

Quantum synchronisation
and the validity of its derivation

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Abstract

In this bachelor thesis, the paper "Classical synchronization indicates persistent entanglement in isolated quantum systems" by *Dirk Witthaut et al.* is looked into and discussed.

First, synchronization in a classic sense is explained. The Kuramoto model is introduced, and a few properties of this model are defined. In the next part of the theoretical background, the creation and annihilation operators are defined. An example is given on how to derive these operators, and how to write the Hamiltonian in the form of creation and annihilation operators. It becomes clear that the Hamiltonian of a vector potential in vacuum does not have any coupling terms. This is why no synchronization will occur here.

Assuming the Hamiltonian has a different form with two-body interactions and a coupling factor, then it will have some sort of interaction between modes. Dirk Witthaut published in his paper a way to derive the Kuramoto equation from this Hamiltonian. First, the time derivative of the expectation value of the \hat{a}_n operator is evaluated by using the Ehrenfest theorem. This returns multiple three point functions. These can also be evaluated by the Ehrenfest theorem, but that will only result in more coupled equations and five point functions. To solve it, a first order mean field approximation is used. According to Witthaut, this results in a series of coupled complex differential equations which can be rewritten into the Kuramoto equations. Witthaut then further elaborates this result in the remainder of his paper.

My calculations point towards a different conclusion. Witthaut made a mistake when calculating the commutation relations that were used in the Ehrenfest theorem. This resulted in a different system of coupled equations, which I couldn't elaborate into the Kuramoto model. This is the conclusion of this bachelor thesis.

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1. Introduction

As quantum mechanics tells us, every particle is simultaneously a wave and a particle. This is in itself a huge statement, and leads to interesting mechanics. A particle can vibrate like a string, flow like a wave, or bounce like a ball. All in all, there are a lot of different processes and every situation leads to different results.

Luckily, there is one equation that predicts quantum movements in every situation: the Schrödinger equation. All you need is a Hamiltonian and an initial state of the system. But which Hamiltonian describes a system? Almost every different problem requires a different Hamiltonian, thus formulating the Hamiltonian is a decisive step.

In the paper "Classical synchronization indicates persistent entanglement in isolated quantum systems" by *Dirk Witthaut et al.*, a certain Hamiltonian is assumed. It is described how the Ehrenfest theorem can be used to derive the Kuramoto model from this Hamiltonian. The paper you're holding is about this derivation.

Firstly, I will look into the theoretical background of classical synchronization and the Kuramoto model. In the following sections, the creation and annihilation operators and their role in the Hamiltonian is elaborated. In this part, I will also make some plots of the expectation value. Finally, the Ehrenfest theorem is applied and a conclusion is made.

This bachelor thesis is part of the double bachelor *Applied Physics* and *Applied Mathematics* on Delft University of Technology.

2. Theoretical background

In the following sections, the necessary theoretical background behind synchronization will be reviewed, after which the problem with quantummechanical synchronization will be discussed. First, the Kuramoto model will be introduced, and mean field coupling will be applied to the model. The following section starts with the Maxwell equations. From here, a quantization of the magnetic vector potential will be derived. Then the Hamiltonian from [1] will be introduced. This Hamiltonian is then used to visualise the system using a matrix representation. Finally, I will investigate the derivation of the Kuramoto equation.

2.1 Classical synchronization

In 1975, Yoshiki Kuramoto introduced the Kuramoto model for N weakly coupled oscillators. This model describes the way the phase of these oscillators interact with each other by the following equation:

$$\frac{d\theta_i}{dt} = \omega_i + \sum_{j=1}^N K_{ij} \sin(\theta_j - \theta_i) \quad (2.1)$$

where θ_i represents the phase of the i th oscillator, ω_i is the natural frequency of this oscillator, and K_{ij} is the "coupling factor", or just "coupling". Kuramoto solved this model for a constant $K_{ij} = \frac{K}{N}$ and $N \rightarrow \infty$ [4][5]. This proof uses the transformation

$$R = \frac{1}{N} \sum_{j=1}^N e^{i\theta_j} = r e^{i\phi} \quad (2.2)$$

where r and ϕ are real. Combining (2.1) and (2.2) gives

$$\frac{d\theta_i}{dt} = \omega_i + K r \sin(\phi - \theta_i) \quad (2.3)$$

which is a simplification of the Kuramoto model, also known as "mean field coupling". A more detailed elaboration of this proof is given in 'Multi-mode Quantum Synchronization'[2] by Lotte van Dongen.

One oscillator will be influenced by all other oscillators, and the magnitude of this influence is dependent on K . When K is small, the oscillators will

have almost no influence on each other; they will oscillate incoherently. When K is sufficiently large, the oscillators will oscillate uniformly, which is called "phase locking". This is a recognisable property of the Kuramoto model.

The Kuramoto model depends on a few assumptions. It is applicable to a system of weakly coupled oscillators that have a certain interaction which depends sinusoidally on the phase difference of each pair of oscillators. In the next section, a quantization of the vector potential is shown, and it is evaluated if the Kuramoto model can be applied.

2.2 Quantization of the vector potential

The Maxwell equations are the known laws of EM-fields, (in Gaussian units) given by:

$$\nabla \cdot \vec{E} = 4\pi\rho \quad (2.4)$$

$$\nabla \cdot \vec{B} = 0 \quad (2.5)$$

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \quad (2.6)$$

$$\nabla \times \vec{B} = \frac{1}{c} (4\pi\vec{J} + \frac{\partial \vec{E}}{\partial t}) \quad (2.7)$$

Given that we have a vacuum, the Maxwell equations become:

$$\nabla \cdot \vec{E} = 0 \quad (2.8)$$

$$\nabla \cdot \vec{B} = 0 \quad (2.9)$$

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \quad (2.10)$$

$$\nabla \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \quad (2.11)$$

where $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$ is the light speed. We can now define the magnetic vector potential \vec{A} , which is:

$$\vec{B} = \nabla \times \vec{A} \quad (2.12)$$

Given (2.10) and (2.12), an expression for the electric field \vec{E} in terms of \vec{A} can also be found:

$$\nabla \times \vec{E} = -\frac{1}{c} \nabla \times \frac{\partial \vec{A}}{\partial t} \Rightarrow \vec{E} = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t} \quad (2.13)$$

These vector potentials are not unique; if \vec{A} gives some fields \vec{E} and \vec{B} , then $\vec{A}' = \vec{A} + \nabla f$ also gives \vec{E} and \vec{B} . f is called the "gauge transformation", and is a twice differentiable function that depends on position and time[7]. For simplification, we use the Coulomb Gauge, for which $\nabla \cdot \vec{A} = 0$. Using (2.11), (2.12), (2.13) and the property of the cross product $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$ gives:

$$\nabla \times (\nabla \times \vec{A}) = -\frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} \quad (2.14)$$

$$\nabla^2 \vec{A} = \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} \quad (2.15)$$

This differential equation has a solution of the form $\vec{A}(\vec{r}, t) = \sum f(\vec{k} \cdot \vec{r} - \omega_{\vec{k}} t)$, also know as a planar wave, with a linear disperion relation $\omega_{\vec{k}} = c|\vec{k}|$. If there are periodic boundary conditions, such that

$$\vec{A}(\vec{r}, t) = \vec{A}(\vec{r} + L\hat{x}, t) = \vec{A}(\vec{r} + L\hat{y}, t) = \vec{A}(\vec{r} + L\hat{z}, t) \quad (2.16)$$

then equation (2.15) gives the following solution:

$$A(\vec{r}, t) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \vec{A}_{\vec{k}}(t) e^{i\vec{k} \cdot \vec{r}} \quad (2.17)$$

with $\vec{k} = \frac{2\pi}{L}(n_x \hat{x} + n_y \hat{y} + n_z \hat{z})$ and $V = L^3$, where $n_x, n_y, n_z \in \mathbb{N}$. Each coefficient $\vec{A}_{\vec{k}}(t)$ is an amplitude for a wave at the stated wave vector[7]. This representation gives an orthonormal basis, because

$$\int e^{i\vec{k}_1 \cdot \vec{r}} (e^{i\vec{k}_2 \cdot \vec{r}})^* d^3 \vec{r} = V \delta_{\vec{k}_1 \vec{k}_2} \quad (2.18)$$

Of course, $\vec{A}(\vec{r}, t)$ is a real function, and (2.17) shows complex modes. This problem can be solved by writing \vec{A} as

$$\vec{A}(\vec{r}, t) = \frac{1}{2\sqrt{V}} \sum_{\vec{k}} (\vec{A}_{\vec{k}}(t) e^{i\vec{k} \cdot \vec{r}} + \vec{A}_{\vec{k}}^*(t) e^{-i\vec{k} \cdot \vec{r}}) \quad (2.19)$$

where another factor of $\frac{1}{2}$ is added to keep the orthonormal basis.

2.3 Independence of the harmonic oscillators

Since (2.15) resulted in a planar wave, it is assumed that $\vec{A}_{\vec{k}}(t) = \vec{a}_{\vec{k}}e^{-i\omega_{\vec{k}}t}$. Using this fact, combining (2.12), (2.13) and (2.19) gives

$$\vec{B} = \frac{1}{2\sqrt{V}} \sum_{\vec{k}} i\vec{k} \times \vec{A}_{\vec{k}}(t)e^{i\vec{k}\cdot\vec{r}} - i\vec{k} \times \vec{A}_{\vec{k}}(t)e^{-i\vec{k}\cdot\vec{r}} \quad (2.20)$$

$$\vec{E} = \frac{1}{2c\sqrt{V}} \sum_{\vec{k}} i\omega_{\vec{k}}\vec{A}_{\vec{k}}(t)e^{i\vec{k}\cdot\vec{r}} - i\omega_{\vec{k}}\vec{A}_{\vec{k}}(t)e^{-i\vec{k}\cdot\vec{r}} \quad (2.21)$$

The Lagrangian density of the free EM field is $\mathcal{L} = \frac{1}{8\pi}(\vec{E}^2 - \vec{B}^2)$, and from here we can work out the canonical momenta in the coordinate system $\vec{q} = \vec{A} = (A_x, A_y, A_z)[7]$. From (2.12) it follows that $\frac{\partial \vec{B}}{\partial \vec{A}_i} = 0$, and (2.13) gives $\frac{\partial E_i}{\partial A_i} = \frac{-1}{c}$. This gives the canonical momenta

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = \frac{\partial(\frac{1}{8\pi}(\vec{E}^2 - \vec{B}^2))}{\partial \dot{A}_i} = \frac{-1}{4\pi c}E_i (= \frac{1}{4\pi c^2} \frac{\partial \vec{A}}{\partial t}) \quad (2.22)$$

