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Signed graphs with maximum nullity two

Marina Arav[a], F. Scott Dahlgren[b], Hein van der Holst[a,*]

[a] *Department of Mathematics and Statistics, Georgia State University, Atlanta, GA 30303, USA*
[b] *Faculty of Electrical Engineering, Mathematics and Computer Science, Delft University of Technology, Delft, the Netherlands*

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A signed graph is a pair (G, Σ), where $G = (V, E)$ is a graph (in which parallel edges are permitted, but loops are not) with $V = \{1, \dots, n\}$ and $\Sigma \subseteq E$. The edges in Σ are called odd and the other edges of E even. If there are parallel edges, then only two edges in each parallel class are permitted, one of which is even and one of which is odd. By $S(G, \Sigma)$ we denote the set of all symmetric $n \times n$ matrices $A = [a_{i,j}]$ with $a_{i,j} < 0$ if i and j are connected by an even edge, $a_{i,j} > 0$ if i and j are connected by an odd edge, $a_{i,j} \in \mathbb{R}$ if i and j are connected by both an even and an odd edge, $a_{i,j} = 0$ if $i \neq j$ and i and j are non-adjacent, and $a_{i,i} \in \mathbb{R}$ for all vertices i.
The maximum nullity $M(G, \Sigma)$ of a signed graph (G, Σ) is the maximum nullity attained by any $A \in S(G, \Sigma)$. Arav et al. gave a combinatorial characterization of 2-connected signed graphs (G, Σ) with $M(G, \Sigma) = 2$. In this paper, we give a complete combinatorial characterization of the signed graphs (G, Σ) with $M(G, \Sigma) = 2$.

* Corresponding author.
E-mail address: hvanderholst@gsu.edu (H. van der Holst).

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1. Introduction

A *signed graph* is a pair (G, Σ), where $G = (V, E)$ is a graph (in which parallel edges are permitted, but loops are not) and $\Sigma \subseteq E$. (We refer to [6] for the notions and concepts in graph theory.) The edges in Σ are called *odd* and the other edges *even*. If there are parallel edges, then only two edges in each parallel class are permitted, one of which is even and one of which is odd. If $V = \{1, 2, \ldots, n\}$, we denote by $S(G, \Sigma)$ the set of all real symmetric $n \times n$ matrices $A = [a_{i,j}]$ with

- $a_{i,j} < 0$ if i and j are connected by an even edge,
- $a_{i,j} > 0$ if i and j are connected by an odd edge,
- $a_{i,j} \in \mathbb{R}$ if i and j are connected by both an odd and an even edge,
- $a_{i,j} = 0$ if $i \neq j$ and i and j are non-adjacent, and
- $a_{i,i} \in \mathbb{R}$ for all vertices i.

For a signed graph (G, Σ), $M(G, \Sigma)$ is the maximum of the nullities of the matrices in $S(G, \Sigma)$. The signed graph parameter $M(G, \Sigma)$ generalizes the graph parameter $M(G)$ in the sense that $M(G) = \max_{\Sigma \subseteq E} M(G, \Sigma)$. See Fallat and Hogben [7] for a survey on the graph parameter $M(G)$. A matrix $A = [a_{i,j}] \in S(G, \Sigma)$ has the SAP if $X = 0$ is the only symmetric matrix $X = [x_{i,j}]$ such that $x_{i,j} = 0$ if i and j are adjacent vertices or $i = j$, and $AX = 0$. The parameter ξ of a signed graph (G, Σ) is defined as the largest nullity of any matrix $A \in S(G, \Sigma)$ satisfying the SAP. It is clear that $\xi(G, \Sigma) \leq M(G, \Sigma)$ for any signed graph (G, Σ). This signed graph parameter ξ is analogous to the parameter ξ for simple graphs introduced by Barioli, Fallat, and Hogben [5].

If $G = (V, E)$ is a graph and $U \subseteq V$, $\delta(U)$ denotes the set of edges of G that have exactly one end in U. The symmetric difference of two sets A and B is the set $A \Delta B = A \setminus B \cup B \setminus A$. If (G, Σ) is a signed graph and $U \subseteq V(G)$, we say that (G, Σ) and $(G, \Sigma \Delta \delta(U))$ are *sign-equivalent* and call the operation $\Sigma \to \Sigma \Delta \delta(U)$ *re-signing on* U. Re-signing on U amounts to performing a diagonal similarity on the matrices in $S(G, \Sigma)$, and hence it does not affect $M(G, \Sigma)$ and $\xi(G, \Sigma)$.

Let (G, Σ) be a signed graph. If H is a subgraph of G, then we say that H is *odd* if $\Sigma \cap E(H)$ has an odd number of elements, otherwise we call H *even*. Zaslavsky showed in [11] that two signed graphs are sign-equivalent if and only if they have the same set of odd cycles. Thus, two signed graphs (G, Σ) and (G, Σ') that have the same set of odd cycles have $M(G, \Sigma) = M(G, \Sigma')$ and $\xi(G, \Sigma) = \xi(G, \Sigma')$.

Contracting an edge e with ends u and v in a graph G means deleting e and identifying the vertices u and v. A graph H is a *minor* of G if H can be obtained from a subgraph of G by contracting edges. If H is isomorphic to a minor of G, we also write that G has an H-minor. A signed graph (H, Γ) is a *weak minor* of a signed graph (G, Σ) if (H, Γ) can be obtained from (G, Σ) by deleting edges and vertices, contracting edges, and re-signing around vertices. We use weak minor to distinguish it from minor in which only even edges are allowed to be contracted (possibly after re-signing around vertices). The parameter ξ has the nice property that if (H, Γ) is a weak minor of the signed graph (G, Σ), then $\xi(H, \Gamma) \leq \xi(G, \Sigma)$.

In [8], Fiedler showed that the paths are the only graphs G for which $M(G) \leq 1$. Johnson et al. [9] characterized all graphs G with $M(G) \leq 2$. Barioli et al. [5] characterized the class of graph G with $\xi(G)$, and Hogen and van der Holst characterized the class of graphs G with $\xi(G) \leq 2$.

For a graph $G = (V, E)$ and a subset $S \subseteq V$, $G - S$ denotes the graph obtained by deleting all vertices in S; we write $G - v$ for $G - \{v\}$. A graph G is connected if for every two vertices u and v of G are connected by a path. A graph $G = (V, E)$ is 2-connected if $|V| > 2$ and $G - v$ is connected for every $v \in V$. Any 2-connected graph contains a cycle.

In [3], Arav et al. showed that a signed graph (G, Σ) has $M(G, \Sigma) \leq 1$ if and only if (G, Σ) is sign-equivalent to a signed graph $(H, \emptyset)$, where H is a path. Furthermore, they showed that a signed graph (G, Σ) has $\xi(G, \Sigma) \leq 1$ if and only if (G, Σ) is sign-equivalent to a signed graph $(H, \emptyset)$, where H is a disjoint union of paths. Observe that in case the signed graph (G, Σ) is connected, $M(G, \Sigma) \leq 1$ if and only if $\xi(G, \Sigma) \leq 1$. In [2], Arav et al. characterized combinatorially the class of 2-connected signed graphs (G, Σ) with $M(G, \Sigma) = 2$, which coincides with the class of 2-connected signed graphs (G, Σ) with $\xi(G, \Sigma) = 2$. In [1], Arav et al. characterized combinatorially the signed graphs (G, Σ) with $\xi(G, \Sigma) \leq 2$. In this paper, we provide a combinatorial characterization of the signed graphs (G, Σ) with $M(G, \Sigma) = 2$.

2. Global structure signed graphs (G, Σ) with $M(G, \Sigma) = 2$

In this section, we provide a global structure of signed graphs (G, Σ) with $M(G, \Sigma) \leq 2$. In the following sections, we then provide the exact structure.

