ON RUCKLE'S CONJECTURE ON ACCUMULATION GAMES*

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Abstract. In an accumulation game, the Hider secretly distributes his given total wealth h among n locations, while the Searcher picks r locations and confiscates the material placed there. The Hider wins if what is left at the remaining $n - r$ locations is at least 1; otherwise the Searcher wins. Ruckle's conjecture says that an optimal Hider strategy is to put an equal amount h/k at k randomly chosen locations for some k. We extend the work of Kikuta and Ruckle by proving the conjecture for several cases, e.g., $r = 2$ or $n-2$; $n \leq 7$; $n = 2r-1$; $h \leq 2+1/(n-r)$ and $n \leq 2r$. The last result uses the Erdős–Ko–Rado theorem. We establish a connection between Ruckle's conjecture and the Hoeffding problem of bounding tail probabilities of sums of random variables.

Key words. accumulation game, optimal strategy, intersecting family, tail probability

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1. Introduction. Accumulation games, as proposed by Ruckle [16, 17] and by Kikuta and Ruckle [11, 12, 13], concern the problem faced by an individual who is forced to stash his wealth at a given number of locations, only to collect it later. We call this individual the Hider. In the meantime an opposing Searcher can search some of these locations and remove all the material that is hidden there. The Hider could be an investor, spreading the risk of the investments, or a hoarder who is caching food to prepare for winter. The Searcher could be nature or an opposing pilferer. The game is played over time. The Hider acquires new wealth and hides it, while the Searcher inspects locations and confiscates the material that is hidden there. The Searcher wins if he confiscates more than a threshold value of wealth; otherwise the Hider wins. Kikuta and Ruckle give several logistical applications regarding human behavior. An example not mentioned in the earlier literature is that of the "scatter hoarder" (e.g., a squirrel) who in the autumn hides food in multiple caches in the hope that enough will remain (after natural disasters or active "pilferage") to survive the winter. The term scatter hoarder was introduced by Morris [15], who initiated what is now a considerable literature in this area of animal behavior.

The game we study here is a special case of a more general dynamic game, which is played over a number of periods. At the beginning of each period i, the Hider distributes some new wealth h_i that he has earned, and during the period the Searcher removes the total wealth (accumulated over time) at a number of locations. The wealth h_i may be nonconstant over time, or it could be stochastic. The Searcher may only find a part of the hidden material at a searched location, or the number of locations that can be searched may vary over time. The game that we study

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here is the simplest type of accumulation game: discrete time, a finite number of locations, the probability of detection is equal to 1, the incoming wealth is constant. In this case, the game essentially reduces to what Kikuta and Ruckle call a one-stage accumulation game. The Hider hides the material only once and the Searcher searches a fixed number of locations.

To model the problem mathematically, we assume a fixed set of locations $\mathcal{N} = \{1, \ldots, n\}$ and a given initial wealth h. The Hider may distribute his total wealth h among these locations in any way he chooses. We call his strategic variable $w = (w_1, \ldots, w_n)$ a *weighting,* where $w_i \geq 0$ is the amount placed at location i. We treat w as a *measure* on $\mathcal{N}$, so that the feasibility condition is $w(\mathcal{N}) = w_1 + \cdots + w_n \leq h$. In the case of equality, we call w a *partition* of h, that is, $h = w_1 + \cdots + w_n$. The Searcher picks any r-subset $I \subset \mathcal{N}$. The Hider wins the game if wealth at the $n - r$ surviving locations $\mathcal{N} - I$ satisfies $w(\mathcal{N} - I) \geq 1$. (The threshold of 1 is a convenient normalization.) Otherwise, the Searcher wins. The parameters r, n, h are all fixed. Interpreting this problem as a (zero-sum, win-lose) game, the value (optimal winning probability of the Hider) and optimal strategies exist by standard minimax results [2]. Although in some instances the game formulation is useful, it will generally be more convenient to analyze the problem as a discrete optimization problem, as already demonstrated by Kikuta and Ruckle. They showed that the Hider has an optimal strategy consisting of picking a weighting w and placing the n weights w_i randomly on the nodes. The Searcher can pick the set I randomly.

Ruckle has made the following remarkable conjecture.

CONJECTURE 1 (Ruckle [17]). *For any parameter values n, r, and h, it is optimal for the Hider to use k equal positive weights and $n - k$ weights of 0 for some $k \leq n$.*

There is no need for the Hider to use weights greater than 1, since he only needs to retrieve mass 1. If the Hider uses $k = \lfloor h \rfloor$ equal positive weights, then he may just as well use k unit weights. In this case we say that the Hider uses *unit weights.* More generally, if the Hider partitions h (that is, if $w(\mathcal{N}) = h$), then of course the positive weights are all h/k, but sometimes it is simpler to use smaller weights. As an example, suppose $n = 6$, $r = 4$, and $5/2 < h < 3$. It turns out that it is optimal for the Hider to place five weights of $1/2$, and one of 0. He does not need to use all the material. Here Ruckle's conjecture holds with $k = 5$.

