Higher Order Elliptic Problems and Positivity

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Proefschrift

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FOR MATHEMATICS

‘Fatti non fummo per viver come bruti,
ma per seguir virtute e conoscenza.’
(*Dante, Divina Commedia, Inf.XXVI, 118-120)*

To my pillars

Preface

This thesis consists of an introduction and five chapters.

- Chapter 2 is an adaptation of:
  A. Dall'Acqua and G. Sweers, Estimates for Green function and Poisson kernels of higher order Dirichlet boundary value problems, J. Differential Equations **205** (2004) 466-487.

- Chapter 3 is an adaptation of:
  A. Dall'Acqua and G. Sweers, The clamped plate equation for the Limaçon, to appear in: Annali di Matematica Pura ed Applicata.

- Chapter 4 is an adaptation of:
  A. Dall'Acqua and G. Sweers, On domains for which the clamped plate system is positivity preserving, Partial Differential Equations and Inverse Problems (eds C. Conca, R. Manasevich, G. Uhlmann, M.S. Vogelius), Contem. Math. 362 (American Mathematical Society, 2004) 133–144.

- Chapter 6 is an adaptation of:
  A. Dall'Acqua, H.-Ch. Grunau and G. Sweers, On a conditioned Brownian motion and a maximum principle in the disk, Journal d'Analyse Mathématique **93** (2004) 309-329.

Contents

Chapter 1

Introduction

The subject of this thesis is the study of positivity for fourth order elliptic problems. By positivity we mean that a positive source term in the differential equation leads to a positive solution. For second order elliptic partial differential equations such a result is known, and usually referred to by the name "maximum principle". It is also well-known that such a maximum principle does not have a straightforward generalization to higher order elliptic equations. Nevertheless, the mechanical models that lead to fourth order elliptic equations, such as the elastic deformation of beams and plates, seem to indicate that some positivity remains. These features we will discuss in the introduction.

In the first section we will present the problem through some models in the one dimensional case. This case is rather simple since everything can be computed explicitly. We will start with the one-dimensional setting to have a gentle introduction of some features that will appear in the higher dimensional case. The core of the present thesis will be the two-dimensional case.

1.1 One dimension

1.1.1 Laundry line

To introduce the setting we would like to start with a rather simple model, namely that of a laundry line. Using x for the horizontal coordinate and u for the deviation from the horizontal, we may consider u as a function of x. A simplified mathematical formulation for the deviation u when this laundry line, of length 2, is loaded by a weight is the following second order problem

$$\begin{cases} -u_{xx}(x) = \dfrac{g}{c} f(x) \text{ with } x \in (-1,1), \\ u(-1) = u(1) = 0. \end{cases} \tag{1.1.1}$$

Here c is a constant that depends on the material and tension of the line, g is the gravity constant and f is the weight density of the laundry hanging on this line. In the model the boundary conditions $u(-1) = 0$ and $u(1) = 0$ appear since the laundry line is fixed at the end-points.

The model in (1.1.1) can be derived from the balance of forces. One could also look at the laundry line from the energy point of view. This energy has two components: one due to the internal tension and the other one due to the weight. The first component is proportional to the increase of length compared with the length at rest, see [17, page 245]:

$$E_s(u) = c \int_{-1}^{1} \left(\sqrt{1 + u_x^2(x)} - 1 \right) dx.$$

One finds that the total energy is given by

$$E_{tot}(u) = c \int_{-1}^{1} \left(\sqrt{1 + u_x^2(x)} - 1 \right) dx - g \int_{-1}^{1} f(x)\ u(x)\ dx.$$

For small deformations we may simplify to

$$\tilde{E}_{tot}(u) = \frac{c}{2} \int_{-1}^{1} u_x^2(x)\ dx - g \int_{-1}^{1} f(x)\ u(x)\ dx. \tag{1.1.2}$$

A physically relevant solution will minimize the energy. If we are looking for a minimizer of (1.1.2) it has to satisfy

$$\frac{\partial}{\partial \tau} \tilde{E}_{tot}(u + \tau\varphi) \bigg|_{\tau=0} = 0, \tag{1.1.3}$$

for all appropriate test functions φ satisfying $\varphi(-1) = \varphi(1) = 0$, that is,

$$c \int_{-1}^{1} u_x(x)\varphi_x(x)\ dx - g \int_{-1}^{1} f(x)\ \varphi(x)\ dx = 0.$$

Integrating by parts, one finds

$$\int_{-1}^{1} (c\ u_{xx}(x) + g\ f(x))\, \varphi(x)\ dx = 0. \tag{1.1.4}$$

Since (1.1.4) has to hold for all appropriate φ we recover the differential equation in (1.1.1).

We are interested in positivity, that is, when $f > 0$ in problem (1.1.1) implies that the solution u is also positive, or in other words, a Positivity Preserving Property.

Positivity Preserving Property *Let $w \in \mathcal{S}$, with $\mathcal{S}$ some set of functions and let L be an operator acting on $\mathcal{S}$. Then $Lw > 0$ implies $w > 0$.*

In the present setting we may take for $\mathcal{S}$ the set of twice continuously differentiable functions in $[-1, 1]$ that equal zero in -1 and 1; $Lu = -u_{xx}$. Problem (1.1.1) has the Positivity Preserving Property. Indeed, notice that f positive in (1.1.1) implies $-u_{xx}$

positive, hence u concave, which in turn gives, together with the boundary conditions, that u itself is positive. Notice that this argument contains a Maximum Principle.

Maximum Principle *Let $w \in \mathcal{S}$, with $\mathcal{S}$ some set of functions and let L be an operator acting on $\mathcal{S}$. Then $Lw > 0$ implies $-w$ cannot attain an interior maximum.*

With the same choice of $\mathcal{S}$ and L as before, one sees that the Maximum Principle implies the Positivity Preserving Property.

Going back to the model we started with, this is clearly what one expects from every day experience:

The line moves down when hanging laundry on it.

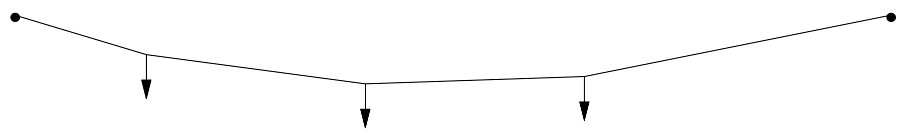

Figure 1.1: *The displacement of the laundry line loaded by three point-masses. From the figure one may guess that the first derivative of the solution is not continuous. Notice that in the figures the positive direction is downward.*

A nice feature of problem (1.1.1) is that we can give an explicit formula for the solution by means of a so-called Green function:

$$G_s(x,y) = \begin{cases} \frac{1}{2}(x+1)(1-y) & \text{for } -1 \le x \le y, \\ \frac{1}{2}(y+1)(1-x) & \text{for } y < x \le 1. \end{cases}$$

Indeed the solution of (1.1.1) is given by

$$u(x) = \frac{g}{c} \int_{-1}^{1} G_s(x,y)\ f(y)\ dy. \tag{1.1.5}$$

Notice that the positivity of the Green function immediately gives a Positivity Preserving Property without going through the Maximum Principle.

1.1.2 Curtain rod

Another simple model where the positivity question appears is that of a beam. One could think of a rod carrying a curtain. Using the same notation x and u, the total energy (see [17, page 245]) is given by

$$E_{tot}(u) = \frac{c}{2} \int_{-1}^{1} \left(\frac{d}{dx} \left(\frac{u_x(x)}{\sqrt{1+u_x^2(x)}} \right) \right)^2 dx - g \int_{-1}^{1} f(x)\ u(x)\ dx.$$

Notice that the energy density due to the bending is proportional to the square of the curvature. Again considering small displacements we find

$$\tilde{E}_{tot}(u) = \frac{c}{2} \int_{-1}^{1} (u_{xx}(x))^2 \, dx - g \int_{-1}^{1} f(x) \; u(x) \; dx.$$

Recalling (1.1.3) a minimizer should satisfy

$$c \int_{-1}^{1} u_{xx}(x) \; \varphi_{xx}(x) \; dx - g \int_{-1}^{1} f(x) \; \varphi(x) \; dx = 0, \tag{1.1.6}$$

for any appropriate test function φ. For this model we may consider two configurations.

(i) We fix the position, that is, we prescribe $u(-1) = u(1) = 0$.

(ii) We fix both position and angle at the boundary, that is, $u(-1) = u(1) = 0$ and $u'(-1) = u'(1) = 0$.

Ad (i) **Supported beam.** The appropriate test functions to be considered for (1.1.6) have to satisfy $\varphi(-1) = \varphi(1) = 0$. Integrating by parts we find

$$\begin{aligned} 0 = & \int_{-1}^{1} (c \; u_{xx}(x) \; \varphi_{xx}(x) - g \; f(x) \; \varphi(x)) \, dx \\ = \; & [c u_{xx}(x) \varphi_x(x)]_{-1}^{1} + \int_{-1}^{1} (-c \; u_{xxx}(x) \; \varphi_x(x) - g \; f(x) \; \varphi(x)) \, dx \\ = \; & c \; u_{xx}(1) \varphi_x(1) - c \; u_{xx}(-1) \varphi_x(-1) + \int_{-1}^{1} (c \; u_{xxxx}(x) - g \; f(x)) \, \varphi(x) dx. \end{aligned}$$

Choosing first test functions φ that disappear at the boundary we find that u has to satisfy the fourth order differential equation

$$u_{xxxx}(x) = \frac{g}{c} f(x) \text{ for } x \in (-1, 1) \, .$$

Next considering test functions φ such that $\varphi_x(1) \neq 0$ and respectively $\varphi_x(-1) \neq 0$ we get that u has to satisfy the so-called natural boundary conditions:

$$u_{xx}(1) = u_{xx}(-1) = 0.$$

Hence the corresponding model is

$$\begin{cases} u_{xxxx}(x) = \dfrac{g}{c} f(x) \text{ with } x \in (-1, 1), \\ u(-1) = u(1) = u_{xx}(-1) = u_{xx}(1) = 0. \end{cases} \tag{1.1.7}$$

System (1.1.7) is called a supported beam.

As in problem (1.1.1) we would like to see if the Positivity Preserving Property holds. Thinking at the rod carrying the curtain, it is obvious that such a feature holds in everyday life. We would like to show that such a property follows from the mathematical model. It gives some evidence that the model corresponds to what happens physically. In order to do that notice that problem (1.1.7) can be written as a system of two differential equations of second order. Indeed, defining $v(x) := -u_{xx}(x)$ problem (1.1.7) is equivalent to

$$\begin{cases} -v_{xx}(x) = \frac{g}{c} f(x) \text{ with } x \in (-1,1), \\ -u_{xx}(x) = v(x) \quad \text{with } x \in (-1,1), \\ u(-1) = u(1) = v(-1) = v(1) = 0. \end{cases} \tag{1.1.8}$$

As before f positive means $-v_{xx}$ positive, hence with the boundary condition v itself is positive and repeating the argument for u we find that u is positive. So, the Positivity Preserving Property for the fourth order problem follows by using twice the maximum principle for second order problems. Going back to the model, we see that:

Hanging the curtain on the supported rod will move the rod downward everywhere.

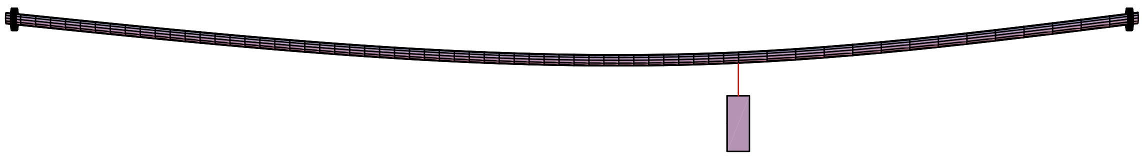

Figure 1.2: *Displacement of a supported beam loaded by a one-point mass. The solution is convex and the first and second derivative are continuous. The positive direction is downward.*

Also this problem allows for a Green function:

$$G_{sb}(x,y) = \begin{cases} \frac{1}{12}(x+1)(1-y)(2-x^2-y^2+2(y-x)) & \text{for } -1 \le x \le y, \\ \frac{1}{12}(y+1)(1-x)(2-y^2-x^2+2(x-y)) & \text{for } y < x \le 1. \end{cases}$$

The solution u is then as in (1.1.5) with $G_s(.,.)$ replaced by $G_{sb}(.,.)$. Notice that again this Green function is positive implying directly the Positivity Preserving Property. We would like to observe that one may compute directly G_{sb} using the equivalence of problem (1.1.7) with system (1.1.8). Indeed, from this it directly follows that:

$$G_{sb}(x,y) = \int_{-1}^{1} G_s(x,z) G_s(z,y)\, dz.$$

Ad (ii) **Clamped beam.** Coming back to the rod with position and angle fixed to be zero at the boundary, we find that in this case the model is

$$\begin{cases} u_{xxxx}(x) = \dfrac{g}{c} f(x) \text{ with } x \in (-1,1), \\ u(-1) = u(1) = u_x(-1) = u_x(1) = 0. \end{cases} \tag{1.1.9}$$

The system in (1.1.9) is called a clamped beam.

If one wants to prove the Positivity Preserving Property for (1.1.9) we cannot use the Maximum Principle for second order problems as before since the boundary conditions do not separate nicely as in the case of the supported beam. However one may still construct a Green function, namely

$$G_{cb}(x,y) = \begin{cases} \frac{1}{24}(x+1)^2(y-1)^2(1-xy+2(y-x)) & \text{for } -1 \le x \le y, \\ \frac{1}{24}(y+1)^2(x-1)^2(1-xy+2(x-y)) & \text{for } y < x \le 1. \end{cases}$$

Notice that since this Green function is positive one finds the Positivity Preserving Property. Also here:

Hanging the curtain on the clamped rod will move the rod downward everywhere.

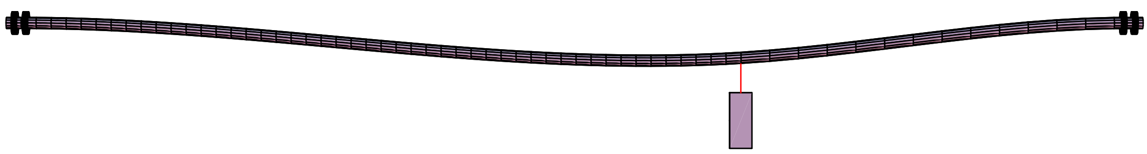

Figure 1.3: *Displacement of a clamped beam loaded by a point-mass. In this case the first and second derivative are still continuous but the solution is not convex.*

We would like to remark that the positivity of the Green function is a sufficient and necessary condition for the Positivity Preserving Property to hold. Indeed if the Green function would be sign changing, say $G(x_0, y_0) < 0$, then taking a point weight at position y_0 will force the solution to be negative at x_0.

A special curtain rod

We would like to give the model of a rod that does not have the Positivity Preserving Property. However we will be able to classify the way in which it may fail to be positive. What does remain is that locally the sign is preserved. We will explain what we mean by that.

Let us consider a rod that is supported in the middle. The corresponding system is as follows

$$\begin{cases} u_{xxxx}(x) = \dfrac{g}{c} f(x) \text{ with } x \in (-1,0) \cup (0,1), \\ u(-1) = u(1) = u_x(-1) = u_x(1) = 0, \\ u(0) = 0 \text{ and } u_x(0^+) = u_x(0^-) \text{ and } u_{xx}(0^+) = u_{xx}(0^-). \end{cases} \tag{1.1.10}$$

Notice that the rod we consider here is clamped in -1 and 1. One may also consider a rod supported in the middle and at the boundary. The behavior, concerning positivity, is the same.

In order to see if problem (1.1.10) have the Positivity Preserving Property we cannot use the second order Maximum Principle. So we may try to proceed through the Green function. The Green function associated to this problem when $0 < y < 1$ is:

$$G_{rp}(x,y) = \begin{cases} \frac{1}{8}\, xy(1+x)^2(1-y)^2 & \text{for } -1 \le x \le 0, \\ \frac{1}{24}\, x(1-y)^2(4x(y-x)+2yx(1-x)+3y(1-x^2)) & \text{for } 0 < x \le y, \\ \frac{1}{24}\, y(1-x)^2(4y(x-y)+2yx(1-y)+3x(1-y^2)) & \text{for } y < x \le 1. \end{cases}$$

For $-1 < y < 0$ one finds a similar formula. This Green function is sign changing. We illustrate this by showing in Figure 1.4 the solution u of problem (1.1.10) with in the right hand side a point-mass at position .3. Indeed, this u will be a multiple of the Green function with y fixed at .3.

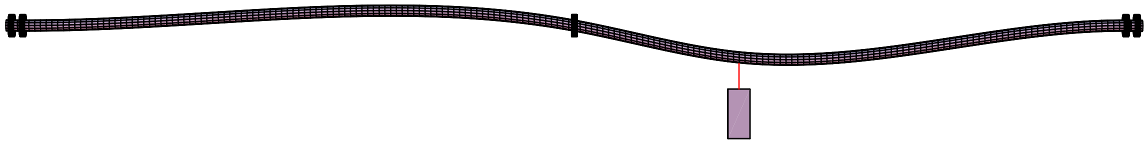

Figure 1.4: *Loading this rod supported in the middle with a point mass in the right half forces the left half of the rod to bend upward. Only a restricted Positivity Preserving Property holds.*

However we can show this local positivity.

Proposition 1.1.1. *If* y *is in* $(0,1)$ *then* $x \mapsto G_{rp}(x,y)$ *is positive in* $(0,1)$ *and negative in* $(-1,0)$.

Proposition 1.1.2. *For any* f *positive there exists* $a \in [-1,1]$ *such that the solution* u *of (1.1.10) with* f *in the right hand side satisfies* $u(x) < 0$ *for* $x \in (\min(a,0), \max(a,0))$ *and* $u(x) \ge 0$ *elsewhere.*

1.2 Two dimensions

The topic of this thesis is concerned with two and higher dimensional problems. For the sake of illustration we will now present some two dimensional models.

1.2.1 Membrane

We consider a membrane spanned over a flat frame and loaded by a weight. An example that we can keep in mind is that of a soap film. Using x_1 and x_2 as coordinates in the plane and u for the deviation from the flat position, we can consider u as a function of x_1 and x_2. Let Ω be the area over which the membrane is spanned.

The energy in this membrane has two components. The one due to the tension is proportional to the change of area ([17, page 247]). When restricting to small deviations we may consider the linearized version and so the total energy of the membrane is given by

$$\tilde{E}_{tot}(u) = \frac{c}{2} \int_\Omega |\nabla u(x_1, x_2)|^2 \, dx_1 dx_2 - g \int_\Omega f(x_1, x_2) \, u(x_1, x_2) \, dx_1 dx_2.$$

By (1.1.3) and considering a membrane fixed at its border we find that the displacement of the membrane is modelled by the following system

$$\begin{cases} -\Delta u &= \frac{g}{c} f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.2.1}$$

Here $\partial\Omega$ denotes the boundary of the membrane and $\Delta u = \frac{\partial^2}{\partial x_1^2} u + \frac{\partial^2}{\partial x_2^2} u$.

As it is well known the boundary value problem (1.2.1) satisfies the Maximum Principle, implying that the Positivity Preserving Property holds. Going back to the example we see that:

Pushing a membrane from below forces the membrane to go upward everywhere.

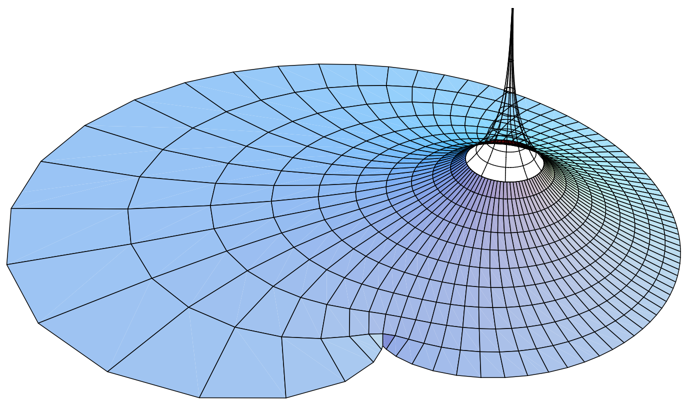

Figure 1.5: *Picture of the solution of (1.2.1) on a Limaçon de Pascal with $a = .49$ (see Figure 1.9) and with a point mass in the right hand side. The solution has a singularity that takes place in the point where the force is applied.*

Explicit formulas for the Green function $G(.,.)$ of problem (1.2.1) are only available for special domains. Nevertheless arguments based on the Maximum Principle and its extensions, such as Harnack's inequality, allow to prove estimates of the Green function for (1.2.1) on smooth Ω of the following type (see [76] and [45]):

$$c_1 \ln\left(1 + \frac{d_\Omega(x) d_\Omega(y)}{|x-y|^2}\right) \leq G(x,y) \leq c_2 \ln\left(1 + \frac{d_\Omega(x) d_\Omega(y)}{|x-y|^2}\right) \text{ for } x, y \text{ in } \Omega. \tag{1.2.2}$$

Here c_1 and c_2 are two positive constants and $d_\Omega(.)$ denotes the distance function to the boundary of Ω, that is

$$d_\Omega(x) := \min_{y\in\partial\Omega} |x-y|. \tag{1.2.3}$$

1.2.2 Plate

In this thesis we will study what remains of positivity for the model of a clamped plate. We will now derive the model starting from the energy functional using the same notation x_1, x_2, Ω and u as before.

In [17, page 250] one finds that the density of the energy due to the tension is a quadratic form of the principal curvatures for the plate. That is, for small displacement u the total energy of a plate loaded by a weight of density f is given by

$$\tilde{E}_{tot}(u) = c\int_\Omega \left(\frac{1}{2}(\Delta u)^2 - (1-\sigma)(u_{x_1x_1}u_{x_2x_2} - u^2_{x_1x_2})\right) dx_1dx_2 - g\int_\Omega f\ u\ dx_1dx_2.$$

Here c and σ are two constants that depend on the elastic properties of the plate. By (1.1.3) a minimizer of the energy should satisfy

$$\begin{aligned} c\int_\Omega (\Delta u\Delta\varphi - (1-\sigma)(u_{x_1x_1}\varphi_{x_2x_2} + \varphi_{x_1x_1}u_{x_2x_2} - 2u_{x_1x_2}\varphi_{x_1x_2}))\, dx_1dx_2 +& \\ -g\int_\Omega f\ \varphi\ dx_1dx_2 = 0,& \end{aligned} \tag{1.2.4}$$

for any appropriate test function φ. As in the case of the rod for this model we may consider two configurations.

(i) Supported plate: we fix the position, that is, $u = 0$ on the boundary of the plate Ω.

(ii) Clamped plate: we fix the position and the angle of $\partial\Omega$, that is, $u = 0$ and, considering ν the exterior normal to the boundary of Ω on the plane x_1x_2, $\frac{\partial}{\partial\nu}u = \nu.\nabla u = 0$ on the boundary of the plate.

Ad (i) **Supported plate.** The test functions to be considered for (1.2.4) have to be zero at the boundary of the plate. Integrating by parts in (1.2.4) one finds that

$$\begin{aligned} 0 \;=\; & c\int_{\partial\Omega}\left(\Delta u\ \frac{\partial}{\partial\nu}\varphi - (1-\sigma)\left(u_{x_1x_1}\varphi_{x_2}\nu_{x_2} + u_{x_2x_2}\varphi_{x_1}\nu_{x_1} - 2u_{x_1x_2}\varphi_{x_1}\nu_{x_2}\right)\right) d\sigma \\ & -c\int_\Omega (\nabla\Delta u.\nabla\varphi - (1-\sigma)(u_{x_1x_1x_2}\varphi_{x_2} + u_{x_2x_2x_1}\varphi_{x_1} - 2u_{x_1x_2x_2}\varphi_{x_1}))\, dx_1\ dx_2 \\ & -g\int_\Omega f\ \varphi\ dx_1\ dx_2 \end{aligned}$$

$$\begin{aligned} &= c\int_{\partial\Omega} \left(\Delta u - (1-\sigma)\left(u_{x_1x_1}\nu_{x_2}^2 + u_{x_2x_2}\nu_{x_1}^2 - 2u_{x_1x_2}\nu_{x_1}\nu_{x_2}\right)\right)\frac{\partial}{\partial\nu}\varphi\, d\sigma \\ &\quad + c\int_\Omega \Delta^2 u\, \varphi\, dx_1\, dx_2 - g\int_\Omega f\, \varphi\, dx_1\, dx_2, \end{aligned}$$

where $\Delta^2 u = \frac{\partial^4}{\partial x_1^4}u + 2\frac{\partial^4}{\partial x_1^2 x_2^2}u + \frac{\partial^4}{\partial x_2^4}u$. Hence the displacement u has to satisfy the differential equation

$$\Delta^2 u = \frac{g}{c} f \text{ in } \Omega,$$

and the natural boundary condition

$$\Delta u - (1-\sigma)(u_{x_1x_1}\nu_{x_2}^2 + u_{x_2x_2}\nu_{x_1}^2 - 2u_{x_1x_2}\nu_{x_1}\nu_{x_2}) = 0 \text{ on } \partial\Omega.$$

So, the model for a supported plate is

$$\begin{cases} \Delta^2 u = \frac{g}{c} f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \Delta u - (1-\sigma)(u_{x_1x_1}\nu_{x_2}^2 + u_{x_2x_2}\nu_{x_1}^2 - 2u_{x_1x_2}\nu_{x_1}\nu_{x_2}) = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2.5}$$

For results concerning positive solutions of (1.2.5) we refer to [54].

One usually studies the following simpler model

$$\begin{cases} \Delta^2 u = \frac{g}{c} f & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2.6}$$

that does correspond to the previous one when $\sigma = 1$.

System (1.2.6) can be written as a system of two differential equations of second order. Indeed, defining $v(x) := -\Delta u(x)$ problem (1.2.6), for smooth Ω, is equivalent to

$$\begin{cases} -\Delta v &= \frac{g}{c} f & \text{in } \Omega, \\ -\Delta u &= v & \text{in } \Omega, \\ u = v &= 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2.7}$$

and since the maximum principle holds for system (1.2.7) it holds also for problem (1.2.6). Also in this case the Positivity Preserving Property for the fourth order problem follows iterating twice the Maximum Principle for the second order problem.

A supported plate loaded by a force moves all in the same direction.

Problem (1.2.6) allows for an explicit Green function $G_{sp}(.,.)$ only for some special Ω's. However, using the same notation as in (1.2.2), one can prove the following estimate for x, y in Ω (see [45])

$$\begin{aligned} &c_1 d_\Omega(x) d_\Omega(y) \ln\left(2 + \frac{1}{|x-y|^2 + d_\Omega(x)d_\Omega(y)}\right) \leq \\ &\qquad \leq G_{sp}(x,y) \leq c_2 d_\Omega(x) d_\Omega(y) \ln\left(2 + \frac{1}{|x-y|^2 + d_\Omega(x)d_\Omega(y)}\right). \end{aligned}$$

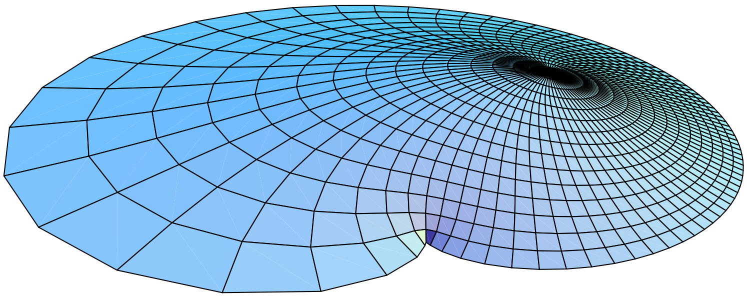

Figure 1.6: *Picture of the solution of (1.2.6) on a Limaçon de Pascal with* $a = .49$ *(see Figure 1.9) and with* f *a point mass. The point mass is located at "the center of the circular axis". Notice that the maximum of* u *is located more to the center of the domain. In this case the solution and also its first derivative are continuous. A discontinuity is appearing in the second derivatives.*

The major tools in the proof are the estimate in (1.2.2) and that the following relation holds between the Green function G for problem (1.2.1) on a domain Ω and the Green function G_{sp} associated to problem (1.2.6) on the same domain:

$$G_{sp}(x,y) = \int_\Omega G(x,z)G(z,y)dz \text{ for } x,y \text{ in } \Omega.$$

Ad (ii) **Clamped plate.** The model for a clamped plate is

$$\begin{cases} \Delta^2 u &= \dfrac{g}{c} f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \\ \frac{\partial}{\partial\nu} u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.2.8}$$

The solution of (1.2.8) gives, for example, the displacement of a flat roof when loaded by a force f. One may think at the force f as the extra weight that the roof has to support because of rain or snow.

As for the previous models we would like to see if the Positivity Preserving Property holds. Problem (1.2.8) does not satisfy the Maximum Principle and we do not have a formula for the Green function associated on a general domain. One may think as Hadamard (see [47] and [31]) at first did, namely

> *If a perpendicular force is applied at some point of a thin, flat elastic plate which is rigidly clamped on its boundary, then the displacement of the plate is of one sign at all points.*

However in general this is not the case. We will illustrate this in Figure 1.7 by showing the graph of a sign-changing solution.

In this thesis we will show that, although the Positivity Preserving Property does not hold, there is a form of local positivity. We will also prove estimates of the Green function for (1.2.8) depending on the distance to the boundary.

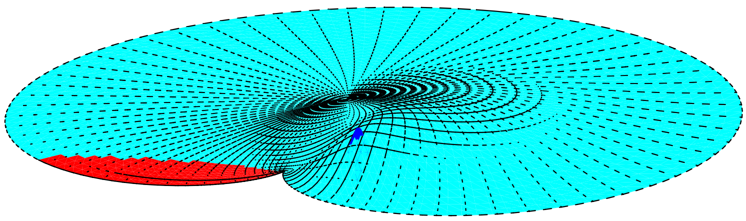

Figure 1.7: *Picture of the solution of (1.2.8) with on the right hand side a point mass and on a Limaçon de Pascal with* $a = .49$*. The dark part in the figure shows the region where the solution becomes negative. The arrow indicates the point where the force is applied. Notice that not always the solution of (1.2.8) is sign changing. Indeed, as it is known, [8], problem (1.2.8) on the unit disk has the Positivity Preserving Property. This figure has been taken from [25].*

A special clamped plate

We consider now a special plate that, to some extent, is close to the rod supported in the middle of the previous section. We consider a plate with the shape of a dumb-bell that is clamped on the boundary.

Problem (1.2.8) on a dumb-bell does not have the Positivity Preserving Property. Indeed the associated Green function is sign-changing. We illustrate this by Figure 1.8 showing the plot of the numerical solution of (1.2.8) with f a point-mass. The

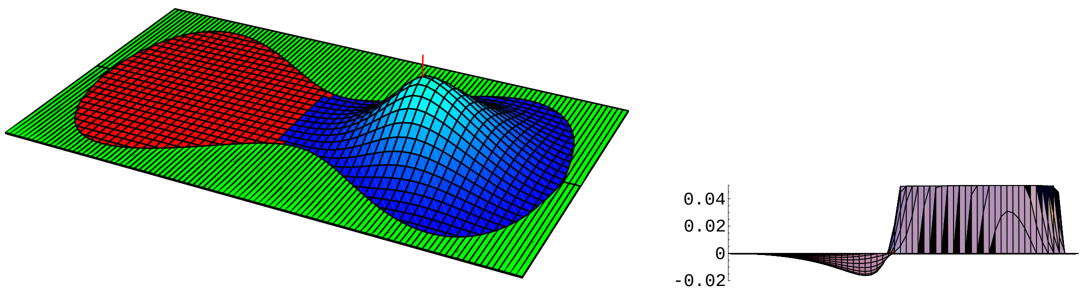

Figure 1.8: *The solution of the clamped plate equation on a dumb-bell with f a point-mass on the right part of it. On the right a view from the side of the solution that shows the change of sign. This figure has been magnified and truncated from above since when it is negative the solution is very small in absolute value. These figures are taken from [70].*

fact that the Green function is sign changing can be explained heuristically as follows. At the center of the dumb-bell the solution gets a strong influence from the boundary and, because of the boundary condition, it is almost forced to go to zero. From Figure 1.8 one intuitively understands that this plate behaves similarly to the special rod of the previous section with the extra boundary point in the middle. (See Figure 1.4).

1.3 Statement of the problem and results

The main subject of this thesis is the study of the sign of the solution of the following fourth order Dirichlet boundary value problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \frac{\partial}{\partial\nu} u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3.1}$$

Here f is a continuous non-negative function on Ω and Ω is a bounded smooth domain in $\mathbb{R}^n$, $n \in \mathbb{N}$ with $n \geq 2$. System (1.3.1) is the model of a clamped plate $\Omega \subset \mathbb{R}^n$. In the previous section we have considered the clamped plate in two dimensions, however the model, from a mathematical point of view, is also interesting in higher dimensions.

This thesis concerns the influence that a positive source f in problem (1.3.1) has on the behavior (sign) of the solution u. We now give the precise definition of what we mean by solution of (1.3.1). We consider several type of solutions. We will call classical the solution in the Hölder spaces setting and strong the one in L^p-spaces setting. For definitions and properties of Hölder, L^p and Sobolev spaces we refer to [2].

Definition 1.3.1. *a: Let $f \in C^\alpha(\Omega)$ with $\alpha \in (0,1)$. We say that $u \in C^{4,\alpha}(\Omega)$ is a classical solution of (1.3.1) if $\Delta^2 u = f$ holds point-wise in Ω and if u and its first derivatives are all zero at the boundary.*

b: Let $f \in L^p(\Omega)$ for $p \in (1,\infty)$. We call $u \in W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega)$ a strong solution of (1.3.1) with right hand side f if $\Delta^2 u = f$ holds in L^p-sense in Ω.

Remark 1.3.2. Sometimes we will also use the notion of weak and half-weak solutions. For $p \in (1,\infty)$ and with $p' := \frac{p}{p-1}$:

i: a weak solution of (1.3.1) is a function $u \in L^p(\Omega)$ that satisfies

$$\int_\Omega u(x)\ \Delta^2 v(x)\ dx = \int_\Omega f(x)\ v(x)\ dx \text{ for every } v \in W^{4,p'}(\Omega) \cap W_0^{2,p'}(\Omega).$$

ii: a one-half weak solution of (1.3.1) is a function $u \in W_0^{2,p}(\Omega)$ such that the following holds

$$\int_\Omega (\Delta u(x))\ (\Delta v(x))\ dx = \int_\Omega f(x)\ v(x)\ dx \text{ for every } v \in W_0^{2,p'}(\Omega).$$

Before focusing on the behavior of the solutions, we recall here a classical result that assures the existence of solutions.

Existence of solution. Existence, uniqueness and regularity theory both for classical and strong solutions of problem (1.3.1) are well known. We refer to the classical work of [3]. The result from [3] that we will use is the following.

Theorem 1.3.3. *(Agmon, Douglis, Nirenberg) Let $\alpha \in (0,1)$ and let Ω be a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}$, with $\partial\Omega \in C^{4,\alpha}$.*

(i) If $f \in C^{\alpha}(\Omega)$ then there exists a unique classical solution $u \in C^{4,\alpha}(\Omega)$ of problem (1.3.1). Moreover the following estimate holds

$$\|u\|_{C^{4,\alpha}(\Omega)} \leq C_1 \|f\|_{C^{\alpha}(\Omega)}.$$

(ii) If $f \in L^p(\Omega)$ then there exists a unique strong solution $u \in W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega)$ of problem (1.3.1) and the following estimate holds

$$\|u\|_{W^{4,p}(\Omega)} \leq C_2 \|f\|_{L^p(\Omega)}.$$

The constants C_1 and C_2 depend on the domain Ω and on the dimension n.

For completeness we recall that another approach to find a solution to problem (1.3.1) is the variational one, that is, to find the function u in $W_0^{2,2}(\Omega)$ that minimizes the energy functional associated to the clamped plate equation, that is

$$E(u) = \int_{\Omega} \tfrac{1}{2} \left((\Delta u)^2 - (1-\sigma) \sum_{i,j=1,\ i\neq j}^{n} (u_{x_i x_i} u_{x_j x_j} - (u_{x_i x_j})^2) - fu \right) dx.$$

Such a minimizer u is a half-weak solution of problem (1.3.1). It is interesting to see that the term $u_{x_i x_i} u_{x_j x_j} - (u_{x_i x_j})^2$ appears when considering the energy functional but not in the differential equation.

In the previous section we have illustrated that problem (1.3.1) in general does not have the Positivity Preserving Property. Since the history of this problem is quite interesting, we briefly present it.

A short history of positivity for the clamped plate equation. Boggio and Hadamard at the beginning of the 20^{th} century conjectured that the Positivity Preserving Property of the clamped plate equation holds true on almost any domain. In 1905 ([8]) Boggio derived the Green function for the clamped plate equation on the unit ball in $\mathbb{R}^n$. From this explicit expression it directly follows that on the ball problem (1.3.1) is positivity preserving.

A first evidence that the conjecture of Boggio and Hadamard was not true in its full generality comes from Hadamard himself. In [47] he states, without giving a proof, that the Green function for problem (1.3.1) in an annulus is sign changing. A proof of Hadamard's claim is in [14].

For a long time the conjecture that the clamped plate equation at least in convex domain has the Positivity Preserving Property stood open. In 1949 Duffin ([31]) showed the first counterexample by an infinite strip. Numerous other counterexamples followed. We recall the one of Garabedian ([34]). He obtained that the Green function of the clamped plate equation on an ellipse with axes having ratio approximately 2 is sign changing. In 1980 Coffman and Duffin ([13]) showed that also in squares and rectangles the clamped plate equation is not positivity preserving. Results concerning the behavior in angles may be found in [58] and [14]. We would like to notice that all counterexamples concern domains in dimension two. In higher dimensions one knows that problem (1.3.1) is positivity preserving in the ball. At the author's knowledge there are no examples of other domains in dimension $n \geq 3$ where the Green function for the clamped plate equation is positivity preserving, neither are there examples of domains where the Green function changes sign.

We may say that in general the clamped plate equation is not positivity preserving. Neither convexity, nor smoothness, nor symmetries of a domain may guarantee the positivity of the Green function of (1.3.1).

Previous results in the literature. In the literature results that are called "maximum principle" for higher order equations do appear. Usually such results state an estimate for or by a functional of u. One such example is [31] where Duffin showed the following principle.

Proposition 1.3.4. *Let w be biharmonic on a region R. Let (a,b) be a point in R and let χ denote a vector with components $x-a$, $y-b$. Then it holds*

$$w(a,b) \leq \max_{\partial R} \left[w - \chi . \nabla w + |\chi|^2 \frac{\Delta w}{4} \right].$$

Nehari in [56] looked for sub-domains of a smooth domain Ω characterized by the positions of the points P and Q and by simple geometric properties of Ω in which the Green function may be shown to be positive. He proved that if the ball of center Q and of radius twice the distance between P and Q is contained in $\Omega \subset \mathbb{R}^3$, then the Green function computed in (P,Q) is positive. For a two-dimensional domain the condition is a bit more complicated. He also showed that the Green function associated to problem (1.3.1) in a smooth three-dimensional domain is more regular than the Green function associated to a smooth two-dimensional domain.

Our results. Our approach will be different. Knowing that the Green function for the clamped plate equation may change sign, we will look at "how much" it may be negative. The idea is that near the singularity the Green function is positive, while far away from the singularity the sign may change but we gain in regularity. The method consists in studying separately the local behavior. We will write the Green function as a sum of two terms: one positive and singular, the other sign-changing and regular. The aim is to separate positivity from regularity. Our results are limited to two-dimensional domains. We expect these to hold true also in higher dimensions.

The main results of the thesis are the following.

Theorem 1.3.5. *Assume that $\Omega \subset \mathbb{R}^2$ is a bounded simply connected domain with $\partial\Omega \in C^{16}$. Then there exist $G_\Omega^{reg}, G_\Omega^{sing} : \bar{\Omega}^2 \to \mathbb{R}$ such that the Green function for (1.3.1) may be written as*

$$G_\Omega(x,y) = G_\Omega^{reg}(x,y) + G_\Omega^{sing}(x,y)$$

and the following is satisfied:

(i) *(a)* $G_\Omega^{sing}(x,y) \geq 0$ *on* $\bar{\Omega}^2$;

(b) $G_\Omega^{sing} \in C^{1,\gamma}\left(\bar{\Omega}^2\right) \cap C_0^1\left(\bar{\Omega}^2\right)$ *for all* $\gamma \in (0,1)$;

(c) $G_\Omega^{sing} \in C^{15,\gamma}\left(\left\{(x,y) \in \bar{\Omega}^2; x \neq y\right\}\right)$ *for all* $\gamma \in (0,1)$;

(ii) *(a)* $G_\Omega^{reg} \in C^{15,\gamma}\left(\bar{\Omega}^2\right) \cap C_0^1\left(\bar{\Omega}^2\right)$ *for all* $\gamma \in (0,1)$.

Moreover there exist positive constants c_1 *and* c_2 *such that the following estimate holds for every* $x, y \in \Omega$

$$-c_1 d_\Omega(x)^2\, d_\Omega(y)^2 \leq G_\Omega(x,y) \leq c_2 d_\Omega(x)\, d_\Omega(y) \min\left\{1, \frac{d_\Omega(x) d_\Omega(y)}{|x-y|^2}\right\}.$$

Here d_Ω *is as in (1.2.3).*

As a consequence we are able to show the following type of maximum principle. In the statement of the next result we use negative Sobolev spaces. For definitions and properties we refer to [2].

Theorem 1.3.6. *Let* $0 < \alpha < 1$ *and* $p \in (1,\infty)$. *Let* Ω *be a bounded simply connected domain in* $\mathbb{R}^2$ *with* $\partial\Omega \in C^{4,\alpha}$ *(see Definition 2.1.3).*

Then for any $q > 2$ *and* $\varepsilon > 0$ *there exists a constant* $c_{q,\Omega,\varepsilon} > 0$ *such that for* $f \in L^p(\Omega)$ *the solution* $u \in W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega)$ *of (1.3.1) satisfies*

$$u(x) \leq c_{q,\Omega,\varepsilon}\left(\left\|f^+\right\|_{L^1(B(x,\varepsilon)\cap\Omega)} + \|u\|_{W^{-1,q}(\Omega)}\right) \text{ for every } x \in \Omega.$$

Here f^+ *denotes the positive part of* f.

1.4 Contents of the thesis

In the first chapters, namely 2 to 5, we focus on the local positivity for the clamped plate equation and we prove Theorems 1.3.5 and 1.3.6. In the last chapter, Chapter 6, we study a problem arising in the study of the Positivity Preserving Property for second order elliptic systems and that has some connections with probability.

We first present the preliminary results that lead to Theorem 1.3.5 and Theorem 1.3.6. Our goal is to prove that the sign preserving effects are much stronger than the opposite ones. Of course this is directly connected with the behavior of the Green function. We expect that the Green function associated to problem (1.3.1) does not have any singularity from below.

We start by showing sharp estimates of the absolute value of the Green function depending on the distance to the boundary. We will do this in Chapter 2 where we consider the following polyharmonic problem

$$\left\{\begin{array}{rcll} (-\Delta)^m u & = & f & \text{in } \Omega, \\ \frac{\partial^i}{\partial\nu^i} u & = & 0 & \text{for } i = 0,\ldots,m-1, \text{ on } \partial\Omega, \end{array}\right. \tag{1.4.1}$$

with Ω a bounded smooth domain in $\mathbb{R}^n$. Using the result of Krasovskiĭ in [51] we prove optimal estimates from above of the Green function (and its derivatives). The precise result is the following.

Theorem 1.4.1. *Let $G_m(x,y)$ be the Green function associated to problem (1.4.1) in a bounded domain $\Omega \subset \mathbb{R}^n$ with $\partial\Omega \in C^{6m+4}$ if $n = 2$ or $\partial\Omega \in C^{5m+2}$ if $n \geq 3$.*

Then the following estimates hold for every $x, y \in \Omega$:

(i) if $2m - n > 0$, then

$$|G_m(x,y)| \leq c_1 d_\Omega(x)^{m-\frac{1}{2}n}\, d_\Omega(y)^{m-\frac{1}{2}n} \min\left\{1, \frac{d_\Omega(x)d_\Omega(y)}{|x-y|^2}\right\}^{\frac{1}{2}n},$$

(ii) if $2m - n = 0$, then

$$|G_m(x,y)| \leq c_2 \log\left(1 + \left(\frac{d_\Omega(x)d_\Omega(y)}{|x-y|^2}\right)^m\right),$$

(iii) if $2m - n < 0$, then

$$|G_m(x,y)| \leq c_3\, |x-y|^{2m-n} \min\left\{1, \frac{d_\Omega(x)d_\Omega(y)}{|x-y|^2}\right\}^m,$$

with c_1, c_2 and c_3 positive constants and with d_Ω defined as in (1.2.3).

This kind of estimates is a useful tool to prove regularity results in spaces involving the behavior at the boundary. Indeed, a direct consequence of Theorem 1.4.1 is that the solution u of (1.3.1) in a domain $\Omega \subset \mathbb{R}^n$, with $n = 2, 3$ satisfies for appropriate f:

$$\left\| d_\Omega(.)^{-2+\theta n} u \right\|_{L^\infty(\Omega)} \leq C^1_{\Omega,2} \left\| d_\Omega(.)^{2-(1-\theta)n} f \right\|_{L^1(\Omega)} \text{ for all } \theta \in [0,1].$$

The main tool for the proof of Theorem 1.4.1 is the result of Krasovskiĭ in [51] . He proves the existence of the Green function associated to problem (1.4.1) and he also gives estimates for the absolute value of this Green function. For the precise statement of the result of Krasovskiĭ see Section 2.3.

A previous result concerning estimates of the Green function in terms of the distance to the boundary is to be found in [41]. In this paper the authors study the Green function associated to the polyharmonic problem with Dirichlet boundary conditions on the unit ball in $\mathbb{R}^n$. They start from the explicit formula of Boggio. They proved optimal two-sided estimates of the Green function depending on the distance to the boundary.

The estimates in Theorem 1.4.1 give a first understanding on how the singularity of the Green functions behaves in relation with the special boundary conditions of problem (1.3.1). These estimates are sharp from above (that is, for positive values of the Green function) but not from below (that is, for negative values of the Green function). The only known sharp estimates from below of the Green function are the one in [41] for the ball. The crucial fact used in that paper is the knowledge of the explicit formula of the Green function due to Boggio [8]. Hence, for general domains we have to find another method.

The idea is to cover a general domain Ω with a finite number of sub-domains that have a positive Green function. Then we will compare the solution of the clamped plate equations in the domain Ω with the sum (with a partition of unity) of the solutions of the clamped plate in each sub-domain.

First we have to find a suitable finite covering of the domain Ω with its boundary: suppose that $\Omega = \cup_{j=1}^{N} E_j$. By suitable we mean that ∂E_j is as regular as $\partial\Omega$, $\partial\Omega \subset \cup_{j=1}^{N} \partial E_j$ and that each E_j is such that the clamped plate equation is positivity preserving in E_j.

So far we have seen that problem (1.3.1) is positivity preserving on balls. This result is not sufficient since a general domain cannot be covered with the boundary by a finite number of balls. In [40] Grunau and Sweers show that in two dimensions on domains that are small $C^{2,\gamma}$ perturbations of the ball the clamped plate equation is positivity preserving. Since the smallness of these perturbation is defined in $C^{2,\gamma}$-norm all these domains are necessarily convex. Also in this case this result is not sufficient since in general a non-convex domain cannot be covered with its boundary by a finite number of convex domains.

The question arises if there are examples of non-convex domains on which the clamped plate equation has the Positivity Preserving Property. Hadamard in [47] states that this property holds for the clamped plate equation on plates having the shape of a Limaçon de Pascal. In this context, the term Limaçon de Pascal refers to a generic element of the family of domains described by the parameter $a \in [0, \frac{1}{2}]$ given by

$$\Omega_a = \left\{ (\rho\cos\varphi, \sin\varphi) \in \mathbb{R}^2 : 0 \le \rho < 1 + 2a\cos\varphi,\ \varphi \in [0, 2\pi) \right\}.$$

These Ω_a are smooth for $a \in [0, \frac{1}{2})$ and convex for $a \in [0, \frac{1}{4}]$. In Chapter 3 we show that the statement of Hadamard is wrong in its full generality but that however, there are non-convex limaçons on which the clamped plate equation has the Positivity Preserving Property.

Theorem 1.4.2. *The clamped plate problem on Ω_a with $a \in \left[0, \frac{1}{2}\right]$ is positivity preserving if and only if $a \in \left[0, \frac{1}{6}\sqrt{6}\right]$.*

The main tool used in the proof is the knowledge of the explicit formula for the Green function for the clamped plate equation on Ω_a. This supplies also the proof of the following optimal estimates from below:

(i) for $a \in [0, \frac{1}{6}\sqrt{6}]$:

$$G_{\Omega_a}(x,y) \ge c_1 \left(\tfrac{1}{6}\sqrt{6} - a \right) d_{\Omega_a}(x)\, d_{\Omega_a}(y)\ \min\left\{ 1, \frac{d_{\Omega_a}(x) d_{\Omega_a}(y)}{|x-y|^2} \right\};$$

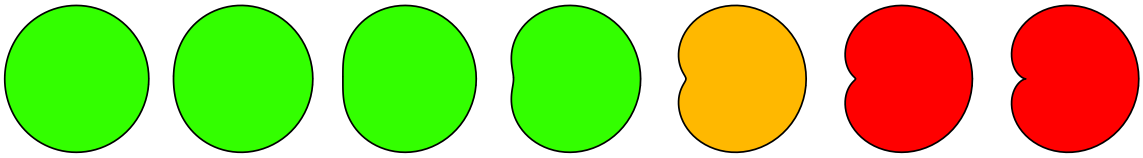

Figure 1.9: *Limaçons for respectively* $a = .1, .175, .25, .325, \frac{1}{6}\sqrt{6}, .45, .5$. *The third one is critical for convexity. The fifth one is critical for the Positivity Preserving Property. For the two last limaçons there are positive* f *with the corresponding* u *solution of the clamped plate equation with* f *in the right hand side that is negative somewhere. This figure has been taken from [26].*

(ii) for $a \in (\frac{1}{6}\sqrt{6}, \bar{a}]$ with $\bar{a} < \frac{1}{2}$:

$$G_{\Omega_a}(x,y) \geq -c_2\left(a - \tfrac{1}{6}\sqrt{6}\right) d_{\Omega_a}(x)^2\, d_{\Omega_a}(y)^2, \tag{1.4.2}$$

with c_1 and c_2 positive constants. We would like to recall that a natural solution space for the clamped plate equation is the Banach lattice

$$\mathcal{C}_e(\bar{\Omega}) := \left\{u \in C(\bar{\Omega}) : \|u\|_e := \sup_{x\in\Omega} \frac{|u(x)|}{d_\Omega^2(x)}\right\}.$$

The estimate in (1.4.2) shows that the function $x \mapsto G_{\Omega_a}(x,y)$ from $\bar{\Omega}_a$ onto $\mathcal{C}_e(\bar{\Omega}_a)$ does not have a singular behavior from below when x goes to the boundary of Ω.

We have now found a family of domains, some convex and others non-convex, on which the Positivity Preserving Property holds. In order to approximate a general domain Ω with sub-domains, we can consider scaled limaçons and we can "play" with the parameter a, for $a \in [0, \bar{a}]$ with $\bar{a} < \frac{1}{6}\sqrt{6}$. However, in order to approximate with the boundary the domain Ω we need a further step. Indeed, choosing the appropriate a and the appropriate scaled limaçon $\Omega_{a,R}$ we can approximate the domain Ω in a boundary point in C^2-sense, but we want it to be equal in a neighborhood of the point. In order to do that, we consider $C^{2,\gamma}$-perturbations of the limaçon Ω_a for $a \in [0, \bar{a}]$ with $\bar{a} < \frac{1}{6}\sqrt{6}$. Using the method in [40] we prove that on these domains the Positivity Preserving Property for the clamped plate equation holds. Moreover, the Green function on these $C^{2,\gamma}$-perturbations of the limaçons is strictly positive in the following sense. Let Ω^* be a $C^{2,\gamma}$-perturbation of a limaçon Ω_a with $a \in [0, \bar{a}]$, $\bar{a} < \frac{1}{6}\sqrt{6}$, then there exists a positive constant c_1 such that for $x, y \in \Omega^*$

$$G_{\Omega^*}(x,y) \geq c_1\left(\tfrac{1}{6}\sqrt{6} - a\right) d_{\Omega^*}(x)\, d_{\Omega^*}(y)\, \min\left\{1, \frac{d_{\Omega^*}(x) d_{\Omega^*}(y)}{|x-y|^2}\right\}.$$

In Chapter 4 one may find the methods presently available to get domains on which the clamped plate equation is positivity preserving. One notices that there is a big difference between the two-dimensional case and the higher-dimensional one. This is due to the fact that the only transformations that do keep the highest order terms polyharmonic are the conformal mappings. In two dimensions there are many of these mappings while in higher dimensions the only conformal mappings are the Möbius transformations.

Using the results of the previous chapters, we show in Chapter 5 that a general $C^{4,\alpha}$ domain Ω can be covered with the boundary by a finite number of sub-domains that are scaled $C^{2,\gamma}$-perturbations of limaçons. For the precise statement of the result see Theorem 5.4.29.

Thanks to this covering we will be able to split the Green function as the sum of a positive term and a sign-changing regular one. This is indeed the result stated in two closely related versions in Theorems 1.3.5 and 1.3.6. We explain now roughly the method that has been used in the proof of these theorems. First, working with a partition of unity we solve the clamped plate equation on each element E_j of the covering of Ω with an appropriate right hand side and with zero boundary conditions. The solution of this problem may be considered as a local approximation of the solution of the clamped plate equation in Ω. What is relevant is that in E_j the Positivity Preserving Property holds and that we have optimal estimates of the Green function both from above and from below. Summing up this local solution via a partition of unity we get a function defined in all the domain Ω and that satisfies the boundary condition of (1.3.1). The solution u of the clamped plate equation in Ω with f in the right hand side can be written as the sum of this function and another term coming from 'patching up' the domains. This component is smooth. In terms of the Green function this reads as: the Green function can be written as the sum of a positive term that gives the local behavior, with a term, that could be sign-changing, that only indirectly depends on the local behavior and hence, that is regular.

The topic of Chapter 6 differs somewhat from the other ones. Studying positivity for general elliptic boundary value problems we encountered some open problems in probability theory.

Some interesting questions concerning positivity arise also in the study of systems of second order elliptic boundary value problems. In [66] the following system is presented as a model problem for the positivity preserving property of systems coupled in a non-cooperative way

$$\left\{\begin{array}{rcll} -\Delta u &=& f-\lambda v & \text{in } \Omega,\\ -\Delta v &=& f & \text{in } \Omega,\\ u=v &=& 0 & \text{on } \partial\Omega. \end{array}\right. \tag{1.4.3}$$

Here Ω is a bounded regular subset of $\mathbb{R}^n$ and $\lambda \in \mathbb{R}^+$. One can show that there exists a value $\lambda_c(\Omega) \in (0,\infty)$ such that for all $f \geq 0$ the solution u of problem (1.4.3) is positive if and only if $\lambda \leq \lambda_c(\Omega)$. The value $\lambda_c(\Omega)$ is defined as follows

$$\lambda_c(\Omega)^{-1} = \sup_{x,y\in\Omega} H_\Omega(x,y),$$

where

$$H_\Omega(x,y) = \int_\Omega \frac{G_\Omega\,(x,z)\,G_\Omega\,(z,y)}{G_\Omega\,(x,y)} dz.$$

Here G_Ω the Green function for

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

Notice that the function $H_\Omega(x,y)$, defined above, is equal to the quotient of the Green function for the supported plate and the one for the membrane.

The function $H_\Omega(x,y)$ has also a probabilistic interpretation. Indeed, it is equal to $\mathbb{E}_x^y(\tau_\Omega)$ the expected lifetime of a conditioned Brownian motion that starts in x, is conditioned to converge to y and that is killed at the boundary of Ω.

There are many open problems regarding this function. In particular, one open question is where the function $H_\Omega(x,y)$ attains its maximum in $\bar{\Omega}\times\bar{\Omega}$. In [37] Griffin, McConnell and Verchota showed that

$$\sup_{x\in\bar{\Omega},\ y\in\partial\Omega} H_\Omega(x,y) \leq \sup_{x,y\in\partial\Omega} H_\Omega(x,y),$$

with Ω a general simply connected domain in $\mathbb{R}^2$. The main tools used in the proof are series expansions and a conformal map that transforms the problem from the general Ω to the unit disk.

In Chapter 6 we study this problem with Ω the unit ball in $\mathbb{R}^n$, $n \geq 2$. What we are interested in is the term $\sup_{x,y\in\bar{\Omega}} H_\Omega(x,y)$. Since now both points might be in the interior the method of Griffin, McConnell and Verchota cannot be used. Our main tool will be the Maximum Principle.

The main result of the chapter is the following.

Theorem 1.4.3. *Let Ω be the unit ball in $\mathbb{R}^n$, $n \geq 2$. For all $y \in \bar{\Omega}$ the function $x \mapsto H_\Omega\,(x,y)$ is*

(i) increasing along 'the hyperbolic geodesics through y' *in increasing Euclidean distance;*

(ii) increasing along the orthogonal trajectories of 'the hyperbolic geodesics through y' *in increasing Euclidean distance.*

A direct consequence of the theorem is that the maximum is attained at opposite boundary points. Moreover, although it shows that the function $H_\Omega(.,y)$ is increasing along the hyperbolic geodesics through y, it also shows that these are not the best increasing directions.

At the end of the chapter we compute the explicit formula for $\lambda_c(\Omega)^{-1}$ and we discuss some remarkable identities involving $\sup_{x,y\in\bar{\Omega}} H_\Omega(x,y)$ and a sum of inverse Dirichlet eigenvalues.

1.5 Ideas for future research

In the last part of this introduction we present some possible extensions and open problems.

1. In Chapter 3 we show that the positivity of the Green function associated to the clamped plate equation on the Limaçon de Pascal is equivalent with the positivity of minus the Bergman kernel on the same domain. It would be interesting to see if the limaçon is a special case or if this result is true in more generality, [39].

2. To the author's knowledge the ball is the only domain in dimension $n \geq 3$ where the Green function for the generalized clamped plate equation is known to be positive. Can one find other domains in $\mathbb{R}^n$ for $n \geq 3$ on which the positivity preserving property holds?

3. It is my dream to be able to generalize the "maximum principle" type result of Chapter 5 to the higher dimensional case. This would be an useful tool in the study of the semilinear problem associated to the generalized clamped plate equation in dimension $n \geq 4$.

Indeed, considering the fourth order boundary value problem

$$\begin{cases} \Delta^2 u + g(x,u) &= f \quad \text{in } \Omega, \\ u = \frac{\partial}{\partial\nu} u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.5.1}$$

by analogy with the second order equation one may think that the hypothesis

$$g(x,t) \cdot t \geq 0 \text{ for every } x \in \Omega \text{ and } t \in \mathbb{R},$$

is sufficient to prove existence of regular solution even without assuming sub-critical growth. This result will depend strongly on a generalization of the "maximum principle" type result of Chapter 5 to the higher dimensional case. See [38] and [53].

4. In Chapter 6 we study where the maximum of $\mathbb{E}_x^y(\tau_\Omega)$ is attained when Ω is the unit ball in $\mathbb{R}^n$. The question is still completely open for a general domain Ω.

5. A direct consequence of the main result in Chapter 6 is that the hyperbolic geodesic through y are not the best "increasing direction" of $x \mapsto \mathbb{E}_x^y(\tau_\Omega)$. It would be interesting to find the curves that give the best increasing direction of $\mathbb{E}_x^y(\tau_\Omega)$ and if there exists a metric such that the best increasing direction of $\mathbb{E}_x^y(\tau_\Omega)$ are geodesics in this metric. See [5].

Chapter 2

Estimates of the Green function

2.1 Introduction

In this chapter we present optimal pointwise estimates for the kernels associated to the following higher order Dirichlet boundary value problem

$$\begin{cases} (-\Delta)^m\, u = \varphi & \text{in } \Omega, \\ u = \psi_0 & \text{on } \partial\Omega, \\ \frac{\partial}{\partial\nu} u = \psi_1 & \text{on } \partial\Omega, \\ \dots & \dots \\ (\frac{\partial}{\partial\nu})^{m-1} u = \psi_{m-1} & \text{on } \partial\Omega, \end{cases} \tag{2.1.1}$$

where $m \in \mathbb{N}^+$ and Ω is an open bounded connected subset of $\mathbb{R}^n$, $n \geq 2$. The regularity of the boundary that we assume depends on the dimension n: for $n = 2$ we assume $\partial\Omega \in C^{6m+4}$ and for $n \geq 3$ $\partial\Omega \in C^{5m+2}$ (see Definition 2.1.3). The Green function G_m and the Poisson kernels K_j are such that the solution of problem (2.1.1), for appropriate φ and ψ_j, can be written as

$$u(x) = \int_\Omega G_m(x,y)\varphi(y)\,dy + \sum_{j=0}^{m-1} \int_{\partial\Omega} K_j(x,y)\psi_j(y)\,d\sigma_y.$$

Our aim will be to prove estimates from above of G_m and K_j depending on the distance to the boundary. For example when $m = 2$ and $n = 2$ we will prove that there is a constant c_Ω such that

$$|G_2(x,y)| \leq c_\Omega d_\Omega(x) d_\Omega(y) \min\left\{1, \frac{d_\Omega(x) d_\Omega(y)}{|x-y|^2}\right\}, \tag{2.1.2}$$

where d_Ω is the distance of x to the boundary $\partial\Omega$:

$$d_\Omega(x) := \inf_{\tilde{x} \in \partial\Omega} |x - \tilde{x}|. \tag{2.1.3}$$

For the sake of an easy statement we have used $L = (-\Delta)^m$ in system (2.1.1) but in fact the estimates that we will derive hold for any uniformly elliptic operator L of order $2m$.

We will focus on the estimates for G_m and K_j. However, we would like to mention that those estimates are the optimal tools for deriving regularity results in spaces that involve the behavior at the boundary. Coming back to the case $m = n = 2$ it follows from (2.1.2) that the solution u of

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega \subset \mathbb{R}^2, \\ u = \frac{\partial}{\partial \nu} u = 0 & \text{on } \partial\Omega, \end{cases}$$

satisfies for appropriate f

$$\left\| \frac{u}{d_\Omega^2} \right\|_{L^\infty(\Omega)} \le c_\Omega \left\| f \right\|_{L^1(\Omega)} \text{ and } \left\| u \right\|_{L^\infty(\Omega)} \le c_\Omega \left\| f \; d_\Omega^2 \right\|_{L^1(\Omega)}.$$

These kinds of estimates, for general m and n, and also L^p-L^q estimates will be addressed in Section 2.7. The estimates are interesting by their own merits. A special case for $m = 1$ appears in [33].

Not only we will derive estimates for those kernels but also for their derivatives. The main tool will be the result of Krasovskiĭ in [51] where he considered general elliptic operators and boundary conditions. The estimates he derived did not involve special growth rates near the boundary. We instead will focus on estimates that contain growth rates near the boundary. These estimates seem to be optimal and indeed, when we consider G_m for $\Omega = B$ a ball in $\mathbb{R}^n$ the growth rates near the boundary are sharp (see e.g. [45]).

For $m = 1$ or $m \ge 2$ and $\Omega = B$ it is known that the Green function is positive and can even be estimated from below by a positive function with the same singular behavior (see [41]). Let us remind the reader that for $m \ge 2$ the Green function in general is not positive. We believe, however that for general domains the optimal behavior in absolute values is captured in our estimates. Sharp estimates for K_{m-1} and K_{m-2} in case of a ball can be found in [43].

Instead of using Krasovskiĭ's result one might use appropriate "heat kernel" estimates. Indeed, integrating pointwise estimates for the parabolic kernel $p(t, x, y)$ with respect to t from 0 to ∞, yields pointwise estimates for the Green function. However, only limited results seem to be available. Barbatis [6] considered higher order parabolic problems on domains and derived pointwise estimates for the kernel using a non-Euclidean metric. Classical estimates by Eidel'man (see e.g. [32]) for higher order parabolic systems do not consider domains with boundary.

For a survey on spectral theory of higher order elliptic operators, including some estimates for the corresponding kernels, we refer to [27].

Finally we would like to remark that we do not pretend that our pointwise estimates are completely new. However we have not been able to find any reference to such estimates for the special type of boundary conditions above.

The chapter is organized as follows. We will complete the first section fixing some notation and giving the main results. We then recall some general properties of the Green function and Poisson kernels for (2.1.1), the results of Krasovskiĭ and some technical lemmas. In the fifth and sixth sections we prove the estimates of the Green function and of the Poisson kernels respectively. In the last section we give some estimates of the solution of (2.1.1) with zero boundary conditions in terms of the distance to the boundary.

2.1.1 Preliminaries and main results

Before stating the main results we fix some notations.

Notation 2.1.1. *(See Grunau and Sweers [41]) Let f and g be functions defined on $\Omega \times \Omega$ with $g \geq 0$. Then we call $f \sim g$ on $\Omega \times \Omega$ if and only if there are $c_1, c_2 > 0$ such that*

$$c_1 f(x,y) \leq g(x,y) \leq c_2 f(x,y) \text{ for all } x, y \in \Omega.$$

We will say $f \preceq g$ on $\Omega \times \Omega$ if and only if there is $c > 0$ such that

$$f(x,y) \leq c\ g(x,y) \text{ for all } x, y \in \Omega.$$

Notation 2.1.2. *Let f a function defined on $\Omega \times \Omega$ and $\alpha, \beta \in \mathbb{N}^n$. Derivatives are denoted*

$$D_x^\alpha D_y^\beta f(x,y) = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} x_2^{\alpha_2} .. \ x_n^{\alpha_n}} \frac{\partial^{|\beta|}}{\partial y_1^{\beta_1} y_2^{\beta_2} .. \ y_n^{\beta_n}} f(x,y),$$

where $|\alpha| = \sum_{k=1}^n \alpha_k$.

In the literature several definitions of $C^{\ell,\alpha}$-domains appear. To avoid any ambiguity we explicitly give the version that we will use in this chapter and also in the following ones.

Definition 2.1.3 (Uniform $C^{\ell,\alpha}$ regularity condition for Ω)**.** *Let $\alpha \in [0,1]$, $\ell \in \mathbb{N}^+$ and Ω be a bounded domain in $\mathbb{R}^n$. The domain Ω satisfies the uniform $C^{\ell,\alpha}$ regularity condition (we write $\partial\Omega \in C^{\ell,\alpha}$) if there exist a positive constant M, a finite open covering $\{U_j\}_{j\in J}$ of $\partial\Omega$, a corresponding collection $\{\varphi_j\}_{j\in J}$ of $C^{\ell,\alpha}$ mappings such that for every $j \in J$:*

(i) $\varphi_j : U_j \to B = \{y \in \mathbb{R}^n : |y| < 1\}$ is a bijection; set $\psi_j = \varphi_j^{inv}$;

(ii) with $(\varphi_{j,1}, \ldots, \varphi_{j,n})$ and $(\psi_{j,1}, \ldots, \psi_{j,n})$ the components of φ_j and ψ_j :

$$\|\varphi_{j,i}\|_{C^{\ell,\alpha}(\bar{U}_j)} \leq M \text{ and } \|\psi_{j,i}\|_{C^{\ell,\alpha}(\bar{B})} \leq M \text{ for } i = 1, \ldots, n;$$

(iii) it holds that $\varphi_j(U_j \cap \Omega) = \{y \in B : y_n > 0\}$;

and moreover, there exists $\delta > 0$ such that

$$\{x \in \Omega : d_\Omega(x) < \delta\} \subset \bigcup_{j \in J} \psi_j \left(\{y \in \mathbb{R}^n : |y| < \tfrac{1}{2}\} \right).$$

Definition 2.1.3 is similar to the uniform C^ℓ regularity condition in [2, Def.4.10 page 84]. In the following $\partial\Omega \in C^\ell$ denotes $\partial\Omega \in C^{\ell,0}$.

We are now ready to state the main results of the chapter.

Theorem 2.1.4. *Let $G_m(x,y)$ be the Green function associated to system (2.1.1) and let $\partial\Omega \in C^{6m+4}$ if $n = 2$ and $\partial\Omega \in C^{5m+2}$ otherwise. The following estimates hold for every $x, y \in \Omega$:*

(i) if $2m - n > 0$, then

$$|G_m(x,y)| \preceq d_\Omega(x)^{m-\frac{1}{2}n}\, d_\Omega(y)^{m-\frac{1}{2}n} \min\left\{1, \frac{d_\Omega(x) d_\Omega(y)}{|x-y|^2}\right\}^{\frac{1}{2}n},$$

(ii) if $2m - n = 0$, then

$$|G_m(x,y)| \preceq \log\left(1 + \left(\frac{d_\Omega(x) d_\Omega(y)}{|x-y|^2}\right)^m\right),$$

(iii) if $2m - n < 0$, then

$$|G_m(x,y)| \preceq |x-y|^{2m-n} \min\left\{1, \frac{d_\Omega(x) d_\Omega(y)}{|x-y|^2}\right\}^m.$$

Theorem 2.1.5. *Let $K_j(x,y)$, for $j = 0, ..., m-1$, be the Poisson kernels associated to system (2.1.1). Suppose furthermore that $\partial\Omega \in C^{6m+4}$ if $n = 2$ and $\partial\Omega \in C^{5m+2}$ if $n \geq 3$. Then the following estimate holds for every $x \in \Omega$ and $y \in \partial\Omega$*

$$|K_j(y,x)| \preceq \frac{d_\Omega(x)^m}{|x-y|^{n-j+m-1}}. \tag{2.1.4}$$

Remark 2.1.6. If $n - 1 < j \leq m - 1$ inequality (2.1.4) gives that on $\partial\Omega \times \Omega$

$$|K_j(y,x)| \preceq d_\Omega(x)^{1+j-n}.$$

The estimates in Theorems 2.1.4 and 2.1.5 hold for $(-\Delta)^m$ replaced by any uniformly elliptic operator of order $2m$. Indeed, the main ingredients are the Dirichlet boundary condition and the estimates of Krasovskiĭ. In the proof one has to use the Dirichlet boundary condition both for the original and the adjoint problem. Although the adjoint problem is different for general elliptic problems the Dirichlet boundary condition will remain. Notice that Krasovskiĭ's derived the estimates for the general case.

In [41] the estimates as in Theorem 2.1.4 are given for the case that Ω is a ball in $\mathbb{R}^n$. There the authors could use the explicit formula of G_m given by Boggio in [8]. We recall that for balls the Green function associated to problem (2.1.1) is positive.

For general domains one cannot expect an explicit formula and instead we will proceed by the estimates of Krasovskiĭ for G_m and K_j given in [51]. For sufficiently regular domains Ω (see Section 2.3) he first proves that the Green function and the Poisson kernels exist and then he gives estimates for these functions.

Our aim will be to prove estimates from above of G_m and K_j depending on the distance to the boundary. We will do so by estimating the j-th derivative through an integration of the $(j+1)$-th derivative along a path to the boundary. The dependence on the distance to the boundary $d_\Omega(x)$ will appear choosing a path which length is proportional to $d_\Omega(x)$. The path will be constructed explicitly in Lemma 2.5.1.

2.2 Green function and Poisson kernels

In this section we recall some of the well known properties of the Green function and the Poisson kernels.

The Green function for (2.1.1)
This function $G_m : \Omega \times \Omega \to \mathbb{R}$ is such that for every $y \in \Omega$ the mapping $x \mapsto G(x,y)$ satisfies (in the sense of distribution)

$$\begin{cases} (-\Delta)^m G_m(\cdot, y) & = & \delta_y(\cdot) & \text{in } \Omega, \\ \left(\frac{\partial}{\partial\nu}\right)^j G_m(\cdot, y) & = & 0 & \text{on } \partial\Omega,\ j = 0, ..., m-1. \end{cases} \tag{2.2.1}$$

Since $(-\Delta)^m$ is selfadjoint on $W^{2m,2}(\Omega) \cap W_0^{m,2}(\Omega) \subset L^2(\Omega)$, the Green function is symmetric. Observe that for $y \in \Omega$ identity (2.2.1) gives for $|s| \le m-1$

$$D_x^s G(x,y) = 0 \text{ for } x \in \partial\Omega. \tag{2.2.2}$$

In fact, taking $j = 0$ in (2.2.1) one finds that $x \mapsto G_m(x,y)$ for $y \in \Omega$ is zero at the boundary. Hence the tangential derivatives of $x \mapsto G_m(x,y)$ of any order, for $y \in \Omega$, are identically zero on $\partial\Omega$. Since the normal derivatives up to order $m-1$ are zero at the boundary, (2.2.2) follows.

The function G_m has a singular behavior on $D_\Omega := \left\{(x,x) : x \in \bar{\Omega}\right\}$. Assuming that $\partial\Omega$ is $C^{4,\alpha}$ one finds that G_m belongs to $C^{4,\alpha}\left(\left(\bar{\Omega}\times\bar{\Omega}\right)\setminus D_\Omega\right)$ and also to $C^\infty((\Omega\times\Omega)\setminus D_\Omega)$.

The Poisson kernels for (2.1.1)
For $j = 0, ..., m-1$, and $y \in \partial\Omega$ the functions $x \mapsto K_j(y,x)$ satisfy (in the sense of distribution)

$$\begin{cases} (-\Delta)^m K_j(y,\cdot) & = & 0 & \text{in } \Omega, \\ \left(\frac{\partial}{\partial\nu}\right)^k K_j(y,\cdot) & = & 0 & \text{on } \partial\Omega,\ \text{for } k \ne j,\ 0 \le k \le m-1, \\ \left(\frac{\partial}{\partial\nu}\right)^j K_j(y,\cdot) & = & \delta_{y,\partial\Omega}(\cdot) & \text{on } \partial\Omega, \end{cases} \tag{2.2.3}$$

where $\delta_{y,\partial\Omega}$ is the delta-function defined on $\partial\Omega$ (that is, the delta-function on an $(n-1)$-dimensional manifold). Moreover, the kernels satisfy for $|s| \leq m-1$ and $j = 0, ..., m-1$

$$D_x^s K_j(y,x) = 0, \text{ for } x, y \in \partial\Omega, x \neq y. \tag{2.2.4}$$

In fact, the mappings $x \mapsto K_j(y,x)$ on $\bar{\Omega} \setminus \{y\}$ with $j = 0, ..., m-1$ are zero on $\partial\Omega \setminus \{y\}$. Hence the tangential derivatives of any order are zero on $\partial\Omega \setminus \{y\}$. Since (2.2.3) implies that the normal derivatives up to order $m-1$ are zero, we find (2.2.4).

The kernels K_j have a singular behavior on $D_{\partial\Omega} = \{(x,x) : x \in \partial\Omega\}$. Assuming that $\partial\Omega$ is C^∞ one finds that K_j belong to $C^\infty\left(\left(\bar{\Omega} \times \bar{\Omega}\right) \setminus D_{\partial\Omega}\right)$.

By an integration by part and by using the explicit order of the singularities of the Green function (for instance from the result of Krasovskiĭ in [51]), one can explicitly write the relation between the Poisson kernels and the Green function. Namely for $j \in \{0, ..., m-1\}$ and y in $\partial\Omega$ the following relation holds in Ω

$$K_j(y,x) = \begin{cases} \frac{\partial}{\partial\nu_y} \left(-\Delta_y\right)^{m-\left(\frac{j}{2}+1\right)} G(x,y) & \text{for } j \text{ even}, \\ \left(-\Delta_y\right)^{m-\frac{j+1}{2}} G(x,y) & \text{for } j \text{ odd}, \end{cases}$$

where ν_y denotes the external normal to $\partial\Omega$ in y.

2.3 The estimates of Krasovskiĭ

We will now recall the theorem in [51] which gives the estimates of the Green function and the Poisson kernels. We first give the main assumption.

Consider the boundary value problem

$$\begin{cases} \mathcal{L}u = \varphi & \text{in } \Omega, \\ B_j u = \psi_j & \text{on } \partial\Omega \text{ for } j = 0, ..., m-1. \end{cases} \tag{2.3.1}$$

The following hypothesis are assumed.

(i) The operator

$$\mathcal{L} := \sum_{|\beta| \leq 2m} a_\beta(x) D^\beta,$$

is uniformly elliptic (see the condition for $\mathcal{L}$ on page 663 of [3]).

(ii) The boundary operators

$$B_j = \sum_{|\beta| \leq m_j} b_{j\beta}(x) D^\beta, \text{ for } j = 0, ..., m-1,$$

satisfy the complementing condition relative to $\mathcal{L}$ (see the complementing condition on page 663 of [3]).

