# MODULO (1,1) PERIODICITY OF CLIFFORD ALGEBRAS AND GENERALIZED (ANTI-)MÖBIUS TRANSFORMATIONS

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# AND
# GENERALIZED (ANTI-)MÖBIUS TRANSFORMATIONS



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## Preface

The contents of this thesis originated from my interest in the Clifford algebra approach to non-Euclidean geometry, roused some four years ago by Vahlen's paper [8]. It was a fortunate coincidence that I got acquainted with Ahlfors' paper [1] on Möbius transformations in Euclidean spaces employing Clifford algebras. For this encouraged me to develop a theory of (anti-)Möbius transformations in a quadratic vector space of general signature.

In the first chapter the reader meets a synoptic introduction in Clifford algebras and spin representations. For more details one is referred to Porteous [6].
A constructive method is given to determine the scalar products on the spinor spaces. The results can also be found in Lounesto [5].

Chapter 2 presents my theory of generalized (anti-)Möbius transformations, based on the modulo (1,1) periodicity of Clifford algebras.

Chapter 3 is a reminiscence of Vahlen's paper [8]. The hyperbolic group is seen to be covered by a subgroup of (anti-)Möbius transformations belonging to a positive definite vector space.

In chapter 4 we discuss the geometry of the Siegel domains of type four. My theory of (anti-)Möbius transformations, also valid for complex vector spaces, makes it possible to give a non-linear representation of the groups of biholomorphic self-mappings.

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# Contents

# CHAPTER 1
# CLIFFORD ALGEBRAS AND SPIN REPRESENTATIONS

Let V(p,q) denote a real n-dimensional vector space equipped with a non-degenerate quadratic form of signature (p,q), i.e., a form Q which can be diagonalized in the following way (n = p + q)

$$Q(v) = v_1^2 + \dots + v_p^2 - (v_{p+1}^2 + \dots + v_n^2), \quad v \in V(p,q).$$

We say that a real associative algebra A with unity $1_A$ is compatible with V(p,q) if a linear injection i: V(p,q) → A exists such that

$$(i(v))^2 + Q(v)\, 1_A = 0 \quad \forall v \in V(p,q).$$

Identifying $\mathbb{R}$ and V(p,q) with their copies in A we simplify the notation

$$v^2 + Q(v) = 0 \ \forall\ v \in V(p,q).$$

The equation Q(v+w) = Q(v) + Q(w) + 2B(v,w), B denoting the associated bilinear form, implies that

$$vw + wv + 2B(v,w) = 0 \quad \forall v,w \in V(p,q).$$

In particular v and w anticommute whenever they are orthogonal. An algebra A which is compatible with V(p,q) is called a *Clifford algebra* for V(p,q) if it is generated by V(p,q) and not by any proper subspace of V(p,q). The existence of such an algebra for any V(p,q) follows by construction, but first we settle the question of uniqueness. Let $A_1$ and $A_2$ be a Clifford algebra for $V_1(p,q)$ and $V_2(p,q)$, respectively, and suppose that $\phi$: $V_1(p,q) \to V_2(p,q)$ is an orthogonal map. Then there is a unique algebra isomorphism $\alpha: A_1 \to A_2$ and a unique algebra anti-isomorphism $\tilde{\alpha}: A_1 \to A_2$ such that the following diagrams commute

$$\begin{array}{ccc} V_1(p,q) & \xrightarrow{\phi} & V_2(p,q) \\ \downarrow \text{inc} & & \downarrow \text{inc} \\ A_1 & \underset{\alpha}{\rightarrow} & A_2 \end{array} \qquad\qquad \begin{array}{ccc} V_1(p,q) & \xrightarrow{\phi} & V_2(p,q) \\ \downarrow \text{inc} & & \downarrow \text{inc} \\ A_1 & \underset{\tilde{\alpha}}{\rightarrow} & A_2 \end{array} .$$

The first diagram expresses the *universal property* in case $V_1(p,q) = V_2(p,q)$ and $\phi$ stands for the identity map.

The Clifford algebra for $V(p,q)$, unique up to isomorphism, will be denoted by $Cl(p,q)$. Now select an orthonormal basis $\{e_1,...,e_n\}$ of $V(p,q)$. Then the Clifford algebra $Cl(p,q)$ is generated over $\mathbb{R}$ by a unity 1 and the n symbols $e_i$ subject to the relations

$$e_1^2 = e_2^2 = ... = e_p^2 = -1,$$
$$e_{p+1}^2 = ... = e_n^2 = 1,$$
$$e_ie_j + e_je_i = 0, \quad i \neq j \in \{1,...,n\}.$$

Every element of $Cl(p,q)$ is an $\mathbb{R}$-linear combination of the $2^n$ basis elements

$$e_1^{i_1} e_2^{i_2} ... e_n^{i_n} \quad \text{with } i_k \in \{0,1\}.$$

In other words, any element of $Cl(p,q)$ is the sum of a scalar, a vector, a bivector, ..., and an n-vector.

The two orthogonal transformations $v \rightarrow \epsilon v$ of $V(p,q)$ ($\epsilon = 1$ or $\epsilon = -1$) are uniquely extended to the anti-automorphisms $\alpha_\epsilon$ of $Cl(p,q)$. Applied to the basis elements of $Cl(p,q)$ these anti-automorphisms read like this

$$\alpha_1 \left(e_1^{i_1} e_2^{i_2} ... e_n^{i_n}\right) = e_n^{i_n} ... e_2^{i_2} e_1^{i_1},$$
$$\alpha_{-1} \left(e_1^{i_1} e_2^{i_2} ... e_n^{i_n}\right) = (-1)^d e_n^{i_n} ... e_2^{i_2} e_1^{i_1} \quad \text{with } d = i_1 + ... + i_n.$$

Hence $\alpha_1$ reverses order and $\alpha_{-1}$ subjects a basis element to reversion combined with plus or minus the identity mapping according as it is a product of an even or odd number of generators. The composite map $\alpha_1 \circ \alpha_{-1} = \alpha_{-1} \circ \alpha_1$, which will henceforth be denoted by ^, is an automorphism of $Cl(p,q)$ inducing the standard $\mathbb{Z}_2$-grading

$$Cl(p,q) = Cl^{o}(p,q) + Cl^{1}(p,q),$$

where $Cl^{o}(p,q) = \left\{a \in Cl(p,q): a = \hat{a}\right\}$ is a subalgebra of dimension $2^{n-1}$, called the *even Clifford algebra* for V(p,q). The even Clifford algebra is a Clifford algebra by itself: the following isomorphisms are easy to prove

$$Cl^{o}(p+1,q) \cong Cl(p,q) \quad \text{and} \quad Cl^{o}(p,q+1) \cong Cl(q,p).$$

Before realizing the Clifford algebras in terms of matrices we discuss their structural properties.

Cl(p,q) and $Cl^{o}(p,q)$ are semi-simple algebras over $\mathbb{R}$ and their centres are given in the following table

| | *Cent Cl(p,q)* | *Cent $Cl^{o}(p,q)$* |
|---|---|---|
| n even | $\mathbb{R}$ | $\{1,j\}_{\mathbb{R}}$ |
| n odd | $\{1,j\}_{\mathbb{R}}$ | $\mathbb{R}$ |

where $j = e_1 e_2 ... e_n$. Thus Cl(p,q) and $Cl^{o}(p,q)$ are central simple algebras over $\mathbb{R}$ when n is even and odd, respectively.

Concerning the other cases, it is clear that $\{1,j\}_{\mathbb{R}}$ is a field if and only if $j^2 = -1$. Hence if $j^2 = -1$, Cl(p,q) and $Cl^{o}(p,q)$ are simple when n is odd and even, respectively. Otherwise, if $j^2 = 1$, they are the direct sum of two mutually annihilating simple ideals $\frac{1}{2}(1 + j)\, Cl^{(o)}(p,q)$ and $\frac{1}{2}(1 - j)\, Cl^{(o)}(p,q)$.

**Spin representations**

We distinguish between the two cases n even, n odd.

(i) *n even:*

Cl(p,q), being simple, admits an irreducible representation $\rho(p,q)$ (unique up to equivalence), which is called the spin representation. The corresponding representation space S(p,q) is called the space of *spinors*. In fact, S(p,q) may be identified with a minimal left ideal of $Cl(p,q)$ and $\rho(p,q)$ with the restriction of the left regular representation to that ideal. The spin representation $\rho(p,q)$ induces a representation $\rho^{o}(p,q)$ of $Cl^{o}(p,q)$ which is irreducible if $j^2 = -1$ ($Cl^{o}(p,q)$ simple) and reducible otherwise. Indeed, if $Cl^{o}(p,q)$ is not simple $\rho^{o}(p,q)$ breaks down into two irreducible inequivalent representations $\rho^{o}_{i}(p,q)$, known as the half-spin representations. They act on the spaces of *half-spinors* $S^{o}_{i}(p,q)$. In this case $S(p,q) = S^{o}_{1}(p,q)\oplus S^{o}_{2}(p,q)$, i.e., every spinor can be written in one and only one way as the sum of two half-spinors.

(ii) *n odd:*

$Cl^{o}(p,q)$, being simple, admits an irreducible representation $\rho^{o}(p,q)$ (unique up to equivalence), acting on the space of spinors $S^{o}(p,q)$. The spin representation $\rho^{o}(p,q)$ can be extended to a representation $\rho(p,q)$ of $Cl(p,q)$ in a unique way if $Cl(p,q)$ is simple and in exactly two ways if $Cl(p,q)$ is not simple. To create a faithful representation of $Cl(p,q)$ take $S(p,q) = S^{o}(p,q)$ or $S(p,q) = S^{o}(p,q)\oplus S^{o}(p,q)$, according as $Cl(p,q)$ is simple or a direct sum of two simple ideals. Concerning the second case, the two components of the space of what we call *double-spinors* are subjected to the two possible extensions of $\rho^{o}(p,q)$.

**Matrix algebras**

Now we proceed to realize the Clifford algebras and their anti-automorphisms in terms of matrices. We start with some low-dimensional cases. In the first place we have the isomorphisms

$$Cl(0,0) \cong \mathbb{R}, \quad Cl(1,0) \cong \mathbb{C}, \quad Cl(2,0) \cong \mathbb{H}.$$

There are no other real associative division algebras than these (Frobenius' theorem). Further, we need

$$Cl(0,1) \cong {}^2\mathbb{R}, \; Cl(1,1) \cong \mathbb{R}(2), \; Cl(3,0) \cong {}^2\mathbb{H}, \; Cl(4,0) \cong \mathbb{H}(2).$$

To convince the reader we list the generating sets of these algebras in the following table.

| | |
|---|---|
| Cl(0,1) | $e_1 = (1,-1)$ |
| Cl(1,1) | $e_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ |
| Cl(3,0) | $e_1 = (i,-i)$, $e_2 = (j,-j)$, $e_3 = (ij,-ij)$ |
| Cl(4,0) | $e_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $e_2 = \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix}$, $e_3 = \begin{pmatrix} ij & 0 \\ 0 & -ij \end{pmatrix}$, $e_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ |

To construct any Cl(p,q) use these isomorphisms together with the following recurrence relations, all of them easy to prove

$$\begin{aligned}
Cl(p+1,q+1) &\cong Cl(p,q)\otimes Cl(1,1) \cong Cl(p,q)\otimes \mathbb{R}(2) \\
Cl(p,q+1) &\cong Cl(q,p+1) \\
Cl(p+4,q) &\cong Cl(p,q)\otimes Cl(4,0) \cong Cl(p,q)\otimes \mathbb{H}(2) \\
Cl(p,q+4) &\cong Cl(p,q)\otimes Cl(0,4)\otimes Cl(p,q)\otimes \mathbb{H}(2) \\
Cl(p+8,q) &\cong Cl(p,q)\otimes \mathbb{R}(16) \qquad (\mathbb{H}(2)\otimes \mathbb{H}(2) \cong \mathbb{R}(16)) \\
Cl(p,q+8) &\cong Cl(p,q)\otimes \mathbb{R}(16) \qquad \text{(the same).}
\end{aligned}$$

Now the following results are readily obtained.

**1.1. Table - Clifford algebras Cl(p,q) for n = p+q < 8**

| p+q \ -p+q | -7 | -6 | -5 | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | | | | | | | | $\mathbb{R}$ | | | | | | | |
| 1 | | | | | | | $\mathbb{C}$ | | $^2\mathbb{R}$ | | | | | | |
| 2 | | | | | | $\mathbb{H}$ | | $\mathbb{R}(2)$ | | $\mathbb{R}(2)$ | | | | | |
| 3 | | | | | $^2\mathbb{H}$ | | $\mathbb{C}(2)$ | | $^2\mathbb{R}(2)$ | | $\mathbb{C}(2)$ | | | | |
| 4 | | | | $\mathbb{H}(2)$ | | $\mathbb{H}(2)$ | | $\mathbb{R}(4)$ | | $\mathbb{R}(4)$ | | $\mathbb{H}(2)$ | | | |
| 5 | | | $\mathbb{C}(4)$ | | $^2\mathbb{H}(2)$ | | $\mathbb{C}(4)$ | | $^2\mathbb{R}(4)$ | | $\mathbb{C}(4)$ | | $^2\mathbb{H}(2)$ | | |
| 6 | | $\mathbb{R}(8)$ | | $\mathbb{H}(4)$ | | $\mathbb{H}(4)$ | | $\mathbb{R}(8)$ | | $\mathbb{R}(8)$ | | $\mathbb{H}(4)$ | | $\mathbb{H}(4)$ | |
| 7 | $^2\mathbb{R}(8)$ | | $\mathbb{C}(8)$ | | $^2\mathbb{H}(4)$ | | $\mathbb{C}(8)$ | | $^2\mathbb{R}(8)$ | | $\mathbb{C}(8)$ | | $^2\mathbb{H}(4)$ | | $\mathbb{C}(8)$ |

Hence any Clifford algebra $Cl(p,q)$ is isomorphic to a matrix algebra over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, ${}^2\mathbb{R}$ or ${}^2\mathbb{H}$. The following consideration shows that the structure of $Cl(p,q)$ only depends on $q-p \pmod 8$. The isomorphism

$$\begin{aligned} Cl(p+1,q+1) &\cong Cl(p,q)\otimes\mathbb{R}(2) && \text{implies,}\\ Cl(p,q) &\cong Cl(p-q,0)\otimes\mathbb{R}(2^q) && \text{if } p \geq q,\\ Cl(p,q) &\cong Cl(0,q-p)\otimes\mathbb{R}(2^p) && \text{if } p \leq q. \end{aligned}$$

Observation of the algebras $Cl(n,0)$ and $Cl(0,n)$ up to $n = 7$ from Table 1.1 then leads to the following result.

**1.2. Table - Clifford algebras $Cl(p,q)$, $k = 2^{[n/2]-1}$**

| $q-p \pmod 8$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $Cl(p,q)$ | $\mathbb{R}(2k)$ | ${}^2\mathbb{R}(2k)$ | $\mathbb{R}(2k)$ | $\mathbb{C}(2k)$ | $\mathbb{H}(k)$ | ${}^2\mathbb{H}(k)$ | $\mathbb{H}(k)$ | $\mathbb{C}(2k)$ |

Having realized the Clifford algebras $Cl(p,q)$ in terms of matrices, we feel obliged to do the same for the anti-involutions $\alpha_\epsilon$. According to Table 1.2 we may identify the space of (double-)spinors $S(p,q)$ with a right module over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, ${}^2\mathbb{R}$ or ${}^2\mathbb{H}$. Any non-degenerate *scalar product* on $S(p,q)$ induces an anti-involution of $Cl(p,q)$, to wit the appropriate *adjoint involution*. The converse is also true: any anti-involution of $Cl(p,q)$ is the adjoint involution belonging to a non-degenerate scalar product on $S(p,q)$ (cf. Porteous [6], chapter 11).

**1.3. Example**

The Clifford algebra $Cl(1,1)$ is isomorphic to $\mathbb{R}(2)$, the generators being $e_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, say. The adjoint belonging to the skew scalar product $\phi\colon \mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}$, $\phi(x,y) = -x_2y_1 + x_1y_2$ is given by the map $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Applied to the generators it yields $e_i \to -e_i$. Hence the anti-involution $\alpha_{-1}$ is the adjoint of the standard skew scalar product on $S(1,1)$. On the other hand, the adjoint of the symmetric scalar product $\psi\colon \mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}$, $\psi(x,y) = x_2y_1 + x_1y_2$ turns out to be the map $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} d & b \\ c & a \end{pmatrix}$. Preserving the generators this anti-involution is the unique realization of $\alpha_1$. □

Now let us classify the scalar products on S(p,q). The classification is up to isomorphism and equivalence. Two scalar products $\phi_1$ and $\phi_2$ on a right IL-module are called equivalent if $\phi_2 = \lambda\phi_1$, for some invertible $\lambda \in \mathbb{L} = \mathbb{R}, \mathbb{C}, \mathbb{H}, {}^2\mathbb{R}$ or ${}^2\mathbb{H}$.
We introduce the following nine classes of scalar products.