From the Lagrangian density, the Hamiltonian density can be derived:

$$\mathcal{H} = \sum_i p_i \dot{q}_i - \mathcal{L} = \vec{p} \cdot \frac{\partial \vec{A}}{\partial t} - \frac{1}{8\pi}(\vec{E}^2 - \vec{B}^2) \quad (2.23)$$

Substituting (2.22) and (2.13) in this equation gives a familiar expression for the Hamiltonian density of the free EM field:

$$\mathcal{H} = \frac{-1}{4\pi c} \vec{E} \cdot (-c\vec{E}) - \frac{1}{8\pi}(\vec{E}^2 - \vec{B}^2) = \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2) \quad (2.24)$$

To find the total energy (in other words: to evaluate the Hamiltonian), we need to work out $\int \vec{E}^2 d^3\vec{r}$ and $\int \vec{B}^2 d^3r$. Working out these integrals will give

$$\int \vec{E}^2 d^3\vec{r} = \sum_{\vec{k}} \frac{\omega_{\vec{k}}^2}{c^2} |\vec{A}_{\vec{k}}|^2 \quad (2.25)$$

$$\int \vec{B}^2 d^3\vec{r} = \sum_{\vec{k}} k^2 |\vec{A}_{\vec{k}}|^2 \quad (2.26)$$

As we can see, the modes have independent influences on the Hamiltonian, which means the different modes don't interfere with each other. Each vector potential of the vacuum can be seen as a linear combination of modes! Because $\omega_{\vec{k}} = c|k|$, we can also conclude that the electric energy is the same as the magnetic energy. Now, we will define a new coordinate system, related to mode \vec{k} :

$$\vec{q}_{\vec{k}} = \vec{A}_{\vec{k}} \quad (2.27) \quad \vec{p}_{\vec{k}} = \frac{1}{4\pi c^2} \frac{\partial \vec{A}_{\vec{k}}}{\partial t} = \frac{-i\omega_{\vec{k}}}{4\pi c^2} \vec{A}_{\vec{k}} \quad (2.28)$$

where at equation (2.28) the assumption is used that $\vec{A}_{\vec{k}}(t) = \vec{a}_{\vec{k}} e^{-i\omega_{\vec{k}} t}$. This means (2.25) and (2.27) give an electric energy, and (2.26) and (2.28) give a magnetic energy, both expressed in the coordinate system $\vec{q}_{\vec{k}}$ and $\vec{p}_{\vec{k}}$. This gives the Hamiltonian of a single mode

$$H = \sum_{\vec{k}} (2\pi c^2 |\vec{p}_{\vec{k}}|^2 + \frac{\omega_{\vec{k}}^2}{8\pi c^2} |\vec{q}_{\vec{k}}|^2) \quad (2.29)$$

It becomes clear that the modes don't interfere with each other. This means synchronization between the modes will not occur here. In the next section, a Hamiltonian will be introduced where synchronization can be possible.

2.4 Hamiltonian operator

Assume there is a system with L modes. For each mode, the ladder operators \hat{a}_j and \hat{a}_j^\dagger lower or raise the number of particles in mode j . Furthermore, the number operator \hat{n} is defined as $\hat{n} = \hat{a}^\dagger \hat{a}$. The eigenstates of the number operator are called the Fock eigenstates. They can be represented by $|n_j\rangle$, where n_j is the number of particles in mode j . The following equations hold:

$$\begin{aligned} \hat{a}_j |n_j\rangle &= \sqrt{n_j} |n_j - 1\rangle & [\hat{a}_j, \hat{a}_k^\dagger] &= \delta_{jk} \\ \hat{a}_j^\dagger |n_j\rangle &= \sqrt{n_j + 1} |n_j + 1\rangle & [\hat{n}_j, \hat{a}_k^\dagger] &= \hat{a}_j^\dagger \delta_{jk} \\ \hat{n} |n_j\rangle &= n_j |n_j\rangle & [\hat{n}_j, \hat{a}_k] &= -\hat{a}_j \delta_{jk} \end{aligned}$$

where δ_{jk} is the Kronecker delta. Now assume the modes interact according to the following Hamiltonian[1]:

$$\hat{H} = \sum_{l=1}^L \omega_l \hat{a}_l^\dagger \hat{a}_l + \frac{U}{2} \hat{a}_l^{\dagger 2} \hat{a}_l^2 + \sum_{k=1}^L \frac{K_{k,l}}{8} [i(\hat{a}_k^\dagger \hat{a}_l - \hat{a}_l^\dagger \hat{a}_k)(\hat{a}_k^\dagger \hat{a}_k - \hat{a}_l^\dagger \hat{a}_l) + h.c.] \quad (2.30)$$

This Hamiltonian is introduced as "the Hamiltonian (...) describing L spatially localised modes $j \in \{1, \dots, L\}$ with on-site two-body interactions of energy scale U ." At first, this Hamiltonian may seem arbitrary, but the terms can be explained one by one.

The first term $\omega_l \hat{a}_l^\dagger \hat{a}_l$ can be written as $\omega_l \hat{n}_l$. This is by a constant equal to the energy levels of a quantum harmonic oscillator, which is $E_n = \hbar\omega(n + \frac{1}{2})$. This means the first term represents the energy needed to have n_l particles in mode l .

The second term $\frac{U}{2} \hat{a}_l^{\dagger 2} \hat{a}_l^2$ can be written out using the commutator $[\hat{n}_l, \hat{a}_l^\dagger] = \hat{a}_l^\dagger$:

$$\frac{U}{2} \hat{a}_l^\dagger \hat{a}_l^\dagger \hat{a}_l \hat{a}_l = \frac{U}{2} \hat{a}_l^\dagger \hat{n}_l \hat{a}_l = \frac{U}{2} (\hat{n}_l \hat{a}_l^\dagger - \hat{a}_l^\dagger) \hat{a}_l = \frac{U}{2} (\hat{n}_l^2 - \hat{n}_l) \quad (2.31)$$

Now, $\frac{1}{2}(n^2 - n)$ can be recognised as the amount of edges in a complete graph with n vertices. A complete graph is a graph where each vertex is connected to all of the other vertices with an undirected edge. This means the second term in the Hamiltonian is representing the on-site two-body interaction; if there are n_l points in mode l , then there are $\frac{1}{2}(n_l^2 - n_l)$ different two-body interactions, and all of those interactions have an energy U . In conclusion, the total contribution of the two-body interaction in mode l is $\frac{U}{2}(n_l^2 - n_l)$.

The third term of the Hamiltonian describes the interaction between different modes. This term is the origin of the synchronization, but it requires some time to fully write it out. First, lets look at just one single term of the sum: $\frac{K_{k,l}}{8} [i(\hat{a}_k^\dagger \hat{a}_l - \hat{a}_l^\dagger \hat{a}_k)(\hat{a}_k^\dagger \hat{a}_k - \hat{a}_l^\dagger \hat{a}_l) + h.c.]$. In this context, $h.c.$ stands for Hermitian conjugate. $\hat{a}_k^\dagger \hat{a}_k$ can be written as \hat{n}_k , so this term is the same as $\frac{K_{k,l}}{8} [i(\hat{a}_k^\dagger \hat{a}_l - \hat{a}_l^\dagger \hat{a}_k)(\hat{n}_k - \hat{n}_l) + h.c.]$. The other term $\hat{t}_{k,l} = (\hat{a}_k^\dagger \hat{a}_l - \hat{a}_l^\dagger \hat{a}_k)$ can be seen as a sum of *hopping operators*, where $\hat{a}_k^\dagger \hat{a}_l$ is the term that represents 1 particle that goes from mode l to mode k , and $\hat{a}_l^\dagger \hat{a}_k$ represents a particle that goes the other way.

2.5 Matrix representation

We want to numerically simulate this system. This can be done by simulating the Schrödinger equation with a matrix representation of the Hamiltonian. Define a basis $B = \{|b_1\rangle, |b_2\rangle, \dots\}$ as a set of states which span the Hilbert space. The state of the system will be an array $\phi = \sum_{n=1}^{\infty} c_n |b_n\rangle$, and the Hamiltonian matrix will contain the elements $\mathbf{H}_{n,m} = \langle b_n | \hat{H} | b_m \rangle$.

To get a matrix representation of the Hamiltonian, it is easier to look at each term of the Hamiltonian on its own. The first term of the Hamiltonian $\sum_{l=1}^L \omega_l \hat{n}_l$ only adds a term to the diagonal of \mathbf{H} , because

$$\langle n_1, n_2, \dots, n_L | \omega_l \hat{n}_l | m_1, m_2, \dots, m_L \rangle = \omega_l n_l \delta_{n_l, m_l} \quad (2.32)$$

This results in a diagonal term of $\omega_l n_l$. The same result can be found in the second term $\sum_{l=1}^L \frac{U}{2} \hat{n}_l (\hat{n}_l - 1)$, which gives a diagonal term of $\frac{U}{2} n_l (n_l - 1)$.

The third term of the sum becomes more clear when it is applied to a quantum state $|\psi\rangle$. For simplification, the quantum state will be written as $|n_k, n_l\rangle$ ($n \neq k$), just to have an idea what the effect is of the k^{th} and l^{th} term of the Hamiltonian on the eigenstate with n_k particles in mode k , n_l particles in mode l and 0 particles in the other modes. It is clear that the addition of the third term will be zero when $k = l$. We will use $(AB)^\dagger = B^\dagger A^\dagger$, $i^\dagger = -i$ and the definition of \hat{a}_l and \hat{a}_l^\dagger :

