Morphological modelling for rivers with non-uniform sediment

J.S. Ribberink

Internal Report no. 1-80

Delft University of Technology
Department of Civil Engineering
Fluid Mechanics Group

Morphological modelling for rivers with non-uniform sediment

J.S.Ribberink

Delft University of Technology

Department of Civil Engineering

Fluid Mechanics Group

Internal Report no. 1-80

Contents

Main symbols

a	mean waterdepth	[L]
A	dimensionless approximated characteristic direction	[-]
$\bar{A}$	$=A.(1-\varepsilon_0).q/\sqrt{\Delta g D_1^3}$	[-]
B	dimensionless approximated characteristic direction	[-]
$\bar{B}$	$=B.(1-\varepsilon_0).q/\sqrt{\Delta g D_1^3}$	[-]
$c_{1,2}$	characteristic directions	[$L\ T^{-1}$]
C	volume concentration of sediment	[-]
C_i	volume concentration of sedimentfraction i	[-]
C_g	Chézy-roughness of the grains	[$L^{\frac{1}{2}}T^{-1}$]
C_t	total Chézy-roughness of the bed	[$L^{\frac{1}{2}}T^{-1}$]
D_i	grain-diameter of sedimentfraction i	[L]
D_m	mean grain-diameter of a sediment-mixture	[L]
Fr	Froude-number	[-]
i	energy-slope	[-]
p_i	probability of sedimentfraction i	[-]
$p_{i_{z_0}}$	probability of sedimentfraction i at the lower boundary of the transportlayer	[-]
p_{i_T}	transportprobability of sedimentfraction i	[-]
q_s	sedimenttransport in volume(real) per unit time and width	[L^2T^{-1}]
q_{s_i}	sedimenttransport of fraction i in volume(real) per unit time and width	[L^2T^{-1}]
R_b	hydraulic radius	[L]
s	sedimenttransport in volume(including pores) per unit time and width	[L^2T^{-1}]
s_i	sedimenttransport of fraction i in volume(including pores) per unit time and width	[L^2T^{-1}]
u	watervelocity averaged over the waterdepth	[$L\ T^{-1}$]
z_0	mean bed-level	[L]
z_{max}	level of the upper boundary of the transportlayer	[L]
α_i	distribution coefficient of sedimentfraction i	[-]
$\bar{\beta}$	quotient of the effective transportlayer-thickness(concentration: $1-\varepsilon_0$) and the actual transportlayer-thickness	[-]
δ	transportlayer-thickness	[L]
Δ	$=(\rho_s-\rho)/\rho$:relative density of the sediment	[-]
ε_0	porosity of the sediment in the bed	[-]
$\phi_{1,2}$	dimensionless characteristic directions	[-]

$\phi_{1,2}$	$= \phi_{1,2}.(1-\varepsilon_0).q/\sqrt{\Delta g D_1^3}$	[-]
ψ	dimensionless transportconcentration parameter	[-]
ψ_i	dimensionless transportconcentration parameter of fraction i	[-]
τ_{c_i}	critical shear stress of sedimentfraction i	$[ML^{-1}T^{-2}]$
τ_{c*_i}	dimensionless critical shear stress of fraction i	[-]
τ_{e*_i}	dimensionless effective shear stress of fraction i	[-]
μ	ripple factor	

In Chapter 4 four simple applications of the mathematical model for two sedimentfractions are treated. The watermotion is simplified in order to be able to carry out the calculations partly by hand.
In Chapter 5 a summary and conclusions are given.

0. Introduction

The mathematical prediction of morphological changes in rivers due to natural causes or human interference has got much attention during the last two decades.

The basic equations for the movement of water and sediment can be decoupled under some restricting assumptions (de Vries, 1959, 1966). For sufficiently low Froude numbers the watermovement can be supposed to be quasi steady. The range of practical problems that can be solved in this way is sufficiently large to use this attractive simplification.

The main restricting assumptions that are used are

(i) The sediment transport is a function of the *local* hydraulic conditions

(ii) The bedroughness is supposed not to vary in time

(iii) The sediment is supposed to be uniform enough leading to a constant representative grainsize both in time and place.

With respect to these restrictions the following remarks can be made

ad.(i) The transport can not be written as a function of the local hydraulic conditions if transport in suspension prevails. More precisely stated such a direct relation can only be assumed if the lengthscale in the morphological process is large compared to the *adaptation length* in the flow direction for the change in a concentration profile due to a change in the hydraulic condition in space (Kerssens, 1974 and Kerssens *et al*, 1979).

ad.(ii) Alluvial roughness predictions for steady uniform flow are not very accurate. For unsteady flow they do not exist. Some attempts have been made to incorporate predicted roughness values in the morphological computations discussed here. (Chollet and Cunge, 1979, 1980).

ad.(iii) Hardly some attention is paid as yet to morphological computations in which the sediment consists of a mixture of different grain-sizes and an interaction exists between composition-changes of the bed and the morphological changes itself.
For example: Downstream of a fixed weir in a river generally erosion of the river-bed takes place caused by the blocked sediment-transport. Because of selective erosion of the finer fractions of the present bed-material, the coarse fractions will stay behind and

eventually can form a layer on top of the river-bed, which gives a protection against further erosion ('armouring').

The objective of this study is to get rid of this last restriction viz. the presence of (nearly) uniform sediment. In order to reduce the complexity of the analysis this study postulates:

- the flow is quasi-steady
- the local transport is a function of the local conditions
- the bedroughness is constant.

In spite of these restrictions the impression is that sufficient practical problems can be tackled in addition to the one that are now (1980) solved on a routine basis by using the original model for (nearly) uniform sediment.

This study does not yet consider the important numerical problems that have to be solved before morphological computations can be carried out on a routine basis. For review of the numerical aspects in the case of (nearly) uniform flow reference can be made to Vreugdenhil (1981).

In Chapter 1 a derivation of the equations and an extension of the mathematical model will be carried out. The sediment-mixture is separated in a number of fractions - each with a representative grain size - and the equations describing the sediment-movement are split up for every fraction separately.
In order to get some insight in the new model in Chapter 2 a restriction will take place to sediment-mixtures consisting of only two sediment-fractions.
As a result only one extra dependent variable *viz.* the probability of one of the fractions, comes into the equations.
Moreover the characteristic directions and relations belonging to the set of partial differential equations will be derived mainly in order to obtain information concerning the time-scales of changes in bedlevel and bedcomposition; also the interaction between these changes and the influence of some determining parameters will be studied.
In Chapter 3 specific calculations will be carried out of the characteristic directions and relations and the influence of two possible concepts for a transportformula per fraction is studied.

1. Basic equations

1.1. General

The existing mathematical model for morphological computations in rivers (see for example de Vries, 1976) consists of two water-equations and two sediment-equations. In principle these equations are a continuity equation and an equation of motion for both phases. The watermovement is described by the one-dimensional long-wave equation and the continuity equation and is assumed to be quasi-steady.

The equation of motion of the sediment is a sedimenttransport formula which implies a direct relationship between sedimenttransport and the depth-averaged watervelocity.
As was stated before this direct relation can be questionable because of a delayed reaction of the sediment on changing hydraulic conditions (hysteresis-effect). In the transportformula the sediment is characterised by one representative grain-diameter.

The continuity-equation for the sediment is rahter simple and it describes the local change of the bed-level caused by a sedimenttransport gradient in the flow-direction.

If the sediment consists of a large-range mixture of grainsizes then the above-described mathematical model is less accurate and it might be useful to set up both sediment equations for every sediment fraction separately. In the next sections these equations and the unchanged water-equations will be treated.

1.2. The watermovement

The watermovement in rivers is generally described by the one-dimensional "long-wave equation" and continuity-equation (see for example Jansen, 1979):

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial a}{\partial x} + g\frac{\partial z_o}{\partial x} = R \tag{1}$$

$$\frac{\partial a}{\partial t} + \frac{\partial z_o}{\partial t} + a\frac{\partial u}{\partial x} + u\frac{\partial a}{\partial x} = 0 \tag{2}$$

in which: u = time-averaged and depth-averaged watervelocity in x-direction
a = waterdepth
z_o = bedlevel
R = friction term.

The riverwidth is supposed to be constant and the waterequations therefore are given per unit width. Important assumptions are the occurence of small bed slopes (i_b << 1) and the validity of the hydrostatic pressure distribution. This confines the problems in which this set of equations can be applied to long-wave problems, in which vertical accelerations are negligible with respect to the acceleration of gravity. This means that for example in regions with sudden bedlevel changes these one-dimensional equations generally are too simple.

The water turbulence which takes places on much smaller time and distance-scales than long waves is included in the friction term R. Before integration in the vertical direction took place this was realised by averaging both equations over a certain time period (much larger than the scale of the turbulent fluctuations but much smaller than the time-scale of the long-wave motion itself) and making use of Reynolds shear stresses.

1.3. The sediment-movement

1.3.1. General

Analogous to the watermovement the sedimentmovement takes place on different scales. In this report the bedlevel-fluctuations because of bedforms are treated in a comparable way as the turbulence in the watermovement.

The rate of sedimenttransport (bed-load) and the particle velocities will vary along a bedform; the process of sedimenttransport considered on bedform scale is therefore often described as being in a "dynamical equilibrium". The sedimentequations which will be treated in this report can only be applied for large-scale processes. This means that only sandwaves (or bedlevel changes) are treated which have a period much larger than the period of the bedforms.

As was mentioned in the previous section the long-wave equation will be applied and the local dissipation of energy behind bedforms ("wakes") can artificially be seen as an energy-loss because of an additional bedroughness and be included in the bedfrictionterm. Analogous to waterturbulence the parameters playing a role in the sediment process will be averaged over a period much larger than the bedformperiods but much smaller than the period of the "long sand-waves" itself.

The derivation of the continuity equation per sedimentfraction is given in Appendix 1. It is shown that the definitions of bedlevel (z_o), transportlayerthickness (δ) and sedimentconcentration per fraction (C_i) are rather important.

It also shows that in contrast with the transport of uniform sediment a term describing the storage of sediment in the transportlayer is included; this term arises from the possibility of exchange of sedimentfractions in the transportlayer in case of sedimentmixtures.

In the first sections of this report the sedimenttransportformula per fraction will be written in a general form. The difficulties in using this formula were mentioned before. Cases in which these difficulties arise will be treated and a general condition for the application of this formula will be given.

1.3.2. Continuity-equation per sedimentfraction

The complete derivation of the continuity-equation per sedimentfraction takes place in Appendix 1.
Before giving the resulting equation the definitions of bedlevel (z_o), transportlayerthickness (δ) and sedimentconcentration per fraction (C_i) as used in the appendix will be treated.

The local bedlevel $z(x,t)$ is defined as the level below which no grainmovement occurs. Because of the existence of bedforms this level will fluctuate and it is therefore possible to define a mean bedlevel:

$$\bar{z}(x,t) = \frac{1}{T} \int_0^T z(x,t)dt \qquad (3)$$

The period of this averaging process is chosen much longer than the period of the bedforms but much smaller than the period of the "long sandwave".
During this period this mean level $\bar{z}(x,t)$ must not change much; the instantaneous bedlevel $z(x,t)$ will be assumed to fluctuate between an upper and a lower boundary $z_{max}(x,t)$ and $z_{min}(x,t)$ (see Fig. 1).

The averaged bottomlevel which will be used in the equations is $z_o(x,t) = z_{min}(x,t)$ because during period T below this level no grainmovement occurs.

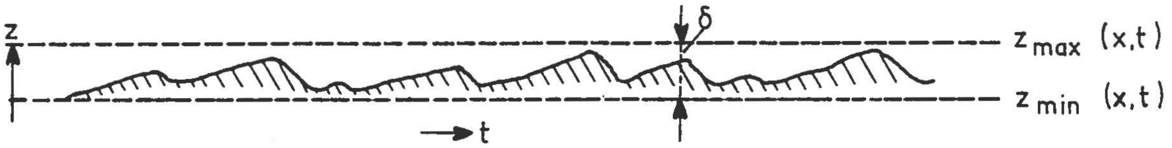


Fig. 1 Registration of z(x,t) during the averaging period T

The transportlayerthickness δ(x,t) (in case of bed-load) is defined as

$$\delta(x,t) = z_{max}(x,t) - z_{min}(x,t) \tag{4}$$

Remark: Experiments showed (Willis and Kennedy (1977)) that the fluctuations of the local bed elevation can be described by a gamma function. The upper and lower boundaries of the transportlayer can then be defined as the levels of the $\frac{1}{2}\alpha$-tails of this probability density function. In the theoretical model, however, it is assumed that all sedimenttransport is occurring between two fixed boundaries.

Because of the existence of bedforms only a small part of the sediment in the transportlayer is moving at one instant. In case of bed-load this happens in a thin layer on top of the bedforms. The propagation of the bedforms in direction of the flow (Fr < 1) finds its cause in a continuous erosion and sedimentation at respectively the front and lee side of the bedform. Because of this phenomenon and the definition of the transportlayer an elementary volume dx-dz (unit width) can contain particles in rest as well as particles in motion.

The concentration (volume) $C_i(x,z,t)$ used in the continuity-equation of sedimentfraction i is defined as the averaged value (period T) of the overall concentration (moving + resting particles) of sedimentfraction i.

A resulting version of the one-dimensional continuity-equation of sedimentfraction i is (see Appendix 1):

$$\bar{p}_{i_{z_o}} \frac{\partial z_o}{\partial t} + \frac{\partial}{\partial t} (\alpha_i \beta \bar{\bar{p}}_i \delta) + \frac{\partial s_i}{\partial x} = 0 \qquad (5)$$

in which $\bar{\bar{p}}_i$ is the double-averaged (over period T and transportlayer-thickness δ) value of the probability of sedimentfraction i. Under certain assumptions (see App. 1) the following relation between $\bar{\bar{p}}_i$ and the double-averaged concentration $\bar{\bar{C}}(x,t)$ is true:

$$\bar{\bar{C}}_i(x,t) = \alpha_i \bar{\bar{C}}(x,t) . \bar{\bar{p}}_i(x,t) \qquad (6)$$

in which α_i is a distribution coefficient which determines the distribution of fraction i in the transportlayer in z-direction.
The double-averaged concentration $\bar{\bar{C}}(x,t)$ of all the sediment in the transportlayer is dependent on "the shape of the bedforms" and the porosity ε_o:

$$\bar{\bar{C}}(x,t) = \beta . (1 - \varepsilon_o) \qquad (7)$$

in which ε_o is supposed to be constant and β is a factor indicating to what extent the transport layer is filled with sediment.

Remark: If the "shape of the bedforms" or the probability density function (p.d.f) of the fluctuations is always constant, β will be constant too. Formally β is defined as the mean value of the cumulative p.d.f. of the bedlevelfluctuations (averaged over transportlayer δ). See Appendix 1.

In Eq. (5) $\bar{p}_{i_{z_o}}$ is the time-averaged value of p_i at the lower boundary of the transportlayer $z = z_o(x,t)$.

With the assumptions that the transportlayerthickness δ as well as factor β (constant p.d.f.) and the distribution-coefficient α_i are constant the second term of Eq. (5) can be written as:

$$\frac{\partial}{\partial t} (\alpha_i \beta \bar{\bar{p}}_i \delta) = \alpha_i \beta \delta \frac{\partial \bar{\bar{p}}_i}{\partial t}$$

Under these assumptions it can also be proven (see App. 1) that $\alpha_i = 1$ for all fractions and Eq. (5) can now be written as:

$$\bar{p}_{i_{z_o}} \cdot \frac{\partial z_o}{\partial t} + \beta\delta \frac{\partial \bar{\bar{p}}_i}{\partial t} + \frac{\partial s_i}{\partial x} = 0 \qquad (8)$$

This form of the continuity-equation of sedimentfraction i will be used in the theoretical model.

1.3.3. Sedimenttransportformula per fraction

Because of the change of hydraulic conditions along a bedform the sedimenttransport (bed-load) is variable. It is said that because of the variations of the transport in time and place, the sedimenttransportprocess is in a dynamical equilibrium.

A sedimenttransportformula only relates the mean value of the transport to the mean watervelocity, the bedroughness and a representative grain-diameter.

Because of the necessary averaging time (T_{av}) of the different parameters the sedimenttransportformula can only be used in very slowly varying circumstances ($T >> T_{av}$).

Besides, in Section 1.1. it was mentioned that the use of a sedimenttransport formula becomes questionable in some cases because of the existence of hysteresis-phenomena:

1. Suspended-load: A sudden change of hydraulic conditions causes a delayed reaction of the sediment. It takes time or distance for the sediment particles to redistribute over the concentration vertical; during this response-time the transportformula cannot be correct.
2. Bed-load: A similar process can happen in case of bed-load when a delayed reaction of the bed-forms takes place on changing hydraulic conditions.

In both cases an adaptation occurs which can be symbolized by an adaptationperiod (T_a) or adaptation length (Kerssens, 1974). Because of these hysteresis-phenomena a second condition for the use of a sedimenttransportformula can be given; this is again the necessity of very slowly varying circumstances ($T >> T_a$).

A general form of a sedimenttransportformula is:

$$s = f(\bar{u}, D_m, C_t) \qquad (9)$$

in which: D_m = mean grain diameter
C_t = Chézy-roughness.

If the sediment is uniform and the bed-roughness is supposed to be fixed Eq. (9) becomes:

$$s = f(\bar{u}) \tag{10}$$

A sedimenttransportformula per fraction based on the above-described transportformula for uniform sediment has similar restrictions.

1. Suspended load: It is a known fact that the different fractions are distributed along the concentration vertical in a different way (Sengupta, 1975). The reactiontimes of the separate distributions on changes in hydraulic conditions must be taken into account. The time or length scale of the adaptation of the finest fraction can be used as a measure of this reaction process.
2. Bed-load: Analogous to suspended load the size-fractions are not necessarily uniformly distributed over the transportlayer (in vertical direction). It is plausible that the"shape of the bed-forms" is important in this respect. A change in hydraulic conditions has its influence on the bedforms and thus the distribution of the different fractions in the bedforms (transportlayer) may be influenced too. Little is known about the adaptation of bedforms or the adaptation of grainsize distributions.

Again the time-scale of the interesting processes (bedlevel and bedcomposition changes) must be an order of magnitude larger than the different adaptation periods T_a ($T >> T_a$).

A general form of a sedimenttransportformula per fraction i is:

$$s_i = f_i(\bar{u}, \bar{\bar{p}}_1, \ldots, \bar{\bar{p}}_i, \ldots, \bar{\bar{p}}_{n-1}, D_1, \ldots, D_n, C_t) \tag{11}$$

$$i = 1, \ldots, n \text{ (fractions)}$$

In this formula the transport of fraction i depends not only on its own (double averaged) probability in the transportlayer but also on the probabilities of the other fractions.

A simplified form is:

$$s_i = \bar{\bar{p}}_i . f'_i(\bar{u}, D_i, C_t) \qquad (12)$$

in which f'_i is the transportrate of fraction i in case of uniform sediment ($D = D_i$) under the same hydraulic conditions.
In this formula the transport of fraction i is a linear function of its probability $\bar{\bar{p}}_i$.

This "basic hypothesis" and other available concepts of bed-load formulae for non-uniform sediment are treated in a literature survey (Ribberink, 1978).

Later in this report a more specific form of the transportformula will be chosen and the influence of this form and some alternatives on the mathematical model will be studied.

1.4. The set of equations

The following set of equations has been described in the previous sections:

equation of motion of water:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial a}{\partial x} + g\frac{\partial z_o}{\partial x} = R \qquad (13)$$

continuity-equation of water:

$$\frac{\partial a}{\partial t} + \frac{\partial z_o}{\partial t} + u\frac{\partial a}{\partial x} + a\frac{\partial u}{\partial x} = 0 \qquad (14)$$

continuity-equation of sedimentfraction i:

$$\frac{\partial s_i}{\partial x} + \beta\delta\frac{\partial \bar{\bar{p}}_i}{\partial t} + \bar{p}_{i_{z_o}} \cdot \frac{\partial z_o}{\partial t} = 0 \qquad (15)$$

$i = 1,\ldots,n$

transportformula of sedimentfraction i:

$$s_i = f_i(\bar{u},\bar{\bar{p}}_1\ldots,\bar{\bar{p}}_i,\ldots,\bar{\bar{p}}_{n-1},D_1,\ldots,D_n,C_t) \qquad (16)$$

$i = 1,\ldots,n$

Remark: The averaging- bars of u and p_i will be omitted in the following.

It will be assumed that the watermovement can be treated as being in a quasi steady state. Because of the bed-level-changes the terms $\partial u/\partial t$ and $\partial a/\partial t$ are not equal to zero but because of the large time-scale of these changes these terms and $\partial z_o/\partial t$ in Eq. (14) can be neglected (quasi steady state).

Equation (14) is obtained by combining the continuity-equation of water and sediment. Because generally the transport of sediment is small compared to the waterdischarge (small transport concentration) it can be stated that Eq. (14) is a good approximation of the continuity-equation of water only.

Because of these simplifications Eqs. (13) and (14) can be combined to:

$$(u - \frac{ga}{u}) \frac{\partial u}{\partial x} + g \frac{\partial z_o}{\partial x} = R \tag{17}$$

which is a differential equation describing a backwatercurve.
With the assumption that the total Chézy-roughness C_t is constant both sedimentequations (15) and (16) can be combined to:

$$\frac{\partial f_i}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f_i}{\partial p_1} \cdot \frac{\partial p_i}{\partial x} + \ldots + \frac{\partial f_i}{\partial p_i} \cdot \frac{\partial p_i}{\partial x} + \ldots + \frac{\partial f_i}{\partial p_{n-1}} \frac{\partial p_{n-1}}{\partial x} +$$

$$+ p_{i_{z_o}} \frac{\partial z_o}{\partial t} + \beta\delta \frac{\partial p_i}{\partial t} = 0 \tag{18}$$

<u>Remark</u>: In the following all differentiations of f_i (generally: $\partial f_i/\partial b$) will be written as f_{i_b}.

The original set of 2n + 2 (n fractions) equations is now reduced to n + 1 equations:

$$(u - \frac{ga}{u}) \frac{\partial u}{\partial x} + g \frac{\partial z_o}{\partial x} = R \tag{19}$$

$$f_{i_u} \frac{\partial u}{\partial x} + f_{i_{p_1}} \frac{\partial p_1}{\partial x} + \ldots + f_{i_{p_i}} \frac{\partial p_i}{\partial x} + \ldots + f_{i_{p_{n-1}}} \frac{\partial p_{n-1}}{\partial x} +$$

$$+ p_{i_{z_o}} \frac{\partial z_o}{\partial t} + \beta\delta \frac{\partial p_i}{\partial t} = 0 \tag{20}$$

$i = 1, \ldots, n$

- quasi-steady watermotion
- small transport concentration
- constant Chézy-roughness

In case of only two sedimentfractions (n = 2) the set of equations becomes more handy and reduces to 3 equations:

$$(u - \frac{ga}{u}) \frac{\partial u}{\partial x} + g \frac{\partial z_o}{\partial x} = R \qquad (21)$$

$$f_{1_u} \frac{\partial u}{\partial x} + f_{1_{p_1}} \frac{\partial p_1}{\partial x} + p_{1_{z_o}} \frac{\partial z_o}{\partial t} + \beta\delta \frac{\partial p_1}{\partial t} = 0 \qquad (22)$$

$$f_{2_u} \frac{\partial u}{\partial x} + f_{2_{p_1}} \frac{\partial p_1}{\partial x} + p_{2_{z_o}} \frac{\partial z_o}{\partial t} - \beta\delta \frac{\partial p_1}{\partial t} = 0 \qquad (23)$$

- quasi-steady watermotion
- small transport concentration
- constant Chézy-roughness
- two fractions (n = 2)

<u>Remark</u>: The last term in eq. (23) is resulting from $p_1 + p_2 = 1$, and therefore $\partial p_2/\partial t = - \partial p_1/\partial t$.

In principle this set of 3 partial differential equations with 3 dependent variables *viz*. u, z_o and p_1, can be solved. To get more insight in this set of equations the characteristics and characteristic relations will be derived in the next chapter.

2. Characteristics of the set of equations

2.1. General

By deriving the characteristics of a set of quasi-linear partial differential equations (p.d.e) it is possible to get more insight in the mathematical character of the set of equations as well as the processes described by these equations.
The mathematical character can give information about the solution method which has to be chosen. In case of real and unequal characteristic directions the set of p.d.e. is hyperbolic and can be solved by integration along characteristics. This leads to a new set of ordinary differential equations in stead of p.d.e. (characteristic relations) which are valid along the characteristics.

The characteristic directions can also be used as estimates of the celerities of the described physical processes. In the original set of p.d.e. describing the unsteady watermovement and the transport of uniform sediment (de Vries, 1965, 1976) three real and unequal characteristic directions were found. Two of these characteristic directions can be very well approximated by the well known expressions resulting form the long-wave watermovement ($\phi_{1,2} = 1 \pm Fr^{-1}$). The third one generally is much smaller then the other two and can be approximated by the expression $\phi_3 = \psi/(1 - Fr^2)$, in which ψ is a dimensionless transportconcentration parameter. This expression is even found exactly if the watermovement is assumed to be quasi-steady; this is a reasonable assumption because of the magnitude of ϕ_1 and $|\phi_2|$ ($>> \phi_3$) which means that the watermovement reacts much faster to disturbances than the sedimentmovement.
If integration along characteristics is used the following charcteristic relation is valid along $\phi = \phi_3$:

$$\frac{dz_o}{dt} = \frac{R}{g} \phi_3 u \tag{24}$$

The left-hand side of this equation can be written as:

$$\frac{dz_o}{dt} = \frac{\partial z_o}{\partial t} + \frac{\phi_3}{u} \cdot \frac{\partial z_o}{\partial x} \tag{25}$$

The right-hand side of Eq. (24) takes care of the damping of the sand-wave (or bottom-level wave). If the bottomfriction is negligible with respect to the other terms in the long-wave equation ($R = 0$) a simple-wave relation results in which $c_3 = \phi_3/u$ represents the propagation velocity (celerity) of a certain bed-level. Generally -also if $R \neq 0$ - $c_3 (=\psi . u/(1 - Fr^2))$ describes the celerity of an infinite small disturbance in the independent variable z_o.

In the new set of p.d.e. describing the quasi-steady watermovement and the sedimentmovement in case of two sedimentfractions a new dependent variable *viz.* p_1 is introduced (see Eqs. (21), (22), (23)).
It is interesting to know whether this set of p.d.e. is hyperbolic again and whether new characteristics can be derived. Moreover it is important to know the order of magnitude of the characteristic directions and whether it is possible to interprete them as celerities of infinite small disturbances in bedlevel (z_o) and bedcomposition (p_1).
In the following sections the characteristic directions and relations will be derived. In order to give some interpretation, approximations are made and it is tried to estimate the order of magnitude of the characteristic directions in a simple way.

2.2. Characteristics and mathematical character

2.2.1. Derivation

Elimination of the term $\partial u/\partial x$ in the Eqs. (22) and (23) using Eq. (21) gives two resulting equations with two dependent variables, p_1 and z_o:

$$p_{1_{z_o}} \frac{\partial z_o}{\partial t} - \frac{gf_{1_u}}{G} \frac{\partial z_o}{\partial x} + \beta\delta \frac{\partial p_1}{\partial t} + f_{1_{p_1}} \frac{\partial p_1}{\partial x} = - \frac{f_{1_u}.R}{G} \qquad (26)$$

$$p_{2_{z_o}} \frac{\partial z_o}{\partial t} - \frac{gf_{2_u}}{G} \frac{\partial z_o}{\partial x} - \beta\delta \frac{\partial p_1}{\partial t} + f_{2_{p_1}} \frac{\partial p_1}{\partial x} = - \frac{f_{2_u}.R}{G} \qquad (27)$$

in which $G = u - ga/u = \frac{ga}{u}(Fr^2 - 1)$.

The variation of $p_1(x,t)$ and $z_o(x,t)$ in any direction c in the x-t plane can be written as:

$$dz_o = \frac{\partial z_o}{\partial t} . dt + \frac{\partial z_o}{\partial x} . dx$$

$$dp_1 = \frac{\partial p_1}{\partial t} . dt + \frac{\partial p_1}{\partial x} . dx$$

or:

$$\frac{dz_o}{dt} = \frac{\partial z_o}{\partial t} + c \frac{\partial z_o}{\partial x} \qquad (28)$$

$$\frac{dp_1}{dt} = \frac{\partial p_1}{\partial t} + c \frac{\partial p_1}{\partial x} \qquad (29)$$

in which $c = \frac{dx}{dt}$ = direction coefficient of the variation.

The equations (26)...(29) can be written in matrix-form as follows

$$\begin{pmatrix} p_{1_{z_o}} & -gf_{1_u}/G & \beta\delta & f_{1_{p_1}} \\ p_{2_{z_o}} & -gf_{2_u}/G & -\beta\delta & f_{2_{p_1}} \\ 1 & c & 0 & 0 \\ 0 & 0 & 1 & c \end{pmatrix} \begin{pmatrix} \frac{\partial z_o}{\partial t} \\ \frac{\partial z_o}{\partial x} \\ \frac{\partial p_1}{\partial t} \\ \frac{\partial p_1}{\partial x} \end{pmatrix} = \begin{pmatrix} -f_{1_u}R/G \\ -f_{2_u}R/G \\ \frac{dz_o}{dt} \\ \frac{dp_1}{dt} \end{pmatrix} \qquad (30)$$

If it is assumed that there is no contradiction in this set of p.d.e. the determinant D of the coefficient matrix can only be zero if the derivatives $\partial z_o/\partial t$, $\partial z_o/\partial x$, $\partial p_1/\partial t$ and $\partial p_1/\partial x$ are indeterminate.
The direction coefficients c belonging to this situation (D = 0) are defined as the characteristic directions of the set of p.d.e. After some simple mathematical operations D = 0 results in the following expression:

$$\beta\delta G c^2 - c\{G(f_{1_{p_1}}.p_{2_{z_o}} - f_{2_{p_1}}.p_{1_{z_o}}) - \beta\delta g(f_{1_u} + f_{2_u})\} +$$

$$- g\{f_{2_u}.f_{1_{p_1}} - f_{1_u}.f_{2_{p_1}}\} = 0 \qquad (31)$$

Because Eq. (31) is a quadratic equation in c, two characteristics can be expected.

Substitution of: $\phi = c/u$ (= dimensionless celerity)

$\psi_i = f_{i_u}/a$ (= dimensionless transportconcentration parameter of fraction i)

and: $G = u - ga/u = \frac{ga}{u}(Fr^2 - 1) = u(1 - Fr^{-2})$

yields:

$$\phi^2 - \phi\left[\frac{f_{1_{p_1}} p_{2_{z_o}} - f_{2_{p_1}}.p_{1_{z_o}}}{\beta\delta u} + \frac{\psi_1 + \psi_2}{1 - Fr^2}\right] - \frac{\psi_1.f_{2_{p_1}} - \psi_2.f_{1_{p_1}}}{\beta\delta u(1 - Fr^2)} = 0$$

Substitution of:

$$A = \frac{f_{1_{p_1}}.p_{2_{z_o}} - f_{2_{p_1}}.p_{1_{z_o}}}{\beta\delta u} \qquad (32)$$

$$B = \frac{\psi_1 + \psi_2}{1 - Fr^2} \qquad (33)$$

$$C = -\frac{\psi_1.f_{2_{p_1}} - \psi_2.f_{1_{p_1}}}{\beta.\delta.u(1 - Fr^2)} \qquad (34)$$

gives: $\phi^2 - \phi(A + B) + C = 0$ (35)

which can be easily solved by the quadratic formula:

$$\phi_{1,2} = \tfrac{1}{2}(A + B \pm [(A + B)^2 - 4C]^{\frac{1}{2}}) \quad (36)$$

This solution can also be written as:

$$\boxed{\phi_{1,2} = \tfrac{1}{2}\{A + B \pm [(A - B)^2 + 4(AB - C)]^{\frac{1}{2}}\}} \quad (37)$$

In the following it will become clear why Eq. (37) is written in this form.

2.2.2. Mathematical character

The mathematical character of the set of p.d.e. is determined by the sign of D'(= $(A + B)^2 - 4C$, see Eq. (36)). If $D' < 0$, $= 0$ or > 0 the characteristic directions are respectively complex, real and equal or real and different and the set of p.d.e. is then elliptic, parabolic or hyperbolic respectively

In Appendix 2.1 an estimation of the sign of D' is given with a simple derivation. The main assumptions which are made in this derivation are:

1. The sedimenttransport of fraction i can be written in the following way:

$$s_i = f_i(u,p_i) = m_i . u^{n_i} \quad (38)$$

in which: m_i = function of p_i and not a function of u.
n_i = not a function of p_i and u.

2. Another way of writing the transport of sedimentfraction i is (see also Appendix 1):

$$s_i = f_i(u,p_i) = \frac{\delta}{1 - \varepsilon_o} \cdot u_{p_i} \cdot C \cdot p_i \quad (39)$$

in which it is assumed that the variables δ (= transportlayer-thickness), u_{p_i} (= double averaged grain velocity of fraction i) and C (= double averaged value of the sedimentconcentration) and porosity ε_o are not a function of p_i.

With these simplifying assumptions it can be proven that (AB - C) can never be negative which is a necessary condition for an elliptic and parabolic character of the set of p.d.e.

The conclusion that can be drawn from this estimation is that the set of p.d.e. is always hyperbolic.

2.2.3. Approximated characteristics

With help of some assumptions and approximations it is possible to write both characteristic directions in a more simple form then the complex expression of Eq. (37).

Equation (33) can be written as:

$$B = \frac{\psi_1 + \psi_2}{1 - Fr^2} = \frac{1}{a}\left(\frac{f_{1_u} + f_{2_u}}{1 - Fr^2}\right) = \frac{f_u/a}{1 - Fr^2} = \frac{\psi}{1 - Fr^2}$$

This corresponds to the characteristic direction as found in the original set of equations (uniform sediment); it can be interpreted in that context as the celerity of an infinite small disturbance in bedlevel (z_o).

However, this expression is not found as one of the exact roots of Eq. (35) unless a condition is fulfilled. From Eq. (37) follows that if AB = C both roots become:

$$\phi_{1,2} = \tfrac{1}{2}(A + B \pm |A - B|) \qquad (40)$$

which means that ϕ_1 is always larger then ϕ_2 and two cases can be considered:

$$\begin{aligned} A > B &: \phi_1 = A \quad \phi_2 = B \\ B < A &: \phi_1 = B \quad \phi_2 = A \end{aligned} \qquad (41)$$

So, if the condition AB = C is fulfilled

$A\ (= \dfrac{p_{2z_o}.f_{1p_1} - p_{1z_o}.f_{2p_1}}{\beta\delta u})$ and $B\ (= \dfrac{\psi_1 + \psi_2}{1 - Fr^2})$ are exact roots of Eq. (35)

In Appendix 2.1 however it is shown with a simple estimation that generally AB - C has a positive sign. In Appendix 2.2 a further interpretation of this condition AB = C is carried out.
Better *conditions for A and B to be good approximations of the exact characteristic directions* must be formulated; two cases can be considered:

1. Clearly hyperbolic case: $\sqrt{D'}$ $(= \{(A - B)^2 + 4(AB - C)\}^{\frac{1}{2}})$ is of the same order of magnitude as A + B. (see Eq. 37)). In this case the condition is:

   $$4(AB - C) << (A - B)^2$$

   which generally means: A >> B or B >> A (42)
2. Approximately parabolic case (but still hyperbolic): $\sqrt{D'}$ is much smaller than A + B. In this case A and B approximately have the same magnitude $(A \simeq B)$ and AB - C approaches zero.

In Appendix 3 simplified expressions for A and B are derived with the aid of similar assumptions (Eqs. (38) and (39)) as have been used for the estimation of the mathematical character of the set of p.d.e. The result of this derivation is:

$$A = \frac{p_{2z_o}.f_{1p_1} - p_{1z_o}.f_{2p_1}}{\beta\delta u} \simeq \frac{p_{2z_o}.u_{p_1} + p_{1z_o}.u_{p_2}}{u} \qquad (43)$$

$$B = \frac{\psi_1 + \psi_2}{1 - Fr^2} \simeq \beta.n.\frac{\delta}{a}\,\frac{(p_1 u_{p_1} + p_2 u_{p_2})}{u} \qquad (44)$$

If it is assumed that the probability of fraction i on the level $z = z_o(x,t)$ is equal to the averaged value of $p_i(x,t)$ over the transportlayer (p_i) and $\beta.n.\frac{\delta}{a}$ is approximately equal to one both expressions become:

$$A \simeq \frac{p_2 . u_{p_1} + p_1 . u_{p_2}}{u} \tag{44}$$

$$B \simeq \frac{p_1 . u_{p_1} + p_2 . u_{p_2}}{u} \tag{45}$$

Expression (4) for B which agrees with the characteristic direction for uniform sediment $(= \frac{\psi}{1 - Fr^2})$, represents a dimensionless sum of grain-velocities weighed with the probabilities p_i in such a way that it can be interpreted as the (dimensionless) mean grain velocity of the sedimentmixture. Expression (45) for A also represents a dimensionless grain-velocity but is weighed in the opposite way. In Fig. 2 the ratio A/B is shown as a function of p_1 for different ratio's of grain-velocities u_{p_1}/u_{p_2}. It is assumed that always $u_{p_1} > u_{p_2}$ which generally means that fraction 1 is the finer sedimentfraction.

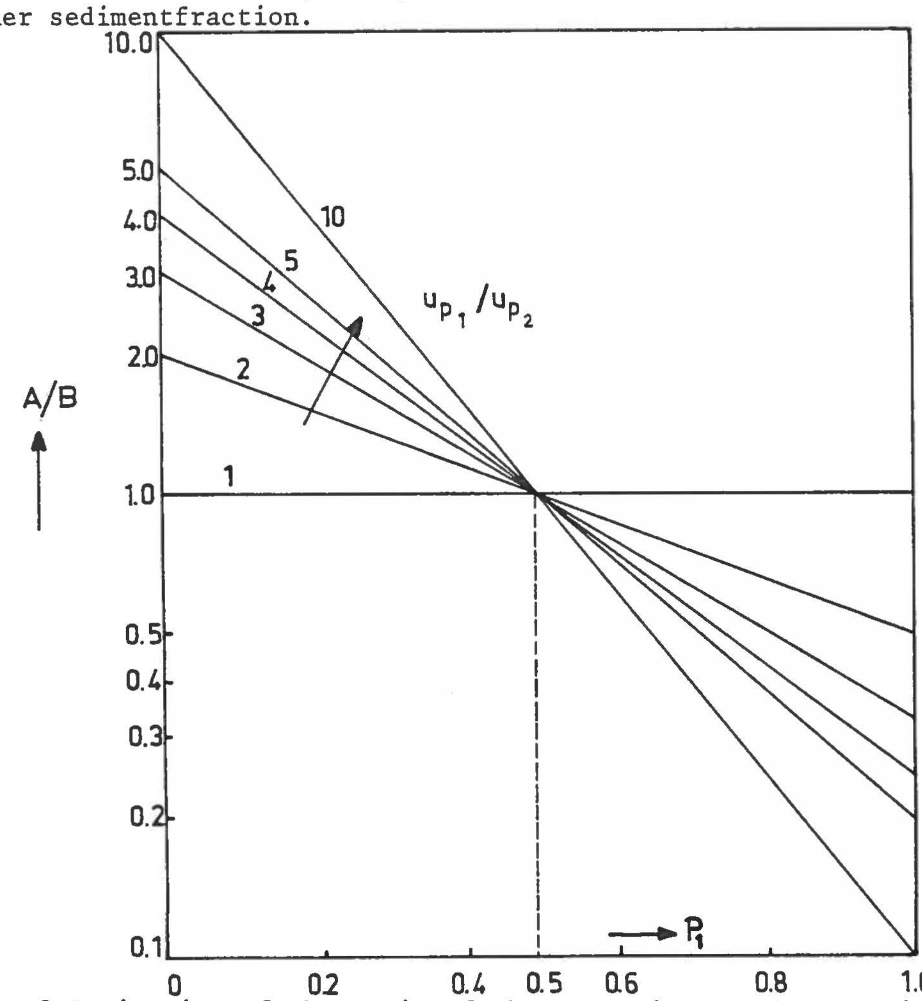


Fig. 2 Estimation of the ratio of the approximated characteristic directions A/B.

