Brownian representations of cylindrical continuous local martingales

Yaroslavtsev, Ivan

DOI
10.1142/S0219025718500133

Publication date
2018

Document Version
Final published version

Published in
Infinite Dimensional Analysis, Quantum Probability and Related Topics

Citation (APA)
Yaroslavtsev, I. (2018). Brownian representations of cylindrical continuous local martingales. *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, *21*(2), 1-25. Article 1850013. https://doi.org/10.1142/S0219025718500133

Infinite Dimensional Analysis, Quantum Probability
and Related Topics
Vol. 21, No. 2 (2018) 1850013 (25 pages)

DOI: 10.1142/S0219025718500133

Brownian representations of cylindrical continuous local martingales

Ivan Yaroslavtsev

*Delft Institute of Applied Mathematics,
Delft University of Technology, P.O. Box 5031,
2600 GA Delft, The Netherlands
I.S.Yaroslavtsev@tudelft.nl*

Received 23 May 2016
Accepted 20 March 2017
Published 8 June 2018
Communicated by Vladimir Igorevich Bogachev

In this paper, we give necessary and sufficient conditions for a cylindrical continuous local martingale to be the stochastic integral with respect to a cylindrical Brownian motion. In particular, we consider the class of cylindrical martingales with closed operator-generated covariations. We also prove that for every cylindrical continuous local martingale M there exists a time change τ such that $M \circ \tau$ is Brownian representable.

Keywords: Brownian representation; cylindrical martingale; quadratic variation; UMD spaces.

AMS Subject Classification: 60H05, 60G44, 47A56, 46E27

1. Introduction

In the fundamental work,[7] Doob showed that a real-valued continuous local martingale can be represented as a stochastic integral with respect to a real-valued Brownian motion if and only if this local martingale has an absolutely continuous quadratic variation. Starting from this point, for a Banach space X the following problem appears: find necessary and sufficient conditions for an X-valued local martingale M in order that there exist a (cylindrical) Brownian motion W and a stochastically integrable function g such that $M = \int_0^\cdot g\,\mathrm{d}W$ (we then call M *Brownian representable.*) For some special instances of Banach spaces X, Brownian representation results are well known. The finite-dimensional version was derived in Refs. 14 and 29 and a generalization to the Hilbert space case was obtained using different techniques in Refs. 4, 18 and 23. But for a general Banach space X some problems arise. For instance, one cannot define a quadratic variation of a Banach space-valued continuous martingale in a proper way (we refer the reader to Ref. 6, where a notion of quadratic variation is defined which, however, is not well-defined for some particular martingales) and therefore it seems difficult to find appropriate necessary conditions for an arbitrary martingale to be Brownian representable. Nevertheless, quite general sufficient conditions (which are also necessary when the Banach space is a Hilbert space) were obtained by Dettweiler in Ref. 5.

In order to generalize the above-mentioned results one can work with so-called cylindrical continuous local martingales (see Refs. 12, 21, 22, 19, 28 and 33), defined for an arbitrary Banach space X as continuous linear mappings from X^* to a linear space of continuous local martingales $\mathcal{M}^{\mathrm{loc}}$ equipped with the ucp topology. Using this approach together with functional calculus arguments, Ondreját[21,22] has shown that if X is a reflexive Banach space, then a cylindrical continuous local martingale M is Brownian representable under appropriate conditions. More precisely, he shows that M is Brownian representable if there exist a Hilbert space H and a scalarly progressively measurable process g with values $\mathcal{L}(H,X)$ such that

$$\int_0^t \|g\|^2 \,\mathrm{d}s < \infty \quad \text{a.s. } \forall\, t > 0, \quad [Mx^*]_t = \int_0^t \|g^*x^*\|^2 \,\mathrm{d}s \quad \text{a.s. } \forall\, t > 0. \tag{1.1}$$

The definition of a quadratic variation of a cylindrical continuous local martingale given in Ref. 33 was inspired by this result.

In this work, we show that if $[Mx^*]$ is absolutely continuous for each $x^* \in X^*$, then there exist a Hilbert space H, an H-cylindrical Brownian motion W_H defined on an enlarged probability space with an enlarged filtration, and a progressively scalarly measurable process G with values in space of (possibly unbounded) linear operators from X^* to H such that $Mx^* = \int_0^\cdot Gx^* \,\mathrm{d}W_H$ for all $x^* \in X^*$. Moreover, necessary and sufficient conditions for a so-called weak Brownian representation of an arbitrary X-valued continuous local martingale are established.

It is well known (see Ref. 13) that each $\mathbb{R}$-, $\mathbb{R}^d$-, or H-valued continuous local martingale M admits a time change τ such that $M \circ \tau$ is Brownian representable. This assertion is tied up to the fact that $[M \circ \tau] = [M] \circ \tau$, which makes the choice of τ evident. The same can be easily shown for a cylindrical continuous local martingale with a quadratic variation (see Ref. 33). Here we prove that such a time change exists for arbitrary cylindrical continuous local martingales.

To conclude this introduction we would like to point out the techniques developed by Ondreját in Refs. 21 and 22, in particular results developing a bounded Borel calculus for bounded operator-valued functions. Quite a reasonable part of this paper is dedicated to applying and extending these techniques to a closed operator-valued g. Of course statements as (1.1) do not make sense then, but thanks to closability of g it is still possible to prove some results on stochastic integrability for such g.

2. Preliminaries

We denote $[0,\infty)$ by $\mathbb{R}_+$.

The *Lebesgue–Stieltjes measure* of a function $F : \mathbb{R}_+ \to \mathbb{R}$ of bounded variation is the finite Borel measure μ_F on $\mathbb{R}_+$ defined by $\mu_F([a, b)) = F(b) - F(a)$ for $0 \leq a < b < \infty$.

Let (S, Σ) be a measurable space and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A mapping $\nu : \Sigma \times \Omega \to [0, \infty]$ will be called a *random measure* if for all $A \in \Sigma$, $\omega \mapsto \nu(A, \omega)$ is measurable and for almost all $\omega \in \Omega$, $\nu(\cdot, \omega)$ is a measure on (S, Σ) and $(S, \Sigma, \nu(\cdot, \omega))$ is separable (i.e. the corresponding L^2-space is separable).

Random measures arise naturally when working with continuous local martingales. Indeed, for almost all $\omega \in \Omega$, the quadratic variation process $[M](\cdot, \omega)$ of a continuous local martingale M is continuous and increasing (see Refs. 13, 19 and 24), so we can associate $\mu_{[M]}(\cdot, \omega)$ with it.

Let (S, Σ, μ) be a measure space. Let X and Y be Banach spaces. An operator-valued function $f : S \to \mathcal{L}(X, Y)$ is called *scalarly measurable* if for all $x \in X$ and $y^* \in Y^*$ the function $s \mapsto \langle y^*, f(s)x\rangle$ is measurable. If Y is separable, by the Pettis measurability theorem this is equivalent to the strong measurability of $s \mapsto f(s)x$ for each $x \in X$ (see Ref. 10).

Often we will use the notation $A \lesssim_Q B$ to indicate that there exists a constant C which only depends on the parameter(s) Q such that $A \leq CB$.

3. Results

3.1. *Cylindrical martingales and stochastic integration*

In this section, we assume that X is a separable Banach space with a dual space X^*. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with filtration $\mathbb{F} := (\mathcal{F}_t)_{t\in\mathbb{R}_+}$ that satisfies the usual conditions, and let $\mathcal{F} := \sigma(\bigcup_{t\geq 0} \mathcal{F}_t)$. We denote the predictable σ-algebra by $\mathcal{P}$.

A scalar-valued process M is called a *continuous local martingale* if there exists a sequence of stopping times $(\tau_n)_{n\geq 1}$ such that $\tau_n \uparrow \infty$ almost surely as $n \to \infty$ and $\mathbf{1}_{\tau_n>0} M^{\tau_n}$ is a continuous martingale.

Let $\mathcal{M}^{\mathrm{loc}}$ be the class of all continuous local martingales. On $\mathcal{M}^{\mathrm{loc}}$ define the translation invariant metric given by

$$\|M\|_{\mathcal{M}^{\mathrm{loc}}} = \sum_{n=1}^{\infty} 2^{-n} \mathbb{E}\left[1 \wedge \sup_{t\in[0,n]} |M|_t\right].$$

One can easily check that the topology generated by this metric coincides with the ucp topology (uniform convergence on compact sets in probability). Moreover, $M_n \to 0$ in $\mathcal{M}^{\mathrm{loc}}$ if and only if for every $T \geq 0$, $[M_n]_T \to 0$ in probability (see Proposition 17.6 of Ref. 13).

Let X be a separable Banach space, $\mathcal{M}^{\mathrm{loc}}(X)$ be the space of X-valued continuous local martingales. $\mathcal{M}^{\mathrm{loc}}(X)$ is complete under the ucp topology generated by the following metric:

$$\|M\|_{\mathcal{M}^{\mathrm{loc}}(X)} = \sum_{n=1}^{\infty} 2^{-n} \mathbb{E}\left[1 \wedge \sup_{t\in[0,n]} \|M\|_t\right].$$

The completeness can be proven in the same way as in Part 3.1 of Ref. 33.

If H is a Hilbert space and $M \in \mathcal{M}^{\mathrm{loc}}(H)$, then we define the quadratic variation $[M]$ as a compensator of $\|M\|^2$, and one can show that a.s.

$$[M] = \sum_{n=1}^{\infty}[\langle M, h_n\rangle],$$

where $(h_n)_{n=1}^{\infty}$ is any orthonormal basis of H. For more details we refer to Chap. 14.3 of Ref. 19.

Remark 3.1. One can show that convergence in the ucp topology on $\mathcal{M}^{\mathrm{loc}}(H)$ is equivalent to convergence of quadratic variation in the ucp topology. This fact can be shown analogously to the scalar case, see Proposition 17.6 of Ref. 13.

Let X be a Banach space. A continuous linear mapping $M : X^* \to \mathcal{M}^{\mathrm{loc}}$ is called a *cylindrical continuous local martingale*. We will write $M \in \mathcal{M}^{\mathrm{loc}}_{\mathrm{cyl}}(X)$. Details on cylindrical martingales can be found in Refs. 12 and 33. For a cylindrical continuous local martingale M and a stopping time τ we define $M^\tau : X^* \to \mathcal{M}^{\mathrm{loc}}$ by $M^\tau x^*(t) = Mx^*(t\wedge\tau)$. In this way $M^\tau \in \mathcal{M}^{\mathrm{loc}}_{\mathrm{cyl}}(X)$ again.

Let Y be a Banach space such that there exists a continuous embedding $j : Y^* \hookrightarrow X^*$ (e.g., X is densely embedded in Y). Then define $M|_Y : Y^* \to \mathcal{M}^{\mathrm{loc}}$ by $y^* \mapsto M(jy^*)$. Obviously $M|_Y \in \mathcal{M}^{\mathrm{loc}}_{\mathrm{cyl}}(Y)$.

Example 3.2. (Cylindrical Brownian motion) Let X be a Banach space and $Q \in \mathcal{L}(X^*, X)$ be a positive self-adjoint operator, i.e. $\forall\, x^*, y^* \in X^*$ it holds that $\langle x^*, Qy^*\rangle = \langle y^*, Qx^*\rangle$ and $\langle Qx^*, x^*\rangle \geq 0$. Let $W^Q : \mathbb{R}_+ \times X^* \to L^0(\Omega)$ be a *cylindrical Q-Brownian motion* (see Chap. 4.1 of Ref. 4), i.e.

- $W^Q(\cdot)x^*$ is a Brownian motion for all $x^* \in X$,
- $\mathbb{E}W^Q(t)x^*\, W^Q(s)y^* = \langle Qx^*, y^*\rangle \min\{t, s\}$, $\forall\, x^*, y^* \in X^*$, $t, s \geq 0$.

The operator Q is called the *covariance operator* of W^Q. (See more in Chap. 1 of Ref. 20 or Chap. 4.1 of Ref. 4 for a Hilbert space-valued case and Chap. 5 of Ref. 32 or of Ref. 25 for the general case). Then $W^Q \in \mathcal{M}^{\mathrm{loc}}_{\mathrm{cyl}}(X)$.

If X is a Hilbert space and $Q = I$ is the identity operator, we call $W_X := W^I$ an *X-cylindrical Brownian motion*.