Lemma 1. *Let (G, Σ) be a disjoint union of (G_1, Σ_1) and (G_2, Σ_2). Then $M(G, \Sigma) = M(G_1, \Sigma_1) + M(G_2, \Sigma_2)$.*

Lemma 2. *Let (G, Σ) be a disconnected signed graph with $M(G, \Sigma) = 2$. Then G consists of two components, each of which is a path.*

The proof of the following lemma follows Formulas 1 and 2 in Arav et al. [4].

Lemma 3. *Let (G, Σ) be a 1-sum of (G_1, Σ_1) and (G_2, Σ_2) at vertex s. Let (H_1, Ω_1) and (H_2, Ω_2) be obtained from (G_1, Σ_1) and (G_2, Σ_2), respectively, by deleting vertex s. Then*

$$M(G, \Sigma) = \max\{M(G_1, \Sigma_1) + M(G_2, \Sigma_2) - 1, M(H_1, \Omega_1) + M(H_2, \Omega_2) - 1\}.$$

If (H_1, Ω_1) and (H_2, Ω_2) are signed graph, then by *attaching* (H_2, Ω_2) to (H_1, Ω_1) we mean identifying a vertex of (H_2, Ω_2) with a vertex of (H_1, Ω). Furthermore, if P is a path with at least one edge, we mean by *attaching a pendant path* P at vertex v to

(H_1, Ω) identifying an end of P with v. Here, we assume that all edges of P are even. Observe that attaching a path (without the adjective pendant) to (H_1, Ω_1) allows an internal vertex of the path to be identified with a vertex of (H_1, Ω_1).

The following lemma follows immediately from Lemma 3.

Lemma 4. *If (G, Σ) is obtained from a signed graph (G_1, Σ_1) by attaching a pendant path at vertex v, then*

$$M(G, \Sigma) = \max\{M(G_1, \Sigma_1), M(G_1 - v, E(G_1 - v) \cap \Sigma_1)\}.$$

In particular,

$$M(G, \Sigma) \geq M(G_1, E(G_1) \cap \Sigma).$$

Lemma 5. *Let (G, Σ) be a connected signed graph containing a cycle. If $M(G, \Sigma) = 2$, then*

1. *(G, Σ) is obtained from a 2-connected signed graph (H, Ω) with $M(H, \Omega) = 2$ by attaching pendant paths at vertices of (H, Ω); or*
2. *(G, Σ) is obtained from an odd cycle with two edges by attaching pendant paths at vertices of this odd cycle.*

Furthermore, at each vertex of H at most two pendant paths can be attached.

Proof. Suppose, for a contradiction, that (G, Σ) is a 1-sum of (H_1, Ω_1) and (H_2, Ω_2), where both H_1 and H_2 contain a cycle. Since $M(H_1, \Omega_1) \geq \xi(H_1, \Omega_1) \geq 2$ and $M(H_2, \Omega_2) \geq \xi(H_2, \Omega_2) \geq 2$, we obtain, by Lemma 3, that

$$M(G, \Sigma) \geq M(H_1, \Omega_1) + M(H_2, \Omega_2) - 1 \geq 2 + 2 - 1 = 3,$$

a contradiction. Therefore, (G, Σ) is obtained from either a 2-connected signed graph (H, Ω) by attaching trees to some vertices of H or from an odd cycle (H, Ω) with two edges by attaching trees to some vertices.

If (G, Σ) is obtained from a 2-connected signed graph (H, Ω) with $M(H, \Omega) \geq 3$, then $M(G, \Sigma) \geq 3$. Thus, $M(H, \Omega) = 2$ in this case.

Let v be a vertex of H that has an attached tree T. If T contain a vertex of degree ≥ 3, then $M(T, \Sigma \cap E(T)) \geq 2$, and hence

$$M(G, \Sigma) \geq M(H, \Omega) + M(T, \Sigma \cap E(T)) - 1 \geq 2 + 2 - 1 = 3.$$

Therefore, (G, Σ) is obtained from (H, Ω) by attaching paths to vertices of H. Furthermore, at each vertex at most two paths can be attached. $\square$

In the next section, we study the structure of 2-connected signed graphs (H, Ω) with $M(H, \Omega) = 2$.

3. Wide partial 2-paths

In this section, we first make some definitions; see [2].

By K_4^i we denote the signed graph $(K_4, \{e\})$, where e is an edge of K_4. A pair $\{e, f\}$ of nonadjacent edges in K_4^i is called *split* if both e and f belong to an even and an odd triangle.

A *sided wide 2-path* $[(G, \Sigma), \mathcal{F}]$ is defined recursively as follows:

1. Let (G, Σ) be an even or odd cycle or a K_4^i. If (G, Σ) is a cycle, let $\mathcal{F}$ be two distinct edges in this cycle. If $(G, \Sigma) = K_4^i$, let $\mathcal{F}$ be a split pair of edges in K_4^i. Then $[(G, \Sigma), \mathcal{F}]$ is a sided wide 2-path.
2. Let $[(G, \Sigma), \mathcal{F}]$ be a sided wide 2-path and let e and f be distinct edges in an even or odd cycle C. If (H, Ω) is obtained from (G, Σ) by identifying the edge f of C with an edge h in $\mathcal{F}$, then $[(H, \Omega), (\mathcal{F} \setminus \{h\}) \cup \{e\}]$ is a sided wide 2-path.
3. Let $[(G, \Sigma), \mathcal{F}]$ be a sided wide 2-path and let $\{e, f\}$ be a split pair of edges in K_4^i. If (H, Ω) is obtained from (G, Σ) by identifying the edge f of K_4^i with an edge h in $\mathcal{F}$, then $[(H, \Omega), (\mathcal{F} \setminus \{h\}) \cup \{e\}]$ is a sided wide 2-path.

The edges in $\mathcal{F}$ are called the sides of the sided wide 2-path. A *wide 2-path* is a signed graph (G, Σ) for which there exists a set $\mathcal{F}$ of two distinct edges of (G, Σ) such that $[(G, \Sigma), \mathcal{F}]$ is a sided wide 2-path. A signed graph (G, Σ) is a *partial wide 2-path* if it is a spanning subgraph of a wide 2-path. Observe that if G is a partial 2-path, then (G, Σ) is a partial wide 2-path for any $\Sigma \subseteq E(G)$.

Let (G, Σ) be a signed graph. A pair $[G_1, G_2]$ of subgraphs of G is a *wide separation* of (G, Σ) if there exists an odd 4-cycle C_4 such that $G_1 \cup C_4 \cup G_2 = G$, $E(G_1) \cap E(C_4) = \emptyset$, $E(G_2) \cap E(C_4) = \emptyset$, $V(G_1) \cap V(G_2) = \emptyset$, $V(G_1) \cap V(C_4) = \{r_1, r_2\}$ and $V(G_2) \cap V(C_4) = \{s_1, s_2\}$, where r_1 and r_2 are nonadjacent vertices of C_4 and s_1 and s_2 are nonadjacent vertices of C_4. We call r_1, r_2 the vertices of attachment of G_1 and s_1, s_2 the vertices of attachment of G_2 in the wide separation. In the definition of sided wide 2-path, we allow the sided wide 2-path be built up from even and odd cycle, and K_4^i; the K_4^i's might yield wide separations in a 2-connected partial wide 2-path.

By K_n^e and K_n^o we denote the signed graphs $(K_n, \emptyset)$ and $(K_n, E(K_n))$, respectively. By $K_n^=$ we denote the signed graph (G, Σ), where G is the graph obtained from K_n by adding to each edge an edge in parallel, and where Σ is the set of edges of K_n. (It is will be clear from the context whether we mean the graph $K_3^=$ or the signed graph $K_3^=$.) By $K_{2,3}^e$, we denote the signed graphs $(K_{2,3}, \emptyset)$.

