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Research paper

Nonlinear fractional-periodic boundary value problems with Hilfer fractional derivative: Existence and numerical approximations of solutions

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A B S T R A C T

We prove conditions for existence of analytical solutions for boundary value problems with the Hilfer fractional derivative, generalizing the commonly used Riemann-Liouville and Caputo operators. The boundary values, referred to in this paper as fractional-periodic, are fractional integral conditions generalizing recurrent solution values for the non-Caputo case of the Hilfer fractional derivative. Analytical solutions to the studied problem are obtained using a perturbation of the corresponding initial value problem with enforced boundary conditions. In general, solutions to the boundary value problem are singular for $t \downarrow 0$. To overcome this singularity, we construct a sequence of converging solutions in a weighted continuous function space. We present a Bernstein spline-based implementation to numerically approximate solutions. We prove convergence of the numerical method, providing convergence criteria and asymptotic convergence rates. Numerical examples show empirical convergence results corresponding with the theoretical bounds. Moreover, the method is able to approximate the singular behavior of solutions and is demonstrated to converge for nonlinear problems. Finally, we apply a grid search to obtain correspondence to the original, non-perturbed system.

1. Introduction

Fractional calculus generalizes classical differentiation and integration to non-integer orders, offering tools to model memory and hereditary effects in dynamical systems [1,2]. Fractional differential equations (FDEs), which involve derivatives of order $\alpha \in \mathbb{R}_{>0}$, are extensively used in fields such as control theory, viscoelasticity and anomalous diffusion [3–7]. In particular, fractional boundary value problems (FBVPs) study the solutions to nonlocal FDEs with nonlocal boundary constraints. Of a particular interest are FBVPs in which solutions attain the same values on all boundaries of the given domain. This is due to the fact that the solutions to FDEs are in general shown to be non-periodic [8–10]. In what follows, we will denote this class of FBVPs as *fractional-periodic* BVPs.

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Two of the most commonly used fractional differential operators are those of the Riemann-Liouville and the Caputo type, as introduced and discussed extensively in e.g. [1,5]. The Caputo fractional derivative provides two beneficial properties for applications in physical systems: it allows for integer-order initial or boundary conditions and it satisfies the integer-order intuition that constant functions are in the operator kernel set of the fractional derivative. Numerous results exist in literature on existence and uniqueness for fractional-periodic BVPs using Caputo derivatives [11–13], including methods for numerical solutions [14]. The Riemann-Liouville operator on the other hand requires fractional-order initial or boundary conditions, but naturally satisfies the property of acting as a left-inverse of the canonical Riemann-Liouville fractional integral operator [5]. Because of the fractional order initial and boundary values, which generally yield singular solutions for nonzero initial values, a common choice for Riemann-Liouville BVPs is a zero boundary value for the left point of the time-domain, with corresponding existence and uniqueness results presented in literature [15,16].

The Hilfer fractional derivative of order $\alpha \in (0,1)$ and type $\beta \in [0,1]$ as introduced by Hilfer [6], interpolates between the Riemann-Liouville ($\beta = 0$) and Caputo fractional derivatives ($\beta = 1$). When applied to FDEs, this operator provides a choice for the fractional order of the initial or boundary conditions and the solution behavior, parametrized by β. We remark that for values of $\beta \in (0,1)$, solution behavior is generally singular as $t \downarrow 0$, where $t = 0$ is the initial point of the time domain for nonzero initial or boundary values. This is similar to the Riemann-Liouville case ($\beta = 0$) as mentioned above. Since β can be seen as an interpolation parameter, the Hilfer fractional derivative can also be applied to homotopy-like methods, taking for instance a Caputo fractional derivative solution as starting point for a solution to the corresponding Riemann-Liouville problem. Moreover, the choice of the generalized Hilfer fractional order offers an additional advantage by interpolating between different memory structures. In particular, the Hilfer derivative enables a choice of the type of *intermediate or mixed memory effects*: no direct *memory effect* of the solution on the initial condition results in the Caputo case of $\beta = 1$, with a stronger solution dependency on the initial condition for lower values of β. This offers a greater modeling flexibility compared to commonly used fractional derivative operators.

Existence and uniqueness for Cauchy-like initial value problems (IVPs) using Hilfer fractional derivatives is proven by Furati et al. [17]. Numerous literature exists on boundary value systems with Hilfer fractional derivatives. Several papers study analytical and numerical results for systems with a zero boundary value at the left boundary [18–23], including fractional-periodic zero-value BVPs [24,25], and their extensions to fractional partial differential equations (FPDEs) [26]. A non-periodic but nonzero left boundary analytical existence and uniqueness result is given by Wang and Zhang [27], where the right boundary point is proportional to the solution, whereas the left boundary point is of fractional order.

To the best of our knowledge, there are no analytical results that prove existence of solutions to Hilfer derivative fractional-periodic BVPs for nonzero boundary values whenever $\beta \in [0,1)$. Moreover, we have found no numerical methods to approximate solutions for the singular solution behavior for Hilfer fractional derivative FBVPs with nonzero left boundary values.

To fill this gap, we present existence conditions for analytical solutions of nonlinear fractional-periodic Hilfer derivative BVPs and construct a numerical method to obtain solution approximations for the full range of boundary values and $\beta \in [0,1]$. For a detailed problem formulation of the FBVP, we refer to Section 3.1. In real-world applications, the FBVP system can be used to model e.g. population dynamics capturing memory effects of for instance past events, where β determines the strength of this memory-effect. In this context, the right hand-side $\mathbf{f}$ can be used to model the cumulative quantitative effects of for example pandemics or migration, whereas the boundary conditions model their seasonal effects (see discussions in [28–30]).

In this work we prove existence of analytical solutions to the FBVP under study with corresponding convergence criteria by following a perturbed IVP approach developed and extended from the thesis work [31]. To construct numerical solutions which accurately capture the singular behavior near $t \downarrow 0$, we utilize a Bernstein splines approach, building on the ε-shifted time domain strategy as used in [32] for Hilfer fractional IVPs. Splines-based methods bring the advantage of providing high accuracy for FDEs by the closed form solution of the fractional integral of their basis functions. Although our method can easily be extended to different choices of polynomial splines basis functions, we remark that the choice of Bernstein basis functions is motivated by their numerical stability properties in equidistantly spaced grids [33]. For a more detailed motivation of the Bernstein-splines numerical method and a comparison to the predictor-corrector method of Diethelm et al. [34], we refer to the results on Hilfer fractional IVPs [32].

Our results generalize existing analytical and numerical results for the classical Caputo fractional derivative case [12,14], allowing for greater flexibility in choice and interpolation of boundary value type and their memory effects on solution. Besides this advantage that our approach offers, we also show explicit convergence requirements and solution convergence for the developed numerical method, with corresponding empirical convergence results for the challenging to model singular solutions. To our knowledge, this provides a novel algorithm to accurately obtain singular solutions for the full range of solutions with singularities for Hilfer fractional derivatives.

The paper is organized as follows. Preliminary results on fractional calculus and Bernstein splines approximation are presented in Section 2. In Section 3, we present the main problem statement of the FBVP existence results for analytical solutions using the perturbed IVP approach. We then present numerical approximations and respective error analysis in Section 4. Section 5 is devoted to numerical results, showing convergence order results for a polynomial example in Section 5.1 and simulation results for a nonlinear system in Section 5.2. Finally, we present conclusions and suggestions for future work in Section 6.

2. Preliminary results

2.1. Fractional calculus

Definition 1 (Left-sided Riemann-Liouville fractional integral, [35]). For $\alpha \geq 0$, $a \leq t \leq b$ and a function $f \in L^p[a,b]$ with $1 \leq p \leq \infty$, define the Riemann-Liouville fractional integral as:

$$ {}_a\mathrm{I}_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s. $$

For additional notational clarity, we will use ${}_a\mathrm{I}_t^\alpha\{f(s)\}(t) := {}_a\mathrm{I}_t^\alpha f(t)$ throughout this paper in cases where confusion about the variable of integration can occur.

Moreover, we will denote ${}_a\mathrm{I}_c^\alpha f(t) := {}_a\mathrm{I}_t^\alpha\{\mathbb{1}_{[a,c]} f(s)\}(t)$ whenever $c \neq t$ with $c > a$, to specify a local domain of integration.

Proposition 2.1 (Semigroup property of fractional integration, [35]). *For $\alpha, \alpha' > 0$, if $f \in L^p[a,b]$ with $1 \leq p \leq \infty$, we have:*

$$ ({}_a\mathrm{I}_b^\alpha\, {}_a\mathrm{I}_b^{\alpha'}) f(t) = {}_a\mathrm{I}_b^{\alpha+\alpha'} f(t), \tag{1} $$

almost everywhere for $t \in [a,b]$. Furthermore, if $\alpha + \alpha' > 1$, equality (1) holds point-wise.

Definition 2 (Beta function, [35]). For $a, b > 0$, denote the (complete) beta function as:

$$ \mathrm{B}(a,b) = \int_0^1 \vartheta^{a-1}(1-\vartheta)^{b-1}\,\mathrm{d}\vartheta. $$

Furthermore, it holds that $\mathrm{B}(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, where $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\,\mathrm{d}t$ denotes the special gamma function.

Definition 3 (Incomplete beta function, [36]). For $a, b > 0$ and $0 \leq z \leq 1$, denote the incomplete beta function as:

$$ \mathrm{B}_z(a,b) = \int_0^z \vartheta^{a-1}(1-\vartheta)^{b-1}\,\mathrm{d}\vartheta. $$

Proposition 2.2 (Fractional integral of a monomial, [1]). *For $\alpha > 0, k > -1$, the α-fractional integral of a monomial of degree k is given by*

$$ {}_0\mathrm{I}_t^\alpha t^k = \frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)} t^{\alpha+k} = \frac{t^{\alpha+k}}{\Gamma(\alpha)} \mathrm{B}(k+1,\alpha). $$

Proposition 2.3 (Fractional integral of monomial with left-local support, [32]). *For $\alpha > 0$, $k > -1$ and $0 < b \leq t$, the α-fractional integral on $(0,b]$ of a monomial of degree k is given by*

$$ {}_0\mathrm{I}_b^\alpha t^k = \frac{t^{\alpha+k}}{\Gamma(\alpha)} \mathrm{B}_{b/t}(k+1,\alpha). $$

Proposition 2.4 (Fractional integral of monomial with right-local support, [32]). *For $\alpha > 0$, $k > -1$ and $0 \leq b \leq t$, the α-fractional integral on $[b,t]$ of a monomial of degree k is given by*

$$ {}_b\mathrm{I}_t^\alpha t^k = \frac{t^{\alpha+k}}{\Gamma(\alpha)} \mathrm{B}_{1-b/t}(\alpha,k+1). $$

We define fractional differentiation in the Hilfer fractional derivative sense as follows:

Definition 4 (Hilfer fractional derivative, [6]). For a function $f : [0,\infty) \to \mathbb{R}$ satisfying ${}_0\mathrm{I}_t^{(1-\beta)(1-\alpha)} f \in C^1[0,\infty)$, the Hilfer fractional derivative of order $0 < \alpha < 1$ and type $0 \leq \beta \leq 1$ for $t \geq 0$ is given as:

$$ {}_0\mathrm{D}_t^{\alpha,\beta} f(t) = {}_0\mathrm{I}_t^{\beta(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} {}_0\mathrm{I}_t^{(1-\beta)(1-\alpha)} f(t). $$

Remark 1 (Interpolation of Caputo and Riemann-Liouville derivative). The Hilfer fractional derivative interpolates two of the most commonly used fractional derivative operators:

$$ {}_0\mathrm{D}_t^{\alpha,\beta} f(t) = \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} {}_0\mathrm{I}_t^{1-\alpha} f(t), & \text{for} \quad \beta = 0, \quad \text{(Riemann-Liouville derivative)}, \\ {}_0\mathrm{I}_t^{1-\alpha} f(t) \frac{\mathrm{d}}{\mathrm{d}t}, & \text{for} \quad \beta = 1, \quad \text{(Caputo derivative)}. \end{cases} $$

To obtain solutions to Hilfer FDEs, Furati et al. [17] introduce the following *weighted space of continuous functions*:

$$ C_{1-\gamma}[0,T] = \{ f : (0,T] \to \mathbb{R} : t^{1-\gamma} f \in C[0,T] \}, $$

together with the corresponding norm

$$ \|f\|_{C_{1-\gamma}[0,T]} = \sup_{t \in (0,T]} |t^{1-\gamma} f(t)|. $$

We note that $C_{1-\gamma}[0,T]$ is complete with the choice of this norm. Furthermore, Furati et al. [17] consider the following fractionally differentiable subspace:

$$ C_{1-\gamma}^{\vartheta}[0,T] = \{ f \in C_{1-\gamma}[0,T],\ {}_0\mathrm{D}_t^{\vartheta,0} f(t) \in C_{1-\gamma}[0,T] \}. $$

Now, consider the following IVP

$$\begin{cases} {}_0\mathrm{D}_t^{\alpha,\beta}\mathbf{x}(t) = \mathbf{f}(t,\mathbf{x}(t)), & 0 < t \leq T, \quad 0 < \alpha < 1, \quad 0 \leq \beta \leq 1, \\ {}_0\mathrm{I}_0^{1-\gamma}\mathbf{x}(0) = \tilde{\mathbf{x}}_0, & \gamma = \alpha + \beta - \alpha\beta. \end{cases} \tag{2}$$

The following existence and uniqueness result holds:

Theorem 2.1 (Existence and uniqueness of solutions to IVPs [17])**.** *If $\mathbf{f} : (0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies:*

1. $\mathbf{f}(\cdot, \mathbf{x}(\cdot)) \in C_{1-\gamma}^{\beta(1-\alpha)}[0,T]$ *for any* $\mathbf{x} \in C_{1-\gamma}[0,T]$*;*
2. $\mathbf{f}$ *is Lipschitz in its second argument, i.e., for some nonnegative matrix* $K \in \mathbb{R}_{\geq 0}^{d\times d}$ *the inequality holds:*

$$|\mathbf{f}(t,\mathbf{u}(t)) - \mathbf{f}(t,\mathbf{v}(t))| \leq K|\mathbf{u}(t) - \mathbf{v}(t)|, \tag{3}$$

for all $t \in (0,T]$, $\mathbf{u}(t), \mathbf{v}(t) \in G \subset \mathbb{R}^d$.