Kikuta and Ruckle [13] showed that the conjecture holds for r equal to 1 and $n - 1$, and they gave examples of other parameter values where it is true. This paper establishes that the conjecture holds for many more parameter values. Using the complementary variable $s = n - r$ which describes the size of the set of unsearched locations, these parameters are $s = 2$ or $n - 2$; $h < 2$ and $n = 0$ or 1 mod s; $n \leq 7$; $n = 2s + 1$; $h < 2 + 1/(s - 1)$ and $n \geq 2s$. The last result uses the well-known Erdős–Ko–Rado theorem [7] on the size of "intersecting families" of s-sets. We also establish a more tenuous connection between Ruckle's conjecture and the difficult Hoeffding problem [9] of bounding tail probabilities of sums of random variables. For general parameters n, r, h, Ruckle's conjecture remains open.

The accumulation games described above are similar to the "number hides game" that has been studied in [3, 19]. Different types of accumulation games have been studied by Kikuta and Ruckle in [11] and [12].

2. Notation. It is notationally easier to analyze the accumulation game from the complementary point of view, in which the pure Searcher strategy is to state the s-set $I \subset \mathcal{N}$ which he leaves unsearched, where $s = n - r$ is a positive integer. Thus the accumulation game may be described by the Hider (secretly) choosing a

weighting w on $\mathcal{N}$ with $w(\mathcal{N}) \leq h$, the Searcher picking an s-subset I of $\mathcal{N}$, and the Hider winning (with payoff 1) if

$$w(I) \geq 1,$$

where $w(I)$ denotes the sum of the weights in I. Otherwise the Searcher wins (with payoff 0). The value of the game is thus the winning probability of the Hider, assuming best play on both sides. As observed by Kikuta and Ruckle (and generalized by Alpern and Fokkink in [1]) it is optimal for the Searcher to pick a random s-set, and hence the Hider faces an optimization problem: Choose w to maximize the number of s-sets I with $w(I) \geq 1$. We say that I is *heavy* if $w(I) \geq 1$, and otherwise it is *light*. To summarize, *an optimal weighting maximizes the number of heavy sets.*

It is useful to restrict the parameter values n, s, h to avoid certain trivial (and exception) cases. If $\frac{sh}{n} \geq 1$, then the Hider can guarantee a win by dividing his material into n equal weights of h/n. If $h < 1$, then obviously the Hider can never win; if $h \geq 1$, then putting all the weight at a single location makes some sets heavy. So we make the following assumption.

STANDING ASSUMPTION: $\frac{sh}{n} < 1$ *and* $h \geq 1$. *So there always exist s-sets that are heavy and s-sets that are light.*

The family of all s-subsets of $\mathcal{N} = \{1, \ldots, n\}$ is a well-known object in combinatorics: it is a hypergraph on $\mathcal{N}$. It is convenient to adopt this terminology, and we say that an s-subset I of $\mathcal{N}$ is an *edge* and that the elements of $\mathcal{N}$ are *nodes*. The Hider orders the nodes by increasing weights: $w_1 \leq \cdots \leq w_n$. (Of course the Searcher doesn't know this ordering.) If an edge I contains nodes $i_1 < \cdots < i_s$ and J contains nodes $j_1 < \cdots < j_s$ under the ordering of the weights, then we write $I \succeq J$ if $i_k \geq j_k$ for $k = 1, \ldots, s$. In particular $w(I) \geq w(J)$ if $I \succeq J$. The family of all heavy edges forms a hypergraph, and the Hider seeks to maximize the number of edges of this hypergraph. We denote the set of heavy edges containing i as $\mathcal{E}_i$ and call its cardinality the *degree* of i, denoted d_i.

3. Bounds on the value of the game. The value of a zero-sum game is often easier to determine, or at least approximate, than the optimal play. That is why we first consider the value of the accumulation game $V(n, s, h)$. It is equal to the maximal number of heavy edges divided by $\binom{n}{s}$. In the proofs in this section, we silently assume that the Hider uses an optimal weighting (whatever it may be).

LEMMA 2. *The degree sequence $d_1 \leq \cdots \leq d_n$ is increasing and $d_1 < d_n$. In particular there exist edges I and J that have $s - 1$ nodes in common such that I is heavy and J is light.*

Proof. Let $w_1 \leq \cdots \leq w_n$ be an optimal weighting. For $0 \leq m \leq n - s$ denote $I_m = \{m + 1, m + 2, \ldots, m + s\}$. By our standing assumption, I_0 is light, I_{n-s} is heavy, and $w(I_m)$ increases with m. Let j be the index such that I_j is light and I_{j+1} is heavy. These edges have $s - 1$ common nodes. For $k < l$, define $\psi = \psi_{k,l}$ to be the set map that replaces k by l when possible (for sets containing k but not l) and is the identity otherwise. Then $w(\psi(I)) \geq w(I)$ and thus ψ preserves heavy edges. Since ψ gives an injection of $\mathcal{E}_k$ into $\mathcal{E}_l$, we have $d_k \leq d_l$. Note that $\psi_{j,j+s} : \mathcal{E}_j \to \mathcal{E}_{j+s}$ is not a surjection, as $I_{j+1} \in \mathcal{E}_{j+s} - \psi_{j,j+s}(\mathcal{E}_j)$. In particular $d_j < d_{j+s}$, and it follows that $d_1 < d_n$. ☐

If $d_i = d_j$ for $j > i$, then the injection $\mathcal{E}_i \to \mathcal{E}_j$ is in fact a bijection. In this case we can reduce w_j to w_i without decreasing the number of heavy edges. Therefore, we may assume that $w_i = w_j$ if and only if $d_i = d_j$. Ruckle's conjecture turns out to

be equivalent to the statement that all the $\leq$ signs in the sequence $d_1 \leq \cdots \leq d_n$ are equalities, except for possibly one inequality.

