Reconfiguration of Graph Colorings

by

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Abstract

For this thesis, we consider two k-colorings of a graph G adjacent if one can recolor one into the other by changing the color of one vertex. The reconfiguration graph of a graph G on k colors $\mathscr{C}_k(G)$ is the graph for which the vertices are the k-colorings of G, and an edge is between two k-colorings if they are adjacent. We further investigate the diameter of the reconfiguration graph on k colors: $\operatorname{diam}(\mathscr{C}_k(G))$.

The general conjecture the thesis is based around says that $\operatorname{diam}(\mathscr{C}_k(G)) = n + \mu(G)$ for every graph G and $k \geq \Delta(G) + 2$. This conjecture is confirmed for various families of graphs, for example the complete graph K_n and complete bipartite $K_{n,m}$. This thesis will prove the lower bound of the conjecture for the family of complete r-partite graphs $G = K_{x_1,x_2,\ldots,x_r}$, utilising an approach from Cambie et al. for the proof. Furthermore we give an algorithm that computes the k-colorings of G, the reconfiguration graph $\mathscr{C}_k(G)$, and its diameter $\operatorname{diam}(\mathscr{C}_k(G))$ and give a few results on this diameter for small graphs.

Layman Abstract

Many problems have multiple possible solutions, and sometimes these solutions may seem similar. To understand how similar they are, we need to define a way to compare two solutions. In this thesis, we will use graphs to represent networks. Graphs consist of vertices (points) and edges (connections between points).

Our specific problem involves graph colorings, which are functions that assign colors to each vertex in such a way that no two connected vertices have the same color. When we use 'k' colors for such a coloring, we call it a 'k-coloring' of the graph.

To analyze these colorings, we consider two k-colorings of a graph to be adjacent if we can change the color of one vertex in one coloring to transform it into the other coloring. The 'reconfiguration graph' of a graph on k colors, denoted as '$\mathscr{C}_k(G)$', is a graph where each vertex represents a k-coloring of the original graph, and an edge is drawn between two k-colorings if they are adjacent.

We can estimate the similarity between two k-colorings by studying the maximum distance between them, known as the 'diameter' of the reconfiguration graph, denoted as 'diam$(\mathscr{C}_k(G))$'. This represents the longest sequence of color changes required to transform one coloring into the other.

The main idea of this thesis revolves around a conjecture, which suggests for any given graph G and k (where k is greater than or equal to the number of connections any vertex in G can have, plus two), that the longest sequence of color changes needed to transform one valid k-coloring of G to another is equal to the number of vertices in G plus a certain parameter associated with the graph called the matching number '$\mu(G)$'.

In this thesis, we will focus on proving the lower bound of this conjecture for a specific family of graphs called complete 'r-partite' graphs, denoted as '$G = K_{x_1,x_2,\ldots,x_r}$'. These graphs have vertices divided into r groups, and every vertex within a group is connected to every vertex in other groups. We will utilize an approach developed by Cambie et al. to prove the lower bound.

Additionally, we will present an algorithm that can compute the different k-colorings of the graph G, as well as the reconfiguration graph $\mathscr{C}_k(G)$. We will also calculate the diameter of this reconfiguration graph: diam$(\mathscr{C}_k(G))$. Finally, we will provide some results on the diameter of the reconfiguration graph for small graphs using our algorithm.

Contents

1 Introduction

In mathematics and many other fields, often certain problems do not have unique solutions. These solutions may be similar to each other in some way, and one may like to analyse their similarities. To more accurately explore the extent of similarity between solutions, we introduce the concept of reconfiguration.

Reconfiguration problems are optimization problems that focus on analyzing the solutions or state spaces of a problem and measuring the distance between them. The notion of distance varies depending on the specific problem being studied. A classic example that illustrates this idea is the Rubik's cube. By rotating the sides of the cube, we can transform it from one arrangement to another, aiming to achieve a solution where each face has a single color. The efficiency of solving the cube is determined by the number of rotations needed to reach the desired state.

In this thesis we will focus on graphs and the reconfiguration of graph colorings. When it comes to defining the distance between two graph colorings, numerous approaches exist, with some seeming more intuitive than others. When we look back at Rubik's cube, we can define a step, or a distance of one, to be a single rotation of a side. After one such step we still have a proper arrangement of the cube. Consequently, the overall distance from any initial cube arrangement to the solution with monochromatic faces is measured by the number of rotations required to reach that state.

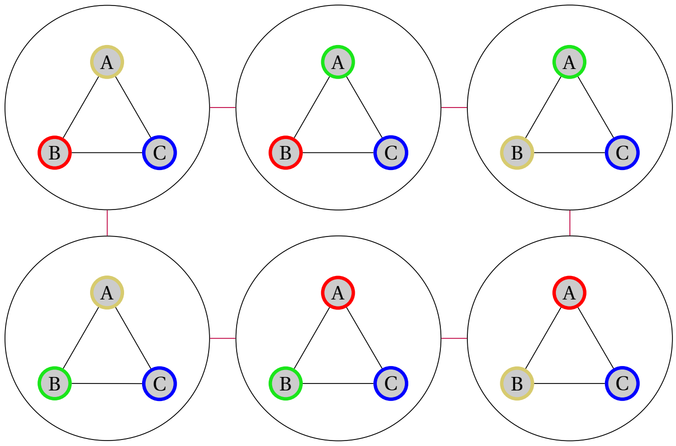
We would like to do something similar to graph colorings of a graph G: by defining a step to be a recoloring of a single vertex, the total distance between two colorings α and β of a graph G is the number of vertex color swaps. Likewise we want to have a proper coloring of the graph every single step as well, hence we are restricted to which vertices we can swap.

We go over a few basic definitions from graph theory. A *graph* is a tuple $G = (V, E)$, where V are called the *vertices* of the graph, and $E \subseteq \binom{V}{2}$ are called the *edges* of the graph, so an edge is between two vertices. A *proper k-coloring* of the graph G is a function $f : V \to \{1, 2, \ldots, k\}$ such that: $uv \in E \implies f(u) \neq f(v)$. From this point onwards, we will refer to proper k-colorings as just "k-colorings".

Currently, the reconfiguration problem lacks a precise definition. To address this, we aim to establish a clear measure of the distance between any two k-colorings of a graph G using mathematical notation to facilitate discussions about the reconfiguration problem. To achieve this, we introduce the concept of a *reconfiguration graph*, denoted as $\mathscr{C}_k(G)$ on k colors. The reconfiguration graph is constructed as follows:

- The vertices of $\mathscr{C}_k(G)$ are the k-colorings of G.
- There is an edge between two vertices of $\mathscr{C}_k(G)$, i.e. two k-colorings of G, if the k-colorings are different in only one vertex.

With this definition, we can directly associate the distance between two k-colorings with the distance between the corresponding nodes on the reconfiguration graph. It is worth noting that the reconfiguration graph consists of vertices that are graphs themselves, which may require visualization for better comprehension. Therefore, let us examine a segment of the cycle graph C_3 along with its reconfiguration graph $\mathscr{C}_4(C_3)$ on 4 colors:



Check for yourself that the edges between the k-colorings in the figure are correct. Note that the total number of vertices of $\mathscr{C}_4(C_3)$ is equal to $4 \cdot 3 \cdot 2 = 24$.

The distance between two k-colorings α and β by itself can vary per k-coloring, hence it makes sense to look at the maximum distance between any two k-colorings, which we can now say is the *diameter* of the reconfiguration graph, i.e. $\operatorname{diam}(\mathscr{C}_k(G))$.

1.1. Defining the main problem

Note that $\mathscr{C}_k(G)$ is only non-empty if $k \geq \chi(G) = \min\{m : \exists \text{ proper } m\text{-coloring}\}$, otherwise known as the *chromatic number*. If we define the degree of a vertex in G to be: $\deg(v) = |\{w \in V(G) : vw \in E(G)\}|$, one can prove that $\chi(G) \leq \Delta(G) + 1$, where $\Delta(G) = \max_{v \in V} \deg(v)$, otherwise known as the maximum degree of G. The proof is based on induction and greedily coloring the vertices of the graph, and is given here:

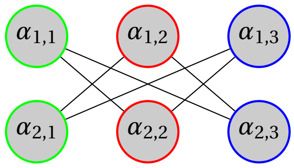
Proof. Start with $|V(G)| = 1$, G is has no edges, hence $\Delta(G) = 0$, and G has to be colored with 1 color, meaning $\chi(G) = 1 \leq 0+1 = \Delta(G)+1$. Suppose that the induction hypothesis, i.e. $\chi(G) \leq \Delta(G)+1$, is true for $|V(G)| = k-1$. Let G now have k vertices, and label the vertices $v_1, v_2, \ldots, v_k$. Then the induced subgraph $G[G-\{v_k\}]$ has $k-1$ vertices and hence can be colored with at most $\chi(G-\{v_k\}) \leq \Delta(G-\{v_k\})+1$ colors.