In the Equations (44) and (45) as well as in Fig. 2 it can be seen that if one of the fractions is hardly present ($p_i \to 0$) A approaches the grain-velocity of this particular fraction and B approaches the grain-velocity of the other fraction j ($p_j \to 1$).
If it is assumed that A and B are good approximations of the exact characteristic directions and that A and B can be interpreted as celerities of infinite small disturbances in p_1 and z_o respectively the following *conclusions* can be drawn:

1. An infinite small disturbance in bedlevel z_o is approximately propagating with the mean grain velocity of the sedimentmixture (two fractions).
   If $p_1 \to 1$ B $(= \frac{\psi_1 + \psi_2}{1 - Fr^2})$ approaches u_{p_1}
   If $p_1 \to 0$ B approaches u_{p_2}
2. An infinite small disturbance in bedcomposition p_1 approaches the grainvelocity of the fraction which is in the minority
   If $p_1 \to 1$ A $(= \frac{p_2 {z_o}.{f_1}_{p_1} - p_1 {z_o}.{f_2}_{p_1}}{\beta\delta u})$ approaches u_{p_2}
   If $p_1 \to 0$ A approaches u_{p_1}
3. If the grain-velocities of both fractions do not differ too much (for example in case of bedload) both celerities have the same order of magnitude.

<u>Remark:</u>

Many assumptions have been made before these conclusions could be drawn. They are:

- The transportformula per sedimentfraction is assumed to be of a certain form (see Eqs. (38),(39)). A specific form of this formula still has to be found.
- A and B are assumed to be good approximations of the exact characteristics; this can only be verified with a specific transportformula.
- The characteristics are interpreted as dimensionless celerities of infinite small disturbances in the dependent variables (p_i and z_o). This assumption depends on the characteristic relations which are true along the characteristics.

In the following section these characteristic relations will be derived.

2.3 Characteristic relations

2.3.1 Derivation

Characteristic relations are ordinary differential equations which are true along the characteristics. They give additional information about the meaning of the characteristics. With help of Eqs. (26) and (27) (which are the resulting p.d.e. after elimination of $\partial u/\partial x$) and the additional equations (28) and (29) the characteristic relations can be derived. The number of equations can be reduced by elimination of $\partial z_o/\partial t$ and $\partial p_1/\partial t$, which results in:

$$(f_{1_{p_1}} - \beta\delta c)\frac{\partial p_1}{\partial x} - (p_{1_{z_o}} c + \frac{g f_{1_u}}{G})\frac{\partial z_o}{\partial x} = -\frac{f_{1_u}.R}{G} - p_{1_{z_o}}.\frac{dz_o}{dt} - \beta\delta\frac{dp_1}{dt} \qquad (46)$$

$$(f_{2_{p_1}} + \beta\delta c)\frac{\partial p_1}{\partial x} - (p_{2_{z_o}} c + \frac{g f_{2_u}}{G})\frac{\partial z_o}{\partial x} = -\frac{f_{2_u}.R}{G} - p_{2_{z_o}}.\frac{dz_o}{dt} + \beta\delta\frac{dp_1}{dt} \qquad (47)$$

With the substitution of $F_i = \frac{g f_{i_u}}{G} \quad i = 1,2$

and $R_i = \frac{R f_{i_u}}{G} \quad i = 1,2$

the equations (46) and (47) can be written in matrix-form:

$$\underbrace{\begin{pmatrix} f_{1_{p_1}} - \beta\delta c & -p_{1_{z_0}} \cdot c - F_1 \\ f_{2_{p_1}} + \beta\delta c & -p_{2_{z_0}} \cdot c - F_2 \end{pmatrix}}_{\text{I}} \begin{pmatrix} \frac{\partial p_1}{\partial x} \\ \frac{\partial z_0}{\partial x} \end{pmatrix} = \underbrace{\begin{pmatrix} -R_1 - p_{1_{z_0}}\frac{dz_0}{dt} & -\beta\delta\frac{dp_1}{dt} \\ -R_2 - p_{2_{z_0}}\frac{dz_0}{dt} & +\beta\delta\frac{dp_1}{dt} \end{pmatrix}}_{\text{II}} \qquad (48)$$

Putting the determinant of matrix I equal to zero yields the characteristic directions again.

The original condition for $\partial z_o/\partial x$ and $\partial p_1/\partial x$ to be indeterminate is that the left-hand side of the eqs. (46) and (47) must be dependent. However, the right-hand side of these equations must be dependent too. This condition can be realised by replacing one of the columns of matrix I by the only column of matrix II and putting the determinant of this new matrix equal to zero again.
The result will be a ordinary differential equation with c as an unknown variable. Substitution of both characteristic directions results in two characteristic relations holding along c_1 and c_2.
Carrying out this procedure with Eq. (48) the following equation is found:

$$\frac{dp_1}{dt}(f_{1_{p_1}} + f_{2_{p_1}}) + \frac{dz_o}{dt}\left(c - \frac{p_{2z_o}.f_{1_{p_1}} - p_{1z_o}.f_{2_p}}{\beta\delta}\right) = -c(R_1 + R_2) + $$
$$+ \frac{f_{1_{p_1}}.R_1 - f_{2_{p_1}}.R_2}{\beta\delta} \qquad (49)$$

With $\phi = c/u$ and the substitutions carried out before Eq. (49) can be written in a dimensionless form:

$$\frac{dp_1}{dt}\left(\frac{f_{1_{p_1}} + f_{2_{p_1}}}{u}\right) + \frac{dz_o}{dt}(\phi - A) = (\phi - A)\frac{uR}{g} \cdot \frac{\psi_1 + \psi_2}{1 - Fr^2} +$$
$$+ \frac{R}{g}\,\frac{(p_{2z_o}.\psi_1 - p_{1z_o}.\psi_2)(f_{1p_1} + f_{2p_1})}{\beta\delta(1 - Fr^2)} \qquad (50)$$

with $\phi = \phi_{1,2}$.

A different expression for this characteristic relation can be found by carrying out the same procedure but in this case replacing the other column of matrix I. The resulting equation is now:

$$\frac{dp_1}{dt}(\phi - B) - \frac{dz_o}{dt}\left(\frac{p_{1_{z_o}}\cdot\psi_2 - p_{2_{z_o}}\cdot\psi_1}{\beta\delta(1 - Fr^2)}\right) = \phi.\frac{uR}{g}\,\frac{(p_{2_{z_o}}\cdot\psi_1 - p_{1_{z_o}}\cdot\psi_2)}{\beta\delta(1 - Fr^2)} \qquad (51)$$

with $\phi = \phi_{1,2}$

It can be shown that the equations (50) and (51) are identical. They show that bedcomposition $p_1(x,t)$ and bedlevel $z_o(x,t)$ are coupled along the characteristics. Therefore generally it cannot be said that infinite small disturbances in bedlevel and bedcomposition are propagating along both characteristics separately. Or in other words the exact characteristic directions cannot be interpreted as celerities of infinite small disturbances in $p_1(x,t)$ and $z_o(x,t)$.
However, under certain conditions approximated characteristic relations can be derived and separate propagation becomes possible.

2.3.2. Approximated characteristic relations

Under the same (exact) condition that the expressions A and B are equal to the exact characteristic directions, the characteristic relations can be replaced by simpler differential equations.
This condition is: AB − C = 0 (see Section 2.2.1) or formulated in another way (see Appendix 2):

$$(p_{2_{z_o}}\psi_1 - p_{1_{z_o}}\psi_2)(f_{1_{p_1}} + f_{2_{p_1}}) = 0$$

which means:

either: $p_{2_{z_o}}\psi_1 - p_{1_{z_o}}\psi_2 = 0$ (52)

or: $f_{1_{p_1}} + f_{2_{p_1}} = 0$ (53)

or: both parts are zero. (54)

In Appendix 2.2. these conditions are interpreted. Applying the conditions (52) and (53) to the exact characteristic directions (Eqs.(40), (41)) and to the exact characteristic relations (Eqs. (50) and (51)) more simple expressions can be derived.
In Table I the results of this approximation are shown. In the first and second column the condition for the approximation and the used characteristic direction (A or B) along which the characteristic relation (fourth column) is true are placed. In the third column the equation used for the characteristic relation (Eq. (50) or (51)) can be found.

Remark: In two cases the approximated characteristic relations cannot be determined with one of the available equations because all terms become zero. In that case the other equation delivers the right equation.

In two cases very simple expressions for the characteristic relations are found:

1. In the last case of Table I (Eq. (58)):

$$\frac{dz_o}{dt} = \frac{uR}{g} \cdot B \text{ , along } \phi = B = \frac{\psi_1 + \psi_2}{1 - Fr^2}$$

   This expression is the same as found in case of uniform sediment (see de Vries, 1976). It means that an infinite small disturbance in bedlevel $z_o(x,t)$ is propagating with (dimensionless) celerity $\phi = B$.
   Under the same condition the other characteristic relation holding along $\phi = A$ is complicated and $z_o(x,t)$ and $p_1(x,t)$ are coupled again.

2. In the first case of Table I (Eq. (55)):

$$\frac{dp_1}{dt} = 0 \text{ , along } \phi = A = \frac{p_{2_{z_o}} \cdot f_{1_{p_1}} - p_{1_{z_o}} \cdot f_{2_{p_1}}}{\beta \delta u}$$

   This means that an infinite small disturbance in bedcomposition $p_1(x,t)$ is propagating without damping with dimensionless celerity $\phi = A$. In this case the characteristic relation along $\phi = B$ is more complicated.

Condition	Characteristic direction (approximated)	Equation used for characteristic relation	Characteristic relation (approximated)
(52) $p_{2_{z_o}}\psi_1 - p_{1_{z_o}}.\psi_2 = 0$	$A = \frac{p_{2_{z_o}}.f_{1_{p_1}} - p_{1_{z_o}}.f_{2_{p_1}}}{\beta\delta u}$	(63)	$\frac{dp_1}{dt} = 0$ (55)
		(64)	$\frac{dp_1}{dt} = 0$
	$B = \frac{\psi_1 + \psi_2}{1 - Fr^2}$	(63)	$\frac{dp_1}{dt}\left(\frac{f_{1_{p_1}} + f_{2_{p_1}}}{u}\right) + \frac{dz_o}{dt}(B - A) = (B - A)\,.$ $.\,\frac{uR}{g}\frac{\psi_1 + \psi_2}{1 - Fr^2} + \frac{R}{g}\frac{(p_{2_{z_o}}\psi_1 - p_{1_{z_o}}\psi_2)(f_{1_{p_1}} + f_{2_{p_1}})}{\beta\delta(1 - Fr^2)}$ (56)
		(64)	undetermined
(53) $f_{1_{p_1}} + f_{2_{p_1}} = 0$	$A = \frac{p_{2_{z_o}}.f_{1_{p_1}} - p_{1_{z_o}}.f_{2_{p_1}}}{\beta\delta u}$	(63)	undetermined (57)
		(64)	$\frac{dp_1}{dt}(A - B) - \frac{dz_o}{dt}\left(\frac{p_{1_{z_o}}\psi_2 - p_{2_{z_o}}\psi_1}{\beta\delta(1 - Fr^2)}\right) =$ $= A\,\frac{uR}{g}\,\frac{p_{2_{z_o}}\psi_1 - p_{1_{z_o}}\psi_2}{\beta\delta(1 - Fr^2)}$
	$B = \frac{\psi_1 + \psi_2}{1 - Fr^2}$	(63)	$\frac{dz_o}{dt} = \frac{uR}{g}\,.\,B$ (58)
		(64)	$\frac{dz_o}{dt} = \frac{uR}{g}\,.\,B$

Table I: Approximated characteristic relations

In the case that both conditions (52) and (53) (See (54)) are true it can be easily seen in table I that both simple characteristic relations ((55) and (56)) are true at the same time.

Remarks:

1. It should be realised that the approximations carried out in the preceding pages are possible in a mathematical sense. Whether the approximated relations have any physical meaning is not clear and has to be investigated.
   For example: the exact condition AB = C which is the basis of the approximated characteristic relations generally is not exactly true. In Appendix 2 a simple consideration with help of some physical laws showed that the sign of AB - C is always positive (hyperbolic character).
   Formally in stead of this condition the conditions (42) and (43) must be used. Nevertheless the approximated characteristic relations of table I show the extreme possibilities of the set of equations.
2. A general conclusion of this chapter can be that generally bedlevel $z_o(x,t)$ is coupled to bedcomposition $p_1(x,t)$. A change of one of these variables has an immediate influence on the other one.
   A simple physical example may illustrate this: Consider a uniform steady watermovement over a plane bed in which a small disturbance in bedcomposition is made (See Fig. 3)

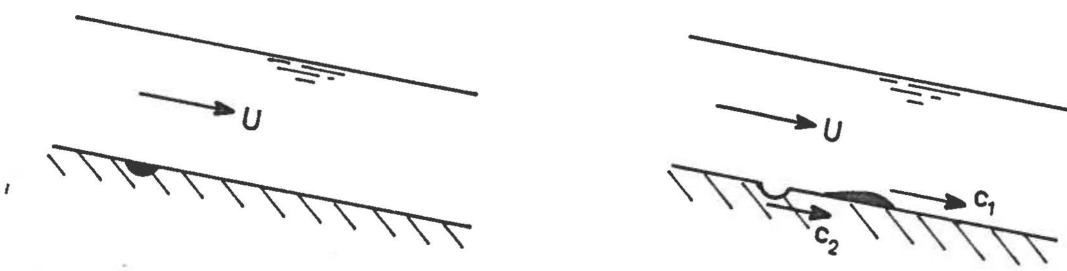


Fig. 3 A small disturbance in bedcomposition and its consequences

If it is assumed that the composition of this disturbance is finer than the surrounding sediment it is possible that this fine sediment moves faster and therefore over the original bed, leaving an erosion hole. The consequences of this disturbance are then:

1. A disturbance in bedcomposition $p_1(x,t)$ and bedlevel $z_o(x,t)$ propagating with celerity c_1
2. A disturbance in bedlevel $z_o(x,t)$ propagating with celerity c_2 ($< c_1$).

3. Specific characteristic directions and relations

3.1. General

In the Chapters 1 and 2 no specific transportformula per sedimentfraction has been used. A general form was taken according to:

$$s_i = f_i \; (u \; , \; D_1, \; \ldots, \; D_n, \; p_1, \; \ldots, \; p_{n-1})$$

(n fractions)

In order to study the behaviour of the derived characterictic directions and relations some simple approximations for a transportformula per fraction were assumed to be true like for example the basic hypothesis:

$$s_i = p_i \; . \; f_i' \; (u, \; D_i)$$

In this Chapter the influence of more specific tranportformulae per fraction on the characteristic directions and relations will be studied. In the first Section a summary is given of a literature survey concerning bed-load formulae for non-uniform sediment which was carried out earlier. After that a specific choice is made and the influence of two bed-load formulae per fraction will be studied in the following sections. First the characteristic directions are calculated and the influence of a number of dimensionless parameters is studied. Items like the absolute as well as relative behaviour of the two exact characteristic directions and the accuracy of the approximated characteristics will then be paid attention to.

Next the characteristic relations will be studied which is of importance because they describe the processes taking place along the characteristics. Questions concerning *the time scales of bed-level and bedcomposition-changes* will come up for discussion. Using some dimensionless parameters several cases will be distinguished in which the characteristic relations are fundamentally different and consequently the processes propagating along the two characteristics are different also.

3.2. Bed-loadformulae per sedimentfraction

3.2.1. A literature survey

In this Section a summary is given of the results of a literature survey which was carried out earlier (Ribberink, 1978). It was shown in the previous sections that the derived mathematical model for morphological computations in case of non-uniform sediment requires a transport-formula for every sedimentfraction separately. The applications of this model are confined to bed-load problems and as a consequence suspended load is not considered in this literature survey.
A well-known example in which the sediment-transport is already separated for every fraction is the bed-loadconcept of Einstein (1950). Other bed-loadformulae per sedimentfraction are based on one of the classical formulae like the formula of Kalinske (1947) and that of Meyer-Peter & Müller (1948). In these formulae the sediment-transport is a function of one representative grain-diameter. A general form of a transportformula per sedimentfraction is:

$$s_i = f_i(u, D_1, \dots, D_n, p_1, \dots, p_{n-1}) \tag{57}$$

The transport of fraction i is described as a function of the flowvelocity, the grain-diameters of all n fractions and the probabiliteis of all fractions minus one (because of $\sum_{i=1}^{n} p_i = 1$).
A bedroughness parameter (for example Chézy-roughness C_t) is as in the complete mathematical model assumed to be constant and therefore not included in Eq. (57).
Several investigators have tried to extend the basic transportformulae for large-range mixtures of sediment. *Pantelopoulos* (1955, 1957) shows an identical derivation as Kalinske (1947) for every sediment fraction separately. In his stochastical consideration the waterturbulence near the bed is included. The resulting formula is:

$$q_{s_i} = 2/3\ p(D_i)\ \Delta D_i\ D_i\ \bar{u}_p(D_i) \tag{58}$$

in which: $p(D_i)\ \Delta D_i$ = part of the unit bed-area occupied by grains with a diameter between D_i and $D_i + \Delta D_i$.

According to Pantelopoulos the mean particle-velocity $\bar{u}_p(D_i)$ is a function of the critical shear-stress of fraction i (τ_{c_i}) and a waterturbulence parameter. However, there is no theoretical expression available and Pantelopoulos only draws some conclusions about τ_{c_i} with help of some experiments. The experimental verification of the whole formula is very restricted. The bedload-formula of *Einstein* (1950) also results from stochastical considerations. On the grounds of earlier experiments Einstein assumes a normal probability distribution of the liftforce acting on bedparticles.

The dimensionless transport of fraction i can be written as:

$$\frac{q_{s_i}}{\sqrt{\Delta g D_i^3}} = \frac{p_i}{A_*} \frac{p}{1 - p} \tag{59}$$

in which A_* is a universal constant and p is called the 'probability of erosion'. This parameter p includes a dimensionless flow-parameter which on its turn includes a *hiding-factor* ξ. This factor must compensate the liftforce for the phenomenon of 'hiding of smaller grains behind larger ones'. With the aid of experiments with 'large range' mixtures of sediment *Einstein and Ning Chien* (1953) empirically modified this hiding-factor ξ. A disadvantage of Einstein's formula appears to be the complex form of it. Many correction-coefficients and figures hardly make it possible to write this formula in a form like expression (57).

Under certain conditions *'a basic hypothesis'* can be derived from Einstein's formula; it can be written in a general form as:

$$s_i = p_i . f'(u, D_i) \tag{60}$$

Comparison of this expression with Eq. (57) shows that the transport of fraction i is proportional to the probability of this fraction (p_i). The influence of the probabilities and grain-diameters of the other fractions has disappeared (the different fractions are transported independently). The term $f'(u, D_i)$ in Eq.(60) represents the transport of fraction i in case of uniform sediment under similar hydraulic conditions. *Antsyferov* (1973) combines Eq. (60) with the transportformula of Engulund & Hansen which however is a total transportformula (bed-load + suspended-load). In case of bed-load only it is also possible to combine the 'basis hypothesis' with a classical formula like for example *Meyer-Peter & Müller* (1948); the result can then be written as:

$$\frac{s_i}{\sqrt{\Delta g D_i^3}} = p_i \; 13.3 \left(\frac{\mu' R_b \, i}{\Delta D_i} - 0.047 \right)^{3/2} \tag{61}$$

The ripple-factor μ is in principle a function of the bedroughness C which can be influenced by p_i. However, the basic-hypothesis does not take this into account. Other investigators which used M.P & M or similar basic concepts realised that this 'basic hypothesis' probably is too simple and that Eq. (61) should be corrected by way of μ or by the constant 0.047. Starting from physical considerations *Egiazaroff* (1965) derives an expression for the dimensionless critical shear stress for every fraction as a part of a sediment mixture (Eq. 62).

$$\tau_{c_{*i}} = \frac{\tau_{c_i}}{(\rho_s - \rho) g\, D_i} = \frac{0.1}{({}^{10}\log 19 \frac{D_i}{D_m})^2} \qquad (62)$$

in which: $D_m = \sum_{i=1}^{n} p_i\, D_i$

The resulting effect of Egiazaroff's expression is a decrease of the critical shear stress of the larger fractions ($D_i > D_m$) and an increase of τ_{c_i} for the smaller fractions ($D_i < D_m$). Egiazaroff substitutes Eq. (62) in a transport-formula (of the type of M.P & M) but does not verify the resulting formula with experiments or rivermeasurements. *Ashida & Michiue* (1973) also combine a transportformula like that of M.P & M (including basic-hypothesis) with the derived expression of Egiazaroff; the result can be written as:

$$\frac{q_{s_i}}{\sqrt{\Delta g\, D_i^{\,3}}} = 17.\, p_i \,.\, \tau_*^{3/2}.\ (1 - \frac{\tau_{c_{*i}}}{\tau_*}).(1 - \frac{u_{*c_i}}{u_*}) \qquad (63)$$

in which τ_* is the dimensionless total shear stress working on the bed $(= \frac{\rho u_*^2}{(\rho_s - \rho) g\, D_i})$ and u_* is the shear velocity.

<u>Remark</u>: Ashida & Michiue (A & M) modify Egiazaroff's expression for $D_i/D_m < 0.4$ which, however, is based on only one experiment.

The total formula (63) is verified by A & M with laboratory-experiments (no bedforms!). They conclude that the bed-load per sedimentfractions is sufficiently described by Eq. (63) except for the coarse part of the sedimentmixture ($D_i/D_m > 1$). *Suzuki* (1976) directly uses the bed-load formula of M.P & M (including 'basic hypothesis') and Egiazaroff's theory. The resulting equation is:

$$\frac{q_{s_i}}{\sqrt{\Delta g\, D_i^{\,3}}} = 8\, p_i\ (\frac{\mu\, R_b\, i}{\Delta D_i} - 0.77\ \tau_{c_{*i}})^{3/2} \qquad (64)$$

with $\tau_{c_{*i}}$ according to Eq. (62).

For $D_i = D_m$ the expression $0.77\ \tau_{c_{*i}} = 0.047$ and Eq. (64) is identical to Eq. (61). Suzuki verifies this formula with a small number of experiments (with bedforms) and finds a reasonable agreement. However, the number of experiments is too small to get a real verification of Eq. (64).

General conclusions from this literature survey are:

(i) There is a lack of verification of the available concepts for a bed-load formula per sedimentfraction.

(ii) The stochastical-empirical approaches are rather complex because of the large number of empirical correction-coefficients and figures. Upmost it hardly seems possible to write these formulae in a suitable form for the mathematical model.

(iii) Use of a bed-load formula per fraction based on the formula of Meyer-Peter & Müller seems to be the most promising way.
Advantages are:
- The majority of the experimental verifications were carried out on this type of formula.
- This formula is written in a rather simple analytical way which can easily be used in the mathematical model.
- Two variations are possible: the formula of M.P & M with and without the use of Egiazaroff's theory.

<u>Remark</u>: In the following Sections 'Egiazaroff's theory' will be abbreviated as 'Eg.'s theory'.

3.2.2. Some properties of two bed-load concepts per fraction

Using the conclusions in the previous section both transportformulae of the type M.P & M (with or without Egiazaroff's theory) will be considered more closely. The first formula (without Eg.'s theory) can be written as:

$$\frac{q_{s_i}}{\sqrt{\Delta g\, D_i^3}} = 8\, p_i \left(\frac{\mu\, R_b\, i}{\Delta D_i} - 0.047 \right)^{3/2} \qquad (65)$$

According to Meyer-Peter & Müller the ripple-factor μ is defined as:

$$\mu = (C_t/C_g)^{3/2} \qquad (66)$$

in which: C_t = 'total' Chézy-roughness of the bed
C_g = Chézy-roughness of the grains

Combination of the Eqs.(65) and (66), usage of the Chézy-equation ($u = C\sqrt{R_b\, i}$) and writing of Eq. (65) in a non-dimensionless form, yields:

$$q_{s_i} = p_i \; 8 \sqrt{\Delta g} \left(\frac{u^2}{C_t^{1/2} \; C_g^{3/2} \; \Delta} - 0.047 \; D_i \right)^{3/2} \qquad (67)$$

Equation (67) shows that the transport of fraction i is strongly influenced by the flowvelocity u and in first view linearly dependent of p_i. However, C_g is dependent of a roughness parameter k_s according to:

$$C_g = 18 \log \frac{12 \; R_b}{k_s + 0.3 \; \delta'} \qquad (68)$$

with: k_s = equivalent sand-roughness of Nikuradse
δ' = thichness of the viscous sub-layer.

In general k_s is set equal to D_{90} of the sedimentmixture (90% is smaller than D_{90}) which in its turn is a function of p_i. However, Eq. (68) shows that this dependency is very small and it will therefore be neglected in the following.

Equation (67) also shows the influence of the total Chêzy-roughness C_t on q_{s_i}. In an earlier stage in this report this parameter was taken constant. However, it must be realised that this assumption is a restriction of the mathematical model because roughness-variations can be considerable in rivers. These variations are influenced by the hydraulic conditions and the type of bedforms which in its turn also is a function of the bedcomposition (p_i).

The second possible bed-loadformula per fraction (with Eg.'s theory) is similar to the concept of Suzuki:

$$\frac{q_{s_i}}{\sqrt{\Delta g \; D_i^3}} = 8 . \; p_i \left(\frac{\mu R_b \; i}{\Delta D_i} - 0.77 \; \tau_{c_{*i}} \right)^{3/2} \qquad (69)$$

with: $\tau_{c_{*i}} = \dfrac{0.1}{({}^{10}\log 19 \frac{D_i}{D_m})^2}$

Concerning $\tau_{e_{*i}}$ ($= \frac{\mu \; R_b \; i}{\Delta D_i}$) the same assumptions are true as in the first formula (Eq. (67)).

In this section a comparison will take place of both formulae (Eqs. (67) and (69)) and the influence of some parameters will be studied. Analogous to the complete mathematical model a restriction will take place to two sediment-fractions.
It will be assumed that:

$$X_i = \frac{q_{s_i}}{\sqrt{\Delta g\, D_i^3}} \quad \text{(= dimensionless transportparameter)} \tag{70}$$

$$\tau_{e_{*i}} = \frac{\mu\, R_b\, i}{\Delta D_i} \quad \text{(= dimensionless flow parameter or dimensionless effective shear stress)} \tag{71}$$

Substitution of Eq. (70) and (71) in both transport concepts (Eqs. (67) and (69) yields:

$$X_1 = 8 \cdot p_1 \cdot (\tau_{e_{*1}} - 0.047)^{3/2}$$

$$X_2 = 8 \cdot p_2 \cdot (\tau_{e_{*2}} - 0.047)^{3/2} \tag{72}$$

Meyer-Peter & Müller without Egiazaroff's theory (M.P & M).

and

$$X_{1_E} = 8 \cdot p_1 \cdot (\tau_{e_{*1}} - 0.77\, \tau_{c_{*1}})^{3/2}$$

$$X_{2_E} = 8 \cdot p_2 \cdot (\tau_{e_{*2}} - 0.77\, \tau_{c_{*2}})^{3/2} \tag{73}$$

Meyer-Peter & Müller with Egiazaroff's theory (M.P & M + Eg.)

<u>Remarks</u>:

(i) In order to distinguish clearly between the dimensionless transport with or without Eg.'s theory X_i will be provided by an index E (X_{i_E}) in case that Eg.'s theory is used.

(ii) Analogous to the foregoing sections it will be assumed that fraction 1 is always the finer fraction.

(iii) The 'Egiazaroff-term' can be written as:

$$\tau_{c_{*i}} = 0.1 \;/\; ({}^{10}\log 19\, D_i \;/\; D_m)^2 = 0.1 \;/\; ({}^{10}\log 19\, D_i / \sum_{i=1}^{n} p_i\, D_i)^2$$

or in case of two fractions (n = 2):

$$\tau_{c_{*1}} = 0.1 \;/\; ({}^{10}\log 19/\{p_1 + (1 - p_1)\, D_2/D_1\})^2$$

$$\tau_{c_{*2}} = 0.1 \;/\; ({}^{10}\log 19/\{p_1\, D_1/D_2 + (1 - p_1)\})^2$$

The influence of three variables *viz.* $\tau_{e_{*i}}$, p_1, D_1/D_2 on the Eqs. (72) and (73) will be studied in the following. In Fig. 4 the dimensionless transport of fraction 1 with and without Eg.'s theory is shown as a function of $\tau_{e_{*1}}$ for different grain-diameter ratio's D_1/D_2 and a fixed p_1. It can be seen that X_{1_E} is smaller than X_1. Moreover this influence is larger when the grain-diameters deviate more ($D_2 >> D_1$) and it becomes smaller when the dimensionless effective shear stress is far from 'initiation of motion' ($\tau_{e_{*1}} >> \tau_{c_{*1}}$). In Fig. 5 an analogous picture is shown for fraction 2. Now X_{2_E} is larger than X_2 and the difference is becoming larger when D_1 becomes smaller and when $\tau_{e_{*2}} \rightarrow \tau_{c_{*2}}$.

Remark: It must be realised that in the Figs. 4 and 5 the dimensionless transport of fraction i (i = 1,2) is shown, which is equal to:

$$X_i = \frac{q_{s_i}}{\sqrt{\Delta g\, D_i^3}}$$

In order to study the influence of D_1/D_2 on the real transport of fraction i (q_{s_i}) by means of these figures D_i must be kept constant (e.g. D_1 = constant in case of studying q_{s_1} and variations of D_1/D_2 are in fact only variations of D_2).

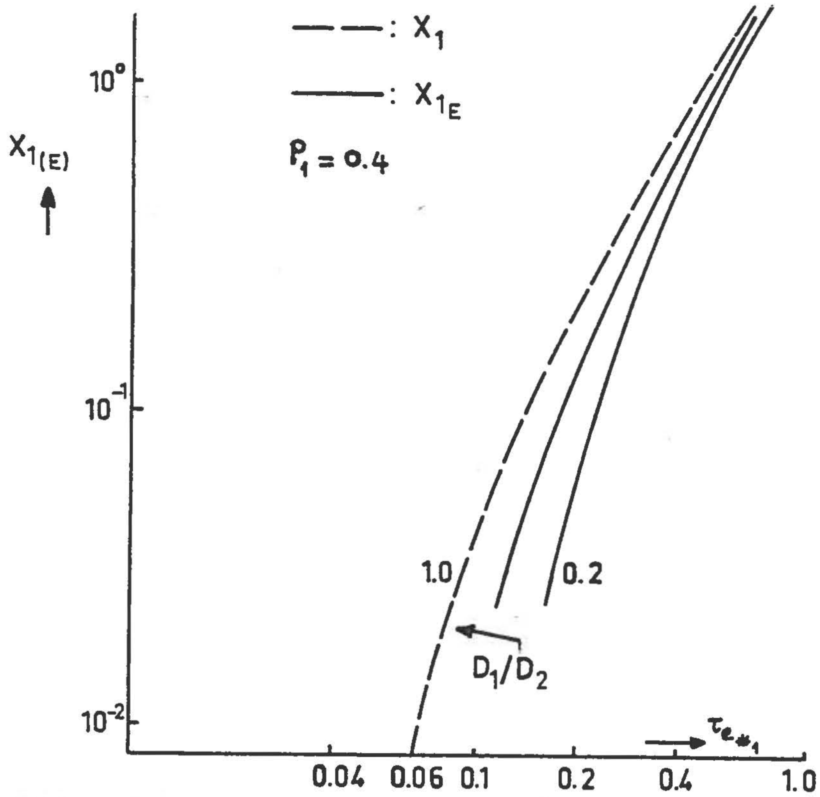


Fig. 4. The dimensionless transport of fraction 1 with/without Eg.'s theory as a function of $\tau_{e_{*1}}$ for different values of D_1/D_2.

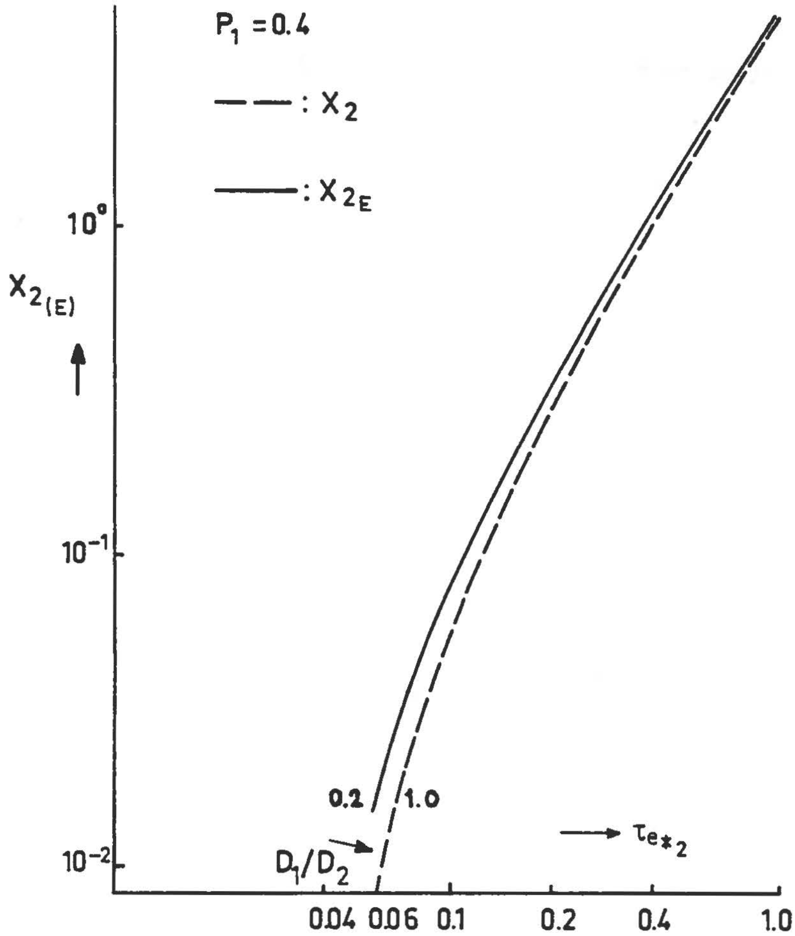


Fig. 5 The dimensionless transport of fraction 2 with/without Eg.'s theory as a function of $\tau_{e_{*2}}$ for different values of D_1/D_2.

In Fig. 6 the influence of Eg.'s theory on the dimensionless transport per fraction is shown in another way. Because $\tau_{e_{*i}} = \tau_{e_i} / (\rho_s - \rho)\ g\ D_i$ the magnitude of $\tau_{e_{*2}}$ must always be a factor D_1/D_2 lower then the magnitude of $\tau_{e_{*1}}$ to let the same effective shear stress (τ_e) work on both fractions and to get comparable dimensionless transports of both fractions. Assuming that $\tau_{e_{*1}} / \tau_{e_{*2}} = D_2/D_1$ also means that the ripple-factor μ is the same for both fractions ($\tau_e = \tau_{e_i} = \mu . \rho g R_b i$). In Fig. 6 $\tau_{e_{*1}}$, D_1/D_2 and therefore also $\tau_{e_{*2}}$ are fixed. The dimensionless transport of both fractions (X_1 and X_2) is linearly dependent of p_1 ('basic hypothesis'). Application of Eg.'s theory again causes a reduced transport of the finer fraction and an increased transport of the coarse fraction.

It can be concluded from these calculations:

(i) The application of Egiazaroff's theory does not fundamentally change the influence of the dimensionless effective shear stress on the dimensionless transport of each fraction. For very large $\tau_{e_{*i}}$ ($>>\tau_{c_{*i}}$) this theory even hardly has some inluence.

(ii) However, because of Eg.'s theory the transport of the finer fraction is smaller and of the coarse fraction is larger. Especially for small grain-diameter ratio's ($D_1/D_2 << 1$) this influence can be considerable.

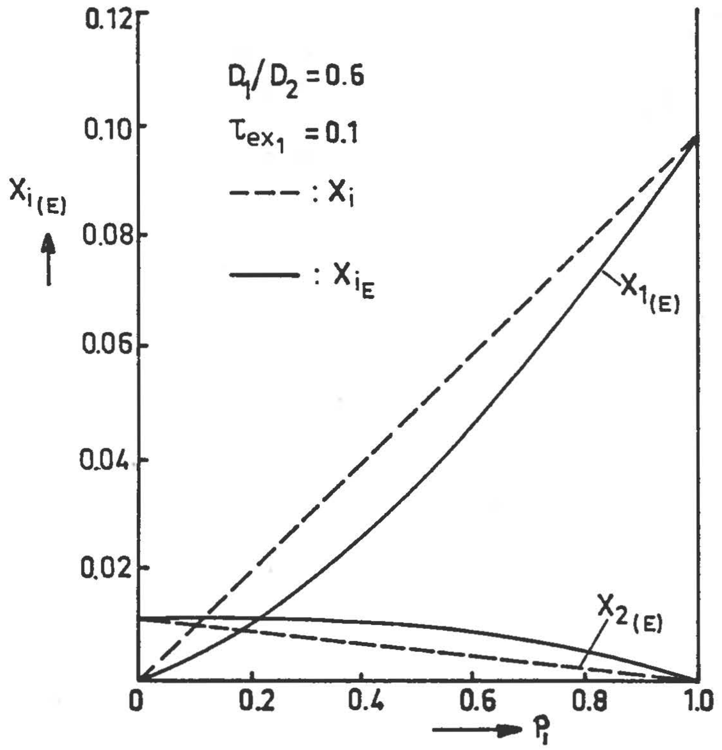


Fig. 6 The dimensionless transport of both fractions with/without Eg.'s theory as a function of p_1 for fixed values of $\tau_{e_{x_1}}$ and D_1/D_2.

Remarks: 1. All calculated values of Chapter 3 can be found in a number of tables in Appendix 7.

2. Transportprobability of fraction i (p_{i_T}) can be defined as:

$$p_{i_T} = q_{s_i} \Big/ \left(\sum_{i=1}^{n} q_{s_i} \right)$$

In case of two fractions and use of the formula of M.P & M without Eg.'s theory the ratio q_{s_1}/q_{s_2} can be written as:

$$\frac{q_{s_1}}{q_{s_2}} = \frac{p_{1_T}}{p_{2_T}} = \frac{p_1 \left\{\frac{\mu R i}{\Delta} - 0.047\, D_1\right\}^{3/2}}{p_2 \left\{\frac{\mu R i}{\Delta} - 0.047\, D_2\right\}^{3/2}}$$

Because $D_2 > D_1$, p_{1_T}/p_{2_T} must always be smaller than p_1/p_2. Or if both fractions are transported with an equal rate ($p_{1_T} = p_{2_T}$) the coarse fraction must dominate in the bed ($p_2 > p_1$). This phenomenon disappears for very large shear-stresses ($\tau_{e_{x_i}} >> 0.047$). Using Eg.'s theory the same phenomenon is present in a less pronounced way.

3.3. Characteristic directions

3.3.1. General

In Chapter 2 the following expressions for exact and approximated characteristic directions of the mathematical model were derived:

$$\phi_{1,2} = \tfrac{1}{2}\left\{A + B \pm \left[(A + B)^2 - 4\,C\right]^{1/2}\right\} \ldots\ldots \textit{exact} \qquad (74)$$

in which:

$$A = \frac{p_{2_{z_0}}\ \partial f_1/\partial p_1 - p_{1_{z_0}}\ \partial f_2/\partial p_1}{\beta\delta\, u} \qquad (75)$$

$$B = \frac{\psi_1 + \psi_2}{1 - Fr^2} \qquad (76)$$

.... *approximated* (Eqs. 75–76)

$$C = \frac{\psi_2\ \partial f_1/\partial p_1 - \psi_1\ \partial f_2/\partial p_1}{\beta\delta\, u\ (1 - Fr^2)} \qquad (77)$$

In order to be able to calculate especially Eq. (75) the following assumptions will be made:

(i) The probability of fraction i at the lower boundary $z = z_0$ of the transportlayer is equal to the averaged value of p_i over this layer:

$$p_{i_{z_0}} = p_i$$

This assumption generally is not true in case of erosion of the bed but is true in case of sedimentation.

