Outer-(J_1, J_2)-lossless factorizations of linear discrete time-varying systems

Xiaode Yu, Jacquelien Scherpen, Allejan van der Veen, and Patrick Dewilde

Delft University of Technology, Dept. of Electrical Engineering,
P.O. Box 5031, 2600 GA Delft, The Netherlands
E-mail: X.Yu@ J.M.A.Scherpen@ A.J.vanderVeen@ P.M.Dewilde@et.tudelft.nl

Abstract

In this paper the outer-J-lossless factorization for linear discrete time-varying systems is treated. Lossless operators and its corresponding J-lossless chain-scattering operators are studied. Then the factorization is treated by first 'taking out' the anticausal part, and then considering the outer-J-lossless factorization of the causal part.

1 Introduction

In reality most physical and economical systems demonstrate a time-varying and/or non-linear behavior. Taking into account the fact that non-linear systems operating around a particular trajectory within their operation envelope can adequately be described as linear, time-varying (LTV) systems, the development of several system theoretical concepts as exist for linear time-invariant systems can be motivated for linear time-varying systems.

In this paper we study the outer-(J_1, J_2)-lossless factorizations of linear discrete time-varying systems. We use the setting of the linear discrete time-varying system theory as is developed in e.g. [2, 4]. We consider the scattering and chain-scattering operators of a time-varying system, where the chain-scattering operator of a system is a useful tool for the design of a controller for the original system. In the time-invariant case we refer to [1], where the J-lossless factorization is studied for chain-scattering representations for the purpose of H_∞ control.

In Section 2 we treat the preliminaries and notation, as is reported in e.g. [4]. Then in Section 3, we study lossless scattering operators and its corresponding J-lossless chain-scattering operators in a time-varying setting. In Section 4 we continue with the development of an outer-J-lossless factorization by separating the system in a causal and anticausal part.

2 Preliminaries

In this section, we introduce the notation used in representing linear Time-Varying (LTV) systems. To be consistent with earlier literature in which this notation was defined, e.g. [4], we think of sequences as row vectors, and of operators as acting on the sequences at the left, so that we will write uT rather than Tu, which is the usual notation for time-invariant systems in the control literature.

A state space realization of the LTV system P to be controlled, is denoted on a local time scale as:

$$\begin{aligned} x_{k+1} &= x_k A_k + u_k B_k \\ y_k &= x_k C_k + u_k D_k \end{aligned} \qquad (1)$$

where x_k, u_k and y_k are (finite dimensional) row vectors in respectively $\mathbb{C}^{N_k}$, $\mathbb{C}^{M_k}$ and $\mathbb{C}^{L_k}$ and the matrices $\{A_k, B_k, C_k, D_k\}$ are bounded matrices of appropriate dimensions. Remark that this notation is compatible with the earlier work on LTV systems as reported in e.g. [4].

To denote the state space representation more compactly, we introduce as done in e.g. [4], the vector sequence space $\mathcal{B}$ (which contains information on the dimensions), $\mathcal{B} = \cdots \times \boxed{\mathcal{B}_0} \times \mathcal{B}_1 \times \cdots$, where $\mathcal{B}_k = \mathbb{C}^{N_k}$ and the square box identifies the space of the 0–th entry. In a similar way, we introduce the dimension space sequence $\mathcal{M}$ and $\mathcal{N}$ from the integer sequences $\{M_k\}$ and $\{L_k\}$. It is allowed that some integers in these sequences are zero. The space of sequences in $\mathcal{B}$ with finite 2-norm will be denoted by $\ell_2^{\mathcal{B}}$. Next we stack the sequence of state vectors x_k, input vectors u_k and output vectors y_k into ∞-dimensional row vectors x, u and y; denoted explicitly for the state vector sequence as, $x = \begin{bmatrix} \cdots & x_{-1} & \boxed{x_0} & x_1 & \cdots \end{bmatrix}$ where the square identifies the position of the 0-th entry. Let $\mathcal{B}^{(-1)}$ denote the shifted dimension space sequence of $\mathcal{B}$, i.e., $\mathcal{B}^{(-1)} = \cdots \times \boxed{\mathcal{B}_1} \times \mathcal{B}_2 \times \cdots$, and let $\mathcal{D}(\mathcal{M}, \mathcal{N})$ denote the Hilbert space of bounded diagonal operators $\ell_2^{\mathcal{M}} \to \ell_2^{\mathcal{N}}$, then we can stack the system operators A_k, B_k, C_k and D_k into the diagonal operators A, B, C and D, as (denoted only explicitly for A):

$$A = diag \begin{bmatrix} \cdots & A_{-1} & \boxed{A_0} & A_1 & \cdots \end{bmatrix} \in \mathcal{D}(\mathcal{B}, \mathcal{B}^{(-1)}),$$

$$C \in \mathcal{D}(\mathcal{B}, \mathcal{N}),\ B \in \mathcal{D}(\mathcal{M}, \mathcal{B}^{(-1)}),\ D \in \mathcal{D}(\mathcal{M}, \mathcal{N}).$$

Let the causal bilateral shift operator on sequences be denoted by Z, such that,

$$\begin{bmatrix} \cdots x_{-1} \boxed{x_0} x_1 \cdots \end{bmatrix} Z = \begin{bmatrix} \cdots x_{-2} \boxed{x_{-1}} x_0 \cdots \end{bmatrix}$$

then a compact notation on a global time scale of the state space representation (1) is:

$$\begin{aligned} xZ^{-1} &= xA + uB \\ y &= xC + uD \end{aligned} \quad \text{or} \quad \mathbf{P} = \begin{bmatrix} A & C \\ B & D \end{bmatrix} \qquad (2)$$

With this notation it is possible to represent a LTV system as an operator. If the system is asymptotically stable, then the inverse of the operator $(I-AZ)$ exists and is in $\mathcal{U}$ and the operator representation of the LTV system P becomes:

$$P = D + BZ(I-AZ)^{-1}C \qquad (3)$$

This transfer operator is *upper* triangular and in general the Hilbert space of bounded upper operators acting from $\ell_2^{\mathcal{M}}$ to $\ell_2^{\mathcal{N}}$ is denoted by $\mathcal{U}(\mathcal{M},\mathcal{N})$ or denoted in short by $\mathcal{U}$. When the dimension N_k of the state vector is finite for all k then the operator represented as in Eq. (3) is *locally finite*. In the same way as $\mathcal{U}$, we denote the space of bounded operators by $\mathcal{X}(\mathcal{M},\mathcal{N})$ and the space of bounded *lower* triangular operators by $\mathcal{L}(\mathcal{M},\mathcal{N})$. In addition to the bounded operator space, we denote by $\mathcal{X}_2(\mathcal{M},\mathcal{N})$ the Hilbert-Schmidt space which is in $\mathcal{X}(\mathcal{M},\mathcal{N})$ and which is additionally bounded in the Hilbert-Schmidt norm. Related spaces in $\mathcal{X}_2$ are the upper, lower and diagonal Hilbert-Schmidt spaces given by $\mathcal{U}_2 = \mathcal{X}_2 \cap \mathcal{U}$, $\mathcal{L}_2 = \mathcal{X}_2 \cap \mathcal{L}$, and $\mathcal{D}_2 = \mathcal{X}_2 \cap \mathcal{D}$. The projection operators of these spaces, $\mathbf{P}$, $\mathbf{P}_{\mathcal{L}_2\mathbf{Z}^{-1}}$, and $\mathbf{P}_0$ denote the projections onto $\mathcal{U}_2$, $\mathcal{L}_2\mathbf{Z}^{-1}$, and $\mathcal{D}$, respectively.
Finally, by an outer operator $T_o \in \mathcal{U}$ we mean that $\overline{\mathcal{U}_2 T_o} = \mathcal{U}_2$, where $\overline{\mathcal{U}_2 T_o}$ is the closure of $\mathcal{U}_2 T_o$. If an outer operator is invertible, then its inverse is also upper.