Let X, Y be Banach spaces, $x^* \in X^*$, $y \in Y$. We denote by $x^* \otimes y \in \mathcal{L}(X, Y)$ a rank-one operator that maps $x \in X$ to $\langle x, x^*\rangle y$.

Remark 3.3. Notice that the adjoint of a rank-one operator is again a rank-one operator and $(x^* \otimes y)^* = y \otimes x^* : Y^* \to X^*$. Also for any Banach space Z and bounded operator $A : Y \to Z$ we have that $A(x^* \otimes y) = x^* \otimes (Ay)$.

The process $f : \mathbb{R}_+ \times \Omega \to \mathcal{L}(X, Y)$ is called *elementary progressive* with respect to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}$ if it is of the form

$$f(t,\omega) = \sum_{n=1}^{N}\sum_{m=1}^{M} \mathbf{1}_{(t_{n-1},t_n]\times B_{mn}}(t,\omega) \sum_{k=1}^{K} x_k^* \otimes y_{kmn},$$

where $0 \leq t_0 < \cdots < t_n < \infty$, for each $n = 1,\dots,N$, $B_{1n},\dots,B_{Mn} \in \mathcal{F}_{t_{n-1}}$, $(x_k^*)_{k=1}^K \subset X^*$ and $(y_{kmn})_{k,m,n=1}^{K,M,N} \subset Y$. For each elementary progressive f we define the stochastic integral with respect to $M \in \mathcal{M}^{\mathrm{loc}}_{\mathrm{cyl}}(X)$ as an element of $L^0(\Omega; C_b(\mathbb{R}_+; Y))$:

$$\int_0^t f(s)\,\mathrm{d}M(s) = \sum_{n=1}^{N}\sum_{m=1}^{M} \mathbf{1}_{B_{mn}} \sum_{k=1}^{K}(M(t_n \wedge t)x_k^* - M(t_{n-1}\wedge t)x_k^*)y_{kmn}. \quad (3.1)$$

Often we will write $f \cdot M$ for the process $\int_0^\cdot f(s)\mathrm{d}M(s)$.

Remark 3.4. Notice that the integral (3.1) defines the same stochastic process for a different form of finite-rank operator $\sum_{k=1}^K x_k^* \otimes y_{kmn}$. Indeed, let $(A_{mn})_{mn}$ be a set of operators from X to Y such that $(A_{mn})_{mn} = (\sum_{k=1}^K x_k^* \otimes y_{kmn})_{mn}$. Then $f\cdot M$ takes its values in a finite-dimensional subspace of Y depending only on $(A_{mn})_{mn}$. Let $Y_0 = \mathrm{span}(\mathrm{ran}(A_{mn}))_{mn}$. For each fixed $y_0^* \in Y_0^*$, m and n one can define $\ell_{mn} \in X^*$, $x \mapsto \langle y_0^*, A_{mn}x\rangle$. In particular, $\ell_{mn}(x) = \sum_{k=1}^K \langle x_k^* \langle y_0^*, y_{kmn}\rangle, x\rangle$. Then because of linearity of M

$$\begin{aligned}\left\langle y_0^*, \int_0^t f(s)\,\mathrm{d}M(s)\right\rangle &= \sum_{n=1}^{N}\sum_{m=1}^{M} \mathbf{1}_{B_{mn}} \sum_{k=1}^{K}(M(t_n \wedge t)x_k^* - M(t_{n-1}\wedge t)x_k^*)\langle y_0^*, y_{kmn}\rangle \\ &= \sum_{n=1}^{N}\sum_{m=1}^{M} \mathbf{1}_{B_{mn}}(M(t_n \wedge t) - M(t_{n-1}\wedge t)) \\ &\quad \times \left(\sum_{k=1}^{K} x_k^*\langle y_0^*, y_{kmn}\rangle\right) \\ &= \sum_{n=1}^{N}\sum_{m=1}^{M} \mathbf{1}_{B_{mn}}(M(t_n \wedge t) - M(t_{n-1}\wedge t))\ell_{mn},\end{aligned}$$

where the last expression does not depend on the form of $(A_{mn})_{mn}$. Then since Y_0 is finite-dimensional, the entire integral does not depend on the form of $(A_{mn})_{mn}$.

We say that $M \in \mathcal{M}^{\mathrm{loc}}_{\mathrm{cyl}}(X)$ is *Brownian representable* if there exist a Hilbert space H, an H-cylindrical Wiener process W_H on an enlarged probability space $(\overline{\Omega}, \overline{\mathbb{F}}, \overline{\mathbb{P}})$ and $g : \mathbb{R}_+ \times \overline{\Omega} \times X^* \to H$ such that $g(x^*)$ is $\overline{\mathbb{F}}$-scalarly progressively measurable and a.s.

$$Mx^* = \int_0^\cdot g^*(x^*)\mathrm{d}W_H, \quad x^* \in X^*. \quad (3.2)$$

We call that $M \in \mathcal{M}^{\mathrm{loc}}_{\mathrm{cyl}}(X)$ is *with an absolutely continuous covariation* (or $M \in \mathcal{M}^{\mathrm{cyl}}_{\mathrm{a.c.c.}}(X)$) if for each $x^*, y^* \in X^*$ the covariation $[Mx^*, My^*]$ is absolutely continuous a.s. Note that by Proposition 17.9 of Ref. 13 this is equivalent to $[Mx^*]$ having an absolutely continuous version for all $x^* \in X^*$.

3.2. *Closed operator representation*

In view of the results of Theorem 2 of Ref. 21, the natural problem presents itself to extend this theorem to the case of an unbounded operator-valued function g. Such a generalization will be proven in the present subsection.

For Banach spaces X, Y define $\mathcal{L}_{\mathrm{cl}}(X,Y)$ as a set of all closed densely defined operators from X to Y, $\mathcal{L}_{\mathrm{cl}}(X) := \mathcal{L}_{\mathrm{cl}}(X,X)$.

Let $M \in \mathcal{M}^{\mathrm{loc}}_{\mathrm{cyl}}(X)$ be Brownian representable. We say that M is *closed operator-Brownian representable* (or simply $M \in \mathcal{M}^{\mathrm{cyl}}_{\mathrm{cl}}(X)$) if there exists $G : \mathbb{R}_+ \times \Omega \to \mathcal{L}_{\mathrm{cl}}(H,X)$ such that for each fixed $x^* \in X^*$, G^*x^* is defined ($\mathbb{P} \times \mathrm{d}s$)-a.s. and it is a version of corresponding $g^*(x^*)$ from (3.2).

Definition 3.5. Let X be a Banach space with separable dual, $M \in \mathcal{M}^{\mathrm{loc}}_{\mathrm{cyl}}(X)$. Then M has *a closed operator-generated covariation* if there exist a separable Hilbert space H and a closed operator-valued function $G : \mathbb{R}_+ \times \Omega \to \mathcal{L}_{\mathrm{cl}}(H,X)$ such that for all $x^* \in X^*$, G^*x^* is defined ($\mathbb{P} \times \mathrm{d}s$)-a.s. and progressively measurable, and for all $x^*, y^* \in X^*$ a.s.

$$[Mx^*, My^*]_t = \int_0^t \langle G^*x^*, G^*y^* \rangle \, \mathrm{d}s, \quad t \geq 0. \tag{3.3}$$

Notice that the last assumption is equivalent to the fact that $[Mx^*] = \int_0^\cdot \|G^*x^*\|^2 \, \mathrm{d}s$ a.s. for all $x^* \in X^*$.

Proposition 3.6. *Let H be a separable Hilbert space, $M \in \mathcal{M}^{\mathrm{loc}}_{\mathrm{cyl}}(H)$ has a closed operator-generated covariation. Let $G : \mathbb{R}_+ \times \Omega \to \mathcal{L}_{\mathrm{cl}}(H)$ be the corresponding covariation family. Then for each scalarly progressively measurable $f : \mathbb{R}_+ \times \Omega \to H$ such that G^*f is defined ($\mathbb{P} \times \mathrm{d}s$)-a.e. and $\|G^*f\| \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$ a.s. one can define the stochastic integral $\int_0^\cdot f \mathrm{d}M$, and a.s.*

$$\left[\int_0^\cdot f \mathrm{d}M\right]_T = \int_0^T \langle G^*(s)f(s), G^*(s)f(s)\rangle \mathrm{d}s, \quad T > 0. \tag{3.4}$$

Proof. Applying the stopping time argument one can restrict the proof to the case $\mathbb{E}[\int_0^\infty \langle G^*f, G^*f\rangle \mathrm{d}s] < \infty$. First of all it is easy to construct the stochastic integral if $\mathrm{ran}(f) \subset H_0$, where H_0 is a fixed finite-dimensional subspace of H, since in this case by redefining G^* one can assume that G^*h is well-defined for all $h \in H_0$, so we just work with a finite-dimensional martingale for which (3.4) obviously holds according to the isometry Chap. 14.6 of Ref. 19.

The general case can be constructed in the following way. Let $(h_i)_{i\geq 1}$ be an orthonormal basis of H. For each $k \geq 1$ set $H_k = \mathrm{span}(h_1, \ldots, h_k)$. Then by Lemma A.5 one can construct scalarly progressively measurable $\tilde{P}_k : \mathbb{R}_+ \times \Omega \to \mathcal{L}(H)$, which is an orthogonal projection onto $G^*(H_k)$, and a scalarly progressively measurable $L_k : \mathbb{R}_+ \times \Omega \to \mathcal{L}(H, H_k)$ such that $G^* L_k = \tilde{P}_k$. Let $f_k = L_k G^* f$. Then from (3.4) for f_k, the fact that $\|G^* f_k\| = \|\tilde{P}_k G^* f\| \nearrow \|G^* f\|$ and $\|G^* f_k - G^* f\| \to 0$ ($\mathbb{P} \times \mathrm{d}s$) a.s., dominated convergence theorem, Proposition 17.6 of Ref. 13 and the fact that $\operatorname{ran} f_k \subset H_k$ one can construct stochastic integral $\int_0^\cdot f \mathrm{d}M$ and (3.4) holds true. □

Remark 3.7. Using the previous theorem one can slightly extend Remark 31 of Ref. 21 in the following way: let $\Psi : \mathbb{R}_+ \times \Omega \to \mathcal{L}(H)$ be progressively scalarly measurable such that $G^*\Psi^* \in \mathcal{L}(H)$ a.s. for all $t \geq 0$ and $\|G^*\Psi^*\| \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$ a.s. Then one can define $N \in \mathcal{M}^{\mathrm{loc}}_{\mathrm{cyl}}(H)$ as follows:

$$Nh = \int_0^\cdot \Psi^* h \, \mathrm{d}M.$$

Moreover, then for each progressively measurable $\phi : \mathbb{R}_+ \times \Omega \to H$ such that $\|G^*\Psi^*\phi\| \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$ a.s. one has that

$$\int_0^\cdot \phi \, \mathrm{d}N = \int_0^\cdot \Psi^* \phi \, \mathrm{d}M. \tag{3.5}$$

Theorem 3.8. *Let X be a separable reflexive Banach space, $M \in \mathcal{M}^{\mathrm{loc}}_{\mathrm{cyl}}(X)$. Then M is closed operator-Brownian representable if and only if it has a closed operator-generated covariation.*

Let $n \geq 1$, $X_1, \ldots, X_n$ be Banach spaces, $A_k \in \mathcal{L}_{\mathrm{cl}}(X_k, X_{k+1})$ for $1 \leq k \leq n-1$. Then we say that $A_{n-1} \ldots A_1$ is *well-defined* if $\operatorname{ran}(A_{k-1} \ldots A_1) \in \operatorname{dom}(A_k)$ for each $2 \leq k \leq n-1$.

Proof. Suppose that M is closed operator-Brownian representable. Let a separable Hilbert space H, an H-cylindrical Brownian motion W_H and $G : \mathbb{R}_+ \times \Omega \to \mathcal{L}_{\mathrm{cl}}(H, X)$ be such that a.s. $Mx^* = \int_0^\cdot G^* x^* \mathrm{d}W_H$ for each $x^* \in X^*$. Then according to Theorem 4.27 of Ref. 4 a.s.

$$[Mx^*, My^*] = \int_0^\cdot \langle G^* x^*, G^* y^* \rangle \, \mathrm{d}s, \quad x^*, y^* \in X^*.$$

To prove the other direction assume that there exist such a separable Hilbert space H and $G : \mathbb{R}_+ \times \Omega \to \mathcal{L}_{\mathrm{cl}}(H, X)$ that (3.3) holds. The proof that M is Brownian representable will be almost the same as the proof of Theorem 2 of Ref. 21, but one has to use Lemma A.1 instead of Proposition 32 of Ref. 21 and apply Proposition 3.6 of Ref. 22 for general Banach spaces.