$\mathbb{R}_+$, symmetric over $\mathbb{R}$.
$<x,y> = <y,x>;\ x,y \in \mathbb{R}^m.$

$\mathbb{R}_-$, skew over $\mathbb{R}$.
$<x,y> = -<y,x>;\ x,y \in \mathbb{R}^m.$

$\mathbb{C}_+$, symmetric over $\mathbb{C}$.
$<x,y> = <y,x>;\ x,y \in \mathbb{C}^m.$

$\mathbb{C}_-$, skew over $\mathbb{C}$.
$<x,y> = -<y,x>;\ x,y \in \mathbb{C}^m.$

$\overline{\mathbb{C}}_+$, symmetric over $\mathbb{C}$ with conjugation $a \to \bar{a}$.
$<x,y> = <y,x>^-;\ x,y \in \mathbb{C}^m.$

$\overline{\mathbb{H}}_+$, symmetric over $\mathbb{H}$ with conjugation $a \to \bar{a}$.
$<x,y> = <y,x>^-;\ x,y \in \mathbb{H}^m.$

$\tilde{\mathbb{H}}_+$, symmetric over $\mathbb{H}$ with reversion $a \to \tilde{a} = j\bar{a}j^{-1}$.
$<x,y> = <y,x>^\sim;\ x,y \in \mathbb{H}^m.$

${}^2\mathbb{R}_+$, symmetric over ${}^2\mathbb{R}$ with swap $(a,b) \to sw(a,b) = (b,a)$.
$<x,y> = sw<y,x>;\ x,y \in {}^2\mathbb{R}^m.$

${}^2\overline{\mathbb{H}}_+$, symmetric over ${}^2\mathbb{H}$ with swap-conjugation $(a,b) \to \overline{sw}(a,b) = (\bar{b},\bar{a})$.
$<x,y> = \overline{sw}<y,x>;\ x,y \in {}^2\mathbb{H}^m.$

**Comment**

$\overline{\mathbb{C}}_-$-scalar products are equivalent to $\overline{\mathbb{C}}_+$-scalar products, for if $\phi$ is $\overline{\mathbb{C}}_+$ then $i\phi$ is $\overline{\mathbb{C}}_-$. In the same way left multiplication by j yields the equivalence of $\overline{\mathbb{H}}_\pm$-scalar products to $\tilde{\mathbb{H}}_\mp$-sclar products. The equation $(1,-1)sw(a,b) = -sw(1,-1)(a,b)$ implies that ${}^2\mathbb{R}_+$-scalar products are equivalent to ${}^2\mathbb{R}_-$-scalar products. For the same reason ${}^2\overline{\mathbb{H}}_+$-scalar products are equivalent to ${}^2\overline{\mathbb{H}}_-$-scalar products. What about ${}^2\tilde{\mathbb{H}}_\pm$-scalar products? Since $(j,j)\overline{sw}(a,b) = (j\bar{b},j\bar{a}) = (\widetilde{bj},\widetilde{aj}) = (-(jb)^\sim,-(ja)^\sim) = \widetilde{sw}(-ja,-jb) = -\widetilde{sw}(j,j)(a,b)$, we may conclude that ${}^2\overline{\mathbb{H}}_\pm$-scalar products are equivalent to ${}^2\tilde{\mathbb{H}}_\mp$-scalar products. One final remark, before we classify the scalar products on S(p,q). Concerning the right modules over the double fields ${}^2\mathbb{R}$ and ${}^2\mathbb{H}$, it is also possible to have scalar products like $\Phi(x,y) = (\phi(x_1,y_1),\phi(x_2,y_2))$, where $\phi$ is a scalar product on $\mathbb{R}^m$ or $\mathbb{H}^m$, respectively. These reducible scalar products, of which the relation between $\Phi(x,y)$ and $\Phi(y,x)$ does *not* involve the swap operation, will be denoted in the most obvious way: $\mathbb{R}_+\times\mathbb{R}_+$, $\overline{\mathbb{H}}_+\times\overline{\mathbb{H}}_+$, etcetera.

□

It will become clear that the scalar products on S(p,q) with adjoint $\alpha_\epsilon$ are completely determined by those on S(n,0) and S(0,n) up to n = 7. These cases are easy to deal with.

**1.4.1. Table - Scalar products on S(n,0) with adjoint $\alpha_\epsilon$**

| $\epsilon$ \ n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 1 | $\mathbb{R}_+$ | $\mathbb{C}_+$ | $\tilde{\mathbb{H}}_+$ | ${}^2\overline{\mathbb{H}}_+$ | $\overline{\mathbb{H}}_+$ | $\mathbb{C}_-$ | $\mathbb{R}_-$ | ${}^2\mathbb{R}_+$ |
| -1 | $\mathbb{R}_+$ | $\overline{\mathbb{C}}_+$ | $\overline{\mathbb{H}}_+$ | $\overline{\mathbb{H}}_+\times\overline{\mathbb{H}}_+$ | $\overline{\mathbb{H}}_+$ | $\overline{\mathbb{C}}_+$ | $\mathbb{R}_+$ | $\mathbb{R}_+\times\mathbb{R}_+$ |

**1.4.2. Table - Scalar products on S(0,n) with adjoint $\alpha_\epsilon$**

| $\epsilon$ \ n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 1 | $\mathbb{R}_+$ | $\mathbb{R}_+\times\mathbb{R}_+$ | $\mathbb{R}_+$ | $\overline{\mathbb{C}}_+$ | $\overline{\mathbb{H}}_+$ | $\overline{\mathbb{H}}_+\times\overline{\mathbb{H}}_+$ | $\overline{\mathbb{H}}_+$ | $\overline{\mathbb{C}}_+$ |
| -1 | $\mathbb{R}_+$ | ${}^2\mathbb{R}_+$ | $\mathbb{R}_-$ | $\mathbb{C}_-$ | $\overline{\mathbb{H}}_+$ | ${}^2\overline{\mathbb{H}}_+$ | $\tilde{\mathbb{H}}_+$ | $\mathbb{C}_+$ |

Note that the product on S(n,0) with adjoint $\alpha_\epsilon$ is of the same type as the one on S(0,8-n) with adjoint $\alpha_{-\epsilon}$. For the classification of the Clifford algebras Cl(p,q) themselves (Table 1.2) we used the isomorphism $Cl(p+1,q+1) \cong Cl(p,q)\otimes Cl(1,1)$. Now we need more information for we consider the pairs $(Cl(p,q),\alpha_\epsilon)$. We prove

### 1.5. Theorem

$$(Cl(p+1,q+1),\alpha_\epsilon) \cong (Cl(p,q),\alpha_{-\epsilon}) \otimes (Cl(1,1),\alpha_\epsilon).$$

*Proof*

It is only a matter of verification.

$\{e_i\}$ and $\{f_j\}$ being a set of generators for $Cl(p,q)$ and $Cl(1,1)$, respectively, take $\{e_i \otimes f_1 f_2,\ 1 \otimes f_j\}$ as a generating set for $Cl(p,q) \otimes Cl(1,1) \cong Cl(p+1,q+1)$. Apply the anti-automorphism $\alpha_{-\epsilon} \otimes \alpha_\epsilon$ to this set, where $\alpha_{-\epsilon}$ and $\alpha_\epsilon$ are of $Cl(p,q)$ and $Cl(1,1)$, respectively. (Although the new notation $\alpha_\epsilon(p,q)$ would be very convenient here, we do not introduce it and thereby undertake a small risk on confusion).

$$(\alpha_{-\epsilon} \otimes \alpha_\epsilon)(e_i \otimes f_1 f_2) = \alpha_{-\epsilon}(e_i) \otimes \alpha_\epsilon(f_1 f_2) = -\epsilon e_i \otimes f_2 f_1 = \epsilon(e_i \otimes f_1 f_2), \text{ and}$$
$$(\alpha_{-\epsilon} \otimes \alpha_\epsilon)(1 \otimes f_j) = \alpha_{-\epsilon}(1) \otimes \alpha_\epsilon(f_j) = 1 \otimes \epsilon f_j = \epsilon(1 \otimes f_j).$$

The generators are multiplied by $\epsilon$.

□

### 1.6. Corollary

The scalar product on $S(p+1,q+1)$ with adjoint $\alpha_1$ is of the same type as the one on $S(p,q)$ with adjoint $\alpha_{-1}$. On the other hand, the scalar product on $S(p,q)$ with adjoint $\alpha_1$ determines the one on $S(p+1,q+1)$ with adjoint $\alpha_{-1}$ in the following way.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $\otimes Cl(1,1)$ | $\alpha_1$ | $\mathbb{R}_\pm(\times\mathbb{R}_\pm)$ | $\mathbb{C}_\pm$ | $\overline{\mathbb{C}}_+$ | $\overline{\mathbb{H}}_+(\times\overline{\mathbb{H}}_+)$ | $\tilde{\mathbb{H}}_+(\times\tilde{\mathbb{H}}_+)$ | ${}^2\mathbb{R}_+$ | ${}^2\overline{\mathbb{H}}_+$ |
| | $\alpha_{-1}$ | $\mathbb{R}_\mp(\times\mathbb{R}_\mp)$ | $\mathbb{C}_\mp$ | $\overline{\mathbb{C}}_+$ | $\tilde{\mathbb{H}}_+(\times\tilde{\mathbb{H}}_+)$ | $\overline{\mathbb{H}}_+(\times\overline{\mathbb{H}}_+)$ | ${}^2\mathbb{R}_+$ | ${}^2\overline{\mathbb{H}}_+$ |

*Proof*

Example 1.3 shows that $\alpha_{\pm 1}$ on $Cl(1,1)$ is the adjoint of an $\mathbb{R}_\pm$-scalar product on $S(1,1)$. Consequently, tensoring $(Cl(p,q),\alpha_{-\epsilon})$ with $(Cl(1,1),\alpha_\epsilon)$ preserves the type of scalar product if $\epsilon = 1$ and swaps symmetric and skew ones if $\epsilon = -1$.

With regard to the second case ($\epsilon = -1$), to obtain the result recall that $\overline{\mathbb{C}}_+ \sim \overline{\mathbb{C}}_-$, $\overline{\mathbb{H}}_\pm \sim \tilde{\mathbb{H}}_\mp$, ${}^2\mathbb{R}_+ \sim {}^2\mathbb{R}_-$ and ${}^2\overline{\mathbb{H}}_+ \sim {}^2\overline{\mathbb{H}}_-$, where $\sim$ denotes equivalence of scalar products.

□

Of course we also need to extend the modulo 8 periodicities $Cl(p+8,q) \cong Cl(p,q+8) \cong$ $\cong Cl(p,q)\otimes\mathbb{R}(16)$ to isomorphisms of algebras with anti-involutions.

**1.7. Theorem**

$$(Cl(p+8,q),\alpha_\epsilon) \cong (Cl(p,q),\alpha_\epsilon)\otimes(Cl(8,0),\alpha_\epsilon),$$

$$(Cl(p,q+8),\alpha_\epsilon) \cong (Cl(p,q),\alpha_\epsilon)\otimes(Cl(0,8),\alpha_\epsilon).$$

*Proof*

We discuss the first isomorphism, the second one being provable in exactly the same way. By verification we show

$$(Cl(p+4,q),\alpha_\epsilon) \cong (Cl(p,q),\alpha_\epsilon)\otimes(Cl(4,0),\alpha_\epsilon).$$

Let $\{e_i\}$ and $\{f_j\}$ generate $Cl(p,q)$ and $Cl(4,0)$, respectively. Then $\{e_i\otimes a, 1\otimes f_j\}$ with $a = f_1f_2f_3f_4$ is a generating set for $Cl(p,q)\otimes Cl(4,0) \cong Cl(p+4,q)$. Since $\alpha_\epsilon(a) = a$ for $\epsilon = \pm 1$, we have

$$(\alpha_\epsilon\otimes\alpha_\epsilon)(e_i\otimes a) = \epsilon\,(e_i\otimes a),$$
$$(\alpha_\epsilon\otimes\alpha_\epsilon)(1\otimes f_j) = \epsilon\,(1\otimes f_j).$$

Hence $\alpha_\epsilon$ on $Cl(p+4,q)$ is $\alpha_\epsilon\otimes\alpha_\epsilon$ on $Cl(p,q)\otimes Cl(4,0)$. Repetition of this argument yields the required result.

□

**1.8. Theorem**

The scalar products on $S(p+8,q)$ and $S(p,q+8)$ with adjoint $\alpha_\epsilon$ are of the same type as the one on $S(p,q)$ with adjoint $\alpha_\epsilon$.

*Proof*

It has to be shown that $\alpha_\epsilon$ on $Cl(8,0) \cong Cl(0,8) \cong \mathbb{R}(16)$ is the adjoint of a symmetric product on $\mathbb{R}^{16}$, this being the case if and only if the dimension of the subspace of $\mathbb{R}(16)$ fixed by the adjoint is equal to $\frac{1}{2}\cdot 16(16+1) = 136$. We shall see. For the basis elements $e_1 e_2 \dots e_k$ of $Cl(p,q)$

$$\alpha_1 (e_1 \dots e_k) = e_1 \dots e_k \quad \text{iff} \quad k \equiv 0,1 \pmod 4$$
$$\alpha_{-1}(e_1 \dots e_k) = e_1 \dots e_k \quad \text{iff} \quad k \equiv 0,3 \pmod 4.$$

Consequently, if $n = 8$ the dimension of the subspace fixed by $\alpha_1$ and $\alpha_{-1}$ is (respectively)

$$\binom{8}{0} + \binom{8}{1} + \binom{8}{4} + \binom{8}{5} + \binom{8}{8} = 136$$
$$\binom{8}{0} + \binom{8}{3} + \binom{8}{4} + \binom{8}{7} + \binom{8}{8} = 136.$$

□

Now we are able to classify the scalar products on $S(p,q)$ with adjoint $\alpha_\epsilon$. Supposing $p \geq q$ we have

$$Cl(p,q) \cong Cl(p-q,0) \otimes \underbrace{Cl(1,1) \otimes \dots \otimes Cl(1,1)}_{q \text{ times}}.$$

According to Theorem 1.5 and its Corollary 1.6 $\alpha_\epsilon$ on $Cl(p,q)$ is determined by $\alpha_\epsilon$ ($\alpha_{-\epsilon}$) on $Cl(p-q,0)$ if $q$ is even (odd), the type of corresponding scalar product being the same (other) if the number of transitions $\alpha_1 \to \alpha_{-1}$ during $q$ times tensoring with $Cl(1,1)$ is even (odd). Hence the result depends on $q$ modulo 4. We begin with $\alpha_1$. Recall Table 1.4.1.

**$p \geq q$. Scalar products on $S(p,q)$ with adjoint $\alpha_1$**

| p−q / q | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | $\mathbb{R}_+$ | $\mathbb{C}_+$ | $\tilde{\overline{\mathbb{H}}}_+$ | $^2\overline{\mathbb{H}}_+$ | $\overline{\mathbb{H}}_+$ | $\mathbb{C}_-$ | $\mathbb{R}_-$ | $^2\mathbb{R}_+$ |
| 1 | $\mathbb{R}_+$ | $\overline{\mathbb{C}}_+$ | $\overline{\mathbb{H}}_+$ | $\overline{\mathbb{H}}_+ \times \overline{\mathbb{H}}_+$ | $\overline{\mathbb{H}}_+$ | $\overline{\mathbb{C}}_+$ | $\mathbb{R}_+$ | $\mathbb{R}_+ \times \mathbb{R}_+$ |
| 2 | $\mathbb{R}_-$ | $\mathbb{C}_-$ | $\overline{\mathbb{H}}_+$ | $^2\overline{\mathbb{H}}_+$ | $\tilde{\mathbb{H}}_+$ | $\mathbb{C}_+$ | $\mathbb{R}_+$ | $^2\mathbb{R}_-$ |
| 3 | $\mathbb{R}_-$ | $\overline{\mathbb{C}}_+$ | $\tilde{\overline{\mathbb{H}}}_+$ | $\tilde{\overline{\mathbb{H}}}_+ \times \tilde{\overline{\mathbb{H}}}_+$ | $\tilde{\overline{\mathbb{H}}}_+$ | $\overline{\mathbb{C}}_+$ | $\mathbb{R}_-$ | $\mathbb{R}_- \times \mathbb{R}_-$ |

Note that p-q has to be taken modulo 8 (Theorem 1.8). If $p \leq q$, needless to say, the starting-point is

$$Cl(p,q) \cong Cl(0,q-p) \otimes \underbrace{Cl(1,1) \otimes \ldots \otimes Cl(1,1)}_{p \text{ times}}.$$

Referring to Table 1.4.2 we give the result.

**$p \leq q$. Scalar products on $S(p,q)$ with adjoint $\alpha_1$**

| q−p / p | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | $\mathbb{R}_+$ | $\mathbb{R}_+ \times \mathbb{R}_+$ | $\mathbb{R}_+$ | $\overline{\mathbb{C}}_+$ | $\overline{\mathbb{H}}_+$ | $\overline{\mathbb{H}}_+ \times \overline{\mathbb{H}}_+$ | $\overline{\mathbb{H}}_+$ | $\overline{\mathbb{C}}_+$ |
| 1 | $\mathbb{R}_+$ | $^2\mathbb{R}_+$ | $\mathbb{R}_-$ | $\mathbb{C}_-$ | $\overline{\mathbb{H}}_+$ | $^2\overline{\mathbb{H}}_+$ | $\tilde{\mathbb{H}}_+$ | $\mathbb{C}_+$ |
| 2 | $\mathbb{R}_-$ | $\mathbb{R}_- \times \mathbb{R}_-$ | $\mathbb{R}_-$ | $\overline{\mathbb{C}}_+$ | $\tilde{\mathbb{H}}_+$ | $\tilde{\mathbb{H}}_+ \times \tilde{\mathbb{H}}_+$ | $\tilde{\mathbb{H}}_+$ | $\overline{\mathbb{C}}_+$ |
| 3 | $\mathbb{R}_-$ | $^2\mathbb{R}_-$ | $\mathbb{R}_+$ | $\mathbb{C}_+$ | $\tilde{\mathbb{H}}_+$ | $^2\overline{\mathbb{H}}_+$ | $\overline{\mathbb{H}}_+$ | $\mathbb{C}_-$ |

These tables are to be united. Compare column j of the first table with column 8-j of the second one. With the new variable p+q mod 8 their true combination emerges as follows.

Consequently, the group of automorphisms which preserve the scalar product $\phi_\epsilon$ is isomorphic to $\mathrm{Inv}_\epsilon(p,q)$.

For the classification of the invariance groups we need the following review of well-known Lie groups (cf. Porteous [6], table 11.53).

| *Scalar product* | *Invariance group* |
|---|---|
| $\mathbb{R}_+$ | $O(r,s)$ |
| $\mathbb{R}_-$ | $Sp(2r,\mathbb{R})$ |
| $\mathbb{C}_+$ | $O(r,\mathbb{C})$ |
| $\mathbb{C}_-$ | $Sp(2r,\mathbb{C})$ |
| $\overline{\mathbb{C}}_+$ | $U(r,s)$ |
| $\tilde{\mathbb{H}}_+$ | $O(r,\mathbb{H}) \cong SO^*(2r)$ |
| $\overline{\mathbb{H}}_+$ | $Sp(r,s) \cong SU^*(2r,2s)$ |
| ${}^2\mathbb{R}_+$ | $GL(r,\mathbb{R})$ |
| ${}^2\overline{\mathbb{H}}_+$ | $GL(r,\mathbb{H})$ |

**Comment**

By definition $SO^*(2r) = O(2r,\mathbb{C}) \cap Sp(2r,\overline{\mathbb{C}})$.