It is convenient to think of $V(n,s,h)$ as a probability. If s numbers $H_1, \ldots, H_s$ are sampled without replacement from an optimal weighting $\{w_1, \ldots, w_n\}$, then $V(n,s,h)$ is the tail probability:

$$V(n,s,h) = \mathbb{P}\left(H_1 + \cdots + H_s \geq 1\right).$$

In the proof below, we denote the sum of random variables by $S_s = H_1 + \cdots + H_s$ or simply by S if the number of samples is clear.

THEOREM 3. $V(n,s,h)$ is nondecreasing in h, decreasing in n, and increasing in s.

Proof. The Hider need not use all the material, so the value is nondecreasing in h. To see that V decreases with n, let $\{w_1, \ldots, w_n\}$ be an optimal weighting for a value V. Note that $\sum d_i = sV \cdot \binom{n}{s}$, so it follows from Lemma 2 that $d_1 < V \cdot \binom{n-1}{s-1}$. The number of heavy edges that do not contain the first node is $V \cdot \binom{n}{s} - d_1 > V \cdot \binom{n-1}{s}$. Hence the weighting $\{w_2, \ldots, w_n\}$ yields a value $> V$, and we conclude that $V(n-1, s, h-w_1) > V(n,s,h)$. Since we have established monotonicity in h we have $V(n-1,s,h) \geq V(n-1,s,h-w_1) > V(n,s,h)$ as claimed.

To see that the value increases with s, we use that $V(n,s,h)$ is the tail probability $\mathbb{P}(S_s \geq 1)$ for an optimal partition. We sample once more to get $V(n,s+1,h) = \mathbb{P}(S_{s+1} \geq 1)$. Let $\mathcal{H}$ be the event that $S_s \geq 1$, where S_s is a sum of random variables. Then

$$\mathbb{P}(S_{s+1} \geq 1) = \mathbb{P}(S_{s+1} \geq 1 \mid \mathcal{H}) \cdot V + \mathbb{P}(S_{s+1} \geq 1 \mid \mathcal{H}^c) \cdot (1 - V),$$

where $V = V(n,s,h)$. Since $\mathbb{P}(S_{s+1} \geq 1 \mid \mathcal{H}) = 1$ it suffices to show that $\mathbb{P}(S_{s+1} \geq 1 \mid \mathcal{H}^c) > 0$. In other words, it suffices to show that there exists a light edge that can be made heavy by adding just one node. This is the content of Lemma 2. □

THEOREM 4.

$$(1) \qquad 1 - e^{-\frac{s\lfloor h \rfloor}{n}} < V(n,s,h) \leq \frac{\lfloor sh \rfloor}{n}.$$

Proof. The Searcher orders the nodes cyclically, in a way that is unrelated to the ordering of the weights, and adopts the strategy of picking edges with nodes that are consecutive in this cyclic order. More specifically, if we number the nodes $\{1, \ldots, n\}$ modulo n, then the Searcher picks a random subset $I_j = \{j+1, \ldots, j+s\}$. Notice that $\sum w(I_j) = sh$ and that there are n intervals. Since the Searcher adopts a strategy of picking intervals only, the value of this "restricted game" is greater than or equal to $V(n,s,h)$. The restriction benefits the Hider. The sum of random variables $S = H_{j+1} + \cdots + H_{j+s}$ has expectation $\mathbb{E}[S] = \frac{sh}{n}$. It follows from Markov's inequality $\mathbb{P}(S \geq 1) \leq \mathbb{E}[S] = \frac{sh}{n}$. There are n intervals, so at most $\lfloor sh \rfloor$ of them can be heavy, which gives the upper bound.

The lower bound follows from the Searcher strategy of placing $\lfloor h \rfloor$ unit weights. The number of heavy edges is $\binom{n}{s} - \binom{n-\lfloor h \rfloor}{s}$ in this case, and if we divide this by the total number of edges $\binom{n}{s}$, we obtain that $1 - \prod_{i=0}^{s-1}\left(1 - \frac{\lfloor h \rfloor}{n-i}\right) \leq V(n,s,h)$. Now observe that $\prod_{i=0}^{s-1}\left(1 - \frac{\lfloor h \rfloor}{n-i}\right) \leq \left(1 - \frac{\lfloor h \rfloor}{n}\right)^s < e^{-\frac{\lfloor h \rfloor}{n}s}$, where we have strict inequality since $0 < h < n$. We conclude that $1 - e^{-\frac{s\lfloor h \rfloor}{n}} < V(n,s,h)$. □

We note that the Azuma–Hoeffding inequality applies to random samples without replacement [9, Thm. 4], so it can also be used to bound the value of the game.

However, it gives a weaker bound than Markov's inequality. If $\frac{s\lfloor h\rfloor}{n}$ is small, then the lower bound in Theorem 4 is $\frac{s\lfloor h\rfloor}{n} - O\big((\frac{s\lfloor h\rfloor}{n})^2\big)$. Under some arithmetic restrictions on s and h, $\frac{s\lfloor h\rfloor}{n}$ is an upper bound.

