(c\Lambda)} f(\delta^{c\Lambda}_{t^{1/c}}(x)) = f^{c\Lambda}_{t^{1/c}}(x), \qquad x \in G. \tag{3.1}$$

Let now $A \in \mathrm{GL}(\mathfrak{g})$ be admissible. Given a Schwartz function $\phi \in \mathcal{S}$ and a tempered distribution $f \in \mathcal{S}'$, the *radial maximal function* $M^0_{\phi,A} f$ of f (with respect to A and ϕ) is

$$M^0_{\phi,A} f(x) = \sup_{t>0} |f * \phi_t^A(x)|, \quad x \in G. \tag{3.2}$$

Suppose now that $\phi \in \mathcal{S}$ is a commutative approximate identity for A, that is, $\int_G \phi \, d\mu = 1$ and $\phi_s^A * \phi_t^A = \phi_t^A * \phi_s^A$ for all $s, t > 0$, cf. [9, 13]. For $p \in (0, 1]$, we define the *Hardy space* H^p_A as

$$H^p_A = \{f \in \mathcal{S}'\colon M^0_{\phi,A} f \in L^p\},$$

endowed with the quasi-norm

$$\|f\|^p_{H^p_A} := \|M^0_{\phi,A} f\|^p_p. \tag{3.3}$$

Here and all throughout, the L^p norms are taken with respect to the Haar measure μ.

We comment on the choice of this definition, among all the others available, in Remark 3.2 below. We first show that with such definition H^p_A is invariant under scaling of the dilation matrix A. This is a straightforward consequence of (3.1), but we state it as a lemma for future reference.

Lemma 3.1. *Let $A \in \mathrm{GL}(\mathfrak{g})$ be admissible and $c > 0$. Then $H^p_A = H^p_{cA}$ with equality of quasi-norms for all $p \in (0, 1]$.*

Proof. It is enough to observe that (3.1) implies $M^0_{\phi,A} f = M^0_{\phi,cA} f$. □

We can now elaborate on this and on our definition of H^p_A.

Remark 3.2. By Lemma 3.1, up to adjusting the dilation matrix A if necessary, it may be assumed that the minimum eigenvalue of A is 1 without affecting the space H^p_A or its norm. Therefore, though the minimum eigenvalue of A being 1 is a standing assumption in [11] which we do not make, several results therein are still valid in our setting. In particular, by combining Lemma 3.1 and [11, Corollary 4.17], one sees that:

(a) a different choice of the commutative approximate identity ϕ originates an equivalent quasi-norm (3.3) of H^p_A;
(b) by [11, Proposition 2.15], H^p_A embeds continuously in $\mathcal{S}'$ for all $p \in (0, 1]$;
(c) by [11, Proposition 2.16], the quasi-norm (3.3) induces a metric on H^p_A which makes it a complete metric space.

Let us emphasize, however, that our definition of admissible matrix does not give rise to new spaces with respect to those of [11], but rather allows (whenever needed) for a larger flexibility in the choice of the matrices which describe the same Hardy space. In view of all this, if one adheres strictly to the setting of [11], i.e., assumes that the minimum eigenvalue of an admissible matrix is 1, then Theorem 1.1 reads as follows: $H^p_A = H^p_B$ *for some (equivalently, all)* $p \in (0, 1]$ *if and only if* $A = B$.

The following lemma will be used repeatedly.

Lemma 3.3. *Suppose $p \in (0, 1]$ and let $A, B \in \mathrm{GL}(\mathfrak{g})$ be admissible. If $H^p_A = H^p_B$, then their quasi-norms are equivalent.*

Proof. By the discussion in Remark 3.2, the maps

$$(f, g) \mapsto \|f - g\|^p_{H^p_A}, \qquad (f, g) \mapsto \|f - g\|^p_{H^p_B}$$

are invariant metrics making H^p_A and H^p_B respectively into complete metric spaces and $H^p_A = H^p_B \hookrightarrow \mathcal{S}'$, whence the map $\iota \colon H^p_A \to H^p_B$, $f \mapsto f$ is well defined and has a closed graph. By the closed graph theorem ι is continuous, so that $\|f\|_{H^p_B} \lesssim \|f\|_{H^p_A}$ for all $f \in H^p_A = H^p_B$. The other inequality follows similarly. □

In the remainder of this section we discuss equivalent characterizations of H^p_A which will be of use to prove Theorem 1.1. In view of Remark 3.2, we shall assume that the minimum eigenvalue of A is 1. We begin with the following simple lemma.

Lemma 3.4. *Suppose A is an admissible matrix with minimum eigenvalue 1. Then there exists $\gamma > 0$ (depending on A) such that, for all homogeneous quasi-norms ρ_A associated with (δ^A_r), the following holds.*

(i) For all $R > 0$ there exist $c_1, c_2 > 0$ (which depend on A, ρ_A and R) satisfying, for all $x \in \overline{\mathcal{B}^A(e, R)}$,

$$c_1 \|x\| \leqslant \rho_A(x) \leqslant c_2 \|x\|^\gamma. \tag{3.4}$$

In particular,

$$\mathcal{B}\Big(e, \Big(\frac{c_1}{c_2}R\Big)^{1/\gamma}\Big) \subseteq \mathcal{B}^A(e, c_1 R) \subseteq \mathcal{B}(e, R). \tag{3.5}$$

(ii) For all $R > 0$ there exists $C > 0$ (which depends on A, ρ_A and R) such that, for all $x, y \in \overline{\mathcal{B}(e, R)}$,

$$\|xy\| \leqslant C\big(\|x\|^\gamma + \|y\|^\gamma\big). \tag{3.6}$$

Proof. Assertion (i) can be proved in the exact same manner as [11, Proposition 1.5], whereas (ii) follows from a combination of (i) and (2.1). □

3.1. **Atomic decompositions.** Assume that $A \in \mathrm{GL}(\mathfrak{g})$ is an admissible matrix whose minimum eigenvalue is 1, and fix an associated homogeneous quasi-norm ρ_A. Denote by $v_1, \dots, v_n$ the eigenvalues of A, listed in increasing order (whence $v_1 = 1$). Given a multiindex $I = (i_1, \dots, i_n) \in \mathbb{N}_0^n$, we define its *isotropic degree* and *homogeneous degree* (associated to A) respectively by

$$|I| = i_1 + \cdots + i_n, \qquad d_A(I) = v_1 i_1 + \cdots + v_n i_n.$$

We denote by Δ_A the sub-semigroup of $\mathbb{R}$ generated by $\{0, v_1, \dots, v_n\}$, that is, $\Delta_A = \{d_A(I) \colon I \in \mathbb{N}^n\}$. Note that $|I| \leqslant d_A(I)$ and $\mathbb{N} \subseteq \Delta_A$ as $v_1 = 1$.

