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Jacques Benders and his decomposition algorithm

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ABSTRACT

Benders is a household name in optimization, but as a person he was hardly known beyond his circle of colleagues and students. In this brief paper, we review his life and work.

1. Benders Day

On May 31, 2024, the optimization group at Eindhoven University of Technology organized Benders Day, a workshop commemorating the centennial of the birth of Jacques Benders. The program covered many aspects of Benders decomposition, including its history, applications, extensions, and implementation [23]. The present paper is based on two talks at the workshop, an overview of the history and significance of the decomposition method by the first author and a personal tribute by the second author. We also have greatly profited from an account of Benders' career by Jaap Wessels [20].

2. Education and research

Jacobus Franciscus Benders was born on June 1, 1924, in Swalmen, a village in Limburg, in the south-east of the Netherlands, where his father and an uncle ran a roof tile factory. Jacques, as he was called, was expected to join the family business, but it soon became clear that his future lay elsewhere.

He finished high school at the Episcopal College in Roermond in 1943. He obtained a teacher qualification for primary schools in 1945 and for mathematics in secondary education in 1946. In the meantime he had entered mathematics and physics studies at Utrecht University, where he obtained a candidate's degree (bachelor) in 1948 and a doctoral degree (master) in 1952. He was a math teacher at a high school in Arnhem from 1949 until 1950 and a teaching assistant at the Mathematical Institute in Utrecht from 1949 until 1953.

His research career started in 1953 at the Rubber Foundation in Delft, a predecessor of TNO, the Dutch organization for applied science and technology. There, he learned statistics in studying the wear and tear of tyres. In 1955 he moved to the Shell Laboratories in Amsterdam, where he and his office mate Guus Zoutendijk were confronted with problems of refinery planning. Visiting their Californian branch, they learned about the potential of linear programming. Back home they started to implement the simplex method and to apply it to production planning in the refinery and petrochemical industry. At the time, the group at the Shell Labs in Amsterdam was the only one in the Netherlands involved in research in optimization at an international level. It was in fact unique in Europe.

With the major advance offered by linear programming came the realization of its limitations. Zoutendijk addressed questions of nonlinearity, which led to his *methods of feasible directions* and to a PhD at the University of Amsterdam in 1960 under the supervision of computer scientist Adriaan van Wijngaarden [21].

Benders tackled the complications caused by the presence of discrete variables. With two colleagues from Shell London, he presented an algorithm for linear programming with 0–1 variables at a RAND symposium in 1959 [7]. They reported computational results on the Ferranti Mark 1 computer, mentioning that its addition time was 1.2 milliseconds. The method combines features of branch-and-bound and dynamic programming; Michel Balinski [1] describes it under the heading of "branch and exclude".

More importantly, Benders developed his decomposition algorithm, which earned him a PhD at Utrecht University in 1960 under the super-

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vision of topologist Hans Freudenthal. The thesis was entitled *Partitioning in Mathematical Programming* [4]. Benders published his algorithm in *Numerische Mathematik* in 1962 [5], a paper that has generated over 17,000 citations. On the occasion of his eightiest birthday, the paper was reprinted in *Computational Management Science*, preceded by a tribute by the editor, István Maros.

3. The decomposition algorithm

Benders decomposition [5] was developed for optimization problems in the following form:

$$z^* = \min\{cx + f(y) \mid Ax + F(y) \geq b,\ x \in \mathbb{R}_p^+,\ y \in Y\}, \tag{1}$$

where $Y \subset \mathbb{R}_q$, f a scalar function and F an m-component vector function, both defined on Y. The presence of the y-variables makes the problem more difficult to solve.

We first give a short description of the algorithm and then sketch the historical setting in which it was developed.

If the variables y are fixed to values $\bar{y}$, Problem (1) becomes a linear programming problem:

$$f(\bar{y}) + \min\{cx \mid Ax \geq b - F(\bar{y}),\ x \in \mathbb{R}_p^+\}.$$

Since this problem may not be feasible for any choice of $\bar{y} \in Y$, we only want to consider vectors y that lie in the set

$$S = \{y \in Y \mid \exists x \text{ such that } Ax \geq b - F(y),\ x \in \mathbb{R}_p^+\}.$$