THEOREM 5. *If $n = 0 \bmod s$ or $n = 1 \bmod s$, then $V(n,s,h) \leq \frac{s\lfloor h\rfloor}{n}$.*

Proof. As in the proof of the previous theorem, the Searcher randomly takes an s-interval, according to some ordering of the nodes. The value of the restricted game is bounded by $\frac{\lfloor sh\rfloor}{n}$ and by our standing assumption $sh < n$, so there exists a light interval. Without loss of generality we may assume that I_{n-1} is light (note that we cannot assume that $w_1 \leq \cdots \leq w_n$ since the Hider adopts a restricted strategy).

To prove the theorem, it suffices to show that there are at most $s\lfloor h\rfloor$ heavy s-intervals. Since $n = 0, 1 \bmod 1$ there exists an integer k such that either $n = ks$ or $n = ks + 1$. Since $I_0 \cup I_s \cup \cdots \cup I_{(k-1)s}$ is a disjoint union, the sum of the weights of these intervals is at most h. So at most $\lfloor h\rfloor$ of these intervals can be heavy. The same argument applies to $I_j \cup I_{s+j} \cup \cdots \cup I_{(k-1)s+j}$, and we find that at most $s\lfloor h\rfloor$ of the intervals I_{j+is} are heavy with $0 \leq j < s$ and $0 \leq i < k$. For the given restrictions on j and i we find all intervals except I_{n-1} for $n = ks + 1$. But I_{n-1} is light.   □

The arithmetic restriction on s and n is necessary. If $n = 5$ and $s = 3$ and $h = 3/2$, then the Hider divides into $\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0\}$, creating 7 heavy s-sets. It follows from Theorem 16 below that this weighting is optimal. The value of the game is $V(5, 3, 3/2) = \frac{7}{10}$, while $\frac{s\lfloor h\rfloor}{n} = \frac{3}{5}$ and $\frac{\lfloor sh\rfloor}{n} = \frac{4}{5}$.

COROLLARY 6. *If $h < 2$ and if $n = 0$ or $1 \bmod s$, then the Hider uses a single unit weight. In particular, Ruckle's conjecture holds.*

Proof. Since $\lfloor h\rfloor = 1$ the lower bound $1 - \prod_{i=0}^{s-1}\big(1 - \frac{\lfloor h\rfloor}{n-i}\big)$ in the proof of Theorem 4 is equal to $\frac{s}{n} = \frac{s\lfloor h\rfloor}{n}$. By Theorem 5 this is the value of the game.   □

COROLLARY 7. *If $n = 0 \bmod s$ and if $h \geq \frac{n-1}{s}$, then Ruckle's conjecture is true.*

Proof. Under these conditions $\lfloor h\rfloor = \frac{n-s}{s}$. By Theorem 5 the number of light edges is at least $\frac{n-s\lfloor h\rfloor}{n} \cdot \binom{n}{s} = \binom{n-1}{s-1}$. Now suppose the Hider puts $n-1$ weights $\frac{1}{s}$. Then the number of light edges is $\binom{n-1}{s-1}$, which is the best possible.   □

These corollaries are typical for the results in our paper. We are able to prove the conjecture only if $\frac{s\lfloor h\rfloor}{n}$ is close to 1 or if it is close to 0. This suggests that there exists a symmetry between sampling s times or $n - s$ times in an accumulation game. We can prove that such a symmetry exists only under a severe restriction on the weights.

THEOREM 8. *Suppose that there exists an optimal weighting with weights bounded by $\frac{h-1}{n-s-1}$ for $s < n - 1$ and $h > 1$. Then $V(n, s, h) = V(n, n - s, \frac{sh+h-n}{h-1})$.*

Proof. Let $w = \{w_1, \ldots, w_n\}$ be an optimal weighting of mass h. Define a new weighting w' with weights $g_i = 1 - \big(\frac{n-s-1}{h-1}\big) \cdot w_i$, which is well defined and has total weight $g = \frac{sh+h-n}{h-1}$. Conversely, any such weighting w' can be transformed to a weighting w with weights $\leq \frac{n-s-1}{h-1}$ by the inverse transformation $w_i = \frac{1-g_i}{n+1-s-g}$. Now compute

$$w_1 + \cdots + w_s \geq 1 \Leftrightarrow w_{s+1} + \cdots + w_n \leq h - 1 \Leftrightarrow$$

$$\frac{n-s-1}{h-1} \cdot w_{s+1} + \cdots + \frac{n-s-1}{h-1} \cdot w_n \leq n - s - 1 \Leftrightarrow$$

$$(1 - g_{s+1}) + \cdots + (1 - g_n) \leq n - s - 1 \Leftrightarrow g_{s+1} + \cdots + g_n \geq 1.$$

In particular, an edge is heavy $w(I) \geq 1$ under the weighting w if and only if its complement is heavy $w'(I^c) \geq 1$ under the weighting w'.   □
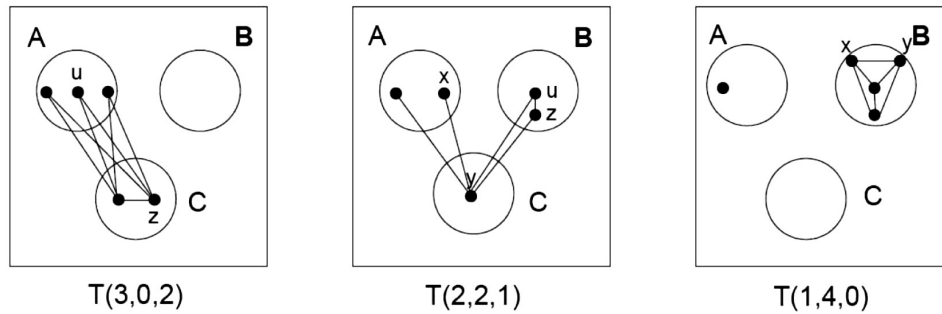
FIG. 1. *Number m of edges in a T-graph on 5 nodes of mass 2.*