Then, there exists a unique solution $\mathbf{x} \in C_{1-\gamma}^{\gamma}[0,T]$ *to IVP (2). Furthermore,* $\mathbf{x}$ *is a solution to (2) if and only if it satisfies the equivalent integral equation:*

$$\mathbf{x}(t) = \frac{\tilde{\mathbf{x}}_0}{\Gamma(\gamma)} t^{\gamma-1} + {}_0\mathrm{I}_t^{\alpha}\mathbf{f}(t,\mathbf{x}(t)), \quad t > 0.$$

Proof. The proof follows from Theorem 23 and 25 of Furati et al. [17], applied component-wise for vector-valued functions in $\mathbb{R}^d$. □

2.2. Polynomial splines approximation

Definition 5 (Bernstein polynomial operator [37])**.** For a function $f : [a,b] \to \mathbb{R}$, the Bernstein operator of polynomial order $q \in \mathbb{N}$ is given as:

$$B^q f(t) = \sum_{j=0}^{q} f(a + j(b-a)/q) \binom{q}{j} \left(\frac{t-a}{b-a}\right)^j \left(\frac{b-t}{b-a}\right)^{q-j}.$$

Definition 6 (Modulus of continuity [37])**.** For a continuous function $f : [a,b] \to \mathbb{R}$, the modulus of continuity ω is defined as:

$$\omega(f;\delta) = \sup_{|t-s|\leq\delta} |f(t) - f(s)|, \quad \text{for } t,s \in [a,b].$$

Theorem 2.2 (Bernstein operator error bound, [37])**.** *For a continuous function* $f : [a,b] \to \mathbb{R}$*, the following error estimate holds:*

$$|f(t) - B^q f(t)| \leq \frac{5}{4}\,\omega\!\left(f; \frac{(b-a)}{\sqrt{q}}\right), \quad \textit{for all } t \in [a,b].$$

Definition 7 (Knot collection)**.** Given a closed interval $[a,b] \subset \mathbb{R}$, we define a knot collection $\mathcal{A}$ of size $k+1$ as the collection of disjoint intervals:

$$\mathcal{A} = \{A_0, \dots, A_k\} = \{[t_0,t_1), \dots, [t_{k-1},t_k), [t_k,t_{k+1}]\},$$

such that $\bigcup_{A_i\in\mathcal{A}} A_i = [a,b]$.

Definition 8 (Bernstein spline operator, [32])**.** For a function $f : [a,b] \to \mathbb{R}$, and a knot collection $\mathcal{A}$ of size $k+1$, we define the order q Bernstein spline operator as:

$$S_{\mathcal{A}}^q f(t) = \sum_{i=0}^{k} \mathbb{1}_{A_i} B^q\{\mathbb{1}_{\bar{A}_i} f\}(t),$$

where $\bar{A}_i$ denotes the closure of A_i.

Moreover, we note that $S_{\mathcal{A}}^q$ as defined above is a continuous operator [32].

3. Existence of analytical solutions

In this section, we first present the main problem statement of the fractional-periodic BVP (4) (see Section 3.1). In Section 3.2, we introduce the corresponding perturbed IVP, which will be used to obtain existence of solutions in Section 3.3. Finally, in Section 3.4, the correspondence of the perturbed IVP solutions to the original BVP is presented. We present a schematic overview of the structure of the results of this section in Fig. 1.

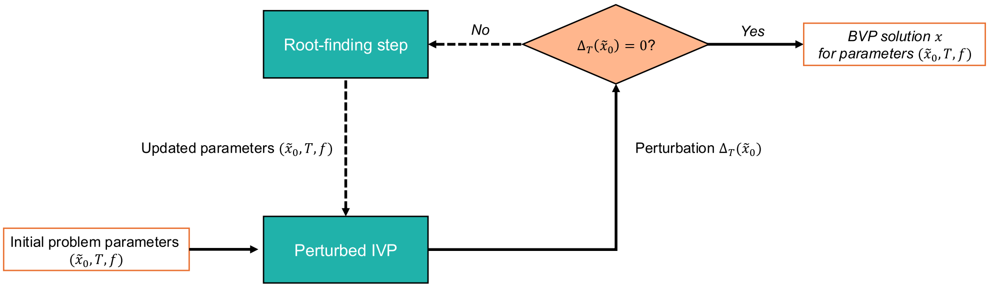


Fig. 1. A schematic overview of the perturbed IVP approach to obtain analytical solutions as presented in Sections 3.2–3.4.

3.1. Boundary value problem statement

Our main problem statement is given by the following fractional-periodic BVP:

$$\begin{cases} {}_0\mathrm{D}_t^{\alpha,\beta}\mathbf{x}(t) = \mathbf{f}(t,\mathbf{x}(t)), & 0 < t < T, \quad 0 < \alpha < 1, \quad 0 \le \beta \le 1, \\ {}_0\mathrm{I}_0^{1-\gamma}\mathbf{x}(0) = {}_0\mathrm{I}_T^{1-\gamma}\mathbf{x}(T), & \gamma = \alpha + \beta - \alpha\beta. \end{cases} \tag{4}$$

Here, $\mathbf{x} : (0,T] \mapsto \mathbb{R}^d$ are the unknown solutions to the system characterized by the nonlinear vector-valued function $\mathbf{f} : (0,T] \times \mathbb{R}^d \mapsto \mathbb{R}^d$. The fractional differential operator ${}_0\mathrm{D}_t^{\alpha,\beta}$ represents the Hilfer fractional derivative of order α and type β, and ${}_0\mathrm{I}_t^{1-\gamma}$ the Riemann-Liouville fractional derivative where we use the convention ${}_0\mathrm{I}_0^{\alpha} f(0) := \lim_{t\downarrow 0} {}_0\mathrm{I}_t^{\alpha} f(t)$. We remark that, unless otherwise specified, the operations ${}_0\mathrm{I}_t^{\alpha}$, ${}_0\mathrm{D}_t^{\alpha,\beta}$, $|\cdot|$, $\|\cdot\|$ and $\le$ are applied component-wise in this paper for vector-valued functions.

In general, because of the nonlinearity of $\mathbf{f}$, it is challenging to obtain apriori guarantees on existence of solutions to (4) for a given set of problem parameters. Hence, to obtain solutions satisfying the boundary conditions, we start from perturbations of the corresponding initial value problem (IVP) on which we enforce the boundary conditions, similar to the existing results on Caputo fractional derivative problems [12,14]. We remark that these perturbed IVP solutions are valid solutions of (4) for a perturbed $\mathbf{f}$. Then, through root-finding, we can obtain the problem parameters for which the perturbed IVP solution is a solution to the original BVP problem formulation and parameters. This approach is outlined in the following sections, with a schematic overview in Fig. 1.

3.2. The perturbed initial value problem

We take a perturbation $v \in \mathbb{R}^d$ of the IVP corresponding to (4):

$$\begin{cases} {}_0\mathrm{D}_t^{\alpha,\beta}\mathbf{x}(t) = \mathbf{f}(t,\mathbf{x}(t)) + v, & 0 < t < T, \\ {}_0\mathrm{I}_0^{1-\gamma}\mathbf{x}(0) = \tilde{\mathbf{x}}_0. \end{cases} \tag{5}$$

We first prove the conditions for which the perturbed IVP (5) satisfies the boundary conditions of BVP (4).

Lemma 3.1 (Perturbed IVP with boundary conditions). *Consider the perturbed IVP (5). Assume the existence and uniqueness requirements of Theorem 2.1 hold for a given* $\mathbf{f}$ *and initial condition* $\tilde{\mathbf{x}}_0$*. Then, solutions* $\mathbf{x} \in C_{1-\gamma}[0,T]$ *of (5) satisfy the boundary conditions of BVP (4) if and only if*

$$v = -\Gamma(\zeta+1)T^{-\zeta}\,{}_0\mathrm{I}_T^{\zeta}\mathbf{f}(T,\mathbf{x}(T)) =: \Delta_T(\tilde{\mathbf{x}}_0), \tag{6}$$

where $\zeta := 1 - \gamma + \alpha$*. Furthermore, the above holds if and only if* $\mathbf{x}$ *satisfies*

$$\mathbf{x}(t) = \frac{\tilde{\mathbf{x}}_0}{\Gamma(\gamma)}t^{\gamma-1} + {}_0\mathrm{I}_t^{\alpha}\mathbf{f}(t,\mathbf{x}(t)) - \frac{\Gamma(\zeta+1)t^{\alpha}}{\Gamma(\alpha+1)T^{\zeta}}\,{}_0\mathrm{I}_T^{\zeta}\mathbf{f}(T,\mathbf{x}(T)). \tag{7}$$

Proof. See Appendix A. □

This result allows us to consider solutions of the perturbed IVP (5) in order to find solutions to BVP (4). To obtain these solutions, we construct the following iterative sequence for integral Eq. (7):

$$\begin{aligned} \mathbf{x}_0(t,\tilde{\mathbf{x}}_0) &= \frac{\tilde{\mathbf{x}}_0}{\Gamma(\gamma)}t^{\gamma-1}, \\ \mathbf{x}_{m+1}(t,\tilde{\mathbf{x}}_0) &= \mathbf{x}_0(t,\tilde{\mathbf{x}}_0) + \{\mathcal{T}\mathbf{f}(t,\mathbf{x}_m(t,\tilde{\mathbf{x}}_0))\}(t), \end{aligned} \tag{8}$$

where $\mathcal{T}$ is a linear operator defined as

$$\{\mathcal{T}\mathbf{y}\}(t) = {}_0\mathrm{I}_t^{\alpha}\mathbf{y}(t) - \frac{\Gamma(\zeta+1)t^{\alpha}}{\Gamma(\alpha+1)T^{\zeta}}\,{}_0\mathrm{I}_T^{\zeta}\mathbf{y}(T), \quad 0 \le t \le T, \tag{9}$$

with $\zeta := 1 - \gamma + \alpha$. First, we establish boundedness of the operator $\mathcal{T}$, which will be used to prove the main convergence result of solutions in Theorem 3.1. Finally, in Theorem 3.2, we show the connection to the original BVP (4).

Lemma 3.2 (Boundedness of $\mathcal{T}$). *Let $\mathbf{y} \in C_{1-\gamma}[0,T]$. Then, for all $t \in (0,T]$ it holds that*

$$|t^{1-\gamma}\{\mathcal{T}\mathbf{y}\}(t)| \le \xi(t)\|\mathbf{y}\|_{1-\gamma},$$

where

$$\xi(t) := \frac{\Gamma(\gamma)t^{\alpha}}{\Gamma(\gamma+\alpha)} + \frac{(t/T)^{\zeta}T^{\alpha}}{\Gamma(\alpha+1)}\left[(1-\gamma-\alpha)\mathrm{B}_{t/T}(\gamma,\zeta) + \zeta\mathrm{B}_{1-t/T}(\zeta,\gamma)\right].$$

Here, B *denotes the (incomplete) beta function as introduced in Definitions 2 and 3. Furthermore, taking the supremum gives*

$$\|\mathcal{T}\mathbf{y}\|_{C_{1-\gamma}[0,T]} \le \Xi\|\mathbf{y}\|_{C_{1-\gamma}[0,T]}, \quad \textit{where} \quad \Xi := \sup_{t\in(0,T]}\xi(t).$$

Proof. See Appendix B. □

3.3. *Existence of solutions to the perturbed IVP*

We will now introduce the notation and assumptions for our main analytical result of Theorem 3.1.

For $D \subseteq \mathbb{R}^d$ closed and bounded, we denote

$$X_D[0,T] = \{\mathbf{u} : (0,T] \to \mathbb{R}^d,\ \mathbf{u} \in C_{1-\gamma}[0,T] : t^{1-\gamma}\mathbf{u}(t) \in D\}.$$

Consider the following conditions for BVP (4) and its corresponding perturbed IVP (5):

A.1 We have $\frac{\tilde{\mathbf{x}}_0}{\Gamma(\gamma)} \in D$ and

$$\left\{\mathbf{u} \in C_{1-\gamma}[0,T] : \left\|\mathbf{u} - \frac{\tilde{\mathbf{x}}_0 t^{\gamma-1}}{\Gamma(\gamma)}\right\|_{C_{1-\gamma}[0,T]} \le \Xi\mathbf{m}\right\} \subseteq X_D[0,T],$$

with $\Xi \in \mathbb{R}$ satisfying $\Xi = \sup_{t\in(0,T]}\xi(t)$ with

$$\xi(t) = \frac{\Gamma(\gamma)t^{\alpha}}{\Gamma(\gamma+\alpha)} + \frac{(t/T)^{\zeta}T^{\alpha}}{\Gamma(\alpha+1)}\left[(1-\gamma-\alpha)\mathrm{B}_{t/T}(\gamma,\zeta) + \zeta\mathrm{B}_{1-t/T}(\zeta,\gamma)\right],$$

where $\zeta = 1-\gamma+\alpha$ and B denotes the (incomplete) beta function as introduced in Definitions 2 and 3.

A.2 The function $\mathbf{f} : (0,T]\times\mathbb{R}^d \mapsto \mathbb{R}^d$ satisfies $\mathbf{f} \in C_{1-\gamma}[0,T]$ and there exists some nonnegative $\mathbf{m} \in \mathbb{R}^d$ such that

$$||t \mapsto \mathbf{f}(\cdot,\mathbf{u})||_{1-\gamma} \le \mathbf{m}$$

for all $\mathbf{u} \in X_D[0,T]$. Furthermore, for all $t \in (0,T]$ and $\mathbf{u},\mathbf{v} \in X_D[0,T]$, we have that $\mathbf{f}$ satisfies the Lipschitz condition (3) in the second coordinate with coefficients $K \in \mathbb{R}^d_{\ge 0}$.

A.3 The matrix Q defined as

$$Q := \Xi K,$$

satisfies the spectral radius requirement

$$\rho(Q) < 1.$$

We use assumptions **A.1** - **A.3** as formulated above to obtain our main analytical results:

Theorem 3.1 (Existence and uniqueness of solutions). *Assume that conditions **A.1** - **A.3** are satisfied. Then, it holds that:*

C.1 *Functions of the sequence (8) are in $C_{1-\gamma}[0,T]$ and satisfy the fractional-periodic boundary conditions of (4):*

$${}_0\mathrm{I}_0^{1-\gamma}x_m(0,\tilde{\mathbf{x}}_0) = {}_0\mathrm{I}_T^{1-\gamma}x_m(T,\tilde{\mathbf{x}}_0).$$

C.2 *The sequence of functions (8) converges in $C_{1-\gamma}[0,T]$ as $m\to\infty$ for $t\in[0,T]$ to the limit function*

$$\mathbf{x}(t,\tilde{\mathbf{x}}_0) := \lim_{m\to\infty}\mathbf{x}_m(t,\tilde{\mathbf{x}}_0),$$

with the following error bound:

$$\|\mathbf{x}_m(t,\tilde{\mathbf{x}}_0) - \mathbf{x}(t,\tilde{\mathbf{x}}_0)\|_{C_{1-\gamma}[0,T]} \le Q^m(I-Q)^{-1}\Xi\mathbf{m}. \tag{10}$$

C.3 *If furthermore $\mathbf{f} \in C_{1-\gamma}^{\beta(1-\alpha)}[0,T]$, the limit function satisfies $\mathbf{x} \in C_{1-\gamma}^{\gamma}[0,T]$ and is the unique solution of integral Eq. (7), which is the corresponding equivalent integral equation to the perturbed IVP (5) for*

$$v = \Delta_T(\tilde{\mathbf{x}}_0) = -\Gamma(\zeta+1)T^{-\zeta}\,{}_0\mathrm{I}_T^{\zeta}\mathbf{f}(T,\mathbf{x}(T)).$$

C.4 *The limit function satisfies the fractional-periodic boundary conditions of (4):*

$${}_0\mathrm{I}_0^{1-\gamma}\mathbf{x}(0,\tilde{\mathbf{x}}_0) = {}_0\mathrm{I}_T^{1-\gamma}\mathbf{x}(T,\tilde{\mathbf{x}}_0).$$

Proof. Statement **C.1** follows from the construction of our approximating sequence (8). Since $\mathbf{x}_0 \in C_{1-\gamma}[0,T]$, Lemma 11 of [17] gives that $\mathbf{x}_m \in C_{1-\gamma}[0,T]$ for all $m \geq 1$. For the boundary conditions, we observe that by the properties of sequence (8) we have

$$\begin{aligned}
&{}_0\mathrm{I}_t^{1-\gamma}\mathbf{x}_m(t,\tilde{\mathbf{x}}_0)\\
&= {}_0\mathrm{I}_t^{1-\gamma}\left\{\frac{\tilde{\mathbf{x}}_0 s^{\gamma-1}}{\Gamma(\gamma)} + {}_0\mathrm{I}_s^{\alpha}\mathbf{f}(s,\mathbf{x}_{m-1}(s,\tilde{\mathbf{x}}_0)) - \frac{\Gamma(\zeta+1)s^{\alpha}}{\Gamma(\alpha+1)T^{\zeta}}\,{}_0\mathrm{I}_T^{\zeta}\mathbf{f}(T,\mathbf{x}_{m-1}(T,\tilde{\mathbf{x}}_0))\right\}(t)\\
&= \tilde{\mathbf{x}}_0 + {}_0\mathrm{I}_t^{\zeta}\mathbf{f}(t,\mathbf{x}_{m-1}(t,\tilde{\mathbf{x}}_0)) - (t/T)^{\zeta}\,{}_0\mathrm{I}_T^{\zeta}\mathbf{f}(T,\mathbf{x}_{m-1}(T,\tilde{\mathbf{x}}_0)).
\end{aligned}$$

And hence, as in Lemma 3.1, direct computation shows that we have satisfied the boundary conditions ${}_0\mathrm{I}_0^{1-\gamma}\mathbf{x}_m(0,\tilde{\mathbf{x}}_0) = {}_0\mathrm{I}_T^{1-\gamma}\mathbf{x}_m(T,\tilde{\mathbf{x}}_0)$.