The isomorphism $O(r,\mathbb{H}) \cong SO^*(2r)$ is based on the unique decomposition of the $\tilde{\mathbb{H}}_+$-scalar product into two complex scalar products. Write $q_i \in \mathbb{H}^r$ as $q_i = a_i + jb_i$ where $a_i, b_i \in \mathbb{C}^{2r}$. Then

$$\tilde{q}_1{}^t q_2 = (a_1{}^t - b_1{}^t j)(a_2 + jb_2) = a_1{}^t a_2 + b_1{}^t b_2 + j(-\bar{b}_1{}^t a_2 + \bar{a}_1{}^t b_2) =$$

$$= (a_1{}^t b_1{}^t)\begin{pmatrix} I_r & 0 \\ 0 & I_r \end{pmatrix}\binom{a_2}{b_2} + j(\bar{a}_1{}^t \bar{b}_1{}^t)\begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}\binom{a_2}{b_2}.$$

On the other hand, let be given the $\overline{\mathbb{H}}_+$-scalar product $(q_1,q_2) \to \bar{q}_1{}^t I_{r,s} q_2$, $I_{r,s}$ being diagonal with r times +1 and s times -1. Then

$$\bar{q}_1{}^t I_{r,s} q_2 = (\bar{a}_1{}^t - \bar{b}_1{}^t j)\, I_{r,s}(a_2 + jb_2) =$$

$$= (\bar{a}_1{}^t + \bar{b}_1{}^t) \begin{pmatrix} I_{r,s} & 0 \\ 0 & I_{r,s} \end{pmatrix} \binom{a_2}{b_2} + j(a_1{}^t b_1{}^t) \begin{pmatrix} 0 & I_{r,s} \\ -I_{r,s} & 0 \end{pmatrix} \binom{a_2}{b_2}.$$

Hence $Sp(r,s) \cong U(2r,2s) \cap Sp(2r,2s;\mathbb{C}) \stackrel{def}{=} SU^*(2r,2s)$. Perhaps surprisingly we also find the general linear groups $GL(r,\mathbb{R})$ and $GL(r,\mathbb{H})$ among the invariance groups. The standard ${}^2\mathbb{R}_+$-scalar product is

$$<x,y> = (x_1{}^t y_2, x_2{}^t y_1), \quad x = (x_1,x_2) \text{ and } y = (y_1,y_2) \in {}^2\mathbb{R}^r.$$

For any $(a,b) \in {}^2\mathbb{R}(r)$ the following relation holds

$$<(a,b)x,y> = <x,(b^t,a^t)y>.$$

Since $(b^t,a^t)(a,b) = (1,1)$ implies $b = (a^t)^{-1}$, the invariance group is isomorphic to $GL(r,\mathbb{R})$. With respect to the standard ${}^2\overline{\mathbb{H}}_+$-scalar product the adjoint of $(a,b) \in {}^2\mathbb{H}(r)$ is $(\bar{b}^t,\bar{a}^t)$. The invariance group consists of the elements $(a,(\bar{a}^t)^{-1})$ with $a \in GL(r,\mathbb{H})$.

□

The scalar products $\phi_{-1}$ and $\phi_1$ are positive definite on S(n,0) and S(0,n), respectively. In all other except the one-dimensional cases $\phi_\epsilon$ is neutral on S(p,q). This follows from the existence of $r \in Cl(p,q)$ with $N_\epsilon(r) = -1$. The result is the following classification of invariance groups (cf. Tables 1.2 and 1.9).

1.10.1. Table - Invariance groups $\mathrm{Inv}_1(p,q)$, $k = 2^{[n/2]-1}$

| p = 0 | q-p / p+q | 0,2 | 1 | 3,7 | 4,6 | 5 |
|---|---|---|---|---|---|---|
| O(2k) | 0 | O(k,k) | | | $SO^*(2k)$ | |
| $^2O(2k)$ | 1 | | $^2O(k,k)$ | $O(2k,\mathbb{C})$ | | $^2SO^*(2k)$ |
| O(2k) | 2 | O(k,k) | | | $SO^*(2k)$ | |
| U(2k) | 3 | | $GL(2k,\mathbb{R})$ | U(k,k) | | $GL(k,\mathbb{H})$ |
| $SU^*(2k)$ | 4 | $Sp(2k,\mathbb{R})$ | | | $SU^*(k,k)$ | |
| $^2SU^*(2k)$ | 5 | | $^2Sp(2k,\mathbb{R})$ | $Sp(2k,\mathbb{C})$ | | $^2SU^*(k,k)$ |
| $SU^*(2k)$ | 6 | $Sp(2k,\mathbb{R})$ | | | $SU^*(k,k)$ | |
| U(2k) | 7 | | $GL(2k,\mathbb{R})$ | U(k,k) | | $GL(k,\mathbb{H})$ |

1.10.2. Table - Invariance groups $\mathrm{Inv}_{-1}(p,q)$, $k = 2^{[n/2]-1}$

| q = 0 | q-p / p+q | 0,2 | 1 | 3,7 | 4,6 | 5 |
|---|---|---|---|---|---|---|
| O(2k) | 0 | O(k,k) | | | $SO^*(2k)$ | |
| *U(2k)* | *1* | | $GL(2k,\mathbb{R})$ | *U(k,k)* | | $GL(k,\mathbb{H})$ |
| $SU^*(2k)$ | 2 | $Sp(2k,\mathbb{R})$ | | | $SU^*(k,k)$ | |
| $^2SU^*(2k)$ | 3 | | $^2Sp(2k,\mathbb{R})$ | $Sp(2k,\mathbb{C})$ | | $^2SU^*(k,k)$ |
| $SU^*(2k)$ | 4 | $Sp(2k,\mathbb{R})$ | | | $SU^*(k,k)$ | |
| U(2k) | 5 | | $GL(2k,\mathbb{R})$ | U(k,k) | | $GL(k,\mathbb{H})$ |
| O(2k) | 6 | O(k,k) | | | $SO^*(2k)$ | |
| $^2O(2k)$ | 7 | | $^2O(k,k)$ | $O(2k,\mathbb{C})$ | | $^2SO^*(2k)$ |

**Spin groups**

The largest group contained in the Clifford algebra $Cl(p,q)$ is the set of invertible elements, to be denoted by $Cl^*(p,q)$. In fact, $Cl(p,q)$ may be regarded as the Lie algebra of $Cl^*(p,q)$, the Lie product being defined as $[r,s] = rs - sr$. We briefly review the subgroups of $Cl^*(p,q)$ covering the orthogonal group $O(p,q)$ and its familiar subgroups $SO(p,q)$ and $O^{\uparrow}(p,q)$. As usual, we start with the *Lipschitz group*

$$\Gamma(p,q) = \left\{ r \in Cl^*(p,q) \colon rV(p,q)\,\hat{r}^{-1} \subset V(p,q) \right\}.$$

For any $r \in \Gamma(p,q)$ the transformation

$$\rho_r \colon V(p,q) \to V(p,q), \quad \rho_r(v) = rv\hat{r}^{-1}$$

belongs to the orthogonal group $O(p,q)$.
$\Gamma(p,q)$ is generated by the set of non-isotropic vectors $v \in V(p,q)$, $\rho_v$ being the reflection in the orthogonal complement of $v$. With $r_1$ and $r_2 \in \Gamma(p,q)$ the identity $\rho_{r_1} = \rho_{r_2}$ is valid if and only if $r_1 = \lambda r_2$ for some $\lambda \in GL(1;\mathbb{R})$. To summarize, we have the exact sequence of groups

$$1 \to GL(1,\mathbb{R}) \to \Gamma(p,q) \xrightarrow{\rho} O(p,q) \to 1.$$

The norm $N_\epsilon$ is real valued on $\Gamma(p,q)$. For let $r = v_1 v_2 \dots v_k \in \Gamma(p,q)$ with $v_j \in V(p,q)$ non-isotropic. Then

$$N_\epsilon(r) = (-\epsilon)^k Q(v_1)Q(v_2)\dots Q(v_k).$$

We use the norm $N_{-1}$ to construct the two-fold covering group

$$Pin(p,q) = \left\{ r \in \Gamma(p,q) \colon N_{-1}(r) \in S^o \right\},$$

where $S^o = \{\pm 1\}$, which fits into the exact sequence

$$1 \to S^o \to Pin(p,q) \xrightarrow{\rho} O(p,q) \to 1.$$

Define the even subgroup

$$\mathrm{Spin}(p,q) = \mathrm{Pin}(p,q) \cap \mathrm{Cl}^{o}(p,q).$$

Since $r \in \mathrm{Spin}(p,q)$ is the product of an even number of non-isotropic vectors, $\rho_r$ is a special orthogonal transformation (even number of hyperplane reflections) for any $r \in \mathrm{Spin}(p,q)$. Hence we have the exact sequence

$$1 \to S^{o} \to \mathrm{Spin}(p,q) \xrightarrow{\rho} SO(p,q) \to 1.$$

Another natural subgroup of $\mathrm{Pin}(p,q)$ is

$$\mathrm{Pin}^{+}(p,q) = \left\{ r \in \mathrm{Pin}(p,q) \colon N_{-1}(r) = 1 \right\}.$$

Let $r = v_1 v_2 \ldots v_k \in \mathrm{Pin}^{+}(p,q)$. Then $N_{-1}(r) = Q(v_1)Q(v_2)\ldots Q(v_k) = 1$ implies that r contains an even number of negative vectors ($Q(v_j) < 0$). Hence $\rho_r$ preserves the orientation of the maximal negative subspaces of $V(p,q)$ for any $r \in \mathrm{Pin}^{+}(p,q)$. Obviously, $\mathrm{Pin}^{+}(p,0) = \mathrm{Pin}(p,0)$. Now we have the exact sequence

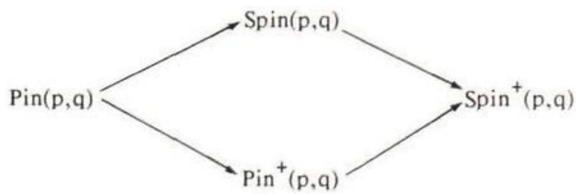
$$1 \to S^{o} \to \mathrm{Pin}^{+}(p,q) \xrightarrow{\rho} O^{\uparrow}(p,q) \to 1.$$

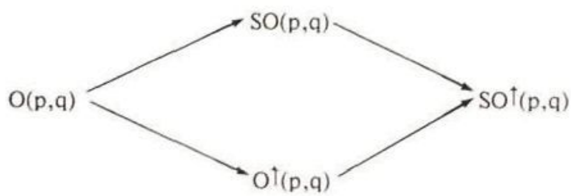
The identity component $SO^{\uparrow}(p,q)$ of the orthogonal group is doubly covered by the group

$$\mathrm{Spin}^{+}(p,q) = \mathrm{Pin}^{+}(p,q) \cap \mathrm{Cl}^{o}(p,q).$$

For any $r \in \mathrm{Spin}^{+}(p,q)$ the transformation $\rho_r$ preserves the orientations of the maximal positive and negative subspaces of $V(p,q)$. The following diagram reviews the groups discussed above, where $\nearrow$ and $\searrow$ symbolize the parity condition, imposed on the number of vectors and the number of negative vectors, respectively.

**1.11. Diagram - Orthogonal groups and their coverings**




By its very definition $\mathrm{Pin}^+(p,q)$ is a subgroup of the invariance group $\mathrm{Inv}_{-1}(p,q)$. With regard to $\mathrm{Spin}^+(p,q)$ we can say more. Since the norms $N_1$ and $N_{-1}$ coincide on the even subgroup, $r \in \mathrm{Spin}^+(p,q)$ implies $N_\epsilon(r) = 1$ for both $\epsilon = 1$ and $\epsilon = -1$. Consequently,

$$\mathrm{Spin}^+(p,q) \subset \mathrm{Inv}_\epsilon(p,q) \quad \text{for } \epsilon = \pm 1.$$

The isomorphisms

$$Cl^0(p+1,q) \cong Cl(p,q) \quad \text{and} \quad Cl^0(p,q+1) \cong Cl(q,p)$$

respect $\alpha_{-1}$ but not $\alpha_1$. To determine the group $\mathrm{Spin}^+(p,q)$ we may therefore use the alternative embeddings

$$\mathrm{Spin}^+(p+1,q) \subset \mathrm{Inv}_{-1}(p,q),$$

$$\mathrm{Spin}^+(p,q+1) \subset \mathrm{Inv}_{-1}(q,p).$$

The following examples show that it is possible to achieve immediate success whenever the dimensions are low enough.

**1.12. Example**

The Clifford algebra $Cl(2,1)$ is isomorphic to $\mathbb{C}(2)$. According to Table 1.10 the invariance groups are

$$Inv_1(2,1) \cong U(1,1),$$
$$Inv_{-1}(2,1) \cong Sp(2,\mathbb{C}) = SL(2,\mathbb{C}).$$

The group $Spin^+(2,1)$ lies in the intersection $SU(1,1)$. Both dimensions equal 3 and therefore

$$Spin^+(2,1) \cong SU(1,1).$$

Alternatively, start from the isomorphism $Cl^0(2,1) \cong Cl(1,1)$. This algebra is isomorphic to $\mathbb{R}(2)$ and

$$Inv_{-1}(1,1) \cong Sp(2,\mathbb{R}) = SL(2,\mathbb{R}).$$

Again this is a three-dimensional group, so

$$Spin^+(2,1) \cong SL(2,\mathbb{R}).$$

□

**1.13. Example**

Take the Clifford algebra $Cl(3,1) \cong \mathbb{H}(2)$.
Its invariance groups are

$$Inv_1(3,1) \cong SU^*(2,2) \cong Sp(1,1),$$
$$Inv_{-1}(3,1) \cong SU^*(2,2) \cong Sp(1,1).$$

Since $\dim Sp(1,1) = 10$ we can say no more than

$$\mathrm{Spin}^{+}(3,1) \subset \mathrm{Sp}(1,1).$$

Proceeding in the alternative way, we need

$$\mathrm{Inv}_{-1}(2,1) \cong \mathrm{SL}(2,\mathbb{C}).$$

But this is a six-dimensional group, so

$$\mathrm{Spin}^{+}(3,1) \cong \mathrm{SL}(2,\mathbb{C}). \qquad \square$$

Paravectors

The vectors from $V(p,q)$, orthogonally transformed by means of the group $\mathrm{Pin}(p,q)$, are not endowed with a real part. In this sense $V(2,0)$ deviates from the standard complex plane, an indigence which is not acceptable for it is especially the Möbius geometry which we are intended to generalize. For that reason we also consider the vector space

$$V_{\pi}(p,q) \overset{\mathrm{def}}{=} \mathbb{R} \oplus V(p,q)$$

in the Clifford algebra $Cl(p,q)$.

The elements of $V_{\pi}(p,q)$, being the sum of a scalar and a vector, are called *paravectors*. For any $w = v_0 + v \in V_{\pi}(p,q)$ we set

$$Q(w) = N_{-1}(w) = v_0^2 + Q(v).$$

Provided with this quadratic form we have

$$V_{\pi}(p,q) \cong V(p+1,q).$$

Naturally enough, we introduce the analogue of the Lipschitz group

$$\Gamma_{\pi}(p,q) = \{s \in Cl^{*}(p,q):\ sV_{\pi}(p,q)\hat{s}^{-1} \subset V_{\pi}(p,q)\}.$$

Though $\Gamma(p,q) \subset \Gamma_{\pi}(p,q)$ there is a substantial difference between the groups $\Gamma(p,q)$ and $\Gamma_{\pi}(p,q)$.

For any $s \in \Gamma_\pi(p,q)$ the transformation

$$\rho_s:\ V_\pi(p,q) \to V_\pi(p,q),\ \rho_s(w) = sw\hat{s}^{-1}$$

belongs to the *special* orthogonal group of $V_\pi(p,q)$, which is isomorphic to $SO(p+1,q)$. Employing the algebra isomorphism

$$Cl(p,q) \cong Cl^o(p+1,q)$$

one easily ascertains the group isomorphism

$$\Gamma_\pi(p,q) \cong \Gamma^o(p+1,q).$$

Any $s \in \Gamma_\pi(p,q)$ is a product $s = w_1 w_2 \dots w_k$ of non-isotropic paravectors $w_j \in V_\pi(p,q)$, $\rho_s$ being special whether k is even or odd. A double covering of $SO(p+1,q)$ is given by the group

$$Pin_\pi(p,q) = \{s \in \Gamma_\pi(p,q):\ N_{-1}(s) \in S^o\}.$$

The identity component $SO^\uparrow(p+1,q)$ is doubly covered by the subgroup

$$Pin^+_\pi(p,q) = \{s \in \Gamma_\pi(p,q):\ N_{-1}(s) = 1\}.$$

Ascertain the isomorphisms

$$Pin^{(+)}_\pi(p,q) \cong Spin^{(+)}(p+1,q).$$

The complete orthogonal group $O(p+1,q)$ may be represented by taking not only the transformations

$$w \to \rho_s(w),\ \text{ but also }\ w \to \rho_s(-\hat{w}),$$

$w \in V_\pi(p,q)$ and $s \in Pin_\pi(p,q)$.

invertible entry in question is placed in the position left above. Such permutations amount to left or right multiplication of the matrix g by $e_{n+2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{Twist}(p,q)$.

□

In the definite cases q = 0 we know that $r\tilde{r} = 0$ if and only if r = 0. Condition (1) combined with the theorem above then affords the property

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Twist}(n) \Rightarrow a,b,c,d \in \Gamma(n) \cup \{0\} \text{ (cf. Ahlfors [1]).}$$

Notably, condition (6) $ad^* - bc^* = \pm 1$ implies that at least two entries of any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Twist}(n)$ are invertible. In the indefinite cases this is not true. For example, the matrix

$$g = \frac{1}{2} \begin{pmatrix} v & 1 \\ 1 & -v \end{pmatrix} \begin{pmatrix} w & -1 \\ 1 & -w \end{pmatrix} \in \mathrm{Twist}(p,q), \text{ where}$$

$Q(v) = -1$, $Q(w) = 1$ and $B(v,w) = 0$, has four non-invertible entries.