(iii) Let $l_1 > \max_j(2m - m_j)$ and $l_0 = \max_j(2m - m_j)$. The coefficients a_β belong to $C^{l_1+1}\left(\bar{\Omega}\right)$ and $b_{j\beta}$ belong to $C^{l_1+1}(\partial\Omega)$;

(iv) The boundary $\partial\Omega$ is C^{l_1+2m+1}.

Theorem 2.3.1. *Let the condition above be satisfied and let l_1 be such that $l_1 > 2(l_0+1)$ for $n = 2$ and $l_1 > \frac{3}{2}l_0$ for $n \geq 3$. If problem (2.3.1) is uniquely solvable then the Green function G_m and the Poisson kernels K_j, with $j = 0, ..., m-1$, for (2.3.1) exist.*

Theorem 2.3.2. *Assume that the conditions of Theorem 2.3.1 are satisfied. Moreover let $\alpha, \beta, \gamma \in \mathbb{N}^n$ with $|\alpha| \leq 2m + l_1 - l_0$, $|\beta| \leq l_1$ and $|\gamma| \leq l_1 - 2m + m_j + 1$.*

Then wherever they are defined, the derivatives of the Green function G_m satisfy:

(i) if $|\alpha| + |\beta| < 2m - n$ then

$$\left|D_x^\alpha D_y^\beta G_m(x,y)\right| \preceq 1,$$

(ii) if $|\alpha| + |\beta| = 2m - n$ then

$$\left|D_x^\alpha D_y^\beta G_m(x,y)\right| \preceq \log\left(\frac{2\,\mathrm{diam}_\Omega}{|x-y|}\right),$$

(iii) if $|\alpha| + |\beta| > 2m - n$ then

$$\left|D_x^\alpha D_y^\beta G_m(x,y)\right| \preceq |x-y|^{2m-n-|\alpha|-|\beta|},$$

and the derivatives of K_j satisfy on $\partial\Omega \times \Omega$:

(i) if $|\alpha| + |\gamma| < m_j - n + 1$ then

$$\left|D_x^\alpha D_y^\gamma K_j(y,x)\right| \preceq 1,$$

(ii) if $|\alpha| + |\gamma| = m_j - n + 1$ then

$$\left|D_x^\alpha D_y^\gamma K_j(y,x)\right| \preceq \log\left(\frac{2\,\mathrm{diam}_\Omega}{|x-y|}\right),$$

(iii) if $|\alpha| + |\gamma| > m_j - n + 1$ then

$$\left|D_x^\alpha D_y^\gamma K_j(y,x)\right| \preceq |x-y|^{m_j-n+1-|\alpha|-|\gamma|}.$$

Here diam_Ω *denotes the diameter of* Ω.

Remark 2.3.3. In case of Dirichlet boundary conditions, hence $l_0 = 2m$, the conditions on l_1 are:

$$\begin{aligned}&\text{for } n \geq 3: \quad l_1 > 3m,\\ &\text{for } n = 2: \quad l_1 > 4m + 2.\end{aligned}$$

Hence one needs $\partial\Omega \in C^{6m+4}$ for $n = 2$ and $\partial\Omega \in C^{5m+2}$ for $n \geq 3$.

Krasovskiĭ has quite strong assumptions on the regularity of the boundary of Ω. This is also due to the fact that he works with general elliptic operators and boundary conditions. One may think that when $\mathcal{L} = (-\Delta)^m$ and with Dirichlet boundary conditions, it would be sufficient the hypothesis $\partial\Omega \in C^{2m,\alpha}$ for $\alpha \in (0,1)$. The assumptions that we have on the boundary of Ω are the ones needed to use the result of Krasovskiĭ.

2.4 Some technical lemmas

In the proof of Theorems 2.1.4 and 2.1.5 we will use some relations involving the term $\min\left\{\frac{d_\Omega(x)d_\Omega(y)}{|x-y|^2}\right\}$. These relations have been studied in [41]. For the sake of convenience we recall here some of their results.

Lemma 2.4.1. *If* $|x-y| \leq \frac{1}{2}\max\{d_\Omega(x), d_\Omega(y)\}$ *then it holds*

$$\tfrac{1}{2}d_\Omega(x) \leq d_\Omega(y) \leq 2d_\Omega(x) \ \text{and}\ 1 \leq \frac{d_\Omega(x)d_\Omega(y)}{|x-y|^2}.$$

Otherwise if $|x-y| \geq \frac{1}{2}\max\{d_\Omega(x), d_\Omega(y)\}$ *then it holds*

$$\frac{d_\Omega(x)}{|x-y|} \leq 2,\ \frac{d_\Omega(y)}{|x-y|} \leq 2 \ \text{and}\ \frac{d_\Omega(x)d_\Omega(y)}{|x-y|^2} \leq 4.$$

Lemma 2.4.2. *Let* $p, q \geq 0$. *The following relations hold on* $\Omega \times \Omega$*:*

$$\begin{aligned}
&i: && \min\left\{1, \tfrac{d_\Omega(x)^p d_\Omega(y)^q}{|x-y|^{p+q}}\right\} \sim \min\left\{1, \tfrac{d_\Omega(x)^p}{|x-y|^p}\right\}\min\left\{1, \tfrac{d_\Omega(y)^q}{|x-y|^q}\right\},\\
&ii: && \log\left(1 + \tfrac{d_\Omega(x)^p d_\Omega(y)^q}{|x-y|^{p+q}}\right) \sim \log\left(2 + \tfrac{d_\Omega(x)}{|x-y|}\right)\min\left\{1, \tfrac{d_\Omega(x)^p d_\Omega(y)^q}{|x-y|^{p+q}}\right\},\\
&iii: && \log\left(2 + \tfrac{d_\Omega(x)}{|x-y|}\right) \sim \log\left(2 + \tfrac{d_\Omega(x)d_\Omega(y)}{|x-y|^2}\right),\\
&iv: && \min\left\{1, \tfrac{d_\Omega(x)^p d_\Omega(y)^q}{|x-y|^{p+q}}\right\} \sim \left(\tfrac{d_\Omega(y)}{d_\Omega(x)}\right)^{\frac{1}{2}(q-p)}\min\left\{1, \tfrac{d_\Omega(x)^{\frac{1}{2}(p+q)}d_\Omega(y)^{\frac{1}{2}(p+q)}}{|x-y|^{p+q}}\right\},\\
&v: && \min\left\{1, \tfrac{d_\Omega(x)d_\Omega(y)}{|x-y|^2}\right\} \sim \min\left\{\tfrac{d_\Omega(y)}{d_\Omega(x)}, \tfrac{d_\Omega(x)}{d_\Omega(y)}, \tfrac{d_\Omega(x)d_\Omega(y)}{|x-y|^2}\right\}.
\end{aligned}$$

2.5 Estimates of the Green function

In this section we will prove Theorem 2.1.4. First we derive an estimate of the $j-$th derivatives of G_m integrating an estimate of the $(j+1)-$th derivative along an appropriate path. We let the path finish at the boundary to benefit from the boundary condition. Moreover, we have to construct the path such that it stays away from the singularity $x=y$ and such that it has a length of the same magnitude as $d_\Omega(x)$.

In the following lemma we state the existence of such a path.

Lemma 2.5.1. *Let $x \in \Omega$ and $y \in \bar{\Omega}$. There exists a curve $\gamma_x^y : [0,1] \to \bar{\Omega}$ with $\gamma_x^y(0) = x$, $\gamma_x^y(1) \in \partial\Omega$ and such that:*

(i) for every $t \in [0,1] : |\gamma_x^y(t) - y| \geq \frac{1}{2}|x-y|$,

(ii) $l \leq (1+\pi)\, d_\Omega(x)$ where l is the length of γ_x^y.

Moreover, letting $\tilde{\gamma}_x^y : [0,l] \to \bar{\Omega}$ be the parametrization by arclength of γ_x^y, it holds that

(iii) $\frac{1}{5}s \leq |x - \tilde{\gamma}_x^y(s)| \leq s$ for $s \in [0,l]$.

Proof. A description on how to define such a path is as follows. One connects x with a straight line to its nearest boundary point $\tilde{x}$ until the straight line possibly gets too close to y. To avoid the neighborhood of y we take a circular route on ∂B with $B = B(y, \frac{1}{2}|x-y|)$. In the case that $\tilde{x} \in B$ one moves on ∂B to some other point on $\partial\Omega$. We will not give the details of the proof but refer to Figure 2.1. $\square$

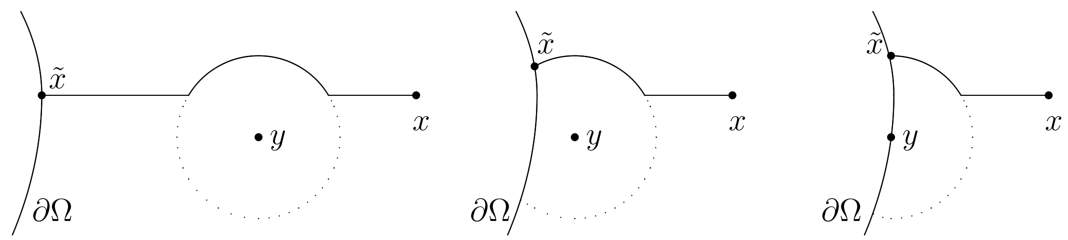

Figure 2.1: *The path γ_x^y for several positions of y.*

We proceed with the proof of Theorem 2.1.4 and start from the estimates in [51] of the m-th derivative of G_m. Integrating this function along the path γ_x^y of Lemma 2.5.1 we find the estimates of the $(m-1)$-th derivative of G_m in terms of the distance to the boundary. Next starting from the new estimates one repeats the argument. Iterating the procedure m times we find the result as stated in Theorem 2.1.4.

There are four cases. Each of the following lemmas will consider one of these cases.

Lemma 2.5.2. *Let $\nu_1, \nu_2, k \in \mathbb{N}_0$, $k \geq 2$. If*

$$|\nabla_x H(x,y)| \preceq |x-y|^{-k} \min\left\{1, \frac{d_\Omega(x)}{|x-y|}\right\}^{\nu_1} \min\left\{1, \frac{d_\Omega(y)}{|x-y|}\right\}^{\nu_2} \quad \textit{for } x,y \in \Omega,$$

and $H(\tilde{x}, y) = 0$ for every $\tilde{x} \in \partial\Omega$ and $y \in \Omega$, then the following inequality holds:

$$|H(x,y)| \preceq |x-y|^{-k+1} \min\left\{1, \frac{d_\Omega(x)}{|x-y|}\right\}^{\nu_1+1} \min\left\{1, \frac{d_\Omega(y)}{|x-y|}\right\}^{\nu_2} \quad \textit{for } x, y \in \Omega.$$

Proof. Let $x, y \in \Omega$ and let γ_x^y the path from x to the boundary from Lemma 2.5.1. Let $\tilde{x} := \gamma_x^y(l)$. Since $\tilde{x} \in \partial\Omega$ one has that

$$H(x,y) = H(\tilde{x}, y) + \int_{\gamma_x^y} \nabla_z H(z,y) \cdot dz = \int_0^l \nabla_x H(\tilde{\gamma}_x^y(s), y) \cdot \tau(s) ds, \tag{2.5.1}$$

with $\tau(s)$ the unit tangent vector. By the hypothesis and Lemma 2.4.2, i) we obtain from (2.5.1) that

$$\begin{aligned}
|H(x,y)| &\preceq \int_0^l |\tilde{\gamma}_x^y(s) - y|^{-k} \min\left\{1, \frac{d_\Omega(\tilde{\gamma}_x^y(s))^{\nu_1} d_\Omega(y)^{\nu_2}}{|\tilde{\gamma}_x^y(s) - y|^{\nu_1+\nu_2}}\right\} ds \preceq \\
&\preceq \int_0^l \big(|x-y| + s\big)^{-k} \min\left\{1, \frac{d_\Omega(x)^{\nu_1} d_\Omega(y)^{\nu_2}}{\big(|x-y| + s\big)^{\nu_1+\nu_2}}\right\} ds \preceq \\
&\preceq |x-y|^{-k+1} \int_0^{\frac{l}{|x-y|}} (1+t)^{-k} \min\left\{1, \frac{d_\Omega(x)^{\nu_1} d_\Omega(y)^{\nu_2}}{|x-y|^{\nu_1+\nu_2} (1+t)^{\nu_1+\nu_2}}\right\} dt.
\end{aligned} \tag{2.5.2}$$

Here we used Lemma 2.5.1 and that $d_\Omega(\tilde{\gamma}_x^y(s)) \preceq d_\Omega(x)$. It is convenient to separate the following two cases.

Case 1, $\frac{d_\Omega(x)}{|x-y|} < 1$: Then $\min\left\{1, \frac{d_\Omega(x)^{\nu_1} d_\Omega(y)^{\nu_2}}{|x-y|^{\nu_1+\nu_2}(1+t)^{\nu_1+\nu_2}}\right\} = \frac{d_\Omega(x)^{\nu_1} d_\Omega(y)^{\nu_2}}{|x-y|^{\nu_1+\nu_2}(1+t)^{\nu_1+\nu_2}}$ and one finds by Lemma 2.4.1 that

$$\begin{aligned}
|H(x,y)| &\preceq \frac{d_\Omega(x)^{\nu_1} d_\Omega(y)^{\nu_2}}{|x-y|^{k+\nu_1+\nu_2-1}} \int_0^{\frac{l}{|x-y|}} \frac{1}{(1+t)^{k+\nu_1+\nu_2}} dt \preceq \\
&\preceq \frac{d_\Omega(x)^{\nu_1+1} d_\Omega(y)^{\nu_2}}{|x-y|^{\nu_1+\nu_2+k}} \preceq |x-y|^{-k+1} \min\left\{1, \frac{d_\Omega(x)^{\nu_1+1}}{|x-y|^{\nu_1+1}} \frac{d_\Omega(y)^{\nu_2}}{|x-y|^{\nu_2}}\right\}.
\end{aligned} \tag{2.5.3}$$

Case 2, $\frac{d_\Omega(x)}{|x-y|} \geq 1$: Since $k \geq 2$ we get again by Lemma 2.4.1 that

$$\begin{aligned}
|H(x,y)| &\preceq |x-y|^{-k+1} \int_0^{\frac{l}{|x-y|}} (1+t)^{-k} dt \preceq |x-y|^{-k+1} \preceq \\
&\preceq |x-y|^{-k+1} \min\left\{1, \frac{d_\Omega(x)^{\nu_1+1}}{|x-y|^{\nu_1+1}} \frac{d_\Omega(y)^{\nu_2}}{|x-y|^{\nu_2}}\right\}.
\end{aligned} \tag{2.5.4}$$

□

Lemma 2.5.3. *Let $\nu_1, \nu_2 \in \mathbb{N}_0$. If*

$$|\nabla_x H(x,y)| \preceq |x-y|^{-1} \min\left\{1, \frac{d_\Omega(x)}{|x-y|}\right\}^{\nu_1} \min\left\{1, \frac{d_\Omega(y)}{|x-y|}\right\}^{\nu_2} \quad \textit{for } x, y \in \Omega,$$

and $H(\tilde{x}, y) = 0$ *for every* $\tilde{x} \in \partial\Omega$ *and* $y \in \Omega$, *then the following inequality holds:*

$$|H(x,y)| \preceq \log\left(2 + \frac{d_\Omega(x) d_\Omega(y)}{|x-y|^2}\right) \ \min\left\{1, \frac{d_\Omega(x)}{|x-y|}\right\}^{\nu_1+1} \min\left\{1, \frac{d_\Omega(y)}{|x-y|}\right\}^{\nu_2},$$

for $x, y \in \Omega$.

Proof. Similarly as in (2.5.2) we find that

$$|H(x,y)| \preceq \int_0^{\frac{l}{|x-y|}} (1+t)^{-1} \min\left\{1, \frac{d_\Omega(x)^{\nu_1} d_\Omega(y)^{\nu_2}}{|x-y|^{\nu_1+\nu_2} (1+t)^{\nu_1+\nu_2}}\right\} dt.$$

Again we will separate the two cases.

Case 1, $\frac{d_\Omega(x)}{|x-y|} < 1$: As in (2.5.3) we obtain

$$\begin{aligned} |H(x,y)| &\preceq \frac{d_\Omega(x)^{\nu_1+1}}{|x-y|^{\nu_1+1}} \frac{d_\Omega(y)^{\nu_2}}{|x-y|^{\nu_2}} \preceq \\ &\preceq \log\left(2 + \frac{d_\Omega(x)}{|x-y|}\right) \min\left\{1, \frac{d_\Omega(x)^{\nu_1+1}}{|x-y|^{\nu_1+1}} \frac{d_\Omega(y)^{\nu_2}}{|x-y|^{\nu_2}}\right\}. \end{aligned}$$

Case 2, $\frac{d_\Omega(x)}{|x-y|} \geq 1$: As in (2.5.4) we get by using Lemma 2.4.2, ii) that

$$\begin{aligned} |H(x,y)| &\preceq \int_0^{\frac{l}{|x-y|}} (1+t)^{-1}\, dt \preceq \\ &\preceq \log\left(1 + \frac{(1+\pi)\, d_\Omega(x)}{|x-y|}\right) \\ &\sim \log\left(2 + \frac{d_\Omega(x)}{|x-y|}\right) \min\left\{1, \frac{d_\Omega(x)^{\nu_1+1}}{|x-y|^{\nu_1+1}} \frac{d_\Omega(y)^{\nu_2}}{|x-y|^{\nu_2}}\right\}. \end{aligned}$$

The claim follows using Lemma 2.4.2, iii). □

Lemma 2.5.4. *Let* $\nu_1, \nu_2 \in \mathbb{N}_0$. *If*

$$|\nabla_x H(x,y)| \preceq \log\left(2 + \frac{d_\Omega(x) d_\Omega(y)}{|x-y|^2}\right) \min\left\{1, \frac{d_\Omega(x)}{|x-y|}\right\}^{\nu_1} \min\left\{1, \frac{d_\Omega(y)}{|x-y|}\right\}^{\nu_2},$$

for $x, y \in \Omega$, *and* $H(\tilde{x}, y) = 0$ *for every* $\tilde{x} \in \partial\Omega$ *and* $y \in \Omega$, *then the following inequality holds:*

$$|H(x,y)| \preceq d_\Omega(x) \min\left\{1, \frac{d_\Omega(x)}{|x-y|}\right\}^{\nu_1} \min\left\{1, \frac{d_\Omega(y)}{|x-y|}\right\}^{\nu_2} \ \textit{for } x, y \in \Omega.$$

Proof. Proceeding as before and using Lemma 2.4.2, iii) one obtains that

$$
\begin{aligned}
|H(x,y)| &\preceq \int_0^l \log\left(2+\frac{d_\Omega(\tilde\gamma_x^y(s))d_\Omega(y)}{|\tilde\gamma_x^y(s)-y|^2}\right)\min\left\{1,\frac{d_\Omega(\tilde\gamma_x^y(s))^{\nu_1}d_\Omega(y)^{\nu_2}}{|\tilde\gamma_x^y(s)-y|^{\nu_1+\nu_2}}\right\}ds \preceq \\
&\preceq |x-y|\int_0^{\frac{l}{|x-y|}}\log\left(2+\tfrac{d_\Omega(x)}{|x-y|(1+t)}\right)\min\left\{1,\tfrac{d_\Omega(x)^{\nu_1}d_\Omega(y)^{\nu_2}}{|x-y|^{\nu_1+\nu_2}(1+t)^{\nu_1+\nu_2}}\right\}dt. \qquad (2.5.5)
\end{aligned}
$$

Case 1, $\frac{d_\Omega(x)}{|x-y|}<1$: From Lemma 2.4.1 it follows that

$$
\begin{aligned}
|H(x,y)| &\preceq \frac{d_\Omega(x)^{\nu_1}d_\Omega(y)^{\nu_2}}{|x-y|^{\nu_1+\nu_2-1}}\int_0^{\frac{l}{|x-y|}}\frac{1}{(1+t)^{\nu_1+\nu_2}}dt \preceq \\
&\preceq \frac{d_\Omega(x)^{\nu_1}d_\Omega(y)^{\nu_2}}{|x-y|^{\nu_1+\nu_2-1}}\frac{d_\Omega(x)}{|x-y|}\sim d_\Omega(x)\min\left\{1,\frac{d_\Omega(x)^{\nu_1}d_\Omega(y)^{\nu_2}}{|x-y|^{\nu_1+\nu_2}}\right\}.
\end{aligned}
$$

Case 2, $\frac{d_\Omega(x)}{|x-y|}\geq 1$: We first observe that $\frac{6d_\Omega(x)}{|x-y|(1+t)}>1$. Indeed since $t\leq\frac{(1+\pi)d_\Omega(x)}{|x-y|}$ we have that

$$
\frac{6d_\Omega(x)}{|x-y|\,(1+t)}\geq\frac{6d_\Omega(x)}{|x-y|+(1+\pi)\,d_\Omega(x)}\geq\tfrac{6}{2+\pi}>1.
$$

Hence from (2.5.5) applying Lemma 2.4.1 we obtain

$$
\begin{aligned}
|H(x,y)| &\preceq |x-y|\int_0^{\frac{l}{|x-y|}}\log\left(6\frac{d_\Omega(x)}{|x-y|\,(1+t)}\right)dt \sim \\
\sim\ & |x-y|\left[\left(1+\tfrac{l}{|x-y|}\right)\log\left(\tfrac{6d_\Omega(x)}{|x-y|\left(1+\frac{l}{|x-y|}\right)}\right)-\log\left(\tfrac{6d_\Omega(x)}{|x-y|}\right)+\tfrac{l}{|x-y|}\right] \preceq \\
\preceq\ & |x-y|\left(1+\frac{(1+\pi)\,d_\Omega(x)}{|x-y|}\right)\log\left(\frac{6\frac{d_\Omega(x)}{|x-y|}}{\frac{(1+\pi)d_\Omega(x)}{|x-y|}+1}\right) \\
& +d_\Omega(x)\left(\frac{|x-y|}{d_\Omega(x)}\log\left(\frac{|x-y|}{6d_\Omega(x)}\right)+1+\pi\right)\sim \\
\sim\ & d_\Omega(x)\sim d_\Omega(x)\min\left\{1,\frac{d_\Omega(x)^{\nu_1}d_\Omega(y)^{\nu_2}}{|x-y|^{\nu_1+\nu_2}}\right\}.
\end{aligned}
$$

□

Lemma 2.5.5. *Let $\nu_1,\nu_2,\alpha_1,\alpha_2\in\mathbb{N}_0$. If*

$$
|\nabla_x H(x,y)|\preceq d_\Omega(x)^{\alpha_1}d_\Omega(y)^{\alpha_2}\ \min\left\{1,\frac{d_\Omega(x)}{|x-y|}\right\}^{\nu_1}\min\left\{1,\frac{d_\Omega(y)}{|x-y|}\right\}^{\nu_2},
$$

for $x,y\in\Omega$, and $H(\tilde x,y)=0$ for every $\tilde x\in\partial\Omega$ and $y\in\Omega$, then the following inequality holds:

$$
|H(x,y)|\preceq d_\Omega(x)^{\alpha_1+1}d_\Omega(y)^{\alpha_2}\min\left\{1,\frac{d_\Omega(x)}{|x-y|}\right\}^{\nu_1}\min\left\{1,\frac{d_\Omega(y)}{|x-y|}\right\}^{\nu_2}\ \textit{for } x,y\in\Omega.
$$

Proof. Proceeding as before, one obtains that

$$\begin{aligned}|H(x,y)| &\preceq \int_0^l d_\Omega(\tilde{\gamma}_x^y(s))^{\alpha_1} d_\Omega(y)^{\alpha_2} \min\left\{1, \frac{d_\Omega(\tilde{\gamma}_x^y(s))^{\nu_1} d_\Omega(y)^{\nu_2}}{|\tilde{\gamma}_x^y(s)-y|^{\nu_1+\nu_2}}\right\} ds \preceq \\ &\preceq |x-y|\, d_\Omega(x)^{\alpha_1} d_\Omega(y)^{\alpha_2} \int_0^{\frac{l}{|x-y|}} \min\left\{1, \frac{d_\Omega(x)^{\nu_1} d_\Omega(y)^{\nu_2}}{|x-y|^{\nu_1+\nu_2}(1+t)^{\nu_1+\nu_2}}\right\} dt.\end{aligned}$$

Again we will separate the two cases.

Case 1, $\frac{d_\Omega(x)}{|x-y|} < 1$: As before it follows that

$$\begin{aligned}|H(x,y)| &\preceq \frac{d_\Omega(x)^{\nu_1+\alpha_1} d_\Omega(y)^{\nu_2+\alpha_2}}{|x-y|^{\nu_1+\nu_2-1}} \int_0^{\frac{l}{|x-y|}} \frac{1}{(1+t)^{\nu_1+\nu_2}}\, dt \preceq \\ &\preceq d_\Omega(x) \frac{d_\Omega(x)^{\nu_1+\alpha_1} d_\Omega(y)^{\nu_2+\alpha_2}}{|x-y|^{\nu_1+\nu_2}} \sim d_\Omega(x)^{\alpha_1+1} d_\Omega(y)^{\alpha_2} \min\left\{1, \frac{d_\Omega(x)^{\nu_1} d_\Omega(y)^{\nu_2}}{|x-y|^{\nu_1+\nu_2}}\right\}.\end{aligned}$$

Case 2, $\frac{d_\Omega(x)}{|x-y|} \geq 1$: We obtain

$$\begin{aligned}|H(x,y)| &\preceq |x-y|\, d_\Omega(x)^{\alpha_1} d_\Omega(y)^{\alpha_2} \int_0^{\frac{l}{|x-y|}} 1\, dt \sim \\ &\sim d_\Omega(x)^{\alpha_1+1} d_\Omega(y)^{\alpha_2} \sim d_\Omega(x)^{\alpha_1+1} d_\Omega(y)^{\alpha_2} \min\left\{1, \frac{d_\Omega(x)^{\nu_1} d_\Omega(y)^{\nu_2}}{|x-y|^{\nu_1+\nu_2}}\right\}.\end{aligned}$$

□

The four lemmas above allow us to prove the following theorem of which Theorem 2.1.4 is a special case.

Theorem 2.5.6. *Let $G_m(x,y)$ be the Green function associated to system (2.1.1) and let $\partial\Omega \in C^{6m+4}$ if $n = 2$ and $\partial\Omega \in C^{5m+2}$ otherwise. Let $k \in \mathbb{N}^n$ with $|k| \leq 4m+3$ if $n = 2$ and $|k| \leq 3m+2$ if $n \geq 3$.*

The following estimates hold for every $x, y \in \Omega$:

(i) For $|k| \geq m$:

(a) if $2m - n - |k| < 0$, then

$$\left|D_x^k G_m(x,y)\right| \preceq |x-y|^{2m-n-|k|} \min\left\{1, \frac{d_\Omega(y)}{|x-y|}\right\}^m,$$

(b) if $2m - n - |k| = 0$, then

$$\begin{aligned}\left|D_x^k G_m(x,y)\right| &\preceq \log\left(1 + \frac{d_\Omega(y)^m}{|x-y|^m}\right) \sim \\ &\sim \log\left(2 + \frac{d_\Omega(y)}{|x-y|}\right) \min\left\{1, \frac{d_\Omega(y)}{|x-y|}\right\}^m,\end{aligned}$$

(c) *if* $2m - n - |k| > 0$, *then*

$$\left|D_x^k G_m(x,y)\right| \preceq d_\Omega(y)^{2m-n-|k|} \min\left\{1, \frac{d_\Omega(y)}{|x-y|}\right\}^{n+|k|-m},$$

(ii) *For* $|k| < m$:

(a) *if* $2m - n - |k| < 0$, *then*

$$\left|D_x^k G_m(x,y)\right| \preceq |x-y|^{2m-n-|k|} \min\left\{1, \frac{d_\Omega(x)}{|x-y|}\right\}^{m-|k|} \min\left\{1, \frac{d_\Omega(y)}{|x-y|}\right\}^{m},$$

(b) *if* $2m - n - |k| = 0$, *then*

$$\left|D_x^k G_m(x,y)\right| \preceq \log\left(1 + \frac{d_\Omega(y)^m d_\Omega(x)^{m-|k|}}{|x-y|^{2m-|k|}}\right) \sim$$
$$\sim \log\left(2 + \frac{d_\Omega(x)}{|x-y|}\right) \min\left\{1, \frac{d_\Omega(y)}{|x-y|}\right\}^{m} \min\left\{1, \frac{d_\Omega(x)}{|x-y|}\right\}^{m-|k|},$$

(c) *if* $2m - n - |k| > 0$, *and moreover*

i. $m - \frac{1}{2}n \le |k|$, *then*

$$\left|D_x^k G_m(x,y)\right| \preceq d_\Omega(y)^{2m-n-|k|} \min\left\{1, \tfrac{d_\Omega(x)}{|x-y|}\right\}^{m-|k|} \min\left\{1, \tfrac{d_\Omega(y)}{|x-y|}\right\}^{n-m+|k|},$$

ii. $|k| < m - \frac{1}{2}n$, *then*

$$\left|D_x^k G_m(x,y)\right| \preceq d_\Omega(y)^{m-\frac{n}{2}} d_\Omega(x)^{m-\frac{n}{2}-|k|} \min\left\{1, \frac{d_\Omega(x) d_\Omega(y)}{|x-y|^2}\right\}^{\frac{n}{2}}.$$

Proof. Let $x, y \in \Omega$. We will start from the estimates of Krasovskiĭ for the higher order derivatives of G_m which are stated in Theorem 2.3.2. The estimates for the lower order derivatives of G_m will be obtained by integrating the higher order estimates along the path γ_x^y from Lemma 2.5.1. Each of the four lemmas above corresponds to one such integration step. Indeed, with $\alpha, \beta \in \mathbb{N}^n$ and $\tilde{x} \in \partial\Omega$ the end point of γ_x^y, we find

$$D_x^\alpha D_y^\beta G_m(x,y) = D_x^\alpha D_y^\beta G_m(\tilde{x},y) + \int_{\gamma_x^y} \nabla_z D_z^\alpha D_y^\beta G_m(z,y) \cdot dz. \tag{2.5.6}$$

If $|\alpha| \le m-1$ then the first term on the right hand side of (2.5.6) equals 0 and we get

$$\left|D_x^\alpha D_y^\beta G_m(x,y)\right| \le \int_0^l \left|\nabla_x D_x^\alpha D_y^\beta G_m(\tilde{\gamma}_x^y(s), y)\right| ds. \tag{2.5.7}$$

If $|\beta| \leq m-1$, then similarly by integrating with respect to y we find

$$\left|D_x^\alpha D_y^\beta G_m(x,y)\right| \leq \int_0^l \left|\nabla_y D_y^\beta D_x^\alpha G_m(x,\tilde{\gamma}_y^x(s))\right| ds. \tag{2.5.8}$$

The explicit estimate coming out of one of such steps depends on which of the four lemmas above we have to use. We take $H(x,y) = D_x^\alpha D_y^\beta G_m(x,y)$ and depending on $|k| = r$ we have to make an appropriate choice for α and β. By hypothesis $r \leq 4m+3$ if $n=2$ while $r \leq 3m+1$ otherwise.

We distinguish the cases as in the statement of the theorem.

Case 1, $r \geq m$: Let $\beta \in \mathbb{N}^n$ with $|\beta| = m-1$. Then proceeding from (2.5.8) with $k=\alpha$ and using the estimate in Theorem 2.3.2, namely $\left|\nabla_y D_x^\alpha D_y^\beta G_m(x,y)\right| \leq |x-y|^{m-n-r}$, three different cases have to be considered.

Case 1(a), $2m-n-r<0$: The claim follows applying m times Lemma 2.5.2.

Case 1(b), $2m-n-r=0$: One gets the estimates by using Lemma 2.5.2 $m-1$ times and Lemma 2.5.3 once.

Case 1(c), $2m-n-r>0$: By first applying Lemma 2.5.2 $n+r-m-1$ times and then Lemma 2.5.3 once we find

$$\left|D_y^{\tilde{\beta}} D_x^k G_m(x,y)\right| \preceq \log\left(1+\frac{d_\Omega(y)^{n+r-m}}{|x-y|^{n+r-m}}\right),$$

with $\tilde{\beta} \in \mathbb{N}^n$, $\tilde{\beta} \leq \beta$ and $\left|\tilde{\beta}\right| = 2m-n-r$. Next one uses Lemma 2.5.4 once and Lemma 2.5.5 $2m-n-r-1$ times.

Case 2, $r<m$: Let $\alpha,\beta \in \mathbb{N}^n$ with $|\alpha| = m-r$ and $|\beta| = m$. One starts from the Krasovskiĭ estimates for $|D_y^\beta D_x^\alpha D_x^k G_m(x,y)|$ and then integrates m times with respect of y and $m-r$ times with respect to x.

Case 2(a), $2m-n-r<0$: The claim follows by applying Lemma 2.5.2 first m times with respect to y and then $m-r$ times with respect to x.

Case 2(b), $2m-n-r=0$: One proves the estimates by using Lemma 2.5.2 m times with respect to y, $m-r-1$ times with respect to x and then Lemma 2.5.3 once with respect to x.

Case 2(c), $2m-n-r>0$: One has to separate the cases $m-r \leq n-1$ and $m-r>n-1$.

Case $m-r \leq n-1$: Applying Lemma 2.5.2 $n-1$ times and Lemma 2.5.3 once we get

$$\left|D_y^{\tilde{\beta}} D_x^k G_m(x,y)\right| \preceq \log\left(1+\frac{d_\Omega(x)^{m-r} d_\Omega(y)^{n-m+r}}{|x-y|^n}\right),$$

with $\tilde{\beta} \in \mathbb{N}^n$, $\tilde{\beta} \leq \beta$ with $\left|\tilde{\beta}\right| = 2m-n-r$. Then using once Lemma 2.5.4 and Lemma 2.5.5 $2m-n-r-1$ times we obtain

$$\left|D_x^k G_m(x,y)\right| \preceq d(y)^{2m-n-r} \min\left\{1,\frac{d_\Omega(x)}{|x-y|}\right\}^{m-r} \min\left\{1,\frac{d_\Omega(y)}{|x-y|}\right\}^{n-m+r}. \tag{2.5.9}$$

The claim follows from (2.5.9) when $m - \frac{1}{2}n \leq r$. Otherwise when $r < m - \frac{1}{2}n$ we rewrite (2.5.9) as

$$\begin{aligned}\left|D_x^k G_m(x,y)\right| &\preceq d_\Omega(y)^{2m-n-r}\left(\frac{d_\Omega(y)}{d_\Omega(x)}\right)^{\frac{n}{2}-m+r}\min\left\{1,\frac{d_\Omega(x)d_\Omega(y)}{|x-y|^2}\right\}^{\frac{n}{2}} \sim \\ &\sim d_\Omega(y)^{m-\frac{n}{2}}d_\Omega(x)^{m-\frac{n}{2}-r}\min\left\{1,\frac{d_\Omega(x)d_\Omega(y)}{|x-y|^2}\right\}^{\frac{n}{2}}.\end{aligned}$$

Here we use Lemma 2.4.2, iv).

Case $m - r > n - 1$: Let $\tilde{\alpha} \in \mathbb{N}^n$, $\tilde{\alpha} \leq \alpha$ with $|\tilde{\alpha}| = m - n - r$. Using Lemma 2.5.2 $n-1$ times and 2.5.3 once we get

$$\left|D_y^\beta D_x^{\tilde{\alpha}} D_x^k G_m(x,y)\right| \preceq \log\left(1+\frac{d_\Omega(x)^n}{|x-y|^n}\right).$$

Then applying once Lemma 2.5.4 and Lemma 2.5.5 $m-1$ times with respect to y and $m-r-n$ times with respect to x, one obtains

$$\begin{aligned}\left|D_x^k G_m(x,y)\right| &\preceq d(y)^m d(x)^{m-r-n}\min\left\{1,\frac{d_\Omega(x)}{|x-y|}\right\}^n \sim \\ &\sim d(y)^{m-\frac{1}{2}n} d(x)^{m-r-\frac{n}{2}}\min\left\{1,\frac{d_\Omega(x)d(y)}{|x-y|^2}\right\}^{\frac{n}{2}},\end{aligned}$$

using again Lemma 2.4.2, iv). Observe that $m - r > n - 1$ implies $r < m - \frac{1}{2}n$ for $n \geq 2$. $\square$

2.6 Estimates of the Poisson kernels

In this section we prove Theorem 2.1.5. The method is similar to the one used for Theorem 2.1.4. A difference is that in this case there is no symmetry between x and y.

In the proof of Theorem 2.1.5 we repeatedly integrate the derivatives of the Poisson kernels along the path constructed in Lemma 2.5.1. The following lemma corresponds to one such integration step.

Lemma 2.6.1. *Let $\nu_1, k \in \mathbb{N}$ with $k \geq 2$. If*

$$|\nabla_x H(y,x)| \preceq |x-y|^{-k}\ d_\Omega(x)^{\nu_1} \text{ for } x \in \Omega,\ y \in \partial\Omega,$$

and $H(y,\tilde{x}) = 0$ for every $\tilde{x} \in \partial\Omega$ with $\tilde{x} \neq y$, then the following inequality holds

$$|H(y,x)| \preceq |x-y|^{-k}\ d_\Omega(x)^{\nu_1+1} \text{ for } x \in \Omega,\ y \in \partial\Omega.$$

Proof. Let $x \in \Omega$ and $y \in \partial\Omega$. Let γ_x^y the path from x to the boundary from Lemma 2.5.1 and let $\tilde{x} := \gamma_x^y(1)$. Since $\tilde{x} \in \partial\Omega$ and $\tilde{x} \neq y$ it holds that

$$H(y,x) = H(y,\tilde{x}) + \int_{\gamma_x^y} \nabla_z H(y,z) \cdot dz = \int_{\gamma_x^y} \nabla_z H(y,z) \cdot dz.$$

By the hypothesis we get that

$$|H(y,x)| \preceq \int_0^l |\nabla_x H(y, \tilde{\gamma}_x^y(s))| \, ds \preceq \int_0^l |\tilde{\gamma}_x^y(s) - y|^{-k} \, d_\Omega(\tilde{\gamma}_x^y(s))^{\nu_1} \, ds.$$

Since $d_\Omega(\tilde{\gamma}_x^y(s)) \preceq d_\Omega(x)$, from Lemma 2.5.1 it follows that

$$\begin{aligned} |H(y,x)| &\preceq d_\Omega(x)^{\nu_1} \int_0^l (|x-y|+s)^{-k} \, ds \preceq \\ &\preceq d_\Omega(x)^{\nu_1} \, |x-y|^{-k+1} \int_0^{\frac{l}{|x-y|}} (1+t)^{-k} \, dt \preceq \frac{d_\Omega(x)^{\nu_1+1}}{|x-y|^k}. \end{aligned}$$

□

The lemma above allows us to prove the following theorem of which Theorem 2.1.5 is a special case.

Theorem 2.6.2. *Let $K_j(y,x)$, for $j = 0, ..., m-1$, be the Poisson kernels associated to system (2.1.1). Suppose furthermore that $\partial\Omega \in C^{6m+4}$ if $n = 2$ and $\partial\Omega \in C^{5m+2}$ otherwise. Let $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq m-1$. The following estimate holds for $x \in \Omega$, $y \in \partial\Omega$*

$$|D_x^\alpha K_j(y,x)| \preceq \frac{d_\Omega(x)^{m-|\alpha|}}{|x-y|^{n-j+m-1}}.$$

Remark 2.6.3. The estimates of $D_x^\alpha K_j(x,y)$ for $|\alpha| \geq m$ can be found in the paper of Krasovskiĭ [51]: for $x \in \Omega$ and $y \in \partial\Omega$

$$|D_x^\alpha K_j(y,x)| \preceq |x-y|^{-n+j-|\alpha|+1}.$$

Proof. Let $x \in \Omega$, $y \in \partial\Omega$, $j \in \{0, \ldots, m-1\}$ and $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq m-1$. We will start from the estimates of Krasovskiĭ for the derivative of order m of K_j which are stated in Theorem 2.3.2. The estimates for the lower order derivatives of K_j will be obtained by integrating the higher order estimates along the path γ_x^y from Lemma 2.5.1. Indeed, with $\beta \in \mathbb{N}^n$, $\beta \geq \alpha$ and $|\beta| = m-1$ we find

$$D_x^\beta K_j(y,x) = D_x^\beta K_j(y, \gamma_x^y(1)) + \int_{\gamma_x^y} \nabla_z D_z^\beta K_j(y,z) \cdot dz = \int_{\gamma_x^y} \nabla_z D_z^\beta K_j(y,z) \cdot dz.$$

Applying Lemma 2.6.1 with $H(y,x) = D_x^\beta K_j(y,x)$ we get

$$\left| D_x^\beta K_j(y,x) \right| \preceq |x-y|^{j-n+1-m} \, d_\Omega(x).$$

The claim follows iterating the procedure $m - |\alpha| - 1$ times. □

2.7 Estimates for the solution with zero boundary conditions

In this section we will derive regularity estimates for

$$\begin{cases} (-\Delta)^m \, u = f & \text{in } \Omega, \\ \quad (\frac{\partial}{\partial \nu})^k u = 0 & \text{on } \partial\Omega \text{ with } 0 \leq k \leq m-1, \end{cases} \tag{2.7.1}$$

where $\Omega \subset \mathbb{R}^n$ is bounded and has the boundary regularity as before. First we recall an estimate involving the Riesz potential (see [36]). Defining $K_\gamma(x) = |x|^{-\gamma}$ and

$$\left(K_\gamma * f\right)(x) := \int_\Omega |x-y|^{-\gamma} f(y) dy,$$

one has:

Lemma 2.7.1. *Let $\Omega \subset \mathbb{R}^n$ be bounded, $\gamma < n$ and $1 \leq p \leq q \leq \infty$. If $\frac{\gamma}{n} < \frac{1}{r} = 1 + \frac{1}{q} - \frac{1}{p}$ then there is $C_{n-\gamma r,\Omega} > 0$ such that for all $f \in L^p(\Omega)$:*

$$\|K_\gamma * f\|_{L^q(\Omega)} \leq C_{n-\gamma r,\Omega} \|f\|_{L^p(\Omega)}. \tag{2.7.2}$$

Proof. This proof is standard, let us recall it for easy reference. Let

$$\frac{\gamma}{n} < \frac{1}{r} = 1 + \frac{1}{q} - \frac{1}{p} = 1 - \delta.$$

Let σ_n denote the surface area of the unit ball in $\mathbb{R}^n$. For $1 < p \leq q < \infty$ one finds by Hölder, setting $c_{n-\gamma r,\Omega} = \frac{1}{n-\gamma r} \sigma_n \left(\mathrm{diam}_\Omega\right)^{n-\gamma r}$,

$$\begin{aligned} \left(K_\gamma * f\right)(x) &= \int_\Omega \frac{1}{|x-y|^{\gamma \frac{r}{q}}} |f(y)|^{\frac{p}{q}} \frac{1}{|x-y|^{\gamma(\frac{p-1}{p})r}} |f(y)|^{p\delta} \, dy \leq \\ &\leq \left(\int_\Omega \frac{1}{|x-y|^{\gamma r}} |f(y)|^p \, dy\right)^{\frac{1}{q}} \left(\int_\Omega \frac{1}{|x-y|^{\gamma r}} dy\right)^{\frac{p-1}{p}} \left(\int_\Omega |f(y)|^p \, dy\right)^{\delta} \\ &\leq \left(c_{n-\gamma r,\Omega}\right)^{\frac{p-1}{p}} \left(\int_\Omega \frac{1}{|x-y|^{\gamma r}} |f(y)|^p \, dy\right)^{\frac{1}{q}} \left(\int_\Omega |f(y)|^p \, dy\right)^{\delta}. \end{aligned}$$

Hence, with a change in the order of integration,

$$\int_\Omega \left|\left(K_\gamma * f\right)(x)\right|^q dx \leq \left(c_{n-\gamma r,\Omega}\right)^{\frac{p-1}{p}q+1} \left(\int_\Omega |f(y)|^p \, dy\right)^{1+\delta q}$$

implying (2.7.2) since $\left(c_{n-\gamma r,\Omega}\right)^{1-\frac{1}{p}+\frac{1}{q}} \leq C_{n-\gamma r,\Omega} := c_{n-\gamma r,\Omega} + 1$.

For $p = 1$ one may skip the middle term in the Hölder estimate; for $q = \infty$ the first term. □

As a consequence of the pointwise estimates and using the lemma above, we next state the optimal L^p-L^q-regularity results mentioned before. Let us recall that $d_\Omega(.)$ is the distance function defined in (2.1.3).

Proposition 2.7.2. *Assume the hypothesis of Theorem 2.5.6. Let* $u \in C^{2m}\left(\bar{\Omega}\right)$ *and* $f \in C\left(\bar{\Omega}\right)$ *satisfy (2.7.1).*

- *If* $2m > n$, *then there exists* $C^1_{\Omega,m} > 0$ *such that for all* $\theta \in [0,1]$

$$\left\| d_\Omega(.)^{-m+\theta n} u \right\|_{L^\infty(\Omega)} \leq C^1_{\Omega,m} \left\| d_\Omega(.)^{m-(1-\theta)n} f \right\|_{L^1(\Omega)}. \tag{2.7.3}$$

- *Let* $1 \leq p \leq q \leq \infty$. *If* $\frac{1}{p} - \frac{1}{q} < \min\left\{\frac{2m}{n}, 1\right\}$, *then taking*

$$\alpha \in \left(\tfrac{1}{p} - \tfrac{1}{q}, \min\left\{1, \tfrac{2m}{n}\right\}\right]$$

there exists $C^2_{\Omega,m,\alpha} > 0$ *such that for all* $\theta \in [0,1]$

$$\left\| d_\Omega(.)^{-m+\theta n\alpha} u \right\|_{L^q(\Omega)} \leq C^2_{\Omega,m,\alpha} \left\| d_\Omega(.)^{m-(1-\theta)n\alpha} f \right\|_{L^p(\Omega)}. \tag{2.7.4}$$

Remark 2.7.3. Notice that the shift in the exponent of $d_\Omega(.)$ between the right and the left hand side of (2.7.4) is $2m - n\alpha$. Hence the shift increases when α goes to $\frac{1}{p} - \frac{1}{q}$.

Remark 2.7.4. The conditions $u \in C^{2m}\left(\bar{\Omega}\right)$ and $f \in C\left(\bar{\Omega}\right)$ may be considerable relaxed for each of the estimates by using a density argument.

Remark 2.7.5. The estimate in (2.7.3) is sharp and does not seem to follow through imbedding results. The estimates in (2.7.4) do need an application of Hölder's inequality. As a consequence the condition $\frac{1}{p} - \frac{1}{q} < \min\left\{\frac{2m}{n}, 1\right\}$ appears with a strict inequality. Such estimates will also follow through regularity results in L^p, Poincaré estimates, Sobolev imbeddings and dual Sobolev imbeddings. See [11].

Remark 2.7.6. In a similar way one may also derive estimates for combinations of boundary behavior and derivatives. For example if $n = m = 2$ one finds with $\theta \in [0,1]$:

$$\left\| d_\Omega(.)^{-1+2\theta} D_x u \right\|_{L^\infty(\Omega)} \leq C^3_{\Omega,m} \left\| d_\Omega(.)^{2\theta} f \right\|_{L^1(\Omega)}.$$

Remark 2.7.7. Fila, Souplet and Weissler in [33, Proposition 2.2] obtained for the case $m = 1$, the following estimate. Assume that $1 \leq p \leq q \leq \infty$ satisfy $\frac{1}{p} - \frac{1}{q} < \frac{2}{n+1}$, then any $u \in W^{1,2}_0(\Omega)$ with $d_\Omega(.)^{\frac{1}{p}} \Delta u \in L^p(\Omega)$ satisfies

$$\left\| d_\Omega(.)^{\frac{1}{q}} u \right\|_{L^q(\Omega)} \leq C^4_\Omega \left\| d_\Omega(.)^{\frac{1}{p}} \Delta u \right\|_{L^p(\Omega)}.$$

This is a special case of (2.7.4). The proof in [33] uses heat kernel estimates.

Proof. In order to consider all the possible splitting between the boundary behavior and the internal regularity we use Lemma 2.4.2, v) to find for all $\alpha \in [0,1]$ and $\sigma \in [-1,1]$ that

$$\min\left\{1, \frac{d_\Omega(x) d_\Omega(y)}{|x-y|^2}\right\} \preceq \left(\frac{d_\Omega(x) d_\Omega(y)}{|x-y|^2}\right)^{1-\alpha} \left(\frac{d_\Omega(x)}{d_\Omega(y)}\right)^{\alpha\sigma}.$$

Hence, for $2m - n > 0$, we may use Theorem 2.1.4, i) to obtain that there exists $C_{\Omega,m} > 0$ such that for $\sigma \in [-1,1]$ and for all $x, y \in \Omega$

$$G_m(x,y) \le C_{\Omega,m}\ d_\Omega(x)^{m-\frac{1}{2}n\alpha} d_\Omega(y)^{m-\frac{1}{2}n\alpha} \frac{1}{|x-y|^{n(1-\alpha)}} \left(\frac{d_\Omega(y)}{d_\Omega(x)}\right)^{\frac{1}{2}n\alpha\sigma}$$
$$\le C_{\Omega,m}\ d_\Omega(x)^{m-\frac{1}{2}n\alpha(1+\sigma)} d_\Omega(y)^{m-\frac{1}{2}n\alpha(1-\sigma)} \frac{1}{|x-y|^{n(1-\alpha)}}. \tag{2.7.5}$$

For $2m - n < 0$ and with $\frac{n}{2m}\alpha \in [0,1]$ we find for $\sigma \in [-1,1]$ and $x, y \in \Omega$

$$G_m(x,y) \le C_{\Omega,m}\ |x-y|^{2m-n} \left(\frac{d_\Omega(x) d_\Omega(y)}{|x-y|^2}\right)^{m(1-\frac{n}{2m}\alpha)} \left(\frac{d_\Omega(y)}{d_\Omega(x)}\right)^{m\frac{n}{2m}\alpha\sigma}$$
$$\le C_{\Omega,m}\ d_\Omega(x)^{m-\frac{1}{2}n\alpha(1+\sigma)} d_\Omega(y)^{m-\frac{1}{2}n\alpha(1-\sigma)} \frac{1}{|x-y|^{n(1-\alpha)}}. \tag{2.7.6}$$

Hence we find

$$d_\Omega(x)^{-m+\frac{1}{2}n\alpha(1+\sigma)} |u(x)| \le C_{\Omega,m} \int_\Omega \frac{1}{|x-y|^{n(1-\alpha)}} d_\Omega(y)^{m-\frac{1}{2}n\alpha(1-\sigma)} |f(y)|\, dy. \tag{2.7.7}$$

The estimate in (2.7.3) follows from (2.7.7) taking $\alpha = 1$ and $\theta = \frac{1}{2}(1+\sigma)$. For the $L^p - L^q$ estimates we may use (2.7.7) and Lemma 2.7.1 to find that:

$$\left\| d^{-m+\frac{1}{2}n\alpha(1+\sigma)} u \right\|_{L^q(\Omega)} \le c \left\| d^{m-\frac{1}{2}n\alpha(1-\sigma)} f \right\|_{L^p(\Omega)},$$

since $\alpha \in \left(\frac{1}{p} - \frac{1}{q}, \min\left\{1, \frac{2m}{n}\right\}\right]$. So with $\theta = \frac{1}{2}(1+\sigma)$ we obtain the estimate in (2.7.3).

In the case that $2m - n = 0$ we may proceed as for (2.7.6) except for a logarithmic term. This term can be taken care of through

$$\log\left(2 + \frac{d_\Omega(x)}{|x-y|}\right) \le C_{\Omega,\varepsilon} \frac{1}{|x-y|^\varepsilon},$$

where we take $\varepsilon = \frac{1}{2}n\left(\alpha - \frac{1}{p} + \frac{1}{q}\right)$. $\square$

Chapter 3

The Clamped Plate Equation for the Limaçon

3.1 Introduction

Hadamard in [47] states that the clamped plate equation for plates having the shape of a *Limaçon de Pascal* is positivity preserving. Positivity preserving for this (linear) equation on $\Omega \subset \mathbb{R}^2$ means that in the fourth order boundary value problem

$$\begin{cases} \Delta^2 u = f & \text{in} \quad \Omega, \\ u = \frac{\partial}{\partial \nu} u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.1.1}$$

the sign of f is preserved by u. Here f is the force (density) and u the deflection of the plate of shape Ω. So the statement reads as, say for $f \in L^1(\Omega)$:

$$f \geq 0 \text{ implies } u \geq 0. \tag{3.1.2}$$

For a precise citation of Hadamard let $\Gamma_A^B = G_\Omega(A, B)$ be the corresponding Green function, that is, $u(x) = \int_\Omega G_\Omega(x, y) f(y)\, dy$ solves (3.1.1). Concerning Γ_A^B Hadamard in [47] writes:

> *M. Boggio, qui a, le premier, noté la signification physique de Γ_A^B, en a déduit l'hypothése que Γ_A^B était toujours positif. Malgré l'absence de démonstration rigoureuse, l'exactitude de cette proposition ne parâit pas douteuse pour les aires convexes. Mais il était l'intéressant d'examiner si elle est vraie pour le cas du Limaçon de Pascal, qui est concave. La réponse est affirmative.*

Let us focus on Hadamard's two claims separately.

Claim 3.1.1. *There is no doubt that Γ_A^B is positive for convex domains.*

This conjecture stood for a long time and only in 1949 a first counterexample, with Ω a long rectangle, was established by Duffin in [31]. This counterexample was soon to be followed by numerous others. A short survey can be found in the introduction of [69]. See also the introduction of Chapter 4. So by now it is well known that convexity is not a sufficient condition.

Let us remind the reader that around 1905 Boggio [8] did prove that (3.1.2) holds in case of a disk. In fact some believed that the disk might be the only domain where (3.1.2) holds. However in [40] it is shown that (3.1.2) also holds in domains that are small perturbations of the disk. Since smallness of these perturbations is defined by a C^2-norm non-convex domains are not included.

Claim 3.1.2. *Γ_A^B is positive for some non-convex domains, namely for the Limaçons de Pascal.*

Hadamard in [47] starts his proof of this claim by observing that:

> *... on constate aisément que, si l'un de ces deux points est très voisin du contour, la partie principale de Γ_A^B est positive.*

Although we are not certain what he meant by '*partie principale*' we know by now that Γ_A^B can be negative when one point is near the boundary. In fact we will show that if the Green function (on a limaçon) is negative somewhere it will be negative for some A and B near the boundary. Hadamard continues his proof by referring to the results in [46]. In this paper he gives an explicit formula for the Green function for (3.1.1) in case of a limaçon. This formula will allow us to show the theorem below. Since there is no explicit proof that his formula indeed gives the Green function we will supply such a proof in the next section.

The domains under consideration are defined for $a \in \left[0, \frac{1}{2}\right]$ by

$$\Omega_a = \left\{(\rho\cos\varphi, \rho\sin\varphi) \in \mathbb{R}^2; 0 \leq \rho < 1 + 2a\cos\varphi\right\}.$$

For $0 \leq a \leq \frac{1}{2}$ the curve $\rho = 1 + 2a\cos\varphi$ is a non self-intersecting limaçon. Special values of the parameter a are the following:

- $a = 0$: Ω_0 is the unit disk;
- $a = \frac{1}{4}$: Ω_a is convex if and only if $a \in \left[0, \frac{1}{4}\right]$;
- $a = \frac{1}{2}$: $\Omega_{\frac{1}{2}}$ is a cardioid.

We will show the following:

Theorem 3.1.3. *The clamped plate problem on Ω_a with $a \in \left[0, \frac{1}{2}\right]$ is positivity preserving if and only if $a \in \left[0, \frac{1}{6}\sqrt{6}\right]$.*

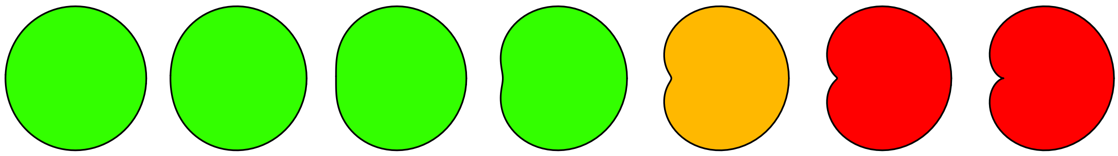

Figure 3.1: *Limaçons for resp.* $a = .1, .175, .25, .325, \frac{1}{6}\sqrt{6}, .45, .5$. *The fifth one with* $a = \frac{1}{6}\sqrt{6}$ *is critical for positivity.*

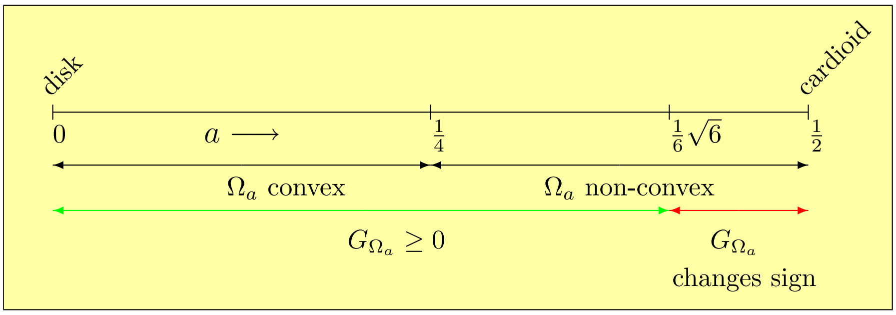

Figure 3.2: *In the graph the critical values of the parameter* a *for convexity of the limaçons and positivity of the Green function. This table has been taken from [23].*

Remark 3.1.4. The limaçon is convex precisely if $0 \leq a \leq \frac{1}{4}$. Notice that $\frac{1}{4} < \frac{1}{6}\sqrt{6}$. So Hadamard is right in the sense that convexity is not a necessary condition. He is wrong in claiming the positivity preserving property for all limaçons.

Remark 3.1.5. One could view this result as a perturbation argument but only the explicit formula allows us to come up with the explicit number $\frac{1}{6}\sqrt{6}$ that is large enough to allow non-convex domains. A small C^2-bound on the perturbation from the unit disk gives a small bound for the curvature κ, namely that $|1-\kappa|$ should be small. Note that $\kappa \geq 0$ means convex.

Remark 3.1.6. A related question is if the first eigenfunction has a fixed sign for all limaçons (compare the Boggio-Hadamard-conjecture versus the Szegö-conjecture in [71], see also [69]). Since one cannot expect an explicit formula for the eigenfunction this seems a much harder question. One does know that the number a where positivity of the first eigenfunction breaks down is strictly larger than the number where (3.1.2) fails. See [44].

We would like to mention some papers that consider explicit solutions for the clamped plate equation. Schot constructed in [61], see also Boggio in [8], an explicit Green function on the disk and on the half-plane. Other publications concerning the clamped plate equation on limaçon are [63] and [30]. Sen considered explicit formula's for the clamped plate equation on other domains in $\mathbb{R}^2$ bounded by fourth order polynomials but only for constant right hand side f. Sen proceeded directly with no hint at Hadamard's result. Dube in [30] gives a series solution for the Green function on a limaçon. He does not refer to Hadamard's explicit formula for the limaçon nor does he consider positivity.

In this chapter we will also prove sharp estimates in term of the distance to the boundary of the Green function for (3.1.1) in case of a limaçon. With respect to the result of Chapter 2 here we have sharp estimates also from below of the Green function. The reason is that in this case we can use the explicit formula of the Green function. It is interesting to see that "boundary singularity"from below does not appear when the Green function becomes somewhere negative.

The quantity $-\Delta_x\Delta_y G_\Omega(x,y)$ is called the Bergman kernel function. Between the sign of this kernel function and the sign of the Green function there is an interesting relation. At the end of this chapter we will study this relation in the case of the limaçons.

The chapter is organized as follows. In the next section we show that the function given by Hadamard is indeed the Green function for problem (3.1.1) in the case of the limaçon. We then prove Theorem 3.1.3 and the estimates of the Green function in terms of the distance to the boundary. In the last section we will study the relation between the sign of the Green function and the Bergman kernel.

3.2 The Green function for the limaçon

This section will contain a proof that the function supplied by Hadamard in [46, Supplement] is indeed the Green function for the limaçons.

Any limaçon can be seen as the image of a circle through the conformal map $z \to z^2$ combined with two shifts. It will be convenient in the following to use complex notation for the unit disk: $B = \{z \in \mathbb{C}\,; |z| < 1\}$. The appropriate conformal map from $B \subset \mathbb{C}$ to $\Omega_a \subset \mathbb{R}^2$ is then given by

$$\begin{array}{rcl} h_a: \ B & \to & \Omega_a, \\ \eta & \mapsto & x = \Big(\mathrm{Re}\,(\eta + a\eta^2)\,, \mathrm{Im}\,(\eta + a\eta^2)\Big)\,. \end{array} \tag{3.2.1}$$

The fact that this conformal map is quadratic, and hence that $\partial\Omega$ is a quartic curve, seems to allow an explicit Green function. This makes the limaçon a special case. For the clamped plate equation with constant f on domains bounded by quartic curves see [63]. In the following x and y denote points in the limaçon while ξ and η denote points in the unit disk.

In [46, Supplement] one finds the explicit formula of the Green function for (3.1.1) on a limaçon Ω_a, which we will denote with G_a. For $x, y \in \Omega_a$ we may rewrite this function as follows

$$G_a\,(x,y) = \tfrac{1}{2}a^2 s^2 r^2 \left[\log\left(\tfrac{r^2}{r_1^2}\right) + \tfrac{r_1^2}{r^2} - 1 - \tfrac{a^2}{1-2a^2}\tfrac{r^2}{s^2}\left(\tfrac{r_1^2}{r^2} - 1\right)^2\right], \tag{3.2.2}$$

where, with $\eta, \xi \in B$ such that $x = h_a(\eta)$ and $y = h_a(\xi)$, the r, r_1 and s are given by

$$r^2 = |\eta - \xi|^2, \quad r_1^2 = \left|1 - \eta\bar{\xi}\right|^2, \quad s^2 = \left|\eta + \xi + \tfrac{1}{a}\right|^2. \tag{3.2.3}$$

In order to show that G_a is the Green function for the limaçon it is convenient to rewrite the function as the sum of the fundamental solution for Δ^2 in $\mathbb{R}^2$ and a biharmonic function. For $x, y \in \mathbb{R}^2$ let $R = |x - y|$. The function $U = R^2 \log(R)$ is the fundamental solution for Δ^2 in $\mathbb{R}^2$: it satisfies $\Delta^2 U(\cdot) = \delta_y(\cdot)$ in $\mathbb{R}^2$. Then writing

$$G_a(x, y) = R^2 \log(R) + J_a(x, y), \tag{3.2.4}$$

the function

$$J_a(x, y) := -R^2 \log(a r_1 s) + \tfrac{a^2}{2} s^2 \left(r_1^2 - r^2\right) - \tfrac{a^4}{2(1-2a^2)} \left(r_1^2 - r^2\right)^2, \tag{3.2.5}$$

should be biharmonic and such that G_a satisfies the boundary condition.

Note that (3.2.4) follows from (3.2.2) using that $ars = R$. In fact one has

$$\begin{aligned} R &= \left|\left(\eta + a\eta^2\right) - \left(\xi + a\xi^2\right)\right| = a\left|\eta^2 + \tfrac{\eta}{a} - \xi^2 - \tfrac{\xi}{a}\right| = \\ &= a\ |\eta - \xi| \left|\eta + \xi + \tfrac{1}{a}\right| = ars. \end{aligned}$$

3.2.1 Boundary condition

Let us rewrite (3.2.4) as

$$G_a(x, y) = \tfrac{1}{2} a^2 s^2 \left[r^2 \log\left(\tfrac{r^2}{r_1^2}\right) + r_1^2 - r^2\right] - \tfrac{a^4}{2(1-2a^2)} \left(r_1^2 - r^2\right)^2. \tag{3.2.6}$$

When $x \in \partial\Omega_a$, then $\eta \in \partial D$ and it holds that $r_1 = r$. It follows from (3.2.6) that $G_a(x, y) = 0$ at the boundary. Now we are interested in $\frac{\partial}{\partial\nu} G_a(x, y)$ on $\partial\Omega_a$. One observes that the term $(r_1^2 - r^2)^2$ gives no contribution because it is a zero of order two at the boundary. The remaining term is a product of two factors: one that is non-zero at the boundary and the other that is identically zero. Hence, when we look at the normal derivative at the boundary the only relevant term will be

$$\tfrac{\partial}{\partial\nu} \left[r^2 \log\left(\tfrac{r^2}{r_1^2}\right) + r_1^2 - r^2\right]. \tag{3.2.7}$$

Using that the term inside the brackets in (3.2.7) is the Green function for the disk, see [8], one gets that also the second Dirichlet boundary condition is satisfied. Notice that we are using that Ω_a is image of the unit disk through a conformal map.

3.2.2 The function $J_a(x, y)$ is biharmonic on Ω_a.

To prove the biharmonicity of J_a it is convenient to consider separately the term with the logarithm and the remaining part.

We first observe that $\log(a r_1 s)$ is a harmonic function on Ω_a. From this, the identity $\Delta^2 \left(R^2 \log(a r_1 s)\right) = 0$ follows using that if v is a harmonic function then $R^2 v$ is biharmonic.

Lemma 3.2.1. *It holds that*

$$\Delta_x^2 \left(s^2 \left(r_1^2 - r^2\right) - \tfrac{a^2}{1-2a^2} \left(r_1^2 - r^2\right)^2\right) = 0.$$

Proof. Next to $h_a : B \subset \mathbb{C} \to \mathbb{R}^2$ we will use $\mathbf{h}_a(\eta) : \mathbb{C} \to \mathbb{C}$ defined by $\mathbf{h}_a(\eta) = \eta + a\eta^2$ with $\eta = \eta_1 + i\eta_2$.

Let us consider

$$\begin{aligned} K(x,y) \quad &:= \quad \left|h_a^{-1}(x) + h_a^{-1}(y) + \tfrac{1}{a}\right|^2 \left(1 - \left|h_a^{-1}(x)\right|^2\right) \\ &\quad\quad - \tfrac{a^2}{1-2a^2} \left(1 - \left|h_a^{-1}(x)\right|^2\right)^2 \left(1 - \left|h_a^{-1}(y)\right|^2\right). \end{aligned}$$

One finds that $s^2 \left(r_1^2 - r^2\right) - \frac{a^2}{1-2a^2} \left(r_1^2 - r^2\right)^2 = (1 - |h_a^{-1}(y)|^2) K(x,y)$, and that

$$\begin{aligned} Y(\eta,\xi) \quad &:= \quad K(h_a(\eta), h_a(\xi)) \\ &= \quad \left|\eta + \xi + \tfrac{1}{a}\right|^2 \left(1 - |\eta|^2\right) - \tfrac{a^2}{1-2a^2} \left(1 - |\eta|^2\right)^2 \left(1 - |\xi|^2\right). \end{aligned}$$

Since h is a conformal map, it holds that:

$$\Delta_\eta Y(\eta,\xi) = \left|\mathbf{h}_a'(\eta)\right|^2 \left(\Delta_x K\right)(h_a(\eta), h_a(\xi)), \tag{3.2.8}$$

$$\begin{aligned} \Delta_\eta^2 Y(\eta,\xi) \quad &= \quad \Delta_\eta \left|\mathbf{h}_a'(\eta)\right|^2 \Delta_x K(h_a(\eta), h_a(\xi)) \\ &\quad + 2\sum_{i=1}^{2} \tfrac{\partial}{\partial \eta_i} \left|\mathbf{h}_a'(\eta)\right|^2 \tfrac{\partial}{\partial \eta_i} \left(\Delta_x K\right)(h_a(\eta), h_a(\xi)) \\ &\quad + \left|\mathbf{h}_a'(\eta)\right|^4 \left(\Delta_x^2 K\right)(h_a(\eta), h_a(\xi)). \end{aligned} \tag{3.2.9}$$

The idea is to use (3.2.9) in order to calculate the term $\left(\Delta_x^2 K\right)(h_a(\eta), h_a(\xi))$ in terms of $\Delta_\eta^2 Y(\eta,\xi)$. Since $\Delta_\eta = 4\frac{\partial}{\partial\bar\eta}\frac{\partial}{\partial\eta}$, one has

$$\begin{aligned} \tfrac{\partial}{\partial\eta} Y(\eta,\xi) \quad &= \quad \left(\bar\eta + \bar\xi + \tfrac{1}{a}\right)(1 - |\eta|^2) - \bar\eta \left|\eta + \xi + \tfrac{1}{a}\right|^2 \\ &\quad + \tfrac{2a^2}{1-2a^2} \bar\eta (1 - |\eta|^2)(1 - |\xi|^2), \\ \tfrac{\partial^2}{\partial\bar\eta\partial\eta} Y(\eta,\xi) \quad &= \quad (1 - |\eta|^2) - \eta\left(\bar\eta + \bar\xi + \tfrac{1}{a}\right) - \left|\eta + \xi + \tfrac{1}{a}\right|^2 - \bar\eta\left(\eta + \xi + \tfrac{1}{a}\right) \\ &\quad + \tfrac{2a^2}{1-2a^2}(1 - |\eta|^2)(1 - |\xi|^2) - \tfrac{2a^2}{1-2a^2}\bar\eta\eta(1 - |\xi|^2), \\ \tfrac{\partial^3}{\partial\eta\partial\bar\eta\partial\eta} Y(\eta,\xi) \quad &= \quad -2\bar\eta - 2\left(\bar\eta + \bar\xi + \tfrac{1}{a}\right) - \tfrac{4a^2}{1-2a^2}\bar\eta(1 - |\xi|^2), \\ \tfrac{\partial^4}{\partial\bar\eta\partial\eta\partial\bar\eta\partial\eta} Y(\eta,\xi) \quad &= \quad -4 - \tfrac{4a^2}{1-2a^2}(1 - |\xi|^2), \end{aligned}$$

which gives

$$\begin{aligned} \Delta_\eta Y(\eta,\xi) \quad &= \quad 4(1 - 3|\eta|^2) - 4\eta\left(\bar\xi + \tfrac{1}{a}\right) - 4\left|\eta + \xi + \tfrac{1}{a}\right|^2 - 4\bar\eta\left(\xi + \tfrac{1}{a}\right) \\ &\quad + \tfrac{8a^2}{1-2a^2}(1 - 2|\eta|^2)(1 - |\xi|^2), \\ \Delta_\eta^2 Y(\eta,\xi) \quad &= \quad -64 - \tfrac{64a^2}{1-2a^2}(1 - |\xi|^2). \end{aligned}$$

By the definition of the conformal map h_a in (3.2.1) and from (3.2.8) we obtain that $|\mathbf{h}'_a(\eta)|^2 = |2a\eta+1|^2$, $\Delta_\eta |\mathbf{h}'_a(\eta)|^2 = 16a^2$ and

$$\begin{aligned}(\Delta_x K)(h_a(\eta), h_a(\xi)) = {} & \tfrac{4}{|2a\eta+1|^2}\left((1-3|\eta|^2) - \eta\left(\bar\xi + \tfrac{1}{a}\right) - \left|\eta+\xi+\tfrac{1}{a}\right|^2\right) \\ & + \tfrac{4}{|2a\eta+1|^2}\left(-\bar\eta\left(\xi+\tfrac{1}{a}\right) + \tfrac{2a^2}{1-2a^2}(1-2|\eta|^2)(1-|\xi|^2)\right).\end{aligned}$$

One finds

$$\begin{aligned} & \sum_{i=1}^{2} \tfrac{\partial}{\partial\eta_i} |\mathbf{h}'_a(\eta)|^2 \tfrac{\partial}{\partial\eta_i}(\Delta_x K)(h_a(\eta), h_a(\xi)) \\ = {} & -\tfrac{64a^2}{|2a\eta+1|^2}\left(-4\eta_1^2 - 4\eta_2^2 - 4\eta_1\xi_1 - \tfrac{4}{a}\eta_1 - 4\eta_2\xi_2 - \tfrac{1}{a^2} - \tfrac{2}{a}\xi_1 + 1 - |\xi|^2\right) \\ & -\tfrac{64a^2}{|2a\eta+1|^2}\tfrac{2a^2}{1-2a^2}(1-2|\eta|^2)(1-|\xi|^2) + \tfrac{32a^2}{|2a\eta+1|^2}\left(-8\eta_2^2 - 4\xi_2\eta_2\right) \\ & +\tfrac{16a}{|2a\eta+1|^2}\left(-8\eta_1 - 4\xi_1 - \tfrac{4}{a} - 16a\eta_1^2 - 8a\eta_1\xi_1 - 8\eta_1\right) \\ & +\tfrac{16a}{|2a\eta+1|^2}\tfrac{2a^2}{1-2a^2}\left(-8a\eta_1^2 - 4\eta_1 - 8a\eta_2^2\right)(1-|\xi|^2)\end{aligned}$$

$$\begin{aligned} = {} & -\tfrac{64a^2}{|2a\eta+1|^2}\left(-2\eta_1\xi_1 - 2\eta_2\xi_2 - \tfrac{1}{a}\xi_1 + 1 - |\xi|^2\right) \\ & -\tfrac{64a^2}{|2a\eta+1|^2}\tfrac{2a^2}{1-2a^2}\left(1+\tfrac{\eta_1}{a}\right)(1-|\xi|^2).\end{aligned}$$

Hence from (3.2.9) we get

$$\begin{aligned} & -1 - \tfrac{a^2}{1-2a^2}(1-|\xi|^2) = \\ = {} & \tfrac{a^2}{|2a\eta+1|^2}\left(-4\eta_1^2 - 4\eta_2^2 - 4\eta_1\xi_1 - \tfrac{4}{a}\eta_1 - 4\eta_2\xi_2 - \tfrac{1}{a^2} - \tfrac{2}{a}\xi_1 + 1 - |\xi|^2\right) \\ & +\tfrac{a^2}{|2a\eta+1|^2}\tfrac{2a^2}{1-2a^2}(1-2|\eta|^2)(1-|\xi|^2) - \tfrac{2a^2}{|2a\eta+1|^2}\tfrac{2a^2}{1-2a^2}\left(1+\tfrac{\eta_1}{a}\right)(1-|\xi|^2) \\ & -\tfrac{2a^2}{|2a\eta+1|^2}\left(-2\eta_1\xi_1 - 2\eta_2\xi_2 - \tfrac{1}{a}\xi_1 + 1 - |\xi|^2\right) \\ & + |\mathbf{h}'_a(\eta)|^4\left(\Delta_x^2 K\right)(h_a(\eta), h_a(\xi)),\end{aligned}$$

$$\begin{aligned} & -1 - \tfrac{a^2}{1-2a^2}(1-|\xi|^2) = \\ = {} & -\tfrac{1}{|2a\eta+1|^2}|2a\eta+1|^2 - \tfrac{2a^2}{|2a\eta+1|^2}\left(2\eta_1\xi_1 + 2\eta_2\xi_2 + \tfrac{1}{a}\xi_1\right) + \tfrac{a^2}{|2a\eta+1|^2}(1-|\xi|^2) \\ & +\tfrac{a^2}{|2a\eta+1|^2}\tfrac{2a^2}{1-2a^2}(1-|\xi|^2) - \tfrac{a^2}{|2a\eta+1|^2}\tfrac{4a^2}{1-2a^2}|\eta|^2(1-|\xi|^2) \\ & -\tfrac{2a^2}{|2a\eta+1|^2}\tfrac{2a^2}{1-2a^2}(1-|\xi|^2) - \tfrac{2a^2}{|2a\eta+1|^2}\tfrac{2a}{1-2a^2}\eta_1(1-|\xi|^2) \\ & -\tfrac{2a^2}{|2a\eta+1|^2}(1-|\xi|^2) - \tfrac{2a^2}{|2a\eta+1|^2}\left(-2\eta_1\xi_1 - 2\eta_2\xi_2 - \tfrac{1}{a}\xi_1\right) \\ & + |\mathbf{h}'_a(\eta)|^4\left(\Delta_x^2 K\right)(h_a(\eta), h_a(\xi)),\end{aligned}$$

$$
\begin{aligned}
& -\tfrac{a^2}{1-2a^2}(1-|\xi|^2) = \\
= \quad & -\tfrac{a^2}{|2a\eta+1|^2}(1-|\xi|^2) - \tfrac{a^2}{|2a\eta+1|^2}\tfrac{1}{1-2a^2}(1-|\xi|^2)\,|2a\eta+1|^2 \\
& +\tfrac{a^2}{|2a\eta+1|^2}\tfrac{1}{1-2a^2}(1-|\xi|^2) - \tfrac{a^2}{|2a\eta+1|^2}\tfrac{2a^2}{1-2a^2}(1-|\xi|^2) \\
& +|\mathbf{h}_a'(\eta)|^4\left(\Delta_x^2 K\right)(h_a(\eta),h_a(\xi)),
\end{aligned}
$$

$$
\begin{aligned}
0 = \quad + \quad & \tfrac{a^2}{|2a\eta+1|^2}(1-|\xi|^2)\left(-1+\tfrac{1}{1-2a^2}-\tfrac{2a^2}{1-2a^2}\right) \\
+ \quad & |\mathbf{h}_a'(\eta)|^4\left(\Delta_x^2 K\right)(h_a(\eta),h_a(\xi)),
\end{aligned}
$$

which gives the claim. □

3.3 Proof of Theorem 3.1.3

In this section we prove Theorem 3.1.3. We will first study the behavior and then the positivity of the Green function.

