On Stackelberg and Inverse Stackelberg Games

&

Their Applications in the Optimal Toll Design Problem,

the Energy Markets Liberalization Problem,

and in the Theory of Incentives

Kateřina Staňková

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Kateřina Staňková
Delft, December 2008.

Contents

Chapter 1

Introduction

This thesis falls within the area of applied mathematics. It raises various problems within the area of game theory and offers mathematical solutions to them.

In this chapter we present the background and the motivation for the research presented in this thesis. In Section 1.1 we first introduce game theory as a theoretical topic of this thesis and as a tool to formalize and solve complex decision making problems. In Section 1.2 we introduce the applications that we consider in this thesis: The optimal toll design problem, the problem of energy market liberalization, and the theory of incentives. We conclude the chapter with an overview and road map of this thesis, and a survey of the contributions to the state of the art in Section 1.3.

1.1 Introduction to game theory

1.1.1 The basics of game theory

What is game theory?

Game theory is a branch of applied mathematics used, among others, in the social sciences (most notably economics), biology, political science, computer science, and philosophy. Game theory attempts to mathematically capture behavior in strategic situations (so–called *games*), in which an individual's success in making choices may depend on the choices of others. Game theory was initially developed in order to analyze competitions in which one individual does better at another's expense (zero sum games, [5]). Later on, game theory was expanded in order to treat a much wider class of interactions.

Traditional applications of game theory attempt to find game equilibria, i.e., sets of strategies in which individuals are unlikely to change their behavior. Many equilibrium concepts have been developed (e.g., the well-known Nash equilibrium [61], the Stackelberg equilibrium [92], and the Pareto equilibrium [93]) in an attempt to capture this idea. These equilibrium concepts are motivated differently depending on the field of application, although they often overlap or coincide.

Game theory has been widely recognized as an important tool in many fields. Eight game theorists have won The Nobel Prize in economics, and John Maynard Smith was awarded the Crafoord prize for his application of game theory to biology.

The established names of “game theory” (developed from approximately 1930) and “theory of differential games” (developed from approximately 1950, parallel to that of optimal control theory) are somewhat unfortunate. “Game theory”, especially, appears to be directly related to board games; of course it is, but the notion that it is only related to such games is far too restrictive. The term “differential game” became a generally accepted name for games in which differential equations play an important role. Nowadays the term “differential game” is also being used for other classes of games for which the more general term “dynamic games” would be more appropriate.

The most widely accepted origin of game theory as stated in the literature is found in the year 1944, when the book *Theory of Games and Economic Behavior* by John von Neumann and Oskar Morgenstern [91] was published. This theory was developed extensively in the 1950s by many scholars. Game theory was later explicitly applied to biology in the 1970s.

Applications of game theory

The applications of “game theory” and the “theory of differential games” mainly deal with economic and political conflict situations, worst-case designs, evolution problems in biology, as well as modeling of war games. However, it is not only the applications in these fields that are important; equally important is the development of suitable concepts to describe and understand conflict situations. It turns out, for instance, that the role of information - what one player knows compared to others - is very crucial in such problems.

Scientifically, dynamic game theory can be viewed as the offspring of game theory and optimal control theory. Its character, however, is much richer than that of its parents, since it involves a dynamic decision process evolving in (discrete or continuous) time, with more than one decision maker, each with his/her own cost function and possibly having access to different information.

Conflict as the origin of game theory

The problems of game theory are often connected with a conflict situation. Although the notion of conflict is as old as mankind, the scientific approach dealing with conflict situations began relatively recently, around the 1930’s, resulting in a still growing stream of scientific publications. More and more scientific disciplines devote time and attention to the analysis of conflict situations. These disciplines include (applied) mathematics, economics, engineering, aeronautics, sociology, politics, and mathematical finance.

In a conflict situation an individual, also called a *player*, *agent*, *decision maker*, *actor*, or simply *person*, has to make a decision and each possible decision may lead to a different outcome, which is valued differently by that individual. This individual may not be the only one deciding in favor of a particular outcome; a series of decisions made by several individuals may be necessary. If some of the individuals value the possible outcomes differently, the seeds of conflict have been sown.

The individuals involved do not always have complete control over the outcome. Sometimes there are uncertainties that influence the outcome in an unpredictable way. Under such circumstances, the outcome is (partly) based on data not yet known and not determined by the other players’ decisions. Sometimes it is said that such data are under the control of "nature" or "God", and that every outcome is caused by the joint or individual actions of human beings and "nature" ("God").

Basic notions

So far we have used terms like "decision" and "strategy" without explaining them properly, assuming that their meaning is intuitively clear. However, some precision is necessary to avoid ambiguities.

In the following simple example the concepts of decision, action, and strategy (also called "decision rule") will be introduced.

Consider a person who has to decide what to do on a Sunday afternoon, and the options are running outdoors or working out in a fitness club. A possible *strategy* of this individual can be framed in these terms: "If the weather is nice, then I will run outside, otherwise I will work out." This is a *strategy* or a *decision rule*: what actually will be done depends on quantities not yet known and not controlled by the decision maker; the decision maker cannot influence the course of the events further, once he/she has fixed his/her strategy. (We assume that the decision maker will stick to his/her strategy.) Any consequence of such a strategy, after the unknown quantities are realized, is called an *action*. In a sense, a constant strategy (such as an irrevocable decision to go running or come what may) coincides with the notion of action.

In the example above, the alternative actions are to run outdoors and to work out, and the actions to be implemented depend on information (the weather), which has to be known at the time it is carried out. In general, such information can be of different types. It can, for instance, comprise the previous actions of all the other players. As an example, consider the following sequence of actions: If he/she is nice to me, I will be nice to him/her. The information can also be of a stochastic nature, such as in the running example. Then, the actual decision (action) is based on data not yet known and not controlled by other players, but instead determined by "nature". If this "nature" plays no role, the problem is deterministic.

Static versus dynamic game theory

There is, in fact, no uniformly accepted line of separation between static games, on the one hand, and dynamic games, on the other. We shall choose to call a game *dynamic* if at least one player is allowed to use a strategy that depends on previous actions of other players or the player herself/himself. If a game is not dynamic, it is *static*.

What does "optimality" mean?

In game theoretic problems, the aim is often to find an optimal strategy for one or more players. Optimality, in itself, is not a well defined concept. In non-cooperative games a solution in terms of the Nash equilibrium is a specific form of optimality. Such a solution is reached if one player cannot improve his/her outcome by altering his/her decision unilaterally.

Another concept of solution is the one that involves a hierarchy in decision making: one or more of the players declare and announce their strategy before the other players choose their strategy and the declaring players are in a position to enforce their own strategy upon the other players. Such games in which one or more players, called the *leaders*, declare their strategy first and impose this strategy upon the other players, called the *followers*, are referred to as *Stackelberg games*.

If, however, the leaders announce their strategy as a mapping from the followers' decision space into their own decision space, we talk about *inverse Stackelberg games*. Examples of inverse Stackelberg games are:

- Think of the leader as the government and of the follower as a citizen. The government fixes how much income tax the citizen has to pay and this tax will depend on the income of the citizen. It is up to the citizen to choose how much money he/she will earn (by working harder or not). The income tax the government will receive is an increasing function of the citizen's earnings, where this tax rule (in many countries piecewise linear) was made known ahead of the citizen's decision as to how hard to work and, hence, how much to earn.

- The leader is a bank and the follower an investor. The investor can buy stocks, the bank acting as an intermediary, with the money he/she has in his/her savings account. Suppose he/she buys stocks worth a certain amount of euro. Then the bank will charge him/her transaction costs depending on this amount. The transaction costs rule has been made known by the bank before the actual transaction takes place.

- The leader is a producer of electricity in a liberalized market and the follower is the market (a group of clients) itself. The price of electricity is set as a function of the amount of electricity traded [64].

- The leader is a road authority and the followers are drivers in the road network. The road authority optimizes system performance by setting tolls on some of the links in the network, the drivers make their travel decisions in order to minimize their perceived travel time. The travel decisions of the drivers determine their traffic flows in the network. If the road authority defines the tolls set in the network as functions of the traffic flow in the network, the problem is of the inverse Stackelberg type.

- The leader is a road authority and drivers in a given road network are the followers. While the leader sets tolls on some links in the network, the drivers make their travel decisions in order to minimize their perceived travel costs. Their travel choices determine the traffic flows in the network. If the link tolls are calculated as mappings of the traffic flows in the network, this game is of the inverse Stackelberg type.

1.2 Introduction to the applications studied in this thesis

In this section the three applications considered in this thesis will be briefly introduced: the optimal toll design problem in Section 1.2.1, the problem of electricity market liberalization in Section 1.2.2, and the theory of incentives in Section 1.2.3.

1.2.1 Optimal toll design

Let us imagine a road network containing cities and routes connecting them. Individual routes have different properties. Some routes, like highways, are wider, have rather high capacities, and have a rather high speed limit, too, whereas other routes, like local ones, are narrower, with lower capacities, and have a rather low speed limit. Clearly, the local roads will clog easier than the highways.

Some routes may be tolled. The toll is set by a road authority, which tries to reach its own goal, by choosing the routes to be tolled and the amount of money the drivers have to pay.

Within the considered network drivers depart from their origin cities to their destination cities, for example, from the city where they live to the city where they work. Each driver chooses among the routes available the one that would be optimal for him/her.

Here the word "optimal" can have a different meaning for different drivers. Some drivers need to depart and arrive within a certain time horizon, and do not mind if they have to pay toll fees, as long as they do not end up on a congested road (e.g., drivers traveling to work everyday, with fixed office hours). For other drivers it is not important when they leave and when they arrive, as long as their total travel time is not too high. There are also drivers for whom the most important consideration is not to pay any tolls, no matter how slow their journey is. Still other drivers pick the most scenic route.

For each of the drivers a dynamic cost function can be defined. This cost function contains a travel-time dependent part, and a part containing the tolls the traveler has to pay when traveling from his/her origin to his/her destination. The cost function can also contain additional terms, like penalty for deviation from the preferred departure time and penalty for deviation from the preferred arrival time. Such a cost function was considered in, e.g., [45]. Each of the travelers chooses his/her route and his/her departure time so as to minimize his/her cost function. The travelers' choices will determine how the traffic spreads over the network. In an equilibrium state, no traveler can improve his/her perceived travel costs by unilateral change of his/her route or departure time. This coincides with the so-called dynamic stochastic user equilibrium [58], or the dynamic deterministic user equilibrium [10, 94] in the complete information case.

The road authority can set tolls in various manners. In this thesis we compare two possible approaches:

- The road authority sets tolls that can vary in time (dynamic toll), but are not directly mapped to the rate of usage of individual routes (traffic-flow invariant toll). The game between the road authority setting tolls to reach its goal and travelers attempting to minimize their perceived travel costs is defined and solved as a Stackelberg game. Many researchers have been dealing with the optimal toll design problem in this setting [46, 53, 84].

- The road authority sets dynamic tolls that are *traffic-flow dependent*. The problem is to find optimal toll mappings that would minimize the total travel time of the system or maximize the total toll revenue. The game between the road authority setting toll mappings to reach its goal and travelers attempting to minimize their travel costs is defined and solved as an inverse Stackelberg game. In the situation with second-best pricing, i.e., when the toll is not set on all links in the network, the concept of the traffic-flow dependent toll is new (See also [74, 79, 81].).

Although we formulate the optimal toll design problem problem in a general manner, such that a solution of the problem exists for wide class of objective functions and user equilibria models

The tolls maximizing the total toll revenue of the system will be much higher than those minimizing the total travel time of the system, as one would intuitively expect.

Both problems mentioned are NP-hard [77, 78], which is why we use advanced heuristic methods, like a neural networks approach [80], to find a satisfying solution.

In some of our case studies the optimal traffic-flow dependent toll is a decreasing function of link traffic flow (rather than increasing as one would assume). This phenomenon is further discussed in Chapters 4 and 5.

1.2.2 Electricity market liberalization

The European electricity market is currently in the midst of drastic transformation from monopolistic, national, and state-owned electricity producers (firms) to a market with competing, private, and often multinational firms. The aim of liberalization is to decrease the sales price of electricity and to bring about more cost efficient electricity production. Little is known about other effects of liberalization, like impacts of this process on environment.

To get more insight into the impacts of liberalization, we introduce a game-theoretic model with electricity producers in various countries as players (see also [75]). Various scenarios of a firms' behavior, depending on the properties and market power of the firms, but also on the strategies of European policy makers, are included in the model. The model encompasses eight European countries: Belgium, Denmark, Finland, France, Germany, The Netherlands, Norway, Sweden.

The firms in individual countries generate electricity by means of different technologies. A producer can own one or more power plants of different types,[1] for which the total capacity for each technology as well as the variable production costs are given. Producers maximize their pay-offs by choosing the amount of electricity to produce with various technologies for various load modes. Firm pay-offs consist in the income from sales of electricity in regional markets minus the (variable) costs of production.

There are limitations on transportation possibilities of electricity, and production capacity of electricity is fixed in the short term. The electricity demand for each country is exogenous. Electricity trade is only feasible with neighboring countries. Emissions are assigned to producers based on the actual technology used and can also be limited.

Real data used for computations are consumers' demands for electricity per region, supply data (generation capacity and cost), trade data (interconnection capacity), data pertaining to distribution losses, and emission factors.

The outcomes of our case studies show that liberalization decreases electricity prices and may decrease production of emissions, provided that restrictions on the electricity production are set well.

The problem of electricity market liberalization is dealt with in Chapter 6.

1.2.3 Theory of incentives

Another application of game theory is so-called theory of incentives. This theory deals with so-called principal-agent models [51, 52, 59, 70], which are an example of the one-leader-one-follower inverse Stackelberg game introduced in Chapter 3.

[1] Each power plant corresponds to one specific technology, but more power plants can be owned by one producer.

Consider a bilateral relationship, in which a *principal* contracts an *agent* to be responsible for the production of some good. The principal has to pay the agent for the good. The salary which the principal offers to the agent for the production of a certain number of products will be decided by the principal. The principal draws up a *contract* in which he specifies the quantity of goods he wants and the salary that he is going to pay to the agent for obtaining the demanded products.

Conflicting objectives and decentralized information are two basic ingredients of incentive theory. The essential paradigm of the analysis of market behavior by economists is one in which economic agents pursue, at least to some extent, their private interests.

The agent can have private information. This private information can be of two types: either the agent can take an action unobserved by the principal (the case of *moral hazard*), or the agent has some private knowledge about his/her cost or valuation that is ignored by the principal (the case of *adverse selection*). In the incentive theory the main problem is to find an optimal strategy for the principal, when he does not have a complete information about the agent.

We will introduce several incentive problems and discuss optimal strategies for the principal with different scenarios of the agent's behavior (See also [76].). These problems fall within the inverse Stackelberg games.

The theory of incentives is dealt with in Chapter 7.

1.3 Overview of this thesis

1.3.1 Thesis outline

This thesis is organized as follows:

- In **Chapter 2** the foundations of classical game theory are introduced.
- In **Chapter 3** we introduce the extension of classical game theory that we use in this thesis, the so-called inverse Stackelberg games.
- In **Chapter 4** we propose an extension of the static optimal toll design problem to a situation with a traffic flow-dependent toll. We develop a neural networks-based algorithm to solve this problem.
- In **Chapter 5** we propose an extension of the dynamic optimal toll design problem to a situation with a traffic-flow dependent toll. Also here we propose a neural networks-based algorithm to solve this problem.
- In **Chapter 6** we define the problem of a liberalized European electricity market. We present various scenarios differing in the electricity producers' behavior and solve these problems analytically or using a numerical algorithm implemented in Matlab.
- In **Chapter 7** we introduce the theory of incentives as a subset of the inverse Stackelberg problems, and we present and solve different principal-agent problems.
- **Chapter 8** summarizes the results of this thesis and outlines directions for future research.

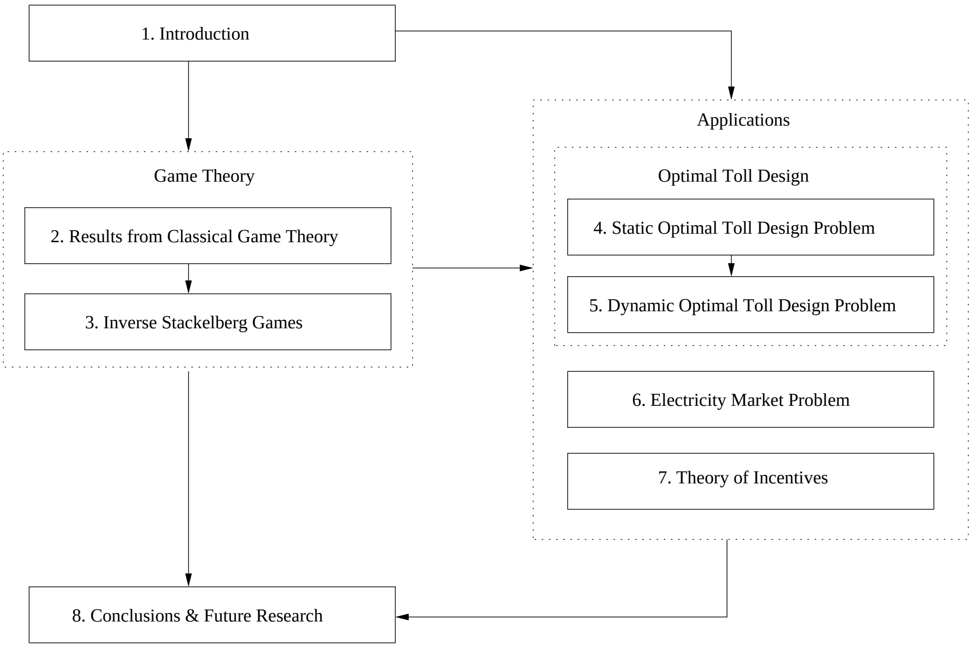

Figure 1.1: Road map. Arrows indicate recommended reading direction

1.3.2 Road map

Figure 1.1 illustrates a grouping of the chapters in related subjects and an ordering in which the chapters can be read. It is suggested to read the chapters in the order as they appear in this thesis. Chapter 1 contains a general introduction to the topic in this thesis, and is therefore suggested to be red first. Chapters 2 and 3 both focus on game theory. Chapter 2 focuses on "classical" game theory and explains its main concepts used in this thesis. Chapter 3 deals with so-called inverse Stackelberg games, and is one of the contributions of this thesis. Chapters 4, 5, 6, and 7 deal with game theory applications studied in this thesis. It is therefore suggested to read Chapters 2 and 3 before Chapters 4, 5, 6, and 7. Both Chapters 4 and 5 focus on bilevel optimal toll design problem, the former on its static version, the latter on its dynamic version. It is suggested to read Chapter 4 before Chapter 5. Chapter 8 summarizes the results of this thesis and gives directions for future research. This chapter should be read as the last chapter.

Main contributions

The main contributions of the research described in this PhD thesis with respect to game theory are the following:

- The concept of an inverse Stackelberg game as generalization of a Stackelberg game is introduced and studied, mainly by means of examples. So far, almost no literature dealing with inverse Stackelberg games exists, thought the concept has been known for some time.

- Possible ways of how to find a (sub-)optimal solution of an inverse Stackelberg game are proposed.

- In a general game theoretical framework, it is shown that under the same initial conditions an inverse Stackelberg game can never bring a worse outcome than a related Stackelberg game, as the Stackelberg strategy is a special case of the Stackelberg strategy.

The main contributions of the research described in this PhD thesis with respect to the game theory application in the optimal toll design problem are the following:

- A concept of a traffic-flow dependent toll in the optimal toll design problem is defined for both the static and the dynamic optimal toll design problem.

- Properties of the optimal toll design problem are discussed.

- A neural-networks based algorithm for solving the optimal toll design problem with a traffic-flow dependent toll is proposed.

- It is shown that the road authority can never be worse-off with a traffic-flow dependent toll than with a traffic-flow invariant toll, since the traffic-flow invariant toll is a trivial case of the traffic-flow dependent toll.

The main contributions of the research described in this PhD thesis with respect to the game theory applications in the energy market liberalization problem are the following:

- A model of a liberalized electricity market, involving 8 European countries, is proposed.

- Different game theory concepts are applied to this model and it is shown that a monopolistic or a duopolistic market yields higher electricity prices than a highly competitive market.

The main contributions of the research described in this PhD thesis with respect to the game theory application in theory of incentives are the following:

- A classical principal-agent model is an inverse Stackelberg game.

- Examples of this game are given and solved analytically.

Chapter 2

Results from Classical Game Theory

In this chapter some classical results from game theory, used in this thesis, will be recapitulated.

2.1 Preliminaries

Definition 2.1 (Game)
A *game* is the interaction among rational, mutually aware players, where the decisions of some players impacts the payoffs of others. A game is described by its players, each player's strategies, and the resulting costs for each outcome. Additionally, in sequential games, the game stipulates the timing (or order) of moves. □

Note that a player's *strategy* in a game is a complete plan of decision (action) for whatever situation might arise; this fully determines the player's behavior. A player's strategy will determine the *decision* (action) the player will take at any stage of the game, for every possible history of play up to that stage. A *strategy profile* is a set of strategies for each player which fully specifies all actions in a game. A strategy profile must include one and only one strategy for every player. A *pure strategy* defines a specific move or action that a player will follow in every possible attainable situation in a game. Such moves may not be random, or drawn from a distribution, as in the case of mixed strategies. A *mixed strategy* is a strategy consisting of possible moves and a probability distribution (collection of weights) which corresponds to how frequently each move is to be played. One can regard a pure strategy as a degenerate case of a mixed strategy, in which that particular pure strategy is selected with probability 1 and every other strategy with probability 0.

We will use the following notation: Let $\mathcal{D}_i$ be a decision space (set of possible decisions) for the i-th player in an n-person noncooperative game. Let $u_i \in \mathcal{D}_i$ be a decision of the i-th player. Let $\mathcal{D} \stackrel{\text{def}}{=} \mathcal{D}_1 \times \mathcal{D}_2 \times \cdots \times \mathcal{D}_n$ be the set of decision spaces. Vector $u \stackrel{\text{def}}{=} (u_1, \ldots, u_n)$ will be called a decision profile, vector $u_{-i} \stackrel{\text{def}}{=} (u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n)$ will be the decision profile without the i-th decision. The objective function for the i-th player will be denoted by $\mathcal{J}_i$, where $\mathcal{J}_i = \mathcal{J}_i(u)$. If player $i \in \{1, \dots, n\}$ chooses decision u_i resulting in decision profile $u = (u_1, \dots, u_n)$, player i obtains outcome $\mathcal{J}_i(u)$. Note that the objective function is individual and depends on the decision profile chosen, i.e., on the decision taken by player i as well as on the decisions of all the other players.

Below we will use the notion u_i not only for the decision of the i-th player, but also for the value of such a decision. In Chapters 4, 5, 6, and 7 different notions for the players' decisions will be introduced to avoid misunderstanding.

2.2 Nash equilibrium

A Nash equilibrium [61] is a set of strategies for finite, non-cooperative games between two or more players whereby no player can improve his/her payoff by changing their strategy. Each player's strategy is an "optimal" response based on the anticipated rational strategy of the other player(s) in the game.

Definition 2.2 (Nash equilibrium)
A decision profile $u^* = (u_1^*, \dots, u_n^*) \in D$ is in a Nash equilibrium (NE) if no unilateral deviation in decision by any single player is profitable for that player, i.e., $\forall i \in \{1, \dots, n\}, \quad u_i \in \mathcal{D}_i, \quad u_i \neq u_i^*$

$$\mathcal{J}_i(u_1^*, \dots, u_{i-1}^*, u_i^*, u_{i+1}^*, \dots, u_n^*) \leq \mathcal{J}_i(u_1^*, \dots, u_{i-1}^*, u_i, u_{i+1}^*, \dots, u_n^*).$$

□

A game can have a pure strategy Nash equilibrium or an Nash equilibrium in its mixed extension. Nash proved that, if we allow mixed strategies (players choose strategies randomly according to pre-assigned probabilities), then every n-player game in which every player can choose from finitely many actions admits at least one Nash equilibrium.

Players are in a Nash equilibrium if each one is making the best decision that he/she can, taking into account the decisions of the others. However, the Nash equilibrium does not necessarily mean the best cumulative payoff for all the players involved; in many cases all the players might improve their payoffs if they could somehow agree on strategies different from the Nash equilibrium.

Remark 2.1 In Chapter 4 the so–called *Wardrop* equilibrium will be introduced, as a limiting case of the Nash equilibrium applied in macroscopic traffic modeling. The Wardrop equilibrium is the Nash equilibrium with a very large number of players. Then the contribution of a single player to the outcome of the game tends to zero. □

2.3 Stackelberg equilibria and terminology

For the sake of simplicity we will consider a game with two players only.

Let us consider two players, called Leader and Follower, respectively, each having his/her own cost function,

$$\mathcal{J}_\mathrm{L}(u_\mathrm{L}, u_\mathrm{F}), \quad \mathcal{J}_\mathrm{F}(u_\mathrm{L}, u_\mathrm{F}),$$

where $u_{\mathrm{L}}, u_{\mathrm{L}} \in \mathbb{R}$. Each player wants to choose his/her own decision variable in such a way as to minimize his/her own cost function. In the Stackelberg equilibrium concept one player, the leader, announces his/her decision u_{L}, which is subsequently made known to the other player, the follower. With this knowledge, the follower chooses his/her u_{F}. Hence u_{F} becomes a function of u_{L}, written as

$$u_{\mathrm{F}} = l_{\mathrm{F}}(u_{\mathrm{L}}),$$

which is determined through the relation

$$\min_{u_{\mathrm{F}}} \mathcal{J}_{\mathrm{F}}(u_{\mathrm{L}}, u_{\mathrm{F}}) = \mathcal{J}_{\mathrm{F}}(u_{\mathrm{L}}, l_{\mathrm{F}}(u_{\mathrm{L}})).$$

Here it is assumed that this minimum exists and that it is unique for each possible choice u_{L} of the leader. The function $l_{\mathrm{F}}(\cdot)$ is sometimes called a reaction function (i.e. it indicates how the follower will react upon the leader's decision). Before the leader announces his/her decision u_{L}, he/she will realize how the follower will react and hence the leader will choose, and subsequently announce, u_{L} such as to minimize $\mathcal{J}_{\mathrm{L}}(u_{\mathrm{L}}, l_{\mathrm{F}}(u_{\mathrm{L}}))$.

Example 2.1
Suppose

$$\mathcal{J}_{\mathrm{L}}(u_{\mathrm{L}}, u_{\mathrm{F}}) = (u_{\mathrm{F}} - 5)^2 + u_{\mathrm{L}}^2, \quad \mathcal{J}_{\mathrm{F}}(u_{\mathrm{L}}, u_{\mathrm{F}}) = u_{\mathrm{L}}^2 + u_{\mathrm{F}}^2 - u_{\mathrm{L}} u_{\mathrm{F}}.$$

The reaction curve l_{F} is given by $u_{\mathrm{F}} = \frac{1}{2}u_{\mathrm{L}}$ (it is easily found by differentiating $\mathcal{J}_{\mathrm{L}}$ with respect to u_{F})and hence u_{L} will be chosen such as to minimize

$$\left(\frac{1}{2}u_{\mathrm{L}} - 5\right)^2 + u_{\mathrm{L}}^2,$$

which immediately results in $u_{\mathrm{L}} = 2$. With this decision of the leader the follower will choose $u_{\mathrm{F}} = 1$. The costs for the leader and follower are given by 20 and 3, respectively. □

Note that the best that the leader can obtain is

$$\min_{u_{\mathrm{L}} \in \mathcal{D}_{\mathrm{L}}, u_{\mathrm{F}} \in \mathcal{D}_{\mathrm{F}}} \mathcal{J}_{\mathrm{L}}(u_{\mathrm{L}}, u_{\mathrm{F}})$$

We will refer to this value as to the *team minimum*.

2.4 Open loop versus closed loop

These concepts appear in games in which time evolution plays a role.

In open-loop information patterns a strategy only depends on the initial state, at the beginning of a game. In closed-loop information patterns the strategy depends on the current state, i.e., the state at the moment that a decision has to be made.

2.5 Tools for one-person optimization

In this section we will introduce some optimization techniques adopted from control system theory and used in this thesis. For more details about individual techniques, see [5]. In Section 2.5.1 the dynamic programming approach for continuous-time systems will be introduced. In Section 2.5.2 the minimum principle will be introduced. Section 2.5.3 deals with affine quadratic continuous-time optimal control problems.

2.5.1 Dynamic programming for continuous-time systems

The method of dynamic programming is based on *the principle of optimality* which states that an optimal strategy has the property that, whatever the initial state and time are, all remaining decisions (from that particular initial state and particular initial time onwards) must also constitute an optimal strategy. To exploit this principle, one has to work backwards in time, starting at all possible final states with the corresponding final times. The dynamic programming approach, when applied to the single criterion optimization problems defined in continuous time, leads to a partial differential equation, known as the Hamilton-Jacobi-Bellman (HJB) equation. We will consider the problem defined as finding decision u minimizing cost $L(u)$, where

$$\dot{x}(t) = f\left(t, x(t), u(t)\right), \quad x(0) = x_0, \quad t \geq 0, \tag{2.1}$$

$$u(t) = \gamma(t, x(t)) \in S, \quad \gamma \in \Gamma, \tag{2.2}$$

$$L(u) = \int_0^T g\left(t, x(t), u(t)\right) \mathrm{d}t + q\left(T, x(T)\right), \tag{2.3}$$

$$T = \min_{t \geq 0} \{t : l\left(t, x(t)\right) = 0\}. \tag{2.4}$$

Here t indicates the time, $\dot{x}$ denotes $\mathrm{d}x/\mathrm{d}t$. The state x of this model evolves in time according to the differential equation $\dot{x} = f\left(t, x(t), u(t)\right)$. In general the state x can be an n-dimensional vector (written as $x \in \mathrm{R}^n$) and $t \in [0, T]$, where $T > 0$ represents the fixed final time. Under suitable conditions on the function f, the time evolution of x is uniquely determined by the differential equation. A scalar function l defines an n-dimensional smooth manifold in the product space $\mathbb{R}^n \times \mathbb{R}_+$, and the class of all admissible closed-loop strategies Γ. The so-called *value function*

$$V(t, x) \stackrel{\text{def}}{=} \min_{\substack{u(s) \\ t \leq s \leq T}} \left[\int_t^T g\left(s, x(s), u(s)\right) \mathrm{d}s + q\left(T, x(T)\right) \right], \tag{2.5}$$

satisfying the boundary condition

$$V(T, x) = q(T, x) \quad \text{along} \quad l(T, x) = 0. \tag{2.6}$$

describes the minimal cost-to-go from any initial state x and any initial time t. If V is continuously differentiable, the principle of optimality yields the following HJB equation:

$$-\frac{\partial V(t, x)}{\partial t} = \min_u \left[\frac{\partial V(t, x)}{\partial x} f(t, x, u) + g(t, x, u) \right], \tag{2.7}$$

which takes (2.6) as the boundary condition.

Theorem 2.2 *(Sufficiency) If a continuously differentiable function $V(t, x)$ can be found that satisfies the HJB equation* (2.7) *subject to the boundary condition* (2.6)*, then it generates the optimal strategy through the static (pointwise) minimization problem defined by the right-hand side of* (2.7).

Proof: See [5]. □

2.5.2 The minimum principle

Let V from (2.7) be twice continuously differentiable. Let function $\tilde{H}$ be defined as

$$\tilde{H}(t,x,u) \stackrel{\text{def}}{=} \frac{\partial V(t,x)}{\partial x} f(t,x,u) + g(t,x,u).$$

Then equation (2.7) for u^* minimizing $\tilde{H}(t,x,u)$ can be written as

$$\frac{\partial V(t,x)}{\partial t} + \tilde{H}(t,x,u^*) = 0. \tag{2.8}$$

Since V is twice continuously differentiable, differentiation of (2.8) with respect to x and t yields

$$\frac{\partial g}{\partial x} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial V}{\partial x}\right) + \frac{\partial V}{\partial x}\frac{\partial f}{\partial x} + \frac{\partial \tilde{H}}{\partial u}\frac{\partial u^*}{\partial x} = 0. \tag{2.9}$$

It can be seen that $\frac{\partial \tilde{H}}{\partial u} = 0$ for $u = u^*$ according to (2.8), if u is not constrained (If there are constraints on u, and u^* happens to be on the boundary, then it can be shown that $\frac{\partial \tilde{H}}{\partial u}\frac{\partial u^*}{\partial x} = 0$.). In all cases, equation (2.9) becomes

$$\frac{\partial g}{\partial x} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial V}{\partial x}\right) + \frac{\partial V}{\partial x}\frac{\partial f}{\partial x} = 0. \tag{2.10}$$

Let x^* denote the state trajectory corresponding to u^*. By introducing the so-called *costate vector*, $\lambda(t) \stackrel{\text{def}}{=} \frac{\partial V(t,x^*(t))}{\partial x}$, (2.10) can be rewritten as

$$\frac{\mathrm{d}p'}{\mathrm{d}t} = -\frac{\partial}{\partial x}\left[g(t,x^*,u^*) + \boldsymbol{\lambda}(t) f(t,x^*,u^*)\right] = -\frac{\partial}{\partial x} H(t,p,x^*,u^*), \tag{2.11}$$

where $H(t,p,x,u) \stackrel{\text{def}}{=} g(t,x,u) + p' f(t,x,u)$. Since $l(T,x) = 0$ for the final time T, T can be regarded as a function of the state, i.e., $T = T(x)$. The boundary condition for $p(t)$ is determined by

$$p'(T) = \frac{\partial V\left(T(x^*),x^*\right)}{\partial x} = \frac{\partial q\left(T(x^*),x^*\right)}{\partial x}. \tag{2.12}$$

Under the assumption that the value function $V(t,x)$ is twice continuously differentiable, the optimal control $u^*(t)$ and corresponding trajectory $x^*(t)$ satisfies the following *canonical equation*:

$$\dot{x}^*(t) = \left(\frac{\partial H}{\partial \boldsymbol{\lambda}}\right)' = f(t,x^*,u^*), \quad x(t_0) = x_0, \tag{2.13}$$

$$\dot{\boldsymbol{\lambda}}(t) = -\frac{\partial H(t,\boldsymbol{\lambda},x^*,u^*)}{\partial x}, \tag{2.14}$$

$$\boldsymbol{\lambda}'(T) = \frac{q(T,x^*)}{\partial x} \quad \text{along} \quad l(T,x) = 0; \tag{2.15}$$

$$H(t,\boldsymbol{\lambda},x,u) \stackrel{\text{def}}{=} g(t,x,u) + \boldsymbol{\lambda} f(t,x,u), \tag{2.16}$$

$$u^*(t) = \arg\min_{u\in S} H(t,\boldsymbol{\lambda},x^*,u). \tag{2.17}$$

In the derivation of (2.13)–(2.17) the controls have been assumed to be functions of time and state. If the control functions are dependent on time only, one obtains the following (Pontryagin) principle.

Theorem 2.3 *(Pontryagin minimum principle) Consider the optimal control problem defined by* (2.1)–(3.13) *and under the open-loop information structure. If the functions* f, g, q, *and* l *are continuously differentiable in* x *and continuous in* t *and* u, *then relations* (2.13)–(2.17) *provide a set of necessary conditions for the optimal control and the corresponding optimal trajectory to satisfy.*

Proof: See [47]. □

2.5.3 Affine quadratic optimal control problems

Let a system be given by

$$\dot{x} = A(t)\,x(t) + B(t)\,u(t) + c(t), \quad x(0) = x_0. \tag{2.18}$$

Let the function to be minimized be defined as

$$L(u) = \frac{1}{2} x'(T)\,Q_f\,x(T) + \frac{1}{2}\int_0^T \left(x'Qx + 2x'p + u'Ru\right)\,\mathrm{d}t, \tag{2.19}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $0 \le t \le T$ and T is fixed. $A(\cdot)$, $B(\cdot)$, $Q(\cdot) \ge 0$, $R(\cdot) > 0$ are matrices of appropriate dimensions and with continuous entries on $[0,T]$. The matrix Q_f is nonnegative-definite, and $c(\cdot)$ and $p(\cdot)$ are continuous vector-valued functions, taking values in $\mathbb{R}^n$. Furthermore, we adopt the feedback information pattern and take a typical control strategy as a continuous mapping $\gamma\colon [0,T]\times\mathbb{R}^n \to \mathbb{R}^m$. The space of all such strategies will be denoted by Γ. The optimal control problem is to find a $\gamma^* \in \Gamma$ such that

$$\mathcal{J}(\gamma^*) \le \mathcal{J}(\gamma), \quad \forall \gamma \in \Gamma, \tag{2.20}$$

where

$$\mathcal{J}(\gamma) \overset{\text{def}}{=} L(u), \quad \text{with} \quad u(\cdot) = \gamma(\cdot, x). \tag{2.21}$$

Since $\mathcal{J}(\gamma)$ is quadratic in x_0 (see [1]) and the minimum cost-to-go, starting from an arbitrary $t \in [0,T]$ at an arbitrary point $x \in \mathbb{R}^n$, is quadratic in x, we can prove that there exists a continuously differentiable value function of the form

$$V(t,x) = \frac{1}{2} x' S(t) x + k'(t) x + m(t) \tag{2.22}$$

that satisfies (2.7). Here S is a symmetric $n \times n$ matrix with continuously differentiable entries, $k(\cdot)$ is a continuously differentiable n-vector, and $m(\cdot)$ is a continuously differentiable function.

Substitution of (2.22) into (2.7) leads to

$$-\frac{1}{2} x'\dot{S}x - x'\dot{k} - \dot{m} = \min_u \left[(Sx+k)'(Ax+Bu+c) + \frac{1}{2} x'Qx + x'p + \frac{1}{2} u'Ru \right]. \tag{2.23}$$

Minimization of the right hand side leads to

$$u^*(t) = \gamma^*(t, x(t)) = -R^{-1}B'\left[S(t)x(t) + k(t)\right]. \tag{2.24}$$

By substituting (2.24) into (2.23) we obtain the following conditions:

$$\dot{S} + SA + A'S - SBR^{-1}B'S + Q = 0, \quad S(T) = Q_f, \tag{2.25}$$

$$\dot{k} + \left(A - BR^{-1}B'S\right)' k + Sc + p = 0, \quad k(T) = 0, \tag{2.26}$$

$$\dot{m} + k'c - \frac{1}{2}k'BR^{-1}B'k = 0, \quad m(T) = 0. \tag{2.27}$$

Proposition 2.5.1 *The affine quadratic continuous-time optimal control problems* (2.18)–(2.19) *admits a unique optimum feedback controller* γ^* *which is given by* (2.24), *where* $S(\cdot)$, $k(\cdot)$, *and* $m(\cdot)$ *uniquely satisfy* (2.25)–(2.27). *The minimum value of the cost function is*

$$\mathcal{J}(\gamma^*) = \frac{1}{2}x_0' S(0)\, x_0 + k'(0)\, x_0 + m(0).$$

Proof: See [5]. □

Chapter 3

Inverse Stackelberg Games

Parts of the research presented in this chapter have been presented in [78]. In Section 3.1 we deal with static inverse Stackelberg games. Dynamic inverse Stackelberg problems are introduced in Section 3.2. Conclusions, possible extensions, and future research are discussed in Section 3.4.

3.1 Static inverse Stackelberg games and equilibria

In Section 3.1.1 the static inverse Stackelberg game with one leader and one follower is introduced, in Section 3.1.2 static inverse Stackelberg problems with one leader and multiple followers will be dealt with.

3.1.1 One leader – one follower games

Let us consider a game with one leader and one follower, each having his/her cost function

$$\mathcal{J}_{\mathrm{L}}\left(u_{\mathrm{L}}, u_{\mathrm{F}}\right), \mathcal{J}_{\mathrm{F}}\left(u_{\mathrm{L}}, u_{\mathrm{F}}\right),$$

to be minimized. In the *inverse Stackelberg game* the leader does not announce the scalar u_{L}, as in the Stackelberg game introduced in Section 2.3, but a function $\gamma_{\mathrm{L}}(\cdot)$, which maps u_{F} into u_{L}.