4. Solution of the conjecture for some special cases. Kikuta and Ruckle proved that Ruckle's conjecture is true if $s = 1$ or $s = n - 1$. Indeed, if $s = 1$, then the Hider divides h into $\lfloor h \rfloor$ parts of weight $h/\lfloor h \rfloor$. If $s = n - 1$, then the Hider puts a single weight h. We prove that Ruckle's conjecture is true if $s = 2$ or $s = n - 2$.

LEMMA 9. *For any partition $h = w_1 + \cdots + w_n$ there exists another partition $g = g_1 + \cdots + g_n$ for $g \leq h$ and all $g_i \in \{0, \frac{1}{2}, 1\}$, such that $w_i + w_j \geq 1$ implies $g_i + g_j \geq 1$.*

Proof. Let $\mathfrak{R}$ be the set of all equations $w_i + w_j \geq 1$ that are satisfied by a weighting $\{w_1, \ldots, w_n\}$. Without loss of generality, we may assume that h minimizes $w_1 + \cdots + w_n$ under the constraints $\mathfrak{R}$ and $w_i \geq 0$. Let $b = \max\{w_i : w_i \neq 1\}$ and $a = \min\{w_i : w_i \neq 0\}$, so that $a \leq b$. Call the weights w_i which are equal to b the "big weights" and those equal to a the "small weights."

If $a + b > 1$, then h could be reduced by decreasing all the big weights to $1 - a$, contradicting our assumption that h is minimal under the constraints $\mathfrak{R}$. Similarly, if $a + b < 1$, then we could decrease all the small weights to zero. Hence $a + b = 1$.

Let α be the number of small weights, let β be the number of big weights, and let ε be the minimum difference between any two weights. If $\beta > \alpha$, we could decrease h by changing the big weights to $b - \varepsilon$ and increasing the small weights to $a + \varepsilon$, contrary to the assumption. If $\beta < \alpha$, then h may be reduced by changing the small weights to $a - \varepsilon$ and the big ones to $\beta + \varepsilon$. We conclude that $\alpha = \beta$.

Minimize $|b - a|$ under the constraints $\mathfrak{R}$ and $h = w_1 + \cdots + w_n$ and $w_i \geq 0$. We claim that $|b - a| = 0$. If not, then we could reduce the big weights to $b - \varepsilon$ and increase the small weights to $a + \varepsilon$ under the constraints, since $\alpha = \beta$, contradicting minimality. We conclude that $a = b = \frac{1}{2}$ and we are done. ∎

DEFINITION 10. *A graph is called a T-graph if its nodes can be partitioned into three sets $A, B,$ and C such that two nodes are connected by an edge if and only if*

  (a) *at least one of the nodes is in C, or*

  (b) *both nodes are in B.*

If the cardinalities of the three node sets are, respectively, $a, b,$ and c, then we write $T(a, b, c)$, and we define the mass *of the graph to be $b + 2c$.*

We say that a T-graph G is *optimal* if its edge number is maximal among all T-graphs that have the same number of nodes as G and that have mass $\leq m(G)$. For instance, the graph $T(3, 0, 2)$ is optimal in Figure 1.

LEMMA 11. *If G is an optimal T-graph, then either $a = 0$ of $c = 0$ or $b \leq 1$.*

Proof. Let H be a T-graph such that $a > 0$ and $c > 0$ and $b > 1$. We show that

H is not optimal by increasing the number of edges while preserving the mass and the nodes. We consider two overlapping cases, $b \geq a$ and $b \leq a$.

$b \geq a$ Since a and c are at least 1, we may move two nodes x and y from A and C, respectively, to B. There will be b more edges incident to x, corresponding to the original nodes of B, and $a - 1$ fewer edges incident to y. Hence there are $b - (a - 1) \geq 1$ more edges in the resulting T-graph.

$b \leq a$ Since $b \geq 2$, we may move two nodes u and z from B into A and C, respectively. In the resulting T-graph there will be $b - 1$ fewer edges incident to u, corresponding to the nodes of B other than u and z, and a more edges incident to z. Hence the resulting T-graph has $a - (b - 2) \geq 1$ more edges.   ☐

THEOREM 12. *Ruckle's conjecture is true if $s = 2$.*

Proof. Suppose that $s = 2$. The optimal weighting maximizes the number of $w_i + w_j \geq 1$, and by Lemma 9 we may suppose that the weights are either 0 or $\frac{1}{2}$ or 1. Hence, the equations $w_i + w_j \geq 1$ correspond to the edges on a T-graph, with A the set of zero weights, B the set of weights $\frac{1}{2}$, and C the set of unit weights. An optimal weighting corresponds to an optimal T-graph, so $a = 0$ or $c = 0$ or $b \leq 1$. If $b = 1$, then there is only one weight $\frac{1}{2}$, which could be changed to a zero weight without losing optimality. We conclude that $\min\{a, b, c\} = 0$. If $a = 0$, then $w_i + w_j \geq 1$ for all weights, so all edges are heavy, contradicting our standing assumption. Therefore, either $b = 0$ or $c = 0$. In other words, either all nonzero weights are $\frac{1}{2}$ or all weights are 1. The conjecture holds.   ☐

THEOREM 13. *Ruckle's conjecture is true if $s = n - 2$.*

Proof. First note that by our standing assumption $h < \frac{n}{s} = 1 + \frac{2}{n-2}$. Since h is only marginally larger than 1, putting a unit weight will be optimal or nearly optimal. Suppose that a weighting contains a weight $> h - 1$. Then an edge is heavy if and only if it contains that weight, so in this case putting a unit weight is optimal. For the rest of the proof, we assume that all weights are $\leq h - 1$.

