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De memristor in feedback control systemen.

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Contents

Abstract

The creation of a passive element with memristive properties by HP has caused a resurgence of research being conducted regarding the memristor. This paper shows how such an element could be used in feedback control systems, with a particular interest in the describing function of such a memristive element. After determining the describing function of a memristor we look at some properties of a memristor and also look at and discuss two applications. The first application is causing a periodic orbit in a feedback control system and the second is creating an MID controller.

1 Introduction

In this section we will present some knowledge about electric forces and feedback control systems required to read this paper. In Section 2 we will explain what a memristor is. In Section 3 we will discuss the memristor developed by Hewlett-Packard. In Section 4 we will talk about the describing function method and develop a describing function for a memristor. In Section 5 we will discuss the properties of a memristor. In Section 6 we will talk about the stability of a feedback control system with a memristor incorporated. In Section 7 we will talk about limit cycles and discuss whether limit cycles can occur in systems with a memristor. In Section 8 we will show how to incorporate the memristor in an industrial controller. In Section 9 a few simulations of systems discussed in the previous sections will be shown and finally in Section 10 conclusions of the research done will be given.

1.1 Electric Variables

When it comes to electricity, there are four main electric variables: Potential (Voltage), Current, Charge and Flux. Where Potential (Voltage) is denoted by V, Current by I, Charge by Q and Flux by ϕ.

Definition 1. *The following mathematical relations between the electic forces exist:*

1. $\frac{dQ}{dt} = I(t)$,
2. $\frac{d\phi}{dt} = V(t)$,
3. $V(t) = RI(t)$,
4. $\phi(t) = LI(t)$ *and*
5. $Q(t) = CV(t)$.

Where R is resistance, C is conductance and L is inductance.

1.2 A basic feedback control system

The fundamental idea behind a control system is that when given an input signal, accompanied by disturbances, it attempts to filter out these disturbances and lets the input signal converge to the desired output signal. Below is a basic representation of a feedback control system in Figure 1:
As shown, there are five fundamental elements. Namely the input, the controller, the plant, the disturbance and the output. Further on in this article there will be more elaboration on a system like this and how a memristor fits into a feedback control system.

2 The Memristor

In this section we will discuss the history and the mathematical derivation of the memristor. Then we will look at the physical memristor developed at Hewlett-Packard (to be named HP from here on).

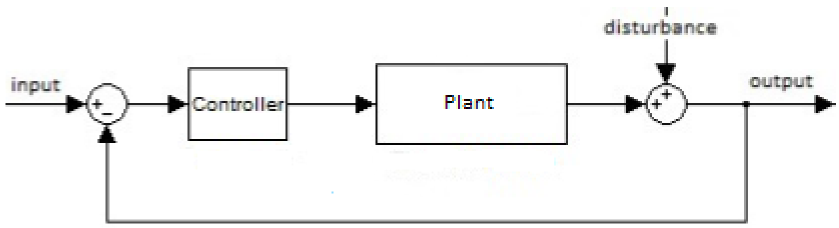

Figure 1: A basic feedback control system.

2.1 The history of the memristor

The existence of a memristor was proposed by Leon O. Chua in 1971 [7], as a fourth basic circuit element, the other three being the resistor, inductor and capacitor. Though it's theoretical existence has been first documented in Leon O. Chua's work, a passive element with memristive properties could not be made. In 2008 an article in Nature [4] was published stating that a team of researchers at HP managed to create a passive element with memristive properties.

2.2 Mathematical derivation

The image below shows the relations of all electrical forces:

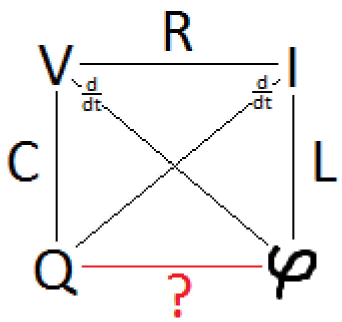

Figure 2: The four main electrical forces and their relations.

From figure 2 it is now clear that the link between Flux and Charge is missing. The mathematical link can be explained by the following theorem:

Theorem 1. *Defining the relation between Charge and Flux as defining Flux to be a function of Charge is equivalent with a resistance which value depends on the amount of Current that went through it over time and its last known value:*

$$V(t) = \hat{R}\left(Q(0) + \int_0^t I(\tau)d\tau\right) I(t). \tag{1}$$

In other words, it is a resistance with memory (hence the term memristor).

Proof. Define the Flux ϕ as the function $\hat{\phi}$ of the charge Q. Since the function should be equal to the Flux for all t, the same must be true if we derive both sides to t, which yields

$$\frac{d\phi}{dt}(t) = \frac{d}{dt}\left(\hat{\phi}\left(Q(t)\right)\right). \tag{2}$$

Using the chain rule of derivatives, this yields $\frac{d\phi}{dt}(t) = \frac{d\hat{\phi}}{dQ}\left(Q(t)\right)\frac{dQ}{dt}(t)$. Now using the known relations from definition 1 this can be written as

$$V(t) = \frac{d\hat{\phi}}{dQ}\left(Q(t)\right) I(t). \tag{3}$$

From (2) - (3) we can see that we have derived something that resembles Ohm's Law. This is why the function $\frac{d\hat{\phi}}{dQ}$ will be redefined as $\hat{R}$. Now we use definition1 again to write $Q(t) = Q(0) + \int_0^t I(\tau)d\tau$. Combining all this yields

$$V(t) = \hat{R}\left(Q(0) + \int_0^t I(\tau)d\tau\right) I(t), \tag{4}$$

showing that equation (1) is correct.

□

2.3 Defining the memristor in a control system

From the previous section we saw that a memristor is basically a function of its initial charge and the integral of the current from the beginning to the present, multiplied by the current at that point in time to yield the potential. Analogous to control systems, it can be said that the current is the input signal and the potential is the output signal. To define a memristive element in a control system, we have the following theorem:

Theorem 2. *An element in a feedback control system with input $u(t)$ and output $y(t)$ defined by the system*

$$\begin{aligned} \dot{x} &= u, \\ y &= f(x)u, \end{aligned} \tag{5}$$

is a memristive element which is described by the equation

$$y(t) = f\left(x(0) + \int_0^t u(\tau)d\tau\right) u(t). \tag{6}$$

Proof. Using equation (1) and setting $I(t) = u(t), Q(0) = x(0), \hat{R} = f$ and $V(t) = y(t)$ we obtain the desired result. □

3 The HP-memristor

As mentioned earlier, HP patentised a passive element which has memristive properties in 2008 [4]. In the following section will explain the physical and mathematical properties of this memristor.

3.1 Introduction of the HP-memristor

The HP-memristor consists of three layers: platinum, titanium oxide and once again platinum. These three layers combined create a thin film of length D, with its proportions on the nanometre scale. This layer is sandwiched in between two thin metal films. This creates a semi-conductor which can be fully doped, undoped or anything inbetween. We call the resistance of this device R_{ON} if it's fully doped and R_{OFF} if it is fully undoped. Next we introduce the function $w(t)$, with the same unit a D. $w(t)$ describes the measure of "dopedness" of the memristor. (In other words: $w(t) \in [0, D]$ and $w(t) = 0$ means the device is fully undoped and if $w(t) = D$ means the device is fully doped. Finally, we have μ_V, which describes ion mobility with the unit $\mathrm{cm}^2\mathrm{s}^{-1}\mathrm{V}^{-1}$.

