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Liang, Fey

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MULTI-TYPE ALGEBRAIC PROOF THEORY

FEI LIANG

TU Delft

Multi-Type Algebraic Proof Theory

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Dissertation

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by

Fei LIANG

Master of Philosophy in Logic
East China Normal University, China
born in Baoji, China

This dissertation has been approved by the promotors:
Dr A. Palmigiano, Prof. dr M. Ma, and Prof. dr ir I. R. van de Poel.

Composition of the doctoral committee:

Rector Magnificus,	Chairman
Prof. dr ir I. R. van de Poel,	Delft University of Technology, promotor
Dr A. Palmigiano,	Delft University of Technology, promotor
Prof. dr M. Ma,	Sun Yat-sen University, China, promotor
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Prof. dr M. A. Moshier,	Chapman University, USA
Prof. dr ir C. G. Chorus,	Delft University of Technology
Other member:	
Dr G. Greco,	Utrecht University

Dr G. Greco of Utrecht University has, as supervisor, contributed significantly to the preparation of this dissertation.

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A thing of beauty is a joy for ever.

John Keats

Contents

Summary

This dissertation pertains to *algebraic proof theory*, a research field aimed at solving problems in structural proof theory using results and insights from algebraic logic, universal algebra, duality and representation theory for classes of algebras. The main contributions of this dissertation involve the very recent theory of *multi-type calculi* on the proof-theoretic side, and the well established theory of *heterogeneous algebras* on the algebraic side.

Given a cut-admissible sequent calculus for a basic logic (e.g. the full Lambek calculus), a core question in structural proof theory concerns the identification of axioms which can be added to the given basic logic so that the resulting axiomatic extensions can be captured by calculi which are again cut-admissible. This question is very hard, since the cut elimination theorem is notoriously a very fragile result. However, algebraic proof theory has given a very satisfactory answer to this question for substructural logics, by identifying a hierarchy $(\mathcal{N}_n, \mathcal{P}_n)$ of axioms in the language of the full Lambek calculus, referred to as the *substructural hierarchy*, and guaranteeing that, up to the level $\mathcal{N}_2$, these axioms can be effectively transformed into special structural rules (called *analytic*) which can be safely added to a cut-admissible calculus without destroying cut-admissibility.

The research program of algebraic proof theory can be exported to arbitrary signatures of *normal lattice expansions*, to the study of the systematic connections between algebraic logic and *display calculi*, and even beyond display calculi, to the study of the systematic connections between the theory of heterogeneous algebras and *multi-type calculi*, a proof-theoretic format generalizing display calculi, which has proven capable to encompass logics which fall out of the scope of the proof-theoretic hierarchy, and uniformly endow them with calculi enjoying the same excellent properties which (single-type) proper display calculi have.

The defining feature of the multi-type calculi format is that it allows entities of different types to coexist and interact on equal ground: each type has its own internal logic (i.e. language and deduction relation), and the interaction between logics of different types is facilitated by special *heterogeneous connectives*, primitive to the language, and treated on a par with all the others. The fundamental insight justifying such a move is the very natural consideration, stemming from the algebraic viewpoint on (unified) correspondence, that the fundamental properties underlying this theory are purely order-theoretic, and that as long as maps or logical connectives have these fundamental properties, there is very little difference whether these maps have one and the same domain and codomain, or bridge different domains and codomains.

This enriched environment is specifically designed to address the problem of expressing the interactions between entities of different types by means of analytic structural rules.

In the present dissertation, we extend the semantic cut elimination and finite model property from the signature of residuated lattices to arbitrary signatures of normal lattice expansions, and build or refine the multi-type algebraic proof theory of three logics, each of which arises in close connection with a well known class of algebras (semi De Morgan algebras, bilattices, and Kleene algebras) and is problematic for standard proof-theoretic methods.

Samenvatting

Dit proefschrift heeft betrekking op *algebraïsche bewijstheorie*, een onderzoeksgebied dat gericht is op het oplossen van problemen in structurele bewijstheorie met behulp van resultaten en inzichten uit de algebraïsche logica, universele algebra, dualiteit en representatietheorie voor klassen van algebra's. De belangrijkste bijdragen van dit proefschrift betreffen, aan de bewijs-theoretische kant, de zeer recente theorie van *multi-type calculi* en, aan de algebraïsche kant, de gevestigde theorie van *heterogene algebra's*.

Gegeven een sequentiële calculus met snede-eliminatiestelling voor een basislogica (b.v. de Lambek-calculus), heeft een kernvraag in de structurele bewijstheorie betrekking op de identificatie van axioma's die kunnen worden toegevoegd aan de gegeven basislogica zodat de resulterende axiomatische uitbreidingen kunnen worden vastgelegd door calculi waarvan de snede opnieuw geëlimineerd kan worden. Deze vraag is moeilijk, omdat de snede-eliminatiestelling bekend staat als een zeer fragiel resultaat. De algebraïsche bewijstheorie heeft echter voor substructurele logica een bevredigend antwoord gegeven op deze vraag, door een hiërarchie $(\mathcal{N}_n, \mathcal{P}_n)$ van axioma's in de taal van de Lambek-calculus te identificeren (die de *substructurele hiërarchie* genoemd wordt), die garanderen dat, tot op het niveau $\mathcal{N}_2$, deze axioma's effectief kunnen worden omgezet in speciale (*analytische*) structurele regels die veilig kunnen worden toegevoegd aan een calculus met snede-eliminatiestelling zonder de snede-eliminatiestelling te vernietigen.

Het onderzoeksprogramma van algebraïsche bewijstheorie kan worden geëxporteerd naar willekeurige talen van *normale tralieuitbreidingen*, naar de studie van de systematische verbindingen tussen algebraïsche logica en *displaycalculi*, en voorbij aan displaycalculi, naar de studie van de systematische verbindingen tussen de theorie van heterogene algebra's en *multi-type calculi* In de voorliggende dissertatie breiden we de semantische cut-eliminatie en de eindige-modeleigenschap uit van de taal van geresidueerde tralies (residuated lattices) naar willekeurige talen van normale tralieuitbreidingen en bouwen of verfijnen de multi-type algebraïsche bewijstheorie van drie logica's.

Het bepalende kenmerk van de aanpak van de multi-type calculi is dat het entiteiten van verschillende types in staat stelt om gelijkwaardig naast elkaar te bestaan en op elkaar in te werken: elk type heeft zijn eigen interne logica (d.w.z. taal- en afleidingsrelatie) en de interactie tussen logica's van verschillende typen wordt gefaciliteerd door speciale *heterogene connectieven*, primitief voor de taal, en gelijkwaardig met alle andere behandeld. Het fundamentele inzicht dat een dergelijke stap rechtvaardigt, is de zeer natuurlijke overweging, voortkomend uit het algebraïsche standpunt over (verenigde) correspondentie, dat de fundamentele eigenschappen die aan deze theorie ten grondslag liggen, zuiver ordetheoretisch zijn. Zolang afbeeldingen of logische verbanden deze fundamentele eigenschappen hebben is er weinig verschil of deze afbeeldingen een en hetzelfde domein en co-domein hebben, danwel verschillende domeinen en co-domeinen overbruggen.

Deze verrijkte omgeving is specifiek ontworpen om het probleem aan te pakken van het uitdrukken van de interacties tussen entiteiten van verschillende typen door middel van analytische structuurregels.

In dit proefschrift bouwen of verfijnen we de multi-type algebraïsche bewijstheorie van drie logica's, elk waarvan ontstaat in nauwe samenhang met één bekende klasse van algebra's (semi De Morgan algebra's, bi-tralies en Kleene algebra's) en problematisch is voor gebruikelijke bewijstheoretische methoden.

Chapter 1

Introduction

This dissertation pertains to *algebraic proof theory*, a research field aimed at solving problems in structural proof theory using results and insights from algebraic logic, universal algebra, duality and representation theory for classes of algebras. The main contributions of this dissertation involve the very recent theory of *multi-type calculi* [32] on the proof-theoretic side, and the well established theory of *heterogeneous algebras* [10] on the algebraic side. In this chapter, we give an overview of algebraic proof theory, discuss the link between the theory of analytic calculi in structural proof theory and algebraic algorithmic correspondence theory, and motivate how this link can be naturally extended to multi-type calculi and their algebraic semantics given by heterogeneous algebras. At this point, we will be in a position to discuss the specific contributions of the following chapters.

1.1 Algebraic proof theory

Algebraic proof theory [14] is a discipline aimed at establishing systematic connections between results and insights in structural proof theory (such as cut elimination theorems) and in algebraic logic (such as representation theorems for classes of algebras). While results of each type have been traditionally formulated and developed independently from the other type, algebraic proof theory aims at realizing a deep integration of these fields. The main results in algebraic proof theory have been obtained for axiomatic extensions of the full Lambek calculus, and, building on the work of many authors [7, 14, 18, 36, 37, 67], establish a systematic connection between a strong form of cut elimination for certain substructural logics (on the proof-theoretic side) and the closure of their corresponding varieties of algebras under MacNeille completions (on the algebraic side). Specifically, given a cut eliminable sequent calculus for a basic logic (e.g. the full Lambek calculus), a core question in structural proof theory concerns the identification of axioms which can be added to the given basic logic so that the resulting axiomatic extensions can be captured by calculi which are again cut eliminable. This question is very hard, since the cut elimination theorem is notoriously a very fragile result. However, algebraic proof theory has given a very satisfactory answer to this question for substructural logics, by identifying a hierarchy $(\mathcal{N}_n, \mathcal{P}_n)$ of axioms in the language of the full Lambek calculus, referred to as the *substructural hierarchy*, and guaranteeing that, up to the level $\mathcal{N}_2$, these axioms can be effectively transformed into special structural rules (called *analytic*) which can be safely added to a cut eliminable calculus without destroying cut elimination. Algebraically, this transformation corresponds to the possibility of transforming equations into equivalent quasiequations, and remarkably, such a transformation (which we will expand on shortly) is also key to proving preservation under MacNeille completions and canonical extensions.

The second major contribution of algebraic proof theory is the identification of the algebraic essence of cut elimination (for cut-free sequent calculi for substructural logics) in the relationship between a certain (polarity-based) relational structure W arising from the given sequent calculus, and a certain ordered algebra W^+ which can be thought of as the complex algebra of W by analogy with modal logic. Specifically, the fact that the calculus is cut-free is captured semantically by W being an *intransitive* structure, while W^+ is by construction an ordered algebra, on which the cut rule is sound. Hence, in this context, cut elimination is encoded in the preservation of validity from W to W^+. For instance, the validity of analytic structural rules/quasiequations is preserved from W to W^+ (cf. [14]), which shows that analytic structural rules can indeed be safely added to the basic Lambek calculus in a way which preserves its cut elimination.

In [15], these results are extended so as to cover the level $\mathcal{P}_3$ of the substructural hierarchy. For this more general class of axioms, sequent calculi are not enough [13], and have been replaced by hypersequent calculi [4].

In Chapter 2, we uniformly extend these notions, constructions and proof-strategies from the basic environment of the (nonassociative) full Lambek calculus to the logics of normal lattice expansions (the *normal LE-logics*).

1.2 Display calculi

The research program of algebraic proof theory can be exported to the study of the systematic connections between algebraic logic and *display calculi* [8, 68]. Display calculi (here we will be particularly interested in the notion of *proper display calculi*, cf. [68]) are yet another generalization of sequent calculi, sharing similarities with hypersequent calculi (cf. [17]).

Like hypersequent calculi (and various other proof-theoretic formats, e.g. labelled calculi [30, 55], nested sequent systems [11, 49]), display calculi have been introduced to capture logics which cannot be captured by Gentzen calculi alone.

The display calculi format is based on the introduction of a special syntax for the constituents of each sequent, which includes more *structural* connectives than the usual ones (i.e. commas and fusion). This richer syntax makes it possible to describe the essentials of syntactic cut elimination with sufficient mathematical precision that a *meta-theorem* can be proved, which gives a set of sufficient conditions for cut elimination, most of which are easily verified by inspection on the shape of the rules (these conditions define the notion of *analytic structural rules* in display calculi). Meta-theorems provide much smoother, robust and modular routes to cut elimination than Gentzen-style proofs. By analogy, cut elimination via meta-theorems is to Gentzen-style cut elimination what canonicity is to completeness, since, just like canonicity provides a *uniform strategy* to achieve completeness, the conditions of Belnap's meta-theorem guarantee that one and the same transformation strategy achieves syntactic cut elimination for any calculus satisfying them. In fact, the results in this dissertation build on and further reinforce the insight that this is much more than just an analogy: in [43], systematic connections have been established between semantic results pertaining to *unified correspondence* theory, and the theory of analyticity in display calculi.

1.3 Unified correspondence

Unified correspondence [20–28, 35, 53, 59, 60] applies insights and techniques from algebra, order-theory and formal topology to strengthen one of the core results in the model theory of classical modal logic - the celebrated Sahlqvist correspondence and canonicity theorem [65] – and extends it from modal logic to a vast array of logical systems which encompasses, among others, all substructural logics [21, 23, 25] and logic with fixed points [19, 20], also multiplying the conceptual significance and ramifications of the original Sahlqvist result.

The starting point of unified correspondence is the insight that the mechanisms underlying Sahlqvist's theorem are algebraic and order-theoretic in their essence [23, 26]. This insight has made it possible to extract the essentials of the original Sahlqvist result and recognize them systematically outside the setting of modal logic, so as to improve on the original Sahlqvist result, extend the definition of Sahlqvist (and the strictly larger class of *inductive*) formulas from modal logic to classes of nonclassical logics algebraically captured by general lattices with arbitrary (normal and regular) operations, and design an algorithm (ALBA) mechanically computing the first-order correspondent of each (generalized) Sahlqvist formula in any of these languages.

In [43], a proper subclass of inductive formulas/inequalities has been identified (referred to as the *analytic inductive inequalities*) the members of which are guaranteed to be effectively transformed into analytic structural rules of display calculi and are shown to be exactly those with this property.[1] Remarkably, this transformation essentially coincides with the one defined in [13] for the FL language (see also [16] for results of analogous strength to those in [43] but formulated in a purely proof-theoretic way), and is effected by the same algorithm ALBA which has been originally designed to compute the first-order correspondents of propositional axioms on Kripke-type structures, and which is also used to prove the canonicity of the same axioms. Finally, the canonical extension environment and the canonicity of (analytic) inductive inequalities are used to give a uniform semantic proof that the display calculi obtained by adding analytic structural rules are conservative w.r.t. the corresponding original logic.

Thus, these results connect unified correspondence with algebraic proof theory, and open up a new research line in algebraic proof theory in which not only MacNeille completions but also canonical extensions can be meaningfully exploited for proof-theoretic purposes. In this dissertation, we will do so mainly in relation with the proof-theoretic

[1] Hence, logics axiomatized with analytic inductive axioms are exactly those which can be presented by a proper display calculus.

environment of *multi-type calculi*, which we discuss below.

1.4 Multi-type calculi

The theory of multi-type calculi [9, 31–34, 44, 45] was developed as a generalization of display calculi, capable to encompass those logics which – like linear logic, dynamic epistemic logic, propositional dynamic logic, and inquisitive logic – fall out of the scope of the characterization given in [43] (cf. Footnote 1), and uniformly endow them with calculi enjoying the same excellent properties which (single-type) proper display calculi have.

The defining feature of the multi-type calculi format is that it allows entities of different types to coexist and interact on equal ground: each type has its own internal logic (i.e. language and deduction relation), and the interaction between logics of different types is facilitated by special *heterogeneous connectives*, primitive to the language, and treated on a par with all the others. The fundamental insight justifying such a move is the very natural consideration, stemming from the algebraic viewpoint on (unified) correspondence, that the fundamental properties underlying this theory are purely order-theoretic, and that as long as maps or logical connectives have these fundamental properties, there is very little difference whether these maps have one and the same domain and codomain, or bridge different domains and codomains. So in particular, all the results and insights of unified correspondence can (and will) be reformulated and used in the multi-type environment. Moreover, the whole theory of single-type (proper) display calculi, from Belnap-style cut elimination metatheorem to the semantic proof of conservativity via canonical extensions, translates smoothly to the multi-type setting. In particular, we will consider the very natural notion of canonical extensions of *heterogeneous algebras* [10] (cf. Definition 1).

This enriched environment is specifically designed to address the problem of expressing the interactions between entities of different types by means of analytic structural rules. Indeed, although each of the logics mentioned above is difficult to treat for its own specific reason, a common core to these difficulties can be identified precisely in the encoding of key interactions between entities of different types. For instance, for dynamic epistemic logic the difficulties lay in the interactions between (epistemic) actions, agents' beliefs, and facts of the world; for linear logic, in the interaction between general resources and reusable resources; for propositional dynamic logic, between general and 'transitive' actions; for inquisitive logic, between general and flat formulas. In each case, precisely the formal encoding of these interactions gives rise to non-analytic axioms in the original formulations of the logics. In each case, the multi-type approach allows to redesign the logics, so as to encode the key interactions into *analytic* structural rules in the multi-type language, and define a multi-type *proper* display calculus for each of them. Metaphorically, adding types is analogous to adding dimensions to the analysis of the interactions, thereby making it possible to unravel these interactions, by reformulating them in analytic terms within a richer language. The multi-type methodology has been also used to design novel logical formalisms focusing on agents' abilities and capabilities to manipulate resources [9]. Conceptually, the multi-type environment can be regarded as the very natural prosecution of Belnap's program, as formulated in [8], of building a unifying environment for "an indefinite number of logics all mixed together including boolean [...], intuitionistic, relevance and (various) modal logics."

1.5 Semi De Morgan Logic, Bilattice Logic and the logic of Kleene algebras

In the present dissertation, we build or refine the multi-type algebraic proof theory of three well known logics, each of which arises in close connection with a class of algebras (although with very different motivations) and is problematic for standard proof-theoretic methods. Before describing the specific contributions of the thesis, in this section we sketch some background and motivation for each of these logics.

Semi De Morgan logic is the selfextensional (cf. [69]) two-valued logic associated with the variety of *semi De Morgan algebras*, introduced by H.P. Sankappanavar [66] as a common abstraction of De Morgan algebras and distributive pseudocomplemented lattices. This logic is a very well known example of a paraconsistent logic [63], that is, a non-classical logic which violates the classical principle $\bot \vdash A$ (*ex falso quodlibet*). Semi De Morgan logic has been studied from a duality-theoretic perspective [46], and from the perspective of canonical extensions [58]. The proof theory of semi De Morgan logic is challenging because, although this logic arises from a variety of normal distributive lattice expansions, hence it falls within the setting of [43], and although its defining axioms are all Sahlqvist, some of these axioms are *not analytic*.

Bilattice logic is the non-selfextensional four-valued logic associated with the algebraic framework of *bilattices*, introduced by M.L. Ginsberg [38] as a unifying framework simultaneously accounting for different approaches to formal reasoning in AI, such as first-order theorem provers, assumption-based truth maintenance systems, and default logic. In a nutshell, a bilattice is a set equipped with two partial orders, such that, for each of which, all finite joins and meets exist. The two partial orders intend to capture two different types of information: one concerning the truth of propositions, and one concerning the information, or evidence, about propositions. Thus, the logic of bilattices can be used to capture inferences from incomplete and inconsistent data. Bilattice logic has been studied from an algebraic and duality-theoretic perspective [48, 54, 64], and from a proof-theoretic perspective [2, 3], and has also been recently integrated into other logical frameworks accounting for agency and epistemic attitudes [5, 6]. The proof theory of bilattice logic is challenging because this logic is not selfextensional, and hence does not fall within the setting of [43]. The most established proposal for a sequent calculus for bilattice logic is due to A. Avron [2]. However, this calculus has no subformula property.

Kleene algebras are the mathematical structures modelling the behaviour of so called *regular expressions* in automata theory, introduced by S.C. Kleene [50] with the same syntactic laws that have been then used to define Kleene algebras. Since then, Kleene algebras have established themselves as one of the most important and best known models of computation, and have been applied to interpret actions in dynamic logic [51, 61], to prove the equivalence of regular expressions and finite automata [1, 29], to give fast algorithms for transitive closure and shortest paths in directed graphs [1], and axiomatize algebras of relations [56, 62].

The proof theory of the logic of Kleene algebras is challenging because the axioms and defining rules of the Kleene star cannot be reduced to an analytic presentation. There are various proposals in the literature for sequent calculi for the logic of Kleene algebras [12, 47, 57, 70], in which there is a tradeoff between calculi with cut elimination and an infinitary rule: from sequent calculi with finitary rules but with a non-eliminable analytic cut [47, 70], to cut-free sequent calculi with infinitary rules [57].

1.6 Original contributions of this dissertation

The original contributions of this dissertation are listed below.

- In Chapter 2, which is based on [39], we extend the research programme in algebraic proof theory from axiomatic extensions of the full Lambek calculus to logics algebraically captured by certain varieties of normal lattice expansions (normal LE-logics). Specifically, we generalise the *residuated frames* in [36] to arbitrary signatures of normal lattice expansions (normal LEs). Such a generalization provides a valuable tool for proving important properties of LE-logics in full uniformity. We prove semantic cut elimination for the display calculi $\mathrm{D.LE}$ associated with the basic normal LE-logics and their axiomatic extensions with analytic inductive axioms. We also prove the finite model property (FMP) for each such calculus $\mathrm{D.LE}$, as well as for its extensions with analytic structural rules satisfying certain additional properties.

- In Chapter 3, which is based on [40], we introduce proper multi-type display calculi for semi De Morgan logic and its extensions which are sound, complete, conservative, and enjoy cut elimination and subformula property. Our proposal builds on an algebraic analysis of semi De Morgan algebras and its subvarieties, and applies the guidelines of the multi-type methodology in the design of display calculi.

- In Chapter 4, which is based on [42], we introduce a proper multi-type display calculus for bilattice logic (with conflation), for which we prove soundness, completeness, conservativity, standard subformula property and cut elimination. Our proposal builds on the product representation of bilattices and applies the guidelines of the multi-type methodology in the design of display calculi.

- In Chapter 5, which is based on [41], we introduce a multi-type calculus for the logic of measurable Kleene algebras, for which we prove soundness, completeness, conservativity, cut elimination and subformula property. Our proposal imports ideas and techniques developed in formal linguistics around the notion of structural control [52].

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Chapter 2

Algebraic proof theory for LE-logics

In this chapter, which is based on [1] *[12], we extend the research programme in algebraic proof theory from axiomatic extensions of the full Lambek calculus to logics algebraically captured by certain varieties of normal lattice expansions (normal LE-logics). Specifically, we generalise the* residuated frames *in [8] to arbitrary signatures of normal lattice expansions (*LE*). Such a generalization provides a valuable tool for proving important properties of* LE*-logics in full uniformity. We prove semantic cut elimination for the display calculi* $\mathrm{D.LE}$ *associated with the basic normal LE-logics and their axiomatic extensions with analytic inductive axioms. We also prove the finite model property (FMP) for each such calculus* $\mathrm{D.LE}$*, as well as for its extensions with analytic structural rules satisfying certain additional properties.*

[1] My specific contributions to this research have been the proof of results, the definition of notions and constructions, the development of examples, and the writing of the first draft of the paper.

2.1 Introduction

In this chapter, we generalize the framework of residuated frames, introduced in [8] to give a semantic proof of cut elimination for various axiomatic extensions of the full Lambek calculus, and applied to the proof of the finite embeddability property and finite model property for some of these. Our generalization concerns two aspects:

1. from the signature of residuated lattices to arbitrary normal lattice expansions; in particular, arbitrary signatures do not need to be closed under the residuals of each connective.

2. from structural rules of so-called simple shape to the more general class of analytic structural rules (cf. [13, Definition 4]) in any signature of normal lattice expansions.

Specifically, for every signature $\mathcal{L}(\mathcal{F}, \mathcal{G})$ of normal lattice expansions (cf. Definition 1) we define the associated notion of functional D-frame, and prove that

1. the cut rule is eliminable in the display calculus $\mathrm{D.LE}$ associated with the basic normal lattice logic in the signature $\mathcal{L}(\mathcal{F}, \mathcal{G})$. We prove this result by suitably generalizing the semantic argument given in the proof of Theorem 3.2 in [8].

2. the cut elimination above transfers to extensions of $\mathrm{D.LE}$ with analytic structural rules generalizing the notion of simple structural rules (cf. Section 5 in [13]). We prove this result by suitably generalizing the argument given in the proof of Theorem 3.10 in [8].

3. the finite model property holds for $\mathrm{D.LE}$ and for extensions of $\mathrm{D.LE}$ with analytic structural rules satisfying certain additional properties. We prove this result by suitably generalizing the argument given in the proof of Theorem 3.15 in [8].

We also discuss how these results recapture the semantic cut elimination results in [4] and apply in a modular way to a range of logics which includes the (nonassociative) full Lambek calculus and its axiomatic extensions, the Lambek-Grishin calculus and its axiomatic extensions, orthologic, and the cyclic involutive full Lambek calculus.

2.2 Syntax and algebraic semantics of LE-logics

This section is based on the Section 2 and 3 in [13].

2.2.1 Basic normal LE-logics and their algebras

Our base language is an unspecified but fixed language $\mathcal{L}_{\mathrm{LE}}$, to be interpreted over lattice expansions of compatible similarity type. This setting uniformly accounts for many well known logical systems, such as (nonassociative) full Lambek calculus, Lambek-Grishin calculus, orthologic, the cyclic involutive full Lambek calculus and other lattice based logics.

In our treatment, we will make heavy use of the following auxiliary definition: an *order-type* over $n \in \mathbb{N}$[2] is an n-tuple $\epsilon \in \{1, \partial\}^n$. For every order type ϵ, we denote its *opposite* order type by ϵ^∂, that is, $\epsilon^\partial_i = 1$ iff $\epsilon_i = \partial$ for every $1 \leq i \leq n$. For any lattice $\mathbb{A}$, we let $\mathbb{A}^1 := \mathbb{A}$ and $\mathbb{A}^\partial$ be the dual lattice, that is, the lattice associated with the converse partial order of $\mathbb{A}$. For any order type ε, we let $\mathbb{A}^\varepsilon := \Pi_{i=1}^n \mathbb{A}^{\varepsilon_i}$.

The language $\mathcal{L}_{\mathrm{LE}}(\mathcal{F}, \mathcal{G})$ (from now on abbreviated as $\mathcal{L}_{\mathrm{LE}}$) takes as parameters: 1) a denumerable set of proposition letters AtProp, elements of which are denoted p, q, r, possibly with indexes; 2) disjoint sets of connectives $\mathcal{F}$ and $\mathcal{G}$.[3] Each $f \in \mathcal{F}$ and $g \in \mathcal{G}$ has arity $n_f \in \mathbb{N}$ (resp. $n_g \in \mathbb{N}$) and is associated with some order-type ε_f over n_f (resp. ε_g over n_g).[4] The terms (formulas) of $\mathcal{L}_{\mathrm{LE}}$ are defined recursively as follows:

$$\phi ::= p \mid \bot \mid \top \mid \phi \wedge \phi \mid \phi \vee \phi \mid f(\overline{\phi}) \mid g(\overline{\phi})$$

where $p \in$ AtProp, $f \in \mathcal{F}$, $g \in \mathcal{G}$. Terms in $\mathcal{L}_{\mathrm{LE}}$ will be denoted either by s, t, or by lowercase Greek letters such as φ, ψ, γ etc. In the context of sequents and proof trees, $\mathcal{L}_{\mathrm{LE}}$-formulas will be denoted by uppercase letters A, B, etc.

Definition 1. *For any tuple $(\mathcal{F}, \mathcal{G})$ of disjoint sets of function symbols as above, a* lattice expansion *(abbreviated as LE) is a tuple $\mathbb{A} = (D, \mathcal{F}^\mathbb{A}, \mathcal{G}^\mathbb{A})$ such that D is a bounded lattice, $\mathcal{F}^\mathbb{A} = \{f^\mathbb{A} \mid f \in \mathcal{F}\}$ and $\mathcal{G}^\mathbb{A} = \{g^\mathbb{A} \mid g \in \mathcal{G}\}$, such that every $f^\mathbb{A} \in \mathcal{F}^\mathbb{A}$ (resp. $g^\mathbb{A} \in \mathcal{G}^\mathbb{A}$) is an n_f-ary (resp. n_g-ary) operation on $\mathbb{A}$. A LE is* normal *if every $f^\mathbb{A} \in \mathcal{F}^\mathbb{A}$ (resp. $g^\mathbb{A} \in \mathcal{G}^\mathbb{A}$) preserves finite joins (resp. meets) in each coordinate with $\epsilon_f(i) = 1$ (resp. $\epsilon_g(i) = 1$) and reverses finite meets (resp. joins) in each coordinate with $\epsilon_f(i) = \partial$ (resp. $\epsilon_g(i) = \partial$).[5] Let $\mathbb{LE}$ be the class of LEs. Sometimes we will refer to certain LEs as $\mathcal{L}_{\mathrm{LE}}$-algebras when we wish to emphasize that these algebras have a compatible signature with the logical language we have fixed.*

In the remainder of the chapter, we will abuse notation and write e.g. f for $f^\mathbb{A}$. Normal LEs constitute the main semantic environment of the present chapter. Henceforth, every LE is assumed to be normal; hence the adjective 'normal' will be typically dropped. The class of all LEs is equational, and can be axiomatized by the usual lattice identities and the following equations for any $f \in \mathcal{F}$ (resp. $g \in \mathcal{G}$) and $1 \leq i \leq n_f$ (resp. for each $1 \leq j \leq n_g$):

[2]Throughout the chapter, order-types will be typically associated with arrays of variables $\vec{p} := (p_1, \ldots, p_n)$. When the order of the variables in $\vec{p}$ is not specified, we will sometimes abuse notation and write $\varepsilon(p) = 1$ or $\varepsilon(p) = \partial$.

[3]It will be clear from the treatment in the present and the following sections that the connectives in $\mathcal{F}$ (resp. $\mathcal{G}$) correspond to those referred to as *positive* (resp. *negative*) connectives in [3]. The reason why this terminology is not adopted in the present chapter is explained later on in Footnote 10. Our assumption that the sets $\mathcal{F}$ and $\mathcal{G}$ are disjoint is motivated by the desideratum of generality and modularity. Indeed, for instance, the order theoretic properties of Boolean negation $\neg$ guarantee that this connective belongs both to $\mathcal{F}$ and to $\mathcal{G}$. In such cases we prefer to define two copies $\neg_\mathcal{F} \in \mathcal{F}$ and $\neg_\mathcal{G} \in \mathcal{G}$, and introduce structural rules which encode the fact that these two copies coincide.

[4]Unary f (resp. g) will be sometimes denoted as $\Diamond$ (resp. $\Box$) if the order-type is 1, and $\downarrow$ (resp. $\uparrow$) if the order-type is ∂.

[5] Normal LEs are sometimes referred to as *lattices with operators* (LOs). This terminology directly derives from the setting of Boolean algebras with operators, in which operators are understood as operations which preserve finite meets in each coordinate. However, this terminology results somewhat ambiguous in the lattice setting, in which primitive operations are typically maps which are operators if seen as $\mathbb{A}^\epsilon \to \mathbb{A}^\eta$ for some order-type ϵ on n and some order-type $\eta \in \{1, \partial\}$. Rather than speaking of lattices with (ε, η)-operators, we then speak of normal LEs.

- if $\varepsilon_f(i) = 1$, then $f(p_1, \ldots, p \vee q, \ldots, p_{n_f}) = f(p_1, \ldots, p, \ldots, p_{n_f}) \vee f(p_1, \ldots, q, \ldots, p_{n_f})$ and $f(p_1, \ldots, \bot, \ldots, p_{n_f}) = \bot$,
- if $\varepsilon_f(i) = \partial$, then $f(p_1, \ldots, p \wedge q, \ldots, p_{n_f}) = f(p_1, \ldots, p, \ldots, p_{n_f}) \vee f(p_1, \ldots, q, \ldots, p_{n_f})$ and $f(p_1, \ldots, \top, \ldots, p_{n_f}) = \bot$,
- if $\varepsilon_g(j) = 1$, then $g(p_1, \ldots, p \wedge q, \ldots, p_{n_g}) = g(p_1, \ldots, p, \ldots, p_{n_g}) \wedge g(p_1, \ldots, q, \ldots, p_{n_g})$ and $g(p_1, \ldots, \top, \ldots, p_{n_g}) = \top$,
- if $\varepsilon_g(j) = \partial$, then $g(p_1, \ldots, p \vee q, \ldots, p_{n_g}) = g(p_1, \ldots, p, \ldots, p_{n_g}) \wedge g(p_1, \ldots, q, \ldots, p_{n_g})$ and $g(p_1, \ldots, \bot, \ldots, p_{n_g}) = \top$.

Each language $\mathcal{L}_{\mathrm{LE}}$ is interpreted in the appropriate class of LEs. In particular, for every LE $\mathbb{A}$, each operation $f^{\mathbb{A}} \in \mathcal{F}^{\mathbb{A}}$ (resp. $g^{\mathbb{A}} \in \mathcal{G}^{\mathbb{A}}$) is finitely join-preserving (resp. meet-preserving) in each coordinate when regarded as a map $f^{\mathbb{A}} : \mathbb{A}^{\varepsilon_f} \to \mathbb{A}$ (resp. $g^{\mathbb{A}} : \mathbb{A}^{\varepsilon_g} \to \mathbb{A}$).

The generic LE-logic is not equivalent to a sentential logic. Hence the consequence relation of these logics cannot be uniformly captured in terms of theorems, but rather in terms of sequents, which motivates the following definition:

Definition 2. *For any language $\mathcal{L}_{\mathrm{LE}} = \mathcal{L}_{\mathrm{LE}}(\mathcal{F}, \mathcal{G})$, the* basic, *or* minimal $\mathcal{L}_{\mathrm{LE}}$-logic *is a set of sequents $\phi \vdash \psi$, with $\phi, \psi \in \mathcal{L}_{\mathrm{LE}}$, which contains the following axioms:*

- *Sequents for lattice operations:*[6]

$$
\begin{array}{llll}
p \vdash p, & \bot \vdash p, & p \vdash \top, & p \vdash p \vee q \\
q \vdash p \vee q, & p \wedge q \vdash p, & p \wedge q \vdash q, &
\end{array}
$$

- *Sequents for additional connectives:*

$$
\begin{array}{l}
f(p_1, \ldots, \bot, \ldots, p_{n_f}) \vdash \bot, \ \text{for } \varepsilon_f(i) = 1, \\
f(p_1, \ldots, \top, \ldots, p_{n_f}) \vdash \bot, \ \text{for } \varepsilon_f(i) = \partial, \\
\top \vdash g(p_1, \ldots, \top, \ldots, p_{n_g}), \ \text{for } \varepsilon_g(i) = 1, \\
\top \vdash g(p_1, \ldots, \bot, \ldots, p_{n_g}), \ \text{for } \varepsilon_g(i) = \partial, \\
f(p_1, \ldots, p \vee q, \ldots, p_{n_f}) \vdash f(p_1, \ldots, p, \ldots, p_{n_f}) \vee f(p_1, \ldots, q, \ldots, p_{n_f}), \ \text{for } \varepsilon_f(i) = 1, \\
f(p_1, \ldots, p \wedge q, \ldots, p_{n_f}) \vdash f(p_1, \ldots, p, \ldots, p_{n_f}) \vee f(p_1, \ldots, q, \ldots, p_{n_f}), \ \text{for } \varepsilon_f(i) = \partial, \\
g(p_1, \ldots, p, \ldots, p_{n_g}) \wedge g(p_1, \ldots, q, \ldots, p_{n_g}) \vdash g(p_1, \ldots, p \wedge q, \ldots, p_{n_g}), \ \text{for } \varepsilon_g(i) = 1, \\
g(p_1, \ldots, p, \ldots, p_{n_g}) \wedge g(p_1, \ldots, q, \ldots, p_{n_g}) \vdash g(p_1, \ldots, p \vee q, \ldots, p_{n_g}), \ \text{for } \varepsilon_g(i) = \partial,
\end{array}
$$

and is closed under the following inference rules:

$$
\frac{\phi \vdash \chi \quad \chi \vdash \psi}{\phi \vdash \psi} \qquad \frac{\phi \vdash \psi}{\phi(\chi/p) \vdash \psi(\chi/p)} \qquad \frac{\chi \vdash \phi \quad \chi \vdash \psi}{\chi \vdash \phi \wedge \psi} \qquad \frac{\phi \vdash \chi \quad \psi \vdash \chi}{\phi \vee \psi \vdash \chi}
$$

$$
\frac{\phi \vdash \psi}{f(p_1, \ldots, \phi, \ldots, p_n) \vdash f(p_1, \ldots, \psi, \ldots, p_n)} \ (\varepsilon_f(i) = 1)
$$

[6] In what follows we will use the turnstile symbol $\vdash$ both as sequent separator and also as the consequence relation of the logic.

$$\frac{\phi \vdash \psi}{f(p_1,\ldots,\psi,\ldots,p_n) \vdash f(p_1,\ldots,\phi,\ldots,p_n)}\ (\varepsilon_f(i) = \partial)$$

$$\frac{\phi \vdash \psi}{g(p_1,\ldots,\phi,\ldots,p_n) \vdash g(p_1,\ldots,\psi,\ldots,p_n)}\ (\varepsilon_g(i) = 1)$$

$$\frac{\phi \vdash \psi}{g(p_1,\ldots,\psi,\ldots,p_n) \vdash g(p_1,\ldots,\phi,\ldots,p_n)}\ (\varepsilon_g(i) = \partial).$$

The minimal $\mathcal{L}_{\mathrm{LE}}$-logic is denoted by $\mathbb{L}_{\mathrm{LE}}$. By an LE-logic *we understand any axiomatic extension of $\mathbb{L}_{\mathrm{LE}}$ in the language $\mathcal{L}_{\mathrm{LE}}$. If all the axioms in the extension are analytic inductive (cf. Definition 7) we say that the given LE-logic is* analytic.

For every LE $\mathbb{A}$, the symbol $\vdash$ is interpreted as the lattice order $\leq$. A sequent $\phi \vdash \psi$ is valid in $\mathbb{A}$ if $h(\phi) \leq h(\psi)$ for every homomorphism h from the $\mathcal{L}_{\mathrm{LE}}$-algebra of formulas over AtProp to $\mathbb{A}$. The notation $\mathbb{LE} \models \phi \vdash \psi$ indicates that $\phi \vdash \psi$ is valid in every LE. Then, by means of a routine Lindenbaum-Tarski construction, it can be shown that the minimal LE-logic $\mathbb{L}_{\mathrm{LE}}$ is sound and complete with respect to its correspondent class of algebras $\mathbb{LE}$, i.e. that any sequent $\phi \vdash \psi$ is provable in $\mathbb{L}_{\mathrm{LE}}$ iff $\mathbb{LE} \models \phi \vdash \psi$.

2.2.2 The "tense" language $\mathcal{L}^*_{\mathrm{LE}}$

Any given language $\mathcal{L}_{\mathrm{LE}} = \mathcal{L}_{\mathrm{LE}}(\mathcal{F},\mathcal{G})$ can be associated with the language $\mathcal{L}^*_{\mathrm{LE}} = \mathcal{L}_{\mathrm{LE}}(\mathcal{F}^*,\mathcal{G}^*)$, where $\mathcal{F}^* \supseteq \mathcal{F}$ and $\mathcal{G}^* \supseteq \mathcal{G}$ are obtained by expanding $\mathcal{L}_{\mathrm{LE}}$ with the following connectives:

1. the n_f-ary connective $f_i^\sharp$ for $0 \leq i \leq n_f$, the intended interpretation of which is the right residual of $f \in \mathcal{F}$ in its ith coordinate if $\varepsilon_f(i) = 1$ (resp. its Galois-adjoint if $\varepsilon_f(i) = \partial$);

2. the n_g-ary connective $g_i^\flat$ for $0 \leq i \leq n_g$, the intended interpretation of which is the left residual of $g \in \mathcal{G}$ in its ith coordinate if $\varepsilon_g(i) = 1$ (resp. its Galois-adjoint if $\varepsilon_g(i) = \partial$).

We stipulate that $f_i^\sharp \in \mathcal{G}^*$ if $\varepsilon_f(i) = 1$, and $f_i^\sharp \in \mathcal{F}^*$ if $\varepsilon_f(i) = \partial$. Dually, $g_i^\flat \in \mathcal{F}^*$ if $\varepsilon_g(i) = 1$, and $g_i^\flat \in \mathcal{G}^*$ if $\varepsilon_g(i) = \partial$. The order-type assigned to the additional connectives is predicated on the order-type of their intended interpretations. That is, for any $f \in \mathcal{F}$ and $g \in \mathcal{G}$,

1. if $\epsilon_f(i) = 1$, then $\epsilon_{f_i^\sharp}(i) = 1$ and $\epsilon_{f_i^\sharp}(j) = (\epsilon_f(j))^\partial$ for any $j \neq i$.

2. if $\epsilon_f(i) = \partial$, then $\epsilon_{f_i^\sharp}(i) = \partial$ and $\epsilon_{f_i^\sharp}(j) = \epsilon_f(j)$ for any $j \neq i$.

3. if $\epsilon_g(i) = 1$, then $\epsilon_{g_i^\flat}(i) = 1$ and $\epsilon_{g_i^\flat}(j) = (\epsilon_g(j))^\partial$ for any $j \neq i$.

4. if $\epsilon_g(i) = \partial$, then $\epsilon_{g_i^\flat}(i) = \partial$ and $\epsilon_{g_i^\flat}(j) = \epsilon_g(j)$ for any $j \neq i$.

2

For instance, if f and g are binary connectives such that $\varepsilon_f = (1, \partial)$ and $\varepsilon_g = (\partial, 1)$, then $\varepsilon_{f_1^\sharp} = (1, 1)$, $\varepsilon_{f_2^\sharp} = (1, \partial)$, $\varepsilon_{g_1^\flat} = (\partial, 1)$ and $\varepsilon_{g_2^\flat} = (1, 1)$.[7]

Definition 3. *For any language $\mathcal{L}_{\mathrm{LE}}(\mathcal{F}, \mathcal{G})$, the* basic "tense" $\mathcal{L}_{\mathrm{LE}}$-logic *is defined by specializing Definition 2 to the language $\mathcal{L}^*_{\mathrm{LE}} = \mathcal{L}_{\mathrm{LE}}(\mathcal{F}^*, \mathcal{G}^*)$ and closing under the following additional rules:*

Residuation rules for $f \in \mathcal{F}$ and $g \in \mathcal{G}$:

$$(\epsilon_f(i) = 1)\ \frac{f(\varphi_1, \ldots, \phi, \ldots, \varphi_{n_f}) \vdash \psi}{\phi \vdash f_i^\sharp(\varphi_1, \ldots, \psi, \ldots, \varphi_{n_f})} \qquad \frac{\phi \vdash g(\varphi_1, \ldots, \psi, \ldots, \varphi_{n_g})}{g_i^\flat(\varphi_1, \ldots, \phi, \ldots, \varphi_{n_g}) \vdash \psi}\ (\epsilon_g(i) = 1)$$

$$(\epsilon_f(i) = \partial)\ \frac{f(\varphi_1, \ldots, \phi, \ldots, \varphi_{n_f}) \vdash \psi}{f_i^\sharp(\varphi_1, \ldots, \psi, \ldots, \varphi_{n_f}) \vdash \phi} \qquad \frac{\phi \vdash g(\varphi_1, \ldots, \psi, \ldots, \varphi_{n_g})}{\psi \vdash g_i^\flat(\varphi_1, \ldots, \phi, \ldots, \varphi_{n_g})}\ (\epsilon_g(i) = \partial)$$

The double line in each rule above indicates that the rule is invertible. Let $\mathbb{L}^*_{\mathrm{LE}}$ be the minimal "tense" $\mathcal{L}_{\mathrm{LE}}$-logic.[8] *For any LE-language $\mathcal{L}_{\mathrm{LE}}$, by a* "tense" LE-logic *we understand any axiomatic extension of the basic "tense" $\mathcal{L}_{\mathrm{LE}}$-logic in $\mathcal{L}^*_{\mathrm{LE}}$.*

The algebraic semantics of $\mathbb{L}^*_{\mathrm{LE}}$ is given by the class of "tense" $\mathcal{L}_{\mathrm{LE}}$-algebras, defined as tuples $\mathbb{A} = (\mathbb{L}, \mathcal{F}^*, \mathcal{G}^*)$ such that $\mathbb{L}$ is a lattice algebra and moreover,

1. for every $f \in \mathcal{F}$ s.t. $n_f \geq 1$, all $a_1, \ldots, a_{n_f} \in D$ and $b \in D$, and each $1 \leq i \leq n_f$,
   - if $\epsilon_f(i) = 1$, then $f(a_1, \ldots, a_i, \ldots a_{n_f}) \leq b$ iff $a_i \leq f_i^\sharp(a_1, \ldots, b, \ldots, a_{n_f})$;
   - if $\epsilon_f(i) = \partial$, then $f(a_1, \ldots, a_i, \ldots a_{n_f}) \leq b$ iff $a_i \leq^\partial f_i^\sharp(a_1, \ldots, b, \ldots, a_{n_f})$.
2. for every $g \in \mathcal{G}$ s.t. $n_g \geq 1$, any $a_1, \ldots, a_{n_g} \in D$ and $b \in D$, and each $1 \leq i \leq n_g$,
   - if $\epsilon_g(i) = 1$, then $b \leq g(a_1, \ldots, a_i, \ldots a_{n_g})$ iff $g_i^\flat(a_1, \ldots, b, \ldots, a_{n_g}) \leq a_i$.
   - if $\epsilon_g(i) = \partial$, then $b \leq g(a_1, \ldots, a_i, \ldots a_{n_g})$ iff $g_i^\flat(a_1, \ldots, b, \ldots, a_{n_g}) \leq^\partial a_i$.

It is also routine to prove using the Lindenbaum-Tarski construction that $\mathbb{L}^*_{\mathrm{LE}}$ (as well as any of its canonical axiomatic extensions) is sound and complete w.r.t. the class of "tense" $\mathcal{L}_{\mathrm{LE}}$-algebras (w.r.t. the suitably defined equational subclass, respectively).

[7]Warning: notice that this notation heavily depends on the connective which is taken as primitive, and needs to be carefully adapted to well known cases. For instance, consider the 'fusion' connective $\circ$ (which, when denoted as f, is such that $\varepsilon_f = (1, 1)$). Its residuals $f_1^\sharp$ and $f_2^\sharp$ are commonly denoted $/$ and $\backslash$ respectively. However, if $\backslash$ is taken as the primitive connective g, then $g_2^\flat$ is $\circ = f$, and $g_1^\flat(x_1, x_2) := x_2/x_1 = f_1^\sharp(x_2, x_1)$. This example shows that, when identifying $g_1^\flat$ and $f_1^\sharp$, the conventional order of the coordinates is not preserved, and depends on which connective is taken as primitive.

[8] Hence, for any language $\mathcal{L}_{\mathrm{LE}}$, there are in principle two logics associated with the expanded language $\mathcal{L}^*_{\mathrm{LE}}$, namely the *minimal* $\mathcal{L}^*_{\mathrm{LE}}$-logic, which we denote by $\mathbb{L}^{\tau}_{\mathrm{LE}}$, and which is obtained by instantiating Definition 2 to the language $\mathcal{L}^*_{\mathrm{LE}}$, and the "tense" logic $\mathbb{L}^*_{\mathrm{LE}}$, defined above. The logic $\mathbb{L}^*_{\mathrm{LE}}$ is the natural logic on the language $\mathcal{L}^*_{\mathrm{LE}}$, however it is useful to introduce a specific notation for $\mathbb{L}^{\tau}_{\mathrm{LE}}$, given that all the results holding for the minimal logic associated with an arbitrary LE-language can be instantiated to the expanded language $\mathcal{L}^*_{\mathrm{LE}}$ and will then apply to $\mathbb{L}^{\tau}_{\mathrm{LE}}$.

Theorem 1. *The logic $\mathbb{L}_{\mathrm{LE}}^*$ is a conservative extension of $\mathbb{L}_{\mathrm{LE}}$, i.e. every $\mathcal{L}_{\mathrm{LE}}$-sequent $\phi \vdash \psi$ is derivable in $\mathbb{L}_{\mathrm{LE}}$ iff $\phi \vdash \psi$ is derivable in $\mathbb{L}_{\mathrm{LE}}^*$.*

Proof. We only outline the proof. Clearly, every $\mathcal{L}_{\mathrm{LE}}$-sequent which is $\mathbb{L}_{\mathrm{LE}}$-derivable is also $\mathbb{L}_{\mathrm{LE}}^*$-derivable. Conversely, if an $\mathcal{L}_{\mathrm{LE}}$-sequent $\phi \vdash \psi$ is not $\mathbb{L}_{\mathrm{LE}}$-derivable, then by the completeness of $\mathbb{L}_{\mathrm{LE}}$ w.r.t. the class of $\mathcal{L}_{\mathrm{LE}}$-algebras, there exists an $\mathcal{L}_{\mathrm{LE}}$-algebra $\mathbb{A}$ and a variable assignment v under which $\phi^{\mathbb{A}} \nleq \psi^{\mathbb{A}}$. Consider the canonical extension $\mathbb{A}^\delta$ of $\mathbb{A}$.[9] Since $\mathbb{A}$ is a subalgebra of $\mathbb{A}^\delta$, the sequent $\phi \vdash \psi$ is not satisfied in $\mathbb{A}^\delta$ under the variable assignment $\iota \circ v$ (ι denoting the canonical embedding $\mathbb{A} \hookrightarrow \mathbb{A}^\delta$). Moreover, since $\mathbb{A}^\delta$ is a perfect $\mathcal{L}_{\mathrm{LE}}$-algebra, it is naturally endowed with a structure of "tense" $\mathcal{L}_{\mathrm{LE}}$-algebra. Thus, by the completeness of $\mathbb{L}_{\mathrm{LE}}^*$ w.r.t. the class of "tense" $\mathcal{L}_{\mathrm{LE}}$-algebras, the sequent $\phi \vdash \psi$ is not derivable in $\mathbb{L}_{\mathrm{LE}}^*$, as required. □

Notice that the algebraic completeness of the logics $\mathbb{L}_{\mathrm{LE}}$ and $\mathbb{L}_{\mathrm{LE}}^*$ and the canonical embedding of LEs into their canonical extensions immediately give completeness of $\mathbb{L}_{\mathrm{LE}}$ and $\mathbb{L}_{\mathrm{LE}}^*$ w.r.t. the appropriate class of perfect LEs.

2.2.3 Inductive and analytic inductive LE-inequalities

In the present subsection, we will report on the definition of *inductive* $\mathcal{L}_{\mathrm{LE}}$-inequalities on which the algorithm ALBA is guaranteed to succeed (cf. [6]).

Definition 4 (**Signed Generation Tree**). *(cf. [13, Definition 14]) The* positive *(resp.* negative*)* generation tree *of any $\mathcal{L}_{\mathrm{LE}}$-term s is defined by labelling the root node of the generation tree of s with the sign $+$ (resp. $-$), and then propagating the labelling on each remaining node as follows:*

- *For any node labelled with $\vee$ or $\wedge$, assign the same sign to its children nodes.*
- *For any node labelled with $h \in \mathcal{F} \cup \mathcal{G}$ of arity $n_h \geq 1$, and for any $1 \leq i \leq n_h$, assign the same (resp. the opposite) sign to its ith child node if $\varepsilon_h(i) = 1$ (resp. if $\varepsilon_h(i) = \partial$).*

Nodes in signed generation trees are positive *(resp.* negative*) if are signed $+$ (resp. $-$).*[10]

[9] The *canonical extension* of a BL (bounded lattice) L is a complete lattice L^δ containing L as a sublattice, such that:

1. *(denseness)* every element of L^δ can be expressed both as a join of meets and as a meet of joins of elements from L;
2. *(compactness)* for all $S, T \subseteq L$, if $\bigwedge S \leq \bigvee T$ in L^δ, then $\bigwedge F \leq \bigvee G$ for some finite sets $F \subseteq S$ and $G \subseteq T$.

It is well known that the canonical extension of a BL L is unique up to isomorphism fixing L (cf. e.g. [10, Section 2.2]), and that the canonical extension of a BL is a *perfect* BL, i.e. a complete lattice which is completely join-generated by its completely join-irreducible elements and completely meet-generated by its completely meet-irreducible elements (cf. e.g. [10, Definition 2.14]). The canonical extension of an $\mathcal{L}_{\mathrm{LE}}$-algebra $\mathbb{A} = (L, \mathcal{F}^{\mathbb{A}}, \mathcal{G}^{\mathbb{A}})$ is the perfect $\mathcal{L}_{\mathrm{LE}}$-algebra $\mathbb{A}^\delta := (L^\delta, \mathcal{F}^{\mathbb{A}^\delta}, \mathcal{G}^{\mathbb{A}^\delta})$ such that $f^{\mathbb{A}^\delta}$ and $g^{\mathbb{A}^\delta}$ are defined as the σ-extension of $f^{\mathbb{A}}$ and as the π-extension of $g^{\mathbb{A}}$ respectively, for all $f \in \mathcal{F}$ and $g \in \mathcal{G}$ (cf. [18, 19]).

[10] The terminology used in [3] regarding 'positive' and 'negative connectives' has not been adopted in the present chapter to avoid confusion with positive and negative nodes in signed generation trees.

2

Skeleton	PIA
Δ-adjoints	SRA
$+ \quad \vee$	$+ \quad \wedge \quad g \quad$ with $n_g = 1$
$- \quad \wedge$	$- \quad \vee \quad f \quad$ with $n_f = 1$
SLR	SRR
$+ \quad f \quad$ with $n_f \geq 1$	$+ \quad g \quad$ with $n_g \geq 2$
$- \quad g \quad$ with $n_g \geq 1$	$- \quad f \quad$ with $n_f \geq 2$

Table 2.1: Skeleton and PIA nodes for DLE.

Signed generation trees will be mostly used in the context of term inequalities $s \leq t$. In this context we will typically consider the positive generation tree $+s$ for the left-hand side and the negative one $-t$ for the right-hand side. We will also say that a term-inequality $s \leq t$ is *uniform* in a given variable p if all occurrences of p in both $+s$ and $-t$ have the same sign, and that $s \leq t$ is *ϵ-uniform* in a (sub)array $\vec{p}$ of its variables if $s \leq t$ is uniform in p, occurring with the sign indicated by ϵ, for every p in $\vec{p}$[11].

For any term $s(p_1, \ldots p_n)$, any order type ϵ over n, and any $1 \leq i \leq n$, an *ϵ-critical node* in a signed generation tree of s is a leaf node $+p_i$ with $\epsilon_i = 1$ or $-p_i$ with $\epsilon_i = \partial$. An *ϵ-critical branch* in the tree is a branch from an ϵ-critical node. The intuition, which will be built upon later, is that variable occurrences corresponding to ϵ-critical nodes are *to be solved for*, according to ϵ.

For every term $s(p_1, \ldots p_n)$ and every order type ϵ, we say that $+s$ (resp. $-s$) *agrees with* ϵ, and write $\epsilon(+s)$ (resp. $\epsilon(-s)$), if every leaf in the signed generation tree of $+s$ (resp. $-s$) is ϵ-critical. In other words, $\epsilon(+s)$ (resp. $\epsilon(-s)$) means that all variable occurrences corresponding to leaves of $+s$ (resp. $-s$) are to be solved for according to ϵ. We will also write $+s' \prec *s$ (resp. $-s' \prec *s$) to indicate that the subterm s' inherits the positive (resp. negative) sign from the signed generation tree $*s$. Finally, we will write $\epsilon(\gamma) \prec *s$ (resp. $\epsilon^\partial(\gamma) \prec *s$) to indicate that the signed subtree γ, with the sign inherited from $*s$, agrees with ϵ (resp. with ϵ^∂).

Definition 5 (**good branch**). *(cf. [13, Definition 15]) Nodes in signed generation trees will be called* Δ-adjoints, syntactically left residual (SLR), syntactically right residual (SRR), *and* syntactically right adjoint (SRA), *according to the specification given in Table 3.1. A branch in a signed generation tree $*s$, with $* \in \{+, -\}$, is called a* good branch *if it is the concatenation of two paths P_1 and P_2, one of which may possibly be of length 0, such that P_1 is a path from the leaf consisting (apart from variable nodes) only of PIA-nodes[12], and P_2 consists (apart from variable nodes) only of Skeleton-nodes.*

Definition 6 (**Inductive inequalities**). *(cf. [13, Definition 16]) For any order type ϵ and any irreflexive and transitive relation $<_\Omega$ on $p_1, \ldots p_n$, the signed generation tree*

[11] The following observation will be used at various points in the remainder of the present chapter: if a term inequality $s(\vec{p}, \vec{q}) \leq t(\vec{p}, \vec{q})$ is ϵ-uniform in $\vec{p}$ (cf. discussion after Definition 28), then the validity of $s \leq t$ is equivalent to the validity of $s(\overrightarrow{\top^{\epsilon(i)}}, \vec{q}) \leq t(\overrightarrow{\top^{\epsilon(i)}}, \vec{q})$, where $\top^{\epsilon(i)} = \top$ if $\epsilon(i) = 1$ and $\top^{\epsilon(i)} = \bot$ if $\epsilon(i) = \partial$.

[12] For explanations of our choice of terminologies here, we refer to [16, Remark 3.24].

$*s$ $(* \in \{-, +\})$ *of a term* $s(p_1, \ldots p_n)$ *is* (Ω, ϵ)-inductive *if*

1. *for all* $1 \leq i \leq n$, *every* ϵ*-critical branch with leaf* p_i *is good (cf. Definition 29);*

2. *every* m*-ary SRR-node occurring in the critical branch is of the form* $\circledast(\gamma_1, \ldots, \gamma_{j-1}, \beta, \gamma_{j+1} \ldots, \gamma_m)$, *where for any* $h \in \{1, \ldots, m\} \setminus j$*:*

   (a) $\epsilon^\partial(\gamma_h) \prec *s$ *(cf. discussion before Definition 29), and*

   (b) $p_k <_\Omega p_i$ *for every* p_k *occurring in* γ_h *and for every* $1 \leq k \leq n$.