By W_4 we denote the graph obtained from C_4 by adding a new vertex v, called the *hub*, and connecting it to each vertex of C_4. The subgraph C_4 in W_4 is called the *rim* of W_4. Any edge between v and a vertex of the rim of W_4 is called a *spoke*. Let e_1, e_2 be two

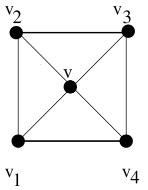

Fig. 1. The signed four-wheel.

nonadjacent edges of the C_4 in W_4. By W_4^o, we denote the signed graph $(W_4, \{e_1, e_2\})$. See Fig. 1 for a picture of W_4^o; here a bold edge is an odd edge and a thin edge an even edge. This signed graph appears as a special case in the characterization of 2-connected signed graphs (G, Σ) with $M(G, \Sigma) = 2$.

In [2], Arav et al. proved the following theorem.

Theorem 6. *Let (G, Σ) be a 2-connected signed graph. Then the following are equivalent:*

(i) $M(G, \Sigma) = 2$,
(ii) $\xi(G, \Sigma) = 2$,
(iii) (G, Σ) *has no weak minor isomorphic to* $K_3^=$, K_4^e, K_4^o, *or* $K_{2,3}^e$.
(iv) (G, Σ) *is a partial wide 2-path or is isomorphic to* W_4^o.

In the next section, we prove that if (G, Σ) is obtained from W_4^o by attaching single pendant paths to some of its vertices, then $M(G, \Sigma) = 2$. In Section 6, we will study the cases where (G, Σ) is obtained from a partial wide 2-path by adding pendant paths.

4. Pendant paths on an odd 4-wheel

Lemma 7. *Let S be a subset of the vertex set of the signed graph W_4^o. If (G, Σ) is obtained from W_4^o by attaching single pendant paths to all the vertices of S, then $M(G, \Sigma) = 2$. If (G, Σ) is obtained from W_4^o by attaching pendant paths to all vertices of S and some of the vertices have two or more pendant paths, then $M(G, \Sigma) > 2$.*

Proof. Suppose first that (G, Σ) is obtained from W_4^o by attaching single pendant paths to the vertices of S. Let

$$\mathcal{G} := \{W_4^o - R \ : \ R \subseteq S\}.$$

Then

$$M(G, \Sigma) = \max\{M(H, \Omega) \ : \ (H, \Omega) \in \mathcal{G}\}.$$

As $M(H, \Omega) \leq 2$ for all $(H, \Omega) \in \mathcal{G}$, $M(G, \Sigma) \leq 2$. Since (G, Σ) has a cycle, $M(G, \Sigma) = 2$.

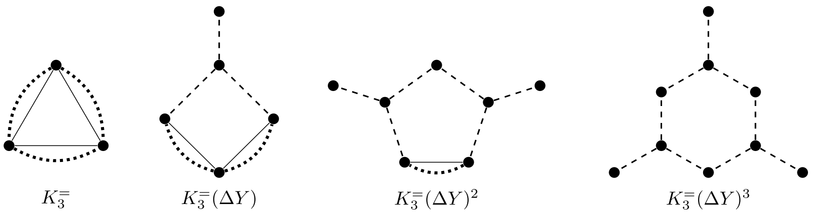

Fig. 2. The $K_3^{=}$-family.

Suppose next that (G, Σ) is obtained from W_4^o by attaching pendant paths to all the vertices of S and there is a vertex $s \in S$ that has two or more attached pendant paths. Then, as the signed graph obtained from (G, Σ) by deleting vertex s contains a cycle and two or more paths, $M(G, \Sigma) \geq 3$. $\square$

5. Signed graphs of the $K_3^{=}$-family

A triangle in a graph is a subgraph isomorphic to K_3. A ΔY-transformation on a triangle T of a signed graph (G, Σ) means deleting the edges T, adding a new vertex v, and connecting v with the vertices of the triangle with edges, giving these new edges any sign. The $K_3^{=}$-family is the family of signed graphs obtained from $K_3^{=}$ by repeatedly subdividing one edge in a parallel class, and then applying a ΔY-transformation on the resulting triangle. See Fig. 2; here, a solid line is an even edge, a dotted line is an odd edge, and a dashed line is either an even or an odd edge.

Lemma 8. *[1] Every member (G, Σ) of the $K_3^{=}$-family has $\xi(G, \Sigma) = 3$.*

Hence, if a signed graph (G, Σ) has a weak minor isomorphic to a signed graph in the $K_3^{=}$-family, then $M(G, \Sigma) \geq \xi(G, \Sigma) \geq 3$.

6. Pendant paths on 2-connected partial wide 2-paths

6.1. *Partial wide 2-paths with two wide separations*

Let (G, Σ) be a signed graph such that the removal of pendant paths yields a 2-connected partial wide 2-path (H, Ω). Suppose $[H_1, H_2]$ and $[H_3, H_4]$ are distinct wide separations of (H, Ω) such that $H_1 \subseteq H_3$ and $H_4 \subseteq H_2$. Let r_1, r_2 be the vertices of attachment of H_2 and let s_1, s_2 be the vertices of attachment of H_3, and let P_1 and P_2 be vertex-disjoint paths between $\{r_1, r_2\}$ and $\{s_1, s_2\}$, where P_i connects r_i and s_i. If P is a path in H, we denote by $l(P)$ the length of P. We call (G, Σ) a *ST-graph* if the following holds:

1. no vertex of H is the end of two or more pendant paths,
2. $l(P_1) + l(P_2) \leq 1$, and

(a) if $l(P_1) + l(P_2) = 1$, then both H_1 and H_2 are disconnected,
(b) if $l(P_1)+l(P_2) = 0$, then exactly one of H_1 and H_2 is disconnected and the other one is a path Q, and if Q has length ≥ 2, then there is at most one pendant path incident with an end of Q, and there are no pendant paths incident with an internal vertex of Q, and

3. exactly one pendant path is incident with a vertex of $P_1 \cup P_2$.

We allow edges between the vertices r_1, r_2 and between the vertices s_1, s_2.

A path in a graph $G = (V, E)$ is *induced* if it is of the form $G[S]$ for some $S \subseteq V$. The *path cover number* of a graph G, denoted $P(G)$, is the minimum number of vertex-disjoint induced paths covering all vertices of G. In the proof of Lemma 10, we use the following result of Sinkovic [10].

Theorem 9. *If G is a partial 2-path, then $M(G) = P(G)$.*

Lemma 10. *If (G, Σ) is a ST-graph, then $M(G, \Sigma) = 2$.*

Proof. Let (G, Σ) be a ST-graph. Let P be the pendant path incident with a vertex p of $P_1 \cup P_2$. Let $(G', \Sigma') := (G, \Sigma) - V(P - p)$ and let $(G'', \Sigma'') := (G, \Sigma) - V(P)$.

As (G', Σ') is a 2-connected partial wide 2-path, $M(G', \Sigma') = 2$, and, as (G'', Σ'') has path cover number 2, $M(G'', \Sigma'') = 2$, by Theorem 9. By Lemma 4, $M(G, \Sigma) = \max\{M(G', \Sigma'), M(G'', \Sigma'')\} = 2$. □

Lemma 11. *Let (G, Σ) be a signed graph such that the removal of pendant paths yields a 2-connected partial wide 2-path (H, Ω) with at least two wide separations. If there is a vertex with at least two pendant paths attached, then $M(G, \Sigma) \geq 3$.*

Proof. Let v be the vertex of H to which at least two paths are attached. If $H - v$ has a component containing a cycle, then, as two pendant paths are attached to v, $M(G, \Sigma) \geq 3$. If $H - v$ has no component containing a cycle, then one component of $H - v$ has a vertex of degree four. Also in this case $M(G, \Sigma) \geq 3$. □

Lemma 12. *Let (G, Σ) be a signed graph such that the removal of pendant paths yields a 2-connected partial wide 2-path (H, Ω). Let $[H_1, H_2]$ and $[H_3, H_4]$ be distinct wide separations of (H, Ω) such that $H_1 \subseteq H_3$ and $H_4 \subseteq H_2$. Let r_1, r_2 be the vertices of attachment of H_2 and let s_1, s_2 be the vertices of attachment of H_3, and let P_1 and P_2 be vertex-disjoint paths between $\{r_1, r_2\}$ and $\{s_1, s_2\}$, where P_i connects r_i and s_i. Suppose a pendant path is incident with a vertex of P_1 or P_2. Then $M(G, \Sigma) = 2$ if and only if (G, Σ) is a ST-graph.*

Proof. Suppose $M(G, \Sigma) = 2$.