3.2 Mathematical properties of the HP-memristor

The equations defining the HP-memristor are defined as follows:

$$V(t) = \left(R_{ON}\frac{w(t)}{D} + R_{OFF}\left(1 - \frac{w(t)}{D}\right)\right) I(t), \tag{7}$$

$$\dot{w}(t) = \frac{\mu_V R_{ON}}{D} I(t). \tag{8}$$

To show that this is a system with memristive properties, define $\dot{h}(t) = D(\mu_V R_{ON})^{-1}\dot{w}(t)$ so that $h(t) = D(\mu_V R_{ON})^{-1}w(t)$ and we can write equations (7) - (8) as the system:

$$V(t) = \left(R_{OFF} + \left(\frac{\mu_V R_{ON}(R_{ON} - R_{OFF})}{D^2}\right) h(t)\right) I(t), \tag{9}$$

$$\dot{h}(t) = I(t). \tag{10}$$

Now the system is in the form of equation (5) which shows that equations (9) - (10) is a system with memristive properties. Other than the article published in Nature, where the assumption $R_{OFF} \gg R_{ON}$ is made, this article chooses not to do so. Integrating equation (8) yields $w(t) = w_0 + \frac{\mu_V R_{ON}}{D} Q(t), w_0 = w(0)$. One should note that in the article published in Nature the variable w_0 is omitted. Substituting this in equation (7) we obtain

$$V(t) = \left(R_{ON}\frac{w_0}{D} + R_{OFF}\left(1 - \frac{w_0}{D}\right) + (R_{ON} - R_{OFF})\frac{\mu_V R_{ON}}{D^2} Q(t)\right) I(t). \tag{11}$$

Which means that according to equation (5) that $\hat{R}(x) = a + bx$ with $a = R_{ON}\frac{w_0}{D} + R_{OFF}\left(1 - \frac{w_0}{D}\right) + (R_{ON} - R_{OFF})\frac{\mu_V R_{ON}}{D^2} Q(0)$ and $b = (R_{ON} - R_{OFF})\frac{\mu_V R_{ON}}{D^2}$. By setting the variables to values obtained from the article in Nature: $R_{ON} = 0.5\Omega$, $R_{OFF} = 190\Omega$, $w_0 = 0$, $D = 10^{-8}m$ and $\mu_V = 10^{-14}m^2 s^{-1} V^{-1}$ and setting the sinusodial current as $7.5 \cdot 10^{-3}\sin(t)$. We use Simulink to plot the input against the output of the memristor so that a pinched hysteresis loop is obtained as shown in Figure 3:

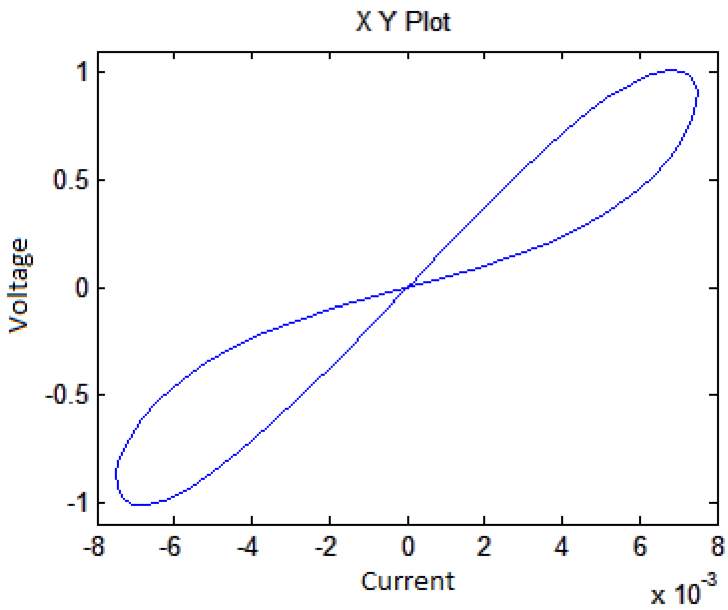

Figure 3: The pinched hysteresis loop produced by the HP-memristor.

3.3 A general memristor

Due to the form of the HP-memristor as seen in equation (11) and using theorem 2, the memristors being used later on in this article will have the following general form:

$$y(t) = \left(a + b\int_0^t u(\tau)d\tau\right) u(t). \tag{12}$$

Where $a = \tilde{a} + \tilde{b}x(0)$ and $b = \tilde{b}$ met $\tilde{a}, \tilde{b} \in \mathbb{R}$.

4 The describing function method

Because the memristor is a nonlinear element an important tool is needed to determine the stability of a system containing a memristor. This tool is called the describing function method. This article will use the four conditions given by Jean-Jacques E. Slotine and Weiping Li [2] which are necessary to define such a describing function.

4.1 Conditions

As mentioned before, the describing function method only works if four conditions are met.

- **Condition 1: There is only one nonlinear element.** In this case, the only nonlinear element we are using is a memristor. So this condition is met.

- **Condition 2: The nonlinear element is time invariant.** For a memristor to be time invariant, it needs to hold that a timeshift in the input $u(t)$ yields the same timeshift in the output $y(t)$. In other words, if $u_s(t) = u(t - t_s)$ is the input, then

$y_s(t) = y(t - ts)$ should be the output with $t_s \in [0, t]$ and so that $t - t_s \in [0, t]$ which means $u_s(t)$ and $y_s(t)$ are well defined in all points where $u(t)$ and $y(t)$ are also well defined. Consider the general memristor defined in equation (12). Using $u_s(t)$ as the input, we obtain

$$\left(a + b\int_0^t u_s(\tau)d\tau\right) u_s(t). \tag{13}$$

The definition of $u_s(t)$ allows us to write (13) as

$$\left(a + b\int_0^t u(\tau - t_s)d\tau\right) u(t - t_s) = \left(a + b\int_{-t_s}^{t-t_s} u(\tau)d\tau\right) u(t - t_s). \tag{14}$$

and since $u(t) = 0$ for all $t < 0$, (14) simplifies to

$$\left(a + b\int_0^{t-t_s} u(\tau)d\tau\right) u(t - t_s) = y(t - t_s) = y_s(t). \tag{15}$$

Equations (13) - (15) prove that a timeshift in the input yields the same timeshift in the output, which means the memristor is time invariant, thus satisfying condition 2.

- **Condition 3: Corresponding to a sinusoidal input $u(t) = A\sin(\omega t)$ and the output $y(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega t) + b_k \sin(k\omega t)$ only the fundamental element $y_1(t) = a_1\cos(\omega t) + b_1\sin(\omega t)$ has to be considered. In other words the plant (which is the function describing the system's behaviour without any form of control) following the memristor has low-pass properties.**

  The plant following the memristor looks like $G(s) = \frac{K}{p_n(s)}$ with K a positive, non-zero constant and $p_n(s) = a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0$ with $a_n \neq 0$ and $n \geq 1$. After substituting $s = j\omega$, this plant has low-pass properties if

$$\|G(j\omega)\| \gg \|G(jk\omega)\|, \forall k \geq 2 \tag{16}$$

  holds. Equation (16) is equivalent to

$$\frac{\|G(jk\omega)\|}{\|G(j\omega)\|} \ll 1, \forall k \geq 2. \tag{17}$$

  By using the definition of G(s), equation (17) is equivalent to

$$\frac{\|p_n(j\omega)\|}{\|p_n(jk\omega)\|} \ll 1, \forall k \geq 2. \tag{18}$$

  For large ω , equation (18) is going to approximate $\|a_n(j\omega)^n\|\|a_n(jk\omega)^n\|^{-1} = k^{-n} \ll 1, \forall k \geq 2$. This shows the chosen plant following the memristor has low-pass properties for a sufficiently large n. Thus showing that plants exist to satisfy condition 3.

- **Condition 4: The nonlinearity is odd.** To elaborate on this condition: The output must be symmetric about the origin with respect to the input. This is so that $a_0 = 0$ in the Fourier expansion of the output. Since condition 3 states $u(t) = A\sin(\omega t)$, it follows that $y(t) = (aA + bA^2\omega^{-1})\sin(\omega t) - (bA^2(2\omega)^{-1})\sin(2\omega t)$ which is an odd function. This means condition 4 is satisfied.

4.2 The describing function of a memristor

If all conditions are satisfied, the general form of a describing function can be determined. Using condition 3 the output of the nonlinear element can be rewritten as $M\sin(\omega t + \phi)$ where $\phi = \tan^{-1}\left(\frac{a_1}{b_1}\right)$ and $M = \sqrt{a_1^2 + b_1^2}$. In the complex plane, the input can be written as $u(t) = Ae^{j\omega t}$ and the output as $y(t) = Me^{j(\omega t+\phi)}$. The general idea behind the describing function method is that the nonlinear element gets replaced by an equivalent gain, here defined as $N(A, \omega)$:

$$Me^{j(\omega t+\phi)} = N(A, \omega)Ae^{j\omega t}. \tag{19}$$

Equation (19) shows that $N(A, \omega)$ could also be seen as the complex ratio of the fundamental component of the nonlinear element by the sinusoid, namely:

$$N(A, \omega) = \frac{Me^{j(\omega t+\phi)}}{Ae^{j\omega t}} = \frac{M}{A}e^{j\phi} = \frac{b_1 + ja_1}{A}. \tag{20}$$