Now v_k has degree $\deg(v_k)$, amd all edges neighbouring v_k are in $G-\{v_k\}$. If $\deg(v_k) \leq \Delta(G-\{v_k\})$, we have $\Delta(G) = \Delta(G-\{v_k\})$ and we use an available color from the $(\Delta(G-\{v_k\})+1)$-coloring of $G-\{v_k\}$ for v_k, which must exist since we are using more colors than neighbours of v_k. This gives a $\Delta(G)+1$ coloring of G, and thus a $(\Delta(G)+1)$-coloring of G. If $\deg(v_k) > \Delta(G-\{v_k\})$, in the worst case scenario, v_k is incident to all colors used in the coloring of $G-\{v_k\}$, which has at most $\Delta(G-\{v_k\})+1$ colors, therefore having to use a new color outside the coloring. This gives at most a $([\Delta(G-v_k)+1]+1)$-coloring of G. Since now $\Delta(G) = \deg(v_k) \geq \Delta(G-\{v_k\})+1$ we find a $(\Delta(G)+1)$-coloring of G.

We find $\chi(G) \leq \Delta(G)+1$ in both instances, and we conclude the proof. □

This upper bound on the number of colors needed to color a graph, means when we color the graph with one more color, i.e. $\Delta(G)+2$ colors, we have a 'free' color for every vertex v; free in the sense that it does not appear on v nor on any neighbor of v. This allows any vertex to be recolored, giving us more insight on the connectedness of the reconfiguration graph with $\Delta(G)+2$ colors. It is good to note that the reconfiguration graph with $\Delta(G)+2$ colors is indeed connected [9], namely that the maximum distance between two k-colorings is proven to be bounded from above by $2n$, where n is the number of vertices. It is possible that this upper bound can be lowered to a more tight bound, one which will be shown later.

In contrast, if we used $k = \Delta(G)+1$ colors for our reconfiguration graph, it might be disconnected. One such case was brought fourth in Cereceda's 2007 paper [7] on page 24. In this paper, the general case of the L_m graph is discussed: L_m is the graph with vertices $\{v_1, v_2, \ldots, v_m, w_1, w_2, \ldots, w_m\}$ with edges between each $v_i w_j$ with $i \neq j$. A smaller example L_3 is shown here for convenience. L_3 is the complete bipartite graph $K_{3,3}$ with a perfect matching removed; now $\Delta(L_3)+1 = (3-1)+1 = 3$. To show that the reconfiguration graph $\mathscr{C}_{\Delta(L_3)+1}(L_3)$ is disconnected, consider the coloring as shown here.



Since we are only allowed to use three colors, here green, blue and red, we cannot go to any other coloring, as any vertex is adjacent to two other colors, hence fixing its color in place. With $\Delta(L_3)+2$ colors, this could be averted.

One more concept needs to be introduced before the main problem can be written down, namely matchings. A matching M of a graph G is a subset of the edges from G, such that no two edges in M share an endpoint. The size of the maximum matching, i.e. the largest matching, in the graph G is called the matching number and is denoted by $\mu(G)$.

For certain types of graphs it is known that the diameter of the reconfiguration graph on $\Delta(G)+2$ colors is bounded from above and below by the quantity $n+\mu(G)$, where n is the amount of vertices of the graph. The following is then conjectured:

Conjecture 1. *For every graph G, it holds that for $k \geq \Delta(G)+2$*

$$\operatorname{diam}(\mathscr{C}_k(G)) = n+\mu(G).$$

This conjecture originally stems from the 2022 paper from Cambie et al. [6], which, if true, is a tight bound on Proposition 5.23 proven in Cereceda's 2007 paper [7]. The main goal of our thesis is to prove the lower bound of this diameter for a family of graphs called complete r-partite graphs. A complete r-partite graph $G = K_{x_1,x_2,\ldots,x_r}$, where $x_1 \leq x_2 \leq \ldots \leq x_r$, is a graph where the vertices are $V(G) = \bigcup_{i=1}^{r} S_i$, where each S_i is an independent set with $|S_i| = x_i$ for each $1 \leq i \leq r$, and the edges are between every two vertices not in the same independent set. The main theorem of this thesis we will prove is formulated as follows:

Theorem 1. *For any complete r-partite graph $G = K_{x_1,x_2,\ldots,x_r}$, we have*

$$\operatorname{diam}\left(\mathscr{C}_{\Delta(G)+2}(G)\right) \geq n+\mu(G).$$

1.2. Motivation and outline

The problem of reconfiguration on graph colorings has some interesting applications in other fields of science. For instance, the connectivity of the reconfiguration graph, i.e. finiteness of the diameter of the reconfiguration graph, is linked to the mixing time of certain Markov chains. Mixing time of a Markov Chain refers to the time needed for the chain to be 'close' to a steady state distribution; when the mixing time converges at most polynomially fast in the size of the problem instance it is called rapidly mixing [10]. The Markov chain used to sample k-colorings of a graph G, denoted by the Glauber dynamics, is rapidly mixing, only if $\mathscr{C}_k(G)$ is connected. Note that the connectivity is only a necessary condition, in that the converse need not be true, which was proven in this paper by Łuczak [12].

Another application comes up in computer science and time complexity [1]. This paper by Belavadi et al. proves that $\mathscr{C}_{k+1}(G)$ is connected for every k-colorable H-free graph G if and only if H is an induced subgraph of the path graphs P_4 or P_3+P_1. It is then also stated that these G are exactly the family of H-free graphs for which figuring out for every k, whether there is a k-coloring of G, is polynomial time solvable [11].

As mentioned, in this thesis, we will prove the lower bound on the diameter $\operatorname{diam}(\mathscr{C}_k(G)) \geq n+\mu(G)$ for r-partite graphs. Before we get to r-partite graphs, we will look into what is known, starting off with a side venture to Cereceda's conjecture in chapter 2, which is an analogue problem which asks the same question on connectivity of the reconfiguration graph, but instead of considering the maximum degree $\Delta(G)$ of G for the colorings of G, it considers the degeneracy $\operatorname{dg}(G)$ of the G. After which we proceed with the known lower bounds for complete graphs and complete bipartite graphs in chapter 3, and technique used to figure out the lower bound. In chapter 4, we prove the desired lower bound for r-partite graphs and look into the specific example $G = K_{1,1,\ldots,1,2,2,\ldots,2}$, which can be viewed as a merge of complete graphs and complete bipartite graphs. Lastly we manually compute the diameter for some small non-complete graphs using Sagemath in chapter 5.

2

Cereceda's conjecture and an easier version

As described in the introduction, we start off with a related problem to using $(\Delta(G)+2)$-colorings for the reconfiguration graph, namely with the degeneracy $\mathrm{dg}(G)$ of G instead of the maximum degree $\Delta(G)$ of G. The degeneracy $\mathrm{dg}(G)$ of G can be defined as follows: $\mathrm{dg}(G) = k$ if one can find an ordering of the vertices $(v_1, v_2, \dots, v_k)$ such that there are at most k edges between $\{v_i\}$ and $\{v_1, v_2, \dots, v_{i-1}\}$ for all $i \in \{2, 3, \dots, k\}$. Note that the number of edges between $\{v_i\}$ and $\{v_1, v_2, \dots, v_{i-1}\}$ for any i is bounded from above by the degree of v_i, therefore: $\mathrm{dg}(G) \le \max_{v \in V} \deg(v) = \Delta(G)$, hence the relation between the two reconfiguration problems. We show Cereceda's conjecture made in his 2007 paper [7] and prove an easier version.

Conjecture 2 (Cereceda's Conjecture)**.** *If a simple graph $G = (V, E)$ has degeneracy* $\mathrm{dg}(G)$*, then*

$$\mathrm{diam}\left(\mathscr{C}_{\mathrm{dg}(G)+2}(G)\right) = \mathcal{O}\left(n^2\right),$$

where $n = |V(G)|$.