Using Farkas' Lemma we can rewrite S by explicit constraints as

$$S = \{y \in Y \mid \tilde{u}^r(b - F(y)) \leq 0,\ r = 1, \ldots, R\}, \tag{2}$$

where $\tilde{u}^r,\ r = 1, \ldots, R$, are the extreme rays of the dual cone $\{u \geq 0 \mid uA \leq 0\}$. If S is empty, Problem (1) is infeasible. Now, consider the case that S is nonempty. We rewrite (1) as

$$\min_{y \in S}\{f(y) + \min\{cx \mid Ax \geq b - F(y),\ x \geq 0\}\}.$$

The inner problem is a linear programming problem and can be replaced by its dual, yielding

$$\min_{y \in S}\{f(y) + \max\{u(b - F(y)) \mid uA \leq c,\ u \geq 0\}\}. \tag{3}$$

Since we assume that S is nonempty, and thereby Problem (1) is feasible, we conclude that, if the dual is infeasible, Problem (1) is unbounded. The dual cannot be unbounded, as that would imply that S is empty.

If indeed the dual problem is bounded, there is an optimal solution in an extreme point of the dual set $\{uA \leq c,\ u \geq 0\}$. Denote the dual extreme points by $u^k,\ k = 1, \ldots, K$. We can now combine the explicit formulation of the set S (2), and Problem (3), written using the dual extreme points, to obtain the so-called *Benders Master Problem* (BMP):

$$\begin{aligned} z^* = z_{\text{BMP}} = \min\ & f(y) + q \\ & q \geq u^k(b - F(y)), \quad k = 1, \ldots, K \\ & 0 \geq \tilde{u}^r(b - F(y)), \quad r = 1, \ldots, R \\ & y \in Y. \end{aligned}$$

Since the BMP has many constraints, one initiates the algorithm by solving an *incomplete master problem* (IMP), in which only some of the constraints are present. One then solves the so-called *Benders Subproblem* (BSP), with the y-vector produced by the IMP as input, to generate more constraints to the IMP if needed:

$$\begin{aligned} z_{\text{BSP}} = f(y) + \max\ & u(b - F(y)) \\ & uA \leq c \\ & u \geq 0. \end{aligned}$$

If the subproblem is infeasible, which will be detected in the first iteration, the original problem is either infeasible or unbounded. If the subproblem is unbounded (bounded), an extreme ray $\tilde{u}^r$ (extreme point u^k) is identified, and an associated constraint is added to the IMP.

Since not all constraints are present in the IMP, it yields a lower bound on the optimal value z^*, whereas the subproblem produces an upper bound due to the fixed values of y. In the basic version of the algorithm, one iterates between the IMP and the BSP until the bounds match or until infeasibility or unboundedness is detected.

Benders' algorithm is referred to as a primal method, since the IMP outputs a primal vector y. Benders mentions the decomposition method for linear programming developed by George Dantzig and Phil Wolfe [12], which is called dual, since its IMP outputs a dual vector and uses the subproblem to generate a new primal vector. The term "Benders decomposition" was probably first used in 1963 by Michel Balinski and Phil Wolfe in a Mathematica working paper [2]. Krarup and Pruzan [17] report that neither Balinski nor Wolfe nor Mathematica were able to produce a copy.

It is often said that Benders decomposition was developed to solve mixed-integer linear programming problems, but Problem (1) is more general. The set Y is an arbitrary subset of the q-dimensional Euclidean vector space $\mathbb{R}_q$. If Y is the set of integer vectors in $\mathbb{R}_q$ and if the function f and the vector function F are linear, we indeed obtain a mixed-integer linear program, but that is only one special case of (1). Nonlinearities in f, F or both are also possibilities. The important point is that the variables y somehow "complicate" the problem, or at least give a natural partitioning of the variables that makes it useful to treat them separately. Benders emphasizes that "the crucial point in the application of the procedures … to the solution of actual problems is the existence of efficient procedures for solving the" IMP and considers "several special cases where this requirement is fulfilled". One of these occurs if Problem (1) is a linear program that has a block structure in the x-variables, where the blocks are linked together by the y-variables. This is the dual problem structure of the formulation considered by Dantzig and Wolfe.