A function $P \colon G \to \mathbb{C}$ is called a *polynomial* on G if $P \circ \exp_G$ is a polynomial on $\mathfrak{g}$. Fix an eigenbasis $\{X_1, \dots, X_n\}$ for A, and let $\{X_1^*, \dots, X_n^*\}$ be the associated dual basis for $\mathfrak{g}^*$. For $j = 1, \dots, n$, set $\eta_{j,A} := X_j^* \circ \exp_G^{-1}$. Then each $\eta_{j,A}$ is a (homogeneous) polynomial on G, and every polynomial P on G can be written uniquely as

$$P = \sum_{I \in \mathbb{N}_0^n} c_I \eta_A^I, \quad \eta_A^I := \eta_{1,A}^{i_1} \cdots \eta_{n,A}^{i_n}, \tag{3.7}$$

where all but finitely many of the coefficients $c_I \in \mathbb{C}$ are zero. The *homogeneous degree* (with respect to A) of a polynomial P as in (3.7) is defined to be $\max\{d_A(I) \colon c_I \neq 0\}$. The set of all polynomials of homogeneous degree at most $N \in \mathbb{N}$ with respect to A is denoted by $\mathcal{P}_N^A$.

Suppose $p \in (0, 1]$. An element $\alpha \in \Delta_A$ is said to be *p-admissible for A* if $\alpha \geqslant \max\{\alpha' \in \Delta_A \colon \alpha' \leqslant \mathrm{tr}(A)(p^{-1} - 1)\}$. A pair (p, α) is said to be *admissible for A* if α is p-admissible for A. Given such a pair (p, α), we say that a function $a \colon G \to \mathbb{C}$ is a (p, α)-*atom* associated to A if it satisfies the conditions:

(a1) $\operatorname{supp} a \subseteq \mathcal{B}^A(x_0, r)$ for some $x_0 \in G$ and $r > 0$,
(a2) $\|a\|_\infty \leqslant \mu(\mathcal{B}^A(x_0, r))^{-\frac{1}{p}}$,
(a3) $\int_G a \cdot P \, d\mu = 0$ for all $P \in \mathcal{P}_\alpha^A$.

We denote by $\mathscr{A}_\alpha^p(A)$ the family of all (p, α)-atoms associated to A.

If $\alpha \in \Delta_A$ is p-admissible, then by [11, Theorem 3.30], the Hardy space H_A^p coincides with the space of all tempered distributions $f \in \mathcal{S}'$ of the form

$$f = \sum\nolimits_j \kappa_j a_j, \qquad \kappa_j \geqslant 0, \ \ (\kappa_j) \in \ell^p, \ \ a_j \in \mathscr{A}_\alpha^p(A),$$

with the equivalence of quasi-norms

$$\|f\|_{H_A^p}^p \asymp \|f\|_{H_\alpha^p(A)}^p := \inf\Big\{ \|(\kappa_j)\|_{\ell^p}^p \colon f = \sum_j \kappa_j a_j, \ a_j \in \mathscr{A}_\alpha^p(A) \Big\}. \tag{3.8}$$

Rather than the classical atoms satisfying conditions (a1)–(a3), we will make use of certain "modified" atoms. Given an admissible pair (p, α) for A and $R > 0$, we shall call *modified* (p, α, R)-atom (associated to A) a function $a \colon G \to \mathbb{C}$ such that

(a1') $\operatorname{supp} a \subseteq x_0 \delta_{e^k}^A(\mathcal{B}(e, R))$ for some $x_0 \in G$ and $k \in \mathbb{Z}$;
(a2') $\|a\|_\infty \leqslant \mu(\delta_{e^k}^A \mathcal{B}(e, R))^{-\frac{1}{p}}$;

(a3) $\int_G a \cdot P \, d\mu = 0$ for all $P \in \mathcal{P}^A_\alpha$.

To show the relation between the above Hardy spaces and those defined by such modified atoms, we use Lemma 3.4 to prove the following simple result.

Lemma 3.5. *Suppose $R > 0$, let $A \in \mathrm{GL}(\mathfrak{g})$ be admissible with minimum eigenvalue 1 and (p, α) be admissible for A. Then H^p_A coincides with the atomic space defined in terms of modified (p, α, R)-atoms associated to A, with equivalence of quasi-norms.*

Proof. By the equivalence of quasi-norms (3.8), it will be enough to show that any (p, α)-atom associated to A is a multiple of a modified (p, α, R) atom associated to A, and viceversa, with uniform constants depending only on p, A and R. As the two proofs are essentially the same, we shall provide the details of the first one only.

Suppose that a is a (p, α)-atom associated to A, supported in a ball $\mathcal{B}^A(x_0, r)$ for which the size condition (a2) holds. Then, for $k = [\ln(\frac{r}{Rc_1})] + 1 \in \mathbb{Z}$,

$$\begin{aligned} \operatorname{supp} a &\subseteq x_0 \delta^A_{\frac{r}{Rc_1}} \mathcal{B}^A(e, c_1 R) \\ &\subseteq x_0 \delta^A_{\mathrm{e}^k} \mathcal{B}^A(e, c_1 R) \subseteq x_0 \delta^A_{\mathrm{e}^k} \mathcal{B}(e, R), \end{aligned}$$

the last inclusion by (3.5). As for the size condition,

$$\|a\|_\infty \leqslant \mu(x_0 \delta^A_{\mathrm{e}^k} \mathcal{B}(e, R))^{-\frac{1}{p}} \left(\frac{\mu(\mathcal{B}^A(x_0, r))}{\mu(x_0 \delta^A_{\mathrm{e}^k} \mathcal{B}(e, R))} \right)^{-\frac{1}{p}},$$

where, by left invariance and (2.3),

$$\frac{\mu(\mathcal{B}^A(x_0, r))}{\mu(x_0 \delta^A_{\mathrm{e}^k} \mathcal{B}(e, R))} = \frac{r^{\mathrm{tr}(A)} \mu(\mathcal{B}^A(e, 1))}{\mathrm{e}^{k \,\mathrm{tr}(A)} \mu(\mathcal{B}(e, R))} \geqslant \left(\frac{r}{\mathrm{e}\, r/(c_1 R)} \right)^{\mathrm{tr}(A)} \frac{\mu(\mathcal{B}^A(e, 1))}{\mu(\mathcal{B}(e, R))} \geqslant C(A, R),$$