We will refer to $<_\Omega$ *as the* dependency order *on the variables. An inequality* $s \leq t$ *is* (Ω, ϵ)-inductive *if the signed generation trees* $+s$ *and* $-t$ *are* (Ω, ϵ)*-inductive. An inequality* $s \leq t$ *is* inductive *if it is* (Ω, ϵ)*-inductive for some* Ω *and* ϵ.

Based on the definition above, in the following definition we adapt the definition of analytic inductive inequalities of [13, Definition 16] to the setting of normal LE-logics.

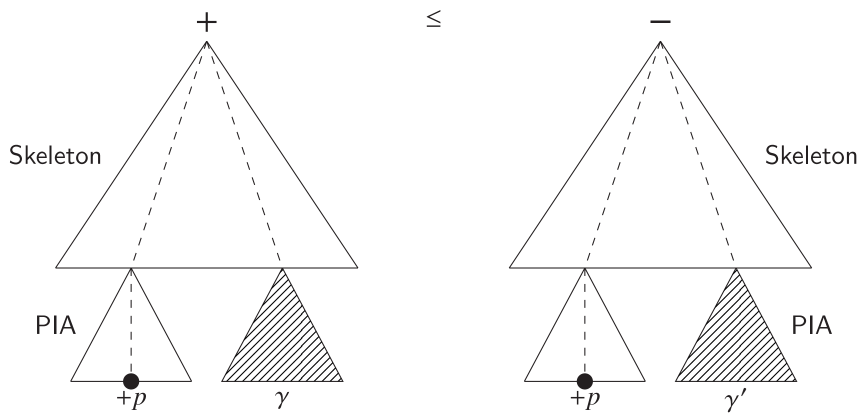
Definition 7 (**Analytic inductive LE-inequalities**). *For any order type* ϵ *and any irreflexive and transitive relation* Ω *on the variables* $p_1, \ldots p_n$*, the signed generation tree* $*s$ $(* \in \{+, -\})$ *of a term* $s(p_1, \ldots p_n)$ *is* analytic (Ω, ϵ)-inductive *if*

1. $*s$ *is* (Ω, ϵ)*-inductive (cf. Definition 6);*

2. *every branch of* $*s$ *is good (cf. Definition 29).*

an inequality $s \leq t$ *is* analytic (Ω, ϵ)-inductive *if* $+s$ *and* $-t$ *are both* (Ω, ϵ)*-analytic inductive. An inequality* $s \leq t$ *is* analytic inductive *if is* (Ω, ϵ)*-analytic inductive for some* Ω *and* ϵ.

The syntactic shape of analytic inductive LE-inequalities can be illustrated by the following picture:



Example 1. $\mathcal{N}_2$ *formulas in the language of full Lambek calculus (cf. [4, Definition 3.1]) are analytic inductive.*

2

2.2.4 Display calculi for basic normal LE-logics

In this section we let $\mathcal{L} = \mathcal{L}(\mathcal{F}, \mathcal{G})$ be a fixed but arbitrary LE-signature (cf. Section 2.2) and define the display calculus $\mathrm{D.LE}$ and its cut-free counterpart $\mathrm{cfD.LE}$ for the basic normal $\mathcal{L}$-logic. Let $S_{\mathcal{F}} := \{\mathsf{F}_f \mid f \in \mathcal{F}^*\}$ and $S_{\mathcal{G}} := \{\mathsf{G}_g \mid g \in \mathcal{G}^*\}$ be the sets of structural connectives associated with $\mathcal{F}^*$ and $\mathcal{G}^*$ respectively (cf. Section 2.2.2). Each such structural connective comes with an arity and an order type which coincide with those of its associated operational connective. For any order type ϵ on n, we let $\mathsf{Str}^{\epsilon}_{\mathcal{F}} := \prod_{i=1}^{n} \mathsf{Str}^{\epsilon(i)}_{\mathcal{F}}$ and $\mathsf{Str}^{\epsilon}_{\mathcal{G}} := \prod_{i=1}^{n} \mathsf{Str}^{\epsilon(i)}_{\mathcal{G}}$, where for all $1 \leq i \leq n$,

$$\mathsf{Str}^{\epsilon(i)}_{\mathcal{F}} = \begin{cases} \mathsf{Str}_{\mathcal{F}} & \text{if } \epsilon(i) = 1 \\ \mathsf{Str}_{\mathcal{G}} & \text{if } \epsilon(i) = \partial \end{cases} \qquad \mathsf{Str}^{\epsilon(i)}_{\mathcal{G}} = \begin{cases} \mathsf{Str}_{\mathcal{G}} & \text{if } \epsilon(i) = 1, \\ \mathsf{Str}_{\mathcal{F}} & \text{if } \epsilon(i) = \partial. \end{cases}$$

Then the calculus $\mathrm{D.LE}$ manipulates both formulas and structures which are defined by the following simultaneous recursions:

$$\begin{array}{rl} \mathsf{Fm} \ni \varphi & ::= p \mid \bot \mid \top \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid f(\varphi_1, \ldots, \varphi_{n_f}) \mid g(\varphi_1, \ldots, \varphi_{n_g}) \\ \mathsf{Str}_{\mathcal{F}} \ni x & ::= \varphi \mid \mathsf{F}_f(\overline{x}) \\ \mathsf{Str}_{\mathcal{G}} \ni y & ::= \varphi \mid \mathsf{G}_g(\overline{y}) \end{array}$$

where, in Fm, $f \in \mathcal{F}$ and $g \in \mathcal{G}$, while in $\mathsf{Str}_{\mathcal{F}}$, $\mathsf{F}_f \in S_{\mathcal{F}}$, and in $\mathsf{Str}_{\mathcal{G}}$, $\mathsf{G}_g \in S_{\mathcal{G}}$, and $\overline{x} \in \mathsf{Str}^{\epsilon_f}_{\mathcal{F}}$, and $\overline{y} \in \mathsf{Str}^{\epsilon_g}_{\mathcal{G}}$.

In what follows, we let

$$\overline{x}^{i} := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

and

$$\overline{x}^{i}_{z} := (x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n)$$

$\overline{y}^{i}$ and $\overline{y}^{i}_{z}$ are defined likewise. The calculus $\mathrm{D.LE}$ consists of the following rules: for any $f \in \mathcal{F}$, $g \in \mathcal{G}$, $\mathsf{F}_f \in S_{\mathcal{F}}$ and $\mathsf{G}_g \in S_{\mathcal{G}}$,

1. Identity and cut rules:

$$(\mathrm{Id})\ p \Rightarrow p \qquad (\mathrm{Cut})\ \frac{x \Rightarrow \varphi \quad \varphi \Rightarrow y}{x \Rightarrow y}$$

2. Display rules:

$$(\epsilon_f(i) = 1)\ \frac{\mathsf{F}_f(\overline{x}) \Rightarrow y}{x_i \Rightarrow \mathsf{G}_{f_i^\sharp}(\overline{x}^{i}_{y})} \qquad \frac{x \Rightarrow \mathsf{G}_g(\overline{y})}{\mathsf{F}_{g_i^\flat}(\overline{y}^{i}_{x}) \Rightarrow y_i}\ (\epsilon_g(i) = 1)$$

$$(\epsilon_f(i) = \partial)\ \frac{\mathsf{F}_f(\overline{x}) \Rightarrow y}{\mathsf{F}_{f_i^\sharp}(\overline{x}^{i}_{y}) \Rightarrow x_i} \qquad \frac{x \Rightarrow \mathsf{G}_g(\overline{y})}{y_i \Rightarrow \mathsf{G}_{g_i^\flat}(\overline{y}^{i}_{x})}\ (\epsilon_g(i) = \partial)$$

2

3. Introduction rules for lattice connectives:

$$(\bot)\ \bot \Rightarrow y \quad (\top)\ x \Rightarrow \top$$

$$(\wedge_{\mathrm{L}})\ \frac{\varphi \Rightarrow y}{\varphi \wedge \psi \Rightarrow y} \quad (\wedge_{\mathrm{L}})\ \frac{\psi \Rightarrow y}{\varphi \wedge \psi \Rightarrow y} \quad (\vee_{\mathrm{R}})\ \frac{x \Rightarrow \varphi}{x \Rightarrow \varphi \vee \psi} \quad (\vee_{\mathrm{R}})\ \frac{x \Rightarrow \psi}{x \Rightarrow \varphi \vee \psi}$$

$$(\wedge_{\mathrm{R}})\ \frac{x \Rightarrow \varphi \quad x \Rightarrow \psi}{x \Rightarrow \varphi \wedge \psi} \quad (\vee_{\mathrm{L}})\ \frac{\varphi \Rightarrow y \quad \psi \Rightarrow y}{\varphi \vee \psi \Rightarrow y}$$

4. Introduction rules for additional connectives:

$$(\mathrm{f_L})\ \frac{\mathsf{F}_f(\overline{\varphi}) \Rightarrow y}{f(\overline{\varphi}) \Rightarrow y} \quad (\mathrm{f_R})\ \frac{\left(x^{\epsilon_f(i)} \Rightarrow \varphi_i \quad \varphi_j \Rightarrow x^{\epsilon_f(j)} \mid 1 \leq i,j \leq n_f, \varepsilon_f(i) = 1 \text{ and } \varepsilon_f(j) = \partial\right)}{\mathsf{F}_f(\overline{x}) \Rightarrow f(\overline{\varphi})}$$

$$(\mathrm{g_L})\ \frac{x \Rightarrow \mathsf{G}_g(\overline{\psi})}{x \Rightarrow g(\overline{\psi})} \quad (\mathrm{g_R})\ \frac{\left(\psi_i \Rightarrow y^{\epsilon_g(i)} \quad y^{\epsilon_g(j)} \Rightarrow \psi_j \mid 1 \leq i,j \leq n_g, \varepsilon_g(i) = 1 \text{ and } \varepsilon_g(j) = \partial\right)}{g(\overline{\psi}) \Rightarrow \mathsf{G}_g(\overline{y})}$$

Let $\mathrm{cfD.LE}$ denote the calculus obtained by removing (Cut) in $\mathrm{D.LE}$. In what follows, we indicate that the sequent $x \Rightarrow y$ is derivable in $\mathrm{D.LE}$ (resp. in $\mathrm{cfD.LE}$) by $\vdash_{\mathrm{D.LE}} x \Rightarrow y$ (resp. by $\vdash_{\mathrm{cfD.LE}} x \Rightarrow y$).

Proposition 1 (**Soundness**). *The calculus* $\mathrm{D.LE}$ *(and hence also* $\mathrm{cfD.LE}$*) is sound w.r.t. the class of complete $\mathcal{L}$-algebras.*

Proof. The soundness of the basic lattice framework is clear. The soundness of the remaining rules uses the monotonicity (resp. antitonicity) of the algebraic connectives interpreting each $f \in \mathcal{F}$ and $g \in \mathcal{G}$, and their adjunction/residuation properties, which hold since any complete $\mathcal{L}$-algebra is an $\mathcal{L}^*$-algebra. □

2.2.5 Proper display calculi and analytic structural rules

In this section, we recall the definition of analytic structural rules which is introduced in [13]. This definition is tightly connected with the notion of *proper display calculus* (cf. [20]), since it is aimed at guaranteeing that adding an analytic structural rule to a proper display calculus preserves cut elimination and subformula property. We start by recalling the conditions C_1-C_8 defining a *proper display calculus*:

C_1: Preservation of formulas. This condition requires each formula occurring in a premise of a given inference to be the subformula of some formula in the conclusion of that inference. That is, structures may disappear, but not formulas. This condition is not included in the list of sufficient conditions of the cut elimination metatheorem, but, in the presence of cut elimination, it guarantees the subformula property of a system. Condition C_1 can be verified by inspection on the shape of the rules. In practice, condition C_1 bans rules in which structure variables occurring in some premise to not occur also in the conclusion, since in concrete derivations these are typically instantiated with (structures containing) formulas which would then disappear in the application of the rule.

C_2: Shape-alikeness of parameters. This condition is based on the relation of *congruence* between *parameters* (i.e., non-active parts) in inferences; the congruence relation is an equivalence relation which is meant to identify the different occurrences of the same formula or substructure along the branches of a derivation [1, Section 4], [17, Definition 6.5]. Condition C_2 requires that congruent parameters be occurrences of the same structure. This can be understood as a condition on the *design* of the rules of the system if the congruence relation is understood as part of the specification of each given rule; that is, each schematic rule of the system comes with an explicit specification of which elements are congruent to which (and then the congruence relation is defined as the reflexive and transitive closure of the resulting relation). In this respect, C_2 is nothing but a sanity check, requiring that the congruence is defined in such a way that indeed identifies the occurrences which are intuitively "the same".[13]

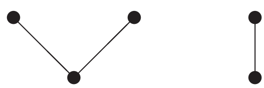
C_3: Non-proliferation of parameters. Like the previous one, also this condition is actually about the definition of the congruence relation on parameters. Condition C_3 requires that, for every inference (i.e. rule application), each of its parameters is congruent to at most one parameter in the conclusion of that inference. Hence, the condition stipulates that for a rule such as the following,

$$\frac{X \vdash Y}{X\,,X \vdash Y}$$

the structure X from the premise is congruent to *only one* occurrence of X in the conclusion sequent. Indeed, the introduced occurrence of X should be considered congruent only to itself. Moreover, given that the congruence is an equivalence relation, condition C_3 implies that, within a given sequent, any substructure is congruent only to itself. In practice, in the general schematic formulation of rules, we will use the same structure variable for two different parametric occurrences if and only if they are congruent, so a rule such as the one above is de facto banned.

Remark 1. *Conditions C_2 and C_3 make it possible to follow the history of a formula along the branches of any given derivation. In particular, C_3 implies that the the history of any formula within a given derivation has the shape of a tree, which we refer to as the* history-tree *of that formula in the given derivation. Notice, however, that the history-tree of a formula might have a different shape than the portion of the underlying derivation corresponding to it; for instance, the following application of the Contraction rule gives rise to a bifurcation of the history-tree of A which is absesent in the underlying branch of the derivation tree, given that Contraction is a unary rule.*

$$\frac{\begin{array}{c}\vdots\\ A\,,A \vdash X\end{array}}{A \vdash X}$$



[13] Our convention throughout the chapter is that congruent parameters are denoted by the same letter. For instance, in the rule

$$\frac{X\,;Y \vdash Z}{Y\,;X \vdash Z}$$

the structures X, Y and Z are parametric and the occurrences of X (resp. Y, Z) in the premise and the conclusion are congruent.

C$_4$: Position-alikeness of parameters. This condition bans any rule in which a (sub)structure in precedent (resp. succedent) position in a premise is congruent to a (sub)structure in succedent (resp. precedent) position in the conclusion.

C$_5$: Display of principal constituents. This condition requires that any principal occurrence (that is, a non-parametric formula occurring in the conclusion of a rule application, cf. [1, Condition C5]) be always either the entire antecedent or the entire consequent part of the sequent in which it occurs. In the following section, a generalization of this condition will be discussed, in view of its application to the main focus of interest of the present chapter.

The following conditions C$_6$ and C$_7$ are not reported below as they are stated in the original paper [1], but as they appear in [20, Section 4.1].

C$_6$: Closure under substitution for succedent parameters. This condition requires each rule to be closed under simultaneous substitution of arbitrary structures for congruent formulas which occur in succedent position. Condition C$_6$ ensures, for instance, that if the following inference is an application of the rule R:

$$\frac{(X \vdash Y)\big([A]_i^{suc} \mid i \in I\big)}{(X' \vdash Y')[A]^{suc}}\ R$$

and $\big([A]_i^{suc} \mid i \in I\big)$ represents all and only the occurrences of A in the premiss which are congruent to the occurrence of A in the conclusion[14], then also the following inference is an application of the same rule R:

$$\frac{(X \vdash Y)\big([Z/A]_i^{suc} \mid i \in I\big)}{(X' \vdash Y')[Z/A]^{suc}}\ R$$

where the structure Z is substituted for A.
This condition caters for the step in the cut elimination procedure in which the cut needs to be "pushed up" over rules in which the cut-formula in succedent position is parametric. Indeed, condition C$_6$ guarantees that, in the picture below, a well-formed subtree $\pi_1[Y/A]$ can be obtained from π_1 by replacing any occurrence of A corresponding to a node in the history tree of the cut-formula A by Y, and hence the following transformation step is guaranteed go through uniformly and "canonically":

$$\begin{array}{c}
\vdots\,\pi_1' \\
X' \vdash A \\
\\
\vdots\,\pi_1 \qquad\qquad \vdots\,\pi_2 \\
\dfrac{X \vdash A \qquad\qquad A \vdash Y}{X \vdash Y}
\end{array}
\quad\rightsquigarrow\quad
\begin{array}{c}
\vdots\,\pi_1' \qquad\qquad \vdots\,\pi_2 \\
\dfrac{X' \vdash A \qquad\qquad A \vdash Y}{X' \vdash Y} \\
\vdots\,\pi_1[Y/A] \\
X \vdash Y
\end{array}$$

if each rule in π_1 verifies condition C$_6$.

[14] Clearly, if $I = \varnothing$, then the occurrence of A in the conclusion is congruent to itself.

C_7: Closure under substitution for precedent parameters. This condition requires each rule to be closed under simultaneous substitution of arbitrary structures for congruent formulas which occur in precedent position. Condition C_7 can be understood analogously to C_6, relative to formulas in precedent position. Therefore, for instance, if the following inference is an application of the rule R:

$$\frac{(X \vdash Y)\big([A]_i^{pre} \mid i \in I\big)}{(X' \vdash Y')[A]^{pre}}\ R$$

then also the following inference is an instance of R:

$$\frac{(X \vdash Y)\big([Z/A]_i^{pre} \mid i \in I\big)}{(X' \vdash Y')[Z/A]^{pre}}\ R$$

Similarly to what has been discussed for condition C_6, condition C_7 caters for the step in the cut elimination procedure in which the cut needs to be "pushed up" over rules in which the cut-formula in precedent position is parametric.

C_8: Eliminability of matching principal constituents. This condition requests a standard Gentzen-style checking, which is now limited to the case in which both cut formulas are *principal*, i.e. each of them has been introduced with the last rule application of each corresponding subdeduction. In this case, analogously to the proof Gentzen-style, condition C_8 requires being able to transform the given deduction into a deduction with the same conclusion in which either the cut is eliminated altogether, or is transformed in one or more applications of cut involving proper subformulas of the original cut-formulas.

Theorem 2. *(cf. [21, Section 3.3, Appendix A]) Any calculus satisfying conditions C_2, C_3, C_4, C_5, C_6, C_7, C_8 enjoys cut elimination. If C_1 is also satisfied, then the calculus enjoys the subformula property.*

Definition 8 (**Analytic structural rules**). *(cf. [5, Definition 3.13]) A structural rule which satisfies conditions C_1-C_7 is an* analytic structural rule.

Proposition 2. *(cf. [13]) Every analytic (Ω, ϵ)-inductive LE-inequality can be equivalently transformed, via an ALBA-reduction, into a set of analytic structural rules.*

2.3 LE-frames and their complex algebras

From now on, we fix an arbitrary normal LE-signature $\mathcal{L} = \mathcal{L}(\mathcal{F}, \mathcal{G})$.

2.3.1 Notation

For any sets A, B and any relation $S \subseteq A \times B$, we let, for any $A' \subseteq A$ and $B' \subseteq B$,

$$S^{\uparrow}[A'] := \{b \in B \mid \forall a(a \in A' \Rightarrow a\ S\ b)\} \text{ and } S^{\downarrow}[B'] := \{a \in A \mid \forall b(b \in B' \Rightarrow a\ S\ b)\}.$$

For all sets $A, B_1, \dots B_n$, and any relation $S \subseteq A \times B_1 \times \cdots \times B_n$, for any $\overline{C} := (C_1, \dots, C_n)$ where $C_i \subseteq B_i$ and $1 \leq i \leq n$ we let $\overline{C}^{i} := (C_1, \dots, C_{i-1}, C_{i+1}, \dots, C_n)$ and for all A',

$\overline{C}^i_{A'} := (C_1 \ldots, C_{i-1}, A', C_{i+1}, \ldots, C_n)$. When $C_i := \{c_i\}$ and $A' := \{a'\}$, we will write $\overline{c}$ for $\overline{\{c\}}$, and $\overline{c}^i$ for $\overline{\{c\}}^i$, and $\overline{c}^i_{a'}$ for $\overline{\{c\}}^i_{\{a'\}}$. We also let:

1. $S^{(0)}[\overline{C}] := \{a \in A \mid \forall \overline{b}(\overline{b} \in \overline{C} \Rightarrow a\ S\ \overline{b})\}$.
2. $S_i \subseteq B_i \times B_1 \times \cdots B_{i-1} \times A \times B_{i+1} \times \cdots \times B_n$ be defined by
$$(b_i, \overline{c}^i_a) \in S_i \text{ iff } (a, \overline{c}) \in S.$$
3. $S^{(i)}[A', \overline{C}^i] := S_i^{(0)}[\overline{C}^i_{A'}]$.

Lemma 1. *If $S \subseteq A \times B_1 \times \cdots \times B_n$ and $\overline{C}$ is as above, then for any $1 \leq i \leq n$,*

$$C_i \subseteq S^{(i)}[S^{(0)}[\overline{C}], \overline{C}^i]. \tag{2.1}$$

Proof. Let $x \in C_i$. Then:

	$\{x\} \subseteq C_i$	assumption
only if	$S^{(0)}[\overline{C}] \subseteq S^{(0)}[\overline{C}^i_x]$	antitonicity of $S^{(0)}[-]$
iff	$(y, \overline{z}^i_x) \in S$ for all $y \in S^{(0)}[\overline{C}]$ and all $\overline{z} \in \overline{C}^i$	definition of $S^{(0)}[-]$
iff	$(x, \overline{z}^i_y) \in S_i$ for all $y \in S^{(0)}[\overline{C}]$ and all $\overline{z} \in \overline{C}^i$	definition of S_i
iff	$x \in S_i^{(0)}[\overline{C}^i_{S^{(0)}[\overline{C}]}]$	definition of $S^{(0)}[-]$
iff	$x \in S^{(i)}[S^{(0)}[\overline{C}], \overline{C}^i]$	definition of $S^{(i)}[-]$

□

2.3.2 LE-frames

Definition 9 (Polarity). *A* polarity *is a structure $\mathbb{W} = (W, U, N)$ where W and U are sets and N is a binary relation from W to U.*

If $\mathbb{L}$ is a lattice, then $\mathbb{W}_L = (L, L, \leq)$ is a polarity. Conversely, for any polarity $\mathbb{W}$, we let $\mathbb{W}^+$ be the complete sub $\bigcap$-semilattice of the Galois-stable sets of the closure operator $\gamma_N : \mathcal{P}(W) \to \mathcal{P}(W)$ defined by the assignment $X \mapsto X^{\uparrow\downarrow}$, where for every $X \subseteq W$ and $Y \subseteq U$, $X^{\uparrow}$ and $Y^{\downarrow}$ are abbreviations for $N^{\uparrow}[X]$ and $N^{\downarrow}[Y]$ respectively. As is well known, $\mathbb{W}^+$ is a complete lattice, in which $\bigvee S := \gamma_N(\bigcup S)$ for any $S \subseteq \gamma_N[\mathcal{P}(W)]$. Moreover, $\mathbb{W}^+$ can be equivalently obtained as the dual lattice of the Galois-stable sets of the closure operator $\gamma'_N : \mathcal{P}(U) \to \mathcal{P}(U)$ defined by the assignment $Y \mapsto Y^{\downarrow\uparrow}$.

From now on, we focus on $\mathcal{L}$-algebras $\mathbb{A} = (L, \wedge, \vee, \bot, \top, \mathcal{F}, \mathcal{G})$.

Definition 10. *An $\mathcal{L}$-frame is a tuple $\mathbb{F} = (\mathbb{W}, \mathcal{R}_{\mathcal{F}}, \mathcal{R}_{\mathcal{G}})$ such that $\mathbb{W} = (W, U, N)$ is a polarity, $\mathcal{R}_{\mathcal{F}} = \{R_f \mid f \in \mathcal{F}\}$, and $\mathcal{R}_{\mathcal{G}} = \{R_g \mid g \in \mathcal{G}\}$ such that for each $f \in \mathcal{F}$ and $g \in \mathcal{G}$, the symbols R_f and R_g respectively denote $(n_f + 1)$-ary and $(n_g + 1)$-ary relations on $\mathbb{W}$,*

$$R_f \subseteq U \times W^{\epsilon_f} \quad \text{and} \quad R_g \subseteq W \times U^{\epsilon_g}, \tag{2.2}$$

where for any order type ϵ on n, we let $W^{\epsilon} := \prod_{i=1}^n W^{\epsilon(i)}$ and $U^{\epsilon} := \prod_{i=1}^n U^{\epsilon(i)}$, where for all $1 \leq i \leq n$,

2

$$W^{\epsilon(i)} = \begin{cases} W & \text{if } \epsilon(i) = 1 \\ U & \text{if } \epsilon(i) = \partial \end{cases} \qquad U^{\epsilon(i)} = \begin{cases} U & \text{if } \epsilon(i) = 1, \\ W & \text{if } \epsilon(i) = \partial. \end{cases}$$

In addition, we assume that the following sets are Galois-stable (from now on abbreviated as stable*) for all $w_0 \in W$, $u_0 \in U$, $\overline{w} \in W^{\epsilon_f}$, and $\overline{u} \in U^{\epsilon_g}$:*

1. $R_f^{(0)}[\overline{w}]$ *and* $R_f^{(i)}[u_0, \overline{w}^i]$*;*
2. $R_g^{(0)}[\overline{u}]$ *and* $R_g^{(i)}[w_0, \overline{u}^i]$.

In what follows, for any order type ϵ on n, we let

$$W^\epsilon \supseteq \overline{X} := (X^{\epsilon(1)}, \ldots, X^{\epsilon(n)}),$$

where $X^{\epsilon(i)} \subseteq W^{\epsilon(i)}$ for all $1 \leq i \leq n$, and let

$$U^\epsilon \supseteq \overline{Y} := (Y^{\epsilon(1)}, \ldots, Y^{\epsilon(n)}),$$

where $Y^{\epsilon(i)} \subseteq U^{\epsilon(i)}$ for all $1 \leq i \leq n$. Moreover, we let $\overline{X}^i$, $\overline{X}^i_Z$, $\overline{Y}^i$ and $\overline{X}^i_Z$ be defined as in Subsection 2.3.1.

Lemma 2. *For any $\mathcal{L}$-frame $\mathbb{F} = (\mathbb{W}, \mathcal{R}_\mathcal{F}, \mathcal{R}_\mathcal{G})$, any $f \in \mathcal{F}$ and any $g \in \mathcal{G}$,*

1. *if $Y_0 \subseteq U$, then $R_f^{(0)}[\overline{X}]$ and $R_f^{(i)}[Y_0, \overline{X}^i]$ are stable sets for all $1 \leq i \leq n_f$;*
2. *if $X_0 \subseteq W$, then $R_g^{(0)}[\overline{Y}]$ and $R_g^{(i)}[X_0, \overline{Y}^i]$ are stable sets for all $1 \leq i \leq n_g$.*

Proof. The second part of item 1 can be proved as follows:

$$\begin{array}{cl} & R_f^{(i)}[Y_0, \overline{X}^i] \\ = & R_f^{(i)}[\bigcup_{u \in Y_0} u, \overline{\bigcup_{w \in X} w}^i] \\ = & \bigcap_{u \in Y_0} \bigcap_{\overline{w}^i \in \overline{X}^i} R_f^{(i)}[u, \overline{w}^i] \quad \text{distributivity of } R_f^{(i)}. \end{array}$$

By Definition 10.1, the last line above is an intersection of stable sets, and hence is stable. The first part of item 1 and item 2 can be verified analogously. □

Lemma 3. 1. *For every $1 \leq i, j \leq n_f$ such that $j \neq i$, let*

$$X_j \in \begin{cases} \gamma_N[\mathcal{P}(W)] & \text{if } \varepsilon_f(j) = 1 \\ \gamma'_N[\mathcal{P}(U)] & \text{if } \varepsilon_f(j) = \partial. \end{cases}$$

(i) *If $\epsilon_f(i) = 1$, then for every $Z \in \mathcal{P}(W)$,*

$$R_f^{(0)}[\overline{X}^i_Z] = R_f^{(0)}[\overline{X}^i_{Z\uparrow\downarrow}]. \tag{2.3}$$

(ii) *If $\epsilon_f(i) = \partial$, then for every $Y \in \mathcal{P}(U)$,*

$$R_f^{(0)}[\overline{X}^i_Y] = R_f^{(0)}[\overline{X}^i_{Y\downarrow\uparrow}]. \tag{2.4}$$

2. *For every* $1 \leq i, j \leq n_g$ *such that* $j \neq i$, *let*

$$Y_j \in \begin{cases} \gamma'_N[\mathcal{P}(U)] & \text{if } \varepsilon_g(j) = 1 \\ \gamma_N[\mathcal{P}(W)] & \text{if } \varepsilon_g(j) = \partial. \end{cases}$$

(i) *If* $\epsilon_g(i) = 1$, *for every* $Z \in \mathcal{P}(U)$,

$$R_g^{(0)}[\overline{Y}^i_Z] = R_g^{(0)}[\overline{Y}^i_{Z^{\downarrow\uparrow}}]. \tag{2.5}$$

(ii) *If* $\epsilon_g(i) = \partial$, *then for every* $X \in \mathcal{P}(W)$,

$$R_g^{(0)}[\overline{Y}^i_X] = R_g^{(0)}[\overline{Y}^i_{X^{\uparrow\downarrow}}]. \tag{2.6}$$

Proof. 1(i) The right-to-left direction follows from $Z \subseteq Z^{\uparrow\downarrow}$ and the antitonicity of $R_f^{(0)}$. The left-to-right direction can be verified as follows:

	$u \in R_f^{(0)}[\overline{X}^i_Z]$	assumption
iff	$Z \subseteq R_f^{(i)}[u, \overline{X}^i]$	Definition of $R_f^{(i)}$
only if	$Z^{\uparrow\downarrow} \subseteq R_f^{(i)}[u, \overline{X}^i]$	Lemma 2
iff	$u \in R_f^{(0)}[\overline{X}^i_{Z^{\uparrow\downarrow}}]$	Definition of $R_f^{(i)}$

1(ii) can be verified analogously. The proofs of 2 are obtained dually. □

2.3.3 Complex algebras of LE-frames

Definition 11. *The* complex algebra *of an* $\mathcal{L}$*-frame* $\mathbb{F} = (\mathbb{W}, \mathcal{R}_{\mathcal{F}}, \mathcal{R}_{\mathcal{G}})$ *is the algebra*

$$\mathbb{F}^+ = (\mathbb{L}, \{f_{R_f} \mid f \in \mathcal{F}\}, \{g_{R_g} \mid g \in \mathcal{G}\}),$$

where $\mathbb{L} := \mathbb{W}^+$ *(cf. Definition 9), and for all* $f \in \mathcal{F}$ *and all* $g \in \mathcal{G}$,

1. *let* $\overline{X^{\epsilon_f}} := (X^{\epsilon_f(1)}, \ldots, X^{\epsilon_f(n_f)})$, *where for all* $1 \leq i \leq n_f$,

$$X_i^{\epsilon_f(i)} = \begin{cases} X_i & \text{if } \epsilon_f(i) = 1, \\ X_i^{\uparrow} & \text{if } \epsilon_f(i) = \partial \end{cases}$$

and $f_{R_f} : \mathbb{L}^{n_f} \to \mathbb{L}$ *is defined by the assignment* $f_{R_f}(\overline{X}) = (R_f^{(0)}[\overline{X^{\epsilon_f}}])^{\downarrow}$.

2. *let* $\overline{X^{\epsilon_g}} := (X^{\epsilon_g(1)}, \ldots, X^{\epsilon_g(n_f)})$, *where for all* $1 \leq i \leq n_g$,

$$X_i^{\epsilon_g(i)} = \begin{cases} X_i^{\uparrow} & \text{if } \epsilon_g(i) = 1, \\ X_i & \text{if } \epsilon_g(i) = \partial \end{cases}$$

and $g_{R_g} : \mathbb{L}^{n_g} \to \mathbb{L}$ *is defined by the assignment,* $g_{R_g}(\overline{X}) = R_g^{(0)}[\overline{X^{\epsilon_g}}]$.

2

Proposition 3. *If $\mathbb{F}$ is an $\mathcal{L}$-frame, then $\mathbb{F}^+$ is a complete $\mathcal{L}$-algebra.*

Proof. We need to prove that for every $f \in \mathcal{F}$ and every $g \in \mathcal{G}$, f_{R_f} is a complete ϵ_f-operator and g_{R_g} is a complete ϵ_g-dual operator. Since the underlying lattice of $\mathbb{F}^+$ is complete, this implies that the residuals of every $f \in \mathcal{F}$ and $g \in \mathcal{G}$ in each coordinate exist.

Let $f \in \mathcal{F}$, and $1 \leq i \leq n_f$ with $\epsilon_f(i) = 1$. Let $\mathcal{X} \subseteq \gamma_N[\mathcal{P}(W)]$, and $\overline{X} := (X_1, \ldots, X_{n_f})$ where $X_j \in \gamma_N[\mathcal{P}(W)]$ for any $1 \leq j \leq n_f$. Then:

$$
\begin{array}{cll}
& \bigvee_{Z\in\mathcal{X}} f_{R_f}(\overline{X}^i_Z) & \\
= & \gamma_N(\bigcup_{Z\in\mathcal{X}} f_{R_f}(\overline{X}^i_Z)) & \text{definition of } \bigvee \text{ in } \gamma_N[\mathcal{P}(W)] \\
= & (\bigcup_{Z\in\mathcal{X}}(R_f^{(0)}[\overline{X^{\epsilon_f}}^i_Z])^{\downarrow})^{\uparrow\downarrow} & \text{definition of } f_{R_f} \text{ and notation for } \gamma_N \\
= & (\bigcap_{Z\in\mathcal{X}}(R_f^{(0)}[\overline{X^{\epsilon_f}}^i_Z])^{\downarrow\uparrow})^{\downarrow} & \text{distributivity of } (\cdot)^{\uparrow} \\
= & (\bigcap_{Z\in\mathcal{X}} R_f^{(0)}[\overline{X^{\epsilon_f}}^i_Z])^{\downarrow\uparrow\downarrow} & \text{distributivity of } (\cdot)^{\downarrow\uparrow} \\
= & (\bigcap_{Z\in\mathcal{X}} R_f^{(0)}[\overline{X^{\epsilon_f}}^i_Z])^{\downarrow} & (\cdot)^{\downarrow\uparrow\downarrow} = (\cdot)^{\downarrow} \\
= & (R_f^{(0)}[\overline{X^{\epsilon_f}}^i_{\bigcup\mathcal{X}}])^{\downarrow} & \text{distributivity of } R_f^{(0)} \\
= & (R_f^{(0)}[\overline{X^{\epsilon_f}}^i_{\gamma_N(\bigcup\mathcal{X})}])^{\downarrow} & \text{Lemma 3.1(i)} \\
= & (R_f^{(0)}[\overline{X^{\epsilon_f}}^i_{\bigvee\mathcal{X}}])^{\downarrow} & \text{definition of } \bigvee \text{ in } \gamma_N[\mathcal{P}(W)] \\
= & f_{R_f}(\overline{X}^i_{\bigvee\mathcal{X}}) & \text{definition of } f_{R_f}
\end{array}
$$

Let $f \in \mathcal{F}$, and $1 \leq i \leq n_f$ with $\epsilon_f(i) = 1$. Let $\mathcal{X} \subseteq \gamma_N[\mathcal{P}(W)]$, and $\overline{X}$ be defined as above. Then:

$$
\begin{array}{cll}
& \bigvee_{Z\in\mathcal{X}} f_{R_f}(\overline{X}^i_Z) & \\
= & \gamma_N(\bigcup_{Z\in\mathcal{X}} f_{R_f}(\overline{X}^i_Z)) & \text{definition of } \bigvee \text{ in } \gamma_N[\mathcal{P}(W)] \\
= & (\bigcup_{Z\in\mathcal{X}}(R_f^{(0)}[\overline{X^{\epsilon_f}}^i_{Z^{\uparrow}}])^{\downarrow})^{\uparrow\downarrow} & \text{definition of } f_{R_f} \text{ and notation for } \gamma_N \\
= & (\bigcap_{Z\in\mathcal{X}}(R_f^{(0)}[\overline{X^{\epsilon_f}}^i_{Z^{\uparrow}}])^{\downarrow\uparrow})^{\downarrow} & \text{distributivity of } (\cdot)^{\uparrow} \\
= & (\bigcap_{Z\in\mathcal{X}} R_f^{(0)}[\overline{X^{\epsilon_f}}^i_{Z^{\uparrow}}])^{\downarrow\uparrow\downarrow} & \text{distributivity of } (\cdot)^{\downarrow\uparrow} \\
= & (\bigcap_{Z\in\mathcal{X}} R_f^{(0)}[\overline{X^{\epsilon_f}}^i_{Z^{\uparrow}}])^{\downarrow} & (\cdot)^{\downarrow\uparrow\downarrow} = (\cdot)^{\downarrow} \\
= & (R_f^{(0)}[\overline{X^{\epsilon_f}}^i_{\bigcup_{Z\in\mathcal{X}} Z^{\uparrow}}])^{\downarrow} & \text{distributivity of } R_f^{(0)} \\
= & (R_f^{(0)}[\overline{X^{\epsilon_f}}^i_{(\bigcup_{Z\in\mathcal{X}} Z^{\uparrow})^{\downarrow\uparrow}}])^{\downarrow} & (2.4) \\
= & (R_f^{(0)}[\overline{X^{\epsilon_f}}^i_{(\bigcap_{Z\in\mathcal{X}} Z^{\uparrow\downarrow})^{\uparrow}}])^{\downarrow} & \text{distributivity of } (\cdot)^{\downarrow} \\
= & (R_f^{(0)}[\overline{X^{\epsilon_f}}^i_{(\bigcap\mathcal{X})^{\uparrow}}])^{\downarrow} & Z^{\uparrow\downarrow} = Z \\
= & f_{R_f}(\overline{X}^i_{\bigcap\mathcal{X}}) & \text{definition of } f_{R_f}
\end{array}
$$

Let $g \in \mathcal{G}$, and $1 \leq i \leq n_g$ with $\epsilon_g(i) = 1$. Let $\mathcal{X} \subseteq \gamma_N[\mathcal{P}(W)]$, and $\overline{X} := (X_1, \ldots, X_{n_g})$ where $X_j \in \gamma_N[\mathcal{P}(W)]$ for any $1 \leq j \leq n_g$. Observe preliminarily that

$$\bigcap\mathcal{X} = \bigcap_{Z\in\mathcal{X}} Z = \bigcap_{Z\in\mathcal{X}} Z^{\uparrow\downarrow} = (\bigcup_{Z\in\mathcal{X}} Z^{\uparrow})^{\downarrow} \tag{2.7}$$

$$
\begin{array}{lll}
 & \bigcap_{Z\in\mathcal{X}} g_{R_g}(\overline{X}^{i}_{Z}) & \\
= & \bigcap_{Z\in\mathcal{X}} \gamma_N(R_g^{(0)}[\overline{X^{\epsilon_g}}^{i}_{Z\uparrow}]) & \text{definition of } g_{R_g} \\
= & \bigcap_{Z\in\mathcal{X}} (R_g^{(0)}[\overline{X^{\epsilon_g}}^{i}_{Z\uparrow}])^{\uparrow\downarrow} & \text{notation for } \gamma_N \\
= & (\bigcap_{Z\in\mathcal{X}} R_g^{(0)}[\overline{X^{\epsilon_g}}^{i}_{Z\uparrow}])^{\uparrow\downarrow} & \text{main distribution property of } (\cdot)^{\uparrow\downarrow} \\
= & (R_g^{(0)}[\overline{X^{\epsilon_g}}^{i}_{\bigcup_{Z\in\mathcal{X}} Z\uparrow}])^{\uparrow\downarrow} & \text{main distribution property of } R_g^{(1)} \\
= & (R_g^{(0)}[\overline{X^{\epsilon_g}}^{i}_{(\bigcup_{Z\in\mathcal{X}} Z\uparrow)\downarrow\uparrow}])^{\uparrow\downarrow} & (2.5) \\
= & (R_g^{(0)}[\overline{X^{\epsilon_g}}^{i}_{(\bigcap \mathcal{X})\uparrow}])^{\uparrow\downarrow} & (2.7) \\
= & g_{R_g}(\overline{X}^{i}_{\bigcap \mathcal{X}}) & \text{definition of } g_{R_g}
\end{array}
$$

Let $g \in \mathcal{G}$, and $1 \leq i \leq n_g$ with $\epsilon_g(i) = \partial$. Let $\mathcal{X} \subseteq \gamma_N[\mathcal{P}(W)]$, and $\overline{X}$ be defined as above.

$$
\begin{array}{lll}
 & \bigcap_{Z\in\mathcal{X}} g_{R_g}(\overline{X}^{i}_{Z}) & \\
= & \bigcap_{Z\in\mathcal{X}} \gamma_N(R_g^{(0)}[\overline{X^{\epsilon_g}}^{i}_{Z}]) & \text{definition of } g_{R_g} \\
= & \bigcap_{Z\in\mathcal{X}} (R_g^{(0)}[\overline{X^{\epsilon_g}}^{i}_{Z}])^{\uparrow\downarrow} & \text{notation for } \gamma_N \\
= & (\bigcap_{Z\in\mathcal{X}} R_g^{(0)}[\overline{X^{\epsilon_g}}^{i}_{Z}])^{\uparrow\downarrow} & \text{main distribution property of } (\cdot)^{\uparrow\downarrow} \\
= & (R_g^{(0)}[\overline{X^{\epsilon_g}}^{i}_{\bigcup_{Z\in\mathcal{X}} Z}])^{\uparrow\downarrow} & \text{main distribution property of } R_g^{(0)} \\
= & (R_g^{(0)}[\overline{X^{\epsilon_g}}^{i}_{(\bigcup_{Z\in\mathcal{X}} Z)\downarrow\uparrow}])^{\uparrow\downarrow} & (2.6) \\
= & (R_g^{(0)}[\overline{X^{\epsilon_g}}^{i}_{\bigvee \mathcal{X}}])^{\uparrow\downarrow} & \text{definition of } \bigvee \text{ in } \gamma_N[\mathcal{P}(W)] \\
= & g_{R_g}(\overline{X}^{i}_{\bigvee \mathcal{X}}) & \text{definition of } g_{R_g}
\end{array}
$$

□

2.4 Functional $\mathbb{D}$-frames

In the present section, we introduce the counterpart, in the setting of LE-logics, of Gentzen frames [8, Section 2].

2.4.1 Definition and main property

Recall that $\mathrm{D.LE}$ and $\mathrm{cfD.LE}$ respectively denote the display calculus for the basic normal $\mathcal{L}$-logic and its cut-free version. Moreover we let $\mathrm{D.LE}'$ and $\mathrm{cfD.LE}'$ denote the extensions of $\mathrm{D.LE}$ and $\mathrm{cfD.LE}$ with some analytic structural rules.

Definition 12. *Let* $\mathrm{D} \in \{\mathrm{D.LE}, \mathrm{D.LE}', \mathrm{cfD.LE}, \mathrm{cfD.LE}'\}$. *A* functional D-frame *is a structure* $\mathbb{F}_{\mathrm{D}} := (W, U, N, \mathcal{R}_{\mathcal{F}}, \mathcal{R}_{\mathcal{G}})$, *where*

1. $W := \mathsf{Str}_{\mathcal{F}}$ *and* $U := \mathsf{Str}_{\mathcal{G}}$*;*
2. *For every* $f \in \mathcal{F}$ *and* $\overline{x} \in W^{\epsilon_f}$, $R_f(y, \overline{x})$ *iff* $\mathsf{F}_f(\overline{x}) N y$*;*
3. *For every* $g \in \mathcal{G}$ *and* $\overline{y} \in U^{\epsilon_g}$, $R_g(x, \overline{y})$ *iff* $x N \mathsf{G}_g(\overline{y})$*;*

2

4. *If*

$$\frac{x_1 \Rightarrow y_1, \ldots, x_n \Rightarrow y_n}{x \Rightarrow y}$$

is a rule in D *(including zero-ary rules), then*

$$\frac{x_1 N y_1, \ldots, x_n N y_n}{x N y}$$

holds in $\mathbb{F}_{\mathrm{D}}$.

It is straightforward to show, by induction on the height of derivations in D, that for every $x \in W$ and $y \in U$ if $\vdash_{\mathrm{D}} x \Rightarrow y$ then xNy.

Proposition 4. *Let* $\mathbb{F}_{\mathrm{D}}$ *be a functional* D*-frame. Then* $\mathbb{F}_{\mathrm{D}}$ *is an* $\mathcal{L}$*-frame.*

Proof. We need to show that the following sets are stable for every $x \in W, y \in U$, $\overline{x} \in W^{\epsilon_f}$ and $\overline{y} \in U^{\epsilon_g}$:

1. $R_f^{(0)}[\overline{x}]$ and $R_f^{(i)}[y, \overline{x}^i]$;

2. $R_g^{(0)}[\overline{y}]$ and $R_g^{(i)}[x, \overline{y}^i]$.

Let us show that $R_f^{(0)}[\overline{x}]$ is stable.

$$\begin{array}{lll} & R_f^{(0)}[\overline{x}] & \\ = & \{y : R_f(y, \overline{x})\} & \text{Definition of } R_f^{(0)} \\ = & \{y : \mathsf{F}_f(\overline{x}) N y\} & \text{Definition of } R_f \\ = & \mathsf{F}_f(\overline{x})^{\uparrow}. & \text{Definition of } (\cdot)^{\uparrow} \end{array}$$

Clearly, $\mathsf{F}_f(\overline{x})^{\uparrow}$ is stable, which proves the claim.
Let us show that $R_f^{(i)}[y, \overline{x}^i]$ is stable when $\epsilon_f(i) = \partial$.

$$\begin{array}{lll} & R_f^{(i)}[y, \overline{x}] & \\ = & \{u \in U : R_f(y, \overline{x})\} & \text{Definition of } R_f^{(i)} \\ = & \{u \in U : \ \mathsf{F}_f(\overline{x}) N y\} & \text{Definition of } R_f \\ = & \{u \in U : \ \mathsf{F}_{f_i^\sharp}(\overline{x}_y^i) N x_i\} & \text{Definition 12.4 and the display rules in } \mathrm{D} \\ = & \mathsf{F}_{f_i^\sharp}(\overline{x}_y^i)^{\uparrow} & \text{Definition of } (\cdot)^{\uparrow} \end{array}$$

Clearly, $\mathsf{F}_{f_i^\sharp}(\overline{x}_y^i)^{\uparrow}$ is stable, which proves the claim. The remaining claims are proven similarly. □

2.4.2 Technical lemmas

Let us introduce the following notation: for any order type ϵ on n, if $X_i \subseteq W^{\epsilon(i)}$ for all $1 \leq i \leq n$, we let

$$X_i^\partial = \begin{cases} X_i^\uparrow & \text{if } \epsilon(i) = 1, \\ X_i^\downarrow & \text{if } \epsilon(i) = \partial \end{cases}$$

and $\overline{X^\partial} := (X_1^\partial, \ldots, X_n^\partial)$. If $Y_i \subseteq U^{\epsilon(i)}$ for all $1 \leq i \leq n$, we let

$$Y_i^\partial = \begin{cases} Y_i^\downarrow & \text{if } \epsilon(i) = 1, \\ Y_i^\uparrow & \text{if } \epsilon(i) = \partial \end{cases}$$

and $\overline{Y^\partial} := (Y_1^\partial, \ldots, Y_n^\partial)$. Moreover we let $N^1 := N$ and N^∂ be the converse of N.

Lemma 4. *For any $f \in \mathcal{F}$ of arity $n_f = n$ (with corresponding structural connective F_f) and any $g \in \mathcal{G}$ of arity $n_g = m$ (with corresponding structural connective G_g):*

1. *If $\overline{x} \in \overline{X} \subseteq W^{\epsilon_f}$, then $\mathsf{F}_f(\overline{x}) \in (R_f^{(0)}[\overline{X}])^\downarrow$. Moreover, if each x_i is a formula for $1 \leq i \leq n$, then $f(\overline{x}) \in (R_f^{(0)}[\overline{X}])^\downarrow$.*

2. *If $X_i \subseteq W^{\epsilon_f(i)}$ and $\overline{x} \in \overline{X^\partial}$, and x_i is a formula for each $1 \leq i \leq n$, then $f(\overline{x}) \in R_f^{(0)}[\overline{X}]$.*

3. *If $\overline{y} \in \overline{Y} \subseteq U^{\epsilon_g}$, then $\mathsf{G}_g(\overline{y}) \in (R_g^{(0)}[\overline{Y}])^\uparrow$. Moreover, if each y_i is a formula for $1 \leq i \leq m$, then $g(\overline{y}) \in (R_g^{(0)}[\overline{Y}])^\uparrow$.*

4. *If $Y_i \subseteq U^{\epsilon_f(i)}$ and $\overline{y} \in \overline{Y^\partial}$, and $y_i \in Y_i$ is a formula for each $1 \leq i \leq m$, then $g(\overline{y}) \in R_g^{(0)}[\overline{Y}]$.*

Proof. 1. In the proof of Proposition 4, we have shown that $(\mathsf{F}_f(\overline{x}))^\uparrow = R_f^{(0)}[\overline{x}]$. Hence, $\mathsf{F}_f(\overline{x}) \in (R_f^{(0)}[\overline{x}])^\downarrow \subseteq (R_f^{(0)}[\overline{X}])^\downarrow$, the inclusion due to the assumption $\overline{x} \in \overline{X}$, which implies $R_f^{(0)}[\overline{X}] \subseteq R_f^{(0)}[\overline{x}]$. This proves the first part of the statement. The clause $\mathsf{F}_f(\overline{x}) \in (R_f^{(0)}[\overline{X}])^\downarrow$ means that $\mathsf{F}_f(\overline{x})Nz$ for all $z \in R_f^{(0)}[\overline{X}]$. Hence if each x_i is a formula, by the rule ($\mathrm{f_L}$) and Definition 12.4, we obtain that $f(\overline{x})Nz$ for all $z \in R_f^{(0)}[\overline{X}]$, which proves that $f(\overline{x}) \in (R_f^{(0)}[\overline{X}])^\downarrow$, as required.

2. The assumption that $\overline{x} \in \overline{X^\partial}$ implies that $z_i N^{\epsilon_f(i)} x_i$ for every $1 \leq i \leq n$ and every $z_i \in X_i$. By the rule ($\mathrm{f_R}$) and Definition 12.4, we obtain that $\mathsf{F}_f(\overline{z}) N f(\overline{x})$. Therefore $R_f(f(\overline{x}), \overline{z})$ holds for every $\overline{z} \in \overline{X}$, which shows that $f(\overline{x}) \in R_f^{(0)}[\overline{X}]$, as required. The proofs of items 3 and 4 are dual.

□

Lemma 5. *For all $\varphi, \psi \in \mathsf{Fm}$,*

1. *If $Y_1, Y_2 \subseteq U$, $\varphi \in Y_1^\downarrow$ and $\psi \in Y_2^\downarrow$, then $\varphi \wedge \psi \in Y_1^\downarrow \cap Y_2^\downarrow$ and $\varphi \vee \psi \in (Y_1 \cap Y_2)^\downarrow$.*

2

2. *If $X_1, X_2 \subseteq W$, $\varphi \in X_1^{\uparrow}$ and $\psi \in X_2^{\uparrow}$, then $\varphi \vee \psi \in X_1^{\uparrow} \cap X_2^{\uparrow}$ and $\varphi \wedge \psi \in (X_1 \cap X_2)^{\uparrow}$.*

Proof. 1. The assumptions $\varphi \in Y_1^{\downarrow}$ and $\psi \in Y_2^{\downarrow}$ are equivalent to $\varphi N y_1$ and $\psi N y_2$ for every $y_1 \in Y$ and $y_2 \in Y_2$. By the rule $(\wedge_{\mathrm{L}})$ and Definition 12.4, this implies that $(\varphi \wedge \psi) N y_1$ and $(\varphi \wedge \psi) N y_2$ for every $y_1 \in Y_1$ and $y_2 \in Y_2$, which shows that $\varphi \wedge \psi \in Y_1^{\downarrow}$ and $\varphi \wedge \psi \in Y_2^{\downarrow}$, i.e. $\varphi \wedge \psi \in Y_1^{\downarrow} \cap Y_2^{\downarrow}$, which proves the first part of the claim. As to the second part, the assumptions imply that $\varphi N y$ and $\psi N y$ for every $y \in Y_1 \cap Y_2$. By the rule $(\vee_{\mathrm{L}})$ and Definition 12.4, we obtain that $(\varphi \vee \psi) N y$ for every $y \in Y_1 \cap Y_2$, therefore $\varphi \vee \psi \in (Y_1 \cap Y_2)^{\downarrow}$ as required. The proof of item 2 is dual. □

Corollary 1. *Let $h : \mathsf{Str}_{\mathcal{F}} \cup \mathsf{Str}_{\mathcal{G}} \to \mathbb{F}_{\mathrm{D}}^{+}$ be the unique homomorphic extension of the assignment $p \mapsto \{p\}^{\downarrow}$. Then $x \in h(x)$ for any $x \in \mathsf{Str}_{\mathcal{F}}$, and $y \in h(y)^{\uparrow}$ for any $y \in \mathsf{Str}_{\mathcal{G}}$.*

Proof. The proof proceeds by simultaneous induction on the complexity of $S \in \mathsf{Str}_{\mathcal{F}} \cup \mathsf{Str}_{\mathcal{G}}$. If S is an atomic proposition p, the statement is immediately true because of the definition of h. If $S = \varphi \vee \psi$ or $S = \varphi \wedge \psi$, then the statement follows from Lemma 5; if $S = f(\overline{\varphi})$ or $S = g(\overline{\varphi})$, then the statement follows from Lemma 4. If $S \in \mathsf{Str}_{\mathcal{F}}$ or $S \in \mathsf{Str}_{\mathcal{G}}$, then the statement follows from Lemma 4.1 and Lemma 4.3, respectively. □

Proposition 5. *For every sequent $x \Rightarrow y$, if $\mathbb{F}_{\mathrm{D}}^{+} \models x \Rightarrow y$ then xNy in $\mathbb{F}_{\mathrm{D}}$.*

Proof. Assume contrapositively that x and y are not N-related, i.e. $x \notin y^{\downarrow}$. We will show that $\mathbb{F}_{\mathrm{D}}^{+} \not\models x \Rightarrow y$. Let $h : \mathsf{Str}_{\mathcal{F}} \cup \mathsf{Str}_{\mathcal{G}} \to \mathbb{F}_{\mathrm{D}}^{+}$ be the unique homomorphic extension of the assignment $p \mapsto \{p\}^{\downarrow}$. By Corollary 1, $x \in h(x)$ and $y \in h(y)^{\uparrow}$, which, since $h(y)$ is stable, implies $h(y) \subseteq y^{\downarrow} \not\ni x$. Hence, $h(x) \nsubseteq h(y)$, which implies $\mathbb{F}_{\mathrm{D}}^{+} \not\models x \Rightarrow y$. □

Let $\mathsf{F}_f[\overline{X}] := \{\mathsf{F}_f(\overline{x}) \mid x_i \in X_i \text{ for all } 1 \leq i \leq n_f\}$ and likewise for $\mathsf{G}_g[\overline{X}]$. For every $f \in \mathcal{F}$ and for every $1 \leq i \leq n_f$, we let

$$\mathsf{H}_{f_i^\sharp} = \begin{cases} \mathsf{G}_{f_i^\sharp} & \text{if } \epsilon(i) = 1, \\ \mathsf{F}_{f_i^\sharp} & \text{if } \epsilon(i) = \partial. \end{cases}$$

For every $g \in \mathcal{G}$ and for every $1 \leq i \leq n_g$, we let

$$\mathsf{H}_{g_i^\flat} = \begin{cases} \mathsf{F}_{g_i^\flat} & \text{if } \epsilon(i) = 1, \\ \mathsf{G}_{g_i^\flat} & \text{if } \epsilon(i) = \partial. \end{cases}$$

Lemma 6. *For any $f \in \mathcal{F}$ of arity $n_f = n$ (with corresponding structural connective F_f) and any $g \in \mathcal{G}$ of arity $n_g = m$ (with corresponding structural connective G_g):*

1. *Let $\overline{X}, \overline{Z} \subseteq W^{\epsilon_f}$. If $X_i^{\partial} \subseteq Z_i^{\partial}$ for all $1 \leq i \leq n$, then $(\mathsf{F}_f[\overline{X}])^{\uparrow} \subseteq R_f^{(0)}[\overline{Z}]$.*

2. *Let $\overline{Y}, \overline{V} \subseteq U^{\epsilon_g}$. If $Y_i^{\partial} \subseteq V_i^{\partial}$ for all $1 \leq i \leq m$, then $(\mathsf{G}_g[\overline{Y}])^{\downarrow} \subseteq R_g^{(0)}[\overline{V}]$.*

Proof. 1. Let $y \in (\mathsf{F}_f[\overline{X}])^{\uparrow}$, i.e. $\mathsf{F}_f(\overline{x})Ny$ for all $\mathsf{F}_f(\overline{x}) \in \mathsf{F}_f(\overline{X})$. Hence, by the display rule and Definition 12.4, this implies that $x_1 N^{\epsilon_f(1)} \mathsf{H}_{f_1^\sharp}(\overline{x}_y^1)$ for every $x_1 \in X_1$, i.e. $\mathsf{H}_{f_1^\sharp}(\overline{x}_y^1) \in X_1^\partial \subseteq Z_1^\partial$. Therefore, $z_1 N^{\epsilon_f(1)} \mathsf{H}_{f_1^\sharp}(\overline{x}_y^1)$ for every $z_1 \in Z_1$, which implies $\mathsf{F}_f(\overline{x}_{z_1}^1)Ny$ for every $z_1 \in Z_1$ by the display rule and Definition 12.4. Reasoning analogously, one shows that $\mathsf{F}_f(z_1, z_2, x_3, \ldots, x_n)Ny$ also holds for all $z_1 \in Z_1$ and $z_2 \in Z_2$. We continue up to n and obtain that $\mathsf{F}_f(\overline{z})Ny$ for all $z_i \in Z_i$, i.e. $y \in R_f^{(0)}[\overline{Z}]$. The proof of item 2 is similar, and hence omitted. □

2.4.3 Soundness of analytic structural rules in complex algebras of functional $\mathbb{D}$-frames

In the present subsection, we show that if $\mathbb{D}$ is obtained by extending the basic calculus $\mathrm{D.LE}$ with analytic structural rules, then these additional rules are sound in the complex algebras of any functional $\mathbb{D}$-frame (cf. Proposition 6). From this, it immediately follows that the analytic inductive inequalities from which these rules arise are valid in these algebras. In what follows, we will need to talk about structural rules, and their shape, as they are given in a display calculus. Typically, structural rules such as the following

$$\frac{Y_1 > (X_1\,;\,X_2) \vdash Y_2}{(Y_1 > X_1)\,;\,X_2 \vdash Y_2}$$

are such that X_1, X_2 and Y_1, Y_2 are meta-variables which range over $\mathsf{Str}_{\mathcal{F}}$, and $\mathsf{Str}_{\mathcal{G}}$ respectively, and $Y_1 > (X_1\,;\,X_2)$ and $(Y_1 > X_1)\,;\,X_2$ are meta-terms. In what follows, we will introduce explicitly a language of meta-variables and meta-terms, which will be useful in the remainder of this section.