3 Lossless and J-lossless operators

Lossless and J-lossless operators (functions) play an important role in system and control engineering because of their many useful and elegant properties. We consider lossless and J-lossless operators and their properties in linear discrete time-varying context for the purpose of H_∞ control.

Definition 3.1 *$\Sigma \in \mathcal{X}$ is an isometry if $\Sigma\Sigma^* = I$, a coisometry if $\Sigma^*\Sigma = I$ and unitary if both $\Sigma\Sigma^* = I$ and $\Sigma^*\Sigma = I$.* □

A special case for an isometric operator or a co-isometric operator occurs when the operator is upper.

Definition 3.2 *An isometric operator Σ is called lossless iff $\Sigma \in \mathcal{U}$. A coisometric operator Σ is called co-lossless iff $\Sigma \in \mathcal{U}$. A unitary operator Σ is called inner iff $\Sigma \in \mathcal{U}$. In this case, Σ is both lossless and co-lossless.* □

The next theorem gives a characterization of isometric and co-isometric operators.

Theorem 3.3 *Let $\Sigma \in \mathcal{U}$ be a locally finite operator with a realization $\{A_\Sigma, B_\Sigma, C_\Sigma, D_\Sigma\}$ and $\ell_{A_\Sigma} < 1$. $\Sigma = D_\Sigma + B_\Sigma \mathbf{Z}(I - A_\Sigma \mathbf{Z})^{-1} C_\Sigma$. Σ is an isometry iff there exists a Hermitian operator $Q \in \mathcal{D}$ such that*

$$\begin{bmatrix} A_\Sigma & C_\Sigma \\ B_\Sigma & D_\Sigma \end{bmatrix} \begin{bmatrix} Q^{(-1)} & \\ & I \end{bmatrix} \begin{bmatrix} A_\Sigma & C_\Sigma \\ B_\Sigma & D_\Sigma \end{bmatrix}^* = \begin{bmatrix} Q & \\ & I \end{bmatrix} \qquad (4)$$

Σ is a coisometry iff there exists a Hermitian operator $P \in \mathcal{D}$ such that:

$$\begin{bmatrix} A_\Sigma & C_\Sigma \\ B_\Sigma & D_\Sigma \end{bmatrix}^* \begin{bmatrix} P & \\ & I \end{bmatrix} \begin{bmatrix} A_\Sigma & C_\Sigma \\ B_\Sigma & D_\Sigma \end{bmatrix} = \begin{bmatrix} P^{(-1)} & \\ & I \end{bmatrix} \qquad (5)$$

Σ is unitary iff both (4) and (5) are satisfied.
If $\{A_\Sigma, B_\Sigma, C_\Sigma, D_\Sigma\}$ is a uniform realization, then $Q \gg 0$ and $P \gg 0$; if Σ is also unitary, then $P = Q^{-1}$.

Proof: *Sufficiency:* Let $\Sigma \in \mathcal{U}$ be a locally finite operator with a realization $\{A_\Sigma, B_\Sigma, C_\Sigma, D_\Sigma\}$ and $\ell_{A_\Sigma} < 1$. Assume that (4) is satisfied. Then $\Sigma\Sigma^* = I$. With the expression of Σ we have

$$\Sigma\Sigma^* = [D_\Sigma + B_\Sigma(\mathbf{Z}^* - A_\Sigma)^{-1}C_\Sigma][D_\Sigma^* + C_\Sigma^*(\mathbf{Z} - A_\Sigma^*)^{-1}B_\Sigma^*]$$

By expanding the right hand side of the above expression and with the conditions given by (4), we can obtain that $\Sigma\Sigma^* = D_\Sigma D_\Sigma^* + B_\Sigma Q^{(-1)} B_\Sigma^* = I$. $\Sigma^*\Sigma = I$ can be proved in a similar way.
Necessity: Let $\Sigma \in \mathcal{U}$ be a locally finite operator with a realization $\{A_\Sigma, B_\Sigma, C_\Sigma, D_\Sigma\}$ and $\ell_{A_\Sigma} < 1$. Assume that $\Sigma\Sigma^* = I$. Then, conditions given by (4) are satisfied. In particular we have $\mathbf{P}_0(\Sigma\Sigma^*) = \Sigma\Sigma^* = I$. Define $\mathbf{F}_o = (I - A_\Sigma \mathbf{Z})^{-1} C_\Sigma$, so that $\Sigma = D_\Sigma + B_\Sigma \mathbf{Z}\mathbf{F}_o$. Hence,

$$\mathbf{P}_0(\Sigma\Sigma^*) = D_\Sigma D_\Sigma^* + B_\Sigma \mathbf{P}_0(\mathbf{Z}\mathbf{F}_o\mathbf{F}_o^*\mathbf{Z}^*) B_\Sigma^* \qquad (6)$$

Let $Q = \mathbf{P}_0(\mathbf{F}_o\mathbf{F}_o^*)$, then $\Sigma\Sigma^* = I$ indicates $D_\Sigma D_\Sigma^* + B_\Sigma Q^{(-1)} B_\Sigma^* = I$ and Q satisfies the recursion $Q = C_\Sigma C_\Sigma^* + A_\Sigma Q^{(-1)} A_\Sigma^*$. Consider

$$\begin{aligned} \mathbf{P}_0(\mathbf{Z}^{-n}\Sigma\Sigma^*) &= \mathbf{P}_0(\mathbf{Z}^{-n} D_\Sigma D_\Sigma^*) + \mathbf{P}_0(\mathbf{Z}^{-n} D_\Sigma \mathbf{F}_o^* \mathbf{Z}^* B_\Sigma^*) \\ &+ \mathbf{P}_0(\mathbf{Z}^{-n} B_\Sigma \mathbf{Z}\mathbf{F}_o D_\Sigma^*) + \mathbf{P}_0(\mathbf{Z}^{-n} B_\Sigma \mathbf{Z}\mathbf{F}_o\mathbf{F}_o^*\mathbf{Z}^* B_\Sigma^*) \end{aligned}$$