and the conclusion follows. □

Lastly, we define a family of auxiliary functions that we will need in the proof of Theorem 1.1. For this, let $\{Y_1^*, \dots, Y_n^*\}$ be the dual basis for $\mathfrak{g}^*$ of the basis $\{Y_1, \dots, Y_n\}$, and define $\eta_j := Y_j^* \circ \exp_G^{-1}$ for $j = 1, \dots, n$. As above, any polynomial P on G can be written uniquely as

$$\sum_{I \in \mathbb{N}_0^n} c_I \eta^I, \quad \eta^I := \eta_1^{i_1} \dots \eta_n^{i_n}, \tag{3.9}$$

for finitely many nonzero coefficients $c_I \in \mathbb{C}$. The *isotropic degree* of a polynomial P as in (3.9) is $\max\{|I| : c_I \neq 0\}$, and we denote the set of all polynomials of isotropic degree at most $N \in \mathbb{N}$ by $\mathcal{P}_N$. Note that if $P \in \mathcal{P}_N$ and $\Lambda \in \mathrm{GL}(\mathfrak{g})$, then also $P \circ \delta^\Lambda_t \in \mathcal{P}_N$ for any $t > 0$. Moreover, a change of basis and the fact that $|I| \leqslant d_A(I)$ imply $\mathcal{P}^A_N \subseteq \mathcal{P}_N$ for all admissible matrices A whose minimum eigenvalue is 1.

Let now $A, B \in \mathrm{GL}(\mathfrak{g})$ be admissible with minimum eigenvalue 1. For $p \in (0, 1]$, choose $\alpha \in \mathbb{N}$ so large that (p, α) is admissible for both A and B. Given $R > 0$, consider the family $\mathcal{F}_{\alpha, p, R}(A, B)$ of functions $f : G \to \mathbb{C}$ such that

(f1) $\operatorname{supp} f \subseteq x_0 \delta^A_{\mathrm{e}^{j_1}} \delta^B_{\mathrm{e}^{j_2}} \mathcal{B}(e, R)$ for some $x_0 \in G$ and $j_1, j_2 \in \mathbb{Z}$.
(f2) $\|f\|_\infty \leqslant \mathrm{e}^{-j_1 \mathrm{tr}(A)/p} \mathrm{e}^{-j_2 \mathrm{tr}(B)/p}$.
(f3) $\int_G f \cdot P \, d\mu = 0$ for all $P \in \mathcal{P}_\alpha$.

The significance of this family for our purposes is provided by the following lemma.

Lemma 3.6. *Let $A, B \in \mathrm{GL}(\mathfrak{g})$ be admissible matrices with minimum eigenvalue 1. Suppose $H^p_A = H^p_B$ for some $p \in (0, 1]$ and that $\alpha \in \mathbb{N}$ is such that (p, α) is admissible for both A and B. Then for all $R > 0$ there is a constant $C' > 0$ such that $\|f\|_{H^p_A} \leqslant C'$ for all $f \in \mathcal{F}_{\alpha, p, R}(A, B)$.*

Proof. Suppose $f \in \mathcal{F}_{\alpha,p,R}(A,B)$. First, by (f1), the function $D^{B,p}_{\mathrm{e}^{j_2}} D^{A,p}_{\mathrm{e}^{j_1}} f$ is supported in

$$\delta^A_{\mathrm{e}^{-j_1}}\, \delta^B_{\mathrm{e}^{-j_2}}(x_0)\, \mathcal{B}(e,R).$$

Second, by (f2),

$$\| D^{B,p}_{\mathrm{e}^{j_2}} D^{A,p}_{\mathrm{e}^{j_1}} f \|_\infty = \mathrm{e}^{\frac{j_2 \operatorname{tr}(B)}{p}} \mathrm{e}^{\frac{j_1 \operatorname{tr}(A)}{p}} \|f\|_\infty \leqslant 1.$$

Lastly, given $P \in \mathcal{P}_\alpha$, it follows by (f3) that

$$\int D^{B,p}_{\mathrm{e}^{j_2}} D^{A,p}_{\mathrm{e}^{j_1}} f \cdot P \, d\mu = \mathrm{e}^{(\frac{1}{p}-1) j_2 \operatorname{tr}(B)} \mathrm{e}^{(\frac{1}{p}-1) j_1 \operatorname{tr}(A)} \int f \cdot P \circ \delta^B_{\mathrm{e}^{-j_2}} \circ \delta^A_{\mathrm{e}^{-j_1}} \, d\mu = 0. \quad (3.10)$$

Hence, the function $\mu(\mathcal{B}(e,R))^{\frac{1}{p}} \cdot D^{B,p}_{\mathrm{e}^{j_2}} D^{A,p}_{\mathrm{e}^{j_1}} f$ is a modified (p,α,R)-atom for both A and B. By Lemma 3.3 and the fact that $D^{A,p}_t$ (resp. $D^{B,p}_t$) is an isometry on H^p_A (resp. H^p_B) gives

$$\|f\|_{H^p_A} = \|D^{A,p}_{\mathrm{e}^{j_1}} f\|_{H^p_A} \lesssim \|D^{A,p}_{\mathrm{e}^{j_1}} f\|_{H^p_B} = \|D^{B,p}_{\mathrm{e}^{j_2}} D^{A,p}_{\mathrm{e}^{j_1}} f\|_{H^p_B}$$

with an implicit constant independent of f. Since $\mu(\mathcal{B}(e,R))^{\frac{1}{p}} \cdot D^{B,p}_{\mathrm{e}^{j_2}} D^{A,p}_{\mathrm{e}^{j_1}} f$ is a multiple of an ordinary (p,α)-atom for B (cf. the proof of Lemma 3.5), an application of [11, Theorem 2.9] yields

$$\|D^{B,p}_{\mathrm{e}^{j_2}} D^{A,p}_{\mathrm{e}^{j_1}} f\|_{H^p_B} \lesssim \mu(\mathcal{B}(e,R))^{-1/p} \lesssim 1$$

where the constants are independent of f. □

3.2. **Grand maximal function.** Let A be an admissible matrix with minimum eigenvalue 1. Given $N \in \mathbb{N}$, the *grand maximal function* associated to the radial maximal function (3.2) is defined by

$$\mathcal{M}^0_{(N),A} f = \sup_{\phi \in \mathcal{S}, \|\phi\|_{(N)} \leqslant 1} M^0_{\phi,A} f,$$

where $\|\phi\|_{(N)}$ is a semi-norm on $\mathcal{S}$ associated to A. By [11, Proposition 2.8 and Theorem 3.30], H^p_A with $p \in (0,1]$ can be characterized as the space of $f \in \mathcal{S}'$ such that $\mathcal{M}^0_{(N),A} f \in L^p$, with the equivalence of semi-norms

$$\|f\|^p_{H^p_A} \asymp \|\mathcal{M}^0_{(N),A} f\|^p_p, \quad (3.11)$$

provided that $N \geqslant \min\{N' \in \mathbb{N} \colon N' \geqslant \min\{\alpha \in \Delta_A \colon \alpha > \operatorname{tr}(A)(p^{-1}-1)\}\}$.