Suppose first that X is a Hilbert space (one then can identify H, X and X^*). Let $\overline{W}_X$ be an independent of M X-cylindrical Wiener process on an enlarged probability space $(\overline{\Omega}, \overline{\mathbb{F}}, \overline{\mathbb{P}})$ with the enlarged filtration $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{t \geq 0}$. Let $(0, \infty) = \bigcup_{n=1}^\infty B_n$ be a decomposition into disjoint Borel sets such that $\operatorname{dist}(B_n, \{0\}) > 0$ for each $n \geq 1$. Define functions $\psi_n(t) = t^{-1}\mathbf{1}_{B_n}$, $t \in \mathbb{R}$, $n \geq 1$, and $\psi_0 = \mathbf{1}_{\{0\}}$. Let us also denote $C_n = \mathbf{1}_{B_n}$, $n \geq 1$, and $C_0 = \psi_0$. By Lemma A.1 for each $n \geq 1$

$\psi_n(G^*G)$ and $C_n(G^*G)$ are $\mathcal{L}(X)$-valued strongly progressively measurable, and since

$$\|G^*G\psi_n(G^*G)\|_{\mathcal{L}(X)} = \|C_n(G^*G)\|_{\mathcal{L}(X)} \leq \|\mathbf{1}_{B_n}\|_{L^\infty(\mathbb{R})} = 1, \quad n \geq 1,$$
$$\|\psi_0(G^*G)\|_{\mathcal{L}(X)} \leq \|\mathbf{1}_{\{0\}}\|_{L^\infty(\mathbb{R})} = 1,$$

then $\operatorname{ran}(\psi_n(G^*G)) \subset \operatorname{dom}(G^*G) \subset \operatorname{dom}(G)$ (see p. 347 of Ref. 26), so $G\psi_n(G^*G)$ is well-defined, and according to Problem III.5.22 of Ref. 15 $G\psi_n(G^*G) \in \mathcal{L}(X)$ $\mathbb{P}\times$ ds-a.s., therefore for each $x^* \in X^*$ by Proposition 3.6 $G\psi_n(G^*G)x^*$ is stochastically integrable with respect to M and one can define

$$W_n(x^*) = \int_0^\cdot G\psi_n(G^*G)x^*\,\mathrm{d}M, \quad n \geq 1, \quad W_0 = \int_0^\cdot \psi_0(G^*G)x^*\,\mathrm{d}\overline{W}_X.$$

Then by (3.4) $[W_n(x^*), W_m(y^*)] = 0$ for each $x^*, y^* \in X^*$ and $0 \leq n < m$ and $[W_n(x^*)] = \int_0^\cdot \|C_n(G^*(s)G(s))x^*\|^2\,\mathrm{d}s$, so for each $t \geq 0$

$$\sum_{n=0}^\infty [W_n(x^*)]_t = \sum_{n=0}^\infty \int_0^t \|C_n(G^*G)x^*\|^2\,\mathrm{d}s = \sum_{n=0}^\infty \int_0^t \langle C_n(G^*G)x^*, x^*\rangle\,\mathrm{d}s$$
$$= \int_0^t \langle x^*, x^*\rangle ds = t\|x^*\|^2.$$

Let us define $W(x^*) = \sum_{n=0}^\infty W_n(x^*)$. Thanks to Proposition 28 of Ref. 21 this sum converges in $C([0,\infty))$ in probability. It is obvious that $W : X^* \to \mathcal{M}^{\mathrm{loc}}$, $x^* \mapsto W(x^*)$ is an H-cylindrical Brownian motion. Moreover, for each $h \in X^*$ one has that by the definition of G^* the H-valued function G^*h is stochastically integrable with respect to W_H, and $[G^*h \cdot W] = [Mh] = \int_0^\cdot G^*(s)h\,\mathrm{d}s$ a.s. So, to prove that Mh and $G^*h\cdot W$ are indistinguishable it is enough to show that a.s. $[G^*h\cdot W, Mh] = \int_0^\cdot G^*(s)h\,\mathrm{d}s$. By Remark 3.7 and the fact that a.s. $\|G^*G\psi_n(G^*G)\| \leq \mathbf{1}_{\mathbb{R}_+} \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$ one can apply (3.5) so for each $n \geq 1$

$$\left[\int_0^\cdot G^*h\,\mathrm{d}W_n, Mh\right] = \left[\int_0^\cdot G\psi_n(G^*G)G^*h\,\mathrm{d}M, \int_0^\cdot h\,\mathrm{d}M\right]$$
$$= \int_0^\cdot \langle G^*G\psi_n(G^*G)G^*h, G^*h\rangle\,\mathrm{d}s.$$

On the other hand $\widetilde{W}$ and M are independent so $[G^*h \cdot W_0, Mh] = 0$. To sum up one has that a.s.

$$[G^*h\cdot W, Mh] = \sum_{n=1}^\infty \int_0^\cdot \langle C_n(G^*G)G^*h, G^*h\rangle = \int_0^\cdot \|G^*h\|^2\,\mathrm{d}s,$$

which finishes the proof for a Hilbert space case.

Now consider a general reflexive Banach space X. Let a separable Hilbert space H, $j : X \to H$ be defined as in p. 154 of Ref. 17. Let W_H be constructed as above but for $M|_H$. Fix $x^* \in X^*$ and find $(h_n)_{n\geq 1}, \subset H$ such that $\lim_{n\to\infty} j^*h_n = x^*$ and $[M(j^*h_n) - M(x^*)]_T$ vanishes almost everywhere for each $T > 0$. By the definition of G one has $\int_0^T \|G^*(j^*h_n - x^*)\|^2 \,\mathrm{d}s \to 0$ in probability for all $T > 0$, so since $M(j^*h_n) \to M(x^*)$ uniformly on all compacts in probability and by Theorem 4.27 of Ref. 4, Proposition 17.6 of Ref. 13 and Proposition 3.6 $M(x^*)$ and $G^*x^* \cdot W_H$ are indistinguishable. □

Thanks to this representation theorem we obtain the following stochastic integrability result.

Theorem 3.9. *Let X be a reflexive separable Banach space, $M \in \mathcal{M}^{\mathrm{loc}}_{\mathrm{cyl}}(X)$ be closed operator-Brownian representable, $G : \mathbb{R}_+ \times \Omega \to \mathcal{L}_{\mathrm{cl}}(H, X)$ be the corresponding operator family. Let $f : \mathbb{R}_+ \times \Omega \to X^*$. Suppose there exist elementary progressive $f_n : \mathbb{R}_+ \times \Omega \to X^*$, $n \geq 1$, such that $f_n \to f$ $(\mathbb{P} \times \mathrm{d}s)$-a.s. Assume also that there exists a limit $N := \lim_{n\to\infty} f_n \cdot M$ in the ucp topology. Then $f \in \mathrm{ran}(G^*)$ $(\mathbb{P} \times \mathrm{d}s)$-a.s., G^*f is progressively measurable and*

$$[N] = \int_0^{\cdot} \|G^*f\|^2 \,\mathrm{d}s. \tag{3.6}$$

We then call f stochastically integrable and define

$$f \cdot M = \int_0^{\cdot} f \,\mathrm{d}M := \lim_{n\to\infty} f_n \cdot M,$$

where the limit is taken in the ucp topology.

Proof. Formula (3.6) is obvious for elementary progressive f by (3.4). Since $[f_n \cdot M]$ is absolutely continuous and $f_n \cdot M$ tends to N in ucp, by Lemma B.1 and the fact that $[N]$ is a.s. continuous one can prove that $[N]$ is absolutely continuous, hence by Lemma 3.10 of Ref. 33 its derivative in time $v : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is progressively measurable. Let a Hilbert space H and an H-cylindrical Brownian motion W_H be constructed for M by Theorem 3.8. Assume that $(h_n)_{n\geq 1} \subset H$ is a dense subset of a unite ball in H. Then $[N, W_H h_n]$ is a.s. absolutely continuous, and has a progressively measurable derivative v_n for each $n \geq 1$. Moreover, since by Proposition 17.9 of Ref. 13 $|\mu_{[N,W_H h_n]}|(I) \leq \mu^{1/2}_{[N]}(I)\mu^{1/2}_{[W_H h_n]}(I)$ for each interval $I \subset \mathbb{R}_+$ and thanks to Theorem 5.8.8 of Ref. 2 one has that $v_n \leq v^{1/2}\|h_n\|$ $(\mathbb{P} \times \mathrm{d}s)$-a.s. Then thanks to linearity, boundedness and denseness of $\mathrm{span}(h_n)_{n\geq 1}$ in H, we obtain that there exists a progressively measurable process $V : \mathbb{R}_+ \times \Omega \to H$ such that $v_n = \langle V, h_n\rangle$ $(\mathbb{P}\times \mathrm{d}s)$-a.s. Let $\tilde{N} = V \cdot W_H$. Then $[N, \Phi \cdot W_H] = [\tilde{N}, \Phi \cdot W_H]$ for each stochastically integrable $\Phi : \mathbb{R}_+ \times \Omega \to H$, hence $[N, \tilde{N}] = [\tilde{N}]$ and $[N, f_n \cdot M] = [\tilde{N}, f_n \cdot M]$ for each natural n. Without loss of generality one can suppose that $\lim_{n\to\infty}[N - f_n \cdot M]_T = 0$ a.s. for each $T > 0$, therefore a.s.

$$\begin{aligned}\lim_{n\to\infty} \int_0^T \|G^*f_n - V\|^2 \,\mathrm{d}s &= \lim_{n\to\infty} [\tilde{N} - f_n \cdot M]_T \\ &= \lim_{n\to\infty} ([\tilde{N}]_T - [\tilde{N}, f_n \cdot M]_T) + ([f_n \cdot M]_T - [N, f_n \cdot M]_T) \\ &= ([\tilde{N}]_T - [\tilde{N}, N]_T) + ([N]_T - [N]_T) = 0,\end{aligned}$$

so, $\tilde{N}$ is a version of N. Also by choosing a subsequence one has that $G^* f_{n_k} \to V$ as $k \to \infty$, which means that $f \in \mathrm{dom}(G^*)$ $(\mathbb{P} \times \mathrm{d}s)$-a.s. and (3.6) holds. $\square$

Remark 3.10. It follows from Theorem 3.9 and Proposition 17.6 of Ref. 13 that for any finite-dimensional subspace $Y_0 \subset Y$ the definition of the stochastic integral can be extended to all strongly progressively measurable processes $\Phi : \mathbb{R}_+ \times \Omega \to \mathcal{L}(X, Y_0)$ that satisfy $(G^*\Phi^*)^* \in L^2(\mathbb{R}_+; \mathcal{L}(H, Y_0))$ a.s. (or equivalently $(G^*\Phi^*)^*$ is scalarly in $L^2(\mathbb{R}_+; H)$ a.s.). Moreover, then $\Phi \cdot M = (G^*\Phi^*)^* \cdot W_H$.

We proceed with a result which is closely related to [31, Theorem 3.6]. In order to state it we need the following terminology.

A Banach space X is called a *UMD Banach space* if for some (or equivalently, for all) $p \in (1, \infty)$ there exists a constant $\beta > 0$ such that for every $n \geq 1$, every martingale difference sequence $(d_j)_{j=1}^n$ in $L^p(\Omega; X)$, and every $\{-1, 1\}$-valued sequence $(\varepsilon_j)_{j=1}^n$ we have

$$\left(\mathbb{E}\left\|\sum_{j=1}^n \varepsilon_j d_j\right\|^p\right)^{\frac{1}{p}} \leq \beta \left(\mathbb{E}\left\|\sum_{j=1}^n d_j\right\|^p\right)^{\frac{1}{p}}.$$

The infimum over all admissible constants β is denoted by $\beta_{p,X}$.

There is a large body of results asserting that the class of UMD Banach spaces is a natural one when pursuing vector-valued generalizations of scalar-valued results in harmonic and stochastic analysis. UMD spaces enjoy many pleasant properties, among them being reflexive. We refer the reader to Refs. 3, 10 and 27 for details.