(ii) Factor $\beta(= \bar{\bar{C}}/(1 - \varepsilon_0)$ is equal to one. This of course is not exactly true because it would mean that the transport-layer is completely filled with sediment without any dunes or throughs. However, this assumption does not change the order of magnitude of Eq. (75) and (77).

Using a simple expression for a transportformula per fraction the approximated characteristic directions turned out to be accurate approximations of the exact characteristic directions when the grain-diameter ratio D_1/D_2 approaches one (see Appendix 2). The general goal of Section 3.3. is to calculate the approximated and exact characteristic directions using the transportformulae per sedimentfraction as described in Section 3.2.2.. The following items can then be verified:

1. The absolute as well as relative magnitude of the two exact characteristic directions.
2. The accuracy of the approximated characteristic directions.
3. The influence of Egiazaroff's theory.
4. The mathematical character of the set of p.d.e.

Remark: In the following Sections some abbreviations will be used:

'exaxt characteristic directions' ='ex. char.'s'.

'approximated characteristic directions' = 'app. char.'s'.

3.3.2. A dimensionless notation

In order to calculate $\phi_{1,2}$, A and B (Eqs. (74)...(77) it is usefull to write these expressions in a different dimensionless form. In that case no absolute values of the mean flow velocity u, the mean waterdepth a, the transportlayer thickness δ and the grain-diameters of both fractions D_1 and D_2 have to be chosen.

Substitution of: $s_i = f_i = q_{s_i} / (1 - \varepsilon_0)$

$$\psi_i = f_{i_u} / a$$

in the Eqs. (75)...(77) yields:

$$A = \left\{ p_2 \frac{q_{s_{1p_1}}}{\sqrt{\Delta g D_1^3}} - p_1 \frac{q_{s_{2p_1}}}{\sqrt{\Delta g D_2^3}} \left(\frac{D_2}{D_1}\right)^{3/2} \right\} \frac{\sqrt{\Delta g D_1^3}}{(1 - \varepsilon_0)\, \delta\, u} \qquad (78)$$

$$B = \left\{ u \frac{q_{s_{1u}}}{\sqrt{\Delta g D_1^3}} + u \frac{q_{s_{2u}}}{\sqrt{\Delta g D_2^3}} \left(\frac{D_2}{D_1}\right)^{3/2} \right\} \frac{\sqrt{\Delta g D_1^3}}{(1-\varepsilon_0)\, a\, u\, (1-Fr^2)} \qquad (79)$$

$$C = \left\{ u \frac{q_{s_{2u}}}{\sqrt{\Delta g D_2^3}} \frac{q_{s_{1p_1}}}{\sqrt{\Delta g D_1^3}} - u \frac{q_{s_{1u}}}{\sqrt{\Delta g D_1^3}} \frac{q_{s_{2p_1}}}{\sqrt{\Delta g D_2^3}} \right\} \frac{\sqrt{\Delta g D_1^3}\ \sqrt{\Delta g D_2^3}}{u^2\, a(1-\varepsilon_0)^2\, \delta\, (1-Fr^2)} \qquad (80)$$

Substitution of the dimensionless derivatives according to:

$$q_{u_i} = \frac{u\, q_{s_{i_u}}}{\sqrt{\Delta g D_i^3}} \qquad i = 1,2 \qquad (81)$$

$$q_{p_i} = \frac{q_{s_{i_{p_1}}}}{\sqrt{\Delta g D_i^3}} \qquad i = 1,2 \qquad (82)$$

and some derivation yields:

$$\bar{A} = \frac{A(1 - \varepsilon_0)\, a\, u}{\sqrt{\Delta g D_1^3}} = \left(p_2\, q_{p_1} - p_1\, q_{p_2} \left(\frac{D_2}{D_1}\right)^{3/2}\right) \frac{a}{\delta} \qquad (83)$$

$$\bar{B} = \frac{B(1 - \varepsilon_0) a\, u}{\sqrt{\Delta g D_1^3}} = \left(q_{u_1} + q_{u_2} \left(\frac{D_2}{D_1}\right)^{3/2}\right) \frac{1}{1 - Fr^2} \qquad (84)$$

$$\bar{C} = \frac{C(1 - \varepsilon_0)^2\, a^2\, u^2}{\sqrt{\Delta g D_1^3}} = \left(q_{u_2}\, q_{p_1} - q_{u_1}\, q_{p_2}\right)\left(\frac{D_2}{D_1}\right)^{3/2} \cdot \frac{a}{\delta} \cdot \frac{1}{1-Fr^2} \qquad (85)$$

$$\overline{\phi}_{1,2} = \frac{\phi_{1,2}(1-\varepsilon_0)\ a\ u}{\sqrt{\Delta g\ D_1^3}} = \tfrac{1}{2}\ \{\overline{A} + \overline{B} \pm \left[(\overline{A}+\overline{B})^2 - 4\ \overline{C}\right]^{1/2}\} \quad (86)$$

The values of the dimensionless derivatives q_{u_i} and q_{p_i} can be calculated using one of the bed-load formulae per fraction. Three dimensionless parameters are necessary for these calculations (see Section 3.2.2.):

$\tau_{e_{*i}}$: dimensionless effective shear stress ($= \frac{\mu R i}{\Delta D_i}$)

D_1/D_2 : grain-diameter ratio (always $\leqslant 1$)

p_1 : probability of fraction 1

For the calculation of the Eqs. (83)...(86) another two parameters must be known:

Fr : Froude number

a/δ : waterdepth-transportlayerthickness ratio

<u>Remarks</u>:

(i) The Eqs. (83), (84) and (86) show that the app. char.'s A and B and the ex. char.'s $\phi_{1,2}$ are made dimensionless in the same way. This means that the *relative* behaviour of A, B and $\phi_{1,2}$ can be studied by considering $\overline{A}$, $\overline{B}$ and $\overline{\phi}_{1,2}$.

(ii) A, B and $\phi_{1,2}$ are made dimensionless by u, a and D_1. These three variables are also included in the parameters to be varied viz. $\tau_{e_{*i}}$, Fr, a/δ, D_1/D_2. Therefore it is not allowed to draw conclusions about the *absolute* behaviour of A, B and $\phi_{1,2}$ from the values of $\overline{A}$, $\overline{B}$, $\overline{\phi}_{1,2}$ as a function of these dimensionless parameters (e.g. studying the behaviour of B with values of $\overline{B}$ as a function of D_1/D_2 is only allowed if D_1 = constant and D_2 varies).

In case that the bed-load formula of M.P & M (without Eg.'s theory) is used, the following dimensionless derivatives can be derived:

$$q_{u_i} = 24\ p_i \cdot (\tau_{e_{*i}} - 0.047)^{1/2} \cdot \tau_{e_{*i}} \qquad i = 1,2 \quad (87)$$

$$q_{p_1} = 8\ (\tau_{e_{*1}} - 0.047)^{3/2} \quad (88)$$

$$q_{p_2} = -8\ (\tau_{e_{*2}} - 0.047)^{3/2} \quad (89)$$

Meyer-Peter & Müller without Egiazaroff's theory ('basic hypothesis')

Use of the same bed-load formula with Egiazaroff's theory yields:

$$q_{u_i} = 24\, p_i\, (\tau_{e_{*i}} - 0.77\, \tau_{c_{*i}})^{1/2}\, \tau_{e_{*i}} \qquad (90)$$

$$q_{p_1} = 8\, (\tau_{e_{*_1}} - 0.77\, \tau_{c_{*_1}})^{3/2} + 8.01\, p_1\, \frac{D_2 - D_1}{D_m}\, \frac{\tau_{c_{*_1}}}{{}^{10}\log \frac{19 D_1}{D_m}}\, (\tau_{e_{*_1}} - 0.77 \tau_{c_{*_1}})^{\frac{1}{2}} \qquad (91)$$

$$q_{p_2} = -8\, (\tau_{e_{*_2}} - 0.77\, \tau_{c_{*_2}})^{3/2} + 8.01\, p_2\, \frac{D_2 - D_1}{D_m}\, \frac{\tau_{c_{*_2}}}{{}^{10}\log (19\, \frac{D_2}{D_m})}\, (\tau_{e_{*_2}} - 0.77 \tau_{c_{*_2}})^{\frac{1}{2}} \qquad (92)$$

Meyer-Peter & Müller with
Egiazaroff's theory

with (see Section 3.2.2.):

$$\tau_{c_{*_1}} = 0.1 / ({}^{10}\log 19 / \{p_1 + (1 - p_1)\, D_2/D_1\})^2$$

$$\tau_{c_{*_2}} = 0.1 / ({}^{10}\log 19 / \{p_1\, D_1/D_2 + (1 - p_1)\})^2$$

In the following section $\overline{A}$, $\overline{B}$ and $\overline{\phi}_{1,2}$ will be calculated by substitution of Eqs (87)...(89) or Eqs. (90)...(92) for different values of the dimensionless parameters Fr, a/δ, $\tau_{e_{*_1}}$, D_1/D_2 and p_1.

3.3.3. Results

3.3.3.1. General

In the first part of this section the influence of the dimensionless parameters Fr, a/δ, p_1, D_1/D_2 and $\tau_{e_{*_1}}$ on the exact and app. char.'s will be studied. Because of the large number of parameters only a limited number of combinations is chosen. The influence of every parameter will be studied at fixed values of the other parameters. During these calculations the bedloadformula of M.P & M including Egiazaroff's theory will assumed to be true. In the second part of this section (3.3.3.3.) the influence of Egiazaroff's theory will be studied while in Section 3.3.3.4. the mathematical character of the set of p.d.e. will be treated.

3.3.3.2. Influence of the dimensionless parameters

The calculation of the dimensionless exact and app.char.'s can be carried out using the Eqs. (83)...(86) and (90)... (92). Table 2 shows the different calculations and the chosen values of the dimensionless parameters. Some values are rather extreme e.g. Fr = 0.8, a/δ = 50, $\tau_{e_{x_1}}$ = 1.0 and D_1/D_2 = 0.2 but they also give information about the extreme possibilities of the mathematical model.

calculation \ dimensionless parameter	$\tau_{e_{x_1}}$	p_1	$\frac{D_1}{D_2}$	a/δ	Fr
influence of $\tau_{e_{x_1}}$	≥ 0.06 ≤ 1.0	0.4	0.4	5	0.2
influence of p_1 and D_1/D_2	0.1 1.0	≥ 0 ≤ 1	0.2 0.4 0.6 0.8	5	0.2
influence of Fr and a/δ	0.1	0.8	0.4	1 5 10 25 50	≥ 0 ≤ 0.8

Table 2: Diagram of calculations and the chosen values of the dimensionless parameters.

Remarks:

(i) Generally not all dimensionless parameters are independent; this means that studying the influence of one of the parameters and fixing the other parameters is not always correct. For instance: changing $\tau_{e_{x_1}}$ may cause a change of the Froude-number Fr which in its turn may have its influence on the a/δ-ratio.Despite this problem a separate variation of every parameter delivers insight in the *relative* behaviour of exact and app. char.'s.

(ii) The calculations are carried out using a CPS-computer programme which is described in Appendix 5.

Dimensionless effective shear stress $\tau_{e_{x_1}}$

Analogous to the assumptions made in section 3.2.2 $\tau_{e_{x_i}}$ $(= \frac{\mu u^2}{C_t^2 \Delta D_i})$ is only a function of u. The ripple-factor μ and the total Chézy-roughness are assumed to be constant. Also the ripple-factor μ is assumed to have the same value for both fractions and consequently the following relation is true:

$$\frac{\tau_{e_{*1}}}{\tau_{e_{*2}}} = \frac{D_2}{D_1}$$

Figure 7 shows $\overline{\phi}_1$, $\overline{\phi}_2$, $\overline{A}$ and $\overline{B}$ as a function of $\tau_{e_{*1}}$.

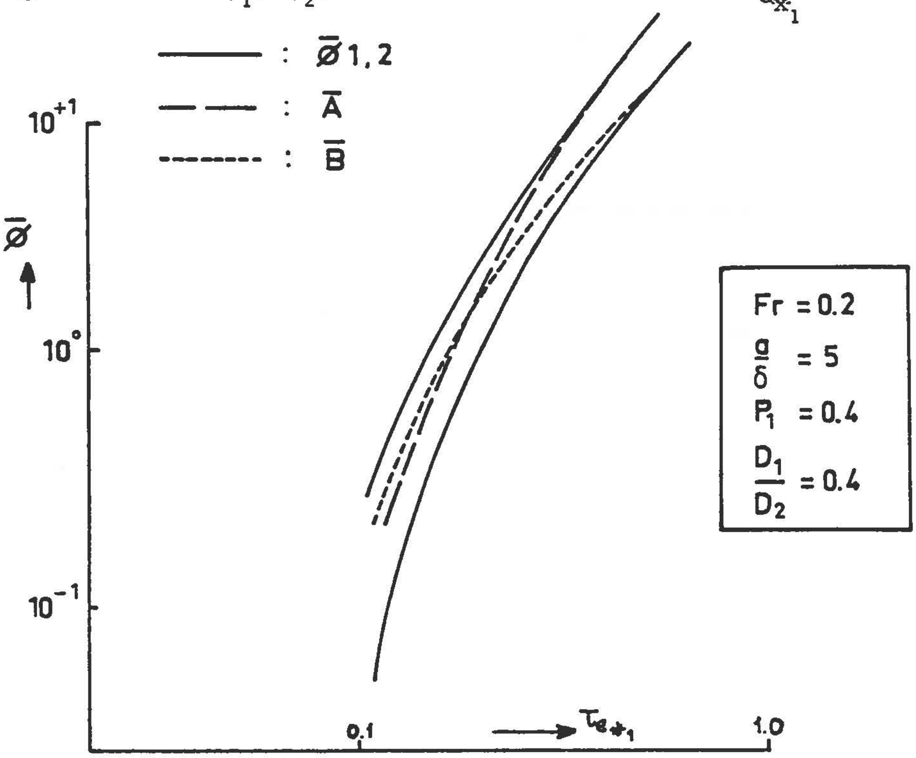


Fig. 7 Exact and approximated char. directions (dimensionless) as a function of $\tau_{e_{*1}}$.

The following conclusions about the relative behaviour of exact and app. char.'s can be drawn:

(i) The ex. char.'s are well approximated by $\overline{A}$ and $\overline{B}$ for large values of the dimensionless effective shear stress (see Remark made below).

(ii) A 'switching point' appears to be present ($\overline{A} = \overline{B}$) down which $\overline{\phi}_1$ (largest root) is best approximated by $\overline{B}$ and above which $\overline{\phi}_1$ is best approximated by $\overline{A}$. For $\overline{\phi}_2$ the contrary is true (switching-effect).

(iii) Both characteristic directions have the same order of magnitude except for small shear stresses when the coarse fraction approaches 'initiation of motion'.

Remark: One of the exact conditions for $\overline{A}$ an $\overline{B}$ to be accurate approximations is (see App. 2):

$$p_2 \psi_1 - p_1\psi_2 = 0 \tag{93}$$

It can be shown that Eq. (93) is true for large values of $\tau_{e_{*1}}$. For $\tau_{e_{*1}} >> \tau_{c_{*1}}$ Eq. (90) can be written as:

$$q_{u_i} \stackrel{\sim}{=} 24\ p_i\ \tau_{e_{*i}}^{3/2}$$

which is identical to:

$$\frac{u\ q_{s_{i_u}}}{\sqrt{\Delta g\ D_i^{3/2}}} \sim 24\ p_i\ \frac{\mu\ R\ i}{\Delta D_i}^{3/2}$$

The ratio $q_{s_{1_u}}/q_{s_{2_u}}$ ($= \psi_1/\psi_2$) is then:

$$\frac{q_{s_{1_u}}}{q_{s_{2_u}}} = \frac{\psi_1}{\psi_2} = \frac{p_1}{p_2}$$

which shows that Eq. (93) is ful-filled.

Probability of the fine fraction p_1 and the ratio of grain-diameters D_1/D_2

Figure 8 and Fig. 9 show the influence of p_1 on the exact and app. char.'s with D_1/D_2 as a parameter and for two different values of $\tau_{e_{*1}}$. Because $\overline{\phi}_{1,2}$, $\overline{A}$ and $\overline{B}$ are not made dimensionless with p_1 and D_2 it is now allowed to draw conclusions from these figures about the behaviour of $\phi_{1,2}$, A and B as a function of p_1 and D_1/D_2 (only variation of D_2!):

(i) A finer sedimentmixture (increase of p_1) causes an increase of both exact and app. char.'s.

(ii) The app. char.'s are more accurate when:
- $p_1 \to 0$ or $p_1 \to 1$
- $D_1/D_2 \to 1$
- $\tau_{e_{*i}} >> \tau_{c_{*i}}$

The second and third condition influence each other; for instance if fraction 2 becomes coarser (increase of D_2) it automatically means that the third condition is fullfilled less easy ($\tau_{e_{*2}} \sim 1/D_2$).

(iii) There appears to be a remarquable difference between Fig. 8 and Fig. 9.
For $\tau_{e_{*1}} = 0.1$ (Fig. 8) : $\overline{B} > \overline{A}$ and $\overline{\phi}_1 \to B$
For $\tau_{e_{*1}} = 1.0$ (Fig. 9) : $\overline{A} > \overline{B}$ and $\overline{\phi}_1 \to A$
Apparently the same 'switching effect' as in Fig. 7 also occurs in these figures.

Remark: Figure 8 shows for $D_1/D_2 = 0.4$ that both lines suddenly stop at $p_1 = 0.4$. The reason is that for lower values of p_1 $(0.2 \leqslant p_1 < 0.4)$ the coarse fraction reaches 'initiation of motion' $(\tau_{e_{*_2}} \rightarrow 0,77\ \tau_{c_{*_2}})$.

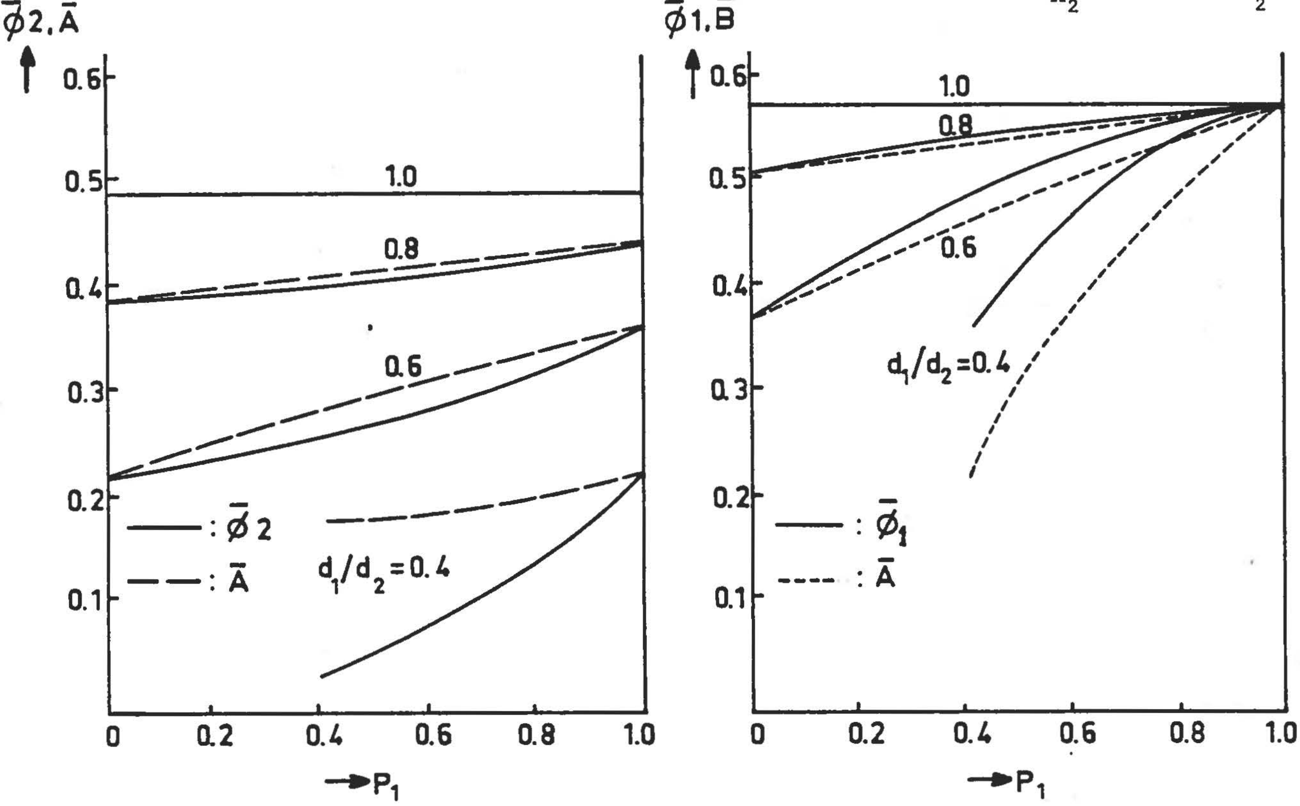


Fig. 8 Dimensionless exact and app. char.'s as a function of p_1 and D_1/D_2
$\tau_{e_{*1}} = 0.1$, Fr = 0.2 , $a/\delta = 5$

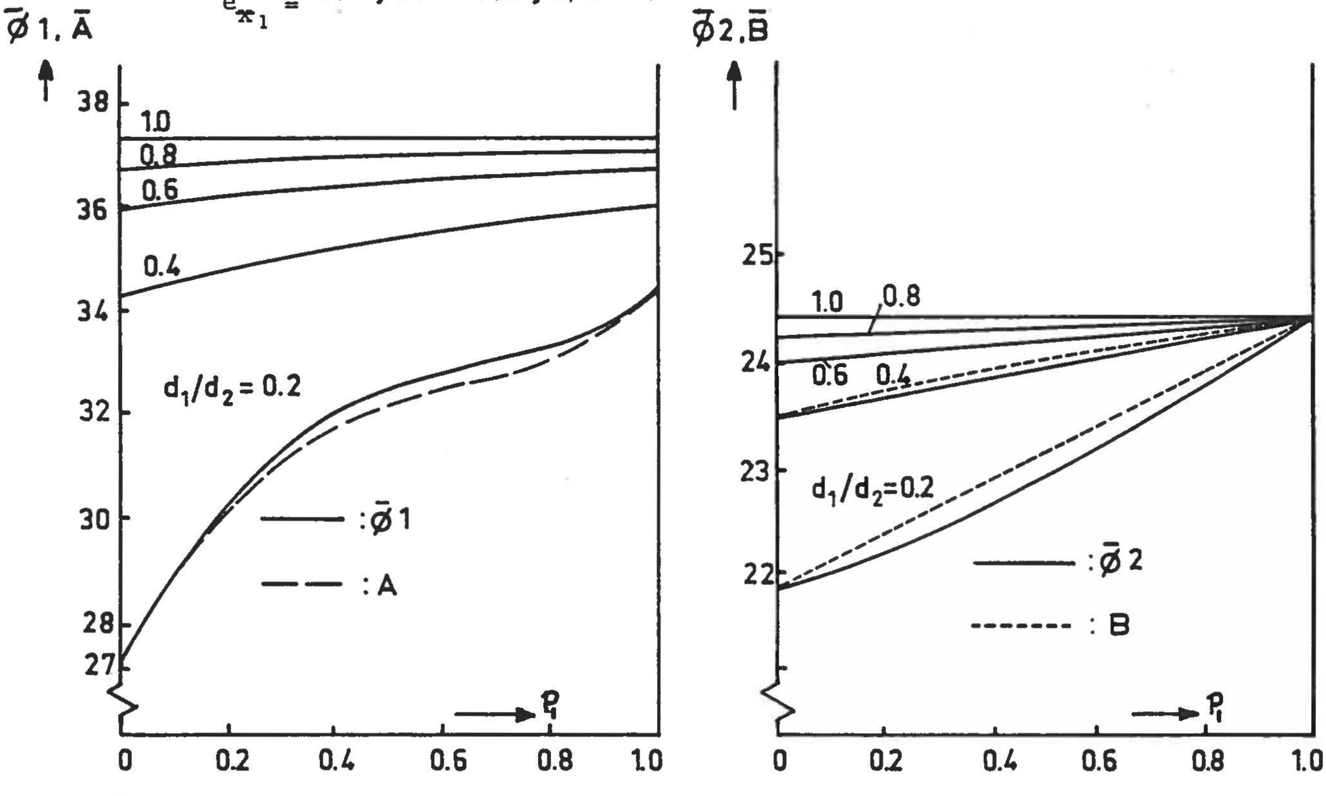


Fig. 9 Dimensionless exact and app. char.'s as a function of p_1 and D_1/D_2
$\tau_{e_{*1}} = 1.0$, Fr = 0.2 , $a/\delta = 5$

Froude-number Fr and waterdepth-transportlayerthickness ratio a/δ

Equations (83) and (84) show the influence of the Froude-number Fr and the a/δ-ratio on the app. char.'s:

$$\overline{A} = (p_2 \, q_{p_1} - p_1 \, q_{p_2} \left(\frac{D_2}{D_1} \right)^{3/2}) \frac{a}{\delta} \tag{94}$$

$$\overline{B} = (q_{u_1} + q_{u_2} \left(\frac{D_2}{D_1} \right)^{3/2}) \frac{1}{1 - Fr^2} \tag{95}$$

Because of the dependency in the parameters and the way $\overline{A}$ and $\overline{B}$ are made dimensionless again no conclusions will be drawn about A and B as a function of a/δ and Fr from the behaviour of $\overline{A}$ and $\overline{B}$. It can be assumed that generally $Fr << 1$ which means that for fixed values of the other parameters ($\tau_{e_{*1}}$, p_1, D_1/D_2) *$\overline{B}$ will be nearly constant while $\overline{A}$ is largely influenced by the transportlayerthickness* δ. Figure 10 and Figure 11 show the exact and app.char.'s as a function of Fr and a/δ.

The following conclusions can be drawn from these figures:

(i) The approximations $\overline{\phi}_1 \to \overline{A}$, $\overline{\phi}_2 \to \overline{B}$ are more accurate when a/δ increases and Fr decreases ($a/\delta \to \infty$, $Fr \to 0$).

(ii) The approximations $\overline{\phi}_1 \to \overline{B}$, $\overline{\phi}_2 \to \overline{A}$ are more accurate when the opposite is true ($a/\delta \to 0$, $Fr \to 1$).

(iii) Apparently the 'switching effect' occurs in these figures which can be explained by the fact that two possibilities exist:

$$\overline{A} >> \overline{B} \quad : \overline{\phi}_1 \to \overline{A}, \quad \overline{\phi}_2 \to \overline{B}$$
$$\overline{B} >> \overline{A} \quad : \overline{\phi}_1 \to \overline{B}, \quad \overline{\phi}_2 \to \overline{A}$$

Remark: If the assumption is made that $\overline{A}$ and $\overline{B}$ represent separate celerities of infinite small disturbances in bedcomposition p_1 resp. bedlevel z_0 it becomes clear that:

- $\overline{A}$ ('composition') increases when the transportlayer becomes thinner ($a/\delta \to \infty$); this could be explained by the fact that in that case the bedcomposition can change faster because of the decreased vertical mixing-length.
- $\overline{B}$ ('level') increases when the Froude- number increases ($Fr \to 1$) because of the changed hydraulic conditions.

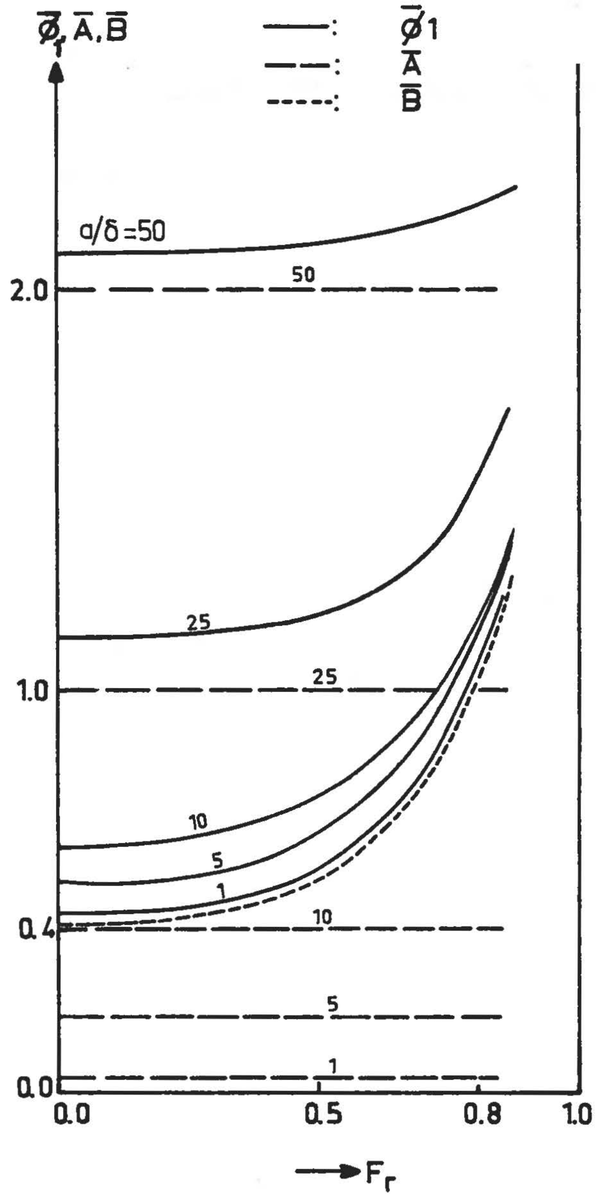


Fig. 10 Largest ex. char. $\overline{\phi}_1$ and $\overline{A}$ and $\overline{B}$

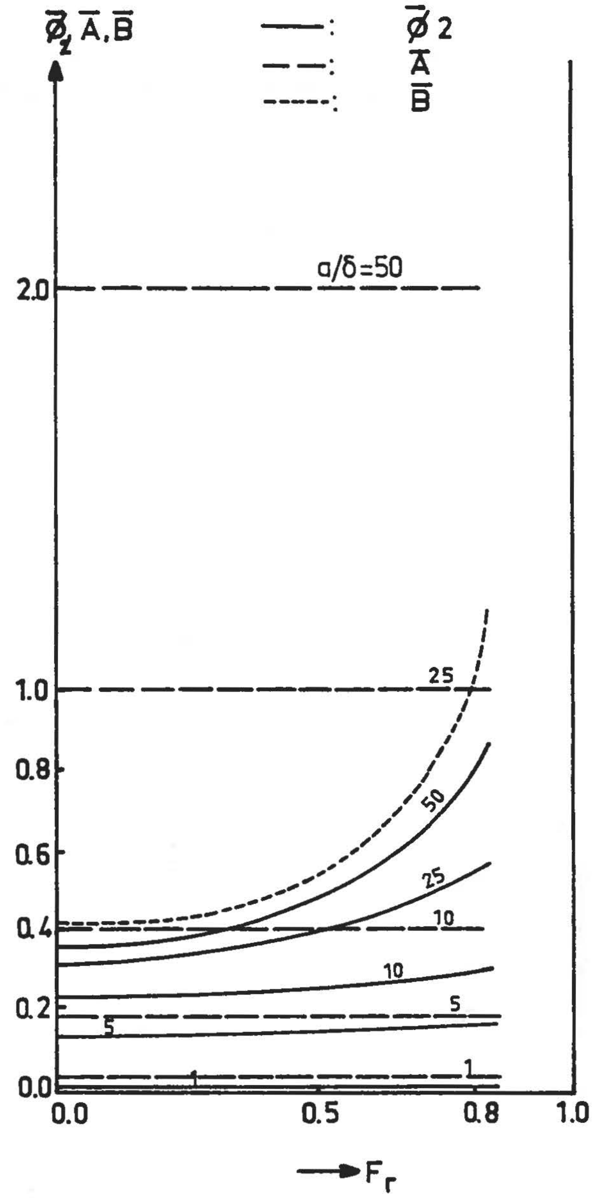


Fig. 11 Smallest ex. char. $\overline{\phi}_2$ and $\overline{A}$ and $\overline{B}$

$\tau_{ex_1} = 0.1$, $D_1/D_2 = 0.4$, $p_1 = 0.8$

3.3.3.3. Influence of Egiazaroff's theory

In Section 3.2.2. the difference between the bed-load concepts with and without Egiazaroff's theory was treated. The main difference is the increased transport of the coarse fraction and the decreased tranpsort of the fine fraction in case Egiazaroff's theory is used. In this section the influence of Eg.'s theory on the exact and app. char.'s is studied with the aid of some calculations.
The different influence of both bed-load concepts can be shown clearly by considering the influence of p_1 and D_1/D_2 while the other parameters are fixed.

Figure 12 shows the ex. char.'s with and without usage of Eg.'s theory. It can be seen that the order of magnitude of both char.'s is not changed. The main difference is that an increase of p_1 causes with the use of Eg.'s theory an increase of both char.'s while without Eg.'s theory a majority of the char.'s is decreasing.

In Figure 13 exact and app. char.'s are shown without usage of Eg.'s theory ('basic-hypothesis') for two values of D_1/D_2. It can be seen that app. char. $\overline{A}$ is decreasing when p_1 increases. When Eg.'s theory is used both app. char.'s $\overline{A}$ and $\overline{B}$ increase (see Section 3.3.2.). This means that especially $\overline{A}$ is influenced by Egiazaroff's theory. This change of $\overline{A}$ is also mainly responsible for the observed differences in Fig. 12.

Moreover Fig. 13 shows the 'switching effect'. Especially for $D_1/D_2 = 0.4$ it can be seen that $\overline{\phi}_1$ is approximated by $\overline{B}$ for small values of p_1 and by $\overline{A}$ for large values of p_1. For $\overline{\phi}_2$ the contrary is true.

This different influence of both bed-load concepts on especially the app. char. $\overline{A}$ may have some consequences. If the assumption is made that $\overline{A}$ can be considered as a dimensionless celerity of an infinite disturbance in bedcomposition p_1 the effect of Eg.'s theory can be illustrated by the following example.

Example:
Consider a riverbed with no bed-level disturbance but with a positive gradient of p_1 in x-direction.. The hydraulic conditions are assumed to be quasi-steady and uniform in x-direction. Figure 14a shows the propagation of a 'bedcomposition-wave' without usage of Eg.'s theory while Fig. 14b shows the same with usage of Eg.'s theory.

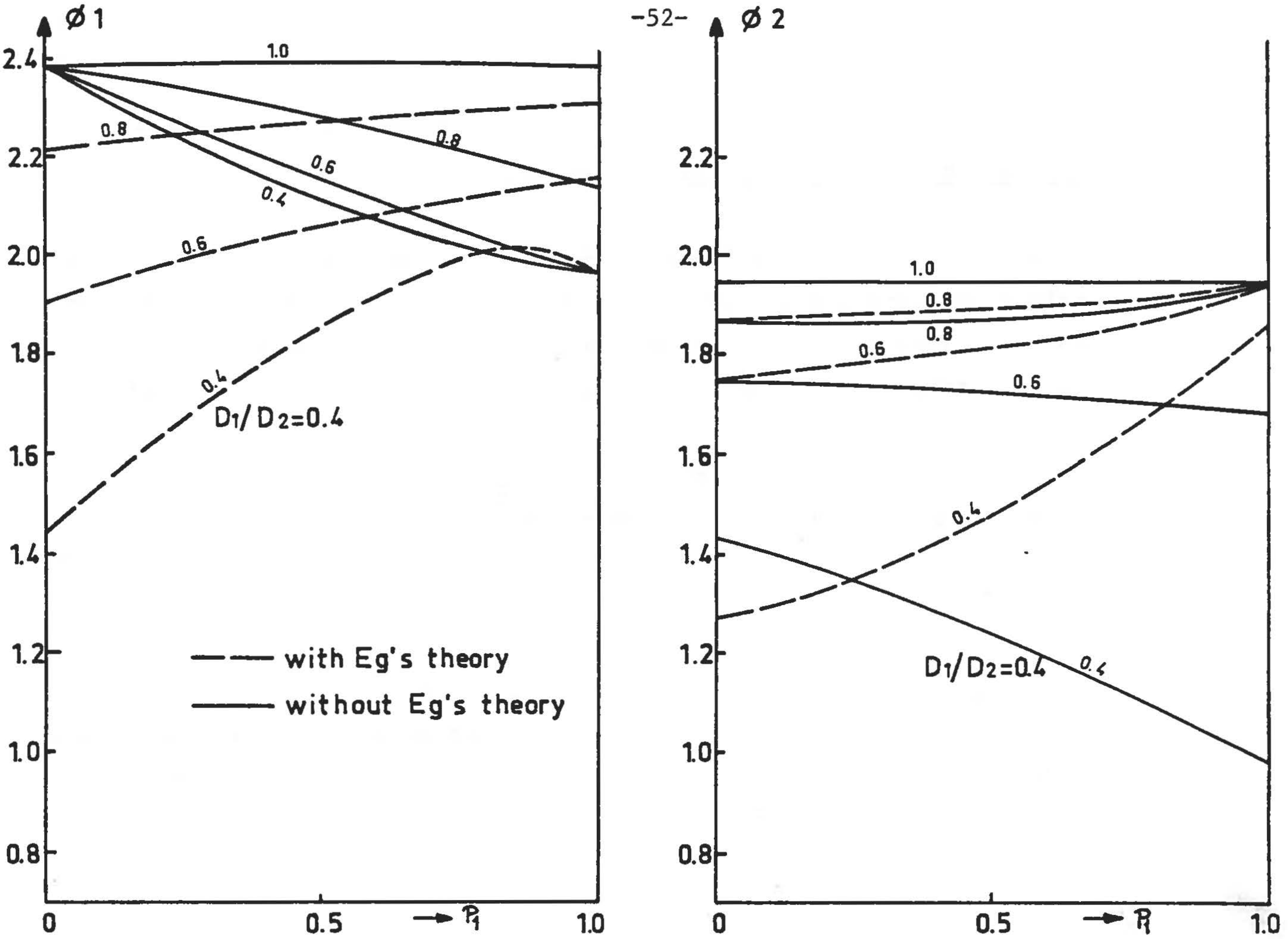


Fig. 12 Influence of Eg.'s theory on both exact char.'s as a function of p_1 for different values of D_1/D_2. Fr=0.2, a/δ =5 , $\tau_{e_{*1}}$ = 0.2

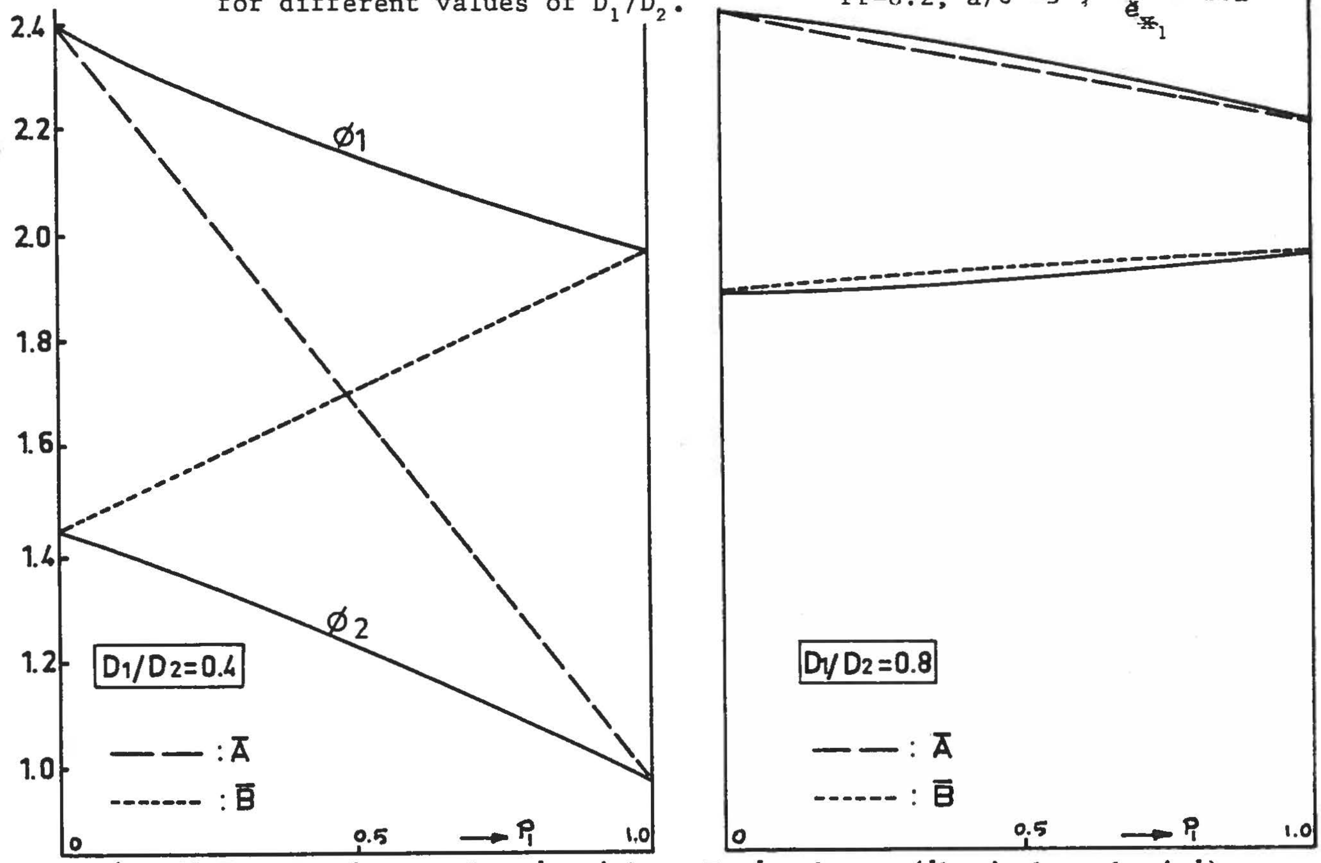
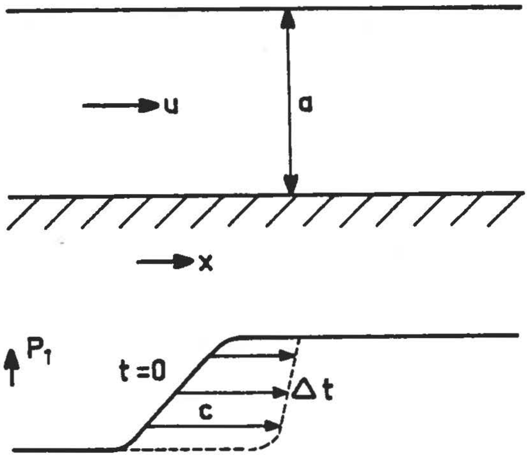
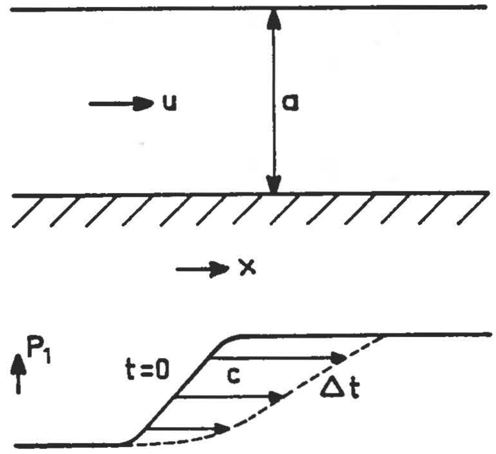


Fig. 13 Exact and app. char.'s without Eg.'s theory ('basic-hypothesis') as a function of p_1 for two values of D_1/D_2.

