El Gawhary, Van Mechelen, and Urbach reply

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DOI
10.1103/PhysRevLett.122.089302

Publication date
2019

Document Version
Final published version

Published in
Physical Review Letters

El Gawhary, Van Mechelen, and Urbach Reply: The criticisms raised in the preceding Comment [1] are unphysical and affected by a simple, though crucial, error in the way the so-called Riemann-Silberstein formalism is handled by the author of the Comment. It is common practice to express the angular momentum (AM) of an electromagnetic field per unit energy (which we indicated as γ in our work), as already done in the work by Allen *et al.* [2], and extensively used from then on (see, e.g., in Ref. [3]). That ratio has indeed the dimension of the inverse of an angular frequency. That is obvious since the AM is proportional to $\hbar$ and the energy of a harmonic oscillator is $\hbar\omega$. It is, however, incorrect to state that the temporal scale is arbitrary. That scale is determined by the photon energy and it is fixed once the angular frequency ω is specified, something that we have done at the very beginning of our work. There is no arbitrariness in the theory by doing that. The definition and interpretation of γ is well established and we have not introduced any new quantities.

As to the second remark, the value of $\gamma = 3/2$ for the spin was not obtained by properly choosing some not better specified weighting coefficients $c_{m,n}$ in the modal expansion used to represent the field. That value is obtained for any field, which is circularly polarized [as in Eq. (7) of Ref. [4]] and it is independent of the presence of an orbital angular momentum because it is only due to the spin carried by the field. Equation (14) of our Letter is clear on that point: the $3\sigma_z/(2\omega)$ term is independent of any coefficient $c_{m,n}$. All calculations to obtain the expressions of the energy and angular momentum of the fields are fully detailed in Ref. [4] and the Supplemental Material published with the Letter. We invite the reader to refer to those documents for the correct expressions and derivations of the field invariants. Here we just point out what is the problem with the calculations presented in the Comment. Given a real-valued electric field $\boldsymbol{\mathcal{E}}(\mathbf{r},t) = [\frac{1}{2}\mathbf{E}(\mathbf{r})\exp(-i\omega t) + \frac{1}{2}\mathbf{E}^\star(\mathbf{r})\exp(i\omega t)]$ and magnetic field $\boldsymbol{\mathcal{H}}(\mathbf{r},t) = [\frac{1}{2}\mathbf{H}(\mathbf{r})\exp(-i\omega t) + \frac{1}{2}\mathbf{H}^\star(\mathbf{r})\exp(i\omega t)]$ the expression of the Riemann-Silberstein vector $\boldsymbol{\mathcal{F}}(\mathbf{r},t)$ reads $\boldsymbol{\mathcal{F}}(\mathbf{r},t) = \sqrt{\varepsilon_0/2}\boldsymbol{\mathcal{E}}(\mathbf{r},t) + i\sqrt{\mu_0/2}\boldsymbol{\mathcal{H}}(\mathbf{r},t)$ that can be written as $\boldsymbol{\mathcal{F}}(\mathbf{r},t) = \boldsymbol{\mathcal{F}}_+(\mathbf{r},t) + \boldsymbol{\mathcal{F}}_-(\mathbf{r},t)$ after defining the positive and negative frequency parts of $\boldsymbol{\mathcal{F}}(\mathbf{r},t)$, $\boldsymbol{\mathcal{F}}_+(\mathbf{r},t) = \{\sqrt{\varepsilon_0/2}[\mathbf{E}(\mathbf{r})/2] + i\sqrt{\mu_0/2}[\mathbf{H}(\mathbf{r})/2]\}\exp(-i\omega t)$ and $\boldsymbol{\mathcal{F}}_-(\mathbf{r},t) = \{\sqrt{\varepsilon_0/2}[\mathbf{E}^\star(\mathbf{r})/2] + i\sqrt{\mu_0/2}[\mathbf{H}^\star(\mathbf{r})/2]\}\times \exp(i\omega t)$. According to the author of the Comment, the time-averaged field energy per unit length (W) should be obtained through the integral [Eq. (6) of the Comment]

$$\begin{aligned}&\int\int[\boldsymbol{\mathcal{F}}_+(\mathbf{r},t)\cdot\boldsymbol{\mathcal{F}}_+^\star(\mathbf{r},t)]dxdy\\&=\int\int\left(\frac{\varepsilon_0}{2}\frac{1}{4}\mathbf{E}(\mathbf{r})\cdot\mathbf{E}^\star(\mathbf{r})+\frac{\mu_0}{2}\frac{1}{4}\mathbf{H}(\mathbf{r})\cdot\mathbf{H}^\star(\mathbf{r})\right.\\&\left.\quad-i\frac{\sqrt{\varepsilon_0\mu_0}}{2}\frac{1}{4}\mathbf{E}(\mathbf{r})\cdot\mathbf{H}^\star(\mathbf{r})+i\frac{\sqrt{\varepsilon_0\mu_0}}{2}\frac{1}{4}\mathbf{H}(\mathbf{r})\cdot\mathbf{E}^\star(\mathbf{r})\right)dxdy\end{aligned} \tag{1}$$

which is manifestly incorrect. The true expression for W is $\int\int\boldsymbol{\mathcal{F}}(\mathbf{r},t)\cdot\boldsymbol{\mathcal{F}}^\star(\mathbf{r},t)dxdy = \int\int[\boldsymbol{\mathcal{F}}_+(\mathbf{r},t)\cdot\boldsymbol{\mathcal{F}}_+^\star(\mathbf{r},t) + \boldsymbol{\mathcal{F}}_-(\mathbf{r},t)\cdot\boldsymbol{\mathcal{F}}_-^\star(\mathbf{r},t)]dxdy$ which, after simple math, gives the correct expression $\int\int[(\varepsilon_0/4)\mathbf{E}(\mathbf{r})\cdot\mathbf{E}^\star(\mathbf{r}) + (\mu_0/4)\mathbf{H}(\mathbf{r})\cdot\mathbf{H}^\star(\mathbf{r})]dxdy$. A similar error affects Eq. (7) of the Comment. The author of the Comment failed to reproduce the proper values for the field invariants (namely, the field energy and the angular momentum) because he confused $\boldsymbol{\mathcal{F}}(\mathbf{r},t)$ with $\boldsymbol{\mathcal{F}}_+(\mathbf{r},t)$ throughout his document [from Eq. (5) of the Comment onwards]. For instance, in case of circularly polarized fields, one obtains $\int\int\boldsymbol{\mathcal{F}}_+(\mathbf{r},t)\cdot\boldsymbol{\mathcal{F}}_+^\star(\mathbf{r},t)dxdy = (\varepsilon_0/4)\int\int|A|^2[1+(k_z/k)]^2d^2\boldsymbol{\xi}$ and $\int\int\boldsymbol{\mathcal{F}}_-(\mathbf{r},t)\cdot\boldsymbol{\mathcal{F}}_-^\star(\mathbf{r},t)dxdy = (\varepsilon_0/4)\int\int|A|^2[1-(k_z/k)]^2d^2\boldsymbol{\xi}$ and only the sum of the two gives the correct field energy $W = (\varepsilon_0/2)\int\int|A|^2[1+(k_z^2/k^2)]d^2\boldsymbol{\xi}$, as given in Eq. (10) of Ref. [4].

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Received 5 December 2018; published 27 February 2019
DOI: 10.1103/PhysRevLett.122.089302

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