Let $\mathsf{MVar} = \mathsf{MVar}_{\mathcal{F}} \uplus \mathsf{MVar}_{\mathcal{G}}$ be a denumerable set of *meta-variables* of sorts $X_1, X_2, \ldots \in \mathsf{MVar}_{\mathcal{F}}$ and $Y_1, Y_2, \ldots \in \mathsf{MVar}_{\mathcal{G}}$. The sets $\mathsf{MStr}_{\mathcal{F}}$ and $\mathsf{MStr}_{\mathcal{G}}$ of the $\mathcal{F}$- and $\mathcal{G}$-*metastructures* are defined by simultaneous induction as follows:

$$\begin{aligned} \mathsf{MStr}_{\mathcal{F}} \ni S \quad &::= X \mid \mathsf{F}_f(\overline{S}) \\ \mathsf{MStr}_{\mathcal{G}} \ni T \quad &::= Y \mid \mathsf{G}_g(\overline{T}) \end{aligned}$$

where $\overline{S} \in \mathsf{MStr}_{\mathcal{F}}^{\epsilon_f}$, and $\overline{T} \in \mathsf{MStr}_{\mathcal{G}}^{\epsilon_g}$, and for any order type ϵ on n, we let $\mathsf{MStr}_{\mathcal{F}}^{\epsilon} := \prod_{i=1}^{n} \mathsf{MStr}_{\mathcal{F}}^{\epsilon(i)}$ and $\mathsf{MStr}_{\mathcal{G}}^{\epsilon} := \prod_{i=1}^{n} \mathsf{MStr}_{\mathcal{G}}^{\epsilon(i)}$, where for all $1 \leq i \leq n$,

$$\mathsf{MStr}_{\mathcal{F}}^{\epsilon(i)} = \begin{cases} \mathsf{MStr}_{\mathcal{F}} & \text{if } \epsilon(i) = 1 \\ \mathsf{MStr}_{\mathcal{G}} & \text{if } \epsilon(i) = \partial \end{cases} \qquad \mathsf{MStr}_{\mathcal{G}}^{\epsilon(i)} = \begin{cases} \mathsf{MStr}_{\mathcal{G}} & \text{if } \epsilon(i) = 1, \\ \mathsf{MStr}_{\mathcal{F}} & \text{if } \epsilon(i) = \partial. \end{cases}$$

We will identify any assignment $h : \mathsf{MVar} \to \mathbb{F}_{\mathbb{D}}^{+}$ with its unique homomorphic extension, and hence write both $h(X)$ and $h(Y)$.

Definition 13. *For any $h : \mathsf{MVar} \to \mathbb{F}_{\mathbb{D}}^{+}$, any $S \in \mathsf{MStr}_{\mathcal{F}}$ and $T \in \mathsf{MStr}_{\mathcal{G}}$, define $h\{S\} \subseteq W$ and $h\{T\} \subseteq U$ by simultaneous recursion as follows:*

1. *If S and T are variables then $h\{S\} = h(S)$ and $h\{T\} = h(T)^{\uparrow}$;*

2

2. $h\{\mathsf{F}_f(\overline{S})\} := \mathsf{F}_f[\overline{h\{S\}}] = \{\mathsf{F}_f(\overline{x})$ *for some* $\overline{x} \in \overline{h\{S\}}\}$;

3. $h\{\mathsf{G}_g(\overline{T})\} := \mathsf{G}_g[\overline{h\{T\}}] = \{\mathsf{G}_g(\overline{y})$ *for some* $\overline{y} \in \overline{h\{T\}}\}$,

where $\overline{S} \subseteq \mathsf{MStr}_{\mathcal{F}}^{\epsilon_f}$, $\overline{h\{S\}} := \prod_{i=1}^{n_f} h\{S^{\epsilon_f(i)}\}$, *such that*

$$S^{\epsilon_f(i)} \in \begin{cases} \mathsf{MStr}_{\mathcal{F}} & \text{if } \epsilon_f(i) = 1 \\ \mathsf{MStr}_{\mathcal{G}} & \text{if } \epsilon_f(i) = \partial \end{cases}$$

and $\overline{T} \subseteq \mathsf{MStr}_{\mathcal{G}}^{\epsilon_g}$, $\overline{h\{T\}} := \prod_{i=1}^{n_g} h\{T^{\epsilon_g(i)}\}$, *such that*

$$T^{\epsilon_g(i)} \in \begin{cases} \mathsf{MStr}_{\mathcal{G}} & \text{if } \epsilon_f(i) = 1 \\ \mathsf{MStr}_{\mathcal{F}} & \text{if } \epsilon_f(i) = \partial. \end{cases}$$

Lemma 7. *For any* $h : \mathsf{MVar} \to \mathbb{F}_{\mathrm{D}}^{+}$ *any* $S \in \mathsf{MStr}_{\mathcal{F}}$ *and* $T \in \mathsf{MStr}_{\mathcal{G}}$,

1. $h\{S\} \subseteq h(S)$ *and* $h\{T\} \subseteq h(T)^{\uparrow}$, *or equivalently* $h(T) \subseteq h\{T\}^{\downarrow}$.

2. $h\{S\}^{\uparrow} \subseteq h(S)^{\uparrow}$ *and* $h\{T\}^{\downarrow} \subseteq h(T)$.

Proof. 1. The proof proceeds by simultaneous induction on S and T. The base case is immediate by Definition 13.1. For the induction step, let S be $\mathsf{F}_f(\overline{S})$ and assume that $h\{S^{\epsilon_f(i)}\} \subseteq h(S^{\epsilon_f(i)})$ for every $1 \leq i \leq n_f$ if $\epsilon_f(i) = 1$, and $h\{S^{\epsilon_f(i)}\} \subseteq h(S^{\epsilon_f(i)})^{\uparrow}$ for every $1 \leq i \leq n_f$ if $\epsilon_f(i) = \partial$. That is, recalling the notation introduced in Definition 11, $\overline{h\{S\}} \subseteq \overline{h(S)^{\epsilon_f}}$. Then:

$$\begin{aligned} h\{\mathsf{F}_f(\overline{S})\} &= \mathsf{F}_f[\overline{h\{S\}}] && \text{Definition 13.2} \\ &\subseteq (R_f^{(0)}[\overline{h(S)^{\epsilon_f}}])^{\downarrow} && (*) \\ &= f_{R_f}(\overline{h(S)}) && \text{Definition 11.1} \\ &= h(\mathsf{F}_f(\overline{S})) && \text{h is a homomorphism} \end{aligned}$$

As to the inclusion marked with $(*)$, any element in $\mathsf{F}_f[\overline{h\{S\}}]$ is of the form $\mathsf{F}_f(\overline{x})$ for some $\overline{x} \in \overline{h\{S\}} \subseteq \overline{h(S)^{\epsilon_f}}$. Hence, by Lemma 4.1, $F(\overline{x}) \in (R_f^{(0)}[\overline{h(S)^{\epsilon_f}}])^{\downarrow}$, as required. The case in which T is of the form $\mathsf{G}_g(\overline{T})$ is shown similarly using Lemma 4.3.

2. The proof proceeds by simultaneous induction on S and T. The base case is immediate from Definition 13.1. For the induction step, let S be $\mathsf{F}_f(\overline{S})$ and assume that $h\{S^{\epsilon_f(i)}\}^{\uparrow} \subseteq h(S^{\epsilon_f(i)})^{\uparrow}$ for every $1 \leq i \leq n_f$ if $\epsilon_f(i) = 1$, and $h\{S^{\epsilon_f(i)}\}^{\downarrow} \subseteq h(S^{\epsilon_f(i)}) = h(S^{\epsilon_f(i)})^{\uparrow\downarrow}$ for every $1 \leq i \leq n_f$ if $\epsilon_f(i) = \partial$. Hence,

$$\begin{aligned} h\{\mathsf{F}_f(\overline{S})\}^{\uparrow} &= \mathsf{F}_f[\overline{h\{S\}}]^{\uparrow} && \text{Definition 13.2} \\ &\subseteq R_f^{(0)}[\overline{h(S)^{\epsilon_f}}] && \text{Lemma 6.1} \\ &= (R_f^{(0)}[\overline{h(S)^{\epsilon_f}}])^{\downarrow\uparrow} && \text{Lemma 2.1} \\ &= (f_{R_f}(\overline{h(S)}))^{\uparrow} && \text{Definition 11.1} \\ &= (h(\mathsf{F}_f(\overline{S})))^{\uparrow} && \text{h is a homomorphism} \end{aligned}$$

The case in which T is of the form $\mathsf{G}_g(\overline{T})$ is shown similarly using Lemma 6.2. $\square$

Lemma 8. *The following are equivalent:*

1. $h(S) \subseteq h(T)$;
2. sNt *for every* $s \in h\{S\}$ *and* $t \in h\{T\}$.

Proof. $1 \Rightarrow 2$. If $h(S) \subseteq h(T)$, by Lemma 7.1, $h\{S\} \subseteq h(S) \subseteq h(T) \subseteq h\{T\}^{\downarrow}$. This means that sNt for every $s \in h\{S\}$ and $t \in h\{T\}$.

$2 \Rightarrow 1$. Since $h(S)$ and $h(T)$ are stable sets, $h(S) \subseteq h(T)$ is equivalent to $h(T)^{\uparrow} \subseteq h(S)^{\uparrow}$. By Lemma 7.2, $h(T)^{\uparrow} \subseteq h\{T\}^{\downarrow\uparrow}$ and $h\{S\}^{\uparrow} \subseteq h(S)^{\uparrow}$. Hence to finish the proof it is enough to show that $h\{T\}^{\downarrow\uparrow} \subseteq h\{S\}^{\uparrow}$. By assumption 2, $h\{T\} \subseteq h\{S\}^{\uparrow}$. Hence $h\{T\}^{\downarrow\uparrow} \subseteq h\{S\}^{\uparrow\downarrow\uparrow} = h\{S\}^{\uparrow}$, as required. □

Proposition 6. *The rules of* D *are sound w.r.t.* $\mathbb{F}^{+}_{\mathrm{D}}$.

Proof. By Proposition 4, $\mathbb{F}_{\mathrm{D}}$ is an $\mathcal{L}$-frame, and hence, by Proposition 3, $\mathbb{F}^{+}_{\mathrm{D}}$ is an $\mathcal{L}^*$-algebra. Therefore, all the rules which D shares with $\mathrm{cfD.LE}$ are sound. Let R be an analytic structural rule of D. Then R has the following shape:

$$\frac{S_1 \vdash T_1 \qquad \cdots \qquad S_n \vdash T_n}{S_0 \vdash T_0}$$

By Definition 12.4,

$$\frac{S_1 N T_1 \qquad \cdots \qquad S_n N T_n}{S_0 N T_0}\ \mathrm{R_N}$$

holds in $\mathbb{F}_{\mathrm{D}}$.

Let $h : \mathsf{MVar} \longrightarrow \mathbb{F}^{+}_{\mathrm{D}}$ be an assignment of metavariables, which we identify with its unique homomorphic extension, and assume that $h(S_1) \subseteq h(T_1), \ldots, h(S_n) \subseteq h(T_n)$. We need to prove that $h(S_0) \subseteq h(T_0)$. By Lemma 8, this is equivalent to showing that $s_0 N t_0$ for every $s_0 \in h\{S_0\}$ and $t_0 \in h\{T_0\}$. Notice that since the rule R is analytic, each metavariable in $S_0 N T_0$ occurs at most once, and hence $h\{S_0\} = \{S'(x_1, \ldots, x_j) \mid x_j \in h(X_j) \text{ for } 1 \leq j \leq k\}$ and $h\{T_0\} = \{T'(y_1, \ldots, y_m) \mid y_j \in h(y_j) \text{ for } 1 \leq j \leq m\}$. Hence each sequent $s_0 N t_0$ is an instance of the conclusion of $\mathrm{R_N}$ and induces a choice function $\eta : \mathsf{MVar} \to \bigcup h[\mathsf{MVar}]$ such that $\eta(X) \in h(X)$. Let $\{s_i N t_i \mid 1 \leq i \leq n\}$ be the corresponding instance of the premises of $\mathrm{R_N}$, in the sense that e.g. each $s_i = S_i(\eta(X_1), \ldots \eta(X_k))$. Because the analyticity of R prevents structural variables to occur both in antecedent and in succedent position, so does $\mathrm{R_N}$. Hence, to prove our claim, it is enough to show that $s_i N t_i$ holds in $\mathbb{F}_{\mathrm{D}}$ for each i. This is guaranteed by the assumption $h(S_i) \subseteq h(T_i)$ and by Lemma 8, since, by Definition 13, $s_i \in h\{S_i\}$ and $t_i \in h\{T_i\}$. □

Let $\mathbb{L}'_{\mathrm{LE}}$ be the analytic LE-logic $\mathbb{L}_{\mathrm{LE}}$ corresponding to the additional rules of D. Proposition 4 and the proposition above immediately imply the following

Theorem 3. *Let* $\mathrm{D} \in \{\mathrm{D.LE}, \mathrm{D.LE}', \mathrm{cfD.LE}, \mathrm{cfD.LE}'\}$. *If* $\mathbb{F}_{\mathrm{D}}$ *is a functional* D-*frame, then* $\mathbb{F}^{+}_{\mathrm{D}}$ *is a complete* $\mathcal{L}$-*algebra (and hence a complete* $\mathcal{L}^*$-*algebra) if* $\mathrm{D} \in \{\mathrm{D.LE}, \mathrm{cfD.LE}\}$, *and is a complete* $\mathbb{L}'_{\mathrm{LE}}$-*algebra if* $\mathrm{D} \in \{\mathrm{D.LE}', \mathrm{cfD.LE}'\}$.

2.5 Semantic cut elimination for normal LE-logics

2.5.1 Semantic cut elimination for basic normal LE-logics

In this subsection, we prove the following generalisation of [8, Theorem 3.2] from the full Lambek calculus to a basic normal LE-logic of fixed but arbitrary signature $\mathcal{L} = \mathcal{L}(\mathcal{F}, \mathcal{G})$ (cf. Section 2.2):

Theorem 4. *For every sequent $x \Rightarrow y$, if $\vdash_{\mathrm{D.LE}} x \Rightarrow y$ then $\vdash_{\mathrm{cfD.LE}} x \Rightarrow y$.*

Proof. Let $\mathbb{F}_{\mathrm{cfD.LE}}$ be the functional $\mathrm{cfD.LE}$-frame in which N is defined as follows: for all $x \in W$ and $y \in U$,

$$xNy \quad \text{iff} \quad \vdash_{\mathrm{cfD.LE}} x \Rightarrow y. \tag{2.8}$$

Our argument is illustrated by the following diagram:

$$\begin{tikzcd}
\vdash_{\mathrm{D.LE}} x \Rightarrow y \arrow[r, dashed] \arrow[d] & \vdash_{\mathrm{cfD.LE}} x \Rightarrow y \arrow[d, leftrightarrow] \\
\mathbb{F}^{+}_{\mathrm{cfD.LE}} \models x \Rightarrow y \arrow[r] & xNy \text{ in } \mathbb{F}_{\mathrm{cfD.LE}}
\end{tikzcd}$$

Propositions 4 and 3 imply that $\mathbb{F}^{+}_{\mathrm{cfD.LE}}$ is a complete $\mathcal{L}$-algebra. By Proposition 1, $\mathrm{D.LE}$ is sound w.r.t. the class of complete $\mathcal{L}$-algebras. Hence, $\vdash_{\mathrm{D.LE}} x \Rightarrow y$ implies that $\mathbb{F}^{+}_{\mathrm{cfD.LE}} \models x \Rightarrow y$, which is the vertical arrow on the left-hand side of the diagram. By Proposition 5, this implies that xNy in $\mathbb{F}_{\mathrm{cf.DLE}}$, which gives the horizontal arrow. By (2.8), xNy is equivalent to $\vdash_{\mathrm{cfD.LE}} x \Rightarrow y$, which yields the vertical bi-implication and completes the proof.

□

2.5.2 Semantic cut elimination for analytic LE-logics

In this subsection, we fix an arbitrary LE-signature $\mathcal{L} = \mathcal{L}(\mathcal{F}, \mathcal{G})$ and show the semantic cut elimination for any analytic LE-logic $\mathbb{L}'$. By Proposition 2, this logic is captured by a display calculus $\mathrm{D.LE}'$ which is obtained by adding analytic structural rules (computed by running ALBA on the additional axioms) to the basic calculus $\mathrm{D.LE}$. By the general theory, $\mathrm{D.LE}'$ is sound and complete w.r.t. the class of complete $\mathcal{L}$-algebras validating the additional axioms. Let $\mathrm{cfD.LE}'$ be the cut-free version of $\mathrm{D.LE}'$.

Theorem 5. *For every sequent $x \Rightarrow y$, if $\vdash_{\mathrm{D.LE}'} x \Rightarrow y$ then $\vdash_{\mathrm{cfD.LE}'} x \Rightarrow y$.*

Proof. Let $\mathbb{F}_{\mathrm{cfD.LE}'}$ be the functional $\mathrm{cfD.LE}'$-frame (cf. Definition 12) in which N is defined as follows: for all $x \in W$ and $y \in U$,

$$xNy \quad \text{iff} \quad \vdash_{\mathrm{cfD.LE}'} x \Rightarrow y. \tag{2.9}$$

The proof strategy is analogous to the one of the previous subsection and is illustrated by the following diagram.

$$\begin{tikzcd}
\vdash_{\mathrm{D.LE'}} x \Rightarrow y \arrow[r, dashed] \arrow[d] & \vdash_{\mathrm{cfD.LE'}} x \Rightarrow y \arrow[d, leftrightarrow] \\
\mathbb{F}^{+}_{\mathrm{cfD.LE'}} \models x \Rightarrow y \arrow[r] & xNy \text{ in } \mathbb{F}_{\mathrm{cfD.LE'}}
\end{tikzcd}$$

The vertical equivalence on the right-hand side of the diagram holds by construction. The horizontal implication follows from Proposition 5. The proof is complete by appealing to Proposition 6. □

2.6 Finite model property

We say that a display calculus has the *finite model property* (FMP) if every sequent $x \Rightarrow y$ that is not derivable in the calculus has a finite counter-model. In this section, we prove the FMP for $\mathrm{D} \in \{\mathrm{D.LE}, \mathrm{D.LE'}\}$ where $\mathrm{D.LE}$ is the display calculus for the basic LE-logic, and $\mathrm{D.LE'}$ is one of its extensions with analytic structural rules subject to certain conditions (see below). For any sequent $x \Rightarrow y$ such that $\nvdash_{\mathrm{D}} x \Rightarrow y$, our proof strategy consists in constructing a functional D-frame $\mathbb{F}_{\mathrm{D}}^{x\Rightarrow y}$ the complex algebra of which is finite. The basic idea to satisfy the requirement of finiteness is provided by the following lemma (the symbol $(\cdot)^c$ denotes the relative complementation).

Lemma 9. *Let $\mathbb{W} = (W, U, N)$ be a polarity. If the set $\{y^{\downarrow} \mid y \in U\}$ is finite, then $\mathbb{W}^{+}$ is finite. Dually, if the set $\{x^{\uparrow} \mid x \in W\}$ is finite, then $\mathbb{W}^{+}$ is finite.*

Proof. Since $\{y^{\downarrow} \mid y \in U\}$ meet-generates $\mathbb{W}^{+}$, an upper bound to the size of $\mathbb{W}^{+}$ is $2^{|\{y^{\downarrow} | y \in U\}|}$. The remaining part of the statement is proven dually. □

Definition 14. *Let D be a display calculus as above. For any sequent $x \Rightarrow y$, let $(x \Rightarrow y)^{\leftarrow}$ be the set of sequents which is defined recursively as follows:*

1. $x \Rightarrow y \in (x \Rightarrow y)^{\leftarrow}$;
2. *if* $\dfrac{x_1 \Rightarrow y_1, \ldots, x_n \Rightarrow y_n}{x_0 \Rightarrow y_0}$ *is an instance of a rule in D, and $x_0 \Rightarrow y_0 \in (x \Rightarrow y)^{\leftarrow}$, then $x_1 \Rightarrow y_1, \ldots, x_n \Rightarrow y_n \in (x \Rightarrow y)^{\leftarrow}$.*

Definition 15. *For any sequent $x \Rightarrow y$, let $\mathbb{F}_{\mathrm{D}}^{x\Rightarrow y}$ denote the structure $(W, U, N, \mathcal{R}_{\mathcal{F}}, \mathcal{R}_{\mathcal{G}})$ such that $W, U, \mathcal{R}_{\mathcal{F}}, \mathcal{R}_{\mathcal{G}}$ are as in Definition 12.1-3, and N is defined as follows:*

$$wNu \quad \textit{iff} \quad \vdash_{\mathrm{D}} w \Rightarrow u \textit{ or } w \Rightarrow u \notin (x \Rightarrow y)^{\leftarrow}. \tag{2.10}$$

Proposition 7. *$\mathbb{F}_{\mathrm{D}}^{x\Rightarrow y}$ is a functional D-frame.*

Proof. We only need to show that $\mathbb{F}_{\mathrm{D}}^{x\Rightarrow y}$ satisfies Definition 12.4, i.e. for every rule R:

$$\frac{x_1 \Rightarrow y_1, \ldots, x_n \Rightarrow y_n}{x_0 \Rightarrow y_0}$$

2

in D,

$$\frac{x_1 N y_1, \ldots, x_n N y_n}{x_0 N y_0}$$

holds. Assume that $x_1 N y_1, \ldots, x_n N y_n$. If $\vdash_{\mathrm{D}} x_1 \Rightarrow y_1, \ldots, \vdash_{\mathrm{D}} x_n \Rightarrow y_n$, then $\vdash_{\mathrm{D}} x_0 \Rightarrow y_0$ by applying R, hence $x_0 N y_0$ by the definition of N. Otherwise, $\nvdash_{\mathrm{D}} x_i \Rightarrow y_i$ for some $1 \leq i \leq n$, and hence (2.10) and the assumption $x_i N y_i$ imply that $x_i \Rightarrow y_i \notin (x \Rightarrow y)^{\leftarrow}$. Hence, $x_0 \Rightarrow y_0 \notin (x \Rightarrow y)^{\leftarrow}$ by Definition 14.2. Therefore, we conclude again that $x_0 N y_0$. □

The above proposition and Theorem 3 imply that the complex algebra of $\mathbb{F}_{\mathrm{D}}^{x \Rightarrow y}$ is a complete $\mathcal{L}$-algebra if D is $\mathrm{D.LE}$ (resp. a complete $\mathbb{L}_{\mathrm{LE}'}$-algebra if D is $\mathrm{D.LE}'$).

Proposition 8. *If $\nvdash_{\mathrm{D}} x \Rightarrow y$, then $(\mathbb{F}_{\mathrm{D}}^{x \Rightarrow y})^+ \nvDash x \Rightarrow y$.*

Proof. Let $h : \mathsf{Str}_{\mathcal{F}} \cup \mathsf{Str}_{\mathcal{G}} \rightarrow (\mathbb{F}_{\mathrm{D}}^{x \Rightarrow y})^+$ be the unique homomorphic extension of the assignment $p \mapsto \{p\}^{\downarrow}$. We will show that $h(x) \nsubseteq h(y)$. Assume that $h(x) \subseteq h(y)$. By Corollary 1, we obtain that $x \in h(x)$ and $h(y) \subseteq y^{\downarrow}$. Hence $x \in y^{\downarrow}$, i.e. xNy, that is $\vdash_{\mathrm{D}} x \Rightarrow y$ or $x \Rightarrow y \notin (x \Rightarrow y)^{\leftarrow}$ by (2.10). Since $x \Rightarrow y \in (x \Rightarrow y)^{\leftarrow}$ by Definition 14.2, we obtain that $\vdash_{\mathrm{D}} x \Rightarrow y$, which contradicts $\nvdash_{\mathrm{D}} x \Rightarrow y$, and hence $h(x) \nsubseteq h(y)$, i.e. $(\mathbb{F}_{\mathrm{D}}^{x \Rightarrow y})^+ \nvDash x \Rightarrow y$. □

Thus, the algebra $(\mathbb{F}_{\mathrm{D}}^{x \Rightarrow y})^+$ is a good candidate for the finite model property, provided we can definite conditions under which it is finite.

Definition 16. *Let $\Phi_{\mathcal{F}}$ denote the following equivalence relation on $\mathsf{Str}_{\mathcal{F}}$: if x and x' are $\mathcal{F}$-structures, $(x, x') \in \Phi_{\mathcal{F}}$ iff the following rule scheme is derivable in D:*

$$\frac{x \Rightarrow Y}{\overline{x' \Rightarrow Y}}$$

An equivalence relation $\Phi_{\mathcal{G}}$ on $\mathsf{Str}_{\mathcal{G}}$ can be defined analogously. In what follows, we will let $[x']_{\Phi_{\mathcal{F}}}$ and $[y']_{\Phi_{\mathcal{G}}}$ denote the equivalence classes induced by $\Phi_{\mathcal{F}}$ and $\Phi_{\mathcal{G}}$ respectively.

Definition 17. *For every sequent $x \Rightarrow y$, let*

$$(x \Rightarrow y)^{\leftarrow}_{\mathcal{F}} := \{x' \in \mathsf{Str}_{\mathcal{F}} \mid x' \Rightarrow y' \in (x \Rightarrow y)^{\leftarrow} \text{ for some } y' \in \mathsf{Str}_{\mathcal{G}}\}$$

$$(x \Rightarrow y)^{\leftarrow}_{\mathcal{G}} := \{y' \in \mathsf{Str}_{\mathcal{G}} \mid x' \Rightarrow y' \in (x \Rightarrow y)^{\leftarrow} \text{ for some } x' \in \mathsf{Str}_{\mathcal{F}}\}.$$

In what follows, we let $y^{\downarrow} := \{x \in W \mid xNy\}$, where N is defined as in (2.10).

Proposition 9. *For all $y' \in \mathsf{Str}_{\mathcal{G}}$ and $x' \in \mathsf{Str}_{\mathcal{F}}$ such that $(y'^{\downarrow})^c \neq \varnothing$ and $(x'^{\uparrow})^c \neq \varnothing$,*

$$(y'^{\downarrow})^c = \bigcup\{[x'']_{\Phi_{\mathcal{F}}} \mid x'' \in A\} \quad \text{and} \quad (x'^{\uparrow})^c = \bigcup\{[y'']_{\Phi_{\mathcal{G}}} \mid y'' \in B\}$$

for some $A \subseteq (x \Rightarrow y)^{\leftarrow}_{\mathcal{F}}$ and $B \subseteq (x \Rightarrow y)^{\leftarrow}_{\mathcal{G}}$.

Proof. Let $y' \in \mathsf{Str}_{\mathcal{G}}$. If $y' \notin (x \Rightarrow y)^{\leftarrow}_{\mathcal{G}}$ then $w \Rightarrow y' \notin (x \Rightarrow y)^{\leftarrow}$ for all $w \in \mathsf{Str}_{\mathcal{F}}$ and therefore, by Definition 15, $y'^{\downarrow} = \mathsf{Str}_{\mathcal{F}}$, i.e. $(y'^{\downarrow})^c = \varnothing$. Therefore, we can assume without loss of generality that $y' \in (x \Rightarrow y)^{\leftarrow}_{\mathcal{G}}$. Let $(x', x'') \in \Phi_{\mathcal{F}}$. Definition 16 implies that for every $u \in \mathsf{Str}_{\mathcal{G}}$

$$\vdash_{\mathrm{D}} x' \Rightarrow u \text{ if and only if } \vdash_{\mathrm{D}} x'' \Rightarrow u \tag{2.11}$$

and

$$x' \Rightarrow u \in (x \Rightarrow y)^{\leftarrow} \text{ if and only if } x'' \Rightarrow u \in (x \Rightarrow y)^{\leftarrow}. \tag{2.12}$$

By Definition 15, (2.11) and (2.12) we obtain

$$x'Ny' \text{ if and only if } x''Ny' \tag{2.13}$$

for every $(x', x'') \in \Phi_{\mathcal{F}}$. Furthermore, by Definition 15, $wN^c y'$ implies that $w \Rightarrow y' \in (x \Rightarrow y)^{\leftarrow}$ and therefore $w \in (x \Rightarrow y)^{\leftarrow}_{\mathcal{F}}$. This combined with (2.13) implies that there exists some $A \subseteq (x \Rightarrow y)^{\leftarrow}_{\mathcal{F}}$ such that $(y'^{\downarrow})^c = \bigcup\{[x'']_{\Phi_{\mathcal{F}}} \mid x'' \in A\}$. The proof for $(x'^{\uparrow})^c$ is shown dually. □

∀ For every sequent $x \Rightarrow y$,

1. if $\{[x']_{\Phi_{\mathcal{F}}} \mid x' \in (x \Rightarrow y)^{\leftarrow}_{\mathcal{F}}\}$ is finite, then $(\mathbb{F}^{x\Rightarrow y}_{\mathrm{D}})^+$ is finite.
2. if $\{[y']_{\Phi_{\mathcal{G}}} \mid y' \in (x \Rightarrow y)^{\leftarrow}_{\mathcal{G}}\}$ is finite, then $(\mathbb{F}^{x\Rightarrow y}_{\mathrm{D}})^+$ is finite.

Proof. By Proposition 9, for every $y' \in \mathsf{Str}_{\mathcal{G}}$, $(y'^{\downarrow})^c = \varnothing$, or $(y'^{\downarrow})^c = \bigcup\{[x'']_{\Phi_{\mathcal{F}}} \mid x'' \in A\}$ for some $A \subseteq (x \Rightarrow y)^{\leftarrow}_{\mathcal{F}}$. If $\{[x']_{\Phi_{\mathcal{F}}} \mid x' \in (x \Rightarrow y)^{\leftarrow}_{\mathcal{F}}\}$ is finite, then $\{(y'^{\downarrow})^c \mid y' \in \mathsf{Str}_{\mathcal{G}}\}$ is finite, so $\{y'^{\downarrow} \mid y' \in \mathsf{Str}_{\mathcal{G}}\}$ is finite, therefore Lemma 9 implies that $(\mathbb{F}^{x\Rightarrow y}_{\mathrm{D}})^+$ is finite. Item 2 is shown analogously. □

Proposition 8 and Corollary 2.6 imply the following:

Theorem 6. *If the calculus* D *verifies one of the assumptions of Corollary 2.6 then FMP holds for* D.

In what follows we will discuss sufficient conditions for the assumptions of Corollary 2.6 to hold.

Proposition 10. *If all rules in* D *applied bottom up decrease or leave unchanged the complexity of sequents, then FMP holds for* D.

Proof. The assumptions imply that the set $(x \Rightarrow y)^{\leftarrow}$ is finite and therefore the assumptions of Corollary 2.6 are satisfied. □

Proposition 11. *1. If $\Phi'_{\mathcal{F}}$ is an equivalence relation such that $\Phi'_{\mathcal{F}} \subseteq \Phi_{\mathcal{F}}$ and moreover $\{[x']_{\Phi'_{\mathcal{F}}} \mid x' \in (x \Rightarrow y)^{\leftarrow}_{\mathcal{F}}\}$ is finite, then the FMP holds for* D.

2. If $\Phi'_{\mathcal{G}}$ is an equivalence relation such that $\Phi'_{\mathcal{G}} \subseteq \Phi_{\mathcal{G}}$ and moreover $\{[x']_{\Phi'_{\mathcal{G}}} \mid x' \in (x \Rightarrow y)^{\leftarrow}_{\mathcal{F}}\}$ is finite, then the FMP holds for D.

Proof. 1. If $\Phi'_{\mathcal{F}} \subseteq \Phi_{\mathcal{F}}$, then every equivalence class of $\Phi_{\mathcal{F}}$ is the union of equivalence classes of $\Phi'_{\mathcal{F}}$. Hence, the assumption that $\{[x']_{\Phi'_{\mathcal{F}}} \mid x' \in (x \Rightarrow y)^{\leftarrow}_{\mathcal{F}}\}$ is finite guarantees that the assumptions of Corollary 2.6 are satisfied, and hence the statement follows by Theorem 6. □

2

The two propositions above provide us with effective strategies to prove the FMP. In particular, while computing $\Phi_{\mathcal{F}}$ or $\Phi_{\mathcal{G}}$ might be practically unfeasible, by Proposition 11 it is in fact enough to produce a suitable refinement of them. We will illustrate this point in the next section.

2.7 Examples

In the present section, we obtain cut elimination and FMP for concrete instances of LE-logics, respectively as a consequence of Theorem 5 and of Propositions 10 and 11.

2.7.1 Full Lambek calculus

The language of the *full Lambek calculus* [9], denoted $\mathcal{L}_{\mathrm{FL}}$, is obtained by instantiating $\mathcal{F} := \{e, \circ\}$ with $n_e = 0$, $n_\circ = 2$, $\varepsilon_\circ = (1,1)$ and $\mathcal{G} := \{\backslash, /\}$ with $n_\backslash = n_/ = 2$, $\varepsilon_\backslash = (\partial, 1)$ and $\varepsilon_/ = (1, \partial)$.

Clearly, Theorem 5 applies to the calculus $\mathrm{D.LE}$ for the basic $\mathcal{L}_{\mathrm{FL}}$-logic and to any calculus $\mathrm{D.LE}'$ obtained by adding any analytic structural rule to $\mathrm{D.LE}$. This result covers the semantic cut elimination for any display calculus for axiomatic extensions of the basic $\mathcal{L}_{\mathrm{FL}}$-logic with $\mathcal{N}_2$ axioms (cf. [4], see also Example 1). Moreover, Proposition 10 applies to $\mathrm{D.LE}$ and any calculus $\mathrm{D.LE}'$ obtained by adding any analytic structural rule to $\mathrm{D.LE}$ such that the complexity of sequents does not increase from bottom to top. This result covers FMP for the display calculi capturing the nonassociative full Lambek calculus (cf. [2]), the full Lambek calculus (which corresponds to $\mathrm{D.LE}$ plus associativity), and its axiomatic extensions with commutativity, weakening, and simple rules that do not increase the complexity of sequents from bottom to top (cf. [8, Theorem 3.15]).

2.7.2 Lambek-Grishin calculus

The language of the *Lambek-Grishin calculus* (cf. [15]), denoted $\mathcal{L}_{\mathrm{LG}}$, is obtained by instantiating $\mathcal{F} := \{\circ, /_\star, \backslash_\star\}$ with $n_\circ = n_{\backslash_\star} = n_{/_\star} = 2$, $\varepsilon_\circ = (1,1)$, $\varepsilon_{\backslash_\star} = (\partial, 1)$, $\varepsilon_{/_\star} = (1, \partial)$ and $\mathcal{G} := \{\star, /_\circ, \backslash_\circ\}$ with $n_\star = n_{/_\circ} = n_{\backslash_\circ} = 2$, $\varepsilon_\star = (1,1)$, $\varepsilon_{\backslash_\circ} = (\partial, 1)$, $\varepsilon_{/_\circ} = (1, \partial)$.

One can explore the space of the axiomatic extensions of the basic $\mathcal{L}_{\mathrm{LG}}$-logic with the following *Grishin interaction principles* [14]:

$$
\begin{array}{llclllcl}
\text{(a)} & (q \star r) \circ p & \vdash & q \star (r \circ p) & \text{(d)} & (p \backslash_\star q) \backslash_\circ r & \vdash & q \backslash_\circ (p \star r) \\
\text{(b)} & p \star (r /_\circ q) & \vdash & (p \star r) /_\circ q & \text{(e)} & (r \circ q) /_\star p & \vdash & r /_\star (p /_\circ q) \\
\text{(c)} & p \backslash_\star (r \circ q) & \vdash & (p \backslash_\star r) \circ q & \text{(f)} & p \circ (r \backslash_\circ q) & \vdash & (p /_\star r) \star q
\end{array} \qquad \text{(I)}
$$

$$
\begin{array}{llclllcl}
\text{(a)} & (q \backslash_\circ r) \circ p & \vdash & q \backslash_\circ (r \circ p) & \text{(d)} & (p \circ q) \backslash_\circ r & \vdash & q \backslash_\circ (p \backslash_\circ r) \\
\text{(b)} & p \backslash_\circ (r /_\circ q) & \vdash & (p \backslash_\circ r) /_\circ q & \text{(e)} & (p /_\circ q) /_\circ r & \vdash & p /_\circ (r \circ q) \\
\text{(c)} & p \circ (r \circ q) & \vdash & (p \circ r) \circ q & \text{(f)} & p \circ (r \backslash_\circ q) & \vdash & (r /_\circ p) \backslash_\circ q
\end{array} \qquad \text{(II)}
$$

$$
\begin{array}{llclllcl}
\text{(a)} & p /_\star (r \star q) & \vdash & (p \backslash_\star r) \star q & \text{(d)} & q /_\star (r \star p) & \vdash & (q /_\star p) /_\star r \\
\text{(b)} & (p \star r) \star q & \vdash & p \star (r \star q) & \text{(e)} & p \backslash_\star (q \backslash_\star r) & \vdash & (q \star p) \backslash_\star r \\
\text{(c)} & (p \backslash_\star r) /_\star q & \vdash & p \backslash_\star (r /_\star q) & \text{(f)} & (r /_\star q) \backslash_\star p & \vdash & q \star (r \backslash_\star p)
\end{array} \qquad \text{(III)}
$$

$$
\begin{array}{llclllcl}
\text{(a)} & (q \backslash_\circ r) /_\star p & \vdash & q \backslash_\circ (r /_\star p) & \text{(d)} & (p \backslash_\circ r) \backslash_\star q & \vdash & r \backslash_\star (p \circ q) \\
\text{(b)} & q \backslash_\circ (r \star p) & \vdash & (q \backslash_\circ r) \star p & \text{(e)} & (p \star q) /_\circ r & \vdash & p /_\circ (r /_\star q) \\
\text{(c)} & p \circ (r /_\star q) & \vdash & (p \circ r) /_\star q & \text{(f)} & p /_\star (q \backslash_\star r) & \vdash & (r /_\circ p) \backslash_\circ q
\end{array} \qquad \text{(IV)}
$$

As observed in [7, Remark 5.3], all these axioms are analytic inductive, and hence they can all be transformed into analytic structural rules (cf. [13]). For instance:

$$
p \backslash_\star (q \circ r) \leq (p \backslash_\star q) \circ r \quad \rightsquigarrow \quad \frac{(y_1 > x_1) \odot x_2 \Rightarrow y_2}{y_1 > (x_1 \odot x_2) \Rightarrow y_2} \quad \rightsquigarrow \quad \frac{y_1 > x_1 \Rightarrow y_2 < x_2}{x_1 \odot x_2 \Rightarrow y_1 \odot y_2}
$$

where the relation between structural and logical connectives in $\mathcal{L}_{\mathrm{LG}}$ is reported in the following table:

Structural symbols	$>$		$<$		$\odot$	
Operational symbols	$\backslash_\star$	$\backslash_\circ$	$/_\star$	$/_\circ$	$\circ$	$\star$

By Theorem 5, any calculus $\mathrm{D.LE}'$ obtained by adding any or more of these rules to the calculus $\mathrm{D.LE}$ for the basic $\mathcal{L}_{\mathrm{LG}}$-logic has semantic cut elimination. Moreover, in each of these rules, the complexity of sequents does not increase from bottom to top. Hence by Proposition 10, FMP holds for any $\mathrm{D.LE}'$. This captures the decidability result of [15].

2.7.3 Orthologic

The language of *Orthologic* (cf. [11]), denoted $\mathcal{L}_{\mathrm{Ortho}}$, is obtained by instantiating $\mathcal{F} := \{\neg\}$ with $n_\neg = 1, \varepsilon_\neg = (\partial)$ and $\mathcal{G} := \{\neg, 0\}$ with $n_0 = 0$, $n_\neg = 1$, $\varepsilon_\neg = (\partial)$.

Orthologic is the axiomatic extension of the basic $\mathcal{L}_{\mathrm{Ortho}}$-logic with the following sequents (cf. [11, Definition 1.1]):

$$
p \wedge \neg p \vdash 0 \quad 0 \vdash p \quad p \vdash \neg\neg p \quad \neg\neg p \vdash p.
$$

These axioms are analytic inductive, and hence, by the procedure outlined in [13], they can be transformed into analytic structural rules:

$$
p \wedge \neg p \vdash 0 \qquad \rightsquigarrow \qquad \frac{x \Rightarrow *x}{x \Rightarrow \mathbb{I}}
$$

$$0 \vdash p \quad \rightsquigarrow \quad \frac{x \Rightarrow \mathbb{I}}{x \Rightarrow y}$$

$$p \vdash \neg\neg p \quad \neg\neg p \vdash p \quad \rightsquigarrow \quad \frac{*x \Rightarrow y}{*y \Rightarrow x} \quad \frac{x \Rightarrow *y}{y \Rightarrow *x} \quad \frac{x \Rightarrow y}{*y \Rightarrow *x} \quad \frac{** x \Rightarrow y}{x \Rightarrow y}$$

where the relation between structural and logical connectives in $\mathcal{L}_{\mathrm{Ortho}}$ is reported in the following table:

Structural symbols	$\mathbb{I}$		$*$	
Operational symbols		0	$\sim$	$\sim$

Let $\mathrm{D.LE}$ be the calculus for the basic $\mathcal{L}_{\mathrm{Ortho}}$-logic, and let $\mathrm{D.LE}'$ be the calculus obtained by adding the rules above to $\mathrm{D.LE}$. Theorem 5 directly applies to $\mathrm{D.LE}'$. In what follows we will show that Proposition 11 can be applied to $\mathrm{D.LE}'$, by defining $\Phi'_{\mathcal{F}}$ (resp. $\Phi'_{\mathcal{G}}$) as follows

$$\Phi'_{\mathcal{F}} := \{(x, *^{(2n)}x), (*^{(2m)}x, x) : n, m \in \mathbb{N} \text{ and } x \in \mathsf{Str}_{\mathcal{F}}\},$$

$$\Phi'_{\mathcal{G}} := \{(y, *^{(2n)}y), (*^{(2m)}y, y) : n, m \in \mathbb{N} \text{ and } y \in \mathsf{Str}_{\mathcal{G}}\}$$

Clearly, $\Phi_{\mathcal{F}}$ and $\Phi_{\mathcal{G}}$ are congruences. The applicability of Proposition 11 is an immediate consequence of the following.

Lemma 10. *1.* $\Phi'_{\mathcal{F}} \subseteq \Phi_{\mathcal{F}}$.

2. For every sequent $x \Rightarrow y$ *the set* $\{[x']_{\Phi'_{\mathcal{F}}} \mid x' \in (x \Rightarrow y)^{\leftarrow}_{\mathcal{F}}\}$ *is finite.*

Proof. 1. By m consecutive applications of the rule

$$\frac{** x \Rightarrow y}{x \Rightarrow y}$$

we obtain the derivability of the following rule

$$\frac{*^{(2m)} x \Rightarrow y}{x \Rightarrow y}$$

Likewise, by n consecutive applications of the following sequence of rules

$$\cfrac{\cfrac{x \Rightarrow y}{*y \Rightarrow *x}}{** x \Rightarrow y}$$

we obtain the derivability of the following rule

$$\frac{x \Rightarrow y}{*^{(2n)} x \Rightarrow y}$$

2. Fix a sequent $x \Rightarrow y$. It is enough to show that if $z \in (x \Rightarrow y)^{\leftarrow}_{\mathcal{F}} \cup (x \Rightarrow y)^{\leftarrow}_{\mathcal{G}}$ then $(z, z') \in \Phi'_{\mathcal{F}} \cup \Phi'_{\mathcal{G}}$ for some structure z' belonging to the following finite set:

$$\Sigma := \mathsf{Sub}(x) \cup \mathsf{sub}(x) \cup \mathsf{Sub}(y) \cup \mathsf{sub}(y) \cup *(\mathsf{Sub}(x) \cup \mathsf{sub}(x) \cup \mathsf{Sub}(y) \cup \mathsf{sub}(y)) \cup \{\mathbb{I}, *\mathbb{I}\},$$

where $\mathsf{Sub}(s)$ is the set of substructures of s, $\mathsf{sub}(s)$ is the set of subformulas of formulas in $\mathsf{Sub}(s)$ and $*A = \{*s \mid s \in A\}$ for any set of structures A. We proceed by induction on the inverse proof-trees. The base case, i.e. $z \in \{x, y\}$, is clear. As to the inductive step, the proof proceeds by inspection on the rules. The cases regarding applications of introduction rules or structural rules of $\mathrm{D.LE}$ which reduce the complexity of sequents when applied bottom-up are straightforward and omitted. Let $w \Rightarrow u \in (x \Rightarrow y)^{\leftarrow}$ and assume that $(w, w'), (u, u') \in \Phi'_{\mathcal{F}} \cup \Phi'_{\mathcal{G}}$ for some $w', u' \in \Sigma$. Then, the bottom-up application of one of the following rules

$$\frac{** x \Rightarrow y}{x \Rightarrow y} \qquad \frac{x \Rightarrow * x}{x \Rightarrow \mathbb{I}} \qquad \frac{x \Rightarrow \mathbb{I}}{x \Rightarrow y}$$

to $w \Rightarrow u$ yields $** w \Rightarrow u$, $w \Rightarrow *w$ and $w \Rightarrow \mathbb{I}$ respectively. Hence, $(** w, w), (w, w') \in \Phi'_{\mathcal{F}}$ and therefore $(** w, w') \in \Phi'_{\mathcal{F}}$, $(*w, *w') \in \Phi'_{\mathcal{G}}$ and $*w' \in \Sigma$, and $\mathbb{I} \in \Sigma$. □

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Chapter 3

Multi-type Display Calculi for semi De Morgan Logic and its extensions

In the present chapter, which is a revised version of the paper [1] [14], we introduce proper multi-type display calculi for semi De Morgan logic and its extensions which are sound, complete, conservative, and enjoy cut elimination and subformula property. Our proposal builds on an algebraic analysis of semi De Morgan algebras and its subvarieties and applies the guidelines of the multi-type methodology in the design of display calculi.

[1] My specific contributions to this research have been the proof of results, the definition of notions and constructions, and the writing of the first draft of the paper.

3.1 Introduction

Semi De Morgan logic, introduced in an algebraic setting by H.P. Sankappanavar [22], is a very well known paraconsistent logic [21], and is designed to capture the salient features of intuitionistic negation in a paraconsistent setting. Semi De Morgan algebras form a variety of normal distributive lattice expansions (cf. [15, Definition 9]), and are a common abstraction of De Morgan algebras and distributive pseudocomplemented lattices. Besides being studied from a universal-algebraic perspective [2, 3, 22], semi De Morgan logic has been studied from a duality-theoretic perspective [18] and from the perspective of canonical extensions [19].

From a proof-theoretic perspective, the main challenge posed by semi De Morgan logic is that, unlike De Morgan logic, its axiomatization is not analytic inductive in the sense of [15, Definition 55]. In [14], an analytic calculus for semi De Morgan logic is introduced which is sound, complete, conservative, and enjoys cut elimination and subformula property. The design of this calculus builds on an algebraic analysis of semi De Morgan algebras, and applies the guidelines of the multi-type methodology, introduced in [5, 7] and further developed in [1, 6, 8, 16, 17]. This methodology guarantees in particular that all the properties mentioned above follow from the general background theory of proper multi-type display calculi (cf. [17, Definition A.1.]).

Due to space constraints, in [14], the proofs of the algebraic analysis on which the design of this calculus is grounded had to be omitted. The present chapter provides the missing proofs, and also extends the results of [14] by explicitly and modularly accounting for the logics associated with the five subvarieties of semi De Morgan algebras introduced in [22]. This modular account is partly made possible by the fact that all but two of these subvarieties correspond to axiomatic extensions of semi De Morgan logic with so-called *analytic inductive* axioms (cf. [15, Definition 55]), and the two remaining ones can be given analytic equivalent presentations in the multi-type setting for the basic calculus. The general theory of proper (multi-type) display calculi provides an algorithm which computes the analytic structural rules corresponding to these axioms, and guarantees that each calculus obtained by adding any subset of these rules to the basic calculus still enjoys cut elimination and subformula property.

Therefore, this chapter introduces a proof-theoretic environment which is suitable to complement, from a proof-theoretic perspective, the investigations on the lattice of axiomatic extensions of semi De Morgan logic, as well as on the connections between the lattices of axiomatic extensions of semi De Morgan logic and of De Morgan logic.

Structure of the chapter. In Section 3.2, we report on the axioms and rules of semi De Morgan logic and its axiomatic extensions arising from the subvarieties of semi De Morgan algebras introduced in [22], and discuss why the basic axiomatization is not amenable to the standard treatment of display calculi. In Section 3.3, we define the algebraic environment which motivates our multi-type approach and prove that this environment is an equivalent presentation of the standard algebraic semantics of semi De Morgan logic and its extensions. Then we introduce the multi-type semantic environment and define translations between the single-type and the multi-type languages of semi De Morgan logic and its extensions. In Section 4.4, we discuss how equivalent analytic (multi-type) reformulations can be given of non-analytic (single-type) axioms in the language of semi De Morgan logic. In Section 3.5, we introduce the display calculi for semi De Morgan logic and its extensions, and in Section 5.6, we discuss their soundness, completeness, conservativity, cut elimination and subformula property.

3.2 Preliminaries

3.2.1 Semi De Morgan logic and its axiomatic extensions

3

Fix a denumerable set Atprop of propositional variables, let p denote an element in Atprop. The language $\mathcal{L}$ of semi De Morgan logic over Atprop is defined recursively as follows:

$$A ::= p \mid \top \mid \bot \mid \neg A \mid A \wedge A \mid A \vee A$$

Definition 18. *Semi De Morgan logic, denoted* SM*, consists of the following axioms:*

$$\bot \vdash A, \quad A \vdash \top, \quad \neg\top \vdash \bot, \quad \top \vdash \neg\bot, \quad A \vdash A, \quad A \wedge B \vdash A, \quad A \wedge B \vdash B,$$

$$A \vdash A \vee B, \quad B \vdash A \vee B, \quad \neg A \vdash \neg\neg\neg A, \quad \neg\neg\neg A \vdash \neg A, \quad \neg A \wedge \neg B \vdash \neg(A \vee B),$$

$$\neg\neg A \wedge \neg\neg B \vdash \neg\neg(A \wedge B), \quad A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)$$

and the following rules:

$$\frac{A \vdash B \qquad B \vdash C}{A \vdash C} \qquad \frac{A \vdash B \qquad A \vdash C}{A \vdash B \wedge C} \qquad \frac{A \vdash B \qquad C \vdash B}{A \vee C \vdash B} \qquad \frac{A \vdash B}{\neg B \vdash \neg A}$$

The following table reports the name of each axiomatic extension of SM *arising from the subvarieties introduced in [22], its acronym, and its characterizing axiom:*

lower quasi De Morgan logic	LQM	$A \vdash \neg\neg A$
upper quasi De Morgan logic	UQM	$\neg\neg A \vdash A$
demi pseudo-complemented lattice logic	DP	$\neg A \wedge \neg\neg A \vdash \bot$
almost pseudo-complemented lattice logic	AP	$A \wedge \neg A \vdash \bot$
weak Stone logic	WS	$\top \vdash \neg A \vee \neg\neg A$

In [15], a characterization is given of the properly displayable (single-type) logics (i.e. those logics that can be captured by a proper display calculus, cf. [23, Chapter 4]). Properly displayable logics are exactly those logics which admit a presentation consisting of analytic inductive axioms (cf. [15, Definition 55]). It is not difficult to verify that $\neg A \vdash \neg\neg\neg A$, $\neg\neg\neg A \vdash \neg A$ and $\neg\neg A \wedge \neg\neg B \vdash \neg\neg(A \wedge B)$ in SM, $\neg\neg A \vdash A$ in UQM, and $\neg A \wedge \neg\neg A \vdash \bot$ in DP are not analytic inductive. To our knowledge, no equivalent axiomatizations have been introduced for semi De Morgan logic and its extensions using only analytic inductive axioms. This provides the motivation for circumventing this difficulty by introducing proper multi-type display calculi for semi De Morgan logic and its extensions.

3.2.2 The variety of semi De Morgan algebras and its subvarieties

We recall the definition of the variety of semi De Morgan algebras and those of its subvarieties introduced in [22, Definition 2.2, Definition 2.6].

Definition 19. *An algebra* $\mathbb{A} = (L, \wedge, \vee, ', \top, \bot)$ *is a* semi De Morgan algebra *(SMA) if for all* $a, b \in L$,

(S1) $(L, \wedge, \vee, 1, 0)$ *is a bounded distributive lattice;*

(S2) $\bot' = \top, \top' = \bot$;

(S3) $(a \vee b)' = a' \wedge b'$;

(S4) $(a \wedge b)'' = a'' \wedge b''$;

(S5) $a' = a'''$.

A lower quasi De Morgan algebra *(LQMA) is an SMA satisfying the following inequality:*

(S6a) $a \leq a''$.

Dually, a upper quasi De Morgan algebra *(UQMA) is an SMA satisfying the following inequality:*

(S6b) $a'' \leq a$.

A demi pseudocomplemented lattice *(DPL) is an SMA satisfying the following equation:*

(S7) $a' \wedge a'' = \bot$.

A almost pseudocomplemented lattice *(APL) is an SMA satisfying the following equation:*

(S8) $a \wedge a' = \bot$.

A weak Stone algebra *(WSA) is an SMA satisfying the following equation:*

(S9) $a' \vee a'' = \top$.

The following proposition is a straightforward consequence of (S8), (S2), (S3) and (S5):

Proposition 12 ([22] see discussion above Corollary 2.7)**.** (S7) *holds in any APL and WSA.*

Definition 20. *An algebra* $\mathbb{D} = (D, \cap, \cup, ^*, 1, 0)$ *is a* De Morgan algebra *(DMA) if for all* $a, b \in D$,

(D1) $(D, \cap, \cup, 1, 0)$ *is a bounded distributive lattice;*

(D2) $0^* = 1, 1^* = 0$;

(D3) $(a \cup b)^* = a^* \cap b^*$;

(D4) $(a \cap b)^* = a^* \cup b^*$;

(D5) $a = a^{**}$.

As is well known, a Boolean algebra (BA) $\mathbb{D}$ is a DMA satisfying one of the following equations:

(B1) $a \vee a^* = 1$;

(B2) $a \wedge a^* = 0$.

The following theorem can be shown using a routine Lindenbaum-Tarski construction.

Theorem 7 (**Completeness**). SM *(resp.* LQM, UQM, DP, AP, WS*) is complete with respect to the class of SMAs (resp. LQMAs, UQMAs, DPLs, APLs, WSAs).*

Definition 21. *A distributive lattice $\mathbb{A}$ is* perfect *(cf. [12, Definition 2.14]) if $\mathbb{A}$ is complete, completely distributive and completely join-generated by the set $J^\infty(\mathbb{A})$ of its completely join-irreducible elements (as well as completely meet-generated by the set $M^\infty(\mathbb{A})$ of its completely meet-irreducible elements).*

A De Morgan algebra (resp. Boolean algebra) $\mathbb{A}$ is perfect *if its lattice reduct is a perfect distributive lattice, and the following distributive laws are valid:*

$$(\bigvee X)^* = \bigwedge X^* \qquad (\bigwedge X)^* = \bigvee X^*.$$

A lattice homomorphism $h : \mathbb{L} \to \mathbb{L}'$ is complete *if for each $X \subseteq \mathbb{L}$,*

$$h(\bigvee X) = \bigvee h(X) \qquad h(\bigwedge X) = \bigwedge h(X).$$

3.3 Towards a multi-type presentation

In the present section, we introduce the algebraic environment which justifies semantically the multi-type approach to semi De Morgan logic and its extensions of Section 3.2.1. In the next subsection, we define the kernel of an SMA (cf. Definition 22) and show that it can be endowed with a structure of DMA (cf. Definition 20). Similarly, we define the kernel of a DPL (cf. Definition 19) and show that it can be endowed with a structure of Boolean algebra. Then we define two maps between the kernel of any SMA (resp. DPL) $\mathbb{A}$ and the lattice reduct of $\mathbb{A}$. These are the main components of the heterogeneous semi De Morgan algebras and the heterogeneous demi p-lattices which we introduce in Subsection 4.3, where we also show that SMAs (resp. DPLs) can be equivalently presented in terms of heterogeneous semi De Morgan algebras (heterogeneous demi p-lattices). Based on these, we can also define the heterogeneous algebras for other subvariety of SMAs we introduced in Section 3.2.2. In Subsection 3.3.3, we apply results pertaining to the theory of canonical extensions to the heterogeneous semi De Morgan algebras and the heterogeneous demi p-lattices.

3.3.1 The kernel of a semi De Morgan algebra

For any semi De Morgan algebra $\mathbb{A} = (L, \wedge, \vee, ', \top, \bot)$, we let $K := \{a'' \mid a \in L\}$, define $h : L \twoheadrightarrow K$ by the assignment $a \mapsto a''$ for any $a \in L$, and let $e : K \hookrightarrow L$ denote the natural embedding. Hence, $eh(a) = a''$ and $h(a) = h(a'')$ for every $a \in L$.