Since $\xi(G, \Sigma) \leq M(G, \Sigma)$, we obtain by Lemma 8 that (G, Σ) has no weak minor isomorphic to a signed graph in the $K_3^{=}$-family.

By Lemma 11, at most one pendant path can be incident with each vertex of H.

Suppose next that a pendant path is incident with an internal vertex of P_1 or P_2. Then (G, Σ) has a weak minor isomorphic to $K_3^{=}(\Delta Y)$. We may therefore assume that every pendant path that is incident with a vertex of $P_1 \cup P_2$ is incident with an end of P_1 or P_2.

We next prove that

$$l(P_1) + l(P_2) \leq 1.$$

By symmetry, we may assume that a pendant path is incident with an end of P_1. If P_1 has at least two edges, then (G, Σ) has a weak minor isomorphic to $K_3^{=}(\Delta Y)^3$. Hence P_1 has at most one edge. If P_2 has at least two edges, then (G, Σ) has a weak minor isomorphic to $K_3^{=}(\Delta Y)^3$. Hence P_2 has at most one edge. If both P_1 and P_2 have exactly one edge, then (G, Σ) has a weak minor isomorphic to $K_3^{=}(\Delta Y)^2$. Hence P_1 or P_2 has length zero.

Suppose first that $l(P_1) + l(P_2) = 1$. By symmetry, we may assume that $l(P_1) = 1$ and $l(P_2) = 0$. If H_1 or H_4 is connected, then (G, Σ) has a weak minor isomorphic to $K_3^{=}(\Delta Y)$ or $K_3^{=}(\Delta Y)^2$. Hence both H_1 and H_4 are disconnected. Suppose now to the contrary that more than one pendant path is incident with vertices of $P_1 \cup P_2$. Let (G', Σ') be obtained from (G, Σ) be removing these pendant paths and their vertices of attachment. Then, as $M(G, \Sigma) \geq M(G', \Sigma')$, and (G', Σ') has path cover number ≥ 3, we obtain that $M(G, \Sigma) \geq 3$; a contradiction. Hence at most one pendant path is incident with $P_1 \cup P_2$. Then (G, Σ) is a ST-graph.

Suppose next that $l(P_1) = l(P_2) = 0$. Then $r_1 = s_1$ and $r_2 = s_2$. If H_1 and H_4 are connected, then (G, Σ) has a weak minor isomorphic to $K_3^{=}(\Delta Y)$. If H_1 and H_4 are disconnected, then the removal of the pendant path with its vertex of attachment yields a signed graph (G', Σ') with $M(G', \Sigma') \geq 3$. Hence $M(G, \Sigma) \geq 3$; a contradiction. By symmetry, we may therefore assume that H_1 is disconnected and H_4 is connected. In the same way as above, there is exactly one pendant path incident with $P_1 \cup P_2$. By symmetry, we may assume that (G, Σ) has a pendant path P incident with P_1.

If H_4 contains a cycle, then (G, Σ) has a weak minor isomorphic to $K_3^{=}(\Delta Y)^2$. Hence, H_4 has no cycle. Let Q be the path in H_4 connecting the vertices of attachment in the wide separation $[H_3, H_4]$. If (G, Σ) has a pendant path incident with an internal vertex of Q, then (G, Σ) has a weak minor isomorphic to $K_3^{=}(\Delta Y)^3$. Hence any pendant path incident with H_4 is incident with one of the vertices of attachment of H_4 in the wide separation $[H_3, H_4]$. If Q has length ≥ 2 and (G, Σ) has pendant paths incident with both ends of Q, then (G, Σ) has a weak minor isomorphic to $K_3^{=}(\Delta Y)^2$. Hence either Q has length 1 or (G, Σ) has only a pendant path attached to one of the ends of Q, if any. Then (G, Σ) is a ST-graph.

The converse implication follows from Lemma 10. □

A signed graph has *two parallel paths* if there exist two pairs of vertices u_1, u_2 and v_1, v_2 such that (G, Σ) is a spanning subgraph of a sided wide 2-path with sides u_1u_2 and v_1v_2, and there exist two disjoint paths connecting u_1 and v_1, and u_2 and v_2, respectively.

Lemma 13. *Let (G, Σ) be a signed graph with two parallel paths. Then $M(G, \Sigma) \leq 2$.*

Proof. The signed graph (G, Σ) is a spanning subgraph of a sided wide 2-path with sides u_1u_2 and v_1v_2. A zero forcing argument starting with the vertex-set $\{u_1, u_2\}$, similar as done in [2], shows that $M(G, \Sigma) \leq 2$. □

Lemma 14. *Let (G, Σ) be a signed graph such that the removal of pendant paths yields a 2-connected partial wide 2-path (H, Ω) with at least two distinct wide separations. Then, $M(G, \Sigma) = 2$ if and only if (G, Σ) has two parallel paths or (G, Σ) is a ST-graph.*

Proof. Suppose $M(G, \Sigma) = 2$. Let $[H_1, H_2]$ and $[H_3, H_4]$ be distinct wide separations of (H, Ω) such that $H_1 \subseteq H_3$ and $H_4 \subseteq H_2$; we take $[H_1, H_2]$ in (H, Ω) such that there is no wide separation $[H_1', H_2']$ with H_1' a proper subgraph of H_1, and similar, we take $[H_3, H_4]$ in (H, Ω) such that there is no wide separation $[H_3', H_4']$ such that H_4' is a proper subgraph of H_4. Let r_1, r_2 be the vertices of attachment of H_2 and let s_1, s_2 be the vertices of attachment of H_3, and let P_1 and P_2 be vertex-disjoint paths between $\{r_1, r_2\}$ and $\{s_1, s_2\}$, where P_i connects r_i and s_i. By Lemma 12, we may assume that no pendant path is incident with a vertex of P_1 or P_2, for otherwise we obtain a ST-graph. Let u_1, u_2 be the vertices of attachment of H_1. By Lemma 11, no two pendant paths of G are incident with a vertex of H.

Suppose H_1 contains a cycle C; we may assume that C is at the end of the partial wide 2-path H, that is, there is a 2-separation (C, F) of H. Let $\{v_1, v_2\} := V(C) \cap V(F)$. Let Q_1 and Q_2 be two vertex-disjoint paths between $\{v_1, v_2\}$ and $\{u_1, u_2\}$, with Q_i connecting v_i and u_i. If a pendant path is incident with a vertex of $Q_1 - v_1$ or $Q_2 - v_2$, then (G, Σ) has a weak $K_3^{=}(\Delta Y)$-minor. Let P be the path obtained from C by removing any edge between v_1 and v_2. If there are two pendant paths incident with nonadjacent vertices of P, then (G, Σ) has a weak $K_3^{=}(\Delta Y)^2$-minor. Hence, at most two pendant paths are incident with vertices of P, and if two pendant paths are incident with vertices of P, then these vertices are adjacent in P.

If H_1 contains no cycle, but H_1 is connected, let P be the path in H_1 connecting u_1 and u_2. If there are two pendant paths incident with nonadjacent vertices of P, then (G, Σ) has a weak $K_3^{=}(\Delta Y)^2$-minor. Hence, at most two pendant paths are incident with vertices of P, and if two pendant paths are incident with vertices of P, then these vertices are adjacent in P.