Since $N(A, \omega)$ as shown in (20) "describes" the input-output relation between $u(t)$ and $y(t)$, $N(A, \omega)$ is called the describing function. The fourier coefficients a_1 and b_1 are defined as

$$a_1 = \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} y(t)\cos(\omega t)dt \tag{21}$$

and

$$b_1 = \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} y(t)\sin(\omega t)dt. \tag{22}$$

Combining (20) - (22) $N(A, \omega)$ becomes

$$N(A, \omega) = \frac{1}{A}\left(\frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} y(t)\sin(\omega t)dt + j\frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} y(t)\cos(\omega t)dt\right). \tag{23}$$

Using that $-j \cdot j = 1$, cos is an even function and sin is an odd function, (23) can be written as

$$N(A, \omega) = \frac{j\omega}{\pi A}\left(\int_0^{\frac{2\pi}{\omega}} y(t)\left(\cos(-\omega t) + j\sin(-\omega t)\right)dt\right). \tag{24}$$

With Euler's relation $e^{jx} = \cos(x) + j\sin(x)$ we get

$$N(A, \omega) = \frac{j\omega}{\pi A}\left(\int_0^{\frac{2\pi}{\omega}} y(t)e^{-j\omega t}dt\right), \tag{25}$$

as the equation for determining describing functions. Interesting to note is that in the article of Gourav Saha [6] which also attempts to determine the describing function of a memristor, he uses the formula $N(A,\omega) = \frac{j\omega}{\pi A}\left(\int_0^{\frac{2\pi}{\omega}} y(t)e^{j\omega t}dt\right)$. The only difference with (25) being the minus sign in the exponential. Using (21) - (22) we can rewrite the equation used by Gourav Saha as $N(A,\omega) = \frac{1}{A}(-b_1 + ja_1)$ which is not the correct equation for determining a describing function. Using equation (25), a theorem for the describing function of a memristor can be postulated:

Theorem 3. *A memristive element with input* $u(t) = A\sin(\omega t)$ *and output* $y(t) = \left(a + b\int_0^t u(\tau)d\tau\right)u(t)$ *has the following describing function:*

$$N(A,\omega) = a + \frac{bA}{\omega}. \tag{26}$$

Proof. Using equation (25), we get

$$N(A,\omega) = \frac{j\omega}{\pi A}\left(\int_0^{\frac{2\pi}{\omega}}\left(a + b\int_0^t A\sin(\omega\tau)d\tau\right)A\sin(\omega t)e^{-j\omega t}dt\right). \tag{27}$$

We determine the inner integral in equation (27) over τ so that there is an integral over t left. Since $\cos(\omega t)\sin(\omega t) = \frac{1}{2}\sin(2\omega t)$ the integral can be split into

$$N(A,\omega) = \left(a + \frac{bA}{\omega}\right)\frac{j\omega}{\pi}\int_0^{\frac{2\pi}{\omega}}\sin(\omega t)e^{-j\omega t}dt - \frac{bA}{2\omega}\frac{j\omega}{\pi}\int_0^{\frac{2\pi}{\omega}}\sin(2\omega t)e^{-j\omega t}dt. \tag{28}$$

Since $\sin(\omega t)e^{-j\omega t} = (2j)^{-1}(1 - e^{-2j\omega t})$ and $\sin(2\omega t)e^{-j\omega t} = (2j)^{-1}(e^{j\omega t} - e^{-3j\omega t})$ we see that

$$\int_0^{\frac{2\pi}{\omega}}\sin(\omega t)e^{-j\omega t}dt = \frac{\pi}{j\omega} \tag{29}$$

and

$$\int_0^{\frac{2\pi}{\omega}}\sin(2\omega t)e^{-j\omega t}dt = 0. \tag{30}$$

Finally, combining equations (27) - (30) results in

$$N(A,\omega) = \left(a + \frac{bA}{\omega}\right)\frac{j\omega}{\pi}\frac{\pi}{j\omega} = a + \frac{bA}{\omega}, \tag{31}$$

showing that the formula of theorem 3 is correct. □

Important to note here is that this describing function is frequency (ω) dependent, which is in contradiction with A. Delgado's article [5]. To demonstrate the validaty of this describing function, the tool Simulink within Matlab was used to compare the actual memristor with the describing function found. Here we used Theorem 2 with $f(x) = a + bx$, $a = 1$ and $b = 1$, varying the amplitude A and the frequency ω of the sinusoidal input signal4:

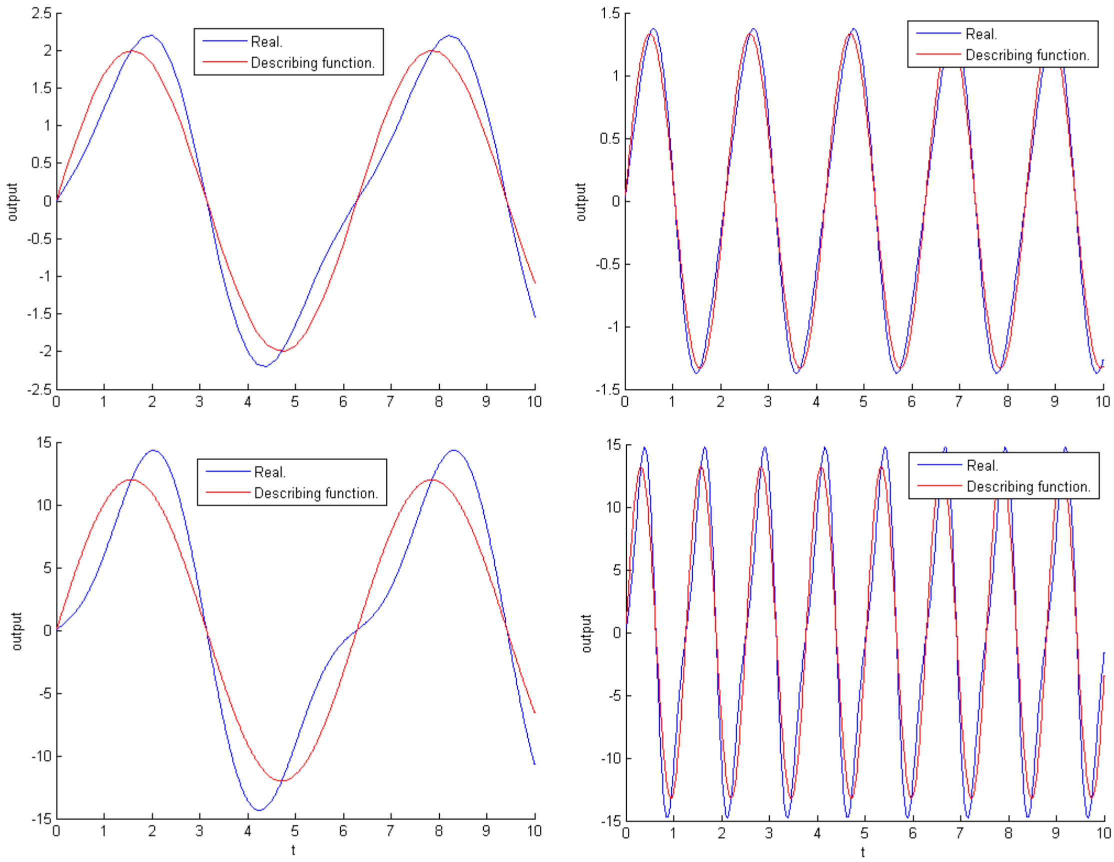

Figure 4: Blue is the actual output, red is the output generated by the describing function. Top left: $A = 1$ and $\omega = 1$. Top right: $A = 1$ and $\omega = 3$. Bottom left: $A = 3$ and $\omega = 1$. Bottom right: $A = 6$ and $\omega = 5$.

4.3 The describing function of the HP-memristor

Since the coefficients a and b were already found in Section 3.2, we can determine the describing function of the HP-memristor. This function is defined as $N_M(A, \omega)$:

$$\begin{aligned} N_M(A,\omega) =& R_{ON}\frac{w_0}{D} + R_{OFF}\left(1-\frac{w_0}{D}\right) + (R_{ON}-R_{OFF})\frac{\mu_V R_{ON}}{D^2}Q(0) \\ &+ (R_{ON}-R_{OFF})\frac{\mu_V R_{ON} A}{D\omega} \end{aligned} \tag{32}$$

5 Properties

In this section, we will look at what the describing function of a memristor means in a physical way. Also, the term passivity will be defined, when it holds for a memristor and why we want it to hold.