If this conjecture is true, this bound is tight, since the family of path graphs P_m was found to have a quadratic lower and upper bound on the diameter of the reconfiguration graph on 3 colors [3]. Furthermore, the reconfiguration graph on 5 colors of planar bipartite graphs was also found to have a quadratic diameter [5].

We can prove an easier version of this conjecture, which was proven in Cereceda's paper:

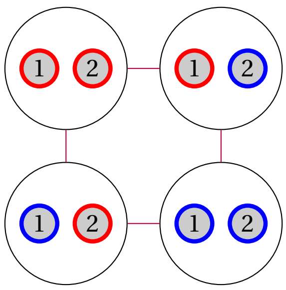
Theorem 2 (Weakened Cereceda's Conjecture)**.** *If a simple graph $G = (V, E)$ has degeneracy* $\mathrm{dg}(G)$*, then*

$$\mathrm{diam}\left(\mathscr{C}_{\mathrm{dg}(G)+2}(G)\right) = \mathcal{O}\left(2^n\right),$$

where $n = |V(G)|$.

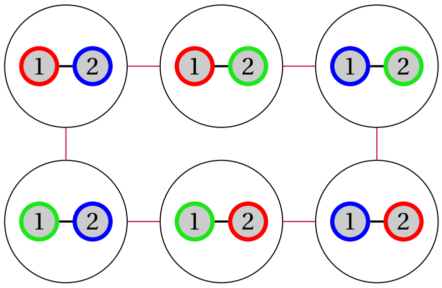
Proof of Theorem 2. We apply induction on the number of vertices $|V(G)| = n$. For the base case we go directly to $n = 2$. For $n = 0$ or $n = 1$ the hypothesis is true, however visually uninteresting, and mostly trivial. For $n = 2$ we do have some neat graphs to show, namely we have two cases:

- If there are no edges, we have two isolated points, and thus $\mathrm{dg}(G) = 0$. Our reconfiguration graph $\mathscr{C}_2(G)$ then consists of the following 4 vertices (graphs).



Thus $\mathrm{diam}\left(\mathscr{C}_2(G)\right) = 2 \le 2^2$.

- If there is an edge between the two vertices, we have $\text{dg}(G) = 1$. Then $\mathscr{C}_3(G)$ consists of the following 6 vertices (graphs).



Thus $\text{diam}\left(\mathscr{C}_3(G)\right) = 3 \leq 2^2$.

Now suppose that for all $m \leq n$, for a simple graph $H = (V, E)$ with $|V(H)| = m$ vertices and degeneracy $\text{dg}(H)$ we have $\text{diam}\left(\mathscr{C}_{\text{dg}(H)+2}(H)\right) = \mathcal{O}\left(2^m\right)$. Consider then a simple graph $G = (V, E)$ with $|V(G)| = n+1$ and degeneracy $\text{dg}(G)$. By definition we can find an ordering $(v_1, v_2, \ldots, v_n, v_{n+1})$ of the vertices of G such that there are at most $\text{dg}(G)$ edges between v_k and $\{v_1, v_2, \ldots, v_{k-1}\}$ for all $k \in \{2, 3, \ldots, n+1\}$.

We now apply the induction hypothesis on the induced subgraph $G[S] \subset G$ of the vertices $S = \{v_1, v_2, \ldots, v_n\}$, i.e. to go from any coloring α' of S to another coloring β' of S, using only $\text{dg}(G)+2$ colors, the amount of steps is bounded by $c \cdot 2^n$ where c is a constant. Define the steps needed to go from α' to β' by $s_1, s_2, \ldots, s_{c \cdot 2^n}$, where each step s_i changes the color of one vertex from $\{v_1, v_2, \ldots, v_n\}$.

If you were to follow these steps on G instead of $G[S]$ you might run into a problem with the color of vertex v_{n+1}. Since there are at most $\text{dg}(G)$ edges going between v_{n+1} and $\{v_1, v_2, \ldots, v_n\}$, and we have $\text{dg}(G)+2$ colors to use, we always have at least

$$\text{dg}(G) + 2 - |N(v_{n+1}) \cup v_{n+1}| = \text{dg}(G) + 2 - |N(v_{n+1})| - |v_{n+1}| \geq \text{dg}(G) + 2 - \text{dg}(G) - 1 = 1$$

free color for v_{n+1} to change to.

To apply one of the steps s_i from the induction hypothesis, vertex v_{n+1} might clash with the new coloring. We know now however that there is always a free color, hence, every time we want to apply one of the steps s_i from the induction hypothesis, we add another step r_i before s_i, which recolors vertex v_{n+1} to the free color, such that no issues arise with coloring. Once every vertex in $\{v_1, v_2, \ldots, v_n\}$ has the correct coloring, if vertex v_{n+1} is not the right color yet, we add a final step f which recolors vertex v_{n+1} to the correct color.

Finally, we find that to get from a coloring α of $V(G)$ to a coloring β of $V(G)$, we need, in order, at most the following steps

$$r_1, s_1, r_2, s_2, \ldots, r_{c \cdot 2^n}, s_{c \cdot 2^n}, f.$$

The total amount of steps needed between any two colorings α and β of $V(G)$ using $\text{dg}(G)+2$, and thus the diameter of the reconfiguration $\mathscr{C}_{\text{dg}(G)+2}(G)$, adds up to be at most

$$\begin{aligned}
&\left|\{r_1, r_2, \ldots, r_{c\cdot 2^n}\} \cup \{s_1, s_2, \ldots, s_{c\cdot 2^n}\} \cup \{f\}\right| \\
&= |\{r_1, r_2, \ldots, r_{c\cdot 2^n}\}| + |\{s_1, s_2, \ldots, s_{c\cdot 2^n}\}| + \left|\{f\}\right| \\
&= c \cdot 2^n + c \cdot 2^n + 1 = c \cdot \left(2^n + 2^n\right) + 1 = c \cdot \left(2 \cdot 2^n\right) + 1 = c \cdot 2^{n+1} + 1 = \mathcal{O}\left(2^{n+1}\right).
\end{aligned}$$

□

Note that an exponential lower bound is nowhere near as good as a quadratic bound, nor any polynomial bound. Far better approximations can be found in the 2019 paper by Bousquet and Heinrich [5], where it is proven that if $k \geq \text{dg}(G) + 2$, then $\text{diam}\left(\mathscr{C}_k(G)\right) = \mathcal{O}\left(n^{\text{dg}(G)+1}\right)$, improving to a polynomial bound, but not necessarily quadratic. However, the paper also proves that if $k \geq \frac{3}{2}(\text{dg}(G)+1)$, then $\text{diam}\left(\mathscr{C}_k(G)\right) = \mathcal{O}\left(n^2\right)$, which is precisely the bound in Cereceda's conjecture, but the condition on k is now not tight enough to prove the conjecture.

Now that we have explored the related problem, we move on to the problem with the maximum degree, which is where our main focus lies.

3

Lower bound on the diameter

We would like to show for any complete graph $G = K_n$ and complete bipartite graph $G = K_{n,m}$ with $n, m \in \mathbb{Z}_{\geq 0}$, a lower bound for the diameter with respect to the maximum degree $\Delta(G)$ of G, i.e. $\operatorname{diam}\left(\mathscr{C}_{\Delta(G)+2}(G)\right) \geq n + \mu(G)$, where $\mu(G)$ is the matching number of G. Aside from that, looking at specific examples might help with understanding the problem when wanting to look at r-partite graphs.

3.1. Helping construction, based on Cambie et al.

To prove the lower bound stated in the introduction of this chapter for certain graphs G, we will be using a certain extension of this graph. This extension was introduced to me in a 2022 paper from Cambie et al. [6], and will be heavily used for finding the lower bound on the diameter in question. Suppose that G is a graph, and take M to be a maximum matching of G. We now construct the extension $\widehat{G} = (V, E)$ as follows:

- $\forall v \in V(G), v \in V(\widehat{G})$.
- if $e \in E(G)$, then $e \in E(\widehat{G})$
- if $a, b, c, d \in V(G)$ and $ab, cd \in M$ and $ac \in E(G)$, then $bd \in E(\widehat{G})$.