We do the same on H_4 if H_4 is connected.

If H_1 and H_4 are connected, then (G, Σ) has two parallel paths. Similarly, if at least one of H_1 and H_4 is disconnected, then (G, Σ) has two parallel paths.

We next prove the converse. If there is a pendant path incident with a vertex of $P_1 \cup P_2$, then the result follows from Lemma 12. If no pendant path is incident with a vertex of $P_1 \cup P_2$, then by the previous lemma $M(G, \Sigma) \leq 2$. Since H is 2-connected, $M(G, \Sigma) = 2$. $\square$

6.2. Partial wide 2-paths with one wide separation

Let (G, Σ) be a signed graph such that the removal of pendant paths yields a partial wide 2-path (H, Ω) with exactly one wide separation $[H_1, H_2]$. Let u_1 and u_2 be the vertices of attachment of H_1 and let w_1 and w_2 be the vertices of attachment of H_2. We call (G, Σ) a *SA-graph* if the following holds:

(a) no vertex of H is the end of two or more pendant paths;
(b) H_2 is a path, and no pendant path is incident with interior vertices of H_2;
(c) the removal of any edge between u_1 and u_2 from H_1, if any, yields a path P with length ≥ 2; if a pendant path is incident with an internal vertex of P, then P has length two;
(d) there is one pendant path incident with u_1 and one pendant path incident with u_2;
(e) if H_2 has an internal vertex and pendant paths are incident with w_1 and w_2, then no pendant path is incident with an internal vertex of P.

Lemma 15. *If (G, Σ) is a SA-graph, then $M(G, \Sigma) = 2$.*

Proof. Let P_1 and P_2 be the pendant paths incident with u_1 and u_2, respectively. Let $(G', \Sigma') := (G, \Sigma) - V(P_1)$ and let $(G'', \Sigma'') := (G, \Sigma) - V(P_1 - u_1)$. Since G' is a partial 2-path with path cover number 2, $M(G', \Sigma') \leq M(G') = 2$.

Suppose first that (G'', Σ'') has a pendant path Q incident with one of the vertices in $\{w_1, w_2\}$; let w be the vertex to which Q is incident. Let $(H, \Omega) := (G'', \Sigma'') - V(Q)$ and let $(H', \Omega') := (G'', \Sigma'') - V(Q - w)$. Since (H, Ω) is a partial 2-path with path cover number 2, $M(H, \Omega) = 2$, by Theorem 9. A zero forcing argument shows that $M(H', \Omega') \leq 2$. Since $M(G'', \Sigma'') = \max\{M(H, \Omega), M(H', \Omega')\}$, we obtain that $M(G'', \Sigma'') = 2$. Since $M(G, \Sigma) = \max\{M(G', \Sigma'), M(G'', \Sigma'')\}$, we see that $M(G, \Sigma) = 2$ if (G, Σ) has a pendant path Q incident with one of the vertices in $\{w_1, w_2\}$.

We may therefore assume that (G'', Σ'') has no pendant paths incident with any vertex in $\{w_1, w_2\}$. Then a zero forcing argument shows that $M(G'', \Sigma'') \leq 2$. Since $M(G, \Sigma) = \max\{M(G', \Sigma'), M(G'', \Sigma'')\}$, we see that $M(G, \Sigma) = 2$. $\square$

Lemma 16. *Let (G, Σ) be a signed graph such that the removal of pendant paths yields a 2-connected partial wide 2-path (H, Ω) with exactly one wide separation $[H_1, H_2]$. Suppose H_1 and H_2 are connected. Then $M(G, \Sigma) = 2$ if and only if (G, Σ) has two parallel paths or (G, Σ) is a SA-graph.*

Proof. Suppose that $M(G, \Sigma) = 2$.

If there is a vertex s of H such that in (G, Σ) at least two pendant paths R_1, R_2 are incident with s, let $(G', \Sigma') := (G, \Sigma) - V(R_1)$. Then (G', Σ') consists of at least two components, one of which contains a cycle, so $M(G', \Sigma') \geq 3$. By Lemma 3, $M(G, \Sigma) \geq 3$. Hence, there is no vertex s of H such that in (G, Σ) at least two pendant paths R_1, R_2 are incident with s.

Let u_1, u_2 be the vertices of attachment of H_1, and let w_1, w_2 be the vertices of attachment of H_2. Suppose first that H_1 contains a cycle C. We may assume that C is at the end of the partial wide 2-path H, that is, there is a 2-separation (C, F) of H. Let $\{v_1, v_2\} := V(C) \cap V(F)$. Let Q_1 and Q_2 be two vertex-disjoint paths between $\{v_1, v_2\}$ and $\{u_1, u_2\}$. If a pendant path of (G, Σ) is incident with H_1, but not with a vertex of C, then the pendant path is incident with a vertex of $Q_1 - v_1$ or a vertex of $Q_2 - v_2$. Then (G, Σ) has a weak $K_3^=(\Delta Y)$-minor, so $M(G, \Sigma) \geq \nu(G, \Sigma) \geq 3$, a contradiction. Hence any pendant path that is incident with a vertex of H_1 is incident with a vertex of C. Let P_1 be the path obtained from C by removing any edge between v_1 and v_2. If there are two pendant paths incident with nonadjacent vertices on P_1 and one is not incident with u_1 or u_2, then (G, Σ) has a weak $K_3^=(\Delta Y)^2$-minor, a contradiction. Hence, one of the following holds:

- at most two pendant paths of (G, Σ) are incident with vertices of P_1, and if two pendant paths of (G, Σ) are incident with vertices of P_1, then these vertices are adjacent,
- three pendant paths of (G, Σ) are incident with vertices of P_1, P_1 has length 2, and the ends of P_1 are u_1 and u_2, or
- two pendant paths of (G, Σ) are incident with the ends of P_1 and the ends are u_1 and u_2.

If H_1 has no cycles, then H_1 is a path P_1 connecting u_1 and u_2. If there are two pendant paths incident with nonadjacent vertices on P_1 and one is not incident with an end of P_1, then (G, Σ) has a weak $K_3^=(\Delta Y)^2$-minor, a contradiction. Hence, one of the following holds:

- at most two pendant paths of (G, Σ) are incident with vertices of P_1, and if two pendant paths of (G, Σ) are incident with vertices of P_1, then these vertices are adjacent,
- three pendant paths of (G, Σ) are incident with vertices of P_1, P_1 has length 2, and the ends of P_1 are u_1 and u_2, or
- two pendant paths of (G, Σ) are incident with the ends of P_1 and the ends are u_1 and u_2.

In the same way, we do the above for H_2.

Suppose that, for $i = 1, 2$, there are at most two pendant paths incident with vertices of P_i and if two pendant paths are incident with vertices of P_i, then these vertices are adjacent in P_i. Then (G, Σ) has two parallel paths. Hence, we may assume that either pendant paths are incident with both ends of P_1, the ends of P_1 are u_1 and u_2, and P_1 has length ≥ 2, or pendant paths are incident with both ends of P_2, the ends of P_2 are w_1 and w_2, and P_2 has length ≥ 2. By symmetry, we may assume that pendant paths are incident with both ends of P_1, the ends of P_1 are u_1 and u_2, and P_1 has length ≥ 2. Then H_2 contains no cycle, for otherwise (G, Σ) has a weak $K_3^=(\Delta Y)^2$-minor. So H_2 is a path connecting w_1 and w_2.

If a pendant path of (G, Σ) is incident with an internal vertex of P_2, then (G, Σ) has a weak $K_3^=(\Delta Y)^3$-minor, a contradiction. Hence, no pendant path of (G, Σ) is incident with an internal vertex of P_2. If P_1 has length > 2, then no pendant path of (G, Σ) is incident with an internal vertex of P_1. If P_1 has length 2, a pendant path of (G, Σ) is incident with an internal vertex of P_1, and P_2 has length ≥ 2, then no pendant path is incident with at least one of the vertices w_1 and w_2, for otherwise (G, Σ) has a weak $K_3^=(\Delta Y)^3$-minor. Thus, (G, Σ) is a SA-graph.