Fr = 0.2 , a/δ = 5 , $\tau_{e_{*1}}$ = 0.2



a without Eg.'s theory



b with Eg.'s theory

Fig. 14 The influence of Eg.'s theory on the propagation of a 'bed composition-wave'.

Figure 14a shows a steepening or compressing wave because of the increased celerity for decreasing values of p_1. Figure 14b shows a gradient which is flattened out (expansion-wave) because of the increased celerity for increasing values of p_1.

3.3.3.4. Mathematical character

As was treated before the mathematical character of the set of p.d.e. is determined by the characteristic directions which can be written as:

$$\bar{\Phi}_{1,2} = \tfrac{1}{2} (\bar{A} + \bar{B} \pm \{(\bar{A} - \bar{B})^2 + 4 (\bar{A}\,\bar{B} - \bar{C})\}^{1/2}) \tag{96}$$

It was shown in Section 2.2.2. that under certain conditions the sign of the expression under the root can never be negative which means an hyperbolic set of p.d.e. In all calculations carried out in the previous sections with two bed-load concepts per sedimentfraction no complex or equal characteristic directions were found. It can therefore be concluded that in the whole area covered by the dimensionless parameters as chosen in the previous sections the set of p.d.e. is *hyperbolic*. By writing Eq. (96) in the original way it can be seen that the smallest second char. $\bar{\Phi}_2$ can be negative only when $\bar{C} < 0$:

$$\phi_2 = \tfrac{1}{2} (\overline{A} + \overline{B} - \{(\overline{A} + \overline{B})^2 - 4\,\overline{C}\}^{1/2})$$

This theoretical possibility never occurred during the calculations which means that both exact char.'s have a *positive sign* and no disturbances in bedlevel or composition can propagate upstream (the Froude-number is restricted to $Fr \lesssim 0.8$!).

3.3.4. Summary

In Table 3 the main results of the calculations of exact and app.char.'s in Section 3.3.3.2. are shown. With the difference between the exact char.'s as a basis several cases are distinguished concerning the accuracy of the approximated char.'s, the difference between them and the behaviour of the dimensionless parameters.
If the behaviour of a dimensionless parameter is not mentioned it means that this parameter does not necessarily have an extreme value. Some remarks will be made for every case:

$\overline{\phi}_1 >> \overline{\phi}_2$:

(i) $\overline{A} >> \overline{B}$: In general the app. char.'s are accurate. This case can be reached especially for a small transportlayerthickness ($a/\delta >> 1$) and small Froude-numbers ($Fr << 1$). If it is assumed that $\overline{\phi}_1$ and $\overline{\phi}_2$ can be considered as dimensionless celerities of infinite small disturbances in p_1 resp. z_0 (see Section 3.4.) it can be concluded that in this case bed-compostition changes propagate faster than bed-level changes.

(ii) $\overline{B} >> \overline{A}$: Also in this case $\overline{A}$ and $\overline{B}$ generally are accurate approximations of $\overline{\phi}_2$ resp. $\overline{\phi}_1$. Especially for a thick transportlayer ($a/\delta \simeq 1$) and large Froude-numbers ($Fr \rightarrow 1$). Under identical assumptions as for $\overline{A} >> \overline{B}$ it can now be concluded that bed-level changes propagate faster than bed-composition changes. Apparently going from $\overline{A} >> \overline{B}$ to $\overline{B} >> \overline{A}$ the 'switching-effect' occurs which coincides with a change of role of both exact char.'s.

(iii) $A \approx B$ with $AB \neq C$: This case can be reached when the Froude-number Fr and a/δ have no extreme values and the coarse fraction 2 comes near 'initation of motion' ($\tau_{e_{*2}} \rightarrow \tau_{c_{*2}}$). This occurs even faster when fraction 2 is becoming coarser ($D_1/D_2 << 1$). The accuracy of the app. char.'s is bad because of $AB \neq C$. This condition is especially true when the grain-diameters and/or grain-velocities of both fractions deviate much (see Appendix 2).

$\overline{\phi}_1 \gtrsim \overline{\phi}_2$:

$A \gtrsim B$ or $B \gtrsim A$ with $AB \simeq C$: In this situation the exact and app. char.'s approximately have the same magnitude. Especially for large shear stresses ($\tau_{e_{*2}} >> \tau_{c_{*2}}$), nearly equal graindiameters of both fractions ($D_1/D_2 \rightarrow 1$) and intermediate values of Fr and a/δ this case can occur. In the asymptotic case that $D_1 \simeq D_2$, the difference between the ex. char.'s is very small and the app. char.'s are very accurate ($\overline{\phi}_1 \simeq \overline{\phi}_2 \simeq \overline{A} \simeq \overline{B}$).Therefore this case is only of importance for large time and distance scales. Else the mathematical model for uniform sediment is probably equally accurate.

The influence of parameter p_1 has not been mentioned in the preceding cases; in general it can be concluded that app. char.'s become more accurate when $p_1 \rightarrow 0$ or $p_1 \rightarrow 1$.

Especially the app. char. $\overline{A}$ is influenced by the usage of a bed-load formula with or without Egiazaroff's theory. Its dependency of p_1 is opposite in both cases (compression-wave or expansion-wave). In all calculations the mathematical character of the set of p.d.e. is hyperbolic and the existence of negative celerities could not be proven.

difference between exact char.'s	difference between the app. char.'s	accuracy of the app. char.'s	behaviour of the dimensionless parameters				
			Fr	a/δ	p_1	$\tau_{e_{*i}}$	D_1/D_2
$\overline{\phi}_1 >> \overline{\phi}_2$	$\overline{A} >> \overline{B}$	$\overline{\phi}_1 \rightarrow \overline{A}$ $\overline{\phi}_2 \rightarrow \overline{B}$	<<1	>> 1			
	$\overline{B} >> \overline{A}$	$\overline{\phi}_1 \rightarrow \overline{B}$ $\overline{\phi}_2 \rightarrow \overline{A}$	→ 1	≈ 1	→ 0 → 1	-	-
	$\overline{A} \simeq \overline{B}$ ($AB \neq C$)	'bad'	-	-	-	$\rightarrow \tau_{c_{*i}}$	<< 1
$\overline{\phi}_1 \gtrsim \overline{\phi}_2$	$\overline{A} \gtrsim \overline{B}$ or $\overline{B} \gtrsim \overline{A}$ ($AB \simeq C$)	$\overline{\phi}_1 \rightarrow \overline{A}, \overline{\phi}_2 \rightarrow \overline{B}$ or $\overline{\phi}_1 \rightarrow \overline{B}, \overline{\phi}_2 \rightarrow \overline{A}$	-	-	-	$>> \tau_{c_{*i}}$	→ 1

Table 3 Results of the calculations of exact and approximated characteristic directions.

3.4 Characteristic relations

3.4.1. General

In the previous section the characteristic directions of the mathematical model were studied without considering the relations which hold along these characteristics. However, the characteristic relations are rather important because they replace the original set of p.d.e. and have to be solved in combination with the characteristic directions in case of some morphological problem (integration along characteristics). Moreover it is interesting to know whether and in which cases the exact characteristic relations (see Section 2.3) can be replaced by more simple relations. In this regard two questions are of importance:

(i) Is it possible that infinite small disturbances in bedlevel resp. bedcomposition propagate along two characteristics separately and what are the conditions for it?

(ii) If 'separate propagation' is possible which of the disturbances is propagating faster? Is the bedlevel adapting faster to changing conditions than the bedcomposition or is it the other way around! What are the conditions for these phenomena?

In Section 3.4.2. a general consideration is given of the behaviour of the exact characteristic relations in some extreme cases and the above-mentioned questions come up for discussion. In Section 3.4.3. some specific calculations are made of the coefficients of the terms in the characteristic relations. The bed-load formula of M.P & M including Egiazaroff's theory will be used and by writing the characteristic relations in a dimensionless form it will be possible to use dimensionless parameters like $\tau_{e_{x_1}}$, p_1, D_1/D_2, Fr and a/δ.

Remark: In the following sections 'characteristic relations' will be abbreviated as 'char. rel.'s.

3.4.2. The 'switching effect' and separate propagation

In Section 2.3. the expressions for exact and approximated char. rel.'s were derived. The app. char. rel.'s are derived from the ex. char. rel.'s with substitution of the app. char.'s ; two cases were distinguished (see Section 2.3.2.).

1. $p_2\psi_1 - p_1\psi_2 = 0$ (97)

2. $f_{1_{p_1}} + f_{2_{p_1}} = 0$ (98)

It was shown that these conditions never are exactly true. As a consequence the exact and app. char.'s are never exactly equal. This means that the app. char. rel.'s cannot be applied without formulating more extensive conditions.
Two (identical) expressions for the char. rel.'s are (see Section 2.3.):

$$\frac{dp_1}{dt}\left(\frac{f_{1p_1}+f_{2p_1}}{u}\right)+\frac{dz_0}{dt}(\phi-A)=(\phi-A)\frac{uR}{g}B+\frac{uR}{g}\frac{(p_2\psi_1-p_1\psi_1)(f_{1p_1}+f_{2p_1})}{\delta u\,(1-Fr^2)}$$

$$\phi=\phi_{1,2} \qquad (99)$$

$$\frac{dp_1}{dt}(\phi-B)-\frac{dz_0}{dt}\frac{p_1\psi_2-p_2\psi_1}{(1-Fr^2)}=\phi\,\frac{uR}{g}\frac{(p_2\psi_1-p_1\psi_2)}{(1-Fr^2)} \qquad (100)$$

$$\phi=\phi_{1,2}$$

Division of Eq. (99) by transportlayerthickness δ and multiplication of Eqs.(99) and (100) by dt yields:

$$dp_1\boxed{F_p}+\frac{1}{\delta}dz_0\boxed{(\phi-A)}=\frac{uR}{g\delta}(\phi-A)B+\frac{(p_2\psi_1-p_1\psi_2)(f_{1p_1}+f_{2p_1})}{\delta u\,(1-Fr^2)}dt$$

$$\phi=\phi_{1,2} \qquad (101)$$

$$dp_1\boxed{(\phi-B)}+\frac{1}{\delta}dz_0\boxed{p\psi}=\frac{uR}{g\delta}\phi\;p\psi\;dt \qquad \phi=\phi_{1,2} \qquad (102)$$

in which: $Fp=(f_{1p_1}+f_{2p_1})/\delta u$ (103)

$$p\psi=(p_2\psi_1-p_1\psi_2)/(1-Fr^2) \qquad (104)$$

If both terms on the left-hand side of Eq.(101) or Eq. (102) have a large difference in magnitude, the dependent variables z_0 and p_1 are propagating independently. Therefore an estimation of the order of magnitude of both terms is necessary.
It is impossible to do this generally because every specific morphological problem will have its specific influence on these terms. Therefore an assumption has to be made:

$$dp_1 \sim \frac{dz}{\delta} \qquad (105)$$

A change of bedcomposition of the order O(1) must then coincide with a bedlevel change of the same order as the transportlayerthickness δ. This assumption may be arbitrary but it gives the possibility to estimate the terms in the char. rel.'s.

This estimation can now be carried out by determination of the coefficients belonging to dp_1 and $\frac{1}{\delta}$ dz in Eq. (101) or Eq. (102):

$$Fp\ (= (f_{1p_1} + f_{2p_1})/\delta u) \text{ and } (\phi - A) \text{ in Eq. (101)} \qquad (106)$$

or:

$$(\phi - B) \text{ and } p\psi\ (= (p_2\psi_1 - p_1\psi_2)/(1 - Fr^2) \text{ in Eq. (102)} \qquad (107)$$

Before treating the specific possible forms of the char. rel.'s first the coefficients from expression (106) and (107) will be considered

(i) $\phi - A$ and $\phi - B$ ($\phi = \phi_{1,2}$) or the deviations between the exact ($\phi_{1,2}$) and app. char.'s (A, B). In Appendix 4 the following form of the exact char.s's is derived:

$$\phi_{1,2} = \tfrac{1}{2}\{A + B \pm |A - B| + \beta\} \qquad (108)$$

in which $\beta/2$ is the deviation between exaxt and approximated characteristic directions. If for example $A > B$, $\phi_1 = A + \frac{1}{2}\beta$ and $\phi_2 = B - \frac{1}{2}\beta$.
Another expression in which β is written implicitly is also derived in Appendix 4:

$$\beta^2 + 2\beta\ |A - B| = 4\alpha \qquad (109)$$

in which: $\alpha = AB - C$.

Equation (109) shows that the magnitude of β is influenced by the relative magnitudes of A and B and the magnitude of AB − C. In the following sections two extreme cases concerning the relative magnitudes of A, B and $|\beta|$ will be considered:

1. $A \gg B$ $\qquad \phi_1 = A + \frac{1}{2}\beta$
$\qquad \phi_2 = B - \frac{1}{2}\beta$

1 accurate app. char. direction $\phi_1 \to A$	2 accurate app. char. directions $\phi_1 \to A$, $\phi_2 \to B$
$\lvert\beta\rvert \sim B \ll A$	$\lvert\beta\rvert \ll B \ll A$

2. B >> A $\phi_1 = B + \frac{1}{2}\beta$

$\phi_2 = A - \frac{1}{2}\beta$

1 accurate app. char. direction $\phi_1 \rightarrow B$	2 accurate app. char. directions $\phi_1 \rightarrow B$ $\phi_2 \rightarrow A$
$\lvert\beta\rvert \simeq A << B$	$\lvert\beta\rvert << A << B$

(ii) The terms Fp and pψ.

It can be derived easily that the product of Fp and pψ is equal to AB - C:

$$Fp.p\psi = AB - C = \frac{(f_{1p_1} + f_{2p_1})}{\delta u} \frac{(p_2\psi_1 - p_1\psi_2)}{(1 - Fr^2)} \tag{110}$$

It is shown in Appendix 4.2 that:

(i) $Fp \approx A, |p\psi| << B$ (111)

(ii) $Fp \approx A, |p\psi| \approx B$ (112)

are two representative combinations and approximations of both coefficients. It has been made clear that when the graindiameters of both fractions deviate more ($D_1/D_2 << 1$) and/or the coarse fraction reaches initiation of motion ($\tau_{e_{x_2}} \rightarrow \tau_{c_{x_2}}$) the first combination gradually shifts to the second one.

Now the magnitudes of the coefficients in the char.rel.'s can all be approximated by the magnitudes of A and B. However, not every combination of ϕ - A, ϕ - B, Fp and pψ is possible. Substitution of Eq. (110) in Eq. (109) yields on extra condition:

$$\beta^2 + 2\beta\ |A - B| = 4\alpha = 4(AB - C) = 4\ Fp.p\psi \tag{113}$$

<u>Example:</u>
Suppose A >> B and $|\beta| \approx B$ (one accurate app. char. direction $\phi_1 \rightarrow A$) then Eq. (113) can be replaced by:

$$|2AB| \approx 4\,|Fp\ p\psi|$$

which means that $Fp \approx A$ must coincide with $|p\psi| \approx B$ (Eq. (112)).

If in the same situation $|\beta| << B$ (two accurate app. char. directions) the choice $Fp \gtrsim A$ must coincide with $|p\psi| << B$ (Eq. (111)).

On the grounds of the previous estimations and choices and the knowledge of the previous section (characteristic directions) the following table, showing app. char. rel.'s in some extreme cases, has been composed (Table 4). It also shows the tendencies of the different dimensionless parameters (Fr, a/δ, $\tau_{e_{*1}}$, D_1/D_2, p_1) for the application of every app. char. relation.

It can be seen that the propagation of disturbances in bedlevel z_0 and bedcomposition p_1 can take place separately (*'separate propagation'*). In all four cases in the largest characteristic direction separate propagation of one of the variables occurs. In the smallest characteristic direction this happens only once and three times a combined change of bedlevel and bedcomposition is propagating along this characteristic. The case of separate propagation is approached better if the accuracy of the approximated char. directions (A and B) becomes better also. Also the meaning of the *'switching-effect'* ($\phi_1 \to A$ or B for $A >> B$ or $B >> A$) becomes clear in this table; apparently this effect is directly connected to which one of the variables (z_0 or p_1) is propagating in the largest characteristic direction and which one in the smallest characteristic direction. If $\phi_1 \to A$ and $\phi_2 \to B$ bedcomposition changes propagate faster than bedlevel changes and for $\phi_1 \to B$ and $\phi_2 \to A$ the reverse is true.

Remarks:

(i) The app. char. rel.'s of Table 4 can be found with Eq. (103) as well as Eq. (104).

(ii) Because of the basic assumption of the previous consideration Table 4 has no general value. For example when $A \gtrsim B$ or $B \gtrsim A$ it is also possible that A and B are accurate approximations and 'separate propagation' and the 'switching effect' are possible also ($\tau_{e_{*i}} >> \tau_{c_{*i}}$, $D_1/D_2 \lesssim 1$). Table 4 only shows some possible cases.

(iii) One of the basic assumptions of the mathematical model is the quasi-steadiness of the water-movement which requires low Froude-numbers. In case of uniform sediment the transition from the quasi-steady to the unsteady set of p.d.e. is noticeable in the third char. direction ϕ_3 ($\approx \psi/(1 - Fr^2)$) for $Fr \geqslant 0.8$ in case of normal transportconcentrations ($\psi \leqslant 0.01$) (Ribberink, 1978). It seems right to use this limit ($Fr < 0.8$) also in case of non-uniform sediment. This means that in Table 4 $Fr \to 1$ must be interpreted as a tendency for the Froude-number with a maximum $Fr = 0.8$.

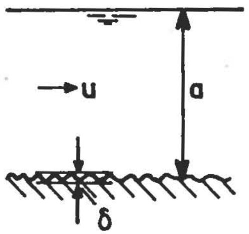
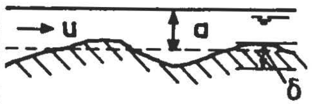
App. char.'s	Fr, a/δ	Accuracy of the app. char.'s	Coefficients Fp, pψ	Characteristic relation along $\phi = \phi_1$ $(\phi_1 > \phi_2)$	Characteristic relation along $\phi = \phi_2$	Tendencies of p_1, D_1/D_2, $\tau_{e_{*1}}$
$A >> B$ 	$Fr \to 0$ $a/\delta >> 1$	1. accurate app. char. $\phi_1 \to A$ $\lvert\beta\rvert \approx B << A$	$Fp \approx A$ $\lvert p\psi\rvert \approx B$	$A\, dp_1 = \gamma_1\, dt$	$A\, dp_1 - A \frac{1}{\delta} dz_0 = \gamma_2\, dt$	$D_1/D_2 << 1$ $\tau_{e_{*i}} \to \tau_{c_{*i}}$
		2. accurate app. char. $\phi_1 \to A$ $\phi_2 \to B$ $\lvert\beta\rvert << B << A$	$Fp \approx A$ $\lvert p\psi\rvert << B$			$D_1/D_2 < 1$ $\tau_{e_{*i}} > \tau_{c_{*i}}$ $p_1 \to 0$, $p_1 \to 1$
$B >> A$	$Fr \to 1$ $a/\delta \approx 1$	1. accurate app. char $\phi_1 \to B$ $\lvert\beta\rvert \approx A << B$	$Fp \approx A$ $\lvert p\psi\rvert \approx B$	$B \frac{1}{\delta} dz_0 = \gamma_1\, dt$	$B\, dp_1 + B \frac{1}{\delta} dz_0 = \gamma_2\, dt$	$D_1/D_2 << 1$ $\tau_{e_{*i}} \to \tau_{c_{*i}}$
		2. accurate app. char.'s $\phi_1 \to B$ $\phi_2 \to A$ $\lvert\beta\rvert << A << B$	$Fp \approx A$ $\lvert p\psi\rvert << B$	$B \frac{1}{\delta} dz_0 = \gamma_1\, dt$	$A\, dp_1 = \gamma_2\, dt$	$D_1/D_2 < 1$ $\tau_{e_{*i}} > \tau_{c_{*i}}$ $p_1 \to 0$, $p_1 \to 1$

Table 4 Approximated characteristic relations in some extreme cases

The right-hand sides of the characteristic relations are written as γ_i dt (for $\phi = \phi_i$, i = 1,2) in all cases. Nevertheless these coefficients γ_i do not have the same value in these cases. In the extreme case of 'sepatate propagation' these *right-hand sides* can be considered more closely:

1. *Separate propagation of the bedlevel along* $\phi = \phi_i$

This can occur in case of a uniform bedcomposition p_1 in x-direction and in case a disturbance in bedlevel z_0 propagates along $\phi = \phi_i$ without influencing the bedcompostition seriously. In the exact characteristic relation (see Eq. (102)) this means that the 'dp_1-term' can be neglected and the following equation remains:

$$dz_0 = \frac{uR}{g} \phi_i \, dt \tag{114}$$

This expression is identical to the characteristic relation of the mathematical model for uniform sediment and its right-hand side generally has a negative sign ($R < 0$ and $\phi_i > 0$ for $Fr < 1$). Substitution of the friction-term in Eq. (114) yields:

$$dz_0 = - \frac{u|u|}{C^2 a} c_i \, dt \tag{115}$$

in which $c_i = \phi_i/u \simeq \frac{1}{u} (\psi_1 + \psi_2)/(1 - Fr^2)$.

In case of a sudden step in bedlevel z_0 (according to Fig. 15) this relation generally shows a damping expansion-wave.

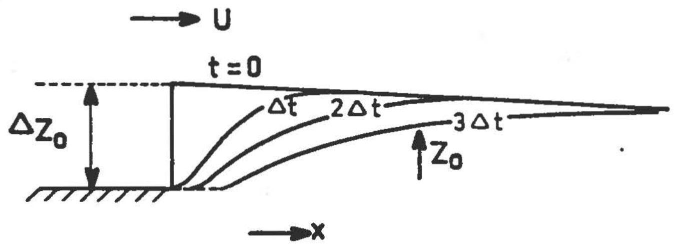


Fig. 15 A damping expansion-wave

The wave is expanding because an increase of z_0 generally causes an increase in watervelocity u which in its turn causes an increased celerity c_i. The wave is damped because an increase of z_0 causes a decrease of $\alpha(= uR\, \phi_i/g$, see Eq. (114)).

2. *Separate propagation of the bed composition along* $\phi = \phi_i$

This case can occur in case of a uniform flow over a sediment-bed in which a disturbance in bedcomposition propagates with no serious influence on the bedlevel z_0. Now the dz_0-term' can be neglected in Eq. (102) and the following relation results:

$$dp_1 = \frac{uR}{g\delta} \frac{p\psi}{\phi_i - B} \phi_i \, dt \qquad (116)$$

However, in this particular case of uniform flow the back-water curve becomes:

$$g \frac{\partial z_0}{\partial x} = R$$

which can be written as:

$$c_i \frac{\partial z_0}{\partial x} = \phi_i \frac{uR}{g}$$

Because the bedlevel is not influenced, $\partial z_0/\partial t$ is very small and consequently this relation can be written as:

$$c_i \frac{\partial z_0}{\partial x} \simeq \frac{\partial z_0}{\partial t} + c_i \frac{\partial z_0}{\partial x} = \frac{dz_0}{dt} = \phi_i \frac{uR}{g}$$

which is identical to Eq. (114).
Consequently neglecting the 'dz_0-term' of Eq. (102) must coincide with neglecting the right-hand side of this equation and Eq. (116) becomes:

$$\frac{dp_1}{dt} = 0 \quad \text{along } \phi = \phi_i \qquad (117)$$

Eq. (117) shows a 'bedcomposition-wave' which is not amplified or damped out but which can be an expansion-wave or a compression-wave (shock!) depending of:
(i) Which bed-load formula has been used (see Section 3.3.3.3.).
(ii) The sign of $\partial p_1/\partial x$ (see Fig. 16).

Fig. 16 Bedcomposition-waves using a bed-load formula including Egiazaroff's theory for $\partial p_1/\partial x > 0$ and $\partial p_1/\partial x < 0$.

3.4.3. Some calculations

In order to calculate the coefficients in the char. rel.'s with the same dimensionless parameters as used for the calculation of the char.'s (viz. $\tau_{e_{*1}}$, p_1, D_1/D_2, Fr, a/δ) the dimensionless form of the char. rel.'s has to be adapted. Because both expressions for the char. rel.'s (Eq. (101) and Eq. (102)) are identical a choice has to be made. Eq.(102) is chosen because of its simpler form:

$$dp_1 \;(\phi - B) + \frac{1}{\delta}\, dz_0 \; p\psi = \frac{uR}{g\delta}\, dt \;\phi\; p\psi \tag{118}$$

with: $p\psi = (p_2\psi_1 - p_1\psi_2)/(1 - Fr^2)$.

Using identical substitutions as in Section 3.3.2. Eq. (118) can be written as:

$$dp_1 \;(\overline{\phi} - \overline{B}) + \frac{1}{\delta}\, dz_0 \,\frac{\overline{Qu}}{1 - Fr^2} = \frac{\sqrt{\Delta g\, D_1^3}\; R}{g(1 - \varepsilon_0)\; a^2}\, \overline{\phi}\; dt\, \frac{a}{\delta}\, \frac{\overline{Qu}}{1 - Fr^2}$$

with: $\overline{Qu} = (p_2\; qu_1 - p_1 \left(\frac{D_2}{D_1}\right)^{3/2} qu_2)$

Division of both sides of this equation by $(\overline{\phi} - \overline{B})$ yields:

$$\boxed{1}\; dp_1 + \boxed{\frac{\overline{Qu}}{(\overline{\phi} - \overline{B})(1 - Fr^2)}}\,\frac{1}{\delta}\, dz_0 = \boxed{\overline{\phi}\, \frac{a}{\delta}\, \frac{\overline{Qu}}{(\overline{\phi} - \overline{B})(1 - Fr^2)}}\; R'\, dt \tag{119}$$

with: $R' = \dfrac{\sqrt{\Delta g\, D_1^3}\; R}{g(1 - \varepsilon_0)\; a^2}$

The marked coefficients can be calculated after the choice of a specific transportformula per sedimentfraction and the dimensionless parameters $\tau_{e_{*1}}$, D_1/D_2, p_1, Fr and a/δ. Again the formula of M.P & M including Egiazaroff's theory will be used. In Table 5 the resulting char. rel.'s in case of two extreme examples are shown as calculated with Eq. (119). Situation 1 can be described by a small Froude-number and a large a/δ-ratio (A >> B).Situation 2 can be described by a large Froude-number and a small a/δ-ratio (B >> A). Analogous to Table 4 both situations are divided in two cases:

(i) One accurate app. char. direction ($\beta \approx$ the smallest app. char. direction).

(ii) Two accurate app. char. directions ($\beta <<$ the smallest app. char. direction).

1. $a/\delta = 50$ $Fr = 0.1$ (A >> B)	$\bar{\phi}_1 \rightarrow \bar{A}$	$\left.\begin{matrix}\bar{\phi}_1 = 1.88 \\ \bar{A} = 1.73\end{matrix}\right\}$ $dp_1 + \frac{1}{\delta} dz_0 \cdot 0.038 = R'.dt\ .\ 3.61$ $\left.\begin{matrix}\bar{\phi}_2 = 0.053 \\ \bar{B} = 0.209\end{matrix}\right\}$ $dp_1 - \frac{1}{\delta} dz_0 \cdot 0.41 = -R'.dt\ .\ 1.1$	$\tau_{e*} = 0.1$	$D_1/D_2 = 0.4$
	$\bar{\phi}_1 \rightarrow \bar{A}$ $\bar{\phi}_2 \rightarrow \bar{B}$	$\left.\begin{matrix}\bar{\phi}_1 = 2.89 \\ \bar{A} = 2.87\end{matrix}\right\}$ $dp_1 + \frac{1}{\delta} dz_0 \cdot 0.006 = R'.dt\ .\ 0.911$ $\left.\begin{matrix}\bar{\phi}_2 = 0.43 \\ \bar{B} = 0.45\end{matrix}\right\}$ $dp_1 - \frac{1}{\delta} dz_0 \cdot 0.697 = -R'.dt.14.84$		$D_1/D_2 = 0.6$
2. $a/\delta = 1$ $Fr = 0.6$ (B >> A)	$\bar{\phi}_1 \rightarrow \bar{B}$	$\left.\begin{matrix}\bar{\phi}_1 = 0.348 \\ \bar{B} = 0.323\end{matrix}\right\}$ $dp_1 + \frac{1}{\delta} dz_0 \cdot 3.88 = R'.dt\ .\ 1.35$ $\left.\begin{matrix}\bar{\phi}_2 = 0.009 \\ \bar{A} = 0.035\end{matrix}\right\}$ $dp_1 - \frac{1}{\delta} dz_0 \cdot 0.32 = R'.dt.\ 0.003$	$p_1 = 0.4$	$D_1/D_2 = 0.4$
	$\bar{\phi}_1 \rightarrow \bar{B}$ $\bar{\phi}_2 \rightarrow \bar{A}$	$\left.\begin{matrix}\bar{\phi}_1 = 0.804 \\ \bar{B} = 0.804\end{matrix}\right\}$ $dp_1 + \frac{1}{\delta} dz_0 \cdot 23.9 = R'.dt.19.2$ $\left.\begin{matrix}\bar{\phi}_2 = 0.081 \\ \bar{A} = 0.081\end{matrix}\right\}$ $dp_1 - \frac{1}{\delta} dz_0 \cdot 0.011 = -R'.dt.\ 0.001$		$D_1/D_2 = 0.8$

Table 5 Characteristic relations and characteristic directions in some extreme cases.

The marked terms in this table are calculated using an earlier mentioned CPS-computer programme (see Appendix 5); in this programme the exact and approximated characteristic directions are calculated also and Fr, a/δ, p_1, D_1/D_2 and $\tau_{e_{*1}}$ are input parameters.

In all the calculated examples of Table 5 $\tau_{e_{*1}}$ and p_1 have the same value ($\tau_{e_{*1}} = 0.1$; $p_1 = 0.4$). As was shown before the case with one accurate app. char. direction shifts to the case with two accurate app. char. directions when the grain-diameter ratio($D_1/D_2 \rightarrow 1$) is increased. If the basic-assumption of Table 4 ($dp_1 \simeq \frac{1}{\delta} dz_0$) is also supposed to be true in Table 5 the marked coefficient of the 'dz_0-term' indicates which term can be neglected. Table 5 shows that this coefficient can be much smaller as well as larger than unity and it can be seen that:

(i) Along $\phi \approx A$ $\left(= \frac{p_2\, f_{1_{p_1}} - p_1\, f_{2_{p_1}}}{\delta u}\right)$ the 'dp_1-term' is dominating or has the same order of magnitude as the 'dz_0-term'.

(ii) Along $\phi \approx B$ $\left(= \frac{\psi_1 + \psi_2}{1 - Fr^2}\right)$ the 'dz_0-term' is dominating or has the same order of magnitude as the 'dp_1-term'.

The difference between situation (i) and situation (ii) in Table 5 is clearly shown. In situation (i) first a fast adaptation of the bed-composition (neglecting the 'dz_0-term' along $\overline{\phi} = \overline{\phi}_1$) followed by a slow combined change of bed-level and bed-compostition. In situation (ii) first a bed-level resp. combined adaptation followed by a slow combined resp. bed-compostition change.

Concerning the right-hand side of the char. rel.'s it can be concluded from Table 5 that generally the magnitude of the coefficient of the right-hand side is simultaneously changing with the coefficient of the 'dz_0-term'. This is in agreement with the consideration of separate propagation of bed-compostition in Section 3.4.2. ($dp_1/dt = 0$).

4. Some applications of the mathematical model in a simplified form

4.1 General

In the following sections integration along characteristics will be used for solving some applications of a simplified mathematical model. In the examples which will be treated the initial condition (t = 0) of the sediment bed is a continuous linear step in bedlevel z_0 and bedcomposition p_1 (see Fig. 17).

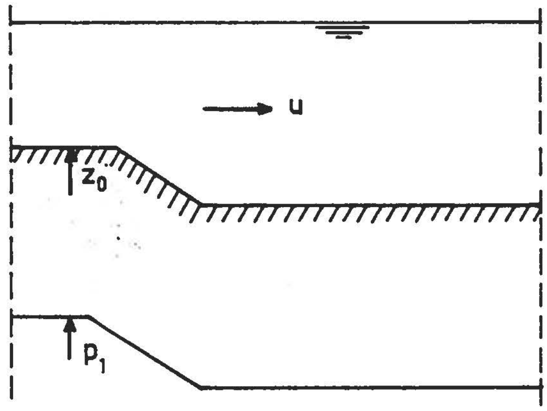


Fig. 17. Initial condition

Two kinds of steps are possible:

(i) Positive steps (causing sedimentation)
(ii) Negative steps (causing erosion).

By carrying out a number of calculation steps the propagation of the 'bedlevel-wave' and the 'bedcomposition-wave' in time and place can be determined.

In principle the mathematical model is only capable in solving sedimentation problems because of the assumption p_1 (composition of the transportlayer) = p_{1z_0} (composition at the lower boundary $z = z_0$ of the transportlayer). If erosion problems are solved with this model this assumption means that the eroded sediment (from below the transportlayer) has the same composition as the sediment in the transportlayer. This of course is not true generally but nevertheless insight can be obtained how the mathematical model works for erosion problems.

Remark:
It will be shown in the next sections that erosion problems have an important advantage *viz.* shocks are less easily formed and the calculation can be carried out during a longer time-period.

In this particular case with two positive characteristic directions integration along characteristics requires the following calculation-procedure:

(i) Calculation of the watervelocities in equally spaced points (along the x-axis) using a backwatercurve-calculation (constant waterdischarge, fixed bed).

(ii) Calculation of the characteristic directions and their characteristic relations in the chosen grid-points.

(iii) Intersection of the characteristics in the x-t plane.

(iv) Calculation of p_1 and z_0 in these points of intersection using a finite difference approximation of the characteristic relations.

The result in the x-t-plane is for example Fig. 18.

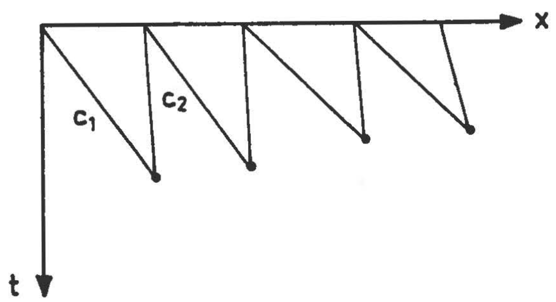


Fig. 18. Points of intersection after one step with integration along characteristics.

The next step in the calculation procedure should be a backwatercurve-calculation again. However, the points of intersection do not have the same time-value t. Therefore an interpolation should be necessary. A more simple and less time-consuming calculation-procedure can be obtained by simplifying the watermotion: the waterlevel is supposed to be fixed on a constant horizontal position or:

$$\frac{\partial h}{\partial x} = \frac{\partial a}{\partial x} + \frac{\partial z_0}{\partial x} = 0 \quad \text{and} \quad \frac{\partial h}{\partial t} = 0$$

in stead of the backwatercurve (constant waterdischarge):

$$(u - \frac{ga}{u}) \frac{\partial a}{\partial x} + g \frac{\partial z_0}{\partial x} = R$$

This assumption includes:

(i) a small Froude-number ($Fr = u/\sqrt{ga} << 1$)

(ii) a small friction-term R.

Now the waterlevel is known in every position of the x-t plane, the backwatercurve-calculation as well as the interpolation is not necessary. Exner (1925) already carried out calculations, in principle using a simplified form of the mathematical model for uniform sediment (neglecting the friction-term R). The result is a simple-wave equation for the bedlevel $z_0(x,t)$ without damping:

$$\frac{\partial z_0}{\partial t} + c(x,t) \frac{\partial z_0}{\partial x} = 0$$

The following subjects will be studied in the examples:

(i) Erosion and sedimentation

(ii) Influence of the transportlayerthickness δ; in this respect two cases are treated:

- $\delta \ll a$ and consequently $A \gg B$
- $\delta \approx a$ and consequently $B \gg A$.

(iii) Influence of Egiazaroff's theory (only in case of sedimentation).

All calculations of the characteristic directions and the characteristic relations are carried out by an earlier mentioned CPS-computerprogramme (see Appendix 5).

4.2 Examples

In the four examples which will be treated the following experimental conditions are identical.

sediment: two fractions $D_1 = 0.4$ mm

$D_2 = 1.0$ mm

porosity $\varepsilon_0 = 0.4$

a constant ripple-factor $\mu = 0.5$

in the flow-parameter $\tau_{e_{i}} = \frac{\mu\ Ri}{Di}$

(when Egiazaroff's theory is not used $\mu = 0.6$)

a constant Chézy-roughness: $C_t = 30\ m^{1/2}s^{-1}$

initial bedcomposition for $z_0 = 0 : p_1 = 0.5$

water: discharge $q = 0.1376\ m^2s^{-1}$

waterdepth for $z_0 = 0 : a = 0.4$ m.