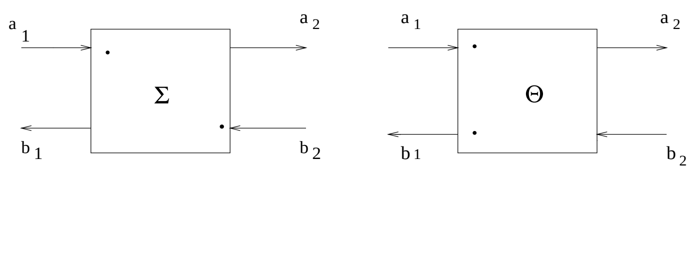
If $n > 0$, the first and second terms on the right hand side of the above equation are equal to zero. The third term yields $\mathbf{P}_0(\mathbf{Z}^{-n} B_\Sigma \mathbf{Z}\mathbf{F}_o D_\Sigma^*) = B_\Sigma^{(n)} A_\Sigma^{\{n-1\}} C_\Sigma D_\Sigma^*$ and the fourth term results in $\mathbf{P}_0(\mathbf{Z}^{-n} B_\Sigma \mathbf{Z}\mathbf{F}_o\mathbf{F}_o^*\mathbf{Z}^* B_\Sigma^*) = B_\Sigma^{(n)} A_\Sigma^{\{n-1\}} A_\Sigma Q^{(-1)} B_\Sigma^*$. Substituting the results of the third and fourth terms back into $\mathbf{P}_0(Z^{-n}\Sigma\Sigma^*)$, we obtain: $\mathbf{P}_0(\mathbf{Z}^{-n}\Sigma\Sigma^*) = B_\Sigma^{(n)} A_\Sigma^{\{n-1\}}(C_\Sigma D_\Sigma^* + A_\Sigma Q^{(-1)} B_\Sigma^*)$. Since $\Sigma\Sigma^*$ is diagonal, $\mathbf{P}_0(Z^{-n}\Sigma\Sigma^*) = 0$ for $n \neq 0$. Then $C_\Sigma D_\Sigma^* + A_\Sigma Q^{(-1)} B_\Sigma^* = 0$. In a similar way, we can prove that if $\Sigma^*\Sigma = I$, then (5) is satisfied. The rest of the proof follows immediately from the definitions of uniform reachability and uniform observability. □

Referring to Figure 1 (a), let Σ be a known operator, mapping input $[\, a_1 \quad b_2 \,]$ to output $[\, a_2 \quad b_1 \,]$, i.e.

$$[\, a_2 \quad b_1 \,] = [\, a_1 \quad b_2 \,]\Sigma = [\, a_1 \quad b_2 \,] \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \qquad (7)$$

In the figure, the variable with a dot stands for an input of the mapping and without a dot stands for an output. If Σ_{22} is invertible, we can derive the mapping from $[\, a_1 \quad b_1 \,]$ to $[\, a_2 \quad b_2 \,]$, denoted by Θ in Figure 1(b) from Σ as

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} = \begin{bmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} & -\Sigma_{12}\Sigma_{22}^{-1} \\ \Sigma_{22}^{-1}\Sigma_{21} & \Sigma_{22}^{-1} \end{bmatrix} \qquad (8)$$



(a) (b)

Figure 1: Scattering operator Σ, chain scattering operator Θ.

Σ is called a scattering operator and Θ is called the corresponding chain scattering operator.
If we introduce a feedback relation $b_1 = a_1 S$ between b_1 and a_1,

then the closed loop mapping from b_2 to a_2, denoted by Φ, is given by

$$\begin{aligned}\Phi &= \Sigma_{21}+\Sigma_{22}(I-S\Sigma_{12})^{-1}S\Sigma_{11} && (9)\\ &= \mathrm{HM}(\Theta;S)=(S\Theta_{12}+\Theta_{22})^{-1}(S\Theta_{11}+\Theta_{21}) && (10)\end{aligned}$$

where HM stands for HoMographic transformation.
In Figure 1, we use a dot to indicate the variables of the input port. The variables with arrows into the block are input variables and with arrows out of the block are output variables.
Let $J_1 \in \mathcal{D}$ be the input port signature and $J_2 \in \mathcal{D}$ the output port signature matrices which are defined as $J_i = \begin{bmatrix} \ddots & & & & \\ & j_{i,-1} & & & \\ & & j_{i,0} & & \\ & & & j_{i,1} & \\ & & & & \ddots \end{bmatrix}$ for $i=1,2$, where the entry $j_{i,k} = \begin{bmatrix} I_{(p+)_k} & \\ & -I_{(p-)_k} \end{bmatrix}$ $(k=-\infty,\cdots,+\infty)$ is determined by the dimensions of the input and output of the ports at time instant k. For a chain scattering operator, the dimension of input variables on the input port is $p+$ and the dimension of output variables on the input port is $p-$. It is reversed on the output port.

Definition 3.4 *Let J_1 and J_2 be the input and output signature operators respectively of a known operator $\Theta \in \mathcal{X}$. Θ is a (J_2,J_1)-isometry (sometimes shortly called a J-isometry) if $\Theta J_2 \Theta^* = J_1$. Θ is a (J_1,J_2)-coisometry (shortly a J-coisometry) if $\Theta^* J_1 \Theta = J_2$ and Θ is J-unitary if both $\Theta J_2 \Theta^* = J_1$ and $\Theta^* J_1 \Theta = J_2$.* □

Theorem 3.5 *Let J_1 and J_2 be the input and output signature operators respectively of a known operator $\Theta \in \mathcal{X}$. Let the operator Σ be isometric, coisometric or unitary. If the corresponding chain scattering operator Θ exists, then it is J-isometric, J-coisometric or J-unitary, respectively. If the corresponding dual chain scattering operator exists, then it is J-isometric, J-coisometric or J-unitary, respectively.*

Proof: For the proof of the first statement we refer to [4]. The second statement is proved in a similar way. □

If $\mathcal{H}$ is a locally finite $\mathcal{D}$-invariant subspace, then it has some strong basis representation $\mathbf{F}$ such that $\mathcal{H} = \mathcal{D}_2\mathbf{F}$. Similar to the definition of a Gramian operator by $\Lambda_F = \mathbf{P}_0(\mathbf{F}\mathbf{F}^*)$, we define the J-Gramian operator of this basis as the diagonal operator: $\Lambda^J_{\mathbf{F}} = \mathbf{P}_0(\mathbf{F}J\mathbf{F}^*) \in \mathcal{D}(\mathcal{B},\mathcal{B})$. The operator $\mathbf{F}$ is J-orthonormal if $\Lambda^J_{\mathbf{F}} = J_{\mathcal{B}}$, where $J_{\mathcal{B}}$ is some signature operator on $\mathcal{B}$. We call $\mathcal{H}$ *regular* if the J-Gramian operator of any strong basis of $\mathcal{H}$ is boundedly invertible. Note that $\Lambda^J_{\mathbf{F}}$ boundedly invertible implies the same for $\Lambda_{\mathbf{F}}$, i.e., $\Lambda_{\mathbf{F}} \gg 0$.

Let $T \in \mathcal{U}$ have a uniformly minimal realization $\{A,B,C,D\}$ with $\ell_A < 1$ and J_1 and J_2 be the input and output signature operators. Then $\mathbf{F}^* = B\mathbf{Z}(I-A\mathbf{Z})^{-1}$ and $\mathbf{F}_o = (I-A\mathbf{Z})^{-1}C$ are strong bases of $\mathcal{H}(T)$ and $\mathcal{H}_o(T)$, respectively. If $\mathbf{P}_0(\mathbf{F}J_1\mathbf{F}^*)$ and $\mathbf{P}_0(\mathbf{F}_oJ_1\mathbf{F}_o^*)$ are invertible, we say the realization $\{A,B,C,D\}$ is *regular*. Regular realizations of bounded lower operators or mixed operators are defined in a similar way.