For the converse, use Lemmas 13 and 15. □

Let (G, Σ) be a signed graph such that the removal of pendant paths yields a 2-connected partial wide 2-path (H, Ω) with exactly one wide separation $[H_1, H_2]$. Let u_1 and u_2 be the vertices of attachment of H_1, suppose the vertices u_1 and u_2 are connected by a path of length ≥ 2 in H_1, and suppose H_2 is disconnected. We call (G, Σ) a *MK-graph* if each of the following hold:

1. no vertex is the end of two or more pendant paths;
2. there is a pendant path at u_1, and no pendant path at u_2;
3. $H_1 - u_2$ is a path;
4. each pendant path P incident with a vertex of the path $H_1 - \{u_1, u_2\}$ is incident with an end of $H_1 - \{u_1, u_2\}$, and if P is incident with the end of $H_1 - \{u_1, u_2\}$ adjacent to u_1, then no edge connects an internal vertex of $H_1 - \{u_1, u_2\}$ with u_2.

Lemma 17. *If (G, Σ) is a MK-graph, then $M(G, \Sigma) = 2$.*

Proof. Let v_1 and v_2 be the vertices of attachment of H_2. Let P be the pendant path at u_1 and let $(G', \Sigma') := (G, \Sigma) - V(P - u_1)$, and let $(G'', \Sigma'') := (G, \Sigma) - V(P)$. By Lemma 4, $M(G, \Sigma) = \max\{M(G', \Sigma'), M(G'', \Sigma'')\}$. Since (G', Σ') is a signed graph with two parallel paths, $M(G', \Sigma') \leq 2$, and hence we may assume that $M(G, \Sigma) = M(G'', \Sigma'')$. In (G'', Σ''), there are two pendant paths attached to u_2. Let Q be one of them. Let $(F, \Psi) := (G'', \Sigma'') - V(Q - u_2)$ and let $(F', \Psi') := (G'', \Sigma'') - V(Q)$. Since (F', Ψ') consists of two disjoint paths, $M(F', \Psi') = 2$. Since (F, Ψ) is a signed graph with two parallel paths, $M(F, \Psi) \leq 2$. By Lemma 4, $M(G'', \Sigma'') = \max\{M(F, \Psi), M(F', \Psi')\} = 2$. □

Lemma 18. *Let (G, Σ) be a signed graph such that the removal of pendant paths yields a 2-connected partial wide 2-path (H, Ω) with exactly one wide separation $[H_1, H_2]$. Suppose* *that H_2 is disconnected and H_1 is connected. Then $M(G,\Sigma) = 2$ if and only if (G,Σ) has two parallel paths or (G,Σ) is a MK-graph.*

Proof. Suppose that $M(G,\Sigma) = 2$. Since $M(G,\Sigma) \geq \xi(G,\Sigma)$, (G,Σ) has no weak minor isomorphic to a graph in the $K_3^=$-family.

Let u_1 and u_2 be the vertices of attachment of H_1. Suppose first that H_1 contains a cycle C. Let Q_1 and Q_2 be vertex-disjoint path between $\{u_1, u_2\}$ and $V(C)$; let v_1 and v_2 be the vertices of Q_1 and Q_2 on C, respectively.

We first assume that a pendant path incident with a vertex of $Q_1 \cup Q_2 - \{v_1, v_2\}$.

If a pendant path is incident with a vertex of $Q_1 \cup Q_2 - \{v_1, v_2, u_1, u_2\}$, then (G,Σ) has a weak $K_3^=(\Delta Y)$-minor. Hence, any pendant path of (G,Σ) incident with a vertex of $Q_1 \cup Q_2 - \{v_1, v_2\}$ is incident with u_1 or u_2. By symmetry, we may assume that a pendant path P_1 of (G,Σ) is incident with u_1. Then Q_1 has length ≥ 1. Then P_1 is the only pendant path incident with u_1, for otherwise $G - V(P_1)$ consists of at least two component, one of which contains a cycle. If Q_2 has length ≥ 1, then (G,Σ) has a weak $K_3^=(\Delta Y)^2$-minor. Hence, Q_2 has length 0, and therefore, $H_1 - u_2$ has no cycle. If a pendant path is incident with u_2, then, as $M(G,\Sigma) \geq M(G - V(P_1), E(G - V(P_1)) \cap \Sigma)$ and $G - V(P_1)$ is a tree with path cover number ≥ 3, $M(G,\Sigma) \geq 3$. Hence, no pendant path is incident with u_2. If a pendant path is incident with an internal vertex of the path $H_1 - \{u_1, u_2\}$, then (G,Σ) contains a weak $K_3^=(\Delta Y)^3$-minor. Hence any pendant path incident with a vertex of the path $H_1 - \{u_1, u_2\}$ is incident with an end of $H_1 - \{u_1, u_2\}$. If an edge connects an internal vertex of $H_1 - \{u_1, u_2\}$ with u_2, then (G,Σ) has a weak $K_3^=(\Delta Y)$-minor. Hence (G,Σ) is a MK-graph.

We may therefore assume that each pendant path incident with a vertex of H_1 is incident with a vertex of C. Let P be the path obtained from C by removing all edges between v_1 and v_2. If two pendant paths are incident with nonadjacent vertices on P and one is not incident with u_1 and u_2, then (G,Σ) has a weak $K_3^=(\Delta Y)^2$-minor. If $u_1 = v_1$ and $u_2 = v_2$, and pendant paths are incident with u_1 and u_2, let P_1 be the pendant path incident with u_1. If $(G',\Sigma') := (G,\Sigma) - V(P_1)$, then (G',Σ') is a tree with path cover number ≥ 3, and hence $M(G,\Sigma) \geq 3$. Hence, there are at most two pendant paths incident with vertices of P and these vertices are adjacent in P_1. Then (G,Σ) has two parallel paths.

We may therefore assume that H_1 has no cycle. Then H_1 is a path P connecting u_1 and u_2. Suppose that there are pendant paths incident with nonadjacent vertices on P. Then the length of P is at least 2. Suppose a pendant path is incident with u_1. If there is a pendant path at u_2, then, as $M(G,\Sigma) \geq M(G - V(P_1), E(G - V(P_1)) \cap \Sigma)$ and $G - V(P_1)$ is a tree with path cover number ≥ 3, $M(G,\Sigma) \geq 3$. Hence, no pendant path is incident with u_2. If a pendant path is incident with an internal vertex of $P - \{u_1, u_2\}$, then (G,Σ) contains a weak $K_3^=(\Delta Y)^3$-minor. Hence, any pendant path is incident with an end of $P - \{u_1, u_2\}$. If a pendant path is incident with the end of $P - \{u_1, u_2\}$ adjacent to u_1 and an edge connects an internal vertex of $P - \{u_1, u_2\}$ with u_2, then (G,Σ) has a weak $K_3^=(\Delta Y)$-minor. Thus, (G,Σ) is a MK-graph. Hence, we may assume that no pendant path is incident with u_1 or u_2. Then (G, Σ) has a weak $K^{=}(\Delta Y)^3$-minor, a contradiction. Hence, we may assume at most two pendant paths are incident with vertices of P and these vertices are adjacent in P. Then (G, Σ) has two parallel paths.