5.1 Physical meaning of the describing function

The describing function as defined in Theorem 3 has the frequency ω in the denominator, so when $\omega = 0$ the describing function of a memristor is undefined. But $\omega = 0$ corresponds to a constant, non-oscillating signal which can only physically be defined with a single input - single output relation. From the hysteresis loop which is characteristic for a memristor we can see that the memristor has two outputs for every input and vice versa. So a memristor is not able to process a constant signal.

5.2 Passivity

As mentioned in Section 3, HP claimed to have found a passive memristor. For passivity in feedback control systems there is the definition as cited in [11]:

Definition 2. *An element in a feedback control system with input u(t) and output y(t) is called passive if the following condition holds:*

$$\int_0^T u(t)y(t) \geq 0 \; \forall T \geq 0.$$

Where $u(t)y(t)$ represents the power of the element. If an element is passive then the element does not need any external sources supplying energy to make the element work. Since

$$u(t)y(t) \geq 0 \; \forall t \geq 0 \Rightarrow \int_0^T u(t)y(t) \geq 0 \; \forall T \geq 0, \tag{33}$$

we can use the stronger definition of passivity which is $u(t)y(t) \geq 0 \; \forall t \geq 0$. Now a memristor as defined in equation (12) is passive when:

$$\left(a + b\int_0^t u(\tau)d\tau\right) u^2(t) \geq 0. \tag{34}$$

Since $u^2(t) \geq 0$, the condition for passivity that needs to be satisfied is

$$\left(a + b\int_0^t u(\tau)d\tau\right) \geq 0. \tag{35}$$

Depending on the value of b we get

$$\int_0^t u(\tau)d\tau \geq -\frac{a}{b}, \; b > 0; \tag{36}$$

or

$$\int_0^t u(\tau)d\tau \leq -\frac{a}{b}, \; b < 0. \tag{37}$$

and of course $a \geq 0$ if $b = 0$, which corresponds to the passivity of a regular memristor. Since $u(t)$ is an undetermined signal in most cases, we need to use equation (34) as a condition when running simulations. This is similiar to the $R_{ON} = 0.5$ of the HP-memristor.

5.3 Pinched hysteresis loop

We have already seen that the HP-memristor produces a double valued Lissajous figure (except when it goes through the origin) in Figure 3. According to [10], a memristive device should always show a pinched hysteresis loop when you plot the input versus the output. Assuming that the input is a sinusoidal current source $A\sin(\omega t)$. We want to show that the input versus output plot of a memristor is a double-valued Lissajous figure, a hysteresis loop. Consider $t_1, t_2 \in (0, 2\pi\omega^{-1})$ with $t_1 \neq t_2$ and the system of equations (4). If the conditions

1. $u(t_1) = u(t_2) \Rightarrow y(t_1) \neq y(t_2)$,
2. $y(t_1) = y(t_2) \Rightarrow u(t_1) \neq u(t_2)$,

are satisfied, then we are dealing with a double valued pinched hysteresis loop. Remember that $y(t) = f(x(t))u(t)$. For the first condition, assume $u(t_1) = u(t_2) \Rightarrow y(t_1) = y(t_2)$. Then $f(x(t_1)) = f(x(t_2))$ and since $f(x) = a + bx$ it must hold that $x(t_1) = x(t_2)$. With the help of equation (5) we see that:

$$0 = x(t_2) - x(t_1) = \int_{t_1}^{t_2} u(\tau)d\tau \tag{38}$$

must hold. Because $t_1 \neq t_2$, either $t_2 = t_1 + 2k\pi\omega^{-1}, k \in \mathbb{Z}$ which cannot hold or $u(t) = 0$ which cannot hold either. This means condition one holds. For condition two to hold, $f(x(t_1)) \neq f(x(t_2))$ should hold. We just proved this fact to show condition one holds, so condition two holds as well. What happens in the origin has already been discussed in Section 5.2. The results found suggest that a memristive device shows a pinched hysteresis loop when one plots the input versus the output.

6 Stability

To determine if a feedback loop system is stable or unstable, this article uses three different tools. These tools are the pole locations, Routh's criterion and the Nyquist criterion of which a definition will be given in this section.

6.1 Pole locations

Feedback loop systems are almost always expressed as their Laplace transformations. The Laplace transformation of for example $u(t)$ is defined as

$$U(s) = \mathcal{L}[u(t)] = \int_0^\infty u(t)e^{-st}dt. \tag{39}$$

This transformation is useful because of the following: Consider a system with input $x(t)$, unit impulse response $h(t)$ and output $y(t)$. Normally one would have to calculate the convolution of $x(t)$ and $h(t)$ to get $y(t)$, but consider their Laplace transformations $X(s), H(s)$ and $Y(s)$. Now $Y(s) = H(s)X(s)$ holds. Now consider $H(s)$ and define it as $H(s) = D(s)^1 N(s)$. A system is said to be stable if the real parts of all the roots of $D(s)$ are less than zero.

6.2 Routh's Criterion

The poles of the closed loop transfer function are the roots of the characteristic equation of the closed loop transfer function. Routh's criterion is defined as follows [12]:

Definition 3. *The roots of the polynomial* $a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_1 s + a_0$*, with* $a_n \neq 0$*, all have a negative real part if and only if the Routh table consists of* $n+1$ *rows and all the elements in the first column of the table have the same sign. (They need to be either all positive or all negative.) The Routh table is defined as*

$$\begin{array}{cccc} a_n & a_{n-2} & a_{n-4} & \cdots \\ a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\ \hline b_{n-2} & b_{n-4} & b_{n-6} & \cdots \\ c_{n-3} & c_{n-5} & c_{n-7} & \cdots \\ d_{n-4} & d_{n-6} & d_{n-8} & \cdots \\ \vdots & \vdots & \vdots & \end{array}$$

Table 1: The definition of a Routh table.

and in the case n *is an even number,* $a_{-1} = 0$*. The* b*'s,* c*'s and* d*s are defined as follows:*

$$b_{n-2} = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}, \; b_{n-4} = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}},$$

$$c_{n-3} = \frac{b_{n-2}a_{n-3} - a_{n-1}b_{n-4}}{b_{n-2}}, \; c_{n-5} = \frac{b_{n-2}a_{n-5} - a_{n-1}b_{n-6}}{b_{n-2}},$$

$$d_{n-4} = \frac{c_{n-3}b_{n-4} - b_{n-2}c_{n-5}}{c_{n-3}}, \; d_{n-6} = \frac{c_{n-3}b_{n-6} - b_{n-2}c_{n-5}}{c_{n-3}},$$

and so on.

6.3 The Nyquist criterion

The Nyquist criterion [3] makes good use of the describing function method because a Laplace transformation on a nonlinear element is impossible to evaluate. The Nyquist criterion is defined as follows:

Definition 4. *Assume* Z *to be the number of roots of the characterics equation* $1 + N(A,\omega)G(s)$ *in the right half-plane,* N *the number of clockwise encirclements of the point* $(-1,0)$ *and* P *the number of poles of the open loop function* $N(A,\omega)G(s)$ *in the right half-plane. Then the system is stable if and only if the equation*

$$Z = N + P$$

holds.

7 Limit cycles

Limit cycles are isolated periodic solutions where the feedback loop system exhibits a self sustained oscillation which has a fixed amplitude. If the amplitude is perturbed the limit cycle will return to its original amplitude over time if it is stable. This behaviour is either desirable in for example a pacemaker, or not desirable in for example the altitude controller of an airplane. In this section, we will show the use of the describing function of a memristor.

7.1 Determining a limit cycle

Consider the following system:

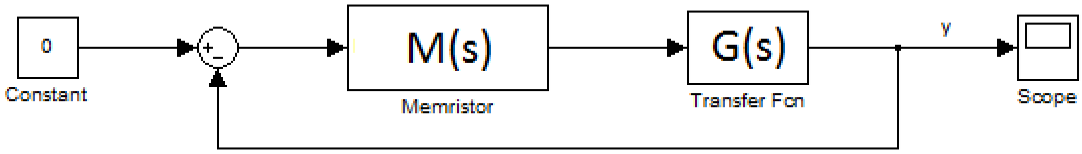

Figure 5: A system with input 0, also called a free system.