$$\begin{aligned} & \frac{K_{k,l}}{8} [i(\hat{a}_k^\dagger \hat{a}_l - \hat{a}_l^\dagger \hat{a}_k)(\hat{n}_k - \hat{n}_l) + (i(\hat{a}_k^\dagger \hat{a}_l - \hat{a}_l^\dagger \hat{a}_k)(\hat{n}_k - \hat{n}_l))^\dagger] |n_k, n_l\rangle \\ &= \frac{K_{k,l}}{8} [i(\hat{a}_k^\dagger \hat{a}_l - \hat{a}_l^\dagger \hat{a}_k)(\hat{n}_k - \hat{n}_l) - i(\hat{n}_k - \hat{n}_l)(\hat{a}_l^\dagger \hat{a}_k - \hat{a}_k^\dagger \hat{a}_l)] |n_k, n_l\rangle \\ &= \frac{iK_{k,l}}{8} [\hat{a}_k^\dagger \hat{a}_l \hat{n}_k - \hat{a}_l^\dagger \hat{a}_k \hat{n}_k - \hat{a}_k^\dagger \hat{a}_l \hat{n}_l + \hat{a}_l^\dagger \hat{a}_k \hat{n}_l - \hat{n}_k \hat{a}_l^\dagger \hat{a}_k + \hat{n}_l \hat{a}_l^\dagger \hat{a}_k + \hat{n}_k \hat{a}_k^\dagger \hat{a}_l - \hat{n}_l \hat{a}_k^\dagger \hat{a}_l] |n_k, n_l\rangle \\ &= \frac{iK_{k,l}}{8} [\sqrt{n_k + 1} \sqrt{n_l} n_k - \sqrt{n_k + 1} \sqrt{n_l} n_l \\ & \quad + (n_k + 1) \sqrt{n_k + 1} \sqrt{n_l} - (n_l - 1) \sqrt{n_k + 1} \sqrt{n_l}] |n_k + 1, n_l - 1\rangle \\ & \quad + \frac{iK_{k,l}}{8} [\sqrt{n_l + 1} \sqrt{n_k} n_l - \sqrt{n_l + 1} \sqrt{n_k} n_k \\ & \quad + (n_l + 1) \sqrt{n_l + 1} \sqrt{n_k} - (n_k - 1) \sqrt{n_l + 1} \sqrt{n_k}] |n_k - 1, n_l + 1\rangle \\ &= \frac{iK_{k,l}}{4} \sqrt{n_k + 1} \sqrt{n_l} (n_k - n_l + 1) |n_k + 1, n_l - 1\rangle + \frac{iK_{k,l}}{4} \sqrt{n_l + 1} \sqrt{n_k} (n_l - n_k + 1) |n_k - 1, n_l + 1\rangle \end{aligned}$$

The effect of the hopping operator can be seen here; the resulting term only has eigenstates where 1 particle has moved from k to l or vice versa.

Furthermore, the elements of the first two terms of the Hamiltonian all depend on the amount of particles in 1 mode (n_l). The third term however is different. The elements of this sum depend on the amount of particles in 2 modes! This means there needs to be a tensor product to determine \mathbf{H} .

This is a problem, because the matrix will be a lot bigger and thus the numerical approximation will be slower. Assume there are modes $1, \dots, L$, and in the numerical approximation the maximum number of particles in a mode is $M - 1$. When every term in the Hamiltonian depends on the amount of particles in only 1 mode, \mathbf{H} has the size of ML . If there are terms in the Hamiltonian that depend on the amount of particles in multiple modes, then a tensor product is used. This will result in a matrix with a size M^L , which is significantly bigger.

To understand the Hamiltonian a bit better, we will look at a system with 2 modes ("1" and "2"). Substituting $L = 2$ in (2.30) gives:

$$\begin{aligned}
\hat{H} &= \omega_1 \hat{a}_1^\dagger \hat{a}_1 + \omega_2 \hat{a}_2^\dagger \hat{a}_2 + \frac{U}{2} \hat{a}_1^{\dagger 2} \hat{a}_1^2 + \frac{U}{2} \hat{a}_2^{\dagger 2} \hat{a}_2^2 \\
&+ \frac{K_{2,1}}{8} [i(\hat{a}_2^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_2)(\hat{a}_2^\dagger \hat{a}_2 - \hat{a}_1^\dagger \hat{a}_1) + i(\hat{a}_2^\dagger \hat{a}_2 - \hat{a}_1^\dagger \hat{a}_1)(\hat{a}_2^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_2)] \\
&+ \frac{K_{1,2}}{8} [i(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1)(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2) + i(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2)(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1)] \\
&= \omega_1 \hat{a}_1^\dagger \hat{a}_1 + \omega_2 \hat{a}_2^\dagger \hat{a}_2 + \frac{U}{2} \hat{a}_1^{\dagger 2} \hat{a}_1^2 + \frac{U}{2} \hat{a}_2^{\dagger 2} \hat{a}_2^2 \\
&+ \frac{i(K_{2,1} + K_{1,2})}{8} [-\hat{a}_1^\dagger \hat{a}_2 \hat{a}_2^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1 \hat{a}_2^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_2 \hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_2 \hat{a}_2^\dagger \hat{a}_1 \\
&- \hat{a}_2^\dagger \hat{a}_1 \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{a}_2 \hat{a}_1^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_1 \hat{a}_2^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1^\dagger \hat{a}_2]
\end{aligned}$$

Now, the effect of the Hamiltonian on a eigenstate $|n_1, n_2\rangle$ will be calculated.

The definition of \hat{a}_l and \hat{a}_l^\dagger is used here.

$$\begin{aligned}
\hat{H} |n_1, n_2\rangle &= [\omega_1 n_1 + \omega_2 n_2 + U(n_1^2 - n_1) + U(n_2^2 - n_2)] |n_1, n_2\rangle \\
&+ \frac{i(K_{2,1} + K_{1,2})}{8} \{ [-\sqrt{n_1 + 1} \sqrt{n_2} n_2 - (n_2 - 1) \sqrt{n_1 + 1} \sqrt{n_2}] \\
&+ \sqrt{n_1 + 1} \sqrt{n_2} n_1 + (n_1 + 1) \sqrt{n_1 + 1} \sqrt{n_2}] |n_1 + 1, n_2 - 1\rangle \\
&+ [-\sqrt{n_1} \sqrt{n_2 + 1} n_1 - (n_1 - 1) \sqrt{n_1} \sqrt{n_2 + 1}] \\
&+ \sqrt{n_1} \sqrt{n_2 + 1} n_2 + (n_2 + 1) \sqrt{n_1} \sqrt{n_2 + 1}] |n_1 - 1, n_2 + 1\rangle \} \\
&= [\omega_1 n_1 + \omega_2 n_2 + U(n_1^2 - n_1) + U(n_2^2 - n_2)] |n_1, n_2\rangle \\
&+ \frac{i(K_{2,1} + K_{1,2}) \sqrt{n_1 + 1} \sqrt{n_2} (n_1 - n_2 + 1)}{4} |n_1 + 1, n_2 - 1\rangle \\
&+ \frac{i(K_{2,1} + K_{1,2}) \sqrt{n_1} \sqrt{n_2 + 1} (n_2 - n_1 + 1)}{4} |n_1 - 1, n_2 + 1\rangle
\end{aligned}$$

As noted before, the basis of the matrix representation of the Hamiltonian can **not** be $B_1 = \{|n_1 = 0\rangle, |n_1 = 1\rangle, |n_1 = 2\rangle, \dots, |n_2 = 0\rangle, |n_2 = 1\rangle, |n_2 = 2\rangle, \dots\}$ ¹. We will at least need a tensor product to sufficiently include the coupling factor into the calculations.

Instead, a basis will be chosen as follows:

$$\begin{aligned}
B_2 &= \{|n_1, n_2\rangle; n_1, n_2 \in \mathbb{N}_0\} \\
&= \{|0, 0\rangle, |1, 0\rangle, |2, 0\rangle, \dots, |0, 1\rangle, |1, 1\rangle, |2, 1\rangle, \dots, |0, 2\rangle, |1, 2\rangle, |2, 2\rangle, \dots\}
\end{aligned}$$

This basis will be used to represent a numerical approximation of the problem, using the Schrödinger equation $i\hbar \frac{\partial}{\partial t} |\phi(t)\rangle = \hat{H} |\phi(t)\rangle$. A matrix representation of this equation will be used in $i\hbar \frac{\partial}{\partial t} \vec{\phi} = \mathbf{H} \vec{\phi}$.

When making this Hamiltonian, it can be noted that $\langle m_1, m_2 | \hat{H} |n_1, n_2\rangle$ will only not be zero when $m_1 + m_2 = n_1 + n_2$. This greatly reduces the size of the needed matrix. Suppose there is a system with 2 modes, and the maximum amount of particles in 1 modes is set on $M - 1$. If all of the possible systems are taken into calculation, then the basis of the Hamiltonian will have the size of M^2 . However, if it is known that there are N particles

¹Trying to write $|\Phi_1\rangle = \frac{1}{\sqrt{2}} |1, 1\rangle + \frac{1}{\sqrt{2}} |2, 2\rangle$ will result in $|\Phi_2\rangle = \frac{1}{\sqrt{2}} (|n_1 = 1\rangle + |n_1 = 2\rangle + |n_2 = 1\rangle + |n_2 = 2\rangle)$. In the equation for $|\Phi_1\rangle$, there will be no interaction between the two modes, because $n_1 = n_2$ in every case. $|\Phi_2\rangle$ however will have a coupling term. There is a difference between the two notations, so B_1 is not a sufficient basis for this Hamiltonian.

devided amongst these two modes, then the basis only has to be of size $N + 1$ ($B_3 = \{|0, N\rangle, |1, N - 1\rangle, \dots, |N, 0\rangle\}$).

In conclusion; the system with this Hamiltonian has eigenstates where $N = n_1 + n_2$ is constant, and these states dont interact with eachother. This is a big conclusion, because it can help with checking the validity of numerical approximations.

With basis B_3 and a matrix representation H , the system can be simulated. Following the article of Witthaut, it is assumed that the phases of $\langle \hat{a}_1 \rangle$ and $\langle \hat{a}_2 \rangle$ synchronize according to the Kuramoto equation. This is also shown in section 2.6. However, a problem occurs when the state of a system is solely 1 eigenstate with $n_1 + n_2 = N$. $\langle \hat{a}_1 \rangle = \langle \phi | \hat{a}_1 | \phi \rangle = 0$, because $\hat{a}_1 | \phi \rangle$ will only consist of states with $N - 1$ particles. Orthogonality of the states gives $\langle \hat{a}_1 \rangle = \langle \hat{a}_2 \rangle = 0$. This is of course not useful, because the values don't synchronize.

This means that it is impossible to simulate synchronization, without taking multiple eigenstates of $n_1 + n_2$ into account. This is why a fourth and final basis for the numerical approximation is made.

$$B_4 = \{|0, 0\rangle, |0, 1\rangle, |1, 0\rangle, |0, 2\rangle, |1, 1\rangle, |2, 0\rangle, |0, 3\rangle, |1, 2\rangle, \dots\} \quad (2.33)$$

This basis will contain every possible combination $|n_1, n_2\rangle$. Furthermore, the first $\frac{(M+1)(M+2)}{2}$ (recognised as the $M + 1$ th pyramid number) will contain all modes with a maximum of M particles spread accros the modes.

2.5.1 Plotting with the Hamiltonian

I used python in combination with numpy and matplotlib to appriximate how a two mode system would behave with this Hamiltonian. The full code can be found in the appendix.