3.3.1 Behavior of the Green function

We want to study when the function G_a is of fixed sign in $\Omega_a \times \Omega_a$. For establishing this positivity we will need to consider the function

$$F(\beta,q) := \log\left(\frac{1}{q}\right) + q - 1 - \beta\frac{(q-1)^2}{q}. \tag{3.3.1}$$

Note that $q = r_1^2/r^2 \geq 1$.

Lemma 3.3.1. *Set* $I_\beta := \{q \geq 1 : F(\beta,q) \leq 0\}$. *It holds that:*

- $I_\beta = \{1\}$ *for* $\beta \in \left[0, \frac{1}{2}\right]$;
- $I_\beta = [1, q_\beta]$ *with* $q_\beta > 1$ *for* $\beta \in \left(\frac{1}{2}, 1\right)$;
- $I_\beta = [1, \infty)$ *for* $\beta \in [1, \infty)$.

Remark 3.3.2. Note that $\beta \mapsto F(\beta,q)$ is decreasing and hence that $\beta \mapsto q_\beta$ is non-decreasing.

It will be convenient to work with functions defined in the disk. If f is a function defined on Ω_a, then $\tilde{f}_a$ will denote the function $\tilde{f}_a := f \circ h_a$ defined on the disk.

We fix the auxiliary function

$$\tilde{H}_a(\eta,\xi) := \tfrac{a^2}{1-2a^2}\frac{r_1^2}{s^2} = \tfrac{a^2}{1-2a^2}\frac{|1-\eta\bar{\xi}|^2}{|\eta+\xi+\frac{1}{a}|^2}, \tag{3.3.2}$$

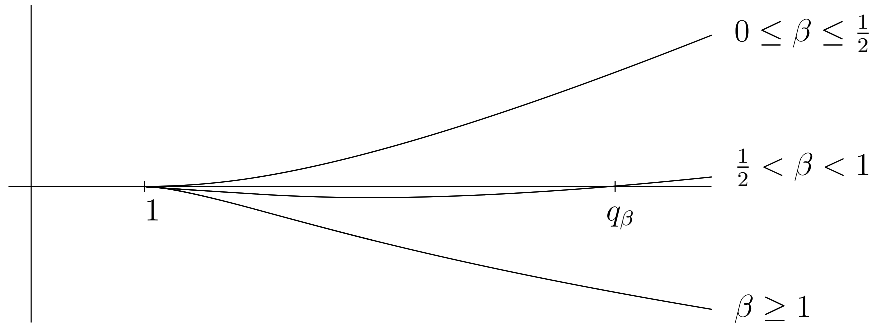

Figure 3.3: *Graphs of* $q \mapsto F(\beta, q)$.

and hence the Green function in (3.2.2) becomes

$$\begin{aligned}\tilde{G}_a(\eta,\xi) &:= \tfrac{1}{2}a^2 s^2 r^2\ F\left(\tilde{H}_a(\eta,\xi), \tfrac{r_1^2}{r^2}\right)\\ &= \tfrac{1}{2}a^2 s^2 r^2\ F\left(\tilde{H}_a(\eta,\xi), \tfrac{\left|1-\eta\bar{\xi}\right|^2}{|\eta-\xi|^2}\right).\end{aligned} \tag{3.3.3}$$

The preceding Lemma 3.3.1 gives that if

$$\sup_{\eta,\xi\in B} \tilde{H}_a(\eta,\xi) \leq \frac{1}{2}, \tag{3.3.4}$$

then F and hence G_a are positive. Note that (3.3.4) gives a condition on the parameter a which is a sufficient condition for the positivity of the function. In the following we will see that this condition is also necessary.

First we will reduce the dimension of the problem. The following lemma states that it is sufficient to study the behavior of $\tilde{H}_a$ for couples of conjugate points.

Lemma 3.3.3. *Let $a < \frac{1}{2}$ and define the sets $\ell_{\eta,\xi}$ for $(\eta,\xi) \in B \times B$ by*

$$\ell_{\eta,\xi} = \left\{\chi = \chi_1 + i\chi_2 \in B : \chi_1 = \tfrac{\eta_1+\xi_1}{2}, |\chi| \geq \max\{|\eta|, |\xi|\}\right\}, \tag{3.3.5}$$

where $\eta = \eta_1 + i\eta_2$ and $\xi = \xi_1 + i\xi_2$.

If $\tilde{H}_a(\eta,\xi) > \frac{1}{2}$, then for every $\chi \in \ell_{\eta,\xi}$ it holds that $\tilde{H}_a(\chi,\bar{\chi}) > \frac{1}{2}$.

Proof. By hypothesis one has:

$$\tilde{H}_a(\eta,\xi) = \frac{a^2}{1-2a^2}\frac{(1-\eta_1\xi_1-\eta_2\xi_2)^2 + (\eta_1\xi_2-\eta_2\xi_1)^2}{\left(\eta_1+\xi_1+\frac{1}{a}\right)^2 + (\eta_2+\xi_2)^2} > \frac{1}{2},$$

which is equivalent to

$$\begin{aligned}&2a^2\left(1+\eta_1^2\xi_1^2+\eta_2^2\xi_2^2-2\eta_1\xi_1-2\eta_2\xi_2+\eta_1^2\xi_2^2+\eta_2^2\xi_1^2\right) >\\ &\qquad\left(1-2a^2\right)\left(\eta_1^2+\xi_1^2+\tfrac{1}{a^2}+2\eta_1\xi_1+\tfrac{2}{a}\eta_1+\tfrac{2}{a}\xi_1+\eta_2^2+\xi_2^2+2\eta_2\xi_2\right),\end{aligned}$$

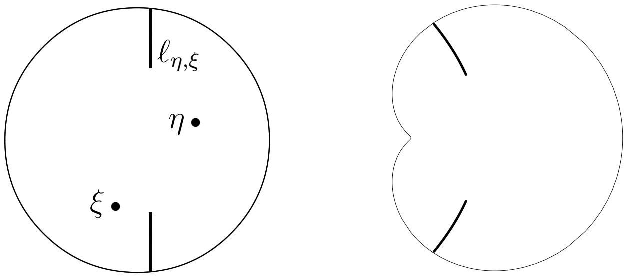

Figure 3.4: *A set* $\ell_{\eta,\xi}$ *and its image within a limaçon.*

or similarly

$$\begin{aligned} &2a^2(1+|\eta|^2)(1+|\xi|^2) > \\ &\quad (\eta_1+\xi_1)^2 + (\eta_2+\xi_2)^2 + \tfrac{1}{a^2} + \tfrac{2}{a}(\eta_1+\xi_1) - 2 - 4a(\eta_1+\xi_1). \end{aligned} \tag{3.3.6}$$

For $\chi \in \ell_{\eta,\xi}$, we have

$$\tilde{H}_a(\chi,\bar{\chi}) - \tfrac{1}{2} = \tfrac{a^2}{1-2a^2}\frac{(1-\chi_1^2+\chi_2^2)^2+4\chi_1^2\chi_2^2}{\left(2\chi_1+\frac{1}{a}\right)^2} - \frac{2\chi_1^2+\frac{1}{2a^2}+\frac{2}{a}\chi_1}{\left(2\chi_1+\frac{1}{a}\right)^2}$$

$$\begin{aligned} &= \tfrac{a^2}{1-2a^2}\tfrac{1+\chi_1^4+\chi_2^4-2\chi_1^2+2\chi_2^2-2\chi_1^2\chi_2^2+4\chi_1^2\chi_2^2-\frac{2}{a^2}\chi_1^2-\frac{1}{2a^4}-\frac{2}{a^3}\chi_1+4\chi_1^2+\frac{1}{a^2}+\frac{4}{a}\chi_1}{\left(2\chi_1+\frac{1}{a}\right)^2} \\ &= \tfrac{1}{1-2a^2}\tfrac{1}{\left(2\chi_1+\frac{1}{a}\right)^2}\left(a^2(1+|\chi|^2)^2-(2\chi_1^2+\tfrac{1}{2a^2}+\tfrac{2}{a}\chi_1-1-4a\chi_1)\right). \end{aligned} \tag{3.3.7}$$

By the definition of $\ell_{\eta,\xi}$ and (3.3.6) it follows that the last term is positive. Indeed:

$$\begin{aligned} (1+|\chi|^2)^2 &\geq (1+|\eta|^2)(1+|\xi|^2) \\ &> \tfrac{1}{2a^2}(\eta_1+\xi_1)^2 + \tfrac{1}{2a^2}(\eta_2+\xi_2)^2 + \tfrac{1}{2a^4} + \tfrac{1}{a^3}(\eta_1+\xi_1) - \tfrac{1}{a^2} - \tfrac{2}{a}(\eta_1+\xi_1) \\ &> \tfrac{1}{a^2}\left(2\chi_1^2 + \tfrac{1}{2a^2} + \tfrac{2}{a}\chi_1 - 1 - 4a\chi_1\right). \end{aligned}$$

□

Remark 3.3.4. Note that (3.3.7) implies: $\tilde{H}_a(\chi,\bar{\chi})$ is increasing in $|\chi_2|$.

We are now able to prove that (3.3.4) also gives a necessary condition for the positivity of F and hence of G_a.

Lemma 3.3.5. *Let* $a < \frac{1}{2}$.

i. If $\tilde{H}_a(v,\bar{v}) > \frac{1}{2}$ *then there is* $\chi \in \ell_{v,\bar{v}}$ *such that*

$$F\left(\tilde{H}_a(\chi,\bar{\chi}), \tfrac{|1-\chi^2|^2}{|\chi-\bar{\chi}|^2}\right) < 0. \tag{3.3.8}$$

ii. If (3.3.8) holds, then $F\left(\tilde{H}_a(z,\bar{z}), \frac{|1-z^2|^2}{|z-\bar{z}|^2}\right) < 0$ *for every* $z \in \ell_{\chi,\bar{\chi}}$.

Proof. First claim: Since the function $\beta \mapsto F(\beta, q)$ is decreasing, see (3.3.1), and the function $\tilde{H}_a(z,\bar{z})$ is increasing in $|z_2|$, by Remark 3.3.4, one gets that

$$F\left(\tilde{H}_a(z,\bar{z}), \frac{|1-z^2|^2}{|z-\bar{z}|^2}\right) \leq F\left(\tilde{H}_a(v,\bar{v}), \frac{|1-z^2|^2}{|z-\bar{z}|^2}\right) \text{ for every } z \in \ell_{v,\bar{v}}. \tag{3.3.9}$$

In $F\left(\tilde{H}_a(v,\bar{v}), \frac{|1-z^2|^2}{|z-\bar{z}|^2}\right)$ the first argument does not depend on z; it is a fixed coefficient which is larger then $1/2$ by hypothesis. Hence, applying Lemma 3.3.1, one has that there exists a $q_{\tilde{H}_a(v,\bar{v})} > 1$ such that

$$F\left(\tilde{H}_a(v,\bar{v}), \frac{|1-z^2|^2}{|z-\bar{z}|^2}\right) < 0, \ \forall z \in \ell_{v,\bar{v}} \text{ with } \frac{|1-z^2|^2}{|z-\bar{z}|^2} < q_{\tilde{H}_a(v,\bar{v})}. \tag{3.3.10}$$

Note that the function $|z_2| \mapsto \frac{|1-z^2|^2}{|z-\bar{z}|^2}$ is decreasing, since

$$\frac{\partial}{\partial z_2}\frac{|1-z^2|^2}{|z-\bar{z}|^2} = -\tfrac{1}{2z_2^3}(1-|z|^2+2z_2^2)(1-|z|^2). \tag{3.3.11}$$

Hence, since $\frac{|1-z^2|^2}{|z-\bar{z}|^2}$ is equal to 1 at the boundary, it follows that there exists $\chi \in \ell_{v,\bar{v}}$ such that

$$\frac{|1-\chi^2|^2}{|\chi-\bar{\chi}|^2} < q_{\tilde{H}_a(v,\bar{v})}. \tag{3.3.12}$$

Combining (3.3.9), (3.3.10) and (3.3.12) the first claim follows.

Second claim: If $F\left(\tilde{H}_a(\chi,\bar{\chi}), \frac{|1-\chi^2|^2}{|\chi-\bar{\chi}|^2}\right) < 0$ we can deduce from Lemma 3.3.1 that

$$\tilde{H}_a(\chi,\bar{\chi}) > \tfrac{1}{2} \text{ and } \frac{|1-\chi^2|^2}{|\chi-\bar{\chi}|^2} < q_{\tilde{H}_a(\chi,\bar{\chi})}. \tag{3.3.13}$$

Since $\tilde{H}_a(z,\bar{z})$ is increasing in $|z_2|$ (Remark 3.3.4) and the function $|z_2| \mapsto \frac{|1-z^2|^2}{|z-\bar{z}|^2}$ is decreasing, see (3.3.11), from (3.3.13) one gets that

$$\tilde{H}_a(z,\bar{z}) > \tfrac{1}{2} \text{ and } \frac{|1-z^2|^2}{|z-\bar{z}|^2} < q_{\tilde{H}_a(\chi,\bar{\chi})} \text{ for every } z \in \ell_{\chi,\bar{\chi}}. \tag{3.3.14}$$

Since $\beta \mapsto q_\beta$ is increasing (Remark 3.3.2), from (3.3.14) we have that

$$\frac{|1-z^2|^2}{|z-\bar{z}|^2} < q_{\tilde{H}_a(z,\bar{z})} \text{ for every } z \in \ell_{\chi,\bar{\chi}}. \tag{3.3.15}$$

By (3.3.14), (3.3.15) and Lemma 3.3.1 it follows that $F\left(\tilde{H}_a(z,\bar{z}), \frac{|1-z^2|^2}{|z-\bar{z}|^2}\right) < 0$ for every $z \in \ell_{\chi,\bar{\chi}}$. □

The previous results show that if the function $G_a(x,y)$ is negative for some $x, y \in \Omega_a$ then G_a will be negative somewhere near opposite boundary points. To be precise:

Corollary 3.3.6. *Suppose that $G_a(x,y) < 0$ for some $x, y \in \Omega_a$, then for all $\varepsilon > 0$ there is $x^\varepsilon \in \Omega_a$ with $d_{\Omega_a}(x^\varepsilon) < \varepsilon$ such that:*

$$G_a((x_1^\varepsilon, x_2^\varepsilon), (x_1^\varepsilon, -x_2^\varepsilon)) < 0.$$

Here $d_\Omega(x)$ denotes the distance of x to the boundary of Ω as defined in (2.1.3).

Proof. If $\tilde{G}_a(\eta,\xi) < 0$, Lemma 3.3.1 gives that necessarily $\tilde{H}_a(\eta,\xi) > \frac{1}{2}$. Hence, one has from Lemma 3.3.3 that $\tilde{H}_a(z,\bar{z}) > \frac{1}{2}$ for every $z \in \ell_{\eta,\xi}$. The claim follows directly from Lemma 3.3.5. □

3.3.2 Positivity of the Green function

Using the results of the previous section, we have seen that the function $\tilde{H}_a$ in (3.3.2) plays a crucial role for the positivity of the Green function. Let us collect this result.

Corollary 3.3.7. *The Green function for the clamped plate equation on a limaçon is positive if and only if*

$$\sup_{\eta,\xi\in B} \tilde{H}_a(\eta,\xi) = \frac{a^2}{1-2a^2} \sup_{\eta,\xi\in B} \frac{\left|1-\eta\bar{\xi}\right|^2}{\left|\eta+\xi+\frac{1}{a}\right|^2} \leq \frac{1}{2}. \tag{3.3.16}$$

Condition (3.3.16) gives an upper bound for the parameter a. In the following lemma we give the explicit value of this upper bound.

Lemma 3.3.8. *Inequality (3.3.16) is satisfied if and only if $a \leq \frac{1}{6}\sqrt{6}$.*

Proof. Lemma 3.3.3 implies that it is sufficient to verify (3.3.16) for couples of conjugate points, that is:

$$\sup_{\chi\in B} \tilde{H}_a(\chi,\bar{\chi}) = \tfrac{a^2}{1-2a^2} \sup_{\chi\in B} \frac{\left|1-\chi^2\right|^2}{\left|\chi+\bar{\chi}+\frac{1}{a}\right|^2} \leq \frac{1}{2}.$$

By (3.3.7) we find

$$\tilde{H}_a(\chi,\bar{\chi}) - \tfrac{1}{2} = \tfrac{1}{1-2a^2} \frac{a^2(1+|\chi|^2)^2 + 1 + 4a\chi_1}{\left(2\chi_1+\frac{1}{a}\right)^2} - \tfrac{1}{1-2a^2}\tfrac{1}{2},$$

which gives

$$\sup_{\chi\in B} \tilde{H}_a(\chi,\bar{\chi}) - \tfrac{1}{2} = \tfrac{1}{1-2a^2} \sup_{\chi\in B} \frac{4a^2 + 1 + 4a\chi_1}{\left(2\chi_1+\frac{1}{a}\right)^2} - \tfrac{1}{1-2a^2}\tfrac{1}{2}. \tag{3.3.17}$$

A straightforward computation shows that the maximum in (3.3.17) is attained for $\chi_1 = -2a$ (and $|\chi| = 1$). We obtain

$$\sup_{\chi\in B} \tilde{H}_a(\chi,\bar{\chi}) - \tfrac{1}{2} = \tfrac{a^2}{1-2a^2} \frac{1}{-4a^2+1} - \tfrac{1}{1-2a^2}\tfrac{1}{2} = \tfrac{1}{1-2a^2} \frac{6a^2-1}{2\left(1-4a^2\right)},$$

which is non-negative for $a > \frac{1}{6}\sqrt{6}$. □

Remark 3.3.9. Let $a \in \left(\frac{1}{6}\sqrt{6}, \frac{1}{2}\right)$. Notice that from Corollary 3.3.6 and Lemma 3.3.8 it follows that the Green function G_{Ω_a} becomes negative for conjugate points near the boundary. More precisely $G_{\Omega_a}((x,y),(x,-y)) < 0$ for (x,y) in a neighborhood of $h_a(-2a, \sqrt{1-4a^2})$. See Figure 3.5.

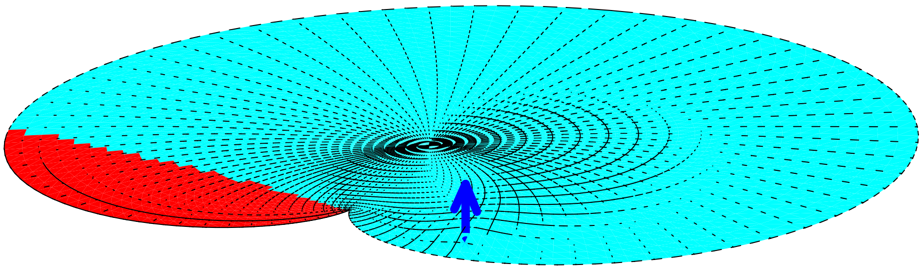

Figure 3.5: *Near strongly inward pointing boundary parts pushing up on one side results in bending downward on the other side due to the stiffness of the plate (fourth order) and the zero normal derivative. The arrow denotes the upward force; the dark part the bending downward. This figure has been taken from [25].*

3.4 Sharp estimates for the Green function

The Green function for the biharmonic problem in two dimensions does not have a singularity in the L^∞-sense: $(x,y) \mapsto G(x,y)$ is uniformly bounded. However, a natural solution space concerning the Dirichlet boundary condition ($u = \frac{\partial}{\partial \nu} u = 0$), see [4], is the Banach lattice (with the natural ordering):

$$C_e(\bar{\Omega}) = \{u \in C(\bar{\Omega}); \|u\|_e := \sup_{x \in \Omega} \left| \frac{u(x)}{d_\Omega^2(x)} \right| < \infty\},$$

where $d_\Omega(.)$ is as in (2.1.3). However $(x \mapsto G(x,.))$ from $\bar{\Omega}$ into $C_e(\bar{\Omega})$ does show 'a singularity' when $x \to \partial\Omega$. Precise information for the singularity of polyharmonic Dirichlet Green functions on balls in $\mathbb{R}^n$, where the Green function is known to be positive, can be found in [45].

The next theorem shows how the estimate of G_a from below changes depending on a. It is interesting to see that although the Green function becomes negative, no 'boundary-singularity' from below appears.

Theorem 3.4.1. *For every $(\eta, \xi) \in B \times B$, the following estimates hold:*

i. for all $a \in \left[0, \frac{1}{2}\right]$ there exists $c_1 > 0$ such that

$$\tilde{G}_a(\eta,\xi) \le c_1 \; d_B(\eta) d_B(\xi) \min\left\{1, \frac{d_B(\eta) d_B(\xi)}{|\eta - \xi|^2}\right\}, \tag{3.4.1}$$

ii. for all $a \in \left[0, \frac{1}{6}\sqrt{6}\right]$ *, there exists* $c_2 > 0$ *such that*

$$\tilde{G}_a(\eta,\xi) \geq c_2 \ \left(\tfrac{1}{6}\sqrt{6} - a\right) d_B(\eta) d_B(\xi) \min\left\{1, \frac{d_B(\eta)d_B(\xi)}{|\eta-\xi|^2}\right\}, \tag{3.4.2}$$

iii. for $a \in \left(\frac{1}{6}\sqrt{6}, \frac{1}{2}\right]$ *there exists* $(\eta^*, \xi^*) \in B \times B$ *such that*

$$\tilde{G}_a(\eta^*, \xi^*) < 0.$$

iv. for all $a \in \left(\frac{1}{6}\sqrt{6}, \frac{1}{2}\right]$ *, there exists* $c_3 > 0$ *such that*

$$\tilde{G}_a(\eta,\xi) \geq -c_3 \ \left(a - \tfrac{1}{6}\sqrt{6}\right) d_B(\eta)^2 d_B(\xi)^2, \tag{3.4.3}$$

where the constants c_1*,* c_2 *and* c_3 *do not depend on* a*.*

Remark 3.4.2. Let us observe that for every $\varepsilon > 0$ there exists two constants m_ε, M such that for every η, $\xi \in B$ and $a \in [0, \frac{1}{2} - \varepsilon]$ it holds:

$$\begin{array}{rcccl} m_\varepsilon \,.\, |\eta - \xi| & \leq & |h_a(\eta) - h_a(\xi)| & \leq & M \,.\, |\eta - \xi|\,, \\ m_\varepsilon \,.\, d(\eta, \partial B) & \leq & d(h_a(\eta), \partial\Omega_a) & \leq & M \,.\, d(\eta, \partial B). \end{array} \tag{3.4.4}$$

Using (3.4.4) one can prove estimates for G_a similar to the one proven for $\tilde{G}_a$ in Theorem 3.4.1. Near the cusp (when $a \to \frac{1}{2}$) the estimate from below in (3.4.4) breaks down.

Remark 3.4.3. One may derive that for $a \in \left[0, \frac{1}{6}\sqrt{6}\right]$ there exist constants c_4, c_5, independently of a, such that

$$c_4 \ \left(\tfrac{1}{6}\sqrt{6} - a\right) D_{\Omega_a}(x,y) \leq G_a\,(x,y) \leq c_5 \ D_{\Omega_a}(x,y),$$

where $D_{\Omega_a}(x,y) = d_{\Omega_a}(x) d_{\Omega_a}(y) \min\left\{1, \frac{d_{\Omega_a}(x) d_{\Omega_a}(y)}{|x-y|^2}\right\}$.

Remark 3.4.4. Note that the Green function is positive on the diagonal. This follows from the eigenfunction expansion and taking $x = y$:

$$G(x,y) = \sum_i \frac{1}{\lambda_i} \varphi_i(x) \varphi_i(y).$$

Here λ_i, φ_i are the eigenvalues/functions of the corresponding eigenvalue problem. Note that $\lambda_i > 0$ holds for all i.

Proof. We will prove the statements separately.

i. One has from (3.2.2) that

$$\begin{aligned}\tilde{G}_a(\eta,\xi) &\le \tfrac{1}{2}a^2s^2\left[-r^2\log\left(\frac{r_1^2}{r^2}\right)+r_1^2-r^2\right]\\ &\le 2\left[-r^2\log\left(\frac{r_1^2}{r^2}\right)+r_1^2-r^2\right]. \qquad (3.4.5)\end{aligned}$$

The term inside the brackets in the right hand side of (3.4.5) is the Green function for the clamped plate equation on the disk. Inequality (3.4.1) follows using the estimate in [41, Prop.2.3(iii)].

ii. Let $a_0 = \frac{1}{6}\sqrt{6}$ and $s_0 = |\eta+\xi+\frac{1}{a_0}|$. We consider first the case $a \in \left[\frac{1}{4}, \frac{1}{6}\sqrt{6}\right]$. Using that s is decreasing in a for all $\eta,\xi \in B$ when $a < \frac{1}{2}$, one finds for $a \in \left[\frac{1}{4}, \frac{1}{6}\sqrt{6}\right]$ that

$$\begin{aligned}\tilde{G}_a(\eta,\xi) &\ge \tfrac{1}{2}a^2\left(s_0^2r^2\log\left(\frac{r^2}{r_1^2}\right)+s_0^2(r_1^2-r^2)-\tfrac{a^2}{1-2a^2}(r_1^2-r^2)^2\right)\\ &= \tfrac{a^4}{1-2a^2}\tfrac{1-2a_0^2}{a_0^4}\tilde{G}_{a_0}(\eta,\xi)\\ &\quad+\tfrac{1}{2}a^2s_0^2\left(1-\tfrac{a^2}{1-2a^2}\tfrac{1-2a_0^2}{a_0^2}\right)\left[-r^2\log\left(\frac{r_1^2}{r^2}\right)+r_1^2-r^2\right]\\ &\ge \tfrac{1}{2}a^2s_0^2\left(1-4\tfrac{a^2}{1-2a^2}\right)\left[-r^2\log\left(\frac{r_1^2}{r^2}\right)+r_1^2-r^2\right],\end{aligned}$$

since $\tilde{G}_{a_0}(\eta,\xi) \ge 0$, see Corollary 3.3.7 and Lemma 3.3.8. For $a \in \left[\frac{1}{4}, \frac{1}{6}\sqrt{6}\right]$ one has $\frac{1}{2}a^2s_0^2(1-4\frac{a^2}{1-2a^2}) \ge \frac{1}{40}(\frac{1}{6}\sqrt{6}-a)$, hence by using [41, Prop.2.3(iii)] one gets

$$\tilde{G}_a(\eta,\xi) \ge c_2\ \left(\tfrac{1}{6}\sqrt{6}-a\right)d_B(\eta)\,d_B(\xi)\min\left\{1,\frac{d_B(\eta)\,d_B(\xi)}{|\eta-\xi|^2}\right\}.$$

For $a \in [0, \frac{1}{4}]$ one finds that

$$\tfrac{1}{2}\log\left(\frac{r^2}{r_1^2}\right)+\tfrac{1}{2}\left(\frac{r_1^2}{r^2}-1\right)-\tfrac{a^2}{1-2a^2}\frac{r^2}{s^2}\left(\frac{r_1^2}{r^2}-1\right)^2 \ge 0. \qquad (3.4.6)$$

Indeed with $q = \frac{r_1^2}{r^2}$ formula (3.4.6) can be written as

$$\begin{aligned}&-\tfrac{1}{2}\log(q)+\tfrac{1}{2}(q-1)-\tfrac{a^2}{1-2a^2}\frac{r_1^2}{s^2}\frac{(q-1)^2}{q}\\ \ge\ &-\tfrac{1}{2}\log(q)+\tfrac{1}{2}(q-1)-\tfrac{a^4}{1-2a^2}\tfrac{4}{|a\xi+a\eta+1|^2}\frac{(q-1)^2}{q}\\ \ge\ &-\tfrac{1}{2}\log(q)+\tfrac{1}{2}(q-1)-\tfrac{a^4}{1-2a^2}\tfrac{4}{(1-2a)^2}\frac{(q-1)^2}{q}\\ \ge\ &-\tfrac{1}{2}\log(q)+\tfrac{1}{2}(q-1)-\tfrac{1}{14}\frac{(q-1)^2}{q},\end{aligned}$$

that is non-negative. Hence from formula (3.2.2) one finds for $a \in [0, \frac{1}{4}]$

$$G_a(x,y) \geq \tfrac{1}{4}a^2 s^2 r^2 \left[\log\left(\frac{r^2}{r_1^2}\right) + \frac{r_1^2}{r^2} - 1\right].$$

The claim follows by using [41, Prop.2.3(iii)].

iii. This claim follows from Corollary 3.3.7 and Lemma 3.3.8.

iv. Let $a_0 = \frac{1}{6}\sqrt{6}$ and $s_0 = |\eta + \xi + \frac{1}{a_0}|$. We have

$$\begin{aligned}
\tilde{G}_a(\eta,\xi) &= \frac{1}{2}a^2\frac{s^2}{s_0^2}\left(r^2 s_0^2 \log\left(\frac{r^2}{r_1^2}\right) + s_0^2(r_1^2 - r^2) - \frac{a_0^2}{1-2a_0^2}\left(r_1^2 - r^2\right)^2\right) \\
&\quad + \frac{1}{2}a^2\left(\frac{s^2}{s_0^2}\frac{a_0^2}{1-2a_0^2} - \frac{a^2}{1-2a^2}\right)\left(r_1^2 - r^2\right)^2 \\
&\geq \frac{1}{2}a^2\left(\frac{s^2}{s_0^2}\frac{1}{4} - \frac{a^2}{1-2a^2}\right)\left(r_1^2 - r^2\right)^2, \qquad (3.4.7)
\end{aligned}$$

since $\tilde{G}_{a_0}$ is positive in the entire domain. Using that $s_0^2 \geq (\sqrt{6}-2)^2$ one gets that

$$\begin{aligned}
&\frac{1}{2}a^2\left(\frac{s^2}{s_0^2}\frac{1}{4} - \frac{a^2}{1-2a^2}\right) = \frac{1}{8}a^2\left(\frac{1-6a^2}{1-2a^2} + \frac{s^2 - s_0^2}{s_0^2}\right) \geq \\
&\geq -\frac{1}{8}a^2\left(\frac{1+\sqrt{6}a}{1-2a^2}\sqrt{6} + \frac{1}{(\sqrt{6}-2)^2}\left(\frac{1}{a} + \sqrt{6} + 4\right)\frac{\sqrt{6}}{a}\right)(a - \frac{1}{6}\sqrt{6}) \\
&\geq -7\left(a - \frac{1}{6}\sqrt{6}\right), \qquad (3.4.8)
\end{aligned}$$

Hence, from (3.4.7) and (3.4.8) it follows that there exists a constant $c_3 > 0$ such that

$$\tilde{G}_a(\eta,\xi) \geq -c_3\left(a - \frac{1}{6}\sqrt{6}\right) d_B(\eta)^2 d_B(\xi)^2,$$

for $a \in (\frac{1}{6}\sqrt{6}, \frac{1}{2})$.

$\square$

3.5 The Bergman kernel

Let Ω be a domain in $\mathbb{R}^n$ and $G_\Omega(.,.)$ be the Green function associated to problem (3.1.1) on Ω. The quantity $-\Delta_x\Delta_y G_\Omega(x,y)$ is called the Bergman kernel function. For the definition and the properties of the Bergman kernel we refer to [7, pages 127-131] and [35, pages 266-268]. What we are interested in is the relation between the sign of the Green function and the sign of the Bergman kernel function. From the Taylor expansion at the boundary of the Green function it follows that

$$G_\Omega(x,y) \geq 0 \text{ in } \Omega^2 \Rightarrow -\Delta_x\Delta_y G_\Omega(x,y) \leq 0 \text{ for } (x,y) \in \partial\Omega \times \partial\Omega \setminus D_{\partial\Omega}. \tag{3.5.1}$$

with $D_{\partial\Omega} := \{(z,z) : z \in \partial\Omega\}$. For relation (3.5.1) see e.g. [34, page 510].

It is an open problem whether the converse of (3.5.1) holds. We will show via a direct computation that in the case of the limaçons this converse does hold. The main result is the following.

Theorem 3.5.1. *The Bergman kernel function $-\Delta_x\Delta_y G_a(x,y)$ is non-positive on $\partial\Omega \times \partial\Omega \setminus D_{\partial\Omega}$ if and only if $a \in [0, \frac{1}{6}\sqrt{6}]$.*

Theorem 3.5.1 shows that the value of the parameter a critical to have positivity of the Green function is also critical for the positivity of minus the Bergman kernel function. We will also show that the points at the boundary where the Green function becomes negative are also the points where minus the Bergman kernel function becomes negative.

In the following technical lemma we compute the Bergman kernel function for the limaçon.

Lemma 3.5.2. *The Bergman kernel function for the limaçon is given for $x, y \in \partial\Omega_a$ by*

$$-\Delta_x\Delta_y G_a(x,y) = 16\,|2a\eta+1|^2\,|2a\xi+1|^2 \left(\frac{a\eta + a\bar{\xi} + a\bar{\eta} + a\xi + 1 + 4a^2}{\eta\bar{\xi} + \bar{\eta}\xi - 2} + \frac{a^2}{1-2a^2} \right),$$

with $\eta, \xi \in \partial B$ such that $h_a(\eta) = x$ and $h_a(\xi) = y$.

Proof. Since it holds

$$-\Delta_x\Delta_y G_a(x,y) = -\,|2a\eta+1|^2\,|2a\xi+1|^2\left(\Delta_\xi\Delta_\eta G_a(h_a(\eta), h_a(\xi))\right), \tag{3.5.2}$$

it remains to study the term $\Delta_\xi\Delta_\eta G_a(h_a(\eta), h_a(\xi))$.

In the following we use complex notation. Notice that $\Delta_\xi = 4\frac{\partial}{\partial\bar{\xi}}\frac{\partial}{\partial\xi}$.

One has

$$\begin{aligned}
&\frac{\partial}{\partial\xi} G_a(h_a(\eta), h_a(\xi)) \\
={}& \tfrac{1}{2}a^2(\bar{\xi} + \bar{\eta} + \tfrac{1}{a}) \left[|\eta-\xi|^2 \log\frac{|\eta-\xi|^2}{|1-\eta\bar{\xi}|^2} + |1-\eta\bar{\xi}|^2 - |\eta-\xi|^2 \right] \\
&+ \tfrac{1}{2}a^2 \left|\xi + \eta + \tfrac{1}{a}\right|^2 \left[-(\bar{\eta}-\bar{\xi}) \log\frac{|\eta-\xi|^2}{|1-\eta\bar{\xi}|^2} + \frac{|\eta-\xi|^2}{1-\bar{\eta}\xi}\bar{\eta} - \bar{\eta}(1-\eta\bar{\xi}) \right] \\
&- \tfrac{a^4}{1-2a^2}\left(|1-\eta\bar{\xi}|^2 - |\eta-\xi|^2 \right)(-\bar{\eta}(1-\eta\bar{\xi}) + \bar{\eta} - \bar{\xi})
\end{aligned}$$

$$
\begin{aligned}
= \;& -\tfrac{1}{2}a^2(\bar{\xi}+\bar{\eta}+\tfrac{1}{a})(\bar{\eta}-\bar{\xi})(2\xi+\tfrac{1}{a})\log\frac{|\eta-\xi|^2}{\left|1-\eta\bar{\xi}\right|^2} \\
& +\tfrac{1}{2}a^2(\bar{\xi}+\bar{\eta}+\tfrac{1}{a})\left[\left|1-\eta\bar{\xi}\right|^2-|\eta-\xi|^2\right]+\tfrac{1}{2}a^2\left|\xi+\eta+\tfrac{1}{a}\right|^2\bar{\eta}\left[\frac{|\eta-\xi|^2}{1-\bar{\eta}\xi}-(1-\eta\bar{\xi})\right] \\
& +\tfrac{a^4}{1-2a^2}\left(\left|1-\eta\bar{\xi}\right|^2-|\eta-\xi|^2\right)\bar{\xi}(1-\bar{\eta}\eta);
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\partial^2}{\partial\bar{\xi}\xi}G_a(h_a(\eta),h_a(\xi)) \\
= \;& \tfrac{1}{2}a^2\left|2\xi+\tfrac{1}{a}\right|^2\log\frac{|\eta-\xi|^2}{\left|1-\eta\bar{\xi}\right|^2}+\tfrac{1}{2}a^2(\bar{\xi}+\bar{\eta}+\tfrac{1}{a})(2\xi+\tfrac{1}{a}) \\
& -\tfrac{1}{2}a^2(\bar{\xi}+\bar{\eta}+\tfrac{1}{a})(\bar{\eta}-\bar{\xi})(2\xi+\tfrac{1}{a})\eta\frac{1}{1-\eta\bar{\xi}}+\tfrac{1}{2}a^2\left[\left|1-\eta\bar{\xi}\right|^2-|\eta-\xi|^2\right] \\
& -\tfrac{1}{2}a^2(\bar{\xi}+\bar{\eta}+\tfrac{1}{a})\xi(1-\eta\bar{\eta})-\tfrac{1}{2}a^2(\xi+\eta+\tfrac{1}{a})\frac{\eta-\xi}{1-\bar{\eta}\xi}(2\bar{\xi}+\tfrac{1}{a})\bar{\eta} \\
& -\tfrac{1}{2}a^2(\xi+\eta+\tfrac{1}{a})\bar{\eta}(1-\eta\bar{\xi})+\tfrac{1}{2}a^2\left|\xi+\eta+\tfrac{1}{a}\right|^2\bar{\eta}\eta-\tfrac{a^4}{1-2a^2}(1-\eta\bar{\eta})^2\xi\bar{\xi} \\
& +\tfrac{a^4}{1-2a^2}\left(\left|1-\eta\bar{\xi}\right|^2-|\eta-\xi|^2\right)(1-\bar{\eta}\eta);
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\partial^3}{\partial\eta\bar{\xi}\xi}G_a(h_a(\eta),h_a(\xi)) \\
= \;& \tfrac{1}{2}a^2\left|2\xi+\tfrac{1}{a}\right|^2\frac{1}{\eta-\xi}+\tfrac{1}{2}a^2\left|2\xi+\tfrac{1}{a}\right|^2\frac{\bar{\xi}}{1-\eta\bar{\xi}} \\
& -\tfrac{1}{2}a^2(\bar{\xi}+\bar{\eta}+\tfrac{1}{a})(\bar{\eta}-\bar{\xi})(2\xi+\tfrac{1}{a})\frac{1}{(1-\eta\bar{\xi})^2} \\
& -\tfrac{1}{2}a^2\bar{\eta}\left[1-\bar{\xi}\xi\right]+\tfrac{1}{2}a^2(\bar{\xi}+\bar{\eta}+\tfrac{1}{a})\xi\bar{\eta}-\tfrac{1}{2}a^2\frac{\bar{\eta}}{1-\bar{\eta}\xi}(2\bar{\xi}+\tfrac{1}{a})(2\eta+\tfrac{1}{a}) \\
& -\tfrac{1}{2}a^2\bar{\eta}(1-\eta\bar{\xi})+\tfrac{1}{2}a^2(\xi+\eta+\tfrac{1}{a})\bar{\eta}\bar{\xi}+\tfrac{1}{2}a^2(\bar{\xi}+\bar{\eta}+\tfrac{1}{a})\bar{\eta}\eta+\tfrac{1}{2}a^2\left|\xi+\eta+\tfrac{1}{a}\right|^2\bar{\eta} \\
& +\tfrac{2a^4}{1-2a^2}(1-\eta\bar{\eta})\bar{\eta}\xi\bar{\xi}-\tfrac{a^4}{1-2a^2}\left(1-\bar{\xi}\xi\right)\bar{\eta}(1-\bar{\eta}\eta)-\tfrac{a^4}{1-2a^2}\left(\left|1-\eta\bar{\xi}\right|^2-|\eta-\xi|^2\right)\bar{\eta}
\end{aligned}
$$

$$
\begin{aligned}
= \;& \tfrac{1}{2}a^2\left|2\xi+\tfrac{1}{a}\right|^2\frac{1}{\eta-\xi}+\tfrac{1}{2}a^2\left|2\xi+\tfrac{1}{a}\right|^2\frac{\bar{\xi}}{1-\eta\bar{\xi}} \\
& -\tfrac{1}{2}a^2(\bar{\xi}+\bar{\eta}+\tfrac{1}{a})(\bar{\eta}-\bar{\xi})(2\xi+\tfrac{1}{a})\frac{1}{(1-\eta\bar{\xi})^2}-\tfrac{1}{2}a^2\bar{\eta}(1-\bar{\xi}\xi) \\
& +\tfrac{1}{2}a^2(\bar{\xi}+\bar{\eta}+\tfrac{1}{a})(\xi+\eta)\bar{\eta}-\tfrac{1}{2}a^2\frac{\bar{\eta}}{1-\bar{\eta}\xi}(2\bar{\xi}+\tfrac{1}{a})(2\eta+\tfrac{1}{a}) \\
& -\tfrac{1}{2}a^2\bar{\eta}(1-\eta\bar{\xi})+\tfrac{1}{2}a^2(\xi+\eta+\tfrac{1}{a})\bar{\eta}\bar{\xi}+\tfrac{1}{2}a^2\left|\xi+\eta+\tfrac{1}{a}\right|^2\bar{\eta} \\
& +\tfrac{2a^4}{1-2a^2}(1-\eta\bar{\eta})\bar{\eta}\xi\bar{\xi}-\tfrac{a^4}{1-2a^2}\left(1-\bar{\xi}\xi\right)\bar{\eta}(1-\bar{\eta}\eta)-\tfrac{a^4}{1-2a^2}\left(\left|1-\eta\bar{\xi}\right|^2-|\eta-\xi|^2\right)\bar{\eta};
\end{aligned}
$$

$$\begin{aligned}
&\frac{\partial^4}{\partial\bar\eta\eta\bar\xi\xi}G_a(h_a(\eta),h_a(\xi))\\
=\ & -\tfrac12 a^2(2\bar\eta+\tfrac1a)(2\xi+\tfrac1a)\frac{1}{(1-\eta\bar\xi)^2}-\tfrac12 a^2(1-\bar\xi\xi)\\
&+\tfrac12 a^2(\xi+\eta)\bar\eta+\tfrac12 a^2(\bar\xi+\bar\eta+\tfrac1a)(\xi+\eta)-\tfrac12 a^2\frac{1}{(1-\bar\eta\xi)^2}(2\bar\xi+\tfrac1a)(2\eta+\tfrac1a)\\
&-\tfrac12 a^2(1-\eta\bar\xi)+\tfrac12 a^2(\xi+\eta+\tfrac1a)\bar\xi+\tfrac12 a^2(\xi+\eta+\tfrac1a)\bar\eta+\tfrac12 a^2\left|\xi+\eta+\tfrac1a\right|^2\\
&-\tfrac{2a^4}{1-2a^2}\eta\bar\eta\xi\bar\xi+\tfrac{2a^4}{1-2a^2}\left(1-\eta\bar\eta\right)\xi\bar\xi-\tfrac{a^4}{1-2a^2}\left(1-\bar\xi\xi\right)\left(1-\bar\eta\eta\right)+\tfrac{a^4}{1-2a^2}\left(1-\bar\xi\xi\right)\bar\eta\eta\\
&+\tfrac{a^4}{1-2a^2}(1-\xi\bar\xi)\eta\bar\eta-\tfrac{a^4}{1-2a^2}\left(\left|1-\eta\bar\xi\right|^2-|\eta-\xi|^2\right)
\end{aligned}$$

$$\begin{aligned}
=\ & -\tfrac12 a^2(2\bar\eta+\tfrac1a)(2\xi+\tfrac1a)\frac{1}{(1-\eta\bar\xi)^2}-\tfrac12 a^2\frac{1}{(1-\bar\eta\xi)^2}(2\bar\xi+\tfrac1a)(2\eta+\tfrac1a)\\
&-\tfrac12 a^2(1-\bar\xi\xi)-\tfrac12 a^2(1-\eta\bar\eta)+\tfrac12 a^2(\xi\bar\eta+\eta\bar\xi)+\tfrac12 a^2\left|\xi+\eta+\tfrac1a\right|^2\\
&+\tfrac12 a^2(\xi+\eta+\tfrac1a)(\bar\xi+\bar\eta)+\tfrac12 a^2(\bar\xi+\bar\eta+\tfrac1a)(\xi+\eta)\\
&-\tfrac{2a^4}{1-2a^2}\eta\bar\eta\xi\bar\xi+\tfrac{2a^4}{1-2a^2}\left(1-\eta\bar\eta\right)\xi\bar\xi-\tfrac{a^4}{1-2a^2}\left(1-\bar\xi\xi\right)\left(1-\bar\eta\eta\right)\\
&+2\tfrac{a^4}{1-2a^2}(1-\xi\bar\xi)\eta\bar\eta-\tfrac{a^4}{1-2a^2}\left(\left|1-\eta\bar\xi\right|^2-|\eta-\xi|^2\right).
\end{aligned}$$

Considering $x,y\in\partial\Omega_a$ and so $\eta,\xi\in\partial B$ we find

$$\begin{aligned}
&\tfrac{1}{16}\Delta_\eta\Delta_\eta G_a(h_a(\eta),h_a(\xi))\\
=\ & -\tfrac12 a^2(2\bar\eta+\tfrac1a)(2\xi+\tfrac1a)\frac{1}{(1-\eta\bar\xi)^2}-\tfrac12 a^2\frac{1}{(1-\bar\eta\xi)^2}(2\bar\xi+\tfrac1a)(2\eta+\tfrac1a)\\
&+\tfrac12 a^2(\xi\bar\eta+\eta\bar\xi)+\tfrac12 a^2\left|\xi+\eta+\tfrac1a\right|^2\\
&+\tfrac12 a^2(\xi+\eta+\tfrac1a)(\bar\xi+\bar\eta)+\tfrac12 a^2(\bar\xi+\bar\eta+\tfrac1a)(\xi+\eta)-\tfrac{2a^4}{1-2a^2}=
\end{aligned}$$

$$\begin{aligned}
=\ & -\tfrac12 a^2(2\bar\eta+\tfrac1a)(2\xi+\tfrac1a)\frac{1}{(1-\eta\bar\xi)^2}-\tfrac12 a^2\frac{1}{(1-\bar\eta\xi)^2}(2\bar\xi+\tfrac1a)(2\eta+\tfrac1a)\\
&+\tfrac12 a^2\left(4\xi\bar\eta+4\bar\xi\eta+6+\frac2a(\xi+\bar\xi)+\frac2a(\eta+\bar\eta)+\frac{1}{a^2}\right)-\tfrac{2a^4}{1-2a^2}\\
=\ & -\tfrac12 a^2(2\bar\eta+\tfrac1a)(2\xi+\tfrac1a)\frac{1}{(1-\eta\bar\xi)^2}-\tfrac12 a^2\frac{1}{(1-\bar\eta\xi)^2}(2\bar\xi+\tfrac1a)(2\eta+\tfrac1a)\\
&+\tfrac12 a^2\left((2\bar\eta+\tfrac1a)(2\xi+\tfrac1a)+(2\bar\xi+\tfrac1a)(2\eta+\tfrac1a)+6-\frac{1}{a^2}\right)-\tfrac{2a^4}{1-2a^2}\\
=\ & -\tfrac12 a^2(2\bar\eta+\tfrac1a)(2\xi+\tfrac1a)\frac{2-\eta\bar\xi}{(1-\eta\bar\xi)^2}\eta\bar\xi-\tfrac12 a^2(2\bar\xi+\tfrac1a)(2\eta+\tfrac1a)\frac{2-\bar\eta\xi}{(1-\bar\eta\xi)^2}\bar\eta\xi\\
&+\tfrac12 a^2(6-\tfrac{1}{a^2})-\tfrac{2a^4}{1-2a^2}=
\end{aligned}$$

$$
\begin{aligned}
&= -\tfrac{1}{2}a^2\left(2\xi+\tfrac{1}{a}\right)\left(2\bar\eta+\tfrac{1}{a}\right)\frac{2-\eta\bar\xi}{\left(1-\bar\eta\xi\right)^2\eta^2\bar\xi^2}\eta\bar\xi \\
&\quad -\tfrac{1}{2}a^2\left(2\bar\xi+\tfrac{1}{a}\right)\left(2\eta+\tfrac{1}{a}\right)\frac{2-\bar\eta\xi}{\left(1-\bar\eta\xi\right)^2}\frac{\bar\eta\xi}{\eta\bar\eta\xi\bar\xi}+\tfrac{1}{2}a^2(6-\tfrac{1}{a^2})-\tfrac{2a^4}{1-2a^2} \\
&= -\tfrac{1}{2}a^2\frac{\left(2\xi+\tfrac{1}{a}\right)\left(2\bar\eta+\tfrac{1}{a}\right)\left(2-\eta\bar\xi\right)+\left(2\bar\xi+\tfrac{1}{a}\right)\left(2\eta+\tfrac{1}{a}\right)\left(2-\bar\eta\xi\right)}{\left(1-\bar\eta\xi\right)^2\eta\bar\xi} \\
&\quad +\tfrac{1}{2}a^2(6-\tfrac{1}{a^2})-\tfrac{2a^4}{1-2a^2}=
\end{aligned}
$$

$$
\begin{aligned}
&= -\tfrac{1}{2}a^2\frac{8\xi\bar\eta+\tfrac{4}{a}\bar\eta+\tfrac{4}{a}\xi+\tfrac{2}{a^2}-4-\tfrac{2}{a}\eta-\tfrac{2}{a}\bar\xi-\tfrac{1}{a^2}\eta\bar\xi}{\left(1-\bar\eta\xi\right)^2\eta\bar\xi} \\
&\quad -\tfrac{1}{2}a^2\frac{8\bar\xi\eta+\tfrac{4}{a}\eta+\tfrac{4}{a}\bar\xi+\tfrac{2}{a^2}-4-\tfrac{2}{a}\bar\eta-\tfrac{2}{a}\xi-\tfrac{1}{a^2}\bar\eta\xi}{\left(1-\bar\eta\xi\right)^2\eta\bar\xi}+\tfrac{1}{2}a^2(6-\tfrac{1}{a^2})-\tfrac{2a^4}{1-2a^2} \\
&= -a^2\left[\frac{a\eta+a\bar\xi+a\bar\eta+a\xi+1+4a^2}{a^2\left(\eta\bar\xi+\bar\eta\xi-2\right)}+\frac{1}{1-2a^2}\right]
\end{aligned}
\tag{3.5.3}
$$

The claim follows directly from (3.5.2) and (3.5.3). □

Proof of Theorem 3.5.1. By the result of Lemma 3.5.2 one sees that in determining the sign of the Bergman kernel the interesting part is the term

$$\frac{a\eta+a\bar\xi+a\bar\eta+a\xi+1+4a^2}{\eta\bar\xi+\bar\eta\xi-2}+\frac{a^2}{1-2a^2}. \tag{3.5.4}$$

We first find where the maximum of (3.5.4) is attained.

We consider

$$\max_{\xi,\eta\in\partial B}\frac{a\eta+a\bar\xi+a\bar\eta+a\xi+1+4a^2}{\eta\bar\xi+\bar\eta\xi-2}=-a\min_{\alpha,\beta\in[0,2\pi]}\frac{\cos(\alpha)+\cos(\beta)+\frac{1+4a^2}{2a}}{1-\cos(\alpha-\beta)}.$$

For sake of conciseness let fix the following function

$$F(\alpha,\beta):=\frac{\cos(\alpha)+\cos(\beta)+\frac{1+4a^2}{2a}}{1-\cos(\alpha-\beta)}.$$

Since,

$$
\begin{aligned}
\partial_\alpha F(\alpha,\beta) &= -\frac{\sin\alpha\left(1-\cos(\alpha-\beta)\right)+\sin(\alpha-\beta)(\cos\alpha+\cos\beta+\frac{1+4a^2}{2a})}{\left(1-\cos(\alpha-\beta)\right)^2}, \\
\partial_\beta F(\alpha,\beta) &= -\frac{\sin\beta\left(1-\cos(\alpha-\beta)\right)+\sin(\beta-\alpha)(\cos\alpha+\cos\beta+\frac{1+4a^2}{2a})}{\left(1-\cos(\alpha-\beta)\right)^2},
\end{aligned}
$$

one sees that the minimum of F is attained at $\alpha=-\beta$ and $\cos(\alpha)=\cos(\beta)=-2a$.

Hence, the maximum is attained at the couple of conjugate points

$$\xi = (-2a, \sqrt{1-4a^2}) \text{ and } \eta = (-2a, -\sqrt{1-4a^2}).$$

Using the result of Lemma 3.5.2 we find that $-\Delta_x \Delta_y G_a(x,y)$ becomes somewhere positive in $\partial\Omega_a \times \partial\Omega_a$ when the term in (3.5.4) becomes somewhere positive in $\partial B \times \partial B$. Considering the maximum one finds

$$\begin{aligned}
&\max_{\xi,\eta\in\partial B}\left(\frac{a\eta + a\bar{\xi} + a\bar{\eta} + a\xi + 1 + 4a^2}{\eta\bar{\xi} + \bar{\eta}\xi - 2} + \frac{a^2}{1-2a^2}\right) \\
= \quad & -a\frac{-4a + \frac{1+4a^2}{2a}}{2-8a^2} + \frac{a^2}{1-2a^2} \\
= \quad & -a\frac{-8a^2 + 1 + 4a^2}{4a(1-4a^2)} + \frac{a^2}{1-2a^2} \\
= \quad & -\tfrac{1}{4} + \frac{a^2}{1-2a^2} = \frac{6a^2-1}{4(1-2a^2)}. \qquad (3.5.5)
\end{aligned}$$

From (3.5.5) it follows that the Bergman kernel function changes sign (becoming somewhere positive) when $a > \frac{1}{6}\sqrt{6}$. □

Remark 3.5.3. Notice that the points where minus the Bergman kernel becomes somewhere negative are exactly the points where the Green function changes sign. Compare with Remark 3.3.9.

Chapter 4

Positivity for the Clamped Plate

4.1 Introduction

In 1905 Boggio in [8] gave an explicit Green formula for the clamped plate equation on a disk, that is, for the boundary value problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \frac{\partial}{\partial \nu} u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1.1}$$

with $\Omega = B = \left\{ x \in \mathbb{R}^2; |x| < 1 \right\}$. As a direct consequence of that formula one finds that (4.1.1) is positivity preserving:

$$f > 0 \text{ implies } u > 0. \tag{4.1.2}$$

Boggio and Hadamard conjectured that such a property holds on almost any (convex) domain. By now this conjecture has numerous counterexamples. Duffin [31] was the first one who in 1949 showed that on the infinite strip a positive f exists for which (4.1.1) has a sign-changing solution u. Garabedian [34] obtained a similar result for an elongated ellipse with axes having ratio 2. Other domains such as non-simply connected ones ([14]) and domains with corners ([58],[13]) followed. It was believed that most non-circular domains failed to have the sign preserving property, or as Hayman and Korenblum stated in [48]: "We are tempted to conjecture that balls are the only domains in $\mathbb{R}^n$". But since they consider the sign not just for biharmonic but for all polyharmonic Green functions they could still be right.

In this chapter we will show that for the biharmonic there are many domains even quite different from the disk where the clamped plate problem is positivity preserving. A first result in this direction was obtained in [40]. The authors did show via a perturbation argument that on domains very close to the disk (4.1.2) remained. The result concerns domains that are close to the disk in $C^{2,\gamma}$-sense and so necessarily convex domains. In Chapter 3, [26], we prove that also on limaçons Ω_a for $a \in [0, \frac{1}{6}\sqrt{6}]$ (4.1.2) holds. This result shows that convexity is not a necessary condition for the positivity preserving property of the clamped plate equation.

Next to showing some more domains for which (4.1.2) holds we aim to survey the limited methods to find such domains that we know to be available presently. In doing so we will also explain that the Möbius transformation plays a special role not only in higher dimensions but also for polyharmonic equations in 2 dimensions.

Different ways of finding domains other than a disk and for which (4.1.2) holds will be addressed in the next sections. Although each of these approaches is known, the combination has not been exploited. The perturbation that we state has a wider range than the version published in [40]. Finally we observe that one ingredient for possible extensions of these results are the optimal estimates from above for the polyharmonic Green functions and its derivatives on general domains proved in Chapter 2. See also [24].

4.2 Direct approaches

The example $(x^3-x)'''' = 0$ immediately shows that for the biharmonic one cannot proceed to a positivity preserving property by way of the local maximum principle as for second order elliptic equations. A way out is to start from a domain with an explicitly known positive Green function and try to transform this to another domain. One may start from the result of Boggio mentioned above or from the result concerning the limaçons.

By the way, Boggio in [8] not only derived the Green function for the clamped plate equation on the disk but did even so for any polyharmonic equation, $(-\Delta)^m u = f$, on a ball in any dimension under zero Dirichlet boundary conditions $u = \frac{\partial}{\partial n} u = \cdots = \left(\frac{\partial}{\partial n}\right)^{m-1} u = 0$. This Green function is as follows:

$$G_B(x,y) = c_{n,m} |x-y|^{2m-n} \int_0^{\frac{(1-|x|^2)(1-|y|^2)}{|x-y|^2}} w^{m-1} (1+w)^{-\frac{1}{2}n} \, dw,$$

with $c_{n,m}$ some explicit constants. The solution of (4.1.1) is $u(x) = \int_B G_B(x,y) f(y) dy$.

One might try to transfer this formula or, in the two-dimensional case, the formula of the Green function for the limaçon given in [46] (see also formula (3.2.2)) to other domains. A necessary condition that such a transformation h at least keeps the highest order terms polyharmonic, that is $(-\Delta)^m (w. (u \circ h)) = \tilde{w}. ((-\Delta)^m u) \circ h + l.o.t.$, is that h is conformal. Without the conformality assumption the transformed differential equation would become anisotropic.

In the following we address such conformal mappings. The two-dimensional case differs considerably from the higher dimensional situation. While for $n = 2$ there are many conformal maps in dimensions $n \geq 3$ the only conformal maps are the Möbius transformations. We first consider the two-dimensional case and then we study the special role played by the Möbius transformations in dimension 2 and higher.

4.2.1 Conformal mappings in two dimensions

For the second order Laplace equation in two dimensions Riemann's Mapping Theorem allows us to solve

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{4.2.1}$$

at least for simply connected Ω by way of a Green function for the disk. Indeed, Riemann's Mapping Theorem says that there exists a conformal map $h : B \to \Omega$ that is bijective. Moreover it holds that

$$\Delta\left(u \circ h\right) = |\mathbf{h}'|^2 \left(\Delta u\right) \circ h, \tag{4.2.2}$$

where $\mathbf{h}(x+iy) = h_1(x,y)+ih_2(x,y)$. Hence, since $|\mathbf{h}'(x+iy)|^2 = J_h(x,y)$, the Jacobian of h, the solution u of (4.2.1) in Ω is given by

$$u(x) = \int_\Omega G_B(h^{-1}(x), h^{-1}(y))f(y)dy.$$

For the biharmonic equation we could try to mimic this approach even if we have to add weight functions. Let us recall that a mapping ϕ is called a similarity if there are $c \in \mathbb{R}^+$, $a \in \mathbb{R}^n$ and an orthogonal matrix F such that $\phi(x) = a + cFx$. In the following boldface is used for the complex alternative.

Lemma 4.2.1. *Let $h \in C^1(\bar{A}; \mathbb{R}^2)$ be a conformal mapping from A to $\Omega \subset \mathbb{R}^2$ and suppose h is not a similarity. Then there exists a meromorphic function $\mathbf{f}$ defined on A and a number c such that for all $u \in C^4(\bar{\Omega})$:*

$$\Delta^2\left(|\mathbf{f}|^2\left(u \circ h\right)\right) = c\ |\mathbf{f}|^2|\mathbf{h}'|^4\left(\Delta^2 u\right) \circ h$$

if and only if h is a Möbius transformation, $c = 1$ and $|\mathbf{f}|^2|\mathbf{h}'|$ is constant.

Proof. Using the complex notation and new independent variables $z = x + iy$ and $\bar{z} = x - iy$ the notations will simplify. Setting $U(x+iy, x-iy) = u(x,y)$ we find

$$\Delta u = 4\partial_{\bar{z}}\partial_z U.$$

Notice that formally $\overline{\mathbf{h}(z)} = \bar{\mathbf{h}}(\bar{z})$ and hence $\partial_{\bar{z}}\overline{\mathbf{h}(z)} = \overline{\mathbf{h}'(z)}$. With $\mathbf{h} = \mathbf{h}(z)$, $\bar{\mathbf{h}} = \bar{\mathbf{h}}(\bar{z})$and $F_1(x_1,x_2) := \frac{\partial}{\partial x_1}F(x_1,x_2)$ and a tedious computation:

$$\partial_{\bar{z}}\partial_z\partial_{\bar{z}}\partial_z\left(\mathbf{f}(z)\bar{\mathbf{f}}(\bar{z})\ U(\mathbf{h}(z),\bar{\mathbf{h}}(\bar{z}))\right) = \tag{4.2.3}$$

$$\begin{aligned}
&= \partial_{\bar{z}}\partial_z\partial_{\bar{z}}\left(\mathbf{f}'\ \bar{\mathbf{f}}\ U + \mathbf{f}\ \bar{\mathbf{f}}\ U_1\ \mathbf{h}'\right) \\
&= \partial_{\bar{z}}\partial_z\left(\mathbf{f}'\ \bar{\mathbf{f}}'\ U + \mathbf{f}'\ \bar{\mathbf{f}}\ U_2\ \bar{\mathbf{h}}' + \mathbf{f}\ \bar{\mathbf{f}}'\ U_1\ \mathbf{h}' + \mathbf{f}\ \bar{\mathbf{f}}\ U_{12}\ \mathbf{h}'\ \bar{\mathbf{h}}'\right) \\
&= \partial_{\bar{z}}\left(\mathbf{f}''\ \bar{\mathbf{f}}'\ U + \mathbf{f}'\ \bar{\mathbf{f}}'\ U_1\ \mathbf{h}' + \mathbf{f}''\ \bar{\mathbf{f}}\ U_2\ \bar{\mathbf{h}}' + \mathbf{f}'\ \bar{\mathbf{f}}\ U_{21}\ \mathbf{h}'\ \bar{\mathbf{h}}'\right) + \\
&\quad +\partial_{\bar{z}}\left(\mathbf{f}'\ \bar{\mathbf{f}}'\ U_1\ \mathbf{h}' + \mathbf{f}\ \bar{\mathbf{f}}'\ U_{11}\ \left(\mathbf{h}'\right)^2 + \mathbf{f}\ \bar{\mathbf{f}}'\ U_1\ \mathbf{h}''\right) + \\
&\quad +\partial_{\bar{z}}\left(\mathbf{f}'\ \bar{\mathbf{f}}\ U_{12}\ \mathbf{h}'\ \bar{\mathbf{h}}' + \mathbf{f}\ \bar{\mathbf{f}}\ U_{121}\ \left(\mathbf{h}'\right)^2\bar{\mathbf{h}}' + \mathbf{f}\ \bar{\mathbf{f}}\ U_{12}\ \mathbf{h}''\ \bar{\mathbf{h}}'\right)
\end{aligned}$$

$$
\begin{aligned}
= \;& \mathbf{f}'' \, \bar{\mathbf{f}}'' \, U + \mathbf{f}'' \, \bar{\mathbf{f}}' \, U_2 \bar{\mathbf{h}}' + \mathbf{f}' \, \bar{\mathbf{f}}'' \, U_1 \, \mathbf{h}' + \mathbf{f}' \, \bar{\mathbf{f}}' \, U_{12} \, \mathbf{h}' \bar{\mathbf{h}}' + \\
& + \mathbf{f}'' \, \bar{\mathbf{f}}' \, U_2 \, \bar{\mathbf{h}}' + \mathbf{f}'' \, \bar{\mathbf{f}} \, U_{22} \, (\bar{\mathbf{h}}')^2 + \mathbf{f}'' \, \bar{\mathbf{f}} \, U_2 \, \bar{\mathbf{h}}'' \\
& + \mathbf{f}' \, \bar{\mathbf{f}}' \, U_{21} \, \mathbf{h}' \, \bar{\mathbf{h}}' + \mathbf{f}' \, \bar{\mathbf{f}} \, U_{212} \, \mathbf{h}' \, (\bar{\mathbf{h}}')^2 + \mathbf{f}' \, \bar{\mathbf{f}} \, U_{21} \, \mathbf{h}' \, \bar{\mathbf{h}}'' \\
& + \mathbf{f}' \, \bar{\mathbf{f}}'' \, U_1 \, \mathbf{h}' + \mathbf{f}' \, \bar{\mathbf{f}}' \, U_{12} \, \mathbf{h}' \bar{\mathbf{h}}' + \mathbf{f} \, \bar{\mathbf{f}}'' \, U_{11} \, \left(\mathbf{h}'\right)^2 + \mathbf{f} \, \bar{\mathbf{f}}' \, U_{112} \, \left(\mathbf{h}'\right)^2 \bar{\mathbf{h}}' \\
& + \mathbf{f} \, \bar{\mathbf{f}}'' \, U_1 \, \mathbf{h}'' + \mathbf{f} \, \bar{\mathbf{f}}' \, U_{12} \, \mathbf{h}'' \bar{\mathbf{h}}' + \mathbf{f}' \, \bar{\mathbf{f}}' \, U_{12} \, \mathbf{h}' \, \bar{\mathbf{h}}' + \mathbf{f}' \, \bar{\mathbf{f}} \, U_{122} \, \mathbf{h}' \, (\bar{\mathbf{h}}')^2 \\
& + \mathbf{f}' \, \bar{\mathbf{f}} \, U_{12} \, \mathbf{h}' \, \bar{\mathbf{h}}'' + \mathbf{f} \, \bar{\mathbf{f}}' U_{121} \, \left(\mathbf{h}'\right)^2 \bar{\mathbf{h}}' + \mathbf{f} \, \bar{\mathbf{f}} \, U_{1212} \, \left(\mathbf{h}'\right)^2 (\bar{\mathbf{h}}')^2 + \mathbf{f} \, \bar{\mathbf{f}} \, U_{121} \, \left(\mathbf{h}'\right)^2 \bar{\mathbf{h}}'' \\
& + \mathbf{f} \, \bar{\mathbf{f}}' \, U_{12} \, \mathbf{h}'' \, \bar{\mathbf{h}}' + \mathbf{f} \, \bar{\mathbf{f}} \, U_{122} \, \mathbf{h}'' \, (\bar{\mathbf{h}}')^2 + \mathbf{f} \, \bar{\mathbf{f}} \, U_{12} \, \mathbf{h}'' \, \bar{\mathbf{h}}''
\end{aligned}
$$

$$
\begin{aligned}
= \;& \mathbf{f}'' \, \bar{\mathbf{f}}'' \, U + \bar{\mathbf{f}}'' \left(2\mathbf{f}' \, \mathbf{h}' + \mathbf{f} \, \mathbf{h}''\right) \, U_1 + \mathbf{f}'' \left(2\bar{\mathbf{f}}' \, \bar{\mathbf{h}}' + \bar{\mathbf{f}} \, \bar{\mathbf{h}}''\right) \, U_2 + \\
& + \mathbf{f} \, \bar{\mathbf{f}}'' \, \left(\mathbf{h}'\right)^2 \, U_{11} + \mathbf{f}'' \, \bar{\mathbf{f}} \, \left(\bar{\mathbf{h}}'\right)^2 \, U_{22} + \left(2\bar{\mathbf{f}}' \, \bar{\mathbf{h}}' + \bar{\mathbf{f}} \, \bar{\mathbf{h}}''\right) \left(2\mathbf{f}' \, \mathbf{h}' + \mathbf{f} \, \mathbf{h}''\right) \, U_{12} + \\
& + \mathbf{f} \, \left(2\bar{\mathbf{f}}' \, \bar{\mathbf{h}}' + \bar{\mathbf{f}} \, \bar{\mathbf{h}}''\right) \left(\mathbf{h}'\right)^2 \, U_{121} + \bar{\mathbf{f}} \, \left(2\mathbf{f}' \, \mathbf{h}' + \mathbf{f} \, \mathbf{h}''\right) \left(\bar{\mathbf{h}}'\right)^2 \, U_{212} + \\
& + \mathbf{f} \, \bar{\mathbf{f}} \, \left(\mathbf{h}'\right)^2 \left(\bar{\mathbf{h}}'\right)^2 \, U_{1212}.
\end{aligned}
$$

In order for the lower order coefficients to cancel we need $\mathbf{f}'' = 0$ and hence find

$$\mathbf{f}(z) = \alpha + \beta z.$$

Plugging this result in we may see that (4.2.3) simplifies to

$$
\begin{aligned}
\ldots \;=\;& \Big(2\bar{\beta} \, \bar{\mathbf{h}}' + \left(\bar{\alpha} + \bar{\beta}\bar{z}\right) \, \bar{\mathbf{h}}''\Big) \Big(2\beta \, \mathbf{h}' + (\alpha + \beta z) \, \mathbf{h}''\Big) U_{12} + \\
& + (\alpha + \beta z) \Big(2\bar{\beta} \, \bar{\mathbf{h}}' + \left(\bar{\alpha} + \bar{\beta}\bar{z}\right) \, \bar{\mathbf{h}}''\Big) \left(\mathbf{h}'\right)^2 \, U_{121} \; + \\
& + \left(\bar{\alpha} + \bar{\beta}\bar{z}\right) \Big(2\beta \, \mathbf{h}' + (\alpha + \beta z) \, \mathbf{h}''\Big) \left(\bar{\mathbf{h}}'\right)^2 \, U_{212} + \\
& + |\alpha + \beta z|^2 \, \left(\mathbf{h}'\right)^2 \left(\bar{\mathbf{h}}'\right)^2 \, U_{1212}.
\end{aligned}
$$

Since $\mathbf{h}' \neq 0$ it follows that the remaining lower order terms vanish if and only if

$$2\beta \, \mathbf{h}' + (\alpha + \beta z) \, \mathbf{h}'' = 0,$$

which implies $\mathbf{h}'' = \beta = 0$ or $\mathbf{h}' = \gamma \left(\alpha + \beta z\right)^{-2}$. The first possibility gives the similarities $\mathbf{h}(z) = \gamma_1 + \gamma_2 z$ and the second one the Möbius transformations

$$\mathbf{h}(z) = \frac{-\gamma/\beta}{\alpha + \beta z} + \delta.$$

Also note that $\mathbf{h}' = \gamma \mathbf{f}^{-2}$. □

The preceding lemma shows that if we want to keep the same biharmonic differential operator it is not sufficient to consider a conformal map but one has to restrict himself to a Möbius transformation, even in dimension 2.

4.2.2 Möbius transformations

In this subsection we study the particular role that the Möbius transformations play concerning the positivity preserving property of the polyharmonic problem with Dirichlet boundary condition in $\Omega \subset \mathbb{R}^n$ for $n \geq 2$.

It is well known that in dimensions 3 and larger very few conformal mappings exist. Except so-called similarities the only ones that exist are the Möbius transformations. This result is due to Liouville around 1850 for $n = 3$. For general dimensions $n \geq 3$ see Theorem 5.10 in [60].

A Möbius transformation can be written as a finite combination of similarities and the inversion $j_0 : x \mapsto |x|^{-2}x$. In fact, see Corollary 4 on page 39 of [60], every Möbius transformation ϕ can be written as

$$\phi = \phi_1 \circ j_0 \circ \phi_2 \tag{4.2.4}$$

with ϕ_1, ϕ_2 similarities and $j_0(x) = |x|^{-2}x$.

Combining polyharmonic equations with similarity transformations one gets an obvious result. Let us address how one may combine biharmonic (and polyharmonic) equations with the inversion j_0. By the way, for $n = 2$ it is common to use notation in $\mathbb{C}$ and to consider the conjugate version $\bar{\mathbf{j}}_0(z) = z^{-1}$.

We shall see that there is only one obvious choice if we want to keep the same polyharmonic differential operator. Since pure powers of $|x|$ remain in this class both under j_0 and Δ it seems reasonable to try with power functions of $|x|$ only.

Lemma 4.2.2. *Let Ω be an open bounded domain in $\mathbb{R}^n \setminus \{0\}$. The numbers $\alpha, \beta, \gamma \in \mathbb{R}$ are such that*

$$\Delta^k \left(|x|^\alpha \left(u \circ j_0\right)(x)\right) = \gamma \ |x|^\beta \left(\Delta^k u\right) \circ j_0(x) \text{ for all } x \in j_0(\Omega),\ \ u \in C^{2k}(\bar{\Omega}),$$

if and only if $\alpha = 2k - n$, $\beta = -2k - n$ and $\gamma = 1$.

Proof. By testing with $u(x) = |x|^\delta$ for $\delta \in \mathbb{R}$, using $\Delta_{rad} = r^{1-n}\partial_r r^{n-1}\partial_r$, one finds:

$$\Delta^k(|x|^\alpha |j_0(x)|^\delta) = \left(\prod_{m=0}^{k-1} (\alpha - \delta - 2m)(n - 2 + \alpha - \delta - 2m)\right) |x|^{\alpha-\delta-2k}, \tag{4.2.5}$$

$$|x|^\beta (\Delta^k |y|^\delta)_{y=j_0(x)} = \left(\prod_{m=0}^{k-1} (\delta - 2m)(n - 2 + \delta - 2m)\right) |x|^{\beta-(\delta-2k)}. \tag{4.2.6}$$

These two expressions are identical for all δ if and only if $\alpha - \delta - 2k = \beta - (\delta - 2k)$, and hence

$$\beta = \alpha - 4k.$$

This leaves us with two coefficients which are polynomials in δ and these are multiples of each other if and only if the roots coincide. For the largest root one finds $n-2+\alpha = 2\,(k-1)$ and hence

$$\alpha = 2k - n.$$

In fact now all roots coincide and one finds that $\gamma = 1$.

To show that

$$\Delta^k \left(|x|^{2k-n} (u \circ j_0)(x)\right) = |x|^{-2k-n} \left(\Delta^k u\right) \circ j_0(x) \tag{4.2.7}$$

holds for all sufficiently smooth functions u we remark that $\Delta = r^{1-n}\partial_r r^{n-1}\partial_r + r^{-2}\Delta_\Gamma$ where Δ_Γ is the Laplace-Beltrami operator on the surface of the unit ball. Let φ denote these angular coordinates. Then

$$\Delta \left(r^\delta \Phi(\varphi)\right) = r^{\delta-2} \left(\delta(\delta + n - 2) + \Delta_\Gamma\right) \Phi(\varphi).$$

So a similar computation as for (4.2.5-4.2.6) leads for $u = r^\delta \Phi(\varphi)$ to

$$\Delta^k (r^{2k-n-\delta}\Phi(\varphi)) = \left(\prod_{m=0}^{k-1} ((2k-\delta-n-2m)(2k-\delta-2m-2) + \Delta_\Gamma)\right) r^{-n-\delta}\Phi(\varphi),$$

$$r^{-2k-n}(\Delta^k s^\delta \Phi(\varphi))_{s=r^{-1}} = \left(\prod_{m=0}^{k-1} ((\delta-2m)(\delta+n-2m-2) + \Delta_\Gamma)\right) r^{-n-\delta}\Phi(\varphi).$$

Both right hand sides are equal so (4.2.7) holds for a dense set of functions in $C^{2k}(\bar\Omega)$ and hence for all $u \in C^{2k}(\bar\Omega)$. □

Proposition 4.2.3. *For any Möbius transformation h in $\mathbb{R}^n$ one finds:*

$$\Delta^k \left(J_h^{\frac{1}{2}-\frac{k}{n}}\; u \circ h\right) = J_h^{\frac{1}{2}+\frac{k}{n}}\; \left(\Delta^k u\right) \circ h, \tag{4.2.8}$$

where $J_h = |\det(\frac{\partial h_i}{\partial x_j})|$ is the Jacobian.