Because the Laplace transformation of the non-linear element $M(s)$ is impossible to compute, we replace $M(s)$ by its describing function $N(A, \omega)$. For Figure 5 to exhibit a limit cycle the output has to be the same each iteration of the loop, so the following equation has to be satisfied:

$$y = -N(A, \omega)G(s)y. \tag{40}$$

Assuming $y(t) \neq 0$, we can divide both sides of equation (40) by y. After adding 1 to both sides the equation simplifies to $1 + N(A, \omega)G(s) = 0$. This form is found in most literature [1][2][3]. For higher order systems this equation becomes very hard to solve by analytical methods, so usually $G(j\omega)$ and $-N(A, \omega)^{-1}$ are plotted in $\mathbb{C}^2$ together to see where they intersect. For a memristor analytical methods can be used, but the graphical method will be shown as well as this method provides some clarity into what is happening in the simulations shown later.

7.2 Frequency of a limit cycle

Assume there exists an ω_1 for which a limit cycle occurs with a memristor. Then $G(j\omega_1) = -N(A, \omega_1)^{-1}$ should hold. Because the describing function of a memristor is real for all A and ω_1, $G(j\omega_1)$ must be real as well. $G(j\omega_1)$ is defined as

$$G(j\omega_1) = \frac{1}{a_3(j\omega_1)^3 + a_2(j\omega_1)^2 + a_1(j\omega_1) + a_0}. \tag{41}$$

Rewriting equation (41) gives

$$G(j\omega_1) = \frac{(a_0 - a_2\omega_1^2) - j\omega_1(a_1 - a_3\omega_1^2)}{(a_0 - a_2\omega_1^2)^2 + \omega_1^2(a_1 - a_3\omega_1^2)^2}. \tag{42}$$

For equation (42) to be real, either $\omega_1 = 0$ has to hold which gives the trivial solution which we are not interested in or $a_1 - a_3\omega_1^2 = 0$ has to hold which means a limit cycle occurs if $a_1 a_3^{-1} > 0$ so that $\omega_1 = \sqrt{a_1 a_3^{-1}}$ which is the frequency of a limit cycle caused by a real valued describing function preceding a plant with a third-order polynomial in the denominator. For higher order plants a graphical approach is most likely needed.

7.3 Predicted amplitude of the limit cycle

Assume $\omega_1 = \sqrt{a_1 a_3^{-1}}$ is the frequency at which the limit cycle occurs. Because the imaginary part of $G(j\omega)$ equals zero at this frequency, $G(j\omega_1)$ simplifies to:

$$G(j\omega_1) = \frac{1}{a_0 - a_2\omega_1^2}. \tag{43}$$

Now we can solve $G(j\omega_1) = -N(A, \omega 1)^{-1}$ for A with (43) and $-N(A, \omega_1)^{-1} = -\omega_1(a\omega_1 + bA)^{-1}$ which gives

$$A = b^{-1}\omega_1(a_2\omega_1^2 - a_0 - a). \tag{44}$$

This is the predicted amplitude of a limit cycle caused by a system as depicted in figure 5.

7.4 Further research

After simulation done in Section 9 later in this article, it is clear that the system in Figure 5 exhibits periodic behaviour after being excited and the predicted frequency is correct. But the predicted amplitude deviates considerably from the simulated amplitude. Furthermore, when the system is excited again, the system keeps exhibiting periodic behaviour but of a different amplitude, which is by definition not a limit cycle.

8 Industrial controllers

Another potential use of a memristor is to combine it with the most commonly used industrial controller, namely a Proportional, Integral and Derivative controller or in short a PID- controller. A PID-controller is mathematically defined as follows:

Definition 5. *A PID-controller with input $u(t)$ has the output*

$$k_P u(t) + k_I \int_0^t u(\tau)d\tau + k_D \frac{du}{dt}(t),$$

where k_P, k_I and k_D are the weights corresponding to respectively the proportional, integral and derivative parts.

In this section a PID-controller will be compared to an MID-controller (with the M standing for Memristive). An MID-controller is slightly different and is defined as follows:

Definition 6. *An MID-controller with input $u(t)$ has the output*

$$k_M\left(a+b\int_0^t u(\tau)d\tau\right)u(t)+k_I\int_0^t u(\tau)d\tau+k_D\frac{du}{dt}(t),$$

where k_M, k_I and k_D are the weights corresponding respectively to the memristive, integral and derivative parts and a, b real valued constants.

Rewriting the definition of the MID-controller found in definition 6 slighty gives:

$$(k_M a)u(t)+(k_M bu(t)+k_I)\int_0^t u(\tau)d\tau+k_D\frac{du}{dt}(t). \tag{45}$$

Which means an MID-controller is equivalent to a PID-controller with a dynamic weight for the integral part of the controller. Consider the system in Figure 6:

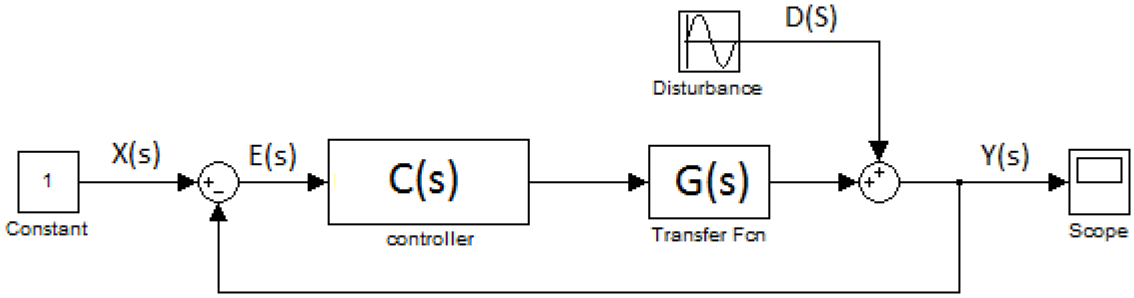

Figure 6: A controlled system with disturbance.

The system's closed loop transfer function is

$$Y(s)=\frac{C(s)G(s)}{1+C(s)G(s)}X(s)+\frac{D(s)}{1+C(s)G(s)}. \tag{46}$$

A system in picture 6 with $D(s)=0$ is called a system without a disturbance. In other cases it is a system with a disturbance. Consider $D(s)=0$, then the stability of this system is determined by calculating the roots of $1+C(s)G(s)$. For a regular (linear) PID-controller this is doable because the Laplace transformation of a PID-controller is easy to compute, but for a (non-linear) MID-controller, we are almost entirely dependent on simulation. When $D(s)\neq 0$, we introduce the sensitivity function:

Definition 7. *The sensitivity function $\epsilon(s)$ is given by*

$$\epsilon(s)=\frac{E(s)}{X(s)-D(s)}=\frac{1}{1+C(s)G(s)}.$$

Proof. Consider the feedback error $E(s)$, this can be written as:

$$\begin{aligned}E(s)=X(s)-Y(s)&=X(s)-\frac{C(s)G(s)}{1+C(s)G(s)}X(s)-\frac{D(s)}{1+C(s)G(s)}\\&=\frac{1}{1+C(s)G(s)}X(s)-\frac{D(s)}{1+C(s)G(s)}=\frac{X(s)-D(s)}{1+C(s)G(s)}.\end{aligned} \tag{47}$$

Then dividing both the left and right side of equation (47) by $X(s) - D(s)$, we get

$$\frac{E(s)}{X(s) - D(s)} = \frac{1}{1 + C(s)G(s)}, \tag{48}$$

which yields the desired result. □

The goal here is to dampen out the disturbance as much as possible, so it is desired that the sensitivity function becomes as small as possible.

9 Examples

Consider the linear plant $G(s) = ((s+1)(s+2)(s+3))^{-1} = (s^3 + 6s^2 + 11s + 6)^{-1}$. The poles of this plant are -1, -2 and -3 so this plant is stable. In this section we are going to use this plant in a free system (no external inputs) to show how a system with this plant and a memristor can exhibit periodic behaviour. We will also show how this plant with a memristor can help converge this system faster and dampen out a disturbance.