Notice that $\widehat{G}$ is dependent on our choice of matching M. If it is clear from the context what our matching is, or our choice of M does not matter, we do not specify the matching. We also notice from the definition that we only add edges to G, so $G \subseteq \widehat{G}$, and hence $\chi(G) \leq \chi(\widehat{G})$. Also note that a proper k-coloring of $\widehat{G}$, is also a proper k-coloring of G. We now prove a sufficient condition for the lower bound we want to achieve.

Lemma 1. *If G has a perfect matching M and for this M we have $\chi(\widehat{G}) \leq k$, then*

$$\operatorname{diam}(\mathscr{C}_k(G)) \geq n + \mu(G).$$

Proof. To prove this lemma, we consider an arbitrary k-coloring α of $\widehat{G}$ and then construct a k-coloring β of $\widehat{G}$ which we can prove is at least $n + \mu(G)$ distance away from α.

Consider the endpoints of the edges of the matching, e.g. $xy \in M$ with $x, y \in V(G)$. We construct the k-coloring β of $\widehat{G}$ by switching the colors on the edges in M. So for instance, for our two vertices x and y, we get $\beta(x) = \alpha(y)$ and $\beta(y) = \alpha(x)$.

To show β is in fact proper, let $x \in V(G)$ be arbitrary. Since M is perfect, let $y \in V(G)$ such that $xy \in M$ and suppose for the sake of contradiction $xz \in E(\widehat{G})$ and $\beta(x) = \beta(z)$. We know $z \neq y$, since $xy \in M$ and thus $\beta(x) \neq \beta(y)$. As M is a perfect matching, let z' be the vertex such that $zz' \in M$, and thus $\beta(z) \neq \beta(z')$. Now as $xz \in E(\widehat{G})$, we have $xz \in E(G)$ or $yz' \in E(G)$ by construction of $\widehat{G}$. If $xz \in E(G)$, then $\beta(x) \neq \beta(z)$, which is contradictory. If $yz' \in E(G)$, then $\alpha(y) \neq \alpha(z')$, but then by definition of β we get $\beta(x) \neq \beta(z)$, again contradictory.

The distance between α and β can be bounded by changing colors along the matching. Consider one edge of the matching, i.e. $xy \in M$, with $\alpha(x) = i, \alpha(y) = j$ (and thus $\beta(x) = j, \beta(y) = i$). If all the other vertices do not interfere, the fastest way to switch the colors of the vertices is to change the color of x to a free color m, then change y to the desired color i and then change x to the desired color j.

Note that if a free color does not exist, the lower bound still stands, since $\infty > n + \mu(G)$.

Hence per edge of the matching we need to change the color of both endpoints, plus at least one extra change of an endpoint to a free color per matching. Thus

$$\operatorname{diam}(\mathscr{C}_k(G)) \geq \sum_{e \in M} (|\{v \in e\}| + 1) = n + \mu(G).$$

This finsishes the proof. □

Since not every graph has a perfect matching, this lemma is only applicable in some cases. We can however modify our proof to show a sufficient condition for all matchings:

Lemma 2. *If at least one of the following is true:*

(a) $\chi(\widehat{G}) \leq k - 1$
(b) $\chi(\widehat{G}) \leq k$ *and* $k \geq \Delta(G) + 2$

then

$$\operatorname{diam}(\mathscr{C}_k(G)) \geq n + \mu(G).$$

Proof. The proof is essentialy the same as Lemma 1.We need one more step to construct β however, since not every vertex is saturated by the matching. For all such vertices v, let $\beta(v)$ be outside of $\alpha(v) \cup \bigcup_{w \in N(v)} \beta(v)$. In situation (b) this is certainly possible, because there is a free color at v. In situation (a) this could mean that $\beta(v)$ introduces a color outside the range of $\chi(\widehat{G})$, and thus makes β a $((k-1)+1)$-coloring of $\widehat{G}$, and thus a k-coloring of G.

The switching argument stays the same, and we also need to change the colors of all vertices v not saturated by M. This means we now have:

$$\operatorname{diam}(\mathscr{C}_k(G)) \geq |\{v : \forall e \in M, v \notin e\}| + \sum_{e \in M} (|\{v \in e\}| + 1) = n + \mu(G),$$

and we are done. □

This is a lot more useful. One immediate consequence of Lemma 2 is a lower bound for complete graphs.

3.2. Results for complete graph and complete bipartite graph

We find for the complete graph K_n (where $n \in \mathbb{Z}_{\geq 0}$) that, since K_n cannot be extended, for any matching M of K_n, $\widehat{K_n} = K_n$, thus $\chi(\widehat{K_n}) = \chi(K_n) = n = (n-1) + 1 = \Delta(K_n) + 1$. Thus we have: $\operatorname{diam}(\mathscr{C}_{n+1}(K_n)) = \operatorname{diam}\left(\mathscr{C}_{\Delta(K_n)+2}(K_n)\right) \geq n + \mu(K_n)$. If you are wondering, the matching number for the complete graph K_n is equal to $\mu(K_n) = \lfloor n/2 \rfloor$, hence the bound is more precisely equal to: $\operatorname{diam}\left(\mathscr{C}_{\Delta(K_n)+2}(K_n)\right) \geq \lfloor \frac{3n}{2} \rfloor$.

Next we aim to prove the lower bound for the complete bipartite graph $G = K_{n,m}$. To prove this we need an extra result, which was proven by Cambie et al. [6], under Proposition 3(1).

Corollary 1. *If* $\chi(G)\left(\chi(G) - 1\right) \leq k - 1$, *then*

$$\operatorname{diam}(\mathscr{C}_k(G)) \geq n + \mu(G).$$

Proof. Suppose $\chi(G)\left(\chi(G) - 1\right) \leq k - 1$. Let α be fixed a $\chi(G)$-colouring of G. Assume that M is a perfect matching of G; if this is not the case, we add an extra edge to every vertex unsaturated by M, and then add the newly created edges to our matching M such that M is now perfect. We require a $(k-1)$-coloring of $\widehat{G}$, hence we give every vertex v the tuple-color $(\alpha(v), \alpha(w))$, where $vw \in M$, call this coloring $\widehat{f}$. We now check that this coloring of $\widehat{G}$ is indeed proper.

Suppose $x, y \in V(G)$, $xy \in E(\widehat{G})$ and $\widehat{f}(x) = \widehat{f}(y)$, then $(\alpha(x), \alpha(x')) = (\alpha(y), \alpha(y'))$, where $xx', yy' \in M$. Then we have $\alpha(x) = \alpha(y)$ and $\alpha(x') = \alpha(y')$. Since $xy \in E(\widehat{G})$, we have $xy \in E(G)$ or $x'y' \in E(G)$ by construction of $\widehat{G}$. If $xy \in E(G)$, then $\alpha(x) \neq \alpha(y)$, and if $x'y' \in E(G)$, then $\alpha(x') \neq \alpha(y')$ by propriety of α. Both cases are contradictory, hence the coloring $\widehat{f}$ is proper.

Since for $vw \in M$ we have $\alpha(v) \neq \alpha(w)$, we have $\chi(G)(\chi(G) - 1)$ different $(\alpha(v), \alpha(w))$ tuples, and thus we have at most a $\chi(G)(\chi(G) - 1)$-coloring of $\widehat{G}$. By combining our first assumption and Lemma 2a, we have proven the lower bound.

□

We then immediately get the following result:

Corollary 2. *For bipartite graphs G, we have*

$$\operatorname{diam}(\mathscr{C}_k(G)) \geq n + \mu(G)$$

for all $k \geq 3$.

Proof. Bipartite graphs G have $\chi(G) = 2$, hence we have for $k \geq 3 : \chi(G)\left(\chi(G) - 1\right) = 2 \cdot 1 = 2 \leq k - 1$, thus by Corollary 1 the lower bound on the diameter holds. □

Now that we have an approach to getting the desired lower bound on the diameter and some results on the complete graph and complete bipartite graphs, we can begin with more general family of graphs, the r-partite graphs, and get results on the lower bound of the diameter for the reconfiguration graph such graphs.