To prove statement **C.2**, we show that the sequence $(\mathbf{x}_m)_{m\geq 1}$ as defined in Eq. (8) is a Cauchy sequence in the Banach space $C_{1-\gamma}[0,T]$ equipped with the norm $\|\cdot\|_{C_{1-\gamma}[0,T]}$, thus showing convergence to the limit function $\mathbf{x}(t,\tilde{\mathbf{x}}_0)$. First, we aim to prove that

$$\mathbf{x}_m(t,\tilde{\mathbf{x}}_0) \in X_D[0,T] \quad \text{for all} \quad m \geq 1. \tag{11}$$

By Assumption **A.1**, it suffices to show that

$$\|\mathbf{x}_m(t,\tilde{\mathbf{x}}_0) - \mathbf{x}_0(t,\tilde{\mathbf{x}}_0)\|_{C_{1-\gamma}[0,T]} \leq \Xi\mathbf{m}.$$

We proceed by mathematical induction. We first take an induction base case of $m = 0$, which is satisfied as $\mathbf{x}_0(t,\tilde{\mathbf{x}}_0) \in X_D[0,T]$ since by assumption **A.1**, $\frac{\tilde{\mathbf{x}}_0}{\Gamma(\gamma)} \in D$. Then, for $m = 1$ we have by Lemma 3.2 and **A.2** that

$$\|\mathbf{x}_1(t,\tilde{\mathbf{x}}_0) - \mathbf{x}_0(t,\tilde{\mathbf{x}}_0)\|_{C_{1-\gamma}[0,T]} = \|\mathcal{T}\mathbf{f}(t,\mathbf{x}_0(t,\tilde{\mathbf{x}}_0)\|_{C_{1-\gamma}[0,T]} \leq \Xi\mathbf{m}, \tag{12}$$

thus satisfying (11) for $m = 1$.

For the induction hypothesis, assume $\|\mathbf{x}_k(t,\tilde{\mathbf{x}}_0) - \mathbf{x}_0(t,\tilde{\mathbf{x}}_0)\|_{C_{1-\gamma}[0,T]} \leq \Xi\mathbf{m}$ for some $k > 1$ and thus $\mathbf{x}_k(t,\tilde{\mathbf{x}}_0) \in X_D[0,T]$. Then, for $k+1$ we have

$$\|\mathbf{x}_{k+1}(t,\tilde{\mathbf{x}}_0) - \mathbf{x}_0(t,\tilde{\mathbf{x}}_0)\|_{C_{1-\gamma}[0,T]} = \|\mathcal{T}\mathbf{f}(t,\mathbf{x}_k(t,\tilde{\mathbf{x}}_0))\|_{C_{1-\gamma}[0,T]} \leq \Xi\mathbf{m},$$

and hence $\mathbf{x}_{k+1}(t,\tilde{\mathbf{x}}_0) \in X_D[0,T]$, proving the induction claim.

To show the sequence defined by (8) converges, we prove that the following estimate holds for $m \geq 1$:

$$\|\mathbf{x}_m(t,\tilde{\mathbf{x}}_0) - \mathbf{x}_{m-1}(t,\tilde{\mathbf{x}}_0)\|_{C_{1-\gamma}[0,T]} \leq \Xi Q^{m-1}\mathbf{m}. \tag{13}$$

We again follow mathematical induction. For the base case $m = 1$, this follows from (12). For the induction hypothesis, assume that statement (13) holds for some $k > 1$. Hence,

$$\|\mathbf{x}_k(t,\tilde{\mathbf{x}}_0) - \mathbf{x}_{k-1}(t,\tilde{\mathbf{x}}_0)\|_{C_{1-\gamma}[0,T]} \leq \Xi Q^{k-1}\mathbf{m}.$$

Then, for $k+1$, we have

$$\begin{aligned}
\|\mathbf{x}_{k+1}(t,\tilde{\mathbf{x}}_0) - \mathbf{x}_k(t,\tilde{\mathbf{x}}_0)\|_{C_{1-\gamma}[0,T]} &= \|\mathcal{T}\mathbf{f}(\mathbf{x}_k(t,\tilde{\mathbf{x}}_0)) - \mathcal{T}\mathbf{f}(\mathbf{x}_{k-1}(t,\tilde{\mathbf{x}}_0))\|_{C_{1-\gamma}[0,T]}\\
&= \|\mathcal{T}\{\mathbf{f}(\mathbf{x}_k(t,\tilde{\mathbf{x}}_0)) - \mathbf{f}(\mathbf{x}_{k-1}(t,\tilde{\mathbf{x}}_0))\}\|_{C_{1-\gamma}[0,T]}\\
&\leq \Xi\|\mathbf{f}(\mathbf{x}_k(t,\tilde{\mathbf{x}}_0)) - \mathbf{f}(\mathbf{x}_{k-1}(t,\tilde{\mathbf{x}}_0))\|_{C_{1-\gamma}[0,T]}\\
&\leq Q\|\mathbf{x}_k(t,\tilde{\mathbf{x}}_0) - \mathbf{x}_{k-1}(t,\tilde{\mathbf{x}}_0)\|_{C_{1-\gamma}[0,T]}\\
&\leq \Xi Q^k\mathbf{m}.
\end{aligned}$$

Then, taking some $j \in \mathbb{N}$, we have by telescoping and the triangle inequality that

$$\begin{aligned}
\|\mathbf{x}_{m+j}(t,\tilde{\mathbf{x}}_0) - \mathbf{x}_m(t,\tilde{\mathbf{x}}_0)\|_{C_{1-\gamma}[0,T]} &= \left\|\sum_{k=1}^{j}\left[\mathbf{x}_{m+k}(t,\tilde{\mathbf{x}}_0) - \mathbf{x}_{m+k-1}(t,\tilde{\mathbf{x}}_0)\right]\right\|_{C_{1-\gamma}[0,T]}\\
&\leq \sum_{k=1}^{j}\|\mathbf{x}_{m+k}(t,\tilde{\mathbf{x}}_0) - \mathbf{x}_{m+k-1}(t,\tilde{\mathbf{x}}_0)\|_{C_{1-\gamma}[0,T]}\\
&\leq \sum_{k=1}^{j}\Xi Q^{m+k-1}\mathbf{m} = \Xi Q^m\sum_{k=0}^{j-1}Q^k\mathbf{m}.
\end{aligned} \tag{14}$$

Now, since by assumption **A.3** the spectral radius $\rho(Q) < 1$, we know that $\lim_{m\to\infty} Q^m = O_d$, where O_d is the d-dimensional zero-matrix. Hence, by taking the limit as $m \to \infty$, we have a Cauchy sequence $(\mathbf{x}_m)_{m\geq 1}$ defined by (8) with uniform converge in $X_D[0,T]$, proving claim 2. For the error estimate (10) of statement **C.2**, note that since $\rho(Q) < 1$, it also holds that

$$\sum_{k=0}^{j-1}Q^k \leq (I_d - Q)^{-1},$$

where I_d is the d-dimensional unit matrix. Combining this with (14) and taking the limit $j \to \infty$, we have

$$\|\mathbf{x}_\infty(t,\tilde{\mathbf{x}}_0) - \mathbf{x}_m(t,\tilde{\mathbf{x}}_0)\|_{C_{1-\gamma}[0,T]} \leq \Xi(I_d - Q)^{-1}Q^m\mathbf{m},$$

providing the error bound.

Consider statement **C.3**. Since by construction of sequence (8) the limit $\mathbf{x}(t,\tilde{\mathbf{x}}_0)$ satisfies

$$\mathbf{x}(t,\tilde{\mathbf{x}}_0) = \frac{\tilde{\mathbf{x}}_0}{\Gamma(\gamma)} t^{\gamma-1} + \{\mathcal{T}\mathbf{f}(t,\mathbf{x}(t,\tilde{\mathbf{x}}_0))\}(t),$$

it follows that $\mathbf{x}(t,\tilde{\mathbf{x}}_0)$ satisfies integral Eq. (7). Then, by Lemma 3.1, we have equivalence to the perturbed IVP with $\mathbf{x} \in C^{\gamma}_{1-\gamma}[0,T]$ and $\nu = \Delta_T(\tilde{\mathbf{x}}_0) = -\frac{\Gamma(\zeta+1)}{T^{\zeta}}\, {}_0\mathrm{I}^{\zeta}_T \mathbf{f}(T,\mathbf{x}(T))$.

For uniqueness, we take two solutions, $\mathbf{x}_a(t)$ and $\mathbf{x}_b(t)$ of integral Eq. (7). Then, by Lemma 3.2 and Assumption **A.2** we derive:

$$\begin{aligned}
\|\mathbf{x}_a - \mathbf{x}_b\|_{C_{1-\gamma}[0,T]} &= \|\mathcal{T}\mathbf{x}_a - \mathcal{T}\mathbf{x}_b\|_{C_{1-\gamma}[0,T]} \\
&= \|\mathcal{T}\{\mathbf{x}_a - \mathbf{x}_b\}\|_{C_{1-\gamma}[0,T]} \\
&\le \Xi \|\mathbf{f}(t,\mathbf{x}_a(t)) - \mathbf{f}(t,\mathbf{x}_b(t))\|_{C_{1-\gamma}[0,T]} \\
&\le Q\|\mathbf{x}_a - \mathbf{x}_b\|_{C_{1-\gamma}[0,T]}.
\end{aligned}$$

Since $\rho(Q) < 1$ by **A.3**, we must have $\|\mathbf{x}_a(t) - \mathbf{x}_b(t)\|_{C_{1-\gamma}[0,T]} = \mathbf{0}$, which yields $\mathbf{x}_a(t) = \mathbf{x}_b(t)$.

Finally, statement **C.4** follows from statement **C.3** and Lemma 3.1. □

3.4. Connection to the original BVP

We will now show the conditions for which the perturbed IVP solutions coincide with the original BVP problem.

Theorem 3.2 (Connection to the original BVP). *Assume conditions **A.1** - **A.3** hold. Then, solution $\mathbf{x}(t,\tilde{\mathbf{x}}_0)$ to the perturbed IVP (5) is a solution of BVP (4) if and only if $\tilde{\mathbf{x}}_0 \in \mathbb{R}^d$ is a solution of*

$$\Delta_T(\tilde{\mathbf{x}}_0) = \mathbf{0},$$

where Δ_T is given in (6).

Proof. The result follows directly from the fact that $\nu = \Delta_T(\tilde{\mathbf{x}}_0)$ in the perturbed problem since the condition of Lemma 3.1 holds as $\mathbf{x}(t,\tilde{\mathbf{x}}_0)$ satisfies the boundary conditions by Theorem 3.1. Hence, we both have ${}_0\mathrm{D}^{\alpha,\beta}_t \mathbf{x}(t) = \mathbf{f}(t,\mathbf{x}(t))$ for $0 < t < T$ and ${}_0\mathrm{I}^{1-\gamma}_0 \mathbf{x}(0) = {}_0\mathrm{I}^{1-\gamma}_T \mathbf{x}(T)$, satisfying BVP (4). □

Remark 2. In practice, solving $\Delta_T(\tilde{\mathbf{x}}_0) = \mathbf{0}$ for $\tilde{\mathbf{x}}_0$ or T can be achieved numerically. A constrained approach such as a grid search can be chosen to guarantee convergence of approximations within the conditions outlined by the assumptions of Theorem 3.1. We will present a numerical example in Section 5.2.

4. Numerical splines approximations

This section establishes a Bernstein splines solutions approach, approximating the sequence (8) with a finite-dimensional splines basis, building upon the work in [32]. Our strategy will be to take the left side of the boundary intervals at $\varepsilon > 0$, easing solution regularity requirements as $t \downarrow 0$. Then, we construct splines approximations in the weighted space starting from $\varepsilon > 0$. We introduce notation and prove preliminary lemma's first in Section 4.1 and present the main numerical convergence results in Section 4.2.

4.1. Preliminary notation and results

First, we introduce the following notation for the $1-\gamma$ weighed norm on a domain starting from $\varepsilon > 0$:

$$\begin{aligned}
&C_{1-\gamma}[\varepsilon,T] = \{f : (\varepsilon,T] \to \mathbb{R} : t^{1-\gamma} f \in C[\varepsilon,T]\},\\
&\|f\|_{C_{1-\gamma}[\varepsilon,T]} = \sup_{t\in(\varepsilon,T]} |t^{1-\gamma} f(t)|.
\end{aligned}$$

We show that continuity on this domain coincides with unweighted continuity:

Proposition 4.1 (Shifted weighed continuous functions are continuous, [32]). *For $0 < \varepsilon < t \le T$, it holds that*

$$C_{1-\gamma}[\varepsilon,T] = C[\varepsilon,T].$$

Now, we introduce an ε-shifted iteration map similar to (9):

$$\{\mathcal{T}^{\varepsilon}\mathbf{y}\}(t) = {}_{\varepsilon}\mathrm{I}^{\alpha}_t \mathbf{y}(t) - \frac{\Gamma(\zeta+1)t^{\alpha}}{\Gamma(\alpha+1)T^{\zeta}}\, {}_{\varepsilon}\mathrm{I}^{\zeta}_T \mathbf{y}(T), \quad \varepsilon < t \le T. \tag{15}$$

We can obtain a similar bound for this operator:

Lemma 4.1 (ε-shifted iteration map bounds). *For $\mathbf{y} \in C_{1-\gamma}[\varepsilon,T]$, the map $\mathcal{T}^{\varepsilon}$ as defined in (15) is bounded, such that*

$$\|\mathcal{T}^{\varepsilon}\mathbf{y}\|_{C_{1-\gamma}[\varepsilon,T]} \le \Xi\|\mathbf{y}\|_{C_{1-\gamma}[\varepsilon,T]},$$

where Ξ is defined in Lemma 3.2.