9.1 Periodic behaviour

Consider the free system in Figure 5. Using the definitions, theorems and equations that came from those in Section 7 we can determine the constants of the memristor used to see whether a limit cycle shows up. To understand when limit cycles occur, a Nyquist plot of the plant is very useful as shown in Figure 7:

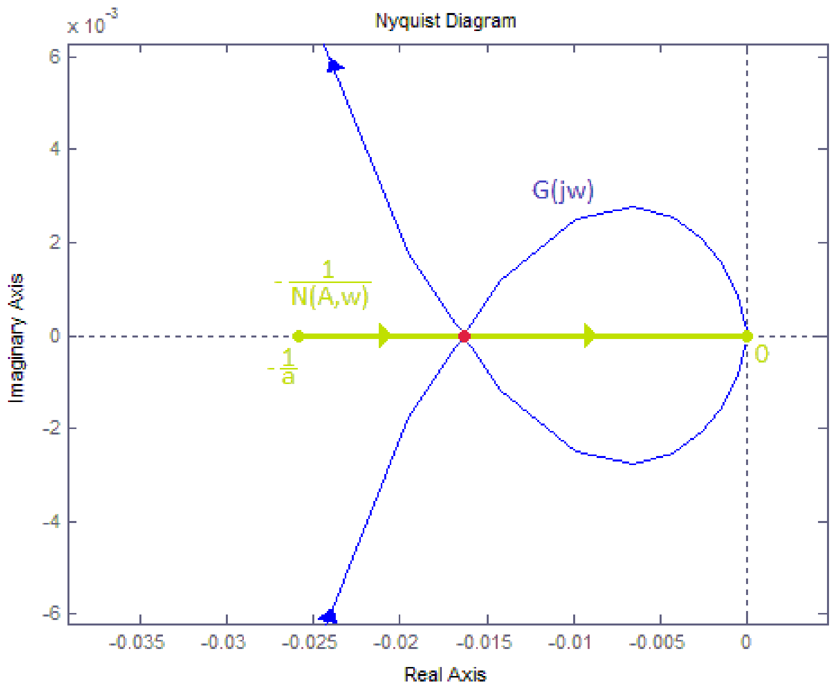

Figure 7: $G(j\omega)$ and $-N(A,\omega)^{-1}$ plotted in $\mathbb{C}^2$.

It is to be noted that the amplitude A of this system always starts at 0 due to the fact that a memristor does not hold any energy (which is why the system is excited beforehand by a pulse of energy). With the $G(s)$ defined earlier, $G(j\omega)$ intersects with the real axis at -60^{-1}. In picture 7 we see that if $a < 60$, the Nyquist criterion as defined in section 6.3 says $N = 0$ because a^{-1} lies outside the encirclement of $G(j\omega)$ and $P = 0$ because the poles of $N(A, \omega_1 = \sqrt{11})G(s)$ are the same as the poles of $G(s)$ which all lie in the left half plane. Z is slighty more complicated, for which we look to the Routh table in Table 9.1 using Table 1 for $1 + N(A, \sqrt{11})G(s)$:

1	11
6	$6 + a + \sqrt{11}^{-1}bA$
$10 - 6^{-1}(a + \sqrt{11}^{-1}bA)$	
$6 + a + \sqrt{11}^{-1}bA$	

Table 2: The Routh table for the poles of $1 + N(A, \sqrt{11})G(s)$.

We look at the third entry which states that all zeros of $1 + N(A, \omega)G(s)$ have a negative real part as long as $60 > a + \sqrt{11}^{-1}bA$ holds. Since A starts at 0 and $a < 60$, $Z = 0$ so $Z = N + P$ holds and the system is stable and dies out as shown in Figure 8:

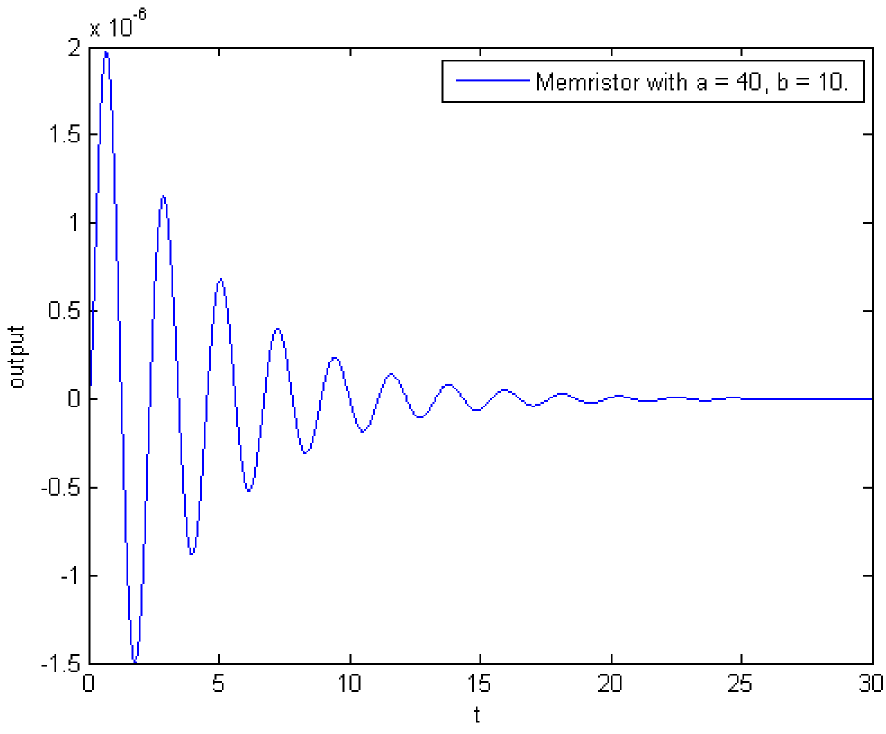

Figure 8: A periodic solution that dies out.

Now consider when $a \geq 60$. Once the system gets excited even a little bit $-N(A, \sqrt{11})^{-1}$ gets encircled by the Nyquist plot once, so $N = 1$. Because the Routh table as defined in Table 1 shows that $Z \neq 0$. The Routh table cannot tell how many zeros of $1 +$

$N(A, \sqrt{11})G(s)$ lie in the right half plane, but simulation shows that once excited the system goes into a periodic orbit instantly if $a = 60$ as shown in Figure 9:

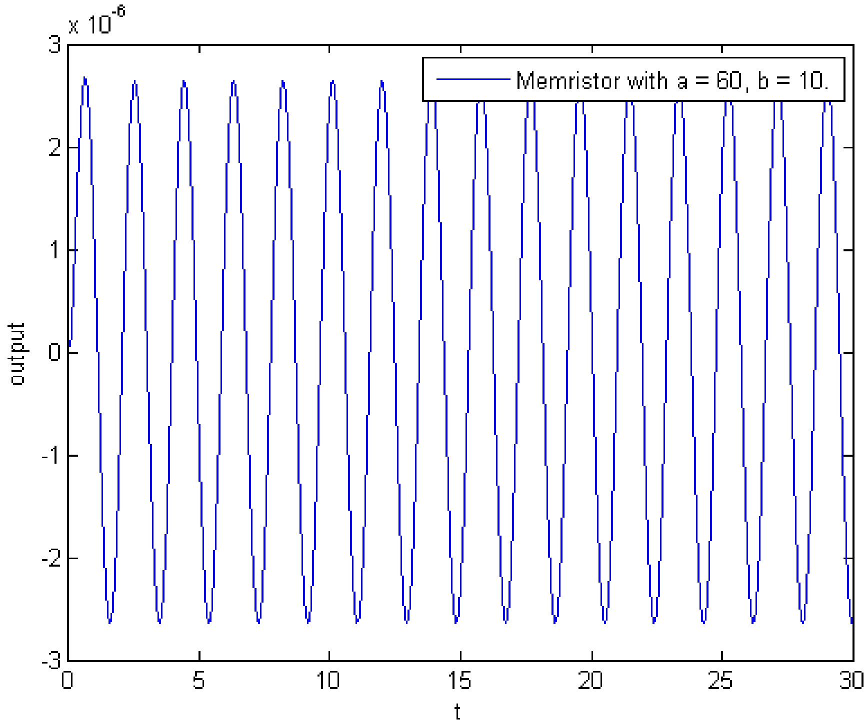

Figure 9: A sustained periodic output.

And the system diverges first, then goes into a periodic output if $a > 60$ as shown in Figure 10:
One last situation for this kind of system that is worth noting: If $a < 60$ but if the pulse used to excite the system is big enough it will end up in a periodic output as seen in Figure 11:

9.1.1 Periodic behaviour conclusions

A system as shown in the previous section is capable of exhibiting periodic outputs, but they can not be called limit cycles as the impulse provided to excite the system is a deciding factor of the amplitude. The amplitude of a limit cycle should always return to its former state when disturbed, but in this case this does not happen.