Given the function $\gamma_{\mathrm{L}}(\cdot)$, the follower's optimal choice of u_{F}, indicated by an asterisk, satisfies

$$u_{\mathrm{F}}^{*} = \arg\min_{u_{\mathrm{F}}} \mathcal{J}_{\mathrm{F}}\left(\gamma_{\mathrm{L}}(u_{\mathrm{F}}), u_{\mathrm{F}}\right). \tag{3.1}$$

The leader, before announcing his/her $\gamma_{\mathrm{L}}(\cdot)$, will realize how the follower will play, and he/she should exploit this knowledge in order to choose the best possible γ_{L}-function, such that ultimately his/her own cost becomes as small as possible. Symbolically this can be written as

$$\gamma_{\mathrm{L}}^{*}(\cdot) = \arg\min_{\gamma_{\mathrm{L}}(\cdot)} \mathcal{J}_{\mathrm{L}}\big(\gamma_{\mathrm{L}}\big(u_{\mathrm{F}}(\gamma_{\mathrm{L}}(\cdot))\big), u_{\mathrm{F}}\big(\gamma_{\mathrm{L}}(\cdot)\big)\big). \tag{3.2}$$

The problem (3.2) belongs to the field of composed functions [50], and is therefore in general very difficult to solve. In general it is very complicated to find an analytical solution of (3.2), if it exists at all.

However, if the leader knows what he/she can achieve (in terms of minimal costs) and what has to be done by all players to reach this outcome, the leader may be able to persuade other players to help him/her to reach this goal (i.e., the value of the leader's cost function obtained if all players minimize it), as shown in Example 3.1. If it is unknown what the leader can achieve in terms of minimal costs, finding the leader's optimal γ_{L}-strategy is generally very difficult.

Example 3.1
Suppose the cost functions are those of Example 2.1, i.e.,

$$\mathcal{J}_{\mathrm{L}}(u_{\mathrm{L}},u_{\mathrm{F}}) = (u_{\mathrm{F}}-5)^2 + u_{\mathrm{L}}^2, \quad \mathcal{J}_{\mathrm{F}}(u_{\mathrm{L}},u_{\mathrm{F}}) = u_{\mathrm{L}}^2 + u_{\mathrm{F}}^2 - u_{\mathrm{L}}u_{\mathrm{F}}.$$

If both the leader and the follower would minimize $\mathcal{J}_{\mathrm{L}}(u_{\mathrm{L}},u_{\mathrm{F}})$, the follower totally disregarding his/her own cost function, the leader would obtain the *team minimum*

$$\min_{u_{\mathrm{L}}\in\mathcal{D}_{\mathrm{L}},u_{\mathrm{F}}\in\mathcal{D}_{\mathrm{F}}} \mathcal{J}_{\mathrm{L}}(u_{\mathrm{L}},u_{\mathrm{F}}) = \mathcal{J}_{\mathrm{L}}(0,5) = 0.$$

To obtain the team minimum in the inverse Stackelberg game the leader should choose the γ_{L}-curve in such a way that the point $(u_{\mathrm{L}},u_{\mathrm{F}}) = (0,5)$ lies on this curve and, moreover, that the set

$$\{(\gamma_{\mathrm{L}}(u_{\mathrm{F}}),u_{\mathrm{F}})\,|u_{\mathrm{F}} \in \mathcal{D}_{\mathrm{F}}\}$$

does not have other points in common with the set

$$\{(u_{\mathrm{L}},u_{\mathrm{F}})\,|\mathcal{J}_{\mathrm{F}}(u_{\mathrm{L}},u_{\mathrm{F}}) < \mathcal{J}_{\mathrm{F}}(0,5)\}.$$

An example of such a curve is $u_{\mathrm{L}} = 2u_{\mathrm{F}} - 10$. Clearly, this is the only linear curve satisfying the requirements.

With this choice of the leader, the best for the follower to do is to minimize

$$\mathcal{J}_{\mathrm{F}}(2u_{\mathrm{F}}-10,u_{\mathrm{F}}),$$

which leads to $u_{\mathrm{F}} = 5$. Then $u_{\mathrm{L}} = 0$ and the leader obtains his/her team minimum in spite of the fact that the follower minimized his/her own cost function (although subject to the constraint $u_{\mathrm{L}} = \gamma_{\mathrm{L}}(u_{\mathrm{F}}) = 2u_{\mathrm{F}} - 10$). □

The following two examples show situations in which the team minimum cannot be reached.

Example 3.2
Let $\mathcal{D}_{\mathrm{L}} = \{\alpha,\beta\}$, $\mathcal{D}_{\mathrm{F}} = \{\gamma,\delta\}$, $\alpha,\beta,\gamma,\delta \in \mathbb{R}$. If the optimal strategies for the leader and for the follower are [1]:

$$u_{\mathrm{L}}^* = \begin{cases} \alpha, & \text{if} \quad u_{\mathrm{F}} = \gamma, \\ \beta, & \text{if} \quad u_{\mathrm{F}} = \delta, \end{cases} \qquad u_{\mathrm{F}}^* = \begin{cases} \gamma, & \text{if} \quad u_{\mathrm{L}} = \beta, \\ \delta, & \text{if} \quad u_{\mathrm{L}} = \alpha. \end{cases}$$

Clearly, in this situation the leader cannot reach the best-possible (team minimum) outcome in the deterministic sense. However, a mixed strategy solution can be found. □

[1] It is easy to define cost functions, corresponding to these optimal strategies.

Example 3.3
Let us consider Example 3.1 with restriction of the decision spaces for both the leader and the follower, $\mathcal{D}_L \stackrel{\text{def}}{=} [-4,3]$, $\mathcal{D}_F \stackrel{\text{def}}{=} [-5,7]$. The worst that can happen to the follower is characterized by $\min_{u_F} \max_{u_L} \mathcal{J}_F$, which is realized for $u_F = -2$, $u_L = -4$ ($\mathcal{J}_F(-4,-2) = 12$). In optimal case the leader obtains

$$\min_{u_L,u_F} \mathcal{J}_L, \quad \text{subject} \quad \text{to } \mathcal{J}_F \le \mathcal{J}_F(-4,-2) = 12.$$

The solution u_L and u_F will be indicated by $u_L^\dagger, u_F^\dagger$. An ε-optimal choice for the leader is

$$u_L = \gamma_L(u_F) = \begin{cases} -4, & \text{for} \quad -5 \le u_F < u_F^\dagger - \varepsilon, \\ u_L^\dagger, & \text{for} \quad u_F^\dagger - \varepsilon \le u_F \le 7, \end{cases}$$

where ε is an arbitrarily small nonnegative number. If $\varepsilon > 0$, the solution is unique, if $\varepsilon = 0$ the follower can respond in a non-unique way. □

Example 3.4 will deal with a situation, in which the leader does not know in advance, what he/she can achieve.

Example 3.4
Let us consider an inverse Stackelberg game, in which the follower minimizes the sum of $f(u_F)$ and $\gamma_L(u_F)$, where $\gamma_L : \mathcal{D}_F \to \mathbb{R}_+^0$, $\gamma_L(0) = 0$, is chosen by the leader and $f : \mathcal{D}_F \to \mathbb{R}$ is a given function, i.e.,

$$u_F^* = \arg \min_{u_F \in \mathcal{D}_F} \left(f(u_F) + \gamma_L(u_F) \right), \tag{3.3}$$

while the leader maximizes $\gamma_L(u_F)$, i.e.,

$$\gamma_L^*(\cdot) = \arg\max_{\gamma_L(\cdot)} \gamma_L(u_F). \tag{3.4}$$

This example can be interpreted as follows: The leader is a bank and the follower is an investor. The investor maximizes his wealth $-f(u_F) - \gamma_L(u_F)$, where u_F [euro] is the investment. For $u_F = 0$ no transaction takes place. Let $\gamma_L(\cdot)$ represent transaction costs function, i.e., if the investor makes investment decision u_F, he has to pay transaction costs of $\gamma_L(u_F)$ [euro]. Since the investor should be secured of a maximum cost $f(0)$ by playing $u_F = 0$, we assume that he/she will only take u_F-values from the set U defined as

$$U \stackrel{\text{def}}{=} \{u_F : f(u_F) \le f(0)\}.$$

In practice the function f would depend on the market situation. Let us consider f defined as follows

$$f(u_F) \stackrel{\text{def}}{=} (u_F - \alpha)^2 + \beta,$$

with $\alpha > 0$. Then $f(0) = \alpha^2 + \beta$ and, therefore, $U = [0, 2\alpha]$.

We will try to find an ε-optimal γ_L-function in an ad-hoc way. Since $\gamma_L(u_F)$ is included in the follower's cost function, we will try to check how different choices of γ_L influence the outcome of the game. Intuitively, it seems to be reasonable to choose γ_L-function defined as follows:

$$\gamma_L(u_F) \stackrel{\text{def}}{=} \begin{cases} (f(0) - f(u_F))(1-\varepsilon), & \text{if} \quad 0 \le u_F \le 2\alpha, \\ \text{nonnegative}, & \text{elsewhere}, \end{cases} \tag{3.5}$$

where $\varepsilon \downarrow 0$.

With the γ_L-function defined by (3.5) the optimal follower's decision is $u_F^* = \alpha$, the follower's costs are $\alpha^2 + \beta - \alpha^2\varepsilon$ and the leader's profit is $(1-\varepsilon)\alpha^2$. The leader (bank) reaps essentially all the follower's (investor's) profits (the latter would have been $\min_{u_F} f(u_F) = \alpha$ if the transaction costs would have been identically zero). Note that the ε-optimal γ_L-function of the leader is non-unique; another choice, similarly advantageous to the bank, would be

$$\gamma_L(u_F) = \begin{cases} \alpha^2 - \varepsilon, & \text{if} \quad u_F \neq 0; \\ 0, & \text{if} \quad u_F = 0, \end{cases}$$

where $\varepsilon \downarrow 0$. Then the outcomes for the leader and the follower are $\alpha^2 - \varepsilon$ and $\alpha^2 + \beta - \varepsilon$, respectively. Note that for $\varepsilon \downarrow 0$ the outcomes of the two games do not differ. □

Note that an upper bound for the leader's profit in Example 3.4 is

$$\mathcal{J}_F(u_F = u_F^*) - \mathcal{J}_F(u_F = 0),$$

where u_F^* is the optimal decision of the follower in absence of transaction costs.

3.1.2 One leader – multiple followers games

If there are two or more followers in the decision problem, the relationship, which determines the solution concept to be adopted between the followers, must be specified. Let $\mathcal{F}$ be the set of all followers and let $|F|$ denotes the number of elements in the set $\mathcal{F}$. Let $\mathcal{D}_L$, $\mathcal{D}_{F_i}$ be decision spaces of the leader and the i-th follower, respectively, $i = 1, \ldots, |F|$.

An inverse Stackelberg strategy for the leader is a mapping $\gamma_L : \mathcal{D}_{F_1} \times \cdots \times \mathcal{D}_{F_{|F|}} \to \mathcal{D}_L$. This mapping can also be a vector-valued function, if $\mathcal{D}_L \in \mathbb{R}^n$, $n \in \mathbb{N}$. Suppose that $u_F^* = \left(u_{F_1}^*, \ldots, u_{F_{|F|}}^*\right)$ is the $|F|$-tuple of the follower's decisions desired by the leader. We say that γ_L is a dominant strategy solution for the leader, if

$$\arg\min_{u_{F_i} \in \mathcal{D}_{F_i}} \mathcal{J}_{F_i}\left(\gamma_L\left(u_{F_1}, \ldots, u_{F_{|F|}}\right), u_{F_1}, \ldots, u_{F_{|F|}}\right) = u_{F_i}^*, \tag{3.6}$$

with arbitrary u_{F_j}, $\forall j \neq i$, $i = 2, \ldots, M$.

If the followers minimize their own cost functions, being noncooperative among themselves, a natural solution concept for their behavior is the Nash equilibrium, introduced in Section 2.2.

Example 3.5 [Followers minimizing their own costs]
Consider three players, the leader $\mathcal{L}$ and two followers $\mathcal{F}_1$, $\mathcal{F}_2$, with decision variables u_L, u_{F_1}, u_{F_2}, respectively. The decision spaces for the followers are the real numbers, i.e.,

$\mathcal{D}_{\mathrm{F}_1} = \mathcal{D}_{\mathrm{F}_2} = \mathbb{R}$, while $\mathcal{D}_{\mathrm{L}} = \mathbb{R}^2$. The followers' cost functions (to be minimized) are given as follows:

$$\mathcal{J}_{\mathrm{F}_1} = \left(u_{\mathrm{L}}^{(1)} - 2\right)^2 + {u_{\mathrm{F}_1}}^2 + u_{\mathrm{F}_2}^2 - u_{\mathrm{L}}^{(1)} u_{\mathrm{F}_1} - u_{\mathrm{L}}^{(1)} u_{\mathrm{F}_2},$$
$$\mathcal{J}_{\mathrm{F}_2} = \left(u_{\mathrm{L}}^{(2)} + 1\right)^2 + {u_{\mathrm{F}_1}}^2 + u_{\mathrm{F}_2}^2 - 2\,u_{\mathrm{L}}^{(2)} u_{\mathrm{F}_1} - 2\,u_{\mathrm{L}}^{(2)} u_{\mathrm{F}_2},$$

and they are playing a Nash game among themselves. The leader has the cost function (to be minimized)

$$\mathcal{J}_{\mathrm{L}} = \left(u_{\mathrm{L}}^{(1)} - 1\right)^2 + \left(u_{\mathrm{L}}^{(2)} + 1\right)^2 + (u_{\mathrm{F}_1} - 1)^2 + (u_{\mathrm{F}_2} + 1)^2 \tag{3.7}$$

It is obvious that the team minimum for the leader is in the point

$$\left(u_{\mathrm{L}}^{(1)}, u_{\mathrm{L}}^{(2)}, u_{\mathrm{F}_1}, u_{\mathrm{F}_2}\right) = (1, -1, 1, -1).$$

An optimal strategy for the leader is to choose linear functions

$$u_{\mathrm{L}}^{(1)} = \frac{u_{\mathrm{F}_1}}{2} + \frac{1}{2}, \quad u_{\mathrm{L}}^{(1)} = \xi(u_{\mathrm{F}_2} + 1) - 1, \quad \xi \in \mathbb{R}. \tag{3.8}$$

Because the parameter ξ can vary, the strategy (3.8) is nonunique. This strategy yields the team minimum for the leader. Outcomes for the leader, the first follower, and the second follower are then 0, 3, and 2, respectively.

□

In some cases the leader can decouple the followers from each other, and, therefore, the leader can control each of the followers' cost functions separately.

Example 3.6 [Example of a decoupling strategy of the leader]
Let us consider three players, one leader $\mathcal{L}$ with $u_{\mathrm{L}} = \left(u_{\mathrm{L}}^{(1)}, u_{\mathrm{L}}^{(2)}\right)$ ($\mathcal{D}_{\mathrm{L}} = (\mathbb{R}_+)^2$) and two followers $\mathcal{F}_1$, $\mathcal{F}_2$ with decision variables u_{F_1} and u_{F_2}, respectively, and decision spaces $\mathcal{D}_{\mathrm{F}_1} = \mathcal{D}_{\mathrm{F}_2} = \mathbb{R}_+$. All players want to minimize their cost functions defined as

$$\mathcal{J}_{\mathrm{L}} = (u_{\mathrm{F}_1} + u_{\mathrm{F}_2})^2,$$
$$\mathcal{J}_{\mathrm{F}_1} = (u_{\mathrm{F}_1} - 1)^2 + u_{\mathrm{F}_2} + u_{\mathrm{L}}^{(1)},$$
$$\mathcal{J}_{\mathrm{F}_2} = (u_{\mathrm{F}_2} - 1)^2 + 2u_{\mathrm{F}_1} + u_{\mathrm{L}}^{(2)}.$$

If the leader applies $u_{\mathrm{L}}^{(1)} = \gamma_{\mathrm{L}}^{(1)}(u_{\mathrm{F}_1}) = 2\,u_{\mathrm{F}_1} - 2\,u_{\mathrm{F}_2}$, it will induce $u_{\mathrm{L}}^{(1)} = 0$ regardless of the value of u_{F_2}, and similarly $u_{\mathrm{L}}^{(2)} = \gamma_{\mathrm{L}}^{(2)}(u_{\mathrm{F}_2}) = 2\,u_{\mathrm{F}_2}$ will induce $u_{\mathrm{L}}^{(2)} = 0$ for all values u_{F_2}, and hence

$$\gamma_{\mathrm{L}}\left(u_{\mathrm{F}_1}, u_{\mathrm{F}_2}\right) = \left(\gamma_{\mathrm{L}}^{(1)}\left(u_{\mathrm{F}_1}\right), \gamma_{\mathrm{L}}^{(2)}\left(u_{\mathrm{F}_2}\right)\right)$$

constitutes a dominant strategy. The leader can control cost functions of each follower separately. However, such a solution does not exist generally, because the cost functions of the followers may not have the required structure. □

Example 3.7 [Followers in a Wardrop equilibrium]
Consider again three players, the leader $\mathcal{L}$ and the followers $\mathcal{F}_1$, $\mathcal{F}_2$, where the decision spaces are defined as $\mathcal{D}_{\mathrm{L}} = \{u_{\mathrm{L}}, u_{\mathrm{L}} = a\,u_{\mathrm{F}_1} + b,\, a, b \in \mathbb{R}\}$, $\mathcal{D}_{\mathrm{F}_1} = \mathcal{D}_{\mathrm{F}_2} = \mathbb{R}^0_+$. Additionally, let the leader's objective function be

$$\mathcal{J}_{\mathrm{L}} = (u_{\mathrm{F}_1})^2 + (u_{\mathrm{F}_2})^2 - u_{\mathrm{F}_1} u_{\mathrm{F}_2} - 3\,u_{\mathrm{F}_1}; \tag{3.9}$$

and let the followers have cost functions

$$\mathcal{J}_{\mathrm{F}_1} = u_{\mathrm{F}_1} + u_{\mathrm{L}},\ \mathcal{J}_{\mathrm{F}_2} = 10\,u_{\mathrm{F}_2}, \tag{3.10}$$

respectively. Additionally, let the following constraints have to be satisfied:

$$u_{\mathrm{F}_1} + u_{\mathrm{F}_2} = 10, \qquad \mathcal{J}_{\mathrm{F}_1} = \mathcal{J}_{\mathrm{F}_2}. \tag{3.11}$$

The leader can in advance compute that the optimal followers' reactions to his $u_{\mathrm{L}} = a\,u_{\mathrm{F}_1} + b$ are

$$u_{\mathrm{F}_1} = -\frac{b-100}{a+11}, \quad u_{\mathrm{F}_2} = \frac{10\,a+b+10}{a+11} \tag{3.12}$$

for any choice of a and b. Hence, the leader minimizes $\mathcal{J}_{\mathrm{L}}$ with u_{F_1} and u_{F_2} given by (3.12), i.e.,

$$J_{\mathrm{L}} = \frac{100\,a^2 - 1100\,a + 33\,ab + 3\,b^2 - 237\,b + 5800}{(11+a)^2}.$$

Minimization of this function with respect to a and b leads to $a = -2/11\,b^* + 79/11$ where b is free. This choice of a yields $u_{\mathrm{L}} = 79/2$ and $\mathcal{J}_{\mathrm{L}} = 37/4$, while the optimal u_{F_1} and u_{F_2} are $11/2$ and $9/2$ and yield $\mathcal{J}_{\mathrm{F}_1} = \mathcal{J}_{\mathrm{F}_2} = 45$. □

Remark 3.1 The interpretation of (3.11) in Example 3.7 is as follows: 10 is the number of drivers traveling from origin o to destination d choosing among two links l_1, l_2, u_{F_1} and u_{F_2} are traffic flows on link l_1 and link l_2, respectively. The travelers' choices of links determine the traffic flows in the network. Link l_1 is tolled with traffic-flow dependent toll $u_{\mathrm{L}} \overset{\mathrm{def}}{=} a\,u_{\mathrm{F}_1} + b$ ($a, b \in \mathbb{R}_+$), $\mathcal{J}_{\mathrm{F}_1}(u_{\mathrm{F}_1}, u_{\mathrm{L}})$ is the cost of using link l_1 and $\mathcal{J}_{\mathrm{F}_2}(u_{\mathrm{F}_2})$ is the cost on the link l_2. Equations (3.11) can be interpreted as Wardrop equilibrium among travelers [94], provided that both links are used. In Chapters 4 and 5 more problems of this type will be considered. While in reality the traffic flows are integer-valued, in our case studies we consider real traffic flows. □

Example 3.8
Consider the game with two followers, with

$$\begin{aligned}\mathcal{J}_{\mathrm{F}_1} &= u_{\mathrm{F}_1}^2 - u_{\mathrm{F}_1} u_{\mathrm{L}} + 2u_{\mathrm{L}}^2,\\ \mathcal{J}_{\mathrm{F}_2} &= u_{\mathrm{F}_2}^2 - 2u_{\mathrm{F}_2} u_{\mathrm{L}} + 5u_{\mathrm{L}}^2,\end{aligned}$$

and one leader with the cost function

$$\mathcal{J}_{\mathrm{L}} = u_{\mathrm{L}}^2 + 2u_{\mathrm{F}_1} u_{\mathrm{L}} + 5u_{\mathrm{F}_2} u_{\mathrm{L}} + u_{\mathrm{F}_1}^2 + u_{\mathrm{F}_2}^2 + 4u_{\mathrm{L}}^2.$$

The team minimum of $\mathcal{J}_{\mathrm{L}}$ would be achieved if

$$u_{\mathrm{F}_1} = -8/25, \quad u_{\mathrm{F}_2} = -20/25, \quad \text{and} \quad u_{\mathrm{L}} = 8/25.$$

The leader will try to obtain his team minimum by right choice of the coefficients α, β, δ in

$$u_{\mathrm{L}} = \gamma_{\mathrm{L}}(u_{\mathrm{F}_1}, u_{\mathrm{F}_2}) = \alpha u_{\mathrm{F}_1} + \beta u_{\mathrm{F}_2} + \delta.$$

If he is successful with linear functions, there is no necessity to consider the larger class of nonlinear functions. We derive three (linear) equations for the coefficients α, β, δ. The first one is obtained by the fact that the absolute minimum must lie on the curve $u_{\mathrm{L}} = \alpha u_{\mathrm{F}_1} + \beta u_{\mathrm{F}_2} + \alpha_3$. The second and third ones are obtained by $\frac{\partial \mathcal{J}_{\mathrm{F}_i}(u_{\mathrm{F}_i}, \gamma_{\mathrm{L}}(u_{\mathrm{F}_1}, u_{\mathrm{F}_2}))}{\partial u_{\mathrm{F}_i}} = 0$, $i = 1, 2$. The equations are

$$\begin{aligned} -8\alpha - 20\beta + 25\delta &= 8, \\ 16\alpha - 20\beta - 25\delta &= 16, \\ -8\alpha + 80\beta - 50\delta &= 40, \end{aligned}$$

which results in

$$\alpha = \frac{3}{5},\ \beta = \frac{7}{15},\ \delta = \frac{332}{375}. \tag{3.13}$$

The γ_{L}-function with coefficients given by (3.13) leads to the leader's team minimum. In other words, he/she cannot do better.

A different approach to find the solution of the problem could be described as follows: Consider the constant level curve $\mathcal{J}_{\mathrm{F}_1}(u_{\mathrm{F}_1}, u_{\mathrm{L}})$ through this point, i.e. $\mathcal{J}_{\mathrm{F}_1}(u_{\mathrm{F}_1}, u_{\mathrm{L}}) = 928/625$. This curve determines u_{L} as a function of u_{F_1}. By taking the total derivative of $\mathcal{J}_{\mathrm{F}_1}(u_{\mathrm{F}_1}, u_{\mathrm{L}}) = 928/625$ with respect to u_{F_1} one obtains $\frac{\partial u_{\mathrm{L}}}{\partial u_{\mathrm{F}_1}} = \frac{3}{5}$ for $(u_{\mathrm{F}_1}, u_{\mathrm{F}_2}, u_{\mathrm{L}}) = (-8/25, -20/25, 8/25)$. By considering the constant level curve $\mathcal{J}_{\mathrm{F}_2}(u_{\mathrm{F}_2}, u_{\mathrm{L}})$ through the same point, one obtains similarly $\frac{\partial u_{\mathrm{L}}}{\partial u_{\mathrm{F}_2}} = \frac{7}{15}$. Hence, if a linear γ_{L} function exists, it must be of the form

$$u_{\mathrm{L}} = \gamma_{\mathrm{L}}(u_{\mathrm{F}_1}, u_{\mathrm{F}_2}) = \alpha u_{\mathrm{F}_1} + \beta u_{\mathrm{F}_2} + \delta$$

with $\alpha = \frac{3}{5}$, $\beta = \frac{7}{15}$. Now δ is obtained by the fact that the curve $u_{\mathrm{L}} = \gamma_{\mathrm{L}}$ must pass through the point $(u_{\mathrm{F}_1}, u_{\mathrm{F}_2}, u_{\mathrm{L}}) = (-8/25, -20/25, 8/25.)$ This yields $\delta = \frac{332}{375}$. □

3.2 Dynamic inverse Stackelberg games and equilibria

The dynamic inverse Stacklelberg game with one leader and one follower is introduced Section 3.2.1. In Section 3.2.2 the inverse Stackelberg problems with one leader and multiple followers will be dealt with. Note that we focus on continuous-time dynamics (as oppose to dynamic problems considered in Chapter 5) although the discretized versions of the problems are used for their solution.

3.2.1 One leader – one follower games

Let us consider a dynamic system defined by

$$\dot{x} = f(x, u_{\mathrm{L}}, u_{\mathrm{F}}), \quad x(0) = x_0.$$

Here t indicates the time, $\dot{x}$ denotes $\mathrm{d}x/\mathrm{d}t$. The state x of this model evolves in time according to the differential equation $\dot{x} = f$.

In general the state x can be an n-dimensional vector (written as $x \in \mathrm{R}^n$), but we restrict ourselves to $n = 1$ and $t \in [0, T]$, where $T > 0$ represents the fixed final time. The quantities $u_i,\ i = \mathrm{L}, \mathrm{F}$, are scalar functions of time possibly restricted to a certain set $u_i(t) \in U_i(t)$, which will be specified later. The function u_i must be chosen in such a way as to minimize the cost function

$$\int_0^T g_i(x, u_{\mathrm{L}}, u_{\mathrm{F}})\,\mathrm{d}t + q_i(x(T));$$

both g_i and q_i are scalar functions and are assumed to satisfy certain regularity conditions, to be introduced later, such that the cost functions are well defined. Under suitable conditions on the function $f(x, u_{\mathrm{L}}, u_{\mathrm{F}})$, the time evolution of x is uniquely determined by the differential equation. There are no restrictions on $x(T)$, it is the so-called *free endpoint problem*. The specific problem on which we will concentrate now is:

$$(P) \begin{cases} \dot{x} = f(x, u_{\mathrm{F}}), \quad x(0) = x_0, \\ \min\limits_{u_{\mathrm{F}}} \mathcal{J}_{\mathrm{F}} = \min\limits_{u_{\mathrm{F}}} \left(q(x(T)) + \int_0^T g(x, u_{\mathrm{F}})\,\mathrm{d}t + \int_0^T \gamma_{\mathrm{L}}(u_{\mathrm{F}}(t))\,\mathrm{d}t \right), \\ \max\limits_{\gamma_{\mathrm{L}}(\cdot)} \mathcal{J}_{\mathrm{L}} = \max\limits_{\gamma_{\mathrm{L}}(\cdot)} \int_0^T \gamma_{\mathrm{L}}(u_{\mathrm{F}}(t))\,\mathrm{d}t. \end{cases}$$

The function γ_{L} is up to the choice of the leader and satisfies

$$\gamma_{\mathrm{L}}(0) = 0,\ \gamma_{\mathrm{L}}(\cdot) \geq 0,\ \gamma_{\mathrm{L}}(u_{\mathrm{F}}) = \gamma_{\mathrm{L}}(-u_{\mathrm{F}}).$$

Similarly, as in Example 3.4, this game can be interpreted as a game between a bank as the leader and an investor as the follower. The investor wants to maximize

$$-q(x(T)) - \int_0^T g(x, u_{\mathrm{F}})\mathrm{d}t - \int_0^T \gamma_{\mathrm{L}}(u_{\mathrm{F}}(t))\,\mathrm{d}t$$

(equivalently wants to minimize $\mathcal{J}_{\mathrm{F}} \stackrel{\mathrm{def}}{=} q(x(T)) + \int_0^T g(x, u_{\mathrm{F}})\mathrm{d}t + \int_0^T \gamma_{\mathrm{L}}(u_{\mathrm{F}}(t))\,\mathrm{d}t$).

The term $-q(x(T))$ represents the wealth of the investor at the final time T and the term $-\int_0^T g(x, u_{\mathrm{F}})\mathrm{d}t$ represents the consumption during the time interval $[0, T]$. The term $u_{\mathrm{F}}(t)$ can be interpreted as a density of the investor's transactions with the bank, i.e., during the time interval $[t, t + \mathrm{d}t]$ the number of transactions equals $u_{\mathrm{F}}(t)\mathrm{d}t$. For $u_{\mathrm{F}} = 0$ no transactions take place ($\gamma_{\mathrm{L}}(0) = 0$). The transactions cost money and we assume that the bank wants to maximize these transaction costs $\mathcal{J}_{\mathrm{L}}$. These costs are included in the costs of the follower $\mathcal{J}_{\mathrm{F}}$. A reasonable restriction on γ_{L} is that $\gamma_{\mathrm{L}}(\cdot)$ is nondecreasing with respect to $|u_{\mathrm{F}}|$. The higher the number of transactions (either buying or selling, one being related to a positive u_{F}, the other one to a negative u_{F}), the higher the costs.

The problem as stated here is obviously a difficult one. We will consider two specific examples of the problem (P) and try to find the optimal γ_{L}-function for the leader.

Example 3.9
This example is a dynamic extension of Example 3.4. The problem is:

$$(P_1)\begin{cases} \dot{x} = u_\mathrm{F}, \quad x(0) = 1, \\ \min\limits_{u_\mathrm{F}} \mathcal{J}_\mathrm{F} = \min\limits_{u_\mathrm{F}} \left(\left(\frac{1}{2}\int_0^1 u_\mathrm{F}^2(t)\mathrm{d}t + \frac{1}{2}x^2(1) \right) + \int_0^1 \gamma_\mathrm{L}\left(u_\mathrm{F}(t)\right)\mathrm{d}t \right), \\ \max\limits_{\gamma_\mathrm{L}(\cdot)} \mathcal{J}_\mathrm{L} = \max\limits_{\gamma_\mathrm{L}(\cdot)} \int_0^T \gamma_\mathrm{L}(u_\mathrm{F}(t))\mathrm{d}t. \end{cases}$$

As in Example 3.4 the transaction costs will be first not considered. Based on the results of the minimization problem of the follower's function without the γ_L-function we will consider different possibilities for γ_L-function to obtain the best-possible outcome for the leader. Therefore, the follower's cost function to be minimized is first defined as

$$\mathcal{J}_\mathrm{F}^0 \stackrel{\text{def}}{=} \left(\frac{1}{2}\int_0^1 u_\mathrm{F}^2(t)\mathrm{d}t + \frac{1}{2}x^2(1) \right). \tag{3.14}$$

The Hamiltonian of the problem given by $\dot{x} = u_\mathrm{F}$, $x(0) = 1$, and by (3.14) is

$$H = \lambda u_\mathrm{F} + \frac{1}{2}u_\mathrm{F}^2.$$

Using the Pontryagin minimum principle we compute that

$$u_\mathrm{F}^*(t) = -\lambda, \quad \dot{\lambda} = 0, \quad \lambda(1) = x(1), \quad u_\mathrm{F}(t) = -x(1), \quad t \in [0,1],$$

and hence

$$u_\mathrm{F}^*(t) = -\frac{1}{2}, \quad x^*(t) = 1 - \frac{1}{2}t, \quad \mathcal{J}_\mathrm{F}\left(u_\mathrm{F} = u_\mathrm{F}^*\right) = \frac{1}{4}, \quad \mathcal{J}_\mathrm{F}(u_\mathrm{F} = 0) = \frac{1}{2}.$$

Mimicking the choice of γ_L in Example 3.4 we will first consider γ_L defined as

$$\gamma_\mathrm{L}(u_\mathrm{F}) \stackrel{\text{def}}{=} -\left(\frac{1}{2} - \varepsilon\right) u_\mathrm{F}\left(1 + u_\mathrm{F}\right)$$

on the interval $[0,1]$ and $\gamma_\mathrm{L}(u_\mathrm{F}) \geq 0$ elsewhere, with $\varepsilon \downarrow 0$. Here $\gamma_\mathrm{L}(\cdot) = 0$ if $\varepsilon = 0$ on $[0,1]$ and

$$H = \lambda u_\mathrm{F} + \frac{1}{2}u_\mathrm{F}^2 - \left(\frac{1}{2} - \varepsilon\right) u_\mathrm{F}\left(1 + u_\mathrm{F}\right).$$

Therefore,

$$u^* = -\frac{\lambda - \frac{1}{2} + \varepsilon}{2\varepsilon}$$

and

$$\dot{\lambda} = 0, \quad \lambda(1) = x(1), \quad x^*(t) = 1 - \frac{1}{2}t.$$

The outcomes of the game for the leader and the follower are

$$\mathcal{J}_\mathrm{L} = \frac{1}{8} - \frac{1}{4}\varepsilon, \quad \mathcal{J}_\mathrm{F} = \frac{3}{8} - \frac{1}{4}\varepsilon.$$

The leader can do better, however, with another choice of quadratic γ_L. Let

$$\gamma_L(u_F) \stackrel{\text{def}}{=} \frac{1}{2}\beta u_F^2 + \alpha u_F,$$

then

$$x^*(1) = \frac{1+\beta-\alpha}{2+\beta}, \quad u_F^*(t) \equiv -\frac{1+\alpha}{2+\beta},$$

where $\beta > -1$ has to be satisfied (second-order conditions). Since $\mathcal{J}_F(u_F = 0) = \frac{1}{2}$, the parameters α and β must satisfy $\mathcal{J}_F(u_F = u_F^*) \leq \frac{1}{2}$, i.e.,

$$\frac{1}{2}\left((u_F^*)^2 + (x^*(1))^2\right) + \gamma_L(u_F^*) = \frac{(2+\beta)-(1+\alpha)^2}{2(2+\beta)} \leq \frac{1}{2},$$

which is always fulfilled for $\beta > -1$. Consider

$$\max_{\alpha,\beta} \gamma_L(u_F^*) = \max_{\alpha,\beta} \frac{1}{2}\left(\beta\left(\frac{1+\alpha}{2+\beta}\right)^2 - 2\alpha\frac{1+\alpha}{2+\beta}\right) = \max_{\alpha,\beta} \frac{\beta - 4\alpha - (4+\beta)\alpha^2}{2(2+\beta)^2}.$$

The maximization with respect to α leads to $\alpha = \frac{-2}{4+\beta}$, which, upon its turn, leads to

$$\max_{\alpha,\beta} \gamma_L(u_F^*) = \max_{\beta} \frac{1}{2(4+\beta)}.$$

Based upon this, the ε-optimal value for β is $\beta = -1+\varepsilon$, where $\varepsilon \downarrow 0$. Subsequently, $\alpha = -\frac{2}{3} + \frac{2}{9}\varepsilon$ up to first order in ε, and with the same accuracy, $u_F^* = -\frac{1}{3} + \frac{1}{9}\varepsilon$. This leads to

$$\mathcal{J}_F = \frac{4}{9} - \frac{1}{27}\varepsilon, \quad \mathcal{J}_L = \frac{1}{6} - \frac{1}{18}\varepsilon,$$

which is a best result for the leader within the class of quadratic γ_L-functions. Without the transaction costs for the investor, its costs would be

$$\mathcal{J}_F - \mathcal{J}_L = \frac{5}{18} + \frac{1}{54}\varepsilon,$$

which is less than what he would have obtained by playing $u_F = 0$. Since $g(u_F^*) > 0$ in a neighborhood of u_F^*, only further away from u_F^* the function γ_L has to be adjusted such that $\gamma_L(\cdot) \geq 0$ everywhere.

We might think that the leader can maximize his/her profit by means of the following nonquadratic choice:

$$\gamma_L(u_F) = \begin{cases} 0, & \text{if} \quad u_F = 0, \\ \frac{1}{4} - \varepsilon, & \text{if} \quad u_F \neq 0, \end{cases}$$

with $\varepsilon \downarrow 0$. It can be derived that the profit of the bank is $\frac{1}{8}$, which is clearly less than what could be obtained with the best quadratic γ_L.

□

To show that with use of the quadratic γ_L-function the profit of the leader is maximized, we will formulate the discretized version of Example 3.9.

Example 3.10 [Discretized version of Example 3.9]
Here we consider the model

$$x^{(i)} = x^{(i-1)} + \frac{1}{N}u^{(i)}, \quad i = 1,2,\ldots,N, \quad x_0 = 1,$$

and the criteria

$$\mathcal{J}_\mathrm{F} = \frac{1}{2N}\sum_{i=1}^{N}\left(u_\mathrm{F}^{(i)}\right)^2 + \frac{1}{2}\left(1+\frac{1}{N}\sum_{i=1}^{N}u_\mathrm{F}^{(i)}\right)^2, \quad \mathcal{J}_\mathrm{L} = \frac{1}{N}\sum_{i=1}^{N}\gamma_\mathrm{L}\left(u_\mathrm{F}^{(i)}\right).$$

The expression to be minimized is then

$$\frac{1}{2}\left[\frac{1}{N}\sum_{k=1}^{N}\left(\sum_{i=1,i\neq k}^{N}\left(u_\mathrm{F}^{(i)}\right)^2\right) + \sum_{k=1}^{N}\left(1+\frac{1}{N}\sum_{i=1,i\neq k}^{N}u_\mathrm{F}^{(i)}\right)^2\right]$$
$$-\frac{1}{2}\sum_{i=1}^{N}\left(u_\mathrm{F}^{(i)}\right)^2 - \frac{1}{2}N\left(1+\frac{1}{N}\sum_{i=1}^{N}u_\mathrm{F}^{(i)}\right)^2.$$

The derivative of this expression with respect to $u_\mathrm{F}^{(1)}$ equals zero. Substitution of $u_\mathrm{F}^{(2)} = \cdots = u_\mathrm{F}^{(N)} = u_\mathrm{F}^{(1)}$ subsequently leads to

$$\left(u_\mathrm{F}^{(i)}\right)^* = -\frac{N}{3N-1},\ i = 1,2,\ldots,N.$$

With $N\to\infty$ we get $\left(u_\mathrm{F}^{(i)}\right)^* = -\frac{1}{3}$, $i = 1,2,\ldots,N$. The profit for the leader is $\frac{N}{2(3N-1)}$. Note that with $\lim_{N\to\infty}\frac{N}{2(3N-1)} = \frac{1}{6}$, as in the case of the best quadratic γ_L-function.