In order to study the extreme possibilities of the mathematical model two types of examples will be treated, one with $A \gg B$ the other with $B \gg A$. These two types will be created by varying the transportlayerthickness δ which especially influences A. In Table 6 the differences between the four examples are shown.

Example	initial / boundary conditions Δz_0 (m)	initial / boundary conditions Δp_1 (-)	δ (m)	usage of Egiazaroff's theory	number of gridpoints along the initial step
1	-0.04	-0.2	0.01	with	2 and 5
2	-0.04	-0.2	0.2	with	2
3	+0.04	+0.2	0.01	with/without	2
4	+0.04	+0.2	0.2	with/without	2

Table 6 Description of the examples

The examples will be treated according to the order of succession in Table 5.

Example 1

Erosion and a negative composition change (coarsening) at a small transportlayer-thickness ($a/\delta >> 1$).

In this example an initial condition according to Fig. 19 is present. At the upstream-side the bedlevel z_0 and bedcomposition p_1 will be kept at a constant value (boundaryconditions: $z_0(o,t) = 0$, $p_1(o,t) = 0.5$).
In order to study the effect of the number of grid points over the initial step this number will be varied from 2 to 5.

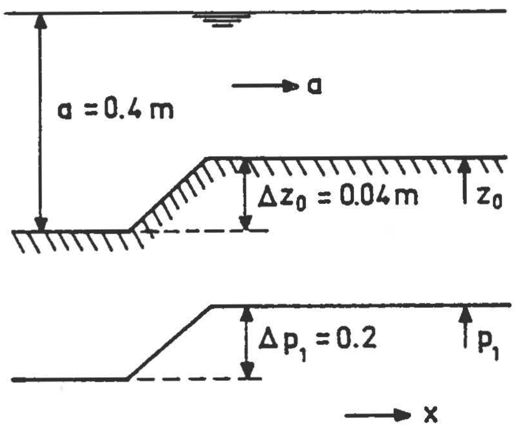


Fig. 19. Initial condition example 1. ($t = 0$)

According to the calculation-procedure as described in Section 3.5.1 the characteristic directions and their characteristic relations are calculated for every grid-point at $t = 0$.

In the points of intersection of the characteristics the values of z_0 and p_1 are determined and the calculation can be repeated for the next time-step. Fig. 20 shows the x-t plane after a number of calculation-steps until $t = 6$ h.

It can be seen that the characteristic directions c_1 and c_2 have a different value (because of $A >> B$) and it is interesting what kind of influences are propagating along these characteristics (see Fig. 21). This figure shows that after a certain interaction-period both initial steps (in z_0 and p_1) are completely separated in two combined influences each propagating with a different celerity:

1. A front consisting of a large step in p_1 (causing a very coarse transport-layer) and a relatively small step in z_0 (causing erosion) propagating with a large celerity ($c = c_1$) and a constant amplitude. The front is expanding, i.e. because the down-stream side of the front propagates faster than the upstream-side of it, the front is more or less stretched out in x-direction.
2. A tail consisting of a small step in p_1 (causing a finer transportlayer again) and a large step in z_0 (causing a large erosion). This combined tail-wave propagates with a smaller celerity ($c = c_2$) a constant amplitude and expands also.

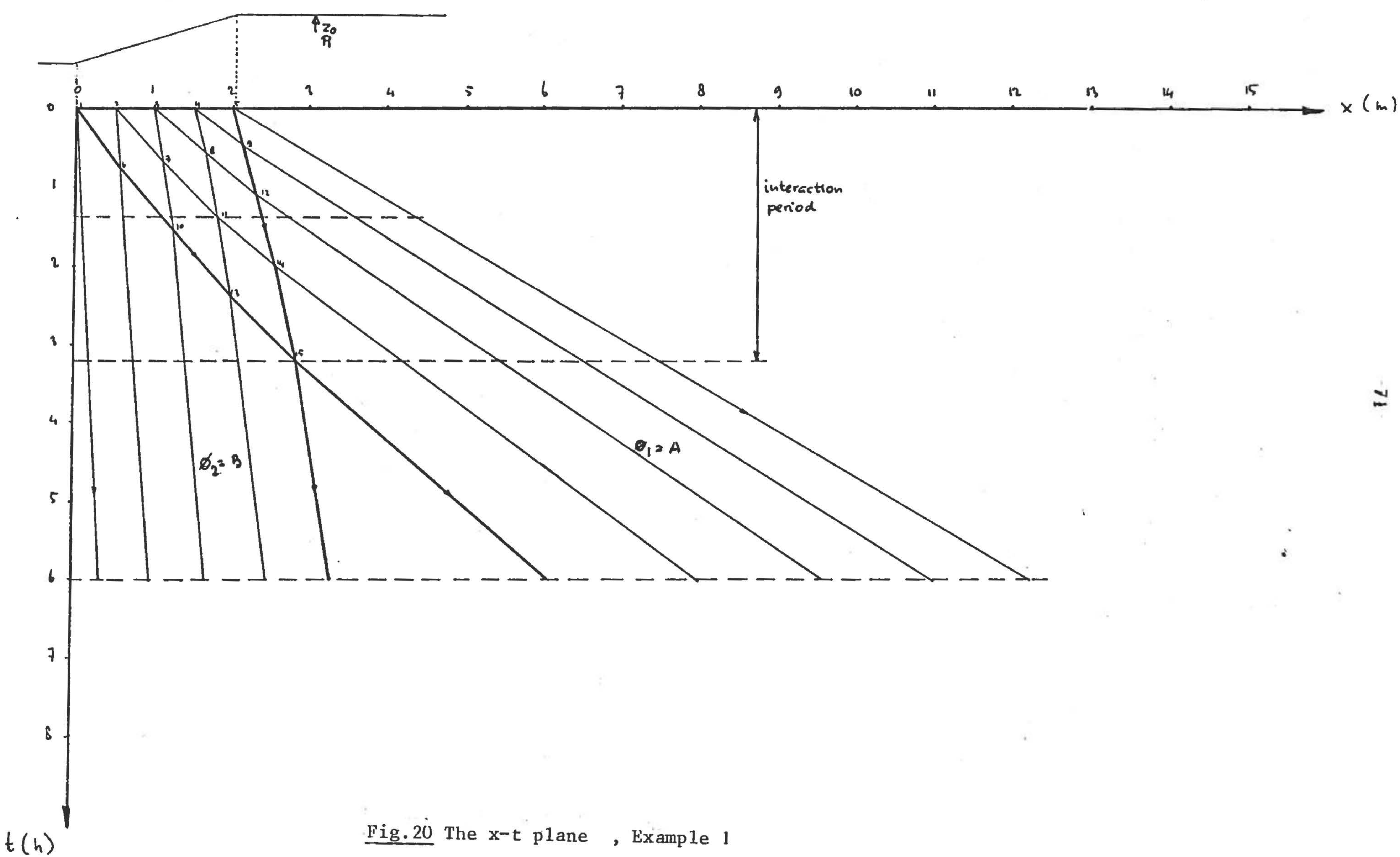
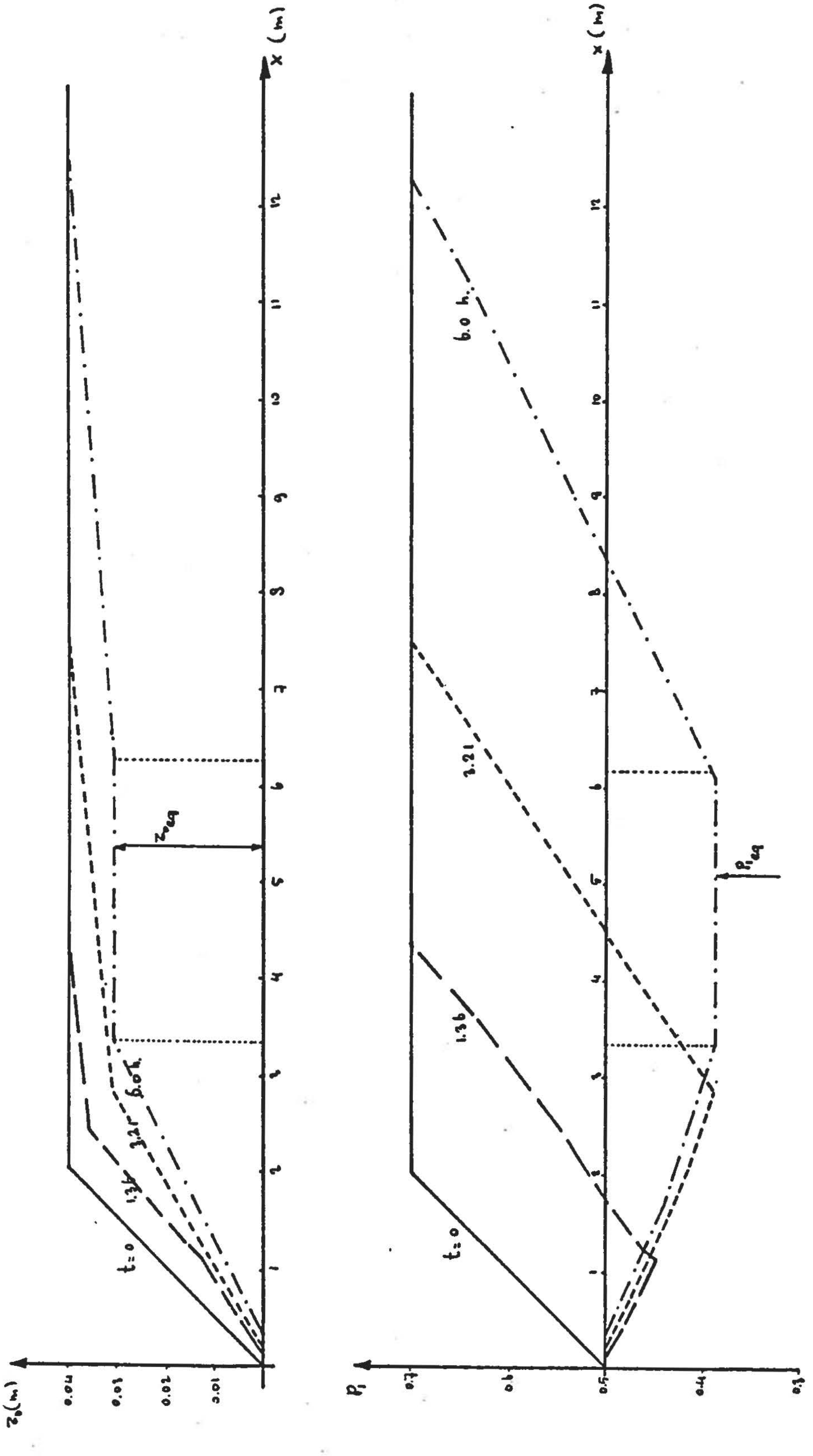


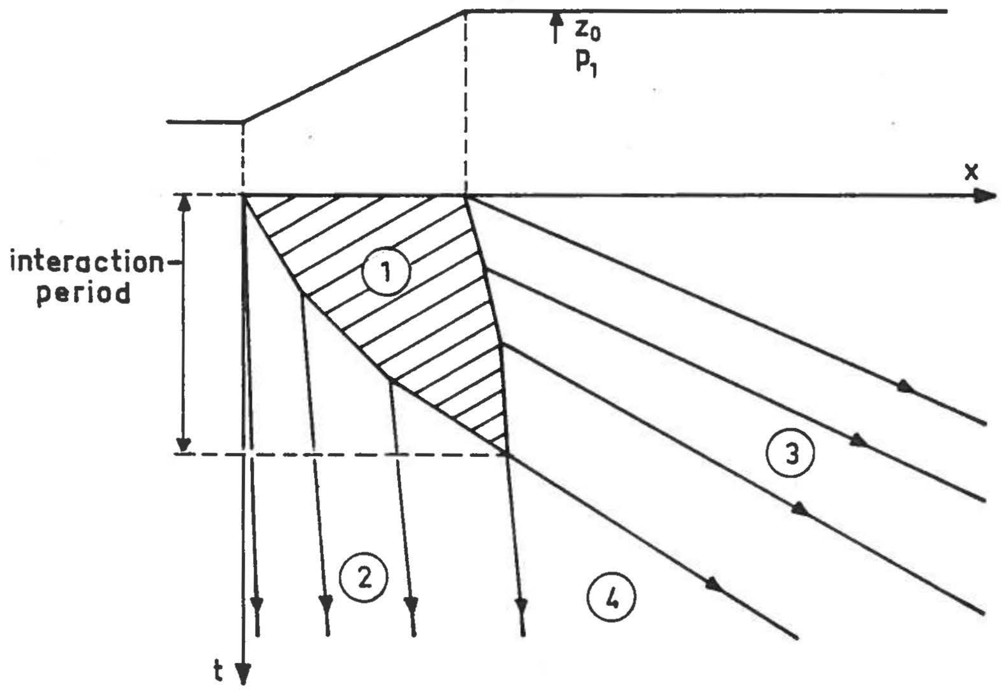
Fig.20 The x-t plane , Example 1



<u>Fig.21</u> Example 1

During the interaction-period both waves are interacting and they cannot be separated clearly but after this period a temporarily equilibrium between both waves develops ($z_{eq} \approx 0.03$, $p_{1_{eq}} \approx 0.4$).

The x-t plane can be devided in a number of areas each with its own meaning (see Fig. 22).



(1) = area of interaction
(2) = a slowly propagating expanding tail
(3) = a fast propagating expanding front
(4) = temporarily equilibrium

Fig. 22 Areas in the x-t plane

It can be shown easily why in area 3 the front propagates along straight characteristics ($c_1 \approx A.u$) and in area 2 the tail propagates along straight characteristics ($c_2 \approx B.u$) (see Section 4.3 and Appendix 6).

Figure 23 shows the influence of the number of grid-points along the initial step. The crossing-point of all the areas in the x-t plane shifts to a large x- and t-coordinate (larger interaction-period) but the values of $p_{1_{eq}}$ and $z_{0_{eq}}$ are nearly the same. Because the only goal of this section is to know the approximate effect of the mathematical model very accurate calculations are not necessary; therefore in the following examples only two grid-points along the initial step will be taken.

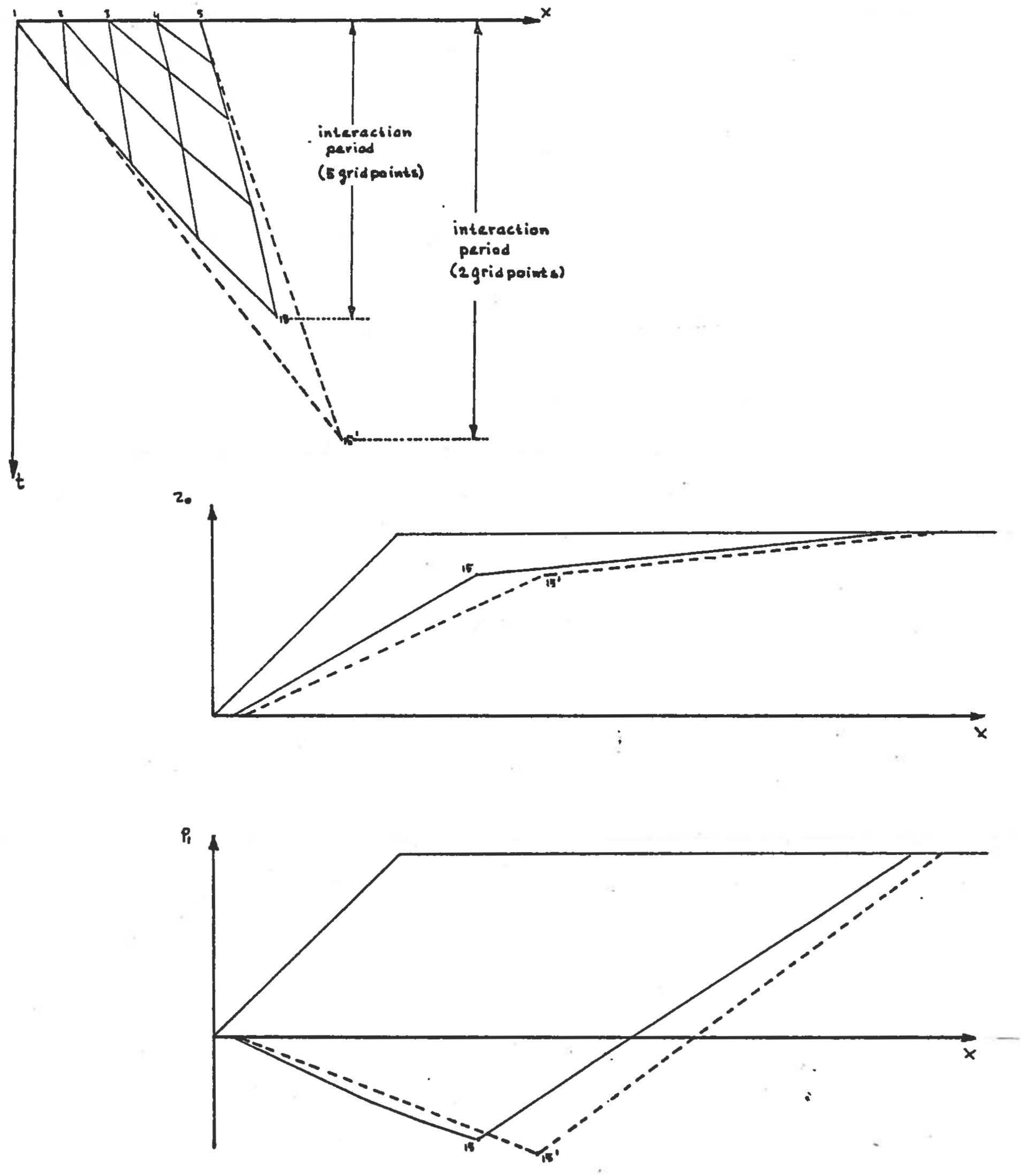


Fig. 23. Influence of the number of grid-points along the initial step.

Tabel 7 and Table 8 show all necessary parameters and results in the points of intersection in the x-t plane during the calculation.

Remark:

α_{z_1} and α_{z_2} represent the coefficients of the characteristic relations:

$$dp_1 + \alpha_{z_1}\, dz_0 = 0 \quad \text{along } c = c_1$$
$$dp_1 + \alpha_{z_2}\, dz_0 = 0 \quad \text{along } c = c_2$$

Point	x (m)	t (h)	z_0 (m)	p_1 (-)	a (m)	u ($m.s^{-1}$)	Fr (-)	a/δ (-)	$\tau_{e_{x_1}}$ (-)	c_1 ($m.h^{-1}$)	c_2 ($m.h^{-1}$)	A ($m.h^{-1}$)	B ($m.h^{-1}$)	α_{z_1} (m^{-1})	α_{z_2} (m^{-1})
1	0	0	0.0	0.500	0.40	0.344	0.174	40	0.10	0.774	0.061	0.685	0.149	4.176	–
2	0,5	0	0.01	0.550	0,39	0.353	0.181	39	0.105	0.914	0.118	0.831	0.202	3.199	-27.242
3	1.0	0	0.02	0.600	0.38	0.362	0.188	38	0.110	1.120	0.184	1.106	0.271	2.391	-28.281
4	1.5	0	0.03	0.650	0.37	0.372	0.195	37	0.116	1.389	0.257	1.329	0.318	1.763	-31.021
5	2.0	0	0.04	0.700	0.36	0.382	0.203	36	0.123	1.717	0.335	1.502	0.385	–	-34.900
6	0.590	0.762	0.0071	0.471	0.393	0.350	0.179	39.3	0.103	0.831	0.078	0.745	0.163	3.850	–
7	0.626	0.685	0.0174	0.526	0.383	0.360	0.186	38.3	0.109	1.028	0.147	0.951	0.225	2.846	-29.140
8	0.649	0.579	0.0278	0.581	0.372	0.370	0.194	37.2	0.115	1.287	0.220	1.220	0.288	2.111	-31.056
9	0.659	0.474	0.0382	0.636	0.362	0.380	0.202	36.2	0.122	1.611	0.301	1.553	0.357	–	-34,534
10	1.255	1.563	0.0145	0.442	0.386	0.357	0.184	38,6	0.107	0.915	0.104	0.835	0.184	3.412	–
11	1.821	1.360	0.0253	0.504	0.375	0.367	0.192	37.5	0.114	1.170	0.181	1.097	0.253	2.495	-31.449
12	2.355	1.128	0.0361	0.564	0.364	0.378	0.200	36.4	0.120	1.491	0.263	1.428	0,325	–	-34.335
13	2.006	2.384	0.0225	0.415	0.378	0.365	0.190	37.8	0.112	1.029	0.137	0.956	0,211	2.944	–
14	2.589	2.017	0.0337	0.483	0.366	0.376	0.198	36.6	0.119	1.354	0.223	1.288	0.290	–	-34.342
15	2.855	3.209	0.0310	0.389	0.369	0.373	0.196	36.9	0.177	1.180	0.177	1.113	0.244	–	–

Table 7. Example 1: Erosion and a *small* transportlayerthickness (5 grid-points over the initial step).

Point	x (m)	t (h)	z_0 (m)	p_1 (-)	a (m)	u ($m.s^{-1}$)	Fr (-)	a/δ (-)	$\tau_{e_{x_1}}$ (-)	c_1 ($m.h^{-1}$)	c_2 ($m.h^{-1}$)	A ($m.h^{-1}$)	B ($m.h^{-1}$)	α_{z_1} (m^{-1})	α_{z_2} (m^{-1})
1	0	0	0.0	0.50	0.40	0.344	0,174	40	0.10	0.774	0.061	0.685	0.149	4.176	–
5	2.0	0	0.04	0.70	0.36	0.382	0.203	36	0.123	1.717	0.335	1.502	0.385	–	-34.900
15	3.53	4.562	0.0306	0.372	0.37	0.372	0.195	37	0.116	1,123	0.163	1.055	0.231		

Table 8. Example 1: Erosion and a *small* transportlayerthickness (2 grid-points over the initial step).

Example 2

Erosion, a negative composition-change ('coarsening') and a large transportlayer-thickness.

This example is similar to Example 1 except that the transportlayerthickness has been increased. Two grid-points are taken along the initial step and the resulting x-t diagram can be seen in Fig. 24. Figure 25 shows the propagation of both initial steps in z_0 and p_1. Again after a certain interaction-period two separated influences propagate with different celerities:

1. A front consisting of a large step in z_0 (erosion below $z_0 = 0$) and a small step in p_1 (a small coarsening of the transportlayer) propagating with a large celerity ($c = c_1$) and a constant amplitude. Again the front expands during its propagation.
2. A tail consisting of smaller step in z_0 (sedimentation until $z_0 = 0$) and a large step in p_1 (large coarsening of the transportlayer) propagating with a small celerity ($c = c_2$) and a constant amplitude. However, in this case the tail is compressing; eventually a *shock* phenomenon will be formed.

Also in this example a temporarily equilibrium situation (between both waves) develops ($z_{eq} \cong -0.03$, $p_{1eq} \cong 0.68$).

Table 9 shows all calculated parameters of Example 2.

Example 3

Sedimentation, a positive composition change (finer transportlayer) and a small transportlayerthickness (influence of Egiazaroff's theory).

In contrast with Example 1 and 2 the initial condition has been reversed (see Fig. 26). At the upstream side the bedlevel z_0 and the bedcomposition p_1 are kept at a constant value (z_0 = 0.04 m, p_1 = 0.7).

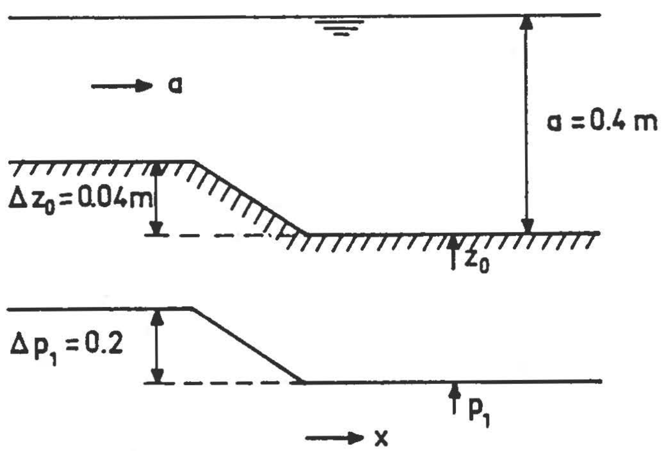


Fig. 26 Initial condition Example 3 (t = 0)

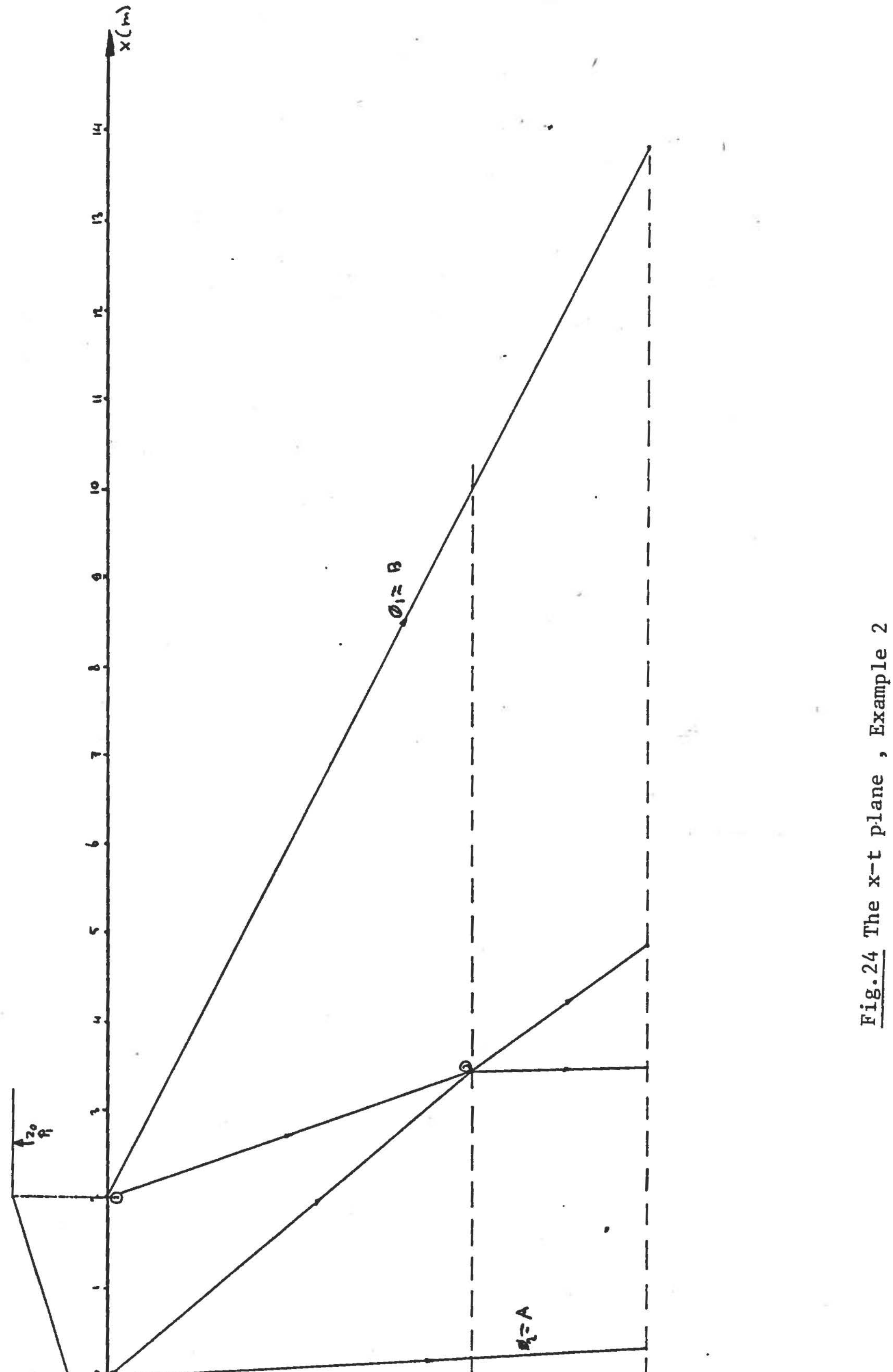


Fig.24 The x-t plane , Example 2

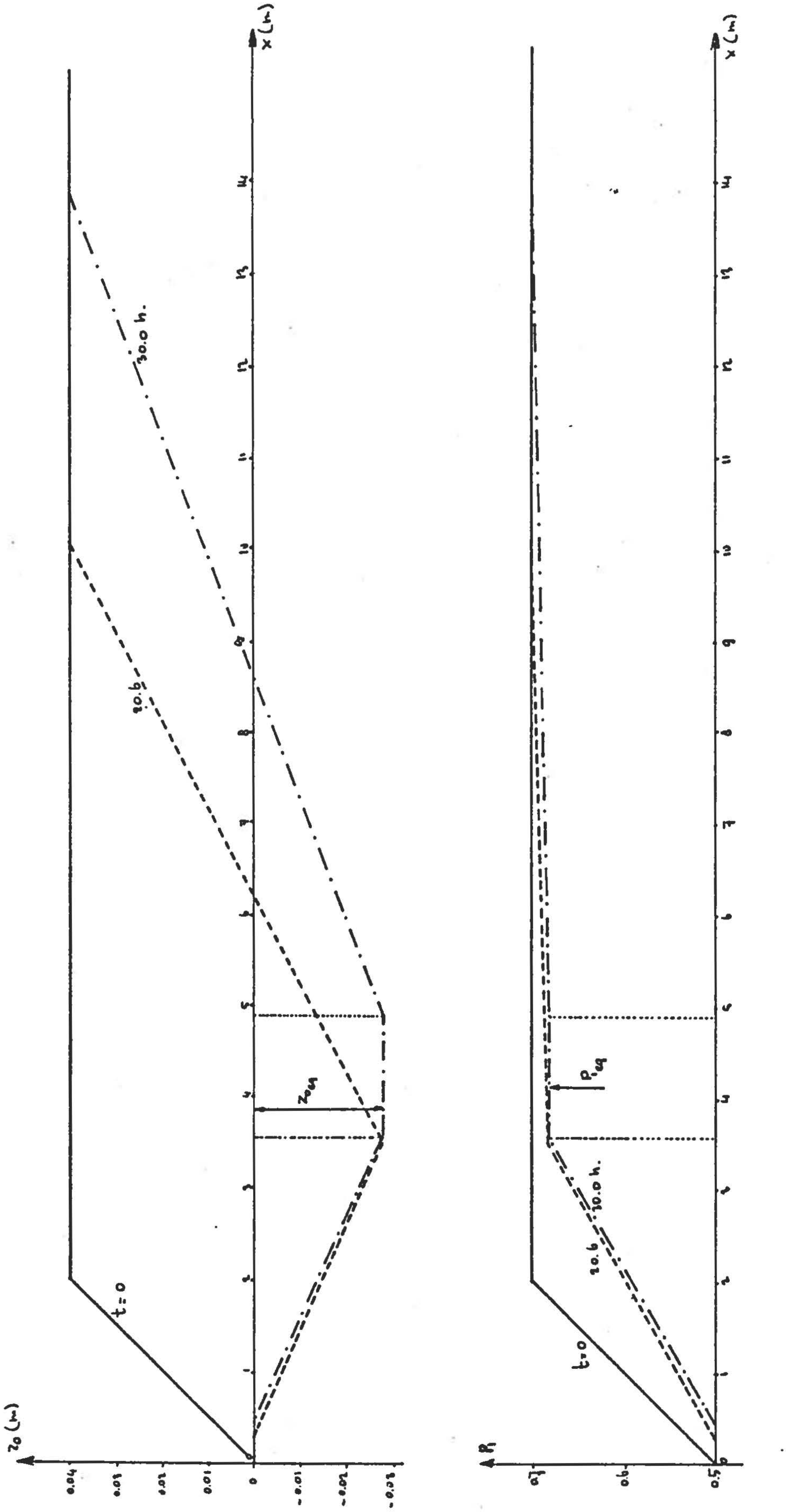


Fig.25 Example 2

The transportlayerthickness is small ($a/\delta >> 1$) and two grid-points are taken along both initial steps in z_0 and p_1; the resulting x-t diagram can be seen in Fig. 27.1.

Figure 28.1 shows the propagation of both steps in z_0 and p_1. It can be seen that again after an interaction-period two separate waves are formed. However, in this case both waves are compressed and *shocks* are formed. The calculation is not continued after the formation of a shock because then the theoretical model cannot be applied anymore.

Both waves can be described as:

1. A front consisting of large step in p_1 (an extra fine transportlayer, $p_1 > 0.7$) and a small step in z_0 (small sedimentation) propagating with a large celerity and a constant amplitude. The front is compressed.
2. A tail with a small step in p_1 (a small coarsening until $p_1 = 0.7$ of the transportlayer) and a large step in z_0 (large sedimentation) propagating with a small celerity and a constant amplitude. The tail is compressing also.
   The temporarily equilibrium situation between these waves can be described with $z_{0eq} \approx 0.01$, $p_{1eq} \approx 0.75$.

In this example also *the influence of Egiazaroff's theory* (used in the transport-formula per fraction) is studied. Figure 27.2 and Fig. 28.2 show the results of the same calculation while Egiazaroff's theory is not used.

<u>Remark:</u>

Unfortunately the ripple-factor had to be increased (from $\mu = 0.5$ to $\mu = 0.6$) in order to avoid a situation in which no transport of the coarse fraction occurs ($\tau_{e_{x_2}} < 0.047$). Therefore this case can only be compared roughly with the preceding one.

It can be seen that the results are roughly similar but that the fast propagating front is an expansion-wave now in stead of a compression-wave.
The 'equilibrium values' of z_0 and p_1 are slightly different: $z_{0eq} \approx 0.005$, $p_{1eq} \approx 0.85$. Table 10 shows the main calculated parameters of Example 3.

<u>Example 4</u>

Sedimentation, a positive composition-change (finer transportlayer) and a small transportlayerthickness (influence of Egiazaroff's theory).

This example is similar to Example 3, except that the transportlayerthickness has been increased ($a/\delta \approx 1$). The resulting x-t diagram and the propagation of both initial steps are shown in Fig. 29.1 and Fig. 30.1 respectively. Again after a certain interaction-period two separated waves are formed:

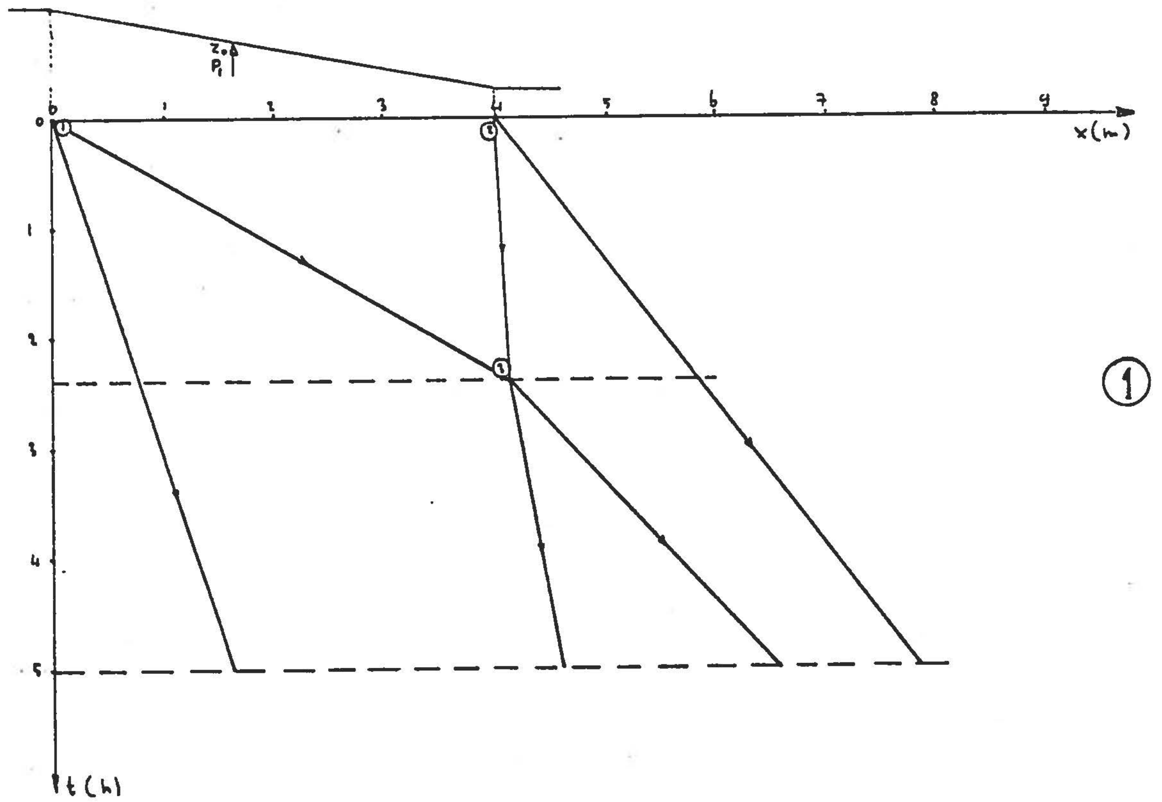
1. A front consisting of a large step in z_0 (sedimentation aboven $z_0 = 0.04$) and a small step in p_1 (finer transportlayer) propagating with a large celerity ($c = c_1 \approx B$) and a constant amplitude. The front is compressing and a shock is formed at t = 16 h.

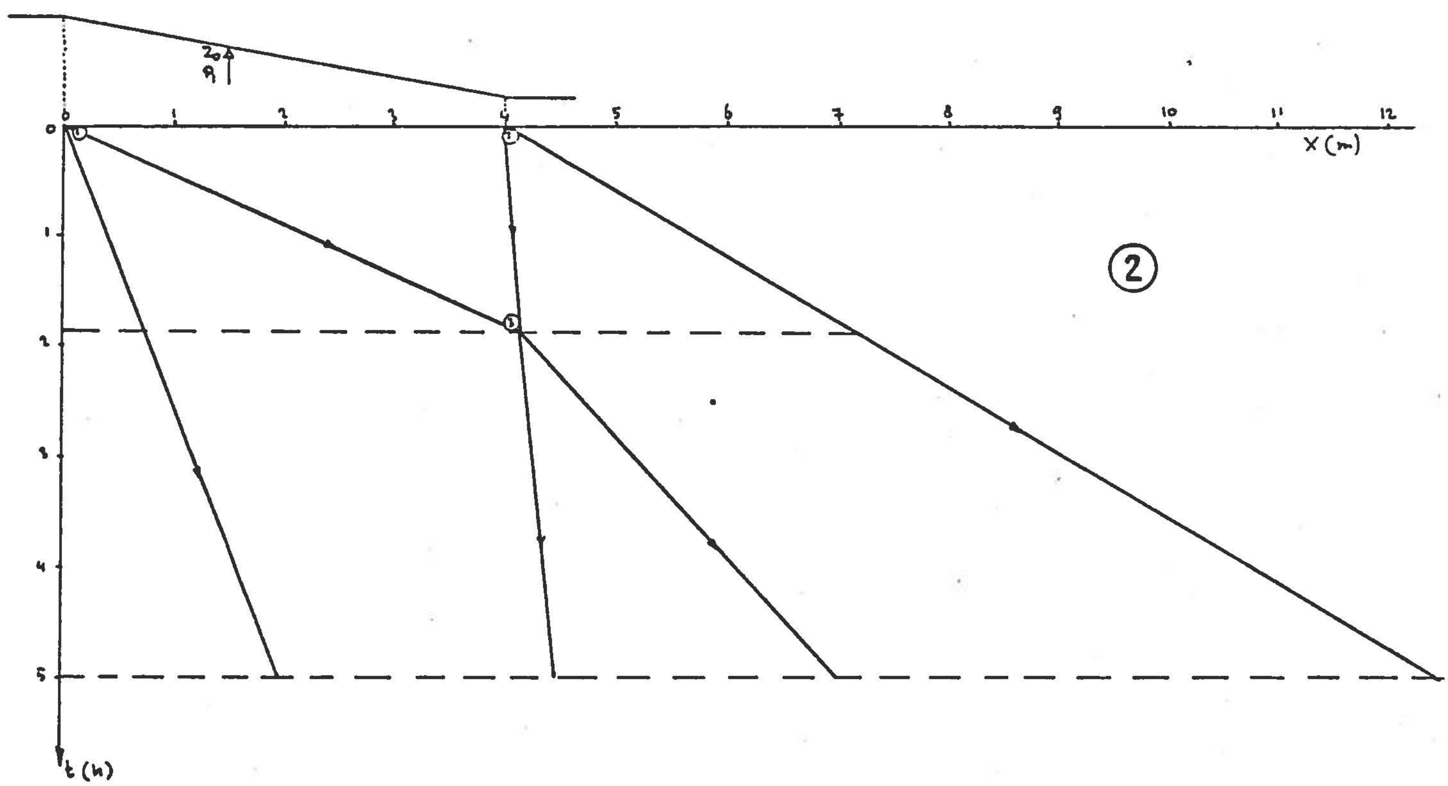
2. A tail consisting of a relatively small step in z_0 (now erosion!) and a large step in p_1 (finer transportlayer) propagating with a small celerity ($c = c_2 \approx A$) and a constant amplitude. The tail is expanding.