Remark 4.2.4. Some special cases:

(i) For $k = 2$ and $n = 2$:

$$\Delta^2 \left(J_h^{-\frac{1}{2}} \cdot (u \circ h)\right) = J_h^{\frac{3}{2}} \cdot \left(\Delta^2 u\right) \circ h. \tag{4.2.9}$$

(ii) For $k \geq 1$ and $n = 2k$:

$$\Delta^k (u \circ h) = J_h \cdot \left(\Delta^k u\right) \circ h. \tag{4.2.10}$$

Proof. It is sufficient to show that (4.2.8) holds for each of the transformations involved. Since scaling, dilation, rotation and reflection give immediately the appropriate changes and since every Möbius transformation can be expressed as (4.2.4), we are left with the inversion j_0. For j_0 one finds

$$\left(\frac{\partial}{\partial x_j} j_{0,i}(x)\right)_{ij} = \left(\frac{\partial}{\partial x_j}\frac{x_i}{|x|^2}\right)_{ij} = \frac{1}{|x|^2}I - \left(\frac{2x_ix_j}{|x|^4}\right)_{ij} = \frac{1}{|x|^2}\left[I - 2\left(\frac{x}{|x|}\right)\left(\frac{x}{|x|}\right)^T\right],$$

using column notation for x. Since the matrix between square brackets describes the reflection in the hyperplane through 0 perpendicular to x, it has determinant -1. Hence the Jacobian of j_0 satisfies:

$$J_{j_0}(x) = \left| \det \left(\frac{\partial}{\partial x_j} j_{0,i}(x) \right)_{ij} \right| = \frac{1}{|x|^{2n}}.$$

$\square$

A direct consequence of Proposition 4.2.3 is that the positivity preserving property for the polyharmonic problem with Dirichlet boundary condition

$$\left\{ \begin{array}{rcll} (-\Delta)^k u &=& f & \text{in } \Omega, \\ \left(\frac{\partial}{\partial \nu}\right)^i u &=& 0 & \text{for } i = 0, \ldots, k-1 \text{ on } \partial\Omega, \end{array} \right. \tag{4.2.11}$$

is invariant under Möbius transformations in $\mathbb{R}^n$. Let give the precise statement.

Corollary 4.2.5. *Suppose that $A, \Omega \subset \mathbb{R}^n$ are bounded domains such that there exists a Möbius transformation from A to Ω. Then (4.2.11) is positivity preserving for Ω if and only if (4.2.11) is positivity preserving for A.*

Proof. Let G_Ω, G_A be the respective Green functions and let us call the Möbius transformation h. A direct computation shows that

$$(J_h(x) J_h(y))^{\frac{k}{n} - \frac{1}{2}} \; G_A(x, y) = G_\Omega(h(x), h(y)). \tag{4.2.12}$$

Indeed, if u is the solution of (4.2.11) on Ω, then it holds $(-\Delta)^k \left(J_h(x)^{\frac{1}{2} - \frac{k}{n}} (u \circ h)(x) \right) = J_h(x)^{\frac{1}{2} + \frac{k}{n}} (f \circ h)(x)$ and since also the boundary conditions go over nicely in the case of zero Dirichlet type:

$$\begin{aligned} (u \circ h)(x) &= J_h(x)^{\frac{k}{n} - \frac{1}{2}} \int_A G_A(x, y) J_h(y)^{\frac{1}{2} + \frac{k}{n}} (f \circ h)(y) dy = \\ &= \int_A J_h(x)^{\frac{k}{n} - \frac{1}{2}} J_h(y)^{\frac{k}{n} - \frac{1}{2}} G_A(x, y) (f \circ h)(y) J_h(y) dy. \end{aligned}$$

On the other hand

$$u(h(x)) = \int_\Omega G_\Omega(h(x), \eta) f(\eta) d\eta = \int_A G_\Omega(h(x), h(y)) (f \circ h)(y) J_h(y) dy.$$

The claim follows from (4.2.12). $\square$

A well-known property of Möbius transformations is that the image of a (generalized) sphere is again a (generalized) sphere (see Theorem 3.4 in [60]). Hence there is no conformal mapping in dimensions ≥ 2 available that could extend Boggio's result to other domains than generalized spheres. By the way, a generalized sphere is either a sphere or a hyperplane.

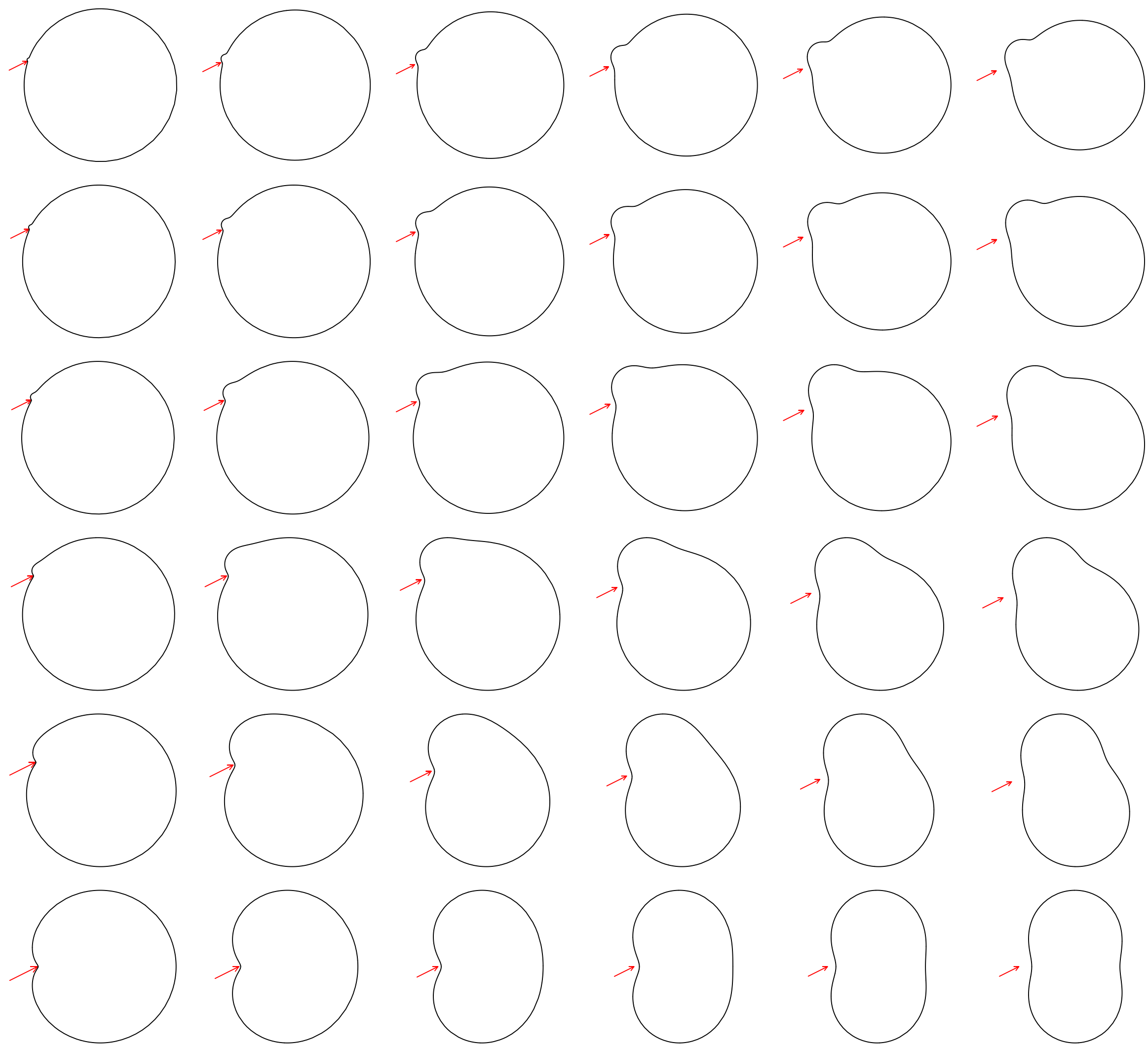

Figure 4.1: *Domains for which the clamped plate system is positivity preserving.*

In dimension 2 some domains quite different from the disk where the clamped plate equation is positivity preserving are obtained considering Möbius transformation of the limaçons Ω_a, $a \in [0, \frac{1}{6}\sqrt{6}]$. For sake of completeness we recall that the clamped plate equation on limaçons Ω_a for $a \in [0, \frac{1}{6}\sqrt{6}]$ has the positivity preserving property. See Theorem 3.1.3.

In particular, combining the Green function for a limaçon Ω_a, $a \in [0, \frac{1}{6}\sqrt{6}]$, with an inversion that has its 'center' just outside of this limaçon one obtains a new domain on which (4.1.1) is positivity preserving. The drawings in Figure 4.1 are transforms of the limaçon in the extreme case $a = \frac{1}{6}\sqrt{6}$ and taking the inversion center just outside that limaçon. Both the angular position and the distance to the limaçon of the inversion center are varied. All graphs are scaled back to unit size. The arrow denotes the inversion center.

4.3 A perturbation argument

In [40] it has been shown that small perturbations of the disk do not destroy property (4.1.2). However, the perturbed domains for which $f > 0$ implies $u > 0$ that were allowed needed a small C^2-bound for the difference between Ω and B. The result of [40] in fact is not restricted to the small perturbations of the disk; only the appropriate estimates for the Green function on the specific domain are needed. Let us give the precise statement.

Here α is a multi-index of non-negative integers, $|\alpha| = \sum \alpha_i$ and $\partial_x^\alpha = \prod \dfrac{\partial^{\alpha_i}}{\partial x_i^{\alpha_i}}$. Here d_Ω denotes the distance function to the boundary as defined in (2.1.3).

Proposition 4.3.1. *Suppose that the Green function for (4.1.1) on Ω satisfies the following estimates:*

(i) from below: $\exists c_\Omega > 0\ \forall x, y \in \Omega$

$$G_\Omega(x,y) \geq c_\Omega d_\Omega(x) d_\Omega(y) \min\left\{1, \tfrac{d_\Omega(x)d_\Omega(y)}{|x-y|^2}\right\}, \tag{4.3.1}$$

(ii) from above: $\exists c_{i,\Omega}\ \forall x, y \in \Omega$:

$$\begin{aligned} & |G_\Omega(x,y)| \leq c_{0,\Omega}\ d_\Omega(x)d_\Omega(y) \min\left\{1, \tfrac{d_\Omega(x)d_\Omega(y)}{|x-y|^2}\right\}, \\ |\alpha| = 1: \quad & |\partial_x^\alpha G_\Omega(x,y)| \leq c_{1,\Omega}\ d_\Omega(y) \min\left\{1, \tfrac{d_\Omega(x)d_\Omega(y)}{|x-y|^2}\right\}, \\ |\alpha| = 2: \quad & |\partial_x^\alpha G_\Omega(x,y)| \leq c_{2,\Omega}\ \log\left(1 + \tfrac{d_\Omega(y)^2}{|x-y|^2}\right), \\ |\alpha| = 3: \quad & |\partial_x^\alpha G_\Omega(x,y)| \leq c_{3,\Omega}\ \tfrac{1}{|x-y|} \min\left\{1, \tfrac{d_\Omega(y)^2}{|x-y|^2}\right\}. \end{aligned}$$

Then there exists $\varepsilon > 0$ such that the following holds. If there is a conformal map $\mathbf{h}$ from $\mathbf{A}$ to $\mathbf{\Omega}$ with $\|\mathbf{h} - \mathbf{Id}\|_{C^2(\bar{\mathbf{A}})} \leq \varepsilon$, then (4.1.1) is also positivity preserving for Ω replaced by A.

Here we used the identity $\mathbf{Id}$ on $\mathbb{C}$: $\mathbf{Id}(z) = z$.

Remark 4.3.2. As in [40] one may show that it is sufficient that there is a $C^{3,\gamma}$-diffeomorphism from A to Ω close to the identity.

Remark 4.3.3. For the estimate from above to hold we expect to need a more regular boundary than just the C^2 from the conformal map. The estimates above are based on results of Krasovskiĭ that would use a C^{16} boundary in the present setting, see Chapter 2.

Remark 4.3.4. If we consider a Möbius transformation of a limaçon Ω_a taking $a < \frac{1}{6}\sqrt{6}$ the resulting domains would allow a (small) perturbation argument without destroying property (4.1.2). Note that for the domains in Figure 4.1 no extra perturbation is allowed since $a = \frac{1}{6}\sqrt{6}$ is critical for positivity.

Proof. Proceeding as in (4.2.3) with $\mathbf{f} = 1$ one obtains since $\Delta|\mathbf{h}'|^2 = 4|\mathbf{h}''|^2$ that

$$\begin{aligned} \Delta^2(u \circ h) &= \Delta\left(|\mathbf{h}'|^2(\Delta u) \circ h\right) = \\ &= |\mathbf{h}'|^2\Delta((\Delta u) \circ h) + 2\nabla|\mathbf{h}'|^2 \cdot \nabla((\Delta u) \circ h) + \Delta|\mathbf{h}'|^2((\Delta u) \circ h) \\ &= |\mathbf{h}'|^4 \left(((\Delta^2 u) \circ h) + \tfrac{2\nabla|\mathbf{h}'|^2}{|\mathbf{h}'|^4} \cdot (\partial_i h_j)(\nabla\Delta u) \circ h) + \tfrac{4|\mathbf{h}''|^2}{|\mathbf{h}'|^4}((\Delta u) \circ h)\right). \end{aligned} \tag{4.3.2}$$

If the L^∞-norms of $\nabla|\mathbf{h}'|^2|\mathbf{h}'|^{-4}$ and $|\mathbf{h}''|^2|\mathbf{h}'|^{-2}$ are sufficiently small then we may use the Green function estimates and the method of [41, Theorem 5.1] to find that the modified fourth order operator in (4.3.2) on Ω has a positive Green function. And indeed, these L^∞-norms become as small as one likes by choosing the ε-bound for the C^2-difference of $\mathbf{h}$ and the identity. For the disk such an approach is found in [40]. If (4.3.2) with Dirichlet boundary conditions on Ω has the positivity preserving property then (4.1.1) is positivity preserving on A. $\square$

Remark 4.3.5. The obvious guess would be to proceed considering $\Delta^2(|\mathbf{h}'|^{-1}u \circ h)$ instead of $\Delta^2(u \circ h)$. However this approach gives lower order terms that contain third order derivatives of $\mathbf{h}$. Indeed one gets

$$\begin{aligned} &\tfrac{1}{16}\Delta^2(|\mathbf{h}'(z)|^{-1}\, u \circ h) \\ =\ &\tfrac{1}{16}\, |\mathbf{h}'|^3\ \left(\Delta^2 u\right) \circ h - \tfrac{1}{2}\left(\bar{\mathbf{h}}'''(z) - \tfrac{3}{2}\left(\bar{\mathbf{h}}''(z)\right)^2 (\bar{\mathbf{h}}'(z))^{-1}\right) (\mathbf{h}'(z))^{\frac{3}{2}}(\bar{\mathbf{h}}'(\bar z))^{-\frac{3}{2}}\ U_{11} \\ &-\tfrac{1}{2}\left(\mathbf{h}'''(z) - \tfrac{3}{2}(\mathbf{h}''(z))^2(\mathbf{h}'(z))^{-1}\right) (\bar{\mathbf{h}}'(\bar z))^{\frac{3}{2}}(\mathbf{h}'(z))^{-\frac{3}{2}}\ U_{22} \\ &+\tfrac{1}{4}\left|\mathbf{h}'''(z) - \tfrac{3}{2}(\mathbf{h}''(z))^2(\mathbf{h}'(z))^{-1}\right|^2\ |\mathbf{h}'|^{-3}\, u \circ h. \end{aligned} \tag{4.3.3}$$

Here we used formula (4.2.3) with $\mathbf{f} = (\mathbf{h}')^{-\frac{1}{2}}$ and $u \circ h(\tilde x, \tilde y) = U(\mathbf{h}(z), \bar{\mathbf{h}}(\bar z))$ with $z = \tilde x + i\tilde y$ and $\bar z = \tilde x - i\tilde y$. Notice that in this case $2\mathbf{f}'\mathbf{h}' + \mathbf{f}\mathbf{h}'' = 0$.

Since with $w = x + iy$ it holds

$$\begin{aligned} \partial_x^2 U(w, \bar w) - \partial_y^2 U(w, \bar w) &= 2\left(U_{11} + U_{22}\right) \\ \partial_{xy}^2 U(w, \bar w) &= i\left(U_{11} - U_{22}\right), \end{aligned}$$

we may rewrite (4.3.3) as

$$\begin{aligned} \Delta^2(|\mathbf{h}'(z)|^{-1}\, u \circ h) &= |\mathbf{h}'|^3 \left(\Delta^2 u\right) \circ h + 8\mathrm{Re}\left(\mathbf{v}(z)\bar{\mathbf{h}}'(\bar z)^{\frac{3}{2}}\right) \left((\partial_x^2 u) \circ h - (\partial_y^2 u) \circ h\right) \\ &\quad +16\ \mathrm{Im}\left(\mathbf{v}(z)\bar{\mathbf{h}}'(\bar z)^{\frac{3}{2}}\ \right) \left(\partial_{xy}^2 u\right) \circ h + 16\left|\mathbf{v}(z)\right|^2 u \circ h, \end{aligned}$$

with

$$\mathbf{v}(z) = -\tfrac{1}{2}\left(\mathbf{h}'''(z) - \tfrac{3}{2}(\mathbf{h}''(z))^2(\mathbf{h}'(z))^{-1}\right) \mathbf{h}'(z)^{-\frac{3}{2}}.$$

One would need C^3-closeness of $\mathbf{h}$ to the identity in order to apply the result in [41].

The three main ingredients of the proof of this proposition are 1) a conformal mapping near the identity from A to Ω, 2) estimates for the Green function from below, and 3) estimates from above for the Green function and its derivatives. By the way, the estimates for the derivatives of the Green function are not the necessary ones for the proposition but are the ones that come out of the Green function itself.

Let us focus on the three ingredients just mentioned.

Using conformal mappings other than Möbius will restrict us to 2-dimensional domains. If estimates would be available for small perturbations in the leading order this could be overcome. We are not aware if such estimates exist in dimensions higher than 2. In two dimensions small perturbations in the leading order terms are allowed since such a differential equation can be transformed to one with bi(poly)harmonic leading order on a disk. See [40].

One can prove estimates from below of the Green function starting from the explicit formula. For optimal estimates from below for the polyharmonic Green function for zero Dirichlet boundary values on the ball in $\mathbb{R}^n$ see [45]. Theorem 3.4.1 gives the estimates from below for the Green function associated to the clamped plate equation on limaçons Ω_a, $a \in [0, \frac{1}{6}\sqrt{6}]$. Also through the perturbation arguments as in [40], [41] one finds optimal estimates from below for the Green function.

The estimates from above for the Green function are known to hold in a much wider range. Indeed, such estimates exists for all polyharmonic systems under zero Dirichlet boundary data, at least when this boundary has sufficient regularity. Theorem 2.5.6 in Chapter 2 gives these estimates for rather general domains starting from the kernel estimates of Krasovskiĭ in [52].

Chapter 5

Separating positivity and regularity

5.1 Introduction and main results

A mayor tool for second order elliptic equations is the maximum principle. The maximum principle not only implies that a positive source will give a positive solution but it helps to obtain a priori estimates and hence to find regularity results. Especially in nonlinear equations such a priori estimates play a crucial role. Several results are referred to by the name maximum principle but the result that we want to refer to is the local result that reads for the laplacian as $\Delta u \geq 0$ in a neighborhood of a implies that u cannot have a strict maximum in a. A serious obstruction for higher order elliptic equations is that one cannot expect a similar result as functions like $\pm x^2$ clearly show.

The situation becomes more complicated when considering a positivity preserving property which is often also named "maximum principle". For the laplacian that is: $-\Delta u \geq 0$ in Ω and $u \geq 0$ on $\partial\Omega$ implies $u \geq 0$ in Ω (with Ω a bounded domain in $\mathbb{R}^n$). This "global maximum principle" also holds for some special higher order problems. Indeed, $\Delta^2 u \geq 0$ in B and $-\frac{\partial}{\partial|x|}u \geq 0$, $u \geq 0$ on ∂B implies $u \geq 0$ in B. Here B is a ball in $\mathbb{R}^n$ with $n \leq 4$. For this special result see [43]. With $\frac{\partial}{\partial|x|}u = u = 0$ on ∂B the result holds for B in any $\mathbb{R}^n$ and goes back 100 years to Boggio ([8]). The restriction to the ball is rather crucial. Since Duffin's counterexample ([31]) it has become well known that on most domains such a positivity preserving property fails (see [42]).

In [56] Nehari looks for subdomains of Ω, characterized by the position of the points x and y and by simple geometric properties of Ω, in which the Green function for the biharmonic problem with Dirichlet boundary condition on Ω may be shown to be positive.

In order to find a priori estimates it is however not necessary to have such a sign preserving result; it is sufficient that the singularity of the solution operator has a fixed sign. This separation of the solution operator in a smooth but sign changing part and a singular part of fixed sign is the main result of this chapter. However, since we are using conformal mappings, our present result is restricted to two dimensional domains. Note that in two dimensions the singularity of the solution operator for the bilaplacian appears in the second derivative. Indeed the fundamental solution is $\frac{-1}{8\pi}|x|^2 \ln |x|$.

Let us be more precise. For Ω an open bounded $C^{4,\alpha}$ domain in $\mathbb{R}^2$ we will show that the solution operator for

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \frac{\partial}{\partial\nu} u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1.1}$$

can be split in the way we just mentioned. Crucial is that we find a uniform behavior of such a splitting even near the boundary. Such a result away from the boundary, that is in compact subsets of Ω, was proven in [45].

We proceed as follows. First, we recall some results of elliptic regularity focusing on the quantities on which the constants in the estimates depend. Secondly, using the results of Chapter 3 that for some family of limaçons the Green function for (5.1.1) is positive, we will show that small perturbations of those limaçons do not destroy the positivity of the corresponding Green function. Thirdly, one may construct a finite number of such slightly perturbed limaçons $\{E_j \subset \mathbb{R}^2\}$ that are such that the boundary of Ω is covered by the boundaries of those perturbed limaçons while these limaçons cover a neighborhood of the boundary of Ω. Together with a covering of the interior one is able to construct the desired splitting of the solution through a separation of unity related with that covering. Roughly explained, for each $x \in \Omega$ there is an element E_j in this finite covering such that the Green function for (5.1.1) can be decomposed as the sum of $G_{E_j}(x,y)$ and a remainder term $G_j^{\text{rest}}(x,y)$ where $G_{E_j}(x,y)$ is positive and $G_j^{\text{rest}}(x,y)$ is without singularity. Note that the choice of E_j depends on x. Since the extension of $G_{E_j}(x,y)$ from E_j^2 to Ω^2 by 0 is not smooth one may guess that the just mentioned decomposition is more involved than just this simple sum.

5.1.1 Main results

In this section we state the two main results of the chapter. First we fix some notation.

The Green function G_Ω is such that the solution of problem (5.1.1) for appropriate f can be written as

$$u(x) = \int_\Omega G_\Omega(x,y) f(y) dy.$$

In the following $d_\Omega(.)$ denotes the distance to the boundary in the domain Ω as defined in (2.1.3).

Two closely related versions of the main result are the following. The first one is a pointwise description which focusses on the splitting of the solution operator.

Theorem 5.1.1. *Assume that $\Omega \subset \mathbb{R}^2$ is a bounded simply connected domain with $\partial\Omega \in C^{16}$. Then there exist $G_\Omega^{reg}, G_\Omega^{sing} : \bar{\Omega}^2 \to \mathbb{R}$ such that the Green function for (5.1.1) can be written as*

$$G_\Omega(x,y) = G_\Omega^{reg}(x,y) + G_\Omega^{sing}(x,y)$$

and the following is satisfied:

(i) *(a)* $G_\Omega^{sing}(x,y) \geq 0$ *on* $\bar{\Omega}^2$;

(b) $G_\Omega^{sing} \in C^{1,\gamma}\left(\bar{\Omega}^2\right) \cap C_0^1\left(\bar{\Omega}^2\right)$ *for all* $\gamma \in (0,1)$;

(c) $G_\Omega^{sing} \in C^{15,\gamma}\left(\left\{(x,y) \in \bar{\Omega}^2; x \neq y\right\}\right)$ *for all* $\gamma \in (0,1)$;

(ii) *(a)* $G_\Omega^{reg} \in C^{15,\gamma}\left(\bar{\Omega}^2\right) \cap C_0^1\left(\bar{\Omega}^2\right)$ *for all* $\gamma \in (0,1)$.

Remark 5.1.2. For the condition $\partial\Omega \in C^{16}$ see Definition 2.1.3.

Remark 5.1.3. Since G_Ω is symmetric one may assume that both G_Ω^{reg} and G_Ω^{sing} are symmetric. If not yet symmetric, then set $G_{\Omega,new}^{\cdots}(x,y) := \frac{1}{2}G_\Omega^{\cdots}(x,y) + \frac{1}{2}G_\Omega^{\cdots}(y,x)$.

The next result is a kind of maximum principle, that is, it gives a pointwise bound from above for the solution in terms of the positive part of the right hand side and a weaker norm of the solution itself. Before we state the result let us recall that the space $W^{-m,p}(\Omega)$ is the dual space of $W_0^{m,p'}(\Omega)$, with $\frac{1}{p}+\frac{1}{p'}=1$, and its norm can be defined as follows

$$\|u\|_{W^{-m,p}(\Omega)} := \sup\left\{u(\varphi); \varphi \in W_0^{m,p'}(\Omega),\ \ \|\varphi\|_{W^{m,p'}(\Omega)} \leq 1\right\}.$$

Theorem 5.1.4. *Let* $0<\alpha<1$ *and* $p \in (1,\infty)$. *Let* Ω *be a bounded simply connected domain in* $\mathbb{R}^2$ *with* $\partial\Omega \in C^{4,\alpha}$.

Then for any $q>2$ *and* $\varepsilon>0$ *there exists a constant* $c_{q,\Omega,\varepsilon}>0$ *such that for* $f \in L^p(\Omega)$ *the solution* $u \in W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega)$ *of (5.1.1) satisfies*

$$u(x) \leq c_{q,\Omega,\varepsilon}\left(\left\|f^+\right\|_{L^1(B(x,\varepsilon)\cap\Omega)} + \|u\|_{W^{-1,q}(\Omega)}\right) \text{ for every } x \in \Omega.$$

Here f^+ *denotes the positive part of* f.

Remark 5.1.5. More precise information on how $c_{q,\Omega,\varepsilon}$ depends on q,Ω and ε can be found in Theorem 5.5.2. For those who want to avoid norms for negative Sobolev spaces we recall that $\|u\|_{W^{-1,q}(\Omega)} \leq c(s,q,\Omega)\|u\|_{L^s(\Omega)}$ for $s > 2q(q+2)^{-1}$.

5.1.2 Some notations

The Hölder spaces $C^r(\bar{\Omega})$ and $C^{r,\gamma}(\bar{\Omega})$ with $r \in \mathbb{N}$ and $\gamma \in (0,1]$ are supplied with the norm:

$$\begin{aligned}\|f\|_{C^r(\bar{\Omega})} &:= \sum_{|\alpha|\leq r} \|D^\alpha f\|_\infty, \\ \|f\|_{C^{r,\gamma}(\bar{\Omega})} &:= \|f\|_{C^r(\bar{\Omega})} + \sum_{|\alpha|=r} [D^\alpha f]_\gamma,\end{aligned}$$

where $[f]_\gamma := \sup\left\{\frac{|f(x)-f(y)|}{|x-y|^\gamma}; x,y \in \bar{\Omega},\ x \neq y\right\}$. Here D^α denotes the derivative of f as fixed in Notation 2.1.2. For convenience we set $C^{r,0}(\bar{\Omega}) := C^r(\bar{\Omega})$.

In the following $C_c^r(\Omega)$ denotes the set of all functions in $C^r(\Omega)$ whose supports are compact subsets of Ω.

For $m \in \mathbb{N}$ and $p \geq 1$, $p \in \mathbb{R}$, $W^{m,p}(\Omega)$ denotes the Sobolev space with the norm

$$\|f\|_{W^{m,p}(\Omega)} = \sum_{|\alpha| \leq m} \|D^\alpha f\|_{L^p(\Omega)} .$$

Notation 5.1.6. *With $p \in (1, \infty)$, p' denotes the conjugate of p, that is:*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

We fix the following notation to point out on which quantities the constants depend.

Notation 5.1.7. *For α, β, $\gamma \in \mathbb{R}$, $C = C(\alpha, \beta, \gamma)$ means that C depends only on α, β and γ, and that C is bounded for bounded values of these parameters.*

Next we will need some notation concerning the domain and its boundary.

Notation 5.1.8 (Relatively open subset of the boundary)**.** *For K a subset of $\partial\Omega \subset \mathbb{R}^n$, set*

$$K^{\circ,\partial\Omega} := (K \cup (\partial\Omega)^c)^\circ \cap \partial\Omega.$$

It will also be convenient to fix the following numbers.

Notation 5.1.9. *Let Ω be a bounded domain with $\partial\Omega \in C^2$.*

(i) We write ρ_Ω for the largest number r such that both Ω and $\mathbb{R}^n \backslash \Omega$ can be filled with balls of radius r. To be precise: for $r > 0$ set $\Omega_r := \{z \in \Omega : d(z, \partial\Omega) \geq r\}$, $\tilde{\Omega}_r := \left\{z \in \mathbb{R}^n \setminus \bar{\Omega} : d(z, \partial\Omega) \geq r\right\}$. Set $\rho_\Omega > 0$ the largest r such that the following holds:

$$\Omega = \bigcup_{z \in \Omega_r} B_r(z) \text{ and } \mathbb{R}^n \setminus \bar{\Omega} = \bigcup_{z \in \tilde{\Omega}_r} B_r(z).$$

(ii) We will also use R_Ω defined as the smallest R such that $\Omega \subset B_R(z)$ for some $z \in \mathbb{R}^2$.

Remark 5.1.10. For most domains we may take $\rho_\Omega = \kappa^{-1}$ where κ denotes the maximal curvature. But notice that ρ_Ω can be strictly smaller than κ^{-1}. For example this happens in the case of a dumb-bell shaped domain with a very narrow passage.

Since some results are quite technical we will use, also in this chapter, Notation 2.1.1 introduced in Chapter 2.

5.2 Elliptic regularity and interpolation

Elliptic regularity results for linear equations can be found in numerous places. However, if one goes beyond second order and if one needs to know how the constants depend on the domain there is no easy reference. For that reason we will collect such type of results in the present section. For the explicit dependence of these constants we will go back to the original source of Agmon, Douglis and Nirenberg ([3]).

This section is organized as follows. First we recall some classical results and the Calderon-Zygmund inequality for $n = 2$. Then we consider a strong and a weak formulation of problem (5.1.1). Finally we study three intermediate versions (between strong and weak) of problem (5.1.1).

Throughout this section the following condition will appear.

Condition 5.2.1. *The number α lies in $(0, 1)$ and Ω is a bounded simply connected domain (open subset) in $\mathbb{R}^2$ satisfying the uniform $C^{4,\alpha}$ regularity condition with constant M.*

5.2.1 Classical results

In this section we recall some results from [36]. For sake of brevity we do not give the most general statements.

Theorem 5.2.2. *[36, Th.9.13] Let Ω be a bounded domain in $\mathbb{R}^n$ satisfying the uniform $C^{1,1}$ regularity condition with constant M. Then it holds*

$$\|u\|_{W^{2,2}(\Omega)} \leq C \left(\|u\|_{L^2(\Omega)} + \|\Delta u\|_{L^2(\Omega)} \right) \text{ for every } u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega),$$

with $C = C(n, M, \rho_\Omega^{-1}, R_\Omega)$.

Remark 5.2.3. The dependence of the constant can be deduced from the proof, see [36, Th.9.13].

We will use the Calderon-Zygmund inequality for $n = 2$. This inequality is usually proved by contradiction. Since we are interested in the dependence of the constant on the domain, we give here a direct proof.

Lemma 5.2.4. *Let Ω be a bounded domain in $\mathbb{R}^2$. Then there is $C = C(R_\Omega)$ such that*

$$\|u\|_{L^2(\Omega)} \leq C \|\Delta u\|_{L^2(\Omega)} \text{ for every } u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega).$$

Proof. Let $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. An application of Aleksandrov's maximum principle ([36, Th.9.1]) and of Theorem IX.17 in [9] yields

$$\sup_\Omega |u| \leq C \|\Delta u\|_{L^2(\Omega)},$$

for some $C = C(R_\Omega)$. Hence we find

$$\|u\|_{L^2(\Omega)} \leq C \|\Delta u\|_{L^2(\Omega)} |\Omega|^{\frac{1}{2}}.$$

$\square$

The following result is a direct consequence of Theorem 5.2.2 and Lemma 5.2.4.

Corollary 5.2.5. *Suppose Ω is a bounded domain in $\mathbb{R}^2$ satisfying the uniform $C^{1,1}$ regularity condition with constant M. Then there is $C = C(M, \rho_\Omega^{-1}, R_\Omega)$ such that*

$$\|u\|_{W^{2,2}(\Omega)} \leq C \|\Delta u\|_{L^2(\Omega)} \text{ for every } u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega).$$

5.2.2 Regularity for strong solutions

The classical regularity result that we like to recall in an explicit statement is the following.

Theorem 5.2.6. *Assume Condition 5.2.1. For every $f \in L^p(\Omega)$ with $p \in (1, \infty)$ there exists a unique solution $u \in W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega)$ of (5.1.1).*

Moreover the solution satisfies

$$\tfrac{1}{2} \|f\|_{L^p(\Omega)} \leq \|u\|_{W^{4,p}(\Omega)} \leq C_s \|f\|_{L^p(\Omega)}, \tag{5.2.1}$$

with $C_s = C_s(p, p', M, \rho_\Omega^{-1}, R_\Omega)$ where C_s satisfies the convention of Notation 5.1.7.

Before proving Theorem 5.2.6 we present some estimates.

Lemma 5.2.7. *Let Ω be a bounded domain in $\mathbb{R}^2$ satisfying the uniform $C^{1,1}$ regularity condition with constant M and let $p \in (1, \infty)$. Then there is $C = C(p, p', M, \rho_\Omega^{-1}, R_\Omega)$ such that*

$$\|u\|_{L^p(\Omega)} \leq C \left\|\Delta^2 u\right\|_{L^p(\Omega)} \text{ for every } u \in W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega).$$

Proof. Since $n = 2$ we find by Sobolev inequalities that

$$\|u\|_{L^p(\Omega)} \leq C_1 \|u\|_{W^{2,2}(\Omega)} \text{ and } \|u\|_{L^{p'}(\Omega)} \leq C_2 \|u\|_{W^{2,2}(\Omega)},$$

for every $u \in W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega)$. Notice that $C_1 = C_1(p, M, \rho_\Omega^{-1}, R_\Omega)$ and $C_2 = C_2(p', M, \rho_\Omega^{-1}, R_\Omega)$. Hence we have by Corollary 5.2.5, by integrating by parts and by Hölder that

$$\begin{aligned} \|u\|_{L^{p'}(\Omega)} \|u\|_{L^p(\Omega)} &\leq C_1 C_2 \|u\|^2_{W^{2,2}(\Omega)} \leq C_3 \int_\Omega |\Delta u|^2 \, dx = \\ &= C_3 \int_\Omega u \; \Delta^2 u \; dx \leq C_3 \left\|\Delta^2 u\right\|_{L^p(\Omega)} \|u\|_{L^{p'}(\Omega)}, \end{aligned}$$

with $C_3 = C_3(p, p', M, \rho_\Omega^{-1}, R_\Omega)$. The claim follows. $\square$

Lemma 5.2.8. *Assume Condition 5.2.1. Then there exists* $C = C(p, p', M, \rho_\Omega^{-1}, R_\Omega)$ *such that*

$$\|u\|_{W^{4,p}(\Omega)} \leq C \left\|\Delta^2 u\right\|_{L^p(\Omega)} \text{ for every } u \in W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega).$$

Remark 5.2.9. Usually Lemma 5.2.8 is proved by contradiction and this does not explain on what the constant depends on. However by using Lemma 5.2.7 we find the explicit quantities.

Proof. The result follows from [3, Th.15.2] and Lemma 5.2.7. The proof of [3, Th.15.2] shows that the dependence of the constant is as given in the statement. □

Proof of Theorem 5.2.6. Uniqueness follows by a standard integration by parts. Indeed, if $\Delta^2 u = 0$ then

$$\int_\Omega |\Delta u|^2 \, dx = \int_\Omega u \ \Delta^2 u \ dx = 0,$$

and with the boundary condition one finds $u \equiv 0$.

Estimate: By definition of the norm in $W^{4,p}(\Omega)$ one finds

$$\tfrac{1}{2} \left\|\Delta^2 u\right\|_{L^p(\Omega)} \leq \|u\|_{W^{4,p}(\Omega)} .$$

The other side of inequality (5.2.1) follows from Lemma 5.2.8.

Existence: For $f \in C^\alpha(\Omega)$ the existence of a solution $u \in C^{4,\alpha}(\Omega) \cap C_0^1(\bar{\Omega})$ is given by [3, Th.12.7]. Such a solution satisfies (5.2.1), ([3, Th.9.3]). The existence in $W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega)$ follows by an approximation argument . □

Remark 5.2.10. The hypothesis $\partial\Omega \in C^{4,\alpha}$ is needed in order to use Theorem 12.7 in [3]. For the rest of the paper it would be sufficient to assume $\partial\Omega \in C^4$.

By Theorem 5.2.6 the solution operator for problem (5.1.1) from the space $L^p(\Omega)$ into $W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega)$ is well defined, surjective and continuous.

For $1 < p < \infty$ we formally fix the operator $T_{4,p}$ by

$$\begin{aligned} D(T_{4,p}) :=& \ W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega), \\ T_{4,p} u :=& \ \Delta^2 u \text{ for } u \in D(T_{4,p}). \end{aligned} \tag{5.2.2}$$

Notice that this operator $T_{4,p}$ is the inverse of the solution operator.

The following result is a consequence of Theorem 5.2.6.

Corollary 5.2.11. *Let* $1 < p < \infty$. *Assuming Condition 5.2.1 the operator* $T_{4,p}$ *defined in (5.2.2) gives an isomorphism from* $W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega)$ *onto* $L^p(\Omega)$. *Moreover one has*

$$\frac{1}{C_s} \leq \|T_{4,p}\|_{\left(W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega) \to L^p(\Omega)\right)} \leq 2,$$

where C_s *is the constant appearing in Theorem 5.2.6.*

5.2.3 Regularity for weak solutions

In the following section we give the explicit definition of what we will call a weak solution for problem (5.1.1) and we recall the classical regularity result in this setting.

Definition 5.2.12. *Let $p \in (1,\infty)$ and $F \in \left(W^{4,p'}(\Omega) \cap W_0^{2,p'}(\Omega)\right)'$. We call $u \in L^p(\Omega)$ a weak solution of problem (5.1.1) with right hand side F if the following holds*

$$\int_\Omega u(x)\ \Delta^2 v(x)\ dx = F(v) \text{ for every } v \in W^{4,p'}(\Omega) \cap W_0^{2,p'}(\Omega).$$

Theorem 5.2.13. *Assume Condition 5.2.1 and let $p \in (1,\infty)$. Then for every $F \in \left(W^{4,p'}(\Omega) \cap W_0^{2,p'}(\Omega)\right)'$ there exists a unique u weak solution of problem (5.1.1) with right hand side F.*

Moreover u satisfies

$$\tfrac{1}{2}\,\|F\|_{\left(W^{4,p'}(\Omega)\cap W_0^{2,p'}(\Omega)\right)'} \le \|u\|_{L^p(\Omega)} \le C_w\,\|F\|_{\left(W^{4,p'}(\Omega)\cap W_0^{2,p'}(\Omega)\right)'},$$

with $C_w = C_w(p, p', M, \rho_\Omega^{-1}, R_\Omega)$.

Proof. Let i_p be the canonical isometry $L^p(\Omega) \to \left(L^{p'}(\Omega)\right)'$, that is, $i_p(u)(v) = \int_\Omega u(x)\ v(x)\ dx$ for every $v \in L^{p'}(\Omega)$.

Existence of u follows by a duality argument. Indeed, by Corollary 5.2.11 we may define

$$U(f) := F(T_{4,p'}^{-1}(f)) \text{ for every } f \in L^{p'}(\Omega).$$

The solution u is given by $u := i_p^{-1}(U)$. Uniqueness and the estimate follow from Corollary 5.2.11. $\square$

For $1 < p < \infty$ let us formally fix the operator $T_{0,p}$ by

$$\begin{aligned} D(T_{0,p}) &:= L^p(\Omega), \\ (T_{0,p}(u))(v) &:= i_p(u)(T_{4,p'}(v)) \text{ for every } v \in W^{4,p'}(\Omega) \cap W_0^{2,p'}(\Omega). \end{aligned} \tag{5.2.3}$$

From Theorem 5.2.13 it follows:

Corollary 5.2.14. *Let $1 < p < \infty$ and assume Condition 5.2.1. The operator $T_{0,p}$ defined in (5.2.3) gives an isomorphism from $L^p(\Omega)$ onto $\left(W^{4,p'}(\Omega) \cap W_0^{2,p'}(\Omega)\right)'$. Moreover one has*

$$\frac{1}{C_w} \le \|T_{0,p}\|_{\left(L^p(\Omega)\to\left(W^{4,p'}(\Omega)\cap W_0^{2,p'}(\Omega)\right)'\right)} \le 2,$$

where C_w is the constant appearing in Theorem 5.2.13.

5.2.4 Regularity between weak and strong

In the following section we consider via interpolation solutions between the 'strong' and the 'weak' ones defined in the previous sections.

We first give the three intermediate notions of solution.

Definition 5.2.15. *Let $p \in (1, \infty)$.*

(i) Let $F \in \left(W_0^{1,p'}(\Omega)\right)'$. We say that $u \in W^{3,p}(\Omega) \cap W_0^{2,p}(\Omega)$ is a "one-quarter weak solution" of problem (5.1.1) with right hand side F if it satisfies

$$-\int_\Omega (\nabla \Delta u(x)).\ (\nabla v(x))\ dx = F(v) \text{ for every } v \in W_0^{1,p'}(\Omega).$$

(ii) Let $F \in \left(W_0^{2,p'}(\Omega)\right)'$. We say that $u \in W_0^{2,p}(\Omega)$ is a "one-half weak solution" of problem (5.1.1) with right hand side F if it satisfies

$$\int_\Omega (\Delta u(x))\ (\Delta v(x))\ dx = F(v) \text{ for every } v \in W_0^{2,p'}(\Omega).$$

(iii) Let $F \in \left(W^{3,p'}(\Omega) \cap W_0^{2,p'}(\Omega)\right)'$. We say that $u \in W_0^{1,p}(\Omega)$ is a "three-quarter weak solution" of problem (5.1.1) with right hand side F if it satisfies

$$-\int_\Omega (\nabla u(x)).\ (\nabla \Delta v(x))\ dx = F(v) \text{ for every } v \in W^{3,p'}(\Omega) \cap W_0^{2,p'}(\Omega).$$

Theorem 5.2.16. *Assume Condition 5.2.1 and let $1 < p < \infty$. Then:*

(i) for every $F \in \left(W_0^{1,p'}(\Omega)\right)'$ there exists a unique u "one-quarter weak solution" of problem (5.1.1) with right hand side F.
Moreover u satisfies

$$\frac{1}{C_1} \|F\|_{\left(W_0^{1,p'}(\Omega)\right)'} \le \|u\|_{W^{3,p}(\Omega)} \le C_1 \|F\|_{\left(W_0^{1,p'}(\Omega)\right)'},$$

with $C_1 = C_1(p, p', M, \rho_\Omega^{-1}, R_\Omega)$.

(ii) for every $F \in \left(W_0^{2,p'}(\Omega)\right)'$ there exists a unique u "one-half weak solution" of problem (5.1.1) with right hand side F.
Moreover u satisfies

$$\frac{1}{C_2} \|F\|_{\left(W_0^{2,p'}(\Omega)\right)'} \le \|u\|_{W^{2,p}(\Omega)} \le C_2 \|F\|_{\left(W_0^{2,p'}(\Omega)\right)'},$$

with $C_2 = C_2(p, p', M, \rho_\Omega^{-1}, R_\Omega)$.

(iii) *for every* $F \in \left(W^{3,p'}(\Omega) \cap W_0^{2,p'}(\Omega)\right)'$ *there exists a unique* u *"three-quarter weak solution" of problem (5.1.1) with right hand side* F.

Moreover u *satisfies*

$$\frac{1}{C_3} \|F\|_{\left(W^{3,p'}(\Omega) \cap W_0^{2,p'}(\Omega)\right)'} \leq \|u\|_{W^{1,p}(\Omega)} \leq C_3 \|F\|_{\left(W^{3,p'}(\Omega) \cap W_0^{2,p'}(\Omega)\right)'},$$

with $C_3 = C_3(p, p', M, \rho_\Omega^{-1}, R_\Omega)$.

Remark 5.2.17. Theorem 5.2.16 part 2 has been studied in [64, Chap.7].

Our aim in giving the proof of Theorem 5.2.16 is to show how the constants in the estimates depend on the domain. We proceed through interpolation: $[\cdot, \cdot]_\theta$ denotes the complex interpolation with parameter $\theta \in (0, 1)$.

For sake of conciseness we fix the following notation:

$$\begin{array}{ll} A_{0,p} := L^p(\Omega) & A_{4,p} := W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega), \\ B_{0,p} := (A_{4,p'})' & B_{4,p} := L^p(\Omega) (\cong (A_{0,p'})'), \end{array}$$

and for $\theta \in (0, 1)$

$$A_{4\theta,p} := [A_{0,p}, A_{4,p}]_\theta \quad \text{and} \quad B_{4\theta,p} := [B_{0,p}, B_{4,p}]_\theta\,.$$

With this notation we have $T_{0,p} : A_{0,p} \to B_{0,p}$ and $T_{4,p} : A_{4,p} \to B_{4,p}$, where $T_{0,p}$ is defined in (5.2.3) and $T_{4,p}$ is defined in (5.2.2).

Lemma 5.2.18. *Assume Condition 5.2.1 and let* $1 < p < \infty$. *The operator* $T_{4,p}$ *is a restriction of* $T_{0,p}$ *to* $A_{4,p}$ *in the following sense:*

$$T_{0,p}(u) \in (B_{4,p'})' \text{ and } T_{0,p}(u) = i_p(T_{4,p}(u)) \text{ for every } u \in A_{4,p}.$$

Proof. Let $u \in A_{4,p}$. For every $v \in A_{4,p'}$ we have

$$(T_{0,p}(u))(v) = \int_\Omega u\ \Delta^2 v\ dx = \int_\Omega v\ \Delta^2 u\ dx = \int_\Omega v\ T_{4,p}(u)\ dx.$$

The claim follows. □

As a consequence of Lemma 5.2.18 in the following lemma we find via interpolation a family of isomorphisms which are extensions of $T_{4,p}$ and restrictions of $T_{0,p}$.

Lemma 5.2.19. *Assume Condition 5.2.1 and fix* $\theta \in (0, 1)$. *Consider the operator* $T_{4\theta,p}$ *such that* $D(T_{4\theta,p}) := A_{4\theta,p}$ *and* $T_{4\theta,p}(u) := T_{0,p}(u)$ *for* $u \in D(T_{4\theta,p})$ *and* $1 < p < \infty$.

Then $T_{4\theta,p}$ *is an isomorphism from* $A_{4\theta,p}$ *onto* $B_{4\theta,p}$ *and moreover*

$$\frac{1}{\max\{C_s, C_w\}} \leq \|T_{4\theta,p}\|_{(A_{4\theta,p} \to B_{4\theta,p})} \leq 2, \tag{5.2.4}$$

where C_s *and* C_w *are the constants appearing in Theorems 5.2.6 and 5.2.13 respectively.*

Proof. The claim follows from Corollaries 5.2.11 and 5.2.14 since the complex interpolation functor is exact and of type θ ([72, Th.1.9.3a]). □

Remark 5.2.20. Notice that (5.2.4) implies that for every $u \in A_{4\theta,p}$ it holds

$$\tfrac{1}{2}\|T_{4\theta,p}(u)\|_{B_{4\theta,p}} \leq \|u\|_{A_{4\theta,p}} \leq \max\{C_s, C_w\}\|T_{4\theta,p}(u)\|_{B_{4\theta,p}}.$$

In the following we consider the operators $T_{1,p}$, $T_{2,p}$ and $T_{3,p}$; i.e. the operators $T_{4\theta_i,p}$ defined in Lemma 5.2.19 with $\theta_i = \frac{1}{4}i$ and $i = 1, 2, 3$. Notice that the solution operator for the "three-quarter weak solution" of problem (5.1.1) is the inverse of $T_{1,p}$. Analogously the solution operator for the "one-half weak solution" of problem (5.1.1) is the inverse of $T_{2,p}$ and the solution operator for the "one-quarter weak solution" of problem (5.1.1) is the inverse of $T_{3,p}$.

For these operators we have that

$$A_{i,p} = W^{i,p}(\Omega) \cap W_0^{\min\{i,2\},p}(\Omega) \text{ with equivalent norms,} \tag{5.2.5}$$

where $A_{i,p} = D(T_{i,p})$. Identity (5.2.5) can be found in Triebel for C^∞-domains. We first show that in order (5.2.5) to hold it is sufficient that $\partial\Omega \in C^{4,\alpha}$. Furthermore we give the dependence on the domain of the constants $D_{1,p,i}$ and $D_{2,p,i}$ that appear in

$$D_{1,p,i}\|u\|_{W^{i,p}(\Omega)\cap W_0^{\min\{i,2\},p}(\Omega)} \leq \|u\|_{A_{i,p}} \leq D_{2,p,i}\|u\|_{W^{i,p}(\Omega)\cap W_0^{\min\{i,2\},p}(\Omega)},$$

for $u \in W^{i,p}(\Omega) \cap W_0^{\min\{i,2\},p}(\Omega)$.

We first recall a classical result from [72].

Proposition 5.2.21. *[72, Th.4.3.3] Let B denote the unit ball in $\mathbb{R}^n$. Then for $i = 1, 2, 3$ and $1 < p < \infty$ one has*

$$\left[L^p(B), W^{4,p}(B) \cap W_0^{2,p}(B)\right]_{\frac{1}{4}i} = W^{i,p}(B) \cap W_0^{\min\{i,2\},p}(B)$$

as Banach spaces. Hence there exist constants $C_{1,p,i}$ and $C_{2,p,i}$ such that for every $u \in W^{i,p}(B) \cap W_0^{\min\{i,2\},p}(B)$ one has

$$C_{1,p,i}\|u\|_{W^{i,p}(B)} \leq \|u\|_{\left[L^p(B), W^{4,p}(B)\cap W_0^{2,p}(B)\right]_{\frac{1}{4}i}} \leq C_{2,p,i}\|u\|_{W^{i,p}(B)}.$$

Theorem 5.2.22. *Let assume Condition 5.2.1. Then for $1 < p < \infty$ and $i = 1, 2, 3$ it holds*

$$[A_{0,p}, A_{4,p}]_{\frac{1}{4}i} = W^{i,p}(\Omega) \cap W_0^{\min\{i,2\},p}(\Omega),$$

as Banach spaces. Hence there exist constants $D_{1,p,i}$ and $D_{2,p,i}$ such that one has for every $u \in W^{i,p}(\Omega) \cap W_0^{\min\{i,2\},p}(\Omega)$

$$D_{1,p,i}\|u\|_{W^{i,p}(\Omega)} \leq \|u\|_{[A_{0,p}, A_{4,p}]_{\frac{1}{4}i}} \leq D_{2,p,i}\|u\|_{W^{i,p}(\Omega)},$$

with $D_{j,p,i} = D_{j,p,i}(p, M, \rho_\Omega^{-1}, R_\Omega)$ for $j = 1, 2$.

Proof. Let S denote a $C^{4,\alpha}$ transformation from $\bar{\Omega}$ onto $\bar{B}$.

Considering the operator

$$E_p : L^p(\Omega) \to L^p(B) \text{ such that } E_p(f) := f \circ S^{-1},$$

one finds that the following properties hold:

- E_p is an isomorphism;
- for $i = 1, \ldots, 4$ the restriction of E_p to $W^{i,p}(\Omega) \cap W_0^{\min\{i,2\},p}(\Omega)$ is an isomorphism onto $W^{i,p}(B) \cap W_0^{\min\{i,2\},p}(B)$;
- there are constants $\bar{C}_{1,p}$ and $\bar{C}_{2,p}$ such that

$$\bar{C}_{1,p} \|E_p(u)\|_{W^{i,p}(B)} \le \|u\|_{W^{i,p}(\Omega)} \le \bar{C}_{2,p} \|E_p(u)\|_{W^{i,p}(B)}, \tag{5.2.6}$$

for every $i = 0, 1, \ldots, 4$ and $u \in W^{i,p}(\Omega) \cap W_0^{\min\{i,2\},p}(\Omega)$. Furthermore the constants $\bar{C}_{1,p}$ and $\bar{C}_{2,p}$ depend only on p, R_Ω, ρ_Ω^{-1} and the M of Condition 5.2.1.

For $\theta \in (0,1)$ the operator E_p induces isomorphisms

$$E_{p,\theta} : A_{4\theta,p} \to \left[L^p(B), W^{4,p}(B) \cap W_0^{2,p}(B)\right]_\theta,$$

and since the complex interpolation functor is exact ([72, Th.1.9.3a]) one has

$$\begin{aligned} \bar{C}_{1,p} \|E_{p,\theta}(u)\|_{\left[L^p(B), W^{4,p}(B)\cap W_0^{2,p}(B)\right]_\theta} &\le \\ \le \|u\|_{A_{4\theta,p}} \le \bar{C}_{2,p} \|E_{p,\theta}(u)\|&_{\left[L^p(B), W^{4,p}(B)\cap W_0^{2,p}(B)\right]_\theta}, \end{aligned} \tag{5.2.7}$$

(See Theorem 1.2.4 in [72]).

Hence, by (5.2.7) and Proposition 5.2.21, we have that

$$\begin{aligned} A_{i,p} &= \left[L^p(\Omega), W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega)\right]_{\frac{1}{4}i} \\ &= \left(E_{p,\frac{1}{4}i}\right)^{-1} \left(\left[L^p(B), W^{4,p}(B) \cap W_0^{2,p}(B)\right]_{\frac{1}{4}i}\right) \\ &= \left(E_{p,\frac{1}{4}i}\right)^{-1} \left(W^{i,p}(B) \cap W_0^{\min\{i,2\},p}(B)\right) \\ &= W^{i,p}(\Omega) \cap W_0^{\min\{i,2\},p}(\Omega). \end{aligned}$$

Furthermore we explicitly find the constants that give the equivalence of the norms. Indeed from (5.2.6), (5.2.7) and Proposition 5.2.21 it follows

$$\begin{aligned} \|u\|_{A_{i,p}} &\le \bar{C}_{2,p} \|E_{p,\frac{1}{4}i}(u)\|_{\left[L^p(B), W^{4,p}(B)\cap W_0^{2,p}(B)\right]_{\frac{1}{4}i}} \\ &\le \bar{C}_{2,p} C_{2,p,i} \|E_{p,\frac{1}{4}i}(u)\|_{W^{i,p}(B)} \le \frac{\bar{C}_{2,p}}{\bar{C}_{1,p}} C_{2,p,i} \|u\|_{W^{i,p}(\Omega)}, \end{aligned}$$

and

$$\begin{aligned}\|u\|_{W^{i,p}(\Omega)} &\leq \bar{C}_{2,p}\|E_p(u)\|_{W^{i,p}(B)} = \bar{C}_{2,p}\|E_{p,\frac{1}{4}i}(u)\|_{W^{i,p}(B)}\\ &\leq \frac{\bar{C}_{2,p}}{C_{1,p,i}}\|E_{p,\frac{1}{4}i}(u)\|_{\left[L^p(B),W^{4,p}(B)\cap W_0^{2,p}(B)\right]_{\frac{1}{4}i}} \leq \frac{\bar{C}_{2,p}}{\bar{C}_{1,p}}\frac{1}{C_{1,p,i}}\|u\|_{A_{i,p}}.\end{aligned}$$

□

Remark 5.2.23. The existence of the $C^{4,\alpha}$ transformation from $\bar{\Omega}$ onto $\bar{B}$ depends upon the regularity of Ω and the fact that Ω is simply connected. This technical assumption can be removed.

Corollary 5.2.24. *Let assume Condition 5.2.1. Then for $1 < p < \infty$ and $i = 1, 2, 3$ it holds*

$$[B_{0,p}, B_{4,p}]_{\frac{1}{4}i} = \left(W^{4-i,p'}(\Omega) \cap W_0^{\min\{4-i,2\},p'}(\Omega)\right)'.$$

Moreover there exist constants $\bar{D}_{j,p,i} = \bar{D}_{j,p,i}(p', M, \rho_\Omega^{-1}, R_\Omega)$ for $j = 1, 2$ such that

$$\bar{D}_{1,p,i}\|u\|_{\left(W^{4-i,p'}(\Omega)\cap W_0^{\min\{4-i,2\},p'}(\Omega)\right)'} \leq \|u\|_{[B_{0,p},B_{4,p}]_{\frac{1}{4}i}}$$

and

$$\bar{D}_{2,p,i}\|u\|_{\left(W^{4-i,p'}(\Omega)\cap W_0^{\min\{4-i,2\},p'}(\Omega)\right)'} \geq \|u\|_{[B_{0,p},B_{4,p}]_{\frac{1}{4}i}},$$

hold for every $u \in \left(W^{4-i,p'}(\Omega) \cap W_0^{\min\{4-i,2\},p'}(\Omega)\right)'$.

Proof. The result follows from Theorem 5.2.22 through duality results for complex interpolation spaces ([72, Th.1.11.3]). □

Corollary 5.2.25. *Assume Condition 5.2.1 and let $1 < p < \infty$.*

Then for $i = 1, 2, 3$ there exist isomorphisms

$$T_{i,p} : W^{i,p}(\Omega) \cap W_0^{\min\{i,2\},p}(\Omega) \to \left(W^{4-i,p'}(\Omega) \cap W_0^{\min\{4-i,2\},p'}(\Omega)\right)',$$

which are restrictions of $T_{0,p}$ and extensions of $T_{4,p}$.

Moreover there exists constants $C_i = C_i(p, p', M, \rho_\Omega^{-1}, R_\Omega)$ such that for every $u \in W^{i,p}(\Omega) \cap W_0^{\min\{i,2\},p}(\Omega)$ it holds

$$\frac{1}{C_i}\|T_{i,p}(u)\|_{\left(W^{4-i,p'}(\Omega)\cap W_0^{\min\{4-i,2\},p'}(\Omega)\right)'} \leq \|u\|_{W^{i,p}(\Omega)},$$

and

$$C_i\|T_{i,p}(u)\|_{\left(W^{4-i,p'}(\Omega)\cap W_0^{\min\{4-i,2\},p'}(\Omega)\right)'} \geq \|u\|_{W^{i,p}(\Omega)}.$$

Proof. The result follows from Lemma 5.2.19, Theorem 5.2.22 and Corollary 5.2.24. □

Theorem 5.2.16 follows directly from the previous corollary.

5.3 Small perturbations of a limaçon

By now we know that problem (5.1.1) is positivity preserving on the disk ([8]) and on limaçons Ω_a for $a \in [0, \frac{1}{6}\sqrt{6}]$ (see Chapter 3). In this chapter we will show that small $C^{2,\gamma}$ perturbations of these domains (Ω_a with $a < \frac{1}{6}\sqrt{6}$) do not destroy this property. In other words, if a domain Ω^* is ε-close in $C^{2,\gamma}$-sense to a limaçon for ε sufficiently small then the clamped plate equation on Ω^* is positivity preserving.

The concept of ε-closeness of domains that we use is the one introduced in [40, Def.1.1]. For sake of completeness we recall the definition.

Definition 5.3.1. *Let $\varepsilon > 0$. We call Ω $\varepsilon-$close in $C^{k,\gamma}$-sense to Ω^* if there exists a $C^{k,\gamma}$-mapping $g : \bar{\Omega}^* \to \bar{\Omega}$ such that $g(\bar{\Omega}^*) = \bar{\Omega}$ and*

$$\|g - Id\|_{C^{k,\gamma}(\bar{\Omega}^*)} \leq \varepsilon.$$

The main result of the section is the following.

Theorem 5.3.2 (Perturbation of the domain). *Let $\bar{a} \in (\frac{1}{4}, \frac{1}{6}\sqrt{6})$ and $\gamma \in (0, 1)$. Then there exist $\varepsilon_0 > 0$ and $c_1, c_2 > 0$ such that for every $\varepsilon \in [0, \varepsilon_0]$ and $a \in [0, \bar{a}]$ the following holds.*

If Ω is ε-close in $C^{2,\gamma}$-sense to Ω_a, then the Green function G_Ω of (5.1.1) satisfies

$$0 < c_1 D_\Omega(x, y) \leq G_\Omega(x, y) \leq c_2 D_\Omega(x, y) \text{ for every } x, y \in \Omega, \tag{5.3.1}$$

where

$$D_\Omega(x, y) = d_\Omega(x) d_\Omega(y) \min\left\{1, \frac{d_\Omega(x) d_\Omega(y)}{|x - y|^2}\right\}. \tag{5.3.2}$$

Remark 5.3.3. In [24] the same estimates from above of G_Ω are given but with more regularity required at the boundary. Thanks to the ε-closeness we get a better estimate from below and the same from above with less assumptions on the boundary.

Remark 5.3.4. Notice that one needs to consider limaçons Ω_a with $a < \frac{1}{6}\sqrt{6}$. Indeed, problem (5.1.1) on a domain that is a small $C^{2,\gamma}$ perturbation of $\Omega_{\frac{1}{6}\sqrt{6}}$ does not have in general the positivity preserving property. In order to allow perturbations of the domain the Green function has to be positive in the strict sense as in (5.3.1). See Chapter 4.

The first result of this kind was proven in [40]. In this paper the authors show that Ω^* ε-close to the unit disk in $C^{3,\gamma}$-sense implies that the clamped plate equation on Ω^* is positivity preserving and moreover, that $G_{\Omega^*}(.,.) \sim D_{\Omega^*}(.,.)$ in $\Omega^* \times \Omega^*$ with $D_{\Omega^*}(.,.)$ defined as in (5.3.2).

The proof of Theorem 5.3.2 consists of several steps and uses similar arguments as in [40] for a disk.

For convenience we summarize here the structure of the section. First we will shortly recall some properties of the limaçons. For technical reason, we will introduce a notation that is slightly different to the one used in Chapter 3. Then we improve Proposition 2.6 in [40]. In this paper the authors show that $C^{2m,\gamma}$-closeness to the disk implies the existence of a conformal map that satisfies the C^{2m-1}-closeness condition. We will show that from $C^{2m,\gamma}$-closeness to the disk one gets the existence of a conformal map that satisfies the $C^{2m,\gamma'}$-closeness condition for $\gamma' \in (0,\gamma)$. Finally in the last subsection we prove Theorem 5.3.2.

5.3.1 Limaçon de Pascal

The Limaçon de Pascal Ω_a with $a \in \left[0, \frac{1}{2}\right]$ is defined as the image of the unit disk through the conformal map

$$h_a(x_1,x_2) = \left(x_1 + 2ax_1x_2, x_2 + ax_2^2 - ax_1^2 + 1 - a\right) \text{ for } a \in \left[0, \tfrac{1}{2}\right]. \tag{5.3.3}$$

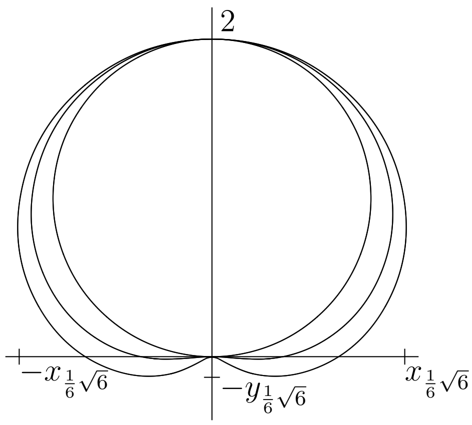

Figure 5.1: *Limaçons Ω_a for respectively $a = 0$, $a = \frac{3}{10}$ and $a = \frac{1}{6}\sqrt{6}$.*

The key ingredient to prove Theorem 5.3.2 is the main result of Chapter 3 (and in [26]). For convenience we recall the result.

Proposition 5.3.5. *The Green function G_{Ω_a} for (5.1.1) with $\Omega = \Omega_a$ and $a \in \left[0, \frac{1}{2}\right]$ is positive if and only if $a \in \left[0, \frac{1}{6}\sqrt{6}\right]$. Moreover, there exist $c_1, c_2 > 0$ such that for $a \in \left[0, \frac{1}{6}\sqrt{6}\right]$ the following estimates hold. Writing $d_a(.) = d_{\Omega_a}(.)$:*

$$\begin{aligned} G_{\Omega_a}(x,y) &\leq c_1 \ d_a(x)d_a(y) \min\left\{1, \frac{d_a(x)d_a(y)}{|x-y|^2}\right\}, \\ G_{\Omega_a}(x,y) &\geq c_2 \ \left(\tfrac{1}{6}\sqrt{6} - a\right) d_a(x)d_a(y) \min\left\{1, \frac{d_a(x)d_a(y)}{|x-y|^2}\right\}. \end{aligned}$$

In the present chapter we will consider limaçons Ω_a for $a \in [0, \bar{a}]$ where $\bar{a}$ is strictly between $\frac{1}{4}$ and $\frac{1}{6}\sqrt{6}$. By taking $\bar{a}$ strictly smaller than $\frac{1}{6}\sqrt{6}$ we will obtain estimates of the Green function $G_{\Omega_a}(.,.)$ which are uniform with respect to a.

We will also need scaled limaçons and we will define these for $R > 0$ by

$$\Omega_{a,R} := \{(Rx, Ry) : (x, y) \in h_a(B_1(0))\},$$

with $B_1(0) = \{(\eta, \xi) \in \mathbb{R}^2 : \eta^2 + \xi^2 < 1\}$. In the following Ω_a denotes $\Omega_{a,1}$.

Some geometrical facts :

(i) For all $a \in \left[0, \frac{1}{2}\right]$ the limaçon $\Omega_{a,R}$ is symmetric with respect to the second axis and both $(0, 0)$ and $(0, 2R)$ lie on $\partial\Omega_{a,R}$.

(ii) Let $[-x_a, x_a] \times [-y_a, 2]$ denote the smallest rectangle that contains $\Omega_{a,1}$. Then

$$a \mapsto x_a \text{ and } a \mapsto y_a \tag{5.3.4}$$

are nondecreasing functions for $a \in \left[0, \frac{1}{2}\right]$ with $1 \leq x_a \leq 1.3$ and $0 \leq y_a \leq 0.25$.

(iii) For $a \in [\frac{3}{16}, \frac{5}{16}]$ we will use $k_{a,R} : [-Rx_a, Rx_a] \to \mathbb{R}$ to describe the lower part of the boundary $\partial\Omega_{a,R}$:

$$k_{a,R}(x) = \inf\{y : (x, y) \in \Omega_{a,R}\}. \tag{5.3.5}$$

In particular in the approximation we will use that the following relations hold:

$$k''_{a,R}(0) = \frac{1}{R}\frac{1 - 4a}{(1 - 2a)^2} \text{ and} \tag{5.3.6}$$

$$\left\|\frac{\partial^i}{\partial x^i} k_{a,R}\right\|_{C^0[-Rx_a^*, Rx_a^*]} \leq \frac{b_i}{R^{i-1}} \text{ for } i = 1, \ldots, 5, \tag{5.3.7}$$

with $x_a^* = \frac{1}{2}(1 - \sqrt{3}a)$. Notice that $x_a^* \in (\frac{1}{5}x_a, \frac{1}{2}x_a)$ where x_a is defined near (5.3.4). The constants b_i can be taken independently of $a \in [\frac{3}{16}, \frac{5}{16}]$.

For sake of completeness we prove formulas (5.3.6) and (5.3.7). The boundary of the limaçon $\Omega_{a,R}$ can be parametrized by

$$\left\{\begin{array}{l} y = R(1 + 2a \sin\theta)\sin\theta + R - 2aR, \\ x = R(1 + 2a \sin\theta)\cos\theta, \end{array}\right. \tag{5.3.8}$$

for $\theta \in [0, 2\pi)$. Notice that $(0, 0)$ is attained by $\theta = \frac{3}{2}\pi$ for every a. From the parametric equations in (5.3.8), we have

$$\left\{\begin{array}{l} \frac{dy}{dx} = R(1 + 4a \sin\theta)\cos\theta \frac{d\theta}{dx}, \\ 1 = -R\left(\sin\theta - 2a + 4a \sin^2\theta\right)\frac{d\theta}{dx}. \end{array}\right.$$

The term $\sin\theta - 2a + 4a \sin^2\theta$ is never zero for θ in a neighborhood of $\frac{3}{2}\pi$ and $a \in [\frac{3}{16}, \frac{5}{16}]$. Hence for θ in this interval we get

$$\frac{dy}{dx} = -\frac{1 + 4a \sin\theta}{\sin\theta - 2a + 4a \sin^2\theta}\cos\theta. \tag{5.3.9}$$

Relation (5.3.7) for $i = 1$ follows directly from (5.3.9). Notice that $(Rx_a^*, k_{a,R}(Rx_a^*))$ is attained for $\theta = \frac{5}{3}\pi$ for every a and that $\theta \mapsto \sin\theta - 2a + 4a\sin^2\theta$ is not zero in $[\frac{4}{3}\pi, \frac{5}{3}\pi]$ for $a \in [\frac{3}{16}, \frac{5}{16}]$.

Differentiating once more one obtains

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{1+4a\sin\theta}{\left(\sin\theta - 2a + 4a\sin^2\theta\right)^2}\cos^2\theta\,(1+8a\sin\theta)\frac{d\theta}{dx} + \frac{\sin\theta + 8a\sin^2\theta - 4a}{\sin\theta - 2a + 4a\sin^2\theta}\frac{d\theta}{dx} \\
&= \frac{d\theta}{dx}\frac{\cos^2\theta + 4a\cos^2\theta\sin\theta + 8a\cos^2\theta\sin\theta + 32a^2\cos^2\theta\sin^2\theta}{\left(\sin\theta - 2a + 4a\sin^2\theta\right)^2} \\
&\quad + \frac{d\theta}{dx}\frac{\sin^2\theta + 8a\sin^3\theta - 4a\sin\theta - 2a\sin\theta - 16a^2\sin^2\theta + 8a^2}{\left(\sin\theta - 2a + 4a\sin^2\theta\right)^2} \\
&\quad + \frac{d\theta}{dx}\frac{4a\sin^3\theta + 32a^2\sin^4\theta - 16a^2\sin^2\theta}{\left(\sin\theta - 2a + 4a\sin^2\theta\right)^2} \\
&= \frac{d\theta}{dx}\frac{1+6a\sin\theta + 8a^2}{\left(\sin\theta - 2a + 4a\sin^2\theta\right)^2} \\
&= -\frac{1}{R}\frac{1+8a^2+6a\sin\theta}{\left(\sin\theta - 2a + 4a\sin^2\theta\right)^3}.
\end{aligned} \tag{5.3.10}$$

Computing in 0 one finds (5.3.6) since

$$\frac{d^2y}{dx^2}(0) = -\frac{1}{R}\frac{1+8a^2-6a}{(-1+2a)^3} = \frac{1}{R}\frac{1-4a}{(1-2a)^2},$$

and moreover, (5.3.10) implies directly (5.3.7) for $i = 2$.

One can prove relation (5.3.7) for $i = 3, 4, 5$ proceeding as before. We skip this proof to avoid too many computations.

5.3.2 Improved ε-closeness to the disk

In this section we show that from $C^{2m,\gamma}$-closeness to the disk one gets the existence of a conformal map that also satisfies the $C^{2m,\gamma'}$-closeness condition for $\gamma' \in (0,\gamma)$, improving [40, Prop.2.6]. We state the result in the following proposition. The proof follows the main steps of the one in [40] except in the last part.

Proposition 5.3.6. *Let $\gamma \in (0,1)$ and $m \in \mathbb{N}$ be given. Then there exist $\varepsilon_0 = \varepsilon_0(m) > 0$ and $c > 0$ such that for $\varepsilon \in [0,\varepsilon_0]$ and $\gamma' \in (0,1)$ with $\gamma' < \gamma$ we have the following.*

If Ω^* is ε-close in $C^{2m,\gamma}$-sense to the disk B, then there exists a biholomorphic mapping $\varphi : \bar{B} \to \bar{\Omega}^*$, with $\varphi \in C^{2m,\gamma'}(\bar{B})$ and $\varphi^{-1} \in C^{2m,\gamma'}(\bar{\Omega}^*)$, such that

$$\|\varphi - Id\|_{C^{2m,\gamma'}(\bar{B})} \le c\,\varepsilon^{\gamma-\gamma'}.$$

Proof. Let $f : \bar{B} \to \bar{\Omega}^*$ be a mapping such that $\|f - Id\|_{C^{2m,\gamma}(\bar{B})} \leq \varepsilon$ with $\varepsilon \leq \varepsilon_0$ small enough. According to [16] (see also [69, Sec.4.2]), a holomorphic mapping $\varphi^{-1} : \Omega^* \to B$, that has the desired qualitative properties, may be constructed in the following way. First set

$$\omega(x) := 2\pi G(x, 0).$$

Here G is the Green function for $-\Delta$ in Ω^* under homogeneous Dirichlet boundary condition. Next define the conjugate harmonic function

$$\omega^*(x) := \int_{1/2}^{x} \left(-\frac{\partial}{\partial \xi_2} \omega(\xi)\, d\xi_1 + \frac{\partial}{\partial \xi_1} \omega(\xi)\ d\xi_2 \right),$$

where the integral is taken with respect to any curve from $\frac{1}{2}$ to x in $\Omega^* \setminus \{0\}$. The function ω^* is well defined up to multiples of 2π. One finds that φ^{-1} is uniquely defined by

$$\varphi^{-1}(x) := \exp(-\omega(x) - i\omega^*(x)) \text{ for } x \in \bar{\Omega}^*,$$

where $\mathbb{R}^2$ and $\mathbb{C}$ are identified. The function φ^{-1} maps 0 onto 0 and the point $\frac{1}{2}$ somewhere into the positive real half-axis. Moreover, for $x \in \partial\Omega^*$ we find that $|\varphi^{-1}(x)| = |\exp(-iw^*(x)| = 1$ and hence $\varphi^{-1}(\partial\Omega^*) \subset \partial B$. For $x \in \Omega^* \setminus \{0\}$ we have $\omega(x) > 0$ and hence $|\varphi^{-1}(x)| < 1$ implying $\varphi^{-1}(\Omega^*) \subset B$.

Setting

$$r(x) = 2\pi G(x, 0) + \log |x| \text{ for } x \in \bar{\Omega}^*$$

one has that r satisfies

$$\begin{cases} -\Delta r &= 0 & \text{in } \Omega^*, \\ r(x) &= \theta(x) := \log|x| & \text{on } \partial\Omega^*. \end{cases} \tag{5.3.11}$$

In order to have that $\|\varphi^{-1} - Id\|_{C^{2m,\gamma'}(\bar{\Omega}^*)} = \mathcal{O}(\varepsilon^{\gamma-\gamma'})$ (and consequently also $\|\varphi - Id\|_{C^{2m,\gamma'}(\bar{B})} = \mathcal{O}(\varepsilon^{\gamma-\gamma'})$) it would be sufficient that

$$\|r\|_{C^{2m,\gamma'}(\bar{\Omega}^*)} = \mathcal{O}(\varepsilon^{\gamma-\gamma'}), \tag{5.3.12}$$

since

$$\varphi^{-1}(x) = x \exp(-r(x) - ir^*(x)) \text{ for } x \in \bar{\Omega}^*,$$

again identifying $\mathbb{R}^2$ and $\mathbb{C}$. The estimate in (5.3.12) follows from the extension of the boundary data $\theta|_{\partial\Omega^*}$ to some $\hat{\theta}$ on $\bar{\Omega}^*$ with

$$\|\hat{\theta}\|_{C^{2m,\gamma'}(\bar{\Omega}^*)} = \mathcal{O}(\varepsilon^{\gamma-\gamma'}). \tag{5.3.13}$$

Indeed, the estimate for $\|r\|_{C^0\left(\bar{\Omega}^*\right)}$ is immediate by the maximum principle applied to (5.3.11). Furthermore, by means of elliptic estimates for second order equations (see [3, Th.7.3] and [36, Chap.6.3-6.4]), we find $\|r\|_{C^{2m,\gamma'}\left(\bar{\Omega}^*\right)} = \mathcal{O}(\varepsilon^{\gamma-\gamma'})$. Note that due to the closeness of Ω^* to B in $C^{2m,\gamma}$-sense, according to Definition 5.3.1, the constants in these estimates may be chosen independently of Ω^*.