Proof. The result follows from observing that ${}_\varepsilon\mathrm{I}_t^\alpha \mathbf{y}(t) = {}_0\mathrm{I}_t^\alpha \mathbb{1}_{[\varepsilon,T]}\mathbf{y}(t)$ and applying Hölder's inequality as in Lemma 3.2 to $\mathbb{1}_{[\varepsilon,T]}\mathbf{y}(t)$. □

We can now bound the difference of the shifted map $\mathcal{T}^\varepsilon$ with the original map $\mathcal{T}$. Moreover, we prove the difference limits to 0 as $\varepsilon \downarrow 0$ and provide convergence rates.

Lemma 4.2 (ε-shifted iteration map difference). *The map $\mathcal{T}^\varepsilon$ as defined in (15) satisfies*

$$\|\mathcal{T}^\varepsilon \mathbf{y} - \mathcal{T}\mathbf{y}\|_{C_{1-\gamma}[\varepsilon,T]} \le \Theta_\varepsilon \|\mathbf{y}\|_{C_{1-\gamma}[0,T]},$$

where $\mathcal{T}$ is given in (9) and

$$\Theta_\varepsilon = \frac{\varepsilon^\alpha}{\Gamma(\alpha)}\mathrm{B}(\gamma,\alpha) + \frac{\zeta T^\alpha}{\Gamma(\alpha+1)}\mathrm{B}_{\varepsilon/T}(\gamma,\zeta).$$

Furthermore, $\Theta_\varepsilon = \mathcal{O}(\varepsilon^\alpha)$ for $\varepsilon \downarrow 0$.

Proof. See Appendix C. □

As a final result for our shifted map $\mathcal{T}^\varepsilon$, we will obtain the modulus of continuity for $\mathcal{T}^\varepsilon$, weighted in the $C_{1-\gamma}[\varepsilon,T]$-norm. This gives a bound on the weighted distance between function values on an interval $[t,t'] \subseteq [\varepsilon,T]$. Moreover, we show the rate for which this bound limits to 0 as $(t'-t) \downarrow 0$.

Lemma 4.3 (ε-shifted iteration map modulus of continuity). *The map $\mathcal{T}^\varepsilon$ of (15) satisfies*

$$|t'^{1-\gamma}\mathcal{T}^\varepsilon \mathbf{y}(t') - t^{1-\gamma}\mathcal{T}^\varepsilon \mathbf{y}(t)| \le \Omega(t,t')\|\mathbf{y}\|_{C_{1-\gamma}[\varepsilon,T]}, \quad \varepsilon \le t < t' \le T,$$

with

$$\Omega(t,t') = \frac{(t'-t)^\alpha}{\Gamma(\alpha+1)}\left(\alpha\mathrm{B}(\gamma,\alpha) + 2\left(\frac{t'}{t}\right)^{1-\gamma}\right) + \frac{(t'-t)^\zeta T^\alpha}{\Gamma(\alpha+1)T^\zeta}\zeta\mathrm{B}(\gamma,\zeta). \tag{16}$$

Furthermore, $\Omega(t,t') = \mathcal{O}\big((t'-t)^\alpha\big)$ for $(t'-t) \downarrow 0$.

Proof. See Appendix D. □

We now construct a splines approximation $\mathcal{T}^{\varepsilon,q}$ of $\mathcal{T}^\varepsilon$. For a knot selection $\mathcal{A}$ on $[\varepsilon,T]$ satisfying Definition 7, we define the shifted splines iteration map $\mathcal{T}^{\varepsilon,q}$ as follows:

$$\{\mathcal{T}^{\varepsilon,q}\mathbf{y}\}(t) = \{t^{\gamma-1}S_{\mathcal{A}}^q t^{1-\gamma}\mathcal{T}^\varepsilon \mathbf{y}\}(t), \quad \varepsilon < t \le T, \tag{17}$$

where $S_{\mathcal{A}}^q$ denotes the Bernstein splines operator as defined in Definition 8. We can now combine the results as obtained above to bound the estimation error introduced in this splines approximation.

Lemma 4.4 (Bernstein splines iteration map error). *For $\mathbf{y} \in C_{1-\gamma}[\varepsilon,T]$, the map $\mathcal{T}^{\varepsilon,q}$ defined in (17) satisfies $\mathcal{T}^{\varepsilon,q}\mathbf{y} \in C_{1-\gamma}[\varepsilon,T]$. Furthermore,*

$$\|\mathcal{T}^{\varepsilon,q}\mathbf{y} - \mathcal{T}^\varepsilon \mathbf{y}\|_{C_{1-\gamma}[\varepsilon,T]} \le \Omega_{\mathcal{A}}^q \|\mathbf{y}\|_{C_{1-\gamma}[\varepsilon,T]},$$

with

$$\Omega_{\mathcal{A}}^q = \frac{5}{4}\sup_{A_i\in\mathcal{A}}\ \sup_{t,t'\in A_i}\ \sup_{|t'-t|\le h_i/\sqrt{q}} \Omega(t',t),$$

where $h_i := t_{i+1} - t_i$ denotes the knot size and Ω is defined in (16).

Proof. See Appendix E. □

4.2. Main convergence result

We will now state the main convergence result, showing converging splines approximations for the perturbed IVP problem (5) corresponding to our BVP (4).

Theorem 4.1 (Convergence of numerical approximations). *For BVP (4) and its corresponding perturbed IVP (5), assume the conditions of existence and uniqueness of Theorem 3.1 hold. Denote $X_D[\varepsilon,T] = \{\mathbf{u} : (\varepsilon,T] \to \mathbb{R}^d,\ \mathbf{u} \in C_{1-\gamma}[\varepsilon,T] : t^{1-\gamma}\mathbf{u}(t) \in D\}$. If additionally*

*A.1** $\left\{\mathbf{u} \in C_{1-\gamma}[\varepsilon,T] : \left\|\mathbf{u} - \frac{\tilde{\mathbf{x}}_0 t^{\gamma-1}}{\Gamma(\gamma)}\right\|_{C_{1-\gamma}[\varepsilon,T]} \le (\Xi + \Omega_{\mathcal{A}}^q)\mathbf{m}\right\} \subseteq X_D[\varepsilon,T],$
where Ξ and $\Omega_{\mathcal{A}}^q$ are given in Theorem 3.1 and Lemma 4.4 respectively;

A.2 boundedness and Lipschitz conditions of **A.2** also hold for $\mathbf{u},\mathbf{v} \in X_D[\varepsilon,T]$.*

A.3 the spectral radius satisfies $\rho\Big((\Xi + \Omega_{\mathcal{A}}^q)K\Big) < 1$,* *then the sequence* $(\mathbf{x}_m^{q,\varepsilon})_{m\geq 0}$ *with* $\mathbf{x}_m^{q,\varepsilon} : [\varepsilon, T] \to \mathbb{R}^d$ *defined by*

$$\begin{aligned} \mathbf{x}_0^{q,\varepsilon}(t, \tilde{\mathbf{x}}_0) &= \frac{\tilde{\mathbf{x}}_0}{\Gamma(\gamma)} t^{\gamma-1}, \\ \mathbf{x}_{m+1}^{q,\varepsilon}(t, \tilde{\mathbf{x}}_0) &= \mathbf{x}_0^{q,\varepsilon}(t, \tilde{\mathbf{x}}_0) + \{\mathcal{T}^{\varepsilon,q}\mathbf{f}(t, \mathbf{x}_m^{q,\varepsilon}(t, \tilde{\mathbf{x}}_0))\}(t), \end{aligned} \tag{18}$$

converges with $\lim_{m\to\infty} \mathbf{x}_m^{q,\varepsilon} =: \mathbf{x}^{q,\varepsilon} \in C_{1-\gamma}[\varepsilon, T]$. *Furthermore, the following statements hold:*

*C.1** ***Continuity of iterations:*** $\mathbf{x}_m^{q,\varepsilon} \in C_{1-\gamma}[\varepsilon, T] = C[\varepsilon, T]$ *for* $m \geq 0$.
*C.2** ***Spline convergence:*** $\lim_{q\to\infty} \mathbf{x}^{q,\varepsilon} = \lim_{h\downarrow 0} \mathbf{x}^{q,\varepsilon} = \mathbf{x}^\varepsilon$ *, where* $h := \sup_{i\in\{0,\dots,k\}} h_i$ *denotes the largest knot size,* $q \in \mathbb{N}$ *the spline polynomial order and* $\mathbf{x}^\varepsilon$ *is the solution to (18) with* $\mathcal{T}^\varepsilon$ *of Lemma 4.2 substituted for* $\mathcal{T}^{\varepsilon,q}$. *Furthermore,* $\|\mathbf{x}^{q,\varepsilon} - \mathbf{x}^\varepsilon\|_{C_{1-\gamma}[\varepsilon,T]} = \mathcal{O}\left(\left(\frac{h}{\sqrt{q}}\right)^\alpha\right)$.
*C.3** ***Convergence in*** ε*:* $\lim_{\varepsilon\downarrow 0} \mathbf{x}^\varepsilon = \mathbf{x}$ *, where* $\mathbf{x} \in C_{1-\gamma}^{\beta(1-\alpha)}[0, T]$ *is the analytical perturbed-IVP solution for BVP (4) as obtained in Theorem 3.1. Furthermore,* $\|\mathbf{x}^\varepsilon - \mathbf{x}\|_{C_{1-\gamma}[\varepsilon,T]} = \mathcal{O}(\varepsilon^\alpha)$.

Proof. First, we prove convergence of the sequence. Rewriting and using Theorem 3.1, Lemmas 4.2 and 4.4 we have

$$\begin{aligned} \|\mathcal{T}^{\varepsilon,q}\mathbf{y}\|_{C_{1-\gamma}[\varepsilon,T]} &= \|\mathcal{T}^{\varepsilon,q}\mathbf{y} - \mathcal{T}^\varepsilon\mathbf{y} + \mathcal{T}^\varepsilon\mathbf{y}\|_{C_{1-\gamma}[\varepsilon,T]} \\ &\leq \|\mathcal{T}^{\varepsilon,q}\mathbf{y} - \mathcal{T}^\varepsilon\mathbf{y}\|_{C_{1-\gamma}[\varepsilon,T]} + \|\mathcal{T}^\varepsilon\mathbf{y}\|_{C_{1-\gamma}[\varepsilon,T]} \\ &\leq \left(\Omega_{\mathcal{A}}^q + \Xi\right)\|\mathbf{y}\|_{C_{1-\gamma}[\varepsilon,T]}. \end{aligned}$$

First, we observe that $\mathbf{x}_0^{q,\varepsilon}(t, \tilde{\mathbf{x}}_0) \in X_D[\varepsilon, T]$. Then, combining assumption A.3* with the linearity of $\mathcal{T}^{\varepsilon,q}\mathbf{y} \in C_{1-\gamma}[\varepsilon, T]$ as presented in Lemma 4.4, we can follow the same line of proof as in Theorem 3.1 for assumption A.1* - A.3*. This gives convergence of the sequence (18) as $n \to \infty$ with iterations $\mathbf{x}_n^{q,\varepsilon}(t, \tilde{\mathbf{x}}_0) \in X_D[\varepsilon, T]$. Furthermore, by Proposition 4.1, $C_{1-\gamma}[\varepsilon, T] = C[\varepsilon, T]$, satisfying claim C.1*.

For claim C.2* and C.3*, note that $\|\mathcal{T}^\varepsilon\mathbf{y}\|_{C_{1-\gamma}[\varepsilon,T]} \leq \Xi\|\mathbf{y}\|_{C_{1-\gamma}[\varepsilon,T]}$ with $\rho(\Xi K) \leq 1$ by Lemma 4.1. Since furthermore $\Omega_{\mathcal{A}}^q \geq \mathbf{0}$ we must also have

$$\{\mathbf{u} \in C_{1-\gamma}[\varepsilon, T] : \|\mathbf{u} - \tilde{\mathbf{x}}_0 t^{\gamma-1}/\Gamma(\gamma)\|_{C_{1-\gamma}[\varepsilon,T]} \leq \Xi\mathbf{m}\} \subseteq X_D[\varepsilon, T].$$

Then, using the same line of proof of Theorem 3.1 we know that the sequence

$$\begin{aligned} \mathbf{x}_0^\varepsilon(t, \tilde{\mathbf{x}}_0) &= \frac{\tilde{\mathbf{x}}_0}{\Gamma(\gamma)} t^{\gamma-1}, \\ \mathbf{x}_{m+1}^\varepsilon(t, \tilde{\mathbf{x}}_0) &= \mathbf{x}_0^\varepsilon(t, \tilde{\mathbf{x}}_0) + \{\mathcal{T}^\varepsilon\mathbf{f}(t, \mathbf{x}_m^\varepsilon(t, \tilde{\mathbf{x}}_0))\}(t) \end{aligned}$$

converges as $m \to \infty$ to a solution $\mathbf{x}^\varepsilon(t, \tilde{\mathbf{x}}_0) \in X_D[\varepsilon, T]$ satisfying the fractional integral equation

$$\mathbf{x}^\varepsilon(t, \tilde{\mathbf{x}}_0) = \frac{\tilde{\mathbf{x}}_0}{\Gamma(\gamma)} t^{\gamma-1} + \{\mathcal{T}^\varepsilon\mathbf{f}(t, \mathbf{x}^\varepsilon(t, \tilde{\mathbf{x}}_0))\}(t), \quad \varepsilon < t \leq T. \tag{19}$$

Using this to prove claim C.2*, we can write

$$\begin{aligned} &\|\mathbf{x}^{q,\varepsilon}(t, \tilde{\mathbf{x}}_0) - \mathbf{x}^\varepsilon(t, \tilde{\mathbf{x}}_0)\|_{C_{1-\gamma}[\varepsilon,T]} = \|\mathcal{T}^{\varepsilon,q}\mathbf{f}(t, \mathbf{x}^{q,\varepsilon}(t, \tilde{\mathbf{x}}_0)) - \mathcal{T}^\varepsilon\mathbf{f}(t, \mathbf{x}^\varepsilon(t, \tilde{\mathbf{x}}_0))\|_{C_{1-\gamma}[\varepsilon,T]} \\ &= \|\{\mathcal{T}^{\varepsilon,q} - \mathcal{T}^\varepsilon\}\mathbf{f}(t, \mathbf{x}^{q,\varepsilon}(t, \tilde{\mathbf{x}}_0)) - \mathcal{T}^\varepsilon\{\mathbf{f}(t, \mathbf{x}^{q,\varepsilon}(t, \tilde{\mathbf{x}}_0)) - \mathbf{f}(t, \mathbf{x}^\varepsilon(t, \tilde{\mathbf{x}}_0))\}\|_{C_{1-\gamma}[\varepsilon,T]} \\ &\leq \Omega_{\mathcal{A}}^q\|\mathbf{f}(t, \mathbf{x}^{q,\varepsilon}(t, \tilde{\mathbf{x}}_0))\|_{C_{1-\gamma}[\varepsilon,T]} + \Xi K\|\mathbf{f}(t, \mathbf{x}^{q,\varepsilon}(t, \tilde{\mathbf{x}}_0)) - \mathbf{f}(t, \mathbf{x}^\varepsilon(t, \tilde{\mathbf{x}}_0))\|_{C_{1-\gamma}[\varepsilon,T]} \\ &\leq \Omega_{\mathcal{A}}^q\mathbf{m} + \Xi K\|\mathbf{x}^{q,\varepsilon}(t, \tilde{\mathbf{x}}_0) - \mathbf{x}^\varepsilon(t, \tilde{\mathbf{x}}_0)\|_{C_{1-\gamma}[\varepsilon,T]}. \end{aligned}$$

Now, using the fact that $\rho(\Xi K) < 1$ we have

$$\|\mathbf{x}^{q,\varepsilon}(t, \tilde{\mathbf{x}}_0) - \mathbf{x}^\varepsilon(t, \tilde{\mathbf{x}}_0)\|_{C_{1-\gamma}[\varepsilon,T]} \leq (I_d - \Xi K)^{-1}\Omega_{\mathcal{A}}^q\mathbf{m},$$

where I_d denotes the d-dimensional identity matrix. Since $\mathcal{O}(\Omega_{\mathcal{A}}^q) = \mathcal{O}\left(\left(\frac{h}{\sqrt{q}}\right)^\alpha\right)$ for $\frac{h}{\sqrt{q}} \downarrow 0$ by Lemmas 4.3 and 4.4, we have

$$\|\mathbf{x}^{q,\varepsilon}(t, \tilde{\mathbf{x}}_0) - \mathbf{x}^\varepsilon(t, \tilde{\mathbf{x}}_0)\|_{C_{1-\gamma}[\varepsilon,T]} = \mathcal{O}\left(\left(\frac{h}{\sqrt{q}}\right)^\alpha\right).$$

Furthermore, since $q \geq 1$, we also have convergence for $h \downarrow 0$ and $q \to \infty$, satisfying the claim.