The inputs of the program are the variables of the system. That is: ω_1 , ω_2 , U , $K_{1,2}$ and $K_{2,1}$. The variable \hbar is also noted, but for simplicity this is set to 1. Finally, the initial state of the system is also given as a variable. The numerical approximation also needs some variables, which are the maximum number of existing particles M , the time step dt and the total time of the simulation t_{max} .

The next simulation is with a small amount of particles and a short time scale, just to show the basics of this simulation (2.1). The plot consists of three subplots. In all three subplots, the horizontal axis is the time axis.

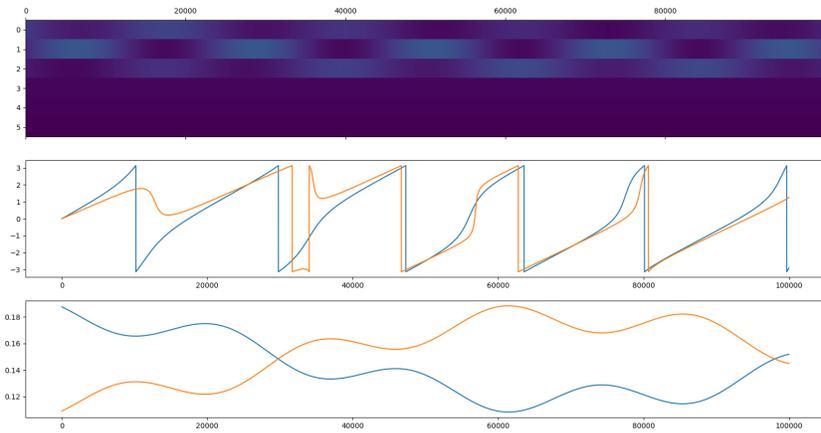


Figure 2.1: $\omega_1 = \omega_2 = K_{1,2} = K_{2,1} = \hbar = 1$, $dt = 0.0001$, $t_{max} = 10$, $M = 2$ and the initial wave was $\phi = \frac{[1,2,3,4,5,6]}{\sqrt{98}}$. Because there are a maximum of 2 particles divided amongst the two modes, there are 6 possible states the system could be in.

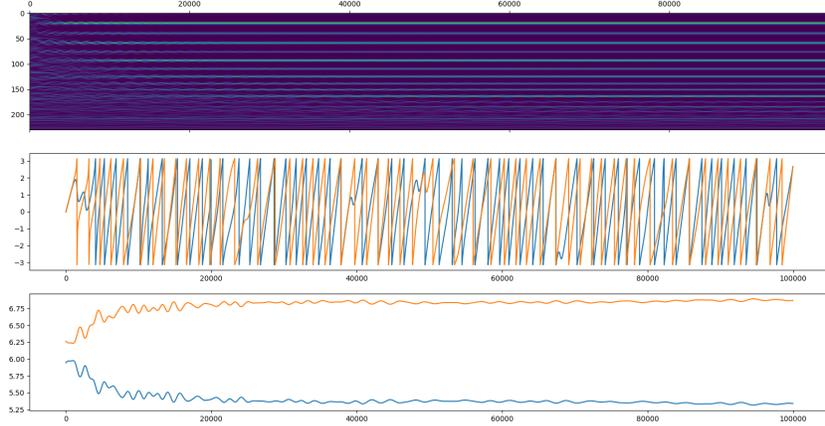


Figure 2.2: $\omega_1 = \omega_2 = K_{1,2} = K_{2,1} = \hbar = 1$, $dt = 0.0001$, $t_{max} = 10$, $M = 20$, and the initial wave was a randomly generated state. The full initial wave can be found in the appendix. It can be seen that the oscillating frequency of \hat{a}_1 and \hat{a}_2 is a lot higher.

This starts at 0, and adds up to the total number of loops. Each dt , one loop passes, so the total will be equal to $\frac{t_{max}}{dt}$. The first plot shows the probability of the system being in a specific state; the lowest is $|0,0\rangle$, above it $|0,1\rangle$, $|1,0\rangle$, $|0,2\rangle$, etc. The lighter the color, the higher the probability. It can be seen that $|0,0\rangle$, $|0,1\rangle$ and $|1,0\rangle$ are completely stationary. This can be derived from the Hamiltonian; these terms have no coupling term and will be constant. The other three states ($|0,2\rangle$, $|1,1\rangle$ and $|2,0\rangle$) oscillate.

The second subgraph shows the complex phase of the expectation value of \hat{a}_1 (blue) and \hat{a}_2 (orange). These are the two variables that should synchronize, according to *Dirk Witthaut et al.* This however is not noticeable in this graph.

The third subgraph shows the expectation value of \hat{n}_1 (blue) and \hat{n}_2 (orange). It can be seen that $\hat{n}_1 + \hat{n}_2$ will always be constant. This confirms the results from the previous chapter.

A more advanced example of this simulation can be seen in the next plot.

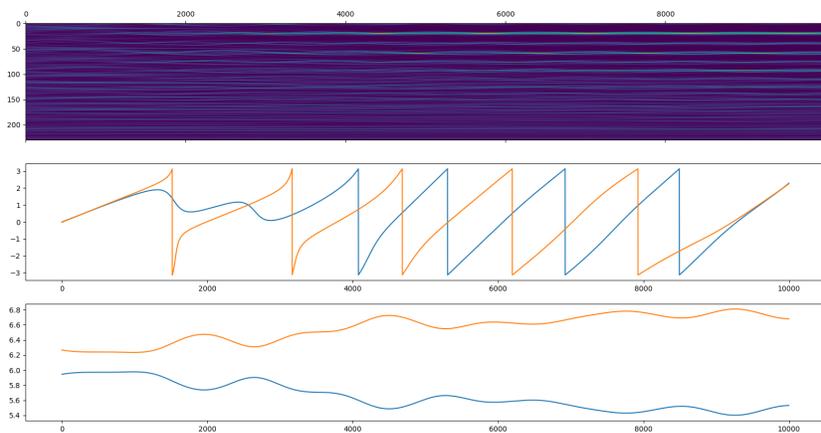


Figure 2.3: $\omega_1 = \omega_2 = K_{1,2} = K_{2,1} = \hbar = 1$, $dt = 0.0001$, $t_{max} = 1$, $M = 20$, and the initial wave is the one used in figure (2.2). A difference between this plot and (2.2) is the t_{max} ; it is 10 times smaller. In this graph, the amount of oscillations is almost the same as in (2.1). This shows that scaling the maximum amount of particles by ten also increases the frequency of the oscillation by roughly ten.

2.6 Deriving the Kuramoto equation

In the following section, the derivation of the Kuramoto equation from the Hamiltonian is presented. We follow the same steps as [1].

From the Schrödinger equation $i\hbar \frac{\partial}{\partial t} |\phi(t)\rangle = \hat{H} |\phi(t)\rangle$, we will derive $\frac{\partial \langle \hat{a}_n \rangle}{\partial t}$. $\langle \hat{a}_n \rangle$ is the expectation value of \hat{a}_n , and is short for $\langle \phi(t) | \hat{a}_n | \phi(t) \rangle$. In the following proof, the Hermitian conjugate of the Schrödinger equation $i\hbar \frac{\partial}{\partial t} \langle \phi(t) | = -\langle \phi(t) | \hat{H}^\dagger = -\langle \phi(t) | \hat{H}$ is also used ($\hat{H}^\dagger = \hat{H}$ because \hat{H} is a hermitian operator). We get:

$$\begin{aligned} \frac{\partial}{\partial t} \langle \hat{a}_n \rangle &= {}_1 \langle \frac{\partial}{\partial t} \phi | \hat{a}_n | \phi \rangle + \langle \phi | \frac{\partial}{\partial t} \hat{a}_n | \phi \rangle + \langle \phi | \hat{a}_n | \frac{\partial}{\partial t} \phi \rangle \\ &= {}_2 \frac{-1}{i\hbar} \langle \phi | \hat{H} \hat{a}_n | \phi \rangle + \langle \frac{\partial \hat{a}_n}{\partial t} \rangle + \frac{1}{i\hbar} \langle \phi | \hat{a}_n \hat{H} | \phi \rangle \\ &= {}_3 \frac{1}{i\hbar} \langle [\hat{a}_n, \hat{H}] \rangle \end{aligned}$$

At step 1, the product rule is applied. At the second step, the Schrödinger equation is applied twice. Step 3 makes the assumption that the operator is constant (and the quantum state is time dependent). From now on, the units will be chosen such that $\hbar = 1$, thus simplifying this equation. This equation is also known as the Ehrenfest theorem.

To calculate the commutator $[\hat{a}_n, \hat{H}]$, the following formulas follow from the basic properties of \hat{a}_n [1]:

$$\hat{a}_n \hat{a}_j^\dagger = \hat{a}_j^\dagger \hat{a}_n + \delta_{n,j} \quad (2.34)$$

$$[\hat{a}_n, \hat{a}_l^\dagger \hat{a}_l] = \delta_{n,2l} \hat{a}_n \quad (2.35)$$

$$[\hat{a}_n, \hat{a}_l^{\dagger 2} \hat{a}_l^2] = 2\delta_{n,l} \hat{a}_l^\dagger \hat{a}_l^2 \quad (2.36)$$

$$[\hat{a}_n, \hat{a}_j^\dagger \hat{a}_l \hat{a}_j^\dagger \hat{a}_j] = \delta_{n,j} (\hat{a}_n^\dagger \hat{a}_l + \hat{a}_l \hat{a}_n^\dagger) \hat{a}_n \quad (2.37)$$

$$[\hat{a}_n, \hat{a}_j^\dagger \hat{a}_l \hat{a}_l^\dagger \hat{a}_l] = \delta_{n,j} \hat{a}_l + \delta_{n,j} \hat{a}_l^\dagger \hat{a}_l^2 + \delta_{n,l} \hat{a}_j^\dagger \hat{a}_l^2 \quad (2.38)$$

At equation (2.37) in [1], it is assumed this equation is equal to $2\delta_{n,j} \hat{a}_n^\dagger \hat{a}_n \hat{a}_j$. This is wrong, but gives the same results after the mean field approximation. A bigger problem occurs at equation (2.38). According to the original paper[1], this would be equal to

$$[\hat{a}_n, \hat{a}_j^\dagger \hat{a}_l \hat{a}_l^\dagger \hat{a}_l] = (1 - \delta_{n,j}) \hat{a}_l^\dagger \hat{a}_l^2 + (1 - \delta_{n,l}) \hat{a}_j^\dagger \hat{a}_l^2 + \delta_{n,j} \hat{a}_l \quad (2.39)$$

This is in no way close to my result. In the next two sections, I will try to get the same results using my calculations for the commutation relations, and afterwards I will show the calculations of Witthaut.