An edge is heavy if and only if its complementary set has weight $\leq h - 1$. Maximizing the number of heavy edges is equivalent to maximizing the number of 2-sets of weight $\leq h - 1$. For a given weighting $\{h_1, \ldots, h_n\}$ let $\mathfrak{R}$ be the set of inequalities $h_i + h_j \leq h - 1$ that correspond to such 2-sets. Suppose that h is minimal and that the weighting satisfies the constraints in $\mathfrak{R}$. For such a minimal h, we take a weighting with a maximal number of weights 0 or $h - 1$.

Let $b = \max\{h_i : h_i \neq h - 1\}$ and $a = \min\{h_i : h_i \neq 0\}$, so that $a \leq b$. Call the weights h_i which are equal to b the "big weights" and those equal to a the "small weights." If $a + b = \lambda < 1$, then we could multiply all weights by λ to obtain a weighting that satisfies the same constraints. This contradicts the minimality of h and therefore $a + b \geq 1$.

Let $\epsilon = \min\{a, h - 1 - b\}$. Increase one of the big weights by ϵ and reduce one of the small weights by ϵ. This is possible as long as there are at least 2 weights between 0 and $h - 1$. The resulting weighting has more weights that are equal to 0 or $h - 1$. We claim that it still satisfies the constraints. To see this, notice that the total weight remains the same, so if one of the constraints in $\mathfrak{R}$ is no longer valid, then it has to involve the big weight that has been increased. However, such a constraint consists of the increased big weight and a zero weight, so it remains valid. Since our weighting maximizes the number of weights that are 0 or $h - 1$, the operation is not possible. We conclude that there exists a weighting that minimizes h under the constraints $\mathfrak{R}$, which contains at most one weight between 0 and $h - 1$.

If there is no weight between 0 and $h - 1$, then we are done. If there exists such an intermediate weight, then we can take away an equal amount from all the weights

that are $h - 1$ and add that to the intermediate weight. The constraints $\mathfrak{R}$ remain satisfied. We can redistribute the weight until all nonzero weights are equal. $\quad\square$

COROLLARY 14. *Ruckle's conjecture is true for $n \leq 6$.*

Proof. Since the conjecture is correct if $s \in \{1, 2, n-2, n-1\}$, it is true for $n \leq 5$. For $n = 6$ the remaining case is $s = 3$, which is settled by Corollary 6. $\quad\square$

THEOREM 15. *Ruckle's conjecture is true if $n = 2s + 1$.*

Proof. If $h < 2$, then Corollary 6 applies. Suppose that $h \geq 2$ and note that $h < 2 + \frac{1}{s}$ by our standing assumption. If the Hider puts two unit weights, then he creates $\binom{2s+1}{s} - \binom{2s-1}{s}$ heavy edges. This is optimal if the number of light edges is always at least $\binom{2s-1}{s}$. Since $\binom{2s-1}{s} = \frac{1}{2}\binom{2s}{s}$ it suffices to show that of the edges with nodes in $\{2, \ldots, 2s + 1\}$, half are light. Suppose that I and J are complementary edges in $\{2, \ldots, 2s + 1\}$ and suppose that both are heavy. Then $sw_1 \geq w(I) \geq 1$ so $w_1 \geq \frac{1}{s}$ and therefore $h \geq w_1 + w(I) + w(J) \geq 2 + \frac{1}{s}$, contradicting our standing assumption. It follows that of each pair of complementary edges, at least one is light. So half of the edges with nodes in $\{2, \ldots, 2s + 1\}$ are light. $\quad\square$

One might expect to dispose of the case $n = 2s - 1$ in a similar manner, but we can only prove this under a restriction.

THEOREM 16. *Ruckle's conjecture is true if $n = 2s - 1$ and $h \geq 2 - \frac{1}{s-1}$.*

Proof. Suppose $h \geq 2 - \frac{1}{s-1}$. We show that the partition into $2s - 3$ weights $\frac{1}{s-1}$ is optimal. This creates $\binom{2s-3}{s-2}$ light edges. Let $w_1 + \cdots + w_n$ be any partition of h with $w_1 \leq \cdots \leq w_n$. Note that $w_k < \frac{1}{s}$; otherwise all edges are heavy. We say that two edges I, J are 1-complementary if $\{1\} = I \cap J$. Two such edges cannot both be heavy since $w(I) + w(J) = h + w_1$, which is < 2 by our standing assumption. So the number of light edges is at least $\frac{1}{2}\binom{2s-2}{s-1} = \binom{2s-3}{s-2}$. $\quad\square$