## Conformal spaces

A first step towards the definition of conformal spaces is the consideration of the generalized stereographic projection. Written homogeneously, this projection maps V(p,q) onto some locus in PV(p+1,q+1), the *projective* space associated with V(p+1,q+1). The point in PV(p+1,q+1) represented by the punctured line $\lambda\underline{v}$ in V(p+1,q+1) will be denoted by $<\underline{v}>$ henceforth.

It is readily seen that the map

$$v \to s(v) = \begin{bmatrix} v & Q(v) \\ 1 & \hat{v} \end{bmatrix}$$

maps any vector $v \in V(p,q)$ to a vector lying on the punctured null-cone

$$\dot{C}(p+1,q+1) \stackrel{\text{def}}{=} \{\underline{0} \neq \underline{v} \in V(p+1,q+1):\ Q(\underline{v}) = 0\}.$$

Consequently, the induced map

$$v \to \langle s(v)\rangle,$$

well-defined for any $v \in V(p,q)$, has its image on the conic section or projective cone

$$PC(p+1,q+1) \overset{\text{def}}{=} \{\langle \underline{v}\rangle \in PV(p+1,q+1):\ Q(\underline{v}) = 0\}.$$

This generalized stereographic projection is injective but not surjective. More precisely, the stereographic projection

$$v \to \langle s(v)\rangle$$

maps $V(p,q)$ onto the part

$$\{\langle \underline{v}\rangle \in PC(p+1,q+1):\ v_{n+1} \neq v_{n+2}\}.$$

Given $\langle \underline{w}\rangle = \langle s(v)\rangle$ for some $v \in V(p,q)$ one easily verifies $v = (w_{n+2} - w_{n+1})^{-1}\, w$. Suggestively, the hyperplane given by the equation $v_{n+1} - v_{n+2} = 0$ is called the hyperplane at infinity. The pole of this hyperplane with respect to $PC(p+1,q+1)$ is $\langle e_{n+1} + e_{n+2}\rangle$.

The aim of this chapter is to develop a formalism, by which the stereographic projection may be extended to some completion of $V(p,q)$ such that the points at infinity on $PC(p+1,q+1)$ are no longer exclusive.

To understand the geometrical meaning of such an extension, let us investigate the intersection of an arbitrary hyperplane in $PV(p+1,q+1)$ with $\langle s(V(p,q))\rangle \subset PC(p+1,q+1)$. The intersection of $\langle s(V(p,q))\rangle$ with the polar hyperplane of $\langle \underline{a}\rangle \in PV(p+1,q+1)$ is given by the equation $B(s(v),\underline{a}) = 0$, i.e., by the equation

$$2B(v,a) + (a_{n+1} - a_{n+2})\, Q(v) - (a_{n+1} + a_{n+2}) = 0 \quad (*).$$

Distinguish between the cases

*(i)* $a_{n+1} - a_{n+2} \neq 0$.
Normalize the pole such that $a_{n+1} - a_{n+2} = 1$. Then we have $a_{n+1} + a_{n+2} = a_{n+1}^2 - a_{n+2}^2 =$ $= Q(\underline{a}) - Q(a)$. This reduces equation (∗) to

$$Q(v + a) = Q(\underline{a}).$$

This is the equation of a *pseudo-sphere* or a *null-cone* in V(p,q) according as $Q(\underline{a}) \neq 0$ or $Q(\underline{a}) = 0$.

*(ii)* $a_{n+1} - a_{n+2} = 0$.
If $a \neq 0$ equation (∗) is reduced to

$$B(v,a) = \frac{1}{2}(a_{n+1} + a_{n+2}),$$

describing a hyperplane in V(p,q).
If $a = 0$ we are concerned with the hyperplane at infinity. Equation (∗) $0.Q(v) =$ $= a_{n+1} + a_{n+2}$ ($\neq 0$) shows that the intersection with <s(V(p,q))> is empty, a fact already known to us. As a remedy, multiply equation (∗) by $Q(v^{-1})$. Then the hyperplane at infinity corresponds with the "locus" $Q(v^{-1}) = 0$ which, tentatively, has to be regarded as a *cone at infinity* added to V(p,q).

To realize this conformal completion of V(p,q), thus bijectively corresponding to the *compact* manifold PC(p+1,q+1), we first introduce the *pre-conformal* space pre-$\overline{V}$(p,q) which may be regarded as the column set of the matrix group Twist(p,q).

According to the modulo (1,1) periodicity of Clifford algebras we may identify Cl(p+1,q+1) with the algebra of right Cl(p,q)-linear endomorphisms of the right Cl(p,q)-module

$$M(p,q) \stackrel{\text{def}}{=} Cl(p,q) \otimes \mathbb{R}^2.$$

Obviously, this module may be realized in the Clifford algebra Cl(p+1,q+1) as the left ideal which is generated by the idempotent $e = \frac{1}{2}(e(1) + e_{n+1}e_{n+2}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

In the sequel we thus identify the element $x = \begin{pmatrix} x_1 & 0 \\ x_2 & 0 \end{pmatrix} \in Cl(p+1,q+1)$ with $x \approx \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in M(p,q)$.

Projection of Twist(p,q) onto M(p,q) affords

### 2.3. Definition

$$\text{pre-}\overline{V}(p,q) = \text{Twist}(p,q)\, e.$$

□

Hence $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \text{pre-}\overline{V}(p,q)$ if and only if $\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \in \text{Twist}(p,q)$ for some $y_1, y_2 \in Cl(p,q)$. According to definition 2.1 it is necessary that $x_i\tilde{x}_i \in \mathbb{R}$ and $x_1\tilde{x}_2 \in$ $\in V(p,q)$. Moreover, if $x_i$ is invertible, then $x_i \in \Gamma(p,q)$ (Theorem 2.2).

The following theorem shows that the pre-conformal space $\text{pre-}\overline{V}(p,q)$ contains a copy of V(p,q).

### 2.4. Theorem

For any $v \in V(p,q)$, $x = \begin{pmatrix} v \\ 1 \end{pmatrix} \in \text{pre-}\overline{V}(p,q)$.

*Proof*

Twist(p,q) is generated by the set of non-isotropic vectors $\underline{v} = \begin{pmatrix} v & \lambda \\ \mu & \tilde{v} \end{pmatrix} \in V(p+1,q+1)$.

Imposing $\mu = 1$ we thus have $\underline{v} = \begin{bmatrix} v & Q(v)^{-1} \\ 1 & \tilde{v} \end{bmatrix} \in \text{Twist}(p,q)$.

Projection onto M(p,q) yields the required result $x = \underline{v}e$.

□

We construct a map from $\text{pre-}\overline{V}(p,q)$ onto the punctured null-cone $\dot{C}(p+1,q+1)$. Let be given any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Twist}(p,q)$. We know that $g\underline{v}g^* \in V(p+1,q+1)$ for all $\underline{v} \in V(p+1,q+1)$. In particular this is true for $\underline{v}_{\infty} = \frac{1}{2}(e_{n+1} + e_{n+2}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. The result $g\underline{v}_{\infty}g^* = \begin{pmatrix} a\tilde{c} & a\tilde{a} \\ c\tilde{c} & c\tilde{a} \end{pmatrix}$ apparently depends only on the first column of $g \in \text{Twist}(p,q)$, *i.e., on the projection* $ge \in \text{pre-}\overline{V}(p,q)$. *This suits nicely to our purpose. For any* $x = ge \in$ $\in \text{pre-}\overline{V}(p,q)$ we are invited to define

$$S(x) = g v_{-\infty} g^{*}.$$

Since $v_{-\infty} \in \dot{C}(p+1,q+1)$ and $\rho_g$ acts transitively on the punctured null-cone, the map

$$S:\ \text{pre-}\overline{V}(p,q) \to \dot{C}(p+1,q+1), \quad S(x) = \begin{pmatrix} x_1\tilde{x}_2 & x_1\tilde{x}_1 \\ x_2\tilde{x}_2 & x_2\tilde{x}_1 \end{pmatrix}$$

is a surjection. However, it is obvious that S is *not* injective.

Recall the map

$$s:\ V(p,q) \to \dot{C}(p+1,q+1), \quad s(v) = \begin{pmatrix} v & Q(v) \\ 1 & \tilde{v} \end{pmatrix}.$$

Identify $v \in V(p,q)$ with $x = \binom{v}{1} \in \text{pre-}\overline{V}(p,q)$. Then the equation $S(x) = s(v)$ shows that the restriction of S to V(p,q) is exactly the map s. We have seen that the stereographic projection

$$V(p,q) \to PC(p+1,q+1), \quad v \to \langle s(v) \rangle$$

is injective but not surjective. For that reason we proceed with the extended map

$$\text{pre-}\overline{V}(p,q) \to PC(p+1,q+1), \quad x \to \langle S(x) \rangle$$

which is surjective but not injective.

Naturally enough, we provide $\text{pre-}\overline{V}(p,q)$ with the equivalence relation $x \sim y$ iff $\langle S(x) \rangle = \langle S(y) \rangle$.

**2.5. Definition**

The set of classes $\langle x \rangle$ thus obtained, denoted by $\overline{V}(p,q)$, is what we call the *conformal* space belonging to V(p,q).

□

Now it is obvious that the map

$$\overline{V}(p,q) \to PC(p+1,q+1), \quad \langle x\rangle \to \langle S(x)\rangle$$

is a bijection. The considerations above are reflected in the following commutative

**2.6. Diagram**

$$\begin{array}{ccc} \text{pre-}\overline{V}(p,q) & \overset{S}{\to} & \dot{C}(p+1,q+1) \\ \text{projection} \downarrow & & \downarrow \text{projection} \\ \overline{V}(p,q) & \underset{\text{bijection}}{\to} & PC(p+1,q+1) \end{array}$$

□

Let us conceive $\overline{V}(p,q)$ in terms of the group Twist(p,q), as we did with regard to pre-$\overline{V}(p,q)$. Suppose $x = g_1 e$, $y = g_2 e \in$ pre-$\overline{V}(p,q)$ with $x \sim y$. We gather

$$\langle \rho_{g_1}(\underset{\sim}{v}_{-\infty})\rangle = \langle \rho_{g_2}(\underset{\sim}{v}_{-\infty})\rangle \Longleftrightarrow \rho_{g_1}\langle(\underset{\sim}{v}_{-\infty})\rangle = \rho_{g_2}\langle(\underset{\sim}{v}_{-\infty})\rangle \Longleftrightarrow$$

$$\Longleftrightarrow \rho_{g_1}\rho_{g_2}^{-1}\langle \underset{\sim}{v}_{-\infty}\rangle = \langle \underset{\sim}{v}_{-\infty}\rangle \Longleftrightarrow \rho_{g_1 g_2^{-1}}\langle \underset{\sim}{v}_{-\infty}\rangle = \langle \underset{\sim}{v}_{-\infty}\rangle.$$

Hence $g_1 g_2^{-1} \in U$, where U denotes the subgroup of Twist(p,q) which represents the stabilizer of $\langle \underset{\sim}{v}_{-\infty}\rangle \in PC(p+1,q+1)$. To determine U, impose $\rho_g \langle \underset{\sim}{v}_{-\infty}\rangle = \langle \underset{\sim}{v}_{-\infty}\rangle$ for arbitrary $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in$ Twist (p,q). The resulting equation $\langle \begin{pmatrix} a\tilde{c} & a\tilde{a} \\ c\tilde{c} & c\tilde{a} \end{pmatrix} \rangle = \langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rangle$ is equivalent to $c = 0$. Consequently, U is the subgroup of *upper triangular* matrices in Twist(p,q). These considerations amount to the following

**2.7. Theorem**

$$\overline{V}(p,q) = (\text{Twist}(p,q) \,/\, U)\, e.$$

□

More honestly, there is a one-to-one correspondence between PC(p+1,q+1) and Twist(p,q) / U. But since $\rho_g(\underset{\sim}{v}_{-\infty})$ depends only on the first column of $g \in$ Twist(p,q), the projection of the factor space Twist(p,q)/U onto the module M(p,q) does not interfere with that bijection. The following characterization of the equivalence relation on pre-$\overline{V}(p,q)$ affording $\overline{V}(p,q)$ is a very useful one.

**2.8. Corollary**

For any pair $x,y \in$ pre-$\overline{V}(p,q)$

$$x \sim y \Longleftrightarrow xr = y \quad \text{for some } r \in \Gamma(p,q).$$

*Proof*

Let be given any $x \in$ pre-$\overline{V}(p,q)$, i.e., any $x = ge$ with $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Twist}(p,q)$. Then $y \in$ pre-$\overline{V}(p,q)$ is equivalent to x if and only if $y = (gu)e$ for some $u = \begin{bmatrix} r & s \\ 0 & \pm(r^*)^{-1} \end{bmatrix} \in U$. We have seen that $r \in \Gamma(p,q)$, necessarily. Hence indeed we have $y = \begin{pmatrix} a \\ c \end{pmatrix} r$ with $r \in \Gamma(p,q)$.

□

This brings us to the following geometrical characterization of the conformal space $\overline{V}(p,q)$. Let $<x> \in \overline{V}(p,q)$ be represented by $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in$ pre-$\overline{V}(p,q)$ such that

(i) $x_2\tilde{x}_2 \neq 0 \qquad (\Longleftrightarrow x_2 \in \Gamma(p,q))$.

We have $x \sim \begin{pmatrix} v \\ 1 \end{pmatrix}$ with $v = x_1x_2^{-1} \in V(p,q)$ in a unique way. Hence it seems natural to identify this part of $\overline{V}(p,q)$ with $V(p,q)$ itself. As we have seen, the restriction of the map S to this part yields the stereographic projection: $<S(x)> = <s(v)>$.

(ii) $x_2\tilde{x}_2 = 0$ and $x_1\tilde{x}_1 \neq 0 \quad (x_1 \in \Gamma(p,q))$.

Here we have $x \sim \begin{pmatrix} 1 \\ w \end{pmatrix}$, again uniquely, where $w = x_2x_1^{-1} \in V(p,q)$ lies on the null-cone $C(p,q)$. This part may be regarded as the image of $C(p,q)$ under the transformation $w \to w^{-1}$, lying on the so-called cone at infinity (mentally added to $V(p,q)$).

(iii) $x_1\tilde{x}_1 = 0$ and $x_2\tilde{x}_2 = 0$.

This part may be viewed as the intersection of the null-cone C(p,q) and the cone at infinity. If $p = 0$ or $q = 0$ this part is empty. Otherwise, x represents the point at infinity of the null-line $\lambda x_1\tilde{x}_2$.

In the definite cases $p = 0$ or $q = 0$ the cone at infinity (like all cones) contains only one point. Only in those cases the conformal compactification is a one-point compactification.

#### Conformal transformations

The group O(p+1,q+1) acts transitively on the punctured null-cone $\dot{C}(p+1,q+1)$ and thereby on the projective manifold PC(p+1,q+1). Referring to the bijection between $\overline{V}(p,q)$ and PC(p+1,q+1), induced by the map

$$S: \text{pre-}\overline{V}(p,q) \to \dot{C}(p+1,q+1),\ S(x) = \begin{pmatrix} x_1\tilde{x}_2 & x_1\tilde{x}_1 \\ x_2\tilde{x}_2 & x_2\tilde{x}_1 \end{pmatrix},$$

we may ask for the corresponding transitive action of O(p+1,q+1) on the conformal space $\overline{V}(p,q)$.

For any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Twist}(p,q)$ the action

$$\underline{v} \to \rho_g(\underline{v}) \quad \text{on } \dot{C}(p+1,q+1)$$

induces the action

$$\langle \underline{v} \rangle \to \rho_g \langle \underline{v} \rangle \overset{\text{def}}{=} \langle \rho_g(\underline{v}) \rangle \quad \text{on } PC(p+1,q+1).$$

On the other hand, let $\lambda_g$ denote the transformation of pre-$\overline{V}(p,q)$, trivially defined by left multiplication

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \lambda_g(x) \overset{\text{def}}{=} gx = \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix}.$$

This action of O(p+1,q+1) on pre-$\overline{V}(p,q)$, easily seen to be transitive, induces the action

$$\langle x\rangle \to \lambda_g\langle(x)\rangle \overset{\text{def}}{=} \langle\lambda_g(x)\rangle \quad \text{on } \overline{V}(p,q).$$

In fact, this representation of $O(p+1,q+1)$ on $\overline{V}(p,q)$ agrees with the representation of left translations on the (left) factor space $\mathrm{Twist}(p,q)/U$.

For any $g \in \mathrm{Twist}(p,q)$ the transformations $\rho_g$ and $\lambda_g$ fit into the following commutative

**2.9. Diagram**

$$\begin{array}{ccc} \overline{V}(p,q) & \xrightarrow{\lambda_g} & \overline{V}(p,q) \\ \text{bijection} \downarrow & & \downarrow \text{bijection} \\ PC(p+1,q+1) & \xrightarrow[\rho_g]{} & PC(p+1,q+1) \end{array}$$

□

Now let us consider the restriction of $\lambda_g$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Twist}(p,q)$, to the vector space $V(p,q)$. Identify $v \in V(p,q)$ with $\langle x\rangle \in \overline{V}(p,q)$ represented by $x = \begin{pmatrix} v \\ 1 \end{pmatrix} \in \text{pre-}\overline{V}(p,q)$. Assume that the image $\langle\lambda_g(x)\rangle$ represented by $\lambda_g(x) = \begin{pmatrix} av + b \\ cv + d \end{pmatrix} \in \text{pre-}\overline{V}(p,q)$ does not lie on the cone at infinity, i.e., assume that $cv + d$ is invertible, i.e., assume that $cv + d \in$ $\in \Gamma(p,q)$. Then we extract the *nonlinear* transformation in $V(p,q)$, given by

$$v \to \mu_g(v) = (av + b)(cv + d)^{-1}$$

by choosing the unique representative

$$\lambda_g(x) \sim \begin{bmatrix} \mu_g(v) \\ 1 \end{bmatrix}.$$

It has to be emphasized that $\mu_g$, called a (generalized) *Möbius* transformation, is not globally defined on $V(p,q)$ unless $c = 0$. More precisely, the locus of singularities, given by the equation $N_{-1}(cv + d) = 0$, is a null-cone or a hyperplane with isotropic normal direction according as $N_{-1}(c) \neq 0$ or $N_{-1}(c) = 0$ $(c \neq 0)$. The open subset of $V(p,q)$, obtained by leaving out that singular locus, will be denoted by $\mathrm{Dom}(\mu_g)$ henceforth.