For the converse, use Lemmas 13 and 17. □

Lemma 19. *Let (G, Σ) be a signed graph such that the removal of pendant paths yields a 2-connected partial wide 2-path (H, Ω) with exactly one wide separation. Then $M(G, \Sigma) = 2$ if and only if (G, Σ) has two parallel paths, (G, Σ) is a SA-graph, or (G, Σ) is a MK-graph.*

Proof. Let $[H_1, H_2]$ be the wide separation of (G, Σ). Suppose that $M(G, \Sigma) = 2$. If H_1 and H_2 are connected, then, by Lemma 16, either (G, Σ) has two parallel paths or (G, Σ) is a SA-graph. Suppose next that H_1 or H_2 is disconnected; we may assume that H_2 is disconnected. Let u_1 and u_2 be the vertices of attachment of H_1. If H_1 has a path of length ≥ 2, then, by Lemma 18, either (G, Σ) has two parallel paths or (G, Σ) is a MK-graph. We may therefore assume that either H_1 consists of only one edge connecting u_1 and u_2, or H_1 is disconnected. In both cases, (G, Σ) has two parallel paths.

The converse follows from Lemmas 13, 15, and 17. □

6.3. Partial wide 2-paths with no wide separations

In [9], Johnson et al. characterized the class of graphs G with $M(G) = 2$. For signed graphs (G, Σ) such that the removal of pendant paths yields a 2-connected partial 2-path with no wide separation, the characterization when $M(G, \Sigma) = 2$ is similar to their result.

Let (G, Σ) be a signed graph such that the removal of pendant paths yields a 2-connected partial 2-path (H, Ω). Suppose C_1 and C_2 are distinct cycles in H such that there exist 2-separations (C_1, H_1) and (C_2, H_2) of H. Let P_1 and P_2 be vertex-disjoint paths between C_1 and C_2, and let u_1, u_2 be the ends of P_1 and P_2, respectively, on C_1, and let v_1, v_2 be the ends of P_1 and P_2, respectively, on C_2. We call (G, Σ) a *SH-graph* if $l(P_1) = 0$ and $l(P_2) \leq 1$, and

1. if $l(P_1) = 0$ (so $u_1 = v_1$) and $l(P_2) = 1$, then
   - there is a single pendant path incident with each end of the paths P_1 and P_2;
   - if the path $C_1 - u_1u_2$ has length 2, then at most one pendant path is incident with the internal vertex of $C_1 - u_1u_2$, and if the path $C_1 - u_1u_2$ has length ≥ 3, then no pendant path is incident with a vertex of $C_1 - \{u_1, u_2\}$;
   - if the path $C_2 - v_1v_2$ has length 2, then at most one pendant path is incident with the internal vertex of $C_2 - v_1v_2$, and if the path $C_2 - v_1v_2$ has length ≥ 3, then no pendant path is incident with a vertex of $C_2 - \{v_1, v_2\}$;
2. if $l(P_1) = l(P_2) = 0$, then
   - there is a single pendant path incident with each end of the paths P_1 and P_2;

- the path $C_1 - u_1u_2$ has length 3, and single pendant paths are incident with the internal vertices of $C_1 - u_1u_2$;
- if the path $C_2 - v_1v_2$ has length 2, then at most one pendant path is incident with the internal vertex of $C_2 - v_1v_2$, and if the path $C_2 - v_1v_2$ has length ≥ 3, then no pendant path is incident with a vertex of $C_2 - \{v_1, v_2\}$.

Lemma 20. *If (G, Σ) is a SH-graph, then $M(G, \Sigma) = 2$.*

Proof. Let P be the pendant path incident with u_1. Let $(G', \Sigma') := (G, \Sigma) - V(P)$ and let $(G'', \Sigma'') := (G, \Sigma) - V(P - u_1)$. By Lemma 4, $M(G, \Sigma) = \max\{M(G', \Sigma'), M(G'', \Sigma'')\}$. Since (G', Σ') is a tree with path cover number 2, $M(G', \Sigma') = 2$, and since (G'', Σ'') has two parallel paths, $M(G'', \Sigma'') \leq 2$. Thus, $M(G, \Sigma) = 2$. $\square$

We call the signed graph obtained from a signed 5-cycle by attaching pendant paths to each of its vertices a *SF-graph*.

The following lemma can be proved in the same way as Lemma 20.

Lemma 21. *If (G, Σ) is a SF-graph, then $M(G, \Sigma) = 2$.*

Lemma 22. *Let (G, Σ) be a signed graph such that the removal of pendant paths yields a 2-connected partial 2-path (H, Ω). Suppose (H, Ω) has no wide separation. Then $M(G, \Sigma) = 2$ if and only if (G, Σ) has two parallel paths, (G, Σ) is a SH-graph, or (G, Σ) is a SF-graph.*

Proof. Suppose $M(G, \Sigma) = 2$.

We first assume that there are at least two distinct cycles in H. Let C_1 and C_2 be distinct cycles such that there exist 2-separations (C_1, L_1) and (C_2, L_2) of H. Let P_1 and P_2 be vertex-disjoint paths between C_1 and C_2, and let u_1, u_2 be the ends of P_1 and P_2, respectively, on C_1, and let v_1, v_2 be the ends of P_1 and P_2, respectively, on C_2.

If a pendant path of G is incident with an internal vertex of P_1 or P_2, then (G, Σ) has a weak $K_3^=(\Delta Y)$-minor, contradicting that $M(G, \Sigma) = 2$. Hence any pendant path is incident with a vertex of C_1 or C_2. If there are no pendant paths incident with nonadjacent vertices of $C_1 - u_1u_2$, and there are no pendant paths incident with nonadjacent vertices of $C_2 - v_1v_2$, then (G, Σ) has two parallel paths. We may therefore assume that there are pendant paths incident with nonadjacent vertices of $C_1 - u_1u_2$, or that there are pendant paths incident with nonadjacent vertices of $C_2 - v_1v_2$. We assume the former, and let w_1, w_2 be the vertices on C_1 to which the pendant paths R_1, R_2, respectively, are incident; we take w_1 and w_2 such that the distance between w_1 and w_2 in $C_1 - u_1u_2$ is maximum. We may assume that w_1 coincides with u_1 or is between u_1 and w_2 on $C_1 - u_1u_2$. If $w_1 \neq v_1$ and $w_2 \neq v_2$, then (G, Σ) has a weak $K_3^=(\Delta Y)^2$-minor. Therefore, $w_1 = v_1$ or $w_2 = v_2$; we may assume that $w_1 = v_1$. Then $u_1 = w_1 = v_1$.

Exactly one pendant path is incident with u_1. For if at least two pendant paths, T_1 and T_2, are incident with u_1, let $(G', \Sigma') := (G, \Sigma) - V(T_1)$. Then (G', Σ') consists of at least two components, one of which is a tree with path cover number ≥ 2, and hence $M(G,\Sigma) \geq M(G',\Sigma') \geq 3$, a contradiction.

Suppose now first that the length of P_2 is at least one.

Suppose a pendant path Q_2 is incident with a vertex of $C_2 - v_1v_2$ that is nonadjacent to u_1 in $C_2 - v_1v_2$. If R_2 is not incident with u_2, then (G,Σ) has a weak $K_3^{=}(\Delta Y)^3$-minor. Hence R_2 is incident with u_2. In the same way, Q_2 is incident with v_2. Then the length of P_1 is one, for otherwise (G,Σ) has a weak $K_3^{=}(\Delta Y)^3$-minor. If $C_1 - u_1u_2$ has length ≥ 3 and a pendant path is incident with a vertex of $C_1 - \{u_1, u_2\}$, then (G,Σ) has either a weak $K_3^{=}(\Delta Y)^3$- or a weak $K_3^{=}(\Delta Y)^2$-minor. In the same way, if $C_2 - v_1v_2$ has length ≥ 3, then no pendant path is incident with a vertex of $C_2 - \{v_1, v_2\}$. Then (G,Σ) is a Sea Horse. We may therefore assume that if a pendant path is incident with a vertex of $C_2 - v_1$, then it is incident with the vertex adjacent to u_1 in $C_2 - v_1v_2$.