4. Equivalence of Hardy spaces

This section is devoted to proving Theorem 1.1. We start with the following lemma.

Lemma 4.1. *Let $A, B \in \mathrm{GL}(\mathfrak{g})$ be admissible and $\varepsilon = \operatorname{tr}(A)/\operatorname{tr}(B)$. Then the following assertions are equivalent:*

(i) $A = cB$ for some $c > 0$;
(ii) $\sup_{j\in\mathbb{Z}} \| \exp(A)^{-j} \exp(B)^{\lfloor \varepsilon j \rfloor} \|_{\mathrm{GL}(\mathfrak{g})}$ is finite.

Proof. Suppose that $A = cB$ for $c > 0$. Then $\varepsilon = c$, and hence, for $j \in \mathbb{Z}$,

$$\exp(A)^{-j} \exp(B)^{\lfloor \varepsilon j \rfloor} = \exp\big((-j + \lfloor jc \rfloor 1/c) A\big) =: \exp(r_j A)$$

where $-1/c \leqslant r_j \leqslant 0$. Therefore,

$$\sup_{j\in\mathbb{Z}} \| \exp(A)^{-j} \exp(B)^{\lfloor \varepsilon j \rfloor} \|_{\mathrm{GL}(\mathfrak{g})} \leqslant \sup_{-1/c \leqslant r \leqslant 0} \| \exp(rA) \|_{\mathrm{GL}(\mathfrak{g})} < \infty,$$

the last fact as $r \mapsto \exp(rA)$ is continuous.

Conversely, suppose that $\exp(A)$ and $\exp(B)$ satisfy (ii). Define the matrix $B' = \frac{\operatorname{tr}(A)}{\operatorname{tr}(B)} B$. Then $\exp(A)$ and $\exp(B')$ satisfy (ii), and $\det(\exp(A)) = \det(\exp(B'))$.

Therefore, an application of [7, Theorem 7.9] yields that $\exp(A) = \exp(B')$. Since the matrix $\exp(A) = \exp(B')$ has only strictly positive eigenvalues, it has a unique real logarithm (cf. [15, Theorem 1.31]), whence $A = B'$. □

We are now ready to prove Theorem 1.1, which we restate for the reader's convenience. The overall method of the proof is inspired by that of [1, Theorem 10.5].

Theorem 4.2. *Let $A, B \in \mathrm{GL}(\mathfrak{g})$ be admissible. Then the following are equivalent:*

(i) $H^p_A = H^p_B$ for some $p \in (0, 1]$;
(ii) $H^p_A = H^p_B$ for all $p \in (0, 1]$;
(iii) $A = cB$ for some $c > 0$.

Proof. By Lemma 3.1, (iii) implies (ii). The fact that (ii) implies (i) is immediate. Hence, it remains to show that (i) implies (iii).

Suppose that $H^p_A = H^p_B$ for some $p \in (0, 1]$. By rescaling A and B if necessary, it may be assumed (cf. Lemma 3.1) that both A and B have minimum eigenvalue 1. Then $A = cB$ if and only if $c = 1$. We argue by contradiction, and suppose that $A \neq B$. Then, by Lemma 4.1, either

$$\limsup_{j\to+\infty} \big\| \exp(A)^j \exp(B)^{-\lfloor \varepsilon j\rfloor} \big\| = \infty, \quad \text{or} \quad \limsup_{j\to-\infty} \big\| \exp(A)^j \exp(B)^{-\lfloor \varepsilon j\rfloor} \big\| = \infty,$$

where in this proof we simply write $\| \cdot \| = \| \cdot \|_{\mathrm{GL}(\mathfrak{g})}$ for the operator norm. Up to passing to a subsequence, we can assume that actually either

$$\lim_{j\to+\infty} \big\| \exp(A)^j \exp(B)^{-\lfloor \varepsilon j\rfloor} \big\| = \infty, \quad \text{or} \quad \lim_{j\to-\infty} \big\| \exp(A)^j \exp(B)^{-\lfloor \varepsilon j\rfloor} \big\| = \infty. \tag{4.1}$$

Note that, for fixed $j \in \mathbb{Z}$,

$$\big\| \exp(A)^j \exp(B)^{-\lfloor \varepsilon j\rfloor - m} \big\| \leqslant \big\| \exp(A)^j \exp(B)^{-\lfloor \varepsilon j\rfloor} \big\| \big\| \exp(B)^{-m} \big\| \to 0$$

as $m \to \infty$, since all eigenvalues of $\exp(B)$ are strictly greater than 1. Therefore, there exists the smallest integer m (and we call it d_j) such that

$$\big\| \exp(A)^j \exp(B)^{-\lfloor \varepsilon j\rfloor - m} \big\| \leqslant 1.$$

Since, by definition of d_j, the above inequality fails for $d_j - 1$,

$$\begin{aligned} \big\| \exp(A)^j \exp(B)^{-\lfloor \varepsilon j\rfloor - d_j} \big\| &\geqslant \big\| \exp(A)^j \exp(B)^{-\lfloor \varepsilon j\rfloor - d_j + 1} \big\| \| \exp(B) \|^{-1} \\ &\geqslant \| \exp(B) \|^{-1}, \end{aligned}$$

whence, for all $j \in \mathbb{Z}$,

$$\| \exp(B) \|^{-1} \leqslant \big\| \exp(A)^j \exp(B)^{-\lfloor \varepsilon j\rfloor - d_j} \big\| \leqslant 1. \tag{4.2}$$

If (d_j) was a bounded sequence, then (4.1) could not hold; therefore, either $d_j \to +\infty$ or $d_j \to -\infty$ as $j \to +\infty$ or $j \to -\infty$. We consider the first case only, the others being analogous.