Lemma 11. *For any semi De Morgan algebra $\mathbb{A}$, and K, h, e defined as above, the following equation holds for any $\alpha \in K$:*

$$he(\alpha) = \alpha \tag{3.1}$$

Proof. Let $\alpha \in K$, and let $a \in L$ such that $h(a) = \alpha$. Hence,

$$\begin{array}{rcll} he(\alpha) & = & heh(a) & \alpha = h(a) \\ & = & h(a'') & eh(a) = a'' \\ & = & h(a) & h(a) = h(a'') \\ & = & \alpha & \text{definition of } h \end{array}$$

□

Definition 22. *For any SMA $\mathbb{A} = (L, \wedge, \vee, \top, \bot, ')$, let the* kernel $\mathbb{K}_{\mathbb{A}} = (K, \cap, \cup, ^*, 1, 0)$ *of $\mathbb{A}$ be defined as follows:*

K1 $K := \{a'' \mid a \in L\}$;

K2 $\alpha \cup \beta := h((e(\alpha) \vee e(\beta))'')$ *for all* $\alpha, \beta \in K$;

K3 $\alpha \cap \beta := h(e(\alpha) \wedge e(\beta))$ *for all* $\alpha, \beta \in K$;

K4 $1 := h(\top)$;

K5 $0 := h(\bot)$;

K6 $\alpha^* := h(e(\alpha)')$.

In what follows, to simplify the notation, we omit as many parentheses as we can without generating ambiguous readings. For example, we write $e(h(a)^*)$ in place of $e((h(a))^*)$, and $eh(a)'$ in place of $(eh(a))'$.

Proposition 13. *If $\mathbb{A} = (L, \wedge, \vee, \top, \bot, ')$ is an SMA, then $\mathbb{K}_{\mathbb{A}}$ is a De Morgan algebra.*

Proof. Let us show that $\mathbb{K}_{\mathbb{A}}$ is a distributive lattice. Associativity and commutativity are straightforwardly verified and their corresponding verification is omitted. To show that the absorption law and the distributive law hold, let $\alpha, \beta, \gamma \in K$, and let $a, b, c \in L$ such that (i) $h(a) = \alpha$, (ii) $h(b) = \beta$ and (iii) $h(a) = \gamma$.

- absorption law:

$$\begin{array}{rll}
& \alpha \cup (\alpha \cap \beta) & \\
= & h((e(\alpha) \vee e(\alpha \cap \beta))'') & \text{K2} \\
= & h((e(\alpha) \vee eh(e(\alpha) \wedge e(\beta)))'') & \text{K3} \\
= & h((e(\alpha) \vee (e(\alpha) \wedge e(\beta))'')'') & eh(a) = a'' \\
= & h((e(\alpha)' \wedge (e(\alpha) \wedge e(\beta))''')') & \text{S3} \\
= & h((e(\alpha)''' \wedge (e(\alpha)'' \wedge e(\beta)'')')') & \text{S5, S4} \\
= & h((e(\alpha)'' \vee (e(\alpha)'' \wedge e(\beta)''))'') & \text{S3} \\
= & h((e(\alpha)'' \vee (e(\alpha)'''' \wedge e(\beta)''''))) & \text{S4} \\
= & h((e(\alpha)'' \vee (e(\alpha)'' \wedge e(\beta)''))) & \text{S5} \\
= & h(e(\alpha)'') & \text{S1} \\
= & he(\alpha) & h(a) = h(a'') \\
= & \alpha & \text{Lemma 11}
\end{array}$$

- distributivity law:

$$\begin{array}{rll}
& \alpha \cap (\beta \cup \gamma) & \\
= & h(e(\alpha) \wedge e(\beta \cup \gamma)) & \text{K3} \\
= & h(e(\alpha) \wedge eh((e(\beta) \vee e(\gamma))'')) & \text{K2} \\
= & h(e(\alpha) \wedge (e(\beta) \vee e(\gamma))'''') & eh(a) = a'' \\
= & h(eh(a) \wedge (e(\beta) \vee e(\gamma))'''') & \text{(i)} \\
= & h(a'' \wedge (e(\beta) \vee e(\gamma))'''') & eh(a) = a'' \\
= & h(a'''' \wedge (e(\beta) \vee e(\gamma))'') & \text{S5} \\
= & h((a'' \wedge (e(\beta) \vee e(\gamma)))'') & \text{S4} \\
= & h(((a'' \wedge e(\beta)) \vee (a'' \wedge e(\gamma)))'') & \text{S1} \\
= & h((a'' \wedge eh(b)) \vee (a'' \wedge eh(c))'') & \text{(ii) and (iii)} \\
= & h(((a'' \wedge b'') \vee (a'' \wedge c''))'') & eh(a) = a'' \\
= & h(((a'''' \wedge b'''') \vee (a'''' \wedge c''''))'') & \text{S5} \\
= & h(((a'' \wedge b'')'' \vee (a'' \wedge c'')'')'') & \text{S4} \\
= & h((eh(eh(a) \wedge eh(b)) \vee eh(eh(a) \wedge eh(c)))'') & eh(a) = a'' \\
= & h((eh(e(\alpha) \wedge e(\beta)) \vee eh(e(\alpha) \wedge e(\gamma)))'') & \text{(i), (ii) and (iii)} \\
= & h(((e(\alpha \cap \beta)) \vee e(\alpha \cap \gamma))'') & \text{K3} \\
= & (\alpha \cap \beta) \cup (\alpha \cap \gamma) & \text{K2}
\end{array}$$

Let us show that $\mathbb{K}_{\mathbb{A}}$ satisfies (D1)-(D5).
As to (D1), we need to show that $\mathbb{K}_{\mathbb{A}}$ is bounded:

$$\begin{array}{rll}
& 0 \cap \alpha & \\
= & h(e(0) \wedge e(\alpha)) & \text{K3} \\
= & h(eh(\bot) \wedge e(\alpha)) & \text{K5} \\
= & h(\bot'' \wedge e(\alpha)) & eh(a) = a'' \\
= & h(\bot \wedge e(\alpha)) & \text{S2} \\
= & h(\bot) & \text{S1} \\
= & 0 & \text{K5}
\end{array}
\qquad
\begin{array}{rll}
& 1 \cup \alpha & \\
= & h((e(1) \vee e(\alpha))'') & \text{K2} \\
= & h((eh(\top)) \vee e(\alpha))'') & \text{K4} \\
= & h((\top'' \vee e(\alpha))'') & eh(a) = a'' \\
= & h((\top \vee e(\alpha))'') & \text{S2} \\
= & h(\top'') & \text{S5} \\
= & 1 & \text{S2, K4}
\end{array}$$

As to (D2):

$$
\begin{array}{rcll@{\qquad}rcll}
0^* & = & h(e(0)') & \mathsf{K6} & 1^* & = & h(e(1)') & \mathsf{K6} \\
& = & h((eh(\bot))') & \mathsf{K5} & & = & h((eh(\top))') & \mathsf{K4} \\
& = & h(\bot''') & eh(a) = a'' & & = & h(\top''') & eh(a) = a'' \\
& = & h(\bot') & \mathsf{S5} & & = & h(\top') & \mathsf{S5} \\
& = & h(\top) & \mathsf{S2} & & = & h(\bot) & \mathsf{S2} \\
& = & 1 & \mathsf{K4} & & = & 0 & \mathsf{K4}
\end{array}
$$

As to (D3):

$$
\begin{array}{rcll}
(\alpha \cup \beta)^* & = & h(e(\alpha \cup \beta)') & \mathsf{K6} \\
& = & h((eh((e(\alpha) \vee e(\beta))'')') & \mathsf{K2} \\
& = & h((e(\alpha) \vee e(\beta))''''') & eh(a) = a'' \\
& = & h((e(\alpha)' \wedge e(\beta)')'''') & \mathsf{S3} \\
& = & h((e(\alpha)''' \wedge e(\beta)''')'') & \mathsf{S5} \\
& = & h((eh(e(\alpha)') \wedge eh(e(\beta)'))'') & eh(a) = a'' \\
& = & h((e(\alpha^*) \wedge e(\beta^*))'') & \mathsf{K6} \\
& = & heh(e(\alpha^*) \wedge e(\beta^*)) & eh(a) = a'' \\
& = & h(e(\alpha^*) \wedge e(\beta^*)) & \text{Lemma 11} \\
& = & \alpha^* \cap \beta^* & \mathsf{K3}
\end{array}
$$

As to (D4):

$$
\begin{array}{rcll}
(\alpha \cap \beta)^* & = & h(e(\alpha \cap \beta)') & \mathsf{K6} \\
& = & h((eh(e(\alpha) \wedge e(\beta)))') & \mathsf{K3} \\
& = & h((e(\alpha) \wedge e(\beta))''') & eh(a) = a'' \\
& = & h((e(\alpha)'' \wedge e(\beta)'')') & \mathsf{S4} \\
& = & h((e(\alpha)' \vee e(\beta)')'') & \mathsf{S3} \\
& = & h((eh(a)' \vee eh(b)')'') & \text{(i) and (ii)} \\
& = & h((a''' \vee b''')'') & eh(a) = a'' \\
& = & h((a''''' \vee b''''')'') & \mathsf{S5} \\
& = & h((eh(eh(a)') \vee eh(eh(b)'))'') & eh(a) = a'' \\
& = & h((eh(e(\alpha)') \vee eh(e(\beta)'))'') & \text{(i) and (ii)} \\
& = & h((e(\alpha^*) \vee e(\beta^*))'') & \mathsf{K6} \\
& = & \alpha^* \cup \beta^* & \mathsf{K2}
\end{array}
$$

As to (D5):

$$
\begin{array}{rcll}
\alpha^{**} & = & h((eh(e(\alpha)'))') & \mathsf{K6} \\
& = & h(e(\alpha)'''') & eh(a) = a'' \\
& = & h((eh(a))'''') & \text{(i)} \\
& = & h(a'''''') & eh(a) = a'' \\
& = & h(a'') & \mathsf{S5} \\
& = & heh(a) & eh(a) = a'' \\
& = & h(a) & \text{Lemma 11} \\
& = & \alpha & \text{(i)}
\end{array}
$$

□

Corollary 2. *If $\mathbb{A} = (L, \wedge, \vee, \top, \bot, ')$ is a DPL, then $\mathbb{K}_{\mathbb{A}}$ is a Boolean algebra.*

Proof. By Proposition 13, $\mathbb{K}_{\mathbb{A}}$ is a De Morgan algebra. Hence, it suffices to show that $\mathbb{K}$ satisfies (B1). For any $\alpha \in \mathbb{K}_{\mathbb{A}}$,

$$
\begin{array}{rcll}
\alpha \cap \alpha^* & = & h(\alpha \cap h(e(\alpha)')) & \text{K3} \\
& = & h(e(\alpha) \wedge eh(e(\alpha)')) & \text{K6} \\
& = & h(e(\alpha) \wedge e(\alpha)''') & eh(a) = a'' \\
& = & h(e(\alpha) \wedge e(\alpha)') & \text{S5} \\
& = & heh(e(\alpha) \wedge e(\alpha)') & \text{Lemma 11} \\
& = & h((e(\alpha) \wedge e(\alpha)')'') & eh(a) = a'' \\
& = & h(e(\alpha)'' \wedge e(\alpha)''') & \text{S4} \\
& = & h(\bot) & \text{S7} \\
& = & 0 & \text{K5}
\end{array}
$$

□

Proposition 14. *Let $\mathbb{A}$ be an SMA (resp. a DPL), and e, h be defined as above. Then h is a lattice homomorphism from $\mathbb{A}$ onto $\mathbb{K}_{\mathbb{A}}$, and for all $\alpha, \beta \in K$,*

$$e(\alpha) \wedge e(\beta) = e(\alpha \cap \beta) \qquad e(1) = \top \qquad e(0) = \bot.$$

Proof. It is an immediate consequence of K1 that h is surjective. We need to show that h is a lattice homomorphism. For any $a, b \in L$,

$$
\begin{array}{rll}
& h(a \wedge b) & \\
= & heh(a \wedge b) & \text{Lemma 11} \\
= & h((a \wedge b)'') & eh(a) = a'' \\
= & h(a'' \wedge b'') & \text{S4} \\
= & h(eh(a) \wedge eh(b)) & eh(a) = a'' \\
= & h(a) \cap h(b) & \text{K3}
\end{array}
\qquad
\begin{array}{rll}
& h(a \vee b) & \\
= & heh(a \vee b) & \text{Lemma 11} \\
= & h((a \vee b)'') & eh(a) = a'' \\
= & h((a' \wedge b')') & \text{S3} \\
= & h(a''' \wedge b''')' & \text{S5} \\
= & h(a'' \vee b'')'' & \text{S3} \\
= & h((eh(a) \vee eh(b))'') & eh(a) = a'' \\
= & h(a) \cup h(b) & \text{K2}
\end{array}
$$

Moreover, $h(\bot) = \bot'' = \bot$ and $h(\top) = \top'' = \top$. This completes the proof that h is a homomorphism from $\mathbb{A}$ to $\mathbb{K}_{\mathbb{A}}$. Next, we show that $e(\alpha) \wedge e(\beta) = e(\alpha \cap \beta)$. For any $\alpha, \beta \in K$,

$$
\begin{array}{rcll}
e(\alpha \cap \beta) & = & eh(e(\alpha) \wedge e(\beta)) & \text{K3} \\
& = & (e(\alpha) \wedge e(\beta))'' & eh(a) = a'' \\
& = & e(\alpha)'' \wedge e(\beta)'' & \text{S4} \\
& = & ehe(\alpha) \wedge ehe(\beta) & eh(a) = a'' \\
& = & e(\alpha) \wedge e(\beta) & \text{Lemma 11}
\end{array}
$$

Finally, $e(0) = eh(\bot) = \bot'' = \bot$ and $e(1) = eh(\top) = \top'' = \top$ are straightforward consequences of (K4), (K5) and (S2). □

In what follows, we will drop the subscript of the kernel whenever it does not cause confusion.

3.3.2 Heterogeneous SMAs as equivalent presentations of SMAs

Definition 23. *A* heterogeneous semi De Morgan algebra *(HSMA) is a tuple* $(\mathbb{L}, \mathbb{D}, e, h)$ *satisfying the following conditions:*

(H1) $\mathbb{L}$ *is a bounded distributive lattice;*

(H2a) $\mathbb{D}$ *is a De Morgan algebra;*

(H3) $e : \mathbb{D} \hookrightarrow \mathbb{L}$ *is an order embedding, and for all* $\alpha_1, \alpha_2 \in \mathbb{D}$,

- $e(\alpha_1) \wedge e(\alpha_2) = e(\alpha_1 \cap \alpha_2)$;
- $e(1) = \top$, $e(0) = \bot$.

(H4) $h : \mathbb{L} \twoheadrightarrow \mathbb{D}$ *is a surjective lattice homomorphism;*

(H5) $he(\alpha) = \alpha$ *for every* $\alpha \in \mathbb{D}$.[2]

A heterogeneous lower quasi De Morgan algebra *(HLQMA) is an HSMA satisfying the following condition:*

(H6a) $a \leq eh(a)$ *for any* $a \in L$.

A heterogeneous upper quasi De Morgan algebra *(HUQMA) is an HSMA satisfying the following condition:*

(H6b) $eh(a) \leq a$ *for any* $a \in L$.

A heterogeneous demi pseudocomplemented lattice *(HDPL) is defined analogously, except replacing (H2a) with the following condition (H2b):*

(H2b) $\mathbb{D}$ *is a Boolean algebra.*

A heterogeneous almost pseudocomplemented lattice *(HAPL) is an HDPL satisfying the following condition:*

(H7) $e(h(a)^*) \wedge a = \bot$ *for all* $a \in \mathbb{L}$.

A heterogeneous weak Stone algebra *(HWSA) is an HDPL satisfying the following condition:*

(H8) $e(\alpha^*) \vee e(\alpha) = \top$ *for all* $\alpha \in \mathbb{A}$.

$$\begin{tikzcd} \mathbb{L} \arrow[r, two heads, "h"'] & \mathbb{D} \arrow[l, bend right, "e"'] \arrow[loop right, "*"] \end{tikzcd}$$

An HSMA (resp. HLQMA,HUQMA, HDPL, HAPL and HWSA) is perfect *if:*

[2]Condition (H5) implies that h is surjective and e is injective.

(PH1) *both $\mathbb{L}$ and $\mathbb{D}$ are perfect as a distributive lattice and De Morgan algebra (or Boolean algebra), respectively (see Definition 21);*

(PH2) *e is an order-embedding and is completely meet-preserving;*

(PH3) *h is a complete homomorphism.*

Definition 24. *For any SMA (resp. LQMA, UQMA, DPL, APL and WSA) $\mathbb{A}$, let*

$$\mathbb{A}^+ := (\mathbb{L}, \mathbb{K}, e, h),$$

where $\mathbb{L}$ is the lattice reduct of $\mathbb{A}$, $\mathbb{K}$ is the kernel of $\mathbb{A}$ (cf. Definition 22), and $e : \mathbb{K} \to \mathbb{L}$ and $h : \mathbb{L} \to \mathbb{K}$ are defined as in the beginning of Section 3.3.1.

Proposition 15. *If $\mathbb{A}$ is an SMA (resp. DPL), then $\mathbb{A}^+$ is an HSMA (resp. HDPL).*

Proof. It immediately follows from Proposition 13 and Proposition 14. □

Corollary 3. *If $\mathbb{A}$ is an LQMA (resp. UQMA, APL and WSA), then $\mathbb{A}^+$ is an HLQMA (resp. HUQMA, HAPL and HWSA).*

Proof. If $\mathbb{A}$ is an LQMA, by Proposition 22, it suffices to show that $\mathbb{A}_+$ satisfies (H6a). By (S6a) and H5, it is easy to see $a \leq e(h(a))$. The argument is dual when $\mathbb{A}$ is a UQMA. If $\mathbb{A}$ is an APL, it suffices to show $\mathbb{A}^+$ satisfies (H7).

$$\begin{array}{lll} & e(h(a)^*) \wedge a & \\ = & eh((eh(a))') \wedge a & \text{K6} \\ = & a''''' \wedge a & eh(a) = a'' \\ = & a' \wedge a & \text{S5} \\ = & \bot & \text{S8} \end{array}$$

If $\mathbb{A}$ is a WSA, it suffices to show $\mathbb{A}^+$ satisfies (H8).

$$\begin{array}{lll} & e(\alpha^*) \vee e(\alpha) & \\ = & eh(e(\alpha)') \vee e(\alpha) & \text{K6} \\ = & eh(e(\alpha)') \vee ehe(\alpha) & \text{Lemma 11} \\ = & e(\alpha)''' \vee ehe(\alpha) & eh(a) = a'' \\ = & e(\alpha)''' \vee e(\alpha)'' & \text{Lemma 11} \\ = & e(\alpha)' \vee e(\alpha)'' & \text{S5} \\ = & \top & \text{S9} \end{array}$$

□

Definition 25. *For any HSMA (resp. HLQMA, HUQMA, HDPL, HAPL and HWSA) $\mathbb{H} = (\mathbb{L}, \mathbb{D}, e, h)$, let*

$$\mathbb{H}_+ := (\mathbb{L}, '),$$

where $' : \mathbb{L} \to \mathbb{L}$ is defined by the assignment $a' \mapsto e(h(a)^*)$.

Proposition 16. *If $\mathbb{H}$ is an HSMA (resp. HDPL), then $\mathbb{H}_+$ is an SMA (resp. DPL). Moreover, $\mathbb{K}_{\mathbb{H}^+} \cong \mathbb{K}$.*

Proof. Since $\mathbb{H}$ is an HSMA by assumption, $\mathbb{L}$ is a bounded distributive lattice, hence it suffices to show that the operation $'$ satisfies (S2)-(S5) (cf. Definition 19).

- As to (S2):

$$
\begin{array}{llll}
\bot' & = & e(h(\bot)^*) & \text{definition of } ' \\
& = & e(0^*) & \text{H3} \\
& = & e(1) & \text{H2a} \\
& = & \top & \text{H3}
\end{array}
\qquad
\begin{array}{llll}
\top' & = & e(h(\top)^*) & \text{definition of } ' \\
& = & e(1^*) & \text{H3} \\
& = & e(0) & \text{H2a} \\
& = & \bot & \text{H3}
\end{array}
$$

- As to (S3):

$$
\begin{array}{llll}
(a \vee b)' & = & e(h(a \vee b)^*) & \text{definition of } ' \\
& = & e((h(a) \cup h(b))^*) & \text{H4} \\
& = & e(h(a)^* \cap h(b)^*) & \text{H2a} \\
& = & e(h(a)^*) \wedge e(h(b)^*) & \text{H3} \\
& = & a' \wedge b' & \text{definition of } '
\end{array}
$$

- As to (S4):

$$
\begin{array}{llll}
(a \wedge b)'' & = & e((he(h(a \wedge b)^*))^*) & \text{definition of } ' \\
& = & e(h(a \wedge b)^{**}) & \text{H5} \\
& = & eh(a \wedge b) & \text{H2a} \\
& = & e(h(a) \cap h(b)) & \text{H4} \\
& = & eh(a) \wedge eh(b) & \text{H3} \\
& = & e(h(a)^{**}) \wedge e(h(b)^{**}) & \text{H2a} \\
& = & e((he(h(a)^*))^*) \wedge e((he(h(b)^*))^*) & \text{H5} \\
& = & a'' \wedge b'' & \text{definition of } '
\end{array}
$$

- As to (S5):

$$
\begin{array}{llll}
a''' & = & e((he((he(h(a)^*))^*))^*) & \text{definition of } ' \\
& = & e(h(a)^{***}) & \text{H5} \\
& = & e(h(a)^*) & \text{H2a} \\
& = & a' & \text{definition of } '
\end{array}
$$

Hence, $(\mathbb{L}, ')$ is a semi De Morgan algebra. If $(\mathbb{L}, \mathbb{D}, e, h)$ is an HDPL, we also need to show that $'$ satisfies (S7):

$$
\begin{array}{llll}
a' \wedge a'' & = & e(h(a)^*) \wedge e((he(h(a)^*))^*) & \text{definition of } ' \\
& = & e(h(a)^*) \wedge e(h(a)^{**}) & \text{H5} \\
& = & e(h(a)^*) \wedge eh(a) & \text{H2a} \\
& = & e(h(a)^* \cap h(a)) & \text{H3} \\
& = & e(0) & \text{H2a} \\
& = & \bot & \text{H3}
\end{array}
$$

which completes the proof that (L, $'$) is a DPL. As to the second part of the statement, let us show preliminarily that the following identities hold:

K2$_{\mathbb{D}}$. $\alpha \cup \beta = h((e(\alpha) \vee e(\beta))'')$ for all $\alpha, \beta \in \mathbb{D}$;

K3$_{\mathbb{D}}$. $\alpha \cap \beta = h(e(\alpha) \wedge e(\beta))$ for all $\alpha, \beta \in \mathbb{D}$;

K4$_{\mathbb{D}}$. $1 = h(\top)$;

K5$_{\mathbb{D}}$. $0 = h(\bot)$;

K6$_{\mathbb{D}}$. $\alpha^* = h(e(\alpha)')$.

As to K2$_{\mathbb{D}}$,

$$\begin{array}{rcll} h((e(\alpha) \vee e(\beta))'') & = & he((he(h(e(\alpha) \vee e(\beta))^*))^*) & \text{definition of } ' \\ & = & (h(e(\alpha) \vee e(\beta)))^{**} & \text{H5} \\ & = & h(e(\alpha) \vee e(\beta)) & \text{H2a} \\ & = & he(\alpha) \cup he(\beta) & \text{H4} \\ & = & \alpha \cup \beta & \text{H5} \end{array}$$

Conditions K3$_{\mathbb{D}}$,K4$_{\mathbb{D}}$ and K5$_{\mathbb{D}}$ easily follow from H4, H5 and H3, and their proofs are omitted.

As to K6$_{\mathbb{D}}$,

$$\begin{array}{rcll} h(e(\alpha)') & = & he((he(\alpha))^*) & \text{definition of } ' \\ & = & \alpha^* & \text{H5} \end{array}$$

To show that $\mathbb{D}$ and $\mathbb{K}$ are isomorphic to each other, notice that the domain of $\mathbb{K}$ is defined as $K :=$ Range($''$) = Range($e \circ^* \circ h \circ e \circ^* \circ h$) = Range($e \circ h$). Since by assumption h is surjective, $K =$ Range(e), and since e is an order embedding, K, regarded as a sub-poset of $\mathbb{L}$, is order-isomorphic to the domain of $\mathbb{D}$ with its lattice order. Let $f : \mathbb{D} \to \mathbb{K}$ denote the order-isomorphism between $\mathbb{D}$ and $\mathbb{K}$. Define $e_k : \mathbb{K} \hookrightarrow \mathbb{L}$ and $h_k : \mathbb{L} \twoheadrightarrow \mathbb{K}$ as as in the beginning of Section 3.3.1. Thus, $e = e_k \circ f$ and $h_k = f \circ h$. We need to show that: for all $\alpha, \beta \in \mathbb{D}$, let $\cap_k, \cup_k, {}^{*_k}$ denote the operations on K,

1. $f(\alpha) \cap_k f(\beta) = f(\alpha \cap \beta)$,

$$\begin{array}{rcll} f(\alpha) \cap_k f(\beta) & = & h_k(e_k f(\alpha) \wedge e_k f(\beta)) & \text{definition of } \cap_k \\ & = & f h(e_k f(\alpha) \wedge e_k f(\beta)) & h_k = f \circ h \\ & = & f h(e(\alpha) \wedge e(\beta)) & e = e_k \circ f \\ & = & f(\alpha \cap \beta) & \mathrm{K3}_{\mathbb{D}} \end{array}$$

2. $f(\alpha) \cup_k f(\beta) = f(\alpha \cap \beta)$,

$$\begin{array}{rcll} f(\alpha) \cup_k f(\beta) & = & h_k((e_k f(\alpha) \vee e_k f(\beta))'') & \text{definition of } \cup_k \\ & = & f h((e_k f(\alpha) \vee e_k f(\beta))'') & h_k = f \circ h \\ & = & f h((e(\alpha) \vee e(\beta))'') & e = e_k \circ f \\ & = & f(\alpha \cup \beta) & \mathrm{K2}_{\mathbb{D}} \end{array}$$

3. $f(\alpha)^{*_k} = f(\alpha^*)$,

$$\begin{array}{rcll}
(f(\alpha))^{*_k} & = & h_k((e_k f(\alpha))') & \text{definition of } {}^{*_k} \\
& = & fh((e_k f(\alpha))') & h_k = f \circ h \\
& = & fh(e(\alpha)') & e = e_k \circ f \\
& = & f(\alpha^*) & \mathrm{K6}_{\mathbb{D}}
\end{array}$$

Hence, $f : \mathbb{D} \longrightarrow \mathbb{K}$ is an isomorphism of De Morgan algebras (resp. Boolean algebras). This completes the proof. □

Corollary 4. *If $\mathbb{H}$ is an HLQMA (resp. HUQMA, HAPL and HWSA), then $\mathbb{A}^+$ is a LQMA (resp. UQMA, APL and WSA).*

Proof. By Proposition 23, if $\mathbb{H}$ is an HLQMA, it suffices to show that $\mathbb{H}_+$ satisfies (S6a).

$$\begin{array}{lll}
& a \leq eh(a) & \text{H6a} \\
\text{iff} & a \leq e(h(a)^{**}) & \text{H2a} \\
\text{iff} & a \leq e((he(h(a)^*))^*) & \text{H5} \\
\text{iff} & a \leq a'' & \text{definition of } '
\end{array}$$

If $\mathbb{H}$ is an HUQMA, the argument is dual. If $\mathbb{H}$ is an HAPL, it is clear that $\mathbb{H}_+$ satisfies (S8) by (H7). If $\mathbb{H}$ is an HWSA, it suffices to show that $\mathbb{H}_+$ satisfies (S6).

$$\begin{array}{rll}
& a' \vee a'' & \\
= & e(h(a)^*) \vee e((he(h(a)^*))^*) & \text{def. of } ' \\
= & e(h(a)^*) \vee e(h(a)^{**}) & \text{Lemma 11} \\
= & e(h(a)^*) \vee eh(a) & \text{H2a} \\
= & \top & \text{H8}
\end{array}$$

□

Proposition 17. *For any SMA (resp. LQMA, UQMA, DPL, APL, and WSA) $\mathbb{A}$ and any HSMA (resp. HLQMA, HUQMA, HDPL, HAPL, and HWSA) $\mathbb{H}$:*

$$\mathbb{A} \cong (\mathbb{A}^+)_+ \quad \textit{and} \quad \mathbb{H} \cong (\mathbb{H}_+)^+.$$

These correspondences restrict appropriately to the relevant classes of perfect algebras and perfect heterogeneous algebras.

Proof. It immediately follows from Proposition 22, Corollary 3, Proposition 23 and Corollary 4. □

3.3.3 Canonical extensions of heterogeneous algebras

Canonicity in the multi-type environment is used both to provide complete semantics for a large class of axiomatic extensions of the basic logic (semi De Morgan logic in the present case), and to prove the conservativity of its associated display calculus (cf. Section 3.6.3). In the present section, we define the canonical extension $\mathbb{H}^\delta$ of any heterogeneous algebra $\mathbb{H}$ introduced in Section 4.3 by instantiating the general definition discussed in [17]. This makes it possible to define the canonical extension of any SMA $\mathbb{A}$ as a perfect SMA $(\mathbb{A}^+)^\delta_+$. We then show that this definition coincides with the definition given in [19, Section 4]. In what follows, we let $\mathbb{L}^\delta$ and $\mathbb{A}^\delta$ denote the canonical extensions of the distributive lattice $\mathbb{L}$ and of the De Morgan algebra (resp. Boolean algebra) $\mathbb{D}$ respectively, and e^π and h^δ denote the π-extensions of e and h[3], respectively. We refer to [9] for the relevant definitions.

Proposition 18. *If* $(\mathbb{L}, \mathbb{D}, e, h)$ *is an HSMA (resp. HDPL, HLQMA, HUQMA, HAPL, and HWSA), then* $(\mathbb{L}^\delta, \mathbb{D}^\delta, e^\pi, h^\delta)$ *is a perfect HSMA (resp. perfect HDPL, perfect HLQMA, perfect HUQMA, perfect HAPL and perfect HWSA).*

$$\begin{tikzcd}[row sep=huge, column sep=huge]
\mathbb{L}^\delta \arrow[rr, bend left=80, "e_\ell"] \arrow[rr, two heads, "h^\delta"] & & \mathbb{D}^\delta \arrow[ll, bend right=50, hook', "e^\pi"'] \arrow[ll, bend right=30, hook', "h_\ell"'] \arrow[ll, bend left=40, hook, "h_r"] \arrow[loop right, "*^\delta"] \\
\mathbb{L} \arrow[u, hook] \arrow[rr, two heads, "h"] & & \mathbb{D} \arrow[u, hook] \arrow[ll, bend right, hook', "e"'] \arrow[loop right, "*"]
\end{tikzcd}$$

Proof. Firstly, $\mathbb{L}^\delta$ and $\mathbb{D}^\delta$ are a perfect distributive lattice and a perfect De Morgan algebra (resp. perfect Boolean algebra) respectively. Secondly, since h is a surjective homomorphism, h is both finitely meet-preserving and finitely join-preserving. Hence, as is well known, h^δ is surjective, and completely meet- (join-) preserving [11, Theorem 3.7]. Since h is also smooth, this shows that $h^\delta = h^\pi = h^\sigma$ is a complete homomorphism. Thirdly, since e is finitely meet-preserving, e^π is completely meet-preserving, and it immediately follows from the definition of π-extension that e^π is an order-embedding [11, Corollary 2.25]. The identity $e^\pi(0) = 0$ clearly holds, since $\mathbb{A}$ is a subalgebra of $\mathbb{A}^\delta$. Moreover, $Id_{\mathbb{D}} = h \circ e$ is canonical by [10, Proposition 14]. This is enough to show that if $(\mathbb{L}, \mathbb{D}, e, h)$ is a SMA (resp. DPL), then $(\mathbb{L}^\delta, \mathbb{D}^\delta, e^\pi, h^\delta)$ is a perfect HSMA (resp. perfect HDPL).

Since (H6a), (H6b), (H7) and (H8) are analytic inductive (cf. Definition 30), they are canonical. So their corresponding heterogeneous algebras are perfect. □

In the environment of perfect heterogeneous algebras, completely join (resp. meet) preserving maps have right (resp. left) adjoints. These adjoints guarantee the soundness of all display rules in the display calculi introduced in the next section.

[3] The order-theoretic properties of h guarantee that the σ-extension and the π-extension of h coincide. This is why we use h^δ to denote the resulting extension.

In [19], C. Palma studied the canonical extensions of semi De Morgan algebras using insights from the Sahlqvist theory of Distributive Modal Logic. She recognized that not all inequalities in the axiomatization of SMA are Sahlqvist, and circumvented this problem by introducing the following term-equivalent presentation of SMAs.

Definition 26 ([19], Definition 4.1.2)**.** *For any SMA $\mathbb{A} = (L, \wedge, \vee, ', \top, \bot)$, let $\mathbb{S}_{\mathbb{A}} = (L, \wedge, \vee, \rhd, \Box, \top, \bot)$ be such that $\Box$ and $\rhd$ are unary operations respectively defined by the assignments $a \mapsto a''$ and $a \mapsto a'$.*

Palma showed that the algebras corresponding to SMAs via the construction above are exactly those $\{\Box, \rhd\}$-reducts of Distributive Modal Algebras satisfying the following additional axioms:

1. $\rhd \top \leq \bot$;
2. $\Box a \leq \rhd\rhd a$;
3. $\rhd\rhd a \leq \Box a$;
4. $\Box \rhd a \leq \rhd a$;
5. $\rhd a \leq \Box \rhd a$.

The axioms above can be straightforwardly verified to be Sahlqvist and hence canonical. This enables Palma to define the canonical extension $\mathbb{A}^\delta$ of $\mathbb{A}$ as the $\{\rhd\}$-reduct of $\mathbb{S}^\sigma_{\mathbb{A}} = (\mathbb{L}^\sigma, \rhd^\pi, \Box^\pi)$. The following lemma immediately implies that $\mathbb{A}^\delta$ coincides with $(\mathbb{A}^{+\delta})_+$.

Lemma 12. *For any SMA $\mathbb{A}$, letting $\mathbb{S}_{\mathbb{A}}$ be defined as above,*

1. $\Box^\pi = e^\pi \circ h^\delta$;

2. $\rhd^\pi = e^\pi \circ *^\delta \circ h^\delta$.

Proof. By the definitions of $\Box, \rhd$, e and h (cf. beginning of Section 3.1),

$$\begin{aligned} \Box^\pi &= ('')^\pi && \text{definition of } \Box \\ &= (e \circ h)^\pi && \text{definitions of } e \text{ and } h \\ &= e^\pi \circ h^\pi && \text{[11, Lemma 3.3, Corollary 2.25]} \\ &= e^\pi \circ h^\delta && h \text{ is smooth} \end{aligned}$$

$$\begin{aligned} \rhd^\pi &= (')^\pi && \text{definition of } \rhd \\ &= (e \circ * \circ h)^\pi && \text{definition of } ' \\ &= e^\pi \circ *^\pi \circ h^\pi && \text{[11, Lemma 3.3, Corollary 2.25]} \\ &= e^\pi \circ *^\delta \circ h^\delta && * \text{ and } h \text{ are smooth} \end{aligned}$$

$\Box$

3.4 Multi-type presentation of semi De Morgan logic and its extensions

In Section 4.3 we showed that heterogeneous semi De Morgan algebras are equivalent presentations of semi De Morgan algebras. This provides a semantic motivation for introducing the multi-type language $\mathcal{L}_{\mathrm{MT}}$, which is naturally interpreted on heterogeneous semi De Morgan algebras. The language $\mathcal{L}_{\mathrm{MT}}$ consists of terms of types DL and K, defined as follows:

$$\begin{array}{rcl} \mathsf{DL} \ni A & ::= & p \mid \Box\alpha \mid \top \mid \bot \mid A \wedge A \mid A \vee A \\ \mathsf{K} \ni \alpha & ::= & \circ A \mid 1 \mid 0 \mid \sim \alpha \mid \alpha \cup \alpha \mid \alpha \cap \alpha \end{array}$$

The interpretation of $\mathcal{L}_{\mathrm{MT}}$-terms into heterogeneous algebras is defined as the easy generalization of the interpretation of propositional languages in universal algebra; namely, the heterogeneous operation e interprets the connective $\Box$, the heterogeneous operation h interprets the connective $\circ$, and DL-terms (resp. K-terms) are interpreted in the first (resp. second) component of heterogeneous algebras.

The toggle between single-type algebras and their heterogeneous counterparts (cf. Sections 4.3) is reflected syntactically by the translations $(\cdot)^\tau : \mathcal{L} \to \mathcal{L}_{\mathrm{MT}}$ defined as follows:

$$\begin{array}{rcl} p^\tau & ::= & p \\ \top^\tau & ::= & \top \\ \bot^\tau & ::= & \bot \\ (A \wedge B)^\tau & ::= & A^\tau \wedge B^\tau \\ (A \vee B)^\tau & ::= & A^\tau \vee B^\tau \\ (\neg A)^\tau & ::= & \Box \sim \circ A^\tau \end{array}$$

Recall that $\mathbb{A}^+$ denotes the heterogeneous algebra associated with the single-type algebra $\mathbb{A}$ (cf. Definition 35). The following proposition is proved by a routine induction on $\mathcal{L}$-formulas.

Proposition 19. *For all $\mathcal{L}$-formulas A and B and any $\mathbb{A} \in \{SMA, LQMA, UQMA, DPL, APL, WSA\}$,*

$$\mathbb{A} \models A \vdash B \quad \textit{iff} \quad \mathbb{A}^+ \models A^\tau \vdash B^\tau.$$

We are now in a position to translate the characteristic axioms of every logic mentioned in Section 3.2.1 into $\mathcal{L}_{\mathrm{MT}}$. Together with Proposition 17, the proposition above guarantees that the translation of each of the axioms below is valid on the corresponding class of heterogeneous algebras.

$$\neg\neg A \wedge \neg\neg B \vdash \neg\neg(A \wedge B) \rightsquigarrow \quad \Box{\sim}\circ\Box{\sim}\circ(A \wedge B)^\tau \vdash \Box{\sim}\circ\Box{\sim}\circ A^\tau \wedge \Box{\sim}\circ\Box{\sim}\circ B^\tau \qquad (i)$$

$$\neg A \vdash \neg\neg\neg A \rightsquigarrow \quad \Box{\sim}\circ A^\tau \vdash \Box{\sim}\circ\Box{\sim}\circ\Box{\sim}\circ A^\tau \qquad (ii)$$

$$\neg\neg\neg A \vdash \neg A \rightsquigarrow \quad \Box{\sim}\circ\Box{\sim}\circ\Box{\sim}\circ A^\tau \vdash \Box{\sim}\circ A^\tau \qquad (iii)$$

$$\neg A \wedge \neg B \vdash \neg(A \vee B) \rightsquigarrow \quad \Box{\sim}\circ(A \vee B)^\tau \vdash \Box{\sim}\circ A^\tau \wedge \Box{\sim}\circ B^\tau \qquad (iv)$$

$$\top \vdash \neg\bot \rightsquigarrow \quad \top \vdash \Box{\sim}\circ\bot \qquad (v)$$

$$\neg\top \vdash \bot \rightsquigarrow \quad \Box{\sim}\circ\top \vdash \bot \qquad (vi)$$

$$A \vdash \neg\neg A \rightsquigarrow \quad A^\tau \vdash \Box{\sim}\circ\Box{\sim}\circ A^\tau \qquad (vii)$$

$$\neg\neg A \vdash A \rightsquigarrow \quad \Box{\sim}\circ\Box{\sim}\circ A^\tau \vdash A^\tau \qquad (viii)$$

$$\neg A \wedge \neg\neg A \vdash \bot \rightsquigarrow \quad \Box{\sim}\circ A^\tau \wedge \Box{\sim}\circ\Box{\sim}\circ A^\tau \vdash \bot \qquad (ix)$$

$$A \wedge \neg A \vdash \bot \rightsquigarrow \quad A^\tau \wedge \Box{\sim}\circ A^\tau \vdash \bot \qquad (x)$$

$$\top \vdash \neg A \vee \neg\neg A \rightsquigarrow \quad \top \vdash \Box{\sim}\circ A^\tau \vee \Box{\sim}\circ\Box{\sim}\circ A^\tau \qquad (xi)$$

Notice that the defining identities of heterogeneous algebras (cf. Definition 23) can be expressed as *analytic inductive* $\mathcal{L}_{\mathrm{MT}}$-inequalities (cf. Definition 30). Hence, these inequalities can be used to generate the analytic rules of the calculus introduced in Section 3.5, with a methodology analogous to the one introduced in [15]. As we will discuss in Section 4.6.2, the inequalities (i)-(xi) are derivable in the calculus obtained in this way.

3.5 Proper Display Calculi for semi De Morgan logic and its extensions

In the present section, we introduce proper multi-type display calculi for semi De Morgan logic and its extensions. The language manipulated by these calculi has types DL and K, and is built up from structural and operational (aka logical) connectives. In the tables of Section 3.5.1, each structural connective corresponding to a logical connective which belongs to the family $\mathcal{F}$ (resp. $\mathcal{G}$, $\mathcal{H}$) defined in Section 3.8 is denoted by decorating that logical connective with ˆ (resp. ˇ, ˜).[4]

[4]For any sequent $x \vdash y$, we define the signed generation trees $+x$ and $-y$ by labelling the root of the generation tree of x (resp. y) with the sign $+$ (resp. $-$), and then propagating the sign to all nodes according to the polarity of the coordinate of the connective assigned to each node. Positive (resp. negative) coordinates propagate the same (resp. opposite) sign to the corresponding child node. Then, a substructure z in $x \vdash y$ is in *precedent* (resp. *succedent*) *position* if the sign of its root node as a subtree of $+x$ or $-y$ is $+$ (resp. $-$).

3.5.1 Language

Structural and operational terms.

$$\mathsf{DL}\begin{cases} A ::= p \mid \top \mid \bot \mid \Box\alpha \mid A \wedge A \mid A \vee A \\ X ::= A \mid \hat{\top} \mid \check{\bot} \mid \check{\Box}\Gamma \mid \hat{\bullet}_\ell \Gamma \mid \check{\bullet}_r \Gamma \mid X \mathbin{\hat{\wedge}} X \mid X \mathbin{\check{\vee}} X \mid X \mathbin{\hat{\succ\!\!-}} X \mid X \mathbin{\check{\rightarrow}} X \end{cases}$$

$$\mathsf{K}\begin{cases} \alpha ::= 1 \mid 0 \mid \circ A \mid {\sim}\,\alpha \mid \alpha \cap \alpha \mid \alpha \cup \alpha \\ \Gamma ::= \alpha \mid \hat{1} \mid \check{0} \mid \tilde{\circ} X \mid \hat{\blacklozenge} X \mid \tilde{*}\Gamma \mid \Gamma \mathbin{\hat{\cap}} \Gamma \mid \Gamma \mathbin{\check{\cup}} \Gamma \mid \Gamma \mathbin{\hat{\supset\!\!-}} \Gamma \mid \Gamma \mathbin{\check{-\!\!\supset}} \Gamma \end{cases}$$

Interpretation of pure-type structural connectives as their logical counterparts[5]:

DL						K							
$\hat{\top}$	$\hat{\wedge}$	$\hat{\succ\!\!-}$	$\check{\bot}$	$\check{\vee}$	$\check{\rightarrow}$	$\hat{1}$	$\hat{\cap}$	$\hat{\supset\!\!-}$	$\check{0}$	$\check{\cup}$	$\check{-\!\!\supset}$	$\tilde{*}$	
$\top$	$\wedge$	$(\succ\!\!-)$	$\bot$	$\vee$	$(\rightarrow)$	1	$\cap$	$(\supset\!\!-)$	0	$\cup$	$(-\!\!\supset)$	$\sim$	$\sim$

Interpretation of heterogeneous structural connectives as their logical counterparts:

DL → K		K → DL	K → DL	K → DL	DL → K
$\tilde{\circ}$		$\hat{\bullet}_\ell$	$\check{\bullet}_r$	$\check{\Box}$	$\hat{\blacklozenge}$
$\circ$	$\circ$	$(\bullet_\ell)$	$(\bullet_r)$	$\Box$	$(\blacklozenge)$

Algebraic interpretation of heterogeneous structural connectives as operations in perfect HSM-algebras (see Lemma 13).

DL → K	K → DL		K → DL	DL → K
$\tilde{\circ}$	$\hat{\bullet}_\ell$	$\check{\bullet}_r$	$\check{\Box}$	$\hat{\blacklozenge}$
h	h_ℓ	h_r	e	e_ℓ

3.5.2 Multi-type display calculi for semi De Morgan logic and its extensions

In what follows, structures of type DL are denoted by the variables X, Y, Z, and W; structures of type A are denoted by the variables Γ, Δ, Θ and Π.

- The proper display calculus for semi De Morgan logic D.SM consists of the following rules:

  - Identity and cut rules

$$\mathsf{Id}\,\frac{}{p \vdash p} \qquad \frac{X \vdash A \qquad A \vdash Y}{X \vdash Y}\,\mathrm{Cut}_{\mathsf{L}} \qquad \frac{\Gamma \vdash \alpha \qquad \alpha \vdash \Delta}{\Gamma \vdash \Delta}\,\mathrm{Cut}_{\mathsf{D}}$$

[5]In the synoptic table, the operational symbols which occur only at the structural level will appear between round brackets.

- Pure DL-type display rules

$$\mathrm{res_L}\ \frac{X \hat{\wedge} Y \vdash Z}{Y \vdash X \check{\rightarrow} Z} \qquad \frac{X \vdash Y \check{\vee} Z}{Y \hat{\succ} X \vdash Z}\ \mathrm{res_L}$$

- Pure K-type display rules

$$\mathrm{res_D}\ \frac{\Gamma \hat{\cap} \Delta \vdash \Theta}{\Delta \vdash \Gamma \check{-\!\!\supset} \Theta} \qquad \frac{\Gamma \vdash \Delta \check{\cup} \Theta}{\Delta \hat{\supset\!\!-} \Gamma \vdash \Theta}\ \mathrm{res_D}$$

$$\mathrm{adj}_*\ \frac{\tilde{*}\Gamma \vdash \Delta}{\tilde{*}\Delta \vdash \Gamma} \qquad \frac{\Gamma \vdash \tilde{*}\Delta}{\Delta \vdash \tilde{*}\Gamma}\ \mathrm{adj}_*$$

- Multi-type display rules

$$\mathrm{adj_{LD}}\ \frac{X \vdash \check{\Box}\Gamma}{\hat{\blacklozenge} X \vdash \Gamma} \qquad \mathrm{adj_{DL}}\ \frac{\tilde{\circ} X \vdash \Gamma}{X \vdash \check{\bullet}_r \Gamma} \qquad \frac{\Gamma \vdash \tilde{\circ} X}{\hat{\bullet}_\ell \Gamma \vdash X}\ \mathrm{adj_{DL}}$$

- Pure DL-type structural rules

$$\hat{\top}\ \frac{X \vdash Y}{X \hat{\wedge} \hat{\top} \vdash Y} \qquad \frac{X \vdash Y}{X \vdash Y \check{\vee} \check{\bot}}\ \check{\bot}$$

$$\mathrm{E_L}\ \frac{X \hat{\wedge} Y \vdash Z}{Y \hat{\wedge} X \vdash Z} \qquad \frac{X \vdash Y \check{\vee} Z}{X \vdash Z \check{\vee} Y}\ \mathrm{E_L}$$

$$\mathrm{A_L}\ \frac{(X \hat{\wedge} Y) \hat{\wedge} Z \vdash W}{X \hat{\wedge} (Y \hat{\wedge} Z) \vdash Z} \qquad \frac{X \vdash (Y \check{\vee} Z) \check{\vee} W}{X \vdash Y \check{\vee} (Z \check{\vee} W)}\ \mathrm{A_L}$$

$$\mathrm{W_L}\ \frac{X \vdash Y}{X \hat{\wedge} Z \vdash Y} \qquad \frac{X \vdash Y}{X \vdash Y \check{\vee} Z}\ \mathrm{W_L}$$

$$\mathrm{C_L}\ \frac{X \hat{\wedge} X \vdash Y}{X \vdash Y} \qquad \frac{X \vdash Y \check{\vee} Y}{X \vdash Y}\ \mathrm{C_L}$$

- Pure K-type structural rules

$$\hat{1}\ \frac{\Gamma \vdash \Delta}{\Gamma \hat{\cap} \hat{1} \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta \check{\cup} \check{0}}\ \check{0}$$

$$\mathrm{E_D}\ \frac{\Gamma \hat{\cap} \Delta \vdash \Theta}{\Delta \hat{\cap} \Gamma \vdash \Theta} \qquad \frac{\Gamma \vdash \Delta \check{\cup} \Theta}{\Gamma \vdash \Theta \check{\cup} \Delta}\ \mathrm{E_D}$$

$$\mathrm{A_D}\ \frac{(\Gamma \hat{\cap} \Delta) \hat{\cap} \Theta \vdash \Pi}{\Gamma \hat{\cap} (\Delta \hat{\cap} \Theta) \vdash \Theta} \qquad \frac{\Gamma \vdash (\Delta \check{\cup} \Theta) \check{\cup} \Pi}{\Gamma \vdash \Delta \check{\cup} (\Theta \check{\cup} \Pi)}\ \mathrm{A_D}$$

$$\mathrm{W_D}\ \frac{\Gamma \vdash \Delta}{\Gamma \hat{\cap} \Theta \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta \check{\cup} \Theta}\ \mathrm{W_D}$$

$$\mathrm{C_D}\ \frac{\Gamma \hat{\cap} \Gamma \vdash \Delta}{\Gamma \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta \check{\cup} \Delta}{\Gamma \vdash \Delta}\ \mathrm{C_D}$$

$$\frac{\Gamma \vdash \Delta}{\tilde{*}\Delta \vdash \tilde{*}\Gamma}\ \text{cont}$$

- Multi-type structural rules

$$\tilde{\circ}\ \frac{X \vdash Y}{\tilde{\circ} X \vdash \tilde{\circ} Y} \qquad \frac{\hat{\bullet}_\ell \Gamma \vdash \check{\bullet}_r \Delta}{\Gamma \vdash \Delta}\ \tilde{\bullet}$$

$$\hat{\blacklozenge}\hat{1}\ \frac{\hat{1} \vdash \Gamma}{\hat{\blacklozenge}\hat{\top} \vdash \Gamma} \qquad \frac{X \vdash \check{\Box}\check{0}}{X \vdash \check{\bot}}\ \check{\Box}\check{0}$$

$$\genfrac{}{}{1.5pt}{}{\Gamma \vdash \tilde{\circ}\check{\Box}\Delta}{\Gamma \vdash \Delta}\ \tilde{\circ}\check{\Box}$$

- Pure DL-type operational rules

$$\top\ \frac{\hat{\top} \vdash X}{\top \vdash X} \qquad \frac{}{\hat{\top} \vdash \top}\ \top$$

$$\bot\ \frac{}{\bot \vdash \check{\bot}} \qquad \frac{X \vdash \check{\bot}}{X \vdash \bot}\ \bot$$

$$\wedge\ \frac{A \mathbin{\hat{\wedge}} B \vdash X}{A \wedge B \vdash X} \qquad \frac{X \vdash A \qquad Y \vdash B}{X \mathbin{\hat{\wedge}} Y \vdash A \wedge B}\ \wedge$$

$$\vee\ \frac{A \vdash X \qquad B \vdash Y}{A \vee B \vdash X \mathbin{\check{\vee}} Y} \qquad \frac{X \vdash A \mathbin{\check{\vee}} B}{X \vdash A \vee B}\ \vee$$

- Pure K-type operational rules

$$1\ \frac{\hat{1} \vdash \Gamma}{1 \vdash \Gamma} \qquad \frac{}{\hat{1} \vdash 1}\ 1$$

$$0\ \frac{}{0 \vdash \check{0}} \qquad \frac{\Gamma \vdash \check{0}}{\Gamma \vdash 0}\ 0$$

$$\cap\ \frac{\alpha \mathbin{\hat{\cap}} \beta \vdash \Gamma}{\alpha \cap \beta \vdash \Gamma} \qquad \frac{\Gamma \vdash \alpha \qquad \Delta \vdash \beta}{\Gamma \mathbin{\hat{\cap}} \Delta \vdash \alpha \cap \beta}\ \cap$$

$$\cup\ \frac{\alpha \vdash \Gamma \qquad \beta \vdash \Delta}{\alpha \cup \beta \vdash \Gamma \mathbin{\check{\cup}} \Delta} \qquad \frac{\Gamma \vdash \alpha \mathbin{\check{\cup}} \beta}{\Gamma \vdash \alpha \cup \beta}\ \cup$$

$$\sim\ \frac{\tilde{*}\alpha \vdash \Gamma}{\sim \alpha \vdash \Gamma} \qquad \frac{\Gamma \vdash \tilde{*}\alpha}{\Gamma \vdash \sim \alpha}\ \sim$$

- Multi-type operational rules

$$\circ\ \frac{\tilde{\circ} A \vdash \Gamma}{\circ A \vdash \Gamma} \qquad \frac{X \vdash \tilde{\circ} A}{X \vdash \circ A}\ \circ$$

$$\Box\ \frac{\alpha \vdash \Gamma}{\Box \alpha \vdash \check{\Box}\Gamma} \qquad \frac{X \vdash \check{\Box}\alpha}{X \vdash \Box \alpha}\ \Box$$

- The proper display calculus D.LQM for lower quasi De Morgan logic consists of all axiom and rules in D.SM plus the following rule:

$$\frac{X \vdash Y}{X \vdash \check{\Box}\tilde{\circ} Y}\ \mathsf{LQM}$$

- The proper display calculus D.UQM for upper quasi De Morgan logic consists of all axiom and rules in D.SM plus the following rule:

$$\frac{\hat{\bullet}_\ell\ \hat{\blacklozenge} X \vdash Y}{X \vdash Y}\ \mathrm{UQM}$$

- The proper display calculus D.DP for demi pseudocomplemented lattice logic consists of all axiom and rules in D.SM plus the following rule:

$$\mathrm{res_B}\ \frac{\Gamma \hat{\cap} \Delta \vdash \Sigma}{\Delta \vdash \tilde{*}\Gamma \check{\cup} \Sigma}$$

- The proper display calculus D.AP for almost pseudocomplemented lattice logic consists of all axiom and rules in D.DP plus the following rule:

$$\frac{X \vdash \check{\Box}\tilde{*}\tilde{\circ} Y}{X \hat{\wedge} Y \vdash \check{\bot}}\ \mathrm{AP}$$

- The proper display calculus D.WS for weak stone logic consists of all axiom and rules in D.DP plus the following rule:

$$\frac{\hat{\blacklozenge} X \vdash \Delta}{\hat{\blacklozenge}(\check{\Box}\tilde{*}\tilde{\circ} X \hat{\succ\!\!-} \hat{\top}) \vdash \Delta}\ \mathrm{WS}$$

3.6 Properties

3.6.1 Soundness

In the present subsection, we outline the verification of the soundness of the rules of D.SM (resp. D.LQM, D.UQM D.DP, D.AP and D.WS) w.r.t. the semantics of *perfect* HSMAs (resp. HQMAs, HDPLs, HAPLs and HWSAs, see Definition 23). The first step consists in interpreting structural symbols as logical symbols according to their (precedent or succedent) position, as indicated at the beginning of Section 3.5. This makes it possible to interpret sequents as inequalities, and rules as quasi-inequalities. For example, the rules on the left-hand side below are interpreted as the quasi-inequalities on the right-hand side:

$$\frac{X \vdash Y}{\tilde{\circ} X \vdash \tilde{\circ} Y} \quad \rightsquigarrow \quad \forall a \forall b[a \leq b \Rightarrow h(a) \leq h(b)]$$

$$\frac{\hat{\blacklozenge} X \vdash \Delta}{\hat{\blacklozenge}(\check{\Box}\tilde{*}\tilde{\circ} X \hat{\succ\!\!-} \hat{\top}) \vdash \Delta} \quad \rightsquigarrow \quad \forall a[e_\ell[e(h(a)^*) \succ\!\!- \top] \leq e_\ell(a)]$$

The proof of the soundness of the rules in these display calculi then consists in verifying the validity of their corresponding quasi-inequalities in the corresponding class of perfect heterogeneous algebras. The verification of the soundness of pure-type rules and of the introduction rules following this procedure is routine, and is omitted. The validity of the quasi-inequalities corresponding to multi-type structural rules follows straightforwardly from the observation that the quasi-inequality corresponding to each rule is obtained by running the algorithm ALBA (cf. Section 3.4 [15]) on some of the defining inequalities of its corresponding heterogeneous algebras.[6] For instance, the soundness of the characteristic rule of D.WS on HWSAs follows from the validity of the inequality (xi) in every HWSA (discussed in Section 4.4) and from the soundness of the following ALBA reduction in every HWSA:

$$
\begin{array}{lll}
 & \forall a[\top \leq e(h(a)^*) \vee e((he(h(a)^*))^*)] & \\
\text{iff} & \forall a \forall b \forall c[b \leq a \,\&\, c \leq e(h(a)^*) \Rightarrow \top \leq e(h(b)^*) \vee e(h(c)^*)] & \\
\text{iff} & \forall a \forall b \forall c[b \leq a \,\&\, a \leq h_r(e_\ell(c)^*) \Rightarrow \top \leq e(h(b)^*) \vee e(h(c)^*)] & \\
\text{iff} & \forall b \forall c[b \leq h_r(e_\ell(c)^*) \Rightarrow \top \leq e(h(b)^*) \vee e(h(c)^*)] & \\
\text{iff} & \forall b \forall c[b \leq h_r(e_\ell(c)^*) \Rightarrow e(h(c)^*) \succ\!\!\!- \top \leq e(h(b)^*)] & \\
\text{iff} & \forall b \forall c[b \leq h_r(e_\ell(c)^*) \Rightarrow b \leq h_r(e_\ell[e(h(c)^*) \succ\!\!\!- \top]^*)] & \\
\text{iff} & \forall c[h_r(e_\ell(c)^*) \leq h_r(e_\ell[e(h(c)^*) \succ\!\!\!- \top]^*)] & \\
\text{iff} & \forall c[e_\ell(c)^* \leq e_\ell[e(h(c)^*) \succ\!\!\!- \top]^*] & h_r \text{ is injective} \\
\text{iff} & \forall c[e_\ell[e(h(c)^*) \succ\!\!\!- \top] \leq e_\ell(c)] & {}^* \text{ is injective}
\end{array}
$$

3.6.2 Completeness

In the present subsection, we show that the translations of the axioms and rules of SM, LQM, UQM, DP, AP and WS are derivable in D.SM, D.LQM, D.UQM, D.DP, D.AP and D.WS, respectively. Then, the completeness of these display calculi w.r.t. the classes of SMAs, LQMAs, UQMAs, DPLs, APLs and WSAs immediately follows from the completeness of SM, LQM, UQM, DP, AP and WS (cf. Theorem 7).