For convergence to the analytical solution as $\varepsilon \downarrow 0$ of C.3*, the same approach yields

$$\begin{aligned} &\|\mathbf{x}^\varepsilon(t, \tilde{\mathbf{x}}_0) - \mathbf{x}(t, \tilde{\mathbf{x}}_0)\|_{C_{1-\gamma}[\varepsilon,T]} = \|\mathcal{T}^\varepsilon\mathbf{f}(t, \mathbf{x}^\varepsilon(t, \tilde{\mathbf{x}}_0)) - \mathcal{T}\mathbf{f}(t, \mathbf{x}(t, \tilde{\mathbf{x}}_0))\|_{C_{1-\gamma}[\varepsilon,T]} \\ &= \|\mathcal{T}^\varepsilon\mathbf{f}(t, \mathbf{x}^\varepsilon(t, \tilde{\mathbf{x}}_0)) - \mathcal{T}^\varepsilon\mathbf{f}(t, \mathbf{x}(t, \tilde{\mathbf{x}}_0)) + \mathcal{T}^\varepsilon\mathbf{f}(t, \mathbf{x}(t, \tilde{\mathbf{x}}_0)) - \mathcal{T}\mathbf{f}(t, \mathbf{x}(t, \tilde{\mathbf{x}}_0))\|_{C_{1-\gamma}[\varepsilon,T]} \\ &\leq \|\mathcal{T}^\varepsilon\{\mathbf{f}(t, \mathbf{x}^\varepsilon(t, \tilde{\mathbf{x}}_0)) - \mathbf{f}(t, \mathbf{x}(t, \tilde{\mathbf{x}}_0))\}\|_{C_{1-\gamma}[\varepsilon,T]} + \|\{\mathcal{T}^\varepsilon - \mathcal{T}\}\mathbf{f}(t, \mathbf{x}(t, \tilde{\mathbf{x}}_0))\|_{C_{1-\gamma}[\varepsilon,T]} \\ &\leq \Xi\|\mathbf{f}(t, \mathbf{x}^\varepsilon(t, \tilde{\mathbf{x}}_0)) - \mathbf{f}(t, \mathbf{x}(t, \tilde{\mathbf{x}}_0))\|_{C_{1-\gamma}[\varepsilon,T]} + \Theta_\varepsilon\|\mathbf{f}(t, \mathbf{x}(t, \tilde{\mathbf{x}}_0))\|_{C_{1-\gamma}[0,T]} \end{aligned}$$

$$\leq \Xi K \|\mathbf{x}^{\varepsilon}(t, \tilde{\mathbf{x}}_0) - \mathbf{x}(t, \tilde{\mathbf{x}}_0)\|_{C_{1-\gamma}[\varepsilon,T]} + \Theta_{\varepsilon}\mathbf{m},$$

which, combined with the fact that $\rho(\Xi K) < 1$ gives after rewriting

$$\|\mathbf{x}^{\varepsilon}(t, \tilde{\mathbf{x}}_0)) - \mathbf{x}(t, \tilde{\mathbf{x}}_0))\|_{C_{1-\gamma}[\varepsilon,T]} \leq (I_d - \Xi K)^{-1}\Theta_{\varepsilon}\mathbf{m}.$$

Then, since $\mathcal{O}(\Theta_{\varepsilon}) = \mathcal{O}(\varepsilon^{\alpha})$ for $\varepsilon \downarrow 0$ as established in Lemma 4.2, we have satisfied the claim. □

5. Numerical results

To find numerical solutions corresponding to sequence (18), we operationalize the splines operator $S^q_{\mathcal{A}}$ of the previous section using the implementation as described in [32]. We apply the convergence results and implementation to two example systems. First, we choose a time-polynomial system with a closed form analytical solution to study the empirical convergence rates of the method in Section 5.1. The second example, as presented in Section 5.2, presents a nonlinear function with an analytical expression for the Lipschitz constant K and function bounds m to validate our convergence results for nonlinear problems. Moreover, we present an example of the root-finding strategy to obtain non-perturbed solutions for $T = T^*$.

5.1. Polynomial example and convergence

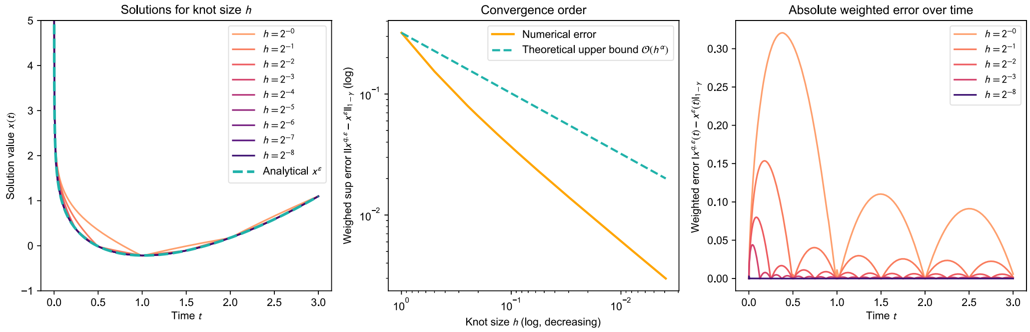
Consider the following BVP:

$$\begin{cases} {}_0\mathrm{D}_t^{\alpha,\beta} x(t) = t^k, & 0 < t < T, \quad 0 < \alpha < 1, \quad 0 \leq \beta \leq 1, \\ {}_0\mathrm{I}_0^{1-\gamma} x(0) = {}_0\mathrm{I}_T^{1-\gamma} x(T), & \gamma = \alpha + \beta - \alpha\beta. \end{cases} \tag{20}$$

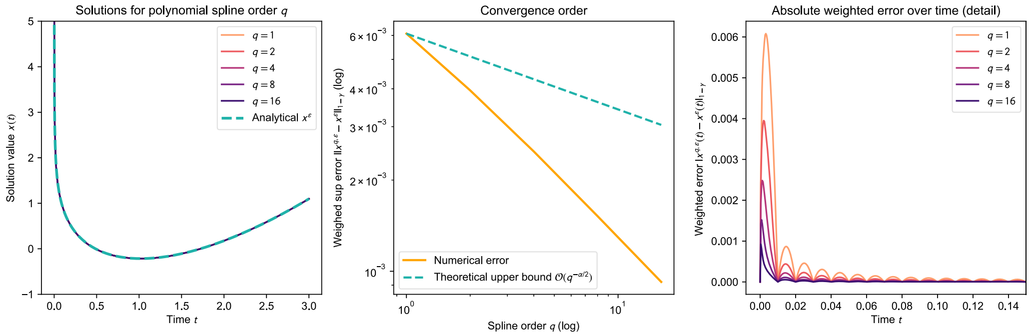
Choosing $\alpha, \beta = 1/2$, $k = 0.9$, $\tilde{x}_0 = 1$ and $T = 3$, we use Theorems 3.1 and 4.1 to obtain numerical approximations $x^{q,\varepsilon}(t, \tilde{x}_0) : (\varepsilon, T] \to \mathbb{R}$ to the perturbed IVP (5) corresponding to (20), satisfying the boundary conditions. For the convergence assumptions, note that we have $\mathbf{m} = \sup_{t\in[0,T]} |f(t)| = 3^{0.9}$. Furthermore, since f is independent of x, the Lipschitz constant satisfies $K = 0$. Hence, we can satisfy Assumption **A.2** and A.2* independently of x, and thus the respective analytical and numerical spaces of convergence $X_{[0,T]}$ and $X_{[\varepsilon,T]}$ are fully determined by the requirement of Assumption **A.1** and A.1*. Finally, since $K = 0$ we have satisfied spectral radius requirements of Assumption **A.3** and A.3*. We thus know a unique perturbed analytical and numerical solution exists for $\tilde{x}_0 = 1$. For numerical implementation, we take an equidistant knot selection grid and a numerical iteration convergence tolerance of $\mathrm{TOL} = 10^{-12}$ (see [32]). This value of 10^{-12} is chosen to be close to the default `np.float64` implementation machine precision of $2.22 \cdot 10^{-16}$ while still empirically providing a reasonable number of convergence iterations for nonlinear problems. For the default splines parameter values, we choose polynomial order $q = 1$, knot size $h = 10^{-2}$ and time domain shift $\varepsilon = 10^{-10}$, serving as a base case with accurate approximations as validated by the convergence results. Errors are computed relative to the analytical functions $x^{\varepsilon}(t, \tilde{x}_0) : (\varepsilon, T] \to \mathbb{R}$ and $x(t, \tilde{x}_0) : (0, T] \to \mathbb{R}$, obtained from integral Eqs. (19) and (7) respectively. We note that $\Delta_T(\tilde{x}_0) = -1.5997$ for $x(t, \tilde{x}_0)$. Results are presented in Tables F.1–F.3 (see Appendix F) and Fig. 2.

Results show convergence to the analytical solution x^{ε} with satisfactory low error already for higher values of h and low values of q. We note that all setups converge in one iteration, corresponding to the results of Theorem 3.1 and 4.1 for $K = 0$. Convergence rates satisfy the analytical upper bounds and all setups show stable behavior for the value of $\Delta_T(\tilde{x}_0)$, with values close to the analytical value. For convergence in h and q, the largest errors are consistently found in the first knots. We conjecture that this is likely caused by numerical rounding errors induced by the $t^{\gamma-1}$ term of $\mathcal{T}^{\varepsilon,q}$. As such, choosing a higher value of $\varepsilon > 0$ reduces this rounding error since $t \geq \varepsilon$, though increasing total error as apparent from Theorem 4.1. In practice, this trade-off could be made by choosing ε such that the multiplication and division by the value $\varepsilon^{\gamma-1}$ does not incur too strong rounding errors given the machine floating point precision used to obtain numerical solutions.

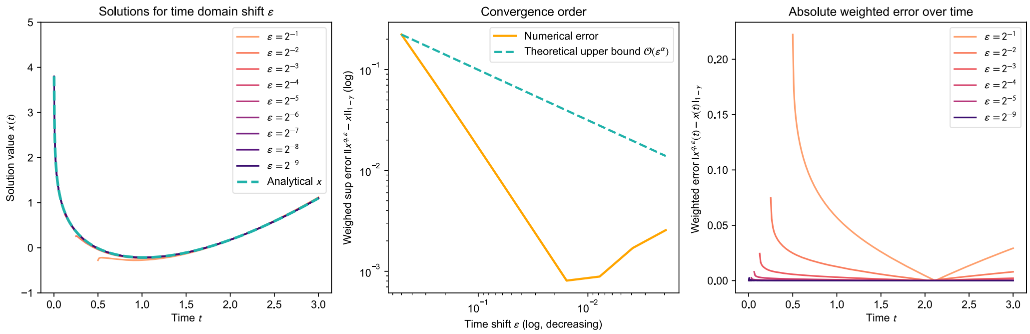
In terms of running time, a smaller knot size h and larger polynomial order q generally increase the total runtime as expected, since more computations are required. Furthermore, decreasing ε increases computational time, possibly a result of taking a larger time domain. Finally, convergence of $x^{q,\varepsilon}(t, \tilde{x}_0)$ to $x(t, \tilde{x}_0)$ is shown as $\varepsilon \downarrow 0$. Here, we note that the error does not reduce after $\varepsilon = 2^{-7}$. We conjecture this effect stems from errors caused by the knot size of $h = 10^{-2}$ dominating the total absolute error. Hence, it is likely that reducing ε does not yield much error reduction after $\varepsilon < h$.



(a) Convergence results for knot size h with polynomial order $q = 1$ and time shift $\varepsilon = 10^{-10}$.



(b) Convergence results for polynomial order q with knot size $h = 10^{-2}$ and time shift $\varepsilon = 10^{-10}$.



(c) Convergence results for ε with polynomial order $q = 1$ and knot size $h = 10^{-2}$.

Fig. 2. Approximations, convergence rates and error as a function of t in knot size h, polynomial order q and time-shift ε for the perturbed IVP corresponding to system (20) with $\alpha, \beta = 1/2$, $k = 0.9$, $\tilde{x}_0 = 1$, $T = 3$.

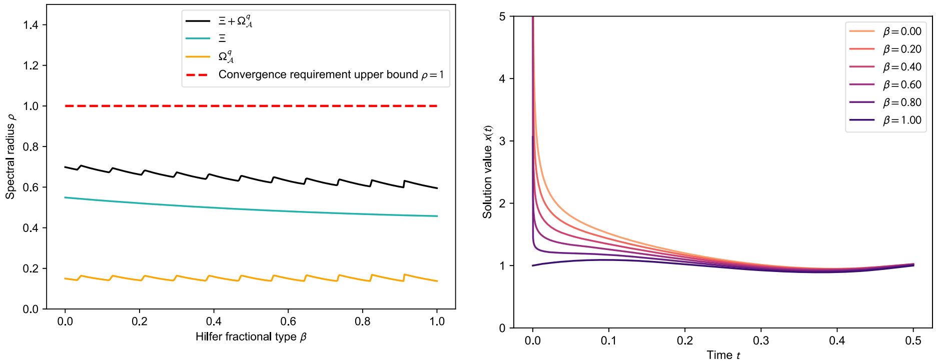
5.2. Nonlinear example

Consider the following nonlinear BVP

$$\begin{cases} {}_0\mathrm{D}_t^{\alpha,\beta} x(t) = \cos(x(t)4\pi t)/2\pi, & 0 < t < T = 1/2, \quad \alpha = 3/4, \quad 0 \le \beta \le 1, \\ {}_0\mathrm{I}_0^{1-\gamma} x(0) = {}_0\mathrm{I}_T^{1-\gamma} x(T), & \gamma = \alpha + \beta - \alpha\beta. \end{cases} \tag{21}$$

For the analytical and numerical domain of convergence of the corresponding perturbed IVP we consider $D = \mathbb{R}$, thus trivially satisfying Assumption **A.1** and A.1*. For Assumption **A.2** and A.2* we first note that

$$\mathbf{m} = \sup_{t\in(0,1/2]} \left| t^{1-\gamma} \cos(x(t)4\pi t)/2\pi \right| \le \frac{(1/2)^{1-\gamma}}{2\pi}.$$



(a) Values of solution convergence values Ξ and $\Omega^q_{\mathcal{A}}$ as given in Theorem 4.1, satisfying spectral radius requirement $\rho(\Xi + \Omega^q_{\mathcal{A}}) < 1$.