It remains to show the existence of some $\hat{\theta}$ that satisfies (5.3.13). This is done as follows. Since Ω^* is $\varepsilon-$close to B in $C^{2m,\gamma}$-sense one may show that $(\theta \circ f)\,|_{\partial B}$ can be extended to θ_f on $\bar{B}$ with $\|\theta_f\|_{C^{2m,\gamma'}(\bar{B})} = \mathcal{O}(\varepsilon^{\gamma-\gamma'})$, provided $\left\|(\theta \circ f)\,|_{\partial B}\right\|_{C^{2m,\gamma'}(\partial B)} = \mathcal{O}(\varepsilon^{\gamma-\gamma'})$. This means that we only have to estimate the tangential derivatives of $(\theta \circ f)\,|_{\partial B}$.

Set $\vartheta(t) := \theta(f(\cos(t), \sin(t)))$. We are done, if we have shown that

$$\max_{j=0,\dots,2m} \max_{t\in[0,2\pi]} \left| \left(\frac{d}{dt}\right)^j \vartheta(t) \right| = \mathcal{O}(\varepsilon^{\gamma}), \tag{5.3.14}$$

$$\left[\left(\frac{d}{dt}\right)^{2m} \vartheta \right]_{\gamma'} = \mathcal{O}(\varepsilon^{\gamma-\gamma'}). \tag{5.3.15}$$

Notice that (5.3.14) was already proved in [40]. The improvement here is that also (5.3.15) holds.

We observe that $\vartheta(t) = \mathcal{O}(\varepsilon)$ since it holds that

$$\log|f(\cos(t), \sin(t))| = \log\left(1 + \mathcal{O}(\varepsilon)\right) = \mathcal{O}(\varepsilon).$$

Let us denote $\tilde{f}(t) := f(\cos(t), \sin(t))$. Then $\tilde{f} = (\tilde{f}_1, \tilde{f}_2) : \mathbb{R} \to \mathbb{R}^2$ and

$$\begin{aligned}
\left(\frac{d}{dt}\right)^j \vartheta &= \left(\frac{d}{dt}\right)^j \left(\theta \circ \tilde{f}\right) \\
&= \sum_{\substack{|\alpha|=1,\\ \alpha\in\mathbb{N}^2}}^{j} \left((D^\alpha\theta)\circ \tilde{f}\right) \left(\sum_{\substack{p_1+\dots+p_{|\alpha|}=j\\ 1\le p_i}} d_{j,\alpha,\vec{p}} \prod_{l=1}^{|\alpha|} \tilde{f}_{\beta_l}^{(p_l)} \right),
\end{aligned}$$

with some suitable coefficients $d_{j,\alpha,\vec{p}}$ and with $\beta_l = 1$ if $1 \le l \le \alpha_1$ and $\beta_l = 2$ otherwise.

We want to compare this with the corresponding expression with f replaced by Id. Writing $\tilde{f}_0(t) = Id \circ (\cos(t), \sin(t))$ we find

$$\begin{aligned}
\left(\frac{d}{dt}\right)^j \vartheta = \sum_{\substack{|\alpha|=1,\\ \alpha\in\mathbb{N}^2}}^{j} &\left(\left((D^\alpha\theta)\circ \tilde{f} - (D^\alpha\theta)\circ \tilde{f}_0\right) + (D^\alpha\theta)\circ\tilde{f}_0\right) \\
&\times \left(\sum_{\substack{p_1+\dots+p_{|\alpha|}=j\\ 1\le p_i}} d_{j,\alpha,\vec{p}} \prod_{l=1}^{|\alpha|} \left(\left(\tilde{f}_{\beta_l}^{(p_l)} - \tilde{f}_{0,\beta_l}^{(p_l)}\right) + \tilde{f}_{0,\beta_l}^{(p_l)}\right) \right).
\end{aligned}$$

Since $\theta(\tilde{f}_0(t)) = \log|(\cos(t), \sin(t))| \equiv 0$, all expressions containing $\tilde{f}_0$ only (and not a difference), sum up to zero. In the remaining sum, every term contains at least one factor of the form

$$(D^\alpha\theta)\circ\tilde{f} - (D^\alpha\theta)\circ\tilde{f}_0 \quad \text{or} \quad \tilde{f}_{\beta_l}^{(p_l)} - \tilde{f}_{0,\beta_l}^{(p_l)},$$

with $|\alpha|, p_l \in \{1,\ldots,2m\}$. For ε small, each of this factors is at most $\mathcal{O}(\varepsilon^\gamma)$. Choosing ε_0 sufficiently small, the other factors remain uniformly bounded with respect to $\varepsilon \in [0,\varepsilon_0)$. This shows (5.3.14). In order to verify (5.3.15) we remark that $\left[\tilde{f}^{(p_l)}_{\beta_l} - \tilde{f}^{(p_l)}_{0,\beta_l}\right]_\gamma = \mathcal{O}(\varepsilon)$ for $p_l \in \{1,\ldots,2m\}$ by the definition of ε-closeness. It remains to study the term $\left[(D^\alpha\theta)\circ\tilde{f} - (D^\alpha\theta)\circ\tilde{f}_0\right]_{\gamma'}$ for $|\alpha| \in \{1,\ldots,2m\}$. One finds

$$\begin{aligned}
&\left[(D^\alpha\theta)\circ\tilde{f} - (D^\alpha\theta)\circ\tilde{f}_0\right]_{\gamma'}\\
&= \sup_{\substack{t,s\in[0,2\pi],\\ |t-s|>\varepsilon}} \frac{\left|(D^\alpha\theta)\circ\tilde{f}(t) - (D^\alpha\theta)\circ\tilde{f}_0(t) - (D^\alpha\theta)\circ\tilde{f}(s) + (D^\alpha\theta)\circ\tilde{f}_0(s)\right|}{|t-s|^{\gamma'}}\\
&+ \sup_{\substack{t,s\in[0,2\pi],\\ |t-s|<\varepsilon}} \frac{\left|(D^\alpha\theta)\circ\tilde{f}(t) - (D^\alpha\theta)\circ\tilde{f}_0(t) - (D^\alpha\theta)\circ\tilde{f}(s) + (D^\alpha\theta)\circ\tilde{f}_0(s)\right|}{|t-s|^{\gamma'}} = \ldots.
\end{aligned}$$

Since $D^\alpha\theta \in C^\gamma$ for $|\alpha| \le 2m$, $\left\|\tilde{f} - \tilde{f}_0\right\|_{C^1(0,2\pi)} = \mathcal{O}(\varepsilon)$ and $\tilde{f},\tilde{f}_0 \in C^1[0,2\pi]$, we get

$$\begin{aligned}
\ldots &\le c \sup_{\substack{t,s\in[0,2\pi],\\ |t-s|>\varepsilon}} \frac{\left|\tilde{f}(t)-\tilde{f}_0(t)\right|^\gamma + \left|\tilde{f}(s)-\tilde{f}_0(s)\right|^\gamma}{|t-s|^{\gamma'}} +\\
&\quad + c \sup_{\substack{t,s\in[0,2\pi],\\ |t-s|<\varepsilon}} \frac{\left|\tilde{f}(t)-\tilde{f}(s)\right|^\gamma + \left|\tilde{f}_0(t)-\tilde{f}_0(s)\right|^\gamma}{|t-s|^{\gamma'}}\\
&\le 2c \sup_{\substack{t,s\in[0,2\pi],\\ |t-s|>\varepsilon}} \frac{\varepsilon^\gamma}{|t-s|^{\gamma'}} + 2c' \sup_{\substack{t,s\in[0,2\pi],\\ |t-s|<\varepsilon}} |t-s|^{\gamma-\gamma'} \le \mathcal{O}(\varepsilon^{\gamma-\gamma'}).
\end{aligned}$$

□

5.3.3 Perturbations from the bilaplacian on a limaçon

In this section we prove Theorem 5.3.2. Since the proof consists of several parts, for convenience we first summarize the main steps here.

We first show that ε-closeness in $C^{2,\gamma}$-sense of Ω to Ω_a implies the existence of a biholomorphic map $\varphi_a : \Omega_a \to \Omega$ such that

$$\|\varphi_a - Id\|_{C^{2,\gamma'}(\bar{\Omega}_a)} \le \delta(\varepsilon) \text{ for } 0 < \gamma' < \gamma. \tag{5.3.16}$$

Next, through this conformal mapping φ_a problem (5.1.1) on Ω is transformed into the following problem on Ω_a:

$$\left\{\begin{array}{rcll} (\Delta^2 + A)\,u &=& \tilde{f} & \text{in } \Omega_a,\\ u &=& 0 & \text{on } \partial\Omega_a,\\ \frac{\partial}{\partial\nu}u &=& 0 & \text{on } \partial\Omega_a, \end{array}\right. \tag{5.3.17}$$

where A is a lower order perturbation of the biharmonic operator; i.e. A can be written as $A = \sum_{|\alpha|\leq 3} A_\alpha(.)D^\alpha$ (see [41, Remark after Theorem 5.1]). From (5.3.16) one also has that there exists a $\delta_1 = \delta_1(\varepsilon) > 0$ such that the coefficients of A in (5.3.17) satisfy

$$\sup_{|\alpha|\leq 3} \|A_\alpha\|_\infty \leq \delta_1.$$

We then see that the positivity of the Green function associated to problem (5.3.17) implies the positivity of the Green function associated to problem (5.1.1) thanks to the properties of conformal maps ([59]). Hence, instead of proving directly Theorem 5.3.2 we prove the following result.

Theorem 5.3.7 (Perturbation of Δ^2 by lower order terms). *Let $a \in [0, \bar{a}]$ with $\bar{a}$ as in Theorem 5.3.2. Consider problem (5.3.17) with $A = \sum_{|\alpha|\leq 3} A_\alpha(x)D^\alpha$, $A_\alpha \in C(\bar{\Omega}_a)$ and let $G_{\Omega_a,A}$ the Green function associated to (5.3.17).*

Then there exists $\eta_0 > 0$ such that, whenever $\|A_\alpha\|_\infty \leq \eta_0$ for all α with $|\alpha| \leq 3$, the Green function associated to (5.3.17) is positive. Moreover, there exist d_1,$d_2 > 0$ such that, with $D_{\Omega_a}(x, y)$ as in (5.3.2), the following holds:

$$d_1 D_{\Omega_a}(x, y) \leq G_{\Omega_a,A}(x, y) \leq d_2 D_{\Omega_a}(x, y). \tag{5.3.18}$$

Theorem 5.3.7 says that if the lower order perturbation of the biharmonic operator is small then the positivity preserving property of system (5.3.17) in Ω_a follows from the positivity preserving property of problem (5.1.1) on the same domain.

A result similar to Theorem 5.3.7 was proven in [41] for the polyharmonic Dirichlet boundary value problem on the unit disk B. The main ingredient of the proof are appropriate estimates of

$$H^k_B(x, y, z) := \frac{G_B(x, z) \left|D^k_z G_B(z, y)\right|}{G_B(x, y)} \text{ with } k \in \mathbb{N}^2,\ |k| \leq 3, \tag{5.3.19}$$

which were proved in [40]. Notice that in [40] one considers Ω being a ball. The only place however where that fact is used is in the explicit estimates of H^k_B. Indeed all the other arguments can be applied to any planar smooth domain Ω whose Green function is positive in the strict sense as in the left hand side of (5.3.18). Hence to prove Theorem 5.3.7 we first show that $H^k_{\Omega_a}$ (that is the quotient in (5.3.19) calculated for G_{Ω_a}) satisfies the same estimates as H^k_B and then refer to the work in [41].

In the next paragraph we construct the conformal mapping from "Ω ε-close to Ω_a" to the limaçon Ω_a and we state the equivalence of Theorem 5.3.2 and Theorem 5.3.7. Then we prove the perturbation result of Theorem 5.3.7.

Conformal transformation

In this subsection we prove that problem (5.1.1) on Ω that is ε-close to Ω_a, corresponds to a problem of the type (5.3.17) on Ω_a with the coefficients of A, the lower order perturbation of Δ^2, being small. Or, to be more precise, there is a function $\varepsilon \mapsto \delta(\varepsilon)$ with $\delta(\varepsilon) \to 0$ when $\varepsilon \downarrow 0$, such that

$$\Omega \ \varepsilon\text{-close in } C^{2,\gamma}\text{-sense to } \Omega_a \Rightarrow \sup_{|\alpha|\leq 3} \|A_\alpha\|_\infty \leq \delta(\varepsilon).$$

Or in other words, that Theorem 5.3.7 implies Theorem 5.3.2.

The first step consist of proving existence of a biholomorphic map from the limaçon to a domain ε-close to the limaçon which is near the identity in $C^{2,\gamma}$-sense.

Proposition 5.3.8. *Let $a \in [0, \bar{a}]$ and $\gamma \in (0,1)$. Then there exist $\bar{\varepsilon} > 0$ and $c = c(\bar{a}) > 0$ such that for $\varepsilon \in [0, \bar{\varepsilon})$ and every $\gamma' \in (0,1)$ with $\gamma' < \gamma$ we have the following.*

If Ω is ε-close in $C^{2,\gamma}$-sense to Ω_a then there is a biholomorphic map $\varphi_a : \bar{\Omega}_a \to \bar{\Omega}$, with $\varphi_a \in C^{2,\gamma'}(\bar{\Omega}_a)$ and $\varphi_a^{-1} \in C^{2,\gamma'}(\bar{\Omega})$, such that

$$\|\varphi_a - Id\|_{C^{2,\gamma'}(\bar{\Omega}_a)} \leq c\, \varepsilon^{\gamma-\gamma'}.$$

The proof of Proposition 5.3.8 consist of the following three lemmas.

Since $a \leq \bar{a} < \frac{1}{6}\sqrt{6} < \frac{1}{2}$ one may check that the map h_a, defined in (5.3.3), is conformal and one-to-one on the domain

$$B_{\sqrt{1.5}} := \left\{ x \in \mathbb{R}^2 : \|x\|_2 < \sqrt{1.5} \right\}.$$

We choose the value $\varepsilon_1 \in (0,1)$ such that, if Ω is ε-close in $C^{2,\gamma}$-sense to Ω_a for $\varepsilon \in (0, \varepsilon_1)$, then $h_a^{-1}(\Omega) \subset B_{\sqrt{1.5}}$. It follows that h_a^{-1} is a conformal map on any domain Ω which is ε-close in $C^{2,\gamma}$-sense to the limaçon for $\varepsilon < \varepsilon_1$.

Lemma 5.3.9. *Let $a \in [0, \bar{a}]$ and $\gamma \in (0,1)$. There exists $c_1 = c_1(\bar{a}) > 0$ such that the following holds. If Ω is ε-close in $C^{2,\gamma}$-sense to Ω_a for $\varepsilon \in (0, \varepsilon_1)$, then $\Omega^* := h_a^{-1}(\Omega)$ is $c_1\varepsilon$-close in $C^{2,\gamma}$-sense to the disk B.*

Proof. Let g be a $C^{2,\gamma}$-mapping, $g : \bar{\Omega}_a \to \bar{\Omega}$, such that $\|g - Id\|_{C^{2,\gamma}(\bar{\Omega}_a)} \leq \varepsilon$. We define the map $f : \bar{B} \to \bar{\Omega}^*$ by

$$f(x) := \left(h_a^{-1} \circ g \circ h_a\right)(x),$$

where $h_a : \bar{B} \to \bar{\Omega}_a$ and $h_a^{-1} : \bar{\Omega} \to \bar{\Omega}^*$ are conformal (see Figure 5.2). Then there exists a positive constant c_1, depending on $\|h_a\|_{C^4}$ and $\|h_a^{-1}\|_{C^4}$, such that $\|f - Id\|_{C^{2,\gamma}(\bar{B})} \leq c_1\varepsilon$. □

In the following Ω^* denotes $h_a^{-1}(\Omega)$.

Lemma 5.3.10. *Let $a \in [0, \bar{a}]$ and $\gamma \in (0,1)$. Then there exist $\varepsilon_2 > 0$ and $c_2 = c_2(\bar{a}) > 0$ such that for every $\varepsilon \in (0, \varepsilon_2)$ and $\gamma' \in (0,1)$ with $\gamma' < \gamma$, the following holds. If Ω is ε-close in $C^{2,\gamma}$-sense to Ω_a, then there exists a biholomorphic mapping $\varphi : \bar{B} \to \bar{\Omega}^*$ with $\varphi \in C^{2,\gamma'}(\bar{B})$, $\varphi^{-1} \in C^{2,\gamma'}(\bar{\Omega}^*)$ and such that*

$$\|\varphi - Id\|_{C^{2,\gamma'}(\bar{B})} \leq c_2 \varepsilon^{\gamma-\gamma'}.$$

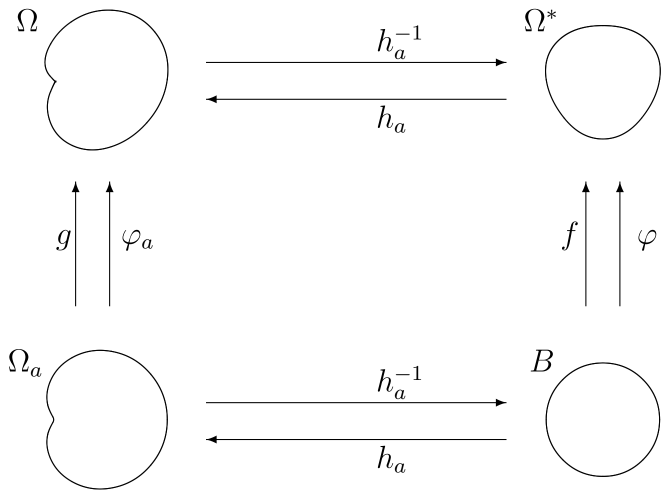

Figure 5.2: *The maps between* Ω, Ω_a, B *and* Ω^*.

Proof. From Lemma 5.3.9 it follows that Ω^* is $c_1\varepsilon$-close to B. Applying Proposition 5.3.6 we have that there exists $\varepsilon_0 > 0$ such that "Ω^* $c_1\varepsilon$-close to B" for $c_1\varepsilon \in (0, \varepsilon_0)$ implies the existence of a biholomorphic mapping $\varphi : \bar{B} \to \bar{\Omega}^*$ with $\varphi \in C^{2,\gamma'}(\bar{B})$, $\varphi^{-1} \in C^{2,\gamma'}(\bar{\Omega}^*)$ and such that it holds

$$\|\varphi - Id\|_{C^{2,\gamma'}(\bar{B})} \leq c_2 \varepsilon^{\gamma-\gamma'},$$

for every $\gamma' \in (0,1)$ with $\gamma' < \gamma$. The claim follows taking $\varepsilon_2 = \min\left\{\varepsilon_1, c_1^{-1}\varepsilon_0\right\}$. □

Lemma 5.3.11. *Let* $a \in [0, \bar{a}]$ *and* $\gamma \in (0,1)$. *There exist* $\varepsilon_3 > 0$ *and* $c_3 = c_3(\bar{a}) > 0$ *such that for every* $\varepsilon \in (0, \varepsilon_3)$ *and* $\gamma' \in (0,1)$ *with* $\gamma' < \gamma$ *the following holds.*

If Ω *is* ε*-close in* $C^{2,\gamma}$*-sense to* Ω_a, *then there exists a biholomorphic mapping* $\varphi_a : \bar{\Omega}_a \to \bar{\Omega}$ *with* $\varphi_a \in C^{2,\gamma'}(\bar{\Omega}_a)$, $\varphi_a^{-1} \in C^{2,\gamma'}(\bar{\Omega})$ *and such that*

$$\|\varphi_a - Id\|_{C^{2,\gamma'}(\bar{\Omega}_a)} \leq c_3 \varepsilon^{\gamma-\gamma'}.$$

Proof. We denote φ_a the map from $\bar{\Omega}_a$ to $\bar{\Omega}$ given by

$$\varphi_a(x) := \left(h_a \circ \varphi \circ h_a^{-1}\right)(x).$$

Here φ is the conformal map of Lemma 5.3.10. The map φ_a is biholomorphic as a composition of biholomorphic maps. Furthermore we have $\varphi_a \in C^{2,\gamma'}(\bar{\Omega}_a)$ and $\varphi_a^{-1} \in C^{2,\gamma'}(\bar{\Omega})$ since $\varphi \in C^{2,\gamma'}(\bar{B})$ and $\varphi^{-1} \in C^{2,\gamma'}(\bar{\Omega}^*)$.

By the way the holomorphic map φ_a is defined one finds that there exists a positive constant K, depending on $\|h_a\|_{C^4}$ and $\|h_a^{-1}\|_{C^4}$, such that

$$\|\varphi_a - Id\|_{C^{2,\gamma'}(\bar{\Omega}_a)} \leq K \|\varphi - Id\|_{C^{2,\gamma'}(\bar{B})}.$$

The claim will follow choosing $c_3 = Kc_2$ and $\varepsilon_3 = \varepsilon_2$ with c_2 and ε_2 as defined in Lemma 5.3.10. □

Remark 5.3.12. Notice that Proposition 5.3.8 follows from Lemma 5.3.11.

We are now ready to prove that the positivity preserving property of problem (5.3.17) with a small perturbation of Δ^2 on Ω_a implies the positivity preserving property of problem (5.1.1) on Ω ε-close in $C^{2,\gamma}$-sense to Ω_a.

Corollary 5.3.13. *Let $\gamma \in (0,1)$. For every $\delta > 0$ small enough and $a \in [0, \bar{a}]$ there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in [0, \varepsilon_0)$ the following holds.*

If Ω is ε-close in $C^{2,\gamma}$-sense to the limaçon Ω_a and the coefficients of the operator A satisfy

$$\sup_{|\alpha|\leq 3} \|A_\alpha\|_\infty \leq \delta, \tag{5.3.20}$$

then the positivity of the Green function associated to problem (5.3.17) on Ω_a implies the positivity of the Green function associated to problem (5.1.1) on Ω.

Proof. To prove the claim we show that problem (5.1.1) on Ω ε-close in $C^{2,\gamma}$-sense to Ω_a can be "transformed" into problem (5.3.17) on Ω_a with the coefficients of the lower order operator A satisfying (5.3.20).

Let u be solution of problem (5.1.1) on Ω. Consider $\delta_0 < \min\left\{\frac{1}{2}, 2^{-7}\delta\right\}$. By Proposition 5.3.8 we know that there exists a $\varepsilon_0 = \varepsilon_0(\delta_0) > 0$ such that for $\varepsilon \in [0, \varepsilon_0)$ and $\gamma' \in (0,1)$ with $\gamma' < \gamma$ we have the following. If Ω is ε-close to Ω_a in $C^{2,\gamma}$-sense then there exists a conformal map $\varphi_a : \bar{\Omega}_a \to \bar{\Omega}$ such that

$$\|\varphi_a - Id\|_{C^{2,\gamma'}(\bar{\Omega}_a)} \leq \delta_0.$$

We define the function $v_a(x) := u \circ \varphi_a(x)$ on Ω_a. Clearly $u > 0$ if and only if $v_a > 0$. Since φ_a is a conformal map, the function v_a satisfies

$$\begin{cases} \Delta^2 v_a - 2\nabla|\varphi_a'|^2 \cdot \nabla \frac{\Delta v_a}{|\varphi_a'|^2} - 4|\varphi_a''|^2 \frac{\Delta v_a}{|\varphi_a'|^2} = |\varphi_a'|^4 f \circ \varphi_a & \text{in } \Omega_a, \\ v_a = 0 & \text{on } \partial\Omega_a, \\ \frac{\partial}{\partial \nu} v_a = 0 & \text{on } \partial\Omega_a, \end{cases} \tag{5.3.21}$$

where φ_a' denotes the complex derivative of φ_a. Hence v_a is solution of a problem as in (5.3.17). The coefficients of the lower order perturbation of Δ^2 in (5.3.21) satisfy (5.3.20) by the choice of δ_0. □

Remark 5.3.14. Notice that since we are working with conformal mappings it is sufficient to have $C^{2,\gamma}$-closeness in order to transform problem (5.1.1) into problem (5.3.17). Working with general transformations fourth order derivatives would appear and $C^{4,\gamma}$-closeness would be necessary.

As a consequence of Corollary 5.3.13, Theorem 5.3.2 will follow from Theorem 5.3.7.

Proof of the perturbation theorem

In [41] Theorem 5.3.7 has been proven in the unit disk (that is Ω_0). We now give a sketch of the proof for Ω_a, $a \in [0, \bar{a}]$, by following similar steps.

First we state some estimates for (5.3.19) with G_B replaced by G_{Ω_a}.

Theorem 5.3.15. *Let $k = (k_1, k_2)$ with $k_1, k_2 \in \mathbb{N}$ and $|k| \leq 3$. The following estimates hold for any $a \in [0, \bar{a}]$ and $x, y, z \in \Omega_a$.*

(i) If $|k| = 3$, then

$$\frac{G_{\Omega_a}(x,z)\left|D_z^k G_{\Omega_a}(z,y)\right|}{G_{\Omega_a}(x,y)} \preceq \frac{1}{|x-z|} + \frac{1}{|y-z|}.$$

(ii) If $|k| = 2$, then

$$\frac{G_{\Omega_a}(x,z)\left|D_z^k G_{\Omega_a}(z,y)\right|}{G_{\Omega_a}(x,y)} \preceq \log\left(\frac{3}{|z-y|}\right).$$

(iii) If $|k| \leq 1$, then

$$\frac{G_{\Omega_a}(x,z)\left|D_z^k G_{\Omega_a}(z,y)\right|}{G_{\Omega_a}(x,y)} \preceq 1.$$

Proof. With the same method as has been used in [41] the result follows from the optimal estimate from below of G_{Ω_a}, which has been proved in Chapter 3 (see [26] and Proposition 5.3.5), and from the estimates of the derivatives of the Green function, which have been proved in Theorem 2.5.6 (see also [24]).

For completeness we give the idea of the proof in the case $|k| = 3$. Using the results in Proposition 5.3.5 and Theorem 2.5.6 we find

$$\begin{aligned}
\frac{G_{\Omega_a}(x,z)\left|D_z^k G_{\Omega_a}(z,y)\right|}{G_{\Omega_a}(x,y)} &\preceq \frac{d_{\Omega_a}(z)\min\left\{1, \frac{d_{\Omega_a}(x)d_{\Omega_a}(z)}{|x-z|^2}\right\}|z-y|^{-1}\min\left\{1, \frac{d_{\Omega_a}(y)}{|y-z|}\right\}^2}{d_{\Omega_a}(y)\min\left\{1, \frac{d_{\Omega_a}(x)d_{\Omega_a}(y)}{|x-y|^2}\right\}} \\
&\preceq \frac{d_{\Omega_a}(z)}{d_{\Omega_a}(y)}\frac{Q(x,y,z)}{|z-y|}\min\left\{1, \frac{d_{\Omega_a}(y)}{|y-z|}\right\},
\end{aligned}$$

where

$$Q(x,y,z) := \frac{\min\left\{1, \frac{d_{\Omega_a}(x)d_{\Omega_a}(z)}{|x-z|^2}\right\}\min\left\{1, \frac{d_{\Omega_a}(y)}{|y-z|}\right\}}{\min\left\{1, \frac{d_{\Omega_a}(x)d_{\Omega_a}(y)}{|x-y|^2}\right\}}.$$

The claim follows since it holds

$$\min\left\{1, \frac{d_{\Omega_a}(y)}{|y-z|}\right\} \preceq \frac{d_{\Omega_a}(y)}{d_{\Omega_a}(z)} \text{ and } Q(x,y,z) \preceq 1 + \frac{|z-y|}{|z-x|},$$

see [40, Lemma 4.3].

The proof for $|k| \leq 2$ follows using similar arguments. □

Let $\mathcal{G}_\Omega$ denote the Green operator associated to problem (5.1.1) in Ω, that is

$$\mathcal{G}_\Omega f(x) := \int_\Omega G_\Omega(x,y) f(y) dy.$$

By the estimate in Theorem 5.3.15 one may observe that the derivatives of the Green function have an integrable singularity. Hence one finds the following two corollaries of Theorem 5.3.15.

Corollary 5.3.16. *There exists $M \in \mathbb{R}^+$ such that for any $0 \leq f \in L^p(\Omega_a)$ with $p \geq 1$ and $k = (k_1, k_2) \in \mathbb{N}^2$ with $0 \leq |k| \leq 3$, the following estimate holds for all $a \in [0, \bar{a}]$*

$$\left| \left(\mathcal{G}_{\Omega_a} D^k \mathcal{G}_{\Omega_a} f \right)(x) \right| \leq M \left(\mathcal{G}_{\Omega_a} f \right)(x) \text{ for all } x \in \Omega_a.$$

Corollary 5.3.17. *Let $a \in [0, \bar{a}]$ and $\eta > 0$ be such that the coefficients of A in (5.3.17) satisfy $\|A_\alpha\|_\infty \leq \eta$ for all $|\alpha| \leq 3$. Then for any $0 \leq f \in L^p(\Omega_a)$ with $p \geq 1$*

$$\left| \left(\mathcal{G}_{\Omega_a} A \mathcal{G}_{\Omega_a} f \right)(x) \right| \leq 10\, M\eta \ \left(\mathcal{G}_{\Omega_a} f \right)(x) \text{ for all } x \in \Omega_a,$$

and furthermore

$$\left| \left(\left(\mathcal{G}_{\Omega_a} A \right)^i \mathcal{G}_{\Omega_a} f \right)(x) \right| \leq (10\, M\, \eta)^i \ \left(\mathcal{G}_{\Omega_a} f \right)(x) \text{ for all } x \in \Omega_a,$$

where M is the constant of Corollary 5.3.16.

For the proofs we refer to [41, Cor.4.2, Lem.5.4-5.5].

Proof of Theorem 5.3.7. Let u be a solution of (5.3.17). Proceeding as in [41, Lemma 5.3] one finds that there exists a $\eta_1 > 0$ such that $\left(\mathcal{I} + \mathcal{G}_{\Omega_a} A \right)^{-1}$ is well defined when the coefficients of A satisfy $\|A_\alpha\|_\infty \leq \eta_1$ for $|\alpha| \leq 3$. We have

$$u = -\mathcal{G}_{\Omega_a} A u + \mathcal{G}_{\Omega_a} f = \left(\mathcal{I} + \mathcal{G}_{\Omega_a} A \right)^{-1} \mathcal{G}_{\Omega_a} f,$$

and may formally write

$$\begin{aligned} \mathcal{G}_{\Omega_a, A} &= \left(\mathcal{I} + \mathcal{G}_{\Omega_a} A \right)^{-1} \mathcal{G}_{\Omega_a} \\ &= \mathcal{G}_{\Omega_a} - \mathcal{G}_{\Omega_a} A \mathcal{G}_{\Omega_a} + \left(\mathcal{G}_{\Omega_a} A \right)^2 \mathcal{G}_{\Omega_a} - \left(\mathcal{G}_{\Omega_a} A \right)^3 \mathcal{G}_{\Omega_a} + \ldots . \end{aligned} \tag{5.3.22}$$

Using Corollary 5.3.17 from (5.3.22) taking $\eta_0 = \min\left\{ \frac{1}{30M}, \eta_1 \right\}$ and $\eta \leq \eta_0$ the series converges and we get

$$\tfrac{1}{2} \mathcal{G}_{\Omega_a} \leq \mathcal{G}_{\Omega_a, A} \leq \tfrac{3}{2} \mathcal{G}_{\Omega_a}. \tag{5.3.23}$$

The estimate in (5.3.18) follows directly from (5.3.23) and Proposition 5.3.5. □

Remark 5.3.18. For the problem

$$\begin{cases} \left(\Delta^2 + A \right) u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \\ \frac{\partial}{\partial\nu} u &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

with Ω ε-close in $C^{2,\gamma}$-sense to Ω_a for $a \in [0, \bar{a}]$ and with A the lower order perturbation of the bilaplacian such that $\|A_\alpha\|_\infty \leq \eta$ for $|\alpha| \leq 3$, the result stated in Theorem 5.3.7 is still valid for ε and η sufficiently small.

5.4 Filling the domain with perturbed limaçons

In this section we prove that a sufficiently smooth bounded two-dimensional domain can be approximated by limaçon-like domains in the sense we want. That is, we will construct a finite number of domains E_j such that:

(i) the union of E_j covers Ω near $\partial\Omega$;

(ii) the union of ∂E_j covers the boundary $\partial\Omega$;

(iii) each E_j is close in $C^{2,\gamma}$-sense to a limaçon $\Omega_{a,R}$ with $a \in [0, \bar{a}]$ in a uniform way;

(iv) the E_j uniformly satisfy the uniform $C^{4,\gamma}$ regularity condition in $a \in [0, \bar{a}]$.

The precise statement is given in Theorem 5.4.29.

5.4.1 Local approximation

We first show that for each z_0 on $\partial\Omega$ there exists a domain ε-close to a limaçon which boundary intersects $\partial\Omega$ in a neighborhood of z_0. In order to do that it will be convenient to use local systems of cartesian coordinates. The following lemma lists some technical results.

Lemma 5.4.1. *Let $\ell \geq 2$ and Ω be a domain in $\mathbb{R}^2$ satisfying the uniform $C^{\ell,\alpha}$ regularity condition, Definition 2.1.3, with constant M and mappings $\varphi_j \in C^{\ell,\alpha}$, $j \in J$. Let ρ_Ω be as in Notation 5.1.9 and set $x_{\rho_\Omega} := \frac{\sqrt{3}}{2}\rho_\Omega$.*

Then for every $z_0 \in \partial\Omega$ there exists a local cartesian coordinates system and a function $g_{z_0} \in C^{\ell,\alpha}$, $g_{z_0} : [-x_{\rho_\Omega}, x_{\rho_\Omega}] \to \mathbb{R}$, such that:

(i) $z_0 = (0,0)$;

(ii) the x-axis is tangential to $\partial\Omega$ in z_0;

(iii) the y-axis has the direction of the internal normal to $\partial\Omega$ in z_0;

(iv) $B_{\frac{1}{2}\rho_\Omega}(z_0) \cap \partial\Omega \subset \{(x,y) : x \in [-x_{\rho_\Omega}, x_{\rho_\Omega}] \text{ and } y = g_{z_0}(x)\}$;

(v) $\|g_{z_0}\|_{C^{\ell,\alpha}[-x_{\rho_\Omega}, x_{\rho_\Omega}]} \leq 2(\ell+1)M$.

Remark 5.4.2. Observe that the function g_{z_0} of Lemma 5.4.1 satisfies $\left|g'_{z_0}(x)\right| \leq \sqrt{3}$.

Remark 5.4.3. Notice that the norm of g_{z_0} grows linearly in ℓ. If we fix the size of the interval of definition of the function g_{z_0} (i.e. $[-x_{\rho_\Omega}, x_{\rho_\Omega}]$) the constant increases when taking more derivatives that is, a bigger ℓ. Instead we can choose a constant independent of ℓ if we let the size of the interval change. For the purpose of an uniform estimate we need to fix the size of the interval. Notice that in our case $\ell = 4$ fixed.

We skip the rather technical proof of Lemma 5.4.1.

In the following theorem we will state that for every point of the boundary of a domain satisfying the uniform $C^{4,\alpha}$ regularity condition there exists a limaçon $\Omega_{a,R}$ that approximates $\partial\Omega$ in the point in C^2-sense. Furthermore we will construct a domain $\tilde{\Omega}$ that is ε-close to the limaçon $\Omega_{a,R}$ and which boundary coincides with $\partial\Omega$ in a neighborhood of that point. By construction $\tilde{\Omega}$ is a domain satisfying the uniform $C^{4,\alpha}$ regularity condition with constant M_1 where M_1 depends only on M and ρ_Ω.

For the purpose of a uniform statement we will have to rescale to limaçons of 'unit' size. In order to do so we define for a given f the scaled function:

$$f^R(x,y) := R^{-1}\, f(Rx, Ry) \text{ for } R \in \mathbb{R}^+. \tag{5.4.1}$$

Theorem 5.4.4. *Assume that the following holds for some $\alpha, \gamma \in (0,1)$:*

(i) $\Omega \subset \mathbb{R}^2$ is a simply connected domain satisfying the uniform $C^{4,\alpha}$ regularity condition with constant M;

(ii) $g_{z_0} \in C^{4,\alpha}$ for $z_0 \in \partial\Omega$ are functions that describe the boundary of Ω as in Lemma 5.4.1 and fix $R := \min\left\{\frac{1}{2}\left(\max_{z_0\in\partial\Omega}\left\|g''_{z_0}\right\|_\infty\right)^{-1}, 1\right\}$;

(iii) $\varepsilon > 0$ is such that for all $\tilde{\Omega}$ which are $\varepsilon-$close to $\Omega_{a,1}$ in $C^{2,\gamma}$ sense with $a \in \left[\frac{3}{16}, \frac{5}{16}\right]$, the Green function associated to problem (5.1.1) on $\tilde{\Omega}$ is positive.

Then there is $\delta = \delta(M, \rho_\Omega^{-1}, \varepsilon, \gamma) \in \left(0, \frac{1}{16}R\right)$ such that the following holds. For every $z_0 \in \partial\Omega$ there exist $a \in \left[\frac{3}{16}, \frac{5}{16}\right]$, a limaçon $\Omega_{a,R}$ and a $C^{4,\alpha}$ mapping $f_{a,R} : \bar{\Omega}_{a,R} \to f_{a,R}\left(\bar{\Omega}_{a,R}\right)$ such that:

i: $\partial\Omega \cap B_\delta(z_0) = \partial\left(f_{a,R}\left(\Omega_{a,R}\right)\right) \cap B_\delta(z_0)$;

ii: the map $f^R_{a,R} := \left(f_{a,R}\right)^R$ is ε-close in $C^{2,\gamma}$-sense to the identity in $\bar{\Omega}_{a,1}$:

$$\left\|f^R_{a,R} - Id\right\|_{C^{2,\gamma}\left(\bar{\Omega}_{a,1}\right)} \le \varepsilon;$$

iii: the map $f^R_{a,R}$ is $C^{4,\alpha}$-bounded by some $A = A(M, \rho_\Omega^{-1}, \varepsilon, \gamma) > 0$:

$$\left\|f^R_{a,R}\right\|_{C^{4,\alpha}\left(\bar{\Omega}_{a,1}\right)} \le A$$

.

Remark 5.4.5. We construct a $C^{4,\alpha}$ mapping $f_{a,R} : \bar{\Omega}_{a,R} \to f_{a,R}\left(\bar{\Omega}_{a,R}\right)$ in order that $f_{a,R}\left(\bar{\Omega}_{a,R}\right)$ is a domain satisfying the uniform $C^{4,\alpha}$ regularity condition with constant M_1 where $M_1 = M_1(M, \rho_\Omega^{-1}, \varepsilon, \gamma)$. Using the result in [28] it should be possible to relax the regularity of the boundary to C^4.

Remark 5.4.6. In order to approximate $\partial\Omega$ with limaçons in $C^{2,\gamma}$-sense it is sufficient that Ω satisfies the uniform $C^{2,\alpha}$ regularity condition for $\alpha > \gamma$.

Remark 5.4.7. R defined in Theorem 5.4.4 depends on Ω via the constant M of the uniform $C^{4,\alpha}$ regularity condition.

Corollary 5.4.8. *Assume Ω, α, γ, ε are such that the hypothesis of Theorem 5.4.4 hold true and let R as defined in that Theorem. Then there is $\delta > 0$ such that for every $z_0 \in \partial\Omega$ there exists a domain E_{z_0} that satisfies the following:*

(i) E_{z_0} satisfies the uniform $C^{4,\alpha}$ regularity condition with constant

$$M_1 = M_1(M, \rho_\Omega^{-1}, \varepsilon, \gamma) > 0;$$

(ii) E_{z_0} is ε-close in $C^{2,\gamma}$-sense to a limaçon $\Omega_{a,R}$ with $a \in \left[\frac{3}{16}, \frac{5}{16}\right]$;

(iii) $z_0 \in \left(\bar{E}_{z_0} \cap \partial\Omega\right)^{\circ,\partial\Omega}$.

Furthermore, letting K_{z_0} be the component of $\left(\bar{E}_{z_0} \cap \partial\Omega\right)^{\circ,\partial\Omega}$ that contains z_0:

(i) $B_\delta(z_0) \cap \partial\Omega = B_\delta(z_0) \cap K_{z_0}$;

(ii) E_{z_0} and Ω have the same outward normal for any $x \in K_{z_0}$.

The proof of Theorem 5.4.4 is divided into several steps. We first present the setting for a fixed $z_0 \in \partial\Omega$.

Let us consider the local system of coordinates in z_0 and the function $g_{z_0} \in C^{4,\alpha}$ given by Lemma 5.4.1 (in this case $l = 4$). In the following we write $g_{z_0} = g$.

Let δ be a positive number such that

$$\delta < \min\left\{1, \frac{x_{\rho_\Omega}}{4}, \frac{R}{16}\left(1 - \frac{5}{16}\sqrt{3}\right)\right\} \text{ and } \delta^{1-\gamma} < \varepsilon\left(C_{10}R^{1+\gamma}\right)^{-1}. \tag{5.4.2}$$

Here C_{10} is a positive constant that depends on M. We remark that δ depends on Ω through ρ_Ω^{-1} and M.

5.4.2 Approximation by a limaçon in one point

There exists $a \in \left[\frac{3}{16}, \frac{5}{16}\right]$ such that $z_0 = (0,0) \in \bar{\Omega}_{a,R}$ and

$$k''_{a,R}(0) = g''(0),$$

where $k_{a,R} \in C^\infty$ is the map that describes, as in (5.3.5), the lower part of the limaçon.

In order to get that $\partial\Omega_{a,R}$ approximates the boundary of Ω in $(0,0)$ up to the second derivative, we have to impose the condition $g''(0) = k''_{a,R}(0)$. Using (5.3.6) this reads as

$$g''(0) = \frac{1}{R}\frac{1-4a}{(1-2a)^2}. \tag{5.4.3}$$

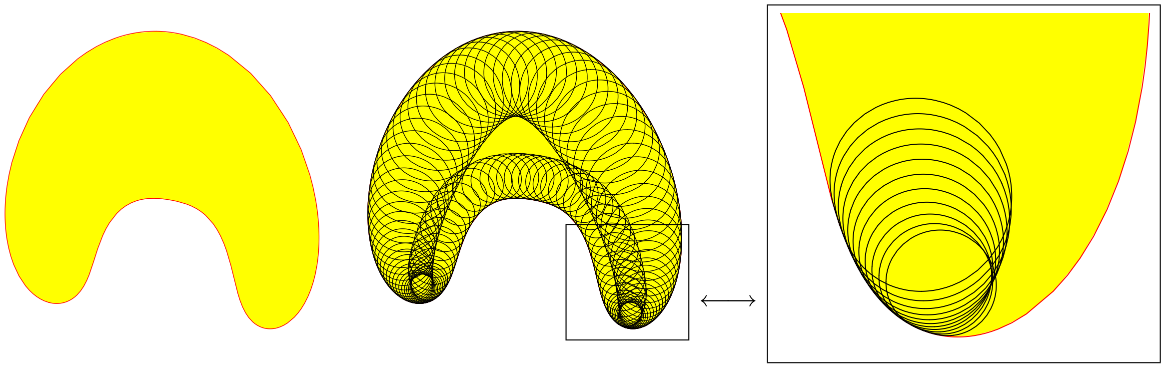

Figure 5.3: *A domain, the finite number of approximating limaçons with their boundaries covering the boundary of the domain, and a zoomed view.*

Since the map $a \mapsto \frac{1-4a}{(1-2a)^2}$ sends the interval $\left[\frac{3}{16}, \frac{5}{16}\right]$ onto $\left[-\frac{1}{2}, \frac{1}{2}\right]$ and $|g''(0)|\, R \leq \frac{1}{2}$ by the definition of R, one finds that $a \in \left[\frac{3}{16}, \frac{5}{16}\right]$ exists such that (5.4.3) holds.

Note that R is fixed and that it is sufficient to play with the parameter a to fit the limaçon $\Omega_{a,R}$ to the domain Ω around z_0.

5.4.3 Construction of the mapping $f_{a,R}$

Again we fix some preliminaries. Let x_a be the number defined in (5.3.4) and let us fix $x_a^* := \frac{1}{2}(1-\sqrt{3}a) \in \left(\frac{1}{5}x_a, \frac{1}{2}x_a\right)$. We introduce two cut-off functions:

(i) $\varphi_{a,R} \in C^\infty(\mathbb{R})$ such that

$$\begin{array}{rll} \varphi_{a,R} \equiv 1 & \text{for} & |x| \leq \frac{1}{2}x_a^* R, \\ \varphi_{a,R} \equiv 0 & \text{for} & |x| \geq x_a^* R, \\ \|\varphi_{a,R}\|_{C^{k,\nu}} \leq \frac{D_{k,\nu}}{R^{k+\nu}} & \text{for} & k = 0, \dots, 4 \text{ and } \nu \in (0,1)\,, \end{array}$$

with $D_{k,\nu}$ some positive constants;

(ii) $\psi_{a,\delta} \in C^\infty(\mathbb{R})$ such that

$$\begin{array}{rll} \psi_{a,\delta} \equiv 1 & \text{for} & |x| \leq \delta, \\ \psi_{a,\delta} \equiv 0 & \text{for} & |x| \geq 2\delta, \\ \|\psi_{a,\delta}\|_{C^{k,\nu}} \leq \frac{D'_{k,\nu}}{\delta^{k+\nu}} & \text{for} & k = 0, \dots, 4 \text{ and } \nu \in (0,1)\,, \end{array}$$

with $D'_{k,\nu}$ some positive constants.

We define a $C^{4,\alpha}$-mapping g_δ on $[-Rx_a, Rx_a]$ that follows the boundary of Ω when $|x| \le \delta$ and the boundary of the limaçon when $Rx_a^* \le |x| \le Rx_a$:

$$g_\delta(x) := \begin{cases} g(x) & \text{for} \quad 0 \le x \le \delta, \\ k_{a,R}(x) + \displaystyle\sum_{i=0}^{2} \frac{1}{i!} \left.(g - k_{a,R})^{(i)}\right|_\delta (x-\delta)^i + & \\ \quad + \psi_{a,\delta}(x) \displaystyle\sum_{i=3}^{4} \frac{1}{i!} \left.(g - k_{a,R})^{(i)}\right|_\delta (x-\delta)^i & \text{for} \quad \delta < x \le 2\delta, \\ k_{a,R}(x) + \displaystyle\sum_{i=0}^{2} \frac{1}{i!} \left.(g - k_{a,R})^{(i)}\right|_\delta (x-\delta)^i & \text{for} \quad 2\delta < x \le \tfrac{1}{2}Rx_a^*, \\ k_{a,R}(x) + \varphi_{a,R}(x) \displaystyle\sum_{i=0}^{2} \frac{1}{i!} \left.(g - k_{a,R})^{(i)}\right|_\delta (x-\delta)^i & \text{for} \quad \tfrac{1}{2}Rx_a^* < x \le Rx_a^*, \\ k_{a,R}(x) & \text{for} \quad Rx_a^* < x \le Rx_a, \end{cases} \tag{5.4.4}$$

and similarly for $x \in [-Rx_a, 0]$.

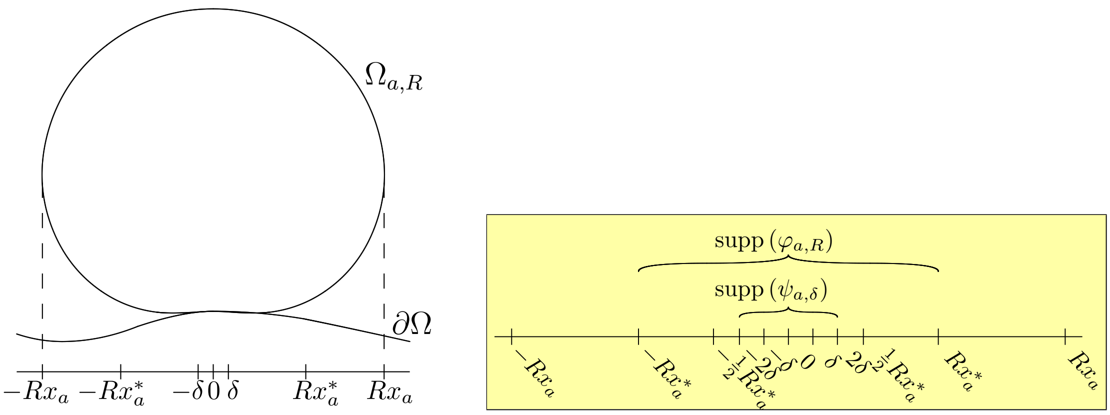

Figure 5.4: *Left: the limaçon that approximates in* $(0,0)$ *the behavior of* $\partial\Omega$ *up to the second derivative.*
Right: scheme for the support of the cut-off functions $\varphi_{a,R}$ *and* $\psi_{a,\delta}$*.*

Remark 5.4.9. In the definition of g_δ we use two cut-off functions. The reason for this construction is that we want g_δ to be close to $k_{a,R}$ in $C^{2,\gamma}$-sense and also to be a $C^{4,\alpha}$-mapping. Indeed, considering $\|g_\delta - k_{a,R}\|_{C^{2,\gamma}(-Rx_a,Rx_a)}$ one sees that the terms $\left.(g - k_{a,R})^{(i)}\right|_\delta$ have a different behavior in the cases $i = 0,1,2$ respectively for $i = 3,4$. One cut-off function can be chosen independent of δ since we will show that for $i = 0,1,2$ the term $\left.(g - k_{a,R})^{(i)}\right|_\delta = O(\delta)$. While for $i = 3,4$ $\left.(g - k_{a,R})^{(i)}\right|_\delta$ will be just bounded, and hence one needs a cut-off function depending on δ in order that the $C^{2,\gamma}$-norm of $g - k_{a,R}$ is an $O(\delta)$. By the way, close in $C^{2,\gamma}$-sense is needed for positivity, $C^{4,\alpha}$ is used in the regularity result.

We define the function $f_{a,R} : \bar{\Omega}_{a,R} \to f_{a,R}\left(\bar{\Omega}_{a,R}\right)$ by

$$\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{f_{a,R}} \begin{pmatrix} x \\ \frac{3R-g_\delta(x)}{3R-k_{a,R}(x)}(y-3R)+3R \end{pmatrix}, \tag{5.4.5}$$

which gives

$$\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{f_{a,R}-Id} \begin{pmatrix} 0 \\ \frac{k_{a,R}(x)-g_\delta(x)}{3R-k_{a,R}(x)}(y-3R) \end{pmatrix}.$$

By construction $f_{a,R} \in C^{4,\alpha}\left(\bar{\Omega}_{a,R}\right)$ and the boundary of $f_{a,R}\left(\Omega_{a,R}\right)$ coincides with $\partial\Omega$ in a neighborhood of $z_0 = (0,0)$ of length at least 2δ.

In the next paragraph we show that $f_{a,R}(\Omega_{a,R})$ is ε-close to $\Omega_{a,R}$ in $C^{2,\gamma}$-sense and that $f_{a,R}(\Omega_{a,R})$ satisfies the uniform $C^{4,\alpha}$ regularity condition.

Remark 5.4.10. Notice that $f_{a,R} \equiv Id$ for $(x,y) \in \bar{\Omega}_{a,R}$ with $|x| \geq Rx_a^*$. While for $|x| < Rx_a^*$ it holds that $f_{a,R} \equiv Id$ for $x = 0$ only. The map $f_{a,R}$ also changes the boundary of $\Omega_{a,R}$ in a neighborhood of the point $(0,2R)$. That is not a problem since one may notice from the expression of $f_{a,R} - Id$ that in the approximation only the term $\frac{k_{a,R}(x)-g_\delta(x)}{3R-k_{a,R}(x)}$ plays a role.

5.4.4 The mapping is close to identity in $C^{2,\gamma}$-sense

In this section we will prove that the function $f_{a,R}^R$, which is the $f_{a,R}$ from (5.4.5) rescaled as in (5.4.1), satisfies

$$\left\| f_{a,R}^R - Id \right\|_{C^{2,\gamma}\left(\bar{\Omega}_{a,1}\right)} \leq \varepsilon. \tag{5.4.6}$$

By the results of section 5.3 and our choice of ε, it then follows that the Green function for the clamped plate equation on $f_{a,R}(\Omega_{a,R})$ is positive.

We first fix some notation. In the following N_1 and N_2 denote respectively

$$N_1 := \left\| \frac{\partial^4}{\partial x^4} k_{a,R} - \frac{\partial^4}{\partial x^4} g \right\|_{C^0[-\delta,\delta]} + \left| \frac{\partial^3}{\partial x^3} k_{a,R}(0) - \frac{\partial^3}{\partial x^3} g(0) \right|, \tag{5.4.7}$$

$$N_2 := \left[\frac{\partial^4}{\partial x^4} k_{a,R} - \frac{\partial^4}{\partial x^4} g \right]_{C^\alpha[-\delta,\delta]} + \left| \frac{\partial^4}{\partial x^4} k_{a,R}(0) - \frac{\partial^4}{\partial x^4} g(0) \right|. \tag{5.4.8}$$

Notice that $N_i = N_i(M)$ for $i = 1,2$. Indeed R depends on M and the dependence of $k_{a,R}$ on a is continuous in $\left[\frac{3}{16}, \frac{5}{16}\right]$ and hence uniform.

We have

$$\begin{aligned} \left\| \frac{\partial^i}{\partial x^i} k_{a,R} - \frac{\partial^i}{\partial x^i} g \right\|_{C^0[-\delta,\delta]} &\leq N_1 \delta^{3-i} \quad \text{for } i = 0,\ldots,3, \\ \left\| \frac{\partial^4}{\partial x^4} k_{a,R} - \frac{\partial^4}{\partial x^4} g \right\|_{C^0[-\delta,\delta]} &\leq N_2. \end{aligned} \tag{5.4.9}$$

In order to prove (5.4.6) one has first to consider the effect of the scaling. In the following general lemma we give the effect on the norms of the scaling defined in (5.4.1).

Lemma 5.4.11. *Let Ω be a subset of $\mathbb{R}^n$ and let $f : \bar{\Omega} \to \bar{\Omega}'$ be a $C^{2,\gamma}$-function. Let f^R be the f rescaled as in (5.4.1). Then it holds*

$$\left\|f^R - Id\right\|_{C^{2,\gamma}\left(R^{-1}\bar{\Omega}\right)} = \frac{1}{R}\left\|f - Id\right\|_{C^0\left(\bar{\Omega}\right)} + \sum_{i=1}^{n}\left\|\frac{\partial}{\partial x_i}\left(f - Id\right)\right\|_{C^0\left(\bar{\Omega}\right)} + \\ + R\sum_{i,j=1}^{n}\left\|\frac{\partial^2}{\partial x_i x_j} f\right\|_{C^0\left(\bar{\Omega}\right)} + R^{1+\gamma}\sum_{i,j=1}^{n}\left[\frac{\partial^2}{\partial x_i x_j} f\right]_{C^\gamma\left(\bar{\Omega}\right)}. \qquad (5.4.10)$$

Lemma 5.4.12. *Let $\gamma \in (0,1)$. The function $f^R_{a,R}$ satisfies*

$$\left\|f^R_{a,R} - Id\right\|_{C^{2,\gamma}\left(\bar{\Omega}_{a,1}\right)} \leq 5\left\|\frac{k_{a,R} - g_\delta}{3R - k_{a,R}}\right\|_{C^0[-Rx^*_a, Rx^*_a]} + 5R\left\|\frac{\partial}{\partial x}\frac{k_{a,R} - g_\delta}{3R - k_{a,R}}\right\|_{C^0[-Rx^*_a, Rx^*_a]} \\ + 9R^2\left\|\frac{\partial^2}{\partial x^2}\frac{k_{a,R} - g_\delta}{3R - k_{a,R}}\right\|_{C^0[-Rx^*_a, Rx^*_a]} + 4R^{2+\gamma}\left[\frac{\partial^2}{\partial x^2}\frac{k_{a,R} - g_\delta}{3R - k_{a,R}}\right]_{C^\gamma[-Rx^*_a, Rx^*_a]}. \quad (5.4.11)$$

Proof. We estimate separately the terms in the right-hand side of (5.4.10) for $f = f_{a,R}$ and $\Omega = \Omega_{a,R}$.

1. Since $-y \leq R$ and $k_{a,R} - g_\delta \equiv 0$ for $|x| \in [Rx^*_a, Rx_a]$ we find
$$\left\|f_{a,R} - Id\right\|_{C^0\left(\bar{\Omega}_{a,R}\right)} \leq 4R\left\|\frac{k_{a,R} - g_\delta}{3R - k_{a,R}}\right\|_{C^0[-Rx^*_a, Rx^*_a]}.$$

2. We also have
$$\sum_{i=1}^{2}\left\|\frac{\partial}{\partial x_i}\left(f_{a,R} - Id\right)\right\|_{C^0\left(\bar{\Omega}_{a,R}\right)} \\ \leq \left\|\frac{k_{a,R} - g_\delta}{3R - k_{a,R}}\right\|_{C^0[-Rx^*_a, Rx^*_a]} + 4R\left\|\frac{\partial}{\partial x}\frac{k_{a,R} - g_\delta}{3R - k_{a,R}}\right\|_{C^0[-Rx^*_a, Rx^*_a]}.$$

3. From the definition of the function $f_{a,R}$ in (5.4.5) we get
$$\sum_{i,j=1}^{2}\left\|\frac{\partial^2}{\partial x_i x_j} f_{a,R}\right\|_{C^0\left(\bar{\Omega}_{a,R}\right)} \\ \leq \left\|\frac{\partial}{\partial x}\frac{k_{a,R} - g_\delta}{3R - k_{a,R}}\right\|_{C^0[-Rx^*_a, Rx^*_a]} + 4R\left\|\frac{\partial^2}{\partial x^2}\frac{k_{a,R} - g_\delta}{3R - k_{a,R}}\right\|_{C^0[-Rx^*_a, Rx^*_a]}.$$

4. One finds
$$\sum_{i,j=1}^{2}\left[\frac{\partial^2}{\partial x_i x_j} f_{a,R}\right]_{C^\gamma\left(\bar{\Omega}_{a,R}\right)} = \left[(x,y) \mapsto (y - 3R)\frac{\partial^2}{\partial x^2}\frac{k_{a,R}(x) - g_\delta(x)}{3R - k_{a,R}(x)}\right]_{C^\gamma\left(\bar{\Omega}_{a,R}\right)} \\ + \left[\frac{\partial}{\partial x}\frac{k_{a,R} - g_\delta}{3R - k_{a,R}}\right]_{C^\gamma[-Rx^*_a, Rx^*_a]} \leq \ldots. \qquad (5.4.12)$$

Since it holds that $[(x,y)\mapsto f(x)g(y)]_{C^\alpha[a,b]^2} \le \|f\|_{C^0[a,b]}[g]_{C^\alpha[a,b]} + \|g\|_{C^0[a,b]}[f]_{C^\alpha[a,b]}$ one gets from (5.4.12) that

$$\begin{aligned} \dots \;\le\; & 3R^{1-\gamma}\left\|\frac{\partial^2}{\partial x^2}\frac{k_{a,R}-g_\delta}{3R-k_{a,R}}\right\|_{C^0[-Rx_a^*,Rx_a^*]} + 4R\left[\frac{\partial^2}{\partial x^2}\frac{k_{a,R}-g_\delta}{3R-k_{a,R}}\right]_{C^\gamma[-Rx_a^*,Rx_a^*]} \\ & + 2R^{1-\gamma}\left\|\frac{\partial^2}{\partial x^2}\frac{k_{a,R}-g_\delta}{3R-k_{a,R}}\right\|_{C^0[-Rx_a^*,Rx_a^*]}, \end{aligned}$$

and the claim follows. $\square$

Proposition 5.4.13. *Let $\gamma\in(0,1)$. There is $C_{10}=C_{10}(M)>0$ such that*

$$\left\|f_{a,R}^R - Id\right\|_{C^{2,\gamma}\left(\bar\Omega_{a,1}\right)} \le C_{10}R^{1+\gamma}\delta^{1-\gamma}. \tag{5.4.13}$$

Relation (5.4.6) follows since the right-hand side in (5.4.13) is less then ε thanks to the choice of δ in (5.4.2).

In order to prove Proposition 5.4.13 we have to estimate the terms in the right hand side of (5.4.11). The technical details of the proof are given in the following paragraph.

Proof of Proposition 5.4.13

We divide the rather technical proof of Proposition 5.4.13 in several lemmas. Using the result of Lemma 5.4.12, to bound $\left\|f_{a,R}^R - Id\right\|_{C^{2,\gamma}\left(\bar\Omega_{a,1}\right)}$ it is sufficient to get the estimates of the terms in the right hand side of (5.4.11) separately. We will do so in the next lemmas.

In the following $C_i = C_i(M) > 0$, for $i=1,\dots,9$. The constants N_i, $i=1,2$ are defined in (5.4.7) and (5.4.8) respectively.

Lemma 5.4.14. *For $k_{a,R}$ and g_δ respectively as in (5.3.5) and (5.4.4) it holds that*

$$\left\|\frac{k_{a,R}-g_\delta}{3R-k_{a,R}}\right\|_{C^0[-Rx_a^*,Rx_a^*]} \le C_1\delta R.$$

Proof. By the definition of g_δ in (5.4.4), and (5.4.9) one has

$$\begin{aligned} \|k_{a,R}-g_\delta\|_{C^0[-Rx_a^*,Rx_a^*]} \;\le\;& \|k_{a,R}-g\|_{C^0[-\delta,\delta]} + \\ +\;& \textstyle\sum_{\sigma=\pm}\left\|\varphi_{a,R}(\cdot)\sum_{i=0}^2\frac{1}{i!}\left(g-k_{a,R}\right)^{(i)}\Big|_{\sigma\delta}(\cdot-\sigma\delta)^i\right\|_{C^0[\sigma\delta,\sigma Rx_a^*]} \\ +\;& \textstyle\sum_{\sigma=\pm}\left\|\psi_{a,\delta}(\cdot)\sum_{i=3}^4\frac{1}{i!}\left(g-k_{a,R}\right)^{(i)}\Big|_{\sigma\delta}(\cdot-\sigma\delta)^i\right\|_{C^0[\sigma\delta,\sigma2\delta]} \\ \le\;& \textstyle N_1\delta^3 + 2\sum_{i=0}^2\frac{1}{i!}N_1\delta^{3-i}R^i + \frac{2}{3!}N_1\delta^3 + \frac{2}{4!}N_2\delta^4 \le C_1\delta R^2. \end{aligned}$$

The claim follows since $|k_{a,R}|\le 2R$. $\square$

Lemma 5.4.15. *Let $k_{a,R}$ and g_δ be given respectively as in (5.3.5) and (5.4.4). Then it holds*

$$\left\| \frac{\partial}{\partial x} \frac{k_{a,R} - g_\delta}{3R - k_{a,R}} \right\|_{C^0[-Rx_a^*, Rx_a^*]} \leq C_2 \delta.$$

Proof. Using Lemma 5.4.14 and (5.3.7) one finds directly

$$\left\| \frac{\partial}{\partial x} \frac{k_{a,R} - g_\delta}{3R - k_{a,R}} \right\|_{C^0[-Rx_a^*, Rx_a^*]} \leq \frac{1}{R} \left\| \frac{\partial}{\partial x} (k_{a,R} - g_\delta) \right\|_{C^0[-Rx_a^*, Rx_a^*]} + C_1 \delta b_1 \leq \dots .$$

By the definition of g_δ and the choice of the cut-off functions $\varphi_{a,R}$ and $\psi_{a,\delta}$ we get

$$\begin{aligned} \dots \quad \leq \quad & \tfrac{N_1}{R}\delta^2 + \tfrac{2}{R} \sum_{i=1}^{2} \tfrac{1}{(i-1)!} N_1 \delta^{3-i} R^{i-1} + \tfrac{2}{R} \tfrac{D_{1,0}}{R} \sum_{i=0}^{2} \tfrac{1}{i!} N_1 \delta^{3-i} R^i \\ & + \tfrac{1}{2!} \tfrac{2}{R} N_1 \delta^2 + \tfrac{1}{3!} \tfrac{2}{R} \tfrac{D'_{1,0}}{\delta} N_1 \delta^3 + \tfrac{1}{3!} \tfrac{2}{R} N_2 \delta^3 + \tfrac{1}{4!} \tfrac{2}{R} \tfrac{D'_{1,0}}{\delta} N_2 \delta^4 + C_1 b_1 \delta \leq C_2 \delta. \end{aligned}$$

Here we used (5.4.9) and that $\delta < R$ and $\delta < 1$. □

Lemma 5.4.16. *For $k_{a,R}$ and g_δ respectively as in (5.3.5) and (5.4.4) it holds that*

$$\left\| \frac{\partial^2}{\partial x^2} \frac{k_{a,R} - g_\delta}{3R - k_{a,R}} \right\|_{C^0[-Rx_a^*, Rx_a^*]} \leq C_3 \frac{\delta}{R}.$$

Proof. Since $\left(\frac{\alpha}{\beta}\right)'' = \frac{1}{\beta}\alpha'' - 2\frac{\beta'}{\beta}\left(\frac{\alpha}{\beta}\right)' - \frac{\beta''}{\beta}\frac{\alpha}{\beta}$, using Lemmas 5.4.14 and 5.4.15 and (5.3.7) one finds

$$\begin{aligned} & \left\| \frac{\partial^2}{\partial x^2} \frac{k_{a,R} - g_\delta}{3R - k_{a,R}} \right\|_{C^0[-Rx_a^*, Rx_a^*]} \\ & \leq \tfrac{1}{R} \left\| \frac{\partial^2}{\partial x^2} (k_{a,R} - g_\delta) \right\|_{C^0[-Rx_a^*, Rx_a^*]} + \tfrac{2}{R} b_1 C_2 \delta + \tfrac{1}{R} \tfrac{b_2}{R} C_1 \delta R \leq \dots . \end{aligned}$$

By the definition of g_δ in (5.4.4) one gets

$$\begin{aligned} \dots \quad \leq \quad & \tfrac{1}{R} N_1 \delta + \tfrac{2}{R} N_1 \delta + \tfrac{4}{R} \tfrac{D_{1,0}}{R} \sum_{i=1}^{2} \tfrac{1}{(i-1)!} N_1 \delta^{3-i} R^{i-1} + \tfrac{2}{R} \tfrac{D_{2,0}}{R^2} \sum_{i=0}^{2} \tfrac{1}{i!} N_1 \delta^{3-i} R^i \\ & + \tfrac{2}{R} N_1 \delta + \tfrac{2}{R} \tfrac{1}{2} 2 \tfrac{D'_{1,0}}{\delta} N_1 \delta^2 + \tfrac{2}{R} \tfrac{1}{3!} N_1 \delta^3 \tfrac{D'_{2,0}}{\delta^2} + \tfrac{2}{R} \tfrac{1}{2} N_2 \delta^2 + \tfrac{2}{R} \tfrac{2}{3!} \tfrac{D'_{1,0}}{\delta} N_2 \delta^3 \\ & + \tfrac{2}{R} \tfrac{1}{4!} \tfrac{D'_{2,0}}{\delta^2} N_2 \delta^4 + \tfrac{1}{R} (2 b_1 C_2 + b_2 C_1) \delta \leq C_3 \frac{\delta}{R}. \end{aligned}$$

The constant C_3 depends on Ω through N_1 and N_2. □

Remark 5.4.17. Notice that the proof also implies that

$$\left\| \frac{\partial^2}{\partial x^2} (k_{a,R} - g_\delta) \right\|_{C^0[-Rx_a^*, Rx_a^*]} \leq C_4 \delta.$$

Lemma 5.4.18. *For $k_{a,R}$ and g_δ respectively as in (5.3.5) and (5.4.4) it holds that*

$$\left[\frac{\partial^2}{\partial x^2}\left(k_{a,R}-g_\delta\right)\right]_{C^\gamma[-Rx_a^*,Rx_a^*]}\leq C_5\delta^{1-\gamma}.$$

Proof. Writing explicitly the function g_δ yields

$$\begin{aligned}&\left[\frac{\partial^2}{\partial x^2}\left(k_{a,R}-g_\delta\right)\right]_{C^\gamma[-Rx_a^*,Rx_a^*]}\\ \leq\;& 2N_1\delta^{1-\gamma}+2\left[\frac{\partial^2}{\partial x^2}\left(\varphi_{a,R}\left(\cdot\right)\sum_{i=0}^{2}\tfrac{1}{i!}\left(g-k_{a,R}\right)^{(i)}\Big|_\delta\left(\cdot-\delta\right)^i\right)\right]_{C^\gamma[\delta,Rx_a^*]}\\ &+2\left[\frac{\partial^2}{\partial x^2}\left(\psi_{a,\delta}\left(\cdot\right)\sum_{i=3}^{4}\tfrac{1}{i!}\left(g-k_{a,R}\right)^{(i)}\Big|_\delta\left(\cdot-\delta\right)^i\right)\right]_{C^\gamma[\delta,2\delta]}.\end{aligned}\tag{5.4.14}$$

It is convenient to study separately the terms on the right-hand side of (5.4.14). In the following $\tilde{C}_i=\tilde{C}_i(M)>0$, $i=1,2$.

1. By (5.4.9) one has

$$\begin{aligned}&\left[\frac{\partial^2}{\partial x^2}\left(\varphi_{a,R}\left(\cdot\right)\sum_{i=0}^{2}\tfrac{1}{i!}\left(g-k_{a,R}\right)^{(i)}\Big|_\delta\left(\cdot-\delta\right)^i\right)\right]_{C^\gamma[\delta,Rx_a^*]}\\ \leq\;&\sum_{i=0}^{2}\tfrac{1}{i!}N_1\delta^{3-i}\left[\left(\cdot-\delta\right)^i\frac{\partial^2}{\partial x^2}\varphi_{a,R}\right]_{C^\gamma[\delta,Rx_a^*]}\\ &+2\sum_{i=1}^{2}\tfrac{1}{(i-1)!}N_1\delta^{3-i}\left[\left(\cdot-\delta\right)^{i-1}\frac{\partial}{\partial x}\varphi_{a,R}\right]_{C^\gamma[\delta,Rx_a^*]}+N_1\delta\left[\varphi_{a,R}\right]_{C^\gamma[\delta,Rx_a^*]}\leq\dots.\end{aligned}$$

Via the definition of the cut-off function $\varphi_{a,R}$ we get

$$\begin{aligned}\dots\;\leq\;&\sum_{i=1}^{2}\tfrac{1}{(i-1)!}N_1\delta^{3-i}R^{i-1}R^{1-\gamma}\tfrac{D_{2,0}}{R^2}+\sum_{i=0}^{2}\tfrac{1}{i!}N_1\delta^{3-i}\tfrac{D_{2,\gamma}}{R^{2+\gamma}}R^i+2N_1\delta R^{1-\gamma}\tfrac{D_{1,0}}{R}\\ &+2\sum_{i=1}^{2}\tfrac{1}{(i-1)!}N_1\delta^{3-i}\tfrac{D_{1,\gamma}}{R^{1+\gamma}}R^{i-1}+N_1\delta\tfrac{D_{0,\gamma}}{R^\gamma}\leq\tilde{C}_1\delta^{1-\gamma}.\end{aligned}$$

2. Since

$$\begin{aligned}&\left[\frac{\partial^2}{\partial x^2}\left(\psi_{a,\delta}\left(\cdot\right)\sum_{i=3}^{4}\tfrac{1}{i!}\left(g-k_{a,R}\right)^{(i)}\Big|_\delta\left(\cdot-\delta\right)^i\right)\right]_{C^\gamma[\delta,2\delta]}\\ \leq\;&\sum_{i=3}^{4}\tfrac{1}{i!}\left(g-k_{a,R}\right)^{(i)}\Big|_\delta\left[\left(\cdot-\delta\right)^i\frac{\partial^2}{\partial x^2}\psi_{a,\delta}(\cdot)\right]_{C^\gamma[\delta,2\delta]}+\end{aligned}$$

$$+\sum_{i=3}^{4} \tfrac{2}{(i-1)!} \left(g-k_{a,R}\right)^{(i)}\Big|_{\delta} \left[(\cdot-\delta)^{i-1} \frac{\partial}{\partial x}\psi_{a,\delta}(\cdot)\right]_{C^{\gamma}[\delta,2\delta]}$$
$$+\sum_{i=3}^{4} \tfrac{1}{(i-2)!} \left(g-k_{a,R}\right)^{(i)}\Big|_{\delta} \left[(\cdot-\delta)^{i-2} \psi_{a,\delta}(\cdot)\right]_{C^{\gamma}[\delta,2\delta]} \leq \ldots,$$

from (5.4.9) and the choice of $\psi_{a,\delta}$ one obtains

$$\begin{aligned} \ldots \;\leq\; & \sum_{i=3}^{4} \tfrac{1}{i!} \left(g-k_{a,R}\right)^{(i)}\Big|_{\delta} \left(\delta^{i} \tfrac{D'_{2,\gamma}}{\delta^{2+\gamma}} + i\delta^{i-1}\delta^{1-\gamma} \tfrac{D'_{2,0}}{\delta^{2}}\right) \\ & +\sum_{i=3}^{4} \tfrac{2}{(i-1)!} \left(g-k_{a,R}\right)^{(i)}\Big|_{\delta} \left(\delta^{i-1} \tfrac{D'_{1,\gamma}}{\delta^{1+\gamma}} + (i-1)\,\delta^{i-2}\delta^{1-\gamma} \tfrac{D'_{1,0}}{\delta}\right) \\ & +\sum_{i=3}^{4} \tfrac{1}{(i-2)!} \left(g-k_{a,R}\right)^{(i)}\Big|_{\delta} \left(\delta^{i-2} \tfrac{D'_{0,\gamma}}{\delta^{\gamma}} + (i-2)\,\delta^{i-3}\delta^{1-\gamma}\right) \leq \tilde{C}_2\delta^{1-\gamma}. \end{aligned}$$

The claim follows. □

Lemma 5.4.19. *Let $k_{a,R}$ and g_δ be given respectively in (5.3.5) and (5.4.4). Then it holds*

$$\left[\frac{\partial^2}{\partial x^2} \frac{k_{a,R}-g_\delta}{3R-k_{a,R}}\right]_{C^{\gamma}[-Rx_a^*,Rx_a^*]} \leq \frac{C_9}{R}\delta^{1-\gamma}.$$

Proof. We have

$$\begin{aligned} \left[\frac{\partial^2}{\partial x^2} \frac{k_{a,R}-g_\delta}{3R-k_{a,R}}\right]_{C^{\gamma}[-Rx_a^*,Rx_a^*]} \leq & \left[\frac{1}{3R-k_{a,R}} \frac{\partial^2}{\partial x^2}\left(k_{a,R}-g_\delta\right)\right]_{C^{\gamma}[-Rx_a^*,Rx_a^*]} \\ & +2\left[\frac{1}{3R-k_{a,R}} \frac{\partial}{\partial x}k_{a,R} \frac{\partial}{\partial x} \frac{k_{a,R}-g_\delta}{3R-k_{a,R}}\right]_{C^{\gamma}[-Rx_a^*,Rx_a^*]} \\ & +\left[\frac{k_{a,R}-g_\delta}{\left(3R-k_{a,R}\right)^2} \frac{\partial^2}{\partial x^2}k_{a,R}\right]_{C^{\gamma}[-Rx_a^*,Rx_a^*]}. \end{aligned} \tag{5.4.15}$$

We study the terms in the right-hand side of (5.4.15) separately.