2.6.1 Ehrenfest theorem following my own calculations

Substituting equations (2.35), (2.36), (2.37) and (2.38) into the Ehrenfest theorem gives the following:

$$\begin{aligned}
\frac{\partial}{\partial t} \langle \hat{a}_n \rangle &= \frac{1}{i} \langle [\hat{a}_n, \hat{H}] \rangle \\
&= \frac{1}{i} \langle [\hat{a}_n, \sum_{l=1}^L \omega_l \hat{a}_l^\dagger \hat{a}_l + \frac{U}{2} \hat{a}_l^{\dagger 2} \hat{a}_l^2 + \sum_{k=1}^L \frac{K_{k,l}}{8} (i(\hat{a}_k^\dagger \hat{a}_l - \hat{a}_l^\dagger \hat{a}_k)(\hat{a}_k^\dagger \hat{a}_k - \hat{a}_l^\dagger \hat{a}_l) + h.c.)] \rangle \\
&= \frac{1}{i} (\langle \omega_n \hat{a}_n \rangle + U \langle \hat{a}_n^\dagger \hat{a}_n^2 \rangle) \\
&+ \langle [\hat{a}_n, \sum_{l=1}^L \sum_{k=1}^L \frac{K_{k,l}}{8} (\hat{a}_k^\dagger \hat{a}_l \hat{a}_k^\dagger \hat{a}_k - \hat{a}_l^\dagger \hat{a}_k \hat{a}_k^\dagger \hat{a}_k - \hat{a}_k^\dagger \hat{a}_l \hat{a}_l^\dagger \hat{a}_l + \hat{a}_l^\dagger \hat{a}_k \hat{a}_l^\dagger \hat{a}_l - h.c.)] \rangle \\
&= \frac{1}{i} (\langle \omega_n \hat{a}_n \rangle + U \langle \hat{a}_n^\dagger \hat{a}_n^2 \rangle) \\
&+ \sum_{l=1}^L \sum_{k=t1}^L \frac{K_{k,l}}{8} < (2\delta_{n,k} \hat{a}_k^\dagger \hat{a}_l \hat{a}_k + \delta_{n,k} \delta_{l,k} \hat{a}_k) - (\delta_{n,l} \hat{a}_k + \delta_{n,l} \hat{a}_k^\dagger \hat{a}_k^2 + \delta_{n,k} \hat{a}_l^\dagger \hat{a}_k^2) \\
&- (\delta_{n,k} \hat{a}_l + \delta_{n,k} \hat{a}_l^2 + \delta_{n,l} \hat{a}_k^\dagger \hat{a}_l^2) + (2\delta_{n,l} \hat{a}_l^\dagger \hat{a}_k \hat{a}_l + \delta_{n,l} \delta_{k,l} \hat{a}_l) \\
&- (\delta_{n,k} \hat{a}_l^\dagger \hat{a}_k^2 + \delta_{n,k} \delta_{k,l} \hat{a}_k + \delta_{n,l} \hat{a}_k^\dagger \hat{a}_k^2) \\
&+ (2\delta_{n,l} \hat{a}_k^\dagger \hat{a}_k \hat{a}_l + \delta_{n,l} \hat{a}_l) + (2\delta_{n,k} \hat{a}_l^\dagger \hat{a}_l \hat{a}_k + \delta_{n,k} \hat{a}_k) - (\delta_{n,l} \hat{a}_k^\dagger \hat{a}_l^2 + \delta_{n,l} \delta_{l,k} \hat{a}_l + \delta_{n,k} \hat{a}_l^\dagger \hat{a}_l^2) > \\
&= \frac{1}{i} (\langle \omega_n \hat{a}_n \rangle + U \langle \hat{a}_n^\dagger \hat{a}_n^2 \rangle) \\
&+ \sum_{l=1}^L \sum_{k=1}^L \frac{K_{k,l}}{8} < \delta_{n,k} (2\hat{a}_k^\dagger \hat{a}_l \hat{a}_k + \delta_{l,k} \hat{a}_k - \hat{a}_l^\dagger \hat{a}_k^2 - \hat{a}_l - \hat{a}_l^\dagger \hat{a}_l^2 - \hat{a}_l^\dagger \hat{a}_k^2 - \delta_{k,l} \hat{a}_k + 2\hat{a}_l^\dagger \hat{a}_l \hat{a}_k + \hat{a}_k - \hat{a}_l^\dagger \hat{a}_l^2) \\
&+ \delta_{n,l} (-\hat{a}_k - \hat{a}_k^\dagger \hat{a}_k^2 - \hat{a}_k^\dagger \hat{a}_l^2 + 2\hat{a}_l^\dagger \hat{a}_k \hat{a}_l + \delta_{k,l} \hat{a}_l - \hat{a}_k^\dagger \hat{a}_k^2 + 2\hat{a}_l^\dagger \hat{a}_k \hat{a}_l + \hat{a}_l - \hat{a}_k^\dagger \hat{a}_l^2 - \delta_{l,k} \hat{a}_l) > \\
&= \frac{1}{i} (\langle \omega_n \hat{a}_n \rangle + U \langle \hat{a}_n^\dagger \hat{a}_n^2 \rangle) \\
&+ \sum_{l=1}^L \sum_{k=1}^L \frac{K_{k,l}}{8} < \delta_{n,k} (2\hat{a}_k^\dagger \hat{a}_l \hat{a}_k - 2\hat{a}_l^\dagger \hat{a}_k^2 - 2\hat{a}_l^\dagger \hat{a}_k^2 + 2\hat{a}_l^\dagger \hat{a}_l \hat{a}_k - \hat{a}_l + \hat{a}_k) \\
&+ \delta_{n,l} (2\hat{a}_l^\dagger \hat{a}_k \hat{a}_l - 2\hat{a}_l^\dagger \hat{a}_l^2 - 2\hat{a}_l^\dagger \hat{a}_k^2 + 2\hat{a}_k^\dagger \hat{a}_k \hat{a}_l - \hat{a}_k + \hat{a}_l) >
\end{aligned}$$

Assuming $K_{k,l} = K_{l,k}$ for every l, k allows us to further elaborate these equations.

$$\begin{aligned}
\frac{\partial}{\partial t} \langle \hat{a}_n \rangle &= \frac{1}{i} (\langle \omega_n \hat{a}_n \rangle + U \langle \hat{a}_n^\dagger \hat{a}_n^2 \rangle) \\
&+ \sum_{k=1}^L \frac{K_{k,n}}{4} (2\hat{a}_n^\dagger \hat{a}_k \hat{a}_n - 2\hat{a}_k^\dagger \hat{a}_n^2 - 2\hat{a}_n^\dagger \hat{a}_k^2 + 2\hat{a}_k^\dagger \hat{a}_k \hat{a}_n - \hat{a}_k + \hat{a}_n)
\end{aligned}$$

Use first order mean field approximation $\langle \hat{a}_j^\dagger \hat{a}_k \hat{a}_l \rangle = \langle \hat{a}_j^\dagger \rangle \langle \hat{a}_k \rangle \langle \hat{a}_l \rangle = c_j^* c_k c_l$, where c_j^* is the complex conjugate of c_j . $c_n^* c_n = \langle \hat{a}_n^\dagger \hat{a}_n \rangle = \langle n_n \rangle$ so it can be

said that $c_n^*c_n = |c_n|^2$ is the amount of particles in mode n . A more detailed explanation about this approximation is given in the next section.

$$\begin{aligned}\frac{\partial c_n}{\partial t} &= \frac{1}{i}(\omega_n c_n + U c_n^* c_n^2) + \sum_{k=1}^L \frac{K_{k,n}}{4} (2c_n^* c_k c_n - 2c_k^* c_n^2 - 2c_n^* c_k^2 + 2c_k^* c_k c_n - c_k + c_n) \\ &= \frac{1}{i}(\omega_n c_n + U |c_n|^2 c_n) + \sum_{k=1}^L \frac{K_{k,n}}{4} (2|c_n|^2 c_k + 2|c_k|^2 c_n - 2c_k^* c_n^2 - 2c_n^* c_k^2 - c_k + c_n)\end{aligned}$$

Now, in [1] it is concluded that the modulus of c_n is constant if all moduli have the same starting value. This equation however gives:

$$\begin{aligned}\frac{\partial |c_n|^2}{\partial t} &= c_n^* \frac{\partial c_n}{\partial t} + c_n \left(\frac{\partial c_n}{\partial t}\right)^* \\ &= \sum_{k=1}^L \frac{K_{k,n}}{4} (2|c_n|^2 c_k c_n^* + 2|c_k|^2 |c_n|^2 - 2c_k^* c_n |c_n|^2 - 2(c_n^*)^2 c_k^2 - c_k c_n^* + |c_n|^2 \\ &\quad + 2|c_n|^2 c_k^* c_n + 2|c_k|^2 |c_n|^2 - 2c_k c_n^* |c_n|^2 - 2c_n^2 (c_k^*)^2 - c_k^* c_n + |c_n|^2) \\ &= \sum_{k=1}^L (|c_n|^2 (4|c_k|^2 + 2) - 2(c_n^*)^2 c_k^2 - 2c_n^2 (c_k^*)^2 - c_k c_n^* - c_k^* c_n)\end{aligned}$$

This is not necessary zero if the moduli have the same starting value. It can be concluded that synchronization is not necessarily existing in this setting, and the method that Witthaut uses is not valid in this situation.