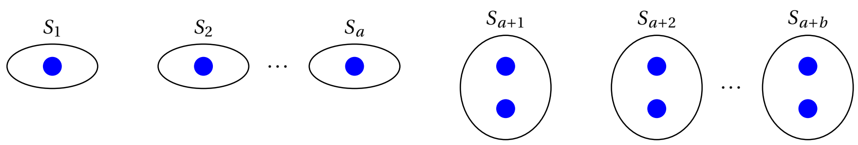
4

On r-partite graphs

We have showed for the complete graph and for the complete bipartite graph, that the lower bound for the diameter of the reconfiguration graph holds. A next step could be to look at a mix, or a merge, of the two. In general, we want to look at r-partite graphs, graphs that be partitioned into r independent sets, i.e. can be colored in r-colors or more. More specifically, we look at complete r-partite graphs of the form $G = K_{x_1,x_2,\ldots,x_r}$, where $x_1 \leq x_2 \leq \ldots \leq x_r$, $\sum_{i=1}^{r} x_i = n$ and each index i belongs to a different ind. set S_i of size x_i, and there is an edge in between every two vertices not belonging to the same ind. set.

4.1. An introduction: $G = K_{1,1,\ldots,1,2,2,\ldots,2}$, with a 1's, and b 2's.

This graph consists of a independent sets of size 1 and b independent sets of size 2, meaning we have $a+2b$ vertices and $\binom{a+b}{2}$ edges. The graph can be visualised as followed, with the edges removed for clarity.



Before we determine a lower bound on the diameter of this graph, we give a few important details. When looking at the degree of a vertex v from an set of size 2, the degree is equal to $a+2b-2$ as there is no edge to itself, and no edge to the other vertex in the set. For a vertex v from an set of size 1, the degree is equal to $a+2b-1$ as there is no edge to itself. Therefore the maximum degree $\Delta(G)$ is equal to $a+2b-1$. By definition, the chromatic number $\chi(G)$ is equal to $a+b$.

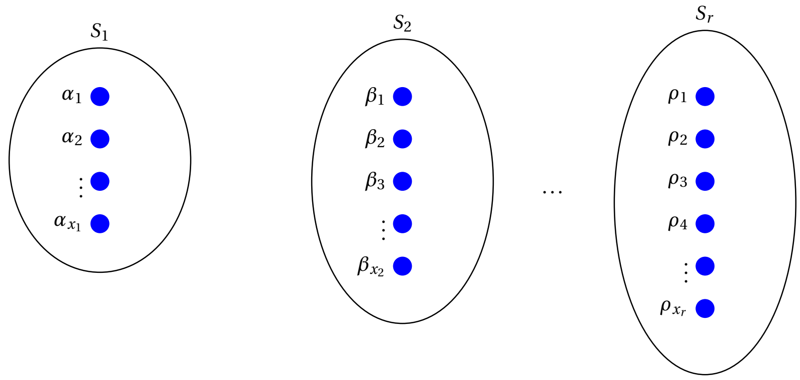
We now apply the construction $\widehat{G}$ on our graph G. Note that for any matching we chose, we cannot extend G past the $a+2b$ vertices, thus $\widehat{G} \subseteq K_{a+2b}$, and thus $\chi(\widehat{G}) \leq \chi(K_{a+2b}) = a+2b$. By Lemma 2a we find immediately

$$\operatorname{diam}(\mathscr{C}_{a+2b+1}(G)) = \operatorname{diam}\left(\mathscr{C}_{\Delta(G)+2}(G)\right) \geq n + \mu(G).$$

Hence the case for $\Delta(G)+2$ for the reconfiguration graph holds. We can however do a lot better; with a specific choice of matching our extension need not have a chromatic number of $a+2b$. A proof for this will be given later. Since we now that we have an idea of how to handle the smaller case of $G = K_{1,1,\ldots,1,2,2,\ldots,2}$, we want to directly tackle the general problem for complete r-partite graphs.

4.2. Complete r-partite graphs

As noted before, we look at $G = K_{x_1,x_2,\ldots,x_r}$, where $x_1 \leq x_2 \leq \ldots \leq x_r$, $\sum_{i=1}^{r} x_i = n$ and each index i belongs to a different ind. set S_i of size x_i, and there is an edge in between every two vertices not belonging to the same ind. set. We again start of with a few facts about this graph.



For starters, by our definition, it is clear that the chromatic number is equal to r. For the maximum degree we need to look at a vertex in the smallest ind. set S_1. Such a vertex has degree $n - x_1$. Notice that $n - x_1 \geq n - x_2 \geq \ldots \geq n - x_r$, so this is the maximum degree.

The chromatic number of the extension $\widehat{G}$ can be at most n, since that is the total amount of vertices. By Lemma 2a, if we assume the worst case scenario where the chromatic number is equal to n, then we have $\operatorname{diam}(\mathscr{C}_{n+1}(G)) \geq n + \mu(G)$. Notice that $\Delta(G) + 2 = n - x_1 + 2$, thus if $x_1 = 1$, we get $\Delta(G) + 2 = n + 1$, but then we get $\operatorname{diam}\left(\mathscr{C}_{\Delta(G)+2}(G)\right) = \operatorname{diam}(\mathscr{C}_{n+1}(G)) \geq n + \mu(G)$, which is precisely what we need.

Hence if $x_1 = 1$ we are already done, but in general x_1 is not equal to 1. From now on we assume $x_1 \geq 2$. We need to refine our extension $\widehat{G}$ such that we do not get the worst case scenario.

4.2.1. Matching number for r-partite graphs $G = K_{x_1, x_2, \ldots, x_r}$

In order to refine the extension, we need to know how to construct a maximum matching inside the r-paritite graph which does not add too many edges for the extension graph $\widehat{G}$. We give the following lemma for the matching number for complete r-partite graphs.

Lemma 3. *For any complete r-partite graph $G = K_{x_1, x_2, \ldots, x_r}$, the matching number is equal to*

$$\mu(G) = \min\{n - x_r, \lfloor n/2 \rfloor\}.$$

Proof. Depending on the size of our independent sets, we have to do our matching differently. It turns out the following two cases are sufficient to find the matching number:

Case 1: When $x_r \leq \lfloor n/2 \rfloor$, we first define a bijection $f : \{1, 2, \ldots, n\} \to V(G)$ where

$$\begin{aligned} S_1 &= f(\{1, 2, \ldots, x_1\}) \\ S_2 &= f(\{x_1 + 1, x_1 + 2, \ldots, x_1 + x_2\}) \\ &\vdots \\ S_r &= f\left(\left\{\sum_{i=1}^{r-1} x_i + 1, \sum_{i=1}^{r-1} x_i + 2, \ldots, n\right\}\right) \end{aligned}$$

We then match $f(i) \leftrightarrow f(i + \lfloor n/2 \rfloor)$ for $1 \leq i \leq \lfloor n/2 \rfloor$. Since $x_i \leq x_r \leq \lfloor n/2 \rfloor$, the matching connects pairs from different independent sets S_i and S_j. Furthermore, if n is even, all vertices are matched, hence we have a perfect matching, and if n is odd, we have $2\lfloor n/2 \rfloor = n - 1$ vertices that are matched, i.e. all but one, hence making it a maximum matching.

Case 2: When $x_r > n/2$, our final independent set S_r is larger than all other independent sets combined, as $\sum_{i=1}^{r-1} |S_i| = \sum_{i=1}^{r-1} x_i = n - x_r < n/2 < x_r = |S_r|$. To this effect, let M be the matching between all vertices from $\bigcup_{i=1}^{r-1} S_i$ to the vertices of S_r, leaving some vertices from S_r unsaturated.

To prove this is a maximum matching, we use Berge's Theorem [2], which states that this matching is maximum, if and only if there is no *augmenting path* with the matching, i.e. there is not a path that starts and ends on unmatched vertices, and alternates between edges in and not in the matching.

In our case, the unmatched vertices are all inside S_r, and S_r is independent, hence any augmenting path starts and ends with an edge outside the matching. Since the augmenting path must alternate between edges in- and outside of the matching, we must have an odd amount of edges in the augmenting path. However, the only unmatched vertices are in S_r, and all edges in the matching are linked to S_r, making this impossible. Thus our matching is indeed maximum.

We find that the matching number in case 1 is of the size $\lfloor n/2 \rfloor$, and in case 2 we have a matching of size $n - x_r$, thus the matching number of any complete r-partite graph G is $\mu(G) = \min\{\lfloor n/2 \rfloor, n - x_r\}$. □

We claim that both constructions given in the proof are beneficial for our extension, in the sense that it proves the lower bound.