1. From (5.3.7), Remark 5.4.17 and Lemma 5.4.18 it follows that

$$\left[\frac{1}{3R-k_{a,R}} \frac{\partial^2}{\partial x^2}\left(k_{a,R}-g_\delta\right)\right]_{C^{\gamma}[-Rx_a^*,Rx_a^*]} \leq \tfrac{1}{R^2}b_1 2R^{1-\gamma}C_4\delta + \tfrac{C_5}{R}\delta^{1-\gamma} \leq \frac{C_6}{R}\delta^{1-\gamma}.$$

2. Using (5.3.7) and Lemmas 5.4.15 and 5.4.16 one obtains

$$\begin{aligned}
&\left[\frac{1}{3R-k_{a,R}}\frac{\partial}{\partial x}k_{a,R}\frac{\partial}{\partial x}\frac{k_{a,R}-g_\delta}{3R-k_{a,R}}\right]_{C^\gamma[-Rx_a^*,Rx_a^*]}\\
\leq\ & b_1C_2\delta\left[\frac{1}{3R-k_{a,R}}\right]_{C^\gamma[-Rx_a^*,Rx_a^*]}+\tfrac{1}{R}C_2\delta\left[\frac{\partial}{\partial x}k_{a,R}\right]_{C^\gamma[-Rx_a^*,Rx_a^*]}\\
&+b_1\tfrac{1}{R}\left[\frac{\partial}{\partial x}\frac{k_{a,R}-g_\delta}{3R-k_{a,R}}\right]_{C^\gamma[-Rx_a^*,Rx_a^*]}\\
\leq\ & \tfrac{1}{R^2}b_1 2R^{1-\gamma}C_2b_1\delta+\tfrac{b_2}{R}2R^{1-\gamma}\tfrac{C_2}{R}\delta+C_3\tfrac{\delta}{R}2R^{1-\gamma}\tfrac{b_1}{R}\leq\frac{C_7}{R}\delta^{1-\gamma}.
\end{aligned}$$

3. Since

$$\begin{aligned}
&\left[\frac{k_{a,R}-g_\delta}{(3R-k_{a,R})^2}\frac{\partial^2}{\partial x^2}k_{a,R}\right]_{C^\gamma[-Rx_a^*,Rx_a^*]}\\
\leq\ & \left[\frac{k_{a,R}-g_\delta}{3R-k_{a,R}}\right]_{C^\gamma[-Rx_a^*,Rx_a^*]}\tfrac{1}{R}\tfrac{b_2}{R}+\left[\frac{1}{3R-k_{a,R}}\right]_{C^\gamma[-Rx_a^*,Rx_a^*]}\tfrac{b_2}{R}C_1\delta R+\\
&+\left[\frac{\partial^2}{\partial x^2}k_{a,R}\right]_{C^\gamma[-Rx_a^*,Rx_a^*]}\tfrac{1}{R}C_1\delta R\leq\dots,
\end{aligned}$$

applying (5.3.7) and Lemmas 5.4.14 and 5.4.15 one finds

$$\dots\leq C_2\delta 2R^{1-\gamma}\tfrac{b_2}{R^2}+\tfrac{1}{R^2}2b_1R^{1-\gamma}b_2C_1\delta+\tfrac{b_3}{R^2}2R^{1-\gamma}C_1\delta\leq\frac{C_8}{R}\delta^{1-\gamma}.$$

The claim follows directly from (5.4.15) using the results of the previous points 1, 2 and 3. □

The proof of Proposition 5.4.13 follows from Lemmas 5.4.14, 5.4.15, 5.4.16 and 5.4.19.

5.4.5 Bounded third and fourth derivative of the mapping

In this section we derive the estimate of $\left\|f_{a,R}^R\right\|_{C^{4,\alpha}\left(\bar\Omega_{a,1}\right)}$. Again this $f_{a,R}^R$ is the $f_{a,R}$ from (5.4.5) rescaled as in (5.4.1). The estimate will imply that $f_{a,R}(\Omega_{a,R})$ satisfies the uniform $C^{4,\alpha}$ regularity condition.

Lemma 5.4.20. *Let $\alpha\in(0,1)$. There is $C_{11}=C_{11}(M)>0$ such that:*

$$\begin{aligned}
\left\|f_{a,R}^R\right\|_{C^{4,\alpha}\left(\bar\Omega_{a,1}\right)}\ \leq\ & x_a+9+5C_{11}\delta R+5R^3\left\|\frac{\partial^3}{\partial x^3}\frac{k_{a,R}-g_\delta}{3R-k_{a,R}}\right\|_{C^0(-Rx_a^*,Rx_a^*)}+\\
&+6R^4\left\|\frac{\partial^4}{\partial x^4}\frac{k_{a,R}-g_\delta}{3R-k_{a,R}}\right\|_{C^0(-Rx_a^*,Rx_a^*)}\\
&+R^{3+\alpha}\left[(x,y)\mapsto(y-3R)\frac{\partial^4}{\partial x^4}\frac{k_{a,R}-g_\delta}{3R-k_{a,R}}\right]_{C^\alpha\left(\bar\Omega_{a,R}\right)}. \qquad (5.4.16)
\end{aligned}$$

Proof. Let $f_{a,R,1}$ and $f_{a,R,2}$ be respectively the first and the second component of $f_{a,R}$. From the definition of $f_{a,R}$ we find: $f_{a,R,1}(x,y) = x$ and

$$f_{a,R,2}(x,y) = \frac{3R - g_\delta(x)}{3R - k_{a,R}(x)}(y - 3R) + 3R.$$

Hence $\|f_{a,R,1}\|_{C^{4,\alpha}\left(\bar{\Omega}_{a,1}\right)} \leq x_a + 1$ and Lemma 5.4.11 yields

$$\begin{aligned} \left\|f^R_{a,R,2}\right\|_{C^{4,\alpha}\left(\bar{\Omega}_{a,1}\right)} &\leq \sum_{\substack{|\beta|=0,\\ \beta\in\mathbb{N}^2}}^{4} R^{|\beta|-1}\left\|D^\beta f_{a,R,2}\right\|_{C^0\left(\bar{\Omega}_{a,R}\right)} + \\ &\quad + R^{3+\alpha}\sum_{\substack{|\beta|=4,\\ \beta\in\mathbb{N}^2}}\left[D^\beta f_{a,R,2}\right]_{C^\alpha\left(\bar{\Omega}_{a,R}\right)} = \ldots. \end{aligned}$$

By observing that

$$\frac{1}{R}\|f_{a,R,2}\|_{C^0\left(\bar{\Omega}_{a,R}\right)} \leq 3 + 4\left\|\frac{3R - g_\delta}{3R - k_{a,R}}\right\|_{C^0(-Rx^*_a, Rx^*_a)},$$

using that $f_{a,R,2}$ is linear in y, and by the definition of x_a in (5.3.4), one finds

$$\begin{aligned} \ldots \leq\ & 3 + 5\sum_{i=0}^{3} R^i\left\|\frac{\partial^i}{\partial x^i}\frac{3R - g_\delta}{3R - k_{a,R}}\right\|_{C^0(-Rx^*_a, Rx^*_a)} + \\ & + 6R^4\left\|\frac{\partial^4}{\partial x^4}\frac{3R - g_\delta}{3R - k_{a,R}}\right\|_{C^0(-Rx^*_a, Rx^*_a)} \\ & + R^{3+\alpha}\left[(x,y) \mapsto (y - 3R)\frac{\partial^4}{\partial x^4}\frac{3R - g_\delta}{3R - k_{a,R}}\right]_{C^\alpha\left(\bar{\Omega}_{a,R}\right)}. \end{aligned}$$

Since

$$\frac{3R - g_\delta}{3R - k_{a,R}} = 1 + \frac{k_{a,R} - g_\delta}{3R - k_{a,R}}$$

the claim follows from Lemmas 5.4.14, 5.4.15 and 5.4.16. □

The estimate we are looking for is then:

Proposition 5.4.21. *Let $\alpha \in (0,1)$. There is $C_{19} = C_{19}(M) > 0$ such that*

$$\left\|f^R_{a,R}\right\|_{C^{4,\alpha}\left(\bar{\Omega}_{a,1}\right)} \leq C_{19}\frac{R^{3+\alpha}}{\delta^{1+\alpha}}.$$

In order to prove Proposition 5.4.21 it is sufficient to estimate the terms in the right hand side of (5.4.16). The technical details are given in the next paragraph.

Proof of Proposition 5.4.21

We also divide the proof of Proposition 5.4.21 in several lemmas.

In the following $C_i = C_i(M) > 0$, for $i = 12, \dots, 18$. The constants N_i, $i = 1, 2$ are as given in (5.4.7) and (5.4.8).

Lemma 5.4.22. *For $k_{a,R}$ and g_δ respectively as in (5.3.5) and (5.4.4) it holds that*

$$\left\| \frac{\partial^3}{\partial x^3} (k_{a,R} - g_\delta) \right\|_{C^0[-Rx_a^*, Rx_a^*]} \le C_{12}.$$

Proof. By the definition of g_δ we have

$$\begin{aligned}
&\left\| \frac{\partial^3}{\partial x^3} (k_{a,R} - g_\delta) \right\|_{C^0[-Rx_a^*, Rx_a^*]} \\
\le\ & \left\| \frac{\partial^3}{\partial x^3} (k_{a,R} - g) \right\|_{C^0[-\delta,\delta]} + 2 \left\| \frac{\partial^3}{\partial x^3} \left(\varphi_{a,R}(\cdot) \sum_{i=0}^{2} \tfrac{1}{i!} \left. (g - k_{a,R})^{(i)} \right|_\delta (\cdot - \delta)^i \right) \right\|_{C^0[\delta, Rx_a^*]} \\
& +2 \left\| \frac{\partial^3}{\partial x^3} \left(\psi_{a,\delta}(\cdot) \sum_{i=3}^{4} \tfrac{1}{i!} \left. (g - k_{a,R})^{(i)} \right|_\delta (\cdot - \delta)^i \right) \right\|_{C^0[\delta, 2\delta]}.
\end{aligned} \tag{5.4.17}$$

It is convenient to study the terms on the right-hand side of (5.4.17) separately. In the following $\bar{C}_i = \bar{C}_i(M) > 0$ for $i = 1, 2$.

1. It follows directly from (5.4.9) that $\left\| \frac{\partial^3}{\partial x^3} (k_{a,R} - g) \right\|_{C^0[-\delta,\delta]} \le N_1$.
2. Via (5.4.9) and the definition of the cut-off function $\varphi_{a,R}$ we get

$$\begin{aligned}
&\left\| \frac{\partial^3}{\partial x^3} \left(\varphi_{a,R}(\cdot) \sum_{i=0}^{2} \tfrac{1}{i!} \left. (g - k_{a,R})^{(i)} \right|_\delta (\cdot - \delta)^i \right) \right\|_{C^0[\delta, Rx_a^*]} \\
\le\ & \sum_{i=0}^{2} \tfrac{1}{i!} N_1 \delta^{3-i} \tfrac{D_{3,0}}{R^3} R^i + 3 \sum_{i=1}^{2} \tfrac{1}{(i-1)!} N_1 \delta^{3-i} \tfrac{D_{2,0}}{R^2} R^{i-1} + 3 N_1 \delta \tfrac{D_{1,0}}{R} \le \tfrac{\bar{C}_1}{R} \delta < \bar{C}_1.
\end{aligned}$$

3. One finds

$$\left\| \frac{\partial^3}{\partial x^3} \left(\psi_{a,\delta}(\cdot) \sum_{i=3}^{4} \tfrac{1}{i!} \left. (g - k_{a,R})^{(i)} \right|_\delta (\cdot - \delta)^i \right) \right\|_{C^0[\delta, 2\delta]} \le$$

$$\begin{aligned}
\le\ & \sum_{i=3}^{4} \tfrac{1}{i!} \left. (g - k_{a,R})^{(i)} \right|_\delta \tfrac{D'_{3,0}}{\delta^3} \delta^i + 3 \sum_{i=3}^{4} \tfrac{1}{(i-1)!} \left. (g - k_{a,R})^{(i)} \right|_\delta \tfrac{D'_{2,0}}{\delta^2} \delta^{i-1} + \\
+\ & 3 \sum_{i=3}^{4} \tfrac{1}{(i-2)!} \left. (g - k_{a,R})^{(i)} \right|_\delta \tfrac{D'_{1,0}}{\delta} \delta^{i-2} + \sum_{i=3}^{4} \left. (g - k_{a,R})^{(i)} \right|_\delta \delta^{i-3} \le \bar{C}_2.
\end{aligned}$$

The claim follows. $\square$

Lemma 5.4.23. *For $k_{a,R}$ and g_δ respectively as in (5.3.5) and (5.4.4) it holds that*

$$\left\| \frac{\partial^3}{\partial x^3} \frac{k_{a,R} - g_\delta}{3R - k_{a,R}} \right\|_{C^0(-Rx_a^*, Rx_a^*)} \leq \frac{C_{13}}{R}.$$

Proof. Since

$$\left(\frac{\alpha}{\beta}\right)''' = \frac{\alpha'''}{\beta} - 3\frac{\beta'}{\beta}\left(\frac{\alpha}{\beta}\right)'' - 3\frac{\beta''}{\beta}\left(\frac{\alpha}{\beta}\right)' - \frac{\beta'''}{\beta}\frac{\alpha}{\beta},$$

using Lemma 5.4.22, (5.3.7) and Lemmas 5.4.16, 5.4.15, 5.4.14 we get

$$\left\| \frac{\partial^3}{\partial x^3} \frac{k_{a,R} - g_\delta}{3R - k_{a,R}} \right\|_{C^0(-Rx_a^*, Rx_a^*)} \leq \tfrac{1}{R} C_{12} + 3\tfrac{b_1}{R}\tfrac{C_3 \delta}{R} + 3\tfrac{b_2}{R^2} C_2 \delta + \tfrac{b_3}{R^3} C_1 \delta R \leq \frac{C_{13}}{R}.$$

□

Lemma 5.4.24. *For $k_{a,R}$ and g_δ respectively as in (5.3.5) and (5.4.4) it holds that*

$$\left\| \frac{\partial^4}{\partial x^4} (k_{a,R} - g_\delta) \right\|_{C^0[-Rx_a^*, Rx_a^*]} \leq \frac{C_{14}}{\delta}.$$

Proof. From the definition of g_δ it follows

$$\begin{aligned}\left\| \frac{\partial^4}{\partial x^4} (k_{a,R} - g_\delta) \right\|_{C^0[-Rx_a^*, Rx_a^*]} &\leq \left\| \frac{\partial^4}{\partial x^4} (k_{a,R} - g) \right\|_{C^0[-\delta,\delta]} + \\ &+2 \left\| \frac{\partial^4}{\partial x^4} \left(\varphi_{a,R}(\cdot) \sum_{i=0}^{2} \tfrac{1}{i!} (g - k_{a,R})^{(i)}\Big|_\delta (\cdot - \delta)^i \right) \right\|_{C^0[\delta, Rx_a^*]} + \\ &+2 \left\| \frac{\partial^4}{\partial x^4} \left(\psi_{a,\delta}(\cdot) \sum_{i=3}^{4} \tfrac{1}{i!} (g - k_{a,R})^{(i)}\Big|_\delta (\cdot - \delta)^i \right) \right\|_{C^0[\delta, 2\delta]}. \end{aligned} \tag{5.4.18}$$

It is convenient to study the terms on the right-hand side of (5.4.18) separately. Here $\tilde{C}_i = \tilde{C}_i(M) > 0$ for $i = 1, 2$.

1. From (5.4.9) it follows directly that $\left\| \frac{\partial^4}{\partial x^4} (k_{a,R} - g) \right\|_{C^0[-\delta,\delta]} \leq N_2$.
2. By (5.4.9) and the definition of the cut-off function $\varphi_{a,R}$ we get that

$$\begin{aligned} &\left\| \frac{\partial^4}{\partial x^4} \left(\varphi_{a,R}(\cdot) \sum_{i=0}^{2} \tfrac{1}{i!} (g - k_{a,R})^{(i)}\Big|_\delta (\cdot - \delta)^i \right) \right\|_{C^0[\delta, Rx_a^*]} \\ \leq\; &\sum_{i=0}^{2} \tfrac{1}{i!} N_1 \delta^{3-i} \tfrac{D_{4,0}}{R^4} R^i + 4 \sum_{i=1}^{2} \tfrac{1}{(i-1)!} N_1 \delta^{3-i} \tfrac{D_{3,0}}{R^3} R^{i-1} + 6 N_1 \delta \tfrac{D_{2,0}}{R^2} \leq \tfrac{\tilde{C}_1}{R^2} \delta < \frac{\tilde{C}_1}{R}. \end{aligned}$$

3. From (5.4.9) and the choice of $\psi_{a,\delta}$ one obtains

$$\left\| \frac{\partial^4}{\partial x^4} \left(\psi_{a,\delta}(\cdot) \sum_{i=3}^{4} \tfrac{1}{i!} \left(g - k_{a,R}\right)^{(i)}\Big|_\delta (\cdot - \delta)^i \right) \right\|_{C^0[\delta,2\delta]}$$

$$\leq \sum_{i=3}^{4} \tfrac{1}{i!} \left(g - k_{a,R}\right)^{(i)}\Big|_\delta \tfrac{D'_{4,0}}{\delta^4}\delta^i + 4\sum_{i=3}^{4} \tfrac{1}{(i-1)!} \left(g - k_{a,R}\right)^{(i)}\Big|_\delta \tfrac{D'_{3,0}}{\delta^3}\delta^{i-1}$$

$$+6\sum_{i=3}^{4} \tfrac{1}{(i-2)!} \left(g - k_{a,R}\right)^{(i)}\Big|_\delta \tfrac{D'_{2,0}}{\delta^2}\delta^{i-2} + 4\sum_{i=3}^{4} \left(g - k_{a,R}\right)^{(i)}\Big|_\delta \tfrac{D'_{1,0}}{\delta}\delta^{i-3}$$

$$+ \left(g - k_{a,R}\right)^{(4)}\Big|_\delta \leq \frac{\tilde{C}_2}{\delta}.$$

The claim follows. □

Lemma 5.4.25. *Let $k_{a,R}$ and g_δ be given respectively in (5.3.5) and (5.4.4). Then it holds that*

$$\left\| \frac{\partial^4}{\partial x^4} \frac{k_{a,R} - g_\delta}{3R - k_{a,R}} \right\|_{C^0(-Rx_a^*, Rx_a^*)} \leq \frac{C_{15}}{\delta R}.$$

Proof. From

$$\left(\frac{\alpha}{\beta}\right)^{(iv)} = \frac{\alpha^{(iv)}}{\beta} - 4\frac{\beta'}{\beta}\left(\frac{\alpha}{\beta}\right)''' - 6\frac{\beta''}{\beta}\left(\frac{\alpha}{\beta}\right)'' - 4\frac{\beta'''}{\beta}\left(\frac{\alpha}{\beta}\right)' - \frac{\beta^{(iv)}}{\beta}\frac{\alpha}{\beta}, \tag{5.4.19}$$

using Lemma 5.4.24, (5.3.7) and Lemmas 5.4.23, 5.4.16, 5.4.15, 5.4.14 we get

$$\left\| \frac{\partial^4}{\partial x^4} \frac{k_{a,R} - g_\delta}{3R - k_{a,R}} \right\|_{C^0(-Rx_a^*, Rx_a^*)} \leq \tfrac{1}{R}\tfrac{C_{14}}{\delta} + 4\tfrac{b_1}{R}\tfrac{C_{13}}{R} + 6\tfrac{b_2}{R^2}C_3\tfrac{\delta}{R} + 4\tfrac{b_3}{R^3}C_2\delta + \tfrac{b_4}{R^4}C_1\delta R \leq \frac{C_{15}}{\delta R}.$$

□

Lemma 5.4.26. *For $k_{a,R}$ and g_δ respectively as in (5.3.5) and (5.4.4) it holds that*

$$\left[\frac{\partial^4}{\partial x^4}\left(k_{a,R} - g_\delta\right) \right]_{C^\alpha[-Rx_a^*, Rx_a^*]} \leq \frac{C_{16}}{\delta^{1+\alpha}}.$$

Proof. From the definition of g_δ one finds

$$\left[\frac{\partial^4}{\partial x^4}\left(k_{a,R} - g_\delta\right) \right]_{C^\alpha[-Rx_a^*, Rx_a^*]} \leq \left[\frac{\partial^4}{\partial x^4}\left(k_{a,R} - g\right) \right]_{C^\alpha[-\delta,\delta]} +$$

$$+2\left[\frac{\partial^4}{\partial x^4} \left(\varphi_{a,R}(\cdot) \sum_{i=0}^{2} \tfrac{1}{i!} \left(g - k_{a,R}\right)^{(i)}\Big|_\delta (\cdot - \delta)^i \right) \right]_{C^\alpha[\delta, Rx_a^*]}$$

$$+2\left[\frac{\partial^4}{\partial x^4} \left(\psi_{a,\delta}(\cdot) \sum_{i=3}^{4} \tfrac{1}{i!} \left(g - k_{a,R}\right)^{(i)}\Big|_\delta (\cdot - \delta)^i \right) \right]_{C^\alpha[\delta, 2\delta]}. \tag{5.4.20}$$

It is convenient to study the terms on the right-hand side of (5.4.20) separately. In the following $\tilde{C}_i = \tilde{C}_i(M) > 0$ for $i = 1, 2, 3$.

1. Since Ω is a $C^{4,\alpha}$ domain with constant M we have

$$\left[\frac{\partial^4}{\partial x^4}\left(k_{a,R} - g\right)\right]_{C^\alpha[-\delta,\delta]} \leq \left[\frac{\partial^4}{\partial x^4} k_{a,R}\right]_{C^\alpha[-\delta,\delta]} + M \leq \tilde{C}_1.$$

Notice that we may choose a constant $\tilde{C}_1$ that depends only on M.

2. One has

$$\begin{aligned}
&\left[\frac{\partial^4}{\partial x^4}\left(\varphi_{a,R}\left(\cdot\right)\sum_{i=0}^{2}\tfrac{1}{i!}\left(g-k_{a,R}\right)^{(i)}\Big|_\delta\left(\cdot-\delta\right)^i\right)\right]_{C^\alpha[\delta,Rx_a^*]}\\
\leq\ & \sum_{i=0}^{2}\tfrac{1}{i!}\left(g-k_{a,R}\right)^{(i)}\Big|_\delta\left[\left(\cdot-\delta\right)^i\frac{\partial^4}{\partial x^4}\varphi_{a,R}(\cdot)\right]_{C^\alpha[\delta,Rx_a^*]}\\
&+4\sum_{i=1}^{2}\tfrac{1}{(i-1)!}\left(g-k_{a,R}\right)^{(i)}\Big|_\delta\left[\left(\cdot-\delta\right)^{i-1}\frac{\partial^3}{\partial x^3}\varphi_{a,R}(\cdot)\right]_{C^\alpha[\delta,Rx_a^*]}\\
&+6\left(g-k_{a,R}\right)^{(2)}\Big|_\delta\left[\frac{\partial^2}{\partial x^2}\varphi_{a,R}\right]_{C^\alpha[\delta,Rx_a^*]} \leq \dots.
\end{aligned}$$

Via (5.4.9) and the definition of the cut-off function $\varphi_{a,R}$ we get

$$\begin{aligned}
\dots\ \leq\ & \sum_{i=0}^{2}\tfrac{1}{i!}N_1\delta^{3-i}\tfrac{D_{4,\alpha}}{R^{4+\alpha}}R^i+\sum_{i=1}^{2}\tfrac{1}{(i-1)!}N_1\delta^{3-i}\tfrac{D_{4,0}}{R^4}R^{i-1}R^{1-\alpha}\\
&+4\sum_{i=1}^{2}\tfrac{1}{(i-1)!}N_1\delta^{3-i}\tfrac{D_{3,\alpha}}{R^{3+\alpha}}R^{i-1}+4N_1\delta\tfrac{D_{3,0}}{R^3}R^{1-\alpha}+6N_1\delta\tfrac{D_{2,\alpha}}{R^{2+\alpha}} \leq \frac{\tilde{C}_2}{R^{2+\alpha}}\delta.
\end{aligned}$$

3. Since

$$\begin{aligned}
&\left[\frac{\partial^4}{\partial x^4}\left(\psi_{a,\delta}\left(\cdot\right)\sum_{i=3}^{4}\tfrac{1}{i!}\left(g-k_{a,R}\right)^{(i)}\Big|_\delta\left(\cdot-\delta\right)^i\right)\right]_{C^\alpha[\delta,2\delta]}\\
\leq\ & \sum_{i=3}^{4}\tfrac{1}{i!}\left(g-k_{a,R}\right)^{(i)}\Big|_\delta\left[\left(\cdot-\delta\right)^i\frac{\partial^4}{\partial x^4}\psi_{a,\delta}(\cdot)\right]_{C^\alpha[\delta,2\delta]}\\
&+\sum_{i=3}^{4}\tfrac{4}{(i-1)!}\left(g-k_{a,R}\right)^{(i)}\Big|_\delta\left[\left(\cdot-\delta\right)^{i-1}\frac{\partial^3}{\partial x^3}\psi_{a,\delta}(\cdot)\right]_{C^\alpha[\delta,2\delta]}\\
&+\sum_{i=3}^{4}\tfrac{6}{(i-2)!}\left(g-k_{a,R}\right)^{(i)}\Big|_\delta\left[\left(\cdot-\delta\right)^{i-2}\frac{\partial^2}{\partial x^2}\psi_{a,\delta}(\cdot)\right]_{C^\alpha[\delta,2\delta]}\\
&+4\sum_{i=3}^{4}\left(g-k_{a,R}\right)^{(i)}\Big|_\delta\left[\left(\cdot-\delta\right)^{i-3}\frac{\partial}{\partial x}\psi_{a,\delta}(\cdot)\right]_{C^\alpha[\delta,2\delta]}\\
&+\left(g-k_{a,R}\right)^{(4)}\Big|_\delta\left[\psi_{a,\delta}\right]_{C^\alpha[\delta,2\delta]} \leq \dots,
\end{aligned}$$

from (5.4.9) and the choice of $\psi_{a,\delta}$ one obtains

$$
\begin{aligned}
\dots \quad \leq \quad & \sum_{i=3}^{4} \tfrac{1}{i!} \left(g-k_{a,R}\right)^{(i)}\Big|_{\delta} \left(\delta^{i} \tfrac{D'_{4,\alpha}}{\delta^{4+\alpha}} + i\delta^{i-1}\delta^{1-\alpha}\tfrac{D'_{4,0}}{\delta^{4}}\right) \\
& + \sum_{i=3}^{4} \tfrac{4}{(i-1)!} \left(g-k_{a,R}\right)^{(i)}\Big|_{\delta} \left(\delta^{i-1} \tfrac{D'_{3,\alpha}}{\delta^{3+\alpha}} + (i-1)\,\delta^{i-2}\delta^{1-\alpha}\tfrac{D'_{3,0}}{\delta^{3}}\right) + \\
& + \sum_{i=3}^{4} \tfrac{6}{(i-2)!} \left(g-k_{a,R}\right)^{(i)}\Big|_{\delta} \left(\delta^{i-2} \tfrac{D'_{2,\alpha}}{\delta^{2+\alpha}} + (i-2)\,\delta^{i-3}\delta^{1-\alpha}\tfrac{D'_{2,0}}{\delta^{2}}\right) \\
& + 4\sum_{i=3}^{4} \left(g-k_{a,R}\right)^{(i)}\Big|_{\delta} \left(\delta^{i-3} \tfrac{D'_{1,\alpha}}{\delta^{1+\alpha}} + (i-3)\,\delta^{i-4}\delta^{1-\alpha}\tfrac{D'_{1,0}}{\delta}\right) \\
& + \left(g-k_{a,R}\right)^{(4)}\Big|_{\delta} \tfrac{D'_{0,\alpha}}{\delta^{\alpha}} \leq \frac{\tilde{C}_3}{\delta^{1+\alpha}}.
\end{aligned}
$$

The claim follows. ◻

Lemma 5.4.27. *For $k_{a,R}$ and g_δ respectively as in (5.3.5) and (5.4.4) it holds that*

$$
\left[\frac{\partial^4}{\partial x^4}\frac{k_{a,R}-g_\delta}{3R-k_{a,R}}\right]_{C^\alpha[-Rx_a^*,Rx_a^*]} \leq \frac{C_{17}}{R\delta^{1+\alpha}}.
$$

Proof. From (5.4.19) by Lemma 5.4.24, (5.3.7) and Lemmas 5.4.26, 5.4.23, 5.4.16, 5.4.15, 5.4.14 one obtains

$$
\begin{aligned}
\left[\frac{\partial^4}{\partial x^4}\frac{k_{a,R}-g_\delta}{3R-k_{a,R}}\right]_{C^\alpha(-Rx_a^*,Rx_a^*)} \leq & \left[\frac{1}{3R-k_{a,R}}\right]_{C^\alpha(-Rx_a^*,Rx_a^*)} \tfrac{C_{14}}{\delta} + \tfrac{1}{R}\tfrac{C_{16}}{\delta^{1+\alpha}} + \\
& +4\left[\frac{1}{3R-k_{a,R}}\right]_{C^\alpha(-Rx_a^*,Rx_a^*)} b_1\tfrac{C_{13}}{R} + 4\left[\frac{\partial}{\partial x}k_{a,R}\right]_{C^\alpha(-Rx_a^*,Rx_a^*)} \tfrac{1}{R}\tfrac{C_{13}}{R} \\
& +4\left[\frac{\partial^3}{\partial x^3}\frac{k_{a,R}-g_\delta}{3R-k_{a,R}}\right]_{C^\alpha(-Rx_a^*,Rx_a^*)} \tfrac{b_1}{R} + 6\left[\frac{1}{3R-k_{a,R}}\right]_{C^\alpha(-Rx_a^*,Rx_a^*)} \tfrac{b_2}{R}C_3\tfrac{\delta}{R} \\
& +6\left[\frac{\partial^2}{\partial x^2}k_{a,R}\right]_{C^\alpha(-Rx_a^*,Rx_a^*)} \tfrac{1}{R}C_3\tfrac{\delta}{R} + 6\left[\frac{\partial^2}{\partial x^2}\frac{k_{a,R}-g_\delta}{3R-k_{a,R}}\right]_{C^\alpha(-Rx_a^*,Rx_a^*)} \tfrac{b_2}{R^2} \\
& +4\left[\frac{1}{3R-k_{a,R}}\right]_{C^\alpha(-Rx_a^*,Rx_a^*)} \tfrac{b_3}{R^2}C_2\delta + 4\left[\frac{\partial^3}{\partial x^3}k_{a,R}\right]_{C^\alpha(-Rx_a^*,Rx_a^*)} \tfrac{1}{R}C_2\delta \\
& +4\left[\frac{\partial}{\partial x}\frac{k_{a,R}-g_\delta}{3R-k_{a,R}}\right]_{C^\alpha(-Rx_a^*,Rx_a^*)} \tfrac{b_3}{R^3} + \left[\frac{1}{3R-k_{a,R}}\right]_{C^\alpha(-Rx_a^*,Rx_a^*)} \tfrac{b_4}{R^3}C_1\delta R \\
& + \left[\frac{\partial^4}{\partial x^4}k_{a,R}\right]_{C^\alpha(-Rx_a^*,Rx_a^*)} \tfrac{1}{R}C_1\delta R + \left[\frac{k_{a,R}-g_\delta}{3R-k_{a,R}}\right]_{C^\alpha(-Rx_a^*,Rx_a^*)} \tfrac{b_4}{R^4}
\end{aligned}
$$

$$\begin{aligned}\le\ & \tfrac{b_1}{R^2}2R^{1-\alpha}\tfrac{C_{14}}{\delta}+\tfrac{1}{R}\tfrac{C_{16}}{\delta^{1+\alpha}}+4\tfrac{b_1}{R^2}2R^{1-\alpha}b_1\tfrac{C_{13}}{R}+4\tfrac{b_2}{R}2R^{1-\alpha}\tfrac{C_{13}}{R^2}+4\tfrac{C_{15}}{\delta R}2R^{1-\alpha}\tfrac{b_1}{R}\\ &+6\tfrac{b_1}{R^2}2R^{1-\alpha}\tfrac{b_2}{R}C_3\tfrac{\delta}{R}+6\tfrac{b_3}{R^2}2R^{1-\alpha}\tfrac{1}{R}C_3\tfrac{\delta}{R}+6\tfrac{C_{13}}{R}2R^{1-\alpha}\tfrac{b_2}{R^2}+4\tfrac{b_1}{R^2}2R^{1-\alpha}\tfrac{b_3}{R^2}C_2\delta\\ &+4\tfrac{b_4}{R^3}2R^{1-\alpha}\tfrac{1}{R}C_2\delta+4C_3\tfrac{\delta}{R}2R^{1-\alpha}\tfrac{b_3}{R^3}+\tfrac{b_1}{R^2}2R^{1-\alpha}\tfrac{b_4}{R^3}C_1\delta R+\tfrac{b_5}{R^4}2R^{1-\alpha}\tfrac{1}{R}C_1\delta R\\ &+C_2\delta 2R^{1-\alpha}\tfrac{b_4}{R^4}\le C_{16}\frac{1}{R\delta^{1+\alpha}}.\end{aligned}$$

The claim follows. ☐

Lemma 5.4.28. *For $k_{a,R}$ and g_δ respectively as in (5.3.5) and (5.4.4) it holds that*

$$\left[(x,y)\mapsto(y-3R)\,\frac{\partial^4}{\partial x^4}\frac{k_{a,R}-g_\delta}{3R-k_{a,R}}\right]_{C^\alpha\left(\bar\Omega_{a,R}\right)}\le\frac{C_{18}}{\delta^{1+\alpha}}.$$

Proof. Since

$$[(x,y)\mapsto f\,(x)\,g\,(y)]_{C^\alpha[a,b]^2}\le[f]_{C^\alpha[a,b]}\,\|g\|_{C^0[a,b]}+\|f\|_{C^0[a,b]}\,[g]_{C^\alpha[a,b]}\,,$$

one finds

$$\begin{aligned}&\left[(x,y)\mapsto(y-3R)\,\frac{\partial^4}{\partial x^4}\frac{k_{a,R}-g_\delta}{3R-k_{a,R}}\right]_{C^\alpha\left(\bar\Omega_{a,R}\right)}\\ \le\ & 3R^{1-\alpha}\left\|\frac{\partial^4}{\partial x^4}\frac{k_{a,R}-g_\delta}{3R-k_{a,R}}\right\|_{C^0(-Rx_a^*,Rx_a^*)}+4R\left[\frac{\partial^4}{\partial x^4}\frac{k_{a,R}-g_\delta}{3R-k_{a,R}}\right]_{C^\alpha(-Rx_a^*,Rx_a^*)}\le\dots.\end{aligned}$$

By Lemma 5.4.25 and Lemma 5.4.27 we get

$$\dots\le 3R^{1-\alpha}\tfrac{C_{15}}{\delta R}+4R\tfrac{C_{17}}{R\delta^{1+\alpha}}<\frac{C_{18}}{\delta^{1+\alpha}}.$$

☐

The boundness of $f_{a,R}$ in $C^{4,\alpha}$-norm follows directly from Lemma 5.4.20 and Lemmas 5.4.23, 5.4.25 and 5.4.28.

5.4.6 The covering

We are now ready to prove that for any domain Ω with $\partial\Omega\in C^{4,\alpha}$ one may find an appropriate covering by finitely many open domains that are ε-close in $C^{2,\gamma}$ sense to some limaçon.

Theorem 5.4.29. *Let Ω, α, γ and ε satisfy the assumptions of Theorem 5.4.4 and let R defined as in that theorem. Then there exist finitely many balls B_j, $j\in J_B$ with $\bar B_j\subset\Omega$, finitely many open domains $E_j\subset\mathbb{R}^2$, $j\in J_E$, and constants $\bar M=\bar M(M,\rho_\Omega^{-1},\varepsilon,\gamma)>0$ and $\delta>0$ such that:*

(i) $\Omega\subset\bigcup_{j\in J_B}B_j\cup\bigcup_{j\in J_E}E_j$;

(ii) $(E_j \cap \partial\Omega)^{\circ,\partial\Omega} \neq \emptyset$ *for all* $j \in J_E$*;*

(iii) *every* E_j *with* $j \in J_E$ *is a domain satisfying the uniform* $C^{4,\alpha}$ *regularity condition with constant* $\bar{M}$*;*

(iv) *each* E_j *is* ε*-close in* $C^{2,\gamma}$ *sense to a limaçon* $\Omega_{a,R}$ *with* $a \in \left[\frac{3}{16}, \frac{5}{16}\right]$*.*

Furthermore, for $K_j = \left(\bar{E}_j \cap \partial\Omega\right)^{\circ,\partial\Omega}$ *with* $j \in J_E$ *it holds:*

5. E_j *and* K_j *have the same outward normal for any* $x \in K_j$*;*

6. $\{K_j\}_{j\in J_E}$ *is a relatively open covering of* $\partial\Omega$*;*

7. *for all* $j \in J_E$ *the diameter of* K_j *is larger than* δ*.*

Proof. According to Corollary 5.4.8 there is a $\delta > 0$ such that for every $z_0 \in \partial\Omega$ there exists a domain E_{z_0} such that the following holds:

- E_{z_0} satisfies the uniform $C^{4,\alpha}$ regularity condition with constant
$$M_{z_0} = M_{z_0}(M, \rho_\Omega^{-1}, \varepsilon, \gamma);$$
- E_{z_0} is ε-close in $C^{2,\gamma}$-sense to a limaçon $\Omega_{a,R}$ with $a \in \left[\frac{3}{16}, \frac{5}{16}\right]$;
- letting K_{z_0} the connected component of $\left(\bar{E}_{z_0} \cap \partial\Omega\right)^{\circ,\partial\Omega}$ that contains z_0, it holds
$$B_\delta(z_0) \cap \partial\Omega = B_\delta(z_0) \cap K_{z_0}.$$

By compactness of $\partial\Omega$ there exist $z_1, \ldots, z_{\tilde{N}} \in \partial\Omega$ such that $\partial\Omega = \bigcup_{j=1}^{\tilde{N}} K_{z_j}$. Setting $E_j := E_{z_j}$ and $\bar{M} = \max\ M_{z_j}$ and K_j accordingly one finds that this family $\{K_j\}_{j=1,\ldots,N}$ satisfies the properties of the last three items.

A straightforward argument implies that $\Omega \backslash \bigcup_{j=1}^{N} (E_j \cap \Omega)$ can be covered by finitely many open balls B_j with $\bar{B}_j \subset \Omega$. $\square$

Remark 5.4.30. In the proof we use that Ω is simply connected. However with a slightly different argument the method would work also for general connected domains.

5.5 Proving the estimates

In this section we prove the main results of the chapter. First we give pointwise estimates for the solution of (5.1.1), and then we prove the splitting of the solution operator between a positive singular part and a sign changing regular part.

5.5.1 A maximum principle type estimate

Before presenting the pointwise estimate we recall a general result about partitions of unity that will be used in the proofs.

Lemma 5.5.1 (Partition of unity with boundary). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $\{D_j\}_{j\in J} \subset \Omega$ be a finite open covering of Ω such that $\partial\Omega \subset \bigcup_{j\in J} (\partial D_j \cap \partial\Omega)^{\circ,\partial\Omega}$. For every $\delta > 0$ there exist finitely many smooth functions $\psi_i \in C^\infty\left(\bar{\Omega}\right)$, $i \in I$, such that:*

(i) $\psi_i \geq 0$ *for all* $i \in I$ *and* $\sum_{i\in I} \psi_i(x) = 1$ *for all* $x \in \bar{\Omega}$*;*

(ii) for every $i \in I$ *there exists* $j = j(i)$ *such that* $\operatorname{supp}(\psi_i) \subset D_j \cup (\partial D_j \cup \partial\Omega)^{\circ,\partial\Omega}$,

(iii) $\operatorname{diam}(\operatorname{supp}\psi_i) \leq \delta$ *for all* $i \in I$.

Proof. Fix $j \in J$ and let $K_j = (\partial D_j \cap \partial\Omega)^{\circ,\partial\Omega}$. Since $D_j \cup K_j$ is relatively open in $\bar{\Omega}$ there exists an open bounded set U_j in $\mathbb{R}^n$ such that $U_j \cap \bar{\Omega} = D_j \cup K_j$. Let Ω_0 denote the set $\cup_{j\in J} U_j$. The domain Ω_0 is open and bounded in $\mathbb{R}^n$ since J is finite.

By compactness of $\bar{\Omega}_0$ it holds

$$\bar{\Omega}_0 \subset \bigcup_{l\in\tilde{J}} B_l,$$

with B_l open balls with diameter δ and $\tilde{J}$ a finite set. Set

$$V_{j,l} = U_j \cap B_l \text{ for } j \in J \text{ and } l \in \tilde{J}.$$

On $\Omega_0 = \cup_{j\in J, l\in\tilde{J}} V_{j,l}$ we have a partition of unity $\{\varphi_i\}_{i\in I}$ of functions in $C_0^\infty(\mathbb{R}^n)$ where the set I may have infinitely many elements, see [75, Page 61].

The set $\{x \in \bar{\Omega} : \varphi_i(x) > 0\}$ is relatively open in $\bar{\Omega}$ and since $\bar{\Omega}$ is compact there are finitely many indices $i_1, \ldots, i_N$ such that

$$\bar{\Omega} = \bigcup_{i\in\{i_1,\ldots,i_N\}} \{x \in \bar{\Omega} : \varphi_i(x) > 0\}.$$

We may assume that $i_1 = 1, \ldots, i_N = N$. We set

$$S(x) = \sum_{i=1}^{N} \varphi_i(x) \quad \text{for every } x \in \bar{\Omega},$$

$$\psi_i(x) := \frac{\varphi_i(x)}{S(x)} \quad \text{for every } x \in \bar{\Omega}.$$

The claim follows. □

The pointwise estimates for the solution of (5.1.1) will be obtained using negative Sobolev spaces. We refer to [2, pages 62-65].

Theorem 5.5.2. *Suppose that the hypothesis of Theorem 5.4.4 hold true with* $0 < \gamma, \alpha < 1$.

Then for any $q > 2$ *and* $\varepsilon \in (0, 4R]$ *there exists* $C = C(\frac{1}{2-q}, M, \rho_\Omega^{-1}, R_\Omega, \varepsilon, \gamma) > 0$ *such that for any* $f \in L^p(\Omega)$, *with* $p \in (1, \infty)$, *the solution* $u \in W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega)$ *of (5.1.1) satisfies*

$$u(x) \le C \left(\|f^+\|_{L^1(B(x,\varepsilon)\cap\Omega)} + \|u\|_{W^{-1,q}(\Omega)} \right) \quad \text{for every } x \in \Omega. \tag{5.5.1}$$

Proof. Let E_j, $j \in J$, be the finite covering of Ω of Theorem 5.4.29. For every $j \in J$ we define $D_j := E_j \cap \Omega$. We first consider the case $\varepsilon = 4R$.

Let ψ_i, $i \in I$, be a partition of unity with boundary associated to the covering $\{D_j\}_{j\in J}$ of Ω obtained applying Lemma 5.5.1 with $\delta = 2R$. We may choose the partition of unity such that it also satisfies for every $i \in I$:

i. $|D^\alpha \psi_i| \le c_\alpha R^{-|\alpha|}$ with $\alpha \in \mathbb{N}^2$, $|\alpha| \le 4$;

ii. $\psi_i \ne 0$ at the boundary only when $\left(\partial E_{j(i)} \cap \partial\Omega\right)^{\circ,\partial\Omega} \ne \varnothing$.

Here $j(i)$ denotes the $j \in J$ such that $\operatorname{supp}(\psi_i) \subset E_j$. By the choice of ψ_i it also holds that $\psi_i \in C_c^\infty(\Omega \cup \left(\partial E_{j(i)} \cap \partial\Omega\right)^{\circ,\partial\Omega})$, $\psi_i \in C_c^\infty(E_{j(i)} \cup \left(\partial E_{j(i)} \cap \partial\Omega\right)^{\circ,\partial\Omega})$ and $\psi_i \ne 0$ only on $\left(E_{j(i)} \cap \Omega\right) \cup \left(\partial E_{j(i)} \cap \partial\Omega\right)^{\circ,\partial\Omega}$. Notice that I is a finite set.

We choose a new family of cut-off functions $\chi_i \in C_c^\infty\left(\Omega \cup \left(\partial E_{j(i)} \cap \partial\Omega\right)^{\circ,\partial\Omega}\right)$, $i \in I$, such that for every $i \in I$:

i. $\operatorname{supp}(\psi_i) \subset \left\{x \in \bar{\Omega} : \chi_i(x) = 1\right\} \subset \operatorname{supp}(\chi_i) \subset \left(E_{j(i)} \cap \Omega\right) \cup \left(E_{j(i)} \cap \partial\Omega\right)^{\circ,\partial\Omega}$;

ii. $0 \le \chi_i(x) \le 1$;

iii. $\|\nabla^\alpha \chi_i\|_\infty \le c_\alpha R^{-|\alpha|}$ for every $\alpha \in \mathbb{N}^2$ with $|\alpha| \le 4$.

Observe that the cut-off function χ_i is not zero only in $\left(E_{j(i)} \cap \Omega\right) \cup \left(\partial E_{j(i)} \cap \partial\Omega\right)^{\circ,\partial\Omega}$.

The functions $\chi_i g$ and $\psi_i g$ denote (with abuse of notation) respectively

$$\chi_i g(x) := \begin{cases} \chi_i(x) g(x) & \text{in } \bar{\Omega}, \\ 0 & \text{otherwise,} \end{cases} \qquad \psi_i g(x) := \begin{cases} \psi_i(x) g(x) & \text{in } \bar{\Omega}, \\ 0 & \text{otherwise.} \end{cases}$$

In the following, if not explicitly stated, every function will be extended by 0 outside its domain of definition.

Let G_{E_j} be the Green function associated to Δ^2 on E_j with zero Dirichlet boundary condition. Let $v_{g,j}$ the function that satisfies

$$\begin{cases} \Delta^2 v_{g,j} &= g \quad \text{in } E_j, \\ v_{g,j} = \frac{\partial}{\partial\nu} v_{g,j} &= 0 \quad \text{on } \partial E_j. \end{cases}$$

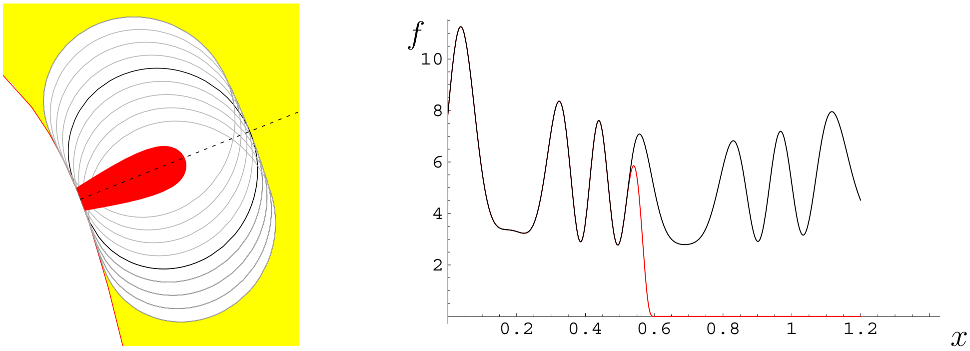

Figure 5.5: *In the picture on the left one finds some E_j's that cover Ω locally. The dark part shows the support of the cut-off function ψ_i. On the right the effect of the multiplication with the cut-off function considered on the dashed line: in black a function f and in red (lighter) the function $\psi_i f$. The scaling is arbitrary but consistent with the one in the following figures.*

We define

$$\tilde{u}_i(x) := \chi_i(x) v_{\psi_i f, j(i)}(x) \text{ and } \tilde{u}(x) := \sum_{i \in I} \tilde{u}_i(x).$$

Here $j(i)$ denotes the $j \in J$ such that $\operatorname{supp}(\psi_i) \subset E_j$.

Since the Green function $G_{E_j}(x,y)$ is positive and bounded on $E_j \times E_j$ (Theorem 5.3.2) we have for some $c_1 = c_1(M, \rho_\Omega^{-1})$

$$\begin{aligned} \tilde{u}_i(x) &= \chi_i(x) \int_{E_{j(i)}} G_{E_{j(i)}}(x,y) \psi_i(y) f(y) \, dy \\ &\leq \chi_i(x) \int_{\operatorname{supp}(\psi_i) \cap \Omega} G_{E_{j(i)}}(x,y) f^+(y) \, dy \leq c_1 \, \chi_i(x) \, \left\| f^+ \right\|_{L^1(\operatorname{supp}(\psi_i) \cap \Omega)} . \end{aligned}$$

Hence with $\varepsilon_R := 4R$ one gets

$$\begin{aligned} \tilde{u}(x) &\leq c_1 \sum_{i \in I} \chi_i(x) \, \left\| f^+ \right\|_{L^1(\operatorname{supp}(\psi_i) \cap \Omega)} \\ &\leq c_2 \left\| f^+ \right\|_{L^1\left(\bigcup_{\substack{i \in I, \\ \chi_i(x) \neq 0}} (\operatorname{supp}(\psi_i) \cap \Omega) \right)} \leq c_2 \left\| f^+ \right\|_{L^1(B(x, \varepsilon_R) \cap \Omega)} . \end{aligned} \tag{5.5.2}$$

We will now estimate the difference $u - \tilde{u}$. For every $i \in I$, one has in $E_{j(i)}$:

$$\Delta^2 v_{\psi_i f, j(i)} = \psi_i \Delta^2 u = \Delta^2 (\psi_i u) - \sum_{\substack{|\alpha+\beta|=4, \\ |\beta| \leq 3}} n_{\alpha,\beta} D^\alpha \psi_i D^\beta u, \tag{5.5.3}$$

where $n_{\alpha,\beta}$ are positive coefficients. From (5.5.3) we find in $E_{j(i)}$ that

$$\Delta^2 \left(v_{\psi_i f, j(i)} - \psi_i u \right) = - \sum_{\substack{|\alpha+\beta|=4, \\ |\beta| \leq 3}} n_{\alpha,\beta} D^\alpha \psi_i D^\beta u.$$

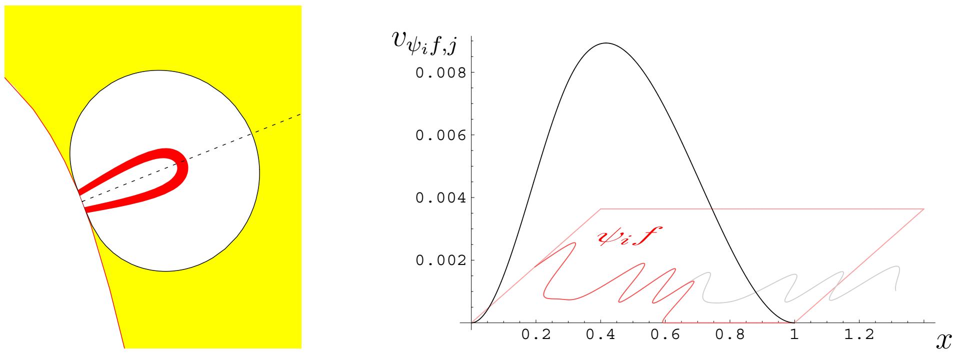

Figure 5.6: *On the left one finds in black the boundary of E_j and in red the set $\{x : \nabla\psi_i(x) \neq 0\}$. In the right one in black the function $v_{\psi_i f, j(i)}$, that is the solution of the clamped plate equation on $E_{j(i)}$ with on the right hand side $\psi_i f$, that is, the truncated f (red in the picture).*

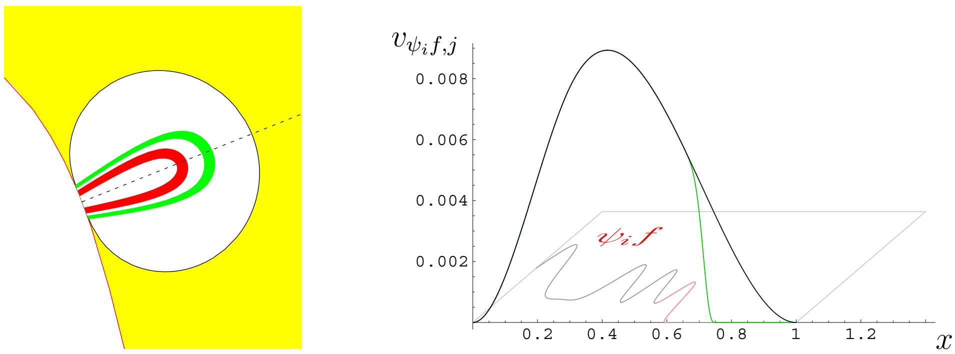

Figure 5.7: *On the left one now also finds in green the set $\{x : \nabla\chi_i(x) \neq 0\}$. On the right in green (lighter) the function $\tilde{u}_i = \chi_i v_{\psi_i f, j(i)}$.*

Furthermore the function $v_{\psi_i f, j(i)} - \psi_i u$ satisfies zero Dirichlet boundary condition on $\partial E_{j(i)}$. Indeed by construction: $u = \frac{\partial}{\partial\nu} u = 0$ on $\partial E_{j(i)} \cap \operatorname{supp}(\psi_i) \subset \partial E_{j(i)} \cap \partial\Omega$ and $\psi_i = \frac{\partial}{\partial\nu}\psi_i = 0$ for $x \in \partial E_{j(i)} \backslash \operatorname{supp}(\psi_i)$.

Hence we may write for $x \in E_{j(i)}$

$$v_{\psi_i f, j(i)}(x) = \psi_i(x) u(x) - R_i(x), \tag{5.5.4}$$

where

$$R_i(x) := \int_{E_{j(i)}} G_{E_{j(i)}}(x,y) \left(\Delta^2(\psi_i(y) u(y)) - \psi_i(y) \Delta^2 u(y) \right) dy.$$

On the other hand we get from (5.5.4)

$$\begin{aligned} \Delta^2 \tilde{u}_i &= \Delta^2 \left(\chi_i v_{\psi_i f, j(i)} \right) = \chi_i \Delta^2 v_{\psi_i f, j(i)} + \left(\Delta^2 \left(\chi_i v_{\psi_i f, j(i)} \right) - \chi_i \Delta^2 v_{\psi_i f, j(i)} \right) \\ &= \chi_i \psi_i f + \left(\Delta^2 \left(\chi_i \psi_i u - \chi_i R_i \right) - \chi_i \Delta^2 \left(\psi_i u - R_i \right) \right). \end{aligned}$$

Since $\operatorname{supp}(\psi_i) \subset \left\{x \in \bar{\Omega} : \chi_i(x) = 1\right\}$ it holds $\Delta^2(\chi_i\psi_i u) = \chi_i \Delta^2(\psi_i u)$. Hence we get

$$\Delta^2 \tilde{u}_i = \chi_i \psi_i f - \Delta^2(\chi_i R_i) + \chi_i \Delta^2 R_i.$$

Notice that this last relation holds in all of Ω. Hence the function $\tilde{u}$ satisfies in Ω

$$\Delta^2 \tilde{u} = f - \sum_{i\in I} \Delta^2(\chi_i R_i) + \sum_{i\in I} \chi_i \Delta^2 R_i.$$

It follows that $u - \tilde{u}$ satisfies

$$\begin{cases} \Delta^2(u-\tilde{u}) = \sum_{i\in I} \Delta^2(\chi_i R_i) - \sum_{i\in I} \chi_i \Delta^2 R_i & \text{in } \Omega, \\ u - \tilde{u} = 0 & \text{on } \partial\Omega, \\ \frac{\partial}{\partial\nu}(u-\tilde{u}) = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.5.5}$$

Here we used that $\tilde{u}_i = \frac{\partial}{\partial\nu}\tilde{u}_i = 0$ on $\partial\Omega$ for every $i \in I$.

Writing

$$\begin{aligned} u(x) &= \tilde{u}(x) + \sum_{i\in I} \int_\Omega G_\Omega(x,y)\left(\Delta^2(\chi_i R_i) - \chi_i\Delta^2 R_i\right)(y)\, dy \\ &= \tilde{u}(x) + \sum_{\substack{i\in I, |\beta'|\le 3. \\ |\alpha'+\beta'|=4}} n_{\alpha',\beta'} \int_\Omega G_\Omega(x,y) D^{\alpha'}\chi_i(y) D^{\beta'} R_i(y)\, dy \\ &= \tilde{u}(x) + \sum_{\substack{i\in I, |\beta|,|\beta'|\le 3, \\ |\alpha'+\beta'|=4, \\ |\alpha+\beta|=4}} n_{\alpha,\beta,\alpha',\beta'} \int_\Omega G_\Omega(x,y) D^{\alpha'}\chi_i(y) D^{\beta'} v_{D^\alpha\psi_i D^\beta u, j(i)}(y) dy, \end{aligned}$$

and using the estimate in (5.5.2) we find

$$\begin{aligned} u(x) &\le c_2 \left\|f^+\right\|_{L^1(B(x,\varepsilon_R)\cap\Omega)} \\ &+ \sum_{\substack{i\in I, |\beta|,|\beta'|\le 3, \\ |\alpha'+\beta'|=4, \\ |\alpha+\beta|=4}} n_{\alpha,\beta,\alpha',\beta'} \left\| \int_\Omega G_\Omega(\cdot,y) D^{\alpha'}\chi_i(y) D^{\beta'} v_{D^\alpha\psi_i D^\beta u, j(i)}(y) dy \right\|_\infty . \end{aligned} \tag{5.5.6}$$

In the following we will estimate the second term in the right hand side of (5.5.6). We fix $i \in I$, $\alpha, \beta, \alpha', \beta' \in \mathbb{N}^2$ with $|\alpha' + \beta'| = |\alpha + \beta| = 4$ and $|\beta'|, |\beta| \le 3$.

We first notice that it is sufficient to prove (5.5.1) for $q > 2$ and near 2. Indeed the result for general $q > 2$ will then follow from the observation that the following inequality holds

$$\|u\|_{W^{-1,q}(\Omega)} \le |\Omega|^{\frac{1}{q}-\frac{1}{\tilde{q}}} \|u\|_{W^{-1,\tilde{q}}(\Omega)} \text{ for any } \tilde{q} > q > 2.$$

Let fix $q > 2$ with $q - 2$ small. The Sobolev Imbedding Theorem yields that for some $c_3 = c_3(\frac{1}{2-q}, \rho_\Omega^{-1}, R_\Omega)$

$$\begin{aligned} &\left\| \int_\Omega G_\Omega(\cdot, y) D^{\alpha'} \chi_i(y) D^{\beta'} v_{D^\alpha \psi_i D^\beta u, j}(y)\, dy \right\|_\infty \le \\ \le\ & c_3 \left\| \int_\Omega G_\Omega(\cdot, y) D^{\alpha'} \chi_i(y) D^{\beta'} v_{D^\alpha \psi_i D^\beta u, j}(y)\, dy \right\|_{W_0^{1,q}(\Omega)} = \dots . \end{aligned}$$

Here and in the following we write simply j instead of $j(i)$.

We proceed using the regularity result for the "three-quarter weak solution" of problem (5.1.1) (see Definition 5.2.15). Indeed by Theorem 5.2.16 the solution operator from $\left(W^{3,q'}(\Omega) \cap W_0^{2,q'}(\Omega)\right)'$ to the space $W_0^{1,q}(\Omega)$ is an isomorphism. Hence we get for some $c_4 = c_4(\frac{1}{2-q}, M, \rho_\Omega^{-1}, R_\Omega)$

$$\begin{aligned} \dots \ \le\ & c_4 \left\| D^{\alpha'} \chi_i(\cdot) D^{\beta'} v_{D^\alpha \psi_i D^\beta u, j}(\cdot) \right\|_{\left(W^{3,q'}(\Omega) \cap W_0^{2,q'}(\Omega)\right)'} \\ =\ & c_4 \sup \Big\{ \langle D^{\alpha'} \chi_i D^{\beta'} v_{D^\alpha \psi_i D^\beta u, j}, \varphi \rangle \Big| \\ & \qquad \varphi \in W^{3,q'}(\Omega) \cap W_0^{2,q'}(\Omega) \text{ with } \|\varphi\|_{W^{3,q'}(\Omega)} \le 1 \Big\} = \dots . \end{aligned}$$

Notice that the constant in Theorem 5.2.16 depends on q and q'. However, since here we consider q near 2 we can choose a constant that depends only on the distance of q to 2.

Next, we consider a restriction from

$$\left(W^{3,q'}(\Omega) \cap W_0^{2,q'}(\Omega)\right)' \text{ onto } \left(W^{3,q'}(E_j) \cap W_0^{2,q'}(E_j)\right)'.$$

One uses that the cut-off function χ_i has support in $(E_j \cap \Omega) \cup (\partial\Omega \cap \partial E_j)^{\circ, \partial\Omega}$. Proceeding formally we take a cut-off function $h_i \in C_c^\infty(\Omega \cup (\partial\Omega \cap \partial E_j)^{\circ,\partial\Omega})$ such that:

i. $\operatorname{supp}(\chi_i) \subset \left\{x \in \bar\Omega : h_i(x) = 1\right\}$;

ii. $\operatorname{supp}(h_i) \subset (E_j \cap \Omega) \cup (\partial E_j \cap \partial\Omega)^{\circ,\partial\Omega}$;

iii. $0 \le h_i \le 1$;

iv. $\|\nabla^\alpha h_i\|_\infty \le c_\alpha R^{-|\alpha|}$ for every $\alpha \in \mathbb{N}^2$, $|\alpha| \le 4$.

Such a cut-off function exists since $\operatorname{supp}(\chi_i) \subset (E_j \cap \Omega) \cup (\partial E_j \cap \partial\Omega)^{\circ,\partial\Omega}$. The function $h_i\varphi$ lies in $W^{3,q'}(E_j) \cap W_0^{2,q'}(E_j)$ for every $\varphi \in W^{3,q'}(\Omega) \cap W_0^{2,q'}(\Omega)$ and moreover it holds

$$\begin{aligned} \langle D^{\alpha'} \chi_i D^{\beta'} v_{D^\alpha \psi_i D^\beta u, j}, \varphi \rangle_\Omega &= \langle D^{\alpha'} \chi_i D^{\beta'} v_{D^\alpha \psi_i D^\beta u, j}, h_i\varphi \rangle_\Omega \\ &= \langle D^{\alpha'} \chi_i D^{\beta'} v_{D^\alpha \psi_i D^\beta u, j}, h_i\varphi \rangle_{Ej}. \end{aligned}$$

Hence using that there exists a constant c_5 such that $\|h_i\varphi\|_{W^{3,q'}(\Omega)} \le c_5 R^{-3}\|\varphi\|_{W^{3,q'}(\Omega)}$ we get

$$\begin{aligned}
\ldots &= c_4 \sup\Big\{ \langle D^{\alpha'}\chi_i D^{\beta'} v_{D^\alpha\psi_i D^\beta u,j}, h_i\varphi\rangle_{E_j} \Big| \\
&\qquad\qquad \varphi \in W^{3,q'}(\Omega)\cap W_0^{2,q'}(\Omega) \text{ with } \|h_i\varphi\|_{W^{3,q'}(\Omega)} \le c_5 R^{-3}\Big\} \\
&\le c_4 \sup\Big\{ \langle D^{\alpha'}\chi_i D^{\beta'} v_{D^\alpha\psi_i D^\beta u,j}, \tilde\varphi\rangle_{E_j} \Big| \\
&\qquad\qquad \tilde\varphi \in W^{3,q'}(E_j)\cap W_0^{2,q'}(E_j) \text{ with } \|\tilde\varphi\|_{W^{3,q'}(E_j)} \le c_5 R^{-3}\Big\} \\
&\le c_6 \sup\Big\{ \langle D^{\beta'} v_{D^\alpha\psi_i D^\beta u,j}, \tilde\varphi\rangle_{E_j} \Big| \\
&\qquad\qquad \tilde\varphi \in W^{3,q'}(E_j)\cap W_0^{2,q'}(E_j) \text{ with } \|\tilde\varphi\|_{W^{3,q'}(E_j)} \le 1\Big\} = \ldots .
\end{aligned}$$

Here $c_6 = c_6(\frac{1}{2-q}, M, \rho_\Omega^{-1}, R_\Omega)$ since R depends on M.

We now proceed integrating by parts. Since $v_{D^\alpha\psi_i D^\beta u,j}$ and $\tilde\varphi$ and their first derivatives are zero on ∂E_j there is no contribution from the boundary. We find

$$\begin{aligned}
\ldots &= c_6 \sup\Big\{ \langle v_{D^\alpha\psi_i D^\beta u,j}, D^{\beta'}\tilde\varphi\rangle_{E_j} \Big| \\
&\qquad\qquad \tilde\varphi \in W^{3,q'}(E_j)\cap W_0^{2,q'}(E_j) \text{ with } \|\tilde\varphi\|_{W^{3,q'}(E_j)} \le 1\Big\} \\
&\le c_6 \sup\big\{ \langle v_{D^\alpha\psi_i D^\beta u,j}, \varphi\rangle_{E_j} \big| \\
&\qquad\qquad \varphi \in W^{3-|\beta'|,q'}(E_j)\cap W_0^{\min\{2,3-|\beta'|\},q'}(E_j), \|\varphi\|_{W^{3-|\beta'|,q'}(E_j)} \le 1\Big\} \\
&\le c_6 \sup\Big\{ \langle v_{D^\alpha\psi_i D^\beta u,j}, \varphi\rangle_{E_j} \Big| \varphi\in L^{q'}(E_j), \|\varphi\|_{L^{q'}(E_j)} \le 1\Big\} \\
&= c_6 \left\| v_{D^\alpha\psi_i D^\beta u,j}\right\|_{L^q(E_j)} = \ldots .
\end{aligned}$$

Next, we apply the regularity result for weak solution of problem (5.1.1) (see Definition 5.2.12). Notice that in order to do that one needs that $\partial E_j \in C^{4,\alpha}$. By the result in Theorem 5.2.13 we get for some $c_7 = c_7(\frac{1}{2-q}, M, \rho_\Omega^{-1}, R_\Omega)$

$$\begin{aligned}
\ldots &\le c_7 \left\| D^\alpha\psi_i D^\beta u\right\|_{\left(W^{4,q'}(E_j)\cap W_0^{2,q'}(E_j)\right)'} \\
&= c_7 \sup\Big\{ \langle D^\alpha\psi_i D^\beta u, \varphi\rangle_{E_j} \Big| \varphi \in W^{4,q'}(E_j)\cap W_0^{2,q'}(E_j), \|\varphi\|_{W^{4,q'}(E_j)} \le 1\Big\} = \ldots .
\end{aligned}$$

Since we consider q near 2 we can choose the dependence on q of the form $\frac{1}{2-q}$ in the constant appearing in the estimate in Theorem 5.2.13.

We now consider an extension from

$$\left(W^{4,q'}(E_j)\cap W_0^{2,q'}(E_j)\right)' \text{ onto } \left(W^{4,q'}(\Omega)\cap W_0^{2,q'}(\Omega)\right)'.$$

Since ψ_i has compact support in $(\Omega\cap E_j)\cup(\partial\Omega\cap\partial E_j)^{\circ,\partial\Omega}$ one has

$$\langle D^\alpha\psi_i D^\beta u, \varphi\rangle_{E_j} = \langle D^\alpha\psi_i D^\beta u, \varphi\rangle_\Omega,$$

which implies

$$\begin{aligned}
\dots &= c_7 \sup \left\{ \langle D^\alpha \psi_i D^\beta u, \varphi \rangle_\Omega \middle| \, \varphi \in W^{4,q'}(E_j) \cap W_0^{2,q'}(E_j), \|\varphi\|_{W^{4,q'}(E_j)} \le 1 \right\} \\
&\le c_7 \sup \left\{ \langle D^\alpha \psi_i D^\beta u, \varphi \rangle_\Omega \middle| \, \varphi \in W^{4,q'}(\Omega) \cap W_0^{2,q'}(\Omega), \|\varphi\|_{W^{4,q'}(\Omega)} \le 1 \right\} \\
&\le c_8 \sup \left\{ \langle D^\beta u, \varphi \rangle_\Omega \middle| \, \varphi \in W^{4,q'}(\Omega) \cap W_0^{2,q'}(\Omega), \|\varphi\|_{W^{4,q'}(\Omega)} \le 1 \right\} = \dots .
\end{aligned}$$

Here $c_8 = c_8(\frac{1}{2-q}, M, \rho_\Omega^{-1}, R_\Omega)$.

The last step is an integration by part. We do not have any contribution from the boundary since u and φ and their first derivative are zero on $\partial\Omega$. Hence one finds

$$\begin{aligned}
\dots &= c_8 \sup \left\{ \langle u, D^\beta \varphi \rangle_\Omega \middle| \, \varphi \in W^{4,q'}(\Omega) \cap W_0^{2,q'}(\Omega), \|\varphi\|_{W^{4,q'}(\Omega)} \le 1 \right\} \\
&\le c_8 \sup \left\{ \langle u, \tilde\varphi \rangle_\Omega \middle| \, \tilde\varphi \in W^{4-|\beta|,q'}(\Omega) \cap W_0^{\min\{2,4-|\beta|\},q'}(\Omega), \|\varphi\|_{W^{4-|\beta|,q'}(\Omega)} \le 1 \right\} \\
&\le c_8 \sup \left\{ \langle u, \tilde\varphi \rangle_\Omega \middle| \, \tilde\varphi \in W_0^{1,q'}(\Omega), \|\varphi\|_{W^{1,q'}(\Omega)} \le 1 \right\} = c_8 \, \|u\|_{\left(W_0^{1,q'}(\Omega)\right)'} .
\end{aligned}$$

The claim follows for $\varepsilon_R = 4R$. For $\varepsilon \in (0, \varepsilon_R]$ one may repeat the same construction with a refinement of the partition of unity ψ_i, $i \in I$. □

Remark 5.5.3. The hypothesis Ω simply connected is required in order to use Theorem 5.2.16. The result can be proved also for general connected domains using a generalization of Theorem 5.2.16.

5.5.2 Green function estimates

In this section we prove Theorem 5.1.1 and we give optimal estimates from below for the Green function of a two-dimensional domain Ω with $\partial\Omega \in C^{16}$. In this section we have to assume more regularity on the boundary of Ω in order to use Theorem 2.5.6 in Chapter 2 (see also [24]). As before, G_Ω denotes the Green function associated to problem (5.1.1) on Ω.

We first present some preliminary lemmas.

Lemma 5.5.4. *Assume that Ω is a bounded domain in $\mathbb{R}^2$ with $\partial\Omega \in C^{16}$. Then*

$$G_\Omega \in W^{3,p}(\Omega^2) \textit{ for any } p \in [1,2).$$

Proof. In Chapter 2 Theorem 2.5.6 (see also [24]) one finds

$$\left| D^\beta G_\Omega(x,y) \right| \preceq |x-y|^{-1} \min \left\{ 1, \frac{d(y)}{|x-y|} \right\}^2 \text{ for any } \beta \in \mathbb{N}^2 \text{ with } |\beta| \le 3. \quad (5.5.7)$$

The result follows directly from (5.5.7). □

Lemma 5.5.5. *Let Ω be a bounded domain in $\mathbb{R}^2$ with $\partial\Omega \in C^{16}$. Then for every $\gamma \in (0,1)$*

$$G_\Omega \in C^{15,\gamma}(\{(x,y) \in \bar{\Omega}^2 : x \neq y\}) \text{ and } G_\Omega \in C^{1,\gamma}(\bar{\Omega}^2) \cap C_0^1(\bar{\Omega}^2)$$

Proof. From general regularity results for elliptic partial differential equations (see [3]) it is well known that $G_\Omega \in C^{15,\gamma}(\{(x,y) \in \bar{\Omega}^2 : x \neq y\})$ for any $\gamma \in (0,1)$. Indeed, in general, given $l \in \mathbb{N}$, $\beta \in [0,1)$ and a bounded domain $\mathcal{D} \in C^{l,\beta}$ then the regularity of $G_\mathcal{D}$ on $\{(x,y) \in \bar{\mathcal{D}}^2 : x \neq y\}$ is as follows:

$$\begin{array}{ll} \text{if } \beta = 0: & G_\mathcal{D} \in C^{l-1,\gamma}\left(\{(x,y) \in \bar{\mathcal{D}}^2 : x \neq y\}\right) \text{ for any } \gamma \in (0,1)\,; \\ \text{if } \beta \neq 0: & G_\mathcal{D} \in C^{l,\beta}\left(\{(x,y) \in \bar{\mathcal{D}}^2 : x \neq y\}\right). \end{array}$$

The result that $G_\Omega \in C^{1,\gamma}(\bar{\Omega}^2)$ for any $\gamma \in (0,1)$ follows directly from Lemma 5.5.4 by the Sobolev imbedding Theorem ([2, Th.4.12 Part 2]). Hence $G_\Omega \in W^{3,p}(\Omega^2) \cap C^{1,\gamma}(\bar{\Omega}^2)$ for $p \in [1,2)$ and $\gamma \in (0,1)$. Moreover the function and its first derivatives are zero on $\partial\Omega \times \Omega$ and on $\Omega \times \partial\Omega$. Hence by continuity and Theorem IX.17 in [9] it follows that $G_\Omega \in C_0^1(\bar{\Omega}^2)$ (and also $G_\Omega \in W_0^{2,p}(\Omega^2)$ for $p \in [1,2)$). □

Proof of Theorem 5.1.1. Following the construction in Theorem 5.5.2, see (5.5.5), one may write the solution of problem (5.1.1) as

$$\begin{aligned} u(x) &= \tilde{u}(x) + \int_\Omega G_\Omega(x,z) \sum_{i\in I} \left(\Delta^2(\chi_i(z)R_i(z)) - \chi_i(z)\Delta^2 R_i(z)\right) dz \\ &= \sum_{i\in I} \chi_i(x) \int_{E_{j(i)}} G_{E_{j(i)}}(x,z)\psi_i(z)f(z)dz + \\ &\quad + \int_\Omega G_\Omega(x,z) \sum_{i\in I} \left(\Delta^2(\chi_i(z)R_i(z)) - \chi_i(z)\Delta^2 R_i(z)\right) dz, \end{aligned}$$

where

$$R_i(z) = \int_{E_{j(i)}} G_{E_{j(i)}}(z,z') \left(\Delta^2(\psi_i(z')u(z')) - \psi_i(z')\Delta^2 u(z')\right) dz',$$

and $j(i)$ denotes the $j \in J$ such that $\operatorname{supp}(\psi_j) \subset E_j$. Considering formally $f(x) = \delta_y(x)$ we get

$$\begin{aligned} G_\Omega(x,y) &= \sum_{i\in I} \chi_i(x) G_{E_{j(i)}}(x,y)\psi_i(y) + \\ &\quad + \int_\Omega G_\Omega(x,z) \sum_{i\in I} \left(\Delta^2(\chi_i(z)R_i(z,y)) - \chi_i(z)\Delta^2 R_i(z,y)\right) dz, \end{aligned}$$

where

$$R_i(z,y) = \int_{E_{j(i)}} G_{E_{j(i)}}(z,z') \left(\Delta^2(\psi_i(z')G_\Omega(z',y)) - \psi_i(z')\Delta^2 G_\Omega(z',y)\right) dz'.$$

We define

$$G_\Omega^{\rm sing}(x,y) := \sum_{i\in I} \chi_i(x) G_{E_{j(i)}}(x,y)\psi_i(y), \tag{5.5.8}$$

$$G_\Omega^{\rm reg}(x,y) := G_\Omega(x,y) - G_\Omega^{\rm sing}(x,y). \tag{5.5.9}$$

From the definition it follows that $G_\Omega^{\rm reg} \in C^{15,\gamma}(\bar\Omega^2)$ for any $\gamma \in (0,1)$. Indeed, writing explicitly R_i and looking at the support of the term inside the integral, we find

$$\begin{aligned} G_\Omega^{\rm reg}(x,y) &= \sum_{i\in I} \sum_{\substack{|\alpha+\beta|=4,\\ |\alpha'+\beta'|=4,\\ |\beta|,|\beta'|\le 3}} n_{\alpha,\beta,\alpha',\beta'} \int_{B_i} G_\Omega(x,z) D^\alpha \chi_i(z) \cdot \\ &\qquad \cdot D^\beta \int_{A_i} G_{E_{j(i)}}(z,z') D^{\alpha'}\psi_i(z') D^{\beta'} G_\Omega(z',y)dz'dz, \end{aligned} \tag{5.5.10}$$

with $n_{\alpha,\beta,\alpha',\beta'}$ some positive coefficients and

$$B_i = \{z\in\Omega : \nabla\chi_i(z) \neq 0\} \text{ and } A_i = \{z\in\Omega : \nabla\psi_i(z)\neq 0\}. \tag{5.5.11}$$

Since by construction $A_i \cap B_i = \varnothing$ one always has $z\neq z'$ in (5.5.10). Hence $G_{E_j}(z,z') \in C^\infty(B_i\times A_i)$. Since the term $D^{\alpha'}\psi_i(z')D^{\beta'}G_\Omega(z',y)$ is integrable it follows that $G_\Omega^{\rm reg}$ is as regular as we want in the interior. The regularity up to the boundary is given by the fact that $\partial\Omega \in C^{16}$.

The positivity of $G_\Omega^{\rm sing}$ follows from the positivity of G_{E_j}. Furthermore by Lemma 5.5.5, the definition of $G_\Omega^{\rm sing}$ and since $G_\Omega^{\rm reg} \in C^{15,\gamma}(\bar\Omega^2)$ for any $\gamma\in(0,1)$, it follows that $G_\Omega^{\rm sing} \in C^{1,\gamma}(\bar\Omega^2)\cap C_0^1(\bar\Omega^2)$ and $G_\Omega^{\rm sing} \in C^{15,\gamma}(\{(x,y)\in\bar\Omega^2 : x\neq y\})$ for any $\gamma\in(0,1)$. Notice that by the boundary condition satisfied by G_Ω and $G_\Omega^{\rm sing}$ we also have that $G_\Omega^{\rm reg} \in C_0^1(\bar\Omega^2)$. □

Remark 5.5.6. The functions $G^{\rm sing}$ and $G^{\rm reg}$ defined in proof of Theorem 5.1.1 are not symmetric. In order to get symmetric functions one may consider $G_{\Omega,new}^{\cdots}(x,y) := \frac12 G_\Omega^{\cdots}(x,y) + \frac12 G_\Omega^{\cdots}(y,x)$.