We provide the manifold V(p,q) with the *pseudo-metric*

$$(ds)^2 = Q(dv),$$

classically regarded as the squared pseudo-length of the infinitesimal vector

$$dv = (dv_1)\, e_1 + \dots + (dv_n)\, e_n \in V(p,q).$$

With respect to this pseudo-metric $\mu_g$ is a *conformal* transformation. We prove

**2.10. Theorem**

For any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Twist}(p,q)$ the induced Möbius transformation

$$\mu_g\colon\ \mathrm{Dom}(\mu_g) \to V(p,q), \quad \mu_g(v) = (av + b)(cv + d)^{-1}$$

respects the pseudo-metric $(ds)^2 = Q(dv)$ up to a (non-zero) magnification factor, according to the equation

$$Q(d\mu_g(v)) = N_{-1}^{-2}(cv + d)\ Q(dv).$$

*Proof*

Given any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Twist}(p,q)$, we draw attention to the equations

$$a^*c = c^*a$$
$$b^*d = d^*b$$
$$a^*d - c^*b = \pm 1,$$

which follow from the normalization $N_{-1}(g) = \pm\, e(1)$.
To express $\mu_g(w) - \mu_g(v)$ in terms of the difference $w - v$, we gather

$$\begin{aligned}\mu_g(w) - \mu_g(v) = \mu_g^*(w) - \mu_g(v) &= \\ &= (wc^* + d^*)^{-1}(wa^* + b^*) - (av + b)(cv + d)^{-1} = \\ &= (wc^* + d^*)^{-1}[(wa^* + b^*)(cv + d) - (wc^* + d^*)(av + b)]\,(cv + d)^{-1} = \\ &= \pm\,(wc^* + d^*)^{-1}(w - v)(cv + d)^{-1}.\end{aligned}$$

Infinitesimally, $d\mu_g(v) = \pm (vc^* + d^*)^{-1}\, dv\, (cv + d)^{-1}$.
Subjecting both sides to the map $N_{-1}$ we arrive at the required result

$$Q(d\mu_g(v)) = N_{-1}^{-2}(cv + d)\, Q(dv).$$

□

It is clear that the Möbius transformations preserve pseudo-angles. Concerning the indefinite cases $pq \neq 0$ it is more significant that $\mu_g$ preserves the *null-structure* of $V(p,q)$, i.e., that any Möbius transformation respects the equation $(ds)^2 = 0$.
To put this invariance in a wider context, we co-ordinate the $\mu_g$-action on $\mathrm{Dom}(\mu_g)$ and the $\rho_g$-action on $\langle s(\mathrm{Dom}(\mu_g))\rangle$. For any $g \in \mathrm{Twist}(p,q)$ the orthogonal transformation $\rho_g$ leaves invariant the set of hyperplanes in $PV(p+1,q+1)$ with isotropic pole.
As we have seen (cf. page 30), any such hyperplane (except the one at infinity) corresponds to a null-cone in $V(p,q)$, possibly degenerated into a hyperplane with isotropic normal direction. Consequently, the induced Möbius transformation $\mu_g$ carries "null-cones" into "null-cones". A similar reasoning shows that $\mu_g$ carries "pseudo-spheres" into "pseudo-spheres" (pseudo-spheres or hyperplanes with non-isotropic normal direction).

Obviously, the set of Möbius transformations $\mu_g$ is *not* a group. To remedy this gap we have to pass to the conformal space $\overline{V}(p,q)$, upon which the linearized Möbius transformations $\lambda_g$ are free from singularities.

### 2.10. Definition

The group of transformations $\lambda_g$ on $\overline{V}(p,q)$ is called the *conformal* group of $V(p,q)$ and will be denoted by $\mathrm{Con}(p,q)$ henceforth.

□

Observe that the covering

$$\mathrm{Twist}(p,q) \to \mathrm{Con}(p,q), \quad g \to \lambda_g,$$

is fourfold, the pre-image of the identity being $\left\{\pm\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \pm\begin{pmatrix}j & 0\\ 0 & -\hat{j}\end{pmatrix}\right\}$ with $j = e_1 e_2 \dots e_n$.

The generating conformal transformations are the *inversions* with respect to "pseudo-spheres". These correspond to the hyperplane reflections in PV(p+1,q+1). Recall that Twist(p,q) is generated by the set of non-isotropic vectors $\underline{a} \in V(p+1,q+1)$.
To examine the Möbius transformation $\mu_{\underline{a}}$ induced by $\underline{a} = \begin{pmatrix} a & \lambda \\ \mu & \hat{a} \end{pmatrix} \in V(p+1,q+1)$, $Q(\underline{a}) \neq 0$, we distinguish between the cases

(i) $\mu \neq 0$.

Assume that $\underline{a}$ has been normalized such that $\underline{a} = \begin{bmatrix} a & Q(\underline{a}) - Q(a) \\ -1 & \hat{a} \end{bmatrix}$. Then we have

$$\mu_{\underline{a}}(v) = (av + Q(\underline{a}) - Q(a))\,(\hat{v} + \hat{a})^{-1} = -a + Q(\underline{a})\,\frac{v + a}{Q(v + a)}\,,$$

which is the inversion with respect to the pseudo-sphere $Q(v + a) = Q(\underline{a})$.

(ii) $\mu = 0$.

It is clear that the Möbius transformation

$$v \rightarrow \mu_{\underline{a}}(v) = (av + \lambda)\,\hat{a}^{-1} = \rho_a(v) + \lambda\,\frac{a}{Q(a)}$$

is the orthogonal reflection with respect to the hyperplane $B(v,a) = \frac{1}{2}\lambda$.

**Subgroups and decompositions**

Dimensional arguments make clear that Con(p,q) is generated by the following well-known subgroups.

*Dilatations*

$$g = \begin{bmatrix} \operatorname{sgn}(\lambda)\sqrt{|\lambda|} & 0 \\ 0 & \sqrt{|\lambda|^{-1}} \end{bmatrix}, \quad \lambda \in GL(1, \mathbb{R}).$$

$\mu_g(v) = \lambda v.$

*Orthogonal transformations*

$$g = \begin{pmatrix} a & 0 \\ 0 & \hat{a} \end{pmatrix}, \qquad a \in \mathrm{Pin}(p,q).$$

$$\mu_g(v) = \rho_a(v).$$

*Translations*

$$g = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \qquad u \in V(p,q).$$

$$\mu_g(v) = v + u.$$

*Transversions*

$$g = \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}, \qquad w \in V(p,q).$$

$$\mu_g(v) = v(wv + 1)^{-1} = (w + v^{-1})^{-1}.$$

Dilatations, orthogonal transformations and translations are represented by upper triangular matrices. We have seen that the subgroup $U \subset \mathrm{Twist}(p,q)$ of upper triangular matrices represents the stabilizer subgroup of $<v_{-\infty}> \in PC(p+1,q+1)$. Consequently, for any $g \in U$ the transformation $\rho_g$ maps the hyperplane at infinity (the polar hyperplane of $<v_{-\infty}>$) onto itself. Concerning the induced Möbius transformation $\mu_g$ this amounts to the property $\mathrm{Dom}(\mu_g) = V(p,q)$ , which is also clear by prompt inspection.
Now consider any $g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in U$.
Necessarily, $d = \pm (a^*)^{-1}$ and $b = au$ for some $u \in V(p,q)$. Then since $a \in \Gamma(p,q)$ admits the polar decomposition

$$a = \lambda a_n, \text{ where } \lambda = \sqrt{|N_{-1}(a)|} \in \mathbb{R}^+ \text{ and } a_n \in \mathrm{Pin}(p,q),$$

$g = \begin{pmatrix} a & au \\ 0 & \pm (a^*)^{-1} \end{pmatrix}$ can be written as the product

$$g = \begin{pmatrix} \lambda & 0 \\ 0 & \pm\lambda^{-1} \end{pmatrix} \begin{pmatrix} a_n & 0 \\ 0 & \hat{a}_n \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}.$$

We thus have proved

**2.11. Theorem**

A Möbius transformation is globally defined on V(p,q) if and only if it is a product of one dilatation, one orthogonal transformation and one translation.

□

Is it possible to decompose any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Twist}(p,q)$ in a similar way, using an additional fourth factor which induces a transversion?
According as $a \in \Gamma(p,q)$ or $d \in \Gamma(p,q)$ we can factorize

$$g = \begin{pmatrix} 1 & 0 \\ ca^{-1} & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & \pm(a^*)^{-1} \end{pmatrix} \quad \text{or}$$

$$g = \begin{pmatrix} \pm(d^*)^{-1} & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ d^{-1}c & 1 \end{pmatrix}.$$

In both cases $\mu_g$ can be written as a product of one dilatation, one orthogonal transformation, one translation and one transversion. Conversely, the assumption that $\mu_g$ is the product in whatever order of such four factors readily leads to the conclusion $a \in \Gamma(p,q)$ or $d \in \Gamma(p,q)$. We thus have cleared up a widespread misunderstanding with regard to a decomposition of the conformal group Con(p,q):

**2.12. Theorem**

The conformal transformation represented by $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Twist}(p,q)$ can be written as a product of one dilatation, one orthogonal transformation, one translation and one transversion if and only if a is invertible or d is invertible.

□

If both a and d are non-invertible a weaker decomposition remains to be considered. Application of the preceding theorem to $e_{n+2}g = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$ yields a factorization which involves one additional inversion (represented by $e_{n+2}$) if and only if b is invertible or c is invertible. If all entries are non-invertible, which is impossible if $q = 0$, repetitional types of factors inevitably occur.

Twistors

The spin representation of the Clifford algebra Cl(p,q) constitutes a representation of the *orthogonal* group O(p,q) on the space of *spinors* S(p,q).
For any $r \in \mathrm{Pin}(p,q)$ the endomorphism

$$S(p,q) \to S(p,q), \quad s \to rs,$$

preserves the scalar product $\phi_\epsilon$ up to sign, as a result of the equation $N_\epsilon(r) = \pm 1$.
Similarly, the covering Twist(p,q) of the *conformal* group Con(p,q) affords a representation of Con(p,q) on the space of *twistors* $T(p,q) \stackrel{\mathrm{def}}{=} S(p,q) \otimes \mathbb{R}^2$. We thus deliberately demythologize the twistor concept. T(p,q) is nothing more than the representation of S(p+1,q+1) as the space of *bispinors* $S(p,q) \otimes \mathbb{R}^2$, this in accordance with the modulo (1,1) periodicity.
Assume that T(p,q) is realized as a subideal of the left ideal M(p,q) in the Clifford algebra Cl(p+1,q+1). Recall the recurrence relation of the anti-automorphism $\alpha_\epsilon$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \alpha_\epsilon(g) = \begin{bmatrix} \alpha_{-\epsilon}(d) & \epsilon\alpha_{-\epsilon}(b) \\ \epsilon\alpha_{-\epsilon}(c) & \alpha_{-\epsilon}(a) \end{bmatrix}.$$

The restriction to T(p,q)

$$x = \begin{bmatrix} x_1 & 0 \\ x_2 & 0 \end{bmatrix} \to \alpha_\epsilon(x) = \begin{bmatrix} 0 & 0 \\ \epsilon\alpha_{-\epsilon}(x_2) & \alpha_{-\epsilon}(x_1) \end{bmatrix}$$

then yields the relation between the scalar products $\phi_\epsilon$ on the twistor space T(p,q) and $\phi_{-\epsilon}$ on the spinor space S(p,q):

$$\phi_\epsilon(x,y) = \phi_{-\epsilon}(x_1,y_2) + \epsilon\phi_{-\epsilon}(x_2,y_1).$$

This scalar product is preserved up to sign by the endomorphism

$$T(p,q) \to T(p,q), \quad x \to gx,$$

for all $g \in \mathrm{Twist}(p,q)$.

**Passage to the space of paravectors**

The complete machinery developed in this chapter applies equally well to the conformal geometry of $V_\pi(p,q)$. Advantageously, the space of paravectors $V_\pi(p,q)$ requires the same Clifford algebra as the space of vectors $V(p,q)$. We shall meet the classical Möbius transformations of the plane, at last, as an example of this *minimal* representation of conformal groups.

Let us briefly comment upon the analogies involved.

$\mathrm{Twist}_\pi(p,q)$

Representation of $\mathrm{Pin}_\pi(p+1,q+1)$ as a group of $(2\times2)$-matrices over $Cl(p,q)$, in accordance with the modulo $(1,1)$ periodicity.

$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Twist}_\pi(p,q)$ if and only if

(1) $a\tilde{a}, b\tilde{b}, c\tilde{c}, d\tilde{d} \in \mathbb{R}$

(2) $a\tilde{c}, b\tilde{d} \in V_\pi(p,q)$

(3) $aw\tilde{b} + b\tilde{\tilde{w}}\tilde{a}, cw\tilde{d} + d\tilde{\tilde{w}}\tilde{c} \in \mathbb{R}$ for all $w \in V_\pi(p,q)$

(4) $aw\tilde{d} + b\tilde{\tilde{w}}\tilde{c} \in V_\pi(p,q)$ for all $w \in V_\pi(p,q)$

(5) $ab^* = ba^*, cd^* = dc^*$

(6) $ad^* - bc^* = \pm 1$.

Recall that $\mathrm{Pin}_\pi(p+1,q+1)$ is a double covering of the *special* orthogonal group of $V_\pi(p+1,q+1)$.

pre-$\overline{V}_\pi(p,q)$

Projection of $\mathrm{Twist}_\pi(p,q)$ onto the left ideal generated by the idempotent $e = \frac{1}{2}(e(1) + e_{n+1}e_{n+2}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

$\dot{C}_\pi(p+1,q+1)$

Punctured null-cone $\{\underline{0} \neq \underline{w} \in V_\pi(p+1,q+1)\colon Q(\underline{w}) = 0\}$.

$PC_\pi(p+1,q+1)$

Projective cone $\{\langle\underline{w}\rangle \in PV_\pi(p+1,q+1)\colon Q(\underline{w}) = 0\}$.

$$S_\pi\colon \text{pre-}\overline{V}_\pi(p,q) \to \dot{C}_\pi(p+1,q+1),\quad S_\pi(x) = \begin{bmatrix} x_1\tilde{x}_2 & x_1\tilde{x}_1 \\ x_2\tilde{x}_2 & x_2\tilde{x}_1 \end{bmatrix}.$$

Map which gives rise to the commutative

**2.13. Diagram**

$$\begin{array}{ccccc} & \text{pre-}\overline{V}_\pi(p,q) & \xrightarrow{S_\pi} & \dot{C}_\pi(p+1,q+1) & \\ \text{projection} & \downarrow & & \downarrow & \text{projection} \\ & \overline{V}_\pi(p,q) & \xrightarrow[\text{bijection}]{} & PC_\pi(p+1,q+1) & \end{array}$$

□

There is a one-to-one correspondence between the conformal space $\overline{V}_\pi(p,q)$ and the quotient space $\mathrm{Twist}_\pi(p,q)/U_\pi$, where $U_\pi$ denotes the subgroup of upper triangular matrices. Geometrically, $\overline{V}_\pi(p,q)$ is the conformal compactification of $V_\pi(p,q)$. Any $\langle x\rangle \in \overline{V}_\pi(p,q)$ with $x_2\tilde{x}_2 = 0$ lies on what is conceived as the cone at infinity.

$\mathrm{Twist}_\pi(p,q)$ acts on $\overline{V}_\pi(p,q)$ according to the commutative

**2.14. Diagram**

$$\begin{array}{ccc} \overline{V}_\pi(p,q) & \xrightarrow{\lambda_g} & \overline{V}_\pi(p,q) \\ \text{bijection} \downarrow & & \downarrow \text{bijection} \\ PC_\pi(p+1,q+1) & \xrightarrow[\rho_g]{} & PC_\pi(p+1,q+1) \end{array}$$

□

For any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Twist}_\pi(p,q)$ the extracted Möbius transformation in $V_\pi(p,q)$

$$w \to \mu_g(w) = (aw + b)(cw + d)^{-1}$$

is conformal with respect to the pseudo-metric

$$(ds)^2 = Q(dw).$$

So far vectors and paravectors are seen to be dealt with in exactly the same way.
To represent the *improper* part of the orthogonal group, as distinct from the vector case we now additionally have to consider the $\mathrm{Twist}_\pi(p,q)$-actions

$$\underline{w} \to \rho_g(-\hat{\underline{w}}) \quad \text{on } V_\pi(p+1,q+1).$$

For any $x \in \text{pre-}\overline{V}_\pi(p,q)$ the equation $S_\pi(\hat{x}) = -\hat{S}_\pi(x)$ holds
$\left(\text{warning: } x = \begin{pmatrix} x_1 & 0 \\ x_2 & 0 \end{pmatrix} \to \hat{x} = \begin{pmatrix} \hat{x}_1 & 0 \\ -\hat{x}_2 & 0 \end{pmatrix}\right).$
Consequently, the $\mathrm{Twist}_\pi(p,q)$-action

$$<\underline{w}> \to \rho_g <-\hat{\underline{w}}> \quad \text{on } PC_\pi(p+1,q+1)$$

corresponds with the action

$$<x> \to \lambda_g <\hat{x}> \quad \text{on } \overline{V}_\pi(p,q).$$

The latter induces what we call the *anti-Möbius* transformation in $V_\pi(p,q)$

$$w \to \mu_g(-\hat{w}) = (-a\hat{w} + b)(-c\hat{w} + d)^{-1}$$

which in fact is a Möbius transformation combined with the reflection in the subspace of vectors $V(p,q)$. Needless to say, any anti-Möbius transformation is also conformal with respect to the pseudo-metric $(ds)^2 = Q(dw)$ on $V_\pi(p,q)$.