If we consider $\mathcal{J}_\mathrm{F}$ as a function of $u_\mathrm{F}^{(1)}$ only and with $u_\mathrm{F}^{(2)} = \ldots = u_\mathrm{F}^{(N)} = -\frac{N}{3N-1}$, then $\mathcal{J}_\mathrm{F}$ can be computed as

$$\mathcal{J}_\mathrm{F}\left(u_\mathrm{F}^{(1)}, u_\mathrm{F}^{(2)},\ldots,u_\mathrm{F}^{(N)} = -\frac{N}{3N-1}\right) =$$
$$\frac{1}{2}\left[\frac{1}{N}\left(u_\mathrm{F}^{(1)}\right)^2 + \frac{N-1}{N}\left(\frac{N}{3N-1}\right)^2 + \left(1+\frac{1}{N}\left(u_\mathrm{F}^{(1)} - \frac{N(N-1)}{3N-1}\right)\right)^2\right]$$

and

$$\mathcal{J}_\mathrm{F}\left(u_\mathrm{F}^{(1)} = 0, u_\mathrm{F}^{(2)},\ldots,u_\mathrm{F}^{(N)} = -\frac{N}{3N-1}\right)$$
$$-\mathcal{J}_\mathrm{F}\left(u_\mathrm{F}^{(1)} = -\frac{N}{3N-1}, u_\mathrm{F}^{(2)},\ldots,u_\mathrm{F}^{(N)} = -\frac{N}{3N-1}\right)$$
$$= \frac{1}{2}\left[\frac{N(N-1)+(2N)^2}{(3N-1)^2} - \frac{N^2+(2N-1)^2}{(3N-1)^2}\right].$$

If we calculate

$$N\left(\mathcal{J}_\mathrm{F}\left(u_\mathrm{F}^{(1)} = 0, u_\mathrm{F}^{(2)},\ldots,u_\mathrm{F}^{(N)} = -\frac{N}{3N-1}\right)\right.$$
$$\left.-\mathcal{J}_\mathrm{F}\left(u_\mathrm{F}^{(1)} = -\frac{N}{3N-1}, u_\mathrm{F}^{(2)},\ldots,u_\mathrm{F}^{(N)} = -\frac{N}{3N-1}\right)\right),$$

the result is $\frac{N}{2(3N-1)}$, which equals the profit of the bank (as already obtained earlier). Apart from an ε-term it is necessary for the leader that

$$\mathcal{J}_{\mathrm{F}}\left(u_{\mathrm{F}}^{(1)}=-\frac{N}{3N-1},u_{\mathrm{F}}^{(2)},\ldots,u_{\mathrm{F}}^{(N)}=-\frac{N}{3N-1}\right)+\frac{1}{N}\gamma_{\mathrm{L}}$$
$$\leq \mathcal{J}_{\mathrm{F}}(u_{\mathrm{F}}^{(1)}=0,u_{\mathrm{F}}^{(2)},\ldots,u_{\mathrm{F}}^{(N)}=-\frac{N}{3N-1}),$$

or with a quadratic ε-term,

$$\mathcal{J}_{\mathrm{F}}+\frac{1}{N}\gamma_{\mathrm{L}}=\mathcal{J}_{\mathrm{F}}\left(u_{\mathrm{F}}^{(1)}=0,u_{\mathrm{F}}^{(2)},\ldots,u_{\mathrm{F}}^{(N)}=-\frac{N}{3N-1}\right)$$
$$+\frac{1}{N}\varepsilon\left[\left(u_{\mathrm{F}}^{(1)}+\frac{N}{3N-1}\right)^2-\left(\frac{N}{3N-1}\right)^2\right].$$

Hence,

$$\frac{1}{N}\gamma_{\mathrm{L}}\left(u_{\mathrm{F}}^{(1)}\right)=\frac{1}{2}\left[\frac{N\,(N-1)+(2N)^2}{(3N-1)^2}-\frac{1}{N}\left(u_{\mathrm{F}}^{(1)}\right)^2-\frac{N-1}{N}\left(\frac{N}{3N-1}\right)^2+\right.$$
$$\left.-\left(1+\frac{1}{N}\left(u_{\mathrm{F}}^{(1)}-\frac{N\,(N-1)}{3N-1}\right)\right)^2\right]$$
$$+\frac{\varepsilon}{N}\left(\left(u_{\mathrm{F}}^{(1)}\right)^2+\frac{2N}{3N-1}u_{\mathrm{F}}^{(1)}\right)=$$
$$=\frac{1}{2}\left[-\frac{1}{N}\left(u_{\mathrm{F}}^{(1)}\right)^2-\frac{1}{N^2}\left(u_{\mathrm{F}}^{(1)}\right)^2-\frac{4}{3N-1}u_{\mathrm{F}}^{(1)}\right]$$
$$+\frac{\varepsilon}{N}\left(\left(u_{\mathrm{F}}^{(1)}\right)^2+\frac{2N}{3N-1}u_{\mathrm{F}}^{(1)}\right).$$

For $N\longrightarrow\infty$ this leads to exactly the optimal quadratic function obtained before. This is at least true for $\varepsilon=0$. The terms linear in ε differ, however. We now write

$$\mathcal{J}_{\mathrm{F}}\left(u_{\mathrm{F}}^{(1)},\ldots,u_{\mathrm{F}}^{(N)}\right)=\mathcal{J}_{\mathrm{F}}^{0}\left(u_{\mathrm{F}}^{(1)},\ldots,u_{\mathrm{F}}^{(N)}\right)$$
$$+\frac{1}{2}\left[\sum_{i=1}^{N}\left(-\frac{1}{N}\left(u_{\mathrm{F}}^{(i)}\right)^2-\frac{1}{N^2}\left(u_{\mathrm{F}}^{(i)}\right)^2-\frac{2}{3N-1}u_{\mathrm{F}}^{(i)}\right)\right.$$
$$\left.+\frac{2\varepsilon}{N}\left(\left(u_{\mathrm{F}}^{(i)}\right)^2+\frac{4N}{3N-1}u_{\mathrm{F}}^{(i)}\right)\right].$$

The Hessian equals

$$\begin{pmatrix} \frac{2\varepsilon}{N} & \frac{1}{N^2} & \cdots & \frac{1}{N^2} \\ \frac{1}{N^2} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{1}{N^2} \\ \frac{1}{N^2} & \cdots & \frac{1}{N^2} & \frac{2\varepsilon}{N} \end{pmatrix}.$$

For $N>\frac{1}{2\varepsilon}$ all eigenvalues lie in the right half plane. For $N\leq\frac{1}{2\varepsilon}$, however, the Hessian is not positive definite. Therefore, for $-\frac{N}{3N-1}\leq u_{\mathrm{F}}\leq 0$, $\gamma_{\mathrm{L}}(u_{\mathrm{F}})$ is as above, and for $u_{\mathrm{F}}<-\frac{N}{3N-1}$ we choose it as a decreasing function. □

Example 3.11
Let us consider the dynamic model defined by

$$\dot{x} = u_F, \quad x(0) = 1,$$

with criterion

$$\min_{u_F} \frac{1}{2}\int_0^1 (x^2 + {u_F}^2)\,\mathrm{d}t + \frac{1}{2}x^2(1).$$

An important difference with the problem of the previous section is that the optimal control is not constant anymore: $u_F^*(t) = -e^{-t}$, which leads to the minimal value $\mathcal{J}_F^* = \frac{1}{2}$. Therefore, in the discretized problem (see the coming subsection) all $\left(u_F^{(i)}\right)^*$ cannot be equal anymore. Consequently $\gamma_L(u_F)$ will have to be specified in the neighborhood of these different $\left(u_F^{(i)}\right)^*$-values.

We will calculate γ_L as in Example 3.9. First we consider a γ-function of the form $\gamma_L(u_F) = \frac{1}{2}\beta {u_F}^2 + \alpha u_F$. The value function, to be minimized with respect to α and β, is (assuming that $x(0) = 1$)

$$\frac{1}{2}S(0) + k(0) + m(0),$$

where $S(t)$, $k(t)$, and $m(t)$ satisfy (see Section 2.5.3)

$$\begin{aligned}
\dot{S} &= \frac{S^2}{1+\beta} - 1, && S(1) = 1;\\
\dot{k} &= \frac{S}{1+\beta}(k+\alpha), && k(1) = 0;\\
\dot{m} &= \frac{1}{1+\beta}(k\alpha + \frac{1}{2}k^2), && m(1) = 0.
\end{aligned}$$

It would be very difficult to proceed in analytical way from here, and, therefore, we will proceed with the discretized version of Example 3.11. □

Example 3.12 [Discretized version of Example 3.9]
The model is

$$x_i = x_{i-1} + \frac{1}{N}u_F^{(i)}, \quad i = 1,2,\ldots,N, \quad x_0 = 1,$$

and the criteria are

$$\begin{aligned}
\mathcal{J}_F &= \frac{1}{2N}\sum_{i=1}^{N}\left(\left(u_F^{(i)}\right)^2 + x_{i-1}^2\right) + \frac{1}{2}x_N^2 =\\
&= \frac{1}{2N}\sum_{i=1}^{N}\left(\left(u_F^{(i)}\right)^2 + \left(1 + \frac{1}{N}\sum_{k=1}^{i-1}u_F^{(k)}\right)^2\right) + \frac{1}{2}\left(1 + \frac{1}{N}\sum_{i=1}^{N}u_F^{(i)}\right)^2;\\
\mathcal{J}_L &= \frac{1}{N}\sum_{i=1}^{N}\gamma_L\left(u_F^{(i)}\right).
\end{aligned}$$

First we want to solve $\min_{u_{\mathrm{F}}} \mathcal{J}_{\mathrm{F}}$ subject to the model equations. This leads to a linear equation in u_{F}-elements:

$$\begin{pmatrix} d+\zeta_1 & \zeta_2 & \zeta_3 & \dots & \zeta_N \\ \zeta_2 & d+\zeta_2 & \zeta_3 & & \vdots \\ \zeta_3 & \zeta_3 & d+\zeta_3 & & \vdots \\ \vdots & & & \ddots & \zeta_N \\ \zeta_N & \dots & \dots & \zeta_N & d+\zeta_N \end{pmatrix} \begin{pmatrix} u_{\mathrm{F}}^{(1)} \\ u_{\mathrm{F}}^{(2)} \\ u_{\mathrm{F}}^{(3)} \\ \vdots \\ u_{\mathrm{F}}^{(N)} \end{pmatrix} = -N \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \vdots \\ \zeta_N \end{pmatrix}, \tag{3.15}$$

where

$$d = \frac{1}{N},\ \zeta_i = \frac{1}{N^3}(N-i) + \frac{1}{N^2}. \tag{3.16}$$

The numerical computations suggest that the solution u_{F} converges towards $-e^{-t}$, as it should, as $N \to \infty$. An upper bound for what the leader can hope for is obtained via the calculation of the maximum (with respect to $u_{\mathrm{F}}^{(i)},\ i = 1,2,\dots,N$) of

$$\begin{array}{rcl} \mathcal{J}_{\mathrm{F}}(0,u_{\mathrm{F}}^{(2)},u_{\mathrm{F}}^{(3)},u_{\mathrm{F}}^{(4)},\dots,u_{\mathrm{F}}^{(N)}) & - & \mathcal{J}_{\mathrm{F}}(u_{\mathrm{F}}^{(1)},u_{\mathrm{F}}^{(2)},u_{\mathrm{F}}^{(3)},\dots,u_{\mathrm{F}}^{(N)})+ \\ \mathcal{J}_{\mathrm{F}}(u_{\mathrm{F}}^{(1)},0,u_{\mathrm{F}}^{(3)},u_{\mathrm{F}}^{(4)},\dots,u_{\mathrm{F}}^{(N)}) & - & \mathcal{J}_{\mathrm{F}}(u_{\mathrm{F}}^{(1)},u_{\mathrm{F}}^{(2)},u_{\mathrm{F}}^{(3)},\dots,u_{\mathrm{F}}^{(N)})+ \\ \mathcal{J}_{\mathrm{F}}(u_{\mathrm{F}}^{(1)},u_{\mathrm{F}}^{(2)},0,u_{\mathrm{F}}^{(4)},\dots,u_{\mathrm{F}}^{(N)}) & - & \mathcal{J}_{\mathrm{F}}(u_{\mathrm{F}}^{(1)},u_{\mathrm{F}}^{(2)},u_{\mathrm{F}}^{(3)},\dots,u_{\mathrm{F}}^{(N)})+ \\ & \vdots & \\ \mathcal{J}_{\mathrm{F}}(u_{\mathrm{F}}^{(1)},u_{\mathrm{F}}^{(2)},u_{\mathrm{F}}^{(3)},\dots,u_{\mathrm{F}}^{(N-1)},0) & - & \mathcal{J}_{\mathrm{F}}(u_{\mathrm{F}}^{(1)},u_{\mathrm{F}}^{(2)},u_{\mathrm{F}}^{(3)},\dots,u_{\mathrm{F}}^{(N)}), \end{array} \tag{3.17}$$

Written out this expression becomes

$$-\left(\frac{1}{2N}+\frac{1}{2N^2}\right)\sum_{l=1}^{N}\left(u_{\mathrm{F}}^{(l)}\right)^2 - \frac{1}{2N}\sum_{l=1}^{N}\sum_{i=l+1}^{N}\left(\frac{1}{N^2}\left(u_{\mathrm{F}}^{(l)}\right)^2 + \frac{2}{N^2}u_{\mathrm{F}}^{(l)}\sum_{k=1,\neq l}^{i-1}u_{\mathrm{F}}^{(k)}\right)+$$

$$-\frac{1}{N^2}\sum_{l=1}^{N}\left(u_{\mathrm{F}}^{(l)}\sum_{i=1,\neq l}^{N}u_{\mathrm{F}}^{(i)}\right) - \frac{1}{2N}\sum_{l=1}^{N}\sum_{i=l+1}^{N}\frac{2}{N}u_{\mathrm{F}}^{(l)} - \frac{1}{N}\sum_{l=1}^{N}u_{\mathrm{F}}^{(l)},$$

Differentiation with respect to $u_{\mathrm{F}}^{(j)}$ leads to

$$\begin{pmatrix} d+\zeta_1 & 2\zeta_2 & 2\zeta_3 & \dots & 2\zeta_N \\ 2\zeta_2 & d+\zeta_2 & 2\zeta_3 & & \vdots \\ 2\zeta_3 & 2\zeta_3 & d+\zeta_3 & & \vdots \\ \vdots & & & \ddots & 2\zeta_N \\ 2\zeta_N & \dots & \dots & 2\zeta_N & d+\zeta_N \end{pmatrix} \begin{pmatrix} u_{\mathrm{F}}^{(1)} \\ u_{\mathrm{F}}^{(2)} \\ u_{\mathrm{F}}^{(3)} \\ \vdots \\ u_{\mathrm{F}}^{(N)} \end{pmatrix} = -N \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \vdots \\ \zeta_N \end{pmatrix}, \tag{3.18}$$

with d and ζ_i defined as in (3.16). If this linear system of equations is symbolically written as $(\frac{1}{N}I + A)u_{\mathrm{F}} = -N\zeta$, I being the identity matrix, then

$$u_{\mathrm{F}} = -(I - NA + (NA)^2 - \cdots)N^2\zeta.$$

Numerical computations suggest that the solution u_{F} resembles the one of (3.15), i.e., the values are negative, $|u(i)|$ decreases with increasing i.

Above it was assumed that all $u_{\mathrm{F}}^{(i)}$-values were different (which also followed from numerical evidence). In case of all $u_{\mathrm{F}}^{(i)}$-values being equal, an upper bound for the best outcome for the leader is the maximum value of δ for which the following inequalities hold for a choice of the α_i's (all being equal):

$$\mathcal{J}_{\mathrm{F}}(\alpha_1, \alpha_2, \ldots, \alpha_N) + \frac{i}{N}\delta < \mathcal{J}_{\mathrm{F}}(\text{“}\ \alpha's\ \textit{are zero},\ \textit{all combinations}\text{”}), \tag{3.19}$$

$i = 1, 2, \ldots, N$. In general the leader will have to deal with a mixture of the extreme cases (3.17) and (3.19) since he/she will not know ahead of time what yields the best results for him/her. Suppose that all optimal $u_{\mathrm{F}}^{(i)}$ are different. Then choose

$$\gamma_{\mathrm{L}}(u_{\mathrm{F}}^{(i)}) = \mathcal{J}_{\mathrm{F}}(u_{\mathrm{F}}^{(1)}, \ldots, u_{\mathrm{F}}^{(i-1)}, 0, u_{\mathrm{F}}^{(i+1)}, \ldots, u_{\mathrm{F}}^{(N)}) - \mathcal{J}_{\mathrm{F}}(u_{\mathrm{F}}^{(1)}, u_{\mathrm{F}}^{(2)}, u_{\mathrm{F}}^{(3)}, \ldots, u_{\mathrm{F}}^{(N)}) - \varepsilon,$$

and choose for all other values of u_{F} γ_{L} large (except for $\gamma_{\mathrm{L}}(0) = 0$). In general this function will not be monotone with respect to $|u_{\mathrm{F}}|$ and its “usefulness” seems questionable for $N \to \infty$. □

3.2.2 One leader – multiple followers games

In this section we will mention a dynamic problem of the inverse Stackelberg type with one leader and two followers.

Example 3.13
Let us consider the following example:

$$\begin{aligned}
x_1(0) = 0, \quad x_2(0) = 0, \quad \dot{x}_1 = u_{\mathrm{F}_1} u_{\mathrm{F}_2}, \quad \dot{x}_2 = u_{\mathrm{F}_2}, \\
\mathcal{J}_{\mathrm{F}_1} = \frac{1}{2}\int_0^1 u_{\mathrm{F}_1}^2(t)\,\mathrm{d}t + \frac{1}{2}x_1^2(1) + \int_0^1 \gamma_{\mathrm{L}}\left(u_{\mathrm{F}_1}(t)\right)\mathrm{d}t, \\
\mathcal{J}_{\mathrm{F}_2} = \frac{1}{2}\int_0^1 u_{\mathrm{F}_2}^2(t)\,\mathrm{d}t + \frac{1}{2}x_2^2(1) + \int_0^1 \gamma_{\mathrm{L}}\left(u_{\mathrm{F}_2}(t)\right)\mathrm{d}t.
\end{aligned}$$

Both $\mathcal{J}_{\mathrm{F}_1}$ and $\mathcal{J}_{\mathrm{F}_2}$ are to be minimized by the followers, while

$$\mathcal{J}_{\mathrm{L}} \stackrel{\mathrm{def}}{=} \int_0^1 \left(\gamma_{\mathrm{L}}\left(u_{\mathrm{F}_1}(t)\right) + \gamma_{\mathrm{L}}\left(u_{\mathrm{F}_2}(t)\right)\right)\mathrm{d}t \tag{3.20}$$

is to be maximized by the leader.

Our problem is to find the optimal $\gamma_{\mathrm{L}}(\cdot)$ maximizing (3.20), while $\mathcal{J}_{\mathrm{F}_1}$ and $\mathcal{J}_{\mathrm{F}_2}$ are minimized by the followers.

Let us first consider $\gamma_{\mathrm{L}}(\cdot)$ defined as

$$\gamma_{\mathrm{L}}(u_{\mathrm{F_i}}) \stackrel{\mathrm{def}}{=} \alpha {u_{\mathrm{F_i}}}^2 + \beta u_{\mathrm{F_i}}, \quad \forall i \in \{1, 2\} \tag{3.21}$$

With this γ_{L} we can compute

$$H_1 = \frac{1}{2}u_{F_1}^2 + \alpha u_{F_1}^2 + \beta u_{F_1} + \lambda_1 u_{F_1} u_{F_2},$$
$$H_2 = \frac{1}{2}u_{F_2}^2 + \alpha u_{F_2}^2 + \beta u_{F_2} + \lambda_1 u_{F_2},$$

and, therefore,

$$u_{F_1}^* = -\frac{2(1+\alpha)(\beta + 2\alpha\beta - 1)}{(5 + 16\alpha + 20\alpha^2 + 8\alpha^3 + 2\beta + \beta^2)},$$
$$u_{F_2}^* = -\frac{1+\beta}{2(1+\alpha)},$$

and

$$\mathcal{J}_{F_1} = \frac{2(1+\alpha)\left(2\alpha^2 - \alpha\beta^2 + 3\alpha + \beta + 1\right)}{5 + 16\alpha + 20\alpha^2 + 8\alpha^3 + 2\beta + \beta^2},$$
$$\mathcal{J}_{F_2} = \frac{1 + 2\alpha - 2\beta - \beta^2}{4(1+\alpha)},$$
$$\mathcal{J}_L = \frac{4\alpha(1+\alpha)^2(\beta + 2\alpha\beta - 1)^2}{y^2} - \frac{2\beta(1+\alpha)(\beta + 2\alpha\beta - 1)}{y} + \frac{\alpha(\beta+1)^2}{4(\alpha+1)^2} - \frac{\beta(\beta+1)}{2(\alpha+1)},$$

with $y \stackrel{\text{def}}{=} 5 + 16\alpha + 20\alpha^2 + 8\alpha^3 + 2\beta + \beta^2$, $y \neq 0, 1$. Minimization of $\mathcal{J}_L$ with respect to α and β gives $\alpha^* \approx 0.696$, $\beta^* \approx -0.111$.

Substituting $\alpha = \alpha^*$ and $\beta = \beta^*$ into $\mathcal{J}_L$, $\mathcal{J}_{F_1}$, and $\mathcal{J}_{F_2}$, respectively, gives us 0.472, 0.383, and 0.076, respectively, and

$$u_{F_1}^* = 0.151, \quad u_{F_2}^* = -0.262.$$

Let us now consider a linear $\gamma_L(\cdot)$, i.e., $\gamma_L(\cdot)$ defined by

$$\gamma_L(x) \stackrel{\text{def}}{=} \alpha x \quad \forall x \in \mathbb{R}.$$

Then using the same approach as in the previous case we obtain

$$\mathcal{J}_{F_1} = \frac{2(1+\alpha)}{5 + 2\alpha + \alpha^2},$$
$$\mathcal{J}_{F_2} = \frac{1}{4} - \frac{1}{2}\alpha - \frac{1}{4}\alpha^2,$$
$$\mathcal{J}_L = -\frac{\alpha\left(1 + 11 + 3\alpha^2 + \alpha^3\alpha\right)}{2(5 + 2\alpha + \alpha^2)}.$$

Minimization of $\mathcal{J}_L$ with respect to α gives

$$\alpha^* \approx -0.047$$

and $\mathcal{J}_{F_1} \approx 0.234 \cdot 10^{-2}$, $\mathcal{J}_{F_2} \approx 0.249$, and $\mathcal{J}_L \approx 0.429$.

This is a slightly worse outcome for the leader than in the previous case. We could not find any better solution with $\gamma_L(\cdot)$ defined as a polynomial of higher degree than 2.

□

3.3 Extension: Two leaders – one follower

In the following example we will study the problem with two leaders and one follower. In this case there is not an obvious point on which both leaders will agree at the outset. Hence we will not try to start with such a point. This example was introduced in [65].

Example 3.14
The follower has cost function

$$\mathcal{J}_{F_1} = u_{F_1}^2 + u_{L_1}^2 + u_{L_2}^2,$$

and the leaders have the cost functions

$$\mathcal{J}_{L_1} = (u_{F_1} - 1)^2 + (u_{L_1} - 1)^2, \quad \mathcal{J}_{L_2} = (u_{F_1} - 2)^2 + (u_{L_2} - 1)^2.$$

respectively. Suppose that the two leaders will choose their functions u_{L_i} as

$$u_{L_1} = \gamma_{L_1}(u_{F_1}) = \alpha_1 u_{F_1} + \alpha_2, \quad u_{L_2} = \gamma_{L_2}(u_{F_1}) = \beta_1 u_{F_1} + \beta_2.$$

In the three dimensional $(u_{F_1}, u_{F_2}, u_{L_1})$ space these two planes have a line of intersection and the follower is forced to choose the best point (i.e. with the minimum value of $\mathcal{J}_{F_1}$) on this line of intersection. This leads to

$$u_{F_1} = -\frac{\alpha_1\alpha_2 + \beta_1\beta_2}{1 + \alpha_1^2 + \beta_1^2}.$$

Realizing this choice of the follower, the two leaders will substitute this choice into their own γ_{L_i}-functions and subsequently into their own cost functions. Thus these cost functions become functions of the parameters α_i and β_i, $i = 1, 2$, only. By setting

$$\frac{\partial \mathcal{J}_{L_1}(\alpha_1, \alpha_2, \beta_1, \beta_2)}{\partial \alpha_i} = 0, \quad \frac{\partial \mathcal{J}_{L_2}(\alpha_1, \alpha_2, \beta_1, \beta_2)}{\partial \beta_i} = 0,$$

i.e. the necessary conditions for a Nash equilibrium, one obtains four equations with four unknowns. The solutions are

$$\alpha_1 = -5, \quad \alpha_2 = 10, \quad \beta_1 = -2, \quad \beta_2 = 5,$$

with corresponding $u_{F_1} = 2, \quad u_{L_1} = 0, \quad u_{L_2} = 1$, and

$$\alpha_1 = -1, \quad \alpha_2 = 2, \quad \beta_1 = 2, \quad \beta_2 = -2,$$

with corresponding $u_{F_1} = 1, \quad u_{L_1} = 1, \quad u_{L_2} = 0$. Besides some other solutions were indicated which result from the roots of a fourth order polynomial.

Let us study the first solution given in more detail. It turns out that the second order conditions are fulfilled. Hence a correct solution has been obtained. It is striking that the resulting u_F-values coincide with the absolute minimum of the second leader (moreover, the second solution obtained corresponds to the absolute minimum of the first leader).

It is claimed now that the solution obtained is only locally optimal. If the second leader sticks to

$$u_{L_2} = \gamma_{L_2}(u_{F_1}) = -2u_{F_1} + 5$$

it is claimed now that the first leader can do better than

$$u_{L_1} = \gamma_{L_1}(u_{F_1}) = -5u_{F_1} + 10,$$

namely by playing

$$u_{L_1} = \gamma_{L_1}(u_{F_1}) = 5\,u_{F_1} - 4.$$

With this choice of γ_{L_1}, while γ_{L_2} remains the same, i.e., $\gamma_{L_2}(u_{F_1}) = -2u_{F_1} + 5$, the first leader obtains his/her team minimum ($u_{F_1} = u_{L_1} = 1$). How has this latter γ_{L_1} function been obtained? Answer: by substituting the function $u_{L_2} = \gamma_{L_2}(u_{F_1}) = -2u_{F_1} + 5$ into the cost functions $\mathcal{J}_{F_1}$ and $\mathcal{J}_{L_1}$. This now is a game for one leader and one follower. The leader can obtain his/her team minimum in this case.

If the first leader chooses $u_{L_1} = \gamma_{L_1}(u_{F_1}) = 5u_{F_1} - 4$, then the second leader might be willing to change his/her γ_{L_2} function. Indeed, that is the case. With $u_{L_1} = \gamma_{L_1}(u_{F_1}) = +5u_{F_1} - 4$ (fixed) and $u_{L_2} = \gamma_{L_2}(u_{F_1}) = -32u_{F_1} + 65$ the resulting $(u_{F_1}, u_{L_1}, u_{L_2})$ coincides with the absolute minimum of the second leader. If the leaders continue with alternately adapting their optimal functions we obtain:

$$\begin{aligned}
u_{L_2} &= \gamma_{L_2}(u_{F_1}) = -2\,u_{F_1} + 5,\\
u_{L_1} &= \gamma_{L_1}(u_{F_1}) = +5\,u_{F_1} - 4,\\
u_{L_2} &= \gamma_{L_2}(u_{F_1}) = -32\,u_{F_1} + 65,\\
u_{L_1} &= \gamma_{L_1}(u_{F_1}) = +1055\,u_{F_1} - 1054,\\
u_{L_2} &= \gamma_{L_2}(u_{F_1}) = -1114082\,u_{F_1} + 2228165,\\
&\text{etc.}
\end{aligned}$$

Obviously this algorithm does not converge, linear γ_L-functions cannot lead to a Nash solution. □

In the following theorem $(u_{F_1,\mathcal{J}_{L_1}}, u_{L_1,\mathcal{J}_{L_1}})$ refers to the pair (u_{F_1}, u_{L_1}) that minimizes $\mathcal{J}_{L_1}$. Similarly, $(u_{F_1,\mathcal{J}_{L_2}}, u_{L_2,\mathcal{J}_{L_2}})$ refers to the pair (u_{F_1}, u_{L_2}) that minimizes $\mathcal{J}_{L_2}$.

Theorem 3.2 *If $u_{F_1,\mathcal{J}_{L_1}} \neq u_{F_1,\mathcal{J}_{L_2}}$, a Nash solution between the leaders does not exist.*

This theorem holds irrespective of the class of $\gamma_{L_i}(u_{F_1})$ functions, $i = 1,2$. These functions are allowed to be discontinuous (even with an infinite number of discontinuity points); the theorem remains true.

Proof: See [65]. □

Example 3.15

Let us consider the cost functions of example 3.14 once more, but now with the constraints $-1 \le u_{F_1} \le +3$, $-1 \le u_{L_i} \le +3$, $i = 1,2$. The roles of the players remain the same. We will let the two leaders alternately minimize their cost functions and see whether this algorithm converges.

We start by assuming γ_{L_1} to be given with $u_{L_1} = \gamma_{L_1}(u_{F_1}) \equiv 0$ (A two-player Stackelberg game results with L_2 as leader and F_1 as follower). Their cost functions are respectively

$$\mathcal{J}_{F_1} = u_{F_1}^2 + u_{L_2}^2,\ \mathcal{J}_{L_2} = (u_{F_1} - 2)^2 + (u_{L_2} - 1)^2.$$

An optimal choice for L_2 is

$$u_{\mathrm{L}_2} = \gamma_{\mathrm{L}_2}(u_{\mathrm{F}_1}) = \begin{cases} 3 & \text{if} \quad u_{\mathrm{F}_1} \neq 2, \\ 1 & \text{if} \quad u_{\mathrm{F}_1} = 2. \end{cases} \tag{3.22}$$

As a result of this choice F_1 will choose $u_{\mathrm{F}_1} = 2$. Subsequently $u_{\mathrm{L}_2} = 1$ and L_2 has realized his/her team minimum. Note that many other choices for γ_{L_1} are possible with the same result, e.g.

$$u_{\mathrm{L}_2} = \gamma_{\mathrm{L}_2}(u_{\mathrm{F}_1}) = \begin{cases} -2u_{\mathrm{F}_1} + 5 & \text{if} \quad 1 \leq u_{\mathrm{F}_1} \leq 3, \\ 3 & \text{if} \quad -1 \leq u_{\mathrm{F}_1} \leq +1. \end{cases}$$

We will continue with the first choice for γ_{L_2} given, i.e. (3.22). Keeping this function fixed, the other leader, L_1, will now choose his/her optimal $\gamma_{\mathrm{L}_1}(u_{\mathrm{F}_1})$ function. Equation (3.22) is substituted into $\mathcal{J}_{\mathrm{F}_1}$ leading to

$$\mathcal{J}_{\mathrm{F}_1} = u_{\mathrm{F}_1}^2 + u_{\mathrm{L}_1}^2 + \begin{cases} 9 & \text{if} \quad u_{\mathrm{F}_1} \neq 2, \\ 1 & \text{if} \quad u_{\mathrm{F}_1} = 2. \end{cases}$$

It is easily verified now that an optimal solution for L_1 is

$$u_{\mathrm{L}_1} = \gamma_{\mathrm{L}_1}(u_{\mathrm{F}_1}) = \begin{cases} 3 & \text{if} \quad u_{\mathrm{F}_1} \neq 1, \\ 1 & \text{if} \quad u_{\mathrm{F}_1} = 1. \end{cases} \tag{3.23}$$

This leads to the team minimum of L_1. Also in this case, the optimal γ_{L_1} is not unique.

We now fix γ_{L_1} as given in (3.23) and study the best answer by L_2. L_2 cannot obtain his/her team minimum anymore, since $\mathcal{J}_{\mathrm{F}_1}$ prefers playing $u_{\mathrm{F}_1} = 1$ to $u_{\mathrm{F}_1} = 2$, whatever the choice of $\gamma_{\mathrm{L}_2}(\cdot)$. The worst that can happen to player F_1 is the outcome 11 which is realized for $u_{\mathrm{F}_1} = 1$, $u_{\mathrm{L}_2} = \gamma_{\mathrm{L}_2}(u_{\mathrm{F}_1} = 1) = 3$, $u_{\mathrm{L}_1} = \gamma_{\mathrm{L}_1}(u_{\mathrm{F}_1} = 1) = 1$. Hence L_2 should consider $\min_{u_{\mathrm{F}_1}, u_{\mathrm{L}_2}} \mathcal{J}_{\mathrm{L}_2}(u_{\mathrm{F}_1}, u_{\mathrm{L}_2})$ subject to $\mathcal{J}_{\mathrm{F}_1}(u_{\mathrm{F}_1}, u_{\mathrm{L}_1} = \gamma_{\mathrm{L}_1}(u_{\mathrm{F}_1}), u_{\mathrm{L}_2}) \leq \mathcal{J}_{\mathrm{F}_1}(u_{\mathrm{F}_1} = 1, u_{\mathrm{L}_1} = 1, u_{\mathrm{L}_2} = 3) = 11$. This leads to

$$u_{\mathrm{L}_2} = \gamma_{\mathrm{L}_2}(u_{\mathrm{F}_1}) = \begin{cases} 3 & \text{if} \quad u_{\mathrm{F}_1} \neq 2(\frac{\sqrt{2}}{\sqrt{5}} - \varepsilon_1), \\ \frac{\sqrt{2}}{\sqrt{5}} - \varepsilon_1 & \text{if} \quad u_{\mathrm{F}_1} = 2(\frac{\sqrt{2}}{\sqrt{5}} - \varepsilon_1). \end{cases} \tag{3.24}$$

as a possible choice for L_2. The value $\varepsilon_1 > 0$ has been added so as to make the choice for F_1 unique after (3.24) has been announced. For $\varepsilon_1 = 0$ player F_1 has two choices, but one of them is preferred by L_2.

In this way we continue, keeping $\gamma_{\mathrm{L}_2}(u_{\mathrm{F}_1})$ fixed again, the new (actually: a new) optimal answer by L_1 turns out to be

$$u_{\mathrm{L}_1} = \gamma_{\mathrm{L}_1}(u_{\mathrm{F}_1}) = \begin{cases} 3 & \text{if} \quad u_{\mathrm{F}_1} \neq \sqrt{1 - \varepsilon_1\sqrt{10}} - \varepsilon_2, \\ \sqrt{1 - \varepsilon_1\sqrt{10}} - \varepsilon_2 & \text{if} \quad u_{\mathrm{F}_1} = \sqrt{1 - \varepsilon_1\sqrt{10}} - \varepsilon_2, \end{cases} \tag{3.25}$$

for a small positive ε_2 such as to make the answer by F_1 unique. If we continue in this way, the algorithm converges to

$$\gamma_{\mathrm{L}_1}(u_{\mathrm{F}_1}) = \gamma_{\mathrm{L}_2}(u_{\mathrm{F}_1}) = \begin{cases} 3 & \text{if} \quad u_{\mathrm{F}_1} \neq 0, \\ 0 & \text{if} \quad u_{\mathrm{F}_1} = 0. \end{cases}$$

This solution leads to the team minimum of the follower (!). Apparently the follower is ”the laughing third party”. □

The problem introduced in this section can be extended to a situation with multiple leaders and multiple followers. For more information about this topic, see [65].

3.4 Conclusions and future research

In this chapter we were dealing with one leader – one follower and one leader – multiple followers inverse Stackelberg games with the aim to find the optimal strategy for the leader.

We showed problems in which the optimal strategy of the leader exists as well as the problems in which the optimal strategy does no exist. Problems with unknown optimal outcome for the leader were elaborated mainly in Sections 3.2.1 and 3.2.2. In such situations an ad-hoc approach was used in order to find the best possible outcome for the leader.

Since the theory of inverse Stackelberg games is still in its infancy, the important phenomena was shown mainly by means of examples. Further investigation of the properties of these games is a subject for future research.

Chapter 4

Static Optimal Toll Design

In this chapter the *static* optimal toll design problem will be dealt with. Here the word static refers to the situation in which the traffic does not evolve over time, i.e., the problem is a one-shot game. In Section 4.1 an overview of existing literature about the static optimal toll design problem will be proposed. In Section 4.2 the problem will be introduced, together with basics from modeling of traffic on road networks. In Section 4.3 two concepts of drivers' behavior will be introduced and explained, together with their properties. In Section 4.4 a problem formulation of the static optimal toll design problem will be given. In Section 4.5 the general problem properties will be discussed. In Section 4.6 possible ways of how to solve the problem will be proposed. In Section 4.7 the proposed solution methods will be illustrated on case studies with the network with one origin–destination pair and on the so-called *Beltway* network, respectively. In Section 4.8 possible extensions of the research presented in this chapter together with a summary of the research of this chapter will be given.

4.1 Introduction and literature overview

The optimal toll design problem is a problem of the Stackelberg type [5, 6], applied to the traffic environment with a road authority as a leader and travelers as followers. The aim of the road authority is to minimize its objective function, which is dependent on the travelers' decisions, by choosing optimal tolls for a subset of links (so-called tollable links), while the travelers minimize their individual travel costs. Their behavior is usually modeled by applying a *traffic assignment* procedure [35, 66].

If it is assumed that all drivers are rational and have complete and perfect information regarding network conditions, the *deterministic user equilibrium* (DUE) applies [94]. With imperfect information and distributed preferences a *probabilistic user equilibrium*, referred as well to as *stochastic user equilibrium* (SUE) takes place, for example, as a *logit-based stochastic equilibrium* (LB-SUE), see [58].

There are two main research streams with respect to definition of the set of tollable links. With so-called first-best tolling (or pricing), all the links in the network are assumed to be tollable [68, 96], with so-called second-best tolling not all links are tollable [85]. The latter concept is clearly more applicable in practice.

In [85] and [84] second-best tolling is considered, travelers are driven by the deterministic user equilibrium (DUE), the objective function of the road authority is defined as the surplus of the network, the traffic demand is elastic, and it is assumed that the link cost functions are increasing with respect to traffic flows. In [60, 66] the lower level of the problem (travelers' minimization of travel costs) is formulated and solved as a variational inequality problem (VIP). Here the travelers are driven by DUE. In [68] a very general Stackelberg model is presented, where the road authority has two decision variables, one of them possibly dependent on traffic flow. The paper itself deals with general mathematical properties of traffic equilibria, however. The tolls are assumed to be constant and the traffic-flow-dependent variable is interpreted as management decision of the road authority.

Following extensive case studies of two-route congestion problems in static networks [24, 86, 88], we have introduced its second-best variant in which the link tolls are functions of link and route flows in the network, for only a proper subset of all links. This formulation fits within a theoretical framework of "inverse Stackelberg problems" [64, 80]. In the inverse Stackelberg game the road authority introduces tolls as mappings of the traffic flows in the network and, therefore, the possible responses of the drivers are taken into account in the first place, while in the classical Stackelberg game the traffic-flow invariant toll is set first and the drivers react as second. In both cases, the road authority is the leader.

This chapter introduces an extension of our recent research to the general problem of optimal design with traffic flow-dependent second-best tolling.

Because the problem is at least NP-hard, advanced optimization techniques, which can be parallelized, should be used in order to speed up the solution process. In this chapter an algorithm using neural networks is proposed as such an optimization technique.

4.2 Preliminaries

Consider a strongly connected road network $\mathcal{G} = (\mathcal{N}, \mathcal{A})$, with a finite nonempty node set $\mathcal{N} = \{1, \dots, n\}$ ($n \in \mathbb{N}$) and a finite nonempty set $\mathcal{A} = \{1, \dots, |\mathcal{A}|\}$ ($|\mathcal{A}| \in \mathbb{N}$) of directed arcs (links). Let $\mathcal{RS} \subset \mathcal{N} \times \mathcal{N}$ be a set of origin-destination pairs in the network. We denote the nonempty set of simple routes (i.e., routes without cycles) from the origin r to the destination s by $\mathcal{P}^{(r,s)}$ and the set of all simple routes of the network by $\mathcal{P}$. Let $\mathcal{T} \subseteq \mathcal{A}$ be a set of so-called *tollable arcs* (links), i.e., the links on which toll can be imposed.

Drivers in the road network $G(\mathcal{N}, \mathcal{A})$ travel from their origins to their destinations, being noncooperative among themselves. When using tollable links, drivers might be obliged to pay a prespecified toll. Drivers choose their routes in order to minimize their travel costs. Each of the travel costs is a combination of travel time and tolls. The travelers' choices will determine the traffic flows in the network.

There is a road authority that sets tolls on the tollable links in the network in order to minimize its objective function. The toll values are assumed to be calculated as functions on traffic flows in the network.

Each change of the tolls will cause change in the travelers' behavior, and vice versa. The optimal toll design problem introduced in this chapter is a one leader–more followers inverse Stackelberg game with the road authority as the leader and drivers as the followers.