(b) Numerical solutions of the system for various values of β.

Fig. 3. Convergence requirement values and solutions for system (21) with knot selection (22). Results are obtained for splines with time-shift $\varepsilon = 10^{-10}$, knot size $h_{\max} = 10^{-2}$ and polynomial order $q = 1$.

The Lipschitz coefficients of f with regards to x can be computed by fixing $t \in (0, 1/2]$ and bounding the derivative w.r.t. $u(t) \in X_D[0, T]$ and $u(t) \in X_D[\varepsilon, T]$ respectively. This gives in both cases that

$$\left| \frac{\partial}{\partial u(t)} f(t, u(t)) \right| = |2t \sin(u(t) 4\pi t)| \leq 1.$$

Finally, for the spectral radius conditions **A.3** and A.3*, take a knot selection satisfying for knot size $h_i := t_{i+1} - t_i$ for $i \in \{0, \dots, k\}$:

$$h_i = \begin{cases} \min\left(h_{\max}, c^{1/(1-\gamma)-1} t_i\right), & \text{if} \quad 0 \leq \gamma < 1, \\ h_{\max} & \text{if} \quad \gamma = 1, \end{cases} \tag{22}$$

ensuring $\left(t_{i+1}/t_i\right)^{1-\gamma} \leq c$. We take $c = 3/2$. As in Section 5.1, we take the choice of parameters $t_0 = \varepsilon = 10^{-10}$ for the time-shift, a maximum knot size of $h_{\max} = 10^{-2}$ and a polynomial order $q = 1$ with an iteration convergence tolerance of $\mathrm{TOL} = 10^{-12}$. Then, numerically computing Ξ and $\Omega^q_{\mathcal{A}}$ gives $\Xi + \Omega_{\mathcal{A}} \leq 0.7064$ for all $\beta \in [0, 1]$, as presented in Fig. 3(a). Combined with the fact that $K \leq 1$ as established above, we thus have satisfied Assumption **A.3** and A.3*. Since all requirements are satisfied, we have a guarantee of existence of a unique analytical and convergence of a numerical solution by Theorem 3.1 and 4.1. Results for various values of β are presented in Table 1 and Fig. 3(b).

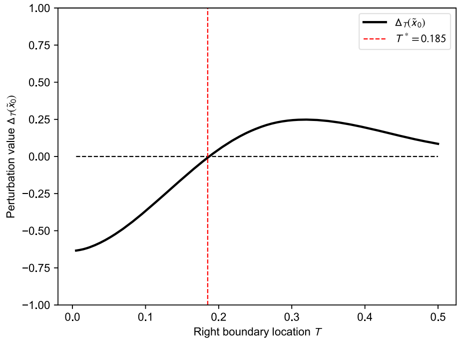
Results show convergence for all values of β, with a continuous transition in solutions from solutions to the Caputo problem of $\beta = 1$ to the Riemann-Liouville solutions for $\beta = 0$. As can be seen in Table 1, the number of knots decreases faster compared to the number of iterations needed for convergence as β increases. This might indicate that convergence knot requirement (22) for $c = 3/2$ poses a stricter condition on knot size than would be required purely for consistent knot iteration computational times.

As discussed in Theorem 3.2 and Remark 2, we can obtain solutions to the original system by satisfying $\Delta_T(\tilde{x}_0) = 0$. Taking the problem above for $\beta = 1/2$ fixed, we apply a grid search strategy, evaluating $\Delta_T(\tilde{x}_0)$ for $T \in \{\Delta T, 2\Delta T, \dots, 1/2\}$ with $\Delta T = 0.005$. We then look for a T^* minimizing $|\Delta_T(\tilde{x}_0)|$, yielding a solution approximation x^* to the original BVP (21). Results of the grid search are presented in Fig. 4(a), with a visualization of several solution values presented in Fig. 4(b).

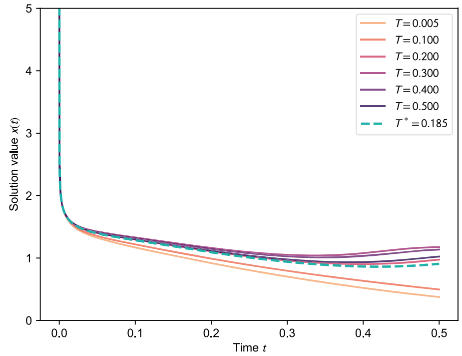
We find $T^* = 0.185$ with corresponding $\Delta_T(\tilde{x}_0) = -0.0066$. Furthermore, we have what appears to be continuous dependence of $\Delta_T(\tilde{x}_0)$ on T, providing means to minimize $\Delta_T(\tilde{x}_0)$ to arbitrary precision for a higher resolution on ΔT.

Table 1
Convergence results for numerical solutions of the perturbed IVP for system (21). Results are obtained for splines with time-shift $\varepsilon = 10^{-10}$, knot size $h_{\max} = 10^{-2}$ and polynomial order $q = 1$.

β	knots	$x(\varepsilon)$	$\Delta_T(\tilde{x}_0)$	time (s)	avg. it. / knot	avg. time / knot
0.00	61	$2.581 \cdot 10^{2}$	$1.360 \cdot 10^{-1}$	$1.114 \cdot 10^{-3}$	$2.951 \cdot 10^{-1}$	$1.827 \cdot 10^{-5}$
0.20	59	$8.589 \cdot 10^{1}$	$1.126 \cdot 10^{-1}$	$8.758 \cdot 10^{-4}$	$3.051 \cdot 10^{-1}$	$1.484 \cdot 10^{-5}$
0.40	56	$2.843 \cdot 10^{1}$	$9.338 \cdot 10^{-2}$	$8.699 \cdot 10^{-4}$	$3.214 \cdot 10^{-1}$	$1.553 \cdot 10^{-5}$
0.60	54	$9.358 \cdot 10^{0}$	$7.767 \cdot 10^{-2}$	$8.091 \cdot 10^{-4}$	$3.148 \cdot 10^{-1}$	$1.498 \cdot 10^{-5}$
0.80	52	$3.066 \cdot 10^{0}$	$6.512 \cdot 10^{-2}$	$7.795 \cdot 10^{-4}$	$3.269 \cdot 10^{-1}$	$1.499 \cdot 10^{-5}$
1.00	50	$1.000 \cdot 10^{0}$	$5.547 \cdot 10^{-2}$	$9.671 \cdot 10^{-3}$	$3.400 \cdot 10^{-1}$	$1.934 \cdot 10^{-4}$



(a) $\Delta_T(\tilde{x}_0)$ obtained in grid search procedure with $T \in \{\Delta T, 2\Delta T, \ldots, 1/2\}$, $\Delta T = 0.005$. $T^* = \arg\min_T |\Delta_T(\tilde{x}_0)|$ gives the corresponding approximating solution x^* to the original system (21).



(b) Solutions of the perturbed IVP for selected values of T as obtained in the grid search.

Fig. 4. Values of IVP perturbation $\Delta_T(\tilde{x}_0)$ and selected solutions in the grid search obtaining the perturbed IVP corresponding to (21) for $\tilde{x}_0 = 1$ and $\beta = 1/2$. Results are obtained for splines with time-shift $\epsilon = 10^{-10}$, knot size $h_{\max} = 10^{-2}$ and polynomial order $q = 1$. .

6. Conclusions

In this paper, we presented convergence requirements and the construction of analytical and numerical solutions to fractional-periodic BVPs with generalized Hilfer fractional derivatives of type $0 \leq \beta \leq 1$ and order $0 < \alpha < 1$ as given in (4). The analytical results were proven in Theorem 3.1, using a perturbed-IVP approach to find solutions in the weighted space $C_{1-\gamma}[0, T]$. Our study generalized previous perturbed-IVP results in literature for Caputo fractional derivative ($\beta = 1$) periodic BVPs to both the corresponding Riemann-Liouville operator problem ($\beta = 0$) and general Hilfer fractional derivative problems ($0 < \beta < 1$).

We implemented the analytical solutions numerically, constructing solution approximations with Bernstein splines. Approximations were obtained on a modified time domain $\epsilon < t < T$, with $\epsilon > 0$ small. Convergence requirements and asymptotic convergence rates to analytical solutions as $\epsilon \downarrow 0$ were presented in Theorem 4.1. Numerical experimentation results showed error convergence rates satisfying the theoretical bounds. Compared to existing methods in literature, our numerical method could numerically capture the singular behavior of solutions as $t \downarrow 0$ and was able to simulate nonlinear problems, as demonstrated in Section 5. Moreover, we gave an example of the apriori calculation of convergence requirements and an example of a grid search to find the corresponding periodic recurrence time T corresponding to the non-perturbed original BVP such that $\Delta_T(\tilde{x}_0) = 0$.

For future work, we note that apriori convergence guarantees to obtain $\Delta_T(\tilde{x}_0) = 0$ would improve the applicability of the method for applied problems, although we expect that this would in general pose stringent requirements for nonlinear systems. Another possible strategy would be to formalize a root-finding procedure on $\Delta_T(\tilde{x}_0)$. Furthermore, a remaining challenge lies in the convergence requirements for the method for higher values of Lipschitz constant K for highly nonlinear problems. In future work, a method with less stringent convergence requirements could be developed, although we remark that this is inherently difficult due to the global $[0, T]$ domain of the boundary solution conditions. Finally, another general extension of the method would be to provide an application for fractional PDEs, incorporating FBVPs by utilizing a multidimensional splines setup.

CRediT authorship contribution statement

Niels Goedegebure: Writing – original draft, Visualization, Validation, Software, Project administration, Methodology, Investigation, Formal analysis, Conceptualization; **Kateryna Marynets:** Writing – review & editing, Supervision, Methodology, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

Code and data is available on https://github.com/ngoedegebure/fractional_splines. The code used in the examples of Section 5 can be found under `/examples/BVP-paper/`.

Appendix A. Proof of Lemma 3.1

Proof. By Theorem 2.1, we have existence and uniqueness of solutions $\mathbf{x} \in C_{1-\gamma}[0,T]$ to the perturbed IVP (5) with the equivalent integral equation

$$\mathbf{x}(t) = \frac{\tilde{\mathbf{x}}_0}{\Gamma(\gamma)} t^{\gamma-1} + {}_0\mathrm{I}_t^{\alpha}[\mathbf{f}(t,\mathbf{x}(t)) + \nu]. \tag{A.1}$$

By assumption, the boundary condition ${}_0\mathrm{I}_0^{1-\gamma}\mathbf{x}(0) = {}_0\mathrm{I}_T^{1-\gamma}\mathbf{x}(T)$ holds. Propositions 2.1 and 2.2 give

$$\begin{aligned} {}_0\mathrm{I}_t^{1-\gamma}\mathbf{x}(t) &= {}_0\mathrm{I}_t^{1-\gamma}\left[\frac{\tilde{\mathbf{x}}_0}{\Gamma(\gamma)} t^{\gamma-1} + {}_0\mathrm{I}_t^{\alpha}[\mathbf{f}(t,\mathbf{x}(t)) + \nu]\right] \\ &= \tilde{\mathbf{x}}_0 + {}_0\mathrm{I}_t^{1-\gamma+\alpha}\mathbf{f}(t,\mathbf{x}(t)) + {}_0\mathrm{I}_t^{1-\gamma+\alpha}\nu \\ &= \tilde{\mathbf{x}}_0 + {}_0\mathrm{I}_t^{\zeta}\mathbf{f}(t,\mathbf{x}(t)) + {}_0\mathrm{I}_t^{\zeta}\nu. \end{aligned} \tag{A.2}$$

We can now compute the limit using Lemma 13 of [17]:

$$\lim_{t\to 0} {}_0\mathrm{I}_t^{1-\gamma}\mathbf{x}(t) = \tilde{\mathbf{x}}_0. \tag{A.3}$$

Then, substituting Eqs. (A.2) and (A.3) in the boundary conditions of (4) yields

$$\tilde{\mathbf{x}}_0 = \tilde{\mathbf{x}}_0 + {}_0\mathrm{I}_T^{\zeta}\mathbf{f}(T,\mathbf{x}(T)) + \frac{\nu T^{\zeta}}{\Gamma(\zeta+1)},$$

where solving for ν gives

$$\nu = -\Gamma(\zeta+1)T^{-\zeta}{}_0\mathrm{I}_T^{\zeta}\mathbf{f}(T,\mathbf{x}(T)),$$

hence satisfying Eq. (6). Substituting this expression back in (A.1) and evaluating ${}_0\mathrm{I}_t^{\alpha}\nu$ using Proposition 2.2 yields the integral Eq. (7).

For the reverse implication, assume ν is of the form (6). Fractionally integrating integral equation (7) then gives

$$\begin{aligned} {}_0\mathrm{I}_t^{1-\gamma}\mathbf{x}(t) &= {}_0\mathrm{I}_t^{1-\gamma}\left\{\frac{\tilde{\mathbf{x}}_0}{\Gamma(\gamma)} s^{\gamma-1} + {}_0\mathrm{I}_s^{\alpha}\mathbf{f}(s,\mathbf{x}(s)) - \frac{\Gamma(\zeta+1)s^{\alpha}}{\Gamma(\alpha+1)T^{\zeta}}{}_0\mathrm{I}_T^{\zeta}\mathbf{f}(T,\mathbf{x}(T))\right\}(t) \\ &= \tilde{\mathbf{x}}_0 + {}_0\mathrm{I}_t^{\zeta}\mathbf{f}(t,\mathbf{x}(t)) - (t/T)^{\zeta}{}_0\mathrm{I}_T^{\zeta}\mathbf{f}(T,\mathbf{x}(T)), \end{aligned}$$

of which direct computation yields ${}_0\mathrm{I}_0^{1-\gamma}\mathbf{x}(0) = \tilde{\mathbf{x}}_0$ and ${}_0\mathrm{I}_T^{1-\gamma}\mathbf{x}(T) = \tilde{\mathbf{x}}_0$. Hence, we have satisfied the boundary conditions of (4), concluding the proof. □

Appendix B. Proof of Lemma 3.2

Proof. First, note that we can rewrite $\mathcal{T}$ as

$$\{\mathcal{T}\mathbf{y}\}(t) = \underbrace{{}_0\mathrm{I}_t^{\alpha}\mathbf{y}(t) - \frac{\Gamma(\zeta+1)t^{\alpha}}{\Gamma(\alpha+1)T^{\zeta}}{}_0\mathrm{I}_t^{\zeta}\mathbf{y}(T)}_{=:\mathcal{I}\mathbf{Y}(t)} - \underbrace{\frac{\Gamma(\zeta+1)t^{\alpha}}{\Gamma(\alpha+1)T^{\zeta}}{}_t\mathrm{I}_T^{\zeta}\mathbf{y}(T)}_{=:\mathcal{J}\mathbf{Y}(t)}.$$

Writing out $\mathcal{I}$ using Definition 1 and the property of the gamma function $\Gamma(z+1) = z\Gamma(z)$ gives

$$\mathcal{I}\mathbf{y}(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathbf{y}(s)\,\mathrm{d}s - \frac{\zeta t^{\alpha}}{\alpha\Gamma(\alpha)T^{\zeta}}\int_0^t (T-s)^{\zeta-1}\mathbf{y}(s)\,\mathrm{d}s.$$