Let $(\gamma_n')_{n\geq 1}$ be a sequence of independent standard Gaussian random variables on a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ and let H be a Hilbert space. A bounded operator $R \in \mathcal{L}(H, X)$ is said to be *γ-radonifying* if for some (or equivalently for each) orthonormal basis $(h_n)_{n\geq 1}$ of H the Gaussian series $\sum_{n\geq 1} \gamma_n' R h_n$ converges in $L^2(\Omega'; X)$. We then define

$$\|R\|_{\gamma(H,X)} := \left(\mathbb{E}'\left\|\sum_{n\geq 1} \gamma_n' R h_n\right\|^2\right)^{\frac{1}{2}}.$$

This number does not depend on the sequence $(\gamma_n')_{n\geq 1}$ and the basis $(h_n)_{n\geq 1}$, and defines a norm on the space $\gamma(H, X)$ of all γ-radonifying operators from H into X. Endowed with this norm, $\gamma(H, X)$ is a Banach space, which is separable if X is separable. For a Hilbert space X, the space $\gamma(H, X)$ is isometrically isomorphic to the space of all Hilbert–Schmidt operators from H to X. If $(S, \mathcal{A}, \mu)$ is a measure space and $X = L^p(S)$, then $\gamma(H, L^p(S)) = L^p(S; H)$ up to equivalence of norms.

For all $R \in \gamma(H, X)$ it holds that $\|R\| \leq \|R\|_{\gamma(H,X)}$. Let $\overline{H}$ be another Hilbert space and let Y be another Banach space. Then the so-called *ideal property* (see Ref. 11) holds true: for all $S \in \mathcal{L}(\overline{H}, H)$ and all $T \in \mathcal{L}(X, Y)$ we have

$TRS \in \gamma(\overline{H}, Y)$ and

$$\|TRS\|_{\gamma(\overline{H},Y)} \leq \|T\|\|R\|_{\gamma(H,X)}\|S\|. \tag{3.7}$$

Let X, Y be Banach spaces, and let $A \in \mathcal{L}_{\mathrm{cl}}(X, Y)$. A linear subspace $X_0 \subset \mathrm{dom}(A)$ is a *core* of A if the closure of an operator $A|_{X_0} : X_0 \to Y$ is A (see details in Ref. 15). Let (S, Σ, μ) be a measure space, and let $F : S \to \mathcal{L}_{\mathrm{cl}}(X, Y)$ be such that Fx is a.s. defined and measurable for each $x \in X$. Then F has a *fixed core* if there exists a sequence $(x_n)_{n\geq 1} \subset X$ such that $\mathrm{span}(x_n)_{n\geq 1}$ is a core of F a.s. (Notice, that in this particular case the core has a countable algebraic dimension).

Theorem 3.11. *Let X be a reflexive Banach space, Y be a UMD Banach space, $M \in \mathcal{M}^{\mathrm{cyl}}_{\mathrm{cl}}(X)$, $G : \mathbb{R}_+ \times \Omega \to \mathcal{L}_{\mathrm{cl}}(H, X)$ be the corresponding operator family. Let G^* have a fixed core. Then for a strongly progressively measurable process $\Phi: \mathbb{R}_+ \times \Omega \to \mathcal{L}(X, Y)$ such that $G^*\Phi^* \in \mathcal{L}(Y^*, H)$ a.s. which is scalarly in $L^2(\mathbb{R}_+; H)$ a.s. the following assertions are equivalent:*

(1) *There exist elementary progressive processes $(\Phi_n)_{n\geq 1}$ such that:*
   (i) *for all $y^* \in Y^*$, $\lim_{n\to\infty} G^*\Phi_n^* y^* = G^*\Phi^* y^*$ in $L^0(\Omega; L^2(\mathbb{R}_+; H))$;*
   (ii) *there exists a process $\zeta \in L^0(\Omega; C_b(\mathbb{R}_+; Y))$ such that*

$$\zeta = \lim_{n\to\infty} \int_0^{\cdot} \Phi_n(t)\, \mathrm{d}M(t) \quad \text{in } L^0(\Omega; C_b(\mathbb{R}_+; Y)).$$

(2) *There exists an a.s. bounded process $\zeta: \mathbb{R}_+ \times \Omega \to Y$ such that for all $y^* \in Y^*$ we have*

$$\langle \zeta, y^* \rangle = \int_0^{\cdot} \Phi^*(t) y^*\, \mathrm{d}M(t) \quad \text{in } L^0(\Omega; C_b(\mathbb{R}_+)).$$

(3) *$(G^*\Phi^*)^* \in \gamma(L^2(\mathbb{R}_+; H), Y)$ almost surely;*

In this case ζ in (1) and (2) coincides and for all $p \in (0, \infty)$ we have

$$\mathbb{E} \sup_{t\in\mathbb{R}_+} \|\zeta(t)\|^p \eqsim_{p,Y} \mathbb{E}\|(G^*\Phi^*)^*\|^p_{\gamma(L^2(\mathbb{R}_+;H),Y)}. \tag{3.8}$$

Proof. Let $\Psi : \mathbb{R}_+ \times \Omega \to \mathcal{L}(H, Y)$ be such that $\Psi^* = G^*\Phi^*$ (recall that UMD spaces are reflexive). Then equivalence of (2) and (3) and the formula (3.8) are just particular cases of corresponding parts of Theorem 5.9 of Ref. 31 and Theorem 5.12 of Ref. 31 for Ψ thanks to Remark 3.7 (for (2) one also has to apply Theorem 3.9).

It remains to prove that (1) for Φ and M and (1) for Ψ and W_H are equivalent. (Notation: $(1, \Phi) \Leftrightarrow (1, \Psi)$).

$(1, \Phi) \Rightarrow (1, \Psi)$ Since $\Phi_n \cdot M$ exists and the range of Φ_n is in a certain fixed finite-dimensional space of Y, $n \geq 1$, then by Remark 3.10 $(G^*\Phi_n^*)^*$ is stochastically integrable with respect to W_H, so by Theorem 5.9 of Ref. 31 there exists a sequence $(\Psi_{nk})_{k\geq 1}$ of elementary progressive $\mathcal{L}(H, Y)$-valued functions such that $\Psi^*_{nk} y^* \to G^*\Phi_n^* y^*$ in $L^0(\Omega; L^2(\mathbb{R}_+; H))$ for each $y^* \in Y^*$, and $\Psi_{nk} \cdot W_H \to \Phi_n \cdot M$

in $L^0(\Omega; C_b(\mathbb{R}_+; Y))$ as $k \to \infty$. Knowing $(1, \Phi)$ one can then find a subsequence $\{\Psi_k\}_{k\geq 1} := \{\Psi_{n_k k}\}_{k\geq 1}$ such that $\lim_{k\to\infty} \Psi_k^* y^* = G^*\Phi^* y^* = \Psi^* y^*$ in $L^0(\Omega; L^2(\mathbb{R}_+; H))$ for each $y^* \in Y^*$ and $\Psi_k \cdot W_H$ converges in $L^0(\Omega; C_b(\mathbb{R}_+; Y))$, which is (1) for Ψ and W_H.

$(1, \Psi) \Rightarrow (1, \Phi)$ Let $(x_k^*)_{k\geq 1} \subset X^*$ be such that $U := \operatorname{span}(x_k^*)_{k\geq 1}$ is a fixed core of G^*. For each $k \geq 1$ define $U_k = \operatorname{span}(x_1^*, \ldots, x_k^*)$. Then due to Lemma A.5 consider $\tilde{P}_k : \mathbb{R}_+ \times \Omega \to \mathcal{L}(H)$ and $L_k : \mathbb{R}_+ \times \Omega \to \mathcal{L}(H, X^*)$ such that $\tilde{P}_k$ is an orthogonal projection onto $G^*(U_k)$ and $G^* L_k = \tilde{P}_k$ $(\mathbb{P}\times \mathrm{d}s)$-a.s. Consider a sequence $(\Psi_n)_{n\geq 1}$ of elementary progressive functions in $L^0(\Omega, \gamma(L^2(\mathbb{R}_+; H), Y))$ constructed thanks to Proposition 2.12 of Ref. 31 such that $\Psi_n \to \Psi$ in $L^0(\Omega, \gamma(L^2(\mathbb{R}_+; H), Y))$. Let $\tilde{P}_0 : \mathbb{R}_+ \times \Omega \to \mathcal{L}(H)$ be an orthogonal projection onto $\overline{\operatorname{ran}(G^*)}$. Then by the ideal property (3.7), a.s. we have

$$
\begin{aligned}
\|(G^*\Phi^*)^* - \Psi_n \tilde{P}_0\|_{\gamma(L^2(\mathbb{R}_+;H),Y)} &= \|((G^*\Phi^*)^* - \Psi_n)\tilde{P}_0\|_{\gamma(L^2(\mathbb{R}_+;H),Y)} \\
&\leq \|(G^*\Phi^*)^* - \Psi_n\|_{\gamma(L^2(\mathbb{R}_+;H),Y)},
\end{aligned}
$$

which means that by Theorem 3.6 of Ref. 31 $\Psi_n \tilde{P}_0$ is stochastically integrable with respect to W_H and $\Psi_n \tilde{P}_0 \cdot W_H \to (G^*\Phi^*)^* \cdot W_H$ in $L^0(\Omega, \gamma(L^2(\mathbb{R}_+; H), Y))$. Therefore, one can define $\widetilde{\Psi}_n := (\tilde{P}_0 \Psi_n^*)^*$.

Set $\Phi_{nk} := (L_k \tilde{P}_k \widetilde{\Psi}_n^*)^*$. Notice that $U = \bigcup_k U_k$ is a core of G^*, so $\overline{\operatorname{ran}(G^*)} = \overline{G^*(U)}$, therefore $\tilde{P}_k \to \tilde{P}_0$ weakly and by Proposition 2.4 of Ref. 31 $(G^*\Phi_{nk}^*)^* = (\tilde{P}_k \widetilde{\Psi}_n^*)^* \to \widetilde{\Psi}_n$ in $L^0(\Omega, \gamma(L^2(\mathbb{R}_+; H), Y))$ as $k \to \infty$, so one can find a subsequence $\Phi_n := \Phi_{nk_n}$ such that $(G^*\Phi_n^*)^* \to \Psi$ in $L^0(\Omega, \gamma(L^2(\mathbb{R}_+; H), Y))$ as $n \to \infty$. Now fix $n \geq 1$. Since Ψ_n is elementary progressive, then it has the following form: for each $t \geq 0$ and $\omega \in \Omega$

$$
\Psi_n(t, \omega) = \sum_{m=1}^{M} \sum_{l=1}^{L} \mathbf{1}_{(t_{m-1}, t_m] \times B_{lm}}(t, \omega) \sum_{j=1}^{J} h_j \otimes y_{jlm}.
$$

Hence by Remark 3.3

$$
\Phi_n(t, \omega) = \sum_{m=1}^{M} \sum_{l=1}^{L} \mathbf{1}_{(t_{m-1}, t_m] \times B_{lm}}(t, \omega) \sum_{j=1}^{J} (L_{k_n} \tilde{P}_{k_n} h_j) \otimes y_{jlm}.
$$

Therefore, Φ_n takes its values in a fixed finite-dimensional subspace Y_n of Y, and so by Remark 3.10 one can construct simple approximations of Φ_n then $(1, \Phi_n)$ holds. This completes the proof. □

Remark 3.12. As the reader can see, the existence of a fixed core of G^* is needed only for (1) in Theorem 3.11. Without this condition one can still show that parts (2) and (3) are equivalent and that estimate (3.8) holds.

3.3. *General case of Brownian representation*

In the preceding subsection, it was shown that a quite general class of cylindrical martingales with absolutely continuous covariation can be represented as stochastic integrals with respect to W_H, and we proved some results on stochastic integrability for them (Proposition 3.6 and Theorem 3.9). Unfortunately, such results do not hold in the general case, and it will be shown in Example 3.15 that G from (3.3) does not always exists. The construction in this example uses two simple remarks on linear operators. For linear spaces X and Y we denote the linear space of all linear operators from X to Y by $L(X,Y)$, $L(X) := L(X,X)$.

Remark 3.13. Let $(x_\alpha)_{\alpha\in\Lambda} \subset X$ be a Hamel (or algebraic) basis of X (see Problem 13.4 of Ref. 16 or Ref. 9). Then one can uniquely determine $A \in L(X,Y)$ only by its values on $(x_\alpha)_{\alpha\in\Lambda}$. Indeed, for each x there exist unique $N \geq 0$, $\alpha_1, \dots, \alpha_N \in \Lambda$, and $c_1, \dots, c_N \in \mathbb{R}$ such that $x = \sum_{n=1}^N c_n x_{\alpha_n}$, so one can define Ax as follows: $Ax := \sum_{n=1}^N c_n A x_{\alpha_n}$.