If a pendant path is incident with a vertex of $C_1 - \{u_1, w_2\}$ that is nonadjacent to w_2 in $C_1 - u_1u_2$, then (G,Σ) has a weak $K_3^{=}(\Delta Y)^2$-minor. Hence, any pendant path incident with a vertex of $C_1 - \{u_1, w_2\}$ is incident with a vertex adjacent to w_2. If a pendant path is incident with the vertex adjacent to u_1 in $C_2 - v_1v_2$, then (G,Σ) has two parallel paths. If no pendant path is incident with a vertex of $C_2 - u_1$, then (G,Σ) has two parallel paths.

Suppose next that the path P_2 has length 0; then $u_2 = v_2$. If no pendant path is incident with u_2 and a pendant path is incident with a vertex $v \neq v_2$ of $C_2 - v_1v_2$ that is nonadjacent to u_1 in $C_2 - v_1v_2$, then (G,Σ) has a weak $K_3^{=}(\Delta Y)^3$-minor (as R_1 and R_2 are at distance ≥ 2 on $C_1 - u_1u_2$). If no pendant path is incident with u_2 and all pendant paths incident with vertices of $C_2 - v_1v_2$ are adjacent to u_1 in $C_2 - v_1v_2$, then (G,Σ) has two parallel paths.

We may therefore assume that a pendant path is incident with u_1 and a pendant path is incident with u_2. If a pendant path is incident with a vertex of C_1 that is nonadjacent to u_1 and u_2 in $C_1 - u_1u_2$, then (G,Σ) has a weak $K_3^{=}(\Delta Y)^3$-minor. Therefore, any pendant path incident with a vertex of $C_1 - \{u_1, u_2\}$ is adjacent to u_1 or u_2 in $C_1 - u_1u_2$. In the same way, any pendant path incident with a vertex of $C_2 - \{u_1, u_2\}$ is adjacent to u_1 or u_2 in $C_2 - u_1u_2$. If Q_1 is a pendant path incident with a vertex of $C_1 - \{u_1, u_2\}$ that is nonadjacent to u_1 in $C_1 - u_1u_2$, and Q_2 is a pendant path incident with a vertex of $C_2 - \{u_1, u_2\}$ that is nonadjacent to u_1 in $C_2 - u_1u_2$, then (G,Σ) has a weak $K_3^{=}(\Delta Y)^3$-minor. In the same way, there are no pendant paths Q_1 and Q_2 with Q_1 incident with a vertex of $C_1 - \{u_1, u_2\}$ that is nonadjacent to u_2 in $C_1 - u_1u_2$ and Q_2 incident with a vertex of $C_2 - \{u_1, u_2\}$ that is nonadjacent to u_2 in $C_2 - u_1u_2$. Hence, if there are two vertices of $C_1 - \{u_1, u_2\}$ with pendant paths attached to them, then $C_1 - u_1u_2$ has length three, and if in addition there is a pendant path incident with a vertex of $C_2 - \{u_1, u_2\}$, then $C_2 - u_1u_2$ has length two. Then (G,Σ) is a Seahorse. The case where there are two vertices of $C_2 - \{u_1, u_2\}$ with pendant paths attached to them is similar.

We may therefore assume that at most one pendant path is incident with a vertex of $C_1 - \{u_1, u_2\}$ and at most one pendant path is incident with a vertex of $C_2 - \{u_1, u_2\}$. By symmetry, we may assume that if a pendant path is incident with a vertex of $C_1 - \{u_1, u_2\}$, then this vertex is adjacent to u_1. Then any pendant path incident with a vertex of $C_2 - \{u_1, u_2\}$ that is adjacent to u_2. Then (G, Σ) has two parallel paths.

We may therefore assume that H contains at most one cycle. As H is 2-connected, H is a cycle with size ≥ 3. If a vertex on H has more than two pendant paths attached to it, then, by Lemma 4, $M(G, \Sigma) \geq 3$. Hence, we may assume that any vertex on H has at most two pendant paths attached to it. Suppose next that there are two pendant paths attached to a vertex v. If a pendant path is incident with vertex that is not adjacent to v, then, by Lemma 4, $M(G, \Sigma) \geq 3$. Hence, any pendant path is either incident with v or incident with a vertex adjacent to v. If there is a vertex adjacent to v with more than one pendant path attached, then, by Lemma 4, $M(G, \Sigma) \geq 3$. Hence to any vertex adjacent to v at most one pendant path is attached. Then (G, Σ) has two parallel paths. Therefore, we may assume that at most one pendant path is incident with each vertex of H.

Let $P_1, \ldots, P_k$ be the pendant paths attached to H, where we assume that $P_1, \ldots, P_k$ are in this order around H. If $k \geq 6$, then (G, Σ) has a weak $K_3^{=}(\Delta Y)^3$-minor. If $k = 5$ and there are pendant path P_i and P_{i+1} (index modulo k) that are at distance ≥ 2 on H, then (G, Σ) has a weak $K_3^{=}(\Delta Y)^3$-minor. If $k = 5$ and there are no consecutive pendant paths at distance ≥ 2, then (G, Σ) is a SF-graph. If $k = 4$ and there is a pendant path that is at distance ≥ 2 from the other pedant paths, then (G, Σ) has a weak $K_3^{=}(\Delta Y)^3$-minor. If $k = 4$ and for all pendant paths, there is a distinct pendant path at distance 1, then (G, Σ) has two parallel paths. If $k = 3$ and the pendant paths in each pair of pendant path are at distance ≥ 2 on H from one another, then (G, Σ) has a weak $K_3^{=}(\Delta Y)^3$-minor. If $k = 3$ and two pendant paths at distance 1 on H, then (G, Σ) has two parallel paths. If $k \leq 2$, then, clearly, (G, Σ) has two parallel paths. □

7. The main result

We now provide a combinatorial characterization of signed graph (G, Σ) with $M(G, \Sigma) = 2$.

Theorem 23. *Let (G, Σ) be a signed graph. Then $M(G, \Sigma) = 2$ if and only if one of the following holds:*

1. *(G, Σ) has two parallel paths, but G is not a path;*
2. *(G, Σ) is a SH-graph;*
3. *(G, Σ) is a SF-graph;*
4. *(G, Σ) is a SA-graph;*
5. *(G, Σ) is a MK-graph;*
6. *(G, Σ) is a ST-graph; or*
7. *(G, Σ) is obtained from W_4^o by attached single pendant paths at some of the vertices of W_4^o.*

Proof. The "if" statement is clear.

We now prove the "only if" statement. Suppose $M(G, \Sigma) = 2$.

If G is disconnected, then G has exactly two components, for otherwise $M(G, \Sigma) \geq 3$. Furthermore, each component is a path, for otherwise $M(G, \Sigma) \geq 3$. Then (G, Σ) is a signed graph with two parallel paths.

We may therefore assume that G is connected. If G has no cycle, then G is a tree. Since $M(G, \Sigma) = M(G)$ when G is a forest, G has path cover number 2. Then (G, Σ) is a signed graph with two parallel paths. We may therefore assume that G has a cycle. By Lemma 5, (G, Σ) can be obtained from a 2-connected signed graph (H, Ω) with $M(H, \Omega) = 2$ or from an odd cycle with two edges by attaching pendant paths. Furthermore, at each vertex of H at most two pendant paths can be attached. In the latter case, (G, Σ) is a signed graph with two parallel paths. We may assume that the former case holds. By Theorem 6, (H, Ω) is either a partial wide 2-path or is isomorphic to W_4^o. If (H, Ω) is isomorphic to W_4^o, then, by Lemma 7, only single pendant paths can be attached at vertices of W_4^o. We may therefore assume that (H, Ω) is a partial wide 2-path. Then the statement follows from Lemmas 14, 19, and 22. □

Declaration of competing interest

None declared.

Data availability

No data was used for the research described in the article.

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