At the time of its publication, Benders' partitioning procedure was very much at the forefront of research in mathematical optimization. The simplex method of George Dantzig [10] for linear programming was known, of course. George Dantzig, Ray Fulkerson and Selmer Johnson [11] had solved a 42-city traveling salesman problem by adding ad-hoc constraints to the linear relaxation of an integer programming formulation, which inspired the use of problem-specific cutting planes. Ralph Gomory had published his cutting plane algorithms for pure integer linear programming [15,16], but little work had been done on more general problems such as mixed-integer linear programming. Martin Beale [3] had developed a procedure that used Gomory's method as a subroutine, and Ailsa Land and Alison Doig [18] had published an enumerative technique that was also applicable to mixed-integer problems. Gomory's results on mixed-integer programming had not yet been published, although they "were known to exist" according to Land and Doig.

The early authors in the field were remarkably careful, sometimes almost apologetic, in presenting their results. For instance, Land and Doig write: "It is not suggested that this method should supersede successful ad hoc methods for particular problems. It may, in fact, be chiefly useful for testing the validity of proposed ad hoc methods for new problems." Similar statements can be found in other papers. The presentations and discussions on optimization at the IBM Symposium on Combinatorial Problems in 1964 [22] bear witness to a fascinating sense of discovery and community building.

When we look at the status of computations, Benders presented several applications in his paper. The instances were of modest size by present-day standards, but in 1962 they were cutting edge. He used top-of-the-line computers and dual simplex, and noticed that adding multiple cuts in one iteration was beneficial. Starting with the work of Dantzig and Wolfe, Gilmore and Gomory [14], and Benders, several types of methods were developed and analyzed in order to solve realistic mixed-integer linear and combinatorial optimization problems. In the mid-1980s, branch-and-bound became the dominant technique, fueled by advanced LP solvers, strong polyhedral bounds, and software that enables the users to add their own problem-specific modules such as heuristics, cutting planes and branching strategies. Bob Bixby [9] gives a comprehensive account of the development of computational linear and mixed-integer programming.

The technique proposed by Benders has been successfully applied to a wide range of difficult optimization problems. It is still used extensively in many situations. Several enhancements of the method have been proposed through the years, and the number of citations of the paper has been growing spectacularly. A review by Ragheb Rahmaniani et al. [19] illustrates the ever widening range of problems and applications handled via Benders decomposition.

4. Benders in Eindhoven

Eindhoven, the city of Philips Electronics and truck company DAF, became the site of the second Dutch University of Technology in 1956. Mathematician Jaap Seidel, who had been advisor to the committee that prepared the start of the university, became chair of the department of mathematics. He set out to develop a curriculum in engineering mathematics and to assemble a faculty of high quality. He wanted to cover new subjects that were not being taught at the established universities, such as numerical analysis, statistics, operations research and computer science. His aim was also a high level of coherence, in which each faculty member was able to teach all courses and would participate in teaching calculus and linear algebra in other programs across the campus.

Seidel's visits to other mathematics groups in the country caused great anxiety, because he was able to entice their best mathematicians away. He appointed Chris Bouwkamp, N.G. (Dick) de Bruijn, Edsger Dijkstra and Jack van Lint, all members of the Royal Netherlands Academy of Arts and Sciences.

In 1961 Seidel came to the Shell Labs in Amsterdam and talked to Jacques Benders. Benders had already taught a course on linear programming in Utrecht in 1960–1961. He was hesitant to move to Eindhoven, however. He was much interested in establishing operations research as a core subject within engineering mathematics, but then he should teach operations research, no calculus and linear algebra. And he was not so fond of the new IBM 1620, Eindhoven's pride. Seidel was a clever strategist and offered Benders an extraordinary appointment for two days a week. Early 1963 Benders started in Eindhoven as the first professor of operations research in the Netherlands. After two years, the parties had gotten used to each other, and as of February 1, 1965, he was full time in Eindhoven.

In his inaugural address [6] Benders positioned models and algorithms as "mathematical equipment" and gave these the same stature as other advanced technological tools. He foresaw that the engineering mathematicians by their knowledge, skills and creativity would contribute much to a world of increasing complexity.

His mission was to teach his students the art of mathematical modeling and the use of optimization techniques in practical settings. He started a "modeling lab" in which students, in their very first semester, were given an unstructured problem and asked to develop modeling and solution techniques. His first PhD students were Israel Herschberg, a chemical engineer, Jaap Wessels, who started in stochastic programming, wrote a thesis on Markov decision problems, and became his *collega proximus*, and Freerk Lootsma, whom he met in his capacity as advisor to the Philips Research Labs. And he carried out dozens of projects with local industry.