Remark 3.14. Any linear functional $\ell : X_0 \to \mathbb{R}$ defined on a linear subspace X_0 of a Banach space X can be extended linearly to X using the fact that X has a Hamel basis (see Part I.11 of Ref. 30, also Ref. 1). The same holds for operators: if $A \in L(X_0, Y)$, where X_0 is a linear subspace of X, one can extend A to a linear operator from X to Y using the Hamel basis of X. Surely this extension is not unique if $X_0 \subsetneq X$.

Example 3.15. We will show that for a Hilbert space H there exists $M \in \mathcal{M}^{\mathrm{loc}}_{\mathrm{cyl}}(H)$ which is not closed operator-Brownian representable. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, $W : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be a Brownian motion. Without loss of generality suppose that $\mathbb{F}$ is generated by W. Let H be a separable Hilbert space with an orthonormal basis $(h_n)_{n\geq 1}$. Let $(\xi_n)_{n\geq 1}$ be the following sequence of random variables: for each fixed $n \geq 1$, $\xi_n \in L^2(\Omega)$ is a measurable integer-valued function of $W(2^{-n+1}) - W(2^{-n})$ such that

$$\mathbb{P}(\xi_n = k) = 2^{-n+1}\mathbf{1}_{2^{n-1}\leq k < 2^n}, \quad k \in \mathbb{N}.$$

It is easy to see that $(\xi_n)_{n\geq 1}$ are mutually independent and each ξ_n is $\mathcal{F}_{2^{-n+1}}$-measurable. For all $n \geq 1$ set $c_n = 2^{\frac{n}{4}}$. Consider linear functional-valued function $\ell : \mathbb{R}_+ \times \Omega \to L(H, \mathbb{R})$ defined as

$$\ell(h) = \sum_{n=1}^{\infty} c_n \langle h, h_{\xi_n}\rangle \tag{3.9}$$

for all $h \in H$ such that $\sum_{n=0}^\infty |c_n\langle h, h_{\xi_n}\rangle|$ converges, and extended linearly to the whole H thanks to Remark 3.14. Fix $h \in H$. Let $\tilde{h} = \sum_{n=1}^\infty |\langle h, h_n\rangle| h_n$ and $a = \sum_{n=0}^\infty \sum_{k=2^{n-1}}^{2^n-1} 2^{-\frac{3n}{4}} h_k \in H$. Then

$$\mathbb{E}\sum_{n=0}^N |c_n\langle h, h_{\xi_n}\rangle| = \sum_{n=0}^N \sum_{k=2^{n-1}}^{2^n-1} \frac{c_n|\langle h, h_k\rangle|}{2^{n-1}} = \sum_{n=0}^N \sum_{k=2^{n-1}}^{2^n-1} \frac{2^{-\frac{3n}{4}}|\langle h, h_k\rangle|}{2} \leq \frac{1}{2}\langle \tilde{h}, a\rangle.$$

Hence by the dominated convergence theorem $\lim_{N\to\infty}\sum_{n=0}^{N}|c_n\langle h, h_{\xi_n}\rangle|$ exists a.s., so, for each fixed $h\in H$ formula (3.9) holds a.s. Consider the stochastic integral

$$M_t(h) = \int_0^t \mathbf{1}_{[1,2]}(s)\ell(h)\mathrm{d}W_s,$$

since integrand is predictable. Moreover, due to the mutual independence of $(\xi_n)_{n\geq 1}$

$$\begin{aligned}
\mathbb{E}\int_{\mathbb{R}_+}\mathbf{1}_{[1,2]}(s)(\ell(h))^2 ds &= \mathbb{E}(\ell(h))^2 \leq \mathbb{E}(\ell(\tilde{h}))^2\\
&= \mathbb{E}\left(\sum_{n=1}^{\infty} c_n|\langle h, h_{\xi_n}\rangle|\right)^2\\
&= \mathbb{E}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_n c_m|\langle h, h_{\xi_n}\rangle||\langle h, h_{\xi_m}\rangle|\\
&= \sum_{n=1}^{\infty} c_n^2 \sum_{k=2^{n-1}}^{2^n-1}\frac{|\langle h, h_k\rangle|^2}{2^{n-1}}\\
&\quad + 2\sum_{n\neq m} c_n c_m \sum_{k=2^{n-1}}^{2^n-1}\sum_{l=2^{m-1}}^{2^m-1}\frac{|\langle h, h_k\rangle||\langle h, h_l\rangle|}{2^{n-1}2^{m-1}}\\
&\leq \|h\|^2 + \sum_{n,m=1}^{\infty}\sum_{k=2^{n-1}}^{2^n-1}\sum_{l=2^{m-1}}^{2^m-1}\frac{|\langle h, h_k\rangle||\langle h, h_l\rangle|}{2^{n-1}2^{m-1}}\\
&= \|h\|^2 + \left(\sum_{n=1}^{\infty}\sum_{k=2^{n-1}}^{2^n-1}\frac{c_n|\langle h, h_k\rangle|}{2^{n-1}}\right)^2\\
&= \|h\|^2 + \frac{1}{4}\langle\tilde{h}, a\rangle^2 \leq \left(1+\frac{1}{4}\|a\|^2\right)\|h\|^2,
\end{aligned}$$

so thanks to Lemma 17.10 of Ref. 13 $M_t(h)$ is an L^2-martingale. But the above computations also show that by Proposition 17.6 of Ref. 13 $M(h)\to 0$ in the ucp topology as $h\to 0$, which means that M is a cylindrical martingale as a continuous linear mapping from H to $\mathcal{M}^{\mathrm{loc}}$.

Now our aim is to prove that M is not closed operator-Brownian representable. Suppose that there exist an H-cylindrical Brownian motion $W_H:\mathbb{R}_+\times\overline{\Omega}\to\mathbb{R}$ on an enlarged probability space $(\overline{\Omega},\overline{\mathcal{F}},\overline{\mathbb{P}})$ with an enlarged filtration $\overline{\mathbb{F}}=(\overline{\mathcal{F}}_t)_{t\geq 0}$ (we may use the same Hilbert space H since all separable infinite-dimensional Hilbert spaces are isometrically isomorphic) and a closed operator-valued H-strongly measurable function $G:\mathbb{R}_+\times\overline{\Omega}\to\mathcal{L}_{\mathrm{cl}}(H)$ such that $M_t(h)=\int_0^t(G^*h)^*\mathrm{d}W_H$.

Since $M|_{[0,1]}=0$ we can assume that $G^*|_{[0,1]}=0$. Because of the structure of M, for each pair of vectors $h,g\in H$ there exist $\overline{\mathcal{F}}_1$-measurable $a,b\in L^0(\Omega)$ such that $a = \ell(g)$ and $b = -\ell(h)$ and $aM_t(h) + bM_t(g) = 0$ a.s. for all $t \geq 0$. As a, b are $\overline{\mathcal{F}}_1$-measurable, then for $t \geq 1$ one can put a, b under the integral:

$$\begin{aligned}0 = aM_t(h) + bM_t(g) &= a\int_1^t (G^*h)^* \mathrm{d}W_H + b\int_1^t (G^*g)^* \mathrm{d}W_H \\ &= \int_1^t (aG^*h + bG^*g)^* \mathrm{d}W_H = \int_0^t (aG^*h + bG^*g)^* \mathrm{d}W_H,\end{aligned}$$

and by the Itô isometry (Proposition 4.20 of Ref. 4) $aG^*h + bG^*g = 0(\overline{\mathbb{P}} \otimes \mathrm{d}s)$-a.s. This means that G^*h and G^*g are collinear $(\overline{\mathbb{P}} \otimes \mathrm{d}s)$-a.s. if a and b are nonzero a.s. If for instance $a = \ell(g) = 0$ on a set of positive measure $A \in \overline{\mathcal{F}}_1$, then $M(g)\mathbf{1}_A = 0$, and consequently $G^*g\mathbf{1}_A = 0$ $(\overline{\mathbb{P}}\otimes \mathrm{d}s)$-a.s., hence G^*h and G^*g are collinear $(\overline{\mathbb{P}}\otimes \mathrm{d}s)$-a.s.

Taking an orthonormal basis $(h_i)_{i\geq 1}$ of H it follows that $(\overline{\mathbb{P}} \otimes \mathrm{d}s)$-a.s. G^*h_i and G^*h_j are collinear for all i, j, and by the closability of G^* one has that $\mathrm{ran}(G^*)$ consists of one vector $(\overline{\mathbb{P}} \otimes \mathrm{d}s)$-a.s. But this means that G^* is a projection on a one-dimensional subspace, so there exist $h'_G, h''_G : \mathbb{R}_+ \times \overline{\Omega} \to H$ such that $G^*h = \langle h, h'_G\rangle h''_G$.

Since the derivative of an absolutely continuous function is defined uniquely, $\overline{\mathbb{P}}$-a.s. for a.e. $t \in [1, 2]$

$$[Mh_i]'_t = \ell(h_i)^2 = \|G^*(t)h_i\|^2 = \|h''_G(t)\|^2\langle h_i, h'_G(t)\rangle^2.$$

But the series of positive functions $\sum_{i=1}^{\infty} \|h''_G\|^2\langle h_i, h'_G\rangle^2 = \|h'_G\|^2\|h''_G\|^2$ converges $(\overline{\mathbb{P}}\otimes \mathrm{d}s)$-a.s., which does not hold true for $\sum_{i=1}^{\infty} \ell(h_i)^2$ (because linear functional ℓ is unbounded $(\mathbb{P} \times \mathrm{d}s)$-a.s. thanks to the choice of $\{c_n\}_{n\geq 1}$).

Remark 3.16. Using the previous example and Example 3.22 of Ref. 33 one can see that in general for a separable Hilbert space H the following proper inclusions hold:

$$\mathcal{M}^{\mathrm{loc}}_{\mathrm{a.c.v.}}(H) \subsetneqq \mathcal{M}^{\mathrm{cyl}}_{\mathrm{bdd}}(H) \subsetneqq \mathcal{M}^{\mathrm{cyl}}_{\mathrm{cl}}(H) \subsetneqq \mathcal{M}^{\mathrm{cyl}}_{\mathrm{a.c.c.}}(H),$$

where $\mathcal{M}^{\mathrm{loc}}_{\mathrm{a.c.v.}}(H)$ is the subspace of $\mathcal{M}^{\mathrm{cyl}}_{\mathrm{loc}}(X)$ with an absolutely continuous quadratic variation (the quadratic variation of a cylindrical continuous local martingale was defined in Ref. 33), and $\mathcal{M}^{\mathrm{cyl}}_{\mathrm{bdd}}(H)$ is the linear space of cylindrical continuous local martingales with a bounded operator-generated covariation (considered for instance in Refs. 21 and 22).

In the general case one can still represent a cylindrical martingale with an absolutely continuous covariation as a stochastic integral with respect to W_H, but then one has to use linear operator-valued functions instead of $\mathcal{L}_{\mathrm{cl}}(H, X)$-valued, and some important properties are lost.

Theorem 3.17. *Let X be a Banach space with separable dual, and let $M \in \mathcal{M}^{\mathrm{loc}}_{\mathrm{cyl}}(X)$. Then M is Brownian representable if and only if it is with an absolutely continuous covariation.*

We will need the following lemma.

Lemma 3.18. *Let H be a separable Hilbert space, and let $M : \mathbb{R}_+ \times \Omega \to H$ be a continuous local martingale such that $\langle M, h\rangle$ has an absolutely continuous variation for all $h \in H$ a.s. Then $[M]$ also has an absolutely continuous version. Moreover, there exists scalarly measurable positive Hilbert–Schmidt operator-valued $\Phi : \mathbb{R}_+ \times \Omega \to \mathcal{L}(H)$ such that a.s. $[\langle M, h\rangle, \langle M, g\rangle] = \int_0^{\cdot} \langle \Phi h, \Phi g\rangle ds$.*

Proof. Since H is a separable Hilbert space one sees that

$$[M]_t = \sum_{n=1}^{\infty} [\langle M, e_n\rangle]_t \quad \text{a.s}\ \forall\, t \geq 0 \tag{3.10}$$

for any given orthonormal basis $(e_n)_{n\geq 1}$ of H. Let $f_n : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ be such that $[\langle M, e_n\rangle]_t = \int_0^t f_n(s) ds$ a.s. $\forall\, n \geq 1, t \geq 0$. Then thanks to (3.10) and the dominated convergence theorem $\sum_{n=1}^{\infty} f_n$ converges in $L^1_{\mathrm{loc}}(\mathbb{R}_+)$ a.e. Let $f := \sum_{n=1}^{\infty} f_n$. Then $[M]_t = \int_0^t f(s) ds$ a.s. for all $t \geq 0$, which means that $[M]_t$ is absolutely continuous a.s.