Optimal estimates from above for the Green function as well as estimates for the absolute value are known. We refer respectively to [52], [41] and [24]. We will next prove optimal estimates from below for G_Ω.

First we prove the following lemma.

Lemma 5.5.7. *Let Ω be a bounded domain in $\mathbb{R}^2$ with $\partial\Omega\in C^{16}$. Then G_Ω satisfies*

$$\|\nabla G_\Omega(\cdot,y)\|_{L^p(\Omega)} \le c'_{p,\Omega} d(y)^2 \textit{ for every } y\in\Omega \textit{ and } p\in[1,2).$$

Proof. Via Theorem 2.5.6 in Chapter 2 (see [24]) one finds

$$\begin{aligned} \|\nabla G_\Omega(\cdot,y)\|_{L^p(\Omega)}^p &\le c_\Omega \int_\Omega d(y)^p \min\left\{1,\frac{d(y)}{|x-y|}\right\}^p \min\left\{1,\frac{d(x)}{|x-y|}\right\}^p dx \\ &\le c_\Omega d(y)^{2p} \int_\Omega \frac{1}{|x-y|^p} dx \le c'_{p,\Omega} d(y)^{2p}, \end{aligned}$$

for $p \in [1, 2)$. $\square$

Theorem 5.5.8. *Let Ω be a bounded domain in $\mathbb{R}^2$ with $\partial\Omega \in C^{16}$. Then there exists $c_\Omega > 0$ such that G_Ω satisfies:*

$$G_\Omega(x, y) \geq -c_\Omega d(x)^2 d(y)^2 \text{ for every } x, y \in \Omega.$$

Proof. Since $G_\Omega(x, y) = G_\Omega^{\text{sing}}(x, y) + G_\Omega^{\text{reg}}(x, y)$, with G_Ω^{sing} and G_Ω^{reg} defined in (5.5.8) and (5.5.9) respectively, and G_Ω^{sing} is positive it holds

$$G_\Omega(x, y) \geq -\left|G_\Omega^{\text{reg}}(x, y)\right| \text{ for every } x, y \in \Omega.$$

Hence in order to prove the result it is sufficient to get an estimate of the absolute value of G_Ω^{reg}.

We first study the $W^{4,p}$-norm of $G_\Omega^{\text{reg}}(\cdot, y)$ for $p \in (1, \infty)$.

Let A_i and B_i as defined in (5.5.11). From (5.5.10) and elliptic regularity theory (see Theorem 5.2.6) it follows that

$$\|G_\Omega^{\text{reg}}(\cdot, y)\|_{W^{4,p}(\Omega)} \leq c \sum_{i\in I} \sum_{\substack{|\alpha+\beta|=4,\\ |\beta|\leq 3}} n_{\alpha,\beta} \left\|D^\alpha \chi_i(\cdot) D^\beta R_i(\cdot, y)\right\|_{L^p(B_i)}.$$

We study separately the term $\left\|D^\alpha \chi_i(\cdot) D^\beta R_i(\cdot, y)\right\|_{L^p(B_i)}$. One has

$$\begin{aligned}&\left\|D^\alpha \chi_i(\cdot) D^\beta R_i(\cdot, y)\right\|_{L^p(B_i)} \leq \\ &\leq c_\Omega \sum_{\substack{|\alpha'+\beta'|=4,\\ |\beta'|\leq 3}} n_{\alpha',\beta'} \left\|D^\beta \int_{A_i} G_{E_{j(i)}}(\cdot, z') D^{\alpha'} \psi_i(z') D^{\beta'} G_\Omega(z', y) dz'\right\|_{L^p(B_i)}.\end{aligned}$$

We first observe that G_{E_j} is non singular in $B_i \times A_i$. Indeed since $\bar{A}_i \cap \bar{B}_i = \varnothing$, the function $G_{E_j}(z, z')$ is in $C^\infty(B_i \times A_i)$ and all its derivatives are bounded by a constant depending only on Ω.

The next step consists in an integration by part. There are no contribution from the boundary since in $\partial A_i \cap \Omega$ the function ψ_i and its derivatives are zero, while in $\partial A_i \cap \partial\Omega$ both G_{E_j} and G_Ω and their first derivatives are zero.

Let $\beta'' \in \mathbb{N}^2$ denote a multiindex such that $\beta'' < \beta'$, $|\beta''| = |\beta'| - 1$. We obtain

$$\begin{aligned}&\left\|D^\alpha \chi_i(\cdot) D^\beta R_i(\cdot, y)\right\|_{L^p(B_i)} \\ \leq\ & c_\Omega \sum_{\substack{|\alpha'+\beta'|=4,\\ |\beta'|\leq 3}} n_{\alpha',\beta'} \left\|D^\beta \int_{A_i} D^{\beta''} \left(G_{E_j}(\cdot, z') D^{\alpha'} \psi_i(z')\right) D^{\beta'-\beta''} G_\Omega(z', y) dz'\right\|_{L^p(B_i)} \\ \leq\ & c_{\Omega,p} \sum_{\substack{|\alpha'+\beta'|=4,\\ |\beta'|=2,3}} n_{\alpha',\beta'} \int_\Omega \left|D^{\beta'-\beta''} G_\Omega(z', y)\right| dz' \leq c'_{\Omega,p} d(y)^2.\end{aligned}$$

In the last step we used Lemma 5.5.7.

Since $G_\Omega^{\mathrm{reg}}(x,y) \in W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega)$ for any $p \in (1,\infty)$, from [12, Lem.5] it follows that

$$\frac{|G_\Omega^{\mathrm{reg}}(x,y)|}{d(x)^2} \leq c_\Omega \left\| G_\Omega^{\mathrm{reg}}(\cdot,y) \right\|_{W^{4,p}(\Omega)} .$$

Hence we obtain

$$|G_\Omega^{\mathrm{reg}}(x,y)| \leq c''_\Omega d(y)^2 d(x)^2.$$

The claim follows. □

Remark 5.5.9. In [12, Lemma 5] the authors consider a bounded domain Ω with $\partial\Omega$ smooth. One can consider a weaker assumption on the boundary. Indeed, in order to apply the Rellich-Kondrachov Theorem, [2, Th.6.3], it is sufficient that Ω is bounded and satisfies the strong Lipschitz condition, [2, Def.4.9]. Notice that if Ω satisfies the uniform C^l regularity condition with $l \geq 2$ then Ω satisfies also the strong Lipschitz condition.

Chapter 6

Brownian motion on the ball in $\mathbb{R}^n$

6.1 Introduction

Let Ω be a Lipschitz domain in $\mathbb{R}^n$ and let G_Ω denote the Green function for

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega, \\ \quad u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{6.1.1}$$

that is, the solution of (6.1.1) is given by $u(x) = \int_\Omega G_\Omega(x,y) f(y) dy$. Let us define

$$H_\Omega(x,y) := \int_\Omega \frac{G_\Omega(x,z) G_\Omega(z,y)}{G_\Omega(x,y)} \text{ for } x, y \in \Omega \times \Omega.$$

The function $H_\Omega(x,y)$ is of some importance in two different areas of mathematics: elliptic partial differential equations and probability.

The topic of this chapter is the study of the function H_Ω with $\Omega = B$ the unit ball in $\mathbb{R}^n$, $n \geq 2$. We will show that for every $y \in \bar{B}$ the function $x \mapsto H_B(x,y)$ is increasing away from y along the hyperbolic geodesics through y and also along a family of trajectories orthogonal to the hyperbolic geodesic through y in increasing Euclidean distance from y. As a consequence we will find that $x \mapsto H_B(x,y)$ has no interior maximum and we will even pinpoint the location of the maximum at the boundary.

Our aim in studying this problem was to supply an answer to some questions left open in [19], [37] and in [49], [50].

6.1.1 The link between analysis and probability

The model problem for the positivity preserving property of systems of second order elliptic boundary value problems that are coupled in a noncooperative way is

$$\begin{cases} -\Delta u = f - \lambda v & \text{in } \Omega, \\ -\Delta v = f & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.1.2}$$

where Ω is a bounded set in $\mathbb{R}^n$ and $\lambda \in \mathbb{R}^+$. One knows, at least for Ω that satisfy some boundary regularity, that there exists $\lambda_c(\Omega) \in (0,\infty)$ such that for all $f \geq 0$ the solution u satisfies $u \geq 0$ if and only if $\lambda \leq \lambda_c(\Omega)$. See [50], [55] and [68]. Since the solution u of (6.1.2) equals

$$u(x) = \int_{y\in\Omega} G_\Omega(x,y)\left(1-\lambda\int_{z\in\Omega}\frac{G_\Omega(x,z)\,G_\Omega(z,y)}{G_\Omega(x,y)}dz\right)f(y)\,dy,$$

one can show that

$$\begin{aligned} \lambda_c(\Omega)^{-1} &= \sup_{x,y\in\Omega}\int_{z\in\Omega}\frac{G_\Omega(x,z)\,G_\Omega(z,y)}{G_\Omega(x,y)}dz \\ &= \sup_{x,y\in\Omega} H_\Omega(x,y). \end{aligned} \tag{6.1.3}$$

For rather general elliptic problems Cranston, Fabes and Zhao in [20] showed that the right hand side of (6.1.3) is finite. For the Laplacian such a bound has been obtained by Cranston in [18] for $n \geq 3$ and with McConnell in [19] for $n = 2$.

The link between (6.1.3) and probability theory is:

$$\mathbb{E}_x^y(\tau_\Omega) = \int_{z\in\Omega}\frac{G_\Omega(x,z)\,G_\Omega(z,y)}{G_\Omega(x,y)}dz, \tag{6.1.4}$$

where $\mathbb{E}_x^y(\tau_\Omega)$ is the expectation of the lifetime of a Brownian motion in Ω starting in x, conditioned to converge to and to be stopped at y and to be killed on exiting Ω.

The famous result from [19] states that there is a $c > 0$ such that

$$\mathbb{E}_x^y(\tau_\Omega) \leq c|\Omega| \text{ for all } \Omega \subset \mathbb{R}^2,$$

where $|\Omega|$ is the Lebesgue measure of Ω.

Some details for identity (6.1.4). A Brownian motion that starts in $x \in \Omega$ and is killed on $\partial\Omega$ has transition density given by $p_\Omega(t,x,y)$ and has expected lifetime given by

$$\mathbb{E}_x(\tau_\Omega) = \int_\Omega G_\Omega(x,z)dz.$$

Here $p_\Omega(t,x,y)$ denotes the heat kernel for

$$\begin{cases} \frac{\partial}{\partial t}u - \Delta u &= 0 \quad \text{in } \mathbb{R}^+ \times \Omega, \\ u &= 0 \quad \text{on } \mathbb{R}^+ \times \partial\Omega, \\ u &= f \quad \text{on } \{0\} \times \Omega, \end{cases} \tag{6.1.5}$$

that is, the solution of (6.1.5) is given by

$$u(t,x) = \int_\Omega p_\Omega(t,x,y)f(y)dy \text{ for } (t,x) \in \mathbb{R}^+ \times \Omega.$$

We would like to recall that the following relation holds in $\Omega \times \Omega$

$$G_\Omega(x,y) = \int_0^\infty p_\Omega(t,x,y)dt.$$

To consider Brownian motion that is conditioned to exit Ω through $\Gamma \subset \partial\Omega$ and stopped at leaving Ω, one uses the transition density $p_\Omega^h(t,x,z) = p_\Omega(t,x,z)\frac{h(z)}{h(x)}$ where h is the solution of

$$\begin{cases} -\Delta h = 0 & \text{in } \Omega, \\ h = 0 & \text{on } \partial\Omega \setminus \Gamma, \\ h = 1 & \text{on } \Gamma. \end{cases}$$

This is a so-called Doob's conditioned Brownian motion, see [29, Part 2, Chap. X]. The expected lifetime is given by

$$\mathbb{E}_x^h(\tau_\Omega) = \int_\Omega G_\Omega(x,z)\frac{h(z)}{h(x)}dz. \tag{6.1.6}$$

We want to consider the expectation for the time that a Brownian motion spends going from x to y and staying inside Ω. This can be approximated by the expected lifetime for the following conditioned Brownian motion. One considers the domains $\Omega_\varepsilon = \Omega \setminus B_\varepsilon(y)$ and the functions $h_{y,\varepsilon}$ such that

$$\begin{cases} -\Delta h_{y,\varepsilon} = 0 & \text{in } \Omega \setminus B_\varepsilon(y), \\ h_{y,\varepsilon} = 1 & \text{on } \partial B_\varepsilon(y), \\ h_{y,\varepsilon} = 0 & \text{on } \partial\Omega, \end{cases}$$

with the expected lifetime given by (6.1.6) replacing h by $h_{y,\varepsilon}$ and G_Ω by G_{Ω_ε}. The expectation of the time we are interested in becomes the expected lifetime of the Brownian motion starting at x and conditioned to leave $\Omega \setminus \{y\}$ at $\{y\}$. This is now given by

$$\mathbb{E}_x^y(\tau_{\Omega\setminus\{y\}}) = \lim_{\varepsilon\to 0} \mathbb{E}_x^{h_{y,\varepsilon}}(\tau_{\Omega_\varepsilon}). \tag{6.1.7}$$

For x and y in the interior, using that

$$\frac{h_{y,\varepsilon}(z)}{h_{y,\varepsilon}(x)} \to \frac{G_\Omega(z,y)}{G_\Omega(x,y)},$$

and that $G_{\Omega_\varepsilon} \to G_\Omega$ holds in dimension $n > 1$, identity (6.1.4) follows from (6.1.6) and (6.1.7).

In the particular case of $y \in \partial\Omega$ a similar procedure leads to

$$\mathbb{E}_x^y\left(\tau_\Omega\right) = \int_{z\in\Omega} G_\Omega\left(x,z\right)\frac{K_\Omega\left(y,z\right)}{K_\Omega\left(y,x\right)}dz, \tag{6.1.8}$$

where $K_\Omega\left(y,\cdot\right)$ is the Poisson kernel for $y \in \partial\Omega$, namely the function such that $u\left(x\right) = \int_{y\in\partial\Omega} K_\Omega\left(x,y\right)g\left(y\right)d\sigma_y$ solves

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

For sufficiently regular domains the expression in (6.1.8) is a continuous extension of (6.1.4) to $\Omega \times \bar{\Omega}$. Note that in the above we have used the analyst's $-\Delta$ instead of $-\frac{1}{2}\Delta$.

6.1.2 Main result

Since in this chapter we work in the unit ball we skip the subscript B and we write $H(x,y) = H_B(x,y)$.

We first extend the definition of H up to the closure $\bar{B} \times \bar{B}$ by using dominated convergence and taking limits. The complete definition of H then reads:

$$H(x,y) = \begin{cases} \dfrac{\int_B G_B(x,z)\, G_B(z,y)\, dz}{G_B(x,y)} & \text{if } x,y \in B \text{ with } x \neq y, \\ 0 & \text{if } x = y \in \bar{B}, \\ \dfrac{\int_B K_B(x,z)\, G_B(z,y)\, dz}{K_B(x,y)} & \text{if } x \in \partial B,\ y \in B, \\ \dfrac{\int_B K_B(y,z)\, G_B(z,x)\, dz}{K_B(y,x)} & \text{if } x \in B,\ y \in \partial B, \\ \frac{n\omega_n}{2} |x-y|^n \int_B K_B(x,z)\, K_B(y,z)\, dz & \text{if } x,y \in \partial B \text{ with } x \neq y, \end{cases} \tag{6.1.9}$$

where $\omega_n = \frac{2\pi^{\frac{n}{2}}}{n\Gamma(\frac{n}{2})}$ is the volume of $B \subset \mathbb{R}^n$.

This function H lies in $C\left(\bar{B} \times \bar{B}\right)$ and is strictly positive on $\bar{B}^2 \backslash \left\{(x,x)\,;\, x \in \bar{B}\right\}$.

A precise formulation of the main result is the following:

Theorem 6.1.1. *For all $y \in \bar{B}$ the function $x \mapsto H(x,y)$ is*

(i) increasing along 'the hyperbolic geodesics through y' *in increasing Euclidean distance;*

(ii) increasing along the orthogonal trajectories of 'the hyperbolic geodesics through y' *in increasing Euclidean distance.*

Remark 6.1.2. For $y \in \partial B$ and in dimension 2 part i of Theorem 6.1.1 has been proved by Griffin, McConnell and Verchota in [37].

Remark 6.1.3. In dimension two 'the hyperbolic geodesics through y' are the circles through y that intersect ∂B perpendicularly. The orthogonal trajectories are again circles. See Figure 6.1.

In dimensions $n \geq 3$ the hyperbolic geodesics in B are the intersection of B with the Euclidean circles that meet ∂B at right angle (see [73, page 66]). See Figure 6.1.

In dimension two it is clear what we mean by orthogonal trajectories to the hyperbolic geodesic through y. It is convenient to explain what we mean by this in higher

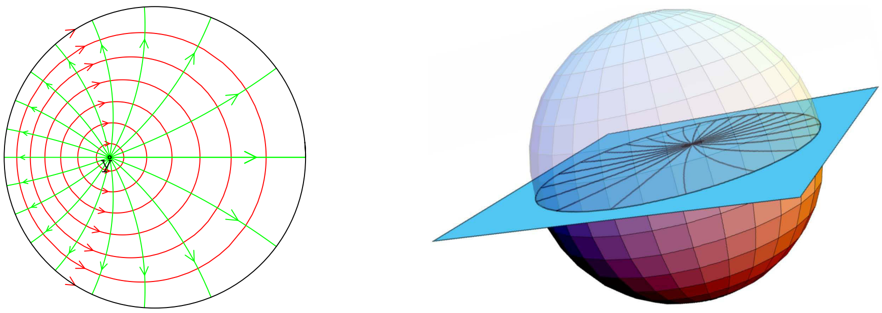

Figure 6.1: *To the left: The geodesics through y in green (light) and the orthogonal trajectories in red (dark) in the ball in dimension two.*
To the right: Some hyperbolic geodesic in the ball in $\mathbb{R}^3$.

dimensions. A generic hyperbolic geodesic through y in $B \subset \mathbb{R}^n$, $n \geq 3$, is obtained in the following way. One considers a generic unit disk in B to which the origin and y belong. Each hyperbolic geodesic through y in this disk is a hyperbolic geodesic through y in $B \subset \mathbb{R}^n$. Then the orthogonal trajectories to this hyperbolic geodesic through y in B are the orthogonal trajectories to the hyperbolic geodesic through y in the disk.

A direct consequence of Theorem 6.1.1 is the following result.

Corollary 6.1.4. *One directly finds that:*

(i) $\sup_{x\in\bar{B}} H(x,y) = H(-y/|y|, y)$ *for any* $y \in \bar{B}\setminus\{0\}$;

(ii) $\sup_{x\in\bar{B}} H(x,0) = H(e_1, 0)$ *with* $e_1 = (1,0,\ldots,0) \in \mathbb{R}^n$;

(iii) and $\sup_{x,y\in\bar{B}} H(x,y) = H(-e_1, e_1)$.

Remark 6.1.5. Since the problem has a rotational symmetry one finds that e_1 above might be replaced by any $a \in \partial B$.

The chapter is organized as follows. We first complete this section presenting some previous results and open questions. In the second section we recall some known properties of conformal transformations focusing on the difference between the two-dimensional and the higher-dimensional case. We also recall a formulation of the maximum principle that will be used. In the third and fourth section we prove Theorem 6.1.1 in the two dimensional case and in the higher dimensional case respectively. The proof is divided in these two parts firstly because the expression of H changes when $n = 2$ and $n \geq 3$ and hence the arguments in the proof are, to some extent, different, and secondly since the two parts appear in two different works ([22] and [21] respectively). In the last section we discuss some identities involving $\lambda_c^{-1}(\Omega)$, defined in 6.1.3, and a sum of inverse Dirichlet eigenvalues.

6.1.3 Earlier related results

Critical numbers related to (6.1.3) have been studied before in a number of papers. Caristi and Mitidieri in [10] considered the radially symmetric case (in any dimension n), that is, system (6.1.2) for radially symmetric functions and hence with $-\Delta$ replaced by $-r^{1-n}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}\right)$. They showed that the corresponding $H_{radial}(r,s)$ is maximal for (r,s) being extremal which means $r=0$ and $s=1$ or vice versa. The critical number that they find for this radial case is as follows:

$$\sup_{r,s\in[0,1]} H_{radial}(r,s) = \frac{1}{2n}.$$

In the one-dimensional case they also considered $\frac{\partial^2}{\partial x^2}+c$ without assuming symmetry.

Maximal lifetime on the disk. Griffin, McConnell and Verchota in [37] considered H for general simply connected 2-dimensional domains Ω but fixed $y\in\partial\Omega$. Two of their main results for such Ω are

$$\sup_{x\in\bar\Omega, y\in\partial\Omega} H(x,y) = \sup_{x,y\in\partial\Omega} H(x,y)$$

and that (with our 'analytic' normalization)

$$\sup_{x,y\in\bar\Omega} H(x,y) \le \frac{1}{2\pi}|\Omega|.$$

For $\Omega=B$ and $y\in\partial B$ they sharpen this estimate:

$$\sup_{x\in\bar B, y\in\partial B} H(x,y) \le 2\log 2 - 1 = \frac{2\log 2-1}{\pi}|B|.$$

The numerical values are $\frac{1}{2\pi} = .159155...$ and $\frac{2\log 2-1}{\pi} = .12296....$ Our result considered for $B\subset\mathbb{R}^2$ improves the last estimate by

$$\sup_{x,y\in\bar B} H(x,y) = \sup_{x\in\bar B, y\in\partial B} H(x,y) \le 2\log 2 - 1,$$

thereby giving an estimate for the lifetime inequality on a disk with a small hole which is sharper than $1/(2\pi)$ (which corresponds to $1/\pi$ in [37, Remark 5.7]). We will also give the explicit formula of $\sup_{x,y\in\bar B} H(x,y)$ in $B\subset\mathbb{R}^n$ for general n.

Domain optimization. In [49] Kawohl and coauthor showed that the disk does not give the smallest bound for H among all convex planar sets of equal area. Indeed, they considered a sector-like domain S, with $|S|=|B|$, and proved that:

$$\sup_{x\in\bar S, y\in\partial S} H(x,y) < \sup_{x\in\bar B, y\in\partial B} H(x,y).$$

The question remains open if

$$\sup_{x,y\in\bar{S}} H(x,y) < \sup_{x,y\in\bar{B}} H(x,y) \ ? \tag{6.1.10}$$

In the present paper we show that $\sup_{x\in\bar{B},y\in\partial B} H(x,y) = \sup_{x,y\in\bar{B}} H(x,y)$ holds. We expect the last identity to hold for all planar domains Ω. Let us put it as a conjecture.

Conjecture 6.1.6. *If Ω is a (simply connected) planar domain, then*

$$\sup_{x,y\in\Omega} H(x,y) = \sup_{x,y\in\partial\Omega} H(x,y) \ .$$

The obvious consequence of this conjecture is (6.1.10). We want to remark that such a result is not likely to hold on a manifold. Consider for example the surface of a ball with a small hole near the pole, see Fig.6.2. Taking y near the north pole one expects the maximum of H to be attained at an interior point near the south pole.

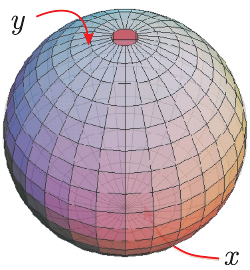

Figure 6.2: *Sphere with a small hole near the north pole.*

Relation with eigenvalues In one dimension critical numbers for sign-changing in (6.1.2) were studied by Schröder [62]. The precise result was revisited in [50]. Due to the fact that in one dimension the boundary consists of isolated points one recovers an eigenvalue problem for the critical number.

A relation between that critical number and the Dirichlet eigenvalues in an interval $I \subset \mathbb{R}$ is

$$\sup_{x,y\in I} H(x,y) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} = 2\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\lambda_k}.$$

Note that for the unit interval $I = (0,1)$ these eigenvalues are $\lambda_k = \pi^2 k^2$, $k \in \mathbb{N}$.

For the disk one finds

$$\sup_{x,y\in B} H(x,y) = 4\sum_{\nu=1}^{\infty} (-1)^{\nu-1} \sum_{k=1}^{\infty} \frac{m_{\nu,k}}{\lambda_{\nu,k}}, \tag{6.1.11}$$

where $\lambda_{\nu,k}$ is the eigenvalue for the eigenfunction with $k-1$ circular nodal lines and ν radial nodal lines, and where $m_{\nu,k}$ is the multiplicity, that is, $m_{\nu,k} = 1$ for $\nu = 0$ and $m_{\nu,k} = 2$ for $\nu \geq 1$. The numbers for the two right hand sides above can be found in [50].

At the end of this chapter we will give an explanation for identity (6.1.11). We will also show that a relation exists also between $\sup_{x,y\in B} H(x,y)$ for $B \subset \mathbb{R}^3$ and a sum of inverse Dirichlet eigenvalues. It is still an open question if such a relation holds for the unit ball in dimension $n \geq 4$.

6.2 Conformal maps and a Maximum Principle

For completeness, we recall here some known properties of conformal maps. We will repeatedly use this kind of mappings in the proof of Theorem 6.1.1. The situation is different in $\mathbb{R}^n$ for $n = 2$ and $n \geq 3$.

Conformal maps are a very useful tool for problems in the plane. The first reason is that there are many conformal maps: every simply connected domain $D \subsetneq \mathbb{R}^2$ can be mapped conformally onto the ball (Riemann Mapping Theorem, [59]). A second important property of conformal maps is the 'invariance' of the Green function. The precise result is stated in the following lemma.

Lemma 6.2.1. *Let A, D be simply connected bounded domains in $\mathbb{R}^2$ and let $\varphi : A \to D$ a conformal map. Let G_A denote the Green function for the Laplace problem with Dirichlet boundary condition in A.*

Then it holds $G_D(\varphi(x), \varphi(y)) = G_A(x,y)$.

In higher dimension the situation is different. The only conformal mappings are the Möbious transforms. Liouville's Theorem, [57], states that every conformal transformation in $\mathbb{R}^n$ with $n \geq 3$ must necessarily reduce to a translation, a magnification, an orthogonal transformation, a reflection through reciprocal radii, or a combination of these elementary transformations. Moreover there is no 'invariance' of the Green function via conformal mappings. However a relation still holds. We write the result in the following lemma.

Lemma 6.2.2. *Let A, D be simply connected bounded domain in $\mathbb{R}^n$, $n \geq 3$, and let $\varphi : A \to D$ be a conformal map. Let J_φ denote the Jacobian of φ. Then it holds that*

$$G_D(\varphi(x), \varphi(y)) = (J_\varphi(x) J_\varphi(y))^{\frac{1}{n}-\frac{1}{2}}\, G_A(x,y).$$

Remark 6.2.3. The result stated in Lemma 6.2.2 holds also if φ is an anti-conformal map since there is only a change in the orientation.

Proof. In Proposition 4.2.3 in Chapter 4 (see also [25, Cor. 2.2]) it is proved that for any Möbious transformation ψ in $\mathbb{R}^n$ and $k \in \mathbb{N}$ it holds

$$\Delta^k (J_\psi^{\frac{1}{2}-\frac{k}{n}}\; u \circ \psi) = J_\psi^{\frac{1}{2}+\frac{k}{n}} \left(\Delta^k u\right) \circ \psi. \tag{6.2.1}$$

In our setting using (6.2.1) with $k = 1$, we get that for any $x \in B$

$$\begin{aligned} u(\varphi(x)) &= J_\varphi^{\frac{1}{n}-\frac{1}{2}}(x) \int_A G_A(x,y) J_\varphi^{\frac{1}{2}+\frac{1}{n}}(y)(\Delta u)(\varphi(y))dy \\ &= \int_A G_A(x,y)(J_\varphi(x)J_\varphi(y))^{\frac{1}{n}-\frac{1}{2}}(\Delta u)(\varphi(y))J_\varphi(y)dy. \end{aligned} \tag{6.2.2}$$

We can also write

$$\begin{aligned} u(\varphi(x)) &= \int_D G_D(\varphi(x),z)\Delta u(z)dz \\ &= \int_A G_D(\varphi(x),\varphi(y))(\Delta u)(\varphi(y))J_\varphi(y)dy. \end{aligned} \tag{6.2.3}$$

The claim follows from (6.2.2) and (6.2.3). □

In order to prove Theorem 6.1.1 in the two-dimensional case we will use the maximum principle. This result will be repeatedly applied to differential operators of which the coefficients become singular on the boundary. We prefer to give the precise formulation of a maximum principle which is appropriate for this situation. For a proof we refer to [36, Sect. 3.1].

Theorem 6.2.4. *Suppose that $\Omega \subset \mathbb{R}^n$ is open, bounded and connected, and that $b \in C(\Omega;\mathbb{R}^n)$ and $c \in C(\Omega;\mathbb{R})$ with $c \geq 0$. Set $L = -\Delta + b \cdot \nabla + c$. If $u \in C^2(\Omega)$ satisfies*

$$\begin{cases} Lu(x) \geq 0 & \text{for } x \in \Omega, \\ \liminf\limits_{\Omega \ni x \to x_\partial} u(x) \geq 0 & \text{for } x_\partial \in \partial\Omega, \end{cases}$$

then $u \geq 0$ in Ω.

6.3 In dimension two

In 2 dimensions the direct relation between conformal maps and Green functions is best exploited using $\mathbb{C}$ instead of $\mathbb{R}^2$. For the sake of clear notation we will use boldface for this complex alternative:

$$\begin{aligned} \text{for } x \in \mathbb{R}^2 \quad &\text{set} \quad \mathbf{x} = x_1 + ix_2, \\ \text{for } h : \mathbb{R}^2 \to \mathbb{R}^2 \quad &\text{set} \quad \mathbf{h}(\mathbf{x}) = h_1(x) + ih_2(x). \end{aligned}$$

The explicit expressions of the Green function and of the Poisson kernel in the disk can now be written as

$$\begin{aligned} G_B(x,y) &= \frac{1}{4\pi} \log\left(\frac{|\bar{\mathbf{y}}\mathbf{x} - 1|^2}{|\mathbf{x} - \mathbf{y}|^2}\right), \text{ where } x, y \in B, \\ K_B(x,y) &= \frac{1}{2\pi} \frac{1 - |\mathbf{y}|^2}{|\mathbf{x} - \mathbf{y}|^2}, \text{ where } x \in \partial B, y \in B. \end{aligned}$$

We first give a brief scheme of the proofs. In subsection 6.3.1 we will consider the case where one of the points lies on the boundary. As mentioned before the case with one point at the boundary has been previously studied by Griffin, McConnell and Verchota in [37]. We will need a more precise characterization of H and in doing so we will recover some of their results. Instead of using power series in $\mathbb{C}$ our basic tools will be conformal mappings, a monotonicity result for a convolution (see Proposition 6.3.1) and the maximum principle.

Since the function under consideration is symmetric, $H(x,y) = H(y,x)$, the behaviour of $x \in B \mapsto H(x,y)$ with $y \in \partial B$ can be used for the behaviour of $x \in \partial B \mapsto H(x,y)$ with $y \in B$. Using such a result on the boundary and by several applications of the maximum principle one is able to transfer a inequality valid on the boundary to the interior. This is done in section 6.3.2 and will lead to our main result. We would like to observe that also because of this use of the symmetries of the ball our result is restricted to the ball.

Most of the steps consist of deriving estimates for some tailor-made functions. Since all these technicalities might blur the line of arguments we hope to clarify our approach by complementing each intermediate result for a increasing direction of $x \mapsto H(x,y)$ (or a related function) by a sketch.

6.3.1 The proof for one point lying on the boundary

In two-dimensions for y fixed at the boundary part (i) of Theorem 6.1.1 has been proven in [37]. We now prove part (ii). In order to do that we will consider a transformation to the half-plane. We first give an explicit expression for $H(x,y)$ for y is fixed at the boundary.

Assuming $y \in \partial B$ we may suppose without loss of generality that $y = e_1 = (1,0)$. The numerator $\int_B K_B(e_1,z)\, G_B(z,x)\, dz$ equals:

$$E(x) := -\frac{1-\mathbf{x}\bar{\mathbf{x}}}{8\pi}\left(\frac{\log(1-\mathbf{x})}{\mathbf{x}} + \frac{\log(1-\bar{\mathbf{x}})}{\bar{\mathbf{x}}} + 1\right) \text{ for } x \in \bar{B}\backslash\{e_1\} \text{ and } E(e_1) = 0.$$

Indeed, since $z \mapsto K_B(e_1,z) \in L^p(B)$ for $p \in [1,2)$ (see Chapter 2) the Dirichlet problem for the Poisson equation $-\Delta u = K_B(e_1,\cdot)$ in B with $u = 0$ on ∂B has a unique solution in $W^{2,p}(B) \cap W_0^{1,p}(B)$ by [36, Theorem 9.15]. Since G_B is the kernel for the solution operator from $L^p(B)$ to $W^{2,p}(B) \cap W_0^{1,p}(B)$ this Dirichlet problem is solved by $u(x) = \int_B K_B(e_1,z)\, G_B(z,x)\, dz$.

Next one checks straightforwardly that E lies in $W^{2,p}(B) \cap C_0\left(\bar{B}\right)$ for $p \in [1,2)$ and by [9, Theorem IX.17] it follows that $E \in W^{2,p}(B) \cap W_0^{1,p}(B)$. Since $-\Delta E = -4\frac{\partial}{\partial \mathbf{x}}\frac{\partial}{\partial \bar{\mathbf{x}}}E = K_B(e_1,\cdot)$ in B one finds $E = u$, the unique solution. The expression for E can also be deduced from an explicit formula for $\int_B G_B(x,z)G_B(z,y)dz$ with $x,y \in B$, which is given in [61].

Dividing $E(x)$ by $K_B(e_1,x)$ yields:

$$H(x,e_1) = -\frac{(1-\mathbf{x})(1-\bar{\mathbf{x}})}{4}\left(\frac{\log(1-\mathbf{x})}{\mathbf{x}} + \frac{\log(1-\bar{\mathbf{x}})}{\bar{\mathbf{x}}} + 1\right), \tag{6.3.1}$$

for $x \in \bar{B}\backslash\{e_1\}$ and by continuity $H(e_1,e_1) = 0$. We remark that log denotes the analytic extension of the standard logarithm to $\mathbb{C}\backslash(-\infty,0]$ and that the function $\mathbf{x} \mapsto \frac{\log(1-\mathbf{x})}{\mathbf{x}}$ is extended by -1 for $\mathbf{x} = 0$.

In the halfplane

We consider the conformal map from the ball B onto the halfplane $\mathbb{R}^+ \times \mathbb{R}$ that maps $(-1,0)$ to $(0,0)$ and $(0,0)$ to $(1,0)$. This map is given by $\mathbf{h}(\mathbf{x}) = \frac{1+\mathbf{x}}{1-\mathbf{x}}$. Note that $h(e_1) = \infty$. We let X denote an element of $\mathbb{R}^+ \times \mathbb{R}$, or in complex notation $\mathbf{X} = \mathbf{X}_1 + i\mathbf{X}_2 \in \mathbb{R}^+ + i\mathbb{R}$. The inverse of $\mathbf{h}$ is also a conformal map and is defined by $\mathbf{h}^{-1}(\mathbf{X}) = \frac{\mathbf{X}-1}{\mathbf{X}+1}$.

It follows from a property of conformal maps (see Lemma 6.2.1) that

$$\begin{aligned} H(x,e_1) &= \int_{\mathbb{R}^+\times\mathbb{R}} \frac{K_B(e_1,h^{-1}(Z))}{K_B(e_1,x)} G_B\left(h^{-1}(Z),x\right) \left|\left(\mathbf{h}^{-1}\right)'(Z_1+iZ_2)\right|^2 dZ_1 dZ_2 \\ &= \int_{\mathbb{R}^+\times\mathbb{R}} \frac{K_B(e_1,h^{-1}(Z))}{K_B(e_1,x)} G_{\mathbb{R}^+\times\mathbb{R}}(Z,h(x)) \left|\left(\mathbf{h}^{-1}\right)'(Z_1+iZ_2)\right|^2 dZ_1 dZ_2, \end{aligned}$$

where $G_{\mathbb{R}^+\times\mathbb{R}}(X,Y) = \frac{1}{4\pi}\log\left(1+\dfrac{4X_1Y_1}{|X-Y|^2}\right)$ is the Green function for the Laplace problem in $\mathbb{R}^+ \times \mathbb{R}$.

Next, by defining the function

$$\tilde{H}(X) := H(x,e_1) \text{ for } X = h(x),$$

one finds

$$\tilde{H}(X) = \frac{1}{4\pi}\int_{\mathbb{R}^+\times\mathbb{R}} \frac{Z_1}{X_1}\log\left(1+\frac{4X_1Z_1}{|X-Z|^2}\right)\frac{4}{\left((1+Z_1)^2+Z_2^2\right)^2} dZ_1 dZ_2.$$

Here we use that $K_B(e_1,x) = K_B(e_1,h^{-1}(X)) = \frac{1}{2\pi}X_1$ and that $\left|\left(\mathbf{h}^{-1}\right)'(Z_1+iZ_2)\right|^2 = 4\left((1+Z_1)^2+Z_2^2\right)^{-2}$.

We want to show that $H(x,e_1)$ is increasing along trajectories orthogonal to the hyperbolic geodesic in B through e_1. This is equivalent to show that

$$X_2 \longmapsto \tilde{H}(X_1,X_2) \text{ is decreasing for } X_2 > 0. \tag{6.3.2}$$

Indeed the image through the map $\mathbf{h}^{-1}$ of the hyperbolic geodesic through e_1 are the lines in the half-plane of the kind $\{(X_1,X_2) \in \mathbb{R}^+ \times \mathbb{R} : X_2 = k\}$ for $k \in \mathbb{R}$. Hence the orthogonal trajectories are the lines $\{(X_1,X_2) \in \mathbb{R}^+ \times \mathbb{R} : X_1 = k\}$ for $k \in \mathbb{R}^+$.

In order to show (6.3.2) we need:

Proposition 6.3.1. *Let* $f, g \in L^2(\mathbb{R})$, $f, g \geq 0$, $f(t) = f(|t|)$, $g(t) = g(|t|)$ *and* f, g *decreasing for* $t > 0$. *Then*

$$t \mapsto \int_{\mathbb{R}} f(x)\, g(x+t)\, dx, \tag{6.3.3}$$

is decreasing on $\mathbb{R}^+$.

Proof. We suppose first that additionally $g \in \mathcal{C}_0^\infty(\mathbb{R})$. One has

$$\frac{\partial}{\partial t} \int_{\mathbb{R}} f(x)\, g(x+t)\, dx = \int_{-\infty}^{+\infty} f(x)\, g'(x+t)\, dx$$

$$= \int_{-\infty}^{-t} f(x)\, g'(x+t)\, dx + \int_{-t}^{+\infty} f(x)\, g'(x+t)\, dx.$$

Using that $g'(x+t) = -g'(-x-t)$, one gets

$$\frac{\partial}{\partial t} \int_{\mathbb{R}} f(x)\, g(x+t)\, dx = -\int_{-\infty}^{-t} f(x)\, g'(-x-t)\, dx + \int_{-t}^{+\infty} f(x)\, g'(x+t)\, dx.$$

Changing the coordinates one obtains

$$\frac{\partial}{\partial t} \int_{\mathbb{R}} f(x)\, g(x+t)\, dx = \int_{+\infty}^{0} f(-y-t)\, g'(y)\, dy + \int_{0}^{+\infty} f(y-t)\, g'(y)\, dy$$

$$= \int_{0}^{+\infty} g'(y)\, (f(y-t) - f(-y-t))\, dy.$$

Now for $t > 0$, one has $|y - t| < |-y - t|$. Hence the function (6.3.3) is decreasing.

The preceding arguments yields the result also for g as in the hypothesis. We observe that such g may be approximated in $L^2(\mathbb{R})$ by $(g_k)_{k \in \mathbb{N}} \subset \mathcal{C}_0^\infty(\mathbb{R})$ having the additional properties above. This is achieved by using an even and in positive x-direction decreasing mollifier in $\mathcal{C}_0^\infty(\mathbb{R})$. □

Corollary 6.3.2. *The relations*

$$\max_{X_2 \in \mathbb{R}} \tilde{H}(X_1, X_2) = \tilde{H}(X_1, 0) \ \textit{ and } X_2 \frac{\partial}{\partial X_2} \tilde{H}(X_1, X_2) \leq 0,$$

hold for every $X_1 \in [0, +\infty)$.

Proof. For every $X_1 \in \mathbb{R}^+$, one has

$$\tilde{H}(X) = \frac{1}{\pi} \int_{\mathbb{R}^+} \frac{Z_1}{X_1} \int_{\mathbb{R}} \log\left(1 + \frac{4X_1 Z_1}{|X - Z|^2}\right) \frac{1}{\left((1+Z_1)^2 + Z_2^2\right)^2} dZ_2 dZ_1.$$

Hence defining

$$
\begin{aligned}
f\left(Z_2\right) &= \log\left(1+\frac{4X_1Z_1}{\left(X_1-Z_1\right)^2+Z_2^2}\right), \\
g\left(Z_2\right) &= \frac{1}{\left(\left(1+Z_1\right)^2+Z_2^2\right)^2},
\end{aligned}
$$

we can write

$$
\int_{\mathbb{R}} \log\left(1+\frac{4X_1Z_1}{|X-Z|^2}\right)\frac{1}{\left(\left(1+Z_1\right)^2+Z_2^2\right)^2}dZ_2 = \int_{\mathbb{R}} f\left(Z_2-X_2\right)g\left(Z_2\right)dZ_2.
$$

Applying Proposition 6.3.1 one gets that the function $\tilde{H}$ is decreasing for X_2 positive and increasing for X_2 negative for every $X_1 \in \mathbb{R}^+$. The claim follows using the regularity of the function. The case $X_1 = 0$ goes similarly by proceeding to the limit. □

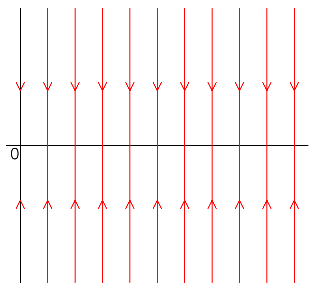

Figure 6.3: *Illustration of Corollary 6.3.2; the arrows denote some increasing directions of* $X \mapsto \tilde{H}(X)$.

Back in the disk

Using the properties of conformal mapping, see [15, Sect. III.3], from the increasing direction of $\tilde{H}$ we get an increasing direction of $H\left(x,e_1\right)$. The lines $\mathbf{h}^{-1}\left(\left\{\mathbf{X}_1 = k_1\right\}\right)$, varying k_1 in $\mathbb{R}^+$, are circles inside the disk which are tangent to ∂B in $(1,0)$ and that are orthogonal to the hyperbolic geodesic in B through e_1. Hence, we have for every (x_1,x_2) that the function H is increasing in the direction

$$
v_{(x_1,x_2)} = \left(-x_2, \frac{2x_1-x_1^2-1+x_2^2}{2\left(1-x_1\right)}\right), \text{ if } x_2 > 0, \tag{6.3.4}
$$

and in the $-v_{(x_1,x_2)}$–direction, if $x_2 < 0$. In particular we obtain that

$$
x_2\frac{\partial}{\partial\theta}H\left(x,e_1\right) := x_2\left(-x_2\frac{\partial}{\partial x_1}+x_1\frac{\partial}{\partial x_2}\right)H\left(x,e_1\right) \geq 0 \text{ when } |x| = 1. \tag{6.3.5}
$$

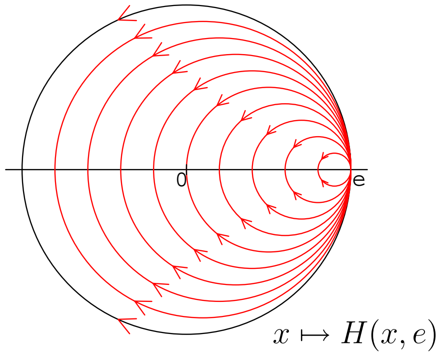

Figure 6.4: *The result of Corollary 6.3.2 transformed back to the disk; arrows denote increasing directions of* $x \mapsto H(x,e)$.

Here we write $x_1 = |x| \cos\theta$ and $x_2 = |x| \sin\theta$.

Collecting together the result of [37] with what just shown Theorem 6.1.1 follows for y fixed at the boundary in the two-dimensional case. Before proceeding with the study of the function H with both points in the interior we will prove that $x_2 \frac{\partial}{\partial\theta} H(x, e_1) \geq 0$ holds in B. This result will be used in the next step of the proof.

Since we will proceed through properties of the differential equation for H let us fix the following formula.

Lemma 6.3.3. *For* $a, b \in C^2$ *with* $b \neq 0$ *the following identity holds*

$$-\Delta\left(\frac{a}{b}\right) - 2\frac{\nabla b}{b} \cdot \nabla\left(\frac{a}{b}\right) + \frac{-\Delta b}{b}\left(\frac{a}{b}\right) = \frac{-\Delta a}{b}.$$

Having $e_1 \in \partial B$ one finds $-\Delta K_B(x, e_1) = 0$ and that

$$-\Delta\left(\int_B G_B(x,z) K_B(z, e_1) dz\right) = K_B(x, e_1) \text{ in } B.$$

These equations together with Lemma 6.3.3 give that the function H satisfies:

$$-\Delta H(x, e_1) - 2\frac{\nabla K_B(x, e_1)}{K_B(x, e_1)} \cdot \nabla H(x, e_1) = 1 \text{ when } x \in B.$$

Let us consider the derivative with respect to the angle $\frac{\partial}{\partial\theta} H$. We first observe that since $\frac{\partial}{\partial\theta} = x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}$, for every f sufficiently regular it holds

$$\nabla \frac{\partial}{\partial\theta} f = \nabla\left(\left(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}\right) f\right) = \left(\mathcal{R} + \frac{\partial}{\partial\theta}\right) \nabla f,$$

with $\mathcal{R} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Since $\frac{\partial}{\partial\theta}$ and Δ commute and since $\mathcal{R}$ is skew-symmetric, one obtains that $\frac{\partial}{\partial\theta} H(x, e_1)$ satisfies the partial differential equation

$$-\Delta \frac{\partial}{\partial\theta} H - 2\nabla \log(K_B) \cdot \nabla \frac{\partial}{\partial\theta} H = -\frac{\partial}{\partial\theta} \Delta H - 2\frac{\nabla K_B}{K_B} \cdot \left(\mathcal{R} + \frac{\partial}{\partial\theta}\right) \nabla H =$$

$$
\begin{aligned}
&= \frac{\partial}{\partial\theta}\left(-\Delta H - 2\frac{\nabla K_B}{K_B}\cdot\nabla H\right) + 2\left(\frac{\partial}{\partial\theta}\frac{\nabla K_B}{K_B}\right)\cdot\nabla H - 2\frac{\nabla K_B}{K_B}\cdot\mathcal{R}\nabla H \\
&= 0 + 2\left(\left(\frac{\partial}{\partial\theta}+\mathcal{R}\right)\nabla\log K_B\right)\cdot\nabla H \\
&= 2\left(\nabla\frac{\partial}{\partial\theta}\log K_B\right)\cdot\nabla H.
\end{aligned}
$$

We now look at which boundary conditions $\frac{\partial}{\partial\theta}H(x,e_1)$ satisfies in B^+ where

$$B^+ := \{x\in B : x_2>0\}.$$

By the symmetry one observes that $\frac{\partial}{\partial\theta}H(x,e_1)=0$ in $\{x\in B : x_2=0\}$. Furthermore it follows from (6.3.5) that $\frac{\partial}{\partial\theta}H(x,e_1)\geq 0$ in $\{x\in\partial B : x_2>0\}$. A priori one knows that $x\mapsto\frac{\partial}{\partial\theta}H(x,e_1)$ is in $\mathcal{C}^2(B^+)\cap\mathcal{C}(\bar{B}^+\backslash\{e_1\})$ and only the behavior near e_1 remains to be studied. Using the explicit formula of $H(x,e_1)$ given by (6.3.1), we will prove the following:

Lemma 6.3.4. *The function $H(x,e_1)$ satisfies* $\lim\limits_{\substack{x\to e_1,\\ x\in B^+}}\frac{\partial}{\partial\theta}H(x,e_1)=0.$

Proof. Since $\frac{\partial}{\partial\theta}=i\left(\mathbf{x}\frac{\partial}{\partial\mathbf{x}}-\bar{\mathbf{x}}\frac{\partial}{\partial\bar{\mathbf{x}}}\right)$, one gets

$$
\begin{aligned}
\frac{\partial}{\partial\theta}H(x,e_1) &= i\frac{(1-\bar{\mathbf{x}})}{4}\left(\log(1-\mathbf{x})+\frac{\mathbf{x}}{\bar{\mathbf{x}}}\log(1-\bar{\mathbf{x}})+\mathbf{x}\right)\\
&\quad -i\frac{(1-\mathbf{x})(1-\bar{\mathbf{x}})}{4}\left(-\frac{\log(1-\mathbf{x})}{\mathbf{x}}-\frac{1}{1-\mathbf{x}}\right)\\
&\quad -i\frac{(1-\mathbf{x})}{4}\left(\frac{\bar{\mathbf{x}}}{\mathbf{x}}\log(1-\mathbf{x})+\log(1-\bar{\mathbf{x}})+\bar{\mathbf{x}}\right)\\
&\quad +i\frac{(1-\mathbf{x})(1-\bar{\mathbf{x}})}{4}\left(-\frac{\log(1-\bar{\mathbf{x}})}{\bar{\mathbf{x}}}-\frac{1}{1-\bar{\mathbf{x}}}\right)
\end{aligned}
$$

$$= i\frac{\log(1-\mathbf{x})}{4\mathbf{x}}(1-2\bar{\mathbf{x}}+\mathbf{x}\bar{\mathbf{x}}) - i\frac{\log(1-\bar{\mathbf{x}})}{4\bar{\mathbf{x}}}(1-2\mathbf{x}+\mathbf{x}\bar{\mathbf{x}}) - i\frac{\bar{\mathbf{x}}-\mathbf{x}}{2}.$$

One observes that

$$\lim_{\substack{x\to e_1,\\ x\in B^+}}\frac{\partial}{\partial\theta}H(x,e_1)=0.$$

□

Hence $\frac{\partial}{\partial\theta}H(\cdot,e_1)\in\mathcal{C}^2(B)\cap\mathcal{C}(\bar{B})$ and $\frac{\partial}{\partial\theta}H(\cdot,e_1)$ satisfies the following boundary value problem

$$
\begin{cases}
-\Delta\frac{\partial}{\partial\theta}H - 2\frac{\nabla K_B}{K_B}\cdot\nabla\frac{\partial}{\partial\theta}H = 2\nabla\frac{\partial}{\partial\theta}\log K_B\cdot\nabla H & \text{in } B^+,\\
\frac{\partial}{\partial\theta}H\geq 0 & \text{on } \partial B^+.
\end{cases}
\tag{6.3.6}
$$

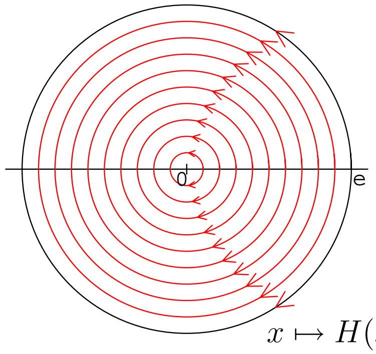

Figure 6.5: *For $y = e_1 \in \partial B$ the function $x \mapsto H(x, y)$ is increasing along semicircles to the left.*

Proposition 6.3.5. *The inequality $x_2 \dfrac{\partial}{\partial \theta} H(x, e_1) \geq 0$ holds for all $x \in B$.*

Proof. Since $K_B(x, e_1) = \dfrac{1 - |x|^2}{|x - e_1|^2}$ we have

$$\begin{aligned} \frac{\partial}{\partial \theta} \log K_B &= -\frac{\frac{\partial}{\partial \theta} |x - e_1|^2}{|x - e_1|^2} = -\frac{\left(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}\right)\left((x_1 - 1)^2 + x_2^2\right)}{|x - e_1|^2} \\ &= 2\frac{x_2 (x_1 - 1) - x_1 x_2}{|x - e_1|^2} = \frac{-2x_2}{|x - e_1|^2}, \end{aligned}$$

and

$$\begin{aligned} \nabla \frac{\partial}{\partial \theta} \log K_B &= \nabla \left(\frac{-2x_2}{|x - e_1|^2} \right) = \frac{-2}{|x - e_1|^2}(0, 1) + \frac{2x_2}{|x - e_1|^4}(2(x_1 - 1), 2x_2) \\ &= \frac{2}{|x - e_1|^4}\left(2x_2 (x_1 - 1), 2x_2^2 - |x - e_1|^2\right) \\ &= 2\frac{\left(2x_2 (x_1 - 1), x_2^2 - (x_1 - 1)^2\right)}{|x - e_1|^4} \\ &= \frac{4(1 - x_1)}{|x - e_1|^4}\left(-x_2, \frac{2x_1 - x_1^2 - 1 + x_2^2}{2(1 - x_1)}\right). \end{aligned}$$

We see that $\nabla \frac{\partial}{\partial \theta} \log K_B (x_1, x_2)$ has the direction of $v_{(x_1, x_2)}$ as defined in (6.3.4) . Hence the term

$$\nabla \frac{\partial}{\partial \theta} \log K_B \cdot \nabla H,$$

is non-negative. Applying Theorem 6.2.4 to (6.3.6) the claim follows. □

6.3.2 The proof for both points in the interior

We consider now y in the interior. Without loss of generality we may suppose that $y = (-s, 0)$ with $s \in (0, 1)$. The case $s = 0$ gives the radial symmetric case which has been considered previously by Caristi and Mitidieri in [19].

In order to prove Theorem 6.1.1 in this case it is convenient to consider a conformal transformation $\mathbf{k}_s$ from the disk onto the disk that maps y into 0:

$$\mathbf{k}_s(\mathbf{x}) = \frac{\mathbf{x} + s}{1 + s\mathbf{x}}.$$

Proceeding as before we will now study the function

$$H^s(x) := H\left(\mathbf{k}_s^{-1}(\mathbf{x}), \mathbf{k}_s^{-1}(\mathbf{0})\right),$$

which due to the behaviour of conformal mappings transforms into

$$H^s(x) = \int_B \frac{G_B(x,z)\, G_B(z,0)}{G_B(x,0)} \left|\left(\mathbf{k}_s^{-1}\right)'(z_1 + iz_2)\right|^2 dz_1 dz_2.$$

We have $\mathbf{k}_s^{-1}(\mathbf{z}) = \frac{\mathbf{z}-s}{1-s\mathbf{z}}$ and $\left|\left(\mathbf{k}_s^{-1}\right)'(z_1 + iz_2)\right|^2 = \frac{\left(1-s^2\right)^2}{|e_1 - sz|^4}$, hence

$$H^s(x) = \int_B \frac{G_B(x,z)\, G_B(z,0)}{G_B(x,0)} \frac{\left(1-s^2\right)^2}{|e - sz|^4} dz_1 dz_2.$$

Once again using Lemma 6.3.3 and since $x \cdot \nabla = r\frac{\partial}{\partial r}$ one finds that the function H^s satisfies for $x \neq 0$

$$-\Delta H^s(x) + \frac{2}{r\,|\log r|}\frac{\partial}{\partial r} H^s(x) = \frac{\left(1-s^2\right)^2}{|e_1 - sx|^4}. \tag{6.3.7}$$

The fact that the function $H(., y)$ is increasing along the hyperbolic geodesic through y is equivalent with the fact that the function $H^s(x)$ is radially increasing. On the other hand, increasing along trajectories orthogonal to the hyperbolic geodesic for the function $H(., y)$ means that the function $H^s(x)$ is tangentially decreasing. Thanks to this observation in order to prove Theorem 6.1.1 for both points in the interior it is sufficient to prove that $H^s(x)$ is tangentially decreasing and radially increasing.

Tangential directions

Let us first fix x at the boundary and consider $H(x, y)$. Let $C_s = \{y : |y| = s\}$. From the previous section it follows that the maximum of $H(x, \cdot)$ in C_s is attained in $y = -sx$. This is equivalent to ask for $y = (-s, 0)$ that $x = (1, 0)$. So using the symmetry of the problem we can say that

$$x_2 \frac{\partial}{\partial \theta} H(x, (-s, 0)) \leq 0 \text{ when } x \in \partial B,$$

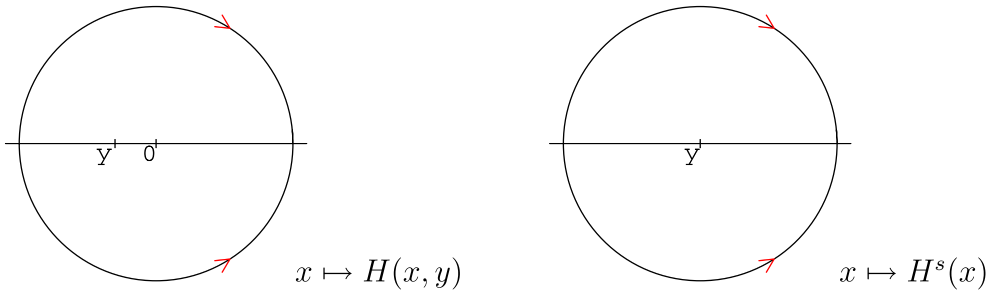

Figure 6.6: *Using the symmetry between x and y we may conclude that for any $y \in B$, the function $x \mapsto H(x,y)$ (left) is increasing along ∂B from the nearest boundary point of y to the most distant boundary point. Putting $y = (-s, 0)$ with $s > 0$ it means increasing to the right along ∂B. Also the function $x \mapsto H^s(x)$ (right) is increasing to the right along ∂B.*

or equivalently

$$x_2 \frac{\partial}{\partial \theta} H^s(x) \leq 0 \text{ when } x \in \partial B. \tag{6.3.8}$$

Proposition 6.3.6. *The inequality $x_2 \frac{\partial}{\partial \theta} H^s(x) \leq 0$ holds for all $x \in B$.*

Proof. By symmetry one may assume $x \in B^+$. We consider the function $\Theta(x) := \frac{\partial}{\partial \theta} H^s(x)$ or to be more specific

$$\Theta(x) = x_1 \frac{\partial}{\partial x_2} H^s(x) - x_2 \frac{\partial}{\partial x_1} H^s(x).$$

Since Δ and $\frac{\partial}{\partial \theta}$ commute, one finds

$$\begin{aligned} -\Delta\Theta(x) &= -\frac{\partial}{\partial \theta} \Delta H^s(x) = \frac{\partial}{\partial \theta}\left[-\frac{2}{r|\log r|}\frac{\partial}{\partial r} H^s(x) + \frac{\left(1-s^2\right)^2}{|e-sx|^4}\right] \\ &= -\frac{2}{r|\log r|}\frac{\partial}{\partial r}\Theta(x) - 4\frac{\left(1-s^2\right)^2}{|e-sx|^6} s x_2. \end{aligned}$$

A priori $\Theta \in \mathcal{C}^2(\bar{B} \setminus \{0\})$ holds and only the behavior of Θ in 0 remains to be studied. We have

$$\frac{\partial}{\partial \theta} H^s(x) = \frac{1}{G_B(x,0)} \frac{\partial}{\partial \theta} R(x),$$

where $R(x)$ satisfies

$$\begin{cases} -\Delta R(x) = -\frac{\left(1-s^2\right)^2}{4\pi|e-sx|^4} \log|x|^2 & \text{in } B, \\ R(x) = 0 & \text{on } \partial B. \end{cases} \tag{6.3.9}$$

Since the right hand side of (6.3.9) is in $L^p(B)$ for every $p \in (1, +\infty)$, one gets $R \in W^{2,p}(B)$ and hence, using the Sobolev imbedding theorem it follows that

$$R \in \mathcal{C}^{1,\alpha}(\bar{B}) \text{ for every } \alpha \in (0,1). \tag{6.3.10}$$

Setting $\Omega = B_{\frac{1}{2}}(0)$, we have $\frac{\partial}{\partial\theta}R$ and $G_B^{-1}(\cdot,0) \in \mathcal{C}(\bar{\Omega})$ (where we extend $G_B^{-1}(\cdot,0)$ in 0 by 0). Hence $\Theta \in \mathcal{C}^2(B^+) \cap \mathcal{C}^0(\bar{B}^+)$.

Using (6.3.8) and the fact that H^s is symmetric in $x_2 = 0$, we find that $\Theta(x) \leq 0$ on ∂B^+. We may summarize:

$$\begin{cases} -\Delta\Theta(x) + \dfrac{2}{r|\log r|}\dfrac{\partial}{\partial r}\Theta(x) = -4\dfrac{(1-s^2)^2}{|e-sx|^6}sx_2 & \text{in } B^+, \\ \Theta(x) \leq 0 & \text{on } \partial B^+. \end{cases}$$

The claim follows applying the maximum principle, see Theorem 6.2.4. $\square$

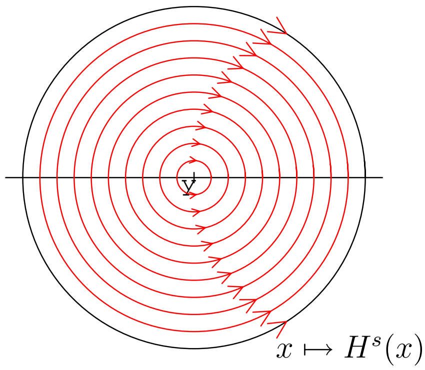

Figure 6.7: *A conformal mapping changed $H(x,y)$ to $H^s(x)$ and put y in the center. By Proposition 6.3.6 the mapping $x \mapsto H^s(x)$ is increasing to the right along all semicircles around* 0*.*

The fact that the function $H(.,y)$ is increasing along trajectories orthogonal to the hyperbolic geodesic through y in increasing Euclidean distance from y follows directly from Proposition 6.3.6

Radial directions

In order to show that the maximum of $H^s(x)$ is attained at the boundary it would be sufficient to show that the function $H^s(x_1, 0)$ is increasing on the interval $(0,1)$. As already said, we will prove something more, that is, the function H^s is increasing in radial direction. The radial directions are the images through the mapping $\mathbf{k}_s$ of the hyperbolic geodesics through y. Theorem 6.1.1 will directly follow from this result and Proposition 6.3.6.

The major tool of the proof is the maximum principle. First we will show that the function H^s satisfies a zero Neumann boundary condition:

Lemma 6.3.7. *The identity $\frac{\partial}{\partial r} H^s(x) = 0$ holds for all $x \in \partial B$.*

Proof. We write

$$H^s(x) = \frac{R(x)}{G_B(x,0)},$$

with $R(x) = \int_B G_B(x,z)\, G_B(z,0) \frac{\left(1-s^2\right)^2}{|e-sz|^4} dz_1 dz_2$ and observe that $R(x) = G_B(x,0) = 0$ for $x \in \partial B$. Moreover

$$-\Delta R(x) = G_B(x,0) \frac{\left(1-s^2\right)^2}{|e-sx|^4} \text{ and } -\Delta G_B(x,0) = 0 \text{ for } x \neq 0,\ x \in B.$$

Since $-\Delta = -\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial^2 \phi}$, we find that at the boundary

$$\begin{aligned} -\frac{\partial^2}{\partial r^2} R(x) &= \frac{\partial}{\partial r} R(x), \\ -\frac{\partial^2}{\partial r^2} G_B(x,0) &= \frac{\partial}{\partial r} G_B(x,0). \end{aligned} \tag{6.3.11}$$

Using the series expansion near the boundary for $R(x)$ and $G_B(x,0)$, we get for $x \in \partial B$:

$$\begin{aligned} \lim_{B \ni \xi \to x} \frac{\partial}{\partial r} H^s(\xi) &= \lim_{B \ni \xi \to x} \frac{\frac{\partial}{\partial r} G_B(\xi,0)}{G_B(\xi,0)} \left(\frac{\frac{\partial}{\partial r} R(\xi)}{\frac{\partial}{\partial r} G_B(\xi,0)} - \frac{R(\xi)}{G_B(\xi,0)} \right) \\ &= \lim_{B \ni \xi \to x} \frac{1}{|\xi|-1} \left(\frac{\frac{\partial}{\partial r} R(x) + (|\xi|-1)\frac{\partial^2}{\partial r^2} R(x) + ..}{\frac{\partial}{\partial r} G_B(x,0) + (|\xi|-1)\frac{\partial^2}{\partial r^2} G_B(x,0) + ..} - \frac{\frac{\partial}{\partial r} R(x) + \frac{|\xi|-1}{2}\frac{\partial^2}{\partial r^2} R(x) + ..}{\frac{\partial}{\partial r} G_B(x,0) + \frac{|\xi|-1}{2}\frac{\partial^2}{\partial r^2} G_B(x,0) + ..} \right) \\ &= \frac{1}{2} \frac{\frac{\partial^2}{\partial r^2} R(x) \frac{\partial}{\partial r} G_B(x,0) - \frac{\partial^2}{\partial r^2} G_B(x,0) \frac{\partial}{\partial r} R(x)}{\left(\frac{\partial}{\partial r} G_B(x,0)\right)^2}, \end{aligned}$$

which is zero by using (6.3.11). $\square$

Proposition 6.3.8. *The inequality $r\dfrac{\partial}{\partial r} H^s(x) \geq 0$ holds for all $x \in B$.*

Proof. By 6.3.7 we know that the function H^s satisfies

$$-\Delta H^s(x) = \frac{\left(1-s^2\right)^2}{|e-sx|^4} + \frac{4}{|x|^2 \left(\log |x|^2\right)} x \cdot \nabla H^s(x). \tag{6.3.12}$$

Let us define $\Xi(x) := r\frac{\partial}{\partial r} H^s(x) = x \cdot \nabla H^s(x)$. One has

$$-\Delta \Xi(x) = -2\Delta H^s(x) - x_1 \frac{\partial}{\partial x_1} \Delta H^s(x) - x_2 \frac{\partial}{\partial x_2} \Delta H^s(x) = \dots,$$

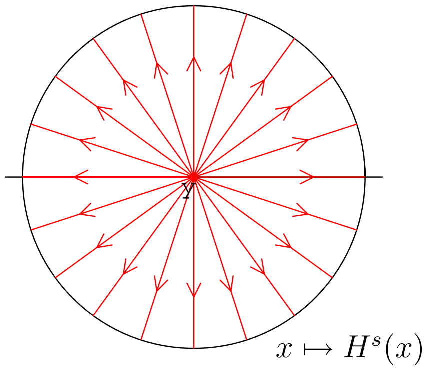

Figure 6.8: *A conformal mapping changed $H(x,y)$ to $H^s(x)$ and, roughly spoken, put y in the center. Here is the result from Proposition 6.3.8: the function $x \mapsto H^s(x)$ is radially increasing. The combination with Figure 6.7 and the inverse conformal mapping lead to the picture on the left of Figure 6.1.*

and by (6.3.12)

$$\begin{aligned} \ldots \quad = \quad & 2\frac{\left(1-s^2\right)^2}{|e-sx|^4} + \frac{8}{|x|^2\left(\log|x|^2\right)} x\cdot\nabla H^s(x) + \\ & +x\cdot\nabla\left(\frac{\left(1-s^2\right)^2}{|e-sx|^4} + \frac{4}{|x|^2\left(\log|x|^2\right)} x\cdot\nabla H^s(x)\right) \end{aligned}$$

$$\begin{aligned} = \quad & 2\frac{\left(1-s^2\right)^2}{|e-sx|^4} + \frac{8}{|x|^2\left(\log|x|^2\right)}\Xi(x) + 4sx_1\frac{\left(1-s^2\right)^2}{|e-sx|^6}(1-sx_1) \\ & + \frac{4x_1}{|x|^2\left(\log|x|^2\right)}\frac{\partial}{\partial x_1}\Xi(x) - \frac{8x_1^2}{|x|^4\left(\log|x|^2\right)}\Xi(x) - \frac{8x_1^2}{|x|^4\left(\log^2|x|^2\right)}\Xi(x) \\ & - 4s^2x_2^2\frac{\left(1-s^2\right)^2}{|e-sx|^6} + \frac{4x_2}{|x|^2\left(\log|x|^2\right)}\frac{\partial}{\partial x_2}\Xi(x) \\ & -\frac{8x_2^2}{|x|^4\log|x|^2}\Xi(x) - \frac{8x_2^2}{|x|^4\left(\log^2|x|^2\right)}\Xi(x), \end{aligned}$$

that gives

$$-\Delta\Xi(x) - \frac{4x\cdot\nabla\Xi(x)}{|x|^2\log|x|^2} + \frac{8\Xi(x)}{|x|^2\left(\log^2|x|^2\right)} = 2\frac{\left(1-s^2\right)^2}{|e-sx|^4}\left(\frac{1-s|x|^2}{|e-sx|^2}\right). \tag{6.3.13}$$

One sees that the right hand side of (6.3.13) is non-negative. Furthermore, since

$$\Xi(x) = \frac{r}{G_B(x,0)}\frac{\partial}{\partial r}R(x) - \frac{R(x)}{\left(G_B(x,0)\right)^2}\, r\frac{\partial}{\partial r}G_B(x,0),$$

with $R \in \mathcal{C}^{1,\alpha}(\bar{B})$ (from (6.3.10)), one has that $\Xi(0) = 0$ and that Ξ is continuous in B. With help of the preceding Lemma 6.3.7 we get that $\Xi \in \mathcal{C}_0(\bar{B})$. Hence, summarizing we have

$$\begin{cases} -\Delta\Xi(x) - \dfrac{4}{|x|^2 \log |x|^2} x \cdot \nabla\Xi(x) + \dfrac{8\Xi(x)}{|x|^2 \left(\log^2 |x|^2\right)} \geq 0 & \text{in } B \setminus \{0\}, \\ \Xi(x) = 0 & \text{on } \partial B \cup \{0\}. \end{cases}$$

The maximum principle stated in Theorem 6.2.4 finally yields $\Xi \geq 0$ in B. □

6.4 In dimensions larger than two

The explicit expression for the Green function in the unit ball in $\mathbb{R}^n$, $n \geq 3$, is for $x, y \in B$,

$$G_B(x,y) = \begin{cases} \frac{1}{n(n-2)\omega_n} \left(|x-y|^{2-n} - \left| x\,|y| - \frac{y}{|y|} \right|^{2-n} \right) & \text{for } y \neq 0, \\ \frac{1}{n(n-2)\omega_n} \left(|x|^{2-n} - 1 \right) & \text{for } y = 0, \end{cases}$$

where $\omega_n = \frac{2\pi^{\frac{n}{2}}}{n\Gamma(\frac{n}{2})}$ is the volume of B.
While the Poisson kernel is given by for $x \in \partial B$ and $y \in B$

$$K_B(x,y) := \frac{1}{n\omega_n} \frac{1 - |y|^2}{|x-y|^n}.$$

In this section we prove Theorem 6.1.1 for the ball in $\mathbb{R}^n$, $n \geq 3$. The method used for the proof is similar to the one used in the two-dimensional case (see [22]) but, to a certain extent, simpler. We look at the differential boundary value problem that the function satisfies and then apply the maximum principle. Compared with the two-dimensional case the proof here is somewhat simplified since, in some cases, we are able to determine the sign of the functions via a geometrical reasoning. In the present setting we have also to study the case $x \mapsto H_\Omega(x,y)$ for y fixed at the boundary since a result as the one in [37] is not available in dimensions $n \geq 3$.

In the following e_i for $i = 1, \ldots, n$ denotes the canonical basis for $\mathbb{R}^n$.

6.4.1 One point fixed at the boundary

In this section we study the function $x \mapsto H(x,y)$ with $y \in \partial B$. Without loss of generality, we can fix $y = e_1$.

Transformation to the half n-space

Instead of studying the problem in the ball it is convenient to consider a transformation to the half n-space. We consider a (anti-)conformal map φ from $S := \mathbb{R}^+ \times \mathbb{R}^{n-1}$, the half n-space, onto B that maps 0 into $-e_1$ and e_1 into 0 given by

$$\varphi(X_1, X_2, .., X_n) = e_1 - 2\frac{QX + e_1}{|X + e_1|^2}, \tag{6.4.1}$$

where $Q_{11} = 1$, $Q_{ii} = -1$ for $i = 2, \ldots, n$ and $Q_{ij} = 0$ for $i, j = 1, \ldots, n$ and $i \neq j$. The map φ is conformal if the dimension n is even, is anti-conformal if the dimension n is odd.

In the following, to avoid ambiguity in the notation, we denote with capital letters the coordinates on the half n-space.

Using the (anti-)conformal transformation φ, we can write

$$\begin{aligned} H(x, e_1) &= \int_B \frac{K_B(e_1, z)G_B(z, x)}{K_B(e_1, x)} dz \\ &= \int_S \frac{K_B(e_1, \varphi(Z))G_B(\varphi(Z), x)}{K_B(e_1, x)} J_\varphi(Z) dZ, \end{aligned}$$

where J_φ is the Jacobian of the transformation φ. We first compute this Jacobian in the following lemma.

Lemma 6.4.1. *Let φ be the (anti-)conformal map defined in (6.4.1). For any $n \geq 3$ it holds that*

$$J_\varphi(X) = \frac{2^n}{|X + e_1|^{2n}}.$$

Proof. By the definition of φ in (6.4.1) it follows

$$\frac{\partial}{\partial X_j}\varphi_i(X) = -2\frac{Qe_i\ \delta_i^j}{|X + e_1|^2} + 4\frac{(QX + e_1)_i}{|X + e_1|^4}(X + e_1)_j,$$

that gives

$$(\partial_j \varphi_i(x))_{i,j} = -\frac{2}{|X + e_1|^2} Q \left(Id - 2\frac{X + e_1}{|X + e_1|} \left(\frac{X + e_1}{|X + e_1|} \right)^T \right),$$

using column notation for $X + e_1$. The claim follows directly since the matrix $Id - 2\frac{X+e_1}{|X+e_1|}\left(\frac{X+e_1}{|X+e_1|}\right)^T$ defines the reflection in the hyperplane through 0 perpendicular to $X + e_1$. □

By the definition of the function φ and Lemma 6.2.2 we find

$$\begin{aligned} H(\varphi(X), e_1) &= \int_S \frac{K_B(e_1, \varphi(Z))G_B(\varphi(Z), \varphi(X))}{K_B(e_1, \varphi(X))} J_\varphi(Z) dZ \\ &= \int_S \frac{K_B(e_1, \varphi(Z))G_S(Z, X)}{K_B(e_1, \varphi(X))} (J_\varphi(Z)J_\varphi(X))^{\frac{1}{n}-\frac{1}{2}} J_\varphi(Z) dZ. \end{aligned} \tag{6.4.2}$$

One may write $K_B(e_1,\varphi(Z))$ in terms of $J_\varphi(Z)$. Indeed it holds

$$\begin{aligned}K_B(e_1,\varphi(Z)) &= \frac{1}{n\omega_n}\frac{1-|\varphi(Z)|^2}{|\varphi(Z)-e_1|^n} = \frac{1}{n\omega_n}\left(1-\left|e_1-2\frac{QZ+e_1}{|QZ+e_1|^2}\right|^2\right)\frac{|Z+e_1|^n}{2^n}\\ &= \frac{1}{n\omega_n}\frac{|Z+e_1|^{n-2}}{2^{n-2}}Z_1 = \frac{1}{n\omega_n}\frac{1}{2^{\frac{n}{2}-1}}J_\varphi(Z)^{\frac{1}{n}-\frac{1}{2}}Z_1.\end{aligned}$$

Hence from (6.4.2) it follows

$$\begin{aligned}H(\varphi(X),e_1) &= \int_S \frac{J_\varphi(Z)^{\frac{1}{n}-\frac{1}{2}}}{J_\varphi(X)^{\frac{1}{n}-\frac{1}{2}}}\frac{Z_1}{X_1}G_S(Z,X)(J_\varphi(Z)J_\varphi(X))^{\frac{1}{n}-\frac{1}{2}}J_\varphi(Z)dZ\\ &= \frac{1}{n(n-2)\omega_n}\int_S\frac{Z_1}{X_1}\left(|X-Z|^{2-n}-|X+QZ|^{2-n}\right)J_\varphi(Z)^{\frac{2}{n}}dZ.\end{aligned}$$

For simplicity of notation we define the function $\tilde{H}$ given by

$$\tilde{H}(X) := \tfrac{1}{n(n-2)\omega_n}\int_S\frac{Z_1}{X_1}\left(|X-Z|^{2-n}-|X+QZ|^{2-n}\right)J_\varphi(Z)^{\frac{2}{n}}dZ.$$

Increasing along the "hyperbolic geodesics" through e_1

In the following section we show that the function $x\mapsto H(x,e_1)$ is increasing along the "hyperbolic geodesics" through e_1. That's equivalent to prove that the function $\tilde{H}(X)$ is decreasing in the X_1 direction. Indeed, the pre-image through the mapping φ, defined in (6.4.1), of a general hyperbolic geodesic in B through e_1 is a straight line in S that intersect the hyperplane $\{X_1=0\}$ perpendicularly.