4.2.2. Lower bound for diameter of complete r-partite graphs $G = K_{x_1,x_2,\ldots,x_r}$

We need a few other results to prove what we need.

Lemma 4. *For any graph G and every maximum matching M of G, we have $\chi(\widehat{G}) \leq 2s + 1$ whenever M can be partitioned into s disjoint sub-matchings $M_1, M_2, \ldots, M_s$, such that for all $1 \leq i \leq s$, M_i induces a bipartite subgraph of G. Furthermore, if M is perfect we have: $\chi(\widehat{G}) \leq 2s$.*

Note: "M_i induces a bipartite subgraph" means that the endpoints of M_i induce a bipartite subgraph.

Proof. Suppose M can be partitioned into s disjoint sub-matchings $M_1, M_2, \ldots, M_s$, such that for all $1 \leq i \leq s$, M_i induces a bipartite subgraph of G. Then every M_i also induces a bipartite subgraph of $\widehat{G}$. Hence in $\widehat{G}$ we only need 2 colors to color the endpoints of each M_i. Therefore we need at most $2s$ colors to color every vertex saturated by M. The vertices that are unsaturated by M form an independent set S, otherwise M would not be a maximum matching. Thus we need at most $2s + 1$ colors to color all vertices of the graph $\widehat{G}$. If M was perfect, S does not exist, and we need only $2s$ colors. □

Theorem 3. *For any complete r-partite graph $G = K_{x_1,x_2,\ldots,x_r}$, we have*

$$\operatorname{diam}\left(\mathscr{C}_{\Delta(G)+2}(G)\right) \geq n + \mu(G).$$

Proof. Reminder that for $x_1 = 1$ we were already done, hence we assume $x_1 \geq 2$. For **Case 2**, where $x_r > \lfloor n/2 \rfloor$, we chose a matching M between the vertices $\bigcup_{i=1}^{r-1} S_i$ to the vertices of S_r. Hence we can partition the matching into disjoint sub-matchings M_i between S_i and S_r, which all induce a bipartite subgraph of G. Hence we have $r - 1$ sub-matchings, and thus by Lemma 4 we find $\chi(\widehat{G}) \leq 2(r-1) + 1$. Since $x_1 \geq 2$ by assumption and $x_1 \leq x_2 \leq \ldots \leq x_r$, we have $n - x_1 = \sum_{i=2}^{r} x_i \geq \sum_{i=2}^{r} x_1 \geq 2(r-1)$, and thus $\chi(\widehat{G}) \leq 2(r-1) + 1 \leq n - x_1 + 1$. By Lemma 2a we have proven the lower bound for $n - x_1 + 2 = \Delta(G) + 2$ colors, which is what we needed.

For **Case 1** it gets more complicated. Let M be the maximum matching described earlier for **Case 1**. Define $N(S_1)$ be the set of vertices connected to S_1 by an edge from M. By our choice of $f(1), \ldots, f(n)$ we know that $N(S_1)$ can intersect with at most two other independent sets S_i and S_j. To this effect, we define the following sets:

$$\begin{aligned}
B_1 &= \{N(S_1) \cap S_i\} \\
B_2 &= \left\{N(S_1) \cap S_j\right\} \\
A_1 &= \{v \in S_1 | \exists w \in B_1 : vw \in M\} \\
A_2 &= \{v \in S_1 | \exists w \in B_2 : vw \in M\}
\end{aligned}$$

Note that the union of A_1 and B_1 induces a bipartite subgraph of G, and if A_2 and B_2 are non-empty, so does the union of A_2 and B_2. Hence we can choose a small enough amount of submatchings s for Lemma 4.

If A_2 is non-empty, we have two submatchings $A_1 \leftrightarrow B_1$ and $A_2 \leftrightarrow B_2$, and we can choose the rest of the edges in M as submatchings, so we get: $s = |M| - (|A_1| - 1) - (|A_2| - 1) = \lfloor n/2 \rfloor - x_1 + 2$. By Lemma 4 we know that $\widehat{G}$ can be colored with $2s + n - 2\lfloor n/2 \rfloor = 2(\lfloor n/2 \rfloor - x_1 + 2) + n - 2\lfloor n/2 \rfloor = n - 2x_1 + 4 \leq n - x_1 + 2$ colors, since $x_1 \geq 2$. If A_2 is empty, we have one submatching $A_1 \leftrightarrow B_1$ and we again let the rest of the edges of M be submatchings, and we get: $s = |M| - (|A_1| - 1) = \lfloor n/2 \rfloor - x_1 + 1$. By Lemma 4, $\widehat{G}$ can be colored with $2s + n - 2\lfloor n/2 \rfloor = [2\lfloor n/2 \rfloor - 2x_1 + 2] + n - 2\lfloor n/2 \rfloor = n - 2x_1 + 2 \leq n - x_1$ colors, again since $x_1 \geq 2$.

In both cases we find $\chi(\widehat{G}) \leq n - x_1 + 2 = \Delta(G) + 2$, and thus by Lemma 2b, we have found the correct lower bound for the diameter of the r-partite graphs. □

4.3. Return to $G = K_{1,1,\ldots,1,2,2,\ldots,2}$

We have proven the general case of the lower bound of the diameter for complete r-partite graphs, however only for k-colorings of G where $k = \Delta(G) + 2$. Through choosing a specific matching, we can reduce the number of colors required while achieving the same lower bound on the diameter. This is not as clear to do for general r-partite graphs, but for specific examples it could be done. Hence we return to our original merge $G = K_{1,1,\ldots,1,2,2,\ldots,2}$ and analyse the possible matchings.

Earlier we assumed the worst case for the extension $\widehat{G}$, namely that it had a chromatic number of $a+2b$; in that case it did not matter since we had an independent set of size 1, which as we saw earlier proves the lower bound regardless. By picking the right matching, we can lower the chromatic number of $\widehat{G}$ by a substantial amount, which in turn gives a better indication of how many colors the reconfiguration graph actually needs.

Lemma 5. *For $G = K_{1,1,\ldots,1,2,2,\ldots,2}$, with a 1's, and b 2's, we have for $k \geq a+b+1$:*

$$\operatorname{diam}(\mathscr{C}_k(G)) \geq n + \mu(G).$$

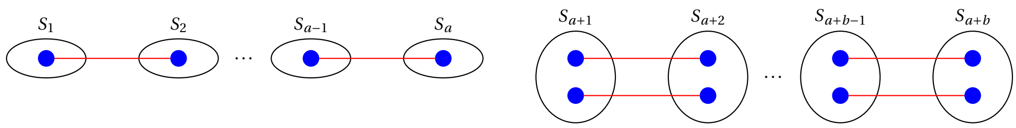
Since the chromatic number of $G = K_{1,1,\ldots,1,2,2,\ldots,2}$ with a 1's and b 2's is equal to $a+b$, every $(a+b)$-coloring in $\mathscr{C}_{a+b}(G)$ is frozen, in that no vertex can be recolored. Hence it makes sense to look at reconfiguration graphs of G on more than $a+b$ colors.

Proof. For this proof, we need to consider the parity of a and b.

Case 1: a is even and b is even.

Then any maximum matching is a perfect matching. Since we have an even amount of sets of size 2, we can split the ind. sets into pairs, and do the following for each pair: If $\{a, b\}$ and $\{c, d\}$ are two ind. sets, then $ac, ad, bc, bd \in E(G)$ by completeness of G, but notably $ab, cd \notin E(G)$ by independence. For the matching then take $ac, bd \in M$, as by the construction of $\widehat{G}$, since $ab \notin E(G) \implies cd \notin E(\widehat{G})$, and vice versa. Hence no edge is added in these ind. sets. Move on to the next pair and do the same until all are covered.

We can apply the same idea for the even amount of ind. sets of size 1, between any pair $\{x\}$ and $\{y\}$ there is $xy \in E(G)$, let $xy \in M$, no edge gets added to $\widehat{G}$ by completeness of G. Move on to the next pair until all are covered.

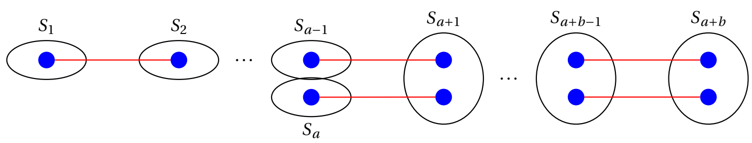


We have now chosen a matching such that no edge gets added in the extension, which means $\widehat{G} = G$ and thus $\chi(\widehat{G}) = \chi(G) = a+b$. Thus by Lemma 1 we find $\operatorname{diam}(\mathscr{C}_{a+b}(G)) \geq n + \mu(G)$.