Rewriting this as one integral and using the substitution $\zeta = 1-\gamma+\alpha$ leads to

$$\begin{aligned} \mathcal{I}\mathbf{y}(t) =& \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathbf{y}(s)\,\mathrm{d}s \\ &- \left(\frac{1-\gamma}{\alpha}+1\right)\frac{t^{\alpha}}{T^{\zeta}}\frac{1}{\Gamma(\alpha)}\int_0^t (T-s)^{\zeta-1}\mathbf{y}(s)\,\mathrm{d}s \\ =& \underbrace{\frac{1}{\Gamma(\alpha)}\int_0^t \left[(t-s)^{\alpha-1} - \frac{t^{\alpha}}{T^{\zeta}}(T-s)^{\zeta-1}\right]\mathbf{y}(s)\,\mathrm{d}s}_{=:\mathcal{I}_1\mathbf{y}} \end{aligned}$$

$$\underbrace{-\left(\frac{1-\gamma}{\alpha}\right)\frac{t^\alpha}{T^\zeta}\frac{1}{\Gamma(\alpha)}\int_0^t (T-s)^{\zeta-1}\mathbf{y}(s)\,\mathrm{d}s}_{=:\mathcal{I}_2\mathbf{y}}.$$

Then, by Hölder's inequality we have:

$$\begin{aligned}|t^{1-\gamma}\mathcal{I}_1\mathbf{y}(t)| &= \left|\frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t \frac{\left[(t-s)^{\alpha-1}-\frac{t^\alpha}{T^\zeta}(T-s)^{\zeta-1}\right]}{s^{1-\gamma}}s^{1-\gamma}\mathbf{y}(s)\,\mathrm{d}s\right|\\ &\le \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t \frac{(t-s)^{\alpha-1}-\frac{t^\alpha}{T^\zeta}(T-s)^{\zeta-1}}{s^{1-\gamma}}\,\mathrm{d}s\|\mathbf{y}\|_{C_{1-\gamma}[0,T]},\end{aligned}$$

since $s\ge 0$ and

$$\begin{aligned}(t-s)^{\alpha-1}-\frac{t^\alpha}{T^\zeta}(T-s)^{\zeta-1} &= (t-s)^{\alpha-1}-T^{\gamma-1}\left(\frac{t}{T}\right)^\alpha (T-s)^{1-\gamma+\alpha-1}\\ &= (t-s)^{\alpha-1}-\left(\frac{T-s}{T}\right)^{1-\gamma}\left(\frac{t}{T}\right)^\alpha (T-s)^{\alpha-1}\\ &\ge (t-s)^{\alpha-1}-\left(\frac{t}{T}\right)^\alpha (T-s)^{\alpha-1}\\ &\ge (t-s)^{\alpha-1}\frac{T-t}{T}\ge 0.\end{aligned}$$

Rewriting and evaluating the integral using the result of Propositions 2.2 and 2.3 results in

$$\begin{aligned}|t^{1-\gamma}\mathcal{I}_1\mathbf{y}(t)| &\le \left[t^{1-\gamma}{}_0\mathrm{I}_t^\alpha\{s^{\gamma-1}\}(t)-\frac{\Gamma(\zeta)t^\zeta}{\Gamma(\alpha)T^\zeta}{}_0\mathrm{I}_t^\zeta\{s^{\gamma-1}\}(T)\right]\|\mathbf{y}\|_{C_{1-\gamma}[0,T]}\\ &= \left[\frac{\Gamma(\gamma)t^\alpha}{\Gamma(\gamma+\alpha)}-\frac{t^\zeta T^\alpha}{\Gamma(\alpha)T^\zeta}\mathrm{B}_{t/T}(\gamma,\zeta)\right]\|\mathbf{y}\|_{C_{1-\gamma}[0,T]}.\end{aligned}$$

Applying the same strategy for $\mathcal{I}_2$, we have

$$|t^{1-\gamma}\mathcal{I}_2\mathbf{y}(t)| \le \left(\frac{1-\gamma}{\alpha}\right)\frac{t^\zeta T^\alpha}{\Gamma(\alpha)T^\zeta}\mathrm{B}_{t/T}(\gamma,\zeta)\|\mathbf{y}\|_{C_{1-\gamma}[0,T]}.$$

For $\mathcal{J}$, proceeding as before but using Proposition 2.4 gives

$$|t^{1-\gamma}\mathcal{J}\mathbf{y}(t)| \le \frac{\zeta t^\zeta T^\alpha}{\Gamma(\alpha+1)T^\zeta}\mathrm{B}_{1-t/T}(\zeta,\gamma)\|\mathbf{y}\|_{C_{1-\gamma}[0,T]}.$$

Finally, combining the bounds using the triangle inequality yields

$$\begin{aligned}&|t^{1-\gamma}\mathcal{T}\mathbf{y}(t)|\\ &= |t^{1-\gamma}\left[\mathcal{I}_1\mathbf{y}(t)-\mathcal{I}_2\mathbf{y}(t)-\mathcal{J}\mathbf{y}(t)\right]|\\ &\le |t^{1-\gamma}\mathcal{I}_1\mathbf{y}(t)|+|t^{1-\gamma}\mathcal{I}_2\mathbf{y}(t)|+|t^{1-\gamma}\mathcal{J}\mathbf{y}(t)|\\ &\le \left(\frac{\Gamma(\gamma)t^\alpha}{\Gamma(\gamma+\alpha)}+\frac{(t/T)^\zeta T^\alpha}{\Gamma(\alpha+1)}\left[(1-\gamma-\alpha)\mathrm{B}_{t/T}(\gamma,\zeta)+\zeta\mathrm{B}_{1-t/T}(\zeta,\gamma)\right]\right)\|\mathbf{y}\|_{C_{1-\gamma}[0,T]}\\ &= \xi(t)\|\mathbf{y}\|_{C_{1-\gamma}[0,T]},\end{aligned}$$

satisfying the inequality and concluding the proof. □

Appendix C. Proof of Lemma 4.2

Proof. Writing out $\mathcal{T}^\varepsilon$ and using Hölder's inequality gives

$$\begin{aligned}t^{1-\gamma}|\mathcal{T}^\varepsilon\mathbf{y}(t)-\mathcal{T}\mathbf{y}(t)| &= t^{1-\gamma}\left|{}_0\mathrm{I}_\varepsilon^\alpha\mathbf{y}(t)-\frac{\Gamma(\zeta+1)t^\alpha}{\Gamma(\alpha+1)T^\zeta}{}_0\mathrm{I}_\varepsilon^\zeta\mathbf{y}(T)\right|\\ &\le \left(t^{1-\gamma}{}_0\mathrm{I}_\varepsilon^\alpha t^{\gamma-1}+\frac{\Gamma(\zeta+1)t^\zeta}{\Gamma(\alpha+1)T^\zeta}{}_0\mathrm{I}_\varepsilon^\zeta T^{\gamma-1}\right)\|\mathbf{y}\|_{C_{1-\gamma}[0,T]}.\end{aligned}$$

By Proposition 2.3, we have

$$t^{1-\gamma}{}_0\mathrm{I}_\varepsilon^\alpha t^{\gamma-1} = \frac{t^\alpha}{\Gamma(\alpha)}\mathrm{B}_{\varepsilon/t}(\gamma,\alpha),$$

and

$$\frac{\Gamma(\zeta+1)t^\zeta}{\Gamma(\alpha+1)T^\zeta}{}_0\mathrm{I}_\varepsilon^\zeta T^{\gamma-1} = \frac{\zeta t^\zeta T^\alpha}{\Gamma(\alpha+1)T^\zeta}\mathrm{B}_{\varepsilon/T}(\gamma,\zeta).$$

Combining both relations then gives rise to the following inequality:

$$t^{1-\gamma}|\mathcal{T}^\varepsilon \mathbf{y}(t) - \mathcal{T}\mathbf{y}(t)| \leq \left(\frac{t^\alpha}{\Gamma(\alpha)}\mathrm{B}_{\varepsilon/t}(\gamma,\alpha) + \frac{\zeta t^\zeta T^\alpha}{\Gamma(\alpha+1)T^\zeta}\mathrm{B}_{\varepsilon/T}(\gamma,\zeta)\right)\|\mathbf{y}\|_{C_{1-\gamma}[0,T]}. \tag{C.1}$$

To obtain an upper bound to this expression independent on t, we start with the first term of the sum of (C.1). Note that

$$t^{1-\gamma}{}_0\mathrm{I}^\alpha_\varepsilon t^{\gamma-1} = \frac{1}{\Gamma(\alpha)}\int_0^\varepsilon t^{1-\gamma}(t-s)^{\alpha-1}\,\mathrm{d}s.$$

Using the product rule yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(t^{1-\gamma}{}_0\mathrm{I}^\alpha_\varepsilon t^{\gamma-1}\right) = \frac{1}{\Gamma(\alpha)}\int_0^\varepsilon (\gamma-1)t^{-\gamma}(t-s)^{\alpha-1} + t^{1-\gamma}(\alpha-1)(t-s)^{\alpha-2}\,\mathrm{d}s.$$

Since $0 \leq s \leq \varepsilon < t$ and furthermore $0 < \alpha < 1$ and $0 < \gamma \leq 1$, this is an integral of a negative function and thus $\frac{\mathrm{d}}{\mathrm{d}t}\left(t^{1-\gamma}{}_0\mathrm{I}^\alpha_\varepsilon t^{\gamma-1}\right) < 0$. Hence, we know $t^{1-\gamma}{}_0\mathrm{I}^\alpha_\varepsilon t^{\gamma-1} = \frac{t^\alpha}{\Gamma(\alpha)}\mathrm{B}_{\varepsilon/t}(\gamma,\alpha)$ is decreasing and must attain its supremum for $t=\varepsilon$.

For the second expression in the sum of (C.1), since $\zeta = 1-\gamma+\alpha$ satisfies $0<\zeta\leq 1$, we have $\frac{\zeta t^\zeta T^\alpha}{\Gamma(\alpha+1)T^\zeta}\mathrm{B}_{\varepsilon/T}(\gamma,\zeta) \leq \frac{\zeta T^\alpha}{\Gamma(\alpha+1)}\mathrm{B}_{\varepsilon/T}(\gamma,\zeta)$. Hence, in total, taking the supremum over t for (C.1) gives

$$\|\mathcal{T}^\varepsilon \mathbf{y} - \mathcal{T}\mathbf{y}\|_{C_{1-\gamma}[\varepsilon,T]} \leq \left(\frac{\varepsilon^\alpha \mathrm{B}(\gamma,\alpha)}{\Gamma(\alpha)} + \frac{\zeta T^\alpha \mathrm{B}_{\varepsilon/T}(\gamma,\zeta)}{\Gamma(\alpha+1)}\right)\|\mathbf{y}\|_{C_{1-\gamma}[0,T]}, \tag{C.2}$$

satisfying the bound. Finally, for the convergence order as $\varepsilon \downarrow 0$, it is clear the first summation term of (C.2) has order $\mathcal{O}(\varepsilon^\alpha)$. For the second term of the bound, write:

$$\begin{aligned}
\mathrm{B}_{\varepsilon/T}(\gamma,\zeta) &= \int_0^{\varepsilon/T} s^{\gamma-1}(1-s)^{\zeta-1}\,\mathrm{d}s \\
&\leq \left(1-\frac{\varepsilon}{T}\right)^{\zeta-1}\int_0^{\varepsilon/T} s^{\gamma-1}\,\mathrm{d}s \\
&= \left(1-\frac{\varepsilon}{T}\right)^{\zeta-1}\frac{\varepsilon^\gamma}{\gamma T^\gamma} \\
&= (1+\mathcal{O}(\varepsilon))\frac{\varepsilon^\gamma}{\gamma T^\gamma} = \mathcal{O}(\varepsilon^\gamma).
\end{aligned}$$

Combining this result with the convergence order of the first term and noting that $\mathcal{O}(\varepsilon^\alpha)$ dominates since $\alpha \leq \gamma \leq 1$, we have $\Theta_\varepsilon = \mathcal{O}(\varepsilon^\alpha)$, proving the final claim. □

Appendix D. Proof of Lemma 4.3

Proof. First, notice that by the construction of $\mathcal{T}_\varepsilon$,

$$\begin{aligned}
&|t'^{1-\gamma}\mathcal{T}^\varepsilon \mathbf{y}(t') - t^{1-\gamma}\mathcal{T}^\varepsilon \mathbf{y}(t)| \\
&\leq |t'^{1-\gamma}{}_\varepsilon\mathrm{I}^\alpha_{t'}\mathbf{y}(t') - t^{1-\gamma}{}_\varepsilon\mathrm{I}^\alpha_t\mathbf{y}(t)| \\
&+\frac{(t'^{1-\gamma+\alpha} - t^{1-\gamma+\alpha})\Gamma(\zeta+1)}{\Gamma(\alpha+1)T^\zeta}|{}_\varepsilon\mathrm{I}^\zeta_T\mathbf{y}(T)|.
\end{aligned} \tag{D.1}$$

Using Definition 1 and the triangle inequality, we can write the first expression of (D.1) as

$$\begin{aligned}
&|t'^{1-\gamma}{}_\varepsilon\mathrm{I}^\alpha_{t'}\mathbf{y}(t') - t^{1-\gamma}{}_\varepsilon\mathrm{I}^\alpha_t\mathbf{y}(t)| \\
&= \frac{1}{\Gamma(\alpha)}\left|\int_\varepsilon^{t'}(t'-s)^{\alpha-1}t'^{1-\gamma}\mathbf{y}(s)\,\mathrm{d}s - \int_\varepsilon^t (t-s)^{\alpha-1}t^{1-\gamma}\mathbf{y}(s)\,\mathrm{d}s\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\left|\int_\varepsilon^t \left[(t'-s)^{\alpha-1}t'^{1-\gamma} - (t-s)^{\alpha-1}t^{1-\gamma}\right]\mathbf{y}(s)\,\mathrm{d}s\right| \\
&+\frac{1}{\Gamma(\alpha)}\left|\int_t^{t'}(t'-s)^{\alpha-1}t'^{1-\gamma}\mathbf{y}(s)\,\mathrm{d}s\right|.
\end{aligned} \tag{D.2}$$

Then, by the same strategy as applied in Lemma 4.2, we have that $\frac{\mathrm{d}}{\mathrm{d}t}(t-s)^{\alpha-1}t^{1-\gamma} < 0$ for all $0 \leq s \leq t$. Hence, we know that $(t-s)^{\alpha-1}t^{1-\gamma} > (t'-s)^{\alpha-1}t'^{1-\gamma}$ since $t' > t$. Thus, using Hölder's inequality once again yields

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\left|\int_\varepsilon^t \left[(t'-s)^{\alpha-1}t'^{1-\gamma} - (t-s)^{\alpha-1}t^{1-\gamma}\right]\mathbf{y}(s)\,\mathrm{d}s\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_\varepsilon^t \left[(t-s)^{\alpha-1}t^{1-\gamma} - (t'-s)^{\alpha-1}t'^{1-\gamma}\right]\left|\mathbf{y}(s)\right|\mathrm{d}s
\end{aligned}$$

$$\le \frac{1}{\Gamma(\alpha)}\int_0^t \left[(t-s)^{\alpha-1}t^{1-\gamma} - (t'-s)^{\alpha-1}t'^{1-\gamma}\right]s^{\gamma-1}\,\mathrm{d}s\,\|\mathbf{y}\|_{C_{1-\gamma}[\varepsilon,T]}.$$