The chain scattering operator of a lossless scattering system is not lossless itself, but has some special features.

Definition 3.6 *If an operator Σ is lossless, then we call the corresponding chain scattering operator Θ is J-lossless.* □

It is easy to obtain a similar characterization of a chain-scattering operator $\Theta \in \mathcal{U}$ being a J isometry as is obtained for of Σ being an isometry in Theorem 3.3. But, contrary to the scattering representation, where lossless operators are always upper by definition, the corresponding chain scattering representation can be lower or mixed. Thus we also need the extension of Theorem 3.3 to lower and mixed operators. It is well known that the cascade connection of J-lossless operators results in a J-lossless operator. In particular, the cascade connection of an upper J-lossless operator and a lower J-lossless operator results in a J-lossless operator which is in general not upper or lower anymore.

Theorem 3.7 *Let $\Theta \in \mathcal{X}$ be a locally finite operator and $\{A_1,B_1,C_1,A_2,B_2,C_2,D_\Theta\}$ be a regular realization with $\ell_{A_1} < 1$ and $\ell_{A_2} < 1$ such that $\Theta = D_\Theta + B_1\mathbf{Z}(I-A_1\mathbf{Z})^{-1}C_1 + B_2\mathbf{Z}^*(I-A_2\mathbf{Z}^*)^{-1}C_2$. Θ is (J_2,J_1)-isometric if there exists a Hermitian operator $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \in \mathcal{D}$ such that:*

$$\left[\begin{array}{cc|c} A_1 & & C_1 \\ & I & C_2 \\ \hline B_1 & & D_\Theta \end{array}\right] \left[\begin{array}{cc|c} Q_{11}^{(-1)} & Q_{12}^{(-1)} & \\ Q_{21}^{(-1)} & Q_{22}^{(-1)} & \\ \hline & & J_2 \end{array}\right] \left[\begin{array}{cc|c} A_1 & & C_1 \\ & I & C_2 \\ \hline B_1 & & D_\Theta \end{array}\right]^*$$

$$= \left[\begin{array}{cc|c} I & & \\ & A_2 & \\ \hline & B_2 & I \end{array}\right] \left[\begin{array}{cc|c} Q_{11} & Q_{12} & \\ Q_{21} & Q_{22} & \\ \hline & & J_1 \end{array}\right] \left[\begin{array}{cc|c} I & & \\ & A_2 & \\ \hline & B_2 & I \end{array}\right]^* \qquad (11)$$

If $Q \gg 0$, then Θ is J-unitary.

Proof: The proof follows straightforwardly from writing out the expressions for $\Theta J_2 \Theta^*$, using (11), and reorganizing the expressions. □

The next theorem reveals an important property of J-lossless operators, since it is a very useful if we want to design an H_∞ controller via the chain-scattering representation.

Theorem 3.8 *Let an operator $\Theta \in \mathcal{X}$ be (J_2,J_1)-lossless and have a partitioning as $\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}$ and let an operator $S \in \mathcal{U}$ be strictly contractive ($\|S\|_\infty < 1$). Let*

$$\Phi = HM(\Theta;S) = (S\Theta_{12}+\Theta_{22})^{-1}(S\Theta_{11}+\Theta_{21}) \qquad (12)$$

Then Φ is upper and $\|\Phi\|_\infty < 1$.

Proof: First, we show the invertibility of $(S\Theta_{12}+\Theta_{22})$. Since Θ is (J_2,J_1)-lossless, the corresponding $\Sigma \in \mathcal{U}$ is lossless and has a partitioning $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$ with Σ_{22} invertible. Under these conditions, $\|\Sigma_{12}\|_\infty < 1$ and $(I-S\Sigma_{12})$ is invertible. With the relation $\Sigma_{12} = -\Theta_{12}\Theta_{22}^{-1}$, we have $(I+S\Theta_{12}\Theta_{22}^{-1})$ invertible and then $(\Theta_{22}+S\Theta_{12})$ invertible.
Φ can be expressed with Σ and S as, $\Phi = \Sigma_{21}+\Sigma_{22}(I-S\Sigma_{12})^{-1}S\Sigma_{11}$.

Using the expansion of $(I-S\Sigma_{12})^{-1}$ yields$\Phi = \Sigma_{21} + \Sigma_{22}S\Sigma_{11} + \Sigma_{22}S\Sigma_{12}S\Sigma_{11} + \cdots$. Under the given conditions, the Neumann series converges to an upper matrix, i.e. Φ is upper. Now, rewrite equation (12) as

$$(S\Theta_{12}+\Theta_{22})\begin{bmatrix} \Phi & I \end{bmatrix} = \begin{bmatrix} S & I \end{bmatrix}\begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \quad (13)$$

Denote $\varphi = (S\Theta_{12}+\Theta_{22})$. Multiplying equation (13) on the right side first with the J operator and then multiplying each side with the conjugate transpose of themselves, we obtain, $\varphi(\Phi\Phi^* - I)\varphi^* = SS^* - I$. From the condition $||S||_\infty < 1$, we then have that $||\Phi||_\infty < 1$. □

Finally, in this section we introduce some notions on spaces that are of interest for the rest of the paper.

Definition 3.9 *Let $T \in \mathcal{U}$. Then we define the input null space as $\mathcal{K}(T) = \{U \in \mathcal{L}_2\mathbf{Z}^{-1} : P(UT) = 0\}$, the input state space as $\mathcal{H}(T) = \mathbf{P}_{\mathcal{L}_2\mathbf{Z}^{-1}}(\mathcal{U}_2T^*)$, the output state space as $\mathcal{H}_o(T) = \mathbf{P}(\mathcal{L}_2\mathbf{Z}^{-1}T)$, and the output null space as $\mathcal{K}_o(T) = \{Y \in \mathcal{U}_2 : \mathbf{P}_{\mathcal{L}_2\mathbf{Z}^{-1}}(YT^*) = 0$.* □

From the above definition it follows that $\overline{\mathcal{H}}(T) \oplus \mathcal{K}(T) = \mathcal{L}_2\mathbf{Z}^{-1}$, and that $\overline{\mathcal{H}_o}(T) \oplus \mathcal{K}_o(T) = \mathcal{U}_2$.

A generalization of a theorem on J-unitary operators (see [4]) to (J_2,J_1)-isometries is given as follows:

Theorem 3.10 *Let $\Theta \in \mathcal{U}$ be a (J_2,J_1)-isometry, i.e., $\Theta J_2\Theta^* = J_1$, then the output null space is given by $\mathcal{K}_o(\Theta) = \mathcal{U}_2 \ominus J_2 \oplus Ker(.\Theta^*|_{\mathcal{U}_2})$.* □

4 J-lossless Factorization

Let us consider the factorization $G = T_o\Theta \in \mathcal{X}$ with T_o invertible and outer, and Θ (J_2,J_1)-lossless in the discrete time-varying context. This kind of factorization is called an outer–J,J'-lossless factorization [1]. Here, we consider the case where the dimension sequence of the output of G is pointwise greater than or equal to the dimension sequence of the input. With T_o invertible this means that Θ should be of the same size as G.