2.6.2 Ehrenfest teorem following the equations of Witthaut

Witthaut uses the following equations in his paper:

$$[\hat{a}_n, \hat{a}_j^\dagger \hat{a}_j] = \delta_{n,j} \hat{a}_n \quad (2.40)$$

$$[\hat{a}_n, \hat{a}_j^{\dagger 2} \hat{a}_j^2] = 2\delta_{n,j} \hat{a}_n^\dagger \hat{a}_n^2 \quad (2.41)$$

$$[\hat{a}_n, \hat{a}_j^\dagger \hat{a}_l \hat{a}_j^\dagger \hat{a}_j] = [\hat{a}_n, \hat{a}_j^\dagger \hat{a}_j \hat{a}_j^\dagger \hat{a}_l] = 2\delta_{n,j} \hat{a}_n^\dagger \hat{a}_n \hat{a}_l \quad (2.42)$$

$$[\hat{a}_n, \hat{a}_j^\dagger \hat{a}_l \hat{a}_l^\dagger \hat{a}_l] = [\hat{a}_n, \hat{a}_l^\dagger \hat{a}_l \hat{a}_j^\dagger \hat{a}_l] = (1 - \delta_{n,j}) \hat{a}_l^\dagger \hat{a}_l^2 + (1 - \delta_{n,l}) \hat{a}_j^\dagger \hat{a}_l^2 + \delta_{n,j} \hat{a}_l \quad (2.43)$$

Substituting this into the Ehrenfest theorem gives the following:

$$\begin{aligned}
\frac{\partial}{\partial t} \langle \hat{a}_n \rangle &= \frac{1}{i} (\langle \omega_n \hat{a}_n \rangle + U \langle \hat{a}_n^\dagger \hat{a}_n^2 \rangle) \\
&+ \langle [\hat{a}_n, \sum_{l=1}^L \sum_{k=1}^L \frac{K_{k,l}}{8} (\hat{a}_k^\dagger \hat{a}_l \hat{a}_k^\dagger \hat{a}_k - \hat{a}_l^\dagger \hat{a}_k \hat{a}_k^\dagger \hat{a}_k - \hat{a}_k^\dagger \hat{a}_l \hat{a}_l^\dagger \hat{a}_l + \hat{a}_l^\dagger \hat{a}_k \hat{a}_l^\dagger \hat{a}_l - h.c.)] \rangle \\
&= \frac{1}{i} (\omega_n \langle \hat{a}_n \rangle + U \langle \hat{a}_n^\dagger \hat{a}_n^2 \rangle) \\
&+ \sum_{j=1}^L \frac{K_{n,j}}{2} (2 \langle \hat{a}_n^\dagger \hat{a}_n \hat{a}_j \rangle - \langle \hat{a}_j^\dagger \hat{a}_n^2 \rangle - \langle \hat{a}_j^\dagger \hat{a}_j^2 \rangle)
\end{aligned}$$

Here, it is assumed that $K_{n,j} = K_{j,n}$. The right hand side of this equation has multiple three point functions of the form $\langle \hat{a}_j^\dagger \hat{a}_k \hat{a}_l \rangle$. The time derivative of these expectation values can also be calculated with the Ehrenfest theorem, but this will only result in more coupled equations with even more variables. To avoid this problem, the three point functions will be approximated using one point functions as $\langle \hat{a}_j^\dagger \rangle \approx \langle \hat{a}_j^\dagger \rangle \langle \hat{a}_k \rangle \langle \hat{a}_l \rangle = c_j^* c_k c_l$. This is of course an approximation, but the error vanishes as $\frac{1}{N}[1][3]$. The previous equation will then yield:

$$i \frac{dc_n}{dt} = \omega_n c_n + U |c_n|^2 c_n + \sum_j^L \frac{K_{n,j}}{2i} (c_j^* c_n^2 + |c_j|^2 c_j - 2 |c_n|^2 c_j) \quad (2.44)$$

Decomposing $c_n = |c_n| e^{-i\phi_n}$ results in two equations, where c_n and ϕ_n are real numbers. Again, $|c_n|^2$ can be seen as the total amount of particles in mode n .

$$\frac{d}{dt} |c_n|^2 = - \sum_{j=1}^L K_{n,j} (|c_j|^2 - |c_n|^2) |c_j| |c_n| \cos(\phi_n - \phi_j) \quad (2.45)$$

$$\frac{d}{dt} \phi_n = \omega_n + U |c_n|^2 + \sum_{j=1}^L \frac{K_{n,j}}{2} \frac{|c_j|}{|c_n|} (3 |c_n|^2 - |c_j|^2) \sin(\phi_j - \phi_n) \quad (2.46)$$

It becomes clear that the amplitudes $|c_n|$ remain constant if all amplitudes have the same value. This means only the phases of these variables will be

time dependant. Assuming $|c_j|^2 = \frac{N}{L}$ for all j , will result in the Kuramoto equation.

$$\frac{d\phi_n}{dt} = \omega_n + U \frac{N}{L} + \sum_{j=1}^L K_{n,j} \frac{N}{L} \sin(\phi_j - \phi_n) \quad (2.47)$$

As I mentioned before, my calculations had an other result. There is an error in the calculations for the commutation relations in [1], which results in a whole different equation for c_n .

3. Conclusion

In this paper, there are two comments that I would like to discuss about the paper of Dirk Witthaut. In this paper, he starts with a Hamiltonian and derives the Kuramoto equation using the Ehrenfest theorem. In the remainder of the paper, this result is further elaborated.

Firstly, the origin of this Hamiltonian is very vague. The paper does give a source [6], but nowhere in this paper is the Hamiltonian given that Witthaut uses. It can be totally possible that this Hamiltonian is in fact derived from an equation in [6], but it would be useful and clarifying to add this derivation.

Secondly and more importantly, there is a mistake in the calculations of the commutation relations. This error results in an incorrect time derivative $\langle \hat{a}_n \rangle$ in the Ehrenfest theorem. When the correct time derivative is calculated, it is concluded that the Kuramoto equation cannot be derived in the way Witthaut derives it. Perhaps synchronization does occur by deriving it in a different way, but that is a totally different problem.

4. Appendix

4.1 Python code for approximating the system

To simulate the system, a numerical approximation of the Schrödinger equation is used. The basis is B_4 , as given in the theoretical background. In the following section, the code will be explained in steps.

```
import numpy as np
import matplotlib.pyplot as plt
from tqdm import tqdm
```

```
M = 5
omega1 = 1.1
omega2 = 1
U = 1
K12 = 1
K21 = 1
hbar = 1
```

```
dt = 0.0001
tmax = 10
```

The python code starts with the necessary imports, and setting the variables. This code approximates a system with 2 modes, and a Hamiltonian according to [1]. This gives the stated variables. M , dt and t_{max} are variables used in the numerical approximation; M is the maximum amount of particles spread across both states, dt is the time step used in the Euler forward method, and t_{max} is the stoptime.

```
def normalize_all(v):
    norm = np.linalg.norm(v)
    if norm == 0:
        return v
    return v / norm
```

```
wave = normalize_all(np.random.rand(int((M+2)*(M+1)/2)) + 1j*np.random.rand(int((M+2)*(M+1)/2)))
```

When there is a maximum of M particles in both states, then there are $\frac{(M+1)(M+2)}{2}$ possible modes. The variable `wave` is the current state of the system, and this is a representation of $\Psi = \sum_i a_i |i\text{-th state of } B_4\rangle$. The i -th element of `wave` is a_i . The wave is then normalized. It is arguable that the initial vector is not at all random. It has a random real part, and a random imaginary part (1j is the imaginary number i in python), but these parts will not be negative, since `np.random.rand` is a function that generates an array of random numbers between 0 and 1.

```
def delta(nr1, nr2):
    if nr1 == nr2:
        return 1
    else:
        return 0
```

```
H = []
basis = []
for Ntot in range(0, M + 1):
```

```

for n1 in range(0, Ntot + 1):
    n2 = Ntot - n1
    basis.append('|' + str(n1) + ', ' + str(n2) + '>')
    row = []
    for Mtot in range(0, M + 1):
        for m1 in range(0, Mtot + 1):
            m2 = Mtot - m1
            value = delta(n1, m1) * delta(n2, m2) * (omega1 * n1 + omega2 * n2 + U * (n1**2 - n1 + n2**2 - n2))
            value += delta(n1 + 1, m1) * delta(n2 - 1, m2) * 1j * (K21 + K12) * np.sqrt(n1 + 1) * np.sqrt(n2) * (n1 - n2 + 1) / 4
            value += delta(n1 - 1, m1) * delta(n2 + 1, m2) * 1j * (K21 + K12) * np.sqrt(n1) * np.sqrt(n2 + 1) * (n2 - n1 + 1) / 4
            row.append(value)
    H.append(row)

H = np.array(H)

```

In this part, the matrix representation of the Hamiltonian is made. This is done by looping over the rows and columns, according to

$$H_{ij} = \langle i\text{-th state of } B_4 | \hat{H} | j\text{-th state of } B_4 \rangle \quad (4.1)$$

The function `delta()` is used as the Kronecker delta function. In the last line, `H` is transformed into a numpy matrix object, which will be useful later.

```

A1 = []
A2 = []
N1 = []
N2 = []

for Ntot in range(0, M + 1):
    for n1 in range(0, Ntot + 1):
        n2 = Ntot - n1
        rowA1 = []
        rowA2 = []
        rowN1 = []
        rowN2 = []
        for Mtot in range(0, M + 1):
            for m1 in range(0, Mtot + 1):
                m2 = Mtot - m1
                valueA1 = delta(n1 - 1, m1) * delta(n2, m2) * np.sqrt(n1)
                rowA1.append(valueA1)
                valueA2 = delta(n1, m1) * delta(n2 - 1, m2) * np.sqrt(n2)
                rowA2.append(valueA2)
                valueN1 = delta(n1, m1) * delta(n2, m2) * n1
                rowN1.append(valueN1)
                valueN2 = delta(n1, m1) * delta(n2, m2) * n2
                rowN2.append(valueN2)
        A1.append(rowA1)
        A2.append(rowA2)
        N1.append(rowN1)
        N2.append(rowN2)

A1 = np.array(A1)
A2 = np.array(A2)
N1 = np.array(N1)
N2 = np.array(N2)

totalmatrix = []
A1s = []
A2s = []
N1s = []
N2s = []

```

In a similar fashion, the matrix representations for \hat{a}_1 , \hat{a}_2 , \hat{n}_1 and \hat{n}_2 are generated. These will be used to calculate the expected values for these variables. The arrays at the end are used to store these expected values.

```

normalisation = [0]
k = 1
i = 0
while i < len(wave):

```

```

normalisation.append(np.linalg.norm(wave[i:i+k]))
i += k
k += 1

def normalize(v):
    i = 0
    k = 1
    while i < len(v):
        norm = np.linalg.norm(v[i:i+k])
        if norm != 0.0:
            v[i:i+k] = v[i:i+k]*normalisation[k-1]/norm
            i += k
            k += 1
    return v

for i in tqdm(range(int(tmax/dt))):
    totalmatrix.append(np.square(np.absolute(wave)))
    dphi = np.dot(H, wave) / (1j * hbar)
    wave = normalize(wave + dt * dphi)
    A1s.append(np.conjugate(wave).dot(A1.dot(wave)))
    A2s.append(np.conjugate(wave).dot(A2.dot(wave)))
    N1s.append(np.conjugate(wave).dot(N1.dot(wave)))
    N2s.append(np.conjugate(wave).dot(N2.dot(wave)))

```

To explain these lines of code, I will have to explain a problem that occurred. Normally, the wave will be normalized. After 1 Euler forward step of the Schrödinger equation ($d\phi_t = \frac{\mathbf{H}\phi_t}{i\hbar}$ and $\phi_{t+dt} = \phi_t + dt \cdot d\phi$), the array in the python code will not be normalised anymore. This is because of small numerical errors in the code. This is no problem, because the vector can just be normalised after every step. The normalisation is the problem here.