During the temporarily equilibrium situation $z_{0_{eq}} \cong 0.06$ and $p_{1_{eq}} \cong 0.55$.

The influence of not using Egiazaroff's theory in this example is shown in Fig. 29.2 and 30.2 (again with a different ripple-factor μ).

It can be seen that Egiazaroff's theory hardly influences the rough results of the calculation. Table 11 shows the main calculated parameters of Example 4.





<u>Fig. 27</u> Example 3. the x-t plane with Egiazaroff's theory (1)
without Egiazaroff's theory (2)

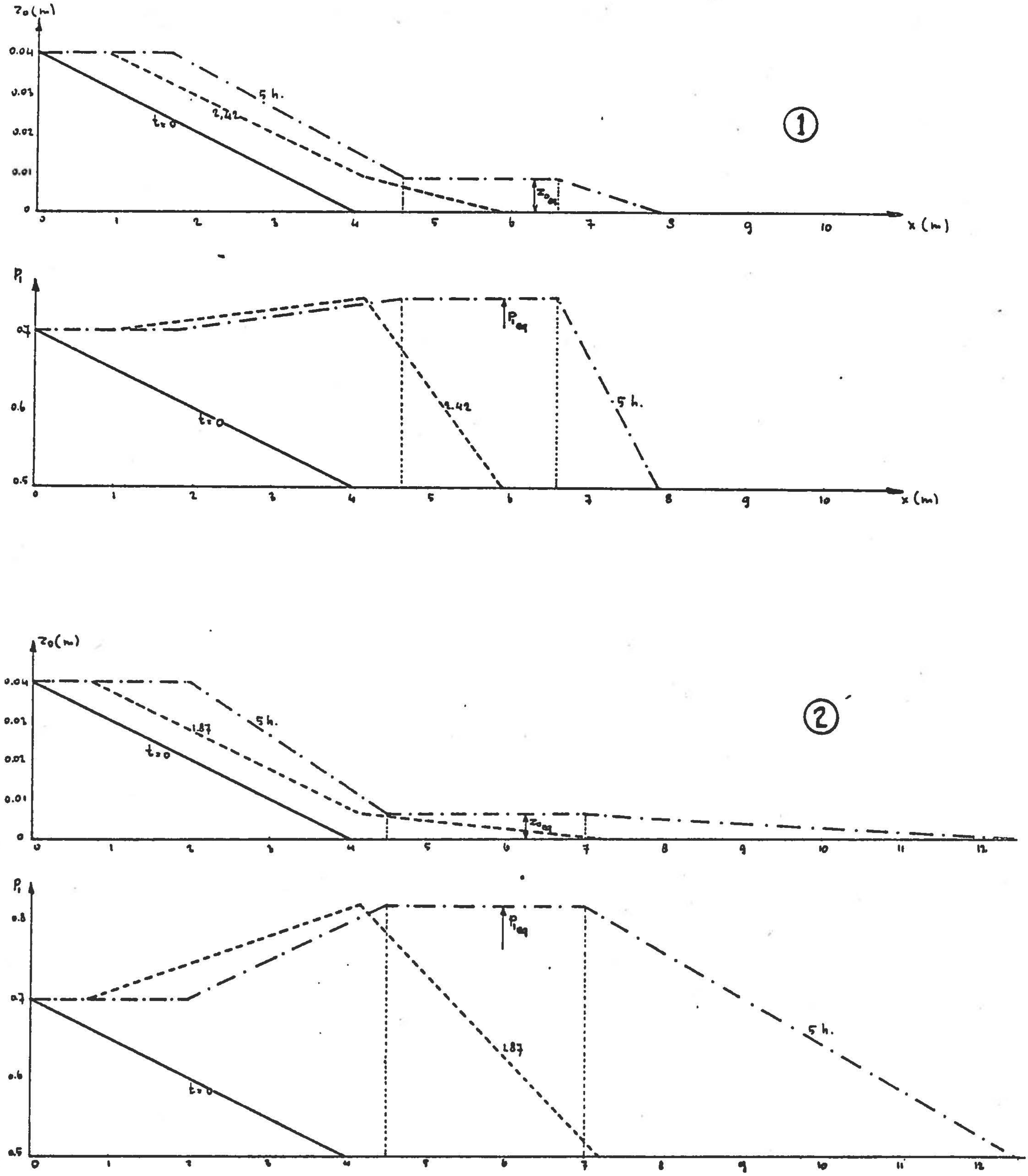
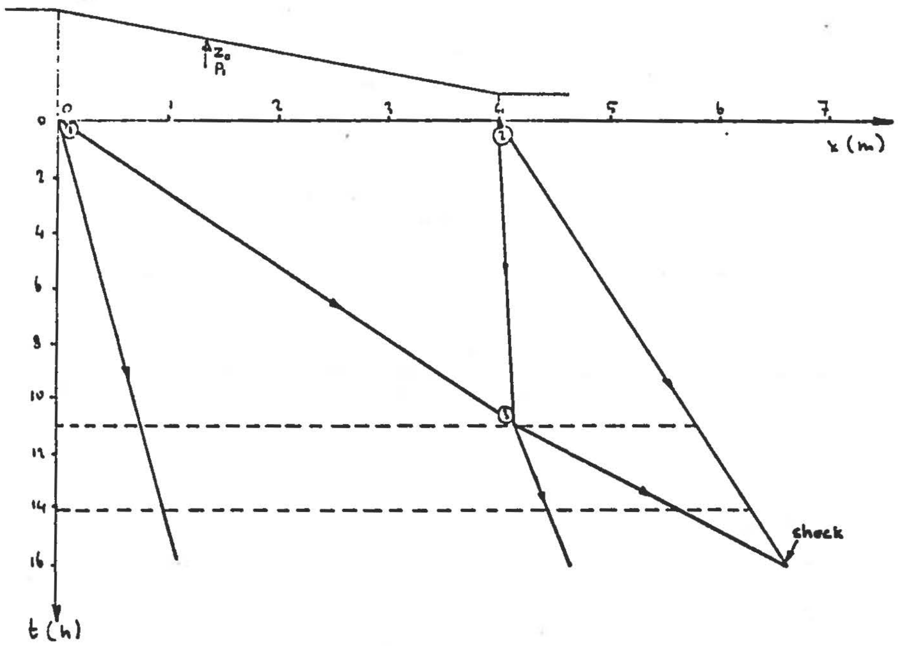
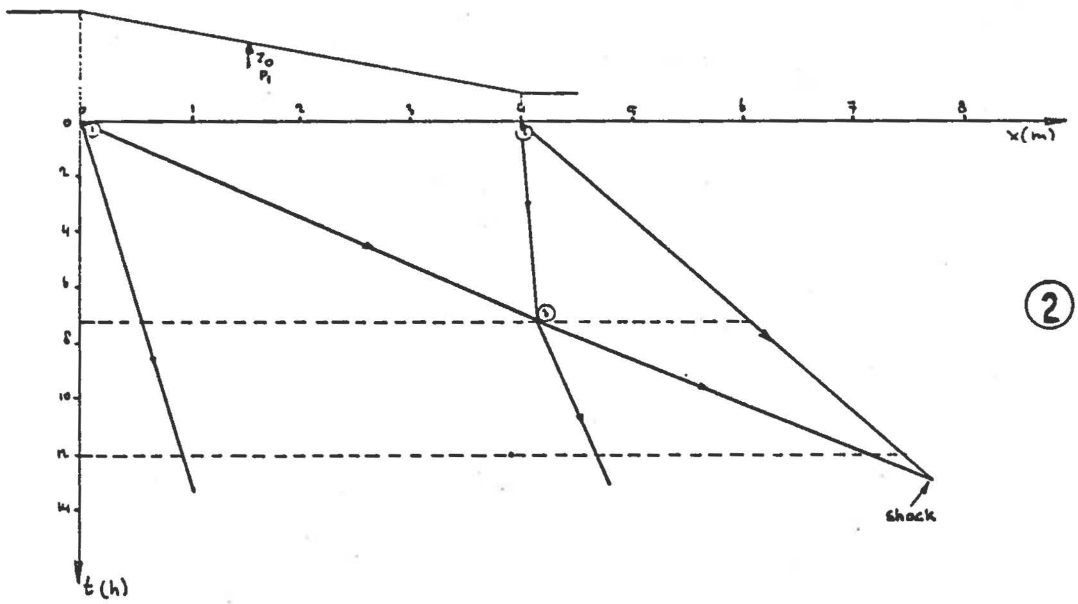


Fig. 28. Example 3. Propagation of the initial step in z_0 and p_1
with Egiazaroff's theory (1)
without Egiazaroff's theory (2)



①



②

Fig. 29. Example 4. The x-t plane with Egiazaroff's theory (1)
without Egiazaroff's theory (2)

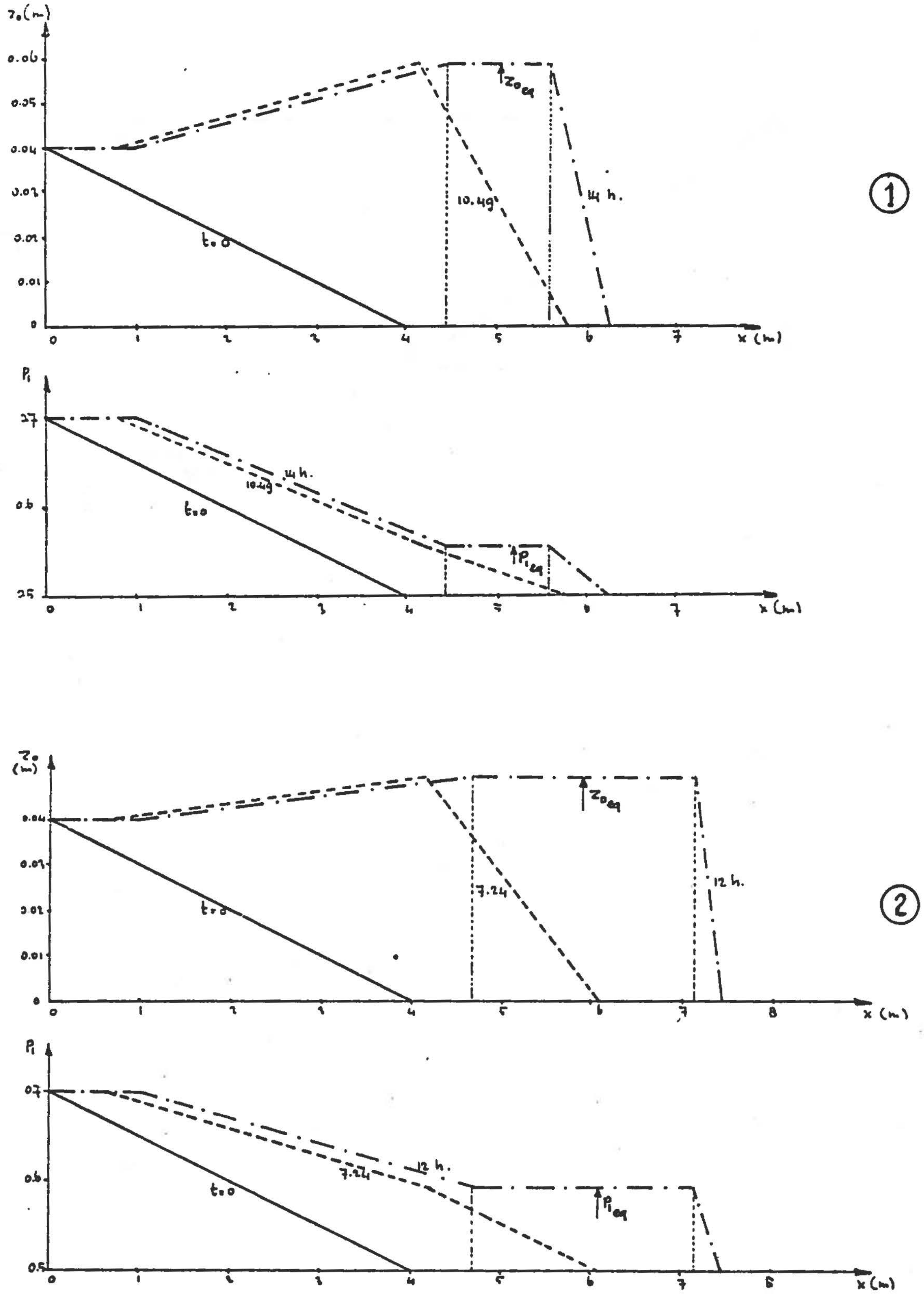


Fig. 30. Example 4. Propagation of the initial step in z_0 and p_1 with Egiazaroff's theory (1) without Egiazaroff's theory (2)

Point	x (m)	t (h)	z_0 (m)	p_1 (-)	a (m)	u ($m.s^{-1}$)	Fr (-)	a/δ (-)	τ_{ex_1} (-)	c_1 ($m.h^{-1}$)	c_2 ($m.h^{-1}$)	A ($m.h^{-1}$)	B ($m.h^{-1}$)	α_{z_1} (m^{-1})	α_{z_2} (m^{-1})
1	0	0	0.0	0.5	0.4	0.344	0.174	2	0.1	0.170	0.014	0.034	0.149	6.385	-
2	2	0	0.04	0.7	0.36	0.382	0.203	1.8	0.123	0.395	0.073	0.083	0.385	-	-0.275
3	3.51	20.62	-0.028	0.681	0.428	0.321	0.157	2.14	0.087	0.138	0.004	0.023	0.118	-	-

Table 9. Example 2: Erosion and a *large* transportlayerthickness (2 grid-points along the initial step).

	Point	x (m)	t (h)	z_0 (m)	p_1 (-)	a (m)	u ($m.s^{-1}$)	Fr (-)	a/δ (-)	τ_{ex_1} (-)	c_1 ($m.h^{-1}$)	c_2 ($m.h^{-1}$)	A ($m.h^{-1}$)	B ($m.h^{-1}$)	α_{z_1} (m^{-1})	α_{z_2} (m^{-1})
3.1	1	0	0	0,04	0.7	0.36	0.382	0.203	36	0.123	1.717	0.335	1.668	0.385	1.287	-
	2	4.0	0	0	0.5	0.40	0.344	0.174	40	0.10	0.774	0.061	0.685	0.149	-	-29.94
	3	4.15	2.42	0.0082	0.741	0.392	0.351	0.179	39.2	0.104	0.948	0.180	0.881	0.247	-	-
3.2	1	0	0	0.04	0.7	0.36	0.382	0.203	36	0.148	2.211	0.398	2.062	0.546	3.586	-
	2	4.0	0	0	0.5	0.40	0.344	0.174	40	0.12	1.700	0.067	1.532	0.229	-	-48.44
	3	4.12	1.87	0.0066	0.819	0.393	0.350	0.178	39.3	0.124	0.919	0.109	0.664	0.365	-	-

Table 10. Example 3: Sedimentation and a *small* transportlayerthickness (2 grid-points)

3.1. Using Egiazaroff's theory

3.2. Not using Egiazaroff's theory

	Point	x (m)	t (h)	z_0 (m)	p_1 (-)	a (m)	u ($m.s^{-1}$)	Fr (-)	a/δ (-)	τ_{ex_1} (-)	c_1 ($m.h^{-1}$)	c_2 ($m.h^{-1}$)	A ($m.h^{-1}$)	B ($m.h^{-1}$)	α_{z_1} (m^{-1})	α_{z_2} (m^{-1})
4.1	1	0	0	0.04	0.7	0.36	0.382	0.203	1.8	0.123	0.395	0.073	0.083	0.385	7.81	-
	2	4.0	0	0	0.5	0.4	0.344	0.174	2.0	0.10	0.170	0.014	0.034	0.149	-	-0.965
	3	4.15	10.5	0.058	0.556	0.342	0.403	0.22	1.71	0.137	0.479	0.099	0.111	0.467	-	-
4.2	1	0	0	0.04	0.7	0.36	0.382	0.203	1.8	0.148	0.572	0.077	0.103	0.546	11.35	-
	2	4.0	0	0	0.5	0.4	0.344	0.174	2.0	0.120	0.286	0.020	0.077	0.229	-	1.88
	3	4.14	7.24	0.049	0.593	0.351	0.393	0.212	1.75	0.156	0.628	0.116	0.148	0.596	-	-

Table 11. Example 4: Sedimentation and a large transportlayerthickness (2 grid-points).

4.1. Using Egiazaroff's theory

4.2. Not using Egiazaroff's theory

4.3 Discussion of results

4.3.1. General results

In all four examples which were treated after a certain *interaction-period* T_{int} two separated waves were formed, *a front wave* and a *tail wave*, each propagating with a different celerity. Both waves propagate without loss of amplitude (because of the simplifications of the mathematical model) and are separated by a *temporarily equilibrium situation* ($z_{0_{eq}}$, $p_{1_{eq}}$) which grows in length during the calculation-time. It was also shown that both waves can be expansion-waves as well as compression-waves (eventually shocks!) which depends on:

1. Whether it regards erosion or sedimentation
2. Whether or not Egiazaroff's theory is used in the transportformula per sedimentfraction

4.3.2. Influence of transportlayer thickness

In case of erosion as well as sedimentation the transportlayerthickness has a considerable influence on the propagation of the waves:

1. For a small transportlayerthickness ($a/\delta >> 1$) the fast propagating front ($c = c_1 \approx A.u$) mainly consists of a change in composition (Δp_1) and the tail ($c = c_2 \approx B.u$) consists of a combined influence (Δp_1 and Δz_0).
2. For a large transportlayerthickness ($a/\delta \approx 1$) the fast propagating front (now $c = c_1 \approx B.u$!) mainly consists of a change in bedlevel (Δz_0) and the tail ($c = c_2 \approx A.u$) consists of a combined influence (Δz_0 and Δp_1).

These results are in agreement with the results of Section 3.3 and 3.4 in which approximated characteristic direcitons and relations were derived for some extreme cases (see for example Table 4).

Example 1 and 2 but also Example 3 and 4 demonstrate the 'switching effect' as was described in Section 3.4.2.

The results of the Examples 1 ... 4 are summarized in Table 12.

Through a direct comparison of Example 1 and 2 (both treating erosion) after $t = 20.7$ h (= T_{int} for Example 2) Figure 31 results.

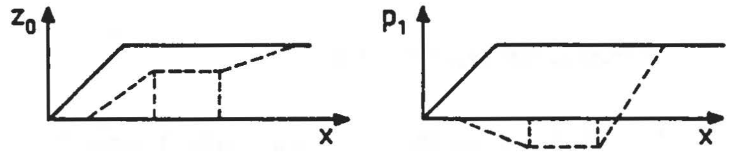
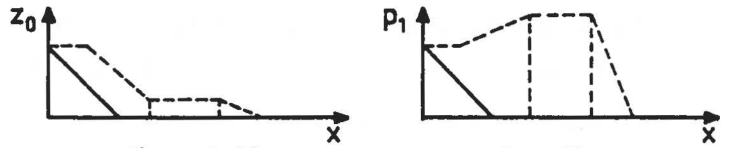
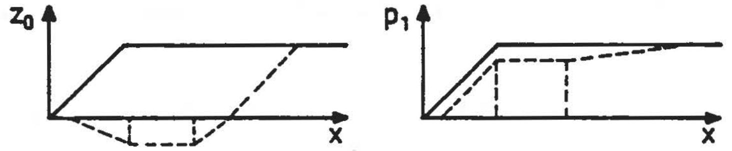
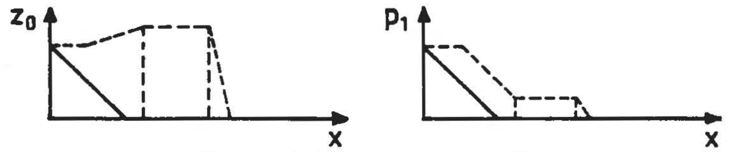
	Example	Results
$a/\delta \gg 1$ ($A \gg B$)	1	 erosion; extra coarse transportlayer
	3	 sedimentation; extra fine transportlayer
$a/\delta \approx 1$ ($A \gg B$)	2	 extra erosion; coarse transportlayer
	4	 extra sedimentation; fine transportlayer

Table 12. Summary of the results of the Examples 1 ... 4.

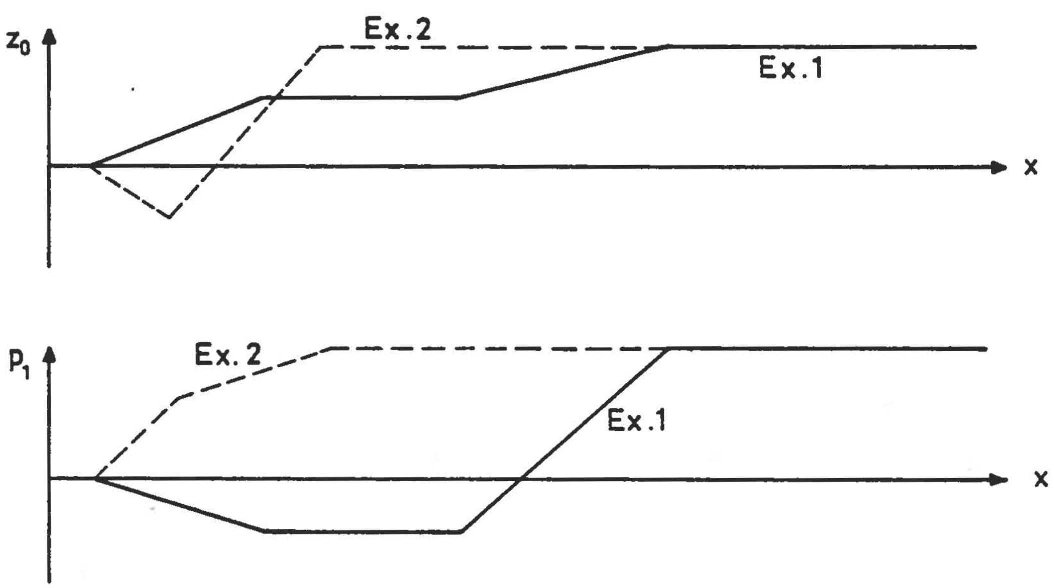


Fig. 31. Comparison of Ex. 1 and 2 at t = 20,7 h.

It can be seen that the bedcomposition is influenced mostly by the different transportlayerthickness δ. This is not surprising because, as was shown before, bed-composition-changes are approximately propagating with $c \approx A.u$ (which is inversely proportional to δ) and bed-level changes are approximately propagating with $c \approx B.u$ (which is not a function of δ at all).

However, the bedlevel is also influenced seriously: in Example 1 a fast front, causing only little erosion, and a temporarily equilibrium situation are present. The tail of Ex. 1 has approximately the same celerity as the front of Ex. 2 ($c_i \approx B$) but in Ex. 2 a serious erosion occurs ($z_0 < 0$).
In Ex. 2 a temporarily equilibrium is still not present and a very slow tail causes sedimentation again. A rough explanation of the differences between both examples can be given.

In both examples a positive gradient in sedimenttransport in x-direction ($\frac{\partial s}{\partial x} > 0$) exists. Because of selective transport of the fine fraction, coarsening of the transportlayer will occur. In case of a thin transportlayer (Ex. 1, $a/\delta >> 1$) this selection-proces does not take much time; therefore in Ex. 1 a fast propagating composition-change is present. Moreover a coarse transportlayer (or bed) does not allow so much erosion as a fine one. Therefore this fast composition-change is combined with a less serious erosion than Example 2 shows.

Because in Example 1 A >> B, in Example 2 B >> A and in both Examples B approximately has the same magnitude, the interaction-period T_{int} is much larger in case of Example 2.

Remark:
For relatively, short ranges of interest (in place or time), long initial steps and only a small difference between the characteristic directions c_1 and c_2, it is possible that a temporarily equilibrium will not occur and consequently the separation in two combined waves (front and tail) will never occur.

A similar comparison can be carried out for Example 3 and 4.

4.3.3. Propagation of front and tail-wave

All four examples showed that outside the area of interaction in the x-t diagram the characteristics are straight lines.

In the front and the tail-wave p_1 and z_0 are constant in resp. the largest characteristic direction ($c = c_1$) and the smallest characteristic direction ($c = c_2$). Consequently the front and the tail-wave propagate without damping or amplification.

It can be shown easily that in these calculations with two positive characteristic direction the above mentioned must be true (see Appendix 6). In the front and the tail the following relations are true:

$$\frac{\partial p_1}{\partial t} + \phi_i \, u \, \frac{\partial p_1}{\partial x} = 0 \tag{120}$$

$$\frac{\partial z_0}{\partial t} + \phi_i \, u \, \frac{\partial z_0}{\partial x} = 0 \tag{121}$$

with: $\phi_i = \phi_1$ in the front-wave

$\phi_i = \phi_2$ in the tail-wave.

Neglecting the friction term R and assuming a horizontal fixed waterlevel the mathematical model can be written as:

$$p_1 \frac{\partial z_0}{\delta t} + u \, \psi_1 \frac{\partial z_0}{\partial x} + \delta \frac{\partial p_1}{\delta t} + f_{1_{p_1}} \frac{\partial p_1}{\partial x} = 0 \tag{122}$$

$$p_2 \frac{\partial z_0}{\delta t} + u \, \psi_2 \frac{\partial z_0}{\partial x} - \delta \frac{\partial p_1}{\partial t} + f_{2_{p_1}} \frac{\partial p_1}{\partial x} = 0 \tag{123}$$

Elimination of $\partial z_0/\partial t$ in Eqs.(122) and (123) and substitution of $\partial p_1/\partial t$ (Eq.(120)) yields:

$$\delta(\phi_i - A) \frac{\partial p_1}{\partial x} = (p_2\psi_1 - p_1\psi_2) \frac{\partial z_0}{\partial x} \tag{124}$$

which is true in the front-wave ($\phi_i = \phi_1$) and in the tail-wave ($\phi_i = \phi_2$). It is also possible to eliminate $\partial p_1/\partial t$ in Egs.(122) and (123) and to substitute $\partial z_0/\partial t$ (Eq. (1)):

$$(f_{1_{p_1}} + f_{2_{p_1}}) \frac{\partial p_1}{\partial x} = u \, (\phi_i - B) \frac{\partial z_0}{\partial x} \tag{125}$$

which is identical to Eq. (124)

Both equations relate the gradients of z_0 and p_1 in x-direction to each other as present in the front or the tail.

<u>Remark:</u>

It can be shown that generally in the front the gradients of z_0 and p_1 have an equal sign; in the tail these signs are opposite. In the extreme situation that $f_{1_{p_1}} + f_{2_{p_1}} \to 0$ and $p_2\psi_1 - p_1\psi_2 \to 0$ (which is true for $\tau_{e_{x_i}} >> \tau_{c_{x_i}}$ or $D_1/D_2 \to 1$) these relations can also be used to show the existence of the asymptotic case of *separate propagation* in an alternative way. For example if $p_2\psi_1 - p_1\psi_2 \to 0$ and $\phi_i \to B$ ($\neq A$) Eq.(124) shows that then $\partial p_1/\partial x \to 0$ and only $\partial z_0/\partial x$ can have a certain magnitude. In other words, *along $\phi_i \to B$ (front or tail) a disturbance in z_0 will propagate*. In a similar way Eq(125) can be used to show that *along $\phi_i \to A$ (front or tail) a disturbance in p_1 will propagate*.

4.3.4. Egiazaroff's theory

In Example 4 ($a/\delta \simeq 1$) Egiazaroff's theory hardly has some influence. In both cases the front (mainly a bedlevel-change) is compressed and the tail (a combined change) is expanding. However, in Example 3 ($a/\delta >> 1$) the front (*mainly a bedcomposition-change*) is influenced considerably. Usage of Egiazaroff's theory causes a compressing front while an expanding front develops if this theory is not used. This completely agrees with the results of Section 3.3.3.3. in which the propagation of a bedcomposition-change with and without usage of Egiazaroff's theory was treated. It can be explained by the fact that use of Egiazaroff's theory causes increased transport of the coarse fraction and a decreased transport of the fine fraction.

4.3.5. Conditions for mathematical models for uniform or non-uniform sediment

It has been shown in the Sections 3.3. and 3.4. that the extreme case of separate propagation is approximated if one or more of the following conditions are fullfilled:

1. A >> B or B >> A
2. $\tau_{e_{x_i}} >> \tau_{c_{x_i}}$
3. $D_1/D_2 \to 1$

As a consequence then the accuracy of the approximated characteristic directions is very good ($\phi_i \to A$, $\phi_j \to B$; $i = 1,2$, $j = 2,1$) and the characteristic relations can be simplified.
In the Examples 1 ... 4 (with R = 0 and a horizontal fixed waterlevel) the characteristic relations in case of separate propagation become:

Along $\phi = \phi_i$ $\quad dz_0 = 0 \quad i = 1,2$

Along $\phi = \phi_j$ $\quad dp_1 = 0 \quad j = 2,1$

Figure 32 compares the propagation of the waves in a similar case as Example 2 with and without separate propagation.

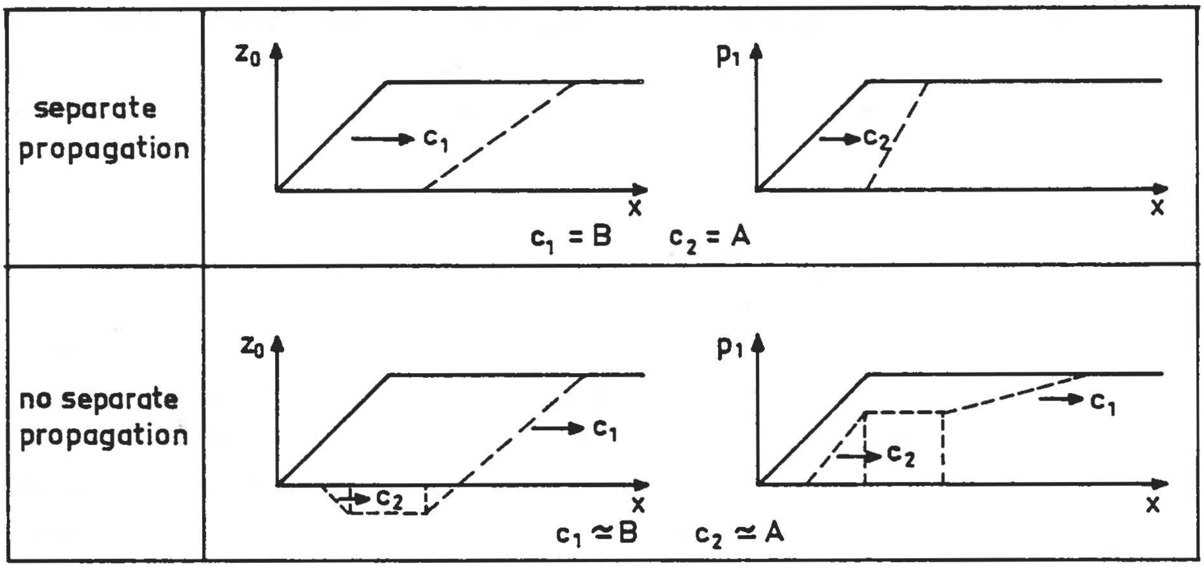


Fig. 32 Example 2 with/without separate propagation

It is shown that in case of separate propagation both waves do not influence each other and do not consist of a combined front or tail wave. In that case the bed-level changes can be predicted by the mathematical model for *uniform* sediment equally well because of the similarity of the resulting characteristic direction and relation (in this particular case with R = 0, Fr << 1 describing the bedlevel propagation):

$$c = B\,u = \psi\,u \qquad dz_0 = 0$$

A difference exists between the case with A >> B and B >> A:
If B >> A (Example 2) and the extreme case of separate propagation occurs the bedlevel changes can be calculated with the model for uniform sediment while p_1 is kept at a constant value. If A >> B and again the extreme case of separate propagation occurs the model for uniform sediment can be used also but the bed-composition is now in a quasi-steady state.

Similar to the water-motion the bedcomposition is assumed to react instantaneously (with a celerity $c \rightarrow \infty$) to bedlevel changes. As a result after every time-step in the calculation:

1. A backwater curve describing the watermotion is determined.
2. A new bedcomposition along the whole range of interest is calculated ($c = \infty$: $dp_1 = 0$ which means that the upstream boundary condition of p_1 is immediately true everywhere along the x-axis).

However, in the general case of 'no separate propagation' extra erosion (below $z_0 = 0$) occurs (see Fig. 32) which can only be predicted by the mathematical model for non-uniform sediment.

5. Summary and conclusions

5.1 Summary

The mathematical model for morphological computations in case of non-uniform sediment as derived and studied in the preceding sections consists of the following equations:

$$u \frac{\partial u}{\partial x} + g \frac{\partial a}{\partial x} + g \frac{\partial z_0}{\partial x} = R \tag{126}$$

$$u \frac{\partial a}{\partial x} + a \frac{\partial u}{\partial x} = \frac{\partial q}{\partial x} = 0 \tag{127}$$

$$\frac{\partial s_1}{\partial x} + \beta\delta \frac{\partial p_1}{\partial t} + p_{1_{z_0}} \frac{\partial z_0}{\partial t} = 0 \tag{128}$$

$$\frac{\partial s_2}{\partial x} - \beta\delta \frac{\partial p_1}{\partial t} + p_{2_{z_0}} \frac{\partial z_0}{\partial t} = 0 \tag{129}$$

$$s_1 = f_1(u, p_1) \tag{130}$$

$$s_2 = f_2(u, p_1) \tag{131}$$

Several assumptions were made during the derivation of this model, *viz.*:

1. Long-wave approximation of the watermotion which includes small transport-concentrations.
2. A quasi-steady water-motion including a small Froude-number.
3. Bed-level variations which proceed slowly with respect to the small-scale bedlevel variations of the bedforms, i.e. $|\partial z_0/\partial t| \ll |\partial z'/\partial t|$
4. A sedimentmixture consisting of 2 fractions (remark: the mathematical model can easily be extended for more fractions).
5. A constant effective transportlayerthickness, i.e. $\overline{\beta}\,\delta$ = constant
6. A small instantaneous transportlayerthickness (on top of the bedforms) with respect to the overall transportlayerthickness ($\delta' \ll \delta$)
7. Uniform vertical distribution of every sedimentfraction over the transport-layer ($\alpha_i = 1$).
8. Constant Chézy-roughness of the bed and constant grain-diameters
9. Constant porosity ε_0 of the sediment in the bed.

The number of equations is reduced by substituting the Eqs.(130) and (131) in the Eqs.(128) and (129) respectively and combining the Eqs.(126) and (127); three partial differential equations remain, *viz.*:

$$(u - \frac{ga}{u})\frac{\partial u}{\partial x} + g\frac{\partial z_0}{\partial x} = R \tag{132}$$

$$f_{1u}\frac{\partial u}{\partial x} + f_{1p_1}\frac{\partial p_1}{\partial x} + p_{1z_0}\frac{\partial z_0}{\partial t} + \beta\delta\frac{\partial p_1}{\delta t} = 0 \tag{133}$$

$$f_{2u}\frac{\partial u}{\partial x} + f_{2p_1}\frac{\partial p_1}{\partial x} + p_{2z_0}\frac{\partial z_0}{\delta t} - \beta\delta\frac{\partial p_1}{\partial t} = 0 \tag{134}$$

In order to get more insight in this set of equations and the mathematical character of it, characteristic directions and the acessory characteristic relations were derived.

The resulting dimensionless characteristic directions are:

$$\phi_{1,2} = \tfrac{1}{2}\left[A + B \pm \{(A - B)^2 + 4(AB - C)\}^{1/2}\right]$$

with: $B = \dfrac{\psi_1 + \psi_2}{1 - Fr^2}$

$A = \dfrac{p_{2z_0} f_{1p_1} - p_{1z_0} f_{2p_1}}{\beta\delta u}$

The expressions A and B are called the approximated characteristic directions (B is identical to the characteristic direction belonging to the mathematical model for uniform sediment, describing the celerity of disturbances in bed-level).

The resulting characteristic relations are:

$$\frac{dp_1}{dt}(\phi_i - B) - \frac{dz_0}{dt}\frac{p_{1z_0}\psi_2 - p_{2z_0}\psi_1}{(1 - Fr^2)} = \phi_i\frac{uR}{g}\frac{p_{2z_0}\psi_1 - p_{1z_0}\psi_2}{(1 - Fr^2)}$$

and are true along $\phi = \phi_i$ (with $i = 1,2$)

These equations can be approximated by more simple expressions in special conditions ('separate propagation'):

Along $\phi_i \to A$ $\quad \dfrac{dp_1}{dt} = 0$

Along $\phi_i \to B$ $\quad \dfrac{dz_0}{dt} = \dfrac{uR}{g} B$

These approximated characteristic directions and relations are true in a mathematical sense, but whether they have any practical meaning is studied in Chapter 3.

The characteristic directions and relations are written in a different dimensionless notation and are a function of five dimensionless parameters, *viz.*:

$\tau_{e_{x_1}}$: dimensionless effective shear stress of the finer fraction 1
D_1/D_2 : grain-diameter ratio
p_1 : probability of the finer fraction
Fr : Froude-number
a/δ : waterdepth-transportlayerthickness ratio.

The general form of the bed-load formulae per sedimentfraction ($s_i = f_i(u, p_1)$) is replaced by two possible bed-load concepts which result from a literature survey:
1. Meyer-Peter & Müller (including 'basic hypothesis')
2. Meyer-Peter & Müller with Egiazaroff's theory.

The influence of each of the above-mentioned dimensionless parameters and of both bed-load concepts on the characteristic directions and relations is studied. In order to simplify the calculations extra assumptions are made:

1. The ripple-factor μ (of the bed-loadformula) is constant in time and place and is identical for both fractions.
2. The probability of the finer fraction at the lower boundary ($z = z_0$) of the transportlayer ($p_{1_{z_0}}$) is equal to the mean probability $\overline{\overline{p}}_1$ in the transport-layer (this is true in case of sedimentation but is not necessarily true in case of erosion!).
3. Factor β ($= \overline{\overline{C}}/(1 - \varepsilon_0)$) is constant and is set equal to unity.

Table 3 is the result of the calculations of the *characteristic directions;* conclusions are:
1. The accuracy of the approximated characteristic directions A and B becomes better if:
   - a large difference in magnitude exists between A and B (A >>> B or B >>> A); this is especially influenced by Fr and a/δ.
   - the coarse fraction is far from initiation of motion ($\tau_{e_{x_2}} >> \tau_{c_{x_2}}$).
   - the grain-diameter ratio D_1/D_2 approaches unity ($D_1/D_2 \rightarrow 1$).
   - the probability of fraction (p_1) approaches zero ($p_1 \rightarrow 0$) or unity ($p_1 \rightarrow 1$).
2. A switching-effect appears to be present. The characteristic directions ϕ_1 and ϕ_2 can be approximated by A resp. B (A > B) but also by B resp. A (B > A), which especially depends of Fr and a/δ.

After these calculations *the characteristic relations* were studied and the magnitude of the terms of these relations were estimated.