The second part is an easy consequence of Theorem 14.3(2) of Ref. 19. □

Proof of Theorem 3.17. One direction is obvious. Now let M be with an absolutely continuous covariation. First suppose that X is a Hilbert space. Consider a Hilbert–Schmidt operator A with zero kernel and dense range. For instance set $Ah_n = \frac{1}{n} h_n$ for some orthonormal basis $(h_n)_{n\geq 1}$ of X. Then according to Theorem A of Ref. 12 $M(A(\cdot))$ admits a local martingale version, namely there exists a continuous local martingale $\widetilde{M} : \mathbb{R}_+ \times \Omega \to X$ such that $M(Ax)$ and $\langle \widetilde{M}, x\rangle$ are indistinguishable for each $x \in X$. Notice that $\widetilde{M}$ has an absolutely continuous quadratic variation by Lemma 3.18. Also by Theorem 8.2 of Ref. 4 there exists an enlarged probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ with an enlarged filtration $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{t\geq 0}$ such that there exist an X-cylindrical Brownian motion $W_X : \mathbb{R}_+ \times \overline{\Omega} \times X \to \mathbb{R}$ and X-strongly progressively measurable $\Phi : \mathbb{R}_+ \times \overline{\Omega} \to \mathcal{L}(X)$ such that $\widetilde{M} = \int_0^{\cdot} \Phi^* \mathrm{d}W_X$.

Now fix $h \notin \operatorname{ran}(A)$. Let $(x_n)_{n\geq 1}$ be a $\mathbb{Q}$-span of $(h_n)_{n\geq 1}$. Denote by f_h, f_{nh} the derivatives of $[Mh]$ and $[Mh, W_X x_n]$ in time, respectively. For each $n \geq 1$ and for each segment $I \subset \mathbb{R}_+$ $|\mu_{[Mh, W_X x_n]}|(I) \leq \mu^{1/2}_{[Mh]}(I)\mu^{1/2}_{[W_X x_n]}(I)$ a.s. by Proposition 17.9 of Ref. 13. So, according to Theorem 5.8.8 of Ref. 2, $|f_{nh}(t)| \leq (f_h(t))^{\frac{1}{2}} \|x_n\|$ a.s. for almost all $t \geq 0$. One can modify f_{nh} on $(x_n)_{n\geq 1}$ in a linear way, so it defines a bounded linear functional on $\operatorname{span}(h_n)_{n\geq 1}$. Therefore, there exists scalarly progressively measurable $a_h : \mathbb{R}_+ \times \Omega \to X$ such that $f_{nh} = \langle a_h, x_n\rangle$ a.s. for almost all $t \geq 0$.

Now consider $N = \int_0^{\cdot} a_h^* \mathrm{d}W_X$. Then $[Mh, W_X g] = [N, W_X g]$ for all $g \in H$, and consequently $[Mh, \Phi \cdot W_H] = [N, \Phi \cdot W_H]$ for each $\overline{\mathbb{F}}$-progressively measurable stochastically integrable $\Phi : \mathbb{R}_+ \times \Omega \to H$, and so $[Mh, Mg] = [N, Mg]$ for all $g \in \operatorname{ran}(A)$ and $[Mh, N] = [N]$. Let $(g_n)_{n\geq 1} \subset \operatorname{ran}(A)$ be such that $g_n \to h$. Then

$[Mh - Mg_n] \to 0$ in ucp, and so

$$\begin{aligned}[N - Mg_n] &= [N] + [Mg_n] - 2[N, Mg_n]\\ &= ([N] - [N, Mg_n]) + ([Mg_n] - [Mh, Mg_n])\\ &\to ([N] - [N, Mh]) + ([Mh] - [Mh]) = 0\end{aligned}$$

in ucp, and therefore N and Mh are indistinguishable.

For a general Banach space X, define a Hilbert space H and a dense embedding $j : X \hookrightarrow H$ as in p. 154 of Ref. 17. Let $(h_\alpha)_{\alpha\in\Lambda}$ be a Hamel basis of H and $(x^*_\beta)_{\beta\in\Delta} \cup (h_\alpha)_{\alpha\in\Lambda}$ be a Hamel basis of X^* (thanks to the embedding $H \hookrightarrow X^*$ and Theorem 1.4.5 of Ref. 8). Let W_H be a Brownian motion constructed as above for $M|_H \in \mathcal{M}^{\mathrm{cyl}}_{\mathrm{loc}}(H)$, i.e. for each $\alpha \in \Lambda$ there exists progressively measurable $a_{h_\alpha} : \mathbb{R}_+ \times \Omega \to H$ such that Mh_α and $\int_0^\cdot a^*_{h_\alpha}\,\mathrm{d}W_H$ are indistinguishable. Using the same technique and the fact that $j^* : H \hookrightarrow X^*$ is a dense embedding, for each $\beta \in \Delta$ one can define progressively measurable $a_{x^*_\beta} : \mathbb{R}_+ \times \Omega \to H$ such that Mx^*_β and $\int_0^\cdot a^*_{x^*_\beta}\,\mathrm{d}W_H$ are indistinguishable. Using Remark 3.13, we can now define an X^*-strongly progressively measurable operator-valued function $F : \mathbb{R}_+ \times \Omega \to L(X^*, H)$ such that for each $x^* \in X^*$ a.s. $Mx^* = \int_0^\cdot (Fx^*)^*\,\mathrm{d}W_H$. $\square$

The next theorem is an obvious corollary of Theorem 3.6 of Ref. 31 and Theorem 5.13 of Ref. 31.

Theorem 3.19. *Let X be a UMD Banach space, H be a Hilbert space, $W_H : \mathbb{R}_+ \times \Omega \times H \to \mathbb{R}$ be an H-cylindrical Brownian motion, $M \in \mathcal{M}^{\mathrm{loc}}_{\mathrm{cyl}}(X)$, and $\Phi : \mathbb{R}_+ \times \Omega \to L(X^*, H)$ be scalarly predictable measurable with respect to filtration $\mathbb{F}_{W_H}$ generated by W_H such that $M = \int_0^\cdot \Phi\,\mathrm{d}W_H$. Then there exists an X-valued continuous local martingale $\widetilde{M} : \mathbb{R}_+ \times \Omega \to X$ such that $Mx^* = \langle \widetilde{M}, x^*\rangle$ for each $x^* \in X^*$ if and only if $\Phi \in \mathcal{L}(X^*, H)$ $(\mathbb{P} \times \mathrm{d}s)$-a.s. and $\Phi^* \in \gamma(L^2(\mathbb{R}_+; H), X)$ a.s.*

3.4. *Time change*

A family $\tau = (\tau_s)_{s\geq 0}$ of finite stopping times is called a *finite random time change* if it is nondecreasing and right-continuous. If $\mathbb{F}$ is right-continuous, then by to Lemma 7.3 of Ref. 13 the *induced filtration* $\mathbb{G} = (\mathcal{G}_s)_{s\geq 0} = (\mathcal{F}_{\tau_s})_{s\geq 0}$ (see Chap. 7 of Ref. 13) is right-continuous as well. $M \in \mathcal{M}^{\mathrm{loc}}_{\mathrm{cyl}}(X)$ is said to be τ-*continuous* if a.s. for each $x^* \in X^*$, Mx^* (and thanks to Problem 17.3 of Ref. 13 equivalently $[Mx^*]$) is a constant on every interval $[\tau_{s-}, \tau_s]$, $s \geq 0$, where we set $\tau_{0-} = 0$.

Remark 3.20. Note that if $M \in \mathcal{M}^{\mathrm{loc}}_{\mathrm{cyl}}(X)$ is τ-continuous for a given time change τ, then $M \circ \tau \in \mathcal{M}^{\mathrm{loc}}_{\mathrm{cyl}}(X)$. Indeed, for each given $x^* \in X^*$ one concludes thanks to Proposition 17.24 of Ref. 13 $Mx^* \circ \tau$ is a continuous local martingale. Also for a given vanishing sequence $(x^*_n)_{n\geq 1} \subset X^*$ one can easily prove that $Mx^*_n \circ \tau \to 0$ in the ucp topology by using the stopping time argument and the fact that $Mx^*_n \to 0$ in the ucp topology by the definition of $\mathcal{M}^{\mathrm{loc}}_{\mathrm{cyl}}(X)$.

The following natural question arise: does there exist a suitable time change making a given $M \in \mathcal{M}^{\mathrm{loc}}_{\mathrm{cyl}}(X)$ Brownian representable? The answer is given in the following theorem.

Theorem 3.21. *Let X be a Banach space with a separable dual space, $M \in \mathcal{M}^{\mathrm{loc}}_{\mathrm{cyl}}(X)$. Then there exists a time change $(\tau_s)_{s\geq 0}$ such that $M \circ \tau$ is with an absolutely continuous covariation, i.e., Brownian representable.*

Proof. Let $(x_n^*)_{n\geq 1} \subset X^*$ be a dense subset of the unit ball of X^*. Consider the increasing predictable process $F : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ given by $F(t) = t(1 + \sum_{n=1}^{\infty} \frac{1}{2^n} \arctan([Mx_n^*]_t))$, and consider the time change $\tau_s = \inf\{t \geq 0 : F_t > s\}$ for $s \geq 0$. This time change is finite since $\lim_{t\to\infty} F(t) = \infty$.

For each fixed $\omega \in \Omega$ and $n \geq 1$ one has $\mu_{[Mx_n^*]} \ll \mu_F$. Then by Lemma B.1 one sees that $\mu_{[Mx^*]} \ll \mu_F$ a.s. for each $x^* \in X^*$, so M is τ-continuous. Moreover, for each $x^* \in X^*$ a.s. one has $\mu_{[Mx^*\circ\tau]} = \mu_{[Mx^*]\circ\tau} \ll \mu_{F\circ\tau}$, where the last measure is a Lebesgue measure on $\mathbb{R}_+$, so by Remark 3.20 $M \circ \tau \in \mathcal{M}^{\mathrm{loc}}_{\mathrm{cyl}}(X)$ is with an absolutely continuous covariation. $\square$

Let X be a separable Banach space, and let $\widetilde{M} \in \mathcal{M}^{\mathrm{loc}}(X)$. Then $\widetilde{M}$ is *weakly Brownian representable* if there exist a Hilbert space H, an H-cylindrical Brownian motion W_H and a function $G : \mathbb{R}_+ \times \Omega \to L(X^*, H)$ such that for each $x^* \in X^*$ the function Gx^* is stochastically integrable with respect to W_H and $\langle \widetilde{M}, x^*\rangle = \int_0^{\cdot} Gx^* \,\mathrm{d}W_H$ a.s. Thanks to Part 3.3 of Ref. 33 there exists an associated cylindrical continuous local martingale $M \in \mathcal{M}^{\mathrm{loc}}_{\mathrm{cyl}}(X)$, so the following corollary of Theorem 3.21 holds.

Corollary 3.22. *Let X be a Banach space with a separable dual space, and let $\widetilde{M} : \mathbb{R}_+ \times \Omega \to X$ be a continuous local martingale. Then there exists a time change $(\tau_s)_{s\geq 0}$ such that $\widetilde{M} \circ \tau$ is weakly Brownian representable.*

Remark 3.23. Unfortunately we do not see a way to prove an analogue of Theorems 3.9–3.11 in the present general case even for an X^*-valued integrand. The main difficulty is the discontinuity of the corresponding operator-valued function. One of course can prove such an analogue for integrands with values in a given finite-dimensional subspace of X^*, but this would amount to a stochastic integral with respect to an $\mathbb{R}^d$-valued continuous local martingale for some $d \geq 1$; the theory for this has been developed by classical works such as e.g., Ref. 19.