The remainder of the proof is split into three steps.

Step 1. (*Auxiliary functions*). We construct a sequence of functions satisfying properties (f1)–(f3) considered in Section 3.1. For this, let $X \in \mathfrak{g}$ be such that $\|X\| = 1$, and define

$$x_1 = \exp_G(X).$$

Notice that $\|x_1\| = 1$. Choose $\epsilon, \theta \in (0, 1)$ small enough so that the balls $\mathcal{B}(x_1, \epsilon)$ and $\mathcal{B}(e, \theta)$ are disjoint. In addition, fix $R > 0$ such that $\mathcal{B}(x_1, \epsilon) \subseteq \mathcal{B}(e, R)$. Even though $\| \cdot \|$ is not a norm, all of these choices are possible by means of Lemma 3.4 (applied to A or B, after an associated homogeneous semi-norm on G is chosen).

Indeed, if $y \in \mathcal{B}(x_1, \epsilon)$, then (3.6) yields a constant $c > 0$ (only depending on A or B) such that

$$\|y\| = \|x_1 \cdot x_1^{-1} \cdot y\| \geqslant \big(c\|x_1\| - \|x_1^{-1} y\|^\gamma\big)^{1/\gamma} \geqslant \big(c - \epsilon\big)^{1/\gamma} > \theta,$$

i.e., $y \in \mathcal{B}(e, \theta)^c$, provided $\epsilon, \theta \in (0,1)$ are sufficiently small. With such choices, we proceed to construct a function a_0 satisfying (f1)–(f3) in Section 3.1 with $x_0 = e$ and $j_1 = j_2 = 0$.

Let $\alpha \in \mathbb{N}$ be so large that (p, α) is admissible for both A and B. Set $n_\alpha := \#\{I \in \mathbb{N}_0^n \colon |I| \leqslant \alpha\}$, and define the map

$$T \colon L^\infty(\mathcal{B}(e,\theta)) \to \mathbb{R}^{n_\alpha}, \quad f \mapsto \left(\int_{\mathcal{B}(e,\theta)} f(x) \eta^I(x) \, d\mu(x) \right)_{|I| \leqslant \alpha}.$$

Then T is surjective, and hence defining $v \in \mathbb{R}^{n_\alpha}$ by $v_I := -\int_{\mathcal{B}(x_1,\epsilon)} \eta^I \, d\mu$, there exists $f \in L^\infty(\mathcal{B}(e,\theta))$ such that $T(f) = v$. Let $\tilde{a}_0 \colon G \to \mathbb{C}$ be defined by

$$\tilde{a}_0(x) = \begin{cases} f(x) & \text{if } x \in \mathcal{B}(e,\theta), \\ 1 & \text{if } x \in \mathcal{B}(x_1,\epsilon), \\ 0 & \text{if } x \notin \mathcal{B}(e,\theta) \cup \mathcal{B}(x_1,\epsilon). \end{cases}$$

Then $\operatorname{supp} \tilde{a}_0 \subseteq \mathcal{B}(e,R)$ and $\int_G \tilde{a}_0 \cdot \eta^I \, d\mu = 0$ for all $|I| \leqslant \alpha$. Choose now $\omega_0 \in (0,1]$ such that, if $a_0 := \omega_0 \, \tilde{a}_0$, then $\|a_0\|_\infty \leqslant 1$. Then $a_0 \in \mathcal{F}_{\alpha,p,R}(A,B)$.

We now construct suitable dilations of a_0. Define Q_j to be the matrix such that

$$\exp(Q_j) = \exp(A)^j \exp(B)^{-\lfloor \varepsilon j \rfloor - d_j}, \qquad j \geqslant 1, \tag{4.3}$$

by using the Baker–Campbell–Hausdorff formula. Then $\exp(Q_j)$ is an automorphism of $\mathfrak{g}$ as it is composition of automorphisms. As such, $\delta_e^{Q_j}$ and $\delta_{1/e}^{Q_j}$ are automorphisms of G.

Then pick $Z_j \in \mathfrak{g}$ such that $\|Z_j\| = 1$ and

$$\|\exp(Q_j) Z_j\| = \|\exp(Q_j)\| =: \tau_j. \tag{4.4}$$

By (4.2) then

$$\|\exp(B)\|^{-1} \leqslant \tau_j \leqslant 1, \tag{4.5}$$

and by taking the determinants in (4.3)

$$\operatorname{tr}(Q_j) = j \operatorname{tr}(A) - (\lfloor \varepsilon j \rfloor + d_j) \operatorname{tr}(B). \tag{4.6}$$

In addition to Q_j, choose a matrix U_j such that $\exp(U_j)$ is unitary and $\exp(U_j) X = Z_j$, and define $z_j = \exp_G(Z_j)$. Then define $a_j := D_{\mathrm{e}^{-1}}^{Q_j,p} D_{\mathrm{e}^{-1}}^{U_j,p} a_0$ for $j \in \mathbb{N}$.

We claim that $a_j \in \mathcal{F}_{\alpha,p,R}(A,B)$ for all $j \in \mathbb{N}$. For this, observe first that since $\exp(U_j)$ is unitary, it holds that $\delta_{\mathrm{e}}^{U_j} \mathcal{B}(e,r) \subseteq \mathcal{B}(e,r)$ for all $r > 0$. Moreover,

$$\delta_{\mathrm{e}}^{U_j} x_1 = \exp_G(\exp(U_j) X) = \exp_G(Z_j) = z_j. \tag{4.7}$$

Since

$$\delta_{\mathrm{e}}^{Q_j} = \delta_{\mathrm{e}^j}^{A} \delta_{\mathrm{e}^{-\lfloor \varepsilon j \rfloor - d_j}}^{B},$$

it holds that $\delta_{\mathrm{e}}^{Q_j} \mathcal{B}(e,R) \subseteq \mathcal{B}(e, \tau_j R)$ by definition (4.4) of τ_j, and

$$\operatorname{supp} a_j \subseteq \delta_{\mathrm{e}}^{Q_j} \mathcal{B}(e,R) = \delta_{\mathrm{e}^j}^{A} \delta_{\mathrm{e}^{-\lfloor \varepsilon j \rfloor - d_j}}^{B} \mathcal{B}(e,R),$$

showing (f1). Moreover, $\operatorname{supp} a_j \subseteq \mathcal{B}(e,R)$ for all j's. More precisely, by (4.7),

$$\delta_{\mathrm{e}}^{Q_j} \delta_{\mathrm{e}}^{U_j} \mathcal{B}(x_1,\epsilon) = \delta_{\mathrm{e}}^{Q_j} \mathcal{B}(z_j,\epsilon),$$

and thus

$$\operatorname{supp} a_j \subseteq \delta_{\mathrm{e}}^{Q_j} (\mathcal{B}(e,\theta) \cup \mathcal{B}(x_1,\epsilon)) \subseteq \mathcal{B}(e, \tau_j \theta) \cup \delta_{\mathrm{e}}^{Q_j} \mathcal{B}(z_j, \epsilon). \tag{4.8}$$