In the theoretical section of this thesis, the Hamiltonian is shown. It can be seen that the absolute values in the Hamiltonian matrix scale linearly and quadratic with n_1 and n_2 . This means that $d\phi_t$ will be substantially bigger when n_1 and n_2 are bigger. When ϕ_{t+dt} is normalised across the whole vector, the expected values of n_1 and n_2 are slowly shifted towards the highest possible number of particles possible. This is shown clearly in figure (4.1).

The all three plots, the time axis is from left to right, and the number under the axis states the amount of loops that the system has looped through. This means the x-axes go from 0 to t_{max}/dt . The first plot shows the state during the whole simulation. Dark blue is a probability of 0 that the system is in that state, and the more yellow the color, the higher the chance. The second plot shows the expected values of the imaginary angle of a_1 and a_2 . The third plot shows the expected values of n_1 and n_2 , and this is the place where we can see the numerical error. It can be seen that $N = n_1 + n_2$ increases over time. However, in the theoretical background, we have concluded that this Hamiltonian has eigenstates of N , that are completely independent. If the system is in a state with a total of N particles, then there will keep being N particles. In conclusion; the trivial normalisation is easy, but leads to wrong

results.

A solution can be found by normalising each eigenstate separately. This means that the chance of finding N particles will be constant throughout the simulation. This is done by creating an array (`normalisation`), where `normalisation[i]` is the chance of being in the eigenstate with $N = i - 1$. After each Euler forward step, the wave will be normalised, using the `normaize` function. This function loops over each eigenstate, consisting of states $|0, N\rangle, |1, N - 2\rangle, \dots, |N, 0\rangle$, and makes sure the probability of being in that eigenstate is the same as that probability in the initial wave. Figure (4.2) shows a simulation with the same variables as figure (4.1), but this time using a correct normalisation function.

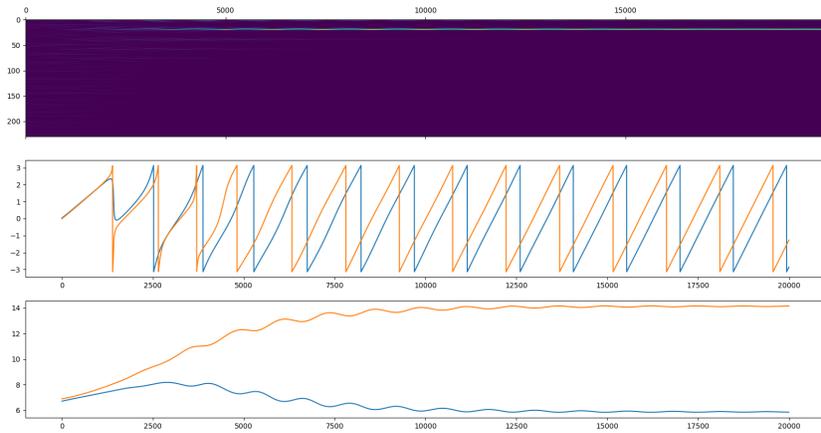


Figure 4.1: Using a trivial normalisation function leads to computational errors. Here, $N = 20$ is the maximum amount of particles. $dt = 0.0001$, $t_{max} = 2$, $\omega_1 = \omega_2 = U = K_{12} = K_{21} = \hbar = 1$

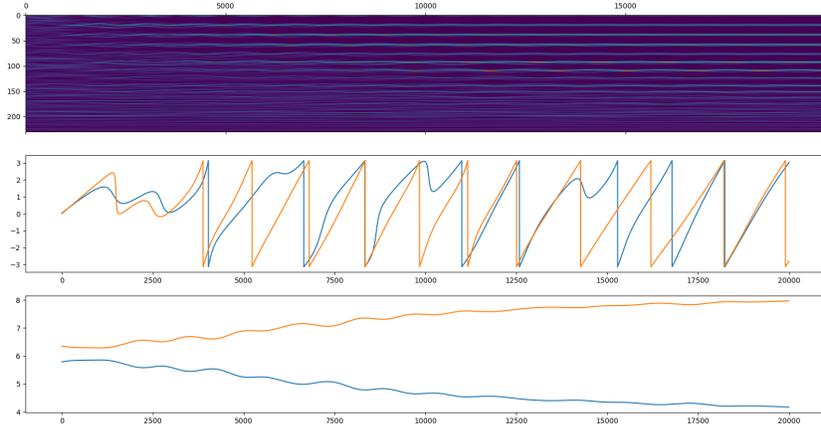


Figure 4.2: In this figure, a correct normalisation function is used after each Euler forward step. This makes sure $n_1 + n_2$ will be constant, just as the theory anticipated. The variables are the same as figure (4.1).

After each normalization, the expected values for a_1 , a_2 , n_1 and n_2 are calculated using the generated matrices. These values are stored in numpy arrays, and finally plotted.

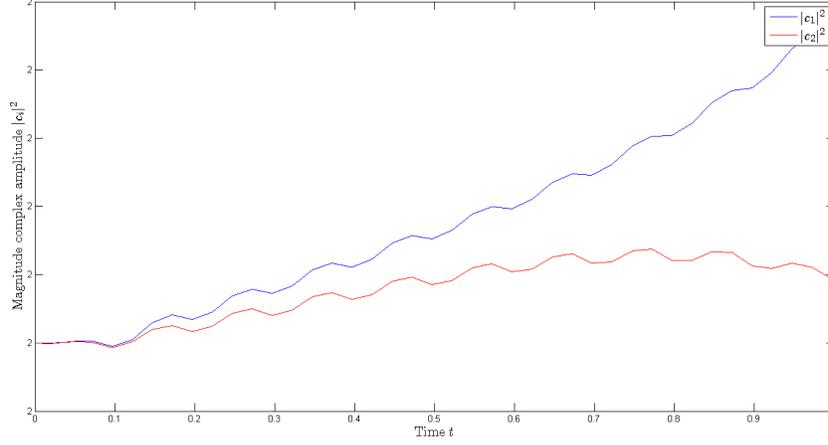


Figure 4.3: A good example of a bad normalisation in programming. In the Bachelor thesis of Lotte van Dongen [2], a plot is made of $|c_1|^2$ and $|c_2|^2$. According to the Hamiltonian, the sum of these two variables must always be constant, because the Hamiltonian has eigenstates of $N = n_1 + n_2$. However, these the sum of these two variables increases over time. This points toward a mistake in the code that was used to plot this graph.

4.2 Commutation relations

$$[\hat{a}_n, \hat{a}_j^\dagger \hat{a}_l \hat{a}_l^\dagger \hat{a}_l] = \hat{a}_n \hat{a}_j^\dagger \hat{a}_l \hat{a}_l^\dagger \hat{a}_l - \hat{a}_j^\dagger \hat{a}_l \hat{a}_l^\dagger \hat{a}_l \hat{a}_n \quad (4.2)$$

$$= (\hat{a}_j^\dagger \hat{a}_n + \delta_{n,j}) \hat{a}_l \hat{a}_l^\dagger \hat{a}_l - \hat{a}_j^\dagger \hat{a}_l \hat{a}_l^\dagger \hat{a}_l \hat{a}_n \quad (4.3)$$

$$= \delta_{n,j} \hat{a}_l \hat{a}_l^\dagger \hat{a}_l + \hat{a}_j^\dagger \hat{a}_l \hat{a}_n \hat{a}_l^\dagger \hat{a}_l - \hat{a}_j^\dagger \hat{a}_l \hat{a}_l^\dagger \hat{a}_l \hat{a}_n \quad (4.4)$$

$$= \delta_{n,j} (\hat{a}_l^\dagger \hat{a}_l + 1) \hat{a}_l + \hat{a}_j^\dagger \hat{a}_l (\hat{a}_l^\dagger \hat{a}_n + \delta_{n,l}) \hat{a}_l - \hat{a}_j^\dagger \hat{a}_l \hat{a}_l^\dagger \hat{a}_l \hat{a}_n \quad (4.5)$$

$$= \delta_{n,j} \hat{a}_l + \delta_{n,j} \hat{a}_l^\dagger \hat{a}_l^2 + \delta_{n,l} \hat{a}_j^\dagger \hat{a}_l^2 \quad (4.6)$$

$$[\hat{a}_n, \hat{a}_l^\dagger \hat{a}_j \hat{a}_l^\dagger \hat{a}_l] = \hat{a}_n \hat{a}_l^\dagger \hat{a}_j \hat{a}_l^\dagger \hat{a}_l - \hat{a}_l^\dagger \hat{a}_j \hat{a}_l^\dagger \hat{a}_l \hat{a}_n \quad (4.7)$$

$$= (\hat{a}_l^\dagger \hat{a}_n + \delta_{n,l}) \hat{a}_j \hat{a}_l^\dagger \hat{a}_l - \hat{a}_l^\dagger \hat{a}_j (\hat{a}_n \hat{a}_l^\dagger - \delta_{n,l}) \hat{a}_l \quad (4.8)$$

$$= \delta_{n,l} (\hat{a}_l^\dagger \hat{a}_j + \hat{a}_j \hat{a}_l^\dagger) \hat{a}_l \quad (4.9)$$

$$= 2\delta_{n,l} \hat{a}_l^\dagger \hat{a}_j \hat{a}_l + \delta_{n,l} \delta_{j,l} \hat{a}_l \quad (4.10)$$

$$[\hat{a}_n, \hat{a}_l^\dagger \hat{a}_l \hat{a}_j^\dagger \hat{a}_l] = \hat{a}_n \hat{a}_l^\dagger \hat{a}_l \hat{a}_j^\dagger \hat{a}_l - \hat{a}_l^\dagger \hat{a}_l \hat{a}_j^\dagger \hat{a}_l \hat{a}_n \quad (4.11)$$

$$= (\hat{a}_l^\dagger \hat{a}_n + \delta_{n,l}) \hat{a}_l \hat{a}_j^\dagger \hat{a}_l - \hat{a}_l^\dagger \hat{a}_l (\hat{a}_n \hat{a}_j^\dagger - \delta_{n,j}) \hat{a}_l \quad (4.12)$$

$$= \delta_{n,l} \hat{a}_l \hat{a}_j^\dagger \hat{a}_l + \delta_{n,j} \hat{a}_l^\dagger \hat{a}_l \hat{a}_l \quad (4.13)$$

$$= \delta_{n,l} \hat{a}_j^\dagger \hat{a}_l^2 + \delta_{n,l} \delta_{l,j} \hat{a}_l + \delta_{n,j} \hat{a}_l^\dagger \hat{a}_l^2 \quad (4.14)$$

$$[\hat{a}_n, \hat{a}_l^\dagger \hat{a}_l \hat{a}_l^\dagger \hat{a}_j] = \hat{a}_n \hat{a}_l^\dagger \hat{a}_l \hat{a}_l^\dagger \hat{a}_j - \hat{a}_l^\dagger \hat{a}_l \hat{a}_l^\dagger \hat{a}_j \hat{a}_n \quad (4.15)$$

$$= (\hat{a}_l^\dagger \hat{a}_n + \delta_{n,l}) \hat{a}_l \hat{a}_l^\dagger \hat{a}_j - \hat{a}_l^\dagger \hat{a}_l (\hat{a}_n \hat{a}_l^\dagger - \delta_{n,l}) \hat{a}_j \quad (4.16)$$

$$= \delta_{n,j} (\hat{a}_l \hat{a}_l^\dagger + \hat{a}_l^\dagger \hat{a}_l) \hat{a}_j \quad (4.17)$$

$$= 2\delta_{n,j} \hat{a}_l^\dagger \hat{a}_l \hat{a}_j + \delta_{n,j} \hat{a}_j \quad (4.18)$$