There is a fixed positive travel demand described by drivers traveling from origin r to destination s: $d^{(r,s)}$ [veh/h]. Let us denote the link traffic flow on link $a \in \mathcal{A}$ by q_a [veh/h] and the route traffic flow on route $p \in \mathcal{P}$ by f_p [veh/h].

Feasibility with respect to the travel demands requires the route flows to satisfy the following conditions [19, 66]:

$$d^{(r,s)} = \sum_{p \in \mathcal{P}^{(r,s)}} f_p, \quad (r,s) \in \mathcal{RS}, \tag{4.1}$$

$$f_p \geq 0, \quad p \in \mathcal{P}^{(r,s)}, \quad (r,s) \in \mathcal{RS}. \tag{4.2}$$

We let $\delta_{p,a}$ be a link-route incidence identifier for $G(\mathcal{N}, \mathcal{A})$, defined as

$$\delta_{p,a} = \begin{cases} 1, & \text{if route } p \in \mathcal{P} \text{ contains link } a \in \mathcal{A}, \\ 0, & \text{otherwise.} \end{cases}$$

The link flow on link a is defined through the route flows in the network by the relation

$$q_a = \sum_{p \in \mathcal{P}} \delta_{p,a} f_p, \quad j \in \{1, \ldots, |\mathcal{A}|\}. \tag{4.3}$$

With each link $a \in \mathcal{A}$ the link cost for travelers entering this link ς_a [euro], defined as

$$\varsigma_a(q_a) = \alpha t_a(q_a) + \theta_a(q_a), \tag{4.4}$$

is associated. Here $\alpha > 0$ [euro/time unit] is the travelers' value of time, $t_a = t_a(q_a)$ [time unit] is the link travel time on link a, and $\theta_a = \theta_a(q_a)$ [euro] is the toll paid by each traveler for using link a.

The link travel time function is assumed to be traffic-flow dependent, continuously differentiable, and increasing with the link traffic flow. The standard way to define the link travel time function is

$$t_a(q_a) = \sum_{k=0}^{|K|} \gamma_k q_a^k, \quad |K| \in \mathbb{N}_0, \quad \text{where} \quad \gamma_k > 0 \quad \forall k. \tag{4.5}$$

If $K = 0$ the link travel time is traffic flow-independent. Another very common link travel time function is the Bureau of Public Roads Delay Function ([17]), defined as

$$t_a(q_a) = t_{a0} \left(1 + \chi_1 \left(\frac{q_a}{C_a} \right)^{\chi_2} \right), \quad \chi_1, \chi_2 > 0, \tag{4.6}$$

where t_{a0} [time unit] is the free-flow travel time on link a, C_a [veh/time unit] is the capacity of link a per time unit.

The route costs c_p, $(p \in \mathcal{P})$ are assumed to be additive, i.e., they are derived from the link costs ς_a $(a \in \mathcal{A})$ through the relation

$$c_p = \sum_{a \in \mathcal{A}} \delta_{p,a} \varsigma_a. \tag{4.7}$$

Let $\mathbf{q}$, $\mathbf{t}$, and $\boldsymbol{\varsigma}$ denote a vector of link flows, a vector of link travel times, and a vector of link travel costs on all links in the network, respectively, i.e,

$$\mathbf{q} \overset{\text{def}}{=} \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_{|A|} \end{pmatrix}, \quad \boldsymbol{t} \overset{\text{def}}{=} \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_{|A|} \end{pmatrix}, \quad \boldsymbol{\varsigma} \overset{\text{def}}{=} \begin{pmatrix} \varsigma_1 \\ \varsigma_2 \\ \vdots \\ \varsigma_{|A|} \end{pmatrix}. \tag{4.8}$$

Similarly, let us define corresponing route vectors:

$$\mathbf{f} \stackrel{\text{def}}{=} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{|\mathcal{P}|} \end{pmatrix}, \quad \boldsymbol{\tau} \stackrel{\text{def}}{=} \begin{pmatrix} \tau_1 \\ \tau_2 \\ \vdots \\ \tau_{|\mathcal{P}|} \end{pmatrix}, \quad \mathbf{c} \stackrel{\text{def}}{=} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{|\mathcal{P}|} \end{pmatrix}.$$

On each link from the set of tollable links $\mathcal{T}$ the road authority can impose a traffic-flow dependent toll. The traffic flow-dependent toll on link $a \in \mathcal{T}$ will be denoted by $\theta_a(\cdot)$. This toll is defined as a polynomial function of the link flow on the same link,[1] i.e.,

$$\theta_a(q_a) = \sum_{m=0}^{M} w_a^{(m)} (q_a)^m, \quad w_a^{(m)} = \begin{cases} 0 & \text{for} \quad a \in \mathcal{A} \setminus \mathcal{T}, \\ \kappa_a^{(m)} & \text{for} \quad a \in \mathcal{T}, \quad \kappa_a^{(m)} \in \mathbb{R}, \end{cases} \tag{4.9}$$

with $M \in \mathbb{N}_0$. The traffic flows for the coming time period can be observed when playing the game repeatedly. If a new toll level scheme is set on the real network, in a finite time (after a finite number of days), during which the travelers try different possibilities of their traveling, the system is assumed to reach an equilibrium state. The road authority is assumed to be perfectly aware of the possible reactions of the drivers to a given vector of toll values and, therefore, can set the toll as defined in (4.9).

By definition,

$$\theta_a(q_a) \begin{cases} = 0 & \text{if} \quad a \in \mathcal{A} \setminus \mathcal{T}, \\ \geq 0 & \text{if} \quad a \in \mathcal{T}. \end{cases} \tag{4.10}$$

This means that the drivers cannot receive rewards when using tolled links. The vector $\boldsymbol{\theta}$ will be a vector of link toll functions[2] and can be symbolically written as follows:

$$\boldsymbol{\theta} \stackrel{\text{def}}{=} \begin{pmatrix} \theta_1(\cdot) \\ \theta_2(\cdot) \\ \vdots \\ \theta_{|\mathcal{A}|}(\cdot) \end{pmatrix}.$$

Additionally, let us define coefficient vectors as follows:

$$\mathbf{w}_a \stackrel{\text{def}}{=} \begin{pmatrix} w_a^{(0)} \\ w_a^{(1)} \\ w_a^{(2)} \\ \vdots \\ w_a^{(M)} \end{pmatrix}, \quad \mathbf{w} \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \vdots \\ \mathbf{w}_{|A|} \end{pmatrix}. \tag{4.11}$$

[1] The motivation for choosing a polynomial toll function is connected with the fact that the polynomial link travel time functions are used in this thesis and, therefore, the first-best toll is a polynomial function of the link flow, too. Other option would be to map the link tolls to the link travel times, as those are best congestion measures.

[2] Note that $_a \equiv 0$ iff $a \in \mathcal{A} \setminus \mathcal{A}$.

Let $w_a^{(m)} \in \left[w^{(m),\min}, w^{(m),\max}\right]$ with $w^{(m),\min}, w^{(m),\max} \in \mathbb{R}$, $w^{(m),\min} < w^{(m),\max}$ for $\forall m \in \{1,\ldots,M\}$ and let sets $\mathbf{W}_a$ and $\mathbf{W}$ be defined as follows:

$$\mathbf{W}_a \stackrel{\text{def}}{=} \left[w^{(0),\min}, w^{(0),\max}\right] \times \left[w^{(1),\min}, w^{(1),\max}\right] \times \cdots \times \left[w^{(M),\min}, w^{(M),\max}\right],$$
$$\mathbf{W} \stackrel{\text{def}}{=} (\mathbf{W}_a)^{|\mathcal{A}|} \quad \forall m \in \{0,\ldots,M\}.$$

Clearly, $\mathbf{W}_a$ is a subset of $\mathbb{R}^M$ and thus $\mathbf{W}_a$ and $\mathbf{W}$ are compact sets. It is assumed that $\mathbf{w}_a \in \mathbf{W}_a$, $\mathbf{w} \in \mathbf{W}$.

With $M = 0$ in equation (4.9) the toll becomes traffic flow-invariant. In that situation the toll on the link a will be set as $w_a^{(0)} \in \mathbb{R}_+^0$, and vector $\boldsymbol{\omega}$ defined as

$$\boldsymbol{\omega} \stackrel{\text{def}}{=} \begin{pmatrix} w_1^{(0)} \\ w_2^{(0)} \\ \vdots \\ w_{|A|}^{(0)} \end{pmatrix}$$

will be a vector of traffic-flow invariant tolls.

4.2.1 Game-theoretic interpretation of the optimal toll design problem

The problem of the optimal toll design can be seen as an inverse Stackelberg game. Two possible interpretations from the game theoretic point of view are possible:

- The drivers, as followers, choose routes from their origins to their destinations so as to minimize their actual or perceived travel costs. Therefore, their decisions are their route choices. Because the traffic flows are dependent upon these decisions and the road authority as the leader sets tolls as functions of the traffic flows in the network, these tolls are also composed functions of the drivers' decisions.

- Because the travelers are uniform, all of them can be seen as one super-player, who is the follower in the one leader – one follower inverse Stackelberg game with the road authority as the leader. The decision of this super-player would establish the traffic flows in the network. The tolls are the functions of the follower's decisions in this game.

In order to model the travelers's behavior (route choices), a traffic assignment model has to be used. In the following section we will discuss such a model.

4.3 Drivers' behavior – static traffic assignment

This section formulates macroscopic *static traffic assignment* (STA) models that describe a way of how individual drivers choose their preferred route from their origin to their destination. The basics of travels' behavior models introduced in this section can be found in, e.g., [66], [67], or [58].

The static traffic assignment contains a path choice model in which all travelers are distributed on all available routes such that a particular static user equilibrium state is reached.

Different variants of network user equilibria can be found in the literature, as the concept of equilibrium is closely related to the properties of the users that give rise to this equilibrium. Network equilibria fall into game-theoretic equilibria discussed in Chapter 2, and derive their properties from those of the participating players (i.e., network users), depending especially on the level of information they have about actions of other players (full information versus partial information or information with perception error), their preferences (the player's cost function), and their behavior (all players are assumed to minimize their own journey costs.).

We consider a single-user class assignment, i.e., all users are assumed to affect the cost of the link they use to other players in the same way and users' tastes in evaluating these costs are identical, although generally users' perception may differ in a random way. The travel demand is inelastic and given.

Various different network traffic equilibria exist (see, e.g., [10]). These equilibria can be seen as specific instances of games, differing in chosen cost functions.

In this chapter two equilibrium concepts will be elaborated upon: The deterministic (Wardrop) user equilibrium (Section 4.3.1) and the logit-based probabilistic (in traffic literature also referred to as stochastic) user equilibrium (Section 4.3.2).

Although the stochastic equilibrium, introduced in Section 4.3.2, represents a more realistic concept of the drivers' behavior than the deterministic user equilibrium, the deterministic equilibria are still widely used, mainly due to its computational advantages and its direct connection to the Nash equilibrium [61].

Note that while a driver is discrete by nature, i.e., half of a driver cannot make a decision, we assume continuous traffic flows, which means that the traffic flows are interpolated by a continuous quantity. This could be justified by the fact that we are interested in average situations and (real-valued) expected traffic flows, in order to compute the optimal tolls for the road authority. A continuous approximation is also acceptable for the large traffic flows.

4.3.1 Deterministic user (Wardrop) equilibrium

The static deterministic traffic equilibrium, or Wardrop equilibrium, is based on the assumption that all road users have complete information about the prevailing traffic conditions, and that they choose the cheapest one among routes available, while taking congestion into account. The Wardrop equilibrium is defined as follows.

Definition 4.1 (Wardrop equilibrium)
For each origin–destination pair, the route travel costs for all users traveling between a specific origin–destination pair are equal, and less than the route travel costs which would be experienced by a single user on any unused feasible route within the same (r,s)-pair, i.e.,

$$f_p\left(c_p - \pi^{(r,s)}\right) = 0, \quad p \in \mathcal{P}^{(r,s)}, \tag{4.12}$$

$$c_p - \pi^{(r,s)} \geq 0, \quad p \in \mathcal{P}^{(r,s)}, \tag{4.13}$$

where $\pi^{(r,s)}$ takes the role of the minimal travel cost of the routes from $\mathcal{P}^{(r,s)}$. □

For alternative formulations of the Wardrop equilibrium (including among others *node-arc optimization* formulation, *arc-route optimization* formulation, or MPEC formulation) and mathematical properties of Wardrop equilibria (including existence and uniqueness), see [60, 66].

4.3.2 Probabilistuc (stochastic) user equilibrium

The main drawback of the Wardrop equilibrium point of view is that each traveler is supposed to have perfect information about the whole network. In more realistic formulations each driver minimizes his or her *perceived* travel costs, i.e., their route valuation is subject to a random error term (either because we do not know their taste, their cost perception is flawed, their knowledge of the least-cost routes is flawed, or because they have information that is unavailable to the road authority). The so-called stochastic user equilibrium applies.

Definition 4.2 (Probabilistic (stochastic) user equilibrium)
For each origin–destination pair, the perceived route travel costs for all users traveling between a specific origin–destination pair are equal, and less than the route travel costs which would be perceived by a single user on any unused feasible route. □

The perceived travel cost from Definition 4.2 is defined as the sum of of the effective route travel cost[3] and a random unobserved component:

$$\tilde{\varsigma}_a(q_a, \varepsilon) = \varsigma_a(q_a) + \varepsilon_a, \tag{4.14}$$

where q_a (the traffic flow on arc a) is considered as a macroscopic deterministic variable, and $E_a(\varepsilon)$ follows some probability distribution, the same for each $a \in \mathcal{A}$, with parameter ε_a (expressing the perception error).

We can distinguish between the case when the user makes the decision for his/her complete route (open-loop game, see Chapter 2), and therefore makes an error in the perceived cost of the route, from the case when he/she takes a new decision at each crossing (closed-loop game, see Chapter 2), and therefore makes multiple perceived cost errors during his/her journey.

When the probabilistic error distribution of error ε is known we can define stochastic assignments.

The *Probit* assignment is an example of the closed-loop model. The errors $E_a(\varepsilon)$ are supposed to be centered Gaussian (normal) random variables. However, the computation of the Probit assignment is difficult and is done using Monte Carlo methods.

In the open-loop situation, with the independent, centered, and Gumbel[4] distributed errors on the perceived costs of the routes (not the arcs) with the same variance (see [27, 58]), the probability that a single driver chooses route $p \in \mathcal{P}^{(r,s)}$ can be computed as follows:

$$P\left\{\tilde{c}_p < \tilde{c}_{\hat{p}}, \quad \forall p \neq \hat{p}, \quad \hat{p} \in \mathcal{P}^{(r,s)} | c\right\} = \frac{\exp(-\mu\, c_p)}{\sum_{\hat{p} \in \mathcal{P}^{(r,s)}} \exp(-\mu c_{\hat{p}})}, \tag{4.15}$$

with the *perception error* μ. This μ is used to calibrate the variance in the cost perception.

[3] The perceived travel costs are additive.

[4] $P\{\mathcal{G} < x\} = e^{-e^{-\mu x - \eta}}$, where η is the Euler constant, the variance of $\mathcal{G}$ is $\pi^2/(6\mu^2)$. The max of an independent Gumbel random variable with the same variances is still a Gumbel variable with the same variance.

The logit-based probabilistic (stochastic) user equilibrium (LB-SUE) conditions can then be characterized by the equations

$$f_p = P\left\{\tilde{c}_p < \tilde{c}_{\hat{p}}, \quad \forall p \neq \hat{p}, \quad \hat{p} \in \mathcal{P}^{(r,s)} | c\right\} d^{(r,s)}, \forall (r,s) \in \mathcal{RS}, \tag{4.16}$$

with $P\left\{\tilde{c}_p < \tilde{c}_{\hat{p}}, \quad \forall \hat{p} \neq p, \quad \hat{p} \in \mathcal{P}^{(r,s)} | c\right\}$ defined by (4.15). The conditions (4.16) are natural, according to the weak law of large numbers (see [38], Chapter 8), i.e., if $d^{(r,s)}$ is large, and if the travelers act independently, then

$$P\left\{\tilde{c}_p < \tilde{c}_{\hat{p}}, \quad \forall \hat{p} \neq p, \quad \hat{p} \in \mathcal{P}^{(r,s)} | c\right\} \approx \frac{f_p}{d^{(r,s)}}.$$

From (4.15) it follows that if the value of μ is large, the perception error is small, and travelers tend to choose routes with minimal cost. Setting $\mu \to \infty$ in (4.15) yields the deterministic user equilibrium (see Section 4.3.1). Therefore, DUE is a special case of LB-SUE and algorithms used to solve LB-SUE can be used also for solving DUE. A small value of μ indicates a large variance in the perception of travel cost, with travelers choosing routes with considerably larger actual travel costs than those being the cheapest. It can also be seen from (4.15) that with $\mu \downarrow 0$ all routes within an (r,s)-pair receive an equal share of the (r,s)-demand.

Remark 4.1 The reason for using the Gumbel distribution is the ease of computing the probability of the maximum of many independent Gumbel random variables and the shape of the distribution, which is close to the normal one. The correlations of the travel costs between the paths are not well represented by logit-based models and probit methods are an attempt to improve the quality of the stochastic models – see [71]. Nevertheless, logit-based methods seem to be the most used ones in traffic engineering.

The logit distribution, obtained from the Gumbel distribution assumption on the perceived travel costs, satisfies a very important property, which justifies its use: *A road having a smaller travel cost than another one has a larger probability of being used than the other one* (see [31]). It can also be seen from equation (4.15) that for all values of μ all routes receive flow, regardless of their travel time. These facts motivate the use of the logit model in our research.

In [31] other properties of the logit distribution (which is in fact the Gibbs distribution of mechanical statistics [39]) are discussed. In particular it minimizes the entropy among all the flow distributions having the same average time. The free parameter μ is a degree of stochasticity. □

For more information about probabilistic (stochastic) equilibria, including derivation of properties of Dial logit equilibrium via Gibbs-Maslov semirings and some well-known mathematical properties, see [58].

4.4 The problem formulation

In Section 4.2.1 we explained that the optimal toll design problem is a game of inverse Stackelberg type, with possible two interpretations.

In this game, the aim of the road authority is to choose w in such a way so as to minimize some given objective function, which can be symbolically written as $Z = Z(q(w),w)$. The problem can be stated as

$$(P)\begin{cases} \text{Find} \\ \quad w^* = \arg\min_{w\in W} Z(q(w),w), \\ \text{subject to} \\ \qquad q = q(w) \in UE(w), \\ \quad \text{with } w\text{'s restricted by condition (4.10).} \end{cases} \tag{4.17}$$

The expression $q \in UE(w)$ reads as "link flow vector q is a result of a used static user equilibrium (UE) model when a polynomial toll function defined by (4.9) with coefficient vector w is used." Here we assume that some equilibrium model, without further assumptions, applies. The "standard" Stackelberg problem is a particular subproblem of (P), defined as

$$(P_0)\begin{cases} \text{Find} \\ \quad \omega^* = \arg\min_{\omega\in W^0} Z(q(\omega),\omega), \\ \quad \text{subject to} \\ \qquad q = q(\omega) \in UE(\omega). \end{cases} \tag{4.18}$$

In the following section the properties of problem (P) will be discussed.

4.5 General problem properties

We will refer to $\boldsymbol{\theta}$ by its coefficient vector w and the pair $(w,q(w))$ will denote a pair containing the vector of chosen toll functions and the link flow vector.

Note that problem (P) is a nonlinear programming problem (NLP) and has at least one solution if a user equilibrium of (4.17) forms a compact set $(w,q(w))$. Also, if for any given w the set $UE(w)$ is a singleton, w determines q uniquely (in general this would not determine the route flows uniquely, though). In this case, the continuity of q in terms of w will guarantee that the constrained set of (P) is closed, which implies the solution existence of (P) since q and w are bounded.

However, since UE denotes a general user equilibrium, it might have multiple solutions in terms of q ($UE(w)$ may not be a singleton). In this case, $UE(w)$ is a point-to-set mapping of w [33].

The following theorem will be used to prove the existence of the solution of the problem (P).

Theorem 4.2 *A set-valued mapping Φ from $\mathbb{R}^n$ to $\mathbb{R}^m$ is closed at any point of $\mathbb{R}^n$ if and only if its graph is a closed set in $\mathbb{R}^n \times \mathbb{R}^m$.*

Proof: See [33]. □

The existence of the solution of problem (P) will depend on the compactness of the graph of $UE(w)$, defined as

$$\Psi(w,q) \stackrel{\text{def}}{=} \{(w,q)\,|q \in UE(w), \forall w \in W\}. \tag{4.19}$$

Theorem 4.3 *(Existence of a solution of problem (P)) Problem (P) has at least one solution if the following conditions are satisfied:*

i. *For all $w \in W$, the set $UE(w)$ is nonempty and compact.*

ii. *Let $w, \overline{w} \in W$ and let $q \in \mathrm{UE}(w)$, $\overline{q} \in \mathrm{UE}(\overline{w})$. For each $\varepsilon > 0$, there exists $\delta > 0$ such that if $||w - \overline{w}|| < \delta$, then*

$$\max_{\forall q \in \mathrm{UE}(w)} \min_{\forall \overline{q} \in \mathrm{UE}(\overline{w})} ||q - \overline{q}|| < \varepsilon.$$

iii. *ς is a continuous function of q.*

Proof: Let $R(0,\varepsilon)$ be an open ball with radius ε. Then $\mathcal{Y} \stackrel{\text{def}}{=} UE(\overline{w}) + R(0,\varepsilon)$ is an open set containing $UE(\overline{w})$. Let us define an other open set $\mathcal{Z} \stackrel{\text{def}}{=} \{w : ||w - \overline{w}|| < \delta\}$. Condition *ii.* in Theorem 4.3 is equivalent to $\cup_{w \in \mathcal{Z}} UE(w) \subseteq \mathcal{Y}$. Thus, under *ii.*, the point-to-set mapping of $UE(w)$ is upper-semicontinuous. Together with condition *i.* it implies that the point-to-set mapping $UE(w)$ is closed on set W. Thus the graph $\Psi(w,q)$ defined in (4.19) is closed by Theorem 4.2. Also, under *i.*, $UE(w)$ is bounded for any $w \in W$. Since W is a bounded set, the graph $\Psi(w,q)$ is bounded as well. Thus, graph $w \in W$ is compact. Together with *iii.* and the fact that W is compact, we can conclude that (P) has at least one solution, since it is a NLP with a continuous objective function defined on a compact set. □

Remark 4.4 Condition *i.* states that for any $w \in W$ the travelers have to respond by at least one q, and that if the solution is not unique, that then the solution set $UE(w)$ must be compact. Condition *ii.* can be roughly stated as "If two toll vectors are very close to each other, then their solution sets are also very close" (if $w \to \overline{w}$, then $UE(w) \to UE(\overline{w})$). This is not satisfied for the deterministic user equilibrium, as shown in Example 4.1, but it holds for many user equilibrium models, including the logit-based probabilistic (stochastic) equilibrium.

Example 4.1 (On properties of Wardrop equilibria)
Let us imagine a one-origin–destination-pair network with two links, i.e., $\mathcal{A} = \{1,2\}$. Link 1 will be tollable, while link 2 is untolled. Let first $\alpha = 1$, let t_1 be defined as $t_1 \stackrel{\text{def}}{=} q_1^5 - 4q_1^3 + 6q_1$, let $t_2 \stackrel{\text{def}}{=} 2$, i.e., it is traffic-flow independent. Let $\theta_1 \stackrel{\text{def}}{=} 1.5q_1^4 - 2.5q_1$, let $q_2 + q_1 + 2$. Then there are two possible solutions in terms of Wardrop equilibrium:

1. $q^{(1)} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$.

2. $q^{(2)} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

It is easy to see that with $\theta \stackrel{\text{def}}{=} \theta_1 \stackrel{\text{def}}{=} 1.5q_1^4 - 2.5q_1 + \varepsilon$ the outcomes would not be "close" to each other, and, therefore, condition *ii.* of Theorem 4.3 is not satisfied. □

Since (P) does not depend on the specific formulation of the user equilibrium, Theorem 4.3 actually establishes the solution existence condition for (P) that can incorporate a broad range of UE models as long as the three conditions in the theorem are satisfied. □

Theorem 4.5 *(NP-hardness of problem (P))*
Problem (P) is strongly NP-hard.

Proof: Problem (P) is a quadratic bilevel programming problem [21]. Even a linear-linear variant (with linear cost functions for the travelers and a linear objective function for the leader) of the problem with traffic-flow invariant tolls was proven to be strongly NP-hard [40, 87]. Therefore, also problem (P) is strongly NP-hard. □

Remark 4.6 Although the solution of the problem (P) yields positive tolls (see equation (4.10)), the toll functions may be decreasing with traffic flow on their own link, as shown in the following simple example. □

Example 4.2 (Toll decreasing with the traffic flow)
Let us consider a problem on a network with three parallel links between one origin–destination pair (r,s), travelers driven by DUE, and the road authority minimizing the total travel time of the system. Let $d^{(r,s)} = D > 0$, let $\alpha = 1$. the link cost and time functions be linear, i.e.,

$$D = q_1 + q_2 + q_3,$$
$$\varsigma_1 = \alpha t_1(q_1) + \theta_1(q_1), \quad \varsigma_2 = \alpha t_2(q_2) + \theta_2(q_2), \quad \varsigma_3 = \alpha t_3(q_3),$$
$$t_1(q_1) = \beta_1 q_1 + \delta_1, \, t_2(q_2) = \beta_2 q_2 + \delta_2, \, t_3(q_3) = \beta_3 q_3 + \delta_3.$$

with $d^{(r,s)} = D$, $\alpha = 1$, $\beta_1 = 1$, $\beta_2 = 2$, $\beta_3 = 0.05$, $\delta_1 = 1.008$, $\delta_2 = 0.672$, $\delta_3 = 2$. Then, the total travel time function can be computed as

$$\begin{aligned} Z(q_1,q_2,q_3) &= \sum_{j=1}^{3} q_j t_j(q_j) \\ &= 1.05 q_1^2 - 0.992 q_1 + 2.05 q_2^2 - 1.328 q_2 + 0.05 D^2 \\ &- 0.1 q_1 D - 0.1 q_2 D + 0.1 q_1 q_2 + 2D. \end{aligned}$$

The global minimum of $Z(q_1,q_2,q_3)$ is in $q_1^* \approx 0.457 + 4.65 \cdot 10^{-2} D$, $q_2^* \approx 0.313 + 2.33 \cdot 10^{-2} D$, $q_3^* \approx 0.930 D - 0.77$ and reaches approximately $-0.435 + 1.993 D + 0.047 D^2$ [time units]. This is the best what the road authority can obtain, given the fixed travel demand (the so-called team minimum).

Let us assume that the road authority introduces the tolls on links l_1 and l_2 as linear functions of the link flows on the same links, i.e., $\theta_1(q_1) = a q_1 + b$, $\theta_2(q_2) = a q_2 + b$, with $\theta_1(\cdot), \theta_2(\cdot) > 0$ on $(0,1)$. With DUE, relation $\varsigma_1 = \varsigma_2 = \varsigma_3$ holds if all three links are used. It can be shown that for any $D > 0.828$ the team minimum for the road authority can be reached (i.e., $q^* = (q_1^*, q_2^*, q_3^*)'$ is that optimal flow pattern for the travelers) and that in such case $a < 0$, while $b > 0$.

The optimal tolls are decreasing with traffic flows on the same links, because link 3 is untolled. Other choices of tollable links would bring toll functions increasing with the traffic flow on the same link. □

Similar phenomena will appear in some of the case studies in Section 4.7.

4.6 Solution of problem (P)

In this section we propose solution methods for solving the problem (P) introduced in Section 4.4.

4.6.1 Analytical solutions

Small problems with drivers driven by the deterministic (Wardrop) user equilibrium can be solved analytically, as a system of equations.

4.6.2 Numerical solutions

With larger problems the analytical solutions become untractable. Standard numerical algorithms for solving the lower level of the problems with DUE are:

- the Frank-Wolfe algorithm and its extensions;
- projection methods;
- relaxation methods;
- the partial lineralization algorithms;
- the column generation algorithms.

More information about these algorithms can be found in, e.g., [66].

To find the solution of the lower level of the problem (the resulting traffic flows for the choice of particular toll) with the drivers driven by the LB-SUE iterative numerical methods are used. The method of *successive averages* (MSA) has been applied to solve the lower level problem. In the MSA algorithm, a search direction is obtained through a stochastic network loading, and the step taken towards that solution corresponds to taking the average of all previously generated solutions, i.e., the step length in iteration k is $1/k$. For more details about the MSA algorithm, see [22, 69].

Solving the upper level of problem (P) (finding optimal toll functions minimizing the objective function) with classical optimization methods may become intractable. If the objective function of the road authority is convex, standard algorithms for convex programming (e.g., conjugate gradient methods, see [13]) can be used. However, in our case the objective function is generally non-convex, as it is usually a polynomial function of the traffic flow (In Example 4.3 a problem with nonconvex objective function is shown.). Therefore, advanced heuristic methods have be used in order to find a solution in a tractable way. We propose to use a a neural-networks approach.

Example 4.3 (Nonconvexity of the objective function)
Let us consider a one-origin–destination pair network with two links, with link 1 tolled with toll defined as a particular polynomial function of the traffic flow, e.g., $\theta_1(q_1) \stackrel{\text{def}}{=} 160 - 25q_1^2 + q_1^4$, and with link 2 untolled. The objective function will be defined as the total toll revenue of the system, i.e., $Z \stackrel{\text{def}}{=} q_1 \cdot \theta_1(q_1) = 160\,q_1 - 25\,q_1^3 + q_1^5$. This function is clearly nonconvex with respect to q_1. □

The solution method for (P) that we propose is a combination of the neural networks approach for the upper level of the problem and a method of sufficient averages for the lower level of the problem. The algorithm will be proposed and discussed in this section.

The concept of neural networks is closely related to the concept of supervised learning, which will be explained below.

4.6.3 Supervised learning

Let function $g : \mathbb{R}^n \to \mathbb{R}^m$ assign a vector $y^i \in \mathbb{R}^m$ to each vector $x^i \in \mathbb{R}^n$, i.e., $y^i = g\left(x^i\right)$. We will refer to the pair $\left(x^i, y^i\right)$ as the i-th *pattern* of the function g. The vector x^i will be called the *input* vector (of g) and the vector y^i will be called the *output* vector (of g). Supervised learning is a way to find an approximation of the function g given a set of o patterns [48].

An artificial neural network (ANN) can be thought of as a simple mathematical formula with parameters called weights [48]. The result of supervised learning is an approximation function g^{app} with an appropriately chosen vector of weights s. The goal of supervised learning with ANN is therefore to find a function $g^{\text{app}} : \mathbb{R}^n \to \mathbb{R}^m$, that is approximating the function g in the "best way". Moreover, it is required that g^{app} has derivatives of all finite orders in the components of x.

There are several criteria that can be used to validate whether the function g^{app} is "close enough" to g. In our approach the so-called *validation error* for each pattern (x^i, y^i), $i = 1, 2, \ldots, o$, is minimized.

The set of o patterns is divided into a set of t training patterns and a set of $o-t$ validation patterns. For a given vector of weights s the training and the validation errors are calculated by

$$\begin{aligned} \varepsilon_t(s) &\overset{\text{def}}{=} \frac{1}{2} \sum_{i=1}^{t} \sum_{k=1}^{m} (g_k^{\text{app}}(x^i; s) - y_k^i)^2, \\ \varepsilon_v(s) &\overset{\text{def}}{=} \frac{1}{2} \sum_{i=t+1}^{o} \sum_{k=1}^{m} (g_k^{\text{app}}(x^i; s) - y_k^i)^2, \end{aligned} \tag{4.20}$$

where g_k^{app} and y_k^i, $k = 1, 2, \ldots, m$, refer to the k-th entry of g^{app} and y^i, respectively. The elements of s are optimized only for t training patterns, while the validation patterns are used to prevent overtraining. Roughly stated: If the training error ε_t becomes small with respect to s, while the validation error ε_v simultaneously grows, the ANN learns the patterns "by heart" and looses its interpolation and extrapolation abilities.

An ANN is trained iteratively, i.e., ε_t is decreased by adaption of s, until ε_v increases for two consecutive iterations (prevention of overtraining). Note that the training stops before a local minimum of ε_t is reached. Weight upgrades $s^{\text{iter}+1} - s^{\text{iter}}$ can be calculated with any minimization algorithm, e.g., a first derivative method such as steepest descent, or a second derivative method such as the Newton's method. For the first derivative methods the iterative sequence

$$s^{\text{iter}+1} = s^{\text{iter}} + \eta\left(\varepsilon_t\left(s^{\text{iter}}\right), \nabla_S \varepsilon_t\left(s^{\text{iter}}\right)\right) \Delta s\left(\varepsilon_t(s^{\text{iter}}), \nabla_S \varepsilon_t\left(s^{\text{iter}}\right)\right), \tag{4.21}$$

with the search direction $\frac{\Delta S}{\|\Delta S\|}$ and with step length η, takes place. Numerical methods implemented within FAUN[5] 1.0 for constrained nonlinear least-squares problems [63] are

[5] Fast Approximation with Universal Neural networks

sequential quadratic programming (SQP) methods and generalized Gauss-Newton (GGN) methods. These methods can exploit the special structure of the Hessian matrix of ε_t [25, 36, 37]. It turns out that in practice SQP and GGN methods can automatically overcome most of the training problems of ANN such as flat spots or steep canyons of the error function ε_t. Advantages of these methods are:

- In comparison to common training methods a more efficient search direction Δs is calculated by use of the so-called *back propagation* (see [16]).
- The step length η is accommodated during the training in contrast to common training methods with fixed step length. The number of learning steps is reduced significantly.
- Only ε_t, $\nabla_S \varepsilon_t$, and ε_v, which can be computed by very fast matrix operations, are required. For other ANN structures, e.g., radial basis functions, an efficient code for $\nabla_S \varepsilon_t$ can be derived by automatic differentiation.
- Maximum and minimum of each weight can be set easily (box constraints).
- The total curvature of the ANN can be constrained (prevention of ANN oscillations).
- Convexity and monotonicity constraints can be set.

4.6.4 Solving the optimal toll design problem

In this section an algorithm for finding the solution of problem (P) is proposed. The flow chart of the solution process is depicted in Figure 4.1.

Below we will describe individual parts of the solution process.

1. Area selection

Initially a set W with very low $w^{(m),\min}$ and very high $w^{(m),\max}$ will be chosen. The area W is changed depending on the outcome of the computation, the algorithm is applied recursively.

2. Computation of sample points of the objective function

This algorithm has two built-in optimization procedures: *outer loop* (corresponding to the upper level of the problem - the decisions of the road authority) and *inner loop* (corresponding to the lower level of the problem - the decision of the drivers) optimization procedures.

In the outer loop of the algorithm a grid search is applied. In each step of the outer algorithm an element of $w \in W$ is selected according to the adopted grid and used as an input for the inner loop. In this way a grid of sample points of the objective function is created.

In the inner loop the *traffic assignment*, including the *route choice model*, aiming to determine the user equilibrium based on the actual travel costs, is applied. To compute new route flow rates in each iteration the *method of successive averages* (MSA) is adopted on

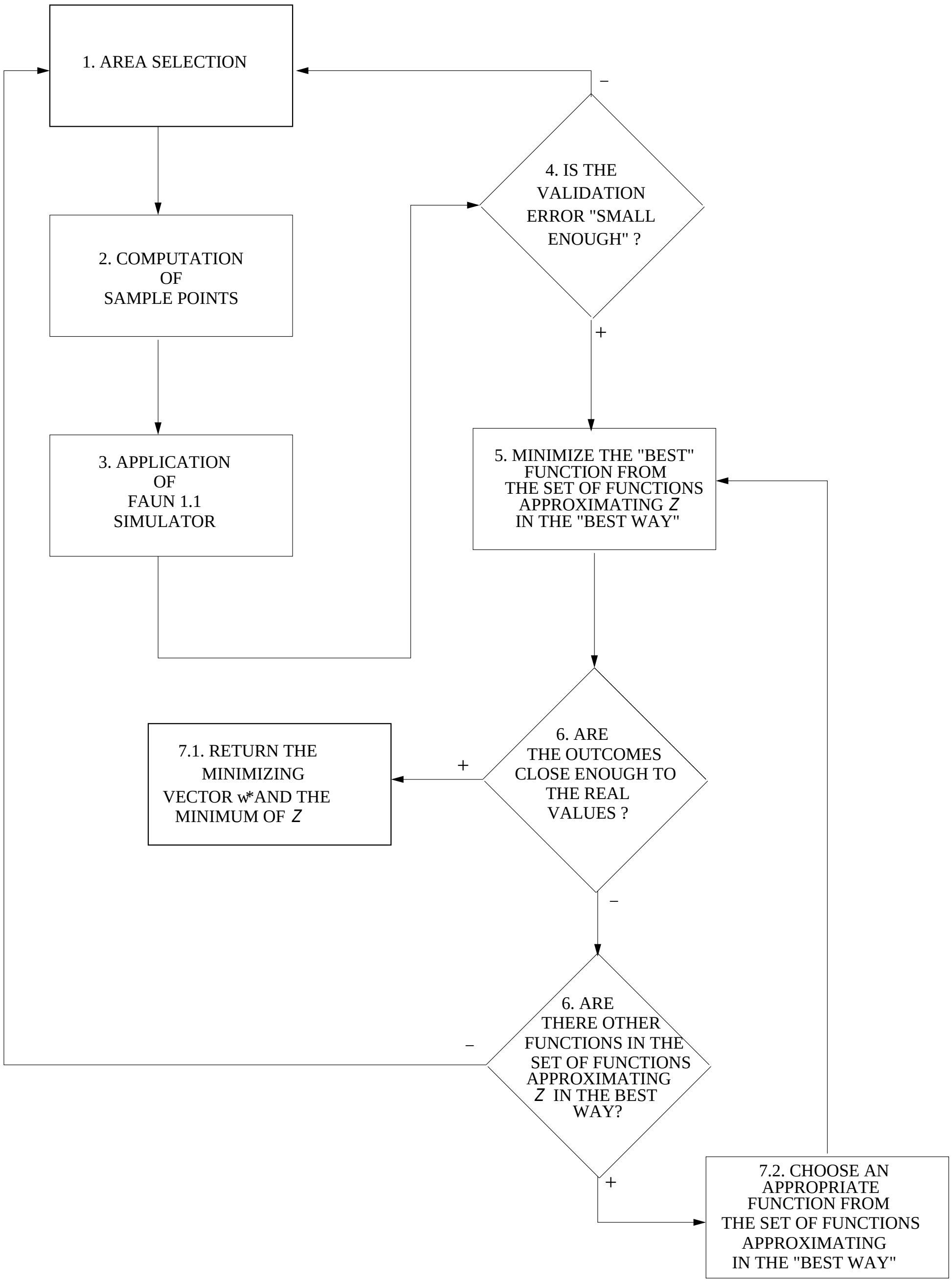

Figure 4.1: Flow chart of the solution process

the route flow level (see [66]). Convergence of the inner loop is verified using the so-called *relative duality gap* $\varepsilon^{(\text{iter})}$, defined as

$$\varepsilon^{(\text{iter})} = \frac{\sum_{(r,s)\in\mathcal{RS}} \sum_{p\in\mathcal{P}^{(r,s)}} \left(c_p^{(r,s),(\text{iter})} - \pi^{(r,s),(\text{iter})}\right) f_p^{(r,s),(\text{iter})}}{\sum_{(r,s)\in\mathcal{RS}} \pi^{(r,s),(\text{iter})} d^{(r,s)}}. \tag{4.22}$$

Here $\pi^{(r,s),(\text{iter})}$ is the minimal route travel time for travelers departing from origin r to destination s as computed in the iteration iter. If the relative duality gaps of two consecutive iterations are close enough, i.e., if $|\varepsilon^{(\text{iter}+1)} - \varepsilon^{(\text{iter})}| < \varepsilon_{\max}$, with a given small positive number $\varepsilon_{\max}$, the algorithm is terminated. Note that $\varepsilon^{(\text{iter})} \downarrow 0$ is the convergence criterium in the case of the deterministic user equilibrium.