For (f2), note that, by (4.6) and $\operatorname{tr}(U_j) = 0$,

$$\|a_j\|_\infty = \mathrm{e}^{-\frac{\operatorname{tr}(Q_j)}{p} - \frac{\operatorname{tr}(U_j)}{p}} \|a_0\|_\infty \leqslant \mathrm{e}^{-j\frac{\operatorname{tr}(A)}{p} + (\lfloor \varepsilon j \rfloor + d_j)\frac{\operatorname{tr}(B)}{p}},$$

as required. In addition, we note that

$$\begin{aligned} a_j(\delta_{\mathrm{e}}^{Q_j}\mathcal{B}(z_j,\epsilon)) &= \mathrm{e}^{-\frac{\operatorname{tr}(Q_j)}{p} - \frac{\operatorname{tr}(U_j)}{p}} a_0(\mathcal{B}(x_1,\epsilon)) \\ &= \mathrm{e}^{-j\frac{\operatorname{tr}(A)}{p} + (\lfloor \varepsilon j \rfloor + d_j)\frac{\operatorname{tr}(B)}{p}} \omega_0 =: \omega_j. \end{aligned} \tag{4.9}$$

Lastly, arguing as in (3.10), one sees that also (f3) holds.

Step 2. (*Case* $p = 1$). Suppose that $p = 1$. By construction, see (4.4) and (4.5), we have that $\|\exp(B)\|^{-1} \leqslant \|\exp(Q_j)\| \leqslant 1$ and $\|Z_j\| = 1$ for $j \in \mathbb{N}$. Hence, by passing to a subsequence if necessary, it may be assumed that $\exp(Q_j) \to Q'$ for some matrix $Q' : \mathfrak{g} \to \mathfrak{g}$ satisfying $\|\exp(B)\|^{-1} \leqslant \|Q'\| \leqslant 1$, and that $Z_j \to Z'$ for some $Z' \in \mathfrak{g}$ with $\|Z'\| = 1$. In addition, it may be assumed that $\exp(U_j) \to U'$ for some unitary matrix $U' \in \mathrm{GL}(\mathfrak{g})$. Since $\varepsilon = \operatorname{tr}(A)/\operatorname{tr}(B)$ and $d_j \to +\infty$, it follows that

$$|\det(\exp(Q_j))| = \mathrm{e}^{\operatorname{tr}(Q_j)} = \mathrm{e}^{j\operatorname{tr}(A) - (\lfloor \varepsilon j \rfloor + d_j)\operatorname{tr}(B)} \leqslant \mathrm{e}^{(1-d_j)\operatorname{tr}(B)} \to 0,$$

as $j \to \infty$. Hence, $|\det(Q')| = 0$, and, in particular, Q' is not surjective.

Next, we show that the sequence $(a_j)_{j\in\mathbb{N}}$ of functions $a_j \in \mathcal{F}_{\alpha,p,R}(A,B)$ constructed in Step 1 converges to a nonzero regular Borel measure a. For this, let $\varphi \in C_b(G)$ be arbitrary. Then a direct calculation entails

$$\begin{aligned} \int_G a_j(x)\varphi(x)\, d\mu(x) &= \mathrm{e}^{-\operatorname{tr}(Q_j)} \int_G (D^{U_j}_{\mathrm{e}^{-1}} a_0)(\delta^{Q_j}_{\mathrm{e}^{-1}}(x))\varphi(\delta_{\mathrm{e}}^{Q_j}\delta^{Q_j}_{\mathrm{e}^{-1}}(x))\, d\mu(x) \\ &= \int_G (D^{U_j}_{\mathrm{e}^{-1}} a_0)(y)\varphi(\delta_{\mathrm{e}}^{Q_j}(y))\, d\mu(y) \\ &= \int_G a_0(z)\varphi(\delta_{\mathrm{e}}^{Q_j}\delta_{\mathrm{e}}^{U_j}(z))\, d\mu(z). \end{aligned}$$

Using that $\varphi(\delta_{\mathrm{e}}^{Q_j}\delta_{\mathrm{e}}^{U_j}(z)) \to \varphi(\exp_G \circ Q' \circ U' \circ \exp_G^{-1}(z))$, together with the dominated convergence theorem, it follows therefore that

$$\begin{aligned} \int_G a_j(x)\varphi(x)\, d\mu(x) \to &\int_G a_0(z)\varphi(\exp_G \circ Q' \circ U' \circ \exp_G^{-1}(z))\, d\mu(z) \\ =: &\int_G \varphi(x)\, da(x) \end{aligned}$$

for a unique regular Borel measure a on G. Since Q' is not surjective, it follows that $\operatorname{supp} a \neq G$, and thus a is singular with respect to the Haar measure μ on G.

We shall show that $a \neq 0$. Given some $z \in \mathcal{B}(e,\theta)$, set $z' = \exp_G \circ Q' \circ U' \circ \exp_G^{-1}(z)$, and set $x_1' = \exp_G \circ Q' \circ U' \circ \exp_G^{-1}(x_1)$. Then

$$\|z'\| = \|Q'U'\exp_G^{-1}(z)\| \leqslant \|Q'\|\|U'\exp_G^{-1}(z)\| \leqslant \theta\|Q'\|,$$

and, using (4.4) and (4.7),

$$\begin{aligned} \|x_1'\| &= \lim_{j\to\infty} \|\exp(Q_j)\exp(U_j)X\| = \lim_{j\to\infty} \|\exp(Q_j)Z_j\| = \lim_{j\to\infty} \|\exp(Q_j)\| \\ &= \|Q'\|. \end{aligned}$$

Therefore, an application of Lemma 3.4 (with $R = 1$) yields a constant $c > 0$ such that

$$\|(x_1')^{-1}z'\| \geqslant \big(c\|x_1'\| - \|z'\|^\gamma\big)^{1/\gamma} \geqslant \big(c\|Q'\| - \theta\|Q'\|^\gamma\big)^{1/\gamma}.$$