Appendix A. Technical Lemmas on Measurable Closed Operator-Valued Functions

The following lemma shows that a Borel bounded function of a closed operator-valued scalarly measurable function is again an operator-valued scalarly measurable function.

Lemma A.1. *Let (S, Σ) be a measurable space, H be a separable Hilbert space, and $f : S \to \mathcal{L}_{\mathrm{cl}}(H)$ be such that $(h_i)_{i=1}^{\infty} \subset \mathrm{dom}(f^*(s))$ for each $s \in S$ and $(f^* h_i)_{i\geq 1}$ are measurable for some fixed orthonormal basis $(h_i)_{i\geq 1}$ of H. Let $g : \mathbb{R} \to \mathbb{R}$ be finite Borel measurable. Then $g(f^*f) : S \to \mathcal{L}(H)$ is well-defined and scalarly measurable, $\|g(f^*f)(s)\| \leq \|g\|_{L^\infty(\mathbb{R})}$ for each $s \in S$.*

To prove this lemma we will need two more lemmas.

Lemma A.2. *Let H be a Hilbert space, and let $T \in \mathcal{L}_{\mathrm{cl}}(H)$. Then $T^*T \in \mathcal{L}_{\mathrm{cl}}(H)$.*

Proof. According to Chap. 118 of Ref. 26 there exists a bounded positive operator $B \in \mathcal{L}(H)$ such that $B = (1 + T^*T)^{-1}$ and $\mathrm{ran}(B) = \mathrm{dom}(T^*T)$. Since $\ker B = \{0\}$ by the construction, T^*T is densely defined. Furthermore since B is closed, by Proposition II.6.3 of Ref. 34 $T^*T = B^{-1} - 1$ is also closed. □

Lemma A.3. *Let H be a Hilbert space, $A \subset \mathcal{L}_{\mathrm{cl}}(H)$ be such that $(h_n)_{n=1}^{\infty} \subset \mathrm{dom}(A^*)$ for a certain orthonormal basis $(h_n)_{n=1}^{\infty}$ of H. For each $n \geq 1$ let $P_n \in \mathcal{L}(H)$ be the orthogonal projection onto* $\mathrm{span}(h_1, \ldots, h_n)$, *and set $A_n := P_n A$. Then*

(i) *the operators $((i + A_n^* A_n)^{-1})_{n\geq 1}$, $(i + A^*A)^{-1}$ are bounded*;
(ii) *$(i + A_n^* A_n)^{-1} h \to (i + A^*A)^{-1} h$ weakly for each $h \in H$.*

Using Problem III.5.26 of Ref. 15 we note that $A_n \subset (A^* P_n)^* \in \mathcal{L}_{\mathrm{cl}}(H)$ for all n.

Proof. The first part is an easy consequence of Theorem XI.8.1 of Ref. 34. To prove the second part we use the formula

$$\begin{aligned}
&((i + A_n^* A_n)^{-1} - (i + A^*A)^{-1})h \\
&\qquad = (i + A^*A)^{-1}(A^*A - A_n^* A_n)(i + A_n^* A_n)^{-1} h, \quad h \in H,
\end{aligned}$$

which follows from the fact that for each $n \geq 1$ there exists $\tilde{h} \in H$ such that $h = (i + A_n^* A_n)\tilde{h}$. Thanks to p. 347 of Ref. 26 $\mathrm{ran}(A^*A - i)^{-1} \subset \mathrm{dom}(A^*A)$, and therefore for each $h, g \in H$ and $n \geq 0$

$$\begin{aligned}
&\langle ((i + A_n^* A_n)^{-1} - (i + A^*A)^{-1})h, g\rangle \\
&\quad = \langle (i + A^*A)^{-1}(A^*A - A_n^* A_n)(i + A_n^* A_n)^{-1} h, g\rangle \\
&\quad = \langle (A^*A - A_n^* A_n)(i + A_n^* A_n)^{-1} h, (A^*A - i)^{-1} g\rangle \\
&\quad = \langle A(i + A_n^* A_n)^{-1} h, A(A^*A - i)^{-1} g\rangle \\
&\qquad - \langle A_n(i + A_n^* A_n)^{-1} h, A_n(A^*A - i)^{-1} g\rangle \\
&\quad = \langle (A - A_n)(i + A_n^* A_n)^{-1} h, A(A^*A - i)^{-1} g\rangle \\
&\qquad - \langle A_n(i + A_n^* A_n)^{-1} h, (A_n - A)(A^*A - i)^{-1} g\rangle
\end{aligned}$$

$$= \langle (i + A_n^* A_n)^{-1} h, (I - P_n) A^* A (A^* A - i)^{-1} g \rangle$$
$$- \langle A_n (i + A_n^* A_n)^{-1} h, (A_n - A)(A^* A - i)^{-1} g \rangle.$$

Let $H_n = \mathrm{span}(h_1, \dots, h_n)$. Note that $(A^* A - i)^{-1} g \in \mathrm{dom}(A^* A) \subset \mathrm{dom}(A)$ (see p. 347 of Ref. 26) and $\mathrm{ran}(A - A_n) \perp H_n$, so $(A_n - A)(A^* A - i)^{-1} g \in \mathrm{ran}(A_n - A) \perp H_n$. Also $A_n(i + A_n^* A_n)^{-1} h \in \mathrm{ran}(A_n) \subset H_n$. Therefore, for each $n \geq 1$

$$\langle A_n (i + A_n^* A_n)^{-1} h, (A_n - A)(A^* A - i)^{-1} g \rangle = 0,$$

and for the sequence $(Q_n)_{n \geq 1} := (I - P_n)_{n \geq 1} \subset \mathcal{L}(H)$ that vanish weakly

$$\langle ((i + A_n^* A_n)^{-1} - (i + A^* A)^{-1}) h, g \rangle = \langle (i + A_n^* A_n)^{-1} h, Q_n A^* A (A^* A - i)^{-1} g \rangle$$
$$\leq \| (i + A_n^* A_n)^{-1} h \| \| Q_n A^* A (A^* A - i)^{-1} g \|,$$

which vanishes as n tends to infinity, where according to Example VIII.1.4 of Ref. 34 $\|(i + A_n^* A_n)^{-1}\| \leq 1$ for each $n \geq 1$. □

Proof of Lemma A.1. First of all, one can construct $g(f^* f)$ (without proving measurability property) by guiding Chap. 120 of Ref. 26 by constructing a spectral family of $f^* f(s)$ for each fixed $s \in S$, and further using bounded calculus (Chap. 126 of Ref. 26) for the corresponding spectral family.

To prove scalar measurability we have to plunge into the construction of the spectral family. Let us first prove that $(i + f^* f)^{-1}$ is scalarly measurable. Notice that by Theorem XI.8.1 of Ref. 34 $(i + f^* f(s))^{-1} \in \mathcal{L}(H)$ for each $s \in S$. We will proceed in two steps.

Step 1. Suppose that $f(s)$ is bounded for all $s \in S$. Fix $k \geq 1$. Consider $\mathrm{span}((i + f^* f) h_i)_{1 \leq i \leq k}$. This is a k-dimensional subspace of H for each $s \in S$ since $i + f^* f$ is invertible. Let $\tilde{P}_k$ be defined as an orthogonal projection onto $\mathrm{span}((i + f^* f) h_i)_{1 \leq i \leq k}$, $(g_i)_{1 \leq i \leq k}$ be obtained from $((i + f^* f) h_i)_{1 \leq i \leq k}$ by the Gram–Schmidt process. These vectors are orthonormal and measurable because $(\langle (i + f^* f) h_i, (i + f^* f) h_j \rangle)_{1 \leq i, j \leq k}$ are measurable, so $\tilde{P}_k$ is scalarly measurable. Moreover, the transformation matrix $C = (c_{ij})_{1 \leq i, j \leq k}$ such that

$$g_i = \sum_{j=1}^{k} c_{ij} (i + f^* f) h_j, \quad 1 \leq i \leq k,$$

has measurable elements and invertible since by Theorem XI.8.1 of Ref. 34 $\ker(i + f^* f) = 0$. So, one can define the scalarly measurable inverse $(i + f^* f)^{-1} \tilde{P}_k$:

$$(i + f^* f)^{-1} \tilde{P}_k g = \sum_{j=1}^{k} d_{ij} \langle g, g_j \rangle h_j,$$

where $D = \{d_{ij}\}_{1 \leq i, j \leq k} = C^{-1}$.

Now fix $s \in S$, $g \in H$. Let $x_k = (i + f^*(s) f(s))^{-1} \tilde{P}_k(s) g$. Since $(i + f^*(s) f(s))^{-1}$ is a bounded operator and $\lim_{k \to \infty} \tilde{P}_k g = g$ (because by Theorem XI.8.1 of Ref. 34

$\operatorname{ran}(i+f^*(s)f(s)) = H)$, then $(i+f^*(s)f(s))^{-1}g = x := \lim_{k\to\infty} x_k$. So $(i+f^*f)^{-1}g$ is measurable as a limit of measurable functions.

Step 2. In general case one can consider the function $f_k = P_k f$ for each $k \geq 1$, where $P_k \in \mathcal{L}(H)$ is an orthogonal projection onto $\operatorname{span}(h_1, \ldots, h_k)$. Then by Lemma A.3 and thanks to Step 1 applied to f_k one can prove that $(i+f^*(s)f(s))^{-1}h$ is a weak limit of measurable functions $(i+f_k^*(s)f_k(s))^{-1}h$, so, since H is separable, it is measurable.

For the same reason $(f^*f - i)^{-1}$ is scalarly measurable, therefore $(1 + (f^*f)^2)^{-1} = (i + f^*f)^{-1}(f^*f - i)^{-1}$ is scalarly measurable.

Now guiding by the construction in Chap. 120 of Ref. 26 one can consider the sequence of orthogonal Hilbert spaces $\{H_i(s)\}_{i\geq 1}$ depending on s such that orthogonal projection P_{H_i} onto H_i is scalarly measurable (thanks to Proposition 32 of Ref. 21 and the fact that $P_{H_i} = \mathbf{1}_{(\frac{1}{i+1},\frac{1}{i}]}((1 + (f^*f)^2)^{-1}))$. Then by Chap. 120 of Ref. 26 $f^*(s)f(s)P_{H_i(s)}$ is bounded for each s. Moreover, $\operatorname{ran}(f^*(s)f(s)P_{H_i(s)}) \subset H_i(s)$ $\forall s \in S$, so $P_{H_i}f^*fP_{H_i} = f^*fP_{H_i}$ and $g(f^*fP_{H_i}) : S \to \mathcal{L}(H, H_i)$ is well-defined and thanks to Proposition 32 of Ref. 21 scalarly measurable. Finally, since $\|g(f^*fP_{H_i})\| \leq \|g\|_{L^\infty}$ and $H = \oplus_i H_i$, then one can define $g(f^*f) := \sum_{i=1}^\infty g(f^*fP_{H_i})$ as in Chap. 120 of Ref. 26, which is scalarly measurable and by Chap. 120 of Ref. 26 $\|g(f^*f)\| \leq \|g\|_{L^\infty}$ as well. □

Corollary A.4. *Let (S, Σ, μ) be a measure space, H be a separable Hilbert space, $f : S \to \mathcal{L}_{\mathrm{cl}}(H)$ be such that f^*h is a.s. defined and measurable for each $h \in H$. Let $g : \mathbb{R} \to \mathbb{R}$ be finite Borel measurable. Then $g(f^*f) : S \to \mathcal{L}(H)$ is well-defined and scalarly measurable, $\|g(f^*f)(s)\| \leq \|g\|_{L^\infty(\mathbb{R})}$ for almost all $s \in S$.*

The following lemma can be proved in the same way as the second part of Lemma A.1 of Ref. 33.

Lemma A.5. *Let (S, Σ, μ) be a measure space, H be a separable Hilbert space, and let X be a Banach space. Let $X_0 \subseteq X$ be a finite dimensional subspace. Let $F : S \to \mathcal{L}_{\mathrm{cl}}(X, H)$ be a function such that Fx is defined a.s. and strongly measurable for each $x \in X$. For each $s \in S$, let $\tilde{P}(s) \in \mathcal{L}(H)$ be the orthogonal projection onto $F(s)X_0$. Then $\tilde{P}$ is strongly measurable. Moreover, there exists a strongly measurable function $L : S \to \mathcal{L}(H, X)$ with values in X_0 such that $FL = \tilde{P}$.*

Appendix B. Lemmas on Absolute Continuity of Quadratic Variations

The main result of this subsection provides a surprising property of a limit in the ucp topology.