Assume that an operator $G \in \mathcal{X}$ is specified by the representation,

$$G = D + B_c\mathbf{Z}(I-A_c\mathbf{Z})^{-1}C_c + B_a\mathbf{Z}^*(I-A_a\mathbf{Z}^*)^{-1}C_a \quad (14)$$

with $\ell_{A_c} < 1$, $\ell_{A_a} < 1$ and the dimension of the output of G is pointwise greater than or equal to the dimension of the input. Suppose that G admits a factorization:

$$G = G_1\Theta_a \quad (15)$$

where the operator $\Theta_a \in \mathcal{L}$ is anticausal and J-lossless (the subscript 'a' stands for anticausal), and G_1 is causal. Furthermore, suppose that G_1 admits a factorization as,

$$G_1 = T_o\Theta_c \quad (16)$$

where $\Theta_c \in \mathcal{U}$ (the subscript 'c' stands for causal) is J-lossless and $T_o \in \mathcal{U}$ is outer. Define

$$\Theta = \Theta_c\Theta_a \quad (17)$$

then, G has an outer–J-lossless factorization $G = T_o\Theta$.
With this strategy, we consider the outer–J-lossless factorization of G in two steps, first we take out the anticausal J-lossless part and then the causal J-lossless part.

4.1 Anticausal J-lossless factorization

Let $G \in \mathcal{X}$ be a given chain scattering operator specified by (14) with $\ell_{A_a} < 1$ and $\ell_{A_c} < 1$, with port signature matrices (J_1,J_2), and with (A_a,C_a) uniformly observable. Let us consider the factorization in equation (15).

Proposition 4.1 *Let $G \in \mathcal{X}$ be a given operator with port signature matrices (J_1,J_2), specified by (14) with $\ell_{A_a} < 1$, $\ell_{A_c} < 1$ and (A_a,C_a) uniformly observable. Let $\mathbf{F}_o^a = (I-A_a\mathbf{Z}^*)^{-1}C_a$. Define a J-unitary operator $\Theta_a \in \mathcal{L}$ with its anticausal output state space $\mathcal{H}_o^a(\Theta_a) = \mathcal{D}_2\mathbf{F}_o^a$. Assume that there is a Hermitian invertible operator $Q \in \mathcal{D}$ such that*

$$A_aQA_a^* - C_aJ_2C_a^* = Q^{(-1)} \quad (18)$$

is satisfied. Under this condition, we embed $[A_a,C_a]$ with a pair $[B_{\Theta_a},D_{\Theta_a}]$ such that:

$$\begin{bmatrix} A_a & C_a \\ B_{\Theta_a} & D_{\Theta_a} \end{bmatrix}\begin{bmatrix} Q & \\ & -J_2 \end{bmatrix}\begin{bmatrix} A_a & C_a \\ B_{\Theta_a} & D_{\Theta_a} \end{bmatrix}^* = \begin{bmatrix} Q^{(-1)} & \\ & -J_2 \end{bmatrix} \quad (19)$$

and

$$\begin{bmatrix} A_a & C_a \\ B_{\Theta_a} & D_{\Theta_a} \end{bmatrix}^*\begin{bmatrix} P^{(-1)} & \\ & -J_2 \end{bmatrix}\begin{bmatrix} A_a & C_a \\ B_{\Theta_a} & D_{\Theta_a} \end{bmatrix} = \begin{bmatrix} P & \\ & -J_2 \end{bmatrix} \quad (20)$$

are satisfied. Define a J_2-unitary operator $\Theta_a = D_{\Theta_a} + B_{\Theta_a}\mathbf{Z}^*(I-A_a\mathbf{Z}^*)^{-1}C_a \in \mathcal{L}$ and let $G_1 = GJ_2\Theta_a^*J_2$. Then, G_1 is upper and has a realization

$$G_1 = D_g + B_g\mathbf{Z}(I-A_g\mathbf{Z})^{-1}C_g \quad (21)$$

where A_g, B_g, C_g and D_g are equal to,

$$A_g = \begin{bmatrix} A_c & C_cJ_2C_a^* \\ & A_a^* \end{bmatrix},\ C_g = \begin{bmatrix} C_cJ_2D_{\Theta_a}^*J_2 \\ B_{\Theta_a}^*J_2 \end{bmatrix} \quad (22)$$

$$B_g = \begin{bmatrix} B_c & DJ_2C_a^* - B_aQA_a^* \end{bmatrix} \quad (23)$$

$$D_g = DJ_2D_{\Theta_a}^*J_2 - B_aQB_{\Theta_a}^*J_2 \quad (24)$$

If $Q \gg 0$, then Θ_a is J_2-lossless and G has a factorization $G = G_1\Theta_a$ with G_1 upper and Θ_a lower and J_2-lossless.

Proof: Rewrite equation (18) as

$$\begin{bmatrix} A_a & C_a \end{bmatrix}\begin{bmatrix} Q & \\ & -J_2 \end{bmatrix}\begin{bmatrix} A_a & C_a \end{bmatrix}^* = Q^{(-1)}$$

For Q invertible, we embed $\begin{bmatrix} A_a & C_a \end{bmatrix}$ with $\begin{bmatrix} B_{\Theta_a} & D_{\Theta_a} \end{bmatrix}$ such that (19) and (20) are satisfied. In this case, $P = Q^{-1}$ and the realization $\{A_a, B_{\Theta_a}, C_a, D_{\Theta_a}\}$ is regular. We construct $\Theta_a = D_{\Theta_a} + B_{\Theta_a}\mathbf{Z}^*(I-A_a\mathbf{Z}^*)^{-1}C_a$. With the lower, and J-unitary version of Theorem 3.3 we know that Θ_a is J_2-unitary. Let $G_1 = GJ_2\Theta_a^*J_2$, then,

$$\begin{aligned} G_1 &= GJ_2\Theta_a^*J_2 = DJ_2D_{\Theta_a}J_2 \\ &+[D+B_c\mathbf{Z}(I-A_c\mathbf{Z})^{-1}C_c]J_2C_a^*(I-\mathbf{Z}A_a^*)^{-1}\mathbf{Z}B_{\Theta_a}^*J_2 \\ &+B_c\mathbf{Z}(I-A_c\mathbf{Z})^{-1}C_cJ_2D_{\Theta_a}^*J_2 + B_a\mathbf{Z}^*(I-A_a\mathbf{Z}^*)^{-1}C_aJ_2D_{\Theta_a}^*J_2 \\ &+B_a\mathbf{Z}^*(I-A_a\mathbf{Z}^*)^{-1}C_aJ_2C_a^*(I-\mathbf{Z}A_a^*)^{-1}\mathbf{Z}B_{\Theta_a}^*J_2 \end{aligned}$$

The first three terms are obviously upper. We can rewrite the last two terms as:

$$B_a\mathbf{Z}^*(I-A_a\mathbf{Z}^*)^{-1}C_aJ_2D^*_{\Theta_a}J_2$$
$$+B_a\mathbf{Z}^*(I-A_a\mathbf{Z}^*)^{-1}C_aJ_2C^*_a(I-\mathbf{Z}A^*_a)^{-1}\mathbf{Z}B^*_{\Theta_a}J_2 =$$
$$-B_aQB_{\Theta_a}J_2 - B_aQA^*_a\mathbf{Z}(I-A^*_a\mathbf{Z})^{-1}B^*_{\Theta_a}J_2$$

Now we see that this part is also upper. Then G_1 is upper. By combining the first three terms with the last two terms of G_1, we derive that G_1 has the realization $\{A_g, B_g, C_g, D_g\}$ of (22), (23), and (24). Since $\Theta^*_a J_2 \Theta_a = J_2$, G admits a factorization $G = G_1\Theta_a$. If $Q \gg 0$, then Θ_a is anticausal and J_2-lossless as (15) requires. □

4.2 Causal J-lossless factorization

In this subsection we continue with the second step, the outer-J,J'-lossless factorization of G_1. We start with a result on the outer part.