Pseudocode for computing sample points of the objective function

(*Initialization*)
Download the network $G(\mathcal{N},\mathcal{A})$, define $\mathcal{RS}$, $\mathcal{P}^{(r,s)}$, $\mathcal{T}$, travel demands,
$\varepsilon_{\max}$ $(1 \gg \varepsilon_{\max} > 0)$;
define μ, n, m, iter $= 0$, W;
set the network empty, compute $\varepsilon^{(0)}$;
(*Outer loop*)
for each w from set W and chosen grid *do*
(*Inner loop*) (Logit-based stochastic traffic assignment)
    iter:=iter+1;
    while $|\varepsilon^{(\text{iter}+1)} - \varepsilon^{(\text{iter})}| > \varepsilon_{\max}$ do
        Compute link costs from (4.4) and route costs from (4.7);
        Determine the route choices of travelers for each (r,s)-pair using (4.15);
        Update route flows using MSA;
        Compute link flows using (4.3);
    end do;
    Compute objective function $Z(q(w),w)$ corresponding to w;
Return $w, Z(q(w),w)$.

Finding the minimal objective function is in this stage of the computation not necessary, since the minimization of the functions given by the best-approximating neural network will take place. Note that in Section 4.6.3 the input of the neurosimulation was vector $x \in \mathbb{R}^n$, which is in the following section replaced by vector w of coefficient vector. It is an $|\mathcal{A}| \cdot M$-vector. Similarly y is replaced by $Z(q(w^*), w^*)$, which is a scalar.

4.6.5 Application of FAUN 1.1 simulator

The grid search produces the values of the objective function at discrete positions in the parameter space. However, the grid search is relatively time consuming. It is desirable to have a function that can be evaluated instantaneously. Furthermore, for every not calculated position in the parameter space the algorithm has to be recomputed. It would speed up the analysis, if the objective function could be computed for arbitrary values of the parameter space. This leads to the following procedure, using only a limited number of sample points and using neurosimulator FAUN to extrapolate the objective function by functions, that can be easily minimized.

Pseudocode for applying ANN to the objective function

(*Initialization*)

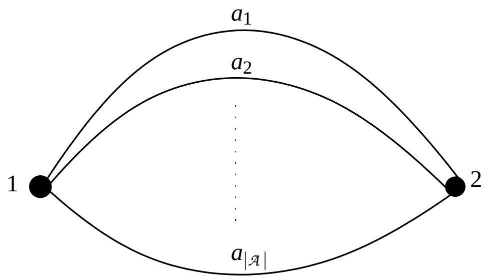

Figure 4.2: Network with one origin–destination pair and multiple links.

Prepare the grid search data for use with FAUN by splitting input and output;
Set appropriate scaling parameters for the data;
Set number of ANN to train successfully S_N;
Set appropriate worst accepted validation quality;
Prepare FAUN for parallel computation.
(*FAUN training*)(Finding appropriate ANN)
do N-times in parallel
  Select random s;
  while ε_v in (4.20) does not grow for two consecutive steps *do*
    reduce ε_t in (4.20) by following the gradient descent on s in (4.21);
  end while
  if ε_v is acceptable
    return and save s;
  else if
    reinitialize s;
  end if
end do
(*Postprocessing*)
Export the best ANN;
Minimize the objective function approximation;
Return w^*, $q(w^*)$, and $Z(q(w^*), w^*)$.

4.7 Case studies

In this section case studies illustrating the solution methods introduced in Section 4.6 will be presented. In Section 4.7.1 the static optimal toll design problem with a network consisting of one origin–destination pair and multiple links is considered, starting with two links and linear link travel time and linear toll function, and proceeding to the problem with more links and quadratic link travel times/tolls. The road authority minimizes the total travel time of the system or maximizes the total toll revenue, the drivers are driven by DUE. The problems are solved analytically. In Section 4.7.2 a larger problem on the so-called *Beltway* network is considered and solved using the algorithm introduced in Section 4.6.2.

4.7.1 One origin–destination pair with multiple parallel links

Let us consider the network with one origin–destination pair as depicted in Figure 5.1 consisting of $|\mathcal{A}| \geq 2$ directed parallel routes (links).

The road authority as the leader sets link tolls as follows: The link $a_{|A|}$ is untolled and each other link is tolled with a toll defined by (4.9), i.e.,

$$\theta_a(q_a) = \sum_{m=0}^{M} w_a^{(m)} (q_a)^m, \quad w_a^{(m)} = \begin{cases} 0 & \text{for} \quad a \in \mathcal{A} \setminus \mathcal{T}, \\ \kappa_a^{(m)} & \text{for} \quad a \in \mathcal{T}, \quad \kappa_a \in \mathbb{R}. \end{cases}$$

Travelers as followers are assumed to have complete information about the network conditions and therefore they are driven by the deterministic user equilibrium defined by Definition 4.1. The link travel time function for a single driver traveling on a link a and the link travel cost function are defined as

$$t_a(q_a) \stackrel{\text{def}}{=} \beta_a q_a + \delta_a, \quad c_a(q_a) \stackrel{\text{def}}{=} \alpha \tau_a(q_a) + \theta_a,$$

respectively. Therefore,

$$c_a(q_a) = \alpha \beta_a q_a + \alpha \delta_a + \theta_a(q_a).$$

We assume a positive inelastic traffic demand $d = d^{(r,s)} > 0$ [veh].

If the process of solving equations that define Wardrop equilibrium leads to negative traffic flows on some links, the link traffic flows on these links can be set to 0 and we might try to solve the new problem problem without some of these links. As shown in Example 4.4, such a problem does not need to have a solution.

In Example 4.1 a situation with nonunique Wardrop equilibria can be seen.

Example 4.4 (Nonexistence of the Wardrop equilibrium solution)
Imagine the game on a three-route (link) network with one origin–destination pair. Let $d^{(1,2)} = 5$ [veh], $\alpha = 10$, and let the route (link) cost functions be given by

$$c_1 = \frac{45}{2} - 2q_1, \tag{4.23}$$

$$c_2 = \frac{35}{3} + \frac{8}{3}q_2, \tag{4.24}$$

$$c_3 = 10 + \frac{5}{2}q_3. \tag{4.25}$$

Then the system of equations

$$c_1 = c_2, \quad c_2 = c_3$$

leads to the following link traffic flows:

$$q_1 = \frac{325}{44}, \quad q_2 = -\frac{65}{44}, \quad q_3 = -\frac{10}{11}.$$

Since q_2 and q_3 are negative, link 2 and/or link 3 will receive zero traffic flow. Therefore, $q_1 = 5$, $q_2 = 0$, $q_3 = 0$ would be the first candidate for the Wardrop solution. If we set only q_3 to 0 and solve equation $c_1 = c_3$, the resulting traffic flows would be $q_1 = \frac{15}{4}$, $q_2 = \frac{5}{4}$, which is the second candidate for the Wardrop solution. If we set only q_2 to 0 and we would solve equation $c_1 = c_3$, the resulting traffic flows would be $q_1 = 0$, $q_3 = 5$, which would determine the third candidate for the Wardrop solution. Unfortunately, none of traffic flows combinations

- $\{q_1, q_2, q_3\} = \{5, 0, 0\}$;

- $\{q_1, q_2, q_3\} = \{\frac{15}{4}, \frac{5}{4}, 0\}$;
- $\{q_1, q_2, q_3\} = \{0, 0, 5\}$.

is in Wardrop equilibrium. □

Remark 4.7 Note that in Example 4.4 one of the route (link) cost functions was decreasing with traffic flow on the same route (link). The Wardrop equilibrium is a reasonable concept only if the route (costs) are increasing with actual traffic flows. Therefore, Example 4.4 is of academic nature only. □

Total travel time minimization on a two-link network with linear link travel time functions

Let us first assume that there are only two routes (links) in the network, i.e., $|\mathcal{A}| = 2$, where the travel demand $d^{(r,s)}$ is fixed. Let only link 1 be tolled. If the road authority minimizes the total travel time of the system, the optimal toll design problem can be written as

$$(P_1) \qquad \begin{cases} \text{Find} \\ \quad \theta_1^*(\cdot) = \arg\min_{\theta_1(\cdot)} q' t, \\ \text{subject to the Wardrop equilibrium constraints,} \\ q = (q_1, q_2)' = (q_1, D - q_1)', t = (t_1, t_2)', \\ \text{and } \sum_{a=1}^{2} q_a = D. \end{cases}$$

Let

$$t_1 \stackrel{\text{def}}{=} \beta_1 q_1 + \delta_1, \quad t_2 \stackrel{\text{def}}{=} \beta_2 q_2 + \delta_2.$$

Since

$$\frac{d^2\,(q' \cdot t(q))}{d\,(q_1)^2} = 2\,\beta_1 + 2\,\beta_2 > 0,$$

the objective function is convex for all q. The total travel time function is minimized for

$$q_1^* = \frac{\delta_2 - \delta_1 - 2\,\beta_2 D}{2\,(\beta_1 + \beta_2)}, \quad q_2^* = \frac{\delta_1 - \delta_2 - 2\,\beta_1 D}{2\,(\beta_1 + \beta_2)}. \tag{4.26}$$

With this choice of q_1 and q_2 the total travel time minimum becomes

$$\frac{4\,\beta_1\beta_2 D^2 + (4\,\beta_1\delta_2 + 4\,\beta_2\delta_1)\,D - (\delta_2 - \delta_1)^2}{4\,(\beta_1 + \beta_2)}.$$

Different strategies for the road authority – An ad-hoc approach

Let us first assume that the road authority sets toll on link 1 as a linear toll function, i.e., $\theta_1(q_1) \stackrel{\text{def}}{=} w_1^{(0)} + w_1^{(1)} q_1$.

If a Wardrop equilibrium with q_1 and q_2 defined by (4.26) applies, an optimal $w_1^{(0)}$ has to satisfy

$$w_1^{(0),*} = \frac{1}{2}\alpha(\delta_2 - \delta_1) + \frac{(\delta_1 - \delta_2 - 2\beta_2 D)}{2\,(\beta_1 + \beta_2)} w_1^{(1)},$$

while $w_1^{(1)}$ is free. Therefore, a linear inverse Stackelberg strategy satisfying

$$\theta_1(q_1) = \frac{1}{2}\alpha(\delta_2 - \delta_1) + \frac{(\delta_1 - \delta_2 - 2\beta_2 D)}{2(\beta_1 + \beta_2)} w_1^{(1)} + q_1 w_1^{(1)} \tag{4.27}$$

yields the optimal flows. Note that setting $w_1^{(1)} = 0$ leads to the optimal Stackelberg strategy $\theta_1 = \frac{1}{2}\alpha(\delta_2 - \delta_1)$, which is independent of the travel demand D.

Obviously, in this simple case there is no necessity to try more complicated strategies for the road authority, since the minimal total travel time can be obtained with the Stackelberg strategy.

Total travel time minimization with linear link travel time functions and multiple parallel links

In the more general case, with $|\mathcal{A}| > 1$ and $\theta_{|A|} = 0$, the total travel time function has the form

$$\begin{aligned}
q^T t &= \sum_{a=1}^{|\mathcal{A}|-1} q_a t_a + q_{|A|} t_{|A|} \\
&= \sum_{a=1}^{|\mathcal{A}|-1} q_a(\beta_a q_a + \delta_a) + (D - \sum_{j=1}^{|\mathcal{A}|-1} q_j)\left(\beta_{|A|}\left(D - \sum_{j=1}^{|\mathcal{A}|-1} q_j\right) + \delta_{|A|}\right) \\
&= \sum_{a=1}^{|\mathcal{A}|-1} \beta_a q_a^2 + \delta_a q_a + \beta_{|A|} D^2 - 2\,\beta_{|A|} D \sum_{a=1}^{|\mathcal{A}|-1} q_a + \delta_{|A|} D \\
&\quad + \beta_{|A|}\left(\sum_{a=1}^{|\mathcal{A}|-1} q_a\right)^2 - \delta_{|A|} \sum_{a=1}^{|\mathcal{A}|-1} q_a .
\end{aligned}$$

The optimal route (link) flow q_a^* ($a \in \{1, \ldots, |A| - 1\}$) for a total travel minimum to be reached has to satisfy

$$q_a^* = \frac{\delta_{|A|} - \delta_a - 2\beta_{|A|} D}{2\beta_a + 2\beta_{|A|}} \tag{4.28}$$

and $q_{|A|}^* = d^{(r,s)} - \sum_{a=1}^{|A|-1} q_a^*$

If $\delta_a > \delta_{|A|} - 2\beta_{|A|} D$ for any $a \in \mathcal{A}$, the q_a^* would be negative, and, therefore, the global minimum of the total travel time cannot be reached and one can try to get as close to the optimum as possible by trying different toll strategies.

If $\delta_a < \delta_{|A|} - 2\beta_{|A|} D$ and $c_a \stackrel{\text{def}}{=} \alpha\tau_a + \theta_a$ for $a \in \{1, \ldots, |A|\}$, (4.28) expresses the link flows minimizing the total travel time of the system. The Wardrop equilibrium in terms of costs yields the following expressions for θ_a in terms of θ_j ($a, j \in \mathcal{A}$, $a \neq j$):

$$\theta_a = \theta_j + \frac{\Psi_4^{a,j}}{2\left(\beta_j + \beta_{|A|}\right)\left(\beta_a + \beta_{|A|}\right)}, \quad j \in \mathcal{A} . \tag{4.29}$$

with $\Psi_4^{a,j} = \alpha\beta_{|A|}(\beta_a\beta_j + 2\beta_{|A|}^2)(\delta_j - \delta_a) + \alpha\beta_{|A|}^2\beta_j(\delta_j - 2\,\beta_{|A|}D + \delta_{|A|} - 2\,\delta_a) + \alpha\beta_a\,\beta_{|A|}^2$ $(2\delta_j - \delta_{|A|} - \delta_a + 2\,\beta_{|A|}D)$. Since one of the links is untolled, the tolls on all other links can be computed by induction. Therefore, if $\Psi_4^{a,j} \geq 0 \quad \forall j \in \mathcal{A}$ the system of linear equations (4.29) yields the Stackelberg solution of the problem.

To recapitulate, if $\delta_j < \delta_{|A|} - 2\beta_{|A|}D$ for $\forall j \in \mathcal{A}$ and $\Psi_4^{a,j} \geq 0$, $(a, j \in \mathcal{A},\ a \neq j)$ the Stackelberg game with one link untolled and traffic flow-invariant toll leads to the optimal total travel time value. In the other cases, the outcomes of the Stackelberg and inverse Stackelberg game may differ.

Total toll revenue maximization, with two parallel links and linear link travel time functions

Clearly, q_1 maximizing the total toll revenue has to satisfy

$$\frac{d\theta_1(q_1)}{d\,q_1} + \theta_1(q_1) = 0$$

and, therefore, it is dependent on the definition of the toll function. We will attempt to maximize the total toll revenue with different toll function definitions. The problem to be solved can be symbolically written as

$$(P_2)\begin{cases} \text{Find} \\ \quad \theta_1^*(\cdot) = \arg\max_{\theta_1(\cdot)} \left(q_1\,\theta_1\left(q_1\right)\right), \\ \text{subject to the deterministic user equilibrium conditions,} \\ \text{with } q = \left(q_1, q_2\right)^{\mathrm{T}}, t = \left(t_1, t_2\right)^{\mathrm{T}}, \text{ and } \sum_{a=1}^{2} q_a = D. \end{cases}$$

Different strategies for the road authority - An ad-hoc approach

With $\theta_1(q_1) = w_1^{(0)} + w_1^{(1)}\,q_1$ the objective function is concave for $w_1^{(1)} < 0$ ($\frac{d^2(\theta_1(q_1)\cdot q_1)}{d\,q_1^2} = 2w_1^{(1)}$). Therefore, the optimal toll has to be decreasing with the traffic flow. Maximization of the total toll revenue function with respect to q_1 leads to the optimal q_1 and q_2 (indicated by superscript *)

$$q_1^* = -\frac{w_1^{(0)}}{2w_1^{(1)}}, \quad q_2^* = D + \frac{w_1^{(0)}}{2w_1^{(1)}}. \tag{4.30}$$

If a Wardrop equilibrium applies, i.e., if $c_1 = c_2$, $w_1^{(0)}$ from (4.30) has to satisfy

$$w_1^{(0)} = \frac{2\,\alpha\,(\delta_1 - \beta_2\,D - \delta_2)\,w_1^{(1)}}{\alpha\beta_1 - w_1^{(1)} + \alpha\,\beta_2}. \tag{4.31}$$

With this choice of $w_1^{(0)}$ the total toll revenue reaches

$$-\frac{\alpha^2\,(\delta_2 - \delta_1 + \beta_2 D)^2\,w_1^{(1)}}{\left(\alpha\,(\beta_1 + \beta_2) - w_1^{(1)}\right)^2} > 0 \quad (w_1^{(1)} < 0). \tag{4.32}$$

It can be shown that the optimal Stackelberg toll is

$$\theta_1 = \alpha(\delta_1 - \delta_2 - \beta_2\,D)$$

and that this toll yields the total toll revenue of

$$\frac{\alpha\,(\delta_2-\delta_1+\beta_2 D)^2}{4\,(\beta_1+\beta_2)}. \tag{4.33}$$

The Stackelberg toll pays off if this expression is positive.

Expression (4.32) has a higher value than expression (4.33) if $w_1^{(1)}$ lies in the interval

$$\left[\frac{\left(\phi_1-2-2\sqrt{1-\phi_1}\right)(\beta_1+\beta_2)\,\alpha}{\phi_1},\frac{\left(\phi_1-2+2\sqrt{1-\phi_1}\right)(\beta_1+\beta_2)\,\alpha}{\phi_1}\right],$$

with $\phi_1=\delta_2-\delta_1+\beta_2 D$ ($1>\phi_1$ has to be satisfied).

If the toll is defined as a quadratic function, i.e., $\theta_1(q_1)\stackrel{\text{def}}{=}w_1^{(0)}+w_1^{(1)}q_1+w_1^{(2)}q_1^2$, the objective function is concave for $w_1>-3w_2 q_1$. The only candidate for optimal q_1 is then

$$q_1=-\frac{w_1-\sqrt{(w_1^{(1)})^2-3w_1^{(0)}w_1^{(2)}}}{3w_1^{(2)}}. \tag{4.34}$$

With q_1 defined by (4.34) and if the Wardrop equilibrium applies, the total toll revenue function has the following form:

$$\frac{(w_1^{(0)})^2\left(2\,w_1^{(0)}+3\,\Psi_1\right)\left(9\,w_1^{(1)}\Psi_1+6\alpha\,w_1^{(0)}\,\Psi_2-4\,w_1^{(0)}\,w_1^{(1)}\right)}{27\left(w_1^{(0)}\alpha\,\Psi_2+2\Psi_1\,w_1^{(1)}-w_1^{(0)}\,w_1^{(1)}\right)^2}, \tag{4.35}$$

with $\Psi_1=\alpha\Phi_1$, $\Psi_2=\beta_1+\beta_2$. It can be seen that the total toll revenue in the quadratic toll case (4.35) reaches higher values than the total toll revenue in the linear toll case (4.32), if $w_1^{(1)}\in\left(w_1^{(1),\min},w_1^{(1),\max}\right)$, with

$$\begin{aligned}w_1^{(1),\max}&=-\frac{\left(16\,(w_1^{(0)})^3-60\,\Psi_1(w_1^{(0)})^2+27\Psi_1^2w_1^{(0)}+54\,\Psi_1^3+\Psi_3\right)\Psi_2 w_1^{(0)}}{27\alpha\,(-\delta_1+\beta_2+\delta_2)^2\left(2\Psi_1 D-w_1^{(0)}\right)^2}\\&\quad+\frac{2\,\Psi_2 w_1^{(0)}\sqrt{2\left(2\,w_1^{(0)}+3\,\Psi_1\right)^5 w_1^{(0)}}}{27\alpha\,(-\delta_1+\beta_2+\delta_2)^2\left(-w_1^{(0)}+2\Psi_1 D\right)^2},\end{aligned}$$

where $\Psi_3=54\,\alpha^3\left(\beta_2{}^3D^3-324\,\beta_2\delta_1\delta_2 D-162\,\delta_2{}^2\right)$, and

$$\begin{aligned}w_1^{(1),\min}&=-\frac{\left(16\,(w_1^{(0)})^3-60\,\Psi_1(w_1^{(0)})^2+27\Psi_1^2w_1^{(0)}+54\,\Psi_1^3+\Psi_3\right)\Psi_2 w_1^{(0)}}{27\alpha\,(-\delta_1+\beta_2+\delta_2)^2\left(-w_1^{(0)}+2\Psi_1 D\right)^2}\\&\quad-\frac{2\sqrt{2\left(2\,w_1^{(0)}+3\,\Psi_1\right)^5 w_1^{(0)}}\,\Psi_2 w_1^{(0)}}{27\alpha\,(-\delta_1+\beta_2+\delta_2)^2\left(-w_1^{(0)}+2\Psi_1 D\right)^2}.\end{aligned}$$

Remark 4.8 The previous example suggests that in the optimal toll design problem defined on one origin–destination pair on the network with 2 parallel links, linear link travel time functions, and with the road authority maximizing the total toll revenue of the problem, the higher degree of the toll polynomial function yields a better outcome for the road authority. □

Total toll revenue maximization on the network with two parallel links and quadratic link travel time functions

Let us consider a two-link problem with the road authority minimizing the total travel time of the system and the link travel time functions defined as

$$t_a \stackrel{\text{def}}{=} \beta_a q_a^2 + \gamma_a q_a + \delta_a,$$

with the link toll function defined as in the previous case study, i.e., $\theta_1(q_1) = w_1^{(0)} + w_1^{(1)} q_1$. Following the same procedure as in the previous case it can be seen that when the Wardrop equilibrium holds, two possible solution flows can be reached (if the transfer condition is satisfied):

$$q_1^{(1)} = -\frac{w_1^{(1)} + \alpha\gamma_1 + \alpha\gamma_2 + 2\alpha\beta_2 D - \sqrt{\Psi_3}}{2\alpha(\beta_1 - \beta_2)}, \tag{4.36}$$

$$\text{or} \quad q_1^{(2)} = -\frac{w_1^{(1)} + \alpha\gamma_1 + \alpha\gamma_2 + 2\alpha\beta_2 D + \sqrt{\Psi_3}}{2\alpha(\beta_1 - \beta_2)}, \tag{4.37}$$

with $\Psi_3 = {w_1^{(1)}}^2 + w_1^{(1)}(2\alpha\gamma_1 + 2\alpha\gamma_2 + 4\alpha\beta_2 D) + w_1^{(0)}(-4\alpha\beta_1 + 4\alpha\beta_2) + \alpha^2{\gamma_1}^2 + 2\alpha^2\gamma_1\gamma_2 + 4\alpha^2\gamma_1\beta_2 D + \alpha^2{\gamma_2}^2 + 4\alpha^2\beta_1\gamma_2 D + 4\alpha^2\beta_1\beta_2 D^2 - 4\alpha^2\beta_1\delta_1 + 4\alpha^2\beta_1\delta_2 + 4\alpha^2\beta_2\delta_1 - 4\alpha^2\beta_2\delta_2$ ($\Psi_3 \geq 0$ is a necessary condition to obtain the optimal traffic flows).

If only one from the traffic flows (4.36) and (4.37) leads to the Wardrop equilibrium, minimization of the total travel time function gives us:

$$w_1^{(0),*} = \frac{1}{3}\alpha(2\delta_2 - 2\delta_1 + \gamma_2 D), \quad w_1^{(1),*} = -\frac{1}{3}\alpha(\gamma_1 + \gamma_2) < 0.$$

Note that this solution is unique. Also, since $\alpha > 0$, $\gamma_1 > 0$, and $\gamma_2 > 0$, the optimal toll will be decreasing with traffic flow, provided that condition (4.10) holds. A necessary condition for this is that $2\delta_2 - 2\delta_1 + \gamma_2 > 0$. Moreover, since $w_1^{(1),*} \neq 0$, the inverse Stackelberg game strategy brings a better outcome for the road authority than the Stackelberg strategy. Since the problem is a second-best problem, link 2 is untolled and, thus, no other possibility for the road authority to get the same outcome with the Stackelberg strategy exists.

The total toll revenue maximization with multiple parallel links and linear link travel time functions

The total toll revenue function has the form

$$q^T \boldsymbol{\theta} = \sum_{a=1}^{|\mathcal{A}|-1} q_a \theta_a(q_a). \tag{4.38}$$

a	β_a	γ_a	δ_a
$1,2,3,5,8,14,17,21,24,26,27,28$	5	$\frac{1}{400}$	$0.15625 \cdot 10^{-5}$
$15,16,34,35$	3	$\frac{1}{30}$	$\frac{1}{900}$
$4,32$	2.7	$\frac{1}{400}$	$0.625 \cdot 10^{-4}$
$9,13,25,37,41,53$	9	$\frac{1}{400}$	$0.625 \cdot 10^{-4}$
$18,19,20$	3	$\frac{1}{100}$	$4 \cdot 10^{-5}$
$10,12$	1.2	$\frac{1}{600}$	$4 \cdot 10^{-5}$
11	4	$\frac{1}{60}$	$4 \cdot 10^{-4}$
others	10	$\frac{1}{30}$	$\frac{1}{900}$

Table 4.1: Coefficients of the travel time function.

Here the toll $\theta_a(\cdot)$ is a function of the link traffic flow on the same link. If $\frac{d^2\theta_a(q_a)}{d(q_a)^2} + \frac{d\theta_a(q_a)}{d\,q_a} < 0$, then the the local maximum of the total toll revenue function is reached for q_a $(a \in \mathcal{A})$ satisfying

$$\frac{d\,\theta_a(q_a)}{d\,q_a} + \theta_a(q_a) = 0. \tag{4.39}$$

Therefore, the structure of the toll functions will influence the possible outcome of the game and, therefore, no claims on optimal strategy for the leader can be made before the structure of the toll function is known.

4.7.2 Beltway network

Let us consider the network depicted in Figure 4.3 with 21 nodes and 56 links. Note that in Figure 4.3 link labels lie on the right-hand side of the driving direction, when going from North to South or from West to East. Nodes $\{1,2,3\}$ will be referred to as to the North nodes, similarly nodes $\{4,9,14\}$ will be referred to as to the West nodes, etc. Initially the set of tollable links will be defined as $\mathcal{T} = \{9, 11, 13, 19, 22, 23, 25, 53\}$, as these are in this network the rather congested links, when the toll is not imposed. [6]

Let the logit-based stochastic equilibrium apply for the lower level of the problem. The set of origins $\mathcal{R}$ contains nodes from North, East, and West, while the set of destinations $\mathcal{S}$ comprises nodes from the South. Let $\mathcal{RS} = \mathcal{R} \times \mathcal{S}$. Therefore, there are 27 origin–destination pairs and 1357 routes in the network. There is a traffic demand of 20 cars for each origin–destination pair and each minute.

The link travel time functions will be defined as $t_a \stackrel{\text{def}}{=} \beta_a + \gamma_a\,q_a + \delta_a\,q_a^2$ for each $a \in \mathcal{A}$, where coefficients β, γ, and δ are depicted in Table 5.23.

The road authority minimizes the total travel time of the system.

The following four problems will be dealt with considering both objectives:

1a) Stackelberg game (defined by (P.0)) with $w_{0,a} = w_0 \in \mathbb{R}^0_+$, $\forall a \in \mathcal{T}$.

[6] Our research does not deal with establishing the optimal set of tollable links. Research on this topic can be found in, e.g., [30].

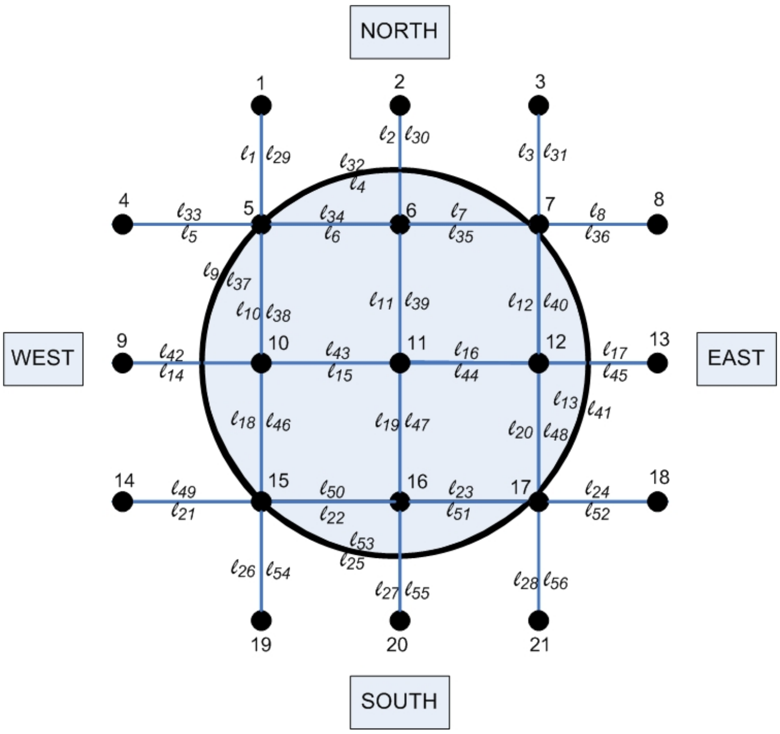

Figure 4.3: Beltway network.

1b) Inverse Stackelberg game (defined in (P)) with $M = 2$, toll defined by (4.9), and with the additional condition that $w_a^{(m)} = w_m \in \mathbb{R}$ for $a \in \mathcal{T}$, i.e., the linear toll functions have link-independent coefficients.

1c) First-best (FB) pricing problem,with toll defined as the marginal link travel time, $\theta_a(q_a) = \frac{\mathrm{d}t_a}{\mathrm{d}q_a} q_a$ for each link $a \in \mathcal{A}$. This presents the best possible outcome of the game if all links are tollable.

1d) Problem with no toll.

To solve these problems the algorithm introduced in Section 4.6, with $\varepsilon_{\max} = 10^{-5}$, $\varepsilon_v = 0.005$, 7500 training patterns, and 2500 validation patterns will be applied.

Note that in 1a) and 1b) the toll functions are identical for all tolled links. It is possible to apply more flexible tolls, but the computations of the optimal tolls would become very lengthy in that case.

Let the road authority minimize the total travel time of the system, i.e.,

$$Z(q(w), w) = (f(q(w), w))^T \cdot \tau(f(q(w), w)),$$

The outcomes of the game with no toll and the game with the first-best tolling are depicted in Table 4.2, while the outcomes of the Stackelberg game together with the outcomes of the inverse Stackelberg game are depicted in Table 4.3.

The optimal toll value for the Stackelberg game is rather high, i.e., 38.2 euro. The optimal values of w_0 and w_1 for the inverse Stackelberg game are 0.083 and 4.57, respectively. The total travel time obtained with the first-best tolling is $1.3286 \cdot 10^4$ [min], the total travel time with use of the inverse Stackelberg game is $1.3698 \cdot 10^4$ [min], and $1.4435 \cdot 10^4$ [min] with use of the Stackelberg game. The total travel time with no tolls is $1.6025 \cdot 10^4$ [min]

Note that to obtain the first-best outcome it is necessary to toll 56 links, while we used only 2 parameters in the toll function to obtain a result rather close to the first-best outcome.

In Table 4.4 outcomes of case studies, that differ in the set of tollable links, are depicted (as Case studies 2–6). These case studies are chosen such that some of the tolled links have very low first-best toll values, thus they should not be tolled in practice. Although outcomes of the Stackelberg game and the inverse Stackelberg game are quite close to each other, the inverse Stackelberg game performs never worse than the Stackelberg game. This is no surprise, since the Stackelberg game is a special case of the inverse Stackelberg game. This becomes clear in Case study 5 in which the optimal inverse Stackelberg strategy is in fact a Stackelberg strategy.

In Case study 3 the optimal tolls on tollable links are decreasing with the link flows on the same links. With increasing traffic flow on the particular tolled link the other links in the network become even more congested, that is why the travelers should still be stimulated to leave the other congested links and switch to the tolled link. This phenomenon would not occur if the links that are more sensitive to congestion would be tolled. Therefore, the flow-dependent tolling can accommodate to the new traffic conditions.

For Case study 1 additional computations with Stackelberg and inverse Stackelberg games were performed. We compare the following games:

- Stackelberg game with

	no tolls			FB tolls		
a	toll	flow	time	toll	flow	time
$1,2,3$	–	60.00	5.16	0.16	60.00	5.16
$4,32$	–	62.45	3.10	0.4483	50.72	2.99
$5,8,14,17,21,24$	–	60.00	5.16	0.16	60.00	5.16
$6,7$	–	42.03	3.34	0.24	26.40	3.17
$9,13$	–	55.09	12.27	7.09	65.95	13.37
$10,12$	–	106.27	1.83	0.66	81.20	1.60
$15,16$	–	65.07	3.69	0.72	50.20	3.45
$18,20$	–	107.48	8.70	7.36	89.88	7.13
$22,23$	–	39.97	13.11	4.18	36.52	12.70
$25,53$	–	38.37	10.88	1.7938	29.18	10.26
$26,27,28$	–	180.00	5.50	0.55	180.00	5.50
$29,30,31,a_{45}$	–	0.00	10.00	0.00	0.00	10.00
$34,35$	–	55.86	3.53	0.56	43.35	3.35
$37,41$	–	13.13	9.44	0.01	0.13	9.00
$38,40$	–	14.41	10.71	0.56	10.06	10.45
39	–	13.46	10.65	0.27	5.89	10.24
47	–	0.55	10.02	0.01	0.01	10.00
$33,36,42,49$	–	0.00	10.00	0.00	0.00	10.00
$43,44$	–	15.47	10.78	0.47	8.87	10.38
$46,48$	–	5.23	10.20	0.01	0.08	10.00
$50,51$	–	15.75	10.80	0.03	0.90	10.03
$52,54,55,56$	–	0.00	10.00	0.00	0.00	10.00
11	–	45.81	5.60	3.07	31.98	4.94
19	–	132.11	11.30	10.55	108.76	8.82

Table 4.2: Link parameters - Stackelberg game (Case study 1)

  – links $9,11,13,19$ tolled with identical toll and links $22,23,25,53$ tolled with identical toll, while these two toll values may differ;
  – links $9,11,13$ tolled with identical toll, links $19,22,23$ tolled with identical toll, and links 25 and 53 tolled with identical toll, while these three toll values may differ;
  – links $9,11$ tolled with identical toll, links $13,19$ tolled with identical toll, $22,23$ tolled with identical toll, and links $25,53$ tolled with identical toll, while these four toll values may differ;

- inverse Stackelberg game with toll set as in equation (4.9)
  – with $M = 1$ and the identical polynomial toll imposed on all tollable links;
  – with $M = 2$ and the identical polynomial toll imposed on all tollable links;
  – with $M = 3$ and the identical polynomial toll imposed on all tollable links.

Results are shown in Table 4.5. It is clear that when comparing Stackelberg and inverse

	SG			ISG - linear		
a	toll	flow	time	toll	flow	time
1,2,3	0.00	60.00	5.16	0.00	60.00	5.16
4	0.00	61.76	3.09	0.00	59.20	3.07
5,8,14,17,21,24	0.00	60.00	5.16	0.00	60.00	5.16
6,7	0.00	26.15	3.16	0.00	24.81	3.15
9,13	38.20	48.13	11.65	10.10	66.61	13.49
10,12	0.00	129.05	2.08	0.00	106.86	1.83
15,16	0.00	94.64	4.31	0.00	81.11	3.99
18,20	38.20	92.50	7.35	11.13	79.03	6.27
22,23	38.20	19.18	11.05	6.80	26.90	11.72
25,53	38.20	12.35	9.40	6.12	18.71	9.69
32	0.00	61.76	3.09	0.00	59.20	3.07
26,27,28	0.00	180.00	5.50	0.00	180.00	5.50
29,30,31,45	0.00	0.00	10.00	0.00	0.00	10.00
34,35	0.00	62.79	3.65	0.00	62.11	3.63
37,41	0.00	6.15	9.18	0.00	0.13	9.00
38,40	0.00	14.39	10.71	0.00	16.04	10.82
39	0.00	13.27	10.64	0.00	15.40	10.78
47	0.00	0.01	10.00	0.00	0.01	10.00
33,36,42,49	0.00	0.00	10.00	0.00	0.00	10.00
43,44	0.00	10.16	10.45	0.00	9.28	10.40
46,48	0.00	2.31	10.08	0.00	0.04	10.00
50,51	0.00	7.03	10.29	0.00	1.42	10.05
52,54,55,56	0.00	0.00	10.00	0.00	0.00	10.00
11	38.20	0.01	4.00	4.64	0.79	4.01
19	38.20	155.69	14.25	15.28	129.04	10.88

Table 4.3: Link parameters - inverse Stackelberg game (Case study 1)

Stackelberg games with the same number of parameters to be optimized, the inverse Stackelberg game performs never worse than the Stackelberg game. Already with 3 parameters the resulting total travel time is very close to the first-best outcome. Therefore, it is profitable for the road authority to calculate the tolls using inverse Stackelberg strategy even when the tolls are set as very simple functions of link flows.

The average computational time with 16 microprocessors was 9.5 minutes for problems with one parameter to optimize, 16.2 minutes with problems with two parameters to be optimized, 25.5 minutes with problems with three parameters, and 40.3 minutes with 4 parameters. The computational time can be reduced with use of more microprocessors.

Discussion

In the presented case studies we suggested how to improve the system performance with use of so-called traffic-flow dependent tolls. It can be seen that the system performance improves even with use of very simple toll functions.

		SG		ISG		FB	no toll
CS	tolled links	toll	*TTT* $(\cdot 10^4)$	$\{w_0, w_1\}$	*TTT* $(\cdot 10^4)$	*TTT* $(\cdot 10^4)$	*TTT* $(\cdot 10^4)$
2	4, 6, 7, 9, 10, 32, 34, 35, 37, 38	1.18	1.51	$\{4.11, -0.20\}$	1.46	1.33	1.60
3	11, 12, 13, 15, 16, 39, 40, 41, 43, 44	1.40	1.49	$\{3.47, -0.10\}$	1.49	1.33	1.60
4	4, 6, 7, 9, 10, 11, 12, 13, 16, 43	1.98	1.46	$\{1.36, 0.03\}$	1.44	1.33	1.60
5	12, 13, 16, 18, 19, 20, 23, 43, 50, 53	1.022	1.52	$\{1.02, 0\}$	1.52	1.33	1.60
6	13, 15, 19, 22, 25, 41, 44, 47, 50, 53	1.16	1.53	$\{7.09, -4.45\}$	1.51	1.33	1.60

Table 4.4: Results of the case studies.

	2 parameters		3 parameters		4 parameters	
link	SG θ_a	ISG w_0, w_1	SG θ_a	ISG $w_0, w_1, w_2(\cdot 10^{-4})$	SG θ_a	ISG $w_0, w_1, w_2(\cdot 10^{-4}), w_3(\cdot 10^{-8})$
9	48.58	4.57, 0.083	43.25	3.27, 0.045, 5.12	41.91	2.81, 0.036, 3.55, 1.22
11	48.58	4.57, 0.083	43.25	3.27, 0.045, 5.12	41.91	2.81, 0.036, 3.55, 1.22
13	48.58	4.57, 0.083	43.25	3.27, 0.045, 5.12	38.54	2.81, 0.036, 3.55, 1.22
19	48.58	4.57, 0.083	37.12	3.27, 0.045, 5.12	38.54	2.81, 0.036, 3.55, 1.22
22	16.24	4.57, 0.083	37.12	3.27, 0.045, 5.12	18.21	2.81, 0.036, 3.55, 1.22
23	16.24	4.57, 0.083	37.12	3.27, 0.045, 5.12	18.21	2.81, 0.036, 3.55, 1.22
25	16.24	4.57, 0.083	16.23	3.27, 0.045, 5.12	15.49	2.81, 0.036, 3.55, 1.22
53	16.24	4.57, 0.083	16.23	3.27, 0.045, 5.12	15.49	2.81, 0.036, 3.55, 1.22
TTT $(\cdot 10^4)$ [min]	1.4123	1.3698	1.4098	1.3537	1.3934	1.3401

Table 4.5: Comparison of different tolling strategies on Case study 1.