COROLLARY 17. *Ruckle's conjecture is true for $n = 7$.*

Proof. The cases $s \in \{1, 2, 5, 6\}$ are settled by the preceding theorem and $s = 3$ is settled by Theorem 15. It remains to settle the case $s = 4$. Theorem 16 settles this if $h \geq 5/3$, so we may assume that $h < 5/3$. We show that in this case $\{0, 0, 0, 0, 1/2, 1/2, 1/2\}$ is an optimal weighting. It has 22 heavy edges and there are 45 edges in total, so we need to argue that any other weighting gives 13 light edges or more. We argue by contradiction. Assume that there exists a weighting with < 13 light edges. Since $I = \{2, 3, 4, 7\}$ has 12 edges that are smaller in the $\succ$ order, it has to be heavy. If I is heavy, then its 3-complementary edge $J = \{1, 3, 5, 6\}$ is light. Indeed, if I and J would both be heavy, then $h + w_3 = w(I) + w(J) \geq 2$, which implies that $w_3 \geq 1/3$. But this is nonsense since then the weights $w_3, w_4, \ldots, w_7$ would add up to $\geq 5/3$. So I is heavy and J is light. There are 8 edges that are smaller than J in the edge order, so they are light also. The edges $\{2, 3, 5, 6\}$ and $\{1, 3, 4, 7\}$ are 3-complementary, so at least one of the following cases holds:

(A) $\{2, 3, 5, 6\}$ is light.

(B) $\{1, 3, 4, 7\}$ is light.

Assume that (A) holds. Since $\{2, 3, 5, 6\}$ is larger than 11 edges in the $\succ$ order, we need just one more light edge to get a contradiction. The edges $\{1, 2, 3, 7\}$ and $\{2, 4, 5, 6\}$ are 2-complementary, so at least one of them is light. This gives 13 light edges, contradicting our assumption that there are at most 12. Assume that (B) holds. There are two edges that are smaller than $\{1, 3, 4, 7\}$ and that are not in the set of 9 light edges that are $\preceq J$ for a total of 12 light edges. The edge $\{2, 3, 4, 6\}$ is not in this set, so it is heavy by our assumption. Its 3-complementary edge $\{1, 3, 5, 7\}$ therefore is light, and this is the 13th edge that is light. We conclude that $\{0, 0, 0, 0, 1/2, 1/2, 1/2\}$ is an optimal weighting if $h < 5/3$.

Now suppose that $h < 3/2$. We claim that it is optimal to put one unit weight, for a total of 20 heavy edges. We argue by contradiction and assume that there are fewer than 15 edges that are light. This implies that $\{3,4,5,6\}4$, $\{2,3,5,7\}$, $\{1,4,5,7\}$ all are heavy, since each has 14 edges that are smaller. By the familiar argument, only one of two 5-complementary edges can be heavy. So the three edges $\{1,2,5,7\}$, $\{1,4,5,6\}$, $\{2,3,5,6\}$ are light, but there are 13 edges that are smaller than one of these three edges, contradicting our assumption.   □

A weaker form of Ruckle's conjecture is that the Hider uses at least $n - sh$ zero weights in an optimal partition. The following result is a step toward settling this weakened conjecture.

THEOREM 18. *For any n, s, h there exists an optimal weighting for the Hider with at least $n - s^2\lfloor h \rfloor$ zero weights.*

Proof. Fix some optimal weighting and let $\{I_1, \ldots, I_k\}$ be a maximal family of disjoint heavy edges, so that $k \leq \lfloor h \rfloor$, and let $\mathcal{I} = I_1 \cup \cdots \cup I_k$. Every heavy edge contains at least one node in $\mathcal{I}$. Let $\mathcal{J} = \{j : w_j > 0\}$ be the nodes of weight > 0 in the Hider's partition. Suppose that $|\mathcal{J}| > s^2\lfloor h \rfloor$. Let ϵ denote the minimum nonzero weight. Reduce the weight on the nodes that are in $\mathcal{J} \setminus \mathcal{I}$ by ϵ and increase the weight on the nodes in $\mathcal{I}$ by $(s-1)\epsilon$. There are more than $s^2\lfloor h \rfloor - s\lfloor h \rfloor$ nodes in $\mathcal{J} \setminus \mathcal{I}$ and there are at most $s\lfloor h \rfloor$ nodes in $\mathcal{I}$, so this operation does not increase the total weight of the partition. It preserves heavy sets, since each heavy set contains a node that increases by $(s-1)\epsilon$. So we can reduce the total weight until $\mathcal{J}$ contains no more than $s^2\lfloor h \rfloor$ nodes.   □

5. Intersecting families. Let $\mathcal{F}$ be a family of subsets of $\{1, \ldots, n\}$. In other words, $\mathcal{F}$ is a hypergraph. It is called an *intersecting family* if no two of its elements are disjoint.

THEOREM 19 (Erdős–Ko–Rado [7]). *Let $\mathcal{F}$ be an intersecting family of s-subsets. If $2s \leq n$, then $\mathcal{F}$ has no more than $\binom{n-1}{s-1}$ elements. In other words, the family of sets with one common element has maximal cardinality.*

The following improves on Corollary 6 and Theorem 15.