Evaluating this expression using Propositions 2.2 and 2.3 then gives

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_0^t \left[(t-s)^{\alpha-1}t^{1-\gamma} - (t'-s)^{\alpha-1}t'^{1-\gamma}\right]s^{\gamma-1}\,\mathrm{d}s\,\|\mathbf{y}\|_{C_{1-\gamma}[\varepsilon,T]}\\
&= \frac{1}{\Gamma(\alpha)}\left|t^{\alpha}\mathrm{B}(\gamma,\alpha) - t'^{\alpha}\mathrm{B}_{t/t'}(\gamma,\alpha)\right|\|\mathbf{y}\|_{C_{1-\gamma}[\varepsilon,T]}\\
&= \frac{1}{\Gamma(\alpha)}\left|t^{\alpha}\mathrm{B}(\gamma,\alpha) - t'^{\alpha}\mathrm{B}(\gamma,\alpha) + t'^{\alpha}\mathrm{B}(\gamma,\alpha) - t'^{\alpha}\mathrm{B}_{t/t'}(\gamma,\alpha)\right|\|\mathbf{y}\|_{C_{1-\gamma}[\varepsilon,T]}\\
&\le \frac{1}{\Gamma(\alpha)}\left((t'-t)^{\alpha}\mathrm{B}(\gamma,\alpha) + t'^{\alpha}\int_{t/t'}^{1} s^{\gamma-1}(1-s)^{\alpha-1}\,\mathrm{d}s\right)\|\mathbf{y}\|_{C_{1-\gamma}[\varepsilon,T]}\\
&\le \frac{1}{\Gamma(\alpha)}\left((t'-t)^{\alpha}\mathrm{B}(\gamma,\alpha) + t'^{\alpha}(t/t')^{\gamma-1}\int_{t/t'}^{1}(1-s)^{\alpha-1}\,\mathrm{d}s\right)\|\mathbf{y}\|_{C_{1-\gamma}[\varepsilon,T]}\\
&= \frac{(t'-t)^{\alpha}}{\Gamma(\alpha+1)}\left(\alpha\mathrm{B}(\gamma,\alpha) + \left(\frac{t'}{t}\right)^{1-\gamma}\right)\|\mathbf{y}\|_{C_{1-\gamma}[\varepsilon,T]}.
\end{aligned}$$

For the second term of (D.2), we write

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\left|\int_t^{t'}(t'-s)^{\alpha-1}t'^{1-\gamma}\mathbf{y}(s)\,\mathrm{d}s\right|\\
&\le \frac{1}{\Gamma(\alpha)}\int_t^{t'}(t'-s)^{\alpha-1}t'^{1-\gamma}s^{\gamma-1}\,\mathrm{d}s\,\|\mathbf{y}\|_{C_{1-\gamma}[\varepsilon,T]}\\
&\le \left(\frac{t'}{t}\right)^{1-\gamma}\frac{1}{\Gamma(\alpha)}\int_t^{t'}(t'-s)^{\alpha-1}\,\mathrm{d}s\,\|\mathbf{y}\|_{C_{1-\gamma}[\varepsilon,T]}\\
&= \left(\frac{t'}{t}\right)^{1-\gamma}\frac{(t'-t)^{\alpha}}{\Gamma(\alpha+1)}\|\mathbf{y}\|_{C_{1-\gamma}[\varepsilon,T]}.
\end{aligned}$$

Combining these two estimates gives

$$|t'^{1-\gamma}{}_{\varepsilon}\mathrm{I}_{t'}^{\alpha}\mathbf{y}(t') - t^{1-\gamma}{}_{\varepsilon}\mathrm{I}_{t}^{\alpha}\mathbf{y}(t)| \le \frac{(t'-t)^{\alpha}}{\Gamma(\alpha+1)}\left(\alpha\mathrm{B}(\gamma,\alpha) + 2\left(\frac{t'}{t}\right)^{1-\gamma}\right)\|\mathbf{y}\|_{C_{1-\gamma}[\varepsilon,T]}.$$

For the second part of our original expression (D.1), it holds that

$$\begin{aligned}
|{}_{\varepsilon}\mathrm{I}_{T}^{\zeta}\mathbf{y}(T)| \le {}_{0}\mathrm{I}_{T}^{\zeta}T^{\gamma-1}\|\mathbf{y}\|_{C_{1-\gamma}[\varepsilon,T]} &= \frac{\Gamma(\gamma)}{\Gamma(\zeta+\gamma)}T^{\zeta+\gamma-1}\|\mathbf{y}\|_{C_{1-\gamma}[\varepsilon,T]}\\
&= \frac{\mathrm{B}(\gamma,\zeta)}{\Gamma(\zeta)}T^{\alpha}\|\mathbf{y}\|_{C_{1-\gamma}[\varepsilon,T]}.
\end{aligned}$$

Thus,

$$\frac{(t'^{1-\gamma+\alpha} - t^{1-\gamma+\alpha})\Gamma(\zeta+1)}{\Gamma(\alpha+1)T^{\zeta}}|{}_{\varepsilon}\mathrm{I}_{T}^{\zeta}\mathbf{y}(T)| \le \frac{(t'-t)^{\zeta}T^{\alpha}}{\Gamma(\alpha+1)T^{\zeta}}\zeta\mathrm{B}(\gamma,\zeta)\|\mathbf{y}\|_{C_{1-\gamma}[\varepsilon,T]},$$

where we use the fact that $\zeta = 1-\gamma+\alpha$ and $\alpha \le \zeta \le 1$. Hence, in total:

$$\begin{aligned}
&|t'^{1-\gamma}\mathcal{T}^{\varepsilon}\mathbf{y}(t') - t^{1-\gamma}\mathcal{T}^{\varepsilon}\mathbf{y}(t)|\\
&\le \left(\frac{(t'-t)^{\alpha}}{\Gamma(\alpha+1)}\left(\alpha\mathrm{B}(\gamma,\alpha) + 2\left(\frac{t'}{t}\right)^{1-\gamma}\right) + \frac{(t'-t)^{\zeta}T^{\alpha}}{\Gamma(\alpha+1)T^{\zeta}}\zeta\mathrm{B}(\gamma,\zeta)\right)\|\mathbf{y}\|_{C_{1-\gamma}[\varepsilon,T]},
\end{aligned}\tag{D.3}$$

satisfying the bound. Finally, the convergence order $\Omega(t,t') = \mathcal{O}\big((t'-t)^{\alpha}\big)$ for $(t'-t)\downarrow 0$ follows from (D.3) since $(t'-t)\downarrow 0$ gives $\left(\frac{t'}{t}\right)^{1-\gamma} \to 1$ and furthermore $\alpha \le \zeta \le 1$. □

Appendix E. Proof of Lemma 4.4

Proof. Weighed continuity $\mathcal{T}^{\varepsilon,q}\mathbf{y} \in C_{1-\gamma}[\varepsilon,T]$ follows from Proposition 4.1, the continuity of $S_{\mathcal{A}}^{q}$ and the fact that $t^{\gamma-1}$ is continuous and bounded for $\varepsilon < t \le T$. For the bound, we first write

$$|t^{1-\gamma}\mathcal{T}^{\varepsilon,q}\mathbf{y}(t) - t^{1-\gamma}\mathcal{T}^{\varepsilon}\mathbf{y}(t)| = |S_{\mathcal{A}}^{q}t^{1-\gamma}\mathcal{T}^{\varepsilon}\mathbf{y}(t) - t^{1-\gamma}\mathcal{T}^{\varepsilon}\mathbf{y}(t)|.$$

For a given $f \in C[\varepsilon, T]$, using the definition of the splines operator, we have

$$|f(t) - S^q_{\mathcal{A}} f(t)| = \left| \sum_{i=0}^{k} \mathbb{1}_{A_i} [\mathbb{1}_{\bar{A}_i} f(t) - B^q\, \mathbb{1}_{\bar{A}_i} f(t)] \right|.$$

Here, $B^q\, \mathbb{1}_{\bar{A}_i}$ denotes the Bernstein polynomial operator of Definition 5 applied to $f : \bar{A}_i \to \mathbb{R}$. Now, using the triangle inequality and Theorem 2.2 gives

$$\begin{aligned}\left| \sum_{i=0}^{k} \mathbb{1}_{A_i} [\mathbb{1}_{\bar{A}_i} f(t) - B^q\, \mathbb{1}_{\bar{A}_i} f(t)] \right| &\le \sum_{i=0}^{k} \mathbb{1}_{A_i}(t) \left| \mathbb{1}_{\bar{A}_i} f(t) - B^q\, \mathbb{1}_{\bar{A}_i} f(t) \right| \\ &\le \sum_{i=0}^{k} \mathbb{1}_{A_i}(t)\, \frac{5}{4}\, \omega\!\left(\mathbb{1}_{\bar{A}_i} f, \frac{h_i}{\sqrt{q}} \right) \\ &\le \frac{5}{4} \sup_{i \in \{0,\ldots,k\}} \omega\!\left(\mathbb{1}_{\bar{A}_i} f, \frac{h_i}{\sqrt{q}} \right).\end{aligned}$$

Choosing $f(t) = t^{1-\gamma} \mathcal{T}^{\varepsilon} \mathbf{y}(t)$ and using the result of Lemma 4.3 results in

$$\begin{aligned}\sup_{t \in (\varepsilon, T]} |S^q_{\mathcal{A}} t^{1-\gamma} \mathcal{T}^{\varepsilon} \mathbf{y}(t) - t^{1-\gamma} \mathcal{T}^{\varepsilon} \mathbf{y}(t)| &\le \frac{5}{4} \sup_{i \in \{0,\ldots,k\}} \omega\!\left(\mathbb{1}_{\bar{A}_i} f, \frac{h_i}{\sqrt{q}} \right) \\ &\le \frac{5}{4} \sup_{A_i \in \mathcal{A}} \sup_{t,t' \in A_i} \sup_{|t'-t| \le h_i/\sqrt{q}} \Omega(t', t),\end{aligned}$$

satisfying the claim. □

Appendix F. Convergence result tables

Table F.1
Convergence results for system (20) in decreasing knot size h with $q = 1$, $\varepsilon = 10^{-10}$.

h	mean weighted error	sup weighted error	approx. $\Delta_T(\tilde{x}_0)$	total time (s)
2^{-0}	$1.172 \cdot 10^{-1}$	$3.208 \cdot 10^{-1}$	$-1.590 \cdot 10^{0}$	$6.113 \cdot 10^{-4}$
2^{-1}	$3.291 \cdot 10^{-2}$	$1.537 \cdot 10^{-1}$	$-1.597 \cdot 10^{0}$	$5.246 \cdot 10^{-4}$
2^{-2}	$9.288 \cdot 10^{-3}$	$8.000 \cdot 10^{-2}$	$-1.599 \cdot 10^{0}$	$5.876 \cdot 10^{-4}$
2^{-3}	$2.637 \cdot 10^{-3}$	$4.402 \cdot 10^{-2}$	$-1.599 \cdot 10^{0}$	$6.440 \cdot 10^{-4}$
2^{-4}	$7.531 \cdot 10^{-4}$	$2.506 \cdot 10^{-2}$	$-1.600 \cdot 10^{0}$	$9.170 \cdot 10^{-4}$
2^{-5}	$2.163 \cdot 10^{-4}$	$1.455 \cdot 10^{-2}$	$-1.600 \cdot 10^{0}$	$1.764 \cdot 10^{-3}$
2^{-6}	$6.241 \cdot 10^{-5}$	$8.537 \cdot 10^{-3}$	$-1.600 \cdot 10^{0}$	$4.119 \cdot 10^{-3}$
2^{-7}	$1.807 \cdot 10^{-5}$	$5.041 \cdot 10^{-3}$	$-1.600 \cdot 10^{0}$	$1.206 \cdot 10^{-2}$
2^{-8}	$5.227 \cdot 10^{-6}$	$2.987 \cdot 10^{-3}$	$-1.600 \cdot 10^{0}$	$4.056 \cdot 10^{-2}$

Table F.2
Convergence results for system (20), increasing splines order q with knot size $h = 10^{-2}$ and time shift $\varepsilon = 10^{-10}$.

q	mean weighted error	sup weighted error	approx. $\Delta_T(\tilde{x}_0)$	total time (s)
1	$2.809 \cdot 10^{-5}$	$6.081 \cdot 10^{-3}$	$-1.600 \cdot 10^{0}$	$4.061 \cdot 10^{-2}$
2	$1.478 \cdot 10^{-5}$	$3.952 \cdot 10^{-3}$	$-1.600 \cdot 10^{0}$	$2.445 \cdot 10^{-2}$
4	$7.486 \cdot 10^{-6}$	$2.484 \cdot 10^{-3}$	$-1.600 \cdot 10^{0}$	$7.687 \cdot 10^{-2}$
8	$3.701 \cdot 10^{-6}$	$1.521 \cdot 10^{-3}$	$-1.600 \cdot 10^{0}$	$2.741 \cdot 10^{-1}$
16	$1.808 \cdot 10^{-6}$	$9.224 \cdot 10^{-4}$	$-1.600 \cdot 10^{0}$	$1.038 \cdot 10^{0}$

Table F.3
Convergence results for system (20) in decreasing time shift values ε with polynomial order $q = 1$ and knot size $h = 1/2$.

ε	mean weighted error	sup weighted error	approx. $\Delta_T(\tilde{x}_0)$	total time (s)	$x^{q,\varepsilon}(\varepsilon)$
2^{-1}	$2.224 \cdot 10^{-1}$	$2.224 \cdot 10^{-1}$	$-1.563 \cdot 10^{0}$	$3.561 \cdot 10^{-2}$	$-2.770 \cdot 10^{-1}$
2^{-2}	$7.479 \cdot 10^{-2}$	$7.479 \cdot 10^{-2}$	$-1.590 \cdot 10^{0}$	$7.863 \cdot 10^{-3}$	$2.569 \cdot 10^{-1}$
2^{-3}	$2.444 \cdot 10^{-2}$	$2.444 \cdot 10^{-2}$	$-1.597 \cdot 10^{0}$	$7.788 \cdot 10^{-3}$	$7.353 \cdot 10^{-1}$
2^{-4}	$7.886 \cdot 10^{-3}$	$7.886 \cdot 10^{-3}$	$-1.599 \cdot 10^{0}$	$7.927 \cdot 10^{-3}$	$1.181 \cdot 10^{0}$
2^{-5}	$2.528 \cdot 10^{-3}$	$2.528 \cdot 10^{-3}$	$-1.600 \cdot 10^{0}$	$7.642 \cdot 10^{-3}$	$1.622 \cdot 10^{0}$
2^{-6}	$8.079 \cdot 10^{-4}$	$8.079 \cdot 10^{-4}$	$-1.600 \cdot 10^{0}$	$7.992 \cdot 10^{-3}$	$2.083 \cdot 10^{0}$
2^{-7}	$8.866 \cdot 10^{-4}$	$8.866 \cdot 10^{-4}$	$-1.600 \cdot 10^{0}$	$7.747 \cdot 10^{-3}$	$2.585 \cdot 10^{0}$
2^{-8}	$1.700 \cdot 10^{-3}$	$1.700 \cdot 10^{-3}$	$-1.600 \cdot 10^{0}$	$7.690 \cdot 10^{-3}$	$3.151 \cdot 10^{0}$
2^{-9}	$2.559 \cdot 10^{-3}$	$2.559 \cdot 10^{-3}$	$-1.600 \cdot 10^{0}$	$7.489 \cdot 10^{-3}$	$3.802 \cdot 10^{0}$

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