In the paravector case the inversion

$$w \to -a + Q(\underline{a}) \frac{w + a}{Q(w + a)}$$

with respect to the pseudo-sphere $Q(w + a) = Q(\underline{a})$, happens to be an anti-Möbius transformation

$$w \to \mu_{\underline{a}}(-\hat{w}) \quad \text{with } \underline{a} = \begin{pmatrix} a & Q(\underline{a}) - Q(a) \\ -1 & \hat{a} \end{pmatrix} \in \mathrm{Twist}_\pi(p,q).$$

Since $\mathrm{Twist}(p,q)$ and $\mathrm{Twist}_\pi(p,q)$ are subsets of the same Clifford algebra $Cl(p+1,q+1) \cong Cl(p,q) \otimes \mathbb{R}(2)$, the conformal groups of $V(p,q)$ and $V_\pi(p,q)$ give rise to the same twistor space $T(p,q) = S(p,q) \otimes \mathbb{R}^2$.

For any $g \in \mathrm{Twist}_\pi(p,q)$, $N_\epsilon(g) = \pm e(1)$ only for $\epsilon = -1$, so the endomorphism

$$T(p,q) \to T(p,q), \quad x \to gx,$$

only preserves the scalar product (up to sign)

$$(x,y) \to \phi_{-1}(x,y) = \phi_1(x_1,y_2) - \phi_1(x_2,y_1).$$

As a reminiscence of the *anti*-Möbius transformations we additionally have to involve the actions

$$T(p,q) \to T(p,q), \quad x \to g\hat{x}$$

on the twistor space (also preserving $\phi_{-1}$ up to sign).

## 2.15. Examples

The Euclidean plane may be realized as the set of paravectors $V_\pi(1,0) = Cl(1,0) \cong \mathbb{C}$. The conformal group is covered by the group $\mathrm{Twist}_\pi(1,0) \cong \{g \in \mathbb{C}(2)\colon \det(g) = \pm 1\}$, inducing the well-known (anti)-Möbius transformations in the complex plane.

Let the Clifford algebra $Cl(0,2) \cong \mathbb{R}(2)$ be generated by $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then the set of paravectors $w = w_0 + w_1 e_1 + w_2 e_2 = \begin{pmatrix} w_0+w_1 & w_2 \\ w_2 & w_0-w_1 \end{pmatrix}$ has to be identified with the set of symmetric matrices. $\mathrm{Twist}^+_\pi(0,2) \cong Sp(4, \mathbb{R})$ induces the well-known linear fractional transformations in the space of symmetric $(2\times 2)$-matrices. The linear action of $Sp(4, \mathbb{R})$ on the twistor space $\mathbb{R}^4$ leaves invariant the standard symplectic form. The remaining part of the conformal group of 3-dimensional Minkowski space is generated by the transformations $w \to -w$ and $w \to \hat{w}$. The corresponding actions on the twistor space $\mathbb{R}^4$ switch the sign of the symplectic form.

The Clifford algebra $Cl(0,3) \cong \mathbb{C}(2)$ may be generated by the so-called Pauli spin matrices $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$.
Thusly, the set of paravectors $w = w_0 + w_1 e_1 + w_2 e_2 + w_3 e_3 = \begin{pmatrix} w_0+w_1 & w_2+iw_3 \\ w_2-iw_3 & w_0-w_1 \end{pmatrix}$ happens to be the set of Hermitian matrices. Conformal transformations in 4-dimensional Minkowski space are linear fractional transformations in the space of Hermitian $(2\times 2)$-matrices, induced by $\mathrm{Twist}^+_\pi(0,3) \cong SU(2,2)$, combined with the transformations $w \to -w$ and $w \to \hat{w}$.
The corresponding linear actions on the twistor space $\mathbb{C}^4$ preserve (up to sign) the pseudo-Hermitian form $(z,z') \to \bar{z}_1 z'_1 + \bar{z}_2 z'_2 - (\bar{z}_3 z'_3 + \bar{z}_4 z'_4)$.

□

# CHAPTER 3
# HYPERBOLIC GEOMETRY

In this chapter we present a straightforward generalization of the Poincaré models of the hyperbolic plane. That is to say, we present two conformal embeddings of hyperbolic k-space in Euclidean k-space: the one onto the interior of a hypersphere, the other *(essentially the same) onto one of two parts separated by a hyperplane.*

Let the Euclidean (n+1)-space be realized as the space of paravectors $V_\pi(n)$ in the Clifford algebra Cl(n) (instead of (n,0) we agree to write (n)).

The conformal geometry of the positive definite space $V_\pi(n)$ is relatively simple. Null-cones in $V_\pi(n)$ are points, the conformal compactification $\overline{V}_\pi(n)$ is a one-point compactification. For any $r \in Cl(n)$ the equivalence $r\tilde{r} = 0 \iff r = 0$ is true. Consequently, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Twist}_\pi(n)$ implies $a \in \Gamma(n)$ or $a = 0$ and the same with regard to the remaining entries b, c, d. The induced Möbius transformation $w \to \mu_g(w) = (aw + b)(cw + d)^{-1}$ in $V_\pi(n)$ has the one-point singularity $w = -c^{-1}d$. The anti-Möbius transformation $w \to \mu_g(-\hat{w})$ is undefined only for $w = \hat{c}^{-1}\hat{d}$. The "image" of those singularities is represented by $x \in \text{pre-}\overline{V}_\pi(n)$ with $x_1 \in \Gamma(n)$ and $x_2 = 0$, the point at infinity which compactifies $V_\pi(n)$.

**Unit ball model**

The first model of (n+1)-dimensional hyperbolic space we wish to discuss, is given by the pair $(B^{n+1}, (ds)^2)$ in the following

**3.1. Definition**

$$B^{n+1} = \{w \in V_\pi(n) : Q(w) < 1\}$$

$$(ds)^2 = 4\,\frac{Q(dw)}{(1 - Q(w))^2}$$

□

In this model, hyperbolic k-spaces ($0 < k < n+1$) are represented by the intersection of $B^{n+1}$ with k-"spheres" orthogonal to the *absolute* hypersphere $S^n = \{w \in V_\pi(n): Q(w) = 1\}$. The latter is called absolute, because it is the infinitely distant horizon with respect to the metric on $B^{n+1}$. Two hyperbolic lines, represented by the 1-"spheres" $S_1$ and $S_2$, say, are said to be parallel to each other if they intersect on the absolute hypersphere $S^n$. In general, a k-space and an $\ell$-space are said to be parallel to each other if they do not intersect in hyperbolic (n+1)-space and do contain two parallel lines.
Since the line element $(ds)^2$ is proportional to the Euclidean line element $(ds)^2 = Q(dw)$, this is a *conformal* model: angles have to be measured in the Euclidean way.
Consequently, the group of hyperbolic isometries must be a group of conformal transformations in $V_\pi(n)$. We thus need to determine a subgroup of $\mathrm{Twist}_\pi(n)$ such that its induced (anti-)Möbius transformations respect the pair $(B^{n+1},(ds)^2)$.
It will turn out that the group of hyperbolic isometries is covered by the following subgroup.

### 3.2. Definition

$$\mathrm{Twist}^*_\pi(n) = \{g \in \mathrm{Twist}_\pi(n): g^* e_{n+1}\, g = e_{n+1}\}$$

□

For any $g \in \mathrm{Twist}^*_\pi(n)$ the transformation

$$\lambda_g: \text{ pre-}\overline{V}_\pi(n) \to \text{pre-}\overline{V}_\pi(n), \quad \lambda_g(x) = gx,$$

leaves invariant the form

$$(x,y) \to x^* e_{n+1}\, y.$$

Let us consider this form more closely.
With $x = \begin{pmatrix} x_1 & 0 \\ x_2 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} y_1 & 0 \\ y_2 & 0 \end{pmatrix} \in \text{pre-}\overline{V}_\pi(n)$ we have

$$x^* e_{n+1}\, y = \begin{bmatrix} 0 & 0 \\ \tilde{x}_2 & \tilde{x}_1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y_1 & 0 \\ y_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \tilde{x}_2 y_2 - \tilde{x}_1 y_1 & 0 \end{bmatrix}.$$

Hence for any $g \in \mathrm{Twist}^*_\pi(n)$, $\lambda_g$ leaves invariant the expression $\tilde{x}_1 y_1 - \tilde{x}_2 y_2$ on pre-$\overline{V}_\pi(n)$.

Since for any $x \in$ pre-$\overline{V}_\pi(n)$ $\tilde{x}_i x_i \in \mathbb{R}$, we are invited to extract the real-valued invariant

$$\psi\colon \text{pre-}\overline{V}_\pi(n) \to \mathbb{R}, \quad \psi(x) \overset{\text{def}}{=} \tilde{x}_1 x_1 - \tilde{x}_2 x_2.$$

Now recall that the conformal space $\overline{V}_\pi(n)$ is the set of classes $\langle x \rangle$, $x \in$ pre-$\overline{V}_\pi(n)$, with respect to the equivalence relation $x \sim y$ iff $y = xr$ for some $r \in \Gamma(n)$. Since $\tilde{r}r > 0$ for all $r \in \Gamma(n)$, the map $\langle x \rangle \to \operatorname{sign} \psi(x)$ is a well-defined class function. What is the geometrical significance of this invariant? We distinguish between the three parts

(i) $\langle x \rangle \in \overline{V}_\pi(n)$ with $\psi(x) < 0$.

Since $0 \le \tilde{x}_1 x_1 < \tilde{x}_2 x_2$ it is impossible that $x_2 = 0$.

Hence we may take the unique representative $x \sim \binom{w}{1}$ with $w = x_1 x_2^{-1} \in V_\pi(n)$. Then $\psi(x) < 0$ amounts to the inequality $Q(w) < 1$, i.e., to the condition $w \in B^{n+1}$.

(ii) $\langle x \rangle \in \overline{V}_\pi(n)$ with $\psi(x) = 0$.

Again it is impossible that $x_2 = 0$. Permissibly, we choose $x \sim \binom{w}{1}$ with $w = x_1 x_2^{-1} \in V_\pi(n)$. Then the equation $\psi(x) = 0$ yields $Q(w) = 1$.

This part of $\overline{V}_\pi(n)$ thus has to be identified with the absolute hypersphere $S^n$.

(iii) $\langle x \rangle \in \overline{V}_\pi(n)$ with $\psi(x) > 0$.

Represents the exterior of $S^n$ together with the point at infinity $\langle x \rangle$ with $x_2 = 0$.

With $g \in \mathrm{Twist}_\pi(n)$ the transformation

$$\overline{V}_\pi(n) \to \overline{V}_\pi(n), \quad \langle x \rangle \to \lambda_g \langle x \rangle,$$

maps the parts distinguished above one-to-one onto themselves if and only if $g \in \mathrm{Twist}^*_\pi(n)$.

Concerning the induced Möbius transformation we arrive at the following conclusion.

**3.3. Theorem**

A Möbius transformation $\mu_g$ maps $B^{n+1}$ and $S^n$ one-to-one onto themselves if and only if $g \in \mathrm{Twist}^*_\pi(n)$.

□

Needless to say, this is also true with regard to the anti-Möbius transformations $w \to \mu_g(-\hat{w})$.
Consequently, for any $g \in \mathrm{Twist}^*_\pi(n)$ the conformal transformations

$$B^{n+1} \to B^{n+1},\ w \to \mu_g(w) \quad \text{and} \quad w \to \mu_g(-\hat{w})$$

respect the set of fragments of k-"spheres" orthogonal to the absolute hypersphere $S^n$. In hyperbolic terms, under these transformations hyperbolic k-spaces are carried over into hyperbolic k-spaces. Also observe that hyperbolic parallelism, defined in terms of intersection on $S^n$, is an invariant property under the (anti-)Möbius transformations induced by $\mathrm{Twist}^*_\pi(n)$.

What remains to be proved, is that $\mu_g$ leaves invariant the line element $(ds)^2 = 4\,\dfrac{Q(dw)}{(1 - Q(w))^2}$ whenever $g \in \mathrm{Twist}^*_\pi(n)$.
To facilitate this task, we first give a more explicit characterization of the group $\mathrm{Twist}^*_\pi(n)$.

**3.4. Theorem**

A necessary and sufficient condition for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Twist}_\pi(n)$ to belong to the subgroup $\mathrm{Twist}^*_\pi(n)$ is

$$d = \pm\,\hat{a} \ \text{ and } \ c = \pm\,\hat{b} \ \text{ according as } ad^* - bc^* = \pm 1.$$

*Proof*

Let be given $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Twist}_\pi(n)$ with $N_{-1}(g) = e(1)$. In terms of the pseudo-determinant: $ad^* - bc^* = 1$. By definition $g \in \mathrm{Twist}^*_\pi(n)$ iff $g^* e_{n+1} g = e_{n+1}$, i.e., iff $e^{-1}_{n+1} g^* e_{n+1} g = e(1)$. Combined with the assumption $N_{-1}(g) = \tilde{g}g = e(1)$, this yields $g \in \mathrm{Twist}^*_\pi(n)$ iff $\tilde{g} = e^{-1}_{n+1} g^* e_{n+1}$.

We have $\tilde{g} = \begin{pmatrix} d^* & -b^* \\ -c^* & a^* \end{pmatrix}$ on the one hand, and $e^{-1}_{n+1} g^* e_{n+1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{d} & \tilde{b} \\ \tilde{c} & \tilde{a} \end{pmatrix} \cdot$

$\cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \tilde{a} & -\tilde{c} \\ -\tilde{b} & \tilde{d} \end{pmatrix}$ on the other. Hence $g \in \mathrm{Twist}^*_\pi(n)$ iff $d = \hat{a}$ and $c = \hat{b}$.

The case $N_{-1}(g) = -e(1)$ may be dealt with either in the same way or by employing the factorization $g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ -c & -d \end{pmatrix}$ with $ad^* - bc^* = -1$.

□

Now it is very easy to prove

**3.5. Theorem**

For any $g \in \mathrm{Twist}^*_\pi(n)$ the conformal transformations

$$B^{n+1} \to B^{n+1}, \quad w \to \mu_g(w) \quad \text{and} \quad w \to \mu_g(-\hat{w}),$$

respect the metric on $B^{n+1}$, defined by the line element $(ds)^2 = 4 \dfrac{Q(dw)}{(1 - Q(w))^2}$.

*Proof*

According to the preceding theorem we may assume $g = \begin{pmatrix} a & b \\ \hat{b} & \hat{a} \end{pmatrix}$ or $g = \begin{pmatrix} a & b \\ -\hat{b} & -\hat{a} \end{pmatrix}$ with $a\tilde{a} = 1 + b\tilde{b}$. The induced Möbius transformations are the same up to sign, so let us restrict ourselves to the first case

$$w \to \mu_g(w) = (aw + b)(\hat{b}w + \hat{a})^{-1}.$$

We already know that $Q(d\mu_g(w)) = \dfrac{Q(dw)}{N^2_{-1}(\hat{b}w + \hat{a})}$ (cf. Theorem 2.10). Further we have

$$1 - Q(\mu_g(w)) = \frac{N_{-1}(\hat{b}w + \hat{a}) - N_{-1}(aw + b)}{N_{-1}(\hat{b}w + \hat{a})} = \frac{N_{-1}(b\hat{w} + a) - N_{-1}(aw + b)}{N_{-1}(\hat{b}w + \hat{a})} =$$

$$= \frac{(N_{-1}(a) - N_{-1}(b))(1 - Q(w))}{N_{-1}(\hat{b}w + \hat{a})} = \frac{1 - Q(w)}{N_{-1}(\hat{b}w + \hat{a})}. \text{ Consequently, } \frac{Q(d\mu_g(w))}{(1 - Q(\mu_g(w)))^2} =$$

$$= \frac{Q(dw)}{(1 - Q(w))^2}.$$

Concerning the anti-Möbius transformations it suffices to observe that the transformation $w \to -\hat{w}$ respects the line element $(ds)^2$ on $B^{n+1}$.

□

Thusly, the group of hyperbolic isometries is seen to be covered by the group $\mathrm{Twist}^*_\pi(n)$. For any $g \in \mathrm{Twist}^*_\pi(n)$ the induced actions on the unit ball model $(B^{n+1}, (ds)^2)$ are

$$B^{n+1} \to B^{n+1}, \quad w \to \mu_g(w) \quad \text{and} \quad w \to \mu_g(-\hat{w}).$$

The covering $\mathrm{Twist}^*_\pi(n) \to \mathrm{Iso}(B^{n+1}, (ds)^2)$ is not faithful. The pre-image of the identity map on $B^{n+1}$ is given by the subgroup $\{\pm\, e(1), \pm\, e(j)e_{n+1}e_{n+2}\}$, to understand in the following way.

In the first place the identity map is trivially represented by $\pm\, e(1) = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in$ $\in \mathrm{Twist}^*_\pi(n)$: $\mu_{\pm e(1)}(w) = w$ for all $w \in B^{n+1}$.

Concerning $\pm\, e(j) = \pm \begin{pmatrix} j & 0 \\ 0 & -\hat{j} \end{pmatrix} \in \mathrm{Twist}^*_\pi(n)$, secondly, we have to distinguish between n odd and n even.

If n is odd then $j = e_1 e_2 \ldots e_n$ lies in the centre of the Clifford algebra $Cl(n)$. Hence $\pm \begin{pmatrix} j & 0 \\ 0 & -\hat{j} \end{pmatrix} = \pm \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}$ induce the Möbius transformation $w \to jwj^{-1} = w$.

If n is even then $rj = j\hat{r}$ for all $r \in Cl(n)$. Then the *anti*-Möbius transformation induced by $\pm \begin{pmatrix} j & 0 \\ 0 & -\hat{j} \end{pmatrix} = \pm \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix}$ happens to be $w \to j(-\hat{w})(-j)^{-1} = w$: the identity map on $B^{n+1}$.

In any case, the subgroup of *sense-preserving* hyperbolic isometries is given by the group of Möbius transformations

$$\mu_g\colon B^{n+1} \to B^{n+1}, \quad \mu_g(w) = (aw + b)(\hat{b}w + \hat{a})^{-1},$$

where $g = \begin{pmatrix} a & b \\ \hat{b} & \hat{a} \end{pmatrix} \in \mathrm{Twist}^{*+}_\pi(n)$ (pseudo-determinant = 1).

Thusly, $\mathrm{Iso}_0(B^{n+1}, (ds)^2) \cong \mathrm{PTwist}^{*+}_\pi(n) \overset{\mathrm{def}}{=} \mathrm{Twist}^{*+}_\pi(n) / \{\pm\, e(1)\}$.