Lemma B.1. *Let $(M_n)_{n\geq 1}$, M be real-valued continuous local martingales, and $F : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be a continuous progressively measurable nondecreasing process* *such that $\mu_{[M_n]} \ll \mu_F$ a.s. for each $n \geq 0$. Let $M_n \to M$ in ucp. Then $\mu_{[M]} \ll \mu_F$ a.s.*

We will need the following lemma.

Lemma B.2. *Let $(M_n)_{n\geq 1}$ be a sequence of real-valued continuous local martingales starting from 0 such that $[M_n]$ is a.s. absolutely continuous for each $n \geq 0$. Then there exist a Hilbert space H, an H-cylindrical Brownian motion W_H on an enlarged probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ and a sequence of functions $(F_n)_{n\geq 1}$, $F_n : \mathbb{R}_+ \times \overline{\Omega} \to H$, $n \geq 1$ such that*

$$M_n = F_n \cdot W_H, \quad n \geq 1.$$

Proof. For each $k \geq 0$ we consider separately $(M_n(t) - M_n(k))_{n\geq 1}$, $k \leq t < k+1$. (The resulting cylindrical Brownian motions W_H^k can be glued together thanks to the independence of Brownian motion increments).

It is enough to consider the case $k = 0$. For each $n \geq 1$ one can find a nonzero real number a_n such that

$$\mathbb{P}\left\{\sup_{0\leq t\leq 1} |a_n M_n(t)| > \frac{1}{2^n}\right\} < \frac{1}{2^n},$$

$$\mathbb{P}\left\{[a_n M_n]_1 > \frac{1}{2^n}\right\} < \frac{1}{2^n}.$$

Without loss of generality redefine $M_n := a_n M_n$. Let H be a separable Hilbert space with an orthonormal basis $(h_n)_{n\geq 1}$. Then $\sum_{n=1}^\infty M_n h_n$ converges uniformly a.s. Therefore, $\sum_{n=1}^\infty M_n h_n$ converges in ucp topology, and $M := \sum_{n=1}^\infty M_n h_n : [0,1] \times \Omega \to H$ is an H-valued continuous local martingale. Moreover, thanks to Lemma 3.18 $[M] = \sum_{n=1}^\infty [M_n]$ a.s. and $[M]$ is absolutely continuous as a countable sum of absolutely continuous nondecreasing functions. Now using H-valued analogue of Brownian representation results one can find an H-cylindrical Brownian motion W_H on an enlarged probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$, operator-valued function $\Phi : \mathbb{R} \times \overline{\Omega} \to \mathcal{L}(H)$ such that

$$\langle M, h\rangle = \Phi h \cdot W_H, \quad h \in H.$$

In particular,

$$M_n = \langle M, h_n\rangle = \Phi h_n \cdot W_H, \quad h \in H. \qquad \square$$

Proof of Lemma B.1. Without loss of generality suppose that $F(0) = 0$ and $F(t) \nearrow \infty$ as $t \to \infty$ a.s. Otherwise redefine

$$F(t) := F(t) - F(0) + t, \quad t \geq 0.$$

Also by choosing a subsequence set M_n converges to M and $[M_n]$ converges to $[M]$ uniformly on compacts a.s. as n goes to infinity.

Let $(\tau_s)_{s\geq\infty}$ be the following time change:

$$\tau_s := \inf\{t \geq 0 : F(t) > s\}, \quad s \geq 0.$$

Then for each $n \geq 1$ by Proposition 17.6 of Ref. 13 and the fact that $\mu_{[M_n]} \ll \mu_F$ a.s., M_n is τ-continuous (see Chap. 7 of Ref. 13). Since M_n is τ-continuous and M_n converges to M uniformly on compacts a.s., then M is τ-continuous, and one can then define local martingales

$$N_n : = M_n \circ \tau, \quad n \geq 1,$$

$$N : = M \circ \tau,$$

which are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with an induced filtration $\mathbb{G} = (\mathcal{G}_s)_{s\geq 0} = (\mathcal{F}_{\tau_s})_{s\geq 0}$ (see Chap. 7 of Ref. 13). Also by Theorem 17.24 of Ref. 13 $[N_n] = [M_n] \circ \tau$ a.s., hence since $\mu_{[M_n]} \ll \mu_F$

$$\mu_{[N_n]} = \mu_{[M_n]\circ\tau} \ll \mu_{F\circ\tau} = \lambda,$$

so by Lemma B.2 there exist a separable Hilbert space H, an H-cylindrical Brownian motion W_H on an enlarged probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ with an enlarged filtration $\overline{\mathbb{G}}$ and a sequence of functions $(F_n)_{n\geq 1}$, $F_n : \mathbb{R}_+ \times \overline{\Omega} \to H$, $n \geq 1$ such that

$$N_n = F_n \cdot W_H, \quad n \geq 1.$$

But we know that $N_n \to N$ uniformly on compacts a.s. since $\tau_s \to \infty$ a.s. as $s \to \infty$, so there exists a progressively measurable function $R : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$, $t \mapsto \sup_{n\geq 1} |N_n|(t) + t$, which is nondecreasing continuous a.s. For each natural k define a stopping time $\rho_k := \inf\{t \geq 0 : R(t) > k\}$. Then $N_n^{\rho_k} \to N^{\rho_k}$ uniformly on compacts a.s. But $N_n^{\rho_k}$, N^{ρ_k} are bounded by k, hence they are L^2-martingales, and the convergence holds in L^2. Hence using the cylindrical case of the Itô isometry (Remark 30 of Ref. 21) one can see that $(F_n \mathbf{1}_{[0,\rho_k]})_{n\geq 1}$ converges to a function F^k in $L^2(\mathbb{R}_+ \times \Omega; H)$. Therefore, $N^{\rho_k} = F^k \cdot W_H$, so $[N^{\rho_k}] = [N]^{\rho_k} = \int_0^\cdot \|F^k(s)\|^2 \, \mathrm{d}s$ is absolutely continuous. Taking k to infinity and using the fact that $\rho_k \to \infty$ as $k \to \infty$ one can see that $[N]$ is absolutely continuous. Then

$$\mu_{[M]} = \mu_{[N]\circ F} \ll \mu_{\lambda\circ F} = \mu_F. \qquad \square$$

Acknowledgments

The author would like to thank Jan van Neerven and Mark Veraar for helpful comments.

References

1. L. M. Arutyunyan and I. S. Yaroslavtsev, Measurable polynomials on infinite-dimensional spaces, *Dokl. Akad. Nauk* **449**(6) (2013) 627–631.
2. V. I. Bogachev, *Measure Theory*, Vols. 1 and 2 (Springer, 2007).
3. D. L. Burkholder, Martingales and singular integrals in Banach spaces, in *Handbook of the geometry of Banach Spaces*, Vol. 1 (North-Holland, 2001), pp. 233–269.

4. G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Encyclopedia of Mathematics and its Applications, Vol. 44 (Cambridge Univ. Press, 1992).
5. E. Dettweiler, Representation of Banach space valued martingales as stochastic integrals, in *Probability in Banach Spaces*, Vol. 7 Progress in Probability, Vol. 21 (Birkhäuser Boston, 1990), pp. 43–62.
6. C. Di Girolami, G. Fabbri and F. Russo, The covariation for Banach space valued processes and applications, *Metrika* **77** (2014) 51–104.
7. J. L. Doob, *Stochastic Processes*, (John Wiley & Sons; Chapman & Hall, 1953).
8. R. E. Edwards, *Functional Analysis. Theory and Applications* (Holt, Rinehart and Winston, 1965).
9. G. Hamel, Eine Basis aller Zahlen und die unstetigen Lösungen der Funktionalgleichung: $f(x+y) = f(x) + f(y)$, *Math. Ann.* **60** (1905) 459–462.
10. T. Hytönen, J. M. A. M. van Neerven, M. C. Veraar and L. Weis, *Analysis in Banach Spaces. Part I: Martingales and Littlewood–Paley Theory*, Ergebmise der Mathematik and ihrer Grenzgebiete. 3. Folge/A Series of Modern suribys in Mathematics, Vol. 63. (Springer, 2016).
11. T. Hytönen, J. M. A. M. van Neerven, M. C. Veraar and L. Weis, *Analysis in Banach Spaces, Part II: Probabilistic Methods and Operator Theory*, Ergebmise der Mathematik and ihrer Grenzgebiete. 3. Folge/A Series of Modern suribys in Mathematics, Vol. 67 (Springer, 2017).
12. A. Jakubowski, S. Kwapień, P. R. de Fitte and J. Rosiński, Radonification of cylindrical semimartingales by a single Hilbert-Schmidt operator, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **5** (2002) 429–440.
13. O. Kallenberg, *Foundations of Modern Probability* 2nd edn., Probability and its Applications New York (Springer, 2002).
14. I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus*, 2nd edn., Graduate Texts in Mathematics, Vol. 113 (Springer, 1991).
15. T. Kato, *Perturbation Theory for Linear Operators*, Die Grundlehren der mathematischen Wissenschaften, Band 132 (Springer, 1966).
16. A. N. Kolmogorov and S. V. Fomīn, *Introductory Real Analysis* (Dover Publications, 1975). Translated from the second Russian edition and edited by Richard A. Silverman, Corrected reprinting.
17. H. H. Kuo, *Gaussian Measures in Banach Spaces*, Lecture Notes in Mathematics, Vol. 463 (Springer, 1975).
18. D. Lépingle and J.-Y. Ouvrard, Martingales browniennes hilbertiennes, *C. R. Acad. Sci. Paris Sér. A-B*, **276** (1973) A1225–A1228.
19. M. Métivier and J. Pellaumail, *Stochastic Integration*, Probability and Mathematical Statistics (Academic Press 1980).
20. D. Nualart, *Malliavin Calculus and its Applications*, CBMS Regional Conference Series in Mathematics, Vol. 110 (Amer. Math. Soc. 2009).
21. M. Ondreját, Brownian representations of cylindrical local martingales, martingale problem and strong Markov property of weak solutions of SPDEs in Banach spaces, *Czechoslovak Math. J.* **55** (2005) 1003–1039.
22. M. Ondreját, Integral representations of cylindrical local martingales in every separable Banach space, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **10** (2007) 365–379.
23. J.-Y. Ouvrard, Représentation de martingales vectorielles de carré intégrable à valeurs dans des espaces de Hilbert réels séparables, *Z. Wahr. Verw. Gebiete* **33** (1975/1976) 195–208.

24. P. E. Protter, *Stochastic Integration and Differential Equations*, 2nd edn., Stochastic Modelling and Applied Probability, Vol. 21 (Springer, 2005) Version 2.1, Corrected third printing.
25. M. Riedle, Cylindrical Wiener processes, In *Séminaire de Probabilités XLIII*, Lecture Notes in Mathematics, Vol. 2006 (Springer, Berlin, 2011) pp. 191–214.
26. F. Riesz and B. Sz.-Nagy, *Functional Analysis* (Frederick Ungar Publishing Company, 1955). Translated by Leo F. Boron.
27. J. L. Rubio de Francia, Martingale and integral transforms of Banach space valued functions, in *Probability and Banach Spaces*, Lecture Notes in Mathematics, Vol. 1221 (Springer, Berlin, 1986) pp. 195–222.
28. L. Schwartz, Le théorème des trois opérateurs, *Ann. Math. Blaise Pascal* **3** (1996) 143–164.
29. D. W. Stroock and S. R. S. Varadhan, *Multidimensional Diffusion Processes*, Classics in Mathematics, (Springer, 2006). Reprint of the 1997 edition.
30. A. E. Taylor and D. C. Lay, *Introduction to Functional Analysis*, 2nd edn. (John Wiley & Sons, 1980).
31. J. M. A. M. van Neerven, M. C. Veraar and L. Weis, Stochastic integration in UMD Banach spacess, *Ann. Probab.* **35** (2007) 1438–1478.
32. J. M. A. M. van Neerven and L. Weis, Stochastic integration of functions with values in a Banach space, *Studia Math.* **166** (2005) 131–170.
33. M. C. Veraar and I. S. Yaroslavtsev, Cylindrical continuous martingales and stochastic integration in infinite dimensions, *Electron. J. Probab.* **21** (2016) Paper No. 59, 53 pp.
34. K. Yosida, *Functional Analysis*, Classics in Mathematics, (Springer, 1995). Reprint of the sixth (1980) edition.