Theorem 4.2 *Let $T \in \mathcal{U}$ with port signature matrices (J_1, J_2). Suppose that there exists a $\Theta \in \mathcal{U}$ which is (J_2, J_1)-isometric with its realization regular, such that $\overline{\mathcal{U}_2TJ_2} = \mathcal{U}_2\Theta J_2$. Then T has a factorization $T = T_o\Theta$ with $T_o \in \mathcal{U}$ outer.*

Proof: Define $T_o = TJ_2\Theta^*J_1$. Then

$$\overline{\mathcal{U}_2T_o} = \overline{\mathcal{U}_2TJ_2\Theta^*J_1} = \overline{\overline{\mathcal{U}_2TJ_2}\Theta^*J_1} = \overline{\mathcal{U}_2\Theta J_2\Theta^*J_1} = \mathcal{U}_2$$

so that T_o is outer. If Θ is J-unitary, $\Theta^{-1} = J_2\Theta^*J_1$, then it is always true that if $T_o = TJ_2\Theta^*J_1$, then $T = T_o\Theta$. In the case that Θ is only (J_2, J_1)-isometric but with its realization regular, there always exists an $\Omega \in \mathcal{U}$ which is the J-complement of Θ such that:

$$\begin{bmatrix} \Omega \\ \Theta \end{bmatrix} J_2 \begin{bmatrix} \Omega^* & \Theta^* \end{bmatrix} = \begin{bmatrix} J_c & \\ & J_1 \end{bmatrix},$$
$$\begin{bmatrix} \Omega^* & \Theta^* \end{bmatrix} \begin{bmatrix} J_c & \\ & J_1 \end{bmatrix} \begin{bmatrix} \Omega \\ \Theta \end{bmatrix} = J_2 \qquad (25)$$

where J_c is called the complement port signature matrix of J_1. Then we have: $\Omega^*J_c\Omega + \Theta^*J_1\Theta = J_2$ or $J_2\Omega^*J_c\Omega = I - J_2\Theta^*J_1\Theta$ and $\Theta J_2\Omega^* = 0$. On the other hand, because $\mathcal{U}_2TJ_2\Omega^* \subset \mathcal{U}_2\Theta J_2\Omega^* = 0$, $TJ_2\Omega^* = 0$. Hence $T = TJ_2\Theta^*J_1\Theta = T_o\Theta$ iff there is an Ω such that the equations of (25) are satisfied. In case of a regular realization of Θ there always exists such an Ω. This proves the theorem. □

We have defined the input and output signature matrices J_1 and J_2 for a chain scattering operator. In general, their entries are time-varying and the relation between J_1 and J_2 can not be given by a simple expression. But in some special cases, J_1 and J_2 are explicitly related. Let us consider the relation of J_1 and J_2 in a special case which is related to the problem we deal with.

Let a chain scattering operator $T \in \mathcal{U}$. The factorization we are interested in is $T = T_o\Theta$ with T_o outer, Θ (J_2, J_1)-lossless and upper. Let Θ be partitioned as $\begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}$ with Θ_{22} invertible. Because Θ is upper, Θ_{22} is upper. On the other hand, the corresponding scattering operator, Σ, is lossless. Thus Θ_{22}^{-1} must be upper as well. Let $\{A_\Theta, B_\Theta, C_\Theta, D_\Theta\}$ be a realization of Θ. Suppose D_Θ is partitioned as $\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}$ following the partitioning of Θ. Since both Θ_{22} and Θ_{22}^{-1} are upper, D_{22} is invertible. The invertibility of D_{22} implies that every entry of D_{22} is invertible and thus square. The dimensions of the negative part of J_1 and J_2 which correspond to the row and column dimensions of D_{22}, are thus equal to each other. This equality in addition with the condition that the dimension of the output is pointwise greater than or equal to the dimension of the input implies that $j_{2,i} = \begin{bmatrix} I & \\ & j_{1,i} \end{bmatrix}$ for $i = \ldots, -1, 0, \ldots$ In the global notation, we denote this as $J_2 = \begin{bmatrix} I & \\ & J_1 \end{bmatrix}$. For the rest of the paper we assume that this relation holds. Note that then J_c in the proof of Theorem 4.2 equals the identity operator.

Let $\Theta \in \mathcal{U}$ be a (J_2, J_1)-isometric operator. Then $\mathcal{K}_o(\Theta) = \mathcal{K}'_o(\Theta) \oplus \mathcal{K}''_o(\Theta)$, where $\mathcal{K}'_o(\Theta) = \mathcal{U}_2\Theta J_2$ and $\mathcal{K}''_o = ker(.\Theta^*|_{\mathcal{U}_2}) = \{\chi \in \mathcal{U}_2, \chi\Theta^* = 0\}$, and $\mathcal{K}_o(\Theta) \oplus \overline{\mathcal{H}}_o(\Theta) = \mathcal{U}_2$. Let $T \in \mathcal{U}$ be an operator with port signature matrices (J_1, J_2). If we find a Θ such that $\mathcal{K}'_o(\Theta) = \overline{\mathcal{U}_2TJ_2}$, then $\overline{\mathcal{U}_2TJ_2} = \mathcal{U}_2\Theta J_2$. We then have the following proposition.

Proposition 4.3 *Let $T \in \mathcal{U}$ be an operator with port signature matrices (J_1, J_2). Let Θ be a (J_2, J_1)-isometric operator such that $\mathcal{K}'_o(\Theta) = \overline{\mathcal{U}_2TJ_2}$. Then, $\overline{\mathcal{H}_o}(\Theta)J_2T^* \subset \overline{\mathcal{H}}(T)$.*

Proof: Since $\mathcal{K}'_o(\Theta) = \overline{\mathcal{U}_2TJ_2} = \mathcal{U}_2\Theta J_2$,

$$\mathcal{U}_2 \ominus \overline{\mathcal{U}_2TJ_2} = \mathcal{U}_2 \ominus \mathcal{U}_2\Theta J_2 = \overline{\mathcal{H}}_o(\Theta) \oplus \mathcal{K}''_o \qquad (26)$$

where $\mathcal{K}''_o(\Theta) = ker(.\Theta^*|_{\mathcal{U}_2})$ and hence, $\overline{\mathcal{U}_2TJ_2} \perp \overline{\mathcal{H}}_o(\Theta) \oplus \mathcal{K}''_o$. For any $\chi \in [\overline{\mathcal{H}_o}(\Theta) \oplus \mathcal{K}''_o]J_2$, $\mathbf{P}_o(\mathcal{U}_2T\chi^*) = 0$. So that $\chi T^* \in \mathcal{L}_2\mathbf{Z}^{-1}$. Together with $\mathcal{U}_2\Theta J_2 \oplus ker(.\Theta^*|_{\mathcal{U}_2}) \oplus \overline{\mathcal{H}}_o(\Theta) = \mathcal{U}_2$ we have:

$$(\overline{\mathcal{H}}_o(\Theta) \oplus \mathcal{K}''_o)J_2 = \{\chi \in \mathcal{U}_2, \chi T^* \in \mathcal{L}_2\mathbf{Z}^{-1}\} \qquad (27)$$