**Upper half space model**

This second model is obtained by subjecting the unit ball model $(B^{n+1},(ds)^2)$ to the so-called *Cayley* transformation, which is nothing more than the Möbius transformation

$$w \to \mu_c(w) = (w + e_n)(e_n w + 1)^{-1}$$

induced by $c = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & e_n \\ e_n & 1 \end{pmatrix} \in \text{Twist}_\pi(n)$.

The singularity of $\mu_c$ is given by the point $w = e_n$, so the Cayley transformation is globally defined on the unit ball $B^{n+1}$. We denote $H^{n+1} = \mu_c(B^{n+1})$.

To determine $H^{n+1}$, start from the inverse $c^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -e_n \\ -e_n & 1 \end{pmatrix}$ inducing the Möbius transformation

$$w \to \mu_{c^{-1}}(w) = (w - e_n)(-e_n w + 1)^{-1} = (w - e_n)(w + e_n)^{-1} e_n.$$

With $\mu_{c^{-1}}(w) \in B^{n+1} \Leftrightarrow Q(w - e_n) < Q(w + e_n)$ it turns out that the image of $B^{n+1}$ under the Cayley transformation $\mu_c$ is the upper half space

$$H^{n+1} = \{w = w_0 + w_1 e_1 + ... + w_n e_n \in V_\pi(n) : w_n > 0\}.$$

For any $x \in B^{n+1}$ there is a unique $w \in H^{n+1}$ such that $x = \mu_{c^{-1}}(w) = (w - e_n) \cdot$ $\cdot (w + e_n)^{-1} e_n$.

With $Q(dx) = 4 \dfrac{Q(dw)}{Q^2(w + e_n)}$ and $1 - Q(x) = \dfrac{Q(w + e_n) - Q(w - e_n)}{Q(w + e_n)} = 4 \dfrac{w_n}{Q(w + e_n)}$

it follows that the pull back of $ds^2(x) = 4 \dfrac{Q(dx)}{(1 - Q(x))^2}$ is given by the line element $ds^2(w) = \dfrac{Q(dw)}{w_n^2}$ on $H^{n+1}$.

Observe that with respect to this metric $V_\pi(n-1)$ $(w_n = 0)$ is the hyperplane at infinity, in agreement with the fact that the Cayley transformation $\mu_c$ maps the absolute hypersphere $S^n$ onto the compactification $\overline{V}_\pi(n-1)$.

The pair $(H^{n+1}, (ds)^2 = \frac{Q(dw)}{w_n^2})$ is our second model of (n+1)-dimensional hyperbolic space.

Hyperbolic k-spaces are represented by k-"spheres" orthogonal to $V_\pi(n-1)$. Hyperbolic parallelism is defined in terms of intersection on the absolute hyperplane $V_\pi(n-1)$. The covering of the hyperbolic group belonging to the upper half space model $(H^{n+1}, (ds)^2)$ is conjugate to $Twist^*_\pi(n)$. For any $g \in Twist^*_\pi(n)$ the commutative

**3.6. Diagram**

$$\begin{array}{ccc} B^{n+1} & \xrightarrow{\mu_g} & B^{n+1} \\ \mu_c \downarrow & & \downarrow \mu_c \\ H^{n+1} & \xrightarrow[\mu_{cgc^{-1}}]{} & H^{n+1} \end{array}$$

□

combined with the commutation rule $-\hat{\mu}_c(w) = \mu_c(-\hat{w})$ yields the hyperbolic isometries

$$H^{n+1} \to H^{n+1}, \quad w \to \mu_{cgc^{-1}}(w) \quad \text{and} \quad w \to \mu_{cgc^{-1}}(-\hat{w}).$$

For the sake of completeness we characterize the group $cTwist^*_\pi(n)\,c^{-1}$, more explicitly.
Let be given any $g = \begin{pmatrix} a & b \\ \hat{b} & \hat{a} \end{pmatrix} \in Twist^{*+}_\pi(n)$ and set $a = a_1 + a_2 e_n$, $b = b_1 + b_2 e_n$: $a_i, b_i \in Cl(n-1)$.
With $a_i e_n = e_n \hat{a}_i$ and $b_i e_n = e_n \hat{b}_i$ it follows that

$$cgc^{-1} = \begin{bmatrix} a_1+b_2 & a_2+b_1 \\ -\hat{a}_2+\hat{b}_1 & \hat{a}_1-\hat{b}_2 \end{bmatrix} \in Twist^+_\pi(n-1).$$

Conversely, for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Twist^+_\pi(n-1)$ we find

$$c^{-1}gc = \begin{bmatrix} p & q \\ \hat{q} & \hat{p} \end{bmatrix} \in Twist^{*+}_\pi(n), \text{ where}$$

$p = a + \hat{d} + (b - \hat{c})\, e_n$ and $q = b + \hat{c} + (a - \hat{d})\, e_n$.
Summarily, $\mathrm{Twist}_\pi^{*+}(n) \cong \mathrm{Twist}_\pi^{+}(n-1)$.
With regard to the remaining part of the group $\mathrm{Twist}_\pi^{*}(n)$ (pseudo-determinant = -1), it suffices to consider its generator $e_{n+1}e_{n+2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Its conjugate

$$c e_{n+1} e_{n+2} c^{-1} = \begin{pmatrix} 0 & -e_n \\ e_n & 0 \end{pmatrix} \in \mathrm{Twist}_\pi(n)$$

induces the transformations

$$H^{n+1} \to H^{n+1}, \quad w \to e_n w^{-1} e_n \text{ and } w \to -e_n \hat{w}^{-1} e_n.$$

The subgroup of sense-preserving hyperbolic isometries is given by the group of Möbius transformations

$$\mu_g\colon H^{n+1} \to H^{n+1}, \quad \mu_g(w) = (aw + b)(cw + d)^{-1},$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Twist}_\pi^{+}(n-1)$.

# CHAPTER 4
# SIEGEL DOMAINS OF TYPE FOUR

In the preceding chapter the complex plane has been conceived as a two-dimensional *real* Clifford algebra. It was that point of view, actually, from which the generalized Poincaré models straightforwardly emerged. Alternatively, in this chapter we generalize the geometries of the upper half plane and the unit disk by considering the complex plane as a one-dimensional *complex* Clifford algebra, in such a way, that we arrive at two Siegel domains of type four. By definition, a subset of $\mathbb{C}^n$ is called a Siegel domain of type four if it is holomorphicly equivalent to the set

$$\{z \in \mathbb{C}^n : |z_1|^2 + \ldots + |z_n|^2 < \frac{1}{2}(1 + |z_1^2 + \ldots + z_n^2|^2) < 1\}.$$

Observe that $n = 1$ yields the unit disk $|z| < 1$.

It is clear that complex Clifford algebras $Cl(p,q;\mathbb{C}) = \mathbb{C} \otimes Cl(p,q)$ are completely determined by the integer $n = p + q$: the signature of a quadratic form is an essentially *real* concept. $Cl(p,q;\mathbb{C})$ is isomorphic to the matrix algebra $\mathbb{C}(2^k)$ or $^2\mathbb{C}(2^k)$ according as $n = 2k$ or $n = 2k + 1$.

If n is even, the spin representation of the even Clifford algebra $Cl^0(p,q;\mathbb{C})$ always breaks down into two (irreducible) half-spin representations. For if $j^2 = -1$ ($j = e_1e_2 \ldots e_n$) just use ij to construct the generating idempotents.

There are three types of scalar products $\phi_\epsilon$ on the spinor spaces $^{(2)}\mathbb{C}^m$, analogous to those on $^{(2)}\mathbb{R}^m$ (cf. page 7), to wit $\mathbb{C}_+$, $\mathbb{C}_-$ and $^2\mathbb{C}_+$. There is no need to discuss this subject as extensively as with the real Clifford algebras. The invariance groups $Inv_\epsilon(p,q;\mathbb{C})$ are given in the following

**4.1. Table - invariance groups $Inv_\epsilon(p,q;\,\mathbb{C})$, $k = 2^{[n/2]}$**

| n (mod 8) | $\epsilon = 1$ | $\epsilon = -1$ |
|---|---|---|
| 0 | $O(k,\mathbb{C})$ | $O(k,\mathbb{C})$ |
| 1 | ${}^2O(k,\mathbb{C})$ | $GL(k,\mathbb{C})$ |
| 2 | $O(k,\mathbb{C})$ | $Sp(k,\mathbb{C})$ |
| 3 | $GL(k,\mathbb{C})$ | ${}^2Sp(k,\mathbb{C})$ |
| 4 | $Sp(k,\mathbb{C})$ | $Sp(k,\mathbb{C})$ |
| 5 | ${}^2Sp(k,\mathbb{C})$ | $GL(k,\mathbb{C})$ |
| 6 | $Sp(k,\mathbb{C})$ | $O(k,\mathbb{C})$ |
| 7 | $GL(k,\mathbb{C})$ | ${}^2O(k,\mathbb{C})$ |

(Lounesto [5], table 13).

Throughout the discussion in chapter 2 (except a few signature-dependent remarks) the employed Clifford algebras may be assumed to be complexified. Thusly, in this chapter the analogous setting of complex conformal geometry is assumed to be extant.
We shall be concerned with two subsets of complexified *Minkowski* space $V_\pi(0,n;\,\mathbb{C})$. The groups of biholomorphic self-mappings of those subsets are proved to be subgroups of the complex conformal group.
For any $g \in Twist_\pi(0,n;\,\mathbb{C})$ the induced transformations $z \to \mu_g(z)$ and $z \to \mu_g(-\hat{z})$ in $V_\pi(0,n;\,\mathbb{C})$ are conformal with respect to the *complex* line element

$$(ds)^2 = Q(dz) = (dz_0)^2 - ((dz_1)^2 + ... + (dz_n)^2).$$

The subsets mentioned above may be endowed with a real (Hermitian) metric which is invariant under any biholomorphic self-mapping.

### The tube domain

Consider the space of complex paravectors

$$z = z_0 + z_1 e_1 + \ldots + z_n e_n \in V_\pi(0,n;\, \mathbb{C}),$$

equipped with the complex quadratic form

$$Q(z) = z_0^2 - (z_1^2 + \ldots + z_n^2).$$

Any $z \in V_\pi(0,n;\, \mathbb{C})$ can be written as a sum $z = x + iy$, where

$$x = x_0 + x_1 e_1 + \ldots + x_n e_n \text{ and } y = y_0 + y_1 e_1 + \ldots + y_n e_n \in V_\pi(0,n)$$

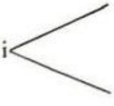
belong to the *real* (n+1)-dimensional Minkowski space.
Consequently, $Q(z) = Q(x) - Q(y) + i2B(x,y)$.
Certain restrictions with regard to the imaginary part bring us to the following

**4.2. Definition**

$$T^n = \{z = x + iy \in V_\pi(0,n;\, \mathbb{C}):\ Q(y) > 0\}$$
$$T^n_+ = \{z = x + iy \in T^n:\ y_0 > 0\}$$
$$T^n_- = \{z = x + iy \in T^n:\ y_0 < 0\}$$

□

It is obvious that $T^n = T^n_+ \cup T^n_-$. $T^0_+$ happens to be Poincaré's upper half plane and, pictorially, for n = 1 we find

$$T^1_+ = \mathbb{R}^2 + i$$ 

*Domains like these are often called tube domains: they are invariant under real trans-*lations. We shall determine the group of biholomorphic self-mappings of the connected component $T^n_+$. In view of that we think it advisable to examine the image of $T^n_+$ under the stereographic projection $z \to \langle s(z) \rangle$. For any $z \in V_\pi(0,n;\, \mathbb{C})$ we have

$$s(z) = e(z) + \frac{1}{2}(Q(z) - 1)\, e_{n+1} + \frac{1}{2}(Q(z) + 1)\, e_{n+2} \in \dot{C}_{\pi}(1,n+1;\ \mathbb{C}).$$

Assume $z = x + iy \in T^n$. Then $B(s(z),\bar{s}(z)) = B(z,\bar{z}) + \frac{1}{4}|Q(z) - 1|^2 - \frac{1}{4}|Q(z) + 1|^2 =$
$= B(z,\bar{z}) - \mathrm{Re}(Q(z)) = 2Q(y) > 0$.

(Remark: the involution $r \to \bar{r}$ on $Cl(0,n;\ \mathbb{C})$ by definition subjects the complex coefficients to complex conjugation.)

Since the sign of the pseudo-Hermitian form $t \to B(t,\bar{t})$ on $V_{\pi}(1,n+1;\ \mathbb{C})$ is a class function, we may conclude

$$z \in T^n \Rightarrow \langle s(z)\rangle \in PC_{\pi}(1,n+1;\ \mathbb{C}) \quad \text{with } B(s(z),\bar{s}(z)) > 0.$$

Conversely, let be given $\langle \underline{w}\rangle \in PC_{\pi}(1,n+1;\ \mathbb{C})$ such that $B(\underline{w},\bar{\underline{w}}) > 0$. The conditions

$$w_0^2 + w_{n+1}^2 - (w_1^2 + \dots + w_n^2 + w_{n+2}^2) = 0 \quad \text{and}$$
$$|w_0|^2 + |w_{n+1}|^2 - (|w_1|^2 + \dots + |w_n|^2 + |w_{n+2}|^2) > 0$$

imply that $\langle \underline{w}\rangle$ does not lie in the hyperplane at infinity.

For the assumption $w_{n+1} = w_{n+2}$ yields

$$|w_0|^2 = |w_1^2 + \dots + w_n^2| > |w_1|^2 + \dots + |w_n|^2$$

which is absurd.

Consequently, $\langle \underline{w}\rangle = \langle s(z)\rangle$ with $z = (w_{n+2} - w_{n+1})^{-1}\, w \in V_{\pi}(0,n;\ \mathbb{C})$.

$$\text{With } Q(z) = \frac{Q(w)}{(w_{n+2} - w_{n+1})^2} = \frac{w_{n+2}^2 - w_{n+1}^2}{(w_{n+2} - w_{n+1})^2} = \frac{w_{n+2} + w_{n+1}}{w_{n+2} - w_{n+1}}$$

$$\text{it follows that } Q(z - \bar{z}) = Q(z) + \overline{Q}(z) - 2B(z,\bar{z}) = -2\,\frac{B(\underline{w},\bar{\underline{w}})}{|w_{n+2} - w_{n+1}|^2} < 0.$$

Hence $z = x + iy \in V_{\pi}(0,n;\ \mathbb{C})$ with $Q(y) > 0$, i.e., $z \in T^n$.

Consequently,

$$\langle s(T^n)\rangle = \{\langle \underline{w}\rangle \in PC_{\pi}(1,n+1;\ \mathbb{C}):\ B(\underline{w},\bar{\underline{w}}) > 0\}.$$

The stereographic projection is a continuous mapping, so the image of $T^n$ consists of the two disjoint connected components $\langle s(T^n_+)\rangle$ and $\langle s(T^n_-)\rangle$.

In order to determine these components, we express the conditions $Q(\underline{w}) = 0$ and $B(\underline{w},\bar{\underline{w}}) > 0$, necessary and sufficient for $\langle\underline{w}\rangle \in PV_\pi(1,n+1;\mathbb{C})$ to belong to $\langle s(T^n)\rangle$, in terms of the real and imaginary part. Set $\underline{w} = \underline{u} + i\underline{v}$. Then $Q(\underline{w}) = 0$ iff $Q(\underline{u}) = Q(\underline{v})$ and $B(\underline{u},\underline{v}) = 0$ while $B(\underline{w},\bar{\underline{w}}) > 0$ iff $Q(\underline{u}) + Q(\underline{v}) > 0$. Hence $\langle\underline{w}\rangle \in \langle s(T^n)\rangle$ iff

$$u_0^2 + u_{n+1}^2 - (u_1^2 + \dots + u_n^2 + u_{n+2}^2) = v_0^2 + v_{n+1}^2 - (v_1^2 + \dots + v_n^2 + v_{n+2}^2) > 0 \text{ and}$$

$$u_0v_0 + u_{n+1}v_{n+1} - (u_1v_1 + \dots + u_nv_n + u_{n+2}v_{n+2}) = 0.$$

In matrix notation:

$$\begin{pmatrix} u_0 & u_{n+1} \\ v_0 & v_{n+1} \end{pmatrix} \begin{pmatrix} u_0 & u_{n+1} \\ v_0 & v_{n+1} \end{pmatrix}^t =$$

$$= \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} u_1 \dots u_n & u_{n+2} \\ v_1 \dots v_n & v_{n+2} \end{pmatrix} \begin{pmatrix} u_1 \dots u_n & u_{n+2} \\ v_1 \dots v_n & v_{n+2} \end{pmatrix}^t$$

for some $\lambda \in \mathbb{R}^+$.

Inferentially, $\det \begin{pmatrix} u_0 & u_{n+1} \\ v_0 & v_{n+1} \end{pmatrix} \neq 0$.

Hence the set of 2-frames consists of two disjoint connected components, one with $\det \begin{pmatrix} u_0 & u_{n+1} \\ v_0 & v_{n+1} \end{pmatrix} > 0$, the other with $\det \begin{pmatrix} u_0 & u_{n+1} \\ v_0 & v_{n+1} \end{pmatrix} < 0$. Returning to complex notation we observe that $\det \begin{pmatrix} u_0 & u_{n+1} \\ v_0 & v_{n+1} \end{pmatrix} = \mathrm{Im}(\bar{w}_0 w_{n+1})$.

Inspection of the image of $i \in T^n_+$, say, brings us to the following conclusion.