Proposition 20. *For every $A \in \mathcal{L}$, the sequent $A^\tau \vdash A^\tau$ is derivable in all display calculi introduced in Section 3.5.2.*

Proof. By inducution on $A \in \mathcal{L}$. The proof of base cases: $A := \top$, $A := \bot$ and $A := p$, are straightforward and are omitted.

Inductive cases:

- as to $A := \neg B$,

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{ind.hyp.}
\UnaryInfC{$B^\tau \vdash B^\tau$}
\LeftLabel{$\tilde{\circ}$}
\UnaryInfC{$\tilde{\circ}\, B^\tau \vdash \tilde{\circ}\, B^\tau$}
\UnaryInfC{$\circ\, B^\tau \vdash \tilde{\circ}\, B^\tau$}
\UnaryInfC{$\circ\, B^\tau \vdash \circ\, B^\tau$}
\RightLabel{cont}
\UnaryInfC{$\tilde{*} \circ B^\tau \vdash \tilde{*} \circ B^\tau$}
\UnaryInfC{$\tilde{*} \circ B^\tau \vdash \sim \circ B^\tau$}
\UnaryInfC{$\sim \circ B^\tau \vdash \sim \circ B^\tau$}
\UnaryInfC{$\Box \sim \circ B^\tau \vdash \check{\Box} \sim \circ B^\tau$}
\UnaryInfC{$\Box \sim \circ B^\tau \vdash \Box \sim \circ B^\tau$}
\end{prooftree}
$$

[6] Indeed, as discussed in [15], the soundness of the rewriting rules of ALBA only depends on the order-theoretic properties of the interpretation of the logical connectives and their adjoints and residuals. The fact that some of these maps are not internal operations but have different domains and codomains does not make any substantial difference.

3

- as to $A := B \vee C$,

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{ind.hyp.}
\UnaryInfC{$B^\tau \vdash B^\tau$}
\RightLabel{W}
\UnaryInfC{$B^\tau \vdash B^\tau \check{\vee} C^\tau$}
\AxiomC{}
\LeftLabel{ind.hyp.}
\UnaryInfC{$C^\tau \vdash C^\tau$}
\RightLabel{W}
\UnaryInfC{$C^\tau \vdash C^\tau \check{\vee} B^\tau$}
\RightLabel{E}
\UnaryInfC{$C^\tau \vdash B^\tau \check{\vee} C^\tau$}
\BinaryInfC{$B^\tau \vee C^\tau \vdash (B^\tau \check{\vee} C^\tau) \check{\vee} (B^\tau \check{\vee} C^\tau)$}
\RightLabel{C}
\UnaryInfC{$B^\tau \vee C^\tau \vdash B^\tau \vee C^\tau$}
\end{prooftree}
$$

- as to $A := B \wedge C$,

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{ind.hyp.}
\UnaryInfC{$B^\tau \vdash B^\tau$}
\LeftLabel{W}
\UnaryInfC{$B^\tau \hat{\wedge} C^\tau \vdash B^\tau$}
\AxiomC{}
\LeftLabel{ind.hyp.}
\UnaryInfC{$C^\tau \vdash C^\tau$}
\LeftLabel{W}
\UnaryInfC{$C^\tau \hat{\wedge} B^\tau \vdash C^\tau$}
\LeftLabel{E}
\UnaryInfC{$B^\tau \hat{\wedge} C^\tau \vdash C^\tau$}
\BinaryInfC{$(B^\tau \hat{\wedge} C^\tau) \hat{\wedge} (B^\tau \hat{\wedge} C^\tau) \vdash B^\tau \wedge C^\tau$}
\LeftLabel{C}
\UnaryInfC{$B^\tau \wedge C^\tau \vdash B^\tau \wedge C^\tau$}
\end{prooftree}
$$

□

Proposition 21. *For every $A, B \in \mathcal{L}$, if $A \vdash B$ is derivable in any logic introduced in 3.2.1, then $A^\tau \vdash B^\tau$ is derivable in its respective display calculus.*

Proof. It is enough to show the statement of the proposition on the axioms. For the sake of readability, in what follows, we suppress the translation symbol $(\cdot)^\tau$. As to the axioms in SM:

- $\neg\top \vdash \bot \quad \rightsquigarrow \quad \Box \sim \circ \top \vdash \bot$,

$$
\begin{prooftree}
\AxiomC{$\hat{\top} \vdash \top$}
\LeftLabel{$\mathrm{W_L}$}
\UnaryInfC{$\hat{\bullet}_\ell \tilde{\ast} \check{0} \hat{\wedge} \hat{\top} \vdash \top$}
\RightLabel{$\hat{\top}$}
\UnaryInfC{$\hat{\bullet}_\ell \tilde{\ast} \check{0} \vdash \top$}
\UnaryInfC{$\tilde{\ast} \check{0} \vdash \tilde{\circ} \top$}
\UnaryInfC{$\tilde{\ast} \check{0} \vdash \circ \top$}
\UnaryInfC{$\tilde{\ast} \circ \top \vdash \check{0}$}
\UnaryInfC{$\sim \circ \top \vdash \check{0}$}
\UnaryInfC{$\Box \sim \circ \top \vdash \check{\Box} \check{0}$}
\RightLabel{$\check{\Box} \check{0}$}
\UnaryInfC{$\Box \sim \circ \top \vdash \check{\bot}$}
\UnaryInfC{$\Box \sim \circ \top \vdash \bot$}
\end{prooftree}
$$

- $\top \vdash \neg\bot \rightsquigarrow \quad \top \vdash \Box \sim \circ \top$,

$$
\begin{prooftree}
\AxiomC{$\bot \vdash \check{\bot}$}
\RightLabel{$\mathrm{W_L}$}
\UnaryInfC{$\bot \vdash \check{\bot} \check{\vee} \check{\bullet}_r \tilde{\ast} \hat{1}$}
\RightLabel{$\check{\bot}$}
\UnaryInfC{$\bot \vdash \check{\bullet}_r \tilde{\ast} \hat{1}$}
\UnaryInfC{$\tilde{\circ} \bot \vdash \tilde{\ast} \hat{1}$}
\UnaryInfC{$\circ \bot \vdash \tilde{\ast} \hat{1}$}
\UnaryInfC{$\hat{1} \vdash \tilde{\ast} \circ \bot$}
\UnaryInfC{$\hat{1} \vdash \sim \circ \bot$}
\LeftLabel{$\hat{\blacklozenge} \hat{\top}$}
\UnaryInfC{$\hat{\blacklozenge} \hat{\top} \vdash \sim \circ \bot$}
\UnaryInfC{$\hat{\top} \vdash \check{\Box} \sim \circ \bot$}
\UnaryInfC{$\hat{\top} \vdash \Box \sim \circ \bot$}
\UnaryInfC{$\top \vdash \Box \sim \circ \bot$}
\end{prooftree}
$$

- $\neg A \vdash \neg\neg\neg A \quad \rightsquigarrow \quad \Box \sim \circ A \vdash \Box \sim \circ \Box \sim \circ \Box \sim \circ A$
  and $\neg\neg\neg A \vdash \neg A \quad \rightsquigarrow \quad \Box \sim \circ \Box \sim \circ \Box \sim \circ A \vdash \Box \sim \circ A$,

$$
\begin{prooftree}
\AxiomC{$A \vdash A$}
\RightLabel{$\tilde{\circ}$}
\UnaryInfC{$\tilde{\circ} A \vdash \tilde{\circ} A$}
\UnaryInfC{$\circ A \vdash \tilde{\circ} A$}
\UnaryInfC{$\circ A \vdash \circ A$}
\RightLabel{cont}
\UnaryInfC{$\tilde{*} \circ A \vdash \tilde{*} \circ A$}
\UnaryInfC{$\tilde{*} \circ A \vdash \sim \circ A$}
\RightLabel{$\tilde{\circ}\check{\Box}$}
\UnaryInfC{$\tilde{*} \circ A \vdash \tilde{\circ} \check{\Box} \sim \circ A$}
\UnaryInfC{$\hat{\bullet}_\ell \tilde{*} \circ A \vdash \check{\Box} \sim \circ A$}
\UnaryInfC{$\hat{\bullet}_\ell \tilde{*} \circ A \vdash \Box \sim \circ A$}
\UnaryInfC{$\tilde{*} \circ A \vdash \tilde{\circ} \Box \sim \circ A$}
\UnaryInfC{$\tilde{*} \circ A \vdash \circ \Box \sim \circ A$}
\UnaryInfC{$\tilde{*} \circ \Box \sim \circ A \vdash \circ A$}
\UnaryInfC{$\sim \circ \Box \sim \circ A \vdash \circ A$}
\UnaryInfC{$\Box \sim \circ \Box \sim \circ A \vdash \check{\Box} \circ A$}
\RightLabel{$\tilde{\circ}$}
\UnaryInfC{$\tilde{\circ} \Box \sim \circ \Box \sim \circ A \vdash \tilde{\circ} \check{\Box} \circ A$}
\UnaryInfC{$\circ \Box \sim \circ \Box \sim \circ A \vdash \tilde{\circ} \check{\Box} \circ A$}
\RightLabel{$\tilde{\circ}\check{\Box}$}
\UnaryInfC{$\circ \Box \sim \circ \Box \sim \circ A \vdash \circ A$}
\RightLabel{cont}
\UnaryInfC{$\tilde{*} \circ A \vdash \tilde{*} \circ \Box \sim \circ \Box \sim \circ A$}
\UnaryInfC{$\tilde{*} \circ A \vdash \sim \circ \Box \sim \circ \Box \sim \circ A$}
\UnaryInfC{$\sim \circ A \vdash \sim \circ \Box \sim \circ \Box \sim \circ A$}
\UnaryInfC{$\Box \sim \circ A \vdash \check{\Box} \sim \circ \Box \sim \circ \Box \sim \circ A$}
\UnaryInfC{$\Box \sim \circ A \vdash \Box \sim \circ \Box \sim \circ \Box \sim \circ A$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{$A \vdash A$}
\RightLabel{$\tilde{\circ}$}
\UnaryInfC{$\tilde{\circ} A \vdash \tilde{\circ} A$}
\UnaryInfC{$\circ A \vdash \tilde{\circ} A$}
\UnaryInfC{$\circ A \vdash \circ A$}
\RightLabel{cont}
\UnaryInfC{$\tilde{*} \circ A \vdash \tilde{*} \circ A$}
\UnaryInfC{$\sim \circ A \vdash \tilde{*} \circ A$}
\UnaryInfC{$\Box \sim \circ A \vdash \check{\Box} \tilde{*} \circ A$}
\RightLabel{$\tilde{\circ}$}
\UnaryInfC{$\tilde{\circ} \Box \sim \circ A \vdash \tilde{\circ} \check{\Box} \tilde{*} \circ A$}
\UnaryInfC{$\circ \Box \sim \circ A \vdash \tilde{\circ} \check{\Box} \tilde{*} \circ A$}
\RightLabel{$\tilde{\circ}\check{\Box}$}
\UnaryInfC{$\circ \Box \sim \circ A \vdash \tilde{*} \circ A$}
\UnaryInfC{$\circ A \vdash \tilde{*} \circ \Box \sim \circ A$}
\UnaryInfC{$\circ A \vdash \sim \circ \Box \sim \circ A$}
\RightLabel{$\tilde{\circ}\check{\Box}$}
\UnaryInfC{$\circ A \vdash \tilde{\circ} \check{\Box} \sim \circ \Box \sim \circ A$}
\UnaryInfC{$\hat{\bullet}_\ell \circ A \vdash \check{\Box} \sim \circ \Box \sim \circ A$}
\UnaryInfC{$\hat{\bullet}_\ell \circ A \vdash \Box \sim \circ \Box \sim \circ A$}
\UnaryInfC{$\circ A \vdash \tilde{\circ} \Box \sim \circ \Box \sim \circ A$}
\UnaryInfC{$\circ A \vdash \circ \Box \sim \circ \Box \sim \circ A$}
\RightLabel{cont}
\UnaryInfC{$\tilde{*} \circ \Box \sim \circ \Box \sim \circ A \vdash \tilde{*} \circ A$}
\UnaryInfC{$\sim \circ \Box \sim \circ \Box \sim \circ A \vdash \tilde{*} \circ A$}
\UnaryInfC{$\sim \circ \Box \sim \circ \Box \sim \circ A \vdash \sim \circ A$}
\UnaryInfC{$\Box \sim \circ \Box \sim \circ \Box \sim \circ A \vdash \check{\Box} \sim \circ A$}
\UnaryInfC{$\Box \sim \circ \Box \sim \circ \Box \sim \circ A \vdash \Box \sim \circ A$}
\end{prooftree}
$$

- $\neg A \wedge \neg B \vdash \neg(A \vee B) \quad \rightsquigarrow \quad \Box \sim \circ A \wedge \Box \sim \circ B \vdash \Box \sim \circ (A \vee B)$,

$$
\begin{prooftree}
\AxiomC{$A \vdash A$}
\RightLabel{$\tilde{\circ}$}
\UnaryInfC{$\tilde{\circ} A \vdash \tilde{\circ} A$}
\UnaryInfC{$\tilde{\circ} A \vdash \circ A$}
\RightLabel{cont}
\UnaryInfC{$\tilde{*} \circ A \vdash \tilde{*} \tilde{\circ} A$}
\UnaryInfC{$\sim \circ A \vdash \tilde{*} \tilde{\circ} A$}
\UnaryInfC{$\Box \sim \circ A \vdash \check{\Box} \tilde{*} \tilde{\circ} A$}
\LeftLabel{$\mathrm{W_L}$}
\UnaryInfC{$\Box \sim \circ A \mathbin{\hat{\wedge}} \Box \sim \circ B \vdash \check{\Box} \tilde{*} \tilde{\circ} A$}
\UnaryInfC{$\Box \sim \circ A \wedge \Box \sim \circ B \vdash \check{\Box} \tilde{*} \tilde{\circ} A$}
\UnaryInfC{$\hat{\blacklozenge} (\Box \sim \circ A \wedge \Box \sim \circ B) \vdash \tilde{*} \tilde{\circ} A$}
\UnaryInfC{$\tilde{\circ} A \vdash \tilde{*} \, \hat{\blacklozenge} (\Box \sim \circ A \wedge \Box \sim \circ B)$}
\UnaryInfC{$A \vdash \check{\bullet}_r \tilde{*} \, \hat{\blacklozenge} (\Box \sim \circ A \wedge \Box \sim \circ B)$}
\AxiomC{$B \vdash B$}
\RightLabel{$\tilde{\circ}$}
\UnaryInfC{$\tilde{\circ} B \vdash \tilde{\circ} B$}
\UnaryInfC{$\tilde{\circ} B \vdash \circ B$}
\RightLabel{cont}
\UnaryInfC{$\tilde{*} \circ B \vdash \tilde{*} \tilde{\circ} B$}
\UnaryInfC{$\sim \circ B \vdash \tilde{*} \tilde{\circ} B$}
\UnaryInfC{$\Box \sim \circ B \vdash \check{\Box} \tilde{*} \tilde{\circ} B$}
\LeftLabel{$\mathrm{W_L}$}
\UnaryInfC{$\Box \sim \circ B \mathbin{\hat{\wedge}} \Box \sim \circ A \vdash \check{\Box} \tilde{*} \tilde{\circ} B$}
\LeftLabel{$\mathrm{E_L}$}
\UnaryInfC{$\Box \sim \circ A \mathbin{\hat{\wedge}} \Box \sim \circ B \vdash \check{\Box} \tilde{*} \tilde{\circ} B$}
\UnaryInfC{$\Box \sim \circ A \wedge \Box \sim \circ B \vdash \check{\Box} \tilde{*} \tilde{\circ} B$}
\UnaryInfC{$\hat{\blacklozenge} (\Box \sim \circ A \wedge \Box \sim \circ B) \vdash \tilde{*} \tilde{\circ} B$}
\UnaryInfC{$\tilde{\circ} B \vdash \tilde{*} \, \hat{\blacklozenge} (\Box \sim \circ A \wedge \Box \sim \circ B)$}
\UnaryInfC{$B \vdash \check{\bullet}_r \tilde{*} \, \hat{\blacklozenge} (\Box \sim \circ A \wedge \Box \sim \circ B)$}
\BinaryInfC{$A \vee B \vdash \check{\bullet}_r \tilde{*} \, \hat{\blacklozenge} (\Box \sim \circ A \wedge \Box \sim \circ B) \mathbin{\check{\vee}} \tilde{\bullet} \tilde{*} \, \hat{\blacklozenge} (\Box \sim \circ A \wedge \Box \sim \circ B)$}
\RightLabel{$\mathrm{C_L}$}
\UnaryInfC{$A \vee B \vdash \tilde{\bullet} \tilde{*} \, \hat{\blacklozenge} (\Box \sim \circ A \wedge \Box \sim \circ B)$}
\UnaryInfC{$\tilde{\circ} (A \vee B) \vdash \tilde{*} \, \hat{\blacklozenge} (\Box \sim \circ A \wedge \Box \sim \circ B)$}
\UnaryInfC{$\circ (A \vee B) \vdash \tilde{*} \, \hat{\blacklozenge} (\Box \sim \circ A \wedge \Box \sim \circ B)$}
\UnaryInfC{$\hat{\blacklozenge} (\Box \sim \circ A \wedge \Box \sim \circ B) \vdash \tilde{*} \circ (A \vee B)$}
\UnaryInfC{$\hat{\blacklozenge} (\Box \sim \circ A \wedge \Box \sim \circ B) \vdash \sim \circ (A \vee B)$}
\UnaryInfC{$\Box \sim \circ A \wedge \Box \sim \circ B \vdash \check{\Box} \sim \circ (A \vee B)$}
\UnaryInfC{$\Box \sim \circ A \wedge \Box \sim \circ B \vdash \Box \sim \circ (A \vee B)$}
\end{prooftree}
$$

3

- $\neg\neg A \wedge \neg\neg B \vdash \neg\neg(A \wedge B) \quad \rightsquigarrow \quad \Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B \vdash \Box \sim \circ \Box \sim \circ (A \wedge B)$,

$$
\begin{array}{cl}
A \vdash A & \\ \hline
\tilde{\circ} A \vdash \tilde{\circ} A & \tilde{\circ} \\ \hline
\circ A \vdash \tilde{\circ} A & \\ \hline
\tilde{\ast} \tilde{\circ} A \vdash \tilde{\ast} \circ A & \mathsf{cont} \\ \hline
\tilde{\ast} \tilde{\circ} A \vdash \sim \circ A & \\ \hline
\tilde{\ast} \tilde{\circ} A \vdash \tilde{\circ} \check{\Box} \sim \circ A & \tilde{\circ}\check{\Box} \\ \hline
\hat{\bullet}_\ell \tilde{\ast} \tilde{\circ} A \vdash \check{\Box} \sim \circ A & \\ \hline
\hat{\bullet}_\ell \tilde{\ast} \tilde{\circ} A \vdash \Box \sim \circ A & \\ \hline
\tilde{\ast} \tilde{\circ} A \vdash \tilde{\circ} \Box \sim \circ A & \\ \hline
\tilde{\ast} \tilde{\circ} A \vdash \circ \Box \sim \circ A & \\ \hline
\tilde{\ast} \circ \Box \sim \circ A \vdash \tilde{\circ} A & \\ \hline
\sim \circ \Box \sim \circ A \vdash \tilde{\circ} A & \\ \hline
\Box \sim \circ \Box \sim \circ A \vdash \check{\Box} \tilde{\circ} A & \\ \hline
\Box \sim \circ \Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ B \vdash \check{\Box} \tilde{\circ} A & \mathrm{W_L} \\ \hline
\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B \vdash \check{\Box} \tilde{\circ} A & \\ \hline
\hat{\blacklozenge}(\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \vdash \tilde{\circ} A & \\ \hline
\hat{\bullet}_\ell \; \hat{\blacklozenge}(\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \vdash A &
\end{array}
\qquad
\begin{array}{cl}
B \vdash B & \\ \hline
\tilde{\circ} B \vdash \tilde{\circ} B & \tilde{\circ} \\ \hline
\circ B \vdash \tilde{\circ} B & \\ \hline
\tilde{\ast} \tilde{\circ} B \vdash \tilde{\ast} \circ B & \mathsf{cont} \\ \hline
\tilde{\ast} \tilde{\circ} B \vdash \sim \circ B & \\ \hline
\tilde{\ast} \tilde{\circ} B \vdash \tilde{\circ} \check{\Box} \sim \circ B & \tilde{\circ}\check{\Box} \\ \hline
\hat{\bullet}_\ell \tilde{\ast} \tilde{\circ} B \vdash \check{\Box} \sim \circ B & \\ \hline
\hat{\bullet}_\ell \tilde{\ast} \tilde{\circ} B \vdash \Box \sim \circ B & \\ \hline
\tilde{\ast} \tilde{\circ} B \vdash \tilde{\circ} \Box \sim \circ B & \\ \hline
\tilde{\ast} \tilde{\circ} B \vdash \circ \Box \sim \circ B & \\ \hline
\tilde{\ast} \circ \Box \sim \circ B \vdash \tilde{\circ} B & \\ \hline
\sim \circ \Box \sim \circ B \vdash \tilde{\circ} B & \\ \hline
\Box \sim \circ \Box \sim \circ B \vdash \check{\Box} \tilde{\circ} B & \\ \hline
\Box \sim \circ \Box \sim \circ B \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A \vdash \check{\Box} \tilde{\circ} B & \mathrm{W_L} \\ \hline
\Box \sim \circ \Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ B \vdash \check{\Box} \tilde{\circ} B & \mathrm{E_L} \\ \hline
\Box \sim \circ \Box \sim \circ B \wedge \Box \sim \circ \Box \sim \circ A \vdash \check{\Box} \tilde{\circ} B & \\ \hline
\hat{\blacklozenge}(\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \vdash \tilde{\circ} B & \\ \hline
\hat{\bullet}_\ell \; \hat{\blacklozenge}(\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \vdash B &
\end{array}
$$

$$
\begin{array}{cl}
\hline
\hat{\bullet}_\ell \; \hat{\blacklozenge}(\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \,\hat{\wedge}\, \hat{\bullet}_\ell \; \hat{\blacklozenge}(\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \vdash A \wedge B & \\ \hline
\hat{\bullet}_\ell \; \hat{\blacklozenge}(\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \vdash A \wedge B & \mathrm{C_L} \\ \hline
\hat{\blacklozenge}(\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \vdash \tilde{\circ}(A \wedge B) & \\ \hline
\hat{\blacklozenge}(\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \vdash \circ(A \wedge B) & \\ \hline
\tilde{\ast} \circ (A \wedge B) \vdash \tilde{\ast}\; \hat{\blacklozenge}(\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) & \mathsf{cont} \\ \hline
\sim \circ (A \wedge B) \vdash \tilde{\ast}\; \hat{\blacklozenge}(\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) & \\ \hline
\Box \sim \circ (A \wedge B) \vdash \check{\Box} \tilde{\ast}\; \hat{\blacklozenge}(\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) & \\ \hline
\tilde{\circ} \Box \sim \circ (A \wedge B) \vdash \tilde{\circ} \check{\Box} \tilde{\ast}\; \hat{\blacklozenge}(\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) & \tilde{\circ} \\ \hline
\tilde{\circ} \Box \sim \circ (A \wedge B) \vdash \tilde{\ast}\; \hat{\blacklozenge}(\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) & \tilde{\circ}\check{\Box} \\ \hline
\circ \Box \sim \circ (A \wedge B) \vdash \tilde{\ast}\; \hat{\blacklozenge}(\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) & \\ \hline
\hat{\blacklozenge}(\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \vdash \tilde{\ast} \circ \Box \sim \circ (A \wedge B) & \\ \hline
\hat{\blacklozenge}(\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \vdash \sim \circ \Box \sim \circ (A \wedge B) & \\ \hline
\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B \vdash \check{\Box} \sim \circ \Box \sim \circ (A \wedge B) & \\ \hline
\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B \vdash \Box \sim \circ \Box \sim \circ (A \wedge B) &
\end{array}
$$

As to the characterizing axioms of LQM and UQM:

- $A \vdash \neg\neg A \quad \rightsquigarrow \quad A \vdash \Box \sim \circ \Box \sim \circ A$ and $\neg\neg A \vdash A \quad \rightsquigarrow \quad \Box \sim \circ \Box \sim \circ A \vdash A$,

$$
\begin{array}{cl}
A \vdash A & \\ \hline
A \vdash \check{\Box}\, \tilde{\circ} A & \text{LQM} \\ \hline
\hat{\blacklozenge} A \vdash \tilde{\circ} A & \\ \hline
\hat{\blacklozenge} A \vdash \circ A & \\ \hline
\tilde{*} \circ A \vdash \tilde{*}\, \hat{\blacklozenge} A & \text{cont} \\ \hline
\sim \circ A \vdash \tilde{*}\, \hat{\blacklozenge} A & \\ \hline
\Box \sim \circ A \vdash \check{\Box}\, \tilde{*}\, \hat{\blacklozenge} A & \\ \hline
\tilde{\circ} \Box \sim \circ A \vdash \tilde{\circ} \check{\Box}\, \tilde{*}\, \hat{\blacklozenge} A & \tilde{\circ} \\ \hline
\tilde{\circ} \Box \sim \circ A \vdash \tilde{*}\, \hat{\blacklozenge} A & \tilde{\circ} \check{\Box} \\ \hline
\circ \Box \sim \circ A \vdash \tilde{*}\, \hat{\blacklozenge} A & \\ \hline
\hat{\blacklozenge} A \vdash \tilde{*} \circ \Box \sim \circ A & \\ \hline
\hat{\blacklozenge} A \vdash \sim \circ \Box \sim \circ A & \\ \hline
A \vdash \check{\Box} \sim \circ \Box \sim \circ A & \\ \hline
A \vdash \Box \sim \circ \Box \sim \circ A &
\end{array}
\qquad
\begin{array}{cl}
A \vdash A & \\ \hline
\tilde{\circ} A \vdash \tilde{\circ} A & \tilde{\circ} \\ \hline
\circ A \vdash \tilde{\circ} A & \\ \hline
\tilde{*} \tilde{\circ} A \vdash \tilde{*} \circ A & \text{cont} \\ \hline
\tilde{*} \tilde{\circ} A \vdash \sim \circ A & \\ \hline
\tilde{*} \tilde{\circ} A \vdash \tilde{\circ} \check{\Box} \sim \circ A & \tilde{\circ} \check{\Box} \\ \hline
\hat{\bullet}_\ell\, \tilde{*} \tilde{\circ} A \vdash \check{\Box} \sim \circ A & \\ \hline
\hat{\bullet}_\ell\, \tilde{*} \tilde{\circ} A \vdash \Box \sim \circ A & \\ \hline
\tilde{*} \tilde{\circ} A \vdash \tilde{\circ} \Box \sim \circ A & \\ \hline
\tilde{*} \tilde{\circ} A \vdash \circ \Box \sim \circ A & \\ \hline
\tilde{*} \circ \Box \sim \circ A \vdash \tilde{\circ} A & \\ \hline
\sim \circ \Box \sim \circ A \vdash \tilde{\circ} A & \\ \hline
\Box \sim \circ \Box \sim \circ A \vdash \check{\Box}\, \tilde{\circ} A & \\ \hline
\hat{\blacklozenge} \Box \sim \circ \Box \sim \circ A \vdash \tilde{\circ} A & \\ \hline
\hat{\bullet}_\ell\, \hat{\blacklozenge} \Box \sim \circ \Box \sim \circ A \vdash A & \\ \hline
\Box \sim \circ \Box \sim \circ A \vdash A & \text{UQM}
\end{array}
$$

As to the characterizing axiom of AP:

- $\neg A \wedge A \vdash \bot \quad \rightsquigarrow \quad \Box \sim \circ A \wedge A \vdash \bot$,

$$
\begin{array}{cl}
A \vdash A & \\ \hline
\tilde{\circ} A \vdash \tilde{\circ} A & \tilde{\circ} \\ \hline
\tilde{\circ} A \vdash \circ A & \\ \hline
\sim \circ A \vdash \tilde{*} \tilde{\circ} A & \\ \hline
\Box \sim \circ A \vdash \check{\Box}\, \tilde{*} \tilde{\circ} A & \\ \hline
A \mathbin{\hat{\wedge}} \Box \sim \circ A \vdash \check{\bot} & \text{AP} \\ \hline
\Box \sim \circ A \vdash A \mathbin{\check{\rightarrow}} \check{\bot} & \\ \hline
A \mathbin{\hat{\wedge}} \Box \sim \circ A \vdash \check{\bot} & \\ \hline
\Box \sim \circ A \mathbin{\hat{\wedge}} A \vdash \check{\bot} & \mathrm{E_L} \\ \hline
\Box \sim \circ A \mathbin{\hat{\wedge}} A \vdash \bot & \\ \hline
\Box \sim \circ A \wedge A \vdash \bot &
\end{array}
$$

As to the characterizing axiom of DP:

- $\neg A \wedge \neg\neg A \vdash \bot \quad \rightsquigarrow \quad \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ A \vdash \bot$,

$$
\begin{array}{rcl}
 & A \vdash A & \\ \hline
 & \tilde{\circ} A \vdash \tilde{\circ} A & \tilde{\circ} \\ \hline
 & \circ A \vdash \tilde{\circ} A & \\ \hline
 & \circ A \vdash \circ A & \\ \hline
 & \circ A \,\hat{\cap}\, \check{0} \vdash \circ A & \mathrm{W_L} \\ \hline
 & \check{0} \vdash \tilde{*} \circ A \,\check{\cup}\, \circ A & \\ \hline
 & \check{0} \vdash \circ A \,\check{\cup}\, \tilde{*} \circ A & \mathrm{E_L} \\ \hline
 & \tilde{*} \circ A \,\hat{\cap}\, \check{0} \vdash \tilde{*} \circ A & \\ \hline
 & \tilde{*} \circ A \vdash \tilde{*} \circ A & \check{0} \\ \hline
 & \tilde{*} \circ A \vdash \sim \circ A & \\ \hline
 & \sim \circ A \vdash \sim \circ A & \\ \hline
 & \sim \circ A \vdash \tilde{\circ} \check{\Box} \sim \circ A & \tilde{\circ}\check{\Box} \\ \hline
 & \hat{\bullet}_\ell \sim \circ A \vdash \check{\Box} \sim \circ A & \\ \hline
 & \hat{\bullet}_\ell \sim \circ A \vdash \Box \sim \circ A & \\ \hline
 & \sim \circ A \vdash \tilde{\circ} \Box \sim \circ A & \\ \hline
 & \sim \circ A \vdash \circ \Box \sim \circ A & \\ \hline
 & \Box \sim \circ A \vdash \check{\Box} \circ \Box \sim \circ A & \\ \hline
 & \hat{\blacklozenge} \Box \sim \circ A \vdash \circ \Box \sim \circ A & \\ \hline
 & \hat{\blacklozenge} \Box \sim \circ A \vdash \circ \Box \sim \circ A \,\check{\cup}\, \check{0} & \mathrm{W_L} \\ \hline
 & \tilde{*} \circ \Box \sim \circ A \,\hat{\cap}\, \hat{\blacklozenge} \Box \sim \circ A \vdash \check{0} & \\ \hline
\mathrm{E_L} & \hat{\blacklozenge} \Box \sim \circ A \,\hat{\cap}\, \tilde{*} \circ \Box \sim \circ A \vdash \check{0} & \\ \hline
 & \tilde{*} \circ \Box \sim \circ A \vdash \tilde{*}\, \hat{\blacklozenge} \Box \sim \circ A \,\check{\cup}\, \check{0} & \\ \hline
 & \sim \circ \Box \sim \circ A \vdash \tilde{*}\, \hat{\blacklozenge} \Box \sim \circ A \,\check{\cup}\, \check{0} & \\ \hline
 & \Box \sim \circ \Box \sim \circ A \vdash \check{\Box} (\tilde{*}\, \hat{\blacklozenge} \Box \sim \circ A \,\check{\cup}\, \check{0}) & \\ \hline
 & \hat{\blacklozenge} \Box \sim \circ \Box \sim \circ A \vdash \tilde{*}\, \hat{\blacklozenge} \Box \sim \circ A \,\check{\cup}\, \check{0} & \\ \hline
 & \hat{\blacklozenge} \Box \sim \circ A \,\hat{\cap}\, \hat{\blacklozenge} \Box \sim \circ \Box \sim \circ A \vdash \check{0} & \\ \hline
 & \hat{\blacklozenge} \Box \sim \circ \Box \sim \circ A \vdash \hat{\blacklozenge} \Box \sim \circ A \,\check{\rightarrow}\, \check{0} & \\ \hline
 & \Box \sim \circ \Box \sim \circ A \vdash \check{\Box} (\hat{\blacklozenge} \Box \sim \circ A \,\check{\rightarrow}\, \check{0}) & \\ \hline
\mathrm{W_L} & \Box \sim \circ \Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ A \vdash \check{\Box} (\hat{\blacklozenge} \Box \sim \circ A \,\check{\rightarrow}\, \check{0}) & \\ \hline
\mathrm{E_L} & \Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A \vdash \check{\Box} (\hat{\blacklozenge} \Box \sim \circ A \,\check{\rightarrow}\, \check{0}) & \\ \hline
 & \hat{\blacklozenge} (\Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A) \vdash \hat{\blacklozenge} \Box \sim \circ A \,\check{\rightarrow}\, \check{0} & \\ \hline
 & \hat{\blacklozenge} \Box \sim \circ A \,\hat{\cap}\, \hat{\blacklozenge} (\Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A) \vdash \check{0} & \\ \hline
 & \hat{\blacklozenge} \Box \sim \circ A \vdash \hat{\blacklozenge} (\Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A) \,\check{\rightarrow}\, \check{0} & \\ \hline
 & \Box \sim \circ A \vdash \check{\Box} (\hat{\blacklozenge} (\Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A) \,\check{\rightarrow}\, \check{0}) & \\ \hline
\mathrm{W_L} & \Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A \vdash \check{\Box} (\hat{\blacklozenge} (\Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A) \,\check{\rightarrow}\, \check{0}) & \\ \hline
 & \hat{\blacklozenge} (\Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A) \vdash \hat{\blacklozenge} (\Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A) \,\check{\rightarrow}\, \check{0}) & \\ \hline
 & \hat{\blacklozenge} (\Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A) \,\hat{\cap}\, \hat{\blacklozenge} (\Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A) \vdash \check{0} & \\ \hline
\mathrm{C_D} & \hat{\blacklozenge} (\Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A) \vdash \check{0} & \\ \hline
 & \Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A \vdash \check{\Box} \check{0} & \\ \hline
 & \Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A \vdash \check{\bot} & \check{\Box}\check{0} \\ \hline
 & \Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A \vdash \bot & \\ \hline
 & \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ A \vdash \bot &
\end{array}
$$

As to the characterizing axiom of WS:

- $\top \vdash \neg\neg A \vee \neg A \quad \rightsquigarrow \quad \top \vdash \Box \sim \circ \Box \sim \circ A \vee \Box \sim \circ A$,

$$
\begin{array}{cl}
A \vdash A & \\ \hline
\tilde{\circ} A \vdash \tilde{\circ} A & \tilde{\circ} \\ \hline
\tilde{\circ} A \vdash \circ A & \\ \hline
\circ A \vdash \circ A & \\ \hline
\sim \circ A \vdash \tilde{\ast} \circ A & \\ \hline
\sim \circ A \vdash \sim \circ A & \\ \hline
\Box \sim \circ A \vdash \check{\Box} \sim \circ A & \\ \hline
\hat{\blacklozenge} \Box \sim \circ A \vdash \sim \circ A & \\ \hline
\hat{\blacklozenge} (\check{\Box} \tilde{\ast} \tilde{\circ} \Box \sim \circ A \mathrel{\hat{\succ}\!\!-} \hat{\top}) \vdash \sim \circ A & \mathrm{WS} \\ \hline
\check{\Box} \tilde{\ast} \tilde{\circ} \Box \sim \circ A \mathrel{\hat{\succ}\!\!-} \hat{\top} \vdash \check{\Box} \sim \circ A & \\ \hline
\check{\Box} \tilde{\ast} \tilde{\circ} \Box \sim \circ A \mathrel{\hat{\succ}\!\!-} \hat{\top} \vdash \Box \sim \circ A & \\ \hline
\hat{\top} \vdash \check{\Box} \tilde{\ast} \tilde{\circ} \Box \sim \circ A \mathrel{\check{\vee}} \Box \sim \circ A & \\ \hline
\Box \sim \circ A \mathrel{\hat{\succ}\!\!-} \hat{\top} \vdash \check{\Box} \tilde{\ast} \tilde{\circ} \Box \sim \circ A & \\ \hline
\hat{\blacklozenge} (\Box \sim \circ A \mathrel{\hat{\succ}\!\!-} \hat{\top}) \vdash \tilde{\ast} \tilde{\circ} \Box \sim \circ A & \\ \hline
\tilde{\circ} \Box \sim \circ A \vdash \tilde{\ast}\, \hat{\blacklozenge} (\Box \sim \circ A \mathrel{\hat{\succ}\!\!-} \hat{\top}) & \\ \hline
\circ \Box \sim \circ A \vdash \tilde{\ast}\, \hat{\blacklozenge} (\Box \sim \circ A \mathrel{\hat{\succ}\!\!-} \hat{\top}) & \\ \hline
\hat{\blacklozenge} (\Box \sim \circ A \mathrel{\hat{\succ}\!\!-} \hat{\top}) \vdash \tilde{\ast} \circ \Box \sim \circ A & \\ \hline
\hat{\blacklozenge} (\Box \sim \circ A \mathrel{\hat{\succ}\!\!-} \hat{\top}) \vdash \sim \circ \Box \sim \circ A & \\ \hline
\Box \sim \circ A \mathrel{\hat{\succ}\!\!-} \hat{\top} \vdash \check{\Box} \sim \circ \Box \sim \circ A & \\ \hline
\Box \sim \circ A \mathrel{\hat{\succ}\!\!-} \hat{\top} \vdash \Box \sim \circ \Box \sim \circ A & \\ \hline
\hat{\top} \vdash \Box \sim \circ \Box \sim \circ A \mathrel{\check{\vee}} \Box \sim \circ A & \\ \hline
\hat{\top} \vdash \Box \sim \circ \Box \sim \circ A \vee \Box \sim \circ A & \\ \hline
\top \vdash \Box \sim \circ \Box \sim \circ A \vee \Box \sim \circ A &
\end{array}
$$

□

3.6.3 Conservativity

To argue that the calculi introduced in Section 3.5 conservatively capture their respective logics (see Section 3.2.1), we follow the standard proof strategy discussed in [13, 15]. Let L be one of the logics of Definition 18, let $\vdash_{\mathsf{L}}$ denote its syntactic consequence relation, and let $\models_{\mathsf{L}}$ (resp. $\models_{\mathsf{HL}}$) denote the semantic consequence relation arising from the class of the perfect (heterogeneous) algebras associated with L. We need to show that, for all $\mathcal{L}$-formulas A and B, if $A^\tau \vdash B^\tau$ is derivable in the display calculus D.L, then $A \vdash_{\mathsf{L}} B$. This claim can be proved using the following facts: (a) the rules of D.L are sound w.r.t. perfect heterogeneous L-algebras (cf. Section 3.6.1); (b) L is complete w.r.t. its associated class of algebras (cf. Theorem 7); and (c) L-algebras are equivalently presented as heterogeneous L-algebras (cf. Section 4.3), so that the semantic consequence relations arising from each type of algebras preserve and reflect the translation (cf. Proposition 25). If $A^\tau \vdash B^\tau$ is derivable in D.L, then by (a), $\models_{\mathsf{HL}} A^\tau \vdash B^\tau$. By (c), this implies that $\models_{\mathsf{L}} A \vdash B$. By (b), this implies that $A \vdash_{\mathsf{L}} B$, as required.

3.6.4 Cut elimination and subformula property

In the present subsection, we briefly sketch the proof of cut elimination and subformula property for all display calculi introduced in Section 3.5.2. As discussed earlier on, proper display calculi have been designed so that the cut elimination and subformula property can be inferred from a meta-theorem, following the strategy introduced by Belnap for display calculi. The meta-theorem to which we will appeal was proved in [6, Theorem 4.1].

Theorem 8. *Cut elimination and subformula property hold for all display calculi introduced in Section 3.5.2.*

Proof. All conditions in [6] except C'_8 are readily satisfied by inspecting the rules. Condition C'_8 requires to check that reduction steps are available for every application of the cut rule in which both cut-formulas are principal, which either remove the original cut altogether or replace it by one or more cuts on formulas of strictly lower complexity. In what follows, we show C'_8 for the unary connectives by induction on the complexity of cut formula.

Pure type atomic propositions:

$$\frac{p \vdash p \qquad p \vdash p}{p \vdash p} \quad \rightsquigarrow \quad p \vdash p$$

Pure type constants:

$$\frac{\hat{\top} \vdash \top \qquad \dfrac{\begin{array}{c}\vdots\, \pi_1 \\ \hat{\top} \vdash X\end{array}}{\top \vdash X}}{\hat{\top} \vdash X} \quad \rightsquigarrow \quad \begin{array}{c}\vdots\, \pi_1 \\ \hat{\top} \vdash X\end{array}$$

The cases for $\bot$, 1, 0 are standard and similar to the one above.

Pure-type unary connectives:

$$\frac{\dfrac{\begin{array}{c}\vdots\, \pi_1 \\ \Gamma \vdash \tilde{*}\alpha\end{array}}{\Gamma \vdash {\sim}\alpha} \qquad \dfrac{\begin{array}{c}\vdots\, \pi_2 \\ \tilde{*}\alpha \vdash \Delta\end{array}}{{\sim}\alpha \vdash \Delta}}{\Gamma \vdash \Delta} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\, \pi_2 \\ \tilde{*}\alpha \vdash \Delta\end{array}}{\tilde{*}\Delta \vdash \alpha} \qquad \dfrac{\begin{array}{c}\vdots\, \pi_1 \\ \Gamma \vdash \tilde{*}\alpha\end{array}}{\alpha \vdash \tilde{*}\Gamma}}{\tilde{*}\Delta \vdash \tilde{*}\Gamma}}{\Gamma \vdash \Delta}\,\mathsf{cont}$$

Pure-type binary connectives:

$$
\dfrac{\dfrac{\overset{\vdots\,\pi_1}{X \vdash A} \quad \overset{\vdots\,\pi_2}{Y \vdash B}}{X \mathbin{\hat{\wedge}} Y \vdash A \wedge B} \quad \dfrac{\overset{\vdots\,\pi_3}{A \mathbin{\hat{\wedge}} B \vdash Z}}{A \wedge B \vdash Z}}{X \mathbin{\hat{\wedge}} Y \vdash Z}
\quad\rightsquigarrow\quad
\dfrac{\dfrac{\dfrac{\overset{\vdots\,\pi_1}{X \vdash A} \quad \dfrac{\dfrac{\dfrac{\dfrac{\overset{\vdots\,\pi_2}{Y \vdash B} \quad \dfrac{\overset{\vdots\,\pi_3}{A \mathbin{\hat{\wedge}} B \vdash Z}}{B \vdash A \mathbin{\check{\rightarrow}} Z}}{Y \vdash A \mathbin{\check{\rightarrow}} Z}}{A \mathbin{\hat{\wedge}} Y \vdash Z}}{Y \mathbin{\hat{\wedge}} A \vdash Z}}{A \vdash Y \mathbin{\check{\rightarrow}} Z}}{X \vdash Y \mathbin{\check{\rightarrow}} Z}}{Y \mathbin{\hat{\wedge}} X \vdash Z}}{X \mathbin{\hat{\wedge}} Y \vdash Z}
$$

The cases for $A \vee B$, $\alpha \cap \beta$, $\alpha \cup \beta$ are standard and similar to the one above.

Multi-type unary connectives:

$$
\dfrac{\dfrac{\overset{\vdots\,\pi_1}{X \vdash \check{\Box}\alpha}}{X \vdash \Box\alpha} \quad \dfrac{\overset{\vdots\,\pi_2}{\alpha \vdash \Delta}}{\Box\alpha \vdash \check{\Box}\Delta}}{X \vdash \check{\Box}\Delta}
\quad\rightsquigarrow\quad
\dfrac{\dfrac{\dfrac{\overset{\vdots\,\pi_1}{X \vdash \check{\Box}\alpha}}{\hat{\blacklozenge} X \vdash \alpha} \quad \overset{\vdots\,\pi_2}{\alpha \vdash \Delta}}{\hat{\blacklozenge} X \vdash \Delta}}{X \vdash \check{\Box}\Delta}
$$

$$
\dfrac{\dfrac{\overset{\vdots\,\pi_1}{\Gamma \vdash \tilde{\circ} A}}{\Gamma \vdash \circ A} \quad \dfrac{\overset{\vdots\,\pi_2}{\tilde{\circ} A \vdash \Delta}}{\circ A \vdash \Delta}}{\Gamma \vdash \Delta}
\quad\rightsquigarrow\quad
\dfrac{\dfrac{\dfrac{\overset{\vdots\,\pi_1}{\Gamma \vdash \tilde{\circ} A}}{\hat{\bullet}_\ell \Gamma \vdash A} \quad \dfrac{\overset{\vdots\,\pi_2}{\tilde{\circ} A \vdash \Delta}}{A \vdash \check{\bullet}_r \Delta}}{\hat{\bullet}_\ell \Gamma \vdash \check{\bullet}_r \Delta}}{\Gamma \vdash \Delta} \bullet
$$

$\Box$

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3

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3.7 Appendix: Proper multi-type display calculi and their meta-theorem

In the present section we report on (an adaptation of) the definitions and results of [6], from which the cut elimination and subformula property can be straightforwardly inferred for the calculi defined in Section 3.5.2.

The calculi defined in Section 3.5.2 satisfy stronger requirements than those for which the cut elimination meta-theorem [6, Theorem 4.1] holds. Hence, below we provide the corresponding restriction of the definition of quasi-proper multi-type calculus given in [6], which applies specifically to the calculi of Section 3.5.2. The resulting definition, given below, is the exact counterpart in the multi-type setting of the definition of proper display calculi introduced in [23] and generalized in [15].

A sequent $x \vdash y$ is *type-uniform* if x and y are of the same type.

Definition 27. Proper multi-type display calculi *are those satisfying the following list of conditions:*

C$_1$: Preservation of operational terms. *Each operational term occurring in a premise of an inference rule* inf *is a subterm of some operational term in the conclusion of* inf*.*

C$_2$: Shape-alikeness and type-alikeness of parameters. *Congruent parameters*[7] *are occurrences of the same structure, and are of the same type.*

[7] The congruence relation between non active-parts in rule-applications is understood as derived from the specification of each rule; that is, we assume that each schematic rule of the system comes with an explicit specification of which elements are congruent to which (and then the congruence relation is defined as the reflexive and transitive closure of the resulting relation). Our convention throughout the chapter is that congruent parameters are denoted by the same structural variables. For instance, in the rule

$$\frac{X ; Y \vdash Z}{Y ; X \vdash Z}$$

the structures X, Y and Z are parametric and the occurrences of X (resp. Y, Z) in the premise and the conclusion are congruent.

C$_3$: Non-proliferation of parameters. *Each parameter in an inference rule* inf *is congruent to at most one constituent in the conclusion of* inf.

C$_4$: Position-alikeness of parameters. *Congruent parameters are either all in precedent position or all in succedent position (cf. Footnote 4).*

C$_5$: Display of principal constituents. *If an operational term a is principal in the conclusion sequent s of a derivation π, then a is in display.*

C$_6$: Closure under substitution for succedent parts within each type. *Each rule is closed under simultaneous substitution of arbitrary structures for congruent operational terms occurring in succedent position,* within each type.

C$_7$: Closure under substitution for precedent parts within each type. *Each rule is closed under simultaneous substitution of arbitrary structures for congruent operational terms occurring in precedent position,* within each type.

C$_8$: Eliminability of matching principal constituents. *This condition requests a standard Gentzen-style checking, which is now limited to the case in which both cut formulas are* principal, *i.e. each of them has been introduced with the last rule application of each corresponding subdeduction. In this case, analogously to the proof Gentzen-style, condition C$_8$ requires being able to transform the given deduction into a deduction with the same conclusion in which either the cut is eliminated altogether, or is transformed in one or more applications of the cut rule, involving proper subterms of the original operational cut-term.*

C$_9$: Type-uniformity of derivable sequents. *Each derivable sequent is type-uniform.*

C$_{10}$: Preservation of type-uniformity of cut rules. *All cut rules preserve type-uniformity.*

Since proper multi-type display calculi are quasi-proper, the following theorem is an immediate consequence of [6, Theorem 4.1]:

Theorem 9. *Every proper multi-type display calculus enjoys cut elimination and subformula property.*

3.8 Appendix: Analytic inductive inequalities

In the present section, we specialize the definition of *analytic inductive inequalities* (cf. [15]) to the multi-type language $\mathcal{L}_{\mathrm{MT}}$, in the types DL and K, defined in Section 4.4 and reported below for the reader's convenience.

$$\mathsf{DL} \ni A ::= p \mid \Box\alpha \mid \top \mid \bot \mid A \wedge A \mid A \vee A$$
$$\mathsf{K} \ni \alpha ::= \circ A \mid 1 \mid 0 \mid {\sim}\,\alpha \mid \alpha \cup \alpha \mid \alpha \cap \alpha$$

We will make use of the following auxiliary definition: an *order-type* over $n \in \mathbb{N}$ is an n-tuple $\epsilon \in \{1, \partial\}^n$. For every order type ϵ, we denote its *opposite* order type by ϵ^∂, that is, $\epsilon^\partial(i) = 1$ iff $\epsilon(i) = \partial$ for every $1 \leq i \leq n$. The connectives of the language above are grouped together into the families $\mathcal{F} := \mathcal{F}_{\mathsf{DL}} \cup \mathcal{F}_{\mathsf{K}} \cup \mathcal{F}_{\mathrm{MT}}$, $\mathcal{G} := \mathcal{G}_{\mathsf{DL}} \cup \mathcal{G}_{\mathsf{K}} \cup \mathcal{G}_{\mathrm{MT}}$, and $\mathcal{H} := \mathcal{H}_{\mathsf{DL}} \cup \mathcal{H}_{\mathsf{K}} \cup \mathcal{H}_{\mathrm{MT}}$ defined as follows:

$$\begin{array}{lll}
\mathcal{F}_{\mathsf{DL}} := \{\wedge, \top\} & \mathcal{G}_{\mathsf{DL}} := \{\vee, \bot\} & \mathcal{H}_{\mathsf{DL}} := \varnothing \\
\mathcal{F}_{\mathsf{K}} := \{\cap, 1\} & \mathcal{G}_{\mathsf{K}} := \{\cup, 0\} & \mathcal{H}_{\mathsf{K}} := \{\sim\} \\
\mathcal{F}_{\mathrm{MT}} := \varnothing & \mathcal{G}_{\mathrm{MT}} := \{\Box\} & \mathcal{H}_{\mathrm{MT}} := \{\circ\}
\end{array}$$

For any $\ell \in \mathcal{F} \cup \mathcal{G} \cup \mathcal{H}$, we let $n_\ell \in \mathbb{N}$ denote the arity of ℓ, and the order-type ϵ_ℓ on n_ℓ indicates whether the ith coordinate of ℓ is positive ($\epsilon_\ell(i) = 1$) or negative ($\epsilon_\ell(i) = \partial$). The order-theoretic motivation for this partition is that the algebraic interpretations of $\mathcal{F}$-connectives (resp. $\mathcal{G}$-connectives), preserve finite joins (resp. meets) in each positive coordinate and reverse finite meets (resp. joins) in each negative coordinate, while the algebraic interpretations of $\mathcal{H}$-connectives, preserve both finite joins and meets in each positive coordinate and reverse both finite meets and joins in each negative coordinate.

For any term $s(p_1, \ldots p_n)$, any order type ϵ over n, and any $1 \leq i \leq n$, an *ϵ-critical node* in a signed generation tree of s is a leaf node $+p_i$ with $\epsilon(i) = 1$ or $-p_i$ with $\epsilon(i) = \partial$. An *ϵ-critical branch* in the tree is a branch ending in an ϵ-critical node. For any term $s(p_1, \ldots p_n)$ and any order type ϵ over n, we say that $+s$ (resp. $-s$) *agrees with* ϵ, and write $\epsilon(+s)$ (resp. $\epsilon(-s)$), if every leaf in the signed generation tree of $+s$ (resp. $-s$) is ϵ-critical. We will also write $+s' \prec *s$ (resp. $-s' \prec *s$) to indicate that the subterm s' inherits the positive (resp. negative) sign from the signed generation tree $*s$. Finally, we will write $\epsilon(s') \prec *s$ (resp. $\epsilon^\partial(s') \prec *s$) to indicate that the signed subtree s', with the sign inherited from $*s$, agrees with ϵ (resp. with ϵ^∂).

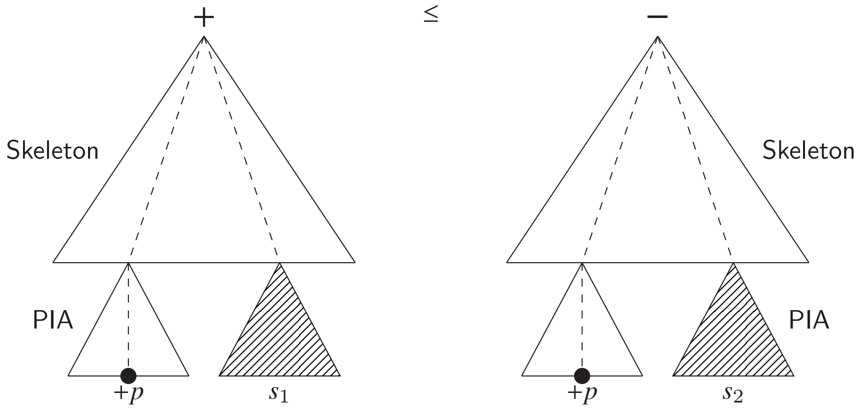
Definition 28 (**Signed Generation Tree**). *The* positive *(resp.* negative*)* generation tree *of any $\mathcal{L}_{MT}$-term s is defined by labelling the root node of the generation tree of s with the sign $+$ (resp. $-$), and then propagating the labelling on each remaining node as follows: For any node labelled with $\ell \in \mathcal{F} \cup \mathcal{G} \cup \mathcal{H}$ of arity n_ℓ, and for any $1 \leq i \leq n_\ell$, assign the same (resp. the opposite) sign to its ith child node if $\epsilon_\ell(i) = 1$ (resp. if $\epsilon_\ell(i) = \partial$). Nodes in signed generation trees are* positive *(resp.* negative*) if are signed $+$ (resp. $-$).*

Definition 29 (**Good branch**). *Nodes in signed generation trees will be called Δ-*adjoints, syntactically left residual (SLR), syntactically right residual (SRR), *and* syntactically right adjoint (SRA), *according to the specification given in Table 3.1. A branch in a signed generation tree $*s$, with $* \in \{+, -\}$, is called a* good branch *if it is the concatenation of two paths P_1 and P_2, one of which may possibly be of length 0, such that P_1 is a path from the leaf consisting (apart from variable nodes) only of PIA-nodes[8], and P_2 consists (apart from variable nodes) only of Skeleton-nodes.*

[8]For an expanded discussion on this definition, see [20, Remark 3.24] and [4, Remark 3.3].

Skeleton	PIA
Δ-adjoints	SRA
$+ \quad \vee \quad \cup$	$+ \quad \wedge \quad \cap \quad \circ \quad \sim \quad \Box$
$- \quad \wedge \quad \cap$	$- \quad \vee \quad \cup \quad \circ \quad \sim$
SLR	SRR
$+ \quad \wedge \quad \cap \quad \circ \quad \sim$	$+ \quad \vee \quad \cup$
$- \quad \vee \quad \cup \quad \circ \quad \sim \quad \Box$	$- \quad \wedge \quad \cap$

Table 3.1: Skeleton and PIA nodes.



Definition 30 (**Analytic inductive inequalities**). *For any order type* ϵ *and any irreflexive and transitive relation* $<_\Omega$ *on* $p_1, \ldots p_n$*, the signed generation tree* $*s$ $(* \in \{-, +\})$ *of an* $\mathcal{L}_{MT}$ *term* $s(p_1, \ldots p_n)$ *is* analytic (Ω, ϵ)-inductive *if*

1. *every branch of* $*s$ *is good (cf. Definition 29);*
2. *for all* $1 \leq i \leq n$*, every SRR-node occurring in any* ϵ*-critical branch with leaf* p_i *is of the form* $\circledast(s, \beta)$ *or* $\circledast(\beta, s)$*, where the critical branch goes through* β *and*
   (a) $\epsilon^\partial(s) \prec *s$ *(cf. discussion before Definition 29), and*
   (b) $p_k <_\Omega p_i$ *for every* p_k *occurring in* s *and for every* $1 \leq k \leq n$*.*

We will refer to $<_\Omega$ *as the* dependency order *on the variables. An inequality* $s \leq t$ *is* analytic (Ω, ϵ)-inductive *if the signed generation trees* $+s$ *and* $-t$ *are analytic* (Ω, ϵ)*-inductive. An inequality* $s \leq t$ *is* analytic inductive *if is analytic* (Ω, ϵ)*-inductive for some* Ω *and* ϵ*.*

In each setting in which they are defined, analytic inductive inequalities are a subclass of inductive inequalities (cf. [15, Definition 16]). In their turn, inductive inequalities are *canonical* (that is, preserved under canonical extensions, as defined in each setting).