From the definition of $\mathcal{H}(T)$ we have: $\chi T^*|_{\chi \in [\overline{\mathcal{H}}_o(\Theta) \oplus \mathcal{K}''_o]J_2} \in \mathcal{H}(T) \subset \overline{\mathcal{H}}(T)$, which implies that $\overline{\mathcal{H}_o}(\Theta)J_2T^* \subset \overline{\mathcal{H}}(T)$ □

Let $T \in \mathcal{U}$ be an operator with port signature matrices (J_1, J_2). Define a (J_2, J_1)-isometric operator Θ such that $\mathcal{K}'_o(\Theta) = \overline{\mathcal{U}_2TJ_2}$. Let $\mathbf{E}_o$ be a J-orthonormal basis representation of $\overline{\mathcal{H}}_o(\Theta)$: $\overline{\mathcal{H}}_o(\Theta) = \mathcal{D}_2\mathbf{E}_o$ and let $\mathbf{F}$ be a basis representation of $\overline{\mathcal{H}}(T)$. Because $\overline{\mathcal{H}}_o(\Theta)J_2T^* \subset \overline{\mathcal{H}}(T)$, we must have $\mathbf{E}_oJ_2T^* = X\mathbf{F}$ for some bounded diagonal operator X which plays an instrumental role in the derivation of a state realization of Θ.

Suppose that $\mathbf{E}_oJ_2$ has a component in $\mathcal{K}''_o$ so that $D\mathbf{E}_oJ_2 \in \mathcal{K}''_o$ for some $D \in \mathcal{D}_2$. Since $\mathcal{K}''_o = ker(.\Theta^*|_{\mathcal{U}_2}) = ker(.T^*|_{\mathcal{U}_2})$ ($T^* = \Theta^*T^*_o$ and $\ker(.T^*_o) = 0$), we have $D\mathbf{E}_oJ_2T^*|_{D\mathbf{E}_oJ_2 \in \mathcal{K}''_o} = DX\mathbf{F}|_{D\mathbf{E}_oJ_2 \in \mathcal{K}''_o} = 0$ so that $D \in ker(.X)$. Hence $\overline{\mathcal{H}}_o(\Theta) = \mathcal{D}_2\mathbf{E}_o$ can be described as the largest subspace $\mathcal{D}_2\mathbf{E}_o$ (and then $\overline{\mathcal{H}}_o(\Theta)J_2 = \mathcal{D}_2\mathbf{E}_oJ_2$ is also the largest subspace) for which: $\mathbf{E}_oJ_2T^* = X\mathbf{F}$ with $ker(.X) = 0$. The two conditions $\mathbf{E}_oJ_2T^* = X\mathbf{F}$ and $ker(.X) = 0$ in addition with the J-losslessnes define a realization of a J-lossless Θ such that $\overline{\mathcal{U}_2TJ_2} = \mathcal{U}_2\Theta J_2$. Then, according to Theorem 4.2, the factorization $T = T_o\Theta$, where T_o is outer and Θ (J_2, J_1)-lossless, exists.

Proposition 4.4 *Let $T \in \mathcal{U}$ be a locally finite transfer operator with port signature matrices (J_1, J_2) such that $J_2 = \begin{bmatrix} I & \\ & J_1 \end{bmatrix}$ and a uniformly reachable realization $\{A, B, C, D\}$ such that $\ell_A < 1$ and $(TJ_2T^*)^{-1}$ exists. T has a factorization $T = T_o\Theta$, where T_o is invertible and outer, and $\Theta \in \mathcal{U}$ is (J_2, J_1)-lossless iff there is a pair*

$\{A_\Theta, C_\Theta\}$ *which corresponds to a J-orthonormal basis representation of* $\overline{\mathcal{H}_o}(\Theta)$*, the output state space of* Θ*, with* $\ell_{A_\Theta} < 1$*, and a diagonal operator* X *such that the following conditions are satisfied,*
(i) $A_\Theta X^{(-1)} A^* + C_\Theta J_2 C^* = X$
(ii) $A_\Theta X^{(-1)} B^* + C_\Theta J_2 D^* = 0$
(iii) $A_\Theta A_\Theta^* + C_\Theta J_2 C_\Theta^* = I$
(iv) $Ker(.X) = 0$
If such an X *exists, it is unique up to a left diagonal unitary factor, i.e,* X^*X *is unique.*

Proof: Let $\mathbf{F} = (I - \mathbf{Z}^* A^*)^{-1} \mathbf{Z}^* B^*$ and $\mathbf{F}_o = (I - A\mathbf{Z})^{-1} C$. Suppose that a pair $\{A_\Theta, C_\Theta\}$ and a diagonal operator X fulfilling $(i)-(iii)$ exist and let $\mathbf{E}_o J_2 = (I - A_\Theta \mathbf{Z})^{-1} C_\Theta J_2$, we have the following equations:

$$\mathbf{E}_o J_2 = C_\Theta J_2 + A_\Theta \mathbf{Z} \mathbf{E}_o J_2 \quad (28)$$
$$\mathbf{Z}\mathbf{F} = B^* + A^* \mathbf{F} \quad (29)$$
$$T^* = D^* + C^* \mathbf{F} \quad (30)$$

As analyzed before, T has a factorization $T = T_o \Theta$ with T_o outer and Θ (J_2, J_1)-lossless, iff the conditions that $\mathbf{E}_o J_2 T^* = X\mathbf{F}$ with $Ker(.X) = 0$ and Θ (J_2, J_1)-lossless are satisfied. Uniform reachability implies that $\mathcal{H}(T) = \mathcal{D}_2 \mathbf{F}$. According to Proposition 4.3, we need to find a (J_2, J_1)-lossless operator Θ such that $\overline{\mathcal{H}_o}(\Theta) J_2 T^* \subset \overline{\mathcal{H}}(T)$. That is $\mathbf{E}_o J_2 T^* = X\mathbf{F}$ for some bounded $X \in \mathcal{D}$. Because $\mathbf{F} \in \mathcal{L}_2 \mathbf{Z}^{-1}$, $P_{\mathcal{L}_2 \mathbf{Z}^{-1}}(\mathbf{E}_o J_2 T^*) = X\mathbf{F}$. With $P_{\mathcal{L}_2 \mathbf{Z}^{-1}}(E_o J_2 T^*) = X\mathbf{F}$ and equation (29), $P_{\mathcal{L}_2 \mathbf{Z}^{-1}}(\mathbf{Z} X \mathbf{F}) = X^{(-1)} P_{\mathcal{L}_2 \mathbf{Z}^{-1}}(\mathbf{Z}\mathbf{F}) = X^{(-1)} A^* \mathbf{F}$. On the other hand

$$A_\Theta P_{\mathcal{L}_2 \mathbf{Z}^{-1}}(\mathbf{Z} \mathbf{E}_o J_2 T^*) = P_{\mathcal{L}\mathbf{Z}^{-1}}([A_\Theta \mathbf{Z} \mathbf{E}_o J_2] T^*)$$
$$= P_{\mathcal{L}_2 \mathbf{Z}^{-1}}(\mathbf{E}_o J_2 T^*) - P_{\mathcal{L}\mathbf{Z}^{-1}}(C_\Theta J_2 T^*) = X\mathbf{F} - C_\Theta J_2 C^* \mathbf{F}$$