Another question to be discussed is the practical relevance of the proposed concept of the traffic-flow dependent tolls. One of the possibilities of how to apply such tolls in practice is to use global positioning systems (GPS) and/or mobile phones for counting the number of cars using specific roads/links in order to compute the traffic-flow dependent tolls.

4.8 Conclusions and future research

In this chapter we have introduced the problem of static optimal toll design with second-best traffic-flow dependent tolling. We have discussed existence of solutions of a very general version of this problem as well as its difficulty and we have proposed a solution algorithm. In the case studies (with specific objective function for the road authority and specific traffic assignment) we have shown both problems solved analytically and problems solved numerically using the proposed algorithm. Some unrealistic assumptions were considered, though, especially inelastic travel demand.

The following topics are subject of our future research:

- **Alternative objectives of the road authority** Although problem (P) was defined in a general way, in the presented case studies the objective function of the road authority was defined as a total travel time or as a total toll revenue of the traffic system. Another option is to define the objective function as the reliability of the network or, for example, as a surplus of the network.

- **Elastic demands** The traffic demand is assumed fixed. The traffic-flow dependent tolls can be implemented also in systems with elastic traffic demands. More about elastic traffic demands can be found in, e.g., [66].

- **Heterogeneous network users** The drivers in the network formed a homogeneous group. In [10] possibilities for defining heterogeneous users are investigated. These possibilities can be incorporated into our problem, too. In that case different toll functions would be imposed for different user groups.

The problems closely related to the research conducted in this chapter, but falling out of this frame, can be listed as follows:

- Finding the best way how to model link and route traffic flow, time, and other link and route traffic variables. We adopted standard methods used in the traffic field.

- Validating of standard ways used to model the traffic variables on the road networks.

- Defining criteria of efficiency of algorithms for solving the problems dealt with in this thesis and comparing different algorithms with respect of such criteria. We tried to develop algorithms that would solve the problems we are dealing with and that can be parallelized. We do not consider any other criteria, like speed and efficiency of the algorithms.

- Finding the best possible toll functions minimizing the objective function of the road authority. We were looking for polynomial toll functions improving the system performance remarkably when compared to outcomes obtained with standard uniform or time-varying tolls.

- Definition of the best possible objective function for the road authority from the practical point of view. While in Chapters 4 and 5 this objective function was not defined and the properties of the problems were discussed with a general objective function, in case studies we used the total travel time (to be minimized) or total toll revenue (to be maximized) as objective functions of the road authority. We are aware of the fact that other objective functions might be more realistic.
- Definition of the best possible way how to define travelers' cost functions. In this thesis the link cost function was defined as a linear combination of link travel time and link toll. There exist other ways how to define the link cost function. Finding of such ways is beyond the scope of this thesis.

Chapter 5

Dynamic Optimal Toll Design

This chapter extends the outcomes of Chapter 4 to the situation in which the problem evolves over time. We then talk about a dynamic optimal toll design problem.

5.1 Introduction and literature overview

There are extensive studies focusing on the *static* optimal toll design problem, i.e., on problems in which decisions of the players (the travelers and the road authority) do not evolve over time (see [68, 85], Chapter 4). Although the static models are still widely used, the theory and practice of *dynamic* models have evolved significantly over the last ten years. In the dynamic version of the optimal toll design problem the *dynamic traffic assignment* (DTA) applies ([10]). DTA models typically describe route choice behavior of travelers on a transportation network and the way in which traffic dynamically propagates through the network.

If all travelers are assumed to have perfect information (i.e., they know the current and future conditions on the network as well as the decisions of the other travelers) and if they are uniform, the *deterministic user equilibrium* (DUE) applies ([10, 94]). Similarly, with imperfect information and distributed travel preferences, a *probabilistic user equilibrium*, in the traffic literature referred often as to a *stochastic user equilibrium* (SUE), applies, in the case studies of this chapter this is often the *logit based stochastic equilibrium* (LB-SUE), see ([58]).

With respect to possible tolling strategies there are two main research streams differing in the definition of the set of tollable links. With so-called first-best-tolling (or first-best pricing) all the links in the network can be tolled ([68, 96]). With the so-called second-best tolling not all links are tollable (see [85]). The latter concept is clearly more applicable in practice.

Dynamic congestion pricing models in which network conditions and link tolls are time-varying, have been addressed in [3], where the effectiveness of various pricing policies (time-varying, uniform, and step tolls) was compared as well. Only one bottleneck or a single origin–destination network was considered there, while the possibility of application of traffic-flow dependent tolls is not discussed here. In [43] and [95] dynamic marginal (first-best) cost pricing models for general transportation networks were developed. As indicated by the authors, the application of their model is limited to destination-specific (rather than route or link-based) tolling strategies, which might complicate its practical application. Moreover, only the first-best pricing is considered here.

In [46] the dynamic optimal toll design problem is considered with a case study on the so-called Chen network. Tolls are assumed uniform or time-varying, but traffic-flow invariant, and the problem of finding the optimal toll is defined, but not solved, although the impact of some specific toll values on travelers' route and departure time choices is presented.

In [85] and [84] second-best tolling is considered, travelers are driven by the deterministic user equilibrium (DUE), the objective function of the road authority is defined as the surplus of the road authority (i.e., amount of money that the road authority receives by imposing tolls minus the investments of the road authority concerning the toll charge), the traffic demand is elastic, and it is assumed that the link cost functions are increasing with respect to traffic flows. In [66] and [60] the lower-level of the problem (travelers' minimization of travel costs) is formulated and solved as a variational inequality problem (VIP). Here the travelers are driven by DUE. In [68] a very general Stackelberg model is presented, where the road authority has two decision variables, one of them possibly traffic-flow dependent. The paper itself deals with general mathematical properties of traffic equilibria, however. The tolls are assumed to be constant and the traffic-flow dependent variable is interpreted as a management decision of the road authority.

This chapter proposes an extension of our research in the field of the static optimal toll design problem to the dynamic problems with both DUE and SUE. Although some authors [3, 14] consider the step-wise second-best tolling, to the best of our knowledge no research dealing with the optimal toll design problem with the second-best tolling, the travelers driven by LB-SUE, and the aim being to find optimal toll defined as a function of the traffic flows in the network has been done before. Since this problem is NP-hard, advanced optimization techniques, which can be parallelized, should be used to solve it. Similarly as in Chapter 4 a neural network-based algorithm as such an optimization technique is implemented. The neurosimulator FAUN has already been employed to solve other problems in the domain of dynamic games [15, 89, 90].

5.2 Preliminaries

Let $G = (\mathcal{N}, \mathcal{A})$ be a strongly connected road network, that means, there exists at least one path connects each (r,s)-pair, where $\mathcal{N}$ and $\mathcal{A}$ are finite nonempty sets of nodes and directed arcs (links), respectively. The set of tollable arcs will be denoted by $\mathcal{T} \subseteq \mathcal{A}$. There is a finite, nonempty set of origin-destination pairs $\mathcal{RS} \subset \mathcal{N} \times \mathcal{N}$ and let the set $\mathcal{K} = \{1, 2, \ldots, |\mathcal{K}|\}$ be a time index set. Here each $k \in \mathcal{K}$ refers to

- the interval $[(k-1.5)\Delta, (k+1.5)\Delta)$ if $k \geq 2$,
- the interval $[0, 0.5\Delta)$ if $k = 1$,

where Δ [h] is the length of each time interval.

For an ordered pair of nodes $(r,s) \in \mathcal{RS}$, where r is an origin and s is a destination, there is a positive number of drivers traveling from r to s and departing during the k-th time interval – so-called travel demand $d^{(r,s),(k)}$ [veh/h].[1] The travel demand is for the sake of simplicity assumed to be time-interval varying but traffic-flow and toll invariant.[2] Let $d^{(r,s)}$ be a $|\mathcal{K}|$-vector of all travel demands from r to s for all time intervals, i.e.,

$$d^{(r,s)} \stackrel{\text{def}}{=} \left(d^{(r,s),(1)}, d^{(r,s),(2)}, \dots, d^{(r,s),(|\mathcal{K}|)}\right)^T.$$

Let (r_1, s_1) denote the first origin-destination pair, (r_2, s_2) denote the second origin-destination pair, etc., let $(r_{|\mathcal{R}|}, s_{|\mathcal{S}|})$ denote the last origin-destination pair. Then,

$$d \stackrel{\text{def}}{=} \begin{pmatrix} d^{(r_1,s_1)} \\ \vdots \\ d^{(r_{|\mathcal{R}|},s_{|\mathcal{S}|})} \end{pmatrix}.$$

In the following text we will denote such a vector by $\left(d^{(r,s)}\right)_{\forall (r,s) \in \mathcal{RS}}$. will be a $|\mathcal{RS}| \cdot |\mathcal{K}|$-vector of all traffic demands for all travel time intervals and all origin–destination pairs in the network.

Let $\mathcal{P}$ be the set of all simple paths (i.e., paths without cycles) in the network and let $\mathcal{P}^{(r,s)} \subset \mathcal{P}$ be the set of all paths between an origin-destination pair (r,s). An element of $\mathcal{P}$ will be denoted by p, while an element of $\mathcal{P}^{(r,s)}$ will be denoted by $p^{(r,s)}$. Each path is formed by one or more directed arcs.[3]

The route flow departure rate[4] on path $p \in \mathcal{P}$ during the k-th time interval will be denoted by $f_p^{(k)}$ ([veh/h]), the arc inflow rate on the link a during the k-th time interval will be denoted by $q_l^{(k)}$ ([veh/h]).

The average route travel cost on the route $p \in \mathcal{P}$ when starting during the k-th time interval will be denoted by $c_p^{(k)}$, the average link travel cost on a link a during the k-th time interval will be denoted by $\varsigma_l^{(k)}$ ([euro]).

The route and link tolls, times, costs, and flows are related through a dynamic route-arc incidence indicator $\delta_{p,a}^{(k),(k')} \in \{0,1\}$, which equals 1, if the travelers entering the route $p \in \mathcal{P}$ during the k-th time interval enter the arc a during the k'-th time interval, and 0 otherwise. We will assume that the route times, costs, and tolls are additive[5], and that the following conservation constraints hold, i.e.,[6]

$$\theta_p^{(k)} = \sum_{k' \in \mathcal{K}} \sum_{a \in \mathcal{A}} \delta_{p,a}^{(k),(k')} \theta_l^{(k')}, \quad \tau_p^{(k)} = \sum_{k' \in \mathcal{K}} \sum_{a \in \mathcal{A}} \delta_{p,a}^{(k),(k')} t_l^{(k')}, \tag{5.1}$$

$$c_p^{(k)} = \sum_{k' \in \mathcal{K}} \sum_{a \in \mathcal{A}} \delta_{p,a}^{(k),(k')} \varsigma_l^{(k')}, \quad q_l^{(k')} = \sum_{k \in \mathcal{K}} \sum_{a \in \mathcal{A}} \delta_{p,a}^{(k),(k')} f_p^{(k)}, \tag{5.2}$$

[1] We do not consider the so-called departure time choice, as our main focus is on the optimal strategy for the road authority. This option is considered in, e.g.,[46].

[2] Elastic demand road pricing models are introduced in, e.g., [84].

[3] Note that the order of links matters, the expression $p = (6,1,4)$ means that route p is formed by three links, where 6 is the first one, 1 is the second one, and 4 is the last one.

[4] In the reminder of this chapter we will use the term “route flow” instead of the “route flow rate”.

[5] In reality, this does not need to be the case. For research dealing with non-additive costs, tolls, or flows we refer the reader to [66].

[6] Since some of the variables have to be rounded off, additional discussion about consistency of these equation is needed. Such a discussion can be found, in, i.e., [74].

$$d^{(r,s),(k)} = \sum_{p \in \mathcal{P}^{(r,s)}} f_p^{(k)}. \tag{5.3}$$

For each link $a \in \mathcal{A}$, the link travel cost $\varsigma_l^{(k)}$ for the k-th time interval is a linear combination of the actual link travel time t_l and the actual link toll θ_a with coefficients α [veh/h] and 1, i.e.,

$$\varsigma_l^{(k)} \stackrel{\text{def}}{=} \alpha t_l^{(k)} + \theta_l^{(k)}, \tag{5.4}$$

where α [euro/time unit] is called the value of time, which is supposed to be independent of q.[7]

Note that the link cost does not need to be strictly increasing with respect to the actual link flows, as the toll functions need not to be (strictly) increasing. It may seem counter-intuitive to have toll functions decreasing with the traffic flow, however, this phenomena was already encountered for the static optimal toll design problem in Chapter 4. In contrast, in , e.g., [85], the link costs are assumed increasing with link flows.

Let $q^{(k)}$, $t^{(k)}$, and $\varsigma^{(k)}$ denote for the k-th time interval a vector of link flow rates on all links, a vector of link flows on all links, and a vector of link costs on all links, respectively, i.e.,

$$q^{(k)} \stackrel{\text{def}}{=} \begin{pmatrix} q_1^{(k)} \\ q_2^{(k)} \\ \vdots \\ q_{|\mathcal{A}|}^{(k)} \end{pmatrix}, \quad \boldsymbol{t}^{(k)} \stackrel{\text{def}}{=} \begin{pmatrix} t_1^{(k)} \\ t_2^{(k)} \\ \vdots \\ t_{|\mathcal{A}|}^{(k)} \end{pmatrix}, \quad \boldsymbol{\varsigma}^{(k)} \stackrel{\text{def}}{=} \begin{pmatrix} \varsigma_1^{(k)} \\ \varsigma_2^{(k)} \\ \vdots \\ \varsigma_{|\mathcal{A}|}^{(k)} \end{pmatrix}. \tag{5.5}$$

Let q, t, and $\boldsymbol{\varsigma}$ denote the vectors of the link flows, the link travel times, and the link travel costs for all time intervals, i.e.,

$$q \stackrel{\text{def}}{=} \begin{pmatrix} q^{(1)} \\ q^{(2)} \\ \vdots \\ q^{(|\mathcal{K}|)} \end{pmatrix} \quad t \stackrel{\text{def}}{=} \begin{pmatrix} t^{(1)} \\ t^{(2)} \\ \vdots \\ t^{(|\mathcal{K}|)} \end{pmatrix}, \quad \boldsymbol{\varsigma} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{\varsigma}^{(1)} \\ \boldsymbol{\varsigma}^{(2)} \\ \vdots \\ \boldsymbol{\varsigma}^{(|\mathcal{K}|)} \end{pmatrix}. \tag{5.6}$$

Similarly, let us define

$$f^{(k)} \stackrel{\text{def}}{=} \begin{pmatrix} f_1^{(k)} \\ f_2^{(k)} \\ \vdots \\ f_{|\mathcal{P}|}^{(k)} \end{pmatrix}, \quad \boldsymbol{\tau}^{(k)} \stackrel{\text{def}}{=} \begin{pmatrix} \tau_1^{(k)} \\ \tau_2^{(k)} \\ \vdots \\ \tau_{|\mathcal{P}|}^{(k)} \end{pmatrix}, \quad c^{(k)} \stackrel{\text{def}}{=} \begin{pmatrix} c_1^{(k)} \\ c_2^{(k)} \\ \vdots \\ c_{|\mathcal{P}|}^{(k)} \end{pmatrix},$$

$$f \stackrel{\text{def}}{=} \begin{pmatrix} f^{(1)} \\ f^{(2)} \\ \vdots \\ f^{(|\mathcal{K}|)} \end{pmatrix}, \quad \boldsymbol{\tau} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{\tau}^{(1)} \\ \boldsymbol{\tau}^{(2)} \\ \vdots \\ \boldsymbol{\tau}^{(|\mathcal{K}|)} \end{pmatrix}, \quad c \stackrel{\text{def}}{=} \begin{pmatrix} c^{(1)} \\ c^{(2)} \\ \vdots \\ c^{(|\mathcal{K}|)} \end{pmatrix}.$$

[7] There are various ways in which the route cost functions can be defined, a common way is based on so-called generalized cost function, as mentioned in [45].

For each link from $\mathcal{T}$ and each time interval a traffic-flow dependent toll can be imposed. The traffic-flow dependent toll on link $a \in \mathcal{T}$ will be denoted by $\theta_l^{(k)}(\cdot)$. Unless stated differently, this toll will be for each k-th time interval defined as a polynomial function of link flow for the same time interval and on the same link, i.e.,

$$\theta_l^{(k)}\left(q_l^{(k)}\right) = \sum_{m=0}^{M} w_l^{(m),(k)} \left(q_l^{(k)}\right)^m, \quad w_l^{(m),(k)} = \begin{cases} 0 & \text{for} \quad a \in \mathcal{A} \setminus \mathcal{T}, \\ \in \mathbb{R} & \text{for} \quad a \in \mathcal{T}, \end{cases} \tag{5.7}$$

with $M \in \mathbb{N}_0$. By definition, w is constrained $\forall q_l^{(k)} \geq 0$ such that

$$\theta_l^{(k)}\left(q_l^{(k)}\right) \begin{cases} = 0 & \text{for} \quad a \in \mathcal{A} \setminus \mathcal{T}, \\ \geq 0 & \text{for} \quad a \in \mathcal{T}. \end{cases} \tag{5.8}$$

More advanced toll functions include traffic flows from previous time period, but we are looking for a very simple scheme improving the system performance, therefore we restrict ourself to toll functions in the form (5.7). Vectors

$$\boldsymbol{\theta}^{(k)} \stackrel{\text{def}}{=} \begin{pmatrix} \theta_1^{(k)}(\cdot) \\ \theta_2^{(k)}(\cdot) \\ \vdots \\ \theta_{|\mathcal{A}|}^{(k)}(\cdot) \end{pmatrix}, \quad \boldsymbol{\theta} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{\theta}^{(1)} \\ \boldsymbol{\theta}^{(2)} \\ \vdots \\ \boldsymbol{\theta}^{(|\mathcal{K}|)} \end{pmatrix}$$

are vectors of link toll functions during the k-th time interval and vectors of all link toll functions for all time periods, respectively. Coefficient vectors will be defined as follows:

$$w_l^{(k)} \stackrel{\text{def}}{=} \begin{pmatrix} w_l^{(1),(k)} \\ w_l^{(2),(k)} \\ \vdots \\ w_l^{(M),(k)} \end{pmatrix}, \quad w^{(k)} \stackrel{\text{def}}{=} \begin{pmatrix} w_1^{(k)} \\ w_2^{(k)} \\ \vdots \\ w_{|\mathcal{A}|}^{(k)} \end{pmatrix}, \quad w \stackrel{\text{def}}{=} \begin{pmatrix} w^{(1)} \\ w^{(2)} \\ \vdots \\ w^{(|\mathcal{K}|)} \end{pmatrix}. \tag{5.9}$$

Let $w_l^{(m),(k)} \in \left[w^{(m),\min}, w^{(m),\max}\right]$ for all m and let sets $W_l^{(k)}$, $W^{(k)}$, and W be defined as follows:

$$W_l^{(k)} \stackrel{\text{def}}{=} \left[w^{(1),\min}, w^{(1),\max}\right] \times \ldots \times \left[w^{(M),\min}, w^{(M),\max}\right], \tag{5.10}$$

$$W^{(k)} \stackrel{\text{def}}{=} \left(W_l^{(k)}\right)^{|\mathcal{A}|}, \quad W \stackrel{\text{def}}{=} \left(W^{(k)}\right)^{|\mathcal{K}|}, \tag{5.11}$$

with $w^{(m),\min}, w^{(m),\max} \in \mathbb{R}$, $w^{(m),\min} < w^{(m),\max}$ for $\forall m \in \{1, \ldots, M\}$. Clearly, $W_l^{(k)}$ is a subset of $\mathbb{R}^M$ and thus $W_l^{(k)}$, $W^{(k)}$, and W are convex and compact sets. It is assumed that $w_l^{(k)} \in W_l^{(k)}$, $w^{(k)} \in W^{(k)}$, $w \in W$ for $\forall k \in \mathcal{K}$, $a \in \mathcal{A}$.

Note that while coefficients $w_l^{(m),(k)}$ can be negative, the toll has to be nonnegative on all links, as stated in (5.8).

With $M = 0$ in equation (5.7) the toll level becomes time-varying, but not directly dependent on traffic flow (although this toll will be influenced by changes in the traffic flow pattern). In that situation the toll on the link a_j will be set as $w_l^{(0),(k)} \in \mathbb{R}_+^0$, and the vectors

$$\omega^{(k)} \stackrel{\text{def}}{=} \begin{pmatrix} w_1^{(0),(k)} \\ w_2^{(0),k)} \\ \vdots \\ w_{|\mathcal{A}|}^{(0),(k)} \end{pmatrix}, \quad \omega \stackrel{\text{def}}{=} \begin{pmatrix} \omega^{(1)} \\ \omega^{(2)} \\ \vdots \\ \omega^{(|\mathcal{A}|)} \end{pmatrix}$$

will be vectors of time-varying, but traffic-flow invariant tolls for the k-th time interval, and of time-varying, but traffic-flow invariant tolls for all time intervals, respectively.

Let W_0 be defined as set W with $M = 0$, i.e., $W_0 \stackrel{\text{def}}{=} \left[w^{(0),\min}, w^{(0),\max}\right]$, with $0 \leq w^{(0),\min} < w^{(0),\max}$. Let $\omega \in W_0$. Clearly W_0 is a subset of $\mathbb{R}_+^0$ and a compact set.

We also introduce the matrix $M = \{0,1\}^{|\mathcal{RS}|\times|\mathcal{P}|}$, which is the origin–destination pair-path incidence matrix. Its element in the (r,s)-th row and p-th column is 1 if the route p starts from origin r and finishes in destination s and 0 otherwise. The traffic flow feasibility is described by

$$M f^{(k)} = d^{(k)}. \tag{5.12}$$

5.2.1 Game-theoretic interpretation of the optimal toll design problem

The problem of the dynamic optimal toll design can be seen as an inverse Stackelberg game. Two possible interpretations from the game theoretic point of view are possible:

- The drivers, as followers, choose in each time period routes from their origins to their destinations so as to minimize their actual or perceived travel costs. Therefore, their decisions are their route choices. Because the average traffic flows are dependent upon these decisions and the road authority as the leader sets dynamic tolls as functions of the average traffic flows in the network, these tolls are also composed functions of the drivers' decisions.

- Because the travelers are uniform, all of them can be seen as one super-player, who is the follower in the one leader – one follower inverse Stackelberg game with the road authority as the leader. The decision of this super-player would establish the average traffic flows in the network. The dynamic tolls are the functions of the follower's decisions in this game.

5.3 Drivers' behavior – dynamic traffic assignment

This section formulates a macroscopic *dynamic traffic assignment (DTA) model* that describes user-optimal flows over a network in which each driver chooses his/her preferred route from origin to destination, based on the time-varying conditions in the network. A driver starting his trip during the k-th time interval will influence the traffic conditions in this interval as well as the traffic conditions during later time intervals. The network conditions in the k-th time period depend on the conditions in previous time time periods. The travel behavior model used in this thesis can be found in, e.g., [10] or [19].

The standard DTA models consist of a *dynamic travel choice* (DTC) model and a *dynamic network loading* (DNL) model.

The DTC contains a path choice model in which all travelers are distributed on all available routes such that some kind of dynamic user equilibrium is achieved. Both deterministic and stochastic equilibria will be considered.

In Section 5.3.1 the dynamic traffic equilibria used in this thesis are defined and discussed. In Section 5.3.2 the dynamic network loading model will be formulated.

5.3.1 Dynamic traffic equilibrium conditions

In the problem of traffic assignment with given traffic demand, each user chooses a certain route from his/her origin to his/her destination. The rules according to which the users decide which route to use have to be specified. The behavioral model used in this thesis is the so-called *Dynamic Traffic Equilibrium*, as stated in, e.g., [10]. We consider both its deterministic and stochastic variants.

Definition 5.1 (Dynamic deterministic traffic equilibrium)
The traffic network is in the dynamic deterministic traffic equilibrium, if for each origin–destination pair, the route travel costs for all users traveling between a specific origin–destination pair and departing during the same time interval are equal, and lower than the route travel costs which would be experienced by a single user on any unused feasible route, i.e., if for all $(r,s) \in \mathcal{RS}$, $p \in \mathcal{P}^{(r,s)}$ the following statement holds:

$$\begin{aligned} &\text{If} \quad f_p^{(k)} > 0, \quad \text{then} \quad c_p^{(k)} = \min_{\hat{p} \in P^{(r,s)}} c_{\hat{p}}^{(k)}, \quad \forall k \in \mathcal{K}, \quad p \in \mathcal{P}^{(r,s)}, \quad (r,s) \in \mathcal{RS}; \\ &\text{if} \quad f_p^{(k)} = 0, \quad \text{then} \quad c_p^{(k)} > \min_{\hat{p} \in P^{(r,s)}} c_{\hat{p}}^{(k)}, \quad \forall k \in \mathcal{K}, \quad p \in \mathcal{P}^{(r,s)}, \quad (r,s) \in \mathcal{RS}. \end{aligned}$$

□

Definition 5.2 (Dynamic stochastic traffic equilibrium)
The traffic network is in the dynamic stochastic traffic equilibrium, if for each origin–destination pair, the perceived route travel costs for all users traveling between a specific origin–destination pair and departing at the same time instant are equal, and lower than the route travel costs which would be experienced by a single user on any unused feasible route, i.e., if for all $(r,s) \in \mathcal{RS}$, $p \in \mathcal{P}^{(r,s)}$ the following statement holds:

$$\begin{aligned} &\text{If} \quad f_p^{(k)} > 0, \quad \text{then} \quad \tilde{c}_p^{(k)} = \min_{\hat{p} \in P^{(r,s)}} \tilde{c}_p^{(k)}, \quad \forall k \in \mathcal{K}, \quad p \in \mathcal{P}^{(r,s)}, \quad (r,s) \in \mathcal{RS}; \\ &\text{if} \quad f_p^{(k)} = 0, \quad \text{then} \quad \tilde{c}_p^{(k)} > \min_{\hat{p} \in P^{(r,s)}} \tilde{c}_p^{(k)}, \quad \forall k \in \mathcal{K}, \quad p \in \mathcal{P}^{(r,s)}, \quad (r,s) \in \mathcal{RS}, \end{aligned}$$

where $\tilde{c}_p^{(k)}$ is the perceived travel cost on the route p. □

As in Chapter 5, Section 4.3 we assume that in equilibrium state, the so-called *logit-based dynamic stochastic equilibrium* takes place. This means that the following equation applies for each $p \in \mathcal{P}^{(r,s)}$, $k \in \mathcal{K}$:

$$f_p^{(k)} = \frac{\exp(-\mu\, c_p^{(k)})}{\sum_{\hat{p} \in \mathcal{P}^{(r,s)}} \exp(-\mu\, c_{\hat{p}})} d^{(r,s),(k)}.$$

5.3.2 The dynamic network loading model

The dynamic network loading (DNL) model is formulated as a system of equations expressing link dynamics, flow conservation, flow propagation, and boundary constraints. The DNL model simulates the progression of the route flows on the network, yielding dynamic link flows, link volumes, and link travel times developing over time. The DNL model used in this thesis is adapted from [10] and can be expressed by the following system of equations:

$$v_{a,p}^{\left(k+\tilde{t}_a^{(k)}\right)} = u_{a,p}^{(k)} \tag{5.13}$$

$$u_{a,p}^{(k)} = \begin{cases} q_p^{(k)}, & \text{if} \quad a \quad \text{is the first link on path} \quad p \in \mathcal{P}^{(r,s)}, \\ v_{a-,p}^{(k)}, & \text{were} \quad a- \quad \text{is the preceding link of} \quad a. \end{cases} \tag{5.14}$$

$$u_a^{(k)} = \sum_{p \in \mathcal{P}^{(r,s)}} u_{a,p}^{(k)} \tag{5.15}$$

$$v_a^{(k)} = \sum_{p \in \mathcal{P}^{(r,s)}} v_{a,p}^{(k)} \tag{5.16}$$

$$x_a^{(k)} = \sum_{k' \leq k} \left(u_a^{(k')} - v_a^{(k')} \right) \Delta, \tag{5.17}$$

where $\tilde{t}_a^{(k)}$ is an approximation of the link travel time. In addition, the link travel time function for the k-th time interval is a nondecreasing and link-specific function of the link volume on the same link for the k-th time interval.

Equation (5.13) is a *flow propagation* equation. It describes the propagation of the inflows $u_{a,p}^{(k)}$ through the link and therefore it determines the outflows $v_{a,p}^{(k)}$. Additionally, it relates the inflows and outflows of link a at the k-th time interval of vehicles traveling on route p from origin r to destination s. The $\tilde{t}_a^{(k)}$ is defined as follows:

$$\tilde{t}_a^{(k)} \stackrel{\text{def}}{=} \chi, \quad \text{if} \quad t_a^{(k)} \in [(\chi - 0.5, \chi + 0.5)\Delta). \tag{5.18}$$

We do not assume explicitly that a FIFO (first-in first-out) condition has to be satisfied.

Equation (5.14) describes the *flow conservation* conditions. If link a is the first link on route p, the inflow rate is equal to the corresponding route flows determined by the route choice model. If link a is not the first link on the route, then the inflow rate $u_{a,p}^{(k)}$ is equal to the link outflow rate $v_{a-,p}^{(k)}$ of the preceding link $a-$.

Equation (5.15) states that the total link inflows are determined by adding all link inflows for all routes that flow into link a at that time interval.

Equation (5.16) states that the total link outflows are determined by adding all link outflows for all routes that flow out of link a at that time interval.

Equation (5.17) defines the link volume $x_a^{(k)}$, i.e., the number of travelers present at the beginning of the k-th time interval on link a.

5.4 The problem formulation

Similarly as in Section 4.5 the goal of the road authority is to choose an optimal w^*, minimizing his/her objective function. As described in Section 5.2.1 the problem is an inverse Stackelberg game. The problem of the total travel time minimization can be formulated as follows:

$$(PD)\begin{cases}\text{Find}\\ w^* = \arg\min_{w\in W} Z\left(\boldsymbol{q}(w), w\right),\\ \quad\text{subject to } q \in DUE\left(w\right), \text{ where } \boldsymbol{\theta} \text{ is defined by (5.7) and (5.8).}\end{cases}$$

The expression $q \in DUE(w)$ reads as “link flow vector q is a result of a used dynamic user equilibrium (DUE) model when a polynomial toll function with coefficient vector w is used.” .

The “standard” Stackelberg problem would be defined as a subproblem of (PD):

$$(PD_0)\begin{cases}\text{Find}\\ w_0^* = \arg\min_{W_0\in W_0} Z\left(\boldsymbol{q}(w_0), w_0\right)\\ \quad\text{subject to } q \in DUE\left(w\right).\end{cases}$$

5.5 General problem properties

Note that problem (PD) is a nonlinear programming problem, similarly as problem (PD) introduced in Section 4.4. Also, the problem (PD) has at least one solution if the DUE constraint represents a compact set of $(w, q(w))$.

If for any given w the set $DUE(w)$ is a singleton, $w \to q$ is a one-to-one mapping. In this case, the continuity of q with respect to w will guarantee that the constrained set of (PD) is closed, which implies the solution existence of (PD) since q and w are bounded.

In general, $DUE(w)$ may have multiple solutions in terms of q and thus $DUE(w)$ may not be a singleton. In this case, $DUE(w)$ is a point-to-set mapping of w ([33]). The solution existence of (PD) will depend on the compactness of the graph $DUE(w)$, defined as

$$\Psi(w,q) = \{(w,q)\,|q \in DUE(w), \forall w \in W\}. \tag{5.19}$$

Theorem 5.1 *The problem (PD) has at least one solution if the following conditions are satisfied:*

i. The set $\mathrm{DUE}(w)$ *is nonempty and compact for* $\forall w \in W$,

ii. Let $w, \overline{w} \in W$ *and let* $q \in \mathrm{DUE}(w)$, $\overline{q} \in \mathrm{DUE}(\overline{w})$. *For each* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that if* $||w - \overline{w}|| < \delta$, *then*

$$\max_{\forall q\in DUE(W)} \min_{\forall \overline{q}\in DUE(W)} ||q - \overline{q}|| < \varepsilon.$$

iii. The link travel cost functions on all links are continous functions of the link flows on the same links.

Proof: Let $R(0,\varepsilon)$ be an open ball with radius ε. Then $\mathcal{Y} \overset{\text{def}}{=} DUE(w) + R(0,\varepsilon)$ is an open set containing $DUE(w)$. Let us define an other open set $\mathcal{Z} \overset{def}{=} \{w : ||w - \overline{w}|| < \delta\}$ containing $\overline{w}$. According to condition *ii.* in Theorem 5.1, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\max_{\forall q \in DUE(w)} \min_{\forall \overline{q} \in DUE(w)} ||q - \overline{q}|| < \varepsilon,$$

which is equivalent to $\cup_{w \in \mathcal{Z}} DUE(w) \subseteq \mathcal{Y}$. Thus, under *ii.*, the point-to-set mapping of $DUE(w)$ is upper-semicontinuous. Together with condition *i.* it implies that the point-to-set mapping $DUE(w)$ is closed on set W. Thus the graph $\Psi(w,q)$ defined in (4.19) is closed by Theorem 4.2. Also, under *i.*, $DUE(w)$ is bounded for any $w \in W$. Since W is a bounded set, the graph $\Psi(w,q)$ is bounded as well. Thus, graph $w \in W$ is compact. Together with *iii.* and the fact that W is compact, we can conclude that (PD) has at least one solution, since it is a nonlinear programming problem with a continuous objective function defined on a compact set. □

Theorem 5.2 *Problem (PD) is strongly NP-hard.*

Proof: The proof follows from the fact that the problem (P) is a special case of the problem (PD) (with $k = 1$) and from Theorem 4.5. □

5.6 Solution methods

The methods used to solve the problem (PD) are those introduced in Section 4.6 adjusted to the dynamic environment. The problems with the drivers driven by the dynamic deterministic user equilibrium can be solved analytically, as long as their scale is not too large. The problems with the drivers driven by the dynamic logit-based (stochastic) equilibrium will be solved by an algorithm containing a neural networks approach for solving the upper level of the problem and the method of the successive averages for the lower level of the problem. Since the dynamic deterministic user equilibrium is a special case of the dynamic logit-based (stochastic) equilibrium, also the deterministic problems can be solved using the neural-network based approach.

5.7 Case studies

In this section the solution methods introduced in Section 4.6 and mentioned in Section 5.6 will be applied on a number of case studies. Problems introduced in Section 5.7.1 are simplified versions of problems (PD) and (PD_0), respectively, with travelers driven by the deterministic dynamic user equilibrium, applied on a two-link network. An analytical solution is given. Problems introduced in Section 5.7.2 are simplified variants of problem (PD) and (PD_0), too, with the drivers driven by stochastic user equilibrium. A numerical solution is given.

5.7.1 Three-links network

In this section problems (PD) and (PD_0) introduced in Section 5.4, played on the three-link (route) parallel road network depicted in Figure 5.1 will be dealt with. Two alternative

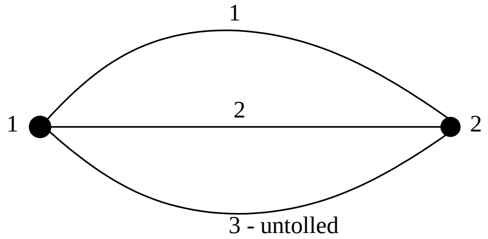

Figure 5.1: One origin–destination pair network with 3 *links.*

objective functions $Z_1 = Z_1(\boldsymbol{q}(w), w)$ and $Z_2 = Z_2(\boldsymbol{q}(w), w)$, defined as

$$Z_1(\boldsymbol{q}(w), w) \stackrel{\text{def}}{=} \boldsymbol{q}^T(w) \cdot t(w), \tag{5.20}$$

$$Z_2(\boldsymbol{q}(w), w) \stackrel{\text{def}}{=} -\boldsymbol{q}^T(w) \cdot \boldsymbol{\theta}(w), \tag{5.21}$$

will be considered.[8]

We assume that $\mathcal{K} = \{1, \ldots, 7\}$, $\Delta = 1$ [h], $d^{(r,s),(1)} = 2000$ [veh], $d^{(r,s),(2)} = 2000$ [veh], $d^{(r,s),(3)} = 3000$ [veh], $d^{(r,s),(4)} = 3000$ [veh], $d^{(r,s),(5)} = 2500$ [veh], $d^{(r,s),(6)} = 2000$ [veh], $d^{(r,s),(7)} = 2000$ [veh], $\alpha = 8$ [euro/h], $t_a^{(k)} \stackrel{\text{def}}{=} \beta_a q_a^{(k)} + \delta_a, \quad a \in \{1, 2, 3\}$, $c_a^{(a)} \stackrel{\text{def}}{=} \alpha t_a^{(k)} + \theta_a^{(k)}$. $\delta_1 = \frac{1}{5}$, $\delta_2 = \frac{1}{4}$, $\delta_3 = \frac{1}{3}$, $\beta_1 = \frac{1}{3000}$, $\beta_2 = \frac{1}{2000}$, $\beta_3 = \frac{1}{2500}$.

5.7.1.1 Total travel time minimization

Let the road authority minimize the total travel time of the network. In the following four games we will consider different toll variants in Stackelberg (traffic-flow invariant) and inverse Stackelberg (traffic-flow dependent) setting to see how different classes of toll functions influence outcomes of the game. To be able to give an objective comparison we will focus on inverse Stackelberg strategies with toll functions having the same number of unknown parameters as Stackelberg strategies to which the inverse Stackelberg strategies are compared to. The goal is to find an inverse Stackelberg strategy that does not increase the problem complexity[9] and that provides better outcomes for the road authority.

In Section 5.7.1 the best possible outcome for the road authority is discussed.

Game 1

Let only link 1 be tolled. Two problems will be compared:

- A problem of total travel time minimization with uniform (constant) toll, i.e., $\theta_1^{(k)} \stackrel{\text{def}}{=} \theta_1 \in \mathbb{R}_+^0$.
- A problem of total travel time minimization with toll defined as a ξ_1-multiple of an actual link traffic flow on link l_1, i.e., $\theta_1^{(k)}\left(q_1^{(k)}\right) \stackrel{\text{def}}{=} \xi_1 \, q_1^{(k)}$, $\xi_1 \in \mathbb{R}_+^0$.

[8] Other possible objective functions for the road authority can be, for example, unreliability of the network [18], negative of the surplus of the network [84], etc.

[9] This is important for possible real-time applications.

The optimal toll for the first problem is $\frac{52}{135} \approx 0.39$ [euro] and yields a total travel time of 9590.79 [h]. A slightly better outcome, 9583.12 [h], can be reached in the second game with an optimal value of ξ_1 equal to $\frac{3809}{10455525} \approx 0.36 \cdot 10^{-3}$. The outcomes of the games are very close to each other because the problem was solved with respect to only one parameter.

Game 2

Let link 1 and link 2 be tolled. Two problems will be compared:

- A problem of total travel time minimization, where toll is uniform, i.e., $\theta_a^{(k)} \stackrel{\text{def}}{=} \theta_a \in \mathbb{R}_+^0$, $a \in \{1,2\}$.
- A problem of total travel time minimization with toll on link a ($a \in \{1,2\}$), defined as a ξ_a-multiple of actual link traffic flow on link a, i.e., $\theta_a^{(k)}\left(q_a^{(k)}\right) \stackrel{\text{def}}{=} \xi_1 \cdot q_a^{(k)}$, $\xi_a \in \mathbb{R}_+^0$, $a \in \{1,2\}$.

For the first problem the optimal tolls on links 1 and 2 are $\frac{8}{15}$ [euro] and $\frac{1}{3}$ [euro], respectively, and yield a total travel time of 9590.79 [h] (the same outcome as in the previous case). The optimal values of ξ_1 and ξ_2 for the second problem are $0.50 \cdot 10^{-3}$ and $0.51 \cdot 10^{-3}$, respectively, and yield the outcome 9578.36 [h]. The traffic-flow dependent toll is accommodating to the traffic conditions in a better way, therefore this toll yields better outcomes.