Some extreme cases were treated resulting in Table 4 and Table 5. Conclusions are:

1. A direct relationship exists between the accuracy of A and B and the occurrence of *separate propagation* (= propagation of each variable p_1 or z_0 along its own characteristic (ϕ_1 or ϕ_2)). If this accuracy becomes worse (see above-mentioned conditions) this asymptotic case gradually shifts to a situation in which combined disturbances of p_1 and z_0 propagate along both characteristics.
2. If 'separate propagation' is reached asymptotically two extreme cases can be distinguished ('switching effect'):
   1. $Fr \to 0$
      $a/\delta >> 1$: in this case a disturbance in bedcomposition propagates faster ($\phi_1 \approx A$) than a disturbance in bedlevel ($\phi_2 \approx B$); or, the time-scale of bedcomposition-changes is smaller than of bedlevel-changes.
   2. $Fr \to 1$
      $a/\delta \approx 1$: in this case the reverse is true ($\phi_1 \approx B$ and $\phi_2 \approx A$).

In Chapter 4 these extreme cases are illustrated by treating some applications of a simplified form of the mathematical model (fixed, horizontal waterlevel!). Because of the difference in magnitude of both characteristic directions in all cases which were treated (erosion/sedimentation, small/large transportlayerthickness, bed-load formula with/without Egiazaroff's theory) after a certain interaction-period two separate waves were formed:

1. A front propagating with a large celerity $\phi = \phi_1$
2. A tail propagating with a small celerity $\phi = \phi_2$.

Between these waves 'a temporarily equilibrium' situation is developed.
In all four examples the case of separate propagation was not reached exactly and *combined* changes of p_1 and z_0 were propagating along the characteristics. However, the tendency of the extreme case of separate propagation was clearly shown and the results of the examples confirm the results of the preceding sections.
In Chapter 3 and Chapter 4 Egiazaroff's theory mainly influences celerity $\phi_i \to A$ ($\approx$ celerity of a disturbance in bedcomposition). Especially Example 4 shows that a compressing (eventually a shock!) or expanding bedcomposition wave develops if Egiazaroff's theory is used or not used respectively.

In all the extreme cases which were treated no complex characteristic directions are found and the set of partial differential equations is always hyperbolic.

5.2 Conclusions

In general the goal of morphological computations in rivers is the prediction of bed-level changes and in many applications the existing mathematical model for uniform sediment can be used. However, under certain conditions an extension of this model to a model for non-uniform sediment is necessary. These conditions are:

1. Not only bed-level changes but also bed-composition changes must be predicted; this seems to be a theoretical case.
2. Large time- and distance scales are present in the problem; even small differences in characteristic directions and relations may have large consequences at a large distance after a long period.
3. A large interaction exists between bed-level changes and bed-composition changes or, in other words, the asymptotic case of *separate propagation* is not reached at all. This is especially true if:
   - (i) The exact characteristic direction $\phi_{1,2}$ are badly approximated by A $(= (p_2 f_{1_{p_1}} - p_1 f_{2_{p_1}})/\beta\delta u)$ and B $(= (\psi_1 + \psi_2)/1 - Fr^2)$.
   - (ii) The approximated characteristic directions A and B approximately have the same magnitude.

   This implies that the mathematical model for non-uniform sediment is especially meaningful if:
   - (i) A large difference in the grain-diameters of both sedimentfractions exists.
   - (ii) Both sedimentfractions are close to 'initiation of motion'.
   - (iii) Nor the coarse, nor the finer fraction is dominating in the transportlayer.
   - (iv) No extremely thin transportlayer combined with a small Froude-number $(a/\delta >> 1, Fr << 1)$ or an extremely thick transportlayer combined with a large Froude-number $(a/\delta \approx 1, Fr \to 1)$ occurs.

5.3 Suggestions for continuation

1. Experiments

Many uncertain factors are still present in the mathematical model for non-uniform sediment e.g. an unknown bed-load formula per sedimentfraction, many

assumptions which are not or hardly verified.
Because an integral verification of the complete model with an unsteady non-uniform experiment is not able to verify each of these uncertainties individually a number of experiments in steady/uniform conditions should be carried out first.
The goal of these experiments will be:

(i) Determination of the influence of varying hydraulic conditions and sedimentmixtures on a number of parameters, *viz.*:
   - The transportlayer thickness
   - The total bed-roughness; this is mainly determined by the bedforms and therefore connected to the transportlayerthickness.
   - The uniformity of the vertical distribution of the sediment-fractions in the transportlayer.

(ii) Determination of empirical relations
   - An important empirical relation is the bed-load formula per sediment-fraction. With the aid of experiments existing concepts can be verified and possibly extended. Recent experiments of Ranga Raju(see Misri , Garde and Ranga Raju ,1980) may be very usefull in this respect.
   - If the assumptions concerning the transportlayerthickness and the total bed-roughness C_t should be seriously inaccurate, the mathematical model must be extended with (empirical) relations in which δ and C_t are a function of the hydraulic and sediment conditions. The necessary experiments could be combined with the preceding ones.

(iii) Determination of the influence of bedforms
The fluctuations of bedlevel z_0' and bedcomposition p_1' because of the existence of bedforms can be serious disturbances of the average bedlevel z_0 and bedcomposition p_1. Experiments under steady uniform conditions are necessary in order to determine:
   - a time-scale of these fluctuations (bedforms) because of the assumption in the mathematical model $|\frac{\partial z_0'}{\delta t}| >> |\frac{\partial z_0}{\delta t}|$ and $|\frac{\partial p_1'}{\delta t}| >> |\frac{\partial p_1}{\delta t}|$.
   - an accurate way of measuring the bedcomposition (taking bedsamples).
   - an accurate statistical treatment of the bedlevel- and bedcomposition-measurements.

Experiments under *unsteady, non-uniform conditions* are necessary in order to carry out the verification of the complete model. Some possibilities exist to create the unsteady non-uniform conditions:

(i) Changing the hydraulic conditions (e.g. lowering of the waterlevel).
(ii) Changing the upstream sediment input (e.g. change of the total sediment input and/or its composition.

2. Numerical model

In this report the method of characteristics was used for the calculation of some simple applications. The water-motion was extremely simplified in order to be able to carry out the calculations (partly) by hand.
In order to know the results of the mathematical model in more general situations a numerical model is necessary. This model can also be used for the experimental verification of the complete theoretical model for two fractions.

3. The assumption $p_1 = p_{1_{z_0}}$.

Using the calculation of characteristic directions and relations the assumption $p_1 = p_{1_{z_0}}$ was made. It means that the double-averaged probability of fraction 1 in the transportlayer (p_1) is equal to the time-averaged probability of this fraction at the lower boundary ($z = z_0$) of the transportlayer. This assumption is correct in case of sedimentation in combination with a uniform vertical distribution of the fractions in the transportlayer. Two cases can be considered in which $p_1 \neq p_{1_{z_0}}$:

(i) In all erosion problems; extra information is necessary about the composition of the bed below the transportlayer.

(ii) In case of sedimentation and a non-uniform vertical distribution of the fractions in the transportlayer extra information is necessary about the shape of this distribution in different circumstances.

In both cases an extra veriable $p_{1_{z_0}}$ must be introduced. Because

$$A \left(= \frac{p_{1_{z_0}} f_{1_{p_1}} - p_{1_{z_0}} f_{2_{p_1}}}{\beta \delta u} \right) \text{ is a function of } p_{1_{z_0}}$$

and B $\left(= \frac{\psi_1 + \psi_2}{1 - Fr^2}\right)$ is not a function of $p_{1_{z_0}}$ it can be expected that especially the propagation of bedcomposition-changes ($\phi \approx A$) will be influenced by $p_{1_{z_0}}$.

4. Extension of the mathematical model for more than two grain-fractions

Every extra fraction means an extra variable $p_i(x,t)$, an extra characteristic direction and as a result a more complicated mathematical model.

5. Extension of the mathematical model to large Froude-numbers

For large Froude-numbers (especially $Fr \approx 1$) the quasi-steady approach of the watermotion is not applicable anymore. The approximated characteristic direction B ($= \frac{\psi_1 + \psi_2}{1 - Fr^2}$) will have the same order of magnitude as the characteristic directions of the watermotion. The approximated characteristic direction A is not directly influenced by Fr and in case the transportlayer is not too thin, and consequently B >> A, in theory a situation is created in which

- bedlevel changes propagate very fast ($\phi \approx B$) and should be calculated with an unsteady watermotion.
- bedcomposition changes propagate slowly ($\phi \approx A$) and the quasi-steady approach of the watermotion can still be used.

Appendix 1 Continuity-equation of sedimentfraction i

1.1. General

Starting from the general two-dimensional mass-balance a derivation will take place of a continuity-equation per sedimentfraction i.

After averaging out the turbulent fluctuations and an integration in vertical direction a one-dimensional form of this equation results, which can be applied in the set of one-dimensional equations (see section 1.4.).

A definition of bed-level, transportlayerthickness (bed-load), concentration and grain-velocity will be found necessary and will be given.

1.2. Two-dimensional form

On the assumption that the mass-density of sediment ρ_s is constant, the general mass balance (per unit width) can be written in the following way:

$$\frac{\partial C}{\partial t} + \frac{\partial u_p C}{\partial x} + \frac{\partial w_p C}{\partial z} = \text{source/sink term} \qquad (1.1)$$

in which: $C = C(x,z,t)$ = sediment concentration
$u_p = u_p(x,z,t)$ = grain-velocity in x-direction
$w_p = w_p(x,z,t)$ = grain-velocity in z-direction.

Because of water-turbulence and the sedimentmovement itself (see Section 1.2.) these dependent variables will fluctuate. It will be assumed that these fluctuations can be averaged out without influencing the large-scale processes which will be considered.

Before carrying out this averaging process Eq.(1.1) will be written per sedimentfraction i:

$$\frac{\partial C_i}{\partial t} + \frac{\partial u_{p_i} \cdot C_i}{\partial x} + \frac{\partial w_{p_i} \cdot C_i}{\partial z} = \text{source/sink of fraction } i \qquad (1.2)$$

in which C_i, u_{p_i}, w_{p_i} are the variables for sedimentfraction i.

The probability of fraction i is defined as:

$$p_i(x,z,t) = C_i(x,z,t)/C(x,z,t) \qquad (1.3)$$

Because of the fluctuations the dependent variables can be written as a sum of an averaged part and a fluctuating part:

$$\begin{aligned} C_i(x,z,t) &= \overline{C_i}(x,z,t) + C_i'(x,z,t) \\ u_{p_i}(x,z,t) &= \overline{u_{p_i}}(x,z,t) + u_{p_i}'(x,z,t) \\ w_{p_i}(x,z,t) &= \overline{w_{p_i}}(x,z,t) + w_{p_i}'(x,z,t) \end{aligned} \qquad (1.4)$$

In these expressions the overlined symbols are averaged over a certain period, according to:

$$\overline{C_i}(x,z,t) = \frac{1}{T}\int_0^T C_i(x,z,\tau)d\tau$$

$$\overline{u_{p_i}}(x,z,t) = \frac{1}{T}\int_0^T u_{p_i}(x,z,\tau)d\tau$$

$$\overline{w_{p_i}}(x,z,t) = \frac{1}{T}\int_0^T w_{p_i}(x,z,\tau)d\tau$$

As was mentioned before, a condition for this averaging process is:

$$\left| \frac{\partial \overline{C_i}}{\partial t} \right| << \left| \frac{\partial C_i'}{\partial t} \right|$$

The same condition can be written for the other variables. Substitution of Eq. (1.4) in Eq. (1.2) results in:

$$\frac{\partial \overline{C_i}}{\partial t} + \frac{\partial C_i'}{\partial t} + \frac{\partial \overline{u_{pi}}.C_i'}{\partial x} + \frac{\partial u_{pi}'.\overline{C_i}}{\partial x} + \frac{\partial \overline{u_{pi}}.\overline{C_i}}{\partial x} + \frac{\partial u_{pi}'.C_i'}{\partial x} +$$

$$+ \frac{\partial \overline{w_{pi}}.C_i'}{\partial z} + \frac{\partial w_{pi}'.\overline{C_i}}{\partial z} + \frac{\partial \overline{w_{pi}}.\overline{C_i}}{\partial z} + \frac{\partial w_{pi}'.C_i'}{\partial z} =$$

$$= \text{source/sink of fraction } i$$

Averaging this equation over period T gives

$$\frac{\partial \overline{C_i}}{\partial t} + \frac{\partial \overline{u_{pi}}.\overline{C_i}}{\partial x} + \frac{\partial \overline{u_{pi}'C_i'}}{\partial x} + \frac{\partial \overline{w_{pi}}.\overline{C_i}}{\partial z} + \frac{\partial \overline{w_{pi}'C_i'}}{\partial z} =$$

$$= \text{source/sink of fraction } i \qquad (1.5)$$

Before integration of this two-dimensional equation over the vertical, definitions of bed-level,transportlayerthickness, concentration and grain-velocity (in case of bed-load) will be given.

1.3. Bed-level, transportlayer, concentration and grainvelocity

The bed-level $z(x,t)$ can be defined as the level below which no grain-movement occurs. Because of the existence of bedforms this level will fluctuate and therefore an averaged bed-level will be defined.

$$\bar{z}(x,t) = \frac{1}{T}\int_0^T z(x,t)dt$$

The instantaneous bed-level will fluctuate around this mean level and can be written as:

$$z(x,t) = \bar{z}(x,t) + z'(x,t)$$

A condition for this averaging process is similar to the one given before *viz.*:

$$\left|\frac{\partial\bar{z}(x,t)}{\partial t}\right| << \left|\frac{\partial z'(x,t)}{\partial t}\right|$$

This means that the large-scale erosion or sedimentation of the mean bottom proceeds much slower than the small scale fluctuations due to the bedforms.

The instantaneous bed-level can be considered as a stochastical variable which is distributed according to a certain probability density function (p.d.f.) in vertical direction (see Fig. 1.1).

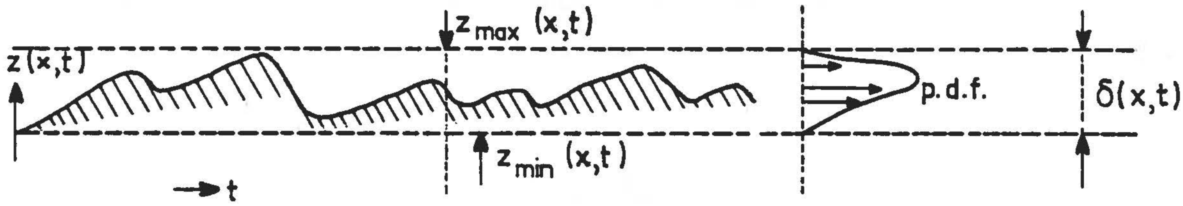


Fig. 1.1 P.d.f. of the instantaneous bottomlevel

It will be assumed that the p.d.f. takes the value zero at two fixed boundaries *viz.* an upper boundary $z_{max}(x,t)$ and a lower boundary $z_{min}(x,t)$.
The definition of bed-level $z_o(x,t)$ which is used in the equations is:

$$z_o(x,t) = z_{min}(x,t) \qquad (1.6)$$

because below this level no grainmovement occurs. In case of bed-load the instantaneous sedimenttransport takes place in a thin layer on top of the bedforms.

The transportlayer $\delta(x,t)$ will be defined as:

$$\delta(x,t) = z_{max}(x,t) - z_{min}(x,t) \qquad (1.7)$$

because it can be assumed that the overall transport takes place between these two levels.

If in this transportlayer an element dx-dz (unit width) is considered it can contain water, moving and resting sedimentparticles (see Fig. 1.2).

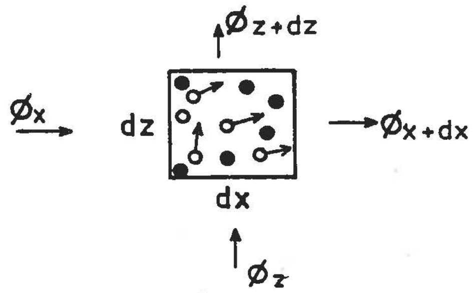


Fig. 1.2. Moving and resting particles in an element dx-dz of the transportlayer.

If: C_{m_i} = volume concentration of the moving particles of fraction i
$u_{p_{m_i}}$ = particle velocity of the moving particles of fraction i
C_{r_i} = volume concentration of the resting particles of fraction i
the following consideration can be given.

The mean velocity of all the particles of fraction i in the element is:

$$u_{p_i}'' = \frac{u_{p_{m_i}} \cdot C_{m_i}}{C_{m_i} + C_{r_i}} \tag{1.8}$$

The total volumeconcentration of all the particles of fraction i is:

$$C_i'' = C_{m_i} + C_{r_i} \tag{1.9}$$

The flux of sedimentfraction i in x-direction is the product of the velocity and concentration of the moving particles:

$$\phi_{x_i} = u_{p_{m_i}} \cdot C_{m_i}$$

which can be written with Eqs. (1.8) and (1.9) as:

$$\phi_{x_i} = u_{p_{m_i}} \cdot C_{m_i} = u_{p_i}'' \cdot C_i''$$

Because of the turbulent watermovement and the irregularities of the bedforms u_{p_i}'' en C_i'' (written as u_{p_i} and C_i in the following) will fluctuate in time.

Carrying out the averaging process (as described in section 1.2) for the total concentration and the mean particle velocity (including restperiods) the overall flux of fraction i in x-direction becomes:

$$\phi_{x_i} = \overline{u_{p_i}} \cdot \overline{C_i} + \overline{u_{p_i}' C_i'}$$

It can be concluded that the variables C_i and u_{p_i} as used in the sediment continuity-equation (Eq. 1.5) can be considered as the <u>total sediment concentration of fraction i</u> and the <u>mean particle velocity</u> (including restperiods) of fraction i.

<u>Remark</u>:

With the assumptions:

1. The thickness of the instantaneous moving layer on top of the bedforms is small compared to the transportlayerthickness $\delta(x,t)$.
2. The porosity of the sedimentmixture ε_o is constant;

it is admissible to determine the concentration distribution in the transport layer (in vertical direction) from the p.d.f. of the instantaneous bedlevel.

The averaged concentration at level z, $\overline{C(z)}$, is then equal to the product of $1 - \varepsilon_o$ and the cumulative p.d.f. (see Fig. 1.3.)

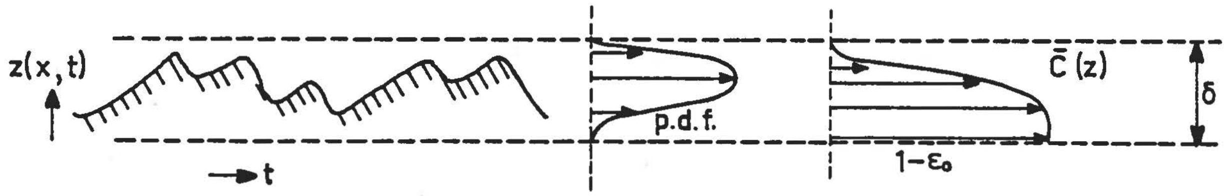


Fig. 1.3. The distribution of the averaged concentration in the transport-layer

Hence:

$$\bar{C}(z) = (1 - \varepsilon_o) \int_z^{\infty} f(\eta).d\eta \qquad (1.10)$$

in which f(z) is the p.d.f. of the instantaneous bedlevel.

1.4. One-dimensional form

The two-dimensional equation (1.5) can now be integrated in vertical direction to get the one-dimensional form. If the integrationboundaries are $-\infty$ and $+\infty$ it is not necessary to make use of the source/sink term for expressing the erosion/sedimentation term. It is therefore neglected in the following.

Integration of every term of Eq.(1.5) gives:

$$\int_{-\infty}^{+\infty} \frac{\partial \overline{C}_i}{\partial t} dz = \frac{\partial}{\partial t} \{ \int_{-\infty}^{+\infty} \overline{C}_i dz\} = \frac{\partial}{\partial t} \{ \int_{-\infty}^{0} \overline{C}_i dz + \int_{0}^{z_o} \overline{C}_i dz + \int_{z_o}^{z_1} \overline{C}_i dz\}$$

$z_1(x,t)$ = the upper boundary of the transportlayer = $= z_{max}(x,t)$. It will be assumed that below $z = 0$ no sedimentmovement occurs which means that the first integral can be dropped; it follows that:

$$\int_{-\infty}^{+\infty} \frac{\partial \overline{C}_i}{\partial t} . dz = \frac{\partial}{\partial t} \{ \int_{0}^{z_o} \overline{C}_i dz + \int_{z_o}^{z_1} \overline{C}_i dz\}$$

$$\int_{-\infty}^{+\infty} \frac{\partial \overline{u}_{p_i} . \overline{C}_i}{\partial x} dz = \frac{\partial}{\partial x} \{ \int_{-\infty}^{+\infty} \overline{u}_{p_i} . \overline{C}_i dz \qquad (1.11)$$

For $z < z_o$ there is no sedimentmovement.
For $z > z_1$ there is no sediment.
So it follows that:

$$\int_{-\infty}^{+\infty} \frac{\partial \overline{u}_{p_i} \overline{C}_i}{\partial x} = \frac{\partial}{\partial x} \{ \int_{z_o}^{z_1} \overline{u}_{p_i} . \overline{C}_i dz\} \qquad (1.12)$$

$$\int_{-\infty}^{+\infty} \frac{\partial \overline{w}_{p_i} . \overline{C}_i}{\partial z} dz = \left[\overline{w}_{p_i} . \overline{C}_i \right]_{-\infty}^{+\infty} = 0 \qquad (1.13)$$

$$\int_{-\infty}^{+\infty} \frac{\partial \overline{u'_{p_i} . C'_i}}{\partial x} dz = \frac{\partial}{\partial x} \{ \int_{-\infty}^{+\infty} \overline{u'_{p_i} . C'_i} dz\} = \frac{\partial}{\partial x} \{ \int_{z_o}^{z_1} \overline{u'_{p_i} . C'_i} dz\} \qquad (1.14)$$

$$\int_{-\infty}^{+\infty} \frac{\partial \overline{w'_{p_i} . C'_i}}{\partial z} dz = \left[\overline{w'_{p_i} . C'_i} \right]_{-\infty}^{+\infty} = 0 \qquad (1.15)$$

Summation of all the integrated terms (Eqs.(1.11)...(1.15)) gives:

$$\frac{\partial}{\partial t}\left\{\int_0^{z_o} \overline{C}_i dz\right\} + \frac{\partial}{\partial t}\left\{\int_{z_o}^{z_1} \overline{C}_i dz\right\} + \frac{\partial}{\partial x}\left\{\int_{z_o}^{z_1} \overline{u}_{pi}\overline{C}_i dz\right\} + \frac{\partial}{\partial x}\left\{\int_{z_o}^{z_1} \overline{u_{pi}'C_i'}\, dz\right\} = 0 \qquad (1.16)$$

In the following all four terms will be written in a simplified form.

The first term of Eq. (1.16) can be written as:

$$\frac{\partial}{\partial t}\left\{\int_0^{z_o} \overline{C}_i dz\right\} = \int_0^{z_o} \frac{\partial \overline{C}_i}{\partial t} dz + \overline{C}_{i_{z=z_o}} \frac{\partial z_o}{\partial t}$$

Because for $z \leq z_o$ no sedimentmovement occurs the first term of the right-hand side is zero; the following expression remains:

$$\frac{\partial}{\partial t}\left\{\int_0^{z_o} \overline{C}_i dz\right\} = \overline{C}_{i_{z=z_o}} \cdot \frac{\partial z_o}{\partial t} \qquad (1.17)$$

This is the erosion/sedimentation term (fraction i); $\overline{C}_{i_{z=z_o}}$ = sediment-concentration of fraction i on boundarylevel $z = z_o$.

The second term in Eq. (1.16) can be written as

$$\frac{\partial}{\partial t}\left\{\int_{z_o}^{z_1} \overline{C}_i dz\right\} = \frac{\partial}{\partial t}(\overline{\overline{C}}_i . \delta) \qquad (1.18)$$

in which $\overline{\overline{C}}_i(x,t)$ is the averaged value of $\overline{C}_i(x,z,t)$ over the transport-layer (vertical direction).

The third term of Eq. (1.16) represents the change in x-direction of that part of the sediment flux ϕ_{x_i} of fraction i, resulting from the average motion.

The fourth term represents another part of ϕ_{x_i}, resulting from the fluctuations.

Because $\phi_{x_i}(x,z,t) = \bar{u}_{p_i}(x,z,t).\bar{C}_i(x,z,t) + \overline{u_{p_i}'(x,z,t).C_i'(x,z,t)}$ the third and fourth term of Eq. (1.16) can be combined to:

$$\frac{\partial}{\partial x}\{\int_{z_o}^{z_1} \overline{u_{p_i}}.\bar{C}_i dz\} + \frac{\partial}{\partial x}\{\int_{z_o}^{z_1} \overline{u_{p_i}'.C_i'}dz\} = \frac{\partial}{\partial x}(\bar{\phi}_{x_i}.\delta) \qquad (1.19)$$

Substitution of Eqs. (1.17)...(1.19) in Eq. (1.16) gives:

$$\bar{C}_{i_{z=z_o}} . \frac{\partial z_o}{\partial t} + \frac{\partial}{\partial t}(\bar{\bar{C}}_i.\delta) + \frac{\partial}{\partial x}(\bar{\phi}_{x_i}.\delta) = 0 \qquad (1.20)$$

The transport of fraction i in real volume per unit width can be written as:

$$q_{s_i}(x,t) = \bar{\phi}_{x_i}(x,t).\delta(x,t) \qquad (1.21)$$

Substitution of Eq. (1.21) in Eq. (1.20) gives:

$$\boxed{\bar{C}_{i_{z=z_o}} \frac{\partial z_o}{\partial t} + \frac{\partial}{\partial t}(\bar{\bar{C}}_i.\delta) + \frac{\partial q_{s_i}}{\partial x} = 0} \qquad (1.22)$$

Through some simplifications this one-dimensional form of the continuity-equation of sedimentfraction i will be written in a form in which it will be used in the set of equations (see Section 1.4).

I.5. Simplifications of the one-dimensional form

Several assumptions can make Eq. (I.22) suitable for use in the set of equations.

The concentration of the sediment in the bedforms will be influenced by several factors:

1. The way of packing of the grains: It is plausible that the packing is different on either the front side or the leeside of a bedform.
2. The grainform
3. Different grain-sizes: In case of a sedimentmixture the pores between the large grains can be filled by small grains.

Despite these factors it is assumed that the sedimentconcentration in the bedforms is equal to $1 - \varepsilon_o$ with a constant porosity ε_o.

If ε_o is not a function of x and t, division of every term of Eq. (I. 22) by $1 - \varepsilon_o$ gives:

$$\frac{\bar{C}_{i_{z=z_o}}}{1 - \varepsilon_o} \frac{\partial z_o}{\partial t} + \frac{\partial}{\partial t} \left(\frac{\bar{\bar{C}}_i . \delta}{1 - \varepsilon_o}\right) + \frac{\partial s_i}{\partial x} = 0 \qquad i = 1,\ldots,n \qquad \text{(I.23)}$$

in which s_i is the transport of fraction i in volume (including pores) per unit width.

According to Eq. (1.3):

$$C_i(x,z,t) = p_i(x,z,t) \; . \; C(x,z,t)$$

In general both p_i and C are stochastic variables; a time-averaged value of C_i can be written in the usual way:

$$\bar{C}_i = \frac{1}{T}\int_0^T C_i dt = \overline{C.p_i} = \overline{(C + C')(p_i + p_i')} = \bar{C}\,.\bar{p}_i + \overline{C'.p_i'}$$

If it is assumed that:

1. The thickness of the instantaneous transportlayer on top of the bedforms is negligible with respect to the overall transportlayer thickness
2. In the bedforms the total sedimentconcentration is $1 - \varepsilon_o$ (ε_o = constant); (this was assumed before)

it can be stated that

(i) In the bedforms C does not fluctuate but p_i can.

(ii) In the troughs between the bedforms where is no sediment: $C = p_i = 0$.

As a direct consequence that: $\overline{C'p_i'} = 0$ and $\bar{C}_i = \bar{C}.\bar{p}_i$ (1.24)

Integration of $\bar{C}_i$ over the transportlayer gives:

$$\bar{\bar{C}}_i = \frac{1}{\delta}\int_{z_o}^{z_1} \bar{C}_i.dz = \frac{1}{\delta}\int_{z_o}^{z_1} \bar{C}.\bar{p}_i.dz = \overline{\bar{C}.\bar{p}_i}$$

Because in general $\bar{C}$ and $\bar{p}_i$ are not uniformly distributed over the transportlayer (vertical direction) it can be stated that:

$$\bar{\bar{C}}_i = \alpha_i \bar{\bar{C}}.\bar{\bar{p}}_i \qquad (1.25)$$

in which α_i = distributioncoefficient ($\alpha_i \neq 1$)

Substitution of Eqs. (1.24) and (1.25) in Eq. (1.23) gives

$$\frac{\bar{C}.\bar{p}_i\big|_{z=z_o}}{1 - \varepsilon_o}\frac{\partial z_o}{\partial t} + \frac{\partial}{\partial t}\left(\frac{\alpha_i \bar{\bar{C}}.\bar{\bar{p}}_i.\delta}{1 - \varepsilon_o}\right) + \frac{\partial s_i}{\partial x} = 0 \qquad i = 1,\ldots,n \quad (1.26)$$

According to Eq. (1.10), which was derived under identical assumptions (see remark of Section 1.3):

$$\bar{C}(z) = (1 - \varepsilon_o) \int_z^\infty f(\eta)d\eta$$

This can be written as:

$$\bar{C}(z) = (1 - \varepsilon_o)\beta(z) \qquad (1.27)$$

in which $\beta(z)$ is the cumulative p.d.f. of the instantaneous bed-level.

Averaging Eq. (1.27) over the transportlayer (vertical direction) gives:

$$\bar{\bar{C}}(z) = (1 - \varepsilon_o)\overline{\beta(z)} \qquad (1.28)$$

in which:

$$\overline{\beta(z)} = \frac{1}{\delta} \int_{z_o}^{z_1} \left(\int_z^\infty f(\eta)d\eta\right)dz \qquad (1.29)$$

which is the mean of the cumulative p.d.f. of the instantaneous bottom-level.

Substitution of Eqs. (1.27) and (1.28) in Eq. (1.26) gives:

$$\bar{p}_{i_{z=z_o}} \cdot \frac{\partial z_o}{\partial t} + \frac{\partial}{\partial t} (\alpha_i \overline{\beta(z)} . \bar{\bar{p}}_i . \delta) + \frac{\partial s_i}{\partial x} = 0 \qquad i=1,\ldots,n \quad (1.30)$$

If it is assumed that:

(i) $\overline{\beta(z)}.\delta$ = constant,i.e. the integrated cumulative p.d.f. of the instantaneous bottomlevel (see Eq. (1.29)) is constant; $\overline{\beta(z)}.\delta$ can be defined as an effective transportlayerthickness $\delta_{eff} = \overline{\beta(z)}.\delta = \delta.\overline{\overline{C}}(z)/1 - \varepsilon_0$.

(ii) α_i = constant, or the distributioncoefficient per fraction i is constant, which automatically means that $\alpha_i = 1$ (see remark),

the following equation results:

$$p_{i_{z_o}} . \frac{\partial z_o}{\partial t} + \overline{\beta(z)}.\delta \frac{\partial \overline{\overline{p}}_i}{\partial t} + \frac{\partial s_i}{\partial x} = 0 \qquad i = 1,\ldots,n \tag{1.31}$$

Remark:

In the following consideration it will be shown that assuption (ii) *viz.* a constant distributioncoefficient per fraction α_i also means that $\alpha_i = 1$ or all fractions are uniformly distributed over the transportlayer (vertical direction).

According to Eq. (1.25): $\overline{\overline{C}}_i = \alpha_i \overline{\overline{C}}.\overline{\overline{p}}_i$

Summation of this expression over all fractions gives:

$$\sum_{i=1}^{n} \overline{\overline{C}}_i = \overline{\overline{C}} = \sum_{i=1}^{n} \alpha_i . \overline{\overline{p}}_i . \overline{\overline{C}}$$

or:

$$\sum_{i=1}^{n} \alpha_i . \overline{\overline{p}}_i = 1 \tag{1.32}$$

This relation links all distributioncoefficients to each other. Also the sum of the probabilities of all fractions is equal to one, so:

$$\sum_{i=1}^{n} \bar{\bar{p}}_i = 1 \qquad (1.33)$$

Consideration of one single fraction (i) and the rest of the mixture as a whole (r) the Eqs. (1.32) and (1.33) can be translated to:

$$\alpha_i . \bar{\bar{p}}_i + \alpha_r \bar{\bar{p}}_r = 1 \qquad (1.34)$$

$$\bar{\bar{p}}_i + \bar{\bar{p}}_r = 1 \qquad (1.35)$$

in which $\bar{\bar{p}}_r$ and α_r are defined as:

$$\bar{\bar{p}}_r = \sum_{\substack{j=1 \\ j \neq i}}^{n} p_j \qquad \alpha_r = \frac{1}{\bar{\bar{p}}_r} \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_j . p_j$$

The only way for α_i and α_r to be constant or to be independent of a change of $\bar{\bar{p}}_i$ is being equal to one. Because fraction i could have been any fraction of the mixture $\alpha_i = 1$ for all fractions of the sediment mixture.

For two sedimentfractions (n=2) Eq. (1.31) can be written as:

$$\bar{p}_{1_{z_o}} \cdot \frac{\partial z_o}{\partial t} + \overline{\beta(z)} . \delta . \frac{\partial \bar{\bar{p}}_1}{\partial t} + \frac{\partial s_1}{\partial x} = 0 \qquad (1.36)$$

$$\bar{p}_{2_{z_o}} \cdot \frac{\partial z_o}{\partial t} - \overline{\beta(z)} . \delta . \frac{\partial \bar{\bar{p}}_1}{\partial t} + \frac{\partial s_2}{\partial x} = 0 \qquad (1.37)$$

In Section 1.3.2. $\overline{\beta(z)}$ will be written as β.

In the derivation of these equations the following assumptions have been made:

- Slow erosion/sedimentation: $\left|\frac{\partial \bar{z}}{\partial t}\right| << \left|\frac{\partial z'}{\partial t}\right|$
- Sedimentconcentration in the bed is constant = $1 - \varepsilon_o$.
- Instantaneous transportlayerthickness $\delta' << \delta$.
- Constant effective transportlayerthickness $\delta_{eff} = \overline{\beta(z)}.\delta$.
- Uniform distribution of every fraction over the transportlayer (δ) or $\alpha_i = 1$.

Appendix 2 Mathematical character and an interpretation of the condition $AB = C$

2.1. Mathematical character

A necessary condition for the elliptic or parabolic character of the set of p.d.e. is:

$$(A + B)^2 - 4C \leq 0$$

or:

$$4C \geq (A + B)^2$$

or:

$$4(C - AB) \geq (A - B)^2$$

Because the righthand side of this inequality is always positive a necessary but not sufficient condition for the elliptic or parabolic character is:

$$AB - C \leq 0 \qquad (2.1)$$

Substitution of the expressions for A, B and C (see Eqs. (32)...(34)) the following form can easily be derived:

$$(p_{2_{z_o}} \cdot \psi_1 - p_{1_{z_o}} \cdot \psi_2)(f_{1_{p_1}} + f_{2_{p_1}}) \leq 0 \qquad (2.2)$$

If it is assumed that the sedimenttransport of fraction i can be written as follows:

$$s_i = f_i(u, p_i) = m_i \cdot u^{n_i} \qquad (2.3)$$

in which: m_i = function of p_i and not a function of u
n_i = not a function of p_i and u.

(This assumption means that for two fractions with given grain diameters the roughness parameter C_t is assumed to be constant).

the derivatives f_{i_u} (i = 1,2) can be written as:

$$f_{1_u} = n_1 . f_1 / u$$

$$f_{2_u} = n_2 . f_2 / u$$

The ratio ψ_1/ψ_2 ($\cong$ ratio of transportconcentrations of both fractions) can then be written as:

$$\frac{\psi_1}{\psi_2} = \frac{f_{1_u}}{f_{2_u}} = \frac{n_1 . f_1}{n_2 . f_2} \simeq \frac{f_1}{f_2} \qquad \text{if } n_1 \simeq n_2 \tag{2.4}$$

If it is assumed that:

$$s_i = f_i(u, p_i) = \frac{1}{1 - \varepsilon_o} . u_{p_i} . C . p_i . \delta \tag{2.5}$$

in which u_{p_i}, C and δ are double-averaged variables and the distribution-coefficients are assumed to be equal to one(see App. 1) and furthermore it is assumed that δ, u_{p_i}, C and ε_o are not a function of p_i, the derivatives $f_{i_{p_1}}$ (i = 1,2) can be written as follows:

$$f_{1_{p_1}} = \frac{1}{1 - \varepsilon_o} . u_{p_1} . C . \delta$$

$$f_{2_{p_1}} = -\frac{1}{1 - \varepsilon_o} . u_{p_2} . C . \delta \tag{2.6}$$

Substitution of (2.4) and (2.6) in the condition (2.2) with the assumption that the composition at level $z_o(x,t)$ ($p_{i_{z_o}}$) is equal to the mean value of p_i in the transportlayer gives the following inequality:

$$\frac{p_1 . \psi_2 . C . \delta . u_{p2}}{1 - \varepsilon_o} \left(\frac{u_{p_1}}{u_{p2}} - 1\right)^2 \leq 0 \qquad (2.7)$$

This expression can never have a negative value which implies that the set of p.d.e. is always hyperbolic.

Remark: It should be realised that this is only a rough estimation with many assumptions made. A specific transportformula per sedimentfraction is necessary to give a more definite conclusion about the mathematical character of this set of p.d.e.

2.2. Interpretation of the condition AB = C

An (exact) condition for the approximated characteristics to be exact is: AB - C = 0. This means as is derived in the first part of this appendix that:

$$(p_{2_{z_o}} \psi_1 - p_{1_{z_o}} \psi_2)(f_{1_{p_1}} + f_{2_{p_1}}) = 0 \qquad (2.8)$$

Three cases can be considered:

either: $p_{2_{z_o}} \psi_1 - p_{1_{z_o}} \psi_2 = 0$ (2.9)

or: $f_{1_{p_2}} + f_{2_{p_1}} = 0$ (2.10)

or: both expressions are zero.

In Section 2.3.2. these equations are used as conditions for the approximated characteristic relations.

In the following the validity of these conditions is discussed.

x <u>Condition (2.9)</u>: $p_{2_{z_o}} \cdot \psi_1 - p_{1_{z_o}} \cdot \psi_2 = 0$

If it is assumed that $p_{i_{z_o}} \simeq p_i$ this can be written as:

$$\frac{\psi_1}{\psi_2} = \frac{p_1}{p_2} \tag{2.12}$$

This means that the ratio of the transportconcentration parameters of both fractions is equal to the ratio of the probabilities of both fractions in the transportlayer.
If the assumptions made in the first part of this appendix are used the expressions (2.4) and (2.5) can be combined to

$$\frac{\psi_1}{\psi_2} = \frac{p_1 \cdot u_{p_1}}{p_2 \cdot u_{p_2}} \tag{2.13}$$

Comparison of Eq. (2.12) and (2.13) learns that condition (2.9) is fulfilled better when the grain-velocities of both fractions approximate each other with generally means that the grain-diameters of both fractions approximate each other.
Another possibility for estimating the ratio ψ_1/ψ_2 is to use a specific simple transportformula per fraction.
In Chapter 1 a "basic hypothesis" is formulated (see Eq. 12) which will be combined in this case with the Engelund-Hansen formula (see also Ribberink, 1978).
The result is then:

$$s_i = f_i(u,p_i,D_i) = p_i \times 0{,}084 \frac{\sqrt{g}}{\Delta^2} (\mu Ri)^{5/2} \frac{1}{D_i}$$

The ratio of the derivatives with respect to u for both fractions becomes:

$$\frac{f_{1_u}}{f_{2_u}} = \frac{\psi_1}{\psi_2} = \frac{p_1 \cdot D_2}{p_2 \cdot D_1} \tag{2.14}$$

Again it can be seen that the grain-diameters have to approximate each other in order to satisfy condition (2.9).

* Condition (2.10) $f_{1_{p_1}} + f_{2_{p_1}} = 0$

If f is defined as the total sedimenttransport this condition can be written as:

$$\frac{\partial f}{\partial p_1} = 0$$

The "basic hypothesis" for a sedimenttransportformula per fraction is used again in its general form:

$$s_i = f_i(u,p_i) = p_i . f'_i$$

in which f'_i is the hypothetical transport of fraction i if the bed consists of this fraction only.
In Fig. 2.1. this formula is illustrated.