Chapter 4

Bilattice Logic Properly Displayed

In the present chapter, which is based on the paper [1] [26], we introduce a proper multi-type display calculus for bilattice logic (with conflation) for which we prove soundness, completeness, conservativity, standard subformula property and cut elimination. Our proposal builds on the product representation of bilattices and applies the guidelines of the multi-type methodology in the design of display calculi.

[1] My specific contributions to this research have been the proof of results, the introduction of notions and constructions, and the writing of the first draft of the paper.

4

4.1 Introduction

Bilattices are algebraic structures introduced in [23] in the context of a multivalued approach to deductive reasoning, and have subsequently found applications in a variety of areas in computer science and artificial intelligence. The basic intuition behind the bilattice formalism, which can be traced back to the work of Dunn and Belnap [3, 4, 14] and even earlier, to Kleene's proposal of a three-valued logic, is to carry out reasoning within a space of truth-values that results from expanding the classical set $\{\mathtt{f}, \mathtt{t}\}$ with a value $\bot$, representing lack of information, and a value $\top$, representing over-defined or contradictory information.

More generally, Ginsberg [23] argued that one could take as space of truth-values a set equipped with *two* lattice orderings (a *bilattice*), reflecting respectively the *degree of truth* and the *degree of information* associated with propositions. The bilattice framework may thus be viewed as an attempt at combining the many-valued approach to vagueness of fuzzy logic with the Dunn-Belnap-Kleene treatment of partial and inconsistent information. In fact, a number of works has shown how bilattice-like structures naturally arise in the context of fuzzy logic when one tries to account for uncertainty, imprecision and incompleteness of information [12, 13, 15, 37].

Negation plays a very special role. Indeed, it is because of this connective that bilattice logics are not *self-extensional* [40] (or, as other authors say, *congruential*), i.e. the inter-derivability relation of the logic is not a congruence of the formula algebra. This means that there are formulas φ and ψ such that $\varphi \dashv\vdash \psi$ and yet $\neg\varphi \not\dashv\vdash \neg\psi$ (this is not the case of the Belnap-Dunn logic, which is self-extensional). In the Gentzen-style calculus for bilattice logic *GBL* introduced in [1, Section 3.2], there are four introduction rules for each binary connective, two of which are standard and introduce it as main connective on the left and on the right of the turnstile, and two are non-standard and introduce the same connective under the scope of negation. From a proof-theoretic perspective, this solution has the disadvantage that the resulting calculus is not fully modular, does not enjoy the standard subformula property, and violates some key criteria about introduction rules for connectives adopted in the literature on display calculi, structural proof theory and dynamic logics on the basis of technical considerations, and in the literature on proof-theoretic semantics on more philosophical ground and concerns (see [19, 36, 38, 39]).

In this chapter, we introduce a *proper multi-type display calculus* for bilattice logic that circumvents all the above-mentioned disadvantages.[2] As a first approximation to the problem of providing a calculus for the full Arieli-Avron logic [1, 9], we shall here focus on its implication-free fragment, which is precisely the logic axiomatized by means of a Hilbert-style calculus in [8]. We consider this to be a reasonable tradeoff: on the one hand because, thanks to the modularity of our calculus, we do not anticipate any major technical difficulties in introducing further rules to account for the implication (this is current work in progress); on the other hand because the characteristic behaviour of the bilattice negation (and the problems that arise in its proof-theoretic treatment) already

[2]The notion of proper display calculus has been introduced in [38]. Properly displayable logics, i.e. those which can be captured by some proper display calculus, have been characterized in a purely proof-theoretic way in [10]. In [27], an alternative characterization of properly displayable logics was introduced which builds on the algebraic theory of unified correspondence [11].

manifest in the context of the implicationless logic. Another natural future project will be providing a display calculus for modal expansions of bilattice logic such as those introduced in [31]–see the concluding remarks in Section 4.7.

The design of our display calculus follows the principles of the *multi-type* methodology introduced in [16–18, 24] for displaying dynamic epistemic logic and propositional dynamic logic, and subsequently applied to displaying several other well known logics (e.g. linear logic with exponentials [29], inquisitive logic [20], semi-De Morgan logic [25], lattice logic [28]) which are not properly displayable in their single-type presentation, and also to design families of novel logical frameworks in a modular and principled way [6]. Our multi-type syntactic presentation of bilattice logic is based on the algebraic insight provided by the product representation theorems (see e.g. [7]) and possesses all the desirable properties of *proper* display calculi. In particular, our calculus enjoys the standard subformula property, supports a proof-theoretic semantics and is fully modular. These features make it possible to prove important results about the logics in a principled way and are key for developing interactive and automated reasoning tools [2].

Structure of the chapter In Section 4.2 we recall basic definitions and results about bilattices and bilattice logics and discuss the general motivations and insights underlying (multi-type) display calculi. Section 4.3 develops an algebraic analysis of bilattices as heterogeneous structures which provides a basis for our multi-type approach to their proof theory. In Section 4.4, we introduce the multi-type bilattice logic which corresponds to heterogeneous bilattices. Our display calculi are introduced in Section 4.5, and we prove its soundness, completeness, conservativity, subformula property and cut elimination in Section 4.6. In Section 4.7 we outline some directions for future work.

4.2 Preliminaries

4.2.1 Bilattices

The definitions and results in this section can be found in [1, 8].

Definition 31. *A* bilattice *is a structure* $\mathbb{B} = (B, \leq_t, \leq_k, \neg)$ *such that* B *is a non-empty set,* $(B, \leq_t)$*,* $(B, \leq_k)$ *are lattices, and* $\neg$ *is a unary operation on* B *having the following properties:*

- *if* $a \leq_t b$*, then* $\neg b \leq_t \neg a$*,*
- *if* $a \leq_k b$*, then* $\neg a \leq_k \neg b$*,*
- $\neg\neg a = a$*.*

We use $\wedge, \vee$ for the lattice operations which correspond to $\leq_t$ and $\otimes, \oplus$ for those that correspond to $\leq_k$. If present, the lattice bounds of $\leq_t$ are denoted by $\mathtt{f}$ and $\mathtt{t}$ (minimum and maximum, respectively) and those of $\leq_k$ by $\bot$ and $\top$. The smallest non-trivial bilattice is the four-element one (called **Four**) with universe $\{\mathtt{f}, \mathtt{t}, \bot, \top\}$.

Fact 1. *The following equations (*De Morgan laws for negation*) hold in any bilattice:*

$$\neg(x \wedge y) = \neg x \vee \neg y, \qquad \neg(x \vee y) = \neg x \wedge \neg y,$$
$$\neg(x \otimes y) = \neg x \otimes \neg y, \qquad \neg(x \oplus y) = \neg x \oplus \neg y.$$

Moreover, if the bilattice is bounded, then

$$\neg t = f, \quad \neg f = t, \quad \neg\top = \top, \quad \neg\bot = \bot.$$

Definition 32. *A bilattice is called* distributive *when all possible distributive laws concerning the four lattice operations, i.e., all identities of the following form, hold:*

$$x \circ (y \bullet z) \approx (x \circ y) \bullet (x \circ z) \quad \text{for all } \circ, \bullet \in \{\wedge, \vee, \otimes, \oplus\}$$

If a distributive bilattice is bounded, then

$$\mathtt{t} \otimes \mathtt{f} = \bot, \quad \mathtt{t} \oplus \mathtt{f} = \top, \quad \top \wedge \bot = \mathtt{f}, \quad \top \vee \bot = \mathtt{t}.$$

In the following, we use B to denote the class of bounded distributive bilattices.

Theorem 10 (Representation of distributive bilattices)**.** *Let $\mathbb{L}$ be a bounded distributive lattice with join $\sqcup$ and meet $\sqcap$. Then the algebra $\mathbb{L} \odot \mathbb{L}$ having as universe the direct product $L \times L$ is a distributive bilattice with the following operations:*

$$\begin{array}{rcl}
\langle a_1, a_2\rangle \wedge \langle b_1, b_2\rangle & := & \langle a_1 \sqcap b_1, a_2 \sqcup b_2\rangle \\
\langle a_1, a_2\rangle \vee \langle b_1, b_2\rangle & := & \langle a_1 \sqcup b_1, a_2 \sqcap b_2\rangle \\
\langle a_1, a_2\rangle \otimes \langle b_1, b_2\rangle & := & \langle a_1 \sqcap b_1, a_2 \sqcap b_2\rangle \\
\langle a_1, a_2\rangle \oplus \langle b_1, b_2\rangle & := & \langle a_1 \sqcup b_1, a_2 \sqcup b_2\rangle \\
\neg\langle a_1, a_2\rangle & := & \langle a_2, a_1\rangle \\
f & := & \langle 0, 1\rangle \\
t & := & \langle 1, 0\rangle \\
\bot & := & \langle 0, 0\rangle \\
\top & := & \langle 1, 1\rangle
\end{array}$$

Theorem 11. *Every distributive bilattice is isomorphic to $\mathbb{L} \odot \mathbb{L}$ for some distributive lattice $\mathbb{L}$.*

Definition 33. *A structure $\mathbb{B} = (B, \leq_t, \leq_k, \neg, -)$ is a* bilattice with conflation *if $(B, \leq_t, \leq_k, \neg)$ is a bilattice and the* conflation *$- : B \longrightarrow B$ is an operation satisfying:*

- *if $a \leq_t b$, then $-a \leq_t -b$;*
- *if $a \leq_k b$, then $-b \leq_k -a$;*
- *$- - a = a$.*

We say that $\mathbb{B}$ is commutative *if it also satisfies the equation: $\neg - x = -\neg x$.*

Fact 2. *The following equations (*De Morgan laws for conflation*) hold in any bilattice with conflation:*

$$-(x \wedge y) = -x \wedge -y \qquad -(x \vee y) = -x \vee -y$$
$$-(x \otimes y) = -x \oplus -y \qquad -(x \oplus y) = -x \otimes -y$$

Moreover, if the bilattice is bounded, then

$$-t = t, \quad -f = f, \quad -\top = \bot, \quad -\bot = \top.$$

We denote by CB the class of bounded commutative distributive bilattices with conflation.

Theorem 12. *Let* $\mathbb{D} = (D, \sqcap, \sqcup, \sim, 0, 1)$ *be a De Morgan algebra, then* $\mathbb{D} \odot \mathbb{D}$ *is a bounded commutative distributive bilattice with conflation where:*

- $(D, \sqcap, \sqcup, 0, 1) \odot (D, \sqcap, \sqcup, 0, 1)$ *is a bounded distributive bilattice;*
- $-(a, b) = (\sim b, \sim a)$*;*

Theorem 13. *Every bounded commutative distributive bilattice with conflation is isomorphic to* $\mathbb{D} \odot \mathbb{D}$ *for some De Morgan algebra* $\mathbb{D}$.

4.2.2 Bilattice logic

In the present subsection we introduce Bilattice Logic (BL) and Bilattice Logic with Conflation (CBL). The language of CBL $\mathcal{L}$ over a denumerable set AtProp $= \{p, q, r, \ldots\}$ of atomic propositions is generated as follows:

$$A ::= p \mid \mathtt{t} \mid \mathtt{f} \mid \top \mid \bot \mid \neg A \mid A \wedge A \mid A \vee A \mid A \otimes A \mid A \oplus A \mid -A.$$

the language of BL is the conflation-free reduct of $\mathcal{L}$, where conflation is the name of the connective '$-$'. Bilattice Logic consists of the following axioms:

$$A \vdash A, \quad \neg\neg A \dashv\vdash A,$$

$$\mathtt{f} \vdash A, \quad A \vdash \mathtt{t}, \quad \bot \vdash A, \quad A \vdash \top,$$

$$A \vdash \neg\mathtt{f}, \quad \neg\mathtt{t} \vdash A, \quad \neg\bot \vdash A, \quad A \vdash \neg\top,$$

$$A \wedge B \vdash A, \quad A \wedge B \vdash B, \quad A \vdash A \vee B, \quad B \vdash A \vee B,$$

$$A \otimes B \vdash A, \quad A \otimes B \vdash B, \quad A \vdash A \oplus B, \quad B \vdash A \oplus B,$$

$$A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C),$$

$$A \otimes (B \oplus C) \vdash (A \otimes B) \vee (A \oplus C),$$

$$\neg(A \wedge B) \dashv\vdash \neg A \vee \neg B, \quad \neg(A \vee B) \dashv\vdash \neg A \wedge \neg B,$$

$$\neg(A \otimes B) \dashv\vdash \neg A \otimes \neg B, \quad \neg(A \oplus B) \dashv\vdash \neg A \oplus \neg B,$$

and the following rules:

$$\frac{A \vdash B \qquad B \vdash C}{A \vdash C}$$

$$\frac{A \vdash B \qquad A \vdash C}{A \vdash B \wedge C} \qquad \frac{A \vdash B \qquad C \vdash B}{A \vee C \vdash B}$$

$$\frac{A \vdash B \qquad A \vdash C}{A \vdash B \otimes C} \qquad \frac{A \vdash B \qquad C \vdash B}{A \oplus C \vdash B}$$

CBL consists of the axioms and rules of BL plus the following axioms:

$$- - A \dashv\vdash A, \quad -\neg A \dashv\vdash \neg - A,$$

$$-\mathtt{f} \vdash A, \quad A \vdash -\mathtt{t}, \quad -\top \vdash A, \quad A \vdash -\bot,$$

$$-(A \wedge B) \dashv\vdash -A \wedge -B, \quad -(A \vee B) \dashv\vdash -A \vee -B,$$

$$-(A \otimes B) \dashv\vdash -A \oplus -B, \quad -(A \oplus B) \dashv\vdash -A \otimes -B.$$

The algebraic semantics of BL (resp. CBL) is given by B (resp. CB). We use $A \models_{\mathsf{B}} C$ (resp. $A \models_{\mathsf{CB}} C$) to mean: for any $\mathbb{B} \in \mathsf{B}$ (resp. $\mathbb{B} \in \mathsf{CB}$), if $A^{\mathbb{B}} \in F_{\mathtt{t}}$ then $C^{\mathbb{B}} \in F_{\mathtt{t}}$. Here $A^{\mathbb{B}}$ and $C^{\mathbb{B}}$ mean the interpretations of A and C in $\mathbb{B}$, respectively; and $F_{\mathtt{t}} = \{a \in B : \mathtt{t} \leq_k a\}$ is the set of designated elements of $\mathbb{B}$ (using the terminology of [1, Definition 2.13], $F_{\mathtt{t}}$ is the *least bifilter* of $\mathbb{B}$).

Soundness of BL (resp. CBL) is straightforward. In order to show completeness, we can prove that every axiom and rule of Arieli and Avron's GBL (resp. GBS, cf. [1]) is derivable in BL (resp. CBL).[3] Then the completeness of BL (resp. CBL) follows from the completeness of GBL (resp. GBS, [1, Theorem 3.7]).

Theorem 14 (Completeness)**.** *$A \vdash_{\mathrm{BL}} C$ iff $A \models_{\mathsf{B}} C$ (resp. $A \vdash_{\mathrm{CBL}} C$ iff $A \models_{\mathsf{CB}} C$).*

4.3 Multi-type algebraic presentation of bilattices

In the present section we introduce the algebraic environment which justifies semantically the multi-type approach to bilattice logic presented in Section 4.5. The main insight is that (bounded) bilattices (with conflation) can be equivalently presented as heterogeneous structures, i.e. tuples consisting of two (bounded) distributive lattices (De Morgan algebras) together with two maps between them.

Multi-type semantic environment

For a bilattice $\mathbb{B}$, let $\mathrm{Reg}(\mathbb{B}) = \{a \in B : a = \neg a\}$ be the set of *regular elements* [7]. It is easy to show that $\mathrm{Reg}(\mathbb{B})$ is closed under $\otimes$ and $\oplus$, hence $(\mathrm{Reg}(\mathbb{B}), \otimes, \oplus)$ is a sublattice of $(B, \otimes, \oplus)$. For every $a \in B$, we let

$$\mathrm{reg}(a) := (a \vee (a \otimes \neg a)) \oplus \neg(a \vee (a \otimes \neg a))$$

be the regular element associated with a. It follows from the representation result of [7, Theorem 3.2] that

$$\mathbb{B} \cong (\mathrm{Reg}(\mathbb{B}), \otimes, \oplus) \odot (\mathrm{Reg}(\mathbb{B}), \otimes, \oplus)$$

where the isomorphism $\pi : B \to \mathrm{Reg}(\mathbb{B}) \times \mathrm{Reg}(\mathbb{B})$ is defined, for all $a \in B$, as $\pi(a) := \langle \mathrm{reg}(a), \mathrm{reg}(\neg a)\rangle$. The inverse map $f : \mathrm{Reg}(\mathbb{B}) \times \mathrm{Reg}(\mathbb{B}) \to B$ is defined, for all $\langle a, b\rangle \in \mathrm{Reg}(\mathbb{B}) \times \mathrm{Reg}(\mathbb{B})$, as

$$f(\langle a, b\rangle) := (a \otimes (a \vee b)) \oplus (b \otimes (a \wedge b)).$$

[3] In order to do this, we view a sequent $\Gamma \Rightarrow \Delta$ of GBL (GBS) as the equivalent sequent $\bigwedge \Gamma \Rightarrow \bigvee \Delta$.

Heterogeneous Bilattices

Definition 34. *A* heterogeneous bilattice *(HBL) is a tuple* $\mathbb{H} = (\mathbb{L}_1,\ \mathbb{L}_2,\ \mathrm{n},\ \mathrm{p})$ *satisfying the following conditions:*

($\mathrm{H1}$*)* $\mathbb{L}_1$, $\mathbb{L}_2$ *are bounded distributive lattices.*

($\mathrm{H2}$*)* $\mathrm{n} : \mathbb{L}_1 \to \mathbb{L}_2$ *and* $\mathrm{p} : \mathbb{L}_2 \to \mathbb{L}_1$ *are mutually inverse lattice isomorphisms.*

We let HBL *denote the class of HBLs. An HBL is* perfect *if:*

($\mathrm{H3}$*) both* $\mathbb{L}_1$ *and* $\mathbb{L}_2$ *are perfect lattices;*[4]

($\mathrm{H4}$*)* p, n *are complete lattice isomorphisms.*

By $(\mathrm{H2})$ we have that np $= \mathrm{Id}_{\mathbb{L}_1}$ and pn $= \mathrm{Id}_{\mathbb{L}_2}$, from which it straightforwardly follows that n and p are both right and left adjoints of each other. The definition of *the heterogeneous bilattice with conflation* (HCBL) is analogous, except that we replace $(\mathrm{H1})$ with the following condition:

$(\mathrm{H1}')$ $\mathbb{L}_1$ and $\mathbb{L}_2$ are De Morgan algebras, with De Morgan negations denoted $\sim_1$ and $\sim_2$ respectively.

We let HCBL denote the class of HCBLs. In what follows, we let $\mathbb{L}^\delta$ denote the canonical extension of the lattice $\mathbb{L}$. The following lemma is an easy consequence of the results in [21, Theorems 2.3 and 3.2].

Lemma 13. *If* $(\mathbb{L}_1, \mathbb{L}_2, \mathrm{n}, \mathrm{p})$ *is an HBL (HCBL), then* $(\mathbb{L}_1^\delta, \mathbb{L}_2^\delta, \mathrm{n}^\delta, \mathrm{p}^\delta)$ *is a perfect HBL (resp. HCBL).*

$$\begin{tikzcd}
\mathbb{L}_1^\delta \arrow[rr, bend right, "\mathrm{n}^\delta"'] & & \mathbb{L}_2^\delta \arrow[ll, bend right, "\mathrm{p}^\delta"'] \\
& & \\
\mathbb{L}_1 \arrow[uu, hook] \arrow[rr, bend right, "\mathrm{n}"'] & & \mathbb{L}_2 \arrow[uu, hook] \arrow[ll, bend right, "\mathrm{p}"']
\end{tikzcd}$$

[4] A distributive lattice $\mathbb{A}$ is *perfect* (cf. [22, Definition 2.14]) if it is complete, completely distributive and completely join-generated by the set $J^\infty(\mathbb{A})$ of its completely join-irreducible elements (as well as completely meet-generated by the set $M^\infty(\mathbb{A})$ of its completely meet-irreducible elements). A lattice isomomorphism $h : \mathbb{L} \to \mathbb{L}'$ is *complete* if it satisfies the following properties for each $X \subseteq \mathbb{L}$:

$$h(\bigvee X) = \bigvee h(X) \qquad h(\bigwedge X) = \bigwedge h(X),$$

Equivalence of the two presentations

The following result is a straightforward verification of Definition 34.

Proposition 22. *For any bounded distributive bilattice* $\mathbb{B}$*, the tuple*

$$\mathbb{B}^+ = (\mathrm{Reg}(\mathbb{B}),\ \mathrm{Reg}(\mathbb{B}),\ \mathrm{Id}_{\mathrm{Reg}(\mathbb{B})},\ \mathrm{Id}_{\mathrm{Reg}(\mathbb{B})})$$

is an HBL, where $\sqcap_1 = \cap = \otimes, \sqcup_1 = \cup = \oplus, 1_1 = 1 = \top$ *and* $0_1 = 0 = \bot$.
For any CB $\mathbb{B}$*, the tuple*

$$\mathbb{B}^+ = ((\mathrm{Reg}(\mathbb{B}), \sim_1),\ (\mathrm{Reg}(\mathbb{B}), \sim_2),\ \mathrm{Id}_{\mathrm{Reg}(\mathbb{B})},\ \mathrm{Id}_{\mathrm{Reg}(\mathbb{B})})$$

4

is an HCBL, where $\sim_2 = \sim_1 = -$.

Proposition 23. *If* $(\mathbb{L}_1,\ \mathbb{L}_2,\ n,\ p)$ *is an HBL (resp. HCBL), then* $L_1 \times L_2$ *is a bilattice (resp. a bilattice with conflation) when endowed with the following structure:*

$$\begin{array}{rcl}
\langle a_1, a_2\rangle \otimes \langle b_1, b_2\rangle & := & \langle a_1 \sqcap_1 b_1,\ a_2 \cap b_2\rangle \\
\langle a_1, a_2\rangle \oplus \langle b_1, b_2\rangle & := & \langle a_1 \sqcup_1 b_1,\ a_2 \cup b_2\rangle \\
\langle a_1, a_2\rangle \wedge \langle b_1, b_2\rangle & := & \langle a_1 \sqcap_1 b_1,\ a_2 \cup b_2\rangle \\
\langle a_1, a_2\rangle \vee \langle b_1, b_2\rangle & := & \langle a_1 \sqcup_1 b_1,\ a_2 \cap b_2\rangle \\
\neg\langle a_1, a_2\rangle & := & \langle p(a_2), n(a_1)\rangle \\
-\langle a_1, a_2\rangle & := & \langle \mathrm{p}(\sim_2 a_2), \mathrm{n}(\sim_1 a_1)\rangle \\
f & := & \langle 0, 1\rangle \\
t & := & \langle 1, 0\rangle \\
\bot & := & \langle 0, 0\rangle \\
\top & := & \langle 1, 1\rangle
\end{array}$$

Proof. Firstly, we show that $\langle L_1 \times L_2, \otimes, \oplus\rangle$ and $\langle L_1 \times L_2, \wedge, \vee\rangle$ are bounded distributive lattices. It is obvious that they are both bounded lattices. We only need to show that the distributivity law holds. We have:

$$\begin{array}{cll}
 & \langle a_1,\ a_2\rangle \otimes (\langle b_1,\ b_2\rangle \oplus (\langle c_1,\ c_2\rangle) & \\
= & \langle a_1,\ a_2\rangle \otimes (\langle b_1 \sqcup_1 c_1,\ b_2 \cup c_2\rangle) & \text{Def. of } \oplus \\
= & \langle a_1 \sqcap_1 (b_1 \sqcup_1 c_1),\ a_2 \cap (b_2 \cup c_2)\rangle & \text{Def. of } \otimes \\
= & \langle (a_1 \sqcap_1 b_1) \sqcup_1 (a_1 \sqcap_1 c_1),\ (a_2 \cap b_2) \cup (a_2 \cap c_2)\rangle & \text{Distributivity of } \mathbb{L}_1 \text{ and } \mathbb{L}_2 \\
= & \langle (a_1 \sqcap_1 b_1),\ (a_2 \cap b_2)\rangle \oplus \langle (a_1 \sqcap_1 c_1),\ (a_2 \cap c_2)\rangle & \text{Def. of } \oplus \\
= & (\langle a_1,\ a_2\rangle \otimes \langle b_1,\ b_2\rangle) \oplus (\langle a_1,\ a_2\rangle \otimes \langle c_1,\ c_2\rangle) & \text{Def. of } \otimes
\end{array}$$

As to $\langle L_1 \times L_2, \wedge, \vee\rangle$, the argument is analogous.

Now we show that the properties of $\neg$ are also met. Assume that $\langle a_1,\ a_2\rangle \leq_t \langle b_1,\ b_2\rangle$, equivalently, $a_1 \leq_1 b_1$ and $b_2 \leq_2 a_2$. By the definition of $\neg$, we have $\neg\langle a_1,\ a_2\rangle = \langle \mathrm{p}(a_2),\ \mathrm{n}(a_1)\rangle$ and $\neg\langle b_1,\ b_2\rangle = \langle \mathrm{p}(b_2),\ \mathrm{n}(b_1)\rangle$. Hence $\mathrm{p}(b_2) \leq_1 \mathrm{p}(a_2)$ and $\mathrm{n}(a_1) \leq_2 \mathrm{n}(b_1)$ by (H2). Thus $\neg\langle b_1,\ b_2\rangle \leq_t \neg\langle a_1,\ a_2\rangle$. A similar reasoning shows that the corresponding property involving $\neg$ and $\leq_k$ also holds. The following argument shows that $\neg$ is involutive.

$$
\begin{array}{lll}
& \neg\neg\langle a_1,\ a_2\rangle & \\
= & \neg\langle \mathrm{p}(a_2),\ \mathrm{n}(a_1)\rangle & \text{Def. of } \neg \\
= & \langle \mathrm{pn}(a_1),\ \mathrm{np}(a_2)\rangle & \text{Def. of } \neg \\
= & \langle a_1,\ a_2\rangle & \mathsf{np} = \mathrm{Id}_{\mathbb{L}_1} \text{ and } \mathsf{pn} = \mathrm{Id}_{\mathbb{L}_2}
\end{array}
$$

As to conflation, assume $\langle a_1,\ a_2\rangle \leq_t \langle b_1,\ b_2\rangle$, equivalently, $a_1 \leq_1 b_1$ and $b_2 \leq_2 a_2$. By the definition of $-$ we have $-\langle a_1,\ a_2\rangle = \langle \mathrm{p}({\sim_2}\ a_2),\ \mathrm{n}({\sim_1}\ a_1)\rangle$ and $-\langle b_1,\ b_2\rangle = \langle \mathrm{p}({\sim_2}\ b_2),\ \mathrm{n}({\sim_1}\ b_1)\rangle$. Hence $\mathrm{p}({\sim_2}\ a_2) \leq_1 \mathrm{p}({\sim_2}\ b_2)$ and $\mathrm{n}({\sim}\ b_1) \leq_2 \mathrm{n}({\sim}\ b_2)$ by (H2). Thus $-\langle a_1,\ a_2\rangle \leq_t -\langle b_1,\ b_2\rangle$. A similar reasoning shows that the corresponding property involving $-$ and $\leq_k$ also holds. The following arguments show that $-$ is involutive and $-$ and $\neg$ are commutative.

$$
\begin{array}{ll}
\ \ -\,-\,\langle a_1,\ a_2\rangle & \\
= -\langle \mathrm{p}({\sim_2}\ a_2),\ \mathrm{n}({\sim_1}\ a_1)\rangle & \text{Def. of } - \\
= \langle \mathrm{p}({\sim_2}\ \mathrm{n}({\sim_1}\ a_1)),\ \mathrm{n}({\sim_1}\ \mathrm{p}({\sim_2}\ a_2))\rangle & \text{Def. of } - \\
= \langle \mathrm{p}({\sim_2}{\sim_2}\ \mathrm{n}(a_1)),\ \mathrm{n}({\sim_1}{\sim_1}\ \mathrm{p}(a_2))\rangle & \text{H2} \\
= \langle \mathrm{pn}(a_1),\ \mathrm{np}(a_2)\rangle & \text{H1} \\
= \langle a_1,\ a_2\rangle & \mathsf{np} = \mathrm{Id}_{\mathbb{L}_1} \text{ and } \mathsf{pn} = \mathrm{Id}_{\mathbb{L}_2}
\end{array}
$$

$$
\begin{array}{ll}
\ \ -\neg\langle a_1,\ a_2\rangle & \\
= -\langle \mathrm{p}(a_2),\ \mathrm{n}(a_1)\rangle & \text{Def. of } \neg \\
= \langle \mathrm{p}({\sim_2}\ \mathrm{n}(a_1)),\ \mathrm{n}({\sim_1}\ \mathrm{p}(a_2))\rangle & \text{Def. of } - \\
= \neg\langle {\sim_1}\ \mathrm{p}(a_2),\ {\sim_2}\ \mathrm{n}(a_1)\rangle & \text{Def. of } \neg \\
= \neg\langle \mathrm{p}({\sim_2}\ a_2),\ \mathrm{n}({\sim_1}\ a_2)\rangle & \text{H2} \\
= \neg\,-\,\langle a_1,\ a_2\rangle & \text{Def. of } -
\end{array}
$$

□

Definition 35. *For any HBL $\mathbb{H} = (\mathbb{L}_1,\ \mathbb{L}_2,\ n,\ p)$, we let $\mathbb{H}_+ = (L_1 \times L_2,\ \wedge,\ \vee,\ \otimes,\ \oplus,\ \neg)$ denote the product algebra where the four lattice operations are defined as in $\mathbb{L}_1 \odot \mathbb{L}_2$ (Theorem 10) and the negation is given by $\neg\langle a_1, a_2\rangle := \langle \mathrm{p}(a_2),\ \mathrm{n}(a_1)\rangle$ for all $\langle a_1, a_2\rangle \in L_1 \times L_2$. If $\mathbb{L}_1$ and $\mathbb{L}_2$ are isomorphic De Morgan algebras, then we define $\mathbb{H}_+ = (L_1 \times L_2,\ \wedge,\ \vee,\ \otimes,\ \oplus,\ \neg,\ -)$ as before, with the conflation given by $-\langle a_1, a_2\rangle := \langle \mathrm{p}({\sim_2}\ a_2),\ \mathrm{n}({\sim_1}\ a_1)\rangle$ for all $\langle a_1, a_2\rangle \in L_1 \times L_2$.*

Proposition 24. *For any $\mathbb{B} \in$ B (resp. $\mathbb{B} \in$ CB) and any HBL (resp. HCBL) $\mathbb{H}$, we have*

$$\mathbb{B} \cong (\mathbb{B}^+)_+ \quad \textit{and} \quad \mathbb{H} \cong (\mathbb{H}_+)^+.$$

Proof. Immediately follows from Propositions 22 and 23. □

4.4 Multi-type bilattice logic

The results of Section 4.3 show that HBL (resp. HCBL) is an equivalent presentation of B (resp. CB), and motivate from a semantic perspective the syntactic shift we take in the present section, from a single-type language to a multi-type language.[5] Indeed,

[5]In what follows, we only introduce the multi-type language associated with HCBL. The language associated with HBL can be obtained by removing the unary operators ${\sim_1}$ and ${\sim_2}$.

heterogeneous algebras provide a natural interpretation for the following multi-type language $\mathcal{L}_{\mathrm{MT}}$ consisting of terms of types L_1 and L_2.

$$\mathsf{L}_1 \ni A_1 ::= p_1 \mid 1_1 \mid 0_1 \mid \mathrm{p}A_2 \mid\sim_1 A_1 \mid A_1 \sqcap_1 A_1 \mid A_1 \sqcup_1 A_1$$

$$\mathsf{L}_2 \ni A_2 ::= p_2 \mid 1 \mid 0 \mid \mathrm{n}A_1 \mid\sim_2 A_2 \mid A_2 \cap A_2 \mid A_2 \cup A_2$$

The interpretation of $\mathcal{L}_{\mathrm{MT}}$-terms into HCBLs is defined as the easy generalization of the interpretation of propositional languages in universal algebra; namely, L_1-terms (resp. $_2$-terms) are interpreted in the first and second De Morgan algebras of any HCBL, respectively.

The toggle between CB and HCBL (cf. Sections 4.3) is reflected syntactically by the translations $t_1(\cdot), t_2(\cdot) : \mathcal{L} \to \mathcal{L}_{\mathrm{MT}}$ defined as follows:

$t_1(p)$	:=	p_1	$t_2(p)$	:=	p_2
$t_1(\mathtt{t})$	:=	1_1	$t_2(\mathtt{t})$	:=	0
$t_1(\mathtt{f})$	:=	0_1	$t_2(\mathtt{f})$	:=	1
$t_1(\top)$	:=	1_1	$t_2(\top)$	:=	1
$t_1(\bot)$	:=	0_1	$t_2(\bot)$	:=	0
$t_1(A \wedge B)$	:=	$t_1(A) \sqcap_1 t_1(B)$	$t_2(A \wedge B)$	:=	$t_2(A) \cup t_2(B)$
$t_1(A \vee B)$	:=	$t_1(A) \sqcup_1 t_1(B)$	$t_2(A \vee B)$	:=	$t_2(A) \cap t_2(B)$
$t_1(A \otimes B)$	:=	$t_1(A) \sqcap_1 t_1(B)$	$t_2(A \otimes B)$	:=	$t_2(A) \cap t_2(B)$
$t_1(A \oplus B)$	:=	$t_1(A) \sqcup_1 t_1(B)$	$t_2(A \oplus B)$	:=	$t_2(A) \cup t_2(B)$
$t_1(\neg A)$	:=	$\mathrm{p}t_2(A)$	$t_2(\neg A)$	:=	$\mathrm{n}t_1(A)$
$t_1(-A)$	:=	$\mathrm{p} \sim_2 t_2(A)$	$t_2(-A)$	:=	$\mathrm{n} \sim_1 t_1(A)$

The translations above are compatible with the toggle between B (resp. CB) and HBL (resp. HCBL). Indeed, recall that $\mathbb{B}^+$ denotes the heterogeneous algebra associated with a given $\mathbb{B} \in \mathsf{B}$ (cf. Definition 35). The following proposition is proved by a routine induction on $\mathcal{L}$-formulas.

Proposition 25. *For all $\mathcal{L}$-formulas A and B and every $\mathbb{B} \in \mathsf{B}$ (resp. $\mathbb{B} \in \mathsf{CB}$),*

$$\mathbb{B} \models A \leq B \quad \textit{iff} \quad \mathbb{B}^+ \models t_1(A) \leq t_1(B).$$

4.5 Multi-type proper display calculus

In this section we introduce the proper display calculus D.BL (D.CBL) for bilattice logic (with conflation).

Language

The language $\mathcal{L}_{\mathrm{MT}}$ of D.CBL is given by the union of the sets $\mathcal{L}_1$ and $\mathcal{L}_2$ defined as follows. $\mathcal{L}_1$ is given by simultaneous induction over the set $\mathsf{AtProp}_1 = \{p_1, q_1, r_1, \ldots\}$ of L_1-type atomic propositions as follows:

$$A_1 ::= p_1 \mid 1_1 \mid 0_1 \mid \mathrm{p}\, A_2 \mid\sim_1 A_1 \mid A_1 \sqcap_1 A_1 \mid A_1 \sqcup_1 A_1$$
$$X_1 ::= A_1 \mid \hat{1}_1 \mid \check{0}_1 \mid \mathrm{P}\, X_2 \mid *_1 X_1 \mid X_1 \mathbin{\hat{\sqcap}_1} X_1 \mid X_1 \mathbin{\check{\sqcup}_1} X_1 \mid X_1 \mathbin{\check{\sqsupset}_1} X_1 \mid X_1 \mathbin{\hat{\sqsubset}_1} X_1$$

$\mathcal{L}_2$ is given by simultaneous induction over the set $\mathsf{AtProp}_2 = \{p_2, q_2, r_2, \ldots\}$ of L_2-type atomic propositions as follows:

$$A_2 ::= p_2 \mid 1 \mid 0 \mid \mathrm{n}\, A_1 \mid \sim_2\, A_2 \mid A_2 \cap A_2 \mid A_2 \cup A_2$$
$$X_2 ::= A_2 \mid \hat{1} \mid \check{0} \mid \mathrm{N}\, X_1 \mid *_2\, X_2 \mid X_2 \mathbin{\hat{\cap}} X_2 \mid X_2 \mathbin{\check{\cup}} X_2 \mid X_2 \mathbin{\check{\sqsupset}_2} X_2 \mid X_2 \mathbin{\hat{\sqsubset}_2} X_2$$

The language of D.BL is the $\{*_1, *_2, \sim_1, \sim_2\}$-free fragment of $\mathcal{L}_{\mathrm{MT}}$.

Rules

For $i \in \{1, 2\}$,

- Pure L_i-type display rules

$$\mathsf{res}\,\frac{X_i \mathbin{\hat{\sqcap}_i} Y_i \vdash Z_i}{X_i \vdash Y_i \mathbin{\check{\sqsupset}} Z_i} \qquad \frac{X_i \vdash Y_i \mathbin{\check{\sqcup}_i} Z_i}{X_i \mathbin{\hat{\sqsubset}_i} Y_i \vdash Z_i}\,\mathsf{res}$$

- Multi-type display rules

$$\mathsf{adj}\,\frac{\mathrm{P}\, X_2 \vdash Y_1}{X_2 \vdash \mathrm{N}\, Y_1} \qquad \frac{\mathrm{N}\, X_1 \vdash Y_2}{X_1 \vdash \mathrm{P}\, Y_2}\,\mathsf{adj}$$

- Pure L_i-type identity and cut rules

$$\mathrm{Id_i}\,\frac{}{p_i \vdash p_i} \qquad \frac{X_i \vdash A_i \qquad A_i \vdash Y_i}{X_i \vdash Y_i}\,\mathsf{Cut}$$

- Pure L_i-type structural rules

$$\hat{1}_i\,\frac{X_i \mathbin{\hat{\sqcap}_i} \hat{1}_i \vdash Y_i}{X_i \vdash Y_i} \qquad \frac{X_i \vdash Y_i \mathbin{\check{\sqcup}_i} \check{0}_i}{X_i \vdash Y_i}\,\check{0}_i$$

$$\mathsf{E}\,\frac{X_i \mathbin{\hat{\sqcap}_i} Y_i \vdash Z_i}{Y_i \mathbin{\hat{\sqcap}_i} X_i \vdash Z_i} \qquad \frac{X_i \vdash Y_i \mathbin{\check{\sqcup}_i} Z_i}{X_i \vdash Z_i \mathbin{\check{\sqcup}_i} Y_i}\,\mathsf{E}$$

$$\mathsf{A}\,\frac{(X_i \mathbin{\hat{\sqcap}_i} Y_i) \mathbin{\hat{\sqcap}_i} Z_i \vdash W_i}{X_i \mathbin{\hat{\sqcap}_i} (Y_i \mathbin{\hat{\sqcap}_i} Z_i) \vdash W_i} \qquad \frac{X_i \vdash (Y_i \mathbin{\check{\sqcup}_i} Z_i) \mathbin{\check{\sqcup}_i} W_i}{X_i \vdash Y_i \mathbin{\check{\sqcup}_i} (Z_i \mathbin{\check{\sqcup}_i} W_i)}\,\mathsf{A}$$

$$\mathsf{W}\,\frac{X_i \vdash Z_i}{X_i \mathbin{\hat{\sqcap}_i} Y_i \vdash Z_i} \qquad \frac{X_i \vdash Y_i}{X_i \vdash Y_i \mathbin{\check{\sqcup}_i} Z_i}\,\mathsf{W}$$

$$\mathsf{C}\,\frac{X_i \mathbin{\hat{\sqcap}_i} X_i \vdash Z_i}{X_i \vdash Z_i} \qquad \frac{X_i \vdash Y_i \mathbin{\check{\sqcup}_i} Y_i}{X_i \vdash Y_i}\,\mathsf{C}$$

- Pure L_i type operational rules

$$1_i \frac{\hat{1}_i \vdash X_i}{1_i \vdash X_i} \qquad \frac{}{\hat{1}_i \vdash 1_i} 1_i$$

$$0_i \frac{}{0_i \vdash \check{0}_i} \qquad \frac{X_i \vdash \check{0}_i}{X_i \vdash 0_i} 0_i$$

$$\sqcap_i \frac{A_i \,\hat{\sqcap}_i\, B_i \vdash X_i}{A_i \sqcap_i B_i \vdash X_i} \qquad \frac{X_i \vdash A_i \qquad Y_i \vdash B_i}{X_i \,\hat{\sqcap}_i\, Y_i \vdash A_i \sqcap_i B_i} \sqcap_i$$

$$\sqcup_i \frac{A_i \vdash X_i \qquad B_i \vdash Y_i}{A_i \sqcup_i B_i \vdash X_i \,\check{\sqcup}_i\, Y_i} \qquad \frac{X_i \vdash A_i \,\check{\sqcup}_i\, B_i}{X_i \vdash A_i \sqcup_i B_i} \sqcup_i$$

- Multi-type structural rules

$$\mathrm{N} \frac{X_1 \vdash Y_1}{\mathrm{N}\, X_1 \vdash \mathrm{N}\, Y_1} \qquad \frac{X_2 \vdash Y_2}{\mathrm{P}\, X_2 \vdash \mathrm{P}\, Y_2} \mathrm{P}$$

$$\mathrm{P}\hat{1} \frac{\hat{1}_1 \vdash X_1}{\mathrm{P}\,\hat{1} \vdash X_1} \qquad \frac{X_1 \vdash \check{0}_1}{X_1 \vdash \mathrm{P}\,\check{0}} \mathrm{P}\check{0}$$

- Multi-type operational rules

$$\mathrm{n} \frac{\mathrm{N}\, A_1 \vdash X_2}{\mathrm{n}\, A_1 \vdash X_2} \qquad \frac{X_2 \vdash \mathrm{N}\, A_1}{X_2 \vdash \mathrm{n}\, A_1} \mathrm{n}$$

$$\mathrm{p} \frac{\mathrm{P}\, A_2 \vdash X_1}{\mathrm{p}\, A_2 \vdash X_1} \qquad \frac{X_1 \vdash \mathrm{P}\, A_2}{X_1 \vdash \mathrm{p}\, A_2} \mathrm{p}$$

The multi-type display calculus D.CBL also includes the following rules:

- Pure L_i display structural rules:

$$\mathsf{adj}* \frac{*_i\, X_i \vdash Y_i}{*_i\, Y_i \vdash X_i} \qquad \frac{X_i \vdash *_i\, Y_i}{Y_i \vdash *_i\, X_i} \mathsf{adj}*$$

- Pure L_i structural rules:

$$\mathsf{cont} \frac{X_i \vdash Y_i}{*_i\, Y_i \vdash *_i\, X_i}$$

- Multi-type structural rules:

$$*_2\mathrm{N} \frac{\mathrm{N} *_1 X_1 \vdash Y_2}{*_2\, \mathrm{N}\, X_1 \vdash Y_2} \qquad \frac{X_2 \vdash \mathrm{N} *_1 Y_1}{X_2 \vdash *_2\, \mathrm{N}\, Y_1} *_2\mathrm{N}$$

- Pure L_i operational rules:

$$\sim_i \frac{*_i\; X_i \vdash Y_i}{\sim_i\; X_i \vdash Y_i} \qquad \frac{X_i \vdash *_i\, Y_i}{X_i \vdash \sim_i\; Y_i} \sim_i$$

An essential feature of our calculus is that the logical rules are standard introduction rules of display calculi. This is key for achieving a canonical proof of cut elimination. The special behaviour of negation is captured by a suitable translation in a multi-type environment, which makes it possible to circumvent the technical difficulties created by the non-standard introduction rules of [1].

4.6 Properties

In this section, we sketch the proofs of the main properties of the calculi D.BL and D.CBL. We only sketch them since these proofs are instances of general facts of the theory of multi-type calculi.

4.6.1 Soundness

We outline the verification of soundness of the rules of D.BL (resp. D.CBL) w.r.t. the semantics of *perfect* HBL (resp. HCBL). The first step consists in interpreting structural symbols as their corresponding logical symbols. This induces a natural interpretation of structural terms as logical / algebraic terms, which we omit. Then we interpret sequents as inequalities, and rules as quasi-inequalities. The verification of soundness of the rules of D.BL (resp. D.CBL) then consists in checking the validity of their corresponding quasi-inequalities in perfect HBL (resp. HCBL). For example, the rules on the left-hand side below are interpreted as the quasi-inequalities on the right-hand side:

$$\frac{\mathrm{P}\, X_2 \vdash Y_1}{X_2 \vdash \mathrm{N}\, Y_1} \rightsquigarrow \forall a_2 \forall b_1 \, [\mathrm{p}(a_2) \leq_1 b_1 \;\Leftrightarrow\; a_2 \leq_2 \mathrm{n}(b_1)]$$

$$\frac{X_i \vdash Y_i}{*_i\, Y_i \vdash *_i\, X_i} \rightsquigarrow \forall a_i \forall b_i \, [a_i \leq_i b_i \;\Leftrightarrow \sim_i b_i \leq_i \sim_i a_i]$$

The verification of soundness of pure-type rules and of the introduction rules following this procedure is routine, and is omitted. The validity of the quasi-inequalities corresponding to multi-type structural rules follows straightforwardly from the observation that the quasi-inequality corresponding to each rule is obtained by running the algorithm ALBA [27, Section 3.4] on one of the defining inequalities of HBL (resp. HCBL).[6] For instance, the soundness of the first rule above is due to p and n being inverse to each other (see discussion after Definition 34).

4.6.2 Completeness

Proposition 26. *For every formula A of BL (resp. CBL), the sequents $t_1(A) \vdash t_1(A)$ and $t_2(A) \vdash t_2(A)$ are derivable in D.BL (resp. D.CBL).*

Proof. By induction on the complexity of the formula A. If A is an atomic formula, the translation of $t_i(A) \vdash t_i(A)$ with $i \in \{1, 2\}$ is $A_i \vdash A_i$, which is derivable using (Id) in L_1 and L_2, respectively. If $A = B \otimes C$, then $t_i(B \otimes C) = t_i(B) \sqcap_i t_i(C)$ and if $A = B \oplus C$, then $t_i(B \oplus C) = t_i(B) \sqcup_i t_i(C)$. By induction hypothesis, $t_i(A_i) \vdash t_i(A_i)$. The following

[6] As discussed in [27], the soundness of the rewriting rules of ALBA only depends on the order-theoretic properties of the interpretation of the logical connectives and their adjoints and residuals. The fact that some of these maps are not internal operations but have different domains and codomains does not make any substantial difference.

derivations complete the proof:

$$
\begin{prooftree}
\AxiomC{$t_i(B) \vdash t_i(B)$}
\LeftLabel{W}
\UnaryInfC{$t_i(B) \,\hat{\sqcap}_i\, t_i(C) \vdash t_i(B)$}
\AxiomC{$t_i(C) \vdash t_i(C)$}
\LeftLabel{W}
\UnaryInfC{$t_i(C) \,\hat{\sqcap}_i\, t_i(B) \vdash t_i(C)$}
\LeftLabel{E}
\UnaryInfC{$t_i(B) \,\hat{\sqcap}_i\, t_i(C) \vdash t_i(C)$}
\BinaryInfC{$(t_i(B) \,\hat{\sqcap}_i\, t_i(C)) \,\hat{\sqcap}_i\, (t_i(B) \,\hat{\sqcap}_i\, t_i(C)) \vdash t_i(B) \sqcap_i t_i(C)$}
\LeftLabel{C}
\UnaryInfC{$t_i(B) \,\hat{\sqcap}_i\, t_i(C) \vdash t_i(B) \sqcap_i t_i(C)$}
\UnaryInfC{$t_i(B) \sqcap_i t_i(C) \vdash t_i(B) \sqcap_i t_i(C)$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$t_i(B) \vdash t_i(B)$}
\RightLabel{W}
\UnaryInfC{$t_i(B) \vdash t_i(B) \,\check{\sqcup}_i\, t_i(C)$}
\AxiomC{$t_i(C) \vdash t_i(C)$}
\RightLabel{W}
\UnaryInfC{$t_i(C) \vdash t_i(C) \,\check{\sqcup}_i\, t_i(B)$}
\RightLabel{E}
\UnaryInfC{$t_i(C) \vdash t_i(B) \,\check{\sqcup}_i\, t_i(C)$}
\BinaryInfC{$t_i(B) \sqcup_i t_i(C) \vdash (t_i(B) \,\check{\sqcup}_i\, t_i(C)) \,\check{\sqcup}_i\, (t_i(B) \,\check{\sqcup}_i\, t_i(C))$}
\RightLabel{C}
\UnaryInfC{$t_i(B) \sqcup_i t_i(C) \vdash t_i(B) \,\check{\sqcup}_i\, t_i(C)$}
\UnaryInfC{$t_i(B) \sqcup_i t_i(C) \vdash t_i(B) \sqcup_i t_i(C)$}
\end{prooftree}
$$

The arguments for $A = B \wedge C$ and $A = B \vee C$ are similar and they are omitted.

If $A = \neg B$, then $t_1(\neg B) = \mathrm{p}t_2(B)$ and $t_2(\neg B) = \mathrm{n}t_1(B)$. By induction hypothesis $t_i(A) \vdash t_i(A)$. Hence, the following derivations complete the proof:

$$
\begin{prooftree}
\AxiomC{$t_2(B) \vdash t_2(B)$}
\RightLabel{P}
\UnaryInfC{$\mathrm{P}\, t_2(B) \vdash \mathrm{P}\, t_2(B)$}
\UnaryInfC{$\mathrm{P}\, t_2(B) \vdash \mathrm{p}\, t_2(B)$}
\UnaryInfC{$\mathrm{p}\, t_2(B) \vdash \mathrm{p}\, t_2(B)$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{$t_1(B) \vdash t_1(B)$}
\LeftLabel{N}
\UnaryInfC{$\mathrm{N}\, t_1(B) \vdash \mathrm{N}\, t_1(B)$}
\UnaryInfC{$\mathrm{N}\, t_1(B) \vdash \mathrm{n}\, t_1(B)$}
\UnaryInfC{$\mathrm{n}\, t_1(B) \vdash \mathrm{n}\, t_1(B)$}
\end{prooftree}
$$

If $A = -B$, then $t_1(-B) = \mathrm{p} \sim_2 t_2(B)$ and $t_2(-B) = \mathrm{n} \sim_1 t_1(B)$. By induction hypothesis $t_i(B) \vdash t_i(B)$. Hence, the following derivations complete the proof:

$$
\begin{prooftree}
\AxiomC{$t_2(B) \vdash t_2(B)$}
\RightLabel{cont}
\UnaryInfC{$*_2\, t_2(B) \vdash *_2\, t_2(B)$}
\UnaryInfC{$*_2\, t_2(B) \vdash \sim_2 t_2(B)$}
\UnaryInfC{$\sim_2 t_2(B) \vdash \sim_2 t_2(B)$}
\RightLabel{P}
\UnaryInfC{$\mathrm{P} \sim_2 t_2(B) \vdash \mathrm{P} \sim_2 t_2(B)$}
\UnaryInfC{$\mathrm{p} \sim_2 t_2(B) \vdash \mathrm{P} \sim_2 t_2(B)$}
\UnaryInfC{$\mathrm{p} \sim_2 t_2(B) \vdash \mathrm{p} \sim_2 t_2(B)$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{$t_1(B) \vdash t_1(B)$}
\RightLabel{cont}
\UnaryInfC{$*_1\, t_1(B) \vdash *_1\, t_1(B)$}
\UnaryInfC{$*_1\, t_1(B) \vdash \sim_1 t_1(B)$}
\UnaryInfC{$\sim_1 t_2(B) \vdash \sim_1 t_1(B)$}
\LeftLabel{N}
\UnaryInfC{$\mathrm{N} \sim_1 t_1(B) \vdash \mathrm{N} \sim_1 t_1(B)$}
\UnaryInfC{$\mathrm{N} \sim_1 t_1(B) \vdash \mathrm{n} \sim_1 t_1(B)$}
\UnaryInfC{$\mathrm{n} \sim_1 t_1(B) \vdash \mathrm{n} \sim_1 t_1(B)$}
\end{prooftree}
$$

□

Proposition 27. *For all formulas A, B of* BL *(resp. CBL), if $A \vdash B$ is derivable in BL (resp. CBL), then $t_1(A) \vdash t_1(B)$ is derivable in D.BL (resp. D.CBL).*

Proof. In what follows we show that the translations of the axioms and rules of BL (resp C.BL) are derivable in D.BL (resp. D.CBL). Since BL (resp C.BL) is complete w.r.t. the class of bilattice algebras (by Theorem 14), and hence w.r.t their associated heterogeneous algebras (by Propositions 22 and 23), this is enough to show the completeness of D.BL (resp. D.CBL). For the sake of readability, although each formula A in precedent (resp. succedent) position should be written as $t_1(A)$, we suppress it in the derivation trees of the axioms.

The Identity axiom $A \vdash A$ is proved in Proposition 26.

The derivations of the binary rules are standard and we omit them.

As to $\mathtt{f} \vdash A$, by the translation, $t_1(\mathtt{f}) = 0_1$, hence we can prove $0_1 \vdash A_1$ in D.BL by the introduction rule of 0_1 on the right side, (W) $(\check{0}_1)$ and the introduction rule of 0_1 on the left side. The proofs of $A \vdash \mathtt{t}$, $\bot \vdash A$ and $A \vdash \top$ are analogous.

As to $A \vdash \neg\mathtt{f}$, by the translation, $t_1(\neg\mathtt{f}) = \mathrm{p}t_2(\mathtt{f}) = \mathrm{p}1_2$, hence we can prove $A_1 \vdash \mathrm{p}1_2$ by the introduction rule of (1_2) on the left side, (W), $(\hat{1})$, (adj), (p) and the introduction rule of (1_2) on the right side. The proofs of $\neg\mathtt{t} \vdash A$, $\neg\bot \vdash A$, $A \vdash \neg\top$ are are analogous.