Hence, by decreasing $\theta \in (0,1]$ if necessary, there exists $\delta > 0$ such that $\|(x_1')^{-1}z'\| > \delta$, that is, $z' \notin \mathcal{B}(x_1', \delta)$. Choose now a non-negative continuous function φ satisfying $\operatorname{supp} \varphi \subseteq \mathcal{B}(x_1', \delta)$ and $\varphi(x_1') = 1$. Then $\varphi(\exp_G \circ Q' \circ U' \circ \exp_G^{-1}(z)) = 0$ for any $z \in \mathcal{B}(e, \theta)$, and, by construction of a_0,

$$\begin{aligned}\int_G \varphi(x)\, da(x) &= \int_G a_0(z)\varphi(\exp_G \circ Q' \circ U' \circ \exp_G^{-1}(z))\, d\mu(z) \\ &= \int_{\mathcal{B}(e,\theta)} a_0(z)\varphi(\exp_G \circ Q' \circ U' \circ \exp_G^{-1}(z))\, d\mu(z) \\ &\qquad + \omega_0 \int_{\mathcal{B}(x_1,\epsilon)} \varphi(\exp_G \circ Q' \circ U' \circ \exp_G^{-1}(z))\, d\mu(z) \\ &= \omega_0 \int_{\mathcal{B}(x_1,\epsilon)} \varphi(\exp_G \circ Q' \circ U' \circ \exp_G^{-1}(z))\, d\mu(z) > 0,\end{aligned}$$

where the inequality follows from the fact that $\varphi \geqslant 0$ is continuous and $\varphi(x_1') = 1$. This shows that a is nonzero, whence in particular $a \notin L^1$.

On the other hand, by Fatou's lemma and the grand maximal characterization (3.11),

$$\begin{aligned}\|a\|_{H^1_A} \asymp \int \mathcal{M}^0_{(N),A} a\, d\mu &= \int \lim_{i\to\infty} \mathcal{M}^0_{(N),A} a_{j_i}\, d\mu \\ &\leqslant \liminf_{i\to\infty} \int \mathcal{M}^0_{(N),A} a_{j_i}\, d\mu \leqslant C \liminf_{i\to\infty} \|a_{j_i}\|_{H^1_A} \leqslant C',\end{aligned}$$

where the last bound follows from Lemma 3.6. Thus $a \in H^1_A$. This contradicts that $a \notin L^1$, and completes the proof for $p = 1$.

Step 3. (*Case* $p < 1$). Suppose that $p < 1$ and set $\sigma := 1/\gamma^2$, where $\gamma > 0$ is that of Lemma 3.4. Pick $\phi \in \mathcal{S}$ such that $\phi = 1$ on $\mathcal{B}(e, \epsilon_1 \|\exp(B)\|^{-\sigma})$ and $\operatorname{supp} \phi \subseteq \mathcal{B}(e, \epsilon_2 \|\exp(B)\|^{-\sigma})$ for some $0 < \epsilon_1 < \epsilon_2 < 1$ to be determined.

For all $z \in G$, by (4.8) and since $\phi(x^{-1}z) = 0$ if $x \notin z\mathcal{B}(e, \epsilon_2\|\exp(B)\|^{-\sigma})$,

$$M^0_{\phi,A} a_j(z) \geqslant \left| \int_{z\mathcal{B}(e,\epsilon_2\|\exp(B)\|^{-\sigma}) \cap (\mathcal{B}(e,\tau_j\theta) \cup \delta_{\mathrm{e}}^{Q_j}\mathcal{B}(z_j,\epsilon))} a_j(x)\phi(x^{-1}z)\, d\mu(x) \right|.$$

Suppose now that $z \in \mathcal{B}(\delta_{\mathrm{e}}^{Q_j}(z_j), \beta\|\exp(B)\|^{-\sigma})$ for some $\beta \in (0,1)$. Then, by (3.6),

$$\begin{aligned}z\mathcal{B}(e, \epsilon_2\|\exp(B)\|^{-\sigma}) &\subseteq \delta_{\mathrm{e}}^{Q_j}(z_j) \cdot \mathcal{B}(e, \beta\|\exp(B)\|^{-\sigma})\mathcal{B}(e, \epsilon_2\|\exp(B)\|^{-\sigma}) \\ &\subseteq \delta_{\mathrm{e}}^{Q_j}(z_j) \cdot \mathcal{B}(e, c_{\beta,\epsilon_2}\|\exp(B)\|^{-1/\gamma}),\end{aligned}$$

with c_{β,ϵ_2} small if β and ϵ_2 are small. Thus, if $x \in z\mathcal{B}(e, \epsilon_2\|\exp(B)\|^{-\sigma})$, then $\|(\delta_{\mathrm{e}}^{Q_j} z_j)^{-1} \cdot x\| \leqslant c_{\beta,\epsilon_2}\|\exp(B)\|^{-1/\gamma}$, so that by (3.6), (4.4) and (4.5) there exists $c > 0$ (independent of $j \in \mathbb{N}$) such that

$$\begin{aligned}\|x\| &= \|\delta_{\mathrm{e}}^{Q_j} z_j \cdot (\delta_{\mathrm{e}}^{Q_j} z_j)^{-1} \cdot x\| \\ &\geqslant \Big(c\|\delta_{\mathrm{e}}^{Q_j} z_j\| - \|(\delta_{\mathrm{e}}^{Q_j} z_j)^{-1} \cdot x\|^\gamma\Big)^{1/\gamma} \geqslant (c\,\tau_j - c_{\beta,\epsilon_2}\tau_j)^{1/\gamma} > \theta\tau_j,\end{aligned}$$

i.e. $x \notin \mathcal{B}(e, \theta\tau_j)$, provided β, ϵ_2 and θ are small enough. Observe that τ_j is bounded above and below away from 0 by (4.5), whence $\tau_j^{1/\gamma} \asymp \tau_j$. The above proves that

$$\begin{aligned}&z\mathcal{B}(e, \epsilon_2\|\exp(B)\|^{-\sigma}) \cap (\mathcal{B}(e, \tau_j\theta) \cup \delta_{\mathrm{e}}^{Q_j}\mathcal{B}(z_j, \epsilon)) \\ &\qquad = z\mathcal{B}(e, \epsilon_2\|\exp(B)\|^{-\sigma}) \cap \delta_{\mathrm{e}}^{Q_j}(z_j) \cdot \delta_{\mathrm{e}}^{Q_j}\mathcal{B}(e, \epsilon),\end{aligned}$$