**4.3. Theorem**

$$\langle s(T^n_+)\rangle = \{\langle\underline{w}\rangle \in PC_\pi(1,n+1;\mathbb{C}):\ B(\underline{w},\bar{\underline{w}}) > 0 \text{ and } \mathrm{Im}(\bar{w}_0 w_{n+1}) > 0\}$$

□

It is clear that $\rho_g \in O(2,n+1;\mathbb{C})$ maps $\langle s(T^n)\rangle$ one-to-one onto itself if and only if

$\rho_g \in U(2,n+1)$. Hence $\rho_g \in O(2,n+1;\ \mathbb{C}) \cap U(2,n+1) = O(2,n+1)$, the *real* orthogonal group. What about the connected component $<s(T^n_+)>$?
Let be given $A = (a_{ij}) \in O(2,n+1)$ leaving invariant the quadratic form

$$Q(u) = u_0^2 + u_{n+1}^2 - (u_1^2 + \dots + u_n^2).$$

It is well-known that A respects the sign of $\det \begin{pmatrix} u_0 & u_{n+1} \\ v_0 & v_{n+1} \end{pmatrix}$ if and only if

$$\det \begin{pmatrix} a_{00} & a_{0\,n+1} \\ a_{n+1\,0} & a_{n+1\,n+1} \end{pmatrix} > 0.$$

After having realized that our arrows ↑↓ indicate the sign of the complementary minor, we recognize

$$SO^{\uparrow}(2,n+1) \cup NO^{\downarrow}(2,n+1)$$

as the subgroup which leaves invariant the sign of $\det \begin{pmatrix} u_0 & u_{n+1} \\ v_0 & v_{n+1} \end{pmatrix}$.
(By definition, $NO(2,n+1) = \{A \in O(2,n+1):\ \det(A) = -1\}$).
Correspondingly, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Twist}_\pi(0,n)$ induces the $T^n_+$ self-mapping $z \to \mu_g(z)$ or $z \to \mu_g(-\hat{z})$ according as $ad^* - bc^* = 1$ or $ad^* - bc^* = -1$. More conveniently, the same group may be given by the transformations $z \to \mu_g(z)$ and $z \to \mu_g(\hat{z})$ with $g \in \mathrm{Twist}^+_\pi(0,n)$ $(ad^* - bc^* = 1)$.
It is obvious that all such self-mappings are biholomorphic. Applying the results of Hirzebruch in [3] we find, conversely, that the group of *all* biholomorphic self-mappings is generated by the orthochronous Lorentz group, the real translations and the involution $z \to -z^{-1}$. We thus arrive at the conclusion

### 4.4. Theorem

The group of biholomorphic $T^n_+$ self-mappings is given by the transformations

$$T^n_+ \to T^n_+,\quad z \to (az + b)(cz + d)^{-1} \ \text{ and } \ z \to (a\hat{z} + b)(c\hat{z} + d)^{-1}$$

with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Twist}^+_\pi(0,n)$. □

**4.5. Examples**

$n = 0$ yields the classical action of $SL(2,\mathbb{R})$ on Poincaré's upper half plane (hoc loco $\hat{z} = z$).

*$Cl(0,2;\mathbb{C}) \cong \mathbb{C}(2)$. We choose $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then the set of paravectors* $z = z_0 + z_1e_1 + z_2e_2$ is exactly the set of symmetric matrices. $z = x + iy$ lies in the tube domain $T_+^2$ if and only if $y_0 > 0$ and $Q(y) > 0$. With $y = \begin{pmatrix} y_0+y_1 & y_2 \\ y_2 & y_0-y_1 \end{pmatrix}$ these conditions amount to a positive trace and determinant of the matrix y. Hence $z = x + iy \in T_+^2$ if and only if y is positive definite: incidentally, $T_+^2$ is a Siegel domain of type three. It is easy to identify $\mathrm{Twist}_\pi^+(0,2) \cong Sp(4,\mathbb{R})$.

Let the real Clifford algebra $Cl(0,3) \cong \mathbb{C}(2)$ be generated by the Pauli spin matrices

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

The set of paravectors $x = x_0 = x_1e_1 + x_2e_2 + x_3e_3$ is the set of Hermitian matrices. The complexification $V_\pi(0,3;\mathbb{C})$ thus may be identified with $\mathbb{C}(2)$, its elements being written as $H_x + iH_y$, where $H_x = \begin{pmatrix} x_0+x_1 & x_2-ix_3 \\ x_2-ix_3 & x_0-x_1 \end{pmatrix}$ and $H_y = \begin{pmatrix} y_0+y_1 & y_2+iy_3 \\ y_2-iy_3 & y_0-y_1 \end{pmatrix}$ are Hermitian. Concerning the tube domain, we find $H_x + iH_y \in T_+^3$ if and only if $H_y$ is positive definite. The group of biholomorphic self-mappings is given by the transformations $z \to \mu_g(z)$ and $z \to \mu_g(\hat{z})$ with $g \in \mathrm{Twist}_\pi^+(0,3) \cong SU(2,2)$.

□

Recall that the metric $(ds)^2 = \dfrac{|dz|^2}{\mathrm{Im}^2(z)}$ is invariant under the $SL(2,\mathbb{R})$-action on the upper half plane. It is a natural desire to generalize this metric on the tube domain $T_+^n$.

Given a real paravector $y \in V_\pi(0,n)$, let $P_y$ denote the matrix representation (along the basis $1,e_1,\dots,e_n$) of the linear transformation $w \to ywy$ in $V_\pi(0,n)$.
One easily verifies that the matrix $P_y$ has the entries

$$P_{00} = 2y_0^2 - Q(y)$$

$$P_{kk} = 2y_k^2 + Q(y), \quad k \in \{1,2,...,n\}$$

$$P_{ij} = 2y_i y_j, \qquad i \neq j \in \{0,1,...,n\}.$$

Obviously, the structure of $P_y$ is the same for all $n \geq 1$.
A continuity argument makes clear that det $(P_y) > 0$ whenever y lies in the convex domain

$$\{y \in V_\pi(0,n): \; y_0 > 0 \; \text{ and } \; Q(y) > 0\}.$$

Recurrently, if y lies in this domain we find det $(P_y^k) > 0$, $k \in \{0,1,...,n\}$, where

$$P_y^k = \begin{pmatrix} P_{00} & \cdots & P_{0k} \\ \vdots & & \vdots \\ P_{k0} & \cdots & P_{kk} \end{pmatrix}.$$

Hence, for any $z = x + iy \in \mathbf{T}_+^n$ the matrix $P_y$ is positive definite.
To generalize the Poincaré metric, we must require an infinitely distant position of the null-cone $Q(y) = 0$. For that reason we employ $P_{y^{-1}}$ (also positive definite) instead of $P_y$.
Now let the metric on $T_+^n$ (Im $z = y$) be given by the line element

$$(ds)^2 = (dz)^t P_{y^{-1}} \overline{dz}.$$

(With impunity we use the same symbol dz to denote both the paravector $dz_0 + (dz_1)e_1 + \ldots + (dz_n)e_n$ and its representation $(dz_0, dz_1, ..., dz_n)^t$). To find the Clifford algebra notation of this line element, we need the relation $<u,v> = B(u,\tilde{v})$ between the Euclidean inner product $<\cdot,\cdot>$ and the bilinear form B on the Minkowski space $V_\pi(0,n)$. The alternative expression (below) enables us to prove the following, expected

**4.6. Theorem**

The biholomorphic $T_+^n$ self-mappings leave invariant the positive definite line element

$$(ds)^2 = B(dz, (y^{-1}\overline{dz}\,y^{-1})^\sim).$$

*Proof*

Consider the self-mapping $z \to \mu_g(z)$, induced by $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Twist}_\pi^+(0,n)$. We set $z = x + iy$ and $w = (az + b)(cz + d)^{-1} = u + iv$. With $p = (cz + d)^{-1} \in \Gamma_\pi(0,n; \mathbb{C})$ we have $dw = p^* dz p$ and $v = p^* y\bar{p} = \bar{p}^* yp$, implying that $v^{-1}\overline{dw}\, v^{-1} = p^{-1}(y^{-1}\overline{dz}\, y^{-1})\, p^{*-1}$. Consequently, $B(dw, (v^{-1}\overline{dw}\, v^{-1})^{\sim}) = B(p^* dz p, \hat{p}^{-1}(y^{-1}\overline{dz} y^{-1})^{\sim} \tilde{p}^{-1}) =$
$= B(p^{*-1} p^* dz p\tilde{p}, p^{*-1}\hat{p}^{-1}(y^{-1}\overline{dz} y^{-1})^{\sim} \tilde{p}^{-1}\tilde{p}) =$
$= B(N_{-1}(p)\, dz, N_{-1}(p^{-1})(y^{-1}\overline{dz} y^{-1})^{\sim}) = B(dz, (y^{-1}\overline{dz} y^{-1})^{\sim})$.

As far as the self-mappings $z \to \mu_g(\hat{z})$ are concerned, it suffices to observe that the transformation $z \to \hat{z}$ respects the line element $(ds)^2$ on the tube domain $T_+^n$.

□

**The bounded domain**

Our final exposure concerns the bounded realization of the tube domain, which is obtained by subjecting $T_+^n$ to the (biholomorphic) Cayley mapping

$$z \to \mu_k(z) = (z - i)(-iz + 1)^{-1}, \text{ induced by}$$

$$k = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \in \mathrm{Twist}_\pi^+(0,n; \mathbb{C}).$$

Observe that $T_+^n$ does not contain any singularity of $\mu_k$: we are allowed to define $D_+^n = \mu_k(T_+^n)$.
Hence $z \in D_+^n$ if and only if $\mu_{k^{-1}}(z) = (z + i)(iz + 1)^{-1} \in T_+^n$.
Since the imaginary part of $\mu_{k^{-1}}(z)$ is

$$(z - i)^{-1}\,(1 - z\bar{z})\,(\bar{z} + i)^{-1}$$

and $Q(z - i)Q(\bar{z} + i) = |Q(z - i)|^2 > 0$ ($i \notin D_+^n$), the requirement $Q(\mathrm{Im}(\mu_{k^{-1}}(z))) > 0$ amounts to the inequality

$$N_{-1}(1 - z\bar{z}) = 1 + |Q(z)|^2 - 2B(\tilde{z}, \bar{z}) > 0.$$

On account of the (in)equality $|Q(z)| \le B(\tilde{z}, \bar{z})$, the condition above implies that

$B(\tilde{z},\bar{z}) < \frac{1}{2}(1 + B^2(\tilde{z},\bar{z}))$. Inferentially, $B(\tilde{z},\bar{z}) < 1$ or $B(\tilde{z},\bar{z}) > 1$.
Since $\mu_k$ is continuous and $\mu_k(i) = 0$ obeys the inequality $B(\tilde{z},\bar{z}) < 1$, we arrive at the determination

$$D_+^n = \{z \in V_\pi(0,n;\mathbb{C}):\ B(\tilde{z},\bar{z}) < \frac{1}{2}(1 + |Q(z)|^2) \text{ and } B(\tilde{z},\bar{z}) < 1\}.$$

With $|Q(z)| \leq B(\tilde{z},\bar{z})$ we find

$$D_+^n = \{z \in V_\pi(0,n;\mathbb{C}):\ B(\tilde{z},\bar{z}) < \frac{1}{2}(1 + |Q(z)|^2) < 1\}.$$

This bounded set is readily seen to be a Siegel domain of type four: just replace $z_0$ by $iz_0$ to obtain

$$\{z \in \mathbb{C}^{n+1}:\ |z_0|^2 + \dots + |z_n|^2 < \frac{1}{2}(1 + |z_0^2 + \dots + z_n^2|^2) < 1\}.$$

We proceed to determine the group of biholomorphic $D_+^n$ self-mappings.
For any $g \in \mathrm{Twist}_\pi^+(0,n)$ the commutative

**4.7. Diagram**

$$\begin{array}{ccc} T_+^n & \xrightarrow{\mu_g} & T_+^n \\ \mu_k \downarrow & & \downarrow \mu_k \\ D_+^n & \xrightarrow[\mu_{kgk^{-1}}]{} & D_+^n \end{array}$$

□

together with the commutation rule $\hat{\mu}_k(z) = \mu_k(\hat{z})$ yields the biholomorphic self-mappings

$$D_+^n \to D_+^n,\quad z \to \mu_{kgk^{-1}}(z) \quad \text{and} \quad z \to \mu_{kgk^{-1}}(\hat{z}).$$

It is readily ascertainable that

$$k\,\mathrm{Twist}_{\pi}^{+}(0,n)\,k^{-1} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Twist}_{\pi}^{+}(0,n;\,\mathbb{C}):\ d = \bar{a} \text{ and } c = \bar{b} \right\}.$$

Following the same procedure as in Chapter 3 (with regard to the group of the unit ball), we find that this subgroup leaves invariant the expression

$$\chi(x) \overset{\text{def}}{=} \bar{x}_1^{*} x_1 - \bar{x}_2^{*} x_2 \text{ on pre-}\overline{V}_{\pi}(0,n;\,\mathbb{C}).$$

In particular, the property $\chi(x) = 0$ happens to be a class invariant. For any $\langle x\rangle \in \overline{V}_{\pi}(0,n;\,\mathbb{C})$ with $\chi(x) = 0$ we may take the unique representative $x \sim \binom{z}{1}$, where $z = x_1 x_2^{-1} \in V_{\pi}(0,n;\,\mathbb{C})$, so that $\chi(x) = 0$ amounts to the condition $\bar{z}z = 1$. Consequently, any biholomorphic $D_{+}^{n}$ self-mapping maps the set of *unitary* elements

$$U^{n} \overset{\text{def}}{=} \{z \in V_{\pi}(0,n;\,\mathbb{C}):\ \bar{z}z = 1\}$$

one-to-one onto itself.

We all know that $U^{0} = S^{1}$ is the closure of $\mu_k(V_{\pi}(0)) = \mu_k(\mathbb{R})$. More generally, we prove

**4.8. Theorem**

$U^{n}$ is the closure of $\mu_k(V_{\pi}(0,n))$.

*Proof*

For any real paravector $x \in V_{\pi}(0,n)$ we have $\bar{\mu}_k(x)\mu_k(x) = (x + i)(x - i)^{-1}(x - i) \cdot (x + i)^{-1} = 1$. Hence $\mu_k(V_{\pi}(0,n)) \subset U^{n}$. Since $U^{n}$ is a closed set, it must also contain the closure of $\mu_k(V_{\pi}(0,n))$.

Conversely, assume $z \in U^{n}$. We distinguish between the following three cases.

(i) $Q(z - i) \neq 0$.

$\mu_{k^{-1}}(z) = (z + i)(iz + 1)^{-1}$ is well-defined. The imaginary part is $(\bar{z} + i)^{-1}(1 - \bar{z}z) \cdot (z - i)^{-1} = 0$, because $\bar{z}z = 1$. Hence $\mu_{k^{-1}}(z) \in V_{\pi}(0,n)$.

(ii) $Q(z - i) = 0$, $Q(z + i) \neq 0$.

Now the same reasoning applies to $(\mu_{k^{-1}}(z))^{-1}$, which means that $\mu_{k^{-1}}(z)$ lies on the cone at infinity in $V_\pi(0,n)$.

(iii) $Q(z - i) = Q(z + i) = 0$.

The image is the point at infinity of the null-line $\lambda(z + i)(i\tilde{z} + 1)$, $\lambda \in \mathbb{C}$ (cf. page 35). We must show that $(z + i)(i\tilde{z} + 1)$ is proportional to a *real* paravector. The conditions $Q(z - i) = Q(z + i) = 0$ and $\bar{z}z = 1$ imply $z_0 = 0$, $z_1^2 + ... + z_n^2 = -1$ and $|z_1|^2 + ... + |z_n|^2 = 1$.
But then z must be of the form $z = iy$, $y \in V_\pi(0,n)$.
Consequently, $(z + i)(i\tilde{z} + 1) = 2i(1 + y)$, the required result.

□

The pull back of $ds^2(z) = B(dz,(y^{-1}\overline{dz}\ y^{-1})^\sim)$, $z = x + iy \in T^n_+$, defines a metric on $D^n_+$ which is invariant under any biholomorphic self-mapping.
We substitute $z = (w + i)(iw + 1)^{-1}$, $w \in D^n_+$.
Then the equations $dz = 2(iw + 1)^{-1}dw(iw + 1)^{-1}$ and

$$y = (w - i)^{-1}(1 - w\bar{w})(\bar{w} + i)^{-1} = (\bar{w} + i)^{-1}(1 - \bar{w}w)(w - i)^{-1}$$

bring us to the line element

$$ds^2(w) = 4B(dw,[(1 - \bar{w}w)^{-1}\overline{dw}(1 - w\bar{w})^{-1}]^\sim) \text{ on } D^n_+.$$

Observe that n = 0 yields the familiar line element

$$ds^2(w) = 4\frac{|dw|^2}{(1 - |w|^2)^2} \text{ on the unit disk.}$$

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## Samenvatting

De kern van dit proefschrift bestaat uit de generalisatie van de (anti-)Möbius transformaties in het complexe vlak. Er wordt een formalisme ontwikkeld waarmee de (anti-)Möbius transformaties in een reële vectorruimte met een kwadratische vorm van willekeurige signatuur kunnen worden beschreven. Een tweede orde periodiciteit van Clifford algebra's stelt ons in staat om de niet-lineaire (anti-)Möbius transformaties in bovenstaande vectorruimte te laten samenhangen met de lineaire orthogonale transformaties in een omhullende projectieve ruimte. De (anti-)Möbius transformaties zijn conforme transformaties wat betreft de door de kwadratische vorm geïnduceerde pseudo-metriek. Vanwege de singulariteiten wordt er een kegel op oneindig aan bovenstaande vectorruimte toegevoegd. Deze compactificatie is een homogene ruimte van de conforme groep.

Als toepassing van de theorie beschouwen we twee generalisaties van de meetkunde van het halfvlak (de cirkelschijf). Een reële generalisatie van de Poincaré-geometrieën verschaft ons twee conforme modellen van de hyperbolische meetkunde van willekeurige dimensie. De hyperbolische groep wordt in beide modellen gerepresenteerd door een ondergroep van de Euclidisch conforme groep. Anderzijds geeft de ophoging van de complexe dimensie ons twee modellen van het Siegel domein van de vierde soort. De groep van biholomorfe bijecties wordt in beide modellen gerepresenteerd door een ondergroep van de complex conforme groep.

**Curriculum Vitae**

Johannes G. Maks was born in 's-Gravenhage on the sixth of December, 1959. He attended the Gymnasium at the "Christelijke Scholengemeenschap Voorburg 't Loo" during the period 1972-1978, before entering the University of Technology in Delft. There he studied pure and applied mathematics. Towards the end of the course his special interest was in subjects like projective geometry, tensor analysis and the theory of group representations. Supervised by Prof.dr.ir. T.H.M. Smits he wrote an essay on Clifford algebras and metrical geometry during the spring of 1985.
After his graduation as a mathematical engineer on the thirtieth of August, 1985, he was appointed as a lecturer of mathematics at the University of Technology in Delft. Since then he has been working at the preparation and, eventually, the completion of this thesis.