4.3 The initial wave in figure (2.3)

[3.45885202e-02+0.07515969j 2.98148705e-03+0.06726983j 7.10326233e-02+0.04392135j 1.57841298e-02+0.04642786j
1.51104093e-02+0.03964125j 3.56448009e-02+0.03385401j 3.93407266e-02+0.05482959j 2.61446035e-02+0.01294644j
5.31776910e-02+0.01321762j 5.31244319e-02+0.05954809j 3.11819716e-03+0.06433536j 1.58737439e-02+0.06877804j
5.25793456e-02+0.04547623j 7.97437039e-02+0.03565731j 3.40583725e-02+0.01506053j 3.68416818e-02+0.02313852j
2.72218478e-02+0.08005924j 8.07522631e-02+0.07715902j 4.61792872e-02+0.05793109j 4.60209773e-02+0.05321337j
1.89037140e-02+0.07443069j 5.12543182e-02+0.05598527j 8.52835544e-03+0.07369857j 1.74797035e-02+0.0134553j
1.55349887e-02+0.01946544j 5.56942476e-02+0.00105079j 2.72051472e-02+0.04377234j 2.41774296e-02+0.0157008j
4.96520967e-02+0.05114932j 3.26102503e-02+0.01503978j 7.54244043e-02+0.03692953j 2.49786092e-03+0.06337932j
2.07554772e-02+0.06447557j 4.98390220e-02+0.01376969j 2.48566306e-02+0.01282293j 4.45914207e-02+0.07858562j
8.87947248e-03+0.00134863j 7.86594200e-02+0.0088705j 1.43233756e-02+0.04638767j 6.13865568e-02+0.01558139j
6.82521763e-02+0.03953123j 5.10299558e-02+0.03876508j 5.95718472e-02+0.02702526j 7.48549100e-02+0.04358307j
2.31805431e-02+0.04721817j 6.67650083e-02+0.01840248j 6.00701442e-02+0.04944478j 9.46835303e-04+0.07469086j
6.84974995e-02+0.0285613j 2.19139471e-02+0.00167038j 4.42984627e-02+0.0450803j 7.93519939e-03+0.01911974j
6.55315799e-02+0.07685478j 8.02379879e-02+0.05523371j 4.02043677e-02+0.0754444j 2.42593241e-02+0.02741492j
4.37970384e-02+0.03905943j 7.81639461e-02+0.04071474j 5.53875373e-02+0.00732287j 3.64467498e-02+0.0367774j
5.10064074e-02+0.06941114j 7.83687728e-03+0.03996477j 4.09031479e-02+0.04318091j 2.85324958e-02+0.04566846j
5.96098410e-02+0.03919946j 6.08842548e-02+0.02685519j 7.10164383e-02+0.0270981j 2.64733285e-03+0.05421414j
7.78932806e-02+0.01934065j 3.35299821e-02+0.06670139j 7.05159651e-02+0.05685867j 2.18117340e-02+0.07599351j
7.39352447e-02+0.04205399j 3.20974921e-02+0.0044804j 2.28124251e-02+0.05936387j 3.26217248e-02+0.0557032j
1.18815375e-02+0.01341601j 6.65877775e-03+0.06534716j 6.98409578e-02+0.01143172j 5.91554862e-02+0.06200594j
6.96495538e-02+0.07428644j 7.77724059e-02+0.01397469j 7.10244122e-02+0.05245068j 4.33852144e-02+0.00194966j
3.73956115e-02+0.07770066j 1.02654625e-03+0.05201261j 7.13718003e-02+0.02278567j 2.88501213e-02+0.02299767j
3.71932192e-02+0.05533938j 1.24379372e-02+0.01950674j 6.35754446e-02+0.03590014j 6.50349474e-02+0.01416203j
7.68115036e-02+0.00058012j 3.06161257e-02+0.03745213j 4.22153878e-03+0.0549986j 2.74593285e-03+0.04098865j
1.04239381e-02+0.06081674j 4.11417083e-02+0.00909395j 3.64099907e-02+0.0189894j 7.64064174e-02+0.03606962j
7.85967698e-03+0.04979434j 1.71569554e-02+0.07976433j 6.21277298e-02+0.00375574j 7.24705387e-02+0.07970613j
5.20570182e-02+0.0614521j 5.59381433e-02+0.02108069j 2.11200599e-03+0.01714199j 4.15259378e-02+0.06047338j
5.11568430e-02+0.00212311j 1.77359468e-02+0.05567617j 1.36191878e-02+0.00113387j 7.26563803e-02+0.06782301j
1.50914272e-02+0.07597185j 3.67422227e-02+0.07346832j 6.02391566e-02+0.06463958j 7.24208565e-02+0.02628145j
4.76325365e-02+0.0767004j 2.49419031e-02+0.02894376j 3.67290433e-02+0.05537815j 1.86790838e-02+0.00926983j
2.59185561e-02+0.00865446j 5.90773004e-02+0.00394439j 3.07155700e-02+0.0428175j 3.74007432e-02+0.03386653j
6.73789698e-02+0.04697928j 5.60676868e-02+0.02508966j 2.49897508e-02+0.07723141j 1.17922622e-02+0.07371479j
2.61999169e-02+0.02390313j 3.53188747e-02+0.01753222j 5.38845809e-02+0.04308834j 4.35795585e-02+0.04238762j
2.85587017e-02+0.01408669j 7.16867092e-02+0.0464283j 1.95458616e-02+0.07275706j 3.68369095e-02+0.04827626j
1.10547775e-03+0.01670854j 7.79835802e-02+0.04163311j 2.57548518e-02+0.08005431j 6.88853231e-02+0.07286309j
6.73055597e-02+0.0282634j 6.51807453e-02+0.01155602j 3.27156618e-02+0.00730385j 5.52200622e-02+0.04814085j
2.64152069e-02+0.02730445j 3.59032324e-02+0.03703152j 1.15368864e-02+0.08050046j 9.64042463e-04+0.01309934j
5.49769609e-02+0.06279358j 3.91163834e-02+0.05982123j 5.08290086e-02+0.0368614j 3.90362191e-02+0.00885441j
4.50998752e-02+0.03577759j 7.22633139e-02+0.04779099j 5.29031595e-02+0.04278719j 7.71834491e-02+0.01023336j
2.73056113e-02+0.07769189j 6.06778066e-02+0.03366697j 6.77367216e-02+0.00507131j 6.18028229e-02+0.0152581j
4.01916583e-02+0.07080361j 2.89284714e-02+0.03937134j 6.90796024e-02+0.01412102j 1.78807116e-02+0.06658602j
2.74012679e-02+0.01689511j 3.02498019e-02+0.03454132j 5.66164712e-02+0.02512054j 2.97521647e-02+0.03545407j
5.69946221e-02+0.02230428j 1.28684360e-02+0.02656809j 7.89923928e-02+0.03513637j 1.87069213e-03+0.03582568j
6.08431610e-02+0.00973584j 7.37778971e-02+0.03917493j 1.04619681e-02+0.01340903j 4.43091428e-02+0.00752083j
6.82978747e-02+0.06566492j 7.49969496e-02+0.01966488j 2.59860663e-02+0.04006293j 1.51363816e-02+0.07749298j
2.99302950e-02+0.00391112j 5.40877120e-02+0.03921135j 3.86881901e-02+0.04919618j 5.63758877e-02+0.04103716j
2.05329749e-02+0.01210835j 1.26602191e-02+0.05582335j 1.62693839e-02+0.0285132j 7.90257199e-03+0.01457794j
4.05173972e-02+0.01175574j 3.63426932e-02+0.02311335j 6.81268508e-02+0.06283355j 2.87416712e-02+0.05764863j
5.59759090e-02+0.04455021j 1.70827855e-03+0.00052351j 1.95835671e-02+0.04793266j 7.73267622e-02+0.07954631j
5.36172046e-02+0.06977101j 1.46493994e-02+0.06060204j 2.38115163e-03+0.01889159j 1.01196657e-02+0.07567414j
1.17617950e-03+0.03204735j 3.61257970e-02+0.02993246j 2.32815469e-02+0.06608145j 1.34302630e-02+0.05938277j
3.60667910e-03+0.04462038j 3.81873259e-05+0.04133417j 2.74887586e-02+0.03219754j 1.14527602e-02+0.06340002j
7.23276318e-02+0.06603592j 7.93621301e-02+0.04373354j 1.74358499e-03+0.01159031j 6.03758453e-02+0.02350391j
1.72584535e-02+0.07853988j 3.76973458e-02+0.04326002j 5.37578029e-02+0.02438022j 3.60057029e-02+0.02637628j
4.97099395e-02+0.05970525j 7.51393327e-02+0.01934366j 7.30170141e-02+0.00766941j 2.27551328e-02+0.04161845j
5.87979766e-02+0.05941727j 5.50440496e-02+0.07931162j 4.14773010e-02+0.02918405j 3.09412709e-02+0.07304922j
4.28878162e-02+0.07521504j 6.74699767e-02+0.06084048j 1.64386037e-03+0.0564201j 1.60192658e-02+0.05605217j
3.52245943e-02+0.02895496j 3.83718908e-02+0.04663567j 1.92913182e-02+0.06109642j]

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