In what follows, we let the sequent on the right side of $\rightsquigarrow$ denote the result of the translation, then we can show the translations of other axioms in BL are also derivable in D.BL as follows:

- $\neg\neg A \dashv\vdash A \quad \rightsquigarrow \quad A_1 \dashv\vdash \mathrm{pn}A_1$,

$$
\begin{prooftree}
\AxiomC{$A_1 \vdash A_1$}
\RightLabel{N}
\UnaryInfC{$\mathrm{N}A_1 \vdash \mathrm{N}A_1$}
\UnaryInfC{$\mathrm{n}A_1 \vdash \mathrm{N}A_1$}
\LeftLabel{adj}
\UnaryInfC{$\mathrm{Pn}A_1 \vdash A_1$}
\UnaryInfC{$\mathrm{pn}A_1 \vdash A_1$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{$A_1 \vdash A_1$}
\LeftLabel{N}
\UnaryInfC{$\mathrm{N}A_1 \vdash \mathrm{N}A_1$}
\UnaryInfC{$\mathrm{N}A_1 \vdash \mathrm{n}A_1$}
\RightLabel{adj}
\UnaryInfC{$A_1 \vdash \mathrm{Pn}A_1$}
\UnaryInfC{$A_1 \vdash \mathrm{pn}A_1$}
\end{prooftree}
$$

- $- - A \dashv\vdash A \quad \rightsquigarrow \quad \mathrm{p} \sim_2 \mathrm{n} \sim_1 A_1 \dashv\vdash A_1$,

$$
\begin{prooftree}
\AxiomC{$A_1 \vdash A_1$}
\LeftLabel{cont}
\UnaryInfC{$*_1A_1 \vdash *_1A_1$}
\UnaryInfC{$*_1A_1 \vdash \sim_1 A_1$}
\LeftLabel{N}
\UnaryInfC{$\mathrm{N} *_1 A_1 \vdash \mathrm{N} \sim_1 A_1$}
\UnaryInfC{$\mathrm{N} *_1 A_1 \vdash \mathrm{n} \sim_1 A_1$}
\LeftLabel{$*_2$N}
\UnaryInfC{$*_2\mathrm{N}A_1 \vdash \mathrm{n} \sim_1 A_1$}
\LeftLabel{adj$*$}
\UnaryInfC{$*_2\mathrm{n} \sim_1 A_1 \vdash \mathrm{N}A_1$}
\UnaryInfC{$\sim_2 \mathrm{n} \sim_1 A_1 \vdash \mathrm{N}A_1$}
\LeftLabel{adj}
\UnaryInfC{$\mathrm{P} \sim_2 \mathrm{n} \sim_1 A_1 \vdash A_1$}
\UnaryInfC{$\mathrm{p} \sim_2 \mathrm{n} \sim_1 A_1 \vdash A_1$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{$A_1 \vdash A_1$}
\LeftLabel{cont}
\UnaryInfC{$*_1A_1 \vdash *_1A_1$}
\UnaryInfC{$\sim_1 A_1 \vdash *_1A_1$}
\LeftLabel{N}
\UnaryInfC{$\mathrm{N} \sim_1 A_1 \vdash \mathrm{N} *_1 A_1$}
\UnaryInfC{$\mathrm{n} \sim_1 A_1 \vdash \mathrm{N} *_1 A_1$}
\RightLabel{$*_2$N}
\UnaryInfC{$\mathrm{n} \sim_1 A_1 \vdash *_2\mathrm{N}A_1$}
\RightLabel{adj$*$}
\UnaryInfC{$\mathrm{N}A_1 \vdash *_2\mathrm{n} \sim_1 A_1$}
\UnaryInfC{$\mathrm{N}A_1 \vdash \sim_2 \mathrm{n} \sim_1 A_1$}
\RightLabel{adj}
\UnaryInfC{$A_1 \vdash \mathrm{P} \sim_2 \mathrm{n} \sim_1 A_1$}
\UnaryInfC{$A_1 \vdash \mathrm{p} \sim_2 \mathrm{n} \sim_1 A_1$}
\end{prooftree}
$$

- $-\neg A \dashv\vdash \neg - A \quad \rightsquigarrow \quad \mathrm{p} \sim_2 \mathrm{n}A_1 \dashv\vdash \mathrm{pn} \sim_1 A_1$,

$$
\begin{prooftree}
\AxiomC{$A_1 \vdash A_1$}
\RightLabel{\textsf{cont}}
\UnaryInfC{$*_1 A_1 \vdash *_1 A_1$}
\UnaryInfC{$*_1 A_1 \vdash \sim_1 A_1$}
\LeftLabel{$*$}
\UnaryInfC{$*_1 \sim_1 A_1 \vdash A_1$}
\LeftLabel{N}
\UnaryInfC{$\mathrm{N} *_1 \sim_1 A_1 \vdash \mathrm{N} A_1$}
\UnaryInfC{$\mathrm{N} *_1 \sim_1 A_1 \vdash \mathrm{n} A_1$}
\LeftLabel{$*_2\mathrm{N}$}
\UnaryInfC{$*_2 \mathrm{N} \sim_1 A_1 \vdash \mathrm{n} A_1$}
\LeftLabel{\textsf{adj}$*$}
\UnaryInfC{$*_2 \mathrm{n} A_1 \vdash \mathrm{N} \sim_1 A_1$}
\UnaryInfC{$\sim_2 \mathrm{n} A_1 \vdash \mathrm{N} \sim_1 A_1$}
\UnaryInfC{$\sim_2 \mathrm{n} A_1 \vdash \mathrm{n} \sim_1 A_1$}
\RightLabel{P}
\UnaryInfC{$\mathrm{P} \sim_2 \mathrm{n} A_1 \vdash \mathrm{Pn} \sim_1 A_1$}
\UnaryInfC{$\mathrm{P} \sim_2 \mathrm{n} A_1 \vdash \mathrm{pn} \sim_1 A_1$}
\UnaryInfC{$\mathrm{p} \sim_2 \mathrm{n} A_1 \vdash \mathrm{pn} \sim_1 A_1$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{$A_1 \vdash A_1$}
\RightLabel{\textsf{cont}}
\UnaryInfC{$*_1 A_1 \vdash *_1 A_1$}
\UnaryInfC{$\sim_1 A_1 \vdash *_1 A_1$}
\RightLabel{\textsf{adj}$*$}
\UnaryInfC{$A_1 \vdash *_1 \sim_1 A_1$}
\LeftLabel{N}
\UnaryInfC{$\mathrm{N} A_1 \vdash \mathrm{N} *_1 \sim_1 A_1$}
\UnaryInfC{$\mathrm{n} A_1 \vdash \mathrm{N} *_1 \sim_1 A_1$}
\RightLabel{$*_2\mathrm{N}$}
\UnaryInfC{$\mathrm{n} A_1 \vdash *_2 \mathrm{N} \sim_1 A_1$}
\RightLabel{\textsf{adj}$*$}
\UnaryInfC{$\mathrm{N} \sim_1 A_1 \vdash *_2 \mathrm{n} A_1$}
\UnaryInfC{$\mathrm{n} \sim_1 A_1 \vdash *_2 \mathrm{n} A_1$}
\UnaryInfC{$\mathrm{n} \sim_1 A_1 \vdash \sim_2 \mathrm{n} A_1$}
\RightLabel{P}
\UnaryInfC{$\mathrm{Pn} \sim_1 A_1 \vdash \mathrm{P} \sim_2 \mathrm{n} A_1$}
\UnaryInfC{$\mathrm{Pn} \sim_1 A_1 \vdash \mathrm{p} \sim_2 \mathrm{n} A_1$}
\UnaryInfC{$\mathrm{pn} \sim_1 A_1 \vdash \mathrm{p} \sim_2 \mathrm{n} A_1$}
\end{prooftree}
$$

- $\neg A \wedge \neg B \dashv\vdash \neg(A \vee B) \quad \rightsquigarrow \quad \mathrm{p}A_2 \sqcap_1 \mathrm{p}B_2 \dashv\vdash \mathrm{p}(A_2 \sqcap_2 B_2)$,

$$
\begin{prooftree}
\AxiomC{$A_2 \vdash A_2$}
\RightLabel{P}
\UnaryInfC{$\mathrm{P}A_2 \vdash \mathrm{P}A_2$}
\UnaryInfC{$\mathrm{p}A_2 \vdash \mathrm{P}A_2$}
\LeftLabel{\textsf{W}}
\UnaryInfC{$\mathrm{p}A_2 \mathbin{\hat{\sqcap}_1} \mathrm{p}B_2 \vdash \mathrm{P}A_2$}
\UnaryInfC{$\mathrm{p}A_2 \sqcap_1 \mathrm{p}B_2 \vdash \mathrm{P}A_2$}
\RightLabel{\textsf{adj}}
\UnaryInfC{$\mathrm{N}(\mathrm{p}A_2 \sqcap_1 \mathrm{p}B_2) \vdash A_2$}
\AxiomC{$B_2 \vdash B_2$}
\RightLabel{P}
\UnaryInfC{$\mathrm{P}B_2 \vdash \mathrm{P}B_2$}
\UnaryInfC{$\mathrm{p}B_2 \vdash \mathrm{P}B_2$}
\LeftLabel{\textsf{W}}
\UnaryInfC{$\mathrm{p}B_2 \mathbin{\hat{\sqcap}_1} \mathrm{p}A_2 \vdash \mathrm{P}B_2$}
\LeftLabel{\textsf{E}}
\UnaryInfC{$\mathrm{p}A_2 \mathbin{\hat{\sqcap}_1} \mathrm{p}B_2 \vdash \mathrm{P}B_2$}
\UnaryInfC{$\mathrm{p}A_2 \sqcap_1 \mathrm{p}B_2 \vdash \mathrm{P}B_2$}
\RightLabel{\textsf{adj}}
\UnaryInfC{$\mathrm{N}(\mathrm{p}A_2 \sqcap_1 \mathrm{p}B_2) \vdash B_2$}
\BinaryInfC{$\mathrm{N}(\mathrm{p}A_2 \sqcap_1 \mathrm{p}B_2) \mathbin{\hat{\sqcap}_2} \mathrm{N}(\mathrm{p}A_2 \sqcap_1 \mathrm{p}B_2) \vdash A_2 \sqcap_2 B_2$}
\LeftLabel{\textsf{C}}
\UnaryInfC{$\mathrm{N}(\mathrm{p}A_2 \sqcap_1 \mathrm{p}B_2) \vdash A_2 \sqcap_2 B_2$}
\RightLabel{\textsf{adj}}
\UnaryInfC{$\mathrm{p}A_2 \sqcap_1 \mathrm{p}B_2 \vdash \mathrm{P}(A_2 \sqcap_2 B_2)$}
\UnaryInfC{$\mathrm{p}A_2 \sqcap_1 \mathrm{p}B_2 \vdash \mathrm{p}(A_2 \sqcap_2 B_2)$}
\end{prooftree}
$$

- $\neg A \otimes \neg B \dashv\vdash \neg(A \otimes B) \quad \rightsquigarrow \quad \mathrm{p}A_2 \sqcap_1 \mathrm{p}B_2 \dashv\vdash \mathrm{p}(A_2 \sqcap_2 B_2)$,

$$
\begin{prooftree}
\AxiomC{$A_2 \vdash A_2$}
\LeftLabel{\textsf{W}}
\UnaryInfC{$A_2 \mathbin{\hat{\sqcap}_2} B_2 \vdash A_2$}
\UnaryInfC{$A_2 \sqcap_2 B_2 \vdash A_2$}
\RightLabel{P}
\UnaryInfC{$\mathrm{P}(A_2 \sqcap_2 B_2) \vdash \mathrm{P}A_2$}
\UnaryInfC{$\mathrm{P}(A_2 \sqcap_2 B_2) \vdash \mathrm{p}A_2$}
\UnaryInfC{$\mathrm{p}(A_2 \sqcap_2 B_2) \vdash \mathrm{p}A_2$}
\AxiomC{$B_2 \vdash B_2$}
\LeftLabel{\textsf{W}}
\UnaryInfC{$B_2 \mathbin{\hat{\sqcap}_2} A_2 \vdash B_2$}
\LeftLabel{\textsf{E}}
\UnaryInfC{$A_2 \mathbin{\hat{\sqcap}_2} B_2 \vdash B_2$}
\UnaryInfC{$A_2 \sqcap_2 B_2 \vdash B_2$}
\RightLabel{P}
\UnaryInfC{$\mathrm{P}(A_2 \sqcap_2 B_2) \vdash \mathrm{P}B_2$}
\UnaryInfC{$\mathrm{P}(A_2 \sqcap_2 B_2) \vdash \mathrm{p}B_2$}
\UnaryInfC{$\mathrm{p}(A_2 \sqcap_2 B_2) \vdash \mathrm{p}B_2$}
\BinaryInfC{$\mathrm{p}(A_2 \sqcap_2 B_2) \mathbin{\hat{\sqcap}_1} \mathrm{p}(A_2 \sqcap_2 B_2) \vdash \mathrm{p}A_2 \sqcap_1 \mathrm{p}B_2$}
\LeftLabel{\textsf{C}}
\UnaryInfC{$\mathrm{p}(A_2 \sqcap_2 B_2) \vdash \mathrm{p}A_2 \sqcap_1 \mathrm{p}B_2$}
\end{prooftree}
$$

- $\neg(A \wedge B) \dashv\vdash \neg A \vee \neg B \quad \rightsquigarrow \quad \mathrm{p}(A_2 \sqcup_2 B_2) \dashv\vdash \mathrm{p}A_2 \sqcup_1 \mathrm{p}B_2$,

$$
\dfrac{
\dfrac{
\mathsf{adj}\,\dfrac{\dfrac{\dfrac{A_2 \vdash A_2}{\mathrm{P}A_2 \vdash \mathrm{P}A_2}\,\mathrm{P}}{\mathrm{P}A_2 \vdash \mathrm{p}A_2}}{\dfrac{\mathrm{P}A_2 \vdash \mathrm{p}A_2 \check{\sqcup}_1 \mathrm{p}B_2}{A_2 \vdash \mathrm{N}(\mathrm{p}A_2 \check{\sqcup}_1 \mathrm{p}B_2)}}\,\mathsf{W}
\qquad
\mathsf{adj}\,\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B_2 \vdash B_2}{\mathrm{P}B_2 \vdash \mathrm{P}B_2}\,\mathrm{P}}{\mathrm{P}B_2 \vdash \mathrm{p}B_2}}{\mathrm{P}B_2 \vdash \mathrm{p}B_2 \check{\sqcup}_1 \mathrm{p}A_2}\,\mathsf{W}}{\mathrm{P}B_2 \vdash \mathrm{p}A_2 \check{\sqcup}_1 \mathrm{p}B_2}\,\mathsf{E}}{B_2 \vdash \mathrm{N}(\mathrm{p}A_2 \check{\sqcup}_1 \mathrm{p}B_2)}
}{
\dfrac{A_2 \sqcup_2 B_2 \vdash \mathrm{N}(\mathrm{p}A_2 \check{\sqcup}_1 \mathrm{p}B_2) \check{\sqcup}_1 \mathrm{N}(\mathrm{p}A_2 \check{\sqcup}_1 \mathrm{p}B_2)}{A_2 \sqcup_2 B_2 \vdash \mathrm{N}(\mathrm{p}A_2 \check{\sqcup}_1 \mathrm{p}B_2)}\,\mathsf{C}
}
}{
\mathsf{adj}\,\dfrac{}{\dfrac{\dfrac{\mathrm{P}(A_2 \sqcup_2 B_2) \vdash \mathrm{p}A_2 \check{\sqcup}_1 \mathrm{p}B_2}{\mathrm{p}(A_2 \sqcup_2 B_2) \vdash \mathrm{p}A_2 \check{\sqcup}_1 \mathrm{p}B_2}}{\mathrm{p}(A_2 \sqcup_2 B_2) \vdash \mathrm{p}A_2 \sqcup_1 \mathrm{p}B_2}}
}
$$

- $\neg(A \oplus B) \dashv\vdash \neg A \oplus \neg B \quad \rightsquigarrow \quad \mathrm{p}(A_2 \sqcup_2 B_2) \dashv\vdash \mathrm{p}A_2 \sqcup_1 \mathrm{p}B_2,$

$$
\dfrac{
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A_2 \vdash A_2}{\mathrm{P}A_2 \vdash \mathrm{P}A_2}\,\mathrm{P}}{\mathrm{p}A_2 \vdash \mathrm{P}A_2}}{\mathrm{Np}A_2 \vdash A_2}\,\mathsf{adj}}{\mathrm{Np}A_2 \vdash A_2 \sqcup_2 B_2}\,\mathsf{W}}{\mathrm{Np}A_2 \vdash A_2 \sqcup_2 B_2}}{\mathrm{p}A_2 \vdash \mathrm{P}(A_2 \sqcup_2 B_2)}\,\mathsf{adj}}{\mathrm{p}A_2 \vdash \mathrm{p}(A_2 \sqcup_2 B_2)}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B_2 \vdash B_2}{\mathrm{P}B_2 \vdash \mathrm{P}B_2}\,\mathrm{P}}{\mathrm{p}B_2 \vdash \mathrm{P}B_2}}{\mathrm{Np}B_2 \vdash B_2}\,\mathsf{adj}}{\mathrm{Np}B_2 \vdash B_2 \sqcup_2 A_2}\,\mathsf{W}}{\mathrm{Np}B_2 \vdash A_2 \sqcup_2 B_2}\,\mathsf{E}}{\mathrm{Np}B_2 \vdash A_2 \sqcup_2 B_2}}{\mathrm{p}B_2 \vdash \mathrm{P}(A_2 \sqcup_2 B_2)}\,\mathsf{adj}}{\mathrm{p}B_2 \vdash \mathrm{p}(A_2 \sqcup_2 B_2)}
}{
\mathrm{p}A_2 \sqcup_1 \mathrm{p}B_2 \vdash \mathrm{p}(A_2 \sqcup_2 B_2) \check{\sqcup}_1 \mathrm{p}(A_2 \sqcup_2 B_2)
}
}{
\mathrm{p}A_2 \sqcup_1 \mathrm{p}B_2 \vdash \mathrm{p}(A_2 \sqcup_2 B_2)
}\,\mathsf{C}
$$

- $-(A \wedge B) \dashv\vdash -A \wedge -B \quad \rightsquigarrow \quad \mathrm{p} \sim_2 (A_2 \sqcup_2 B_2) \dashv\vdash \mathrm{p} \sim_2 A_2 \sqcap_1 \mathrm{p} \sim_2 B_2,$

$$
\mathsf{C}\,\dfrac{
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathsf{cont}\,\dfrac{\dfrac{\dfrac{A_2 \vdash A_2}{A_2 \vdash A_2 \sqcup_2 B_2}\,\mathsf{W}}{A_2 \vdash A_2 \sqcup_2 B_2}}{*_2(A_2 \sqcup_2 B_2) \vdash *_2 A_2}}{*_2(A_2 \sqcup_2 B_2) \vdash \sim_2 A_2}}{\sim_2 (A_2 \sqcup_2 B_2) \vdash \sim_2 A_2}}{\mathrm{P} \sim_2 (A_2 \sqcup_2 B_2) \vdash \mathrm{P} \sim_2 A_2}\,\mathrm{P}}{\mathrm{P} \sim_2 (A_2 \sqcup_2 B_2) \vdash \mathrm{p} \sim_2 A_2}}{\mathrm{p} \sim_2 (A_2 \sqcup_2 B_2) \vdash \mathrm{p} \sim_2 A_2}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathsf{cont}\,\dfrac{\dfrac{\dfrac{\dfrac{B_2 \vdash B_2}{B_2 \vdash B_2 \sqcup_2 A_2}\,\mathsf{W}}{B_2 \vdash A_2 \sqcup_2 B_2}\,\mathsf{E}}{B_2 \vdash A_2 \sqcup_2 B_2}}{*_2(A_2 \sqcup_2 B_2) \vdash *_2 B_2}}{*_2(A_2 \sqcup_2 B_2) \vdash \sim_2 B_2}}{\sim_2 (A_2 \sqcup_2 B_2) \vdash \sim_2 B_2}}{\mathrm{P} \sim_2 (A_2 \sqcup_2 B_2) \vdash \mathrm{P} \sim_2 B_2}\,\mathrm{P}}{\dfrac{\mathrm{P} \sim_2 (A_2 \sqcup_2 B_2) \vdash \mathrm{p} \sim_2 B_2}{\mathrm{p} \sim_2 (A_2 \sqcup_2 B_2) \vdash \mathrm{p} \sim_2 B_2}}
}{
\mathrm{p} \sim_2 (A_2 \sqcup_2 B_2) \hat{\sqcap}_1 \mathrm{p} \sim_2 (A_2 \sqcup_2 B_2) \vdash \mathrm{p} \sim_2 A_2 \sqcap_1 \mathrm{p} \sim_2 B_2
}
}{
\mathrm{p} \sim_2 (A_2 \sqcup_2 B_2) \vdash \mathrm{p} \sim_2 A_2 \sqcap_1 \mathrm{p} \sim_2 B_2
}
$$

$$
\begin{prooftree}
\AxiomC{$A_2 \vdash A_2$}
\RightLabel{cont}
\UnaryInfC{$*_2 A_2 \vdash *_2 A_2$}
\UnaryInfC{$\sim_2 A_2 \vdash *_2 A_2$}
\RightLabel{P}
\UnaryInfC{$\mathrm{P} \sim_2 A_2 \vdash \mathrm{P} *_2 A_2$}
\UnaryInfC{$\mathrm{p} \sim_2 A_2 \vdash \mathrm{P} *_2 A_2$}
\LeftLabel{W}
\UnaryInfC{$\mathrm{p} \sim_2 A_2 \,\hat{\sqcap}_1\, \mathrm{p} \sim_2 B_2 \vdash \mathrm{P} *_2 A_2$}
\RightLabel{adj}
\UnaryInfC{$\mathrm{N}(\mathrm{p} \sim_2 A_2 \,\hat{\sqcap}_1\, \mathrm{p} \sim_2 B_2) \vdash *_2 A_2$}
\RightLabel{adj*}
\UnaryInfC{$B_2 \vdash *_2 \mathrm{N}(\mathrm{p} \sim_2 A_2 \,\hat{\sqcap}_1\, \mathrm{p} \sim_2 A_2)$}
\AxiomC{$B_2 \vdash B_2$}
\RightLabel{cont}
\UnaryInfC{$*_2 B_2 \vdash *_2 B_2$}
\UnaryInfC{$\sim_2 B_2 \vdash *_2 B_2$}
\RightLabel{P}
\UnaryInfC{$\mathrm{P} \sim_2 B_2 \vdash \mathrm{P} *_2 B_2$}
\UnaryInfC{$\mathrm{p} \sim_2 B_2 \vdash \mathrm{P} *_2 B_2$}
\LeftLabel{W}
\UnaryInfC{$\mathrm{p} \sim_2 B_2 \,\hat{\sqcap}_1\, \mathrm{p} \sim_2 A_2 \vdash \mathrm{P} *_2 B_2$}
\LeftLabel{E}
\UnaryInfC{$\mathrm{p} \sim_2 A_2 \,\hat{\sqcap}_1\, \mathrm{p} \sim_2 B_2 \vdash \mathrm{P} *_2 B_2$}
\RightLabel{adj}
\UnaryInfC{$\mathrm{N}(\mathrm{p} \sim_2 A_2 \,\hat{\sqcap}_1\, \mathrm{p} \sim_2 B_2) \vdash *_2 B_2$}
\RightLabel{adj*}
\UnaryInfC{$B_2 \vdash *_2 \mathrm{N}(\mathrm{p} \sim_2 A_2 \,\hat{\sqcap}_1\, \mathrm{p} \sim_2 B_2)$}
\BinaryInfC{$A_2 \sqcup_2 B_2 \vdash *_2 \mathrm{N}(\mathrm{p} \sim_2 A_2 \,\hat{\sqcap}_1\, \mathrm{p} \sim_2 B_2) \sqcup_2 *_2 \mathrm{N}(\mathrm{p} \sim_2 A_2 \,\hat{\sqcap}_1\, \mathrm{p} \sim_2 B_2)$}
\RightLabel{C}
\UnaryInfC{$A_2 \sqcup_2 B_2 \vdash *_2 \mathrm{N}(\mathrm{p} \sim_2 A_2 \,\hat{\sqcap}_1\, \mathrm{p} \sim_2 B_2)$}
\RightLabel{adj*}
\UnaryInfC{$\mathrm{N}(\mathrm{p} \sim_2 A_2 \,\hat{\sqcap}_1\, \mathrm{p} \sim_2 B_2) \vdash *_2 (A_2 \sqcup_2 B_2)$}
\UnaryInfC{$\mathrm{N}(\mathrm{p} \sim_2 A_2 \,\hat{\sqcap}_1\, \mathrm{p} \sim_2 B_2) \vdash \sim_2 (A_2 \sqcup_2 B_2)$}
\RightLabel{adj}
\UnaryInfC{$\mathrm{p} \sim_2 A_2 \,\hat{\sqcap}_1\, \mathrm{p} \sim_2 B_2 \vdash \mathrm{P} \sim_2 (A_2 \sqcup_2 B_2)$}
\UnaryInfC{$\mathrm{p} \sim_2 A_2 \,\hat{\sqcap}_1\, \mathrm{p} \sim_2 B_2 \vdash \mathrm{p} \sim_2 (A_2 \sqcup_2 B_2)$}
\LeftLabel{C}
\UnaryInfC{$\mathrm{p} \sim_2 A_2 \sqcap_1 \mathrm{p} \sim_2 B_2 \vdash \mathrm{p} \sim_2 (A_2 \sqcup_2 B_2)$}
\end{prooftree}
$$

- $-(A \otimes B) \dashv\vdash -A \oplus -B \quad \rightsquigarrow \quad \mathrm{p} \sim_2 (A_2 \sqcap_2 B_2) \dashv\vdash \mathrm{p} \sim_2 A_2 \sqcup_1 \mathrm{p} \sim_2 B_2,$

$$
\begin{prooftree}
\AxiomC{$A_2 \vdash A_2$}
\LeftLabel{cont}
\UnaryInfC{$*_2 A_2 \vdash *_2 A_2$}
\UnaryInfC{$*_2 A_2 \vdash \sim_2 A_2$}
\RightLabel{P}
\UnaryInfC{$\mathrm{P} *_2 A_2 \vdash \mathrm{P} \sim_2 A_2$}
\UnaryInfC{$\mathrm{P} *_2 A_2 \vdash \mathrm{p} \sim_2 A_2$}
\RightLabel{W}
\UnaryInfC{$\mathrm{P} *_2 A_2 \vdash \mathrm{p} \sim_2 A_2 \,\check{\sqcup}_1\, \mathrm{p} \sim_2 B_2$}
\UnaryInfC{$\mathrm{P} *_2 A_2 \vdash \mathrm{p} \sim_2 A_2 \sqcup_1 \mathrm{p} \sim_2 B_2$}
\LeftLabel{adj}
\UnaryInfC{$*_2 A_2 \vdash \mathrm{N}(\mathrm{p} \sim_2 A_2 \sqcup_1 \mathrm{p} \sim_2 B_2)$}
\LeftLabel{adj*}
\UnaryInfC{$*_2 \mathrm{N}(\mathrm{p} \sim_2 A_2 \sqcup_1 \mathrm{p} \sim_2 B_2) \vdash A_2$}
\AxiomC{$B_2 \vdash B_2$}
\LeftLabel{cont}
\UnaryInfC{$*_2 B_2 \vdash *_2 B_2$}
\UnaryInfC{$*_2 B_2 \vdash \sim_2 B_2$}
\RightLabel{P}
\UnaryInfC{$\mathrm{P} *_2 B_2 \vdash \mathrm{P} \sim_2 B_2$}
\UnaryInfC{$\mathrm{P} *_2 B_2 \vdash \mathrm{p} \sim_2 B_2$}
\RightLabel{W}
\UnaryInfC{$\mathrm{P} *_2 B_2 \vdash \mathrm{p} \sim_2 B_2 \,\check{\sqcup}_1\, \mathrm{p} \sim_2 A_2$}
\RightLabel{E}
\UnaryInfC{$\mathrm{P} *_2 B_2 \vdash \mathrm{p} \sim_2 A_2 \,\check{\sqcup}_1\, \mathrm{p} \sim_2 B_2$}
\UnaryInfC{$\mathrm{P} *_2 B_2 \vdash \mathrm{p} \sim_2 A_2 \sqcup_1 \mathrm{p} \sim_2 B_2$}
\LeftLabel{adj}
\UnaryInfC{$*_2 B_2 \vdash \mathrm{N}(\mathrm{p} \sim_2 A_2 \sqcup_1 \mathrm{p} \sim_2 B_2)$}
\LeftLabel{adj*}
\UnaryInfC{$*_2 \mathrm{N}(\mathrm{p} \sim_2 A_2 \sqcup_1 \mathrm{p} \sim_2 B_2) \vdash B_2$}
\BinaryInfC{$*_2 \mathrm{N}(\mathrm{p} \sim_2 A_2 \sqcup_1 \mathrm{p} \sim_2 B_2) \,\hat{\sqcap}_2\, *_2 \mathrm{N}(\mathrm{p} \sim_2 A_2 \sqcup_1 \mathrm{p} \sim_2 B_2) \vdash A_2 \sqcap_2 B_2$}
\LeftLabel{C}
\UnaryInfC{$*_2 \mathrm{N}(\mathrm{p} \sim_2 A_2 \sqcup_1 \mathrm{p} \sim_2 B_2) \vdash A_2 \sqcap_2 B_2$}
\LeftLabel{adj*}
\UnaryInfC{$*_2 (A_2 \sqcap_2 B_2) \vdash \mathrm{N}(\mathrm{p} \sim_2 A_2 \sqcup_1 \mathrm{p} \sim_2 B_2)$}
\UnaryInfC{$\sim_2 (A_2 \sqcap_2 B_2) \vdash \mathrm{N}(\mathrm{p} \sim_2 A_2 \sqcup_1 \mathrm{p} \sim_2 B_2)$}
\LeftLabel{adj}
\UnaryInfC{$\mathrm{P} \sim_2 (A_2 \sqcap_2 B_2) \vdash \mathrm{p} \sim_2 A_2 \sqcup_1 \mathrm{p} \sim_2 B_2$}
\UnaryInfC{$\mathrm{p} \sim_2 (A_2 \sqcap_2 B_2) \vdash \mathrm{p} \sim_2 A_2 \sqcup_1 \mathrm{p} \sim_2 B_2$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$A_2 \vdash A_2$}
\LeftLabel{cont}
\UnaryInfC{$*_2 A_2 \vdash *_2 A_2$}
\UnaryInfC{$\sim_2 A_2 \vdash *_2 A_2$}
\RightLabel{P}
\UnaryInfC{$\mathrm{P} \sim_2 A_2 \vdash \mathrm{P} *_2 A_2$}
\UnaryInfC{$\mathrm{p} \sim_2 A_2 \vdash \mathrm{P} *_2 A_2$}
\RightLabel{adj}
\UnaryInfC{$\mathrm{Np} \sim_2 A_2 \vdash *_2 A_2$}
\RightLabel{adj$*$}
\UnaryInfC{$A_2 \vdash *_2 \mathrm{Np} \sim_2 A_2$}
\LeftLabel{W}
\UnaryInfC{$A_2 \hat{\sqcap}_2 B_2 \vdash *_2 \mathrm{Np} \sim_2 A_2$}
\UnaryInfC{$A_2 \sqcap_2 B_2 \vdash *_2 \mathrm{Np} \sim_2 A_2$}
\RightLabel{adj$*$}
\UnaryInfC{$\mathrm{Np} \sim_2 A_2 \vdash *_2 (A_2 \sqcap_2 B_2)$}
\UnaryInfC{$\mathrm{Np} \sim_2 A_2 \vdash \sim_2 (A_2 \sqcap_2 B_2)$}
\RightLabel{adj}
\UnaryInfC{$\mathrm{p} \sim_2 A_2 \vdash \mathrm{P} \sim_2 (A_2 \sqcap_2 B_2)$}
\UnaryInfC{$\mathrm{p} \sim_2 A_2 \vdash \mathrm{p} \sim_2 (A_2 \sqcap_2 B_2)$}
\AxiomC{$B_2 \vdash B_2$}
\LeftLabel{cont}
\UnaryInfC{$*_2 B_2 \vdash *_2 B_2$}
\UnaryInfC{$\sim_2 B_2 \vdash *_2 B_2$}
\RightLabel{P}
\UnaryInfC{$\mathrm{P} \sim_2 B_2 \vdash \mathrm{P} *_2 B_2$}
\UnaryInfC{$\mathrm{p} \sim_2 B_2 \vdash \mathrm{P} *_2 B_2$}
\RightLabel{adj}
\UnaryInfC{$\mathrm{Np} \sim_2 B_2 \vdash *_2 B_2$}
\RightLabel{adj$*$}
\UnaryInfC{$B_2 \vdash *_2 \mathrm{Np} \sim_2 B_2$}
\LeftLabel{W}
\UnaryInfC{$B_2 \hat{\sqcap}_2 A_2 \vdash *_2 \mathrm{Np} \sim_2 B_2$}
\LeftLabel{E}
\UnaryInfC{$A_2 \hat{\sqcap}_2 B_2 \vdash *_2 \mathrm{Np} \sim_2 B_2$}
\UnaryInfC{$A_2 \sqcap_2 B_2 \vdash *_2 \mathrm{Np} \sim_2 B_2$}
\RightLabel{adj$*$}
\UnaryInfC{$\mathrm{Np} \sim_2 B_2 \vdash *_2 (A_2 \sqcap_2 B_2)$}
\UnaryInfC{$\mathrm{Np} \sim_2 B_2 \vdash \sim_2 (A_2 \sqcap_2 B_2)$}
\RightLabel{adj}
\UnaryInfC{$\mathrm{p} \sim_2 B_2 \vdash \mathrm{P} \sim_2 (A_2 \sqcap_2 B_2)$}
\UnaryInfC{$\mathrm{p} \sim_2 B_2 \vdash \mathrm{p} \sim_2 (A_2 \sqcap_2 B_2)$}
\BinaryInfC{$\mathrm{p} \sim_2 A_2 \sqcup_1 \mathrm{p} \sim_2 B_2 \vdash \mathrm{p} \sim_2 (A_2 \sqcap_2 B_2) \check{\sqcup}_1 \mathrm{p} \sim_2 (A_2 \sqcap_2 B_2)$}
\RightLabel{C}
\UnaryInfC{$\mathrm{p} \sim_2 A_2 \sqcup_1 \mathrm{p} \sim_2 B_2 \vdash \mathrm{p} \sim_2 (A_2 \sqcap_2 B_2)$}
\end{prooftree}
$$

□

4.6.3 Conservativity

To argue that the calculus introduced in Section 4.5 is conservative w.r.t. BL (resp. CBL), we follow the standard proof strategy discussed in [24, 27]. Denote by $\vdash_{\mathrm{BL}}$ (resp. $\vdash_{\mathrm{CBL}}$) the consequence relation defined by the calculus for BL (resp. CBL) introduced in Section 4.2, and by $\models_{\mathsf{HBL}}$ (resp. $\models_{\mathsf{HCBL}}$) the semantic consequence relation arising from the class of (perfect) HBLs (resp. HCBLs). We need to show that, for all formulas A and B of the original language of BL (resp. CBL), if $t_1(A) \vdash t_1(B)$ is a D.BL-derivable (resp. D.CBL-derivable) sequent, then $A \vdash_{\mathrm{BL}} B$ (resp. $A \vdash_{\mathrm{CBL}} B$). This can be proved using the following facts: (a) the rules of D.BL (resp. D.CBL) are sound w.r.t. perfect HBLs (resp. HCBLs); (b) BL (resp. CBL) is complete w.r.t. B (resp. CB); and (c) B (resp. CB) are equivalently presented as HBL (resp. HCBL, cf. Section 4.3), so that the semantic consequence relations arising from each type of structures preserve and reflect the translation (cf. Propositions 26 and 27). Let then A, B be formulas of the original language of BL (resp. CBL). If $t_1(A) \vdash t_1(B)$ is a D.BL (resp. D.CBL)-derivable sequent, then, by (a), $t_1(A) \models_{\mathsf{HBL}} t_1(B)$ (resp. $t_1(A) \models_{\mathsf{HCBL}} t_1(B)$). By (c) and Proposition 25, this implies that $A \models_{\mathsf{B}} B$ (resp. $A \models_{\mathsf{CB}} B$). By (b), this implies that $A \vdash_{\mathrm{BL}} B$ (resp. $A \vdash_{\mathrm{CBL}} B$), as required.

4.6.4 Subformula property and cut elimination

Let us briefly sketch the proof of cut elimination and subformula property for D.BL (resp. D.CBL). As discussed earlier on, proper display calculi have been designed so that the cut elimination and subformula property can be inferred from a meta-theorem, following the strategy introduced by Belnap for display calculi [5]. The meta-theorem to which we will appeal for D.BL (resp. D.CBL) was proved in [17].

All conditions in [17, Theorem 4.1] except C_8' are readily seen to be satisfied by inspection of the rules. Condition C_8' requires to check that reduction steps are available for every application of the cut rule in which both cut-formulas are principal, which either remove the original cut altogether or replace it by one or more cuts on formulas of strictly lower complexity. In what follows, we only show C_8' for the unary connectives $\sim$ and n (the proof for p is analogous). The cases of lattice connectives are standard and hence omitted.

L_i-type connectives

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\pi_1\\ X_i \vdash *_i A_i\end{matrix}}{X_i \vdash \sim_i A_i} \qquad \dfrac{\begin{matrix}\vdots\,\pi_2\\ *_i A_i \vdash Y_i\end{matrix}}{\sim_i A_i \vdash Y_i}}{X_i \vdash Y_i} \qquad \rightsquigarrow \qquad {\scriptstyle\text{cont}}\,\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\,\pi_2\\ *_i A_i \vdash Y_i\end{matrix}}{*_i Y_i \vdash A_i} \qquad \dfrac{\begin{matrix}\vdots\,\pi_1\\ X_i \vdash *_i A_i\end{matrix}}{A_i \vdash *_i X_i}}{*_i Y_i \vdash *_i X_i}}{X_i \vdash Y_i}$$

Multi-type connectives

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\pi_1\\ X_2 \vdash \mathrm{N}A_1\end{matrix}}{X_2 \vdash \mathrm{n}A_1} \qquad \dfrac{\begin{matrix}\vdots\,\pi_2\\ \mathrm{N}A_1 \vdash Y_2\end{matrix}}{\mathrm{n}A_1 \vdash Y_2}}{X_2 \vdash Y_2} \qquad \rightsquigarrow \qquad {\scriptstyle\mathrm{P}}\,\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\,\pi_1\\ X_2 \vdash \mathrm{N}A_1\end{matrix}}{\mathrm{P}X_2 \vdash A_1} \qquad \dfrac{\begin{matrix}\vdots\,\pi_2\\ \mathrm{N}A_1 \vdash Y_2\end{matrix}}{A_1 \vdash \mathrm{P}Y_2}}{\mathrm{P}X_2 \vdash \mathrm{P}Y_2}}{X_2 \vdash Y_2}$$

4.7 Conclusions and future work

The modular character of proper multi-type display calculi makes it possible to easily extend our formalism so as to capture axiomatic extensions (e.g. the logic of *classical bilattices with conflation* [1, Definition 2.11]) as well as language expansions of the basic bilattice logics treated in the present chapter. Expansions of bilattice logic have been extensively studied in the literature as early as in [1], which introduces an implication enjoying the deduction-detachment theorem (see also [9]). More recently, modal operators have been added to bilattice logics, motivated by potential applications to computer science and in particular verification of programs [31, 34]; as well as dynamic modalities, motivated by applications in the area of dynamic epistemic logic [32, 33].

Yet more recently, bilattices with a negation not necessarily satisfying the involution law ($\neg\neg a = a$) have been introduced with motivations of domain theory and topological duality (see [30]), and the study of the corresponding logics has been started [35]. These logics are weaker than the one considered in the present chapter, and so adapting our display calculus formalism to them might prove a more challenging task (in particular, the translations introduced in Section 4.6 may need to be redefined, as they rely on the maps p and n being lattice isomorphisms, which is no longer true in the non-involutive case).

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Chapter 5

Kleene algebras, adjunction and structural control

In the present chapter, which is based on[1] [8], we introduce a multi-type calculus for the logic of measurable Kleene algebras, for which we prove soundness, completeness, conservativity, cut elimination and subformula property. Our proposal imports ideas and techniques developed in formal linguistics around the notion of structural control [17].

[1] My specific contributions to this research have been the proof of results, the introduction of notions and constructions, the development of examples, and the writing of the first draft of the paper.

5.1 Introduction

A general pattern. In this chapter, we are going to explore the proof-theoretic ramifications of a pattern which recurs, with different motivations and guises, in various branches of logic, mathematics, theoretical computer science and formal linguistics. Since the most immediate application we intend to pursue is related to the issue of *structural control* in categorial grammar [17], we start by presenting this pattern in a way that is amenable to make the connection with structural control. The pattern we focus on features two *types* (of logical languages, of mathematical structures, of data structures, of grammatical behaviour, etc.), a General one and a Special one. Objects of the Special type can be regarded as objects of the General type; moreover, each General object can be approximated both "from above" and "from below" by Special objects. That is, there exists a natural notion of order such that the collection of special objects order-embeds into that of general objects; moreover, for every general object the smallest special object exists which is greater than or equal to the given general one, and the greatest special object exists which is smaller than or equal to the given general one. The situation just described can be captured order-theoretically by stipulating that a given order-embedding $e : \mathbb{A} \hookrightarrow \mathbb{B}$ has both a left adjoint $f : \mathbb{B} \twoheadrightarrow \mathbb{A}$ and a right adjoint $g : \mathbb{B} \twoheadrightarrow \mathbb{A}$ such that $fe = ge = id_{\mathbb{A}}$. From these conditions it also follows that the endomorphisms ef and eg on $\mathbb{B}$ are respectively a *closure operator* $\gamma : \mathbb{B} \to \mathbb{B}$ (mapping each general object to the smallest special object which is greater than or equal to the given one) and an *interior operator* $\iota : \mathbb{B} \to \mathbb{B}$ (mapping each general object to the greatest special object which is smaller than or equal to the given one).

5

Examples. A prime example of this situation in logic is the natural embedding map e of the Heyting algebra $\mathbb{A}$ of the up-sets of a poset $\mathbb{W}$, understood as an intuitionistic Kripke structure, into the Boolean algebra $\mathbb{B}$ of the subsets of the domain of the same Kripke structure. This embedding is a complete lattice homomorphism, and hence both its right adjoint and its left adjoint exist. This adjunction situation is the mechanism semantically underlying the celebrated McKinsey-Gödel-Tarski translation of intuitionistic logic into the classical normal modal logic S4 (cf. [2] for an extended discussion). Another example arises from the theory of quantales [19] (order-theoretic structures arising as "noncommutative" generalizations of locales, or pointfree topologies). For every unital quantale, its two-sided elements[2] form a locale, which is embedded in the quantale, and this embedding has both a left and a right adjoint, so that every element of the quantale is approximated from above and from below by two-sided elements. A third example arises from approximation spaces[3] in rough set theory (cf. [7, 21]), in which the natural embedding of the Boolean algebra generated by the equivalence classes into the power set algebra of the domain of the given approximation space has both a left adjoint and a right adjoint.

Structural control. These and other similar adjunction situations provide a promising semantic environment for a line of research in formal linguistics, started in [17], and

[2] I.e. those elements x such that $x \cdot 1 \leq x$ and $1 \cdot x \leq x$.

[3] An approximation space is a structure (X, R) with X a set and R an equivalence relation on X.

aimed at establishing systematic forms of communication between different grammatical regimes. In [17], certain well known extensions of the Lambek calculus are studied as logics for reasoning about the grammatical structure of linguistic resources, in such a way that the requirement of grammatical correctness on the linguistic side is matched by the requirement of derivability on the logical side. In this regard, the various axiomatic extensions of the Lambek calculus correspond to different grammatical regimes which become progressively laxer (i.e. recognize progressively more constructions as grammatically correct) as their associated logics become progressively stronger. In this context, the basic Lambek calculus incarnates the most general grammatical regime, and the 'special' behaviour of its extensions is captured by additional analytic structural rules. A systematic two-way communication between these grammatical regimes is captured by introducing extra pairs of adjoint modal operators (the *structural control* operators), which make it possible to import a degree of flexibility from the special regime into the general regime, and conversely, to endow the special regime with enhanced 'structural discrimination' coming from the general regime. The control operators are normal modal operators inspired by the exponentials of linear logic [6] but are not assumed to satisfy the modal S4-type conditions that are satisfied by the linear logic exponentials. Interestingly, in linear logic, precisely the S4-type axioms guarantee that the 'of course' exponential $!$ is an *interior operator* and the 'why not' exponential $?$ is a *closure operator*, and hence each of them can be reobtained as the composition of adjoint pairs of maps between terms of the linear (or general) type and terms of the classical/(co-)intuitionistic (or special) type, which are section/(co-)retraction pairs. Instead, in [17], the adjunction situation is taken as primitive, and the structural control adjoint pairs of maps are not section/(co-)retraction pairs. In [10], a multi-type environment for linear logic is introduced in which the Linear type encodes the behaviour of general resources, and the Classical/Intuitionistic type encodes the behaviour of special (renewable) resources. The special behaviour is captured by additional analytic rules (weakening and contraction), and is exported in a controlled form into the general type via the pairs of adjoint connectives which account for the well known controlled application of weakening and contraction in linear logic. This approach has made it possible to design the first calculus for linear logic in which all rules are closed under uniform substitution (within each type), so that its cut elimination result becomes straightforward. In [10] it is also observed that the same underlying mechanisms can be used to account for the controlled application of other structural rules, such as associativity and exchange. Since these are precisely the structural analytic rules capturing the special grammatical regimes in the setting of [17], this observation strengthens the connection between linear logic and the structural control approach of [17].

Kleene algebras: similarities and differences. In this chapter, we focus on the case study of Kleene algebras in close relationship with the ideas of structural control and the multi-type approach illustrated above. Kleene algebras have been introduced to formally capture the behaviour of programs modelled as relations [15, 16]. While general programs are encoded as arbitrary elements of a Kleene algebra, the Kleene star makes it possible to access the special behaviour of reflexive and transitive programs and to import it in a controlled way within the general environment. Hence, the role

5

played by the Kleene star is similar to the one played by the exponential $?$ in linear logic, which makes it possible to access the special behaviour of renewable resources, captured proof-theoretically by the analytic structural rules of weakening and contraction, and to import it, in a controlled way, into the environment of general resources. Another similarity between the Kleene star and $?$ is that their axiomatizations guarantee that their algebraic interpretations are closure operators, and hence can be obtained as the composition of adjoint maps in a way which provides the approximation "from above" which is necessary to instantiate the general pattern described above, and use it to justify the soundness of the controlled application of the structural rules capturing the special behaviour. However, in the general setting of Kleene algebras there is no approximation "from below", as e.g. it is easy to find examples in the context of Kleene algebras of relations in which more than one reflexive transitive relation can be maximally contained in a given general relation. Our analysis (cf. Section 5.4) identifies the lack of such an approximation "from below" as the main hurdle preventing the development of a smooth proof-theoretic treatment of the logic of general Kleene algebras, which to date remains very challenging.

Extant approaches to the logic of Kleene algebras and PDL. The difficulties in the proof-theoretic treatment of the logic of Kleene algebras propagate into the difficulties in the proof-theoretic treatment of Propositional Dynamic Logic (PDL) [4, 12, 22]. Indeed, PDL can be understood (cf. [4]) as an expansion of the logic of Kleene algebras with a Formula type. Heterogeneous binary operators account for the connection between the action/program types and the Formula type. The properties of these binary operators are such that their proof-theoretic treatment is per se unproblematic. However, the PDL axioms encoding the behaviour of the Kleene star are *non analytic*, and in the literature several approaches have been proposed to tackle this hurdle, which always involve some trade-off: from sequent calculi with finitary rules but with a non-eliminable analytic cut [12, 13], to cut-free sequent calculi with infinitary rules [20, 22].

Measurable Kleene algebras. In this chapter, we introduce a subclass of Kleene algebras, referred to as *measurable* Kleene algebras,[4] which are Kleene algebras endowed with a *dual* Kleene star operation, associating any element with its reflexive transitive *interior*. Similar definitions have been introduced in the context of dioids (cf. e.g. [11] and [18]; in the latter, however, the order-theoretic behaviour of the dual Kleene star is that of a second closure operator rather than that of an interior operator). In measurable Kleene algebras, the defining properties of the dual Kleene star are those of an *interior* operator, which then provides the approximation "from below" which is missing in the setting of general Kleene algebras. Hence measurable Kleene algebras are designed to provide yet another instance of the pattern described in the beginning of the present introduction. In this chapter, this pattern is used as a semantic support of a proper display calculus for the logic of measurable Kleene algebras, and for establishing a conceptual and technical connection between Kleene algebras and structural control which is potentially beneficial for both areas.

[4]The name is chosen by analogy with measurable sets in analysis, which are defined in terms of the existence of approximations "from above" and "from below".

Structure of the chapter. In Section 5.2, we collect preliminaries on (continuous) Kleene algebras and their logics, introduce the notion of measurable Kleene algebra, and propose an axiomatization for the logic corresponding to this class. In Section 5.3, we introduce the heterogeneous algebras corresponding to (continuous, measurable) Kleene algebras and prove that each class of Kleene algebras can be equivalently presented in terms of its heterogeneous counterpart. In Section 5.4, we introduce multi-type languages corresponding to the semantic environments of heterogeneous Kleene algebras, define a translation from the single-type languages to the multi-type languages, and analyze the proof-theoretic hurdles posed by Kleene logic with the lenses of the multi-type environment. This analysis leads to our proposal, introduced in Section 5.5, of a proper display calculus for the logic of measurable Kleene algebras. In Section 5.6 we verify that this calculus is sound, complete, conservative and has cut elimination and subformula property.

5.2 Kleene algebras and their logics

5.2.1 Kleene algebras and continuous Kleene algebras

Definition 36. *A* Kleene algebra *[14] is a structure* $\mathbb{K} = (K, \cup, \cdot, ()^*, 1, 0)$ *such that:*

K1 $(K, \cup, 0)$ *is a join-semilattice with bottom element* 0*;*

K2 $(K, \cdot, 1)$ *is a monoid with unit* 1*, moreover* $\cdot$ *preserves* $\cup$ *in each coordinate, and* 0 *is an annihilator for* $\cdot$*;*

K3 $1 \cup \alpha \cdot \alpha^* \leq \alpha^*$*,* $1 \cup \alpha^* \cdot \alpha \leq \alpha^*$*, and* $1 \cup \alpha^* \cdot \alpha^* \leq \alpha^*$*;*

K4 $\alpha \cdot \beta \leq \beta$ *implies* $\alpha^* \cdot \beta \leq \beta$*;*

K5 $\beta \cdot \alpha \leq \beta$ *implies* $\beta \cdot \alpha^* \leq \beta$*.*

A Kleene algebra is continuous *[14] if:*[5]

K1' $(K, \cup, 0)$ *is a* complete *join-semilattice;*

K2' $\cdot$ *is* completely *join-preserving in each coordinate;*

K6 $\alpha^* = \bigcup \alpha^n$ *for* $n \geq 0$*.*

Lemma 14. *[15, Section 2.1] For any Kleene algebra* $\mathbb{K}$ *and any* $\alpha, \beta \in K$*,*

1. $\alpha \leq \alpha^*$*;*
2. $\alpha^* = \alpha^{**}$*;*
3. *if* $\alpha \leq \beta$*, then* $\alpha^* \leq \beta^*$*.*

By Lemma 14, the operation $* : K \to K$ is a closure operator on K seen as a poset.

[5] For any $n \in \mathbb{N}$ let α^n be defined by induction as follows: $\alpha^0 := 1$ and $\alpha^{n+1} := \alpha^n \cdot \alpha$.

Lemma 15. *For any continuous Kleene algebra $\mathbb{K}$ and any $\alpha, \beta \in K$,*

$$\text{If } \alpha \leq \beta \text{ and } 1 \leq \beta \text{ and } \beta \cdot \beta \leq \beta \text{ then } \alpha^* \leq \beta.$$

Next, we introduce a subclass of Kleene algebras endowed with both a Kleene star and a dual Kleene star. To our knowledge, this definition has not appeared as such in the literature, although similar definitions have been proposed in different settings (cf. [1, 18]).

Definition 37. *A* measurable Kleene algebra *is a structure* $\mathbb{K} = (K, \cup, \cdot, ()^*, ()^\star, 1, 0)$ *such that:*

MK1 $(K, \cup, \cdot, ()^*, 1, 0)$ *is a continuous Kleene algebra;*

MK2 $()^\star$ *is a monotone unary operation;*

MK3 $1 \leq \alpha^\star$, *and* $\alpha^\star \cdot \alpha^\star \leq \alpha^\star$;

MK4 $\alpha^\star \leq \alpha$ *and* $\alpha^\star \leq \alpha^{\star\star}$;

MK5 $\beta \leq \alpha$ *and* $1 \leq \beta$ *and* $\beta \cdot \beta \leq \beta$ *implies* $\beta \leq \alpha^\star$.

5

Lemma 16. *For any measurable Kleene algebra $\mathbb{K}$ and any $\alpha \in K$, if $1 \leq \alpha$ and $\alpha \cdot \alpha \leq \alpha$, then*

$$\alpha^* = \alpha = \alpha^\star.$$

Hence,

$$\text{Range}(*) = \text{Range}(\star) = \{\beta \in K \mid 1 \leq \beta \text{ and } \beta \cdot \beta \leq \beta\}.$$

Proof. By MK4 $\alpha^\star \leq \alpha$; the converse direction follows by MK5 with $\beta := \alpha$. By Lemma 14, $\alpha \leq \alpha^*$; the converse direction follows from Lemma 15. This completes the proof of the first part of the statement, and of the inclusion of the set of the βs with the special behaviour into Range($*$) and Range($\star$). The converse inclusions immediately follow from K3 and MK3. □

5.2.2 The logics of Kleene algebras

Fix a denumerable set Atprop of propositional variables, the elements of which are denoted a, b possibly with sub- or superscripts. The language KL over Atprop is defined recursively as follows:

$$\alpha ::= a \mid 1 \mid 0 \mid \alpha \cup \alpha \mid \alpha \cdot \alpha \mid \alpha^*$$

In what follows, we use α, β, γ (with or without subscripts) to denote formulas in KL.

Definition 38. *Kleene logic, denoted* $\mathrm{S.KL}$, *is presented in terms of the following axioms*

$$0 \vdash \alpha, \quad \alpha \vdash \alpha, \quad \alpha \vdash \alpha \vee \beta, \quad \beta \vdash \alpha \vee \beta, \quad 0 \cdot \alpha \dashv\vdash \alpha \cdot 0, \quad 0 \cdot \alpha \dashv\vdash 0,$$

$$1 \cdot \alpha \dashv\vdash \alpha \cdot 1, \quad 1 \cdot \alpha \dashv\vdash \alpha, \quad \alpha \cdot (\beta \cup \gamma) \dashv\vdash (\alpha \cdot \beta) \cup (\alpha \cdot \gamma), \quad (\beta \cup \gamma) \cdot \alpha \dashv\vdash (\beta \cdot \alpha) \cup (\gamma \cdot \alpha)$$

$$(\alpha \cdot \beta) \cdot \gamma \dashv\vdash \alpha \cdot (\beta \cdot \gamma), \quad 1 \cup \alpha \cdot \alpha^* \vdash \alpha^*, \quad 1 \cup \alpha^* \cdot \alpha \vdash \alpha^*, \quad 1 \cup \alpha^* \cdot \alpha^* \vdash \alpha^*$$

and the following rules:

$$\frac{\alpha \vdash \beta \qquad \beta \vdash \gamma}{\alpha \vdash \gamma} \qquad \frac{\alpha \vdash \gamma \qquad \beta \vdash \gamma}{\alpha \vee \beta \vdash \gamma} \qquad \frac{\alpha_1 \vdash \beta_1 \qquad \alpha_2 \vdash \beta_2}{\alpha_1 \cdot \alpha_2 \vdash \beta_1 \cdot \beta_2}$$

$$\text{K4}\ \frac{\alpha \cdot \beta \vdash \beta}{\alpha^* \cdot \beta \vdash \beta} \qquad \frac{\beta \cdot \alpha \vdash \beta}{\beta \cdot \alpha^* \vdash \beta}\ \text{K5}$$

Continuous Kleene logic, denoted $\mathrm{S.KL}_\omega$*, is the axiomatic extension of* $\mathrm{S.KL}$ *determined by the following axioms:*

$$\alpha \cdot (\bigcup{}_{i\in\omega}\, \beta_i) \dashv\vdash \bigcup{}_{i\in\omega}\, (\alpha \cdot \beta_i), \qquad \bigcup{}_{i\in\omega}\, \beta_i \cdot \alpha \dashv\vdash \bigcup{}_{i\in\omega}\, (\beta_i \cdot \alpha),$$

$$\bigcup{}_{n\geq 0}\, \beta \cdot \alpha^n \cdot \gamma \dashv\vdash \beta \cdot \alpha^* \cdot \gamma$$

Theorem 15. *[15]* $(\mathrm{S.KL}_\omega)$ $\mathrm{S.KL}$ *is complete with respect to (continuous) Kleene algebras.*

The language MKL over Atprop is defined recursively as follows:

$$\alpha ::= a \mid 1 \mid 0 \mid \alpha \cup \alpha \mid \alpha \cdot \alpha \mid \alpha^* \mid \alpha^\star.$$

Definition 39. *Measurable Kleene logic, denoted* $\mathrm{S.MKL}$*, is presented in terms of the axioms and rules of* $\mathrm{S.KL}$ *plus the following axioms:*

$$1 \vdash \alpha^\star \quad \alpha^\star \cdot \alpha^\star \vdash \alpha^\star \quad \alpha^\star \vdash \alpha \quad \alpha^\star \vdash (\alpha^\star)^\star$$

and the following rules:

$$\frac{\alpha \vdash \beta}{\alpha^\star \vdash \beta^\star} \qquad \frac{\beta \vdash \alpha \qquad 1 \vdash \beta \qquad \beta \cdot \beta \vdash \beta}{\beta \vdash \alpha^\star}$$

5.3 Multi-type semantic environment for Kleene algebras

In the present section, we introduce the algebraic environment which justifies semantically the multi-type approach to the logic of measurable Kleene algebras which we develop in Section 5.2.2. In the next subsection, we take Kleene algebras as starting point, and expand on the properties of the image of the algebraic interpretation of the Kleene star, leading to the notion of 'kernel'. In the remaining subsections, we show that (continuous, measurable) Kleene algebras can be equivalently presented in terms of their corresponding heterogeneous algebras.

5.3.1 Kleene algebras and their kernels

By Lemma 14, for any Kleene algebra $\mathbb{K}$, the operation $()^* : K \to K$ is a closure operator on K seen as a poset. By general order-theoretic facts (cf. [3, Chapter 7]) this means that

$$()^* = e\gamma,$$

where $\gamma : K \twoheadrightarrow \mathrm{Range}(*)$, defined by $\gamma(\alpha) = \alpha^*$ for every $a \in K$, is the left adjoint of the natural embedding $e : \mathrm{Range}(*) \hookrightarrow K$, i.e. for every $\alpha \in K$, and $\xi \in \mathrm{Range}(*)$,

$$\gamma(\alpha) \leq \xi \quad \text{iff} \quad \alpha \leq e(\xi).$$

In what follows, we let S be the subposet of K identified by $\mathrm{Range}(*) = \mathrm{Range}(\gamma)$. We will also use the variables α, β, possibly with sub- or superscripts, to denote elements of K, and π, ξ, χ, possibly with sub- or superscripts, to denote elements of S.

Lemma 17. *For every Kleene algebra $\mathbb{K}$ and every $\xi \in S$,*

$$\gamma(e(\xi)) = \xi. \tag{5.1}$$

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Proof. By adjunction, $\gamma(e(\xi)) \leq \xi$ iff $e(\xi) \leq e(\xi)$, which always holds. As to the converse inequality $\xi \leq \gamma(e(\xi))$, since e is an order-embedding, it is enough to show that $e(\xi) \leq e(\gamma(e(\xi)))$, which by adjunction is equivalent to $\gamma(e(\xi)) \leq \gamma(e(\xi))$, which always holds. □

Definition 40. *For any Kleene algebra $\mathbb{K} = (K, \cup, \cdot, ()^*, 1, 0)$, let the* kernel *of $\mathbb{K}$ be the structure $\mathbb{S} = (S, \sqcup, 0_S)$ defined as follows:*

KK1. $S := \mathrm{Range}(*) = \mathrm{Range}(\gamma)$, *where $\gamma : K \twoheadrightarrow S$ is defined by letting $\gamma(\alpha) = \alpha^*$ for any $\alpha \in K$;*

KK2. $\xi \sqcup \chi := \gamma(e(\xi) \cup e(\chi))$;

KK3. $0_S := \gamma(0)$.