As part of his mission, he became involved in computational issues. He had a strong opinion on choosing a successor to the IBM 1620 and, given his experience at Shell, he insisted on a high professional level of organization and service. With the four K's – Kerbosch, Keulemans, Kool and Kroep – he developed NATHALIE, a flexible LP solver.

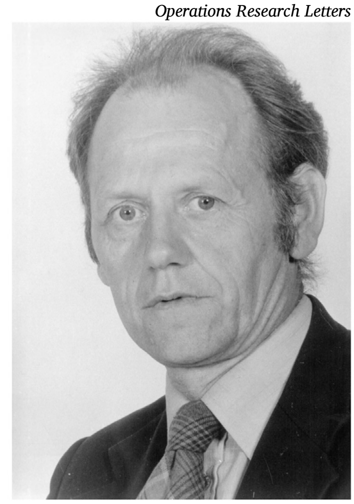

Fig. 1. Jacques Benders in 1989.

He could not avoid administrative responsibilities. After several years as vice-chair of the department, he became its chair in 1969, in a period of emerging students' democracy, administrative reforms in academia, and increasing tensions among the faculty. The episode took a heavy toll on his personal life, including an extended sick leave.

In the 1970's and 1980's Benders again focused on the work he liked. He taught classes on linear and integer programming, relying on his associate Jan de Jong for nonlinear optimization. He supervised 67 master's students (the second author of this paper being number 42) and three more PhD students: Joop Evers, Robert van der Vet, and Jaap Koene. He paid much attention to the interface between the mathematical tools and their users and worked on a matrix generator generator, a modeling language for linear programming, and decision support systems. He had a strong influence on practice as advisor to Shell, Philips and the Centre for Quantitative Methods, a Philips spin-off.

With his colleague Jo van Nunen, Benders worked on a location-allocation problem that occurred within a Dutch brewery. They formulated a large mixed-integer linear program and observed that, in the solution to its linear relaxation, the large majority of the 0–1 variables took on an integral value, which enabled them to find excellent feasible solutions by applying rounding heuristics. This led to a brief paper in *Operations Research Letters* [8]. They consider the generalized assignment problem: given are m bins i of capacity b_i $(i = 1, \ldots, m)$ and n items j $(j = 1, \ldots, n)$; when item j is assigned to bin i, it occupies a capacity a_{ij} and incurs a cost c_{ij} $(i = 1, \ldots, m,\ j = 1, \ldots, n)$; find a feasible assignment of the items to the bins of minimum total cost. They prove that, for the standard formulation of this problem in mn 0–1 variables, any basic feasible solution to its LP relaxation has no more split items than the number of bins that are used to full capacity. This is good news when m is small and n is large. They extend their observation to more complicated mixed-integer LPs involving assignment and packing constraints, like the brewery problem. Benders again applied a decomposition approach, solving a master problem in the first phase and, in this case, resolving the infeasibilities heuristically in the second phase. We note that the same observation is the basis of approximation algorithms with good performance guarantees for the generalized assignment problem and the related problem of minimizing makespan on unrelated parallel machines.

In Eindhoven Benders' most important contribution was his direct influence on students, colleagues and people in industry, which changed the way they thought about mathematics and its role in society. He will be remembered as a gold mine of knowledge and experience.

5. Retirement

Benders retired in the summer of 1989. (See Fig. 1.) On June 14, his colleagues organized a farewell symposium and presented him with a *liber amicorum* [13], containing scientific as well as personal contributions. Benders left academia behind. He was succeeded by Jan Karel Lenstra (1989–2003), Gerhard Woeginger (2004–2016), and Frits Spieksma (since 2018).

At its symposium in Atlanta in 2000, the Mathematical Programming Society honored eleven pioneers of optimization with a founders award. Zoutendijk was the only non-American among them. Benders' seminal paper did not meet the cut-off year of 1960. He was awarded the EURO Gold Medal in 2009, a belated recognition for, indeed, one of the founding fathers of optimization, who never sought the limelight.

Jacques Benders passed away in Eindhoven on January 9, 2017.

CRediT authorship contribution statement

Karen Aardal: Writing – original draft, Methodology, Investigation, Conceptualization. **Cor Hurkens:** Writing – original draft, Methodology, Investigation, Conceptualization. **Jan Karel Lenstra:** Writing – original draft, Methodology, Investigation, Conceptualization.

Data availability

No data was used for the research described in the article.

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