Since $P_{\mathcal{L}_2 \mathbf{Z}^{-1}}(\mathbf{Z} \mathbf{E}_o J_2 T^*) = P_{\mathcal{L}\mathbf{Z}^{-1}}(\mathbf{Z} X \mathbf{F})$, we have $A_\Theta X^{(-1)} A^* \mathbf{F} = X\mathbf{F} - C_\Theta J_2 C^* \mathbf{F}$. The uniform reachability yields $A_\Theta X^{(-1)} A^* + C_\Theta J_2 C^* = X$, i.e., condition (i).
Condition (ii) is derived from the condition that $\mathbf{E}_o J_2 T^* = X\mathbf{F} \in \mathcal{L}_2 \mathbf{Z}^{-1}$ as follows

$$\mathbf{P}_0(\mathbf{E}_o J_2 T^*) = C_\Theta J_2 D^* + A_\Theta \mathbf{P}_0(\mathbf{Z} \mathbf{E}_o J_2 T^*) =$$
$$C_\Theta J_2 D^* + A_\Theta \mathbf{P}_0(\mathbf{Z} X \mathbf{F}) = C_\Theta J_2 D^* + A_\Theta X^{(-1)} B^* = 0$$

Condition (iii) is given by the fact that $\mathbf{E}_o J_2$ is a J-orthonormal basis representation of the output state space of a J-lossless operator and condition (iv) has been derived before.
Conversely, if the conditions $(i)-(iv)$ are satisfied, then the conditions for the existence of the outer–(J_1, J_2)-lossless factorization $T = T_o \Theta$ are satisfied. Substitution of the conditions $(i)-(ii)$ into $\mathbf{E}_o J_2 T^*$ yields that $\mathbf{E}_o J_2 T^* = X\mathbf{F}$ and that the conditions $(iii)-(iv)$ are the same in both directions.
With the same strategy given by Theorem 3.28 in [4] we can prove that $.H_T^* = \mathbf{P}_o(.\mathbf{F}_o^*)\mathbf{F}$. Hence $\mathbf{P}_{\mathcal{L}_2 \mathbf{Z}^{-1}}(\mathbf{E}_o J_2 T^*) = \mathbf{P}_o(\mathbf{E}_o J_2 \mathbf{F}_o^*)\mathbf{F}$. Since T is uniformly reachable, $X = \mathbf{P}_o(\mathbf{E}_o J_2 \mathbf{F}_o^*)$. X^*X is obtained as:

$$X^*X = \mathbf{P}_o(\mathbf{F}_o J_2 \mathbf{E}_o^*) \mathbf{P}_o(\mathbf{E}_o J_2 \mathbf{F}_o^*) = \mathbf{P}_o(\mathbf{P}_o(\mathbf{F}_o J_2 \mathbf{E}_o^*) \mathbf{E}_o J_2 \mathbf{F}_o^*)$$
$$= \mathbf{P}_o(\mathbf{P}_{\mathcal{H}}^{J_2}(\mathbf{F}_o) J_2 \mathbf{F}_o^*) \qquad (\mathbf{P}_{\mathcal{H}}^{J_2}(.) = \mathbf{P}_o(.J_2 \mathbf{E}_o^*)\mathbf{E}_o)$$

This implies that X^*X is unique. □

In order to obtain X in a unique manner, we can choose X_k at every step to be in an upper triangular form with all its diagonal entries positive. If we have found X such that the conditions $(i)-(iv)$ are satisfied, then we have the pair $\{A_\Theta, C_\Theta\}$ which corresponds to a realization of a (J_2, J_1)-lossless operator Θ. Embedding $\{A_\Theta, C_\Theta\}$ with $\{B_\Theta, D_\Theta\}$ such that,

$$\begin{bmatrix} A_\Theta & C_\Theta \\ B_\Theta & D_\Theta \end{bmatrix} \begin{bmatrix} I & \\ & J_2 \end{bmatrix} \begin{bmatrix} A_\Theta & C_\Theta \\ B_\Theta & D_\Theta \end{bmatrix}^* = \begin{bmatrix} I & \\ & J_1 \end{bmatrix}$$

then, $\Theta = D_\Theta + B_\Theta \mathbf{Z}(I - A_\Theta \mathbf{Z})^{-1} C_\Theta$ and $\Theta J_2 \Theta^* = J_1$. With $T = T_o \Theta$, the outer operator T_o is derived as follows

$$T_o = T J_2 \Theta^* J_1 = D J_2 D_\Theta^* J_1 + B\mathbf{Z}(I - A\mathbf{Z})^{-1} C J_2 D_\Theta^* J_1 + T J_2 C_\Theta^* (I - \mathbf{Z}^* A_\Theta^*)^{-1} \mathbf{Z}^* B_\Theta^* J_1 \quad (31)$$

After rewriting the third term of the above equation we get

$$T J_2 C_\Theta^* (I - \mathbf{Z}^* A_\Theta^*)^{-1} \mathbf{Z}^* B_\Theta^* J_1 =$$
$$B X^{(-1)*} B_\Theta^* J_1 + B\mathbf{Z}(I - A\mathbf{Z})^{-1} A X^{(-1)*} B_\Theta^* J_1.$$

By substituting into (31), we obtain the realization of T_o given by

$$T_o = \left[\frac{A \mid C J_2 D_\Theta^* J_1 + A X^{(-1)*} B_\Theta^* J_1}{B \mid D J_2 D_\Theta^* J_1 + B X^{(-1)*} B_\Theta^* J_1} \right] \quad (32)$$

The invertibility of T_o follows from condition of the invertibility of $T J_2 T^*$.

If we rewrite the above results together with a special case of Lemma 5.16 in [4] in an algorithm, a problem that remains is the initialization of X. For a finite operator the dimension of the states after time instant 0 is zero, i.e., $X_0 = [.]$. For a system which is time-varying until time-instant 0, and time invariant after time instant 0, the initial condition is determined by the solution of the time invariant system. For a periodic system, the initial condition is determined by the solution of the equivalent time invariant system within one period. The time invariant system solution can be obtained from an analysis of the eigen space of a corresponding Riccati equation. See e.g. [5].

5 Concluding remarks

In this paper we have treated the outer-J-lossless factorization of a linear discrete time-varying system mostly in an operator setting, i.e., the characterization is given in terms of Lyapunov-type of equations. The proposed strategy can be used as a tool for the development of a solution to the H_∞ control problem. This has been done in [10].

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