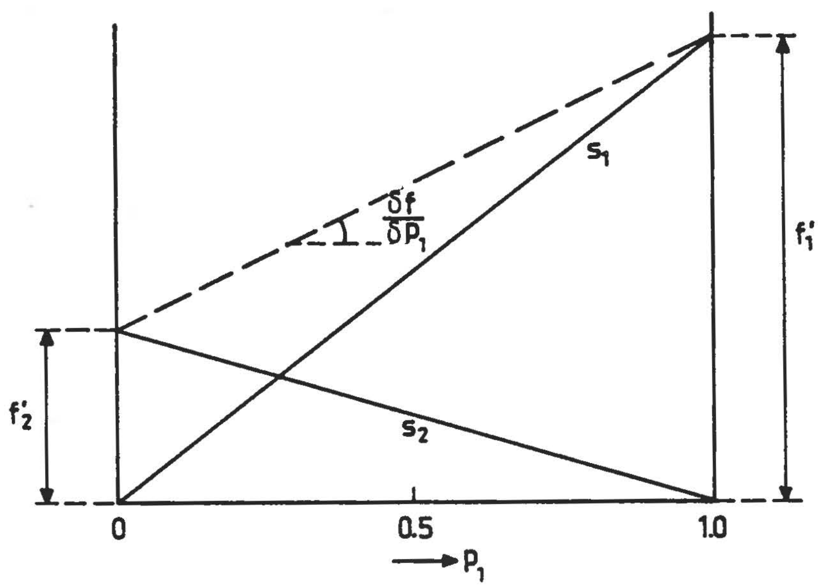


Fig. 2.1 "Basic hypothesis"

The derivative $\partial f/\partial p_1$ is approaching zero when $\partial f_1/\partial p_1 \rightarrow \partial f_2/\partial p_2$ which is identical to $f_1' \rightarrow f_2'$.
This result also means that condition (2.10) is satisfied better when the grain-diameters of both fractions approach each other.

It can be concluded from this consideration (including all its assumptions) that the condition AB = C (for the approximated characteristics and characteristic relations to be exact) is satisfied better when the graindiameters of both fractions approach each other.
It may be expected in that case that also both exact characteristic directions approach each other.

Appendix 3: A simple form of the approximated characteristic directions

Using some assumptions the expressions A and B (approximated characteristic directions) will be written in a simple form which can be interpreted more easily.
For Froude-numbers Fr << 1 expression B can be written as:

$$B = \frac{\psi_1 + \psi_2}{1 - Fr^2} \simeq \psi_1 + \psi_2$$

Because $\psi_i = f_{i_u}/a$, this becomes:

$$B \simeq (f_{1_u} + f_{2_u})/a = f_u/a \qquad (3.1)$$

Using a simple general transportformula:

$$f = m.u^n$$

in which m and n are no function of u, the derivative f_u can be written as:

$$f_u = m.n.u^{n-1} = n \cdot \frac{f}{u} \qquad (3.2)$$

Substitution of Eq. (3.2) into Eq. (3.1) gives:

$$B \simeq n \cdot \frac{f}{u.a} \qquad (3.3)$$

Analogous to the assumption made in Appendix 2 it will be assumed that the total transport can be written as:

$$f = \frac{1}{1 - \varepsilon_o} \cdot u_{p_m} \cdot C \cdot \delta \qquad (3.4)$$

in which: $u_{p_m} = \sum_{i=1}^{2} p_i . u_{p_i} = p_1 . u_{p_1} + p_2 . u_{p_2}$ (3.5)

Substitution of Eqs. (3.4) and (3.5) into Eq. (3.3) gives:

$$B \approx \frac{nC}{1 - \varepsilon_o} \cdot \frac{\delta}{a} \frac{(p_1 . u_{p_1} + p_2 . u_{p_2})}{u}$$

Remark: The parameters C, u_{p_i} and p_i are time-averaged as well as averaged over the transportlayer and should have been written formally as $\bar{\bar{C}}$, $\bar{\bar{u}}_{p_i}$ and $\bar{\bar{p}}_i$ (see Appendix 1).

With $C = (1 - \varepsilon_o) . \beta$ (see also Appendix 1) this becomes:

$$B \approx \beta \, . \, n \, . \, \frac{\delta}{a} \, . \, \frac{(p_1 . u_{p_1} + p_2 u_{p_2})}{u} \qquad (3.6)$$

If it is assumed that the transport per sedimentfraction can be written as (see also Appendix 2):

$$s_i = f_i = \frac{1}{1 - \varepsilon_o} \, . \, p_i \, . \, u_{p_i} \, . \, C \, . \, \delta \qquad (i = 1,2)$$

in which f_i is a linear function of p_i ("basic hypothesis") and δ, ε_o, u_{p_i} and C are no function of p_i, the expression A can also be simplified. The derivatives $f_{i_{p_1}}$ (i = 1,2) can then be written as:

$$f_{1_{p_1}} = \frac{1}{1 - \varepsilon_o} \, . \, u_{p_1} \, . \, C \, . \, \delta$$

$$f_{2_{p_1}} = - \frac{1}{1 - \varepsilon_o} \, . \, u_{p_2} \, . \, C \, . \, \delta$$

Substitution in the expression for A gives:

$$A = \frac{p_{2_{z_o}} \cdot f_{1_{p_1}} - p_{1_{z_o}} \cdot f_{2_{p_1}}}{\beta \delta u} \simeq \frac{C}{(1 - \varepsilon_o)\beta . u} (p_{2_{z_o}} \cdot u_{p_1} + p_{1_{z_o}} \cdot u_{p_2})$$

With $C = \beta(1 - \varepsilon_o)$ and the assumption that p_i at level $z = z_o$ is equal to the averaged value of p_i in the transportlayer this becomes:

$$\boxed{A \simeq \frac{(p_2 . u_{p_1} + p_1 . u_{p_2})}{u}} \qquad (3.7)$$

Appendix 4 The coefficients in the characteristic relations

4.1. Deviation of the approximated and exact characteristic directions

It is shown in Section 3.4.2. that the terms $\phi - A$ ($\phi = \phi_{1,2}$) and $\phi - B$ ($\phi = \phi_{1,2}$) are of importance for the estimation of the order of magnitude of the terms in both identical expressions for the characteristic relations. After a general derivation some possible approximations of these terms will be considered. The exact characteristic directions as derived in Section 2.2.1. can be written as:

$$\phi_{1,2} = \tfrac{1}{2} \{A + B \pm \left[(A - B)^2 + 4(AB - C)\right]^{1/2}\} \qquad (4.1)$$

with $\alpha = AB - C$ this becomes:

$$\phi_{1,2} = \tfrac{1}{2} \{A + B \pm \left[(A - B)^2 + 4\alpha\right]^{1/2}\} \qquad (4.2)$$

By assuming that:

$$(A - B)^2 + 4\alpha^{\ 1/2} = |A - B| + \beta \qquad (4.3)$$

Equation (4.2) becomes:

$$\phi_{1,2} = \tfrac{1}{2} \{A + B \pm \ |A - B| + \beta \} \qquad (4.4)$$

Equation (4.4) shows that $\beta/2$ is the deviation between the exact characteristic directions and their approximations.

Example: Suppose $A > B$ then Eq. (4.4) yields: $\phi_1 = A + \tfrac{1}{2}\beta$
$\phi_2 = B - \tfrac{1}{2}\beta$

The magnitude of $|\beta|$ with respect to A and B determines the accuracy of the app. char.'s.

With help of Eq. (4.3) α can be expressed in β:

$$\alpha = \tfrac{1}{2}\beta\ |A - B| + \tfrac{1}{4}\beta^2 \qquad (4.5)$$

The following relation is true also:

$$\alpha = AB - C$$

$$\alpha = \frac{f_{1p_1} + f_{2p_1}}{\delta u} \cdot \frac{p_2\psi_1 - p_1\psi_2}{(1 - Fr^2)}$$

or: $$\alpha = Fp \,.\, p\psi \qquad (4.6)$$

Substitution of Eq. (4.6) in Eq. (4.5) yields:

$$Fp . p\psi = \tfrac{1}{2}\beta \, |A - B| + \tfrac{1}{4}\beta^2 \qquad (4.7)$$

It is shown in Section 3.4.2. that also the terms Fp and $p\psi$ are important for the estimation of the terms in the char. rel.'s. Equation (4.7) yields a condition for the product of Fp and $p\psi$ when the values A, B and β are known.

<u>Example:</u> If A >> B it generally means that A and B are accurate approximations of ϕ_1 and ϕ_2 (see Section 2.2.3.). However, it is still possible that $|\beta|$ has the same order of magnitude as B. In that case according to Eq. (4.4) it follows that:

$$\phi_1 = A + \tfrac{1}{2}\beta \approx A + \tfrac{1}{2}B \approx A$$

$$\phi_2 = B - \tfrac{1}{2}\beta \approx B$$

and consequently A is an accurate approximation of ϕ_1 ; B has the same order of magnitude as ϕ_2. Equation (4.7) yields a condition for Fp . $p\psi$:

$$Fp \,.\, p\psi \approx \tfrac{1}{2}B \,.\, A$$

Another possibility is that $|\beta| << B$; again:

$$\phi_1 \approx A$$
$$\phi_2 \approx B$$

and Eq. (4.7) now becomes:

$$Fp \; p\psi \approx \tfrac{1}{2}A \,.\, \beta$$

In the next part of this Appendix the behaviour of Fp and $p\psi$ will be studied.

4.2. The terms Fp and pψ

According to Eq. (4.6) the expressions Fp and pψ can be written as:

$$Fp = \frac{f_{1_{p_1}} + f_{2_{p_1}}}{\delta u} \qquad p\psi = \frac{p_2\psi_1 - p_1\psi_2}{1 - Fr^2}$$

The numerators of these expressions were discussed in Appendix 2. According to this consideration both terms become smaller when the grain-diameters and/or grain-velocities of both fractions approximate each other. With a specific transportformula per fraction like M.P & M including Eg.'s theory,Fp and pψ become smaller when:

$$D_1/D_2 \rightarrow 1$$
$$\tau_{e_{*i}} >> \tau_{c_{*i}}$$

The last condition is of importance because despite a small difference in grain-diameters near 'initiation of motion' of the coarse fraction there probably is a large difference between the behaviour of both fractions (e.g. grain-velocities).

In the following the magnitudes of Fp and pψ will be considered with respect to A and B.

1. *Magnitude of Fp with respect to A*

It will be assumed in this estimation that the 'basic-hypothesis' can be used as a transportformula per fraction (see Fig. 4.1).

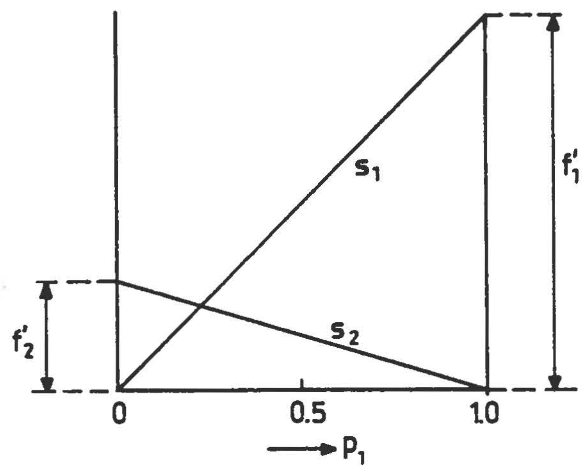


Fig. 4.1 the 'basic-hypothesis': $s_i = p_i \cdot f'_i$.

In that case:

$$f_{1_{p_1}} = \frac{\partial s_1}{\partial p_1} = f_1'$$

$$f_{2_{p_1}} = \frac{\partial s_2}{\partial p_1} = - f_2'$$

and A and Fp can be written as:

$$A = \frac{p_2\ f_1' - p_1 f_2'}{\delta u} \qquad (4.8)$$

$$Fp = \frac{f_1' - f_2'}{\delta u} \qquad (4.9)$$

Three cases will be distinguished:

1. $|f_{1_{p_1}}| \simeq |f_{2_{p_1}}|$

   This case is reached when $D_1/D_2 \rightarrow 1$ and $\tau_{e_{*_2}} >> \tau_{c_{*_2}}$. Equation (4.8) and (4.9) can be approximated by:

$$A = \frac{p_2 f_1' + p_1 f_1'}{\delta u} = \frac{f_1'}{\delta u}$$

$$Fp \rightarrow 0$$

   which means that : Fp << A.
2. $|f_{1_{p_1}}| > |f_{2_{p_1}}|$

   A small shift takes place to : $D_1/D_2 < 1$ and $\tau_{e_{*_2}} > \tau_{c_{*_2}}$. In this case A and Fp have the same order of magnitude: $Fp \approx A$.
3. $|f_{1_{p_1}}| >> |f_{2_{p_1}}|$.

   This case can be reached for very large grain-diameter differences ($D_1/D_2 << 1$) and the coarse fraction near 'initiation of motion' ($\tau_{e_{*_2}} \rightarrow \tau_{c_{*_2}}$).
   Two extreme cases can be distinguished:
   $p_1 \rightarrow 0$ $\quad Fp \simeq A \simeq f_1'/\delta u$
   $p_1 \rightarrow 1$ $\quad Fp >> A \simeq f_2'/\delta u$

   Apparently Fp can be smaller, equal and larger than A. The choice $Fp \approx A$ seems to be a representative estimate.

2. Magnitude of $p\psi$ with respect to B

Because ψ_1 (= f_{1_u}/a) and ψ_2 (= f_{2_u}/a) always have a positive sign, $p\psi$ is always smaller than B:

$$p\psi = \frac{p_2\psi_1 - p_1\psi_2}{1 - Fr^2} < B = \frac{\psi_1 + \psi_2}{1 - Fr^2}$$

If $D_1/D_2 \rightarrow 1$ and $\tau_{e_{x_2}} >> \tau_{c_{x_2}}$ both fractions will behave rather similar; according to the 'basic hypothesis':

$$s_i = p_i \cdot f_i'$$

and:

$$\frac{s_1}{s_2} = \frac{p_1}{p_2}\frac{f_1'}{f_2'}$$

The quotient ψ_1/ψ_2 becomes:

$$\frac{\psi_1}{\psi_2} = \frac{f_{1u}}{f_{2u}} = \frac{p_1}{p_2}\frac{f_{1u}'}{f_{2u}'}$$

If both fractions behave rather similar this becomes:

$$\frac{\psi_1}{\psi_2} \approx \frac{p_1}{p_2}$$

or in other words, $p\psi$ appproaches zero and:

$$|p\psi| << B$$

When this situation gradually shifts to $D_1/D_2 << 1$ and $\tau_{e_{x_2}} \rightarrow \tau_{c_{x_2}}$ it is possible that $\psi_2 >> \psi_1$, because near initiation of motion a small velocity change has a large influence on the transport (large power of the M.P & M formula near initiation of motion!)

In this situation two cases can be distinguished:

$$p_1 \rightarrow 0 \qquad |p\psi| \approx \frac{\psi_1}{1 - Fr^2} << B \approx \frac{\psi_2}{1 - Fr^2}$$

$$p_1 \rightarrow 1 \qquad |p\psi| \approx \frac{\psi_2}{1 - Fr^2} \approx B$$

From this consideration two choices for a representative estimate of $p\psi$ will be made:

1. $|p\psi| \approx B$
2. $|p\psi| << B$

Appendix 5 CPS-computerprogramme for the determination of dimensionless characteristic directions and relations

The exact and approximated characteristic directions and the char.rel.'s are:

ex. char.'s : $\phi_{1,2} = \frac{1}{2}\{A + B \pm [(A - B)^2 + 4(AB - C)]^{1/2}\}$ (5.1)

app.char.'s : $A = (p_2 f_{1_{p_1}} - p_1 f_{2_{p_1}})/\delta u$ (5.2)

char. rel.'s: $\frac{dp_1}{dt} \cdot \frac{f_{1_{p_1}} + f_{2_{p_1}}}{\delta u} + \frac{dz_0}{dt} \cdot (\phi - A) = (\phi - A)\frac{uR}{g} \cdot B +$

$+ \frac{uR}{g}(AB - C)$ (5.3)

$\phi = \phi_{1,2}$

Equation (5.3) is one of the two possible char.rel.'s which are available. With the help of some substitutions (see Sections 3.3.2. and 3.4.3.) Eqs. (5.1)...(5.3) can be written in a different dimensionless form:

ex.char.'s : $\overline{\phi}_{1,2} = \frac{1}{2}\{\overline{A} + \overline{B} \pm [(\overline{A} - \overline{B})^2 + 4(\overline{AB} - \overline{C})]^{1/2}\}$

$$\overline{\phi}_{1,2} = \frac{\phi_{1,2}(1 - \varepsilon_0)a.u}{\sqrt{\Delta g\, D_1^3}}$$

app.char.'s : $\overline{A} = \frac{A(1 - \varepsilon_0)a.u}{\sqrt{\Delta g\, D_1^3}} = \left(p_2 \cdot qp_1 - p_1 \cdot qp_2\left(\frac{D_2}{D_1}\right)^{3/2}\right) \cdot \frac{a}{\delta}$

$$\overline{B} = \frac{B(1 - \varepsilon_0)a.u}{\sqrt{\Delta g\, D_1^3}} = \left(qu_1 + qu_2\left(\frac{D_2}{D_1}\right)^{3/2}\right) \cdot \frac{1}{1 - Fr^2}$$

char.rel.'s: $dp_1 + \frac{1}{\delta} \cdot dz_0 \cdot \frac{\delta}{a} \cdot (zi) = dt \,.\, R' \,.\, (Ri)$

$i = 1,2$

with: $R' = \frac{R\sqrt{\Delta g\, D_1^3}}{g(1 - \varepsilon_0)a^2}$

$$zi = \frac{\overline{\phi}_i - \overline{A}}{\left(qp_1 + \left(\frac{D_2}{D_1}\right)^{3/2} qp_2\right)} \quad \text{along } \overline{\phi}_i = \overline{\phi}_{1,2}$$

$$Ri = \frac{(\overline{\phi}_i - \overline{A}).\overline{B}}{\left(qp_1 + \left(\frac{D_2}{D_1}\right)^{3/2} qp_2\right)} + \frac{\left(p_2\, qu_1 - p_1\left(\frac{D_2}{D_1}\right)^{3/2} qu_2\right)}{\delta/a \,.\, (1 - Fr^2)}$$

along $\overline{\phi}_i = \overline{\phi}_{1,2}$

With the help of a transportformula per fraction and values of $\tau_{e_{*1}}$, D_1/D_2, p_1, δ/a and Fr the ex.char.'s $\overline{\phi}_{1,2}$, the app. char.'s $\overline{A}$ and $\overline{B}$ and the coefficients z_1, z_2, R_1 and R_2 of the char. rel.'s can be calculated.

In the computer-programme the formula of M.P & M including Egiazaroff's theory has been used. The required input-parameters, their synonyms in the programme and the number of these parameters which can be processed every run are shown in Table 5.1.

Fr	f (line number 340)	1
a/δ	h (line number 320)	1
$\tau_{e_{*1}}$	tau	5
D_1/D_2	rd	5
p_1	p	6

Table 5.1 Input-parameters

The synonyms of the output-parameters are shown in Table 5.2.

$\overline{\phi}_1$	01
$\overline{\phi}_2$	02
$\overline{A}$	A
$\overline{B}$	B
z1	z1
z2	z2
R1	R1
R2	R2

Table 5.2 Output-parameters

A listing of the CPS-programme is given below.

```
          DECLARE tau(5),rd(5),p(6);
          GET LIST(tau);
          GET LIST(rd);
          GET LIST(p);
          PUT IMAGE('o1','o2','A','B','z1','z2','R1','R2')(Im1);
          DO ld=1 TO 5;
          PUT LIST('d1/d2=',rd(ld));
          DO lt=1 TO 5;
          PUT LIST('taue1=',tau(lt));
          DO lp=1 TO 6;
          arg1=19/(p(lp)+(1-p(lp))/rd(ld));
          arg2=19/(p(lp)*rd(ld)+1-p(lp));
          tauc1=.1/log10(arg1)**2;
          tauc2=.1/log10(arg2)**2;
          r1=tau(lt)-.768*tauc1;
          IF r1<0 THEN GO TO con3;
con1:     ap1=8*r1**1.5+8.01*p(lp)*(1-rd(ld))/(p(lp)*rd(ld)+1-p(lp))*tauc1/log10(arg1)*r1**.5;
          au1=24*p(lp)*tau(lt)*r1**.5;
          r2=rd(ld)*tau(lt)-.768*tauc2;
          IF r2>=0 THEN GO TO con2;
con3:     o1=.999E33;
          o2=.999E33;
          a=.999E33;
          b=.999E33;
          ze1=.999E33;
          ze2=.999E33;
          Re1=.999E33;
          Re2=.999E33;
          GO TO a2;
con2:     ap2=-8*r2**1.5+8.01*(1-p(lp))*(1-rd(ld))/(p(lp)*rd(ld)+1-p(lp))*tauc2/log10(arg2)*r2**.5;
          au2=24*(1-p(lp))*rd(ld)*tau(lt)*r2**.5;
          h=1.9;
          a=h*((1-p(lp))*ap1-p(lp)*ap2*(1/rd(ld))**1.5);
          f=.347;
          b=(au1+au2*(1/rd(ld))**1.5)/(1-f**2);
          c=h/(1-f**2)*(au2*ap1-au1*ap2)*(1/rd(ld))**1.5;
          w1=(a-b)**2;
          w2=4*(a*b-c);
          v1=sqrt(w1);
          v2=sqrt(w1+w2);
          z1=ap1+ap2*(1/rd(ld))**1.5;
          z2=(1-p(lp))*au1-p(lp)*au2*(1/rd(ld))**1.5;
          o1=(a+b)/2+v2/2;
          o2=(a+b)/2-v2/2;
          ze1=(o1-a)/z1;
          ze2=(o2-a)/z1;
          Re1=((o1-a)*b+z1*z2*h/(1-f**2))/z1;
          Re2=((o2-a)*b+z1*z2*h/(1-f**2))/z1;
a2:       PUT IMAGE(o1,o2,a,b,ze1,ze2,Re1,Re2)(Im2);
          END ;
          PUT LIST('');
          END ;
          END ;
          PUT LIST('');
Im2:      IMAGE;
--,----- --,----- --,----- --,----- ----,----- ----,----- ----,----- ----,-----
Im1:      IMAGE;
--        --      -       -       --         --         --         --
          STOP ;
```

Appendix 6 Propagation of the front and the tail-wave

It will be shown in this Appendix that the front and the tail-wave propagate along straight characteristics without damping or amplification with celerity:

1. $c = c_1$ in the front
2. $c = c_2$ in the tail.

In Fig. 6.1 the x-t diagram is shown and the different areas can be seen.

The front-wave:

According to the initial condition point 2 is identical to point 1. Because point 4 is determined by the characteristics originating from these points, point 4 is identical to point 1 also.

Point 3 satisfies the characteristic relation (with $c = c_2$) originating from point 1 but then also the characteristic relation (with $c = c_2$) from point 4. Ofcourse point 3 also satisfies the characteristic relation ($c = c_1$) originating from point 3 itself. Result: Point 5, satisfying the characteristic relation originating from point 3 and 4 must be identical to point 3.

In other words: The front-wave propagates without damping or amplification with celerity $c = c_1$.

The tail-wave:

An analogous consideration can be given for the tail-wave. The only difference is that a minus-sign must be added to indicate the point-numbers and c_1 and c_2 must be replaced by c_2 and c_1 respectively.

Result: The tail-wave propagates without damping or amplification with celerity $c = c_2$.

After the interaction-period point 6 is reached and the front and tail-wave are intersecting. As a result p_1 and z_0 of point 6 are constant along both characteristics originating from point 6 and a *temporarily equilibrium situation* develops with $z_{0_{eq}} = z_{0_6}$ and $p_{1_{eq}} = p_{1_6}$.

Remark: It must be realised that this specific propagation of front and tail in only possible because of the simplification of the watermotion in the mathematical model:

1. Neglecting the friction-term R (→ 0) results in characteristic relations with a zero right-hand side (so no damping or amplification during one step of the calculation-procedure).

2. Assuming small Froude-numbers (Fr << 1, no backwatercurve ∂h/∂x = 0) and a fixed waterlevel (∂h/∂t = 0) causes that the characteristic directions and relations are completely determined by z_0 and p_1 only. Consequently the characteristics, along which z_0 and p_1 are constant, will be straight lines.

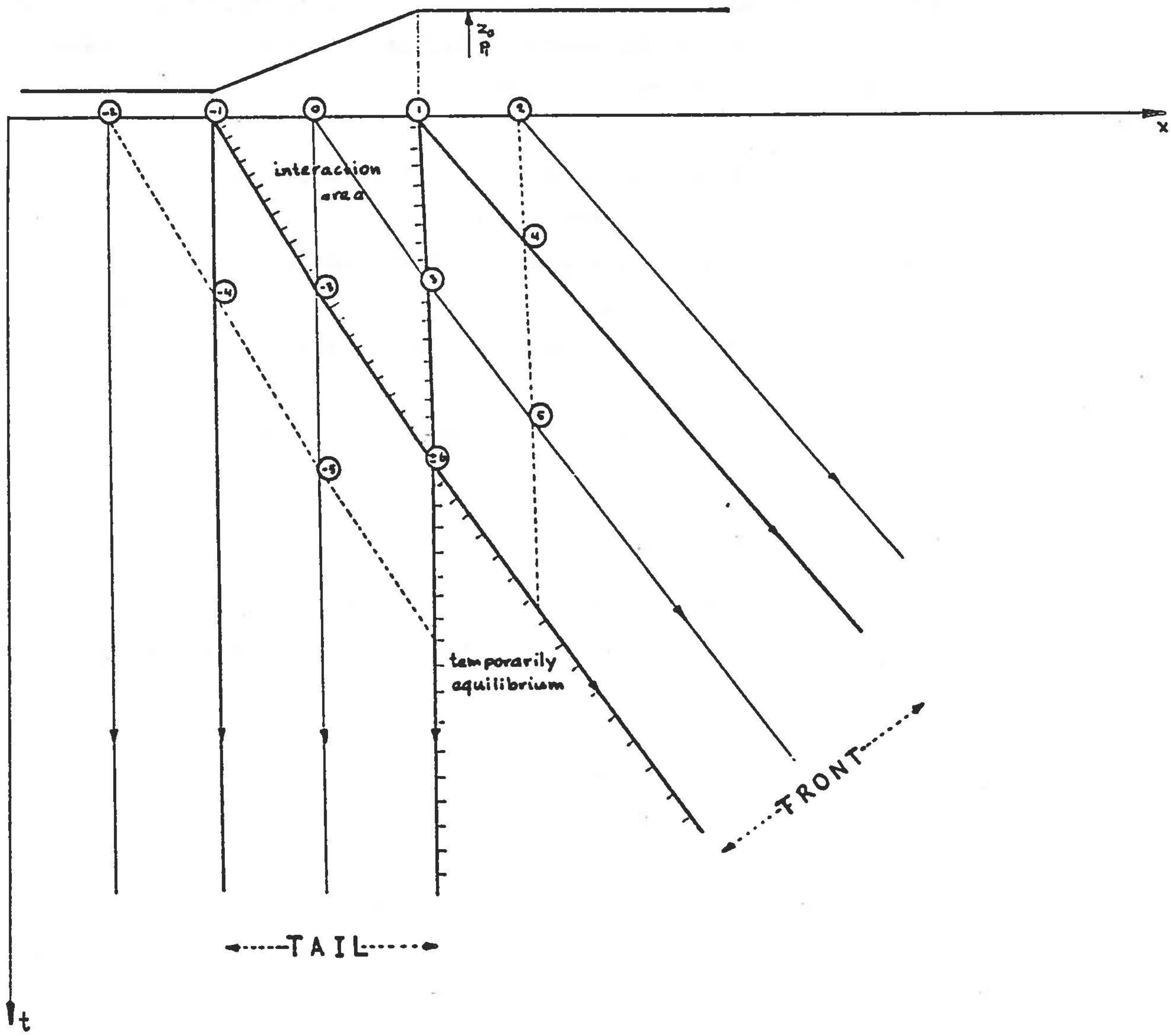


Fig.6.1.Propagation in the x-t plane

Appendix 7 Tables of Chapter 3 .

D_1/D_2	$\tau_{e_{*i}}$	X_{1_E}	X_{2_E}	X_1	X_2
0.2	0.06	-	0.0171		
	0.1	-	0.0764		
	0.2	0.0499	0.3167	↓	
	0.4	0.4303	1.0510		
	1.0	2.5631	4.5381		
0.6	0.06	-	0.0177	0.0047	0.0071
	0.1	0.0257	0.0673	0.0390	0.0586
	0.2	0.1679	0.3019	0.1915	0.2873
	0.4	0.6348	1.0287	0.6711	1.0067
	1.0	2.917	4.5018	2.9771	4.4656
1.0	0.06	0.0047	0.0071		
	0.1	0.0390	0.0586	↑	
	0.2	0.1915	0.2873		
	0.4	0.6711	1.0067		
	1.0	2.9771	4.4656		

Table 7.1. The dimensionless transport of fraction 1 (fine) and fraction 2 with/without Egiazaroff's theory as a function of $\tau_{e_{*i}}$ for different values of D_1/D_2 and for p_1=0.4 (Fig.4 and Fig.5)

p_1	X_{1_E}	X_{2_E}	X_1	X_2
0	0	0.0977	0	0.0977
0.2	0.0108	0.0839	0.0195	0.0781
0.4	0.0257	0.0673	0.0390	0.0586
0.6	0.0450	0.0479	0.0586	0.0390
0.8	0.0689	0.0254	0.0781	0.0195
1.0	0.0977	0	0.0977	0

Table 7.2. The dimensionless transport of fraction 1 and 2 with/without Egiazaroff's theory as a function of p_1 for $\tau_{e_{*i}}$=0.1 , D_1/D_2=0.6 (Fig.6)

$\tau_{e_{*1}}$	$\bar{\phi}_1$	$\bar{\phi}_2$	$\bar{A}$	$\bar{B}$
0.06	-	-	-	-
0.1	0.359	0.029	0.173	0.215
0.2	1.866	1.384	1.602	1.659
0.4	7.180	5.502	7.112	5.570
1.0	35.084	23.828	35.059	23.852

Table 7.3. Dimensionless approximated and exact characteristic directions as a function of $\tau_{e_{*i}}$ for Fr=0.2 , a/δ = 5, D_1/D_2=0.4 and p_1=0.4 (Fig.7)

Fr	a/	1	2	$\bar{A}$	$\bar{B}$
0.0	1		0.031	0.040	0.470
	5	0.529	0.140	0.199	"
	10	0.634	0.234	0.398	"
	25	1.140	0.326	0.995	"
	50	2.108	0.352	1.990	"
0.2	1	0.499	0.031	0.040	0.490
	5	0.548	0.141	0.199	"
	10	0.650	0.238	0.398	"
	25	1.148	0.337	0.995	"
	50	2.114	0.366	1.990	"
0.4	1		0.031	0.04	0.560
	5	0.615	0.144	0.199	"
	10	0.708	0.249	0.398	"
	25	1.181	0.374	0.995	"
	50	2.197	0.528	1.990	"
0.6	1	0.743	0.031	0.040	0.735
	5	0.786	0.147	0.199	"
	10	0.864	0.268	0.398	"
	25	1.275	0.455	0.995	"
	50	2.197	0.528	1.990	"
0.8	1	1.314	0.031	0.040	1.306
	5	1.353	0.152	0.199	"
	10	1.412	0.292	0.398	"
	25	1.692	0.609	0.995	"
	50	2.457	0.839	1.990	"

Table 7.4. Dimensionless exact and approximated characteristic directions as a function of Fr and a/δ for $\tau_{e_{*1}}$=0.1 , D_1/D_2=0.4 and p_1=0.8 (Fig.10 and Fig.11) .

	$\tau_{e_{*1}}$	0.1				1.0			
D_1/D_2	p_1	$\bar{\Phi}_1$	$\bar{\Phi}_2$	$\bar{A}$	$\bar{B}$	$\bar{\Phi}_1$	$\bar{\Phi}_2$	$\bar{A}$	$\bar{B}$
0.2	0.0	-	-	-	-	27.107	21.868	27.107	21.868
	0.2	-	-	-	-	30.393	22.235	30.313	22.315
	0.4	-	-	-	-	32.033	22.661	31.854	22.840
	0.6	-	-	-	-	32.770	23.174	32.557	23.387
	0.8	-	-	-	-	33.250	23.776	33.108	23.918
	1.0	0.576	0.003	0.003	0.576	34.256	24.406	34.256	24.406
0.4	0.0	-	-	-	-	34.210	23.486	34.210	23.486
	0.2	-	-	-	-	34.719	23.651	34.704	23.667
	0.4	0.359	0.029	0.173	0.215	35.084	23.828	35.059	23.852
	0.6	0.482	0.084	0.185	0.382	35.368	24.104	35.343	24.040
	0.8	0.548	0.141	0.199	0.489	35.638	24.208	35.621	24.225
	1.0	0.576	0.226	0.226	0.576	35.973	24.406	35.973	24.406
0.6	0.0	0.368	0.221	0.221	0.368	35.946	24.001	35.946	24.001
	0.2	0.436	0.237	0.257	0.417	36.106	24.079	36.103	24.082
	0.4	0.489	0.259	0.287	0.461	36.246	24.159	36.242	24.163
	0.6	0.528	0.286	0.312	0.502	36.376	24.240	36.371	24.245
	0.8	0.557	0.318	0.335	0.540	36.503	24.322	36.499	24.325
	1.0	0.576	0.359	0.359	0.576	36.638	24.406	36.638	24.406
0.8	0.0	0.508	0.382	0.382	0.508	36.747	24.255	36.747	24.255
	0.2	0.526	0.390	0.394	0.522	36.798	24.285	36.798	24.285
	0.4	0.542	0.400	0.406	0.536	36.849	24.315	36.848	24.315
	0.6	0.555	0.411	0.416	0.549	36.898	24.345	36.898	24.346
	0.8	0.567	0.423	0.427	0.563	36.946	24.375	36.946	24.376
	1.0	0.576	0.437	0.437	0.576	36.995	24.406	36.995	24.406
1.0	0.0								
	0.2								
	0.4	0.576	0.489	0.489	0.576	37.215	24.406	37.215	24.406
	0.6								
	0.8								
	1.0								

Table 7.5. Dimensionless exact and approximated characteristic directions as a function of p_1 and D_1/D_2 for two values of $\tau_{e_{*1}}$, Fr=0.2 and a/δ =5 . (Fig.8 and Fig.9)

D_1/D_2	p_1	$\overline{\phi}_1$	$\overline{\phi}_2$	$\overline{A}$	$\overline{B}$
0.4	0.0	2.394	1.436	2.394	1.436
	0.2	2.269	1.375	2.105	1.540
	0.4	2.163	1.297	1.815	1.644
	0.6	2.076	1.198	1.526	1.748
	0.8	2.008	1.081	1.237	1.852
	1.0	1.956	0.948	0.948	1.956
0.6	0.0	2.394	1.744	2.394	1.744
	0.2	2.300	1.741	2.254	1.786
	0.4	2.208	1.736	2.115	1.829
	0.6	2.119	1.728	1.976	1.871
	0.8	2.033	1.717	1.837	1.913
	1.0	1.956	1.698	1.698	1.956
0.8	0.0	2.394	1.879	2.394	1.879
	0.2	2.347	1.887	2.339	1.894
	0.4	2.298	1.897	2.286	1.910
	0.6	2.247	1.910	2.232	1.925
	0.8	2.191	1.927	2.178	1.940
	1.0	2.123	1.956	2.123	1.956
1.0	0.0				
	0.2				
	0.4	2.394	1.956	2.394	1.956
	0.6				
	0.8				
	1.0				

Table 7.6. Dimensionless exact and approximated characteristic directions as a function of p_1 and D_1/D_2 for Fr=0.2 , a/δ =5 and $\tau_{e\ast_1}$=0.2 , without Egiazaroff's theory (Fig.12 and Fig.13)

Literature

Antsyferov, S.M. (1973) - Computation of the transport of sediment of non-uniform particle-size composition, Oceanology 13 (1973), 3, pp. 394-401.

Ashida, K. and Michiue, M. (1973) - Studies on bedload transportrate in open channel flows, Symp. IAHR Bangkok, Jan. 1973.

Chollet, J.P. and Cunge, J.A. (1979) - New interpretation of some head loss-flow velocity relationships for deformable movable beds, Jour. of Hyd. Res. 17, no. 1, 1979.

Chollet, J.P. and Cunge, J.A. (1980) - Simulation of unsteady flow in alluvial streams, Int. Symp. on River Sedimentation, March 24-29 1980, Chinese Soc. of Hydr. Eng., Beijing, China.

Egiazaroff, Par. I. (1965) - Calculation of non-uniform sediment concentrations, Proc. ASCE, HY4, July 1965.

Einstein, H.A. (1950) - The bed-load function for sediment transportation in open channel flows, US Soil Conservation Service, Tech. Bulletin no. 1025, sept. 1950.

Einstein, H.A. and Ning Chien (1953) - Transport of sediment mixtures with large ranges of grain sizes, Univ. of California, Miss. Riv. Division, Sediment Series no. 2.

Exner, F.M. (1925) - Über die Wechselwirkung zwischen Wasser und Geschiebe in Flüssen, Sitzber. Akad. Wiss. Wien, pt. IIa, Bd. 134.

Jansen, P.Ph. (Ed.) (1979) - Principles of riverengineering, Pitman, London, 1979.

Kalinske, A.A. (1947) - Movement of sediment as bed-load in rivers, Trans. Am. Geophysical Union, vol. 28, no. 4.

Kerssens, P.J.M. (1974) - Adaptation length of the vertical suspended sand profiles, (in Dutch), Delft University of Technology, Dept. of Civil Eng., Aug. 1974.

Kerssens, P.J.M., Prins, A. and Van Rijn, L.C. (1979) - Model for suspended sediment transport, Proc. A.S.C.E., Journ. of Hyd. Div., vol. 105, no. HY5, May 1979.

Misri, R.L., Garde, R.J. and Ranga Raju, K.G. (1980) - Bed load transport of non-uniform sediment, Proc. IAHR Belgrade 1980.

Meyer-Peter, E. and Müller, R. (1948) - Formulas for bedload transport, Proc. IAHR, Stockholm, 1948.

Pantalopoulos, J. (1957) - Etude experimentale du mouvement par charriage de fond d'un mélange de materiaux; recherches sur la similitude au charriage, Proc. IAHR, Lisbon 1957.

Ribberink, J.S. (1978) - Bed-load formulae for non-uniform sediment, Delft University of Technology, Dept. of Civil Engineering, Fluid Mechanics Group, Internal Report no. 4-78.

Ribberink, J.S. (1978) - Basic equations for sediment transport (in Dutch) Delft University of Technology, Dept. of Civil Engineering, Fluid Mechanics Group, Internal Report no. 3-78.

Sengupta, S. (1975) - Size-sorting during suspension-transportation; lognormality and other characteristics, Sedimentology 22, pp. 257-273, 1975.

Suzuki, K. (1976) - On the propagation of a disturbance in the bed composition of an open channel, Delft University of Technology, Dept. of Civil Eng., Fluid Mechanics Group, Report R/1976/09/L.

Vreugdenhil, C.B. (1981) - Numerical effects in models for river morphology, to be published in a book in homage to A. Preissmann, Pitman, London, 1981.

Vries, M. de (1965) - Considerations on non-steady bed-load transport in open channels, Proc. IAHR Leningrad, pp. 3.8.1.-3.8.8. (also Delft Hydr. Lab. Publ. no. 36).

Vries, M. de (1976) - Morphological computations, Lecture notes, Delft University of Technology, Dept. of Civil Eng., Fluid Mechanics Group.

Willes, J.C. and Kennedy, J.F. (1977) - Sediment discharge of alluvial streams calculated from bed-form statistics, Iowa Inst. of Hydr. Research, Report no. 202, The University of Iowa, june 1977.