COROLLARY 20. *Ruckle's conjecture is true if $h \leq 2 + \frac{1}{s}$ and $n \geq 2s$.*

Proof. If $h < 2$, then the heavy edges form an intersecting family, and by the Erdős–Ko–Rado theorem the Hider puts one unit weight. Consider the case that $h \geq 2$. Obviously $w_n \geq \frac{1}{s}$; otherwise no edge can be heavy. If $w_n = \frac{1}{s}$, then all other nonzero weights can be taken to be $\frac{1}{s}$ as well; otherwise they do not contribute to any heavy edge. So in this case Ruckle's conjecture is true. It remains to consider the case that $h - w_n < 2$. In this case the family of heavy edges that do not contain node n form an intersecting family $\mathcal{F}$. By the Erdős–Ko–Rado theorem $|\mathcal{F}| \leq \binom{n-2}{s-1}$. There are $\binom{n-1}{s-1}$ edges that contain node n, so the number of heavy edges is bounded by

$$\binom{n-2}{s-1} + \binom{n-1}{s-1} = \binom{n}{s} - \binom{n-2}{s},$$

which is the number of heavy edges if the Hider puts two unit weights.   □

The Erdős–Ko–Rado theorem is a celebrated result and a starting point of the theory of hypergraphs [4]. It has been extended in many ways. For integers n, s, k the number $f(n, s, k)$ is defined as the largest possible collection of s-sets, no k of which are pairwise disjoint, that can be chosen from a set of size n.[1] Obviously, if $k = \lfloor h \rfloor$,

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[1] This number is also denoted by $f(n, s, k, 0)$, where 0 represents empty intersection and Erdős in [6] denotes it by $f(n, s, k) - 1$. In other papers, the number $f(n, s, k)$ represents the maximum cardinality of a union of k intersecting families.

then the maximal number of heavy edges is bounded by $f(n, s, k)$.

THEOREM 21 (Erdős [6]). *For each $s \geq 2$ there exists a constant $c(s)$ depending only on s such that*

$$f(n, s, k) = \binom{n}{s} - \binom{n - k + 1}{s} \quad \textit{for } n > c(s)k.$$

The value of $f(n, s, k)$ in this theorem is attained by the family of all s-subsets that contain at least one element of a given $k-1$-subset. In other words, it is attained if the Hider uses unit weights.

COROLLARY 22. *Ruckle's conjecture is true if $n > c(s)\lfloor h \rfloor$, and in this case the Hider uses $\lfloor h \rfloor$ unit weights.*

The best known estimate of the constant in Theorem 21, due to Bollobás, Daykin, and Erdős [5], is $c(s) \leq 2s^3$.

6. Hoeffding's problem. Ruckle's conjecture is related to the work of Hoeffding and others in probability. Suppose that the Searcher samples randomly and *with replacement*, so he may pick the same weight twice. Unlike in Ruckle's accumulation game, it is not easy to give a real-life interpretation of this game, but it does simplify the random variables. In particular, the samples $X_1, \ldots, X_s$ now are *independent* and the Hider wants to maximize the tail probability $\mathbb{P}(X_1 + \cdots + X_s \geq 1)$ for i.i.d. random variables. This is related to a probability problem that was proposed by Hoeffding [8] and studied by Hoeffding and Shrikhande [10]. Hoeffding's problem is to find nonnegative i.i.d. random variables that maximize $\mathbb{P}(X_1 + \cdots + X_s \geq 1)$ for a given $\mathbb{E}[X_i] = \alpha$.

THEOREM 23 (Hoeffding–Shrikhande). *If $s = 2$ and if $2\alpha < 1$, then the tail probability is maximized by either $X_i \in \{0, \frac{1}{2}\}$ or $X_i \in \{0, 1\}$.*

Note that the random variable X_i is well defined, since it takes only two values and since its expectation is known. The Hoeffding–Shrikhande theorem is similar to our Theorem 12.

Hoeffding's problem has been proposed in several contexts. The problem satisfies a common rule: $s = 1$ is trivial, $s = 2$ can be solved with a reasonable amount of work, and $s \geq 3$ is hard; see [14]. There is no conjectured solution to Hoeffding's problem, but the general idea seems to be that the tail probability can be maximized by a random variable that takes on only two values. The only result on Hoeffding's problem apart from the Hoeffding–Shrikhande theorem is the following asymptotic result.

THEOREM 24 (Samuels [18]). *Let X_i be i.i.d. and nonnegative for $1 \leq i \leq s$. If $\max\{4sh/n, (s-1)sh/n\} < 1$, then the tail probability is maximized by $X_i \in \{0, 1\}$.*

In particular, if $2s^2h < n$ and if h is an integer, then a weighting by unit weights is optimal. Note the similarity with our Corollary 22, and also note that the order s^2 is sharper than s^3, which follows from the results of Bollobás, Daykin, and Erdős.

Hoeffding's problem is not exactly the same as the problem of finding an optimal weighting in an accumulation game with replacement. For instance, if $s = 2$ and $n = 4$ and $h = 3/2$, then $\mathbb{E}[X_i] = 3/8$. By the Hoeffding–Shrikhande theorem the tail probability is maximized by random variables $X_i \in \{0, 1\}$ (which give a greater tail probability under these conditions than $X_i \in \{0, \frac{1}{2}\}$). However, these random variables cannot be created by a weighting on 4 locations. The optimal weighting is $\{0, 0, \frac{1}{2}, 1\}$. Suppose we double the number of locations $n = 8$ and the mass $h = 3$, keeping the expectation at $\mathbb{E}[X_i] = 3/8$; then it is possible to create the optimal random variables by the weighting $\{0, 0, 0, 0, 0, 1, 1, 1\}$, which therefore is optimal.

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