9.2 Comparing a PD- and an MD-controller

A PD- and MD-controller are the same as a PID- and MID-controller respectively, but with $k_I = 0$. To compare a PD-controller with an MD-controller, we will be using the system as shown in Figure 6. First, we compare the convergence with a PD- and MD-controller with D(s) = 0 Then, we compare the convergence with an added sinusoidal disturbance ($D(s) \neq 0$). To give a fair comparison, the weights as defined in 5 and 6 will be the same. That means $k_P = k_M$ and both derivative weights will be the same as

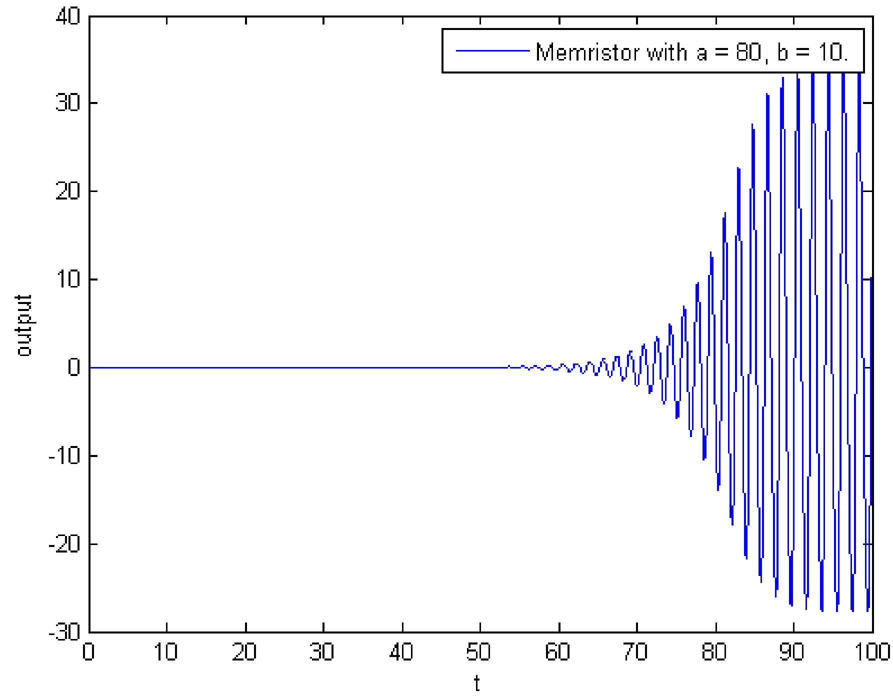

Figure 10: A periodic output that diverges, then stabilizes.

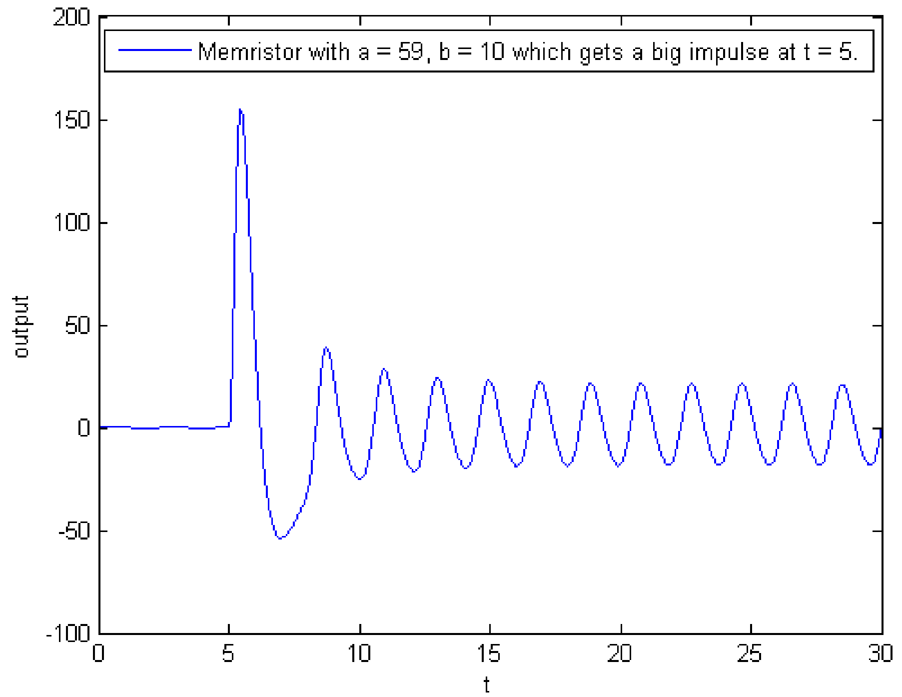

Figure 11: A sustained oscillation after a second big impulse.

well. In picture 12, 13 and 14 the weights were set as $k_P = 60$, $k_M = 60$ and $k_D = 60$. For the memristor variables we choose $a = 0$ and $b = 1$ so that the controller is an M.I.D.-controller. First we compare the two controllers without disturbance, so $D(s) = 0$ as shown in Figure 12:

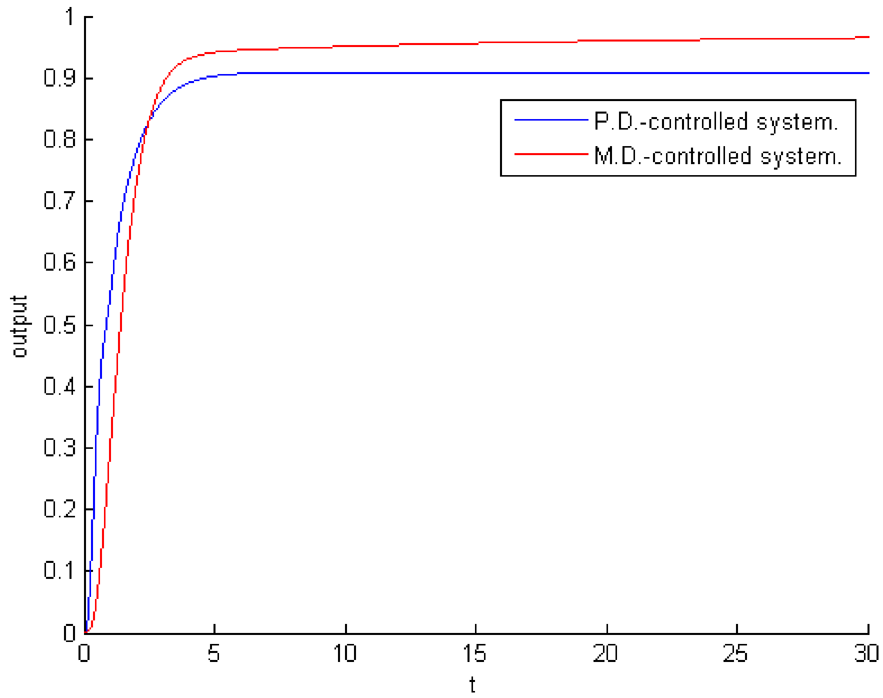

Figure 12: Comparing a PD- and MD-controller without disturbance.

And comparing the two controllers with disturbance as shown in Figure 13:
As we notice, without a disturbance it does not matter much whether one uses a PD- or MD-controller, but with a disturbance an MD-controller is significantly better. However,with simulation for extended periods of time, the MD-controlled system diverges when reaching its equilibrium point as seen in Figure 14:
This occurs later in time when the derivative weight is increased, but it happens nonetheless. This has to be considered when further researching these kind of systems.

9.3 Comparing a PID- and an MID-controller

To fix the problem found in Section 9.2 , we set $k_I \neq 0$ and we give k_I the same value in both systems. Because it is already known that a PID-controller is very efficient when it comes to controlling a feedback loop system without any disturbance, this section will only compare the PID- and MID-controllers in a system with disturbance. Simulation shows that if k_I is kept relatively small, the system doesn't diverge anymore and an MID-controller dampens out the disturbance better than a PID-controller. Figure 15 shows a simulation of an MID-controller and PID-controller with $k_I = 1$ as shown in Figure 15:
But when k_I is too large it is detrimental for the MID-controller's performance as shown in Figure 16:

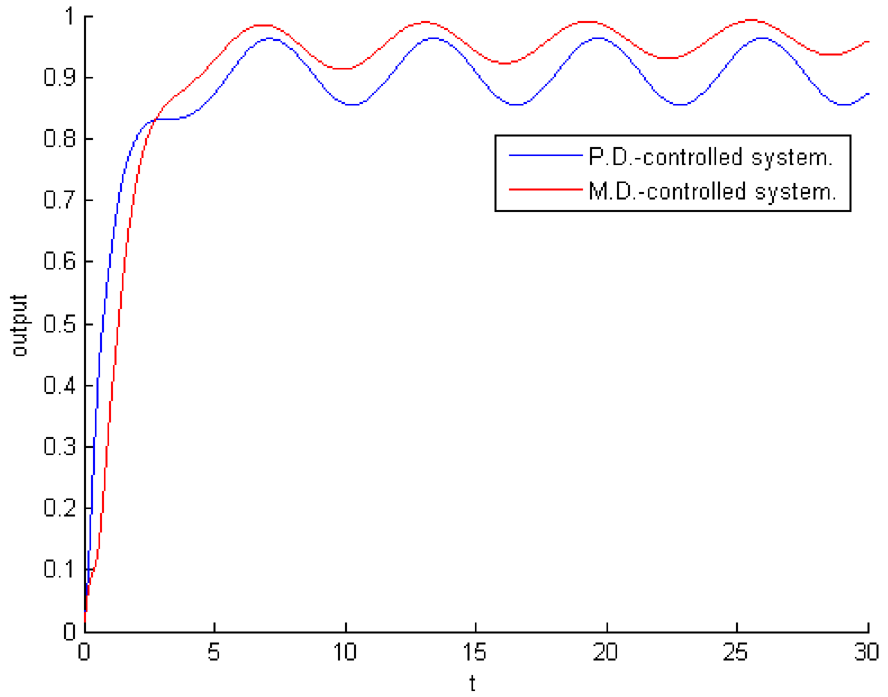

Figure 13: Comparing a PD- and MD-controller with disturbance.

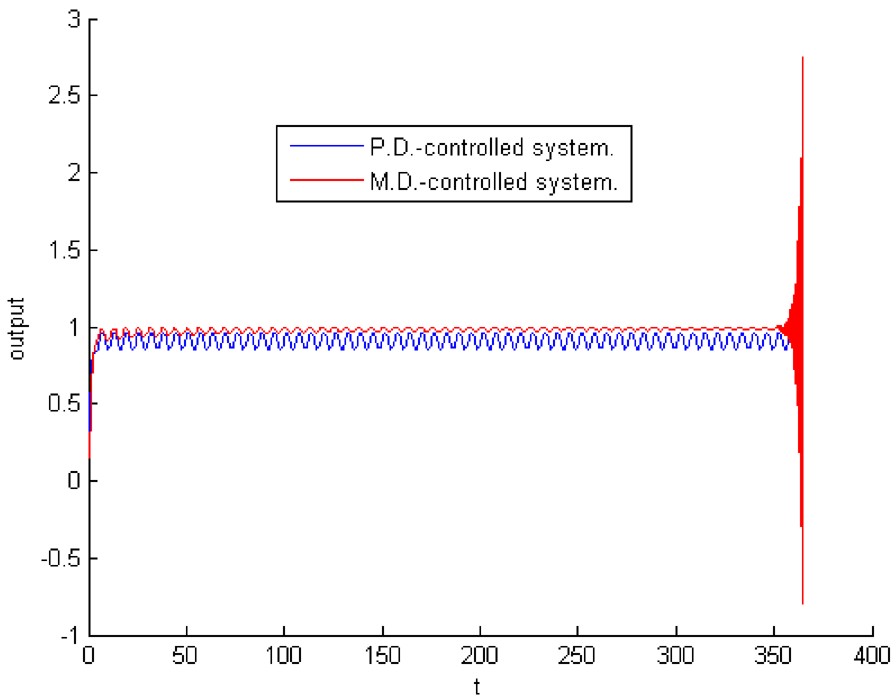

Figure 14: Comparing a PD- and MD-controller.

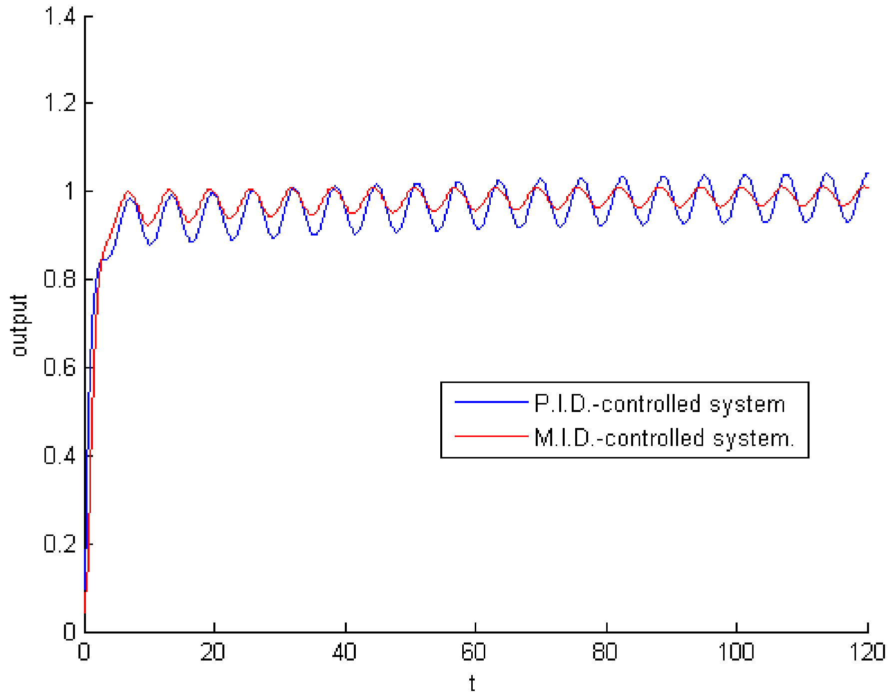

Figure 15: Comparing a PD- and MD-controller.

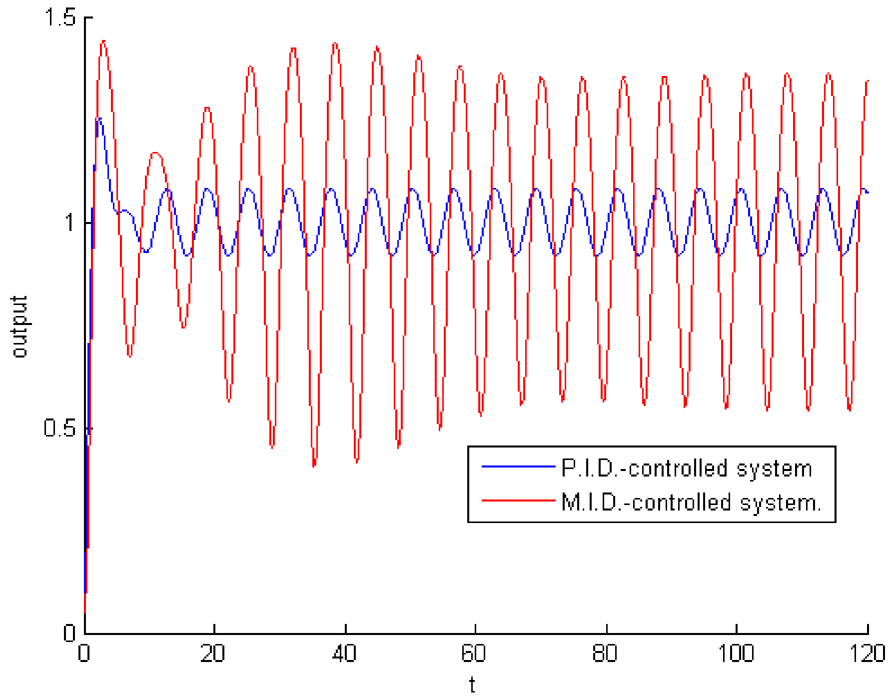

Figure 16: Comparing a PD- and MD-controller.

10 Conclusion

The memristor developed by HP has made a large impact on the fields of research which deal with memristive behaviour. We have developed an accurate describing function for the memristor, although in a control system it is unable to determine the amplitude correctly. This is because the memristor is capable of exhibiting periodic orbits in a control system, although those are not limit cycles due to their dependancy on the initial conditions of the system. There is also merit in replacing the proportional part in a PD-controller with a memristive part so that it becomes an MD-controller. Without any disturbances in the system the memristive element does not yield many advantages, but in a system with disturbances it yields more advantages since the memristive part is good at dampening out those disturbances. The MD-controlled system explodes after some time, but this can be fixed by making it into an MID-controller.

11 Future Research

This paper leaves a few questions unanswered. The first question is why the amplitude of a limit cycle caused by a memristor doesn't return to it's initial amplitude. The second question is why does a system controlled by an MD-controller diverge once it gets close to its point of convergence. Both questions could be an interesting subject for future papers.

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