Case 2: a is even and b is odd

Then any maximum matching is a perfect matching; we can try as before with creating a matching which does not add edges in $\widehat{G}$, however, for this parity, it is not possible to do so.

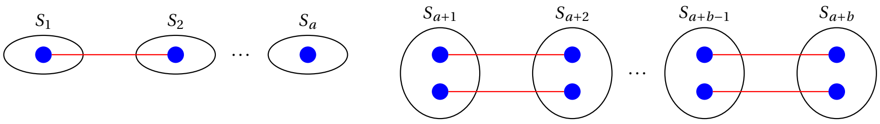
Add the edges to the matching M in the same way as above for all but two ind. sets of size 1, and all but one ind. set of size 2. For these sets $\{x, y\}, \{a\}$ and $\{b\}$, take $xa, yb \in M$. Since $ab \in E(G)$, xy will be added in $\widehat{G}$. This means $\widehat{G}$ has $a+2$ ind. sets of size 1 and $b-1$ ind. sets of size 2, giving a chromatic number for $\widehat{G}$ of $(a+2)+(b-1) = a+b+1$.



By Lemma 1 we then get $\operatorname{diam}(\mathscr{C}_{a+b+1}(G)) \geq n + \mu(G)$.

Case 3: a is odd and b is even

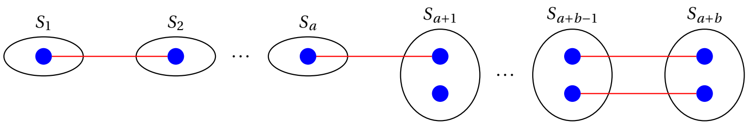
We have an odd amount of vertices so there will be one vertex not covered by the maximum matching M. By pairing all the independent 2-sets together, and all but one independent 1-set together as before, we do not add edges and we get $\chi(\widehat{G}) = a + b$.



Then using Lemma 2a we get diam$(\mathscr{C}_{a+b+1}(G)) \geq n + \mu(G)$.

Case 4: a is odd and b is odd

We have an odd amount of vertices so there will be one vertex not covered by the maximum matching M. By pairing all but one independent 2-sets together, and all but one independent 1-set together as before, and adding one edge between the leftover ind. 1- and 2-sets, we get no added edges, so $\chi(\widehat{G}) = a + b$.



Then using Lemma 2a we get diam$(\mathscr{C}_{a+b+1}(G)) \geq n + \mu(G)$.

$\square$

In general we have found that we can bring down the chromatic number of $\widehat{G}$, and therefore the number of colors for the reconfiguration graph can be brought down to $a + b + 1$ and the lower bound on the diameter will still hold. We notice that the choice of matching is crucial to getting the best lower bound.

5

Finding diameter using Sagemath

While having proven the lower bound for r-partite graphs is good progress, a bigger issue is that most, if not almost all graphs are not complete r-partite graphs. These graphs are not great to work with analytically, hence some computer work is in order. For this chapter we will write the code for making the reconfiguration graph, which is not trivial, we then analyse some graphs and give some results on the diameter. For this we use the extension Sagemath, since it has many packages related to graph theory; the code written can be found in appendix A.

5.1. The algorithm

To check the diameter of the reconfiguration graph of a certain graph with an algorithm, we first of all need to define the reconfiguration graph, i.e. both vertices and edges have to be coded in. After that we can compute the diameter with an already implemented command.

In Sagemath, a coloring of a graph is a dictionary, where the keys are vertices, and calling "`coloring[vertex]`" gives the color used for that vertex in the coloring. We can hence check if two colorings are adjacent by checking in how many vertices the coloring differs, which is what the following code does:

```
def is_adjacent(coloring1, coloring2):
    diff_count = 0
    for vertex in coloring1.keys():
        if vertex in coloring2:
            if coloring1[vertex] != coloring2[vertex]:
                diff_count += 1
                if diff_count > 1:
                    return False
        else:
            return False
    return diff_count == 1
```

We can then add the edges for the reconfiguration graph, by going over all k-colorings and adding an edge when two colors are adjacent. Defining the k-colorings we want, is not as easy. There is a function in Sagemath called "`all_n_colorings`" but it has limitations. The biggest limitation is when asking for all k-colorings, where k is larger than the amount of vertices $|V|$: then the function gives back 0 since it does not consider $|V|$-colorings as valid k-colorings. This essentially means we have to define our own valid colorings in order for our algorithm to work.

```
def all_extended_colorings(graph, max_color):
all_colorings = []
chi = graph.chromatic_number()
for m in range(chi, max_color + 1):
    color_dict = {v: None for v in graph.vertices()}  # Initialize with empty colors
    stack = [(0, color_dict)]  # Start with the first vertex and the initial color dictionary

    while stack:
        vertex_index, current_color_dict = stack.pop()

        if vertex_index == len(graph.vertices()):
            if len(set(current_color_dict.values())) == m:  # Check if all colors are used
                all_colorings.append(current_color_dict.copy())
            continue

        vertex = graph.vertices()[vertex_index]
        neighbors = graph.neighbors(vertex)
        neighbor_colors = [current_color_dict[n] for n in neighbors \
                           if current_color_dict[n] is not None]
        available_colors = [color for color in range(1, max_color + 1) \
                            if color not in neighbor_colors]

        for color in available_colors:
            new_color_dict = current_color_dict.copy()
            new_color_dict[vertex] = color
            stack.append((vertex_index + 1, new_color_dict))
return all_colorings
```

Since this excerpt of the code is bulky, we go through the code step by step:

1. We initialize with an empty list `all_colorings` for the colorings we want.
2. We go through the colorings where exactly m colorings are used, where m ranges from the chromatic number to our maximum amount of colors k.
3. In the main `for` loop we initialize with an empty dictionary for a coloring where every vertex is a key and first gets the color "None".
4. We then create a stack data structure, which starts with a tuple containing the vertex index “0" and the initial (empty) color dictionary. The stack will store pairs of vertex indices and color dictionaries for backtracking. We then do a `while` loop which continues until the stack is empty, i.e. it processes vertices and their color dictionaries until all possible colorings have been explored.
5. We then pop from top of the stack a vertex index and its corresponding color dictionary, and we check if vertex index is the last one, and all m distinct colors are used. If so, we add a copy of the coloring to `all_colorings`.
6. We then continue with our code; we look at the vertex assigned to the vertex index, the vertex’ neighbours and their colors which are already assigned, and check which colors are still available.
7. We then iterate over the available colors, create a copy of the current color dictionary to avoid modifying it directly, assign the current color from the loop in the new color dictionary, and finally pushes a tuple containing the incremented vertex index and the new color dictionary onto the stack. It simulates the recursive call by adding the next vertex and the updated color dictionary for future exploration.
8. Finally, after the main loop, the function returns the `all_colorings` list containing all the valid extended colorings of the hypercube graph.

This way the code explores all possible combinations of colors for each vertex, and we get all colorings as desired. The rest of the code is straight forward and can be found in Appendix A.

5.2. Results on smaller graphs

We analyse a few classes of graphs. We will also give the total number k-colorings of G, for this we will utilise the chromatic polynomial $P(G,k)$ of G [8], which calculates just that. To start off we have the cycle graphs C_n. The maximum degree of these graphs are equal to 2 for all n. Plugging in the code for `G = graph.CycleGraph(n)` with $\Delta(C_n)+2=4$ colors will give us the following table.

→ graph	C_3	C_4	C_5	C_6	C_7	C_8
→ diameter	4	6	7	9	10	12
→ #4-colorings	24	84	240	732	2184	6564

This is exactly what we expect, since one can show $\mu(C_n) = \lfloor n/2 \rfloor$ and one can show that every diameter in the table conforms to $n+\mu(G)$. In fact, Cambie et al. proved that for cycles C_n the upper and lower bound of $n+\mu(G)$ on the diameter diam$(\mathscr{C}_k(C_n))$ holds (see Lemma 21 [6]). Note that $n=9$ takes a very long time to load, it is actually too much for the windows sage cell. The polynomial for the cycle graph C_n is equal to $P(C_n,k) = (k-1)^n + (-1)^n(k-1)$. Therefore $P(C_9,4) = 19680$, $P(C_{10},4) = 59052$, and so on. The number of k-colorings on C_n increases exponentially when increasing n, but should increase polynomially when increasing k. In both cases the diameter will take an even longer time to compute.