Game 3

Let only link 1 be tolled. Two problems to be compared are:

- Find $\theta_1^{(k)}$ minimizing the total travel time of the system, where

$$\theta_1^{(k)} \stackrel{\text{def}}{=} \begin{cases} \theta_1, & k \in \{1,2,6,7\}, \\ \tilde{\theta}_1, & k \in \{3,4,5\}. \end{cases}$$

- Find $\theta_1^{(k)}\left(q_1^{(k)}\right)$ minimizing the total travel time of the system, where

$$\theta_1^{(k)}\left(q_1^{(k)}\right) \stackrel{\text{def}}{=} \begin{cases} \xi_1 q_1^{(k)}, & k \in \{1,2,6,7\}, \\ \tilde{\xi}_1 q_1^{(k)}, & k \in \{3,4,5\}. \end{cases}$$

The optimal values of θ_1 and $\tilde{\theta}_1$ in the first game are $\frac{52}{135} \approx 0.39$ [euro] and $\frac{52}{135} \approx 0.39$ [euro], respectively, and yield the total travel time 9590.79 [h]. The optimal values of ξ_1 and $\tilde{\xi}_1$ are $\frac{13}{29925} \approx 0.43 \cdot 10^{-3}$ and $\frac{1}{225} \approx 0.44 \cdot 10^{-2}$, respectively, and yield the total travel time of 9582.68 [h].

Game 4

Let links 1 and 2 be tolled. Two problems to be solved are:

k	$q_1^{(k)}$	$\tau_1^{(k)}$	$q_2^{(k)}$	$\tau_2^{(k)}$	$q_3^{(k)}$	$\tau_3^{(k)}$
1	$\frac{2660}{3}$	$\frac{223}{450}$	$\frac{4870}{9}$	$\frac{937}{1800}$	$\frac{5150}{9}$	$\frac{253}{450}$
2	$\frac{2660}{3}$	$\frac{223}{450}$	$\frac{4870}{9}$	$\frac{937}{1800}$	$\frac{5150}{9}$	$\frac{253}{450}$
3	$\frac{3860}{3}$	$\frac{283}{450}$	$\frac{7270}{9}$	$\frac{1177}{1800}$	$\frac{8150}{9}$	$\frac{313}{450}$
4	$\frac{3860}{3}$	$\frac{283}{450}$	$\frac{7270}{9}$	$\frac{1177}{1800}$	$\frac{8150}{9}$	$\frac{313}{450}$
5	$\frac{3260}{3}$	$\frac{253}{450}$	$\frac{6070}{9}$	$\frac{1057}{1800}$	$\frac{6650}{9}$	$\frac{283}{450}$
6	$\frac{2660}{3}$	$\frac{223}{450}$	$\frac{4870}{9}$	$\frac{937}{1800}$	$\frac{5150}{9}$	$\frac{253}{450}$
7	$\frac{2660}{3}$	$\frac{223}{450}$	$\frac{4870}{9}$	$\frac{937}{1800}$	$\frac{5150}{9}$	$\frac{253}{450}$

Table 5.1: The optimal link traffic flows [veh/h] and the link travel times [h] for the problem of total travel time minimization.

- Find $\theta_1^{(k)}, \theta_2^{(k)}$, minimizing the total travel time of the system, where

$$\theta_a^{(k)} \stackrel{\text{def}}{=} \begin{cases} \theta_a \in \mathbb{R}_+^{(0)}, & k \in \{1,2,6,7\}, \quad a \in \{1,2\}, \\ \tilde{\theta}_a \in \mathbb{R}_+^{(0)}, & k \in \{3,4,5\}, \quad a \in \{1,2\}. \end{cases}$$

- Find $\theta_1^{(k)}, \theta_2^{(k)}$, minimizing the total travel time of the system, where

$$\theta_a^{(k)}\left(q_a^{(k)}\right) \stackrel{\text{def}}{=} \begin{cases} \xi_a q_a^{(k)}, & k \in \{1,2,6,7\}, \quad a \in \{1,2\} \\ \tilde{\xi}_a q_a^{(k)} & k \in \{3,4,5\}, \quad a \in \{1,2\}. \end{cases}$$

The optimal values of θ_1, θ_2, $\tilde{\theta}_1$, and $\tilde{\theta}_2$ for the first problem are $\frac{8}{15}$ [euro], $\frac{8}{15}$ [euro], $\frac{1}{3}$ [euro], and $\frac{1}{3}$ [euro], respectively, and yield a total travel time of 9649.51 [h]. The optimal values of ξ_1, ξ_2, $\tilde{\xi}_1$, and $\tilde{\xi}_2$ for the second problem are $0.77 \cdot 10^{-2}$, $1.29 \cdot 10^{-2}$, $0.78 \cdot 10^{-2}$, and $1.26 \cdot 10^{-2}$, respectively, and yield a total travel time of 9577.38 [h].

General outcome

Minimization of the total travel time function with respect to the traffic flows yields the link traffic flows and the link travel times as depicted in Table 5.1.

If these traffic flows and travel times are the travelers' response to the tolls, minimal total travel time $\frac{1034347}{108} \approx 9577.29$ [h] will be obtained. This means that the second strategy from Game 4 yields a total travel time close to the optimal outcome. The optimal outcome 9577.29 [h] can be reached if more parameters in the toll functions are included. In Table 5.2 we find the optimal linear inverse Stackelberg strategy ($\theta_a^{(k)} \stackrel{\text{def}}{=} w_a^{(1),(k)} q_a^{(k)} + w_a^{(0),(k)}$) and the optimal Stackelberg strategy, minimizing the total travel time of the system. Since for the inverse Stackelberg strategy parameters $w_a^{(1),(k)}$ are free (and, therefore, the solution of the inverse Stackelberg game is nonunique), clearly 7 parameters in the toll function are needed to obtain the optimal outcome. Obviously, by setting $w_a^{(1),(k)}$ to 0 the optimal Stackelberg strategy will be reached. Therefore, it seems that with enough toll parameters the outcomes of the two strategies would be the same for general networks. However, it is difficult to

k	$w_1^{(0),(k)}$	$w_2^{(0),(k)}$
1	$\frac{8}{15} - \frac{2660}{3} w_1^{(1),(1)}$	$\frac{1}{3} - \frac{4870}{9} w_2^{(1),(1)}$
2	$\frac{8}{15} - \frac{2660}{3} w_1^{(1),(2)}$	$\frac{1}{3} - \frac{4870}{9} w_2^{(1),(2)}$
3	$\frac{8}{15} - \frac{3860}{3} w_1^{(1),(3)}$	$\frac{1}{3} - \frac{7270}{9} w_2^{(1),(3)}$
4	$\frac{8}{15} - \frac{3860}{3} w_1^{(1),(4)}$	$\frac{1}{3} - \frac{7270}{9} w_2^{(1),(4)}$
5	$\frac{8}{15} - \frac{3260}{3} w_1^{(1),(5)}$	$\frac{1}{3} - \frac{6070}{9} w_2^{(1),(5)}$
6	$\frac{8}{15} - \frac{2660}{3} w_1^{(1),(6)}$	$\frac{1}{3} - \frac{4870}{9} w_2^{(1),(6)}$
7	$\frac{8}{15} - \frac{2660}{3} w_1^{(1),(7)}$	$\frac{1}{3} - \frac{4870}{9} w_2^{(1),(7)}$

Table 5.2: The optimal link toll function coefficients for the inverse Stackelberg game (total travel time minimization). The optimal tolls for the Stackelberg game are $\theta_1^{(k)} = 8/15$ *[euro] and* $\theta_2^{(k)} = 8/15$ *[euro].*

compute optimal time-varying toll on each link and for each time period in the real time, if the number of toll parameters is very high. That is why it is important to find strategy working better even with low number of toll parameters.

5.7.1.2 Total toll revenue maximization

Let us again consider the network depicted in Figure 5.1, where the road authority maximizes the total toll revenue of the system. The traffic flows imposing the best outcome for the road authority are unknown here, because the total toll revenue toll changes with change of toll strategy.

Game 1

We will first assume that only link 1 is tolled. Two problems will be compared:

- The problem of total toll maximization, where the toll is uniform, i.e., $\theta_1^{(k)} \stackrel{\text{def}}{=} \theta_1 \in \mathbb{R}_+^0$.
- The problem of total toll maximization with toll defined as a ξ_1-multiple of actual link traffic flow on link l_1, i.e., $\theta_1^{(k)}\left(q_1^{(k)}\right) \stackrel{\text{def}}{=} \xi_1 \cdot q_1^{(k)}$, $\xi_1 \in \mathbb{R}_+^0$.

The optimal toll for the first problem is $\frac{2344}{945} \approx 2.48$ [euro] and yields a total toll revenue of 9690.19 [euro]. The solution of the second problem is $\xi_1 = \frac{1}{225}$ and yields a total toll revenue of 9931.46 [euro].

Game 2

Let link 1 and link 2 be tolled. We will compare two problems:

- The problem of total toll maximization, where the toll is uniform, i.e., $\theta_a^{(k)} \stackrel{\text{def}}{=} \theta_a \in \mathbb{R}_+^0, \quad a \in \{1,2\}$.

- The problem of total toll maximization, with toll defined as a ξ_a-multiple of actual link traffic flow on link a, i.e.,

$$\theta_a^{(k)}\left(q_1^{(k)}\right) \stackrel{\text{def}}{=} \xi_a \cdot q_a^{(k)}, \quad \xi_a \in \mathbb{R}_+^0, \quad a \in \{1,2\}.$$

The optimal tolls for the first problem are $\frac{452}{105} \approx 4.30$ [euro] and $\frac{431}{105} \approx 4.10$ [euro] for links 1 and 2, respectively, yielding the total toll revenue 26071.23 [euro]. For the second problem the optimal values of ξ_1 and ξ_2 are $0.77 \cdot 10^{-2}$ and $1.27 \cdot 10^{-2}$, respectively. The resulting total toll revenue is 26794.74 [euro].

Game 3

Let only link 1 be tolled. We will compare two problems:

- Find $\theta_1^{(k)}$ maximizing the total toll revenue of the system, where

$$\theta_1^{(k)} \stackrel{\text{def}}{=} \begin{cases} \theta_1, & k \in \{1,2,6,7\} \\ \tilde{\theta}_1, & k \in \{3,4,5\}. \end{cases}$$

- Find $\theta_1^{(k)}\left(q_1^{(k)}\right)$ maximizing the total toll revenue of the system, where

$$\theta_1^{(k)}\left(q_1^{(k)}\right) \stackrel{\text{def}}{=} \begin{cases} \xi_1 q_1^{(k)}, & k \in \{1,2,6,7\} \\ \tilde{\xi}_1 q_1^{(k)}, & k \in \{3,4,5\}. \end{cases}$$

The optimal values of θ_1 and $\tilde{\theta}_1$ for the first problem are $\frac{292}{135} \approx 2.16$ [euro] and $\frac{392}{135} \approx 2.90$ [euro], respectively, and yield the total toll revenue 9901.83 [euro]. The optimal values of ξ_1 and $\tilde{\xi}_1$ for the second problem are both equal to $\frac{1}{225}$, and yield a total toll revenue of 9931.46 [euro].

Game 4

Let both link 1 and 2 be tolled. We will compare two problems:

- Find $\theta_1^{(k)}$ and $\theta_2^{(k)}$ maximizing total toll revenue of the system, where

$$\theta_a^{(k)} \stackrel{\text{def}}{=} \begin{cases} \theta_a \in \mathbb{R}_+^{(0)}, & k \in \{1,2,6,7\}, \quad a \in \{1,2\}, \\ \tilde{\theta}_a \in \mathbb{R}_+^{(0)}, & k \in \{3,4,5\}, \quad a \in \{1,2\}. \end{cases}$$

- Find $\theta_1^{(k)}\left(q_1^{(k)}\right)$, $\theta_2^{(k)}\left(q_2^{(k)}\right)$, maximizing the total toll revenue of the system, where

$$\theta_a^{(k)}\left(q_a^{(k)}\right) \stackrel{\text{def}}{=} \begin{cases} \xi_a q_a^{(k)}, & k \in \{1,2,6,7\}, \quad a \in \{1,2\}, \\ \tilde{\xi}_a q_a^{(k)} & k \in \{3,4,5\}, \quad a \in \{1,2\}. \end{cases}$$

k	$q_1^{(k)}$	$q_2^{(k)}$	$q_3^{(k)}$
1	$-\frac{w_1^{(0),(1)}}{2w_1^{(1),(1)}}$	$-\frac{w_2^{(0),(1)}}{2w_2^{(1),(1)}}$	$\frac{4000w_2^{(1),(1)}w_1^{(1),(1)}+w_2^{(1),(1)}w_1^{(0),(1)}+w_1^{(1),(1)}w_2^{(0),(1)}}{2w_2^{(1),(1)}w_1^{(1),(1)}}$
2	$-\frac{w_1^{(0),(2)}}{2w_1^{(1),(2)}}$	$-\frac{w_2^{(0),(2)}}{2w_2^{(1),(2)}}$	$\frac{4000w_2^{(1),(2)}w_1^{(1),(2)}+w_2^{(1),(2)}w_1^{(0),(2)}+w_1^{(1),(2)}w_2^{(0),(2)}}{2w_2^{(1),(2)}w_1^{(1),(2)}}$
3	$-\frac{w_1^{(0),(3)}}{2w_1^{(1),(3)}}$	$-\frac{w_2^{(0),(3)}}{2w_2^{(1),(3)}}$	$\frac{6000w_2^{(1),(3)}w_1^{(1),(3)}+w_2^{(1),(3)}w_1^{(0),(3)}+w_1^{(1),(3)}w_2^{(0),(3)}}{2w_2^{(1),(3)}w_1^{(1),(3)}}$
4	$-\frac{w_1^{(0),(4)}}{2w_1^{(1),(4)}}$	$-\frac{w_2^{(0),(4)}}{2w_2^{(1),(4)}}$	$\frac{6000w_2^{(1),(4)}w_1^{(1),(4)}+a_2{}^{(4)}w_1^{(0),(4)}+w_1^{(1),(4)}w_2^{(0),(4)}}{2w_2^{(1),(4)}w_1^{(1),(4)}}$
5	$-\frac{w_1^{(0),(5)}}{2w_1^{(1),(5)}}$	$-\frac{w_2^{(0),(5)}}{2a_2{}^5}$	$\frac{5000w_2^{(1),(5)}w_1^{(1),(5)}+w_2^{(1),(5)}w_1^{(0),(5)}+w_1^{(1),(5)}w_2^{(0),(5)}}{2w_2^{(1),(5)}w_1^{(1),(5)}}$
6	$-\frac{w_1^{(0),(6)}}{2w_1^{(1),(6)}}$	$-\frac{w_2^{(0),(6)}}{2w_2^{(1),(6)}}$	$\frac{4000w_2^{(1),(6)}w_1^{(1),(6)}+w_2^{(1),(6)}w_1^{(0),(6)}+w_1^{(1),(6)}w_2^{(0),(6)}}{2w_2^{(1),(6)}w_1^{(1),(6)}}$
7	$-\frac{w_1^{(0),(7)}}{2q_1^{(7)}}$	$-\frac{w_2^{(0),(7)}}{2w_2^{(1),(7)}}$	$\frac{4000w_2^{(1),(7)}w_1^{(1),(7)}+w_2^{(1),(7)}w_1^{(0),(7)}+w_1^{(1),(7)}w_2^{(0),(7)}}{2w_2^{(1),(7)}w_1^{(1),(7)}}$

Table 5.3: Optimal link flows: Total toll revenue maximization with linear toll functions.

The optimal values of θ_1, θ_2, $\tilde{\theta}_1$, and $\tilde{\theta}_2$ for the first problem are $\frac{56}{15} \approx 3.73$ [euro], $\frac{73}{15} \approx 5.07$ [euro], $\frac{53}{15} \approx 3.53$ [euro], and $\frac{73}{15} \approx 4.87$ [euro], respectively, and yield a total toll revenue of 26706.15 [euro]. The optimal values of ξ_1, ξ_2, $\tilde{\xi}_1$, and $\tilde{\xi}_2$ for the second problem are $0.77 \cdot 10^{-2}$, $1.29 \cdot 10^{-2}$, $0.78 \cdot 10^{-2}$, and $1.26 \cdot 10^{-2}$, respectively, and yield a total toll revenue of 26795.01 [euro].

Since the total toll revenue function will vary depending on the chosen structure of the toll functions, it is impossible to get the maximal total toll revenue before knowing the toll structure used. In the following game the optimal value of the total toll revenue with linear tolls will be computed, as this toll brought the best possible outcome when various polynomial toll functions were tested.

Game 5

We will consider the situation, in which the road authority maximizes the total toll revenue of the system by setting tolls defined as follows:

$$\theta_1^{(k)}\left(q_1^{(k)}\right) \stackrel{\text{def}}{=} w_1^{(1),(k)} q_1^{(k)} + w_1^{(0),(k)}, \quad \theta_2^{(k)}\left(q_1^{(k)}\right) \stackrel{\text{def}}{=} w_2^{(1),(k)} q_2^{(k)} + w_2^{(0),(k)}. \tag{5.22}$$

Provided that coefficients $w_j^{(1),(k)}$, $j = 1,2, \quad k \in \{1,\dots,7\}$ are negative, local maxima of the total toll revenue function with respect to the link traffic flows will be reached with flows depicted in Table 5.3. These traffic flows are dependent on $w_j^{(1),(k)}$ and $w_j^{(0),(k)}$ ($j = 1,2,3$, $k = 1,\dots,7$).

The road authority maximizing the total toll revenue, aiming to influence the travelers such that the traffic flows depicted in Table 5.3 will be obtained, has to take into account the dynamic deterministic user equilibrium conditions. If all three links are used, these conditions will yield coefficients $b_1^{(k)}$ and $b_2^{(k)}$ as depicted in Table 5.4.

k	$w_1^{(0),(k)}$	$w_2^{(0),(k)}$
1	$\frac{80\,w_1^{(1),(1)}\left(-73+17500\,w_2^{(1),(1)}\right)}{3-550\,w_2^{(1),(1)}-675\,w_1^{(1),(1)}+93750\,w_1^{(1),(1)}\,w_2^{(1),(1)}}$	$\frac{40\,w_2^{(1),(1)}\left(-247+99375\,w_1^{(1),(1)}\right)}{3\left(3-550\,w_2^{(1),(1)}-675\,w_1^{(1),(1)}+93750\,w_1^{(1),(1)}w_2^{(1),(1)}\right)}$
2	$\frac{80\,w_1^{(1),(2)}\left(-73+17500\,w_2^{(1),(2)}\right)}{3-550\,w_2^{(1),(2)}-675\,w_1^{(1),(2)}+93750\,w_1^{(1),(2)}\,w_2^{(1),(2)}}$	$\frac{40\,w_2^{(1),(2)}\left(-247+99375\,w_1^{(1),(2)}\right)}{3\left(3-550\,w_2^{(1),(2)}-675\,w_1^{(1),(2)}+93750\,w_1^{(1),(2)}w_2^{(1),(2)}\right)}$
3	$\frac{80\,w_1^{(1),(3)}\left(-103+25000\,w_2^{(1),(3)}\right)}{3-675\,w_1^{(1),(3)}-550\,w_2^{(1),(3)}+93750\,w_2^{(1),(3)}\,w_1^{(1),(3)}}$	$\frac{40\,w_2^{(1),(3)}\left(-367+144375\,w_1^{(1),(3)}\right)}{3\left(3-675\,w_1^{(1),(3)}-550\,w_2^{(1),(3)}+93750\,w_2^{(1),(3)}\,w_1^{(1),(3)}\right)}$
4	$\frac{80\,w_1^{(1),(4)}\left(-103+25000\,w_2^{(1),(4)}\right)}{3-675\,w_1^{(1),(4)}-550\,w_2^{(1),(4)}+93750\,w_2^{(1),(4)}w_1^{(1),(4)}}$	$\frac{40\,w_2^{(1),(4)}\left(-367+144375\,w_1^{(1),(4)}\right)}{3\left(3-675\,w_1^{(1),(4)}-550\,w_2^{(1),(4)}+93750\,w_2^{(1),(4)}\,w_1^{(1),(4)}\right)}$
5	$\frac{160\,w_1^{(1),(5)}\left(10625\,w_2^{(1),(5)}-44\right)}{3-675\,w_1^{(1),(5)}-550\,w_2^{(1),(5)}+93750\,w_1^{(1),(5)}w_2^{(1),(5)}}$	$\frac{40\,w_2^{(1),(5)}\left(-307+121875\,w_1^{(1),(5)}\right)}{3\left(3-675\,w_1^{(1),(5)}-550\,w_2^{(1),(5)}+93750\,w_1^{(1),(5)}\,w_2^{(1),(5)}\right)}$
6	$\frac{80\,w_1^{(1),(6)}\left(-73+17500\,w_2^{(1),(6)}\right)}{3-675\,w_1^{(1),(6)}-550\,w_2^{(1),(6)}+93750\,w_1^{(1),(6)}w_2^{(1),(6)}}$	$\frac{40\,w_2^{(1),(6)}\left(-247+99375\,w_1^{(1),(6)}\right)}{3\left(3-675\,w_1^{(1),(6)}-550\,w_2^{(1),(6)}+93750\,w_1^{(1),(6)}\,w_2^{(1),(6)}\right)}$
7	$\frac{80\,w_1^{(1),(7)}\left(-73+17500\,w_2^{(1),(7)}\right)}{3-675\,w_1^{(1),(7)}-550\,w_2^{(1),(7)}+93750\,w_2^{(1),(7)}\,w_1^{(1),(7)}}$	$\frac{40\,w_2^{(1),(7)}\left(-247+99375\,w_1^{(1),(7)}\right)}{3\left(3-675\,w_1^{(1),(7)}-550\,w_2^{(1),(7)}+93750\,w_2^{(1),(7)}\,w_1^{(1),(7)}\right)}$

Table 5.4: Coefficient of linear toll functions yielding local maximum of the total toll revenue.

k	$w_1^{(1),(k)}$	$w_1^{(0),(k)}$	$w_2^{(1),(k)}$	$w_2^{(0),(k)}$	$\theta_1^{(k)}$	$\theta_2^{(k)}$
1	−0.0077	7.4795	−0.0129	7.0757	3.7397	3.5379
2	−0.0077	7.4795	−0.0129	7.0757	3.7397	3.5379
3	−0.0078	10.6860	−0.0126	10.2769	5.3430	5.1384
4	−0.0078	10.6860	−0.0126	10.2768	5.3430	5.1384
5	−0.0077	9.0533	−0.0127	8.6608	4.5267	4.3304
6	−0.0077	7.4795	−0.0129	7.0757	3.7397	3.5379
7	−0.0077	7.4795	−0.0129	7.0757	3.7397	3.5379

Table 5.5: Optimal toll function coefficients and resulting tolls [euro]: Total toll revenue maximization.

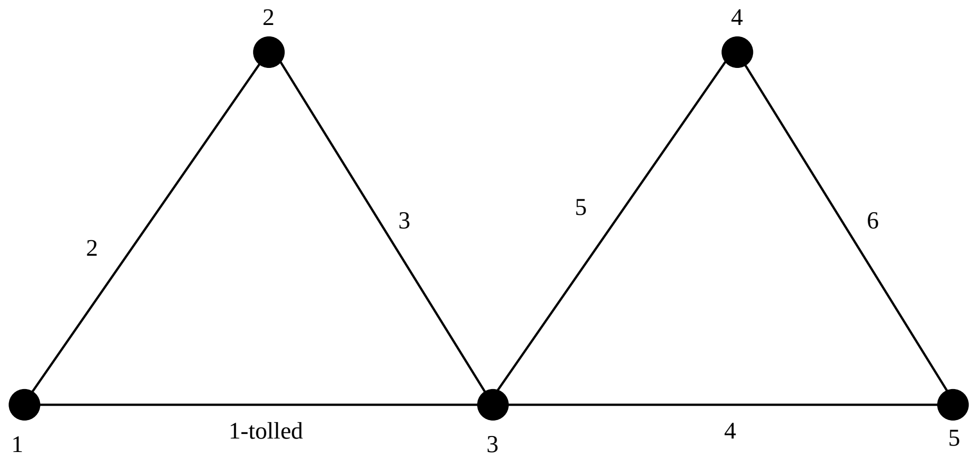

Figure 5.2: Chen network.

Here $w_1^{(1),(k)}$, $w_2^{(1),(k)}$, $k = 1,\dots,7$, are free. However, after substituting $w_1^{(0),(k)}$, $w_2^{(0),(k)}$, $k = 1,\dots,7$, from Table 5.4 into the total toll revenue function and maximizing the obtained function with respect to $w_1^{(1),(k)}$, $w_2^{(1),(k)}$, $k = 1,\dots,7$, the values of the coefficients of the toll function can be obtained. These coefficients are depicted in Table 5.5 and yield the maximal toll revenue $2.6795 \cdot 10^4$ [euro]. Substituting the coefficients $w_1^{(1),(k)}$, $w_1^{(0),(k)}$, $w_2^{(1),(k)}$, and $w_2^{(0),(k)}$ from Table 5.5 into (5.22) will result in toll values $\theta_1^{(k)}$ and $\theta_2^{(k)}$ as depicted in the same table. Obviously, toll defined as a polynomial function (of the actual link flow) of degree higher than 1 will not lead to a system performance improvement (with respect to the system performance when the classical Stackelberg strategy is employed), thus this is the best outcome achieved. This means that also the second strategy from Game 4 is the best strategy that we found.

5.7.2 Chen network

In this section case studies with the Chen network consisting of 6 links, 2 origin–destination pairs $(1,5)$ and $(3,5)$, and 6 routes will be investigated (depicted in Figure 5.2). The traffic on each link flows from the node with lower identification number into the node with higher identification number. Only link 1 is tollable, the toll is defined as

$$\theta_a^{(k)} \stackrel{\text{def}}{=} w_a^{0,(k)} + w_a^{1,(k)} x_a^{(k)},$$

with the traffic volume $x_a^{(k)}$. Nonnegativity condition applies here, too.

For each directed arc $a \in \mathcal{A}$ the following parameters are initially given: link length s_a [km], maximum speed $\vartheta_a^{\max}$ [km/h], minimum speed $\vartheta_a^{\min}$ [km/h], critical speed $\vartheta_a^{\text{crit}}$ [km/h], jam density J_a^{jam} [pcu/km], where pcu denotes passenger car units, and the unrestricted link capacity C_a [pcu/h]. Dynamic link travel time for an individual user entering link a during k-th time interval ($k \in \mathcal{K}$) is defined as

$$\tau_a^{(k)} = \frac{s_a}{\vartheta_a^{(k)}}, \tag{5.23}$$

where the link speed $\vartheta_a^{(k)}$ [km/h] can be computed using *Smulders speed-density function* (see [73]):

$$\vartheta_a^{(k)} = \begin{cases} \vartheta_a^{\max} + \frac{\vartheta_a^{\text{crit}} - \vartheta_a^{\max}}{J_a^{\text{crit}}} J_a^{(k)}, & \text{if} \quad J_a^{(k)} \le J_a^{\text{crit}}, \\ J_a^{\text{jam}} + \left(\vartheta_a^{\text{crit}} - \vartheta_a^{\min}\right) \frac{\left(J_a^{(k)}\right)^{-1} - \left(J_a^{\text{jam}}\right)^{-1}}{\left(J_a^{\text{crit}}\right)^{-1} - \left(J_a^{\text{jam}}\right)^{-1}}, & \text{if} \quad J_a^{\text{crit}} \le J_a^{(k)} \le J_a^{\text{jam}}, \\ \vartheta_a^{\min} & \text{if} \quad J_a^{(k)} \ge J_a^{\text{jam}}, \end{cases} \tag{5.24}$$

with critical density J_a^{crit} [pcu/km] defined as $J_a^{\text{crit}} = C_a / \vartheta_a^{\text{crit}}$.

The road authority minimizes the total travel time of the system, i.e.,

$$Z(w, q(w)) \stackrel{\text{def}}{=} \sum_{k \in \mathcal{K}} \sum_{(r,s) \in \mathcal{RS}} \sum_{p \in \mathcal{P}^{(r,s)}} f_p \cdot \tau_p^{(k)}.$$

We assume that the logit-based dynamic stochastic equilibrium applies for the drivers.

Case Study 1

Four time intervals are considered, i.e., $\mathcal{K} = \{1, 2, 3, 4\}$, $\Delta = 1$ [h]. The link properties and the travel demands are depicted in Table 5.6. The other parameters are set as: $\mu = 0.2$, $\varepsilon = 0.05$, $\alpha = 8$ [euro/h], $w_a^{(1),\min} = -10$, $w_a^{(0),\min} = -5$, $w_a^{(1),\max} = 10$, $w_a^{(0),\max} = 5$.

a	s_a	$\vartheta_a^{\max}$	$\vartheta_a^{\text{crit}}$	$\vartheta_a^{\min}$	J_a^{jam}	C_a
1	7.5	150	90	20	50	1500
2	15	120	70	10	150	3500
3	15	120	70	10	150	3500
4	10	150	90	20	50	1500
5	15	120	70	10	150	3500
6	15	120	70	10	150	3500

(r,s)	$d^{(r,s),(1)}$	$d^{(r,s),(2)}$	$d^{(r,s),(3)}$	$d^{(r,s),(4)}$
$(1,5)$	2000	8000	8000	3000
$(3,5)$	1000	1500	2000	1500

Table 5.6: Link properties and travel demands in Case study 1.

The neural-networks based algorithm was applied, with 33620 training data, 13297 validation data, and worst accepted validation error equal to 1.1%. Sixteen processors were used to compute the problem in a parallel way, where both grid search and neurosimulation were parallelized. The neural ANN function that approximates the total travel time function in the "best way" (See Section 4.6.3 for explaining this best way.) is smooth, twice differentiable, with more than one local minimum (shown numerically) and one global minimum. The minimum 14173 [h] is found at $[w_1^{(1),(1)}, w_1^{(0),(1)}, w_1^{(1),(2)}, w_1^{(0),(2)}, w_1^{(1),(3)}, w_1^{(0),(3)}, w_1^{(1),(4)}, w_1^{(0),(4)}] = [-0.50, 0.20, -0.03, 1.19, 0, 0, -0.04, 3.96]$. Note that for the first and forth interval the optimal toll is decreasing with the current traffic volume. This phenomenon appears when other links are congested than the tolled link and the aim is to attract the travelers to the tolled link.

With no toll the total travel time reaches 19542 [h] the optimal time-varying (but traffic-flow independent) tolls are $\theta_1^{(1)} = 2.3$ [euro],$\theta_2^{(1)} = 6.6$ [euro], $\theta_2^{(3)} = 9.5$ [euro], $\theta_1^{(4)} = 7.4$ [euro], and yield total travel time of $1.7844 \cdot 10^4$ [h].

The computational time of the FAUN simulator was 10.23 hours, the computational time of the grid search was 35.21 hours. This time can be decreased by using more processors to solve the problem.

Game 2

In this case study the number of time intervals will be increased to 8, with travel demands depicted in Table 5.7. Also, there are no boundaries on parameters of linear toll functions

(r,s)	$d^{(r,s),(1)}$	$d^{(r,s),(2)}$	$d^{(r,s),(3)}$	$d^{(r,s),(4)}$	$d^{(r,s),(5)}$	$d^{(r,s),(6)}$	$d^{(r,s),(7)}$	$d^{(r,s),(8)}$
$(1,5)$	2000	4000	6000	8000	8000	6000	4000	2000
$(3,5)$	1000	2000	3000	4000	4000	3000	2000	1000

Table 5.7: Travel demands in Case study 2.

and only 14122 training data and 9301 validation data were used. The worst accepted validation quality was set to 1.1%. The best-trained neural network was minimized using Matlab again. The approximation function is again twice differentiable, with multiple local minima, and one global minimum 29149.00 at $[w_1^{(1),(0)}, w_1^{(0),(0)}, w_1^{(1),(1)}, w_1^{(0),(1)}, w_1^{(1),(2)}, w_1^{(0),(2)}, w_1^{(1),(3)}, w_1^{(0),(3)}, w_1^{(1),(4)}, w_1^{(0),(4)}, w_1^{(1),(5)}, w_1^{(0),(5)}, w_1^{(1),(6)}, w_1^{(0),(6)}, w_1^{(1),(7)}, w_1^{(0),(7)}, w_1^{(1),(8)}, w_1^{(0),(8)}] = [-0.02, 2.62, -0.04, 3.20, 0.4, -0.93, 0.01, -1.32, 0.01, 0.99, 0.05, 0.40, 0, 0, 0.02, -0.24]$.

Optimal toll decreasing with the current traffic volume appears in the first time interval and in the second time interval. With no toll the total travel time reaches 39659.20 hours. The optimal time-varying (but traffic-flow independent) tolls yield a total travel time of 34822.60 hours.

The computational time of the FAUN simulator was 7.15 [h], the grid search took 26.11 [h]. This time can be decreased by using more processors to solve the problem. From the tests made after the computation it follows that the obtained solution is very accurate in its neighborhood (with an error of 1%), although a lower number of training and validation data was used.

Discussion

In both case studies the traffic-volume (and hence traffic-flow) dependent toll improved the system performance remarkably. Also, phenomena of the toll decreasing with traffic volume was observed. The natural explanation for this phenomena is that the traffic should be attracted to the tolled link.

The grid search is very time consuming, although the network used is very small. The speed of the solution process can be increased by further parallelization of both phases of the solution process.

Generally, the time-varying but traffic-flow invariant toll can never lead to a better outcome than the traffic-flow dependent toll. This follows from the fact that the dynamic optimal toll design problem with traffic-flow invariant toll is a special case of (PD). See [74] for further explanation.

5.8 Conclusions and future research

In this chapter we have introduced the concept of the dynamic optimal toll design problem with second-best traffic-flow dependent toll. We discussed existence of solution of this problem as well as its difficulty and proposed a solution algorithm, based on the algorithm used for the solution of the static version of the problem. In the case studies we have shown both problems solved analytically and problems solved numerically using the proposed algorithm.

All the topics mentioned in Section 4.8, extended to the dynamic environment, can be subjects of future research. Additionally, departure time choice of the travelers is a topic calling for the further investigation. In [45] the departure time choice of the travelers was considered, without the traffic-flow dependent toll and finding the optimal solution of the problem.

Chapter 6

Electricity Market Problem

In this chapter we propose a simple formalization of the electricity market problem. The model aims to see the differences between the prices in the perfectly competitive market, in the market with one leader, and in the market with two leaders, playing Nash among themselves.

6.1 Introduction

The European electricity market is currently being transformed from a market with monopolistic, national, and state-owned producers to a market with competing, private, and often multinational firms. This transformation is called *liberalization*. The speed and current state of this process vary among different European countries, from a near monopoly in the Czech Republic to highly competitive markets in Norway, Sweden, and the Netherlands [54].

Main aims of the liberalization are to bring benefit to consumers by lowering electricity prices and to cause more cost efficient electricity production. Little is known about the environmental consequences of liberalization. On the one hand, more cost efficient production may be beneficial for the environment, while, on the other hand, lower market prices imply higher electricity demand that may increase the burden on the environment. Moreover, in a highly competitive market an incentive to produce electricity with cheap, but often not environmental-friendly means, is increased. It is also assumed that quite recent developments, such as the implementation of the European union's CO_2 emission trading system in 2005, may have major environmental impacts.

In the liberalization process, with various competing firms, the market and its rules are no longer fixed. The effects of liberalization on market structure can be illustrated by the recent development in Germany. Following liberalization, the initial 30 relatively small electricity producers were merged into four large producers in only few years of time. These firms have market power on the German market, but they also face competition from neighboring countries. The extent of international competition is limited by the transmission capacities between countries, but it is also affected by the market structure in these neighboring countries. Also, the character of electricity markets in countries that are neighbors of Germany will be changed by trade with Germany.

To get more insights into how the liberalization can influence the European electricity market, we will view the electricity producers in eight European countries: Belgium, Denmark, Finland, France, Germany, The Netherlands, Norway, and Sweden as players in a game. We chose these countries because of the following reasons:

- There exists a model including these countries (in [56]) with different ways to model the firms' behavior, thus we can compare our results with already existing results.

- Real data about electricity prices, emission factors, and electricity productions are available [32].

In the game that we consider the electricity producers choose technologies for electricity production as well as amounts of electricity to be produced for different load periods. Only the game among electricity producers of different countries is played, the consumers' demand of electricity is exogenous. This approach is reasonable in the situation, in which the selling price of electricity in each country has to be the same for each producer, i.e., the consumers cannot choose "cheaper" electricity from different producers.

The number of producers per country is given, as well as parameters like electricity production costs and electricity production capacities, and the emission factors per country and per technology. These initial data are derived from real data and were taken from existing literature and electronic sources [32, 56]. Additionally, shadow prices on emissions per energy producer can be set.

Different game theoretic scenarios of firms behavior will be formulated, namely a perfect competition, a Stackelberg game, in which in each country one firm acts as a first-moving Stackelberg market leader during dispatch, and a Stackelberg game with two leaders per country acting as first-moving Stackelberg market leaders, playing Nash among themselves. Extension of the model to the dynamic game, in which investments can take place over a longer time horizon is formulated and explored as well. In all considered scenarios the possibility of electricity transmission between neighboring countries will be considered.

Extensive studies of static energy models have been carried out. In [62] the Belgian, Dutch, French, and German electricity market were considered and the effect of market power among three static models was compared. One of these model, the nodal pricing static equilibrium model COMPETES, was additionally studied in [41, 42]. In [29] and [49] the consequences of market power in the Nordic electricity market are considered. In [11] and [28] a static game theoretic model of the European gas market is presented. In [20] emission permit trading to a nodal pricing model to study strategic effects of holding NO_X permits is added. In [56] the electricity market with eight European countries was considered. However, the decision variable of the individual electricity producers was the so-called market power mark-up, unlike the quantities of electricity to be produced, as it is in our research.

The contributions of the research presented in this chapter can be listed as follows:

- A new game theoretical model of electricity market of eight European countries is developed. Our approach differs from those presented in the existing literature, in which other types of markets, electricity market with less countries included in the model, or different decision variables for the electricity producers are considered.

- Various scenarios, like a perfect competition case or a Stackelberg game with one leading producer and the rest of the producers being perfectly competitive, are included in the model.

- Most of the input data for our model come from real measurements, presented in existing literature. Therefore, the improved version of the model can help to explain some recent changes in the real European electricity market.

- Extension of the existing model to a dynamic model with the possibility of investments is proposed. Such a dynamic model is applicable for real data to analyze the current and future phenomena in the European electricity market.

This chapter is organized as follows. In Section 6.2 a static model of electricity producers in eight European countries will be formulated. In Section 6.3 various case studies are performed. In Section 6.4 an extension of the static model into a dynamic model with the possibility of investments will be introduced. The outcomes of the case studies, their relation to the situation in the current European electricity market, and possible future research directions are discussed in Section 6.5. Our research helps to understand such a complicated process, as the liberalization of the electricity market is.

6.2 Games of the European electricity market

Let us consider a game with electricity producers (also called firms or companies) in different countries (also called regions) as players. The following countries are considered in the model: Belgium (BEL), Denmark (DEN), Finland (FIN), France (FRA), Germany (GER), The Netherlands (NLD), Norway (NOR), and Sweden (SWE). The number of producers for each country is given. Firms generate electricity through different technologies; each producer chooses technologies among technologies available to him/her and the amounts of electricity produced by the chosen technologies. A producer can own several power plants of different types, of which the total capacity for each, as well as variable production costs, is given. Producers maximize their payoffs by choosing the amount of electricity to produce with various technologies for various load modes[1]. Producer's payoffs consist of the income from sales of electricity in regional markets minus the variable costs of production. There are limitations in transportation capacity of electricity and moreover, production capacity of electricity is fixed on the short term. The electricity demand curve for each country is exogenous and comes from real measurement [56]. Trade is only feasible with neighboring countries and includes netting, which means that bi-directional flows between two countries are permitted, as long as trade constraints are not violated. Emissions are assigned to producers based on the actual technology used. These emissions can be restricted, too.