where we used that $\delta_{\mathrm{e}}^{Q_j}\mathcal{B}(z_j,\epsilon) = \delta_{\mathrm{e}}^{Q_j}(z_j)\cdot\delta_{\mathrm{e}}^{Q_j}\mathcal{B}(e,\epsilon)$. Therefore, by the above, (4.9) and since $(\delta_{\mathrm{e}}^{Q_j}(z_j))^{-1}z \in \mathcal{B}(e,\beta\|\exp(B)\|^{-\sigma})$, if β and ϵ_1 are small enough then

$$\begin{aligned}
M^0_{\phi,A}a_j(z) &\geqslant \omega_j\,\mu(\mathcal{B}(z,\epsilon_2\|\exp(B)\|^{-\sigma}) \cap \delta_{\mathrm{e}}^{Q_j}(z_j)\cdot\delta_{\mathrm{e}}^{Q_j}\mathcal{B}(e,\epsilon)) \\
&= \omega_j\,\mu((\delta_{\mathrm{e}}^{Q_j}(z_j))^{-1}z\cdot\mathcal{B}(e,\epsilon_2\|\exp(B)\|^{-\sigma}) \cap \delta_{\mathrm{e}}^{Q_j}\mathcal{B}(e,\epsilon)) \\
&\geqslant \omega_j\,\mu(\mathcal{B}(e,\epsilon_1\|\exp(B)\|^{-\sigma}) \cap \delta_{\mathrm{e}}^{Q_j}\mathcal{B}(e,\epsilon)) \\
&= \omega_j\,\mathrm{e}^{\mathrm{tr}(Q_j)}\mu(\delta_{1/\mathrm{e}}^{Q_j}\mathcal{B}(e,\epsilon_1\|\exp(B)\|^{-\sigma}) \cap \mathcal{B}(e,\epsilon)). \qquad (4.10)
\end{aligned}$$

Observe now that $\delta_{1/\mathrm{e}}^{Q_j}\mathcal{B}(e,\epsilon_1\|\exp(B)\|^{-\sigma}) \supseteq \mathcal{B}(e,\tau_j^{-1}\epsilon_1\|\exp(B)\|^{-\sigma})$. Thus, if ϵ_1 is small enough so that $\tau_j^{-1}\epsilon_1\|\exp(B)\|^{-\sigma} < \epsilon$, then by (4.10)

$$\begin{aligned}
M^0_{\phi,A}a_j(z) &\geqslant \omega_j\,\mathrm{e}^{\mathrm{tr}(Q_j)}\mu(\mathcal{B}(e,\tau_j^{-1}\epsilon_1\|\exp(B)\|^{-\sigma})) \\
&= \omega_j\,\mathrm{e}^{\mathrm{tr}(Q_j)}\mu(\delta^I_{\tau_j^{-1}\epsilon_1\|\exp(B)\|^{-\sigma}}\mathcal{B}(e,1)) \geqslant c\,\omega_j\,\mathrm{e}^{\mathrm{tr}(Q_j)}.
\end{aligned}$$

Thus, by (4.6) and (4.9),

$$M^{0,A}_{\varphi}a_j(z) \geqslant c\,\mathrm{e}^{(\frac{1}{p}-1)[\lfloor\varepsilon j\rfloor\,\mathrm{tr}(B) - j\,\mathrm{tr}(A)]}\mathrm{e}^{(\frac{1}{p}-1)\,\mathrm{tr}(B)d_j}.$$

Since

$$0 = \varepsilon j\,\mathrm{tr}(B) - j\,\mathrm{tr}(A) \geqslant \lfloor\varepsilon j\rfloor\,\mathrm{tr}(B) - j\,\mathrm{tr}(A) \geqslant (\varepsilon j - 1)\,\mathrm{tr}(B) - j\,\mathrm{tr}(A) = -\,\mathrm{tr}(B)$$

for all $j \in \mathbb{N}$, we conclude

$$M^0_{\phi,A}a_j(z) \geqslant c'\mathrm{e}^{(\frac{1}{p}-1)\,\mathrm{tr}(B)d_j},$$

from which it follows that

$$\begin{aligned}
\int_G |M^0_{\phi,A}a_j|^p\,d\mu &\geqslant \int_{\mathcal{B}(\delta_{\mathrm{e}}^{Q_j}z_j,\beta\|\exp(B)\|^{-\sigma})} |M^0_{\phi,A}a_j|^p\,d\mu \\
&\geqslant C\mu(\mathcal{B}(e,\beta\|\exp(B)\|^{-\sigma}))\,\mathrm{e}^{(\frac{1}{p}-1)\,\mathrm{tr}(B)d_j} \to \infty
\end{aligned}$$

as $j \to \infty$, which is a contradiction by Lemma 3.6 and the grand maximal function characterization (3.11) of the Hardy space seminorm. This completes the proof. □

Lastly, we have the following simple consequence on equivalence of the dual of the Hardy spaces H^1_A and H^1_B. These spaces can be identified with BMO spaces; see [2, 11] for definitions and precise details. In particular, $(H^1_A)^*, (H^1_B)^* \hookrightarrow \mathcal{S}'/\mathcal{P}_0$ by [11, Proposition 5.9].

Corollary 4.3. *Let $A, B \in \mathrm{GL}(\mathfrak{g})$ be admissible. Then $(H^1_A)^* = (H^1_B)^*$ if and only if $A = cB$ for some $c > 0$.*

Proof. Suppose that $(H^1_A)^* = (H^1_B)^*$. By arguing as in Lemma 3.3, one sees that $\|f\|_{(H^1_A)^*} \asymp \|f\|_{(H^1_A)^*}$ holds for all $f \in (H^1_A)^* = (H^1_B)^*$. By duality then

$$\|h\|_{H^1_A} = \sup_{\substack{f\in(H^1_A)^* \\ \|f\|_{(H^1_A)^*}\leqslant 1}} |f(h)| \asymp \sup_{\substack{f\in(H^1_B)^* \\ \|f\|_{(H^1_B)^*}\leqslant 1}} |f(h)| = \|h\|_{H^1_B}$$

for all $h \in H^1_A = H^1_B$. An application of Theorem 4.2 therefore yields $A = cB$.

Conversely, if $A = cB$, then $H^1_A = H^1_B$ by Proposition 3.1, and the result follows by duality. □

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