When we use 3 colors instead of 4, we get the following table:

→ graph	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}
→ diameter	$+\infty$	6	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$
→ #3-colorings	6	18	30	66	126	258	510	1026	2046	4098

We can get more values for the diameter with using less colors, but most diameters are infinite. Interestingly, C_4 does not conform with the other cycles, in that the diameter of its reconfiguration graph for 3 colors is finite. We can show that the 4-cycle graph C_4 is equivalent to the complete bipartite graph $K_{2,2}$, i.e. $C_4 \cong K_{2,2}$. According to the Cambie et al. paper we cited before we can prove for complete bipartite graphs that the diameter of the reconfiguration graph on 3 colors is bounded from above by $n+\mu(G)$ (see Theorem 20 in [6]).

For hypercubes Q_n we have results for $n=2$ and $n=3$, but Q_2 is just equal to the cycle graph C_4, and we already have results for this graph. To obtain the 3-dimensional cube graph, we can use the command `G = CubeGraph(3)`, and we get the following table

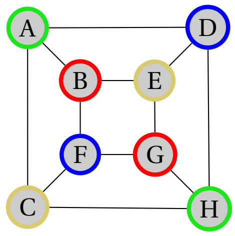
→ k=colors	3	4	5
→ diameter	12	$+\infty$	12
→ #k-colorings	114	2652	29660

One can figure out the chromatic polynomial of Q_3, the cube graph, in Maple or Wolframalpha and find out that $P(Q_3,k) = (k-1)k(133-290k+282k^2-159k^3+55k^4-11k^5+k^6)$. The next item in the table would have $k=6$ and thus $P(Q_3,6) = 198030$, which is almost a factor 10 larger than the previous item. Due to time restrains, figuring out the diameter for $k=6$ could not be done. Interestingly $k=4$ gives a disconnected reconfiguration graph, however, this is not the case for $k=5$ or more interestingly $k=3$. In the code in Appendix A there is an extra bit of code written to find two disconnected colorings if the diameter is infinite:

```
if diameter == +Infinity:
    # Find the connected components in the reconfiguration graph
    components = connected_components(reconfig_graph)

    # Print the disconnected colorings
    disconnected_colorings = []
    counter = 0
    for component in components:
        disconnected_colorings.append(component[0])
        counter += 1
        if counter == 2:
            break
    print("Disconnected Colorings:")
    for coloring in disconnected_colorings:
        print(coloring)
```

By letting the code run over the case where $k = 4$ we find a coloring of Q_3 which is frozen in its coloring.



Every single vertex in this coloring is adjacent to 3 other colors, which forces its own color to be the 4th color. Note that the cube graph Q_3 is equivalent to the L_4 graph mentioned in the introduction. Apparently, coloring with three colors does not give a disconnected reconfiguration graph.

6
Conclusion

After starting with a variant of the main problem, i.e. Cereceda's conjecture, and going over the known proofs of the diameter for complete graphs and complete bipartite graphs, which gave us a useful tool, the extension graph $\widehat{G}$, we were able to prove the lower bound on the diameter for r-partite graphs G: $\operatorname{diam}\left(\mathscr{C}_{\Delta(G)+2}(G)\right) \geq n + \mu(G)$. For this we had to figure out a working matching for the r-partite graph, in order to get as few extra edges as possible in the extension. Furthermore we determined the diameter of the reconfiguration graph for a few smaller graphs.

The original conjecture was that for all graphs G we have $\operatorname{diam}\left(\mathscr{C}_{\Delta(G)+2}(G)\right) = n + \mu(G)$, meaning we still have an upper bound to consider. An indication on how to attempt such a proof can be found in the 2018 paper made by Bonamy and Bousquet [4] under Lemma 5, which proves an upper of $2n$ bound for the reconfiguration graph of K_n on k colors: $\operatorname{diam}(\mathscr{C}_k(K_n))$. For this proof you have two colorings α and β, and one creates a new directed graph $D_{\alpha,\beta}$ in which $V(D_{\alpha,\beta}) = V(K_n)$ and $xy \in E(D_{\alpha,\beta})$ iff $\alpha(y) = \beta(x)$. This directed graph has only cycles and paths, both of which can be resolved via careful recoloring. This bound can be lowered to $\lfloor 3n/2 \rfloor$ according to Cambie et al. [6] using the same directed graph argument. The author of this thesis was able to prove this improved upper bound for complete graphs using an inductive method, also using the same approach as above, but has not attempted yet on generalising the argument for complete r-partite graphs. It would be interesting to see whether such a generalisation is possible or not.

For our algorithm, the number of k-colorings of a graph G increases quite quickly for certain graphs. In our case, the hypercube grows insanely fast, where even an increase of 1 color can increase the amount of colorings by a factor 10, which might not seem like much, but our algorithm is not a polynomial time algorithm.

Future research can therefore look into the upper bound of $\operatorname{diam}(\mathscr{C}_k(G))$ and come up with a proof for complete r-partite graphs. The tactic used for proving the lower bound, which was heavily based on that of Cambie et al., proved to be fruitful for complete r-partite graphs, and perhaps it may also be helpful for proving the lower bound for non-complete r-partite graphs, although they are harder to work with due to their non-completeness. Lastly the author is sure a more elegant version of the algorithm can be coded which can run larger reconfiguration graphs, and they encourage anyone to optimise the code.

A
Sagemath code

```
from itertools import combinations
from sage.graphs.connectivity import connected_components

def all_extended_colorings(graph, max_color):
    all_colorings = []
    chi = graph.chromatic_number()
    for m in range(chi, max_color + 1):
        color_dict = {v: None for v in graph.vertices()}  # Initialize with empty colors
        stack = [(0, color_dict)]  # Start with the first vertex and the initial color dictionary

        while stack:
            vertex_index, current_color_dict = stack.pop()

            if vertex_index == len(graph.vertices()):
                if len(set(current_color_dict.values())) == m:  # Check if all colors are used
                    all_colorings.append(current_color_dict.copy())
                continue

            vertex = graph.vertices()[vertex_index]
            neighbors = graph.neighbors(vertex)
            neighbor_colors = [current_color_dict[n] for n in neighbors \
                               if current_color_dict[n] is not None]
            available_colors = [color for color in range(1, max_color + 1) \
                                if color not in neighbor_colors]

            for color in available_colors:
                new_color_dict = current_color_dict.copy()
                new_color_dict[vertex] = color
                stack.append((vertex_index + 1, new_color_dict))

    return all_colorings

def is_adjacent(coloring1, coloring2):
    # Check if two colorings are adjacent by differing in exactly one vertex
    diff_count = 0
    for vertex in coloring1.keys():
        if vertex in coloring2:
            if coloring1[vertex] != coloring2[vertex]:
                diff_count += 1
                if diff_count > 1:
```

```
                    return False
        else:
            return False
    return diff_count == 1


# Specify the value of k
k = 4

# Construct the hypercube
G = graphs.CubeGraph(3)

# Generate all possible vertex colorings with k colors
valid_colorings = all_extended_colorings(G, k)

# Create a new graph for reconfiguration
reconfig_graph = Graph()

# Add vertices to the reconfiguration graph
for coloring in valid_colorings:
    reconfig_graph.add_vertex(str(coloring))

# Add edges to the reconfiguration graph
for i, j in combinations(valid_colorings, 2):
    if is_adjacent(i, j):
        reconfig_graph.add_edge(str(i), str(j))

# Calculate the diameter of the reconfiguration graph
diameter = reconfig_graph.diameter()

if diameter == +Infinity:
    # Find the connected components in the reconfiguration graph
    components = connected_components(reconfig_graph)

    # Print the disconnected colorings
    disconnected_colorings = []
    counter = 0
    for component in components:
        disconnected_colorings.append(component[0])
        counter += 1
        if counter == 2:
            break
    print("Disconnected Colorings:")
    for coloring in disconnected_colorings:
        print(coloring)

else:
    # Print the diameter
    print("Diameter:", diameter)
```

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