Let $\tilde{H}_{X_1}$ denote $\frac{\partial}{\partial X_1}\tilde{H}(X)$. We proceed studying the differential boundary value problem that $\tilde{H}_{X_1}$ satisfies in order to apply the maximum principle.

Since ∂S is composed of two parts, $\partial S=\{Z\in\mathbb{R}^n : Z_1=0\}\cup\{\infty\}$, we treat those separately. In the following $\{Z_1=0\}$ denotes the hyperplane $\{Z\in\mathbb{R}^n : Z_1=0\}$.

Lemma 6.4.2. *It holds that $\tilde{H}_{X_1}(X)=0$ for $X\in\{X_1=0\}$.*

Proof. Writing $\tilde{H}(X)=\frac{1}{X_1}\tilde{R}(X)$ with

$$\tilde{R}(X) := \tfrac{1}{n(n-2)\omega_n}\int_S Z_1\left(|X-Z|^{2-n}-|X+QZ|^{2-n}\right)J_\varphi(Z)^{\frac{2}{n}}dZ,$$

one finds

$$\tilde{H}_{X_1}(X)=\frac{1}{X_1}\left(\frac{\partial}{\partial X_1}\tilde{R}(X)-\frac{\tilde{R}(X)}{X_1}\right).$$

We first notice that since $\tilde{R}(X)=0$ for $X\in\{X_1=0\}$ and $-\Delta\tilde{R}(X)=X_1J_\varphi(X)^{\frac{2}{n}}$, one finds that $\frac{\partial^2}{\partial X_1^2}\tilde{R}(X)=0$ on $\{X_1=0\}$. Hence using the series expansion near

$X \in \{X_1 = 0\}$ one has

$$
\begin{aligned}
\lim_{S \ni Y \to X} \tilde{H}_{X_1}(Y) &= \lim_{S \ni Y \to X} \frac{1}{Y_1} \left(\frac{\partial}{\partial X_1} \tilde{R}(X) + Y_1 \frac{\partial^2}{\partial X_1^2} \tilde{R}(X) + .. \right. \\
&\qquad \left. \cdots - \frac{\partial}{\partial X_1} \tilde{R}(X) - \frac{1}{2} Y_1 \frac{\partial^2}{\partial X_1^2} \tilde{R}(X) + .. \right) \\
&= \frac{1}{2} \frac{\partial^2}{\partial X_1^2} \tilde{R}(X) = 0.
\end{aligned}
$$

The claim follows. $\square$

Lemma 6.4.3. *It holds that* $\lim_{|X| \to \infty} \tilde{H}_{X_1}(X) = 0$.

Proof. Since

$$
\begin{aligned}
\tilde{H}_{X_1}(X) &= -\tfrac{1}{n\omega_n} \int_S \frac{Z_1}{X_1} \left(\frac{X_1 - Z_1}{|X - Z|^n} - \frac{X_1 + Z_1}{|X + QZ|^n} \right) \frac{2^2}{|Z + e_1|^4} dZ \\
&\quad - \tfrac{1}{n(n-2)\omega_n} \int_S \frac{Z_1}{X_1^2} \left(|X - Z|^{2-n} - |X + QZ|^{2-n} \right) \frac{2^2}{|Z + e_1|^4} dZ,
\end{aligned}
$$

and it holds $|X - Z| < |X + QZ|$, one has

$$
\begin{aligned}
\left| \tilde{H}_{X_1}(X) \right| &\leq \tfrac{2^3}{n\omega_n} \int_S \frac{Z_1}{X_1} \frac{1}{|X - Z|^{n-1}} \frac{1}{|Z + e_1|^4} dZ \\
&\quad + \tfrac{2^3}{n(n-2)\omega_n} \int_S \frac{Z_1}{X_1^2} \frac{1}{|X - Z|^{n-2}} \frac{1}{|Z + e_1|^4} dZ \\
&\leq \tfrac{2^3}{n\omega_n} \frac{1}{X_1} \int_S \frac{1}{|X - Z|^{n-1}} \frac{1}{|Z + e_1|^3} dZ \\
&\quad + \tfrac{2^3}{n(n-2)\omega_n} \frac{1}{X_1^2} \int_S \frac{1}{|X - Z|^{n-2}} \frac{1}{|Z + e_1|^3} dZ.
\end{aligned} \tag{6.4.3}
$$

We now proceed studying separately the terms in the right hand side of (6.4.3). For the first term one finds

$$
\begin{aligned}
\int_S \frac{1}{|X - Z|^{n-1}} \frac{1}{|Z + e_1|^3} dZ &= \int_{S \cap B_{\frac{|X|}{2}}(X)} \frac{1}{|X - Z|^{n-1}} \frac{1}{|Z + e_1|^3} dZ + \\
+ \int_{\substack{S \setminus B_{\frac{|X|}{2}}(X), \\ |Z| < 2|X|}} \frac{1}{|X - Z|^{n-1}} \frac{1}{|Z + e_1|^3} dZ &+ \int_{\substack{S \setminus B_{\frac{|X|}{2}}(X), \\ |Z| > 2|X|}} \frac{1}{|X - Z|^{n-1}} \frac{1}{|Z + e_1|^3} dZ = \ldots.
\end{aligned}
$$

One observes that $|Z + e_1| > |Z| \geq \frac{|X|}{2}$ if $Z \in B_{\frac{|X|}{2}}(X)$. While if $Z \notin B_{\frac{|X|}{2}}(X)$ it holds

$|X-Z| > \frac{|X|}{2}$ and even more $|X-Z| > \frac{|Z|}{2}$ if also $|Z| > 2\,|X|$. Hence we get

$$\begin{aligned}
\dots \quad &\leq \frac{2^3}{|X|^3}\int_{S\cap B_{\frac{|X|}{2}}(X)} \frac{1}{|X-Z|^{n-1}}dZ + \frac{2^{n-1}}{|X|^{n-1}}\int_{S,|Z|<2|X|}\frac{1}{|Z+e_1|^3}dZ \\
&\quad +2^{n-1}\int_{S,|Z|>2|X|}\frac{1}{|Z|^{n-1}}\frac{1}{|Z+e_1|^3}dZ \\
&\leq \frac{C_1}{|X|^2} + \frac{2^{n-1}}{|X|^{n-1}}\int_{|Z|<2|X|}\frac{1}{|Z|^2}dZ + 2^{n-1}\int_{|Z|>2|X|}\frac{1}{|Z|^{n+2}}dZ \\
&\leq \frac{C_1}{|X|^2} + \frac{C_2}{|X|^{n-1}}|X|^{n-2} + \frac{C_3}{|X|^2},
\end{aligned}$$

that goes to zero when $|X|$ goes to infinity.

For the second term proceeding similarly one finds

$$\begin{aligned}
&\int_S \frac{1}{|X-Z|^{n-2}}\frac{1}{|Z+e_1|^3}dZ = \int_{S\cap B_{\frac{|X|}{2}}(X)}\frac{1}{|X-Z|^{n-2}}\frac{1}{|Z+e_1|^3}dZ + \\
&+\int_{\substack{S\setminus B_{\frac{|X|}{2}}(X),\\ |Z|<2|X|}}\frac{1}{|X-Z|^{n-2}}\frac{1}{|Z+e_1|^3}dZ + \int_{\substack{S\setminus B_{\frac{|X|}{2}}(X),\\ |Z|>2|X|}}\frac{1}{|X-Z|^{n-2}}\frac{1}{|Z+e_1|^3}dZ \\
&\leq \frac{2^3}{|X|^3}\int_{S\cap B_{\frac{|X|}{2}}(X)}\frac{1}{|X-Z|^{n-2}}dZ + \frac{2^{n-2}}{|X|^{n-2}}\int_{\substack{S\setminus B_{\frac{|X|}{2}}(X),\\ |Z|<2|X|}}\frac{1}{|Z+e_1|^3}dZ \\
&+2^{n-2}\int_{\substack{S\setminus B_{\frac{|X|}{2}}(X),\\ |Z|>2|X|}}\frac{1}{|Z|^{n-2}}\frac{1}{|Z+e_1|^3}dZ \leq \frac{C_1}{|X|}.
\end{aligned}$$

The claim follows. ☐

Proposition 6.4.4. *The function $\tilde{H}(X)$ is decreasing in the X_1 direction.*

Proof. Since it holds

$$-\Delta\tilde{H}(X) = J_\varphi(X)^{\frac{2}{n}} + \frac{2}{X_1}\frac{\partial}{\partial X_1}\tilde{H}(X),$$

one gets

$$-\Delta\tilde{H}_{X_1}(X) - \frac{2}{X_1}\frac{\partial}{\partial X_1}\tilde{H}_{X_1}(X) + \frac{2}{X_1^2}\tilde{H}_{X_1}(X) = \frac{\partial}{\partial X_1}J_\varphi(X)^{\frac{2}{n}} = -2^4\frac{X_1+1}{|X+e_1|^6} \leq 0.$$

Hence the function $\tilde{H}_{X_1}$ satisfies

$$\begin{cases} -\Delta\tilde{H}_{X_1}(X) - \dfrac{2}{X_1}\dfrac{\partial}{\partial X_1}\tilde{H}_{X_1}(X) + \dfrac{2}{X_1^2}\tilde{H}_{X_1}(X) \leq 0 & \text{in } S, \\ \tilde{H}_{X_1} = 0 & \text{on } \partial S. \end{cases}$$

Applying the maximum principle we find that $\tilde{H}_{X_1} \le 0$ on S. $\square$

By the result in the previous proposition and using that the hyperbolic geodesics are transformed onto hyperbolic geodesics by Möbious transformations, we get the following.

Corollary 6.4.5. *The function $x \mapsto H(x, e_1)$ is increasing along the "hyperbolic geodesics" through e_1 in increasing Euclidean distance.*

Behavior at the boundary

In this section we study the behavior of $x \mapsto H(x, e_1)$ on ∂B. Indeed, since by the result of the previous section we already know that

$$\max_{x \in \bar{B}} H(x, e_1) = \max_{x \in \partial B} H(x, e_1),$$

it only remains to find the location on ∂B of this maximum. Also in this case it is convenient to use the transformation φ, defined in (6.4.1), and to work in the half n-space.

Proposition 6.4.6. *For any $i \in \{2, \dots, n\}$ it holds that $X_i \frac{\partial}{\partial X_i} \tilde{H}(X) \le 0$ on $\{X_1 = 0\}$.*

Proof. We find that for $X \in \{X_1 = 0\}$

$$\tilde{H}(X) = \frac{2}{n\omega_n} \int_S \frac{Z_1^2}{|X - Z|^n} J_\varphi(Z)^{\frac{2}{n}} dZ.$$

Fix $i \in \{2, \dots, n\}$ and $X \in \{X_1 = 0\}$. We have

$$\frac{\partial}{\partial X_i} \tilde{H}(X) = \frac{2^3}{\omega_n} \int_S \frac{Z_1^2}{|X - Z|^{n+2}} \frac{Z_i - X_i}{|Z + e_1|^4} dZ. \tag{6.4.4}$$

We will now determine the sign of the integral in (6.4.4). Let

$$S_{p,i} := \{Z \in S : Z_i - X_i > 0\} \text{ and } S_{n,i} := \{Z \in S : Z_i - X_i < 0\}.$$

Let $P \in S_{p,i}$ and let P' the unique element in $S_{n,i}$ such that: $P_j = P'_j$ for $j \in \{1, \dots, n\}$ with $j \ne i$, and $|X - P| = |X - P'|$. By the choice it follows that

$$\frac{P_1^2}{|X - P|^{n+2}} (P_i - X_i) = -\frac{P_1'^2}{|X - P'|^{n+2}} (P'_i - X_i).$$

We notice that the term

$$\frac{P_1^2}{|X - P|^{n+2}} \frac{P_i - X_i}{|P + e_1|^4} + \frac{P_1'^2}{|X - P'|^{n+2}} \frac{P'_i - X_i}{|P' + e_1|^4},$$

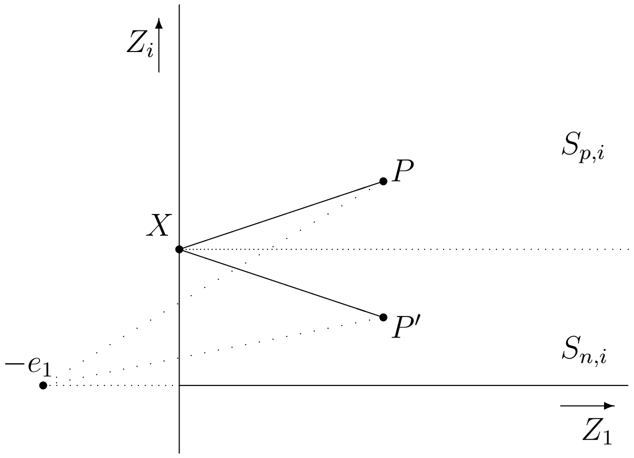

Figure 6.9: *The sets $S_{p,i}$, $S_{n,i}$ and the distance to $-e_1$.*

is positive if $X_2 < 0$, is negative if $X_2 > 0$ and is zero if $X_2 = 0$. This follows from the observation that

$$\begin{aligned} |P' + e_1| &> |P + e_1| \text{ if } X_2 < 0, \\ |P' + e_1| &= |P + e_1| \text{ if } X_2 = 0, \\ |P' + e_1| &< |P + e_1| \text{ if } X_2 > 0, \end{aligned}$$

(see Figure 6.9).

The claim follows repeating the same reasoning for every $P \in S_{p,i}$ and for every $i \in \{2, \dots, n\}$. $\square$

Corollary 6.4.7. *The function $X \mapsto \tilde{H}(X)$ for $X \in \bar{S}$ attains its maximum in $X = \vec{0}$.*

By the result of the previous corollary it directly follows that the maximum of $H(x, e_1)$ is attained at $x = -e_1$.

Orthogonal trajectories

In order to prove Theorem 6.1.1 it remains to show that $H(x, e_1)$ is increasing, in increasing Euclidean distance from e_1, along a family of trajectories that are orthogonal to the hyperbolic geodesic through e_1. This result will follow from the proof that the function $\tilde{H}(X)$ satisfies

$$X_i \frac{\partial}{\partial X_i} \tilde{H}(X) \leq 0 \text{ in } S \text{ for every } i = 2, \dots, n. \tag{6.4.5}$$

Indeed, by definition of hyperbolic geodesics in $B \subset \mathbb{R}^n$ and by symmetry it is sufficient to consider a family of hyperbolic geodesic through e_1 belonging to a unit disk D in B. Notice that the origin belongs to D. The pre-image via φ of this disk D in the half n-space is a plane Π to which the half-axes $\{X_2 = \dots = X_n = 0\}$ belongs and that is generated by e_1 and by a unitary vector $v \in \mathbb{R}^n$ orthogonal to e_1. The pre-image via φ of the hyperbolic geodesics through e_1 are the lines in the plane Π with directions e_1 and the pre-image of the orthogonal trajectories are the lines in Π parallel to v. A general line $l(t)$ in Π parallel to v has equation

$$l(t) = vt + k_1 e_1 \text{ for } t \in \mathbb{R} \text{ and a fixed } k_1 > 0.$$

The fact that $H(., e_1)$ is increasing along trajectories orthogonal to the hyperbolic geodesic is equivalent to $t\frac{d}{dt}\tilde{H}(l(t)) \leq 0$ and this will follow directly from (6.4.5).

We now prove (6.4.5) in the next proposition using the method of Proposition 6.4.6.

Proposition 6.4.8. *For any $i \in \{2, \ldots, n\}$ and every $X \in S$ the following holds true*

$$X_i \frac{\partial}{\partial X_i} \tilde{H}(X) \leq 0.$$

Proof. Let fix $i \in \{2, \ldots, n\}$ and $X \in S$. Starting from the explicit expression of $\tilde{H}(X)$ one finds

$$X_i \frac{\partial}{\partial X_i} \tilde{H}(X) = \tfrac{1}{n\omega_n} \frac{X_i}{X_1} \int_S Z_1(Z_i - X_i) \left(|X - Z|^{-n} - |X + QZ|^{-n}\right) \frac{2^2}{|Z + e_1|^4} dZ. \tag{6.4.6}$$

We recall that $|X - Z|^{-n} > |X + QZ|^{-n}$.

In order to determine the sign of the integral in (6.4.6) we set

$$S_{p,i} := \{Z \in S : Z_i - X_i > 0\} \text{ and } S_{n,i} := \{Z \in S : Z_i - X_i < 0\}.$$

Let $P \in S_{p,i}$ and let P' the unique element in $S_{n,i}$ such that: $P_j = P'_j$ for $j \in \{1, \ldots, n\}$ with $j \neq i$, and $|X - P| = |X - P'|$. By the choice it follows that

$$(P_i - X_i) \left(|X - P|^{-n} - |X + QP|^{-n}\right) = -(P'_i - X'_i) \left(|X - P'|^{-n} - |X + QP'|^{-n}\right).$$

The claim follows proceeding as in the proof of Proposition 6.4.6. $\square$

Theorem 6.1.1 for y fixed at the boundary and in dimension $n \geq 3$ follows directly from Propositions 6.4.4 and 6.4.8.

6.4.2 One point fixed in the interior

In the following section we study the function $x \mapsto H(x, y)$ with y fixed in B. Without loss of generality, we can fix $y = -se_1$ with $s \in (0, 1)$ and $e_1 = (1, 0, .., 0) \in \mathbb{R}^n$.

Transformation to the center

Instead of studying directly the function $x \mapsto H(x, -se_1)$ it is convenient to consider a transformation. We consider a (anti-)conformal map h_s from B onto B that maps 0 into $y = -se_1$ and e_1 into e_1 given by

$$\begin{aligned} h_s(x_1, x_2, .., x_n) &= \frac{(Id + s^2 Q)\, x}{|sx - e_1|^2} - s\frac{1 + |x|^2}{|sx - e_1|^2} e_1 \\ &= -\frac{1}{s} e_1 - \frac{1 - s^2}{s} \frac{sQx - e_1}{|sx - e_1|^2}, \end{aligned} \tag{6.4.7}$$

where $Q_{11} = 1$, $Q_{ii} = -1$ for $i = 2, \dots, n$ and $Q_{ij} = 0$ for $i, j = 1, \dots, n$ and $i \neq j$. Notice that h_s is conformal if the dimension n is even, is anti-conformal if the dimension n is odd. One can also see h_s as the combination of the following mappings

$$x \longmapsto Qx - \tfrac{1}{s} e_1 \longmapsto \tfrac{Qx - \frac{1}{s} e_1}{\left|Qx - \frac{1}{s} e_1\right|^2} \longmapsto -\tfrac{1-s^2}{s} \tfrac{sQx - e_1}{|sx - e_1|^2} \longmapsto -\tfrac{1}{s} e_1 - \tfrac{1-s^2}{s} \tfrac{sQx - e_1}{|sx - e_1|^2}.$$

Using the (anti-)conformal transformation h_s, we can write

$$\begin{aligned} H(\tilde{x}, y) &= \int_B \frac{G_B(\tilde{x}, z) G_B(z, y)}{G_B(\tilde{x}, y)} dz \\ &= \int_B \frac{G_B(\tilde{x}, h_s(z')) G_B(h_s(z'), y)}{G_B(\tilde{x}, y)} J_{h_s}(z') dz', \end{aligned}$$

where J_{h_s} is the Jacobian of the transformation h_s. We now compute J_{h_s} in the following lemma.

Lemma 6.4.9. *Let h_s the (anti-)conformal map defined in (6.4.7). For any $n \geq 3$ it holds that*

$$J_{h_s}(x) = \frac{(1 - s^2)^n}{|sx - e_1|^{2n}}.$$

Proof. The proof is similar to the one of Lemma 6.4.1. One uses that by the definition of h_s in (6.4.7) it holds

$$(\partial_j h_{s,i}(x))_{i,j} = -\frac{(1 - s^2)}{|sx - e_1|^2} Q \left(Id - 2\frac{sx - e_1}{|sx - e_1|} \left(\frac{sx - e_1}{|sx - e_1|} \right)^T \right),$$

using column notation for $sx - e_1$. □

By the definition of the function h_s and Lemma 6.2.2 we find

$$\begin{aligned} H(h_s(x), h_s(0)) &= \int_B \frac{G_B(h_s(x), h_s(z')) G_B(h_s(z'), h_s(0))}{G_B(h_s(x), h_s(0))} J_{h_s}(z') dz' \\ &= \int_B \frac{G_B(x, z') G_B(z', 0)}{G_B(x, 0)} J_{h_s}^{\frac{2}{n}}(z') dz'. \end{aligned} \tag{6.4.8}$$

For simplicity of notation we define on B the function H^s given by

$$H^s(x) := \int_B \frac{G_B(x,z)G_B(z,0)}{G_B(x,0)} J_{h_s}^{\frac{2}{n}}(z)dz.$$

Using again Lemma 6.3.3 one sees that the function H^s satisfies in $B\backslash\{0\}$ the equation

$$-\Delta_x H^s(x) - 2\frac{\nabla_x G_B(x,0)}{G_B(x,0)} \cdot \nabla_x H^s(x) = J_{h_s}^{\frac{2}{n}}(x),$$

that we can rewrite as

$$-\Delta_x H^s(x) = 2(2-n)\frac{|x|^{-n}}{|x|^{2-n}-1} x \cdot \nabla_x H^s(x) + J_{h_s}^{\frac{2}{n}}(x), \tag{6.4.9}$$

using the explicit formula of the Green function.

The radial direction

In the following section we show that the function H^s is increasing in radial direction. The method consists in studying the differential boundary value problem that $\frac{\partial}{\partial r}H^s$ satisfies and then apply the maximum principle.

We first prove that H^s satisfies zero Neumann boundary condition.

Lemma 6.4.10. *Let $s \in (0,1)$. It holds that $\frac{\partial}{\partial r}H^s(x) = 0$ for every $x \in \partial B$.*

Proof. Let $R^s(x)$ denote the numerator of $H^s(x)$; that is

$$R^s(x) := \int_B G_B(x,z)G_B(z,0)J_{h_s}^{\frac{2}{n}}(z)dz. \tag{6.4.10}$$

One has that $R^s(x) = 0$ for $x \in \partial B$ and that it holds

$$-\Delta R^s(x) = G_B(x,0)J_{h_s}^{\frac{2}{n}}(x).$$

Since $-\Delta = -r^{1-n}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}\right) - r^{-2}\Delta_\Gamma$ where Δ_Γ is the Laplace-Beltrami operator on the surface of the unit ball, we find that at the boundary

$$\begin{aligned} \tfrac{\partial}{\partial r^2}R^s(x) &= -(n-1)\tfrac{\partial}{\partial r}R^s(x), \\ \tfrac{\partial}{\partial r^2}G_B(x,0) &= -(n-1)\tfrac{\partial}{\partial r}G_B(x,0). \end{aligned} \tag{6.4.11}$$

Hence from the series expansion near the boundary of $R^s(\cdot)$ and $G_B(\cdot,0)$ one gets for $x \in \partial B$

$$\lim_{B\ni\xi\to x}\frac{\partial}{\partial r}H^s(\xi) = \lim_{B\ni\xi\to x}\frac{\frac{\partial}{\partial r}G_B(\xi,0)}{G_B(\xi,0)}\left(\frac{\frac{\partial}{\partial r}R^s(\xi)}{\frac{\partial}{\partial r}G_B(\xi,0)} - \frac{R^s(\xi)}{G_B(\xi,0)}\right)$$

$$= \lim_{B\ni\xi\to x} \frac{(2-n)}{|\xi|^{2-n}-1} \left(\frac{\frac{\partial}{\partial r} R^s(x) + (|\xi|-1)\frac{\partial^2}{\partial r^2} R^s(x) + ..}{\frac{\partial}{\partial r} G_B(x,0) + (|\xi|-1)\frac{\partial^2}{\partial r^2} G_B(x,0) + ..} - \frac{\frac{\partial}{\partial r} R^s(x) + \frac{|\xi|-1}{2}\frac{\partial^2}{\partial r^2} R^s(x) + ..}{\frac{\partial}{\partial r} G_B(x,0) + \frac{|\xi|-1}{2}\frac{\partial^2}{\partial r^2} G_B(x,0) + ..} \right)$$

$$= \frac{1}{2} \frac{\frac{\partial^2}{\partial r^2} R^s(x) \frac{\partial}{\partial r} G_B(x,0) - \frac{\partial}{\partial r} R^s(x) \frac{\partial^2}{\partial r^2} G_B(x,0)}{\left(\frac{\partial}{\partial r} G_B(x,0)\right)^2}. \tag{6.4.12}$$

The claim follows from (6.4.12) using (6.4.11). □

We now show that $r\frac{\partial}{\partial r} H^s(x)$ is well defined in 0.

Lemma 6.4.11. *Let $s \in (0,1)$. Then $\lim_{x\to 0} r\frac{\partial}{\partial r} H^s(x) = 0$.*

Proof. With R^s defined as in (6.4.10) one finds

$$r\frac{\partial}{\partial r} H^s(x) = x.\nabla H^s(x) = \frac{x}{G_B(x,0)}.\nabla R^s(x) - \frac{R^s(x)}{G_B(x,0)} \frac{x}{G_B(x,0)}.\nabla G_B(x,0). \tag{6.4.13}$$

Since

$$\frac{x}{G_B(x,0)}.\nabla G_B(x,0) = \frac{(2-n)|x|^{2-n}}{|x|^{2-n}-1} = \frac{2-n}{1-|x|^{n-2}},$$

and since from Lemma 6.4.9 and [69, Sec.5] (see Remark 6.4.12) it follows that

$$\frac{R^s(x)}{G_B(x,0)} \le \frac{(1-s^2)^2}{(s-1)^4} \frac{1}{G_B(x,0)} \int_B G_B(x,z) G_B(z,0) dz \le \frac{(1-s^2)^2}{(s-1)^4} c_\Omega |x|,$$

we get

$$\lim_{x\to 0} \left(\frac{R^s(x)}{G_B(x,0)} \frac{x}{G_B(x,0)}.\nabla G_B(x,0) \right) = 0.$$

The other term in (6.4.13) is given by

$$\frac{x}{G_B(x,0)}.\nabla R^s(x) = -\frac{1}{n\omega_n} \frac{|x|^{n-2}}{1-|x|^{n-2}}.$$

$$\cdot x. \int_B \left(|x-z|^{-n}(x-z) - \left| x|z| - \tfrac{z}{|z|} \right|^{-n} (x|z| - \tfrac{z}{|z|})|z| \right) \left(|z|^{2-n}-1\right) J_{h_s}^{\frac{2}{n}}(z) dz.$$

One sees directly that

$$\lim_{x\to 0} \frac{|x|^{n-2}}{1-|x|^{n-2}} x. \int_B \left| x|z| - \tfrac{z}{|z|} \right|^{-n} (x|z| - \tfrac{z}{|z|})|z| (|z|^{2-n}-1) J_{h_s}^{\frac{2}{n}}(z) dz = 0.$$

Hence to show that $\lim_{x\to 0} \frac{x}{G_B(x,0)}.\nabla R^s(x) = 0$ it is sufficient to prove that the limit for x going to 0 of the modulus of

$$\frac{|x|^{n-2}}{1-|x|^{n-2}} x. \int_B |x-z|^{-n}(x-z)(|z|^{2-n}-1) J_{h_s}^{\frac{2}{n}}(z) dz,$$

is zero. One has

$$\begin{aligned}
&\lim_{x\to 0}\frac{|x|^{n-2}}{1-|x|^{n-2}}\left|x.\int_B |x-z|^{-n}(x-z)(|z|^{2-n}-1)J_{h_s}^{\frac{2}{n}}(z)dz\right| \\
\leq\ & 4\frac{(1-s^2)^2}{(1-s)^4}\lim_{x\to 0}\left(|x|^{n-1}\int_B |x-z|^{1-n}|z|^{2-n}dz\right).
\end{aligned}$$

We study separately the integral term. Writing

$$\begin{aligned}
|x|^{n-1}\int_B |x-z|^{1-n}|z|^{2-n}dz &= |x|^{n-1}\int_{|z|<\frac{|x|}{2}} |x-z|^{1-n}|z|^{2-n}dz \\
&\quad + |x|^{n-1}\int_{B\setminus\left\{|z|<\frac{|x|}{2}\right\}} |x-z|^{1-n}|z|^{2-n}dz = \dots,
\end{aligned}$$

since $|x-z|\geq\frac{|x|}{2}$ for $|z|<\frac{|x|}{2}$, one finds

$$\dots\leq 2^{n-1}\int_{|z|<\frac{|x|}{2}}|z|^{2-n}dz + 2^{n-2}|x|\int_{B\setminus\left\{|z|<\frac{|x|}{2}\right\}}|x-z|^{1-n}dz,$$

that goes to zero for x going to 0. □

Remark 6.4.12. In [68] it is proved that for $x,y\in\Omega$ it holds

$$\begin{aligned}
H_\Omega(x,y) &\leq c_\Omega\left(\ln\frac{C_\Omega}{|x-y|}\right)^{-1} \text{ for } n=2, \\
H_\Omega(x,y) &\leq c_\Omega|x-y| \text{ for } n\geq 3, \\
H_\Omega(x,y) &\leq c_{\Omega,\varepsilon}|x-y|^{2-\varepsilon} \text{ for } n\geq 4 \text{ and } \varepsilon>0.
\end{aligned}$$

Notice that there is a different behavior for $n=2$ and $n\geq 3$ but also between the case $n=3$ and $n\geq 4$.

Proposition 6.4.13. *For every $x\in B$ it holds that $r\frac{\partial}{\partial r}H^s(x)\geq 0$.*

Proof. Let Σ denote $r\frac{\partial}{\partial r}H^s(x)$(which is equal to $x.\nabla H^s(x)$). By definition of Σ and (6.4.9) one has that

$$\begin{aligned}
-\Delta\Sigma(x) &= -2\Delta_x H^s(x) - x.\nabla\Delta_x H^s(x) \\
&= 4(2-n)\frac{|x|^{-n}}{|x|^{2-n}-1}\Sigma(x) + 2J_{h_s}^{\frac{2}{n}}(x) \\
&\quad +2(2-n)x.\nabla\left(\frac{|x|^{-n}}{|x|^{2-n}-1}\Sigma(x)\right) + x.\nabla J_{h_s}^{\frac{2}{n}}(x) \\
&= 4(2-n)\frac{|x|^{-n}}{|x|^{2-n}-1}\Sigma(x) + 2(2-n)\frac{|x|^{-n}(n-2|x|^{2-n})}{(|x|^{2-n}-1)^2}\Sigma(x) \\
&\quad +2(2-n)\frac{|x|^{-n}}{|x|^{2-n}-1}x.\nabla(\Sigma(x)) + 2J_{h_s}^{\frac{2}{n}}(x) + x.\nabla J_{h_s}^{\frac{2}{n}}(x).
\end{aligned}$$

Hence Σ satisfies

$$\begin{aligned}-\Delta\Sigma(x) - 2(2-n)\frac{|x|^{-n}}{|x|^{2-n}-1}x.\nabla\Sigma(x) + 2(n-2)^2\frac{|x|^{-n}}{(|x|^{2-n}-1)^2}\Sigma(x)\\ = 2J_{h_s}^{\frac{2}{n}}(x) + x.\nabla J_{h_s}^{\frac{2}{n}}(x),\end{aligned} \tag{6.4.14}$$

and the right hand side in (6.4.14) is positive. Indeed from Lemma 6.4.9 and since $s \in (0,1)$ it holds for $x \in B$

$$\begin{aligned}2J_{h_s}^{\frac{2}{n}}(x) + x.\nabla J_{h_s}^{\frac{2}{n}}(x) &= (1-s^2)^2\left(\frac{2}{|sx-e_1|^4} + x.\nabla\frac{1}{|sx-e_1|^4}\right)\\ &= 2(1-s^2)^2\left(\frac{1}{|sx-e_1|^4} - \frac{(sx-e_1).2sx}{|sx-e_1|^6}\right)\\ &= -2(1-s^2)^2\frac{(sx-e_1).(sx+e_1)}{|sx-e_1|^6}\\ &= 2(1-s^2)^2\frac{1-s^2|x|^2}{|sx-e_1|^6} > 0.\end{aligned}$$

Using the result of Lemmas 6.4.10 and 6.4.11 one finds

$$\begin{cases} -\Delta\Sigma(x) - 2(2-n)\frac{|x|^{-n}}{|x|^{2-n}-1}x.\nabla\Sigma(x) + 2(n-2)^2\frac{|x|^{-n}}{(|x|^{2-n}-1)^2}\Sigma(x) \geq 0 & \text{in } B\setminus\{0\},\\ \Sigma(x) = 0 & \text{on } \partial B \cup \{0\}\,. \end{cases}$$

The claim follows by the maximum principle. □

Behavior at the boundary

In the previous section we have shown that $x \mapsto H^s(x)$ is radially increasing. In order to find in which point the maximum of $H^s(x)$ is attained it is sufficient to study the behavior at the boundary of this function.

For $x \in \partial B$ one finds

$$\begin{aligned}H^s(x) &= \int_B \frac{K_B(x,z)G_B(z,0)}{K_B(x,0)}J_{h_s}^{\frac{2}{n}}(z)dz\\ &= \tfrac{(1-s^2)^2}{n(n-2)\omega_n}\int_B \frac{1-|z|^2}{|x-z|^n}\frac{|z|^{2-n}-1}{|sz-e_1|^4}dz.\end{aligned} \tag{6.4.15}$$

Lemma 6.4.14. *It holds that* $\max_{x\in\partial B} H^s(x) = H^s(e_1)$.

Proof. We first notice that by symmetry it is sufficient to consider $x = (x_1, x_2, \vec{0})$ with $\vec{0} \in \mathbb{R}^{n-2}$ and $x_1^2 + x_2^2 = 1$. Then in order to see how the function $H^s(x)$ varies when x belongs to this circumference we consider

$$\frac{\partial}{\partial\theta}H^s(x) = -x_2\frac{\partial}{\partial x_1}H^s(x) + x_1\frac{\partial}{\partial x_2}H^s(x).$$

From (6.4.15) one finds

$$\begin{aligned}\frac{\partial}{\partial\theta}H^s(x) &= \tfrac{-(1-s^2)^2}{(n-2)\omega_n}\int_B(1-|z|^2)\frac{|z|^{2-n}-1}{|sz-e_1|^4}\frac{-x_2(x_1-z_1)+x_1(x_2-z_2)}{|x-z|^{n+2}}dz\\ &= \tfrac{(1-s^2)^2}{(n-2)\omega_n}\int_B(1-|z|^2)\frac{|z|^{2-n}-1}{|sz-e_1|^4}\frac{x_1z_2-x_2z_1}{|x-z|^{n+2}}dz.\end{aligned}$$

We now study the sign of the integral. Let

$$B_p := \{z\in B : x_1z_2-x_2z_1>0\} \text{ and } B_n := \{z\in B : x_1z_2-x_2z_1<0\}.$$

One sees that if $\xi\in B_p$ then $-\xi\in B_n$ and that the intersection of the closure of B_p and B_n is a hyperplane in $\mathbb{R}^n$ going through x and the origin.

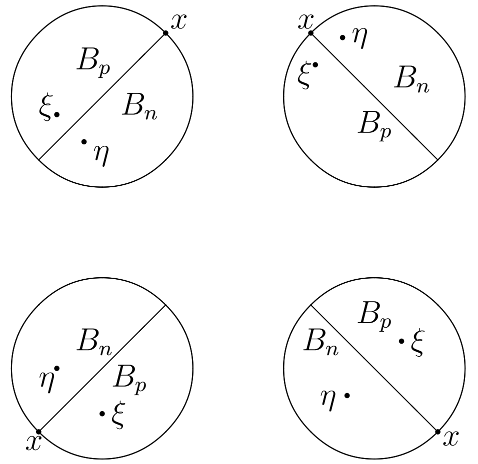

Figure 6.10: *The sets B_p and B_n for different positions of x.*

Let $\xi\in B_p$ and let η the unique element in B_n such that: $|\xi|=|\eta|$, $\xi_i=\eta_i$ for every $i\geq 3$ and $|x-\xi|=|x-\eta|$. By the choice it follows that

$$(1-|\xi|^2)(|\xi|^{2-n}-1)\frac{x_1\xi_2-x_2\xi_1}{|x-\xi|^{n+2}} = -(1-|\eta|^2)(|\eta|^{2-n}-1)\frac{x_1\eta_2-x_2\eta_1}{|x-\eta|^{n+2}}.$$

We notice that the term

$$(1-|\xi|^2)\frac{|\xi|^{2-n}-1}{|s\xi-e_1|^4}\frac{x_1\xi_2-x_2\xi_1}{|x-\xi|^{n+2}}+(1-|\eta|^2)\frac{|\eta|^{2-n}-1}{|s\eta-e_1|^4}\frac{x_1\eta_2-x_2\eta_1}{|x-\eta|^{n+2}},$$

is positive if $x_2<0$, is negative if $x_2>0$ and is zero if $x_2=0$. This follows from the observation that

$$\begin{aligned} s\left|\xi-\tfrac{1}{s}e_1\right| &< s\left|\eta-\tfrac{1}{s}e_1\right| \text{ if } x_2<0,\\ s\left|\xi-\tfrac{1}{s}e_1\right| &= s\left|\eta-\tfrac{1}{s}e_1\right| \text{ if } x_2=0,\\ s\left|\xi-\tfrac{1}{s}e_1\right| &> s\left|\eta-\tfrac{1}{s}e_1\right| \text{ if } x_2>0,\end{aligned}$$

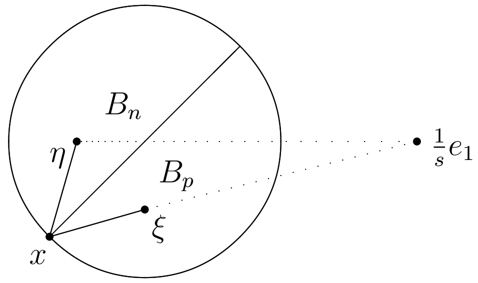

Figure 6.11: *The distances* $|\eta - x|$, $|\xi - x|$, $|\xi - \frac{1}{s}e_1|$ *and* $|\eta - \frac{1}{s}e_1|$.

(See Figure 6.11).

Repeating the same reasoning for every $\xi \in B_p$ we get that $x_2 \frac{\partial}{\partial\theta} H^s(x) \leq 0$ for every $x \in \partial B$ with $x = (x_1, x_2, \vec{0})$. Hence, by symmetry it follows that $\sup_{x\in\partial B} H^s(x) = H^s(e_1)$. $\square$

Corollary 6.4.15. *Let* $s \in (0,1)$*. The function* $H^s(x)$ *is radially increasing in* B *and*

$$\max_{x\in\bar{B}} H^s(x) = H^s(e_1).$$

Orthogonal trajectories

In this paragraph we show that the function $x \mapsto H(x,y)$ is increasing, in increasing Euclidean distance from y, along trajectories that are orthogonal to the hyperbolic geodesic through y. As before we fix $y = -se_1$ with $s \in (0,1)$.

By the definition of hyperbolic geodesic in the ball in $\mathbb{R}^n$, $n \geq 3$, and since we are interested to the orthogonal trajectories it is sufficient to work in a generic disk of radius 1 in B to which 0 and y belong. By symmetries of the ball without loss in generality we may consider the disk in the x_1, x_2-plane, that is $D := \{x \in \mathbb{R}^n : x_i = 0 \text{ for } i = 3, \ldots, n \text{ and } x_1^2 + x_2^2 \leq 1\}$. The image through h_s^{-1} of this disk D is D itself with y sent to 0 and e_1 sent to e_1. Hence the fact that $H(.,y)$ is increasing along trajectories orthogonal to the hyperbolic geodesic through y is equivalent to

$$\frac{\partial}{\partial\theta} H^s(x) := x_2 \left(-x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} \right) H^s(x) \leq 0 \text{ for every } x \in D.$$

Proposition 6.4.16. *The function* $H^s(x)$ *satisfies* $\frac{\partial}{\partial\theta} H^s(x) \leq 0$ *for every* $x \in D$.

Proof. We use the method of Lemma 6.4.14. We have

$$
\begin{aligned}
\frac{\partial}{\partial\theta}H^s(x) &= -\frac{(1-s^2)^2}{n\omega_n}\int_B \Big(|x-z|^{-n}(-x_2(x_1-z_1)+x_1(x_2-z_2))+ \\
&\qquad -\left|x|z|-\frac{z}{|z|}\right|^{-n}\left(-x_2\left(x_1|z|-\frac{z_1}{|z|}\right)+x_1\left(x_2|z|-\frac{z_2}{|z|}\right)\right)\Big)\cdot \\
&\qquad \cdot\frac{|z|^{2-n}-1}{|x|^{2-n}-1}\frac{1}{|sz-e_1|^4}dz \\
&= \frac{(1-s^2)^2}{n\omega_n}\int_B (z_2x_1-x_2z_1)\left(|x-z|^{-n}-\left|x|z|-\frac{z}{|z|}\right|^{-n}\frac{1}{|z|}\right)\cdot \\
&\qquad \cdot\frac{|z|^{2-n}-1}{|x|^{2-n}-1}\frac{1}{|sz-e_1|^4}dz.
\end{aligned}
$$

We now study the sign of the integral. Let

$$B_p := \{z\in B : x_1z_2-x_2z_1>0\} \text{ and } B_n := \{z\in B : x_1z_2-x_2z_1<0\}.$$

Let $\xi\in B_p$ and let η the unique element in B_n such that: $|\xi|=|\eta|$, $\xi_i=\eta_i$ for every $i\geq 3$ and $|x-\xi|=|x-\eta|$. By the choice it follows that

$$
\begin{aligned}
&(\xi_2x_1-x_2\xi_1)\left(|x-\xi|^{-n}-\left|x|\xi|-\frac{\xi}{|\xi|}\right|^{-n}\frac{1}{|\xi|}\right)\frac{|\xi|^{2-n}-1}{|x|^{2-n}-1} = \\
&\quad = -(\eta_2x_1-x_2\eta_1)\left(|x-\eta|^{-n}-\left|x|\eta|-\frac{\eta}{|\eta|}\right|^{-n}\frac{1}{|\eta|}\right)\frac{|\eta|^{2-n}-1}{|x|^{2-n}-1}.
\end{aligned}
$$

The claim follows proceeding as in Lemma 6.4.14. □

Theorem 6.1.1 follows from Proposition 6.4.13 and Proposition 6.4.16.

6.5 Relation with the eigenvalues

6.5.1 Previous results

In [50] the authors show that there exists a relation between the inverse of $\lambda_c(\Omega)$, defined in (6.1.3), and the Dirichlet eigenvalues for two choices of $\Omega : \Omega=[0,1]\subset\mathbb{R}$ (see also [67]) and Ω the unit disk. In an interval $I=[0,1]\subset\mathbb{R}$ the following identities hold

$$\frac{1}{\lambda_c(I)}=\sum_{m=1}^{\infty}\frac{1}{\lambda_m}=2\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{\lambda_m},$$

with $\lambda_m=\pi^2m^2$, $m\in\mathbb{N}$. For the disk D it holds

$$\frac{1}{\lambda_c(D)}=4\sum_{m=0}^{\infty}(-1)^{m-1}\sum_{i=1}^{\infty}\frac{\nu_{m,i}}{\lambda_{m,i}}, \tag{6.5.1}$$

where $\nu_{0,i} = 1$ and $\nu_{m,i} = 2$ for $m \geq 1$. The eigenvalue $\lambda_{m,i}$ corresponds to the eigenfunction with $i-1$ circular nodal lines and m radial nodal lines.

We are now able to give an explanation to identity (6.5.1). A complete orthonormal set of eigenfunctions for (6.1.1) on the disk is given by, writing $x = re^{i\varphi}$:

$$\begin{aligned}
\varphi_{0,i}(x) &= \frac{1}{\sqrt{2\pi}} \frac{J_0(j_{0,i}r)}{\frac{1}{\sqrt{2}}|J_0'(j_{0,i})|} \text{ for } i \in \mathbb{N}, \\
\varphi_{e,m,i}(x) &= \frac{\cos(m\varphi)}{\sqrt{\pi}} \frac{J_m(j_{m,i}r)}{\frac{1}{\sqrt{2}}|J_m'(j_{m,i})|} \text{ for } m, i \in \mathbb{N}, \\
\varphi_{o,m,i}(x) &= \frac{\sin(m\varphi)}{\sqrt{\pi}} \frac{J_m(j_{m,i}r)}{\frac{1}{\sqrt{2}}|J_m'(j_{m,i})|} \text{ for } m, i \in \mathbb{N},
\end{aligned}$$

with eigenvalues

$$\lambda_{0,i} = j_{0,i}^2 \text{ and } \lambda_{e,m,i} = \lambda_{o,m,i} = j_{m,i}^2 \text{ for } i, m \in \mathbb{N}.$$

Here, as usual, J_m denotes the m-th Bessel function of the first kind and $j_{m,i}$ denotes the i-th zero of J_m. For the normalization of the Bessel function see [74, 5.11 (11)]. By orthonormality one finds

$$\begin{aligned}
\frac{1}{\lambda_c(D)} &= \sup_{x,y\in D} \frac{1}{G_D(x,y)} \left[\sum_{i=1}^{\infty} \frac{1}{j_{0,i}^4} \frac{J_0(j_{0,i}r)J_0(j_{0,i}\rho)}{\pi J_0'^2(j_{0,i})} + \right. \\
&\left. + \sum_{m=1}^{\infty} \frac{1}{\pi}(\cos(m\varphi)\cos(m\varphi') + \sin(m\varphi)\sin(m\varphi')) \sum_{i=1}^{\infty} \frac{2}{j_{m,i}^4} \frac{J_m(j_{m,i}r)J_m(j_{m,i}\rho)}{J_m'^2(j_{m,i})} \right] \\
&= \lim_{\substack{x\to e_1,\\ y\to -e_1}} \frac{1}{\pi G_D(x,y)} \left[\sum_{i=1}^{\infty} \frac{1}{j_{0,i}^4} \frac{J_0(j_{0,i}r)J_0(j_{0,i}\rho)}{J_0'^2(j_{0,i})} + \right. \\
&\left. + 2\sum_{m=1}^{\infty} \frac{1}{\pi}(\cos(m\varphi)\cos(m\varphi') + \sin(m\varphi)\sin(m\varphi')) \sum_{i=1}^{\infty} \frac{1}{j_{m,i}^4} \frac{J_m(j_{m,i}r)J_m(j_{m,i}\rho)}{J_m'^2(j_{m,i})} \right] = \ldots .
\end{aligned}$$

Differentiating with respect to ρ and computing for $y = -e_1$, we get

$$\begin{aligned}
\ldots &= \lim_{x\to e_1} \frac{1}{\pi K_D(x,-e_1)} \left[\sum_{i=1}^{\infty} \frac{1}{j_{0,i}^3} \frac{J_0(j_{0,i}r)}{J_0'(j_{0,i})} \right. \\
&\qquad \left. + 2\sum_{m=1}^{\infty} (-1)^m \cos(m\varphi) \sum_{i=1}^{\infty} \frac{1}{j_{m,i}^3} \frac{J_m(j_{m,i}r)}{J_m'(j_{m,i})} \right] = \ldots ,
\end{aligned}$$

and differentiating with respect to r and computing for $x = e_1$

$$\cdots = \frac{1}{\pi} \frac{\sum_{i=1}^{\infty} \frac{1}{j_{0,i}^2} + 2\sum_{m=1}^{\infty}(-1)^m \sum_{i=1}^{\infty} \frac{1}{j_{m,i}^2}}{\frac{1}{4\pi}} = 4\left(\sum_{i=1}^{\infty} \frac{1}{j_{0,i}^2} + 2\sum_{m=1}^{\infty}(-1)^m \sum_{i=1}^{\infty} \frac{1}{j_{m,i}^2} \right).$$

In [50] the numbers $\nu_{m,i}$ in (6.5.1) were interpreted as the multiplicity of the eigenvalue $\lambda_{m,i}$, since it holds $\nu_{0,i} = 1$ and $\nu_{m,i} = 2$ for $m \geq 1$. Instead, from the derivation of the formula it seems that what plays a role is the different normalization of the eigenfunctions.

6.5.2 The identity for the critical value

We will now show that an identity holds also between $\lambda_c^{-1}(B)$ and a sum of Dirichlet eigenvalues when B is the unit ball in $\mathbb{R}^3$. We first compute the value of $\lambda_c^{-1}(B)$ for $B \subset \mathbb{R}^n$ with $n \geq 3$. By Theorem 6.1.1 and (6.1.9) one has that it holds

$$\frac{1}{\lambda_c(B)} = H(-e_n, e_n) = \frac{2^{n-1}}{n\omega_n} \int_B \frac{(1-|z|^2)^2}{|z-e_n|^n\,|z+e_n|^n} dz.$$

Via a C.A.S. (computer algebra system) one finds the following

$$\frac{1}{\lambda_c(B)} = \frac{\sqrt{\pi}\left(2\Gamma(\frac{n}{2}) - (2+n)\Gamma(1+n)\ {}_2F_1(2+\frac{1}{2}n, n; 3+\frac{1}{2}n; -1)\right)}{4(n-1)\Gamma(\frac{1}{2}(n-1))},$$

where $\Gamma(\cdot)$ denotes the Gamma function and ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ denotes the Gauss hypergeometric function (see [1, Chap.6 and 15]). In the following table we collect the values of $\lambda_c^{-1}(B)$ with $B \subset \mathbb{R}^n$ for $n \leq 5$.

n	$\lambda_c^{-1}(B)$
1	$\frac{2}{3} \simeq 0.6666$
2	$2\log 2 - 1 \simeq 0.3862$
3	$2(\pi - 3) \simeq 0.2831$
4	$3 - 2\log(4) \simeq 0.2274$
5	$\frac{1}{3}(10 - 3\pi) \simeq 0.1917$

On the unit ball in $\mathbb{R}^3$ a complete orthonormal set of eigenfunctions is given in polar coordinates (r, φ, θ) by:

- with $m = 0$, $k \in \mathbb{N}_0$ and $i \in \mathbb{N}$,

$$u_{0,k,i}(r,\varphi,\theta) = \sqrt{\tfrac{2k+1}{4\pi}} P_k(\cos(\theta)) \frac{j_k(j_{\frac{1}{2}+k,i}r)}{\frac{1}{\sqrt{2}} j_k'(j_{\frac{1}{2}+k,i})},$$

- with $m, k, i \in \mathbb{N}$ and $k \geq m$,

$$\begin{aligned} u_{e,m,k,i}(r,\varphi,\theta) &= \sqrt{\tfrac{2k+1}{2\pi}} \sqrt{\tfrac{(k-|m|)!}{(k+|m|)!}} \cos(m\varphi)\, P_k^m(\cos(\theta)) \frac{j_k(j_{k+\frac{1}{2},i}r)}{\frac{1}{\sqrt{2}} j_k'(j_{k+\frac{1}{2},i})}, \\ u_{o,m,k,i}(r,\varphi,\theta) &= \sqrt{\tfrac{2k+1}{2\pi}} \sqrt{\tfrac{(k-|m|)!}{(k+|m|)!}} \sin(m\varphi)\, P_k^m(\cos(\theta)) \frac{j_k(j_{k+\frac{1}{2},i}r)}{\frac{1}{\sqrt{2}} j_k'(j_{k+\frac{1}{2},i})}, \end{aligned}$$

(see [65, App. A]). We use the usual convention: $0 \leq r \leq 1$, $0 \leq \varphi < 2\pi$ and $0 \leq \theta \leq \pi$. Here $P_k^m(\cdot)$ denotes the Legendre function, j_k denotes the fractional Bessel function of first kind and $j_{k+\frac{1}{2},i}$ denotes the i-th zero of j_k (see [1, Chap.8 and 10] and [74]). We choose this notation for the i-th zero of j_k since it coincides with the i-th zero of $J_{k+\frac{1}{2}}$. Notice that $j_k(z) = \frac{1}{\sqrt{z}} J_{k+\frac{1}{2}}(z)$.

The associated eigenvalues are

$$\lambda_{0,0,i} = \frac{1}{j^2_{\frac{1}{2},i}} \text{ and } \lambda_{0,k,i} = \lambda_{e,m,k,i} = \lambda_{o,m,k,i} = \frac{1}{j^2_{k+\frac{1}{2},i}} \text{ with } m,k,i \in \mathbb{N} \text{ and } k \geq m.$$

Notice that each eigenvalue has multiplicity $2k+1$. For simplicity of notation we fix

$$\mu_{k,i} = \frac{1}{j^2_{k+\frac{1}{2},i}} \text{ for } k \in \mathbb{N}_0 \text{ and } i \in \mathbb{N}. \tag{6.5.2}$$

Hence, $\mu_{k,i}$ for $k \in \mathbb{N}_0$ and $i \in \mathbb{N}$ are the eigenvalues for problem (6.1.1) on B the unit ball in $\mathbb{R}^3$ counted without multiplicity.

Lemma 6.5.1. *For $k \in \mathbb{N}_0$ and $i \in \mathbb{N}$ let $\mu_{k,i}$ as defined in (6.5.2). Then it holds that*

$$\frac{1}{\lambda_c(B)} = 4\sum_{k=0}^{\infty}(-1)^{k+1}\nu_k \sum_{i=1}^{\infty}\frac{1}{\mu_{k,i}}, \tag{6.5.3}$$

with $\nu_0 = 1$ and $\nu_k = 4$ for $k \geq 1$.

Proof. By [74, 15.51] one gets for $k \in \mathbb{N}_0$

$$\sum_{i=1}^{\infty}\frac{1}{\mu_{k,i}} = \sum_{i=1}^{\infty}\frac{1}{j^2_{k+\frac{1}{2},i}} = \frac{1}{4(k+\frac{3}{2})}.$$

Hence it holds

$$\begin{aligned} 4\sum_{k=0}^{\infty}(-1)^{k+1}\nu_k \sum_{i=1}^{\infty}\frac{1}{\mu_{k,i}} &= -4\sum_{i=1}^{\infty}\frac{1}{\mu_{0,i}} + 16\sum_{k=1}^{\infty}(-1)^{k+1}\sum_{i=1}^{\infty}\frac{1}{\mu_{k,i}} \\ &= -\frac{1}{\frac{3}{2}} + 4\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k+\frac{3}{2}} \\ &= -\frac{2}{3} + \frac{4}{6}(3\pi - 8) = 2(\pi - 3). \end{aligned}$$

The claim follows. $\square$

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Summary

The main subject of this thesis concerns positivity for fourth order elliptic problems. By positivity we mean that a positive source term in the differential equation leads to a positive solution. For second order elliptic partial differential equations such a result is known and referred to by the name "maximum principle". It is also well known that such a maximum principle does not have a straightforward generalization to higher order elliptic equations.

The difference between second order and higher order elliptic problems appears also when looking at the mechanical model related to this two kinds of elliptic equations. A physical problem described by a second order elliptic equation is the displacement of a membrane loaded by a weight. From everyday experience one knows that when we apply a force to a membrane all of it will move in the same direction. A fourth order elliptic equation instead describes the displacement of an elastic plate loaded by some weight. One may think of the displacement of a flat roof and the weight which it has to support due to rain or snow. In general the displacement is not everywhere in one direction. However, the mechanical model seems to indicate that some positivity remains. In this thesis we will study this aspect.

We will consider the mathematical model of an elastic plate loaded by a force f and clamped at its edges, that is, the following fourth order Dirichlet boundary value problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \frac{\partial}{\partial\nu} u = 0 & \text{on } \partial\Omega. \end{cases} \tag{S.1}$$

Here Ω is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}$ and $n \geq 2$, with $\partial\Omega \in C^{4,\alpha}$, $\alpha \in (0,1)$ and f is a continuous non-negative function defined on $\bar{\Omega}$.

The Green function associated to problem (S.1) on a general Ω may be sign changing. However, thinking of the physical problem, we expect that the sign preserving effects are much stronger than the opposite ones.

The main result of this thesis is the splitting of the Green function associated to (S.1) as the sum of two terms: a positive singular term and a sign-changing regular one, both satisfying the zero Dirichlet boundary conditions. The positive term describes the local behavior while the other, that could be sign-changing, depends only indirectly on the local behavior and hence is regular. As a consequence we prove that the sign preserving effects are much stronger than the opposite ones. Our results are presently

limited to the two-dimensional case.

A first understanding on how the singularity of the Green function behaves in relation with the Dirichlet boundary conditions is studied in Chapter 2. There we prove sharp estimates for the absolute value of the Green function and for its derivatives depending on the distance to the boundary. This kind of estimates are a useful tool to prove regularity results in spaces involving the behavior at the boundary.

In Chapters 3 and 4 we study for which domains the clamped plate equation is positivity preserving. For a long time it was an open problem if convexity of the domain is necessary for positivity. Hadamard in [47] states that the clamped plate equation is positivity preserving on the Limaçons de Pascal. The Limaçons de Pascal is a one-parameter family of domains varying continuously from the ball to the cardioid. We prove that Hadamard's statement is wrong in its full generality but that however, there are non-convex limaçons on which the clamped plate equation has the positivity preserving property. In Chapter 4 one may find the methods presently available to find domains on which the clamped plate equation has the positivity preserving property.

The estimates of the Green function proved in Chapter 2 are sharp from above but not from below. In order to estimate the behavior of the Green function from below we construct a covering of the domain Ω and its boundary by a finite number of sub-domains that have a positive Green function. It is possible to construct a finite covering of the domain and of its boundary since there are convex and also non-convex domains in which the clamped plate equation has the positivity preserving property and since we can show that small $C^{2,\gamma}$ perturbations of the domain do not destroy this property. Thanks to this covering we will be able to split the Green function as the sum of a positive singular term and a sign-changing regular one. This is done in Chapter 5.

The topic of Chapter 6 differs somewhat from the other ones. Studying positivity for elliptic boundary value problems we encounter some open problems in probability theory.

Some interesting questions concerning positivity arise also in the study of systems of second order elliptic boundary value problems. In [66] the following elliptic system is presented as a model problem for the positivity preserving property of systems coupled in a non-cooperative way

$$\begin{cases} -\Delta u &= f - \lambda v & \text{in } \Omega, \\ -\Delta v &= f & \text{in } \Omega, \\ u = v &= 0 & \text{on } \partial\Omega. \end{cases}$$

Here Ω is a bounded regular subset of $\mathbb{R}^n$ and $\lambda \in \mathbb{R}^+$. One can show that there exists a value $\lambda_c(\Omega)$ such that for all $f \geq 0$ the solution u is positive if and only if $\lambda \leq \lambda_c(\Omega)$ and that

$$\lambda_c(\Omega)^{-1} = \sup_{x,y\in\Omega} H_\Omega(x,y),$$

where

$$H_\Omega(x,y) = \int_\Omega \frac{G_\Omega\left(x,z\right) G_\Omega\left(z,y\right)}{G_\Omega\left(x,y\right)} dz.$$

Here G_Ω is the Green function for the Laplace problem in Ω with Dirichlet boundary conditions.

The function $H_\Omega(x,y)$, defined above, has also a probabilistic interpretation. Indeed, $H_\Omega(x,y) = \mathbb{E}_x^y(\tau_\Omega)$ the expected lifetime of a conditioned Brownian motion that starts in x, is conditioned to converge to y and that is killed at the boundary.

There are many open problems regarding this expected lifetime. In particular, one open question is where $H_\Omega(x,y)$ attains its maximum in $\bar{\Omega} \times \bar{\Omega}$. In Chapter 6 we study this problem for Ω the unit ball in $\mathbb{R}^n$, $n \geq 2$.

Our result states that the function $x \mapsto H_\Omega(x,y)$ is increasing along the hyperbolic geodesics through y in increasing Euclidean distance and along the orthogonal trajectories of the hyperbolic geodesics through y. A direct consequence is that the maximum is attained at opposite boundary points.

At the end of the chapter we compute an explicit formula for $\lambda_c(\Omega)^{-1}$ and we discuss some remarkable identities involving this quantity and a sum of inverse Dirichlet eigenvalues.

Samenvatting

Hogere Orde Elliptische Vergelijkingen en Positiviteit

van Anna Dall'Acqua

Dit proefschrift behandelt positiviteit van vierde orde elliptische problemen. Met positiviteit wordt bedoeld dat een positief rechterlid een positieve oplossing van de differentiaalvergelijking geeft. Voor tweede orde elliptische partiële differentiaalvergelijkingen is dit feit bewezen en staat het bekend onder de naam "maximum principe". Dit maximum principe heeft echter geen directe uitbreiding naar hogere orde elliptische differentialvergelijkingen.

Het verschil tussen tweede en hogere orde elliptische problemen komt ook tot uiting in het mechanische model dat wordt beschreven door deze vergelijkingen. Een tweede orde elliptische vergelijking beschrijft de verplaatsing van een membraan dat wordt belast en uit alledaagse ervaring weet men dat dit membraan in het geheel in de richting van de kracht beweegt. Een vierde orde elliptische vergelijking beschrijft daarentegen de verplaatsing van een elastische plaat die wordt belast. In praktijk kan worden gedacht aan een plat dak waarop regenwater of sneeuw druk uitoefent. In het algemeen zullen niet alle delen van de plaat onder invloed van de last in dezelfde richting bewegen ten opzichte van de ruststand. Het mechanische model geeft echter aanleiding te veronderstellen dat enige positiviteit behouden blijft. Dit aspect zal nader worden bestudeerd in dit proefschrift.

We beschouwen het model dat een elastische plaat beschrijft die is vastgeklemd aan de rand en waarop een kracht f wordt uitgeoefend. Dit model wordt beschreven door het volgende vierde orde probleem met Dirichlet randvoorwaarden

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = 0 & \text{op } \partial\Omega, \\ \frac{\partial}{\partial\nu} u = 0 & \text{op } \partial\Omega. \end{cases} \tag{S.1}$$

Hierin is Ω een begrensd gebied in $\mathbb{R}^n$ voor zekere $n \in \mathbb{N}$, $n \geq 2$ met $\partial\Omega \in C^{4,\alpha}$, $\alpha \in (0,1)$, en f is een continue, niet-negatieve functie die is gedefinieerd op $\bar{\Omega}$.

De Greense functie die kan worden geassocieerd met (S.1) voor algemene Ω kan een tekenwisseling hebben. Gezien de fysische interpretatie verwachten we echter dat de tekenbehoudende effecten veel sterker zijn.

Het voornaamste resultaat van dit proefschrift is het opsplitsen van de Greense functie voor (S.1) als een som van twee termen: een positieve singuliere term en een tekenveranderende reguliere term die echter beide wel voldoen aan de Dirichlet randvoorwaarden. De positieve term beschrijft het lokale gedrag terwijl de andere term, die mogelijk van teken wisselt, alleen indirect afhangt van het lokale gedrag en dus regulier is. Als gevolg hiervan bewijzen we dat de tekenbehoudende effecten veel sterker zijn dan de tekenveranderende. Momenteel zijn onze resultaten beperkt tot het twee-dimensionale geval.

In Hoofdstuk 2 zetten we een eerste stap in de bestudering van de relatie tussen het gedrag van de singulariteit van de Greense functie en de Dirichlet randvoorwaarden. We bewijzen scherpe afschattingen voor de absolute waarde van de Greense functie en haar afgeleiden in termen die afhankelijk zijn van de afstand tot de rand. Dit soort afschattingen vormen een belangrijk hulpmiddel in het bewijzen van regulariteitseigenschappen in de omgeving van de rand.

In Hoofdstuk 3 en 4 bestuderen we voor welke gebieden de vergelijking van de geklemde plaat positiviteit behoudend is. Lange tijd was onbekend of convexiteit van het domein een noodzakelijke voorwaarde is voor positiviteit. Hadamard sprak in [47] het vermoeden uit dat de vergelijking voor de geklemde plaat positiviteit behoudt op gebieden met de vorm van een Limaçon de Pascal. De Limaçons de Pascal zijn een één-parameter familie van gebieden die continu overgaan van cirkel naar cardioïde. We bewijzen dat dit vermoeden in het algemeen onjuist is. Er bestaan echter wel niet-convexe limaçons waarop de vergelijking van de geklemde plaat positiviteit behoudt.

De afschattingen voor de Greense functie zijn scherp van boven maar niet van beneden. Om een ondergrens voor de Greense functie te bepalen construeren we een overdekking van het gebied Ω en haar rand door een eindig aantal deelgebieden die ieder een positieve Greense functie hebben. Zo'n overdekking van het gebied en haar rand bestaat omdat er zowel convexe als niet-convexe gebieden bestaan waarop de vergelijking van de geklemde plaat positiviteit bewaart en omdat we kunnen aantonen dat kleine $C^{2,\gamma}$ verstoringen van het gebied deze eigenschap niet verstoren. Dankzij deze overdekking kunnen we de Greense functie opsplitsen in een positief, singulier deel en een tekenveranderend regulier deel. Dit wordt bewezen in Hoofdstuk 5.

Hoofdstuk 6 staat enigszins los van de overige hoofdstukken. In de bestudering van elliptische randwaardeproblemen stuiten we op enkele open problemen in de waarschijnlijkheidsrekening.

Ook voor stelsels van tweede orde elliptische randwaardeproblemen bestaan enkele interessante vragen betreffende positiviteit. In [66] wordt het volgende stelsel elliptische differentiaalvergelijkingen geïntroduceerd als model-probleem voor de positiviteit van een stelsel niet-coöperatief gekoppelde differentiaalvergelijkingen

$$\begin{cases} -\Delta u &= f - \lambda v \quad \text{in } \Omega, \\ -\Delta v &= f \quad \text{in } \Omega, \\ u = v &= 0 \quad \text{op } \partial\Omega. \end{cases}$$

Hierin is Ω een begrensde reguliere deelverzameling van $\mathbb{R}^n$ en $\lambda \in \mathbb{R}^+$. Er kan worden bewezen dat er een waarde $\lambda_c(\Omega)$ bestaat zodanig dat voor alle $f \geq 0$ de oplossing u positief is dan en slechts dan als $\lambda < \lambda_c(\Omega)$ en dat

$$\lambda_c(\Omega)^{-1} = \sup_{x,y\in\Omega} H_\Omega(x,y),$$

met

$$H_\Omega(x,y) = \int_\Omega \frac{G_\Omega(x,z)\, G_\Omega(z,y)}{G_\Omega(x,y)} dz.$$

Hierin is G_Ω the Greense functie voor het Laplace-probleem in Ω met Dirichlet randvoorwaarden.

De functie $H_\Omega(x,y)$ is ook gelijk aan de verwachte levensduur $\mathbb{E}_x^y(\tau_\Omega)$ van de Brownse beweging die geconditioneerd is zodanig dat zij begint in x, eindigt in y en niet meetelt als zij de rand overschrijdt.

Er zijn veel open problemen betreffende de verwachte levensduur. Een van de open vragen is waar $H_\Omega(x,y)$ zijn maximum aanneemt in $\bar{\Omega} \times \bar{\Omega}$. In Hoofdstuk 6 wordt dit probleem behandeld in het geval dat het gebied Ω de eenheidsbol in $\mathbb{R}^n$ is met $n \geq 2$.

We bewijzen dat de functie $x \mapsto H_\Omega(x,y)$ stijgend is langs de hyperbolische geodeten door y in de richting van toenemende Euclidische afstand en ook langs de orthogonale trajectoriën van deze geodeten. Een direct gevolg hiervan is dat het maximum wordt aangenomen in tegenoverliggende randpunten.

Aan het einde van het hoofdstuk leiden we een expliciete formule af voor $\lambda_c(\Omega)^{-1}$ en we bespreken enkele opmerkelijke relaties van deze uitdrukking met een som van inverse Dirichlet eigenwaarden.

Sommario

Problemi Ellittici di Ordine Superiore e Positività

di Anna Dall'Acqua

L'argomento principale di questa tesi è lo studio della positività dei problemi ellittici di quarto ordine. Qui il termine positività si riferisce alla proprietà che un termine positivo nel membro destro dell'equazione differenziale porti a soluzioni positive. Nel caso di equazioni alle derivate parziali ellittiche di secondo ordine tale risultato è noto come "principio di massimo". Com'è ben noto, il principio di massimo non si può generalizzare ad equazioni ellittiche di ordine superiore.

La differenza fra i problemi ellittici di secondo ordine e quelli di ordine superiore appare evidente anche guardando ai problemi fisici descritti da questi due diversi tipi di equazioni ellittiche. Un'equazione ellittica del secondo ordine descrive la posizione d'equilibrio assunta da una membrana caricata di un peso. L'esperienza quotidiana indica che una membrana, su cui è stata applicata una forza, si muove tutta nella stessa direzione. Un'equazione ellittica di quarto ordine descrive, invece, la posizione d'equilibrio assunta da una piastra elastica caricata di un peso. Si può pensare, per esempio, alla posizione di un tetto orizzontale e al peso che questo deve sostenere per neve o pioggia. In generale la piastra non si muove tutta nella stessa direzione. Ciò nonostante dal modello meccanico appare evidente che della positività rimane. In questa tesi studieremo questo aspetto.

Noi considereremo il modello matematico di una piastra elastica su cui è applicata una forza f e che è incastrata sul bordo, cioè il seguente problema ellittico del quarto ordine con condizioni al contorno di Dirichlet:

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = 0 & \text{su } \partial\Omega, \\ \frac{\partial}{\partial\nu} u = 0 & \text{su } \partial\Omega. \end{cases} \tag{S.1}$$

Qui Ω è un dominio limitato in $\mathbb{R}^n$, $n \in \mathbb{N}$ e $n \geq 2$, con $\partial\Omega \in C^{4,\alpha}$, $\alpha \in (0,1)$, e f è una funzione definita su $\bar{\Omega}$ continua e non-negativa. Il problema (S.1) è noto come "l'equazione della piastra incastrata".

In generale la funzione di Green associata a (S.1) in un dominio Ω può cambiare segno. Ciò nonostante, pensando al problema fisico, ci aspettiamo che la tendenza a preservare il segno sia più forte di quella opposta.

Il maggior risultato di questa tesi consiste nello scrivere la funzione di Green associata a (S.1) come la somma di due termini che soddisfano le condizioni al contorno di Dirichlet: un termine positivo e singolare e un termine regolare ma di segno variabile. Il termine positivo descrive il comportamento locale mentre l'altro termine, che potrebbe diventare negativo, dipende dal comportamento locale solo indirettamente e quindi è regolare. Da questo risultato segue che la tendenza a preservare il segno è più forte di quella opposta. I nostri risultati sono, fino a questo momento, limitati al caso bidimensionale.

Un primo approfondimento su come la singolarità della funzione di Green si comporta in relazione alle condizioni al contorno di Dirichlet è riportato nel Capitolo 2. Il risultato principale presentato nel secondo capitolo consiste nelle stime ottimali, dipendenti dalla distanza dalla frontiera del dominio, del valore assoluto della funzione di Green associata a (S.1) e delle sue derivate. Queste stime sono utili per dimostrare risultati di regolarità in spazi che considerano il comportamento della funzione alla frontiera.

Nel terzo e quarto capitolo studiamo su quali dominii l'equazione della piastra incastrata preservi la positività. Il seguente problema è stato per molto tempo irrisolto: la convessità del dominio è una condizione necessaria per la positività? In [47], Hadamard dice che il problema (S.1) su una piastra con la forma di un "Limaçon de Pascal" preserva la positività. Ricordiamo brevemente che il termine "Limaçons de Pascal" indica una famiglia di dominii dipendente da un parametro e che variano con continuità dal disco alla cardioide. Nel terzo capitolo dimostriamo che l'affermazione di Hadamard è sbagliata, ma che comunque ci sono "limaçons" non-convessi su cui l'equazione della piastra incastrata ha la proprietà di preservare la positività. Nel quarto capitolo sono presentati i metodi, conosciuti fino a questo momento, per trovare dominii su cui l'equazione della piastra incastrata preserva la positività.

Le stime della funzione di Green ottenute nel secondo capitolo sono ottime dal di sopra ma non dal di sotto. Il nostro metodo per ottenere delle stime ottimali del comportamento della funzione di Green dal di sotto consiste nello costruire un ricoprimento del dominio Ω e della sua frontiera con un numero finito di dominii che hanno una funzione di Green positiva. Questa costruzione è possibile perchè ci sono dominii convessi e anche non-convessi su cui l'equazione della piastra incastrata ha la proprietà di mantenere la positività e inoltre perchè si può dimostrare che questa proprietà non si perde con piccole $C^{2,\gamma}$ perturbazioni del dominio. Grazie a questo ricoprimento è possibile scrivere la funzione di Green come la somma di un termine positivo e singolare e di uno regolare ma di segno variabile. Questo risultato è presentato nel quinto capitolo.

Il sesto capitolo differisce negli argomenti dai precedenti. Studiando la positività nei problemi ellittici abbiamo incontrato alcuni problemi aperti nella teoria della probabilità.

Nello studio della positività nei sistemi di equazioni ellittiche di secondo ordine con condizioni al contorno si incontrano molte questioni interessanti. In [66] il seguente sistema ellittico

$$\begin{cases} -\Delta u &= f - \lambda v \quad \text{in } \Omega, \\ -\Delta v &= f \qquad\quad\ \text{in } \Omega, \\ u = v &= 0 \qquad\quad\ \text{su } \partial\Omega, \end{cases}$$

è presentato come un problema modello per lo studio della positività nei sistemi accoppiati in modo non-cooperativo. Qui Ω è un dominio in $\mathbb{R}^n$ limitato e regolare e $\lambda \in \mathbb{R}^+$. Si può dimostrare che esiste $\lambda_c(\Omega)$ tale che per ogni $f \geq 0$ la soluzione u è positiva se e solo se $\lambda \leq \lambda_c(\Omega)$. Inoltre si ha che

$$\lambda_c(\Omega)^{-1} = \sup_{x,y\in\Omega} H_\Omega(x,y),$$

dove

$$H_\Omega(x,y) = \int_\Omega \frac{G_\Omega\,(x,z)\,G_\Omega\,(z,y)}{G_\Omega\,(x,y)} dz.$$

Qui G_Ω indica la funzione di Green per il problema di Laplace con condizioni al contorno di Dirichlet in Ω.

La funzione $H_\Omega(x,y)$ definita precedentemente ha anche un'interpretazione probabilistica. Infatti, $H_\Omega(x,y) = \mathbb{E}_x^y(\tau_\Omega)$, che rappresenta la durata di vita attesa di un moto Browniano che inizia in x, è condizionato a convergere ad y ed è ucciso sulla frontiera.

Ci sono molti problemi aperti che riguardano questa attesa durata di vita. Uno fra questi è il determinare dove la funzione $x \mapsto H_\Omega(x,y)$ raggiunge il suo massimo in $\bar{\Omega} \times \bar{\Omega}$. Nel sesto capitolo studiamo questo problema sulla palla unitaria in $\mathbb{R}^n$, $n \geq 2$, e dimostriamo che la funzione $H_\Omega(x,y)$ è crescente lungo le geodesiche iperboliche passanti per y e allontanandosi da questo punto, e inoltre lungo le traiettorie ortogonali a queste, sempre allontanandosi dal punto. Una conseguenza diretta di questa caratteristica è che il massimo di $H_\Omega(x,y)$ è raggiunto al bordo per x e y diametralmente opposti.

Alla fine del capitolo calcoliamo la formula esplicita di $\lambda_c(\Omega)^{-1}$ e discutiamo alcune identità che coinvolgono questa quantità e una somma di inversi degli autovalori di Dirichlet.

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Curriculum Vitae

Anna Dall'Acqua was born on October 28, 1978 in Conegliano, Italy, third child of Ferdinando and Eugenia and beloved sister of Angelo and Mario.

In 1997 she completed her high school education at the "Liceo Scientifico G. Marconi" in Conegliano and started her studies in Mathematics at the University of Triest, Italy. Under the supervision of Prof.dr. E.L. Mitidieri, Anna obtained her M.Sc. degree "cum laude" on July 11, 2001 with the thesis "Esistenza di soluzioni positive per sistemi di tipo reazione-diffusione" (Existence of positive solutions for reaction-diffusion systems).

In September 2001 she started her PhD research at the Functional Analysis Group at the Delft University of Technology under the supervision of Prof.dr. Ph. Clément and Dr. G. Sweers.