Proposition 28. *If $\mathbb{K}$ is a (continuous) Kleene algebra, then its kernel $\mathbb{S}$ defined as above is a (complete) join-semilattice with bottom element.*

Proof. By KK1, S is a subposet of K. Let $\xi, \chi \in S$. Using KK2 and Lemma 5.1, one shows that $\xi \sqcup \chi$ is a common upper bound of ξ and χ w.r.t. the order S inherits from K. Since e and γ are monotone, $\xi \leq \pi$ and $\chi \leq \pi$ imply that $\xi \sqcup \chi = \gamma(e(\xi) \cup e(\chi)) \leq \gamma(e(\pi)) = \pi$, the last equality due to Lemma 5.1. This shows that $\xi \sqcup \chi$ is the least upper bound of ξ and χ w.r.t. the inherited order. Analogously one shows that, if $\mathbb{K}$ is continuous and $Y \subseteq S$, $\bigsqcup Y := \gamma(\bigcup e[Y])$ is the least upper bound of Y. Finally, $\gamma(0)$ being the bottom element of S follows from 0 being the bottom element of K and the monotonicity and surjectivity of γ. □

Remark 2. *We have proved a little more than what is stated in Proposition 28. Namely, we have proved that all (finite) joins exist w.r.t. the order that S inherits from K, and hence the join-semilattice structure of S is also in a sense inherited from K. However, this does not mean or imply that S is a sub-join-semilattice of K, since joins in S are 'closures' of joins in K, and hence $\sqcup$ is certainly not the restriction of $\cup$ to S.*

5.3.2 Measurable Kleene algebras and their kernels

The results of Section 5.3.1 apply in particular to measurable Kleene algebras, where in addition, by definition, the operation $()^{\star} : K \to K$ is an interior operator on K seen as a poset. By general order-theoretic facts (cf. [3, Chapter 7]) this means that

$$()^{\star} = e'\iota,$$

where $\iota : K \twoheadrightarrow \mathrm{Range}(\star)$, defined by $\iota(\alpha) = \alpha^{\star}$ for every $a \in K$, is the right adjoint of the natural embedding $e' : \mathrm{Range}(\star) \hookrightarrow K$, i.e. for every $\alpha \in K$ and $\xi \in \mathrm{Range}(\star)$,

$$e'(\xi) \leq \alpha \quad \text{iff} \quad \xi \leq \iota(\alpha).$$

Moreover, Lemma 16 guarantees that

$$\mathrm{Range}(*) = \mathrm{Range}(\star) = \{\beta \in K \mid 1 \leq \beta \text{ and } \beta \cdot \beta \leq \beta\}.$$

Hence, e' coincides with the natural embedding $e : \mathrm{Range}(*) \hookrightarrow K$, which is then endowed with both the left adjoint and the right adjoint.

In what follows, we let S be the subposet of K identified by

$$\mathrm{Range}(*) = \mathrm{Range}(\gamma) = \mathrm{Range}(\iota) = \mathrm{Range}(\star).$$

We will use the variables α, β, possibly with sub- or superscripts, to denote elements of K, and π, ξ, χ, possibly with sub- or superscripts, to denote elements of S.

Lemma 18. *For every measurable Kleene algebra $\mathbb{K}$ and every $\xi \in S$,*

$$\gamma(e(\xi)) = \xi = \iota(e(\xi)). \tag{5.2}$$

Proof. The first identity is shown in Lemma 5.1. As to the second one, by adjunction, $\xi \leq \iota(e(\xi))$ iff $e(\xi) \leq e(\xi)$, which always holds. As to the converse inequality $\iota(e(\xi)) \leq \xi$, since e is an order-embedding, it is enough to show that $e(\iota(e(\xi))) \leq e(\xi)$, which by adjunction is equivalent to $\iota(e(\xi)) \leq \iota(e(\xi))$, which always holds. □

Definition 41. *For any measurable Kleene algebra $\mathbb{K} = (K, \cup, \cdot, ()^*, ()^{\star}, 1, 0)$, let the* kernel *of $\mathbb{K}$ be the structure $\mathbb{S} = (S, \sqcup, 0_s)$ defined as follows:*

KK1. $S := \mathrm{Range}(*) = \mathrm{Range}(\gamma) = \mathrm{Range}(\iota) = \mathrm{Range}(\star)$*;*

KK2. $\xi \sqcup \chi := \gamma(e(\xi) \cup e(\chi))$*;*

KK3. $0_s := \gamma(0)$.

$$\begin{tikzcd}
\mathbb{S} \arrow[rr, hook, "e"] & & \mathbb{K} \arrow[ll, bend right=50, "\gamma"'] \arrow[ll, bend left=50, "\iota"] \\
\end{tikzcd}$$

5.3.3 Heterogeneous Kleene algebras

Definition 42. *A* heterogeneous Kleene algebra *is a tuple* $\mathbb{H} = (\mathbb{A}, \mathbb{S}, \otimes_1, \otimes_2, \gamma, e)$ *verifying the following conditions:*

H1 $\mathbb{A} = (A, \sqcup, \cdot, 1_s, 0)$ *is such that* $(A, \sqcup, 0)$ *is a join-semilattice with bottom element* 0 *and* $(A, \cdot, 1_s)$ *is a monoid with unit* 1*, moreover* $\cdot$ *preserves finite joins in each coordinate, and* 0 *is an annihilator for* $\cdot$*;*

H2 $\mathbb{S} = (S, \sqcup, 0_s)$ *is a join-semilattice with bottom element* 0_s*;*

H3 $\otimes_1 : \mathbb{S} \times \mathbb{A} \to \mathbb{A}$ *preserves finite joins in its second coordinate, is monotone in its first coordinate, and has unit* 1 *in its second coordinate, and* $\otimes_2 : \mathbb{A} \times \mathbb{S} \to \mathbb{A}$ *preserves finite joins in its first coordinate, is monotone in its second coordinate, and has unit* 1 *in its first coordinate. Moreover, for all* $\alpha \in \mathbb{A}$ *and* $\xi \in S$*,*

$$\xi \otimes_1 \alpha = e(\xi) \cdot \alpha \quad \text{and} \quad \alpha \otimes_2 \xi = \alpha \cdot e(\xi); \tag{5.3}$$

H4 $\gamma : \mathbb{A} \twoheadrightarrow \mathbb{S}$ *and* $e : \mathbb{S} \hookrightarrow \mathbb{A}$ *are such that* $\gamma \dashv e$ *and* $\gamma(e(\xi)) = \xi$ *for all* $\xi \in S$*;*

H5 $1 \leq e(\xi)$*, and* $e(\xi) \cdot e(\xi) \leq e(\xi)$ *for any* $\xi \in \mathbb{S}$*;*

H6 $\alpha \cdot \beta \leq \beta$ *implies* $\gamma(\alpha) \otimes_1 \beta \leq \beta$*, and* $\beta \cdot \alpha \leq \beta$ *implies* $\beta \otimes_2 \gamma(\alpha) \leq \beta$ *for all* $\alpha, \beta \in \mathbb{A}$*.*

A heterogeneous Kleene algebra is continuous *if*

H1' $(A, \sqcup, 0)$ *is a* complete *join-semilattice and* $\cdot$ *preserves* arbitrary *joins in each coordinate;*

H2' $\mathbb{S} = (S, \sqcup, 0_s)$ *is a* complete *join-semilattice;*

H7 $e(\gamma(\alpha)) = \bigcup \alpha^n$ *for any* $n \in \mathbb{N}$*.*

Definition 43. *For any Kleene algebra* $\mathbb{K} = (K, \cup, \cdot, ()^*, 1, 0)$*, let*

$$\mathbb{K}^+ = (\mathbb{A}, \mathbb{S}, \otimes_1, \otimes_2, \gamma, e)$$

be the structure defined as follows:

1. $\mathbb{A} := (K, \cup, \cdot, 1, 0)$ *is the* $()^*$*-free reduct of* $\mathbb{K}$*;*
2. $\mathbb{S}$ *is the kernel of* $\mathbb{K}$ *(cf. Definition 40);*
3. $\gamma : \mathbb{A} \twoheadrightarrow \mathbb{S}$ *and* $e : \mathbb{S} \hookrightarrow \mathbb{A}$ *are defined as the maps into which the closure operator* $()^*$ *decomposes (cf. discussion before Lemma 5.1);*
4. $\otimes_1$ *(resp.* $\otimes_2$*) is the restriction of* $\cdot$ *to* $\mathbb{S}$ *in the first (resp. second) coordinate.*

Proposition 29. *For any (continuous) Kleene algebra* $\mathbb{K}$*, the structure* $\mathbb{K}^+$ *defined above is a (continuous) heterogeneous Kleene algebra.*

Proof. Since $\mathbb{K}$ verifies by assumption K1 and K2, $\mathbb{K}^+$ verifies H1. Condition H2 (resp. H2') is verified by Proposition 28. Condition H3 immediately follows from the definition of $\otimes_1$ and $\otimes_2$ in $\mathbb{K}^+$. Condition H4 holds by Lemma 14 and 5.1. Condition H5 follows from $\mathbb{K}$ verifying assumption K3. Condition H6 follows from $\mathbb{K}$ verifying assumption K4 and K5. If $\mathbb{K}$ is continuous, then $\mathbb{K}$ verifies conditions K1', K2' and K6, which guarantee that $\mathbb{K}^+$ verifies H1' and H7. □

Definition 44. *For any heterogeneous Kleene algebra $\mathbb{H} = (\mathbb{A}, \mathbb{S}, \otimes_1, \otimes_2, \gamma, e)$, let $\mathbb{H}_+ := (\mathbb{A}, ()^*)$, where $()^* : \mathbb{A} \to \mathbb{A}$ is defined by $\alpha^* := e(\gamma(\alpha))$ for every $\alpha \in \mathbb{A}$.*

Proposition 30. *For any (continuous) heterogeneous Kleene algebra $\mathbb{H} = (\mathbb{A}, \mathbb{S}, \otimes_1, \otimes_2, \gamma, e)$, the structure $\mathbb{H}_+$ defined above is a (continuous) Kleene algebra. Moreover, the kernel of $\mathbb{H}_+$ is join-semilattice-isomorphic to $\mathbb{S}$.*

Proof. As to the first part of the statement, we only need to show that $()^*$ satisfies conditions K3-K5 (resp. K1', K2' and K6) of Definition 36. Condition K3 easily follows from assumption H5 and the proof is omitted. As to K4, let $\alpha, \beta \in \mathbb{A}$ such that $\alpha \cdot \beta \leq \beta$.

$$\begin{array}{llll} \alpha \cdot \beta \leq \beta & \Rightarrow & \gamma(\alpha) \otimes_1 \beta \leq \beta & \text{(H6)} \\ & \Rightarrow & e(\gamma(\alpha)) \cdot \beta \leq \beta & \text{(H3)} \\ & \Rightarrow & \alpha^* \cdot \beta \leq \beta & \text{(definition of } ()^*) \end{array}$$

The proof of K5 is analogous. Conditions K1', K2' and K6 readily follow from assumptions H1' and H7.

This completes the proof of the first part of the statement. As to the second part, let us show preliminarily that the following identities hold:

AK2. $\xi \sqcup \chi := \gamma(e(\xi) \cup e(\chi))$ for all $\xi, \chi \in \mathbb{S}$;

AK3. $0_s := \gamma(0)$.

Being a left adjoint, γ preserves existing joins. Hence, $\gamma(0) = 0_s$, which proves (AK2), and, using H4, $\gamma(e(\xi) \cup e(\chi)) = \gamma(e(\xi)) \sqcup \gamma(e(\chi)) = \xi \sqcup \chi$, which proves (AK3). To show that the kernel of $\mathbb{H}_+$ and $\mathbb{S}$ are isomorphic as (complete) join-semilattices, notice that the domain of the kernel of $\mathbb{H}_+$ is defined as $K_* :=$ Range$(()^*) =$ Range$(e \circ \gamma) =$ Range(e). Since e is an order-embedding (which is easily shown using H4), this implies that K_*, regarded as a sub-poset of $\mathbb{A}$, is order-isomorphic to the domain of $\mathbb{S}$ with its join-semilattice order. Let $i : \mathbb{S} \to \mathbb{K}_*$ denote the order-isomorphism between $\mathbb{S}$ and $\mathbb{K}_*$. To show that $\mathbb{S} = (S, \sqcup_{\mathbb{S}}, 0_s)$ and $\mathbb{K}_* = (K_*, \sqcup_{\mathbb{K}_*}, 0_{s*})$ are isomorphic as join-semilattices, we need to show that for all $\xi, \chi \in \mathbb{S}$,

$$i(\xi \sqcup_{\mathbb{S}} \chi) = i(\xi) \sqcup_{\mathbb{K}_*} i(\chi) \quad \text{and} \quad i(0_s) = 0_{s*}.$$

Let $e' : \mathbb{K}_* \hookrightarrow \mathbb{A}$ and $\gamma' : \mathbb{A} \twoheadrightarrow \mathbb{K}_*$ be the pair of adjoint maps arising from $*$. Thus, $e = e'i$ and $\gamma' = i\gamma$, and so,

$$\begin{array}{rcll} i(\xi) \sqcup_{\mathbb{K}_*} i(\chi) & = & \gamma'(e'(i(\xi)) \cup e'(i(\chi))) & (\text{definition of } \sqcup_{\mathbb{K}_*}) \\ & = & \gamma'(e(\xi) \cup e(\chi)) & (e = e'i) \\ & = & i(\gamma(e(\xi) \cup e(\chi))) & (\gamma' = i\gamma) \\ & = & i(\xi \sqcup_{\mathbb{S}} \chi). & (\text{AK2}) \end{array}$$

$$\begin{array}{rcll} 0_{s*} & = & \gamma'(0) & (\text{KK3}) \\ & = & i(\gamma(0)) & (\gamma' = i\gamma) \\ & = & i(0_s) & (\text{AK3}) \end{array}$$

□

The following proposition immediately follows from Propositions 29 and 30:

Proposition 31. *For any Kleene algebra $\mathbb{K}$ and heterogeneous Kleene algebra $\mathbb{H}$,*

$$\mathbb{K} \cong (\mathbb{K}^+)_+ \quad \text{and} \quad \mathbb{H} \cong (\mathbb{H}_+)^+.$$

Moreover, these correspondences restrict to continuous Kleene algebras and continuous heterogeneous Kleene algebras.

5

5.3.4 Heterogeneous measurable Kleene algebras

The extra conditions of measurable Kleene algebras allow for their 'heterogeneous presentation' (encoded in the definition below) being much simpler than the one for Kleene algebras:

Definition 45. *A* heterogeneous measurable Kleene algebra *is a tuple $\mathbb{H} = (\mathbb{A}, \mathbb{S}, \iota, \gamma, e)$ verifying the following conditions:*

HM1 $\mathbb{A} = (A, \sqcup, \cdot, 1_s, 0)$ *is such that* $(A, \sqcup, 0)$ *a complete join-semilattice with bottom element* 0 *and* $(A, \cdot, 1_s)$ *a monoid with unit* 1, *moreover* $\cdot$ *preserves arbitrary joins in each coordinate, and* 0 *is an annihilator for* $\cdot$*;*

HM2 $\mathbb{S} = (S, \sqcup, 0_s)$ *is a complete join-semilattice with bottom element* 0_s*;*

HM3 $e(\gamma(\alpha)) = \bigcup \alpha^n$ *for any* $n \in \mathbb{N}$.

HM4 $\gamma : \mathbb{A} \twoheadrightarrow \mathbb{S}$ *and* $\iota : \mathbb{A} \twoheadrightarrow \mathbb{S}$ *and* $e : \mathbb{S} \hookrightarrow \mathbb{A}$ *are such that* $\gamma \dashv e \dashv \iota$ *and* $\gamma(e(\xi)) = \xi = \iota(e(\xi))$ *for all* $\xi \in S$*;*

HM5 $1 \leq e(\xi)$, *and* $e(\xi) \cdot e(\xi) \leq e(\xi)$ *for any* $\xi \in \mathbb{S}$*;*

HM6 *For any* $\beta \in \mathbb{A}$, *if* $1 \leq \beta$ *and* $\beta \cdot \beta \leq \beta$, *then* $\gamma(\beta) \leq \iota(\beta)$.

Definition 46. *For any measurable Kleene algebra* $\mathbb{K} = (K, \cup, \cdot, ()^*, ()^\star, 1, 0)$, *let*

$$\mathbb{K}^+ = (\mathbb{A}, \mathbb{S}, \iota, \gamma, e)$$

be the structure defined as follows:

1. $\mathbb{A} := (K, \cup, \cdot, 1, 0)$ *is the* $\{()^*, ()^\star\}$*-free reduct of* $\mathbb{K}$*;*

2. *$\mathbb{S}$ is the kernel of $\mathbb{K}$ (cf. Definition 41);*

3. *$\gamma : \mathbb{A} \twoheadrightarrow \mathbb{S}$ and $e : \mathbb{S} \hookrightarrow \mathbb{A}$ are defined as the maps into which the closure operator $()^*$ decomposes, and $\iota : \mathbb{A} \twoheadrightarrow \mathbb{S}$ and $e : \mathbb{S} \hookrightarrow \mathbb{A}$ are defined as the maps into which the interior operator $()^\star$ decomposes (cf. discussion before Lemma 18).*

Proposition 32. *For any measurable Kleene algebra $\mathbb{K}$, the structure $\mathbb{K}^+$ defined above is a heterogeneous measurable Kleene algebra.*

Proof. Since $\mathbb{K}$ verifies by assumption K1', K2, and K6, $\mathbb{K}^+$ verifies HM1. Condition HM2 is verified by Proposition 28. Condition HM3 immediately follows from the definition of $()^*$ and assumption K6. Condition HM4 holds by Lemma 18. Condition HM5 follows from $\mathbb{K}$ verifying assumption K3. As to condition HM6, if $1 \leq \beta$ and $\beta \cdot \beta \leq \beta$, then by Lemma 16, $e(\gamma(\beta)) = \beta^* = \beta^\star = e(\iota(\beta))$, which implies, since e is injective, that $\gamma(\beta) \leq \iota(\beta)$, as required. □

Definition 47. *For any heterogeneous measurable Kleene algebra $\mathbb{H} = (\mathbb{A}, \mathbb{S}, \iota, \gamma, e)$, let $\mathbb{H}_+ := (\mathbb{A}, ()^*, ()^\star)$, where $()^* : \mathbb{A} \to \mathbb{A}$ and $()^\star : \mathbb{A} \to \mathbb{A}$ are respectively defined by $\alpha^* := e(\gamma(\alpha))$ and $\alpha^\star := e(\iota(\alpha))$ for every $\alpha \in \mathbb{A}$.*

Proposition 33. *For any heterogeneous measurable Kleene algebra $\mathbb{H} = (\mathbb{A}, \mathbb{S}, \iota, \gamma, e)$, the structure $\mathbb{H}_+$ defined above is a measurable Kleene algebra. Moreover, the kernel of $\mathbb{H}_+$ is join-semilattice-isomorphic to $\mathbb{S}$.*

Proof. The part of the statement which concerns the verification of axioms K1', K2', K3-K6 is accounted for as in the proof of Proposition 30. Let us verify that $()^\star$ satisfies conditions MK2-MK5 of Definition 37. Conditions MK2 and MK4 easily follow from the assumption that $e \dashv \iota$ (HM4). Condition MK3 follows from the surjectivity of ι and assumption HM5. As to MK5, it is enough to show that if $\alpha, \beta \in K$ such that $\beta \leq \alpha$ and $1 \leq \beta$ and $\beta \cdot \beta \leq \beta$, then $\beta \leq e(\iota(\alpha))$. Since $\beta \leq \alpha$ by assumption and e and ι are monotone, it is enough to show that $\beta \leq e(\iota(\beta))$. By adjunction, this is equivalent to $\gamma(\beta) \leq \iota(\beta)$, which holds by assumption HM6. This completes the proof of the first part of the statement. The proof of the second part is analogous to the corresponding part of the proof of Proposition 30, and is omitted. □

The following proposition immediately follows from Propositions 32 and 33:

Proposition 34. *For any measurable Kleene algebra $\mathbb{K}$ and heterogeneous measurable Kleene algebra $\mathbb{H}$,*

$$\mathbb{K} \cong (\mathbb{K}^+)_+ \quad \text{and} \quad \mathbb{H} \cong (\mathbb{H}_+)^+.$$

5.4 Multi-type presentations for Kleene logics

In Section 5.3.3, (continuous) heterogeneous (measurable) Kleene algebras have been introduced (cf. Definitions 42 and 45) and shown to be equivalent presentations of (continuous, measurable) Kleene algebras. These constructions motivate the multi-type presentations of Kleene logics we introduce in the present section. Indeed, heterogeneous

Kleene algebras are natural models for the following multi-type language $\mathcal{L}_{\mathrm{MT}}$, defined by simultaneous induction from a set AtAct of atomic actions (the elements of which are denoted by letters a, b):

$$\text{Special} \ni \xi ::= \blacklozenge\alpha$$
$$\text{General} \ni \alpha ::= a \mid 1 \mid 0 \mid \Box\xi \mid \alpha \cup \alpha$$

while heterogeneous measurable Kleene algebras are natural models for the following multi-type language $\mathcal{L}_{\mathrm{MT}}$, defined by simultaneous induction from AtAct:

$$\text{Special} \ni \xi ::= \blacklozenge\alpha \mid \blacksquare\alpha$$
$$\text{General} \ni \alpha ::= a \mid 1 \mid 0 \mid \Box\xi \mid \alpha \cup \alpha$$

where, in any heterogeneous (measurable) Kleene algebra, the maps γ and e (and ι) interpret the heterogeneous connectives $\blacklozenge$, $\Box$ (and $\blacksquare$) respectively. The interpretation of $\mathcal{L}_{\mathrm{MT}}$-terms into heterogeneous algebras is defined as the straightforward generalization of the interpretation of propositional languages in algebras of compatible signature, and is omitted.

5

The toggle between Kleene algebras and heterogeneous Kleene algebras is reflected syntactically by the following translation $(\cdot)^t : \mathcal{L} \to \mathcal{L}_{\mathrm{MT}}$ between the original language $\mathcal{L}$ of Kleene logic and the language $\mathcal{L}_{\mathrm{MT}}$ defined above:

$$\begin{array}{rcl} a^t & = & a \\ 1^t & = & 1 \\ 0^t & = & 0 \\ (\alpha \cup \beta)^t & = & \alpha^t \cup \beta^t \\ (\alpha \cdot \beta)^t & = & \alpha^t \cdot \beta^t \\ (\alpha^*)^t & = & \Box\blacklozenge\alpha^t \\ (\alpha^\star)^t & = & \Box\blacksquare\alpha^t \end{array}$$

The following proposition is proved by a routine induction on $\mathcal{L}$-formulas.

Proposition 35. *For all $\mathcal{L}$-formulas A and B and every Kleene algebra $\mathbb{K}$,*

$$\mathbb{K} \models \alpha \leq \beta \quad \textit{iff} \quad \mathbb{K}^+ \models \alpha^t \leq \beta^t.$$

The general definition of *analytic inductive* inequalities can be instantiated to inequalities in the $\mathcal{L}_{\mathrm{MT}}$-signature according to the order-theoretic properties of the algebraic interpretation of the $\mathcal{L}_{\mathrm{MT}}$-connectives in heterogeneous (measurable) Kleene algebras. In particular, all connectives but $\otimes_1$ and $\otimes_2$ are normal. Hence, we are now in a position to translate the axioms and rules describing the behaviour of $()^*$ and $()^\star$ from the single-type languages into $\mathcal{L}_{\mathrm{MT}}$ using $(\cdot)^t$, and verify whether the resulting translations are analytic inductive.

$$1 \cup \alpha \leq \alpha^* \rightsquigarrow \begin{cases} 1 \cup \alpha^t \leq \Box\blacklozenge\alpha^t & (i) \\ \Box\blacklozenge\alpha^t \leq 1 \cup \alpha^t & (ii) \end{cases}$$

$$1 \cup \alpha^* = \alpha^* \rightsquigarrow \begin{cases} 1 \cup \Box\blacklozenge\alpha^t \leq \Box\blacklozenge\alpha^t & (iii) \\ \Box\blacklozenge\alpha^t \leq 1 \cup \Box\blacklozenge\alpha^t & (iv) \end{cases}$$

$$\alpha \cdot \beta \leq \beta \text{ implies } \alpha^* \cdot \beta \leq \beta \rightsquigarrow \begin{cases} \alpha^t \cdot \beta^t \leq \beta^t \text{ implies } \Box\blacklozenge\alpha^t \cdot \beta^t \leq \beta^t & (v) \end{cases}$$

$$\beta \cdot \alpha \leq \beta \text{ implies } \beta \cdot \alpha^* \leq \beta \rightsquigarrow \begin{cases} \beta^t \cdot \alpha^t \leq \beta^t \text{ implies } \beta^t \cdot \Box\blacklozenge\alpha^t \leq \beta^t & (vi) \end{cases}$$

Notice that, relative to the order-theoretic properties of their interpretations on heterogeneous Kleene algebras, $\cdot$, 1, $\blacklozenge$ are $\mathcal{F}$-connectives, while $\Box$ is a $\mathcal{G}$-connective. However, relative to the order-theoretic properties of their interpretations on heterogeneous measurable Kleene algebras, $\cdot$, 1, $\blacklozenge$ are $\mathcal{F}$-connectives, while $\Box$ is both an $\mathcal{F}$-connective and a $\mathcal{G}$-connective. Hence, it is easy to see that, relative to the first interpretation, (i) is the only analytic inductive inequality of the list above, due to the occurrences of the McKinsey-type nesting $\Box\blacklozenge\alpha^t$ in antecedent position. However, relative to the second interpretation, the same nesting becomes harmless, since the occurrences of $\Box$ in antecedent position are part of the Skeleton.

Likewise, it is very easy to see that the conditions HM1-HM6 in the definition of heterogeneous measurable Kleene algebras do not violate the conditions on nesting of analytic inductive inequalities. However, some of these conditions do not consist of inequalities taken in isolation but are given in the form of quasi-inequalities. When embedded into a quasi-inequality, the proof-theoretic treatment of an inequality such as $\beta \cdot \beta \leq \beta$ (which in isolation would be unproblematic) becomes problematic, since the translation of the quasi-inequality into a logically equivalent rule would not allow to 'disentangle' the occurrences of β in precedent position from the occurrences of β in succedent position, thus making it impossible to translate the quasi-inequality directly as an analytic structural rule. This is why the calculus defined in the following section features an infinitary rule, introduced to circumvent this problem.

5

5.5 The proper multi-type display calculus D.MKL

5.5.1 Language

In the present section, we define a *multi-type language* for the proper multi-type display calculus for measurable Kleene logic. As usual, this language includes constructors for both logical (operational) and structural terms.

- Structural and operational terms:

$$\text{General} \begin{cases} \alpha ::= a \mid 1 \mid 0 \mid \Box\xi \mid \alpha \cup \alpha \mid \alpha \cdot \alpha \\ \\ \Gamma ::= \Phi \mid \circ\Pi \mid \Gamma \odot \Gamma \mid \Gamma < \Gamma \mid \Gamma > \Gamma \end{cases}$$

$$\text{Special} \begin{cases} \xi ::= \blacklozenge\alpha \mid \blacksquare\alpha \\ \\ \Pi ::= \bullet\Gamma \end{cases}$$

In what follows, we reserve α, β, γ (with or without subscripts) to denote General-type operational terms, and ξ, χ, π (with or without subscripts) to denote formulas in Special-type operational terms. Moreover, we reserve Γ, Δ, Θ (with or without subscripts) to denote General-type structural terms, and Π, Ξ, Λ (with or without subscripts) to denote Special-type structural terms.

- Structural and operational terms:

General								S → G		G → S	
Φ		$\odot$		$<$		$>$		$\circ$		$\bullet$	
1	0	$\cdot$			$(/)$		$(\backslash)$	$\Box$	$\Box$	$\blacklozenge$	$\blacksquare$

Notice that, for the sake of minimizing the number of structural symbols, we are assigning the same structural connective $\bullet$ to $\blacklozenge$ and $\blacksquare$ although these modal operators are not dual to one another, but are respectively interpreted as the left adjoint and the right adjoint of $\Box$, which is hence both an $\mathcal{F}$-operator and a $\mathcal{G}$-operator, and can therefore correspond to the structural connective $\circ$ both in antecedent and in succedent position.

5.5.2 Rules

In the rules below, the symbols Γ, Δ and Θ denote structural variables of general type, and Σ, Π and Ξ structural variables of special type. The calculus D.MKL consists the following rules:

- Identity and cut rules:

$$\mathsf{Id}\ \frac{}{a \vdash a}$$

$$\frac{\Gamma \vdash \alpha \qquad \alpha \vdash \Delta}{\Gamma \vdash \Delta}\ \mathsf{Cut}_g \qquad \frac{\Pi \vdash \xi \qquad \xi \vdash \Xi}{\Pi \vdash \Xi}\ \mathsf{Cut}_s$$

- General type display rules:

$$\mathsf{res}\ \frac{\Gamma \odot \Delta \vdash \Theta}{\Delta \vdash \Gamma > \Theta} \qquad \frac{\Gamma \odot \Delta \vdash \Theta}{\Gamma \vdash \Theta < \Delta}\ \mathsf{res}$$

- Multi-type display rules:

$$\mathsf{adj}\ \frac{\Gamma \vdash \circ\Xi}{\bullet\Gamma \vdash \Xi} \qquad \frac{\circ\Xi \vdash \Gamma}{\Xi \vdash \bullet\Gamma}\ \mathsf{adj}$$

- General type structural rules:

$$\Phi_L \frac{\Gamma \vdash \Delta}{\Phi \odot \Gamma \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \odot \Phi \vdash \Delta} \Phi_R$$

$$\text{assoc} \frac{(\Gamma_1 \odot \Gamma_2) \odot \Gamma_3 \vdash \Delta}{\Gamma_1 \odot (\Gamma_2 \odot \Gamma_3) \vdash \Delta} \qquad \frac{\Gamma \vdash \Phi}{\Gamma \vdash \Delta} \Phi\text{-W}$$

- Multi-type structural rules:[6]

$$\text{one} \frac{}{\Phi \vdash \circ\Pi} \qquad \frac{\Gamma \vdash \circ\Pi \qquad \Delta \vdash \circ\Pi}{\Gamma \odot \Delta \vdash \circ\Pi} \text{abs}$$

$$\text{b-bal} \frac{\Pi \vdash \Sigma}{\bullet \circ \Pi \vdash \bullet \circ \Sigma} \qquad \frac{\Pi \vdash \Xi}{\circ\Pi \vdash \circ\Xi} \text{w-bal}$$

$$\omega \frac{(\Gamma^{(n)} \vdash \Delta \mid n \geq 1)}{\circ \bullet \Gamma \vdash \Delta} \qquad \frac{\circ\Pi \odot \circ\Pi \vdash \Delta}{\circ\Pi \vdash \Delta} \circ\text{-C}$$

- General type operational rules: in what follows, $i \in \{1, 2\}$,

$$1 \frac{\Phi \vdash \Delta}{1 \vdash \Delta} \qquad \frac{}{\Phi \vdash 1} 1$$

$$0 \frac{}{0 \vdash \Phi} \qquad \frac{\Gamma \vdash \Phi}{\Gamma \vdash 0} 0$$

$$\cup \frac{\alpha_1 \vdash \Delta \qquad \alpha_2 \vdash \Delta}{\alpha_1 \cup \alpha_2 \vdash \Delta} \qquad \frac{\Gamma \vdash \alpha_i}{\Gamma \vdash \alpha_1 \cup \alpha_2} \cup$$

$$\cdot \frac{\alpha \odot \beta \vdash \Delta}{\alpha \cdot \beta \vdash \Delta} \qquad \frac{\Gamma \vdash \alpha \qquad \Delta \vdash \beta}{\Gamma \odot \Delta \vdash \alpha \cdot \beta} \cdot$$

- Multi-type operational rules:

$$\blacklozenge \frac{\bullet\alpha \vdash \Pi}{\blacklozenge\alpha \vdash \Pi} \qquad \frac{\Gamma \vdash \alpha}{\bullet\Gamma \vdash \blacklozenge\alpha} \blacklozenge$$

$$\blacksquare \frac{\alpha \vdash \Gamma}{\blacksquare\alpha \vdash \bullet\Gamma} \qquad \frac{\Pi \vdash \bullet\alpha}{\Pi \vdash \blacksquare\alpha} \blacksquare$$

$$\Box \frac{\circ\xi \vdash \Gamma}{\Box\xi \vdash \Gamma} \qquad \frac{\Gamma \vdash \circ\xi}{\Gamma \vdash \Box\xi} \Box$$

The following fact is proven by a straightforward induction on α and ξ. We omit the details.

Proposition 36. *For every $\alpha \in$ General and $\xi \in$ Special, the sequents $\alpha \vdash \alpha$ and $\xi \vdash \xi$ are derivable in D.MKL.*

[6]Let $\Gamma^{(n)}$ be defined by setting $\Gamma^{(1)} := \Gamma$ and $\Gamma^{(n+1)} := \Gamma \odot \Gamma^{(n)}$.

5.6 Properties

5.6.1 Soundness

In the present subsection, we outline the verification of the soundness of the rules of D.MKL w.r.t. heterogenous measurable Kleene algebras (cf. Definition 45). The first step consists in interpreting structural symbols as logical symbols according to their (precedent or succedent) position, as indicated in the synoptic table of Section 5.5.1. This makes it possible to interpret sequents as inequalities, and rules as quasi-inequalities. For example, (modulo standard manipulations) the rules on the left-hand side below correspond to the (quasi-)inequalities on the right-hand side:

$$\frac{}{\Phi \vdash \circ\Pi} \quad \rightsquigarrow \quad \forall\xi[1 \leq \Box\xi]$$

$$\frac{\Gamma \vdash \circ\Pi \qquad \Delta \vdash \circ\Pi}{\Gamma \odot \Delta \vdash \circ\Pi}\ \textsf{abs} \quad \rightsquigarrow \quad \forall\alpha\forall\beta[\blacklozenge(\alpha \cdot \beta) \leq \blacklozenge\alpha \sqcup \blacklozenge\beta]$$

$$\textsf{b-bal}\ \frac{\Pi \vdash \Sigma}{\bullet \circ \Pi \vdash \bullet \circ \Sigma} \quad \rightsquigarrow \quad \forall\xi[\blacklozenge\Box\xi \leq \blacksquare\Box\xi]$$

$$\frac{\Pi \vdash \Xi}{\circ\Pi \vdash \circ\Xi}\ \textsf{w-bal} \quad \rightsquigarrow \quad \forall\xi\forall\pi[\pi \leq \xi \Leftrightarrow \Box\pi \leq \Box\xi]$$

$$\omega\ \frac{(\Gamma^{(n)} \vdash \Delta \mid n \geq 1)}{\circ \bullet \Gamma \vdash \Delta} \quad \rightsquigarrow \quad \forall\alpha[\Box\blacklozenge\alpha \leq \bigcup_{n\in\omega} \alpha^n]$$

$$\frac{\circ\Pi \odot \circ\Pi \vdash \Delta}{\circ\Pi \vdash \Delta}\ \circ\textsf{-C} \quad \rightsquigarrow \quad \forall\xi[\Box\xi \leq \Box\xi \cdot \Box\xi]$$

Then, the verification of the soundness of the rules of D.MKL boils down to checking the validity of their corresponding quasi-inequalities in heterogenous measurable Kleene algebras. This verification is routine and is omitted.

5.6.2 Completeness

In the present section, we show that the translations – by means of the map $()^t$ defined in Section 5.4 – of the axioms and rules of S.MKL (cf. Section 5.2.2) are derivable in the calculus D.MKL. For the reader's convenience, here below we report the recursive definition of $()^t$:

$$\begin{array}{rcl}
a^t & ::= & a \\
1^t & ::= & 1 \\
0^t & ::= & 0 \\
(\alpha \cdot \beta)^t & ::= & \alpha^t \cdot \beta^t \\
(\alpha \cup \beta)^t & ::= & \alpha^t \cup \beta^t \\
(\alpha^*)^t & ::= & \Box\blacklozenge\alpha^t \\
(\alpha^\star)^t & ::= & \Box\blacksquare\alpha^t
\end{array}$$

Proposition 37. *For every $\alpha \in$ S.KL, the sequent $\alpha^t \vdash \alpha^t$ is derivable in D.MKL.*

Let $\alpha^{(n)}$ be defined by setting $\alpha^{(1)} := \alpha$ and $\alpha^{(n+1)} := \alpha \odot \alpha^{(n)}$.

Lemma 19 (Omega). *If $\alpha \odot \beta \vdash \beta$ (resp. $\beta \odot \alpha \vdash \beta$) is derivable, then $\alpha^{(n)} \odot \beta \vdash \beta$ (resp. $\beta \odot \alpha^{(n)} \vdash \beta$) is derivable for every $n \geq 0$.*

Proof. Let us show that for any $n \geq 1$, if $\alpha^{(n)} \odot \beta \vdash \beta$ is derivable, then $\alpha^{(n+1)} \odot \beta \vdash \beta$ is derivable (the proof that $\beta \odot \alpha^{(n)} \vdash \beta$ is derivable from $\beta \odot \alpha^{(n)} \vdash \beta$ is analogous and it is omitted). Indeed:

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha \vdash \alpha \qquad \overset{\text{hyp}}{\alpha^{(n)} \odot \beta \vdash \beta}}{\alpha \odot (\alpha^{(n)} \odot \beta) \vdash \alpha \cdot \beta}}{(\alpha \odot \alpha^{(n)}) \odot \beta \vdash \alpha \cdot \beta}}{\alpha^{(n+1)} \odot \beta \vdash \alpha \cdot \beta} \qquad \overset{\text{assump}}{\alpha \cdot \beta \vdash \beta}}{\alpha^{(n+1)} \odot \beta \vdash \beta}\ \text{cut}$$

Hence, the sequent $\alpha^{(n)} \odot \beta \vdash \beta$ for any n is obtained from a proof of $\alpha \odot \beta \vdash \beta$ by concatenating n derivations of the shape shown above. □

As to the rule K4 (cf. Definition 38), if $\alpha \cdot \beta \vdash \beta$ is derivable in D.MKL, then $\alpha \odot \beta \vdash \beta$ is derivable in D.MKL[7], hence by Lemma 19 so are the sequents $\alpha^{(n)} \odot \beta \vdash \beta$ for any $n \geq 1$. By applying the appropriate display postulate to each such sequent, we obtain derivations of $\alpha^{(n)} \vdash \beta < \beta$ for any $n \geq 1$. Hence:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\omega\ \dfrac{(\alpha^{(n)} \vdash \beta < \beta \mid 1 \leq n)}{\circ \bullet \alpha \vdash \beta < \beta}}{\bullet \alpha \vdash \bullet(\beta < \beta)}}{\blacklozenge \alpha \vdash \bullet(\beta < \beta)}}{\circ \blacklozenge \alpha \vdash \beta < \beta}}{\Box \blacklozenge \alpha \vdash \beta < \beta}$$

$$\dfrac{\dfrac{\Box \blacklozenge \alpha \vdash \beta < \beta}{\Box \blacklozenge \alpha \odot \beta \vdash \beta}}{\Box \blacklozenge \alpha \cdot \beta \vdash \beta}$$

The proof that the rule K5 is derivable is analogous and we omit it. As to the axioms of Definition 38 in which $()^*$-terms occur,

$$\dfrac{\dfrac{\dfrac{\text{one}\ \dfrac{}{\Phi \vdash \circ \blacklozenge \alpha}}{\Phi \vdash \Box \blacklozenge \alpha}}{1 \vdash \Box \blacklozenge \alpha} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \vdash \alpha}{\bullet \alpha \vdash \blacklozenge \alpha}}{\alpha \vdash \circ \blacklozenge \alpha} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\alpha \vdash \alpha}{\bullet \alpha \vdash \blacklozenge \alpha}}{\blacklozenge \alpha \vdash \blacklozenge \alpha}}{\circ \blacklozenge \alpha \vdash \circ \blacklozenge \alpha}\ \text{w-bal}}{\Box \blacklozenge \alpha \vdash \circ \blacklozenge \alpha}}{\alpha \odot \Box \blacklozenge \alpha \vdash \circ \blacklozenge \alpha}\ \text{abs}}{\alpha \odot \Box \blacklozenge \alpha \vdash \Box \blacklozenge \alpha}}{\alpha \cdot \Box \blacklozenge \alpha \vdash \Box \blacklozenge \alpha}}{1 \cup \alpha \cdot \Box \blacklozenge \alpha \vdash \Box \blacklozenge \alpha}$$

[7] This is due to the fact that $\cdot$ is a normal $\mathcal{F}$-operator, and in proper display calculi the left introduction rules of $\mathcal{F}$-operators are invertible.

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{one}
\UnaryInfC{$\Phi \vdash \circ\blacklozenge\alpha$}
\UnaryInfC{$\Phi \vdash \Box\blacklozenge\alpha$}
\UnaryInfC{$1 \vdash \Box\blacklozenge\alpha$}
\AxiomC{$\alpha \vdash \alpha$}
\UnaryInfC{$\bullet\alpha \vdash \blacklozenge\alpha$}
\UnaryInfC{$\blacklozenge\alpha \vdash \blacklozenge\alpha$}
\RightLabel{w-bal}
\UnaryInfC{$\circ\blacklozenge\alpha \vdash \circ\blacklozenge\alpha$}
\UnaryInfC{$\Box\blacklozenge\alpha \vdash \circ\blacklozenge\alpha$}
\AxiomC{$\alpha \vdash \alpha$}
\UnaryInfC{$\bullet\alpha \vdash \blacklozenge\alpha$}
\UnaryInfC{$\blacklozenge\alpha \vdash \blacklozenge\alpha$}
\RightLabel{w-bal}
\UnaryInfC{$\circ\blacklozenge\alpha \vdash \circ\blacklozenge\alpha$}
\UnaryInfC{$\Box\blacklozenge\alpha \vdash \circ\blacklozenge\alpha$}
\RightLabel{abs}
\BinaryInfC{$\Box\blacklozenge\alpha \odot \Box\blacklozenge\alpha \vdash \circ\blacklozenge\alpha$}
\UnaryInfC{$\Box\blacklozenge\alpha \odot \Box\blacklozenge\alpha \vdash \Box\blacklozenge\alpha$}
\UnaryInfC{$\Box\blacklozenge\alpha \cdot \Box\blacklozenge\alpha \vdash \Box\blacklozenge\alpha$}
\BinaryInfC{$1 \cup \Box\blacklozenge\alpha \cdot \Box\blacklozenge\alpha \vdash \Box\blacklozenge\alpha$}
\end{prooftree}
$$

The translations of $0 \cdot \alpha \dashv\vdash 0$ are derivable as follows:

$$
\begin{prooftree}
\AxiomC{$0 \vdash \Phi$}
\RightLabel{Φ-W}
\UnaryInfC{$0 \vdash \Phi < \alpha$}
\UnaryInfC{$0 \odot \alpha \vdash \Phi$}
\UnaryInfC{$0 \cdot \alpha \vdash \Phi$}
\UnaryInfC{$0 \cdot \alpha \vdash 0$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{$0 \vdash \Phi$}
\RightLabel{Φ-W}
\UnaryInfC{$0 \vdash \alpha \cdot 0$}
\end{prooftree}
$$

The translation of $0^* \vdash 1$ is derivable as follows:

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{one}
\UnaryInfC{$\Phi \vdash \circ \bullet 1$}
\LeftLabel{Φ}
\UnaryInfC{$\Phi \odot 0 \vdash \circ \bullet 1$}
\LeftLabel{Φ}
\UnaryInfC{$0 \vdash \circ \bullet 1$}
\UnaryInfC{$\bullet 0 \vdash \bullet 1$}
\UnaryInfC{$\blacklozenge 0 \vdash \bullet 1$}
\UnaryInfC{$\circ\blacklozenge 0 \vdash 1$}
\UnaryInfC{$\Box\blacklozenge 0 \vdash 1$}
\end{prooftree}
$$

The translation of $1 \vdash 0^*$ is derivable as follows:

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{one}
\UnaryInfC{$\Phi \vdash \circ\blacklozenge 0$}
\UnaryInfC{$\Phi \vdash \Box\blacklozenge 0$}
\UnaryInfC{$1 \vdash \Box\blacklozenge 0$}
\end{prooftree}
$$

The translation of $1^* \vdash 1$ is derivable applying the rule $\circ \bullet \Phi$ (that is derivable using the ω-rule):

$$
\begin{prooftree}
\AxiomC{$\Phi \vdash 1$}
\LeftLabel{$\circ \bullet \Phi$}
\UnaryInfC{$\circ \bullet \Phi \vdash 1$}
\UnaryInfC{$\bullet \Phi \vdash \bullet 1$}
\UnaryInfC{$\Phi \vdash \circ \bullet 1$}
\UnaryInfC{$1 \vdash \circ \bullet 1$}
\UnaryInfC{$\bullet 1 \vdash \bullet 1$}
\UnaryInfC{$\blacklozenge 1 \vdash \bullet 1$}
\UnaryInfC{$\circ\blacklozenge 1 \vdash 1$}
\UnaryInfC{$\Box\blacklozenge 1 \vdash 1$}
\end{prooftree}
$$

The derivations of the translations of the remaining axioms are standard and are omitted. Below, we derive the translations of the axioms of Definition 39.

$$\dfrac{\dfrac{\overline{\quad\Phi \vdash \circ\blacksquare\alpha\quad}\ \text{one}}{\Phi \vdash \Box\blacksquare\alpha}}{1 \vdash \Box\blacksquare\alpha}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \vdash \alpha}{\blacksquare\alpha \vdash \bullet\alpha}}{\blacksquare\alpha \vdash \blacksquare\alpha}}{\circ\blacksquare\alpha \vdash \circ\blacksquare\alpha}\,\text{w-bal}}{\Box\blacksquare\alpha \vdash \circ\blacksquare\alpha}
\quad
\dfrac{\dfrac{\dfrac{\dfrac{\alpha \vdash \alpha}{\blacksquare\alpha \vdash \bullet\alpha}}{\blacksquare\alpha \vdash \blacksquare\alpha}}{\circ\blacksquare\alpha \vdash \circ\blacksquare\alpha}\,\text{w-bal}}{\Box\blacksquare\alpha \vdash \circ\blacksquare\alpha}}{\Box\blacksquare\alpha \odot \Box\blacksquare\alpha \vdash \circ\blacksquare\alpha}\,\text{abs}}{\Box\blacksquare\alpha \odot \Box\blacksquare\alpha \vdash \Box\blacksquare\alpha}$$

$$\dfrac{\dfrac{\dfrac{\alpha \vdash \alpha}{\blacksquare\alpha \vdash \bullet\alpha}}{\circ\blacksquare\alpha \vdash \alpha}}{\Box\blacksquare\alpha \vdash \alpha}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \vdash \alpha}{\blacksquare\alpha \vdash \bullet\alpha}}{\blacksquare\alpha \vdash \blacksquare\alpha}}{\bullet\circ\blacksquare\alpha \vdash \bullet\circ\blacksquare\alpha}\,\text{b-bal}}{\circ\bullet\circ\blacksquare\alpha \vdash \circ\blacksquare\alpha}}{\circ\bullet\circ\blacksquare\alpha \vdash \Box\blacksquare\alpha}}{\bullet\circ\blacksquare\alpha \vdash \bullet\Box\blacksquare\alpha}}{\bullet\circ\blacksquare\alpha \vdash \blacksquare\Box\blacksquare\alpha}}{\circ\blacksquare\alpha \vdash \circ\blacksquare\Box\blacksquare\alpha}}{\circ\blacksquare\alpha \vdash \Box\blacksquare\Box\blacksquare\alpha}}{\Box\blacksquare\alpha \vdash \Box\blacksquare\Box\blacksquare\alpha}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \vdash \alpha}{\blacksquare\alpha \vdash \bullet\alpha}}{\blacksquare\alpha \vdash \blacksquare\alpha}}{\circ\blacksquare\alpha \vdash \circ\blacksquare\alpha}\,\text{w-bal}}{\Box\blacksquare\alpha \vdash \circ\blacksquare\alpha}}{\Box\blacksquare\alpha \vdash \Box\blacksquare\alpha}$$

Finally, let us derive the translation of the ternary rule of Definition 39. Assume that the translations of $\beta \vdash \alpha$, and $1 \vdash \beta$ and $\beta \cdot \beta \vdash \beta$ are derivable. Hence, by the invertibility of the introduction rules of $\mathcal{F}$-connectives in proper display calculi, $\Phi \vdash \beta$ and $\beta \odot \beta \vdash \beta$ are derivable. By Lemma 19, $\beta^{(n)} \odot \beta \vdash \beta$ is derivable. Therefore, we can derive the following sequents for any $n \geq 1$:

$$\dfrac{\dfrac{\dfrac{(\beta^{(n)} \vdash \beta < \beta^{(n)} \mid 1 \leq n)}{\circ\bullet\beta \vdash \beta < \beta^{(n)}}\,\omega}{\circ\bullet\beta \odot \beta^{(n)} \vdash \beta}}{\beta^{(n)} \vdash \circ\bullet\beta > \beta}$$

Hence:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(\beta^{(n)} \vdash \circ\bullet\beta > \beta \mid 1 \leq n)}{\circ\bullet\beta \vdash \circ\bullet\beta > \beta}\,\omega}{\circ\bullet\beta \odot \circ\bullet\beta \vdash \beta}}{\circ\bullet\beta \vdash \beta}\,\circ\text{-C} \quad \beta \vdash \alpha}{\circ\bullet\beta \vdash \alpha}\,\text{Cut}}{\bullet\beta \vdash \bullet\alpha}}{\bullet\beta \vdash \blacksquare\alpha}}{\beta \vdash \circ\blacksquare\alpha}}{\beta \vdash \Box\blacksquare\alpha}$$

5.6.3 Conservativity

For any heterogeneous measurable Kleene algebra $\mathbb{H} = (\mathbb{A}, \mathbb{S}, \gamma, \iota, e)$, the algebra $\mathbb{A}$ is a complete join-semilattice, and $\cdot$ distributes over arbitrary joins in each coordinate. This implies that the right residuals exist of $\cdot$ in each coordinate, which we denote / and \:

$$\alpha\backslash\beta := \bigcup\{\alpha' : \alpha\cdot\alpha' \leq \beta\}, \quad \beta/\alpha := \bigcup\{\alpha' : \alpha'\cdot\alpha \leq \beta\}.$$

From here on, the proof of conservativity proceeds in the usual way as detailed in [9].

5.6.4 Cut elimination and subformula property

The cut elimination of D.MKL follows from the Belnap-style meta-theorem proven in [5], of which a restriction to proper multi-type display calculi is stated in [10]. The proof consists in verifying conditions C_1-C_{10} of [10, Section 6.4]. Most of these conditions are easily verified by inspection on rules; the most interesting one is condition C'_8, concerning the principal stage in the cut elimination, on which we expand in the lemma below.

Lemma 20. $\mathrm{D.MKL}$ *satisfies* C'_8.

Proof. By induction on the shape of the cut formula.

Atomic propositions:

$$\frac{a \vdash a \qquad a \vdash a}{a \vdash a} \quad\rightsquigarrow\quad a \vdash a$$

Constants:

$$\frac{\Phi \vdash 1 \qquad \dfrac{\begin{array}{c}\vdots\,\pi_1\\ \Phi \vdash \Delta\end{array}}{1 \vdash \Delta}}{\Phi \vdash \Delta} \quad\rightsquigarrow\quad \begin{array}{c}\vdots\,\pi_1\\ \Phi \vdash \Delta\end{array}$$

The cases for 0, 0_s are standard and similar to the one above.

Unary connectives: As to $\blacklozenge\alpha$,

$$\frac{\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \Gamma \vdash \alpha\end{array}}{\bullet\Gamma \vdash \blacklozenge\alpha} \qquad \dfrac{\begin{array}{c}\vdots\,\pi_2\\ \bullet\alpha \vdash \Xi\end{array}}{\blacklozenge\alpha \vdash \Xi}}{\bullet\Gamma \vdash \Xi} \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \Gamma \vdash \alpha\end{array} \qquad \dfrac{\begin{array}{c}\vdots\,\pi_2\\ \bullet\alpha \vdash \Xi\end{array}}{\alpha \vdash \circ\Xi}}{\Gamma \vdash \circ\Xi}}{\bullet\Gamma \vdash \Xi}$$

As to $\Box\alpha$,

$$\frac{\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \Gamma \vdash \circ\xi\end{array}}{\Gamma \vdash \Box\xi} \qquad \dfrac{\begin{array}{c}\vdots\,\pi_2\\ \xi \vdash \Xi\end{array}}{\Box\xi \vdash \circ\Xi}}{\Gamma \vdash \circ\Xi} \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \Gamma \vdash \circ\xi\end{array}}{\bullet\Gamma \vdash \xi} \qquad \begin{array}{c}\vdots\,\pi_2\\ \xi \vdash \Xi\end{array}}{\bullet\Gamma \vdash \Xi}}{\Gamma \vdash \circ\Xi}$$

Binary connectives: As to $\alpha_1 \cup \alpha_2$,

$$
\dfrac{\dfrac{\begin{array}{c}\vdots\, \pi_1 \\ \Gamma \vdash \alpha_1\end{array}}{\Gamma \vdash \alpha_1 \cup \alpha_2} \qquad \dfrac{\begin{array}{c}\vdots\, \pi_2 \\ \alpha_1 \vdash \Delta\end{array} \qquad \begin{array}{c}\vdots\, \pi_3 \\ \alpha_2 \vdash \Delta\end{array}}{\alpha_1 \cup \alpha_2 \vdash \Delta}}{\Gamma \vdash \Delta}
\quad \rightsquigarrow \quad
\dfrac{\begin{array}{c}\vdots\, \pi_1 \\ \Gamma \vdash \alpha_1\end{array} \qquad \begin{array}{c}\vdots\, \pi_2 \\ \alpha_1 \vdash \Delta\end{array}}{\Gamma \vdash \Delta}
$$

$\square$

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Chapter 6

Conclusion

The present dissertation contributes to the research program in algebraic proof theory in the following respects:

1. by extending some core results and techniques (e.g. semantic cut elimination, finite model property) in algebraic proof theory from the setting of (non-associative) full Lambek calculus to the much broader setting of *normal lattice expansions* logics (normal LE-logics). Besides full Lambek calculus, this setting includes all normal modal expansions of Lambek calculus, the Lambek-Grishin calculus, the multiplicative-additive fragment of linear logic, orthologic, bunched logic, and the list could go on;

2. by transferring notions, insights and techniques to the setting of *multi-type calculi* and their semantic environment given by *heterogeneous algebras*. This move makes it possible to further extend the reach of the insights and results of algebraic proof theory beyond analytic inductive axiomatizations. In the thesis we provide three very different examples of non-analytic logics (i.e. logics no known axiomatization of which consists only of analytic inductive axioms) which can be associated with a *proper* display calculus via an equivalent reformulation into a suitable multi-type calculus.

The scientific significance of these contributions is partly grounded on the significance of the concrete logics the theory of which has been advanced by the present contributions. In particular, as discussed in the previous chapters, semi De Morgan logic is a paraconsistent logic designed to capture aspects of bounded rationality such as being able to entertain locally inconsistent reasoning in some areas which does not extend to global inconsistency [12]; Bilattice logic is a logical formalism motivated by the need of integrating in a principled way incomplete or inconsistent information coming from different sources [4]; the logic of Kleene algebras has been a very important tool in program verification [6], and we have also built a connection between Kleene algebras and issues in linguistics connected with the formalization of exceptions to grammatical rules [7].

Moreover, having successfully applied the multi-type methodology to these three case studies has also contributed to sharpen and refine the multi-type methodology itself, and thus make it a better tool to address the challenges that come from the formalization of real-life situations involving social behaviour.

Finally, recent developments [2, 3] have proposed a conceptual interpretation of some LE-logics as epistemic logics of categories, and used them and their semantics to formalize certain core notions of categorization theory in management science [5]. Further ongoing research stemming from these developments suggests that LE-logics can be understood as an overarching logical framework for entities such as categories [9], concepts [10], theories [8], questions [1, 11], which are of great interest in a broad range of fields including cognition, epistemology, linguistics, and AI. Embedded in a modular way as parts of specific multi-type environments, LE-logics show great potential as tools for analyzing social behaviour.

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Curriculum Vitæ

Fei LIANG

10-06-1988	Born in Baoji, China.

Education

2003 – 2016	Shanjiu High school Baoji, China
2006 – 2010	B.E., Electrical Engineering and Automation Yanshan University, Qinhuangdao, China
2011 – 2014	M.A., Philosophy East China Normal University, Shanghai, China
2014 – 2017	Ph.D., Philosophy Institute of Logic and Cognition, Sun Yat-sen University, Guangzhou, China
2016 – 2018	Ph.D., Logic Applied Logic Group, Delft University of Technology, the Netherlands

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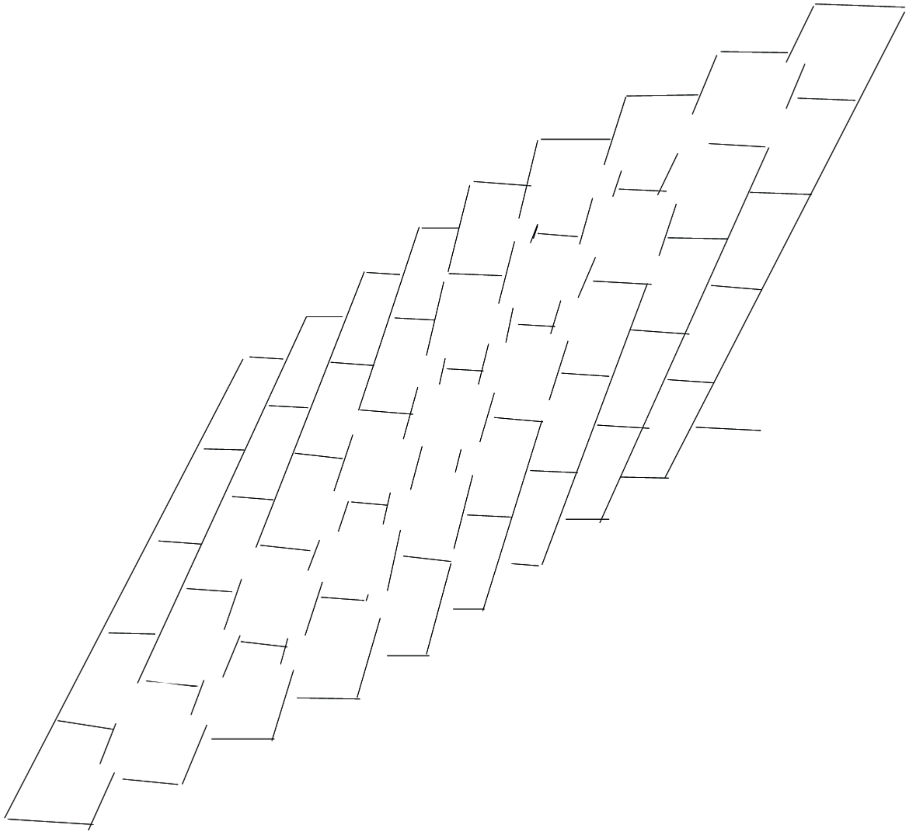

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