Let F and R be a set of firms and a set of countries included in the model, respectively. Let $F_r \in F$ be a set of firms located in region r, and let $|F_r|$ be their number. Let I be a set of possible technologies for electricity generation, i.e.,

$$I \stackrel{\text{def}}{=} \{\text{wind energy}, \text{nuclear energy}, \ldots\},$$

[1] The set of load modes in our research contains two possible loads: base load and peak load.

let $I_r \subset I$ be a set of technologies that are available in region $r \in R$, and let I_f be a set of technologies available to firm f. Let L be a set of possible load modes, e.g., base load or peak load. Let K be an emission type set, i.e., $K \stackrel{\text{def}}{=} \{\text{acid}, \text{smog}, \ldots\}$.

Firm $f \in F_r$ maximizes its surplus $\mathcal{I}_f$ [euro] defined as

$$\mathcal{I}_f \stackrel{\text{def}}{=} \sum_{l \in L} h_l \sum_{r' \in R} \left(p_{r',l} s_{f,r',l}\right) - \sum_{l \in L} h_l \sum_{r' \in R} \left(\sum_{i \in I} c_{i,r} q_{i,f,r',l}\right), \tag{6.1}$$

where $s_{f,r',l}$ [GW] denotes the supply of electricity of load mode l from firm f into region r' and $p_{r',l}$ [euro] denotes the market electricity price for region r' and load mode l. Let h_l [h] denote the number of hours of a particular load l per year, $c_{i,r}$ [euro/MWh] represents the variable production costs with technology i in region r, in which firm f is located, while $q_{i,f,r',l}$ [GW] is the production of firm f with technology i for region r' for load mode l. The supply of electricity of load mode l per firm f to region r' denoted by $s_{f,r',l}$ can be defined as

$$s_{f,r',l} \stackrel{\text{def}}{=} (1 - \lambda_{r'}) \sum_{i \in I} q_{i,f,r',l}, \tag{6.2}$$

where $\lambda_{r'} \in [0,1]$ is the loss of electricity due to transport to region r', initially given. Moreover, the electricity supply is additive, i.e., the total supply of electricity $S_{r',l}$ [GW] for load mode l per region r' can be computed as

$$S_{r',l} = \sum_{f \in F_{r'}} s_{f,r',l},$$

where $S_{r',l}$ [GW] is a total electricity supply to region r' during load mode l.

We assume that the price $p_{r,l}$ is dependent on the total supply of electricity $S_{r,l}$ and follows a constant elasticity of distribution (CED) [2, 4] with elasticity $-\varepsilon_{r,l}$ ($\varepsilon_{r,l} > 0$) dependent on region r and load mode l, the reference demand for electricity $d^0_{r,l}$ [GW], and the reference price of electricity $p^0_{r,l}$ [euro], and can be computed using the following equation:

$$d^0_{r,l} \left(\frac{p_{r,l}}{p^0_{r,l}}\right)^{-\varepsilon_{r,l}} = \sum_{f \in F} s_{f,r,l} = S_{r,l}. \tag{6.3}$$

The firm's regional market share $\pi_{f,r,l} \in [0,1]$ can be then computed as

$$\pi_{f,r,l} = \frac{s_{f,r,l}}{\sum_{f' \in F_r} s_{f',r,l}}. \tag{6.4}$$

Firms can trade electricity with other countries as well. The amount of electricity traded $x_{r,r',l}$ [GW] is defined as the exported amount of electricity from region r to region r' minus the imported amount of electricity entering region r from region r' ($f \in F_r$), i.e.,

$$x_{r,r',l} = \sum_{f \in F_r} \sum_{i \in I} q_{i,f,r',l} - \sum_{f' \in F_{r'}} \sum_{i \in I} q_{i,f',r,l}. \tag{6.5}$$

There is a restriction on the maximum production capacity of each firm f, the technology i, and the load mode l. The maximum production capacity is complementary to the shadow price $\mu_{i,f,l}$ [euro/MWh], which has a nonnegative value if the production with technology i, by firm f during load mode l, is equal to the maximum available production capacity:

$$\mu_{i,f,l}\left(\sum_{r'\in R} q_{i,f,r',l} - q^{\max}_{i,f}\right) \geq 0, \quad \mu_{i,f,l} > 0. \tag{6.6}$$

The amount of electricity traded is also restricted. It is is complementary to the shadow price $\tau_{r,r',l}$. This shadow price obtains a nonnegative value, when the trade restriction is binding:

$$\tau_{r,r',l}\left(x_{r,r',l} - x^{\max}_{r,r'}\right) \geq 0, \quad \tau_{r,r',l} > 0, \tag{6.7}$$

with the maximum amount of electricity traded between regions r and r' denoted by $x^{\max}_{r,r'}$ [GW].

Emissions can also be limited. Due to the Kyoto protocol, firms have to reduce the amount of emissions, where the shadow price of emission constraint κ^k [euro/MWh] is nonzero as soon as the current amount of emissions is equal to a permissible emission ceiling E^k [g],

$$\kappa^k\left(\sum_{l\in L} h_l \sum_{r'\in R}\sum_{i\in I}\sum_{f\in F} \sigma^k_{i,r} q_{i,f,r',l} - E^k\right) \geq 0, \quad \kappa^k > 0. \tag{6.8}$$

Emission factors $\sigma^k_{i,r}$ [g/MWh] are associated with the region, in which firm f produces electricity.

If we include constraints (6.6), (6.7), and (6.8) into the problem of maximizing the surplus function (6.1), firm f maximizes L_f defined as

$$\begin{aligned}
L_f \stackrel{\text{def}}{=} & \sum_{l\in L}\sum_{r'\in R}\sum_{i\in I}\left(p_{r',l}\left(1-\lambda_{r'}\right) q_{i,f,r',l} - c_{i,r} q_{i,f,r',l}\right) \\
& - \sum_{l\in L} h_l \sum_{i\in I} \mu_{i,f,l}\left(\sum_{r'\in R} q_{i,f,r',l} - q^{\max}_{i,f}\right) \\
& - \sum_{l\in L} h_l \sum_{\substack{r'\in R \\ r'\neq r}} \tau_{r,r',l}\left(\sum_{f'\in F_r}\sum_{i\in I} q_{i,f',r',l} - \sum_{f'\in F_{r'}}\sum_{i\in I} q_{i,f',r,l} - x^{\max}_{r,r'}\right) \\
& - \sum_{k\in K} \kappa^k \left(\sum_{l\in L} h_l \sum_{r'\in R}\sum_{i\in I}\sum_{f''\in F} \sigma^k_{i,r} q_{i,f'',r',l} - E^k\right).
\end{aligned} \tag{6.9}$$

Karush-Kuhn-Tucker (KKT) conditions for maximizing the objective function can be derived from equation (6.9) by taking the derivative with respect to the production of electricity $q_{i,f,r',l}$ for firm $f \in F_r$:

$$\begin{aligned}
& 0 \leq q_{i,f,r',l}\left(c_{i,r} + \mu_{i,f,l} + \tau_{r,r',l} + \sum_{k\in K} \kappa_k \sigma_{i,r,l} - (1-\lambda_{r'}) p_{r',l}\left[1 - \frac{\pi_{f,r',l}}{\varepsilon_{r',l}}\right]\right), \\
& 0 < q_{i,f,r',l}.
\end{aligned} \tag{6.10}$$

Market share $\pi_{f,r,l}$ is defined by equation (6.4). Inequalities (6.10) can be interpreted as follows: As long as the marginal revenues from electricity sales are not lower than the marginal costs of production, a power company is willing to produce electricity.

Marginal costs can be derived from equation (6.10) as follows:

$$c^m_{i,f,r',l} = c_{i,r} + \mu_{i,f,l} + \tau_{r,r',l} + \sum_{k\in K} \kappa^k \sigma^k_{i,r}. \tag{6.11}$$

The four components of the marginal costs can be interpreted as follows. The first term are the costs of the producing electricity. The second and third term are the scarcity price of maximum production capacity per technology and the transmission price related to trade, respectively. The fourth term represents the addition to the production costs due to an emission constraint.

We substitute the marginal costs defined in equation (6.11) into equation (6.10) to obtain the following necessary condition for firm f to produce electricity:

$$q_{i,f,r',l}\left(c^m_{i,f,r',l} - (1-\lambda_{r'})\, p_{r',l}\left[1 - \frac{\pi_{f,r',l}}{\varepsilon_{r',l}}\right]\right) \geq 0, \quad q_{i,f,r',l} > 0. \tag{6.12}$$

6.2.1 Game formulations

We will consider three possible games among the electricity producers: a perfect competition (PC), a Stackelberg game with one leader per country (SG), in which the rest of the producers is perfectly competitive, and a Stackelberg game with two leaders per country (competitive among themselves), where the rest of the producers are perfectly competitive, too (NSG). We will denote an optimal quantity produced by firm $f \in F_r$ for region r', load mode l, and technology $i \in I$ as follows:

- $q^{f,P}_{i,f,r',l}$ for perfect competition (PC);
- $q^{f,S}_{i,f,r',l}$ for Stackelberg game with one leader per region (SG);
- $q^{f,NS}_{i,f,r',l}$ for Stackelberg game with two leaders playing (Nash game among themselves) per region (NSG).

Perfect competition

We consider a situation with uniform players having the same technologies available and the same restrictions on capacities. In a perfectly competitive market the companies enter the game if their utility from the game is nonnegative, i.e., the problem of producer f in region r can be written as:

$$(PC)\begin{cases} \text{Find} \\ \left(q^{f,P}_{i,f,r',l}\right)^* \in [0, \frac{q^{\max}_{ir}}{|F_r|}], \\ \quad \text{such that} \\ \sum_{l\in L} h_l \sum_{r'\in R} \left(p_{r',l} s_{f,r',l}\right) - \sum_{l\in L} h_l \sum_{r'\in R} \left(\sum_{i\in I} c_{i,r}\left(q^{f,P}_{i,f,r',l}\right)^*\right) = 0. \end{cases}$$

Since the solution of (PC) might be nonunique, we will additionally assume that

$$\left(q_{i,f,r',l}^{f,P}\right)^* = \arg\min_{q_{i,f,r',l}^{f,P}} p_{r',l}\left(\cdot\right),$$

i.e., in a perfectly competitive market the firms choose the quantities to be produced in order to minimize the selling costs, while their utilities are nonnegative.

Stackelberg game with one leader per region

We assume that there is one leading firm in each region acting as the first player, choosing $\left(q_{i,f,r',l}^{f,S}\right)_{i\in I_r,r'\in R,l\in L}$ so as to maximize L_f defined in (6.9), whereas other producers, which are the followers, are perfectly competitive. The leader can in advance determine how the other producers will react to his/her decision and with this information the leader can choose his/her optimal $\left(q_{i,f,r',l}^{f,S}\right)_{i\in I_r,r'\in R,l\in L}$. For each region, the means of electricity production are available to the leader of this game only, and are initially given.

Stackelberg game with two leaders per region

We assume that there are two leading firms f, f' in each region r acting first, playing Nash game between each other and choosing $\left(q_{i,f,r',l}^{f,NS}\right)_{i\in I_r,r'\in R,l\in L}$, $\left(q_{i,f',r'',l}^{f',NS}\right)_{i\in I_{r'},r''\in R,l\in L}$ so as to minimize their profits L_f and $L_{f'}$. Other producers, which are perfectly competitive, choose their production amounts per load and technology after the leaders have made their choice.

In each of the three games we are interested not only in the payoffs for individual players, but also in how their behavior influences emission levels, what technologies to produce electricity would be preferred, and what amounts of electricity will be traded among neighboring countries.

Data used for computations are consumers' demand of electricity per region, supply data (generation capacity and cost), trade data (interconnection capacity), distribution losses data, emission factors. These data are taken from [32, 56, 83], and are presented and discussed below.

6.2.2 Model specifications

Demand and supply side of the model

Within the electricity markets of the considered countries we distinguish 34 different electricity producers or firms, as presented, together with net losses λ_r and values for reference demands d^0, in Table 6.1.

The demand side of the model consists of one sector per national market. However, there are different markets for peak load (high demand) and base load (moderate demand). We consider two load periods, namely a peak period (20% of the year) and a base load (80% of the year). We additionally assume that demand at peak hours requires 90 % of the total available capacity:

$$d_{\text{peak}}^0 = 0.9\sum_{i\in I} q_{i,f}^{\max}.$$

	BEL	DEN	FIN	FRA	GER	NLD	NOR	SWE
no. of firms	2	3	3	2	5	5	7	7
net losses	4.5%	6.5%	3.5%	6.8%	4.7%	3.9%	8.9%	8.2%
avg. demand (GW)	9.04	3.75	8.72	46.88	54.45	11.48	12.66	15.46

Table 6.1: Characteristics of eight European electricity markets.

	BEL	DEN	FIN	FRA	GER	NLD	NOR	SWE
BEL				2500		1400		
DEN					1750		950	1900
FIN							70	1450
FRA	2850				1150			
GER		1350		1750		3300		
NLD	1400				3300			
NOR		950	70					3035
SWE		1840	2050		550		3035	

Table 6.2: Transmission capacities between the countries (MW).

The price elasticity of the demand is assumed to be set to -0.4, as in [56]. The intuition behind this relatively high price elasticity is that it reflects the alternatives for consumers to choose their electricity supplier [83].

The model encompasses 12 different production technologies, which can be listed as follows:

- conventional thermal power technologies: nuclear (N), coal (C), gas (G), lignite (L), oil (O).
- combined heat and power production (CHP) technologies: gas (CHP-G), coal (CHP-C), oil (CHP-O), biomass (CHP-B), and other fuels (CHP-X).
- renewable technologies: hydro (H) and wind power (W).

Due to varying fuel and production taxes across countries the variable production costs differ across regions and technologies, but not across producers within each country. A summary of total production capacities in the countries included in the model is given in Table 6.3.

Trade and distribution losses

Firms in our model are assigned to a specific country. Hence, no crossborder ownership is permitted. There is an opportunity to trade electricity among countries, with the following restrictions:

- Trade via imports and exports to countries outside the considered countries is ignored.
- A firm can only trade with neighboring countries.

	BEL	DEN	FIN	FRA	GER	NLD	NOR	SWE
nuclear	5.71		2.64	63.18	21.37	0.45		9.46
coal	2.95	5.10	2.29	12.69	17.86	4.05		
lignite					18.97			
gas	3.50	0.04	0.90	1.89	13.82	7.17		
oil	1.20	0.79	1.24	12.23	8.11	0.99		4.64
CHP-gas	0.58	2.58	1.80		0.99	4.66		0.13
CHP-coal		1.13	1.47		6.96			0.56
CHP-oil	0.10		0.16		0.30			0.65
CHP-bio	0.29	0.23	1.04			0.64		0.46
CHP-other			1.44	6.64			0.20	1.00
hydro	1.40	0.01	2.88	25.60	11.61	0.04	27.46	16.33
wind	0.01	2.42	0.04	0.08	0.36	0.44	0.01	0.25
total	15.74	12.30	15.89	122.31	100.33	18.44	27.67	33.48

Table 6.3: Electricity production capacities in 2000 (GW).

	BEL	DEN	FIN	FRA	GER	NLD	NOR	SWE
nuclear	6.14		6.14	6.14	6.14	6.14		7.50
coal	16.94	13.83	13.97	15.19	14.42	16.83		
lignite					15.50			
gas	24.22	23.81	20.28	23.83	29.04	23.25		
oil	36.42	35.21	35.21	38.84	38.70	41.21		39.83
CHP-gas	13.29	13.08	11.21		15.85	12.78		13.52
CHP-coal		7.57	7.63		7.84			11.73
CHP-oil	19.58	19.58	19.58		21.43			21.58
CHP-bio	19.94	19.94	19.94			19.94		19.94
CHP-other			14.59	16.69			16.69	16.69
hydro	0.00	0.00	0.00	5.84	0.00	0.00	0.00	1.18
wind	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

Table 6.4: Variable cost (euro/MWh) per technology in 2000.

	BEL	DEN	FIN	FRA	GER	NLD	NOR	SWE
coal	920.0	972.2	915.9	915.9	970.0	915.9		
lignite					1219.7			
gas	388.0	327.2	348.9	401.9	348.9	411.0		
oil	877.3	692.6	877.3	756.8	877.3	877.3		877.3
CHP-gas	330.6	673.9	528.3		327.1	327.1		327.1
CHP-coal		948.9	776.1		33.1			733.1
CHP-oil	503.4		503.4		503.4			503.4
CHP-bio	0.0	81.9	2.1			0.0		0.0
CHP-other			1296.1	401.6			403.4	403.4

Table 6.5: Greenhouse gas emission factors (kg CO_2 equivalents/MWh) per technology.

	BEL	DEN	FIN	FRA	GER	NLD	NOR	SWE
coal	31.594	20.699	23.310	31.549	23.307	28.365		
lignite					33.896			
gas	5.901	2.174	4.522	15.435	4.522	6.783		
oil	21.821	2.486	21.821	25.610	21.821	21.821		21.821
CHP-gas	2.174	19.833	6.848		2.174	2.174		2.174
CHP-coal		20.217	32.459		2.649			2.649
CHP-oil	2.486		2.486		2.486			2.486
CHP-bio	7.160	31.692	46.726			7.160		12.288
CHP-other			83.071	15.435			3.736	3.736

Table 6.6: Emission factors for acidifying emissions (g acid equivalent/MWh) per technology.

	BEL	DEN	FIN	FRA	GER	NLD	NOR	SWE
coal	80.0	57.0	172.9	170.0	66.0	17.0		
lignite					96.0			
gas	0.0	0.0	0.0	0.0	0.0	0.0		
oil	21.0	1.0	3.0	130.0	2.0	2.0		21.0
CHP-gas	0.0	0.0	0.0		0.0	0.0		0.0
CHP-coal		57.0	150.0		10.0			10.0
CHP-oil	1.0		2.0		2.0			2.0
CHP-bio	30.0	0.0	21.0			30.0		233.0
CHP-other			195.0	0.0			1.0	1.0

Table 6.7: Emission factors for smog formation (g fine particles/MWh) per technology.

The transmission capacity within a country is unrestricted. The interconnection capacity among countries of the electricity network is restricted and the data is derived from ETSO data [32].

Electricity distribution after transmission through the electricity grid is accompanied by losses. These distribution losses are generally much larger than the transmission losses, which are ignored in the model. The distribution losses λ_r ($r \in R$) occur when the electricity is distributed within a country, and differ across countries, as depicted in Table 6.1.

Emission factors

Three environmental effects can be taken into account in the game: greenhouse gas emissions, acidification, and smog formation due to emissions of fine particles.

Information about emission factors for all technologies per country is taken from [83] and [56] and is depicted in Table 6.5, Table 6.6, and Table 6.7. For all technologies, the specific emissions of the 8 considered countries due to the electricity generation were determined. Emissions due to construction and deconstruction of power plants, mining, extraction, and transportation have been disregarded, as these emissions, including emissions of extraction and transportation, are rather small, and in the same range of those for wind or hydroelectric power. Consequently emissions of hydroelectric, nuclear, and wind power are set to zero, CO_2 emissions of biomass power are also set to zero.

6.3 Case studies

For each situation (PC), (SG), (NSG) the following three scenarios will be considered.

- There is only one country in the model (The Netherlands); electricity transmissions with other countries are not considered.
- There are two countries in the model (The Netherlands and Belgium), electricity transmissions between these two countries can be considered; transmissions with other countries are not considered.
- All eight countries are included in the model; the electricity transmissions among these 8 countries can be considered.

In Table 6.8 and Table 6.9 schemes of the case studies are depicted. The first tabular refers to the games without emission constraints. The second tabular refers to the problems with emission constraints consideration. We set the permissible emission ceiling E^k [g] is set to 50% of the average of the resulting emissions of the eight countries when game E1.8 is played.

In the Stackelberg game we will assume that the leaders have access to the means of electricity production depicted in Table 6.10,

For games with 2 and more countries there will be variant (c), denoting that the cross-border transactions are considered.

6.3.1 Games with one country

The problems involving only The Netherlands will be solved analytically.

without emissions	SG	PC	NSG
1 country	E1.1	E1.2	E1.3
2 countries	E1.4	E1.5	E1.6
8 countries	E1.7	E1.8	E1.9

Table 6.8: Scheme of case studies with no emission restrictions.

with emissions	SG	PC	NSG
1 country	E2.1	E2.2	E2.3
2 countries	E2.4	E2.5	E2.6
8 countries	E2.7	E2.8	E2.9

Table 6.9: Scheme with case studies with emission restrictions.

Game	SG	NSG
BEL	wind, hydro, nuclear	wind, hydro, nuclear, CHP-gas
DEN	wind, hydro	wind, hydro, CHP-coal, CHP-gas
FIN	wind, hydro, nuclear	wind, hydro, nuclear, CHP-gas
FRA	wind, hydro	wind, hydro
GER	wind, hydro	wind, hydro
NLD	wind, hydro, nuclear, CHP-gas	wind, hydro, nuclear, CHP-gas, coal
NOR	wind, hydro	wind, hydro
SWE	wind, hydro	wind, hydro

Table 6.10: The available means of electricity production for leaders in Stackelberg game.

Games E1.2 and E2.2

Maximization of the utility functions with respect to the quantities produced gives the following outcome. With perfect competition (Game E1.2) and with all producers having equal access to the means of electricity production, the selling price of electricity is 17.23 [euro/MWh].

When the emission constraints are considered (Game E2.2), the selling price of electricity is 19.13 [euro/MWh].

Games E1.1 and E2.1

Let the leading producer have access to the means of production depicted in Table 6.10 as the only producer. In game E1.1 maximization of his/her profit with respect to the constraint of nonnegative profit for other producers leads to a selling cost of 25.98 [euro/MWh], yielding him/her a profit of 55182.92 [euro], while the utility of all other producers is zero.

When the emission constraints are considered (Game E2.1), the selling price of electricity is 30.10 [euro/MWh] and the profit for the leader is 49819.10 [euro], while the other producers obtain a zero profit.

Games E1.3 and E2.3

Let the two leading producers as only producers have (symmetric) access to the means of production depicted in Table 6.10. Then maximization of their profit with respect to the constraint of nonnegative profit for other producers leads to a selling cost of 20.31 [euro/MWh] and an average profit of 44632.41 [euro], while all other producers have a zero profit.

With emission constraints included the selling cost is 26.15 [euro/MWh]. This cost yields profit of 41023.24 [euro] for each of the leading producers, while all other producers have a zero profit.

Discussion

In the case studies with only one country (The Netherlands), the selling price is remarkably higher in the Stackelberg games than in the perfectly competitive market, especially when the leading producers have access to more ecological means for electricity production than the rest of the producers. While the resulting prices of our case studies are still about 25 % smaller when compared to the actual situation in the electricity market, the influence of the type of competition on the electricity prices matches the trends presented in [56] very well.

6.3.2 Games with two countries

Games E1.4 and E2.4

If game E1.4 is played, the electricity price in The Netherlands is 19.42 [euro/MWh] and yields a profit of 50244.12 [euro] for the leader; the electricity price in Belgium is 17.99 [euro/MWh]; the profit of the leading producer will be 61213.24 [euro].

If game E1.4(c) is played, the electricity price in The Netherlands is 18.35 [euro/MWh] and yields a profit of 46001.21 [euro] for the leader; the electricity price in Belgium is 15.85

[euro/MWh]; the profit of the leading producer is 57192.91 [euro]. Belgian firms will sell 1400 [MW] to the Netherlands.

If game E2.4 is played, the electricity price in The Netherlands is 22.32 [euro/MWh] and yields a profit of 44115.23 [euro] for the leader; the electricity price in Belgium is 18.56 [euro/MWh]; the profit of the leading producer is 57234.11 [euro].

If game E2.4(c) is played, the electricity price in the Netherlands will be 20.15 [euro/MWh] and yields a profit of 37125.24 [euro] for the leader; the electricity price in Belgium is 17.12 [euro/MWh]; the profit of the leading producer is 50259.44 [euro].

Game E1.5 and Game E2.5

If both Belgium and The Netherlands are considered in the perfect competition case (Game E1.5), 10.41 [euro/MWh] and 18.12 [euro/MWh] are the selling prices in Belgium and The Netherlands, respectively.

If emission restrictions are included, the prices are 12.99 [euro/MWh] and 19.99 [euro/MWh], respectively.

Game E1.6 and Game E2.6

If both Belgium and The Netherlands have two leading producers, playing Nash among themselves,[2] and cross-border electricity transmissions are prohibited (Game E1.6), the game does not have a solution. The two electricity producers in Belgium cannot cover the demand on electricity. Together they can produce only 7.70 [GW] of electricity, while the initial electricity demand in Belgium is 9.04. If the demand would not need to be satisfied, the optimal strategy for the identical leaders would be to set the price of electricity infinitely high.

If Game E1.6(c) is played, the situation is solvable. Moreover, the electricity producers in Belgium cannot set the electricity prices arbitrary high, as they are limited by the electricity prices in The Netherlands. A solution to the problem is as follows: The electricity price in both Belgium and The Netherlands is 18.25 [euro], the average profit of the Dutch producers is 73140.23 [euro], the average profit of Belgian producers is 23095.18 [euro]. If Game E2.6(c) is played, the selling price of electricity for both Belgium and the Netherlands will be 19.31 [euro] and the average profits for the Dutch and Belgian producers will be 65232.13 [euro] and 18123 [euro], respectively.

Discussion

As in the game with one country, the perfect competitive market yields much lower electricity prices. The prices will be remarkably increased if emission restrictions are included.

6.3.3 Games with eight countries

Since the analytical computation of the solution of the problem with eight countries would be extremely time-consuming, the problem was implemented and solved numerically in Matlab.

[2] In Belgium these two producers are only players.

Game	E1.7	E1.7 (c)	E1.8	E1.8(c)	E1.9	E1.9(c)
BEL	20.73	18.41	10.04	8.23	15.98	13.21
DEN	20.03	19.97	5.98	5.44	14.98	14.72
FIN	20.32	20.28	7.81	5.23	15.05	14.88
FRA	20.35	19.44	8.20	6.87	17.21	16.88
GER	18.95	17.95	7.87	6.62	11.22	10.99
NLD	21.13	19.25	15.85	12.01	19.54	18.22
NOR	13.21	12.59	0.01	0.01	9.25	9.01
SWE	17.32	15.45	1.37	1.08	14.21	13.54

Table 6.11: Resulting selling costs for base load period for games with eight countries.

Game	E1.7(c)	E1.8(c)	E1.9(c)
BEL-FRA	1320/1500	0/2850	1440/1410
BEL-NLD	890/50	1400/0	1000/25
DEN-GER	1460/1300	1750/0	1500/750
DEN-NOR	60/800	0/950	50/900
DEN-SWE	210/800	0/950	100/880
FIN-NOR	20/65	0/70	10/68
FIN-SWE	320/1800	0/2050	200/1900
FRA-GER	850/275	1150/0	910/105
GER-NLD	2950/1500	3300/0	3005/545
GER-SWE	200/455	0/550	150/505
NOR-SWE	1420/2650	0/3035	720/2810

Table 6.12: Electricity traded between neighboring countries [MW].

For each of the three games we will consider both variants with and without electricity transmissions between neighboring countries.

The variants of the games, in which cross-border electricity transmissions are allowed, will be denoted by (c). The resulting prices for the base load period are mentioned in Table 6.11, whereas the amounts of electricity traded between the neighboring countries are depicted in Table 6.12. In this table, 1320/1500 in column BEL-FRA illustrates that 1320 [MW] of electricity from Belgian firms will be sold in France, while 1500 [MW] of electricity units will be sold in Belgium by French firms. The amounts of acid particles per firm in a country ([g]) , the amounts of CO particles per firm in a country ([g]), and the resulting amounts of smog particles per firm in a country ([g]) for game E1.8 are depicted in Table 6.11.

Discussion

The resulting electricity prices are in our case studies quite lower than the prices appearing in the actual European electricity market (see [32]). The trends in the prices coincides with the actual situation in the European electricity market, though. The emission levels were

Game E1.8	acid	CO	smog
BEL	$3.5961 \cdot 10^4$	$1.4191 \cdot 10^6$	$3.5961 \cdot 10^4$
DEN	$1.3304 \cdot 10^3$	$3.1320 \cdot 10^5$	$1.3304 \cdot 10^3$
FIN	$9.7799 \cdot 10^4$	$2.0925 \cdot 10^6$	$9.7799 \cdot 10^4$
FRA	0	0	0
GER	$8.2164 \cdot 10^4$	$3.3121 \cdot 10^6$	$8.2164 \cdot 10^4$
NLD	$4.3591 \cdot 10^4$	$1.6014 \cdot 10^6$	$4.3591 \cdot 10^4$
NOR	0	0	0
SWE	$6.2383 \cdot 10^3$	$1.1744 \cdot 10^6$	$6.2383 \cdot 10^3$
total	$2.6708 \cdot 10^5$	$9.9127 \cdot 10^6$	$2.6708 \cdot 10^5$

Table 6.13: Game E1.8: Emission acid particles (g), emission CO particles (g), and emission smog particles (g), in different countries per firm

not compared to those from actual measurements.

6.4 Extension: Dynamic model

In this section we propose a dynamic extension of the model introduced in Section 6.2.

Let us consider the case that the time horizon of the model is extended one time period ahead. In that case, firms can aim to maximize their *discounted* payoff by choosing the amount of electricity to produce with various technologies for each time period. In addition to the utility from sales of the electricity in the regional markets minus the average variable costs of production, firms also have to accommodate the fixed costs by financing the purchase of new production capacity via investments. The following equation expresses the payoffs in the next time period:

$$\tilde{\jmath}_f = \frac{1}{1+\beta}\left[\sum_{l\in L} h_l \sum_{r'\in R} \tilde{p}_{r',l}\,\tilde{s}_{f,r',l} - \sum_{l\in L} h_l \sum_{r'\in R}\left(\sum_{i\in I} c_{i,r}\tilde{q}_{i,f,r',l}\right) + \sum_{i\in I} V_i\,\tilde{q}^{\max}_{i,f}\right] - \sum_{i\in I} V_i\, q^{\text{new}}_{i,f}, \tag{6.13}$$

with $\tilde{q}^{\max}_{i,f}$ defined later by (6.15).

The variables in (6.13) are defined as in Section 6.2, with "$\sim$" identifying the variable for the next time period. Hence, the investment decision of the firm in the current period depends on the expected outcome in the next period. That is why the prices, the supply, the generation, and the production capacity are assigned to the next period and discounted with interest rate β. In addition to serving the market in the next period, firms also need to consider the value of installed capacity in the next period, while new investments are needed to keep the production capacity at a desired level. Parameter V_i [euro/kW] represents the value of technology i, while $q^{\text{new}}_{i,f}$ [GW] denotes the amount of the new production capacity of electricity (chosen by the producer). This means that the maximum production capacity is no longer fixed. The firms make their investment decisions in every time period based on the most recent information (cf. the feedback information structure, see [5]). Equation (6.14) describes the capacity expansion, in which capital is depreciated at technology-specific depreciation rate δ_i:

$$\overline{q}_{i,f}^{\max} = (1-\delta_i)\, q_{i,f}^{\max} + q_{i,f}^{\text{new}}. \tag{6.14}$$

Because of environmental considerations with respect to nuclear waste and physical limitations on the capacity of the used technologies (e.g., it is natural to restrict the capacity of coal power plants), the following inequality is included into the model:

$$\tilde{\varphi}_i \left(\sum_{f\in F} q_{i,f}^{\text{new}} + (1-\delta_i) \sum_{f\in F} q_{i,f}^{\max} - q_i^{\max\text{abs}} \right) \geq 0, \quad \tilde{\varphi}_i > 0. \tag{6.15}$$

Here the shadow price $\tilde{\varphi}_i$ might become nonzero, once the planned expansion of capacity of a certain technology reaches the maximum allowable installed capacity $q_i^{\max\text{abs}}$ [GW].

The producer maximizes the net profit L_f^{DYN}, by a joint choice of the investment decision and his/her production of technologies for possible regions and both load types in the next period. Therefore, the net profit can be defined as:

$$\begin{aligned}
L_f^{\text{DYN}} = {} & \frac{1}{1+\beta} \left[\sum_{l\in L} h_l \sum_{r'\in R} \left(\tilde{p}_{r',l} \tilde{s}_{f,r',l} \right) - \sum_{l\in L} h_l \sum_{r'\in R} \left(\sum_{i\in I} c_{i,r}\, \tilde{q}_{i,f,r',l} \right) \right. \\
& \left. + \sum_{i\in I} V_i \tilde{q}_{i,f}^{\max} \right] - \sum_{l\in L} V_i q_{i,f}^{\text{new}} - \frac{1}{1+\beta} \sum_{l\in L} h_l \sum_{i\in I} \tilde{\mu}_{i,f,l} \left(\sum_{r'\in R} \tilde{q}_{i,f,r',l} - \tilde{q}_{i,f}^{\max} \right) \\
& - \frac{1}{1+\beta} \sum_{l\in L} h_l \sum_{r'\in R} \sum_{r\in R} \tilde{\tau}_{r',r,l} \left(\sum_{f'\in F_r} \sum_{i\in I} \tilde{q}_{i,f',r',l} - \sum_{g\in F_r} \sum_{i\in I} \tilde{q}_{i,g,r,l} - \tilde{x}_{r',r}^{\max} \right) \\
& - \frac{1}{1+\beta} \sum_{k\in K} \tilde{\kappa}_k \left(\sum_{l\in L} h_l \sum_{r'\in R} \sum_{i\in I} \sum_{f''\in F} \sigma_{i,k} \tilde{q}_{i,f'',r',l} - \tilde{E}_k \right) \\
& - \frac{1}{1+\beta} \sum_{i\in I} \tilde{\varphi}_i \left(\sum_{f\in F} q_{i,f}^{\text{new}} + (1-\delta_i) \sum_{f\in F} q_{i,j}^{\max} - q_i^{\max\text{abs}} \right).
\end{aligned} \tag{6.16}$$

The derivative of (6.16) with respect to production leads to the following first-order condition for the next period, which is equivalent to (6.10) in the static case:

$$\begin{aligned}
& 0 \leq \tilde{q}_{i,f,r',l} \left(c_{i,r} + \tilde{\mu}_{i,f,l} + \tilde{\tau}_{r',r,l} - \sum_{k\in K} \tilde{\kappa}_k \sigma_{i,r}^k - (1-\lambda_{r'})\, \tilde{p}_{r',l} \left[1 - \frac{\tilde{\pi}_{f,r',l}}{\varepsilon_{r',l}} \right] \right), \\
& 0 < \tilde{q}_{i,f,r',l}.
\end{aligned} \tag{6.17}$$

The derivative of (6.16) with respect to the investment in new capacity, leads to the following additional first-order condition:

$$\begin{aligned}
q_{i,f}^{\text{new}} \left(\sum_{l\in L} h_l\, \tilde{\mu}_{i,f,l} - \beta V_i - (1+\beta)\, \tilde{\varphi}_i \right) & \geq 0, \\
q_{i,f}^{\text{new}} & > 0.
\end{aligned} \tag{6.18}$$

Equation (6.18) shows that firm f makes an investment $\left(q_{i,f}^{\text{new}}\right)_{i \in I}$ as long as the marginal return on capital $\sum_{l \in L} h_l \tilde{\mu}_{i,f,l}$ exceeds the marginal cost of capital $\beta V_i + (1+\beta)\tilde{\varphi}_i$.

From an economical point of view, the value of the shadow price of the capacity usage $\tilde{\mu}_{i,f,l}$ provides the signal to what extent a firm would like to use a particular technology during a particular load period.

The amount of production capacity is no longer constant in the dynamic model and equation (6.6) can be rewritten in terms of the decision variables as follows:

$$\tilde{\mu}_{i,f,l} \left(\sum_{r' \in R} \tilde{q}_{i,f,r',l} - (1-\delta_i)\, q_{i,f}^{\max} - q_{i,f}^{\text{new}} \right) \geq 0, \quad \tilde{\mu}_{i,f,l} > 0. \tag{6.19}$$

6.5 Conclusions and future research

Conclusions

We have proposed a model of the liberalized European electricity market, consisting of eight European countries. In the model emission limitations can be set as well as maximal transmission capacities between the neighboring countries. The aim has been to see how different the electricity prices will be in a monopolistic, a duopolistic, and a perfectly competitive situation.

Although the considered model is rather simple, some interesting phenomena can be observed:

- The electricity prices become lower when cross-border electricity transmissions are allowed.
- In the monopolistic and the duopolistic situation the electricity prices are higher than in the situation with perfectly competitive market.
- Generally in the perfect competition the producers tend to use cheaper and non-environmentally friendly means of electricity production. The emission restrictions are needed to motivate the electricity producers to act more ecologically. This increases the electricity prices, though, especially in the countries with a low number of hydro and wind power plants.

The extended variant of the model can be used for monitoring and predicting the behavior of the European electricity market.

Model limitations and future research

The major limitations of the model can be listed as follows:

- Only three possible games were considered in each of the case studies: perfect competition, Stackelberg game with one leader, Stackelberg game with two leaders. Although the aim of liberalization is to obtain a highly competitive market, it will never be perfectly competitive. Situations with noncooperative electricity producers, in non-perfect competition have to be considered to obtain more realistic outcome.

- Only eight countries were included in the model.
- Cross-border ownerships of the electricity producers are not allowed in the model, while in reality they appear more and more often.
- The electricity price is assumed to be constant within one country, while in reality this price might differ per the electricity producer [83].
- The model is very simple, while some of the data used for the modeling are real, the assumptions on the players' behavior are very strong.

These limitations will be resolved in our future research.

Practical relevance of the outcomes of the research presented in this chapter

The liberalization of the European electricity market draws the attention of many researchers. There are numerous attempts to model the current situation in order to predict the possible consequences of liberalization.

In this chapter we have tried to model the electricity market in eight European countries. The model is quite simple and does not take into account all factors that can influence the liberalization process throughout Europe. Still, outcomes of our case studies coincide with practical observations. Although the resulting prices of our case studies are remarkably lower than in the current European electricity market,[3] their structure of prices coincides with other relevant studies in the same field [29, 42, 56]. The dynamic model, describing the current situation in a more realistic way, is being developed.

[3] We assume that there are some other factors, not included into our modeling, influencing the electricity price.

Chapter 7

Theory of Incentives

In this chapter we will introduce some problems from the so-called *theory of incentives* and view them as specific problems of the inverse Stackelberg type.

7.1 Introduction

Incentive theory emerges with the division of labor and exchange. The division of labor induces the need for delegation. The first contracts probably appeared in an agricultural setting, when landlords contracted their tenants. Adam Smith [72] recognized the contractual nature of the relationship between the masters and the workers. He recognized the conflicting interests of those two kinds of players and recognized that bargaining power was not evenly distributed among them; the masters generally had all the bargaining power. Smith also stressed the agent's participation constraint, which limits what the principal can ask from the agent. Although Smith did not have a vision of economic actors as long-time maximizers of utility, his work was important as a headstone of incentive theory, since he discussed the problems associated with price-rate contracts in the industry.

Barnard [8] is the one who can be credited with the first attempt to define a general theory of incentives in management. Even much earlier, Hume [44] wrote the first explicit statement of the so-called free-rider problem. With the beginning of the theory of voting, the issue of strategic voting as a principal-agents problem was noticed [23]. The first attempt to address the issue of strategic voting can be found in [12].

The notion of moral hazard, i.e., the ability of insured agents to affect the probabilities of insured events, was well-known in the insurance profession [26, 34].

In [57] the regulation literature was put in the framework of the principal-agent literature with adverse selection by stressing the lack of information of the regulator. The problem was transformed into the second-best problem by weighting the firm's profit with a smaller weight than consumers' surplus in the social welfare function maximized by the regulated firm [9]. In [52] the model featuring both adverse selection and moral hazard was introduced. The ex-post observability of cost made the model technically an adverse selection model, though.

7.2 Preliminaries

Principal-agent models fall within the economic theory of incentives or contracts [51, 59, 70], which forms a subset of the one-leader-one-follower inverse Stackelberg games introduced in Chapter 3.

Let us consider a bilateral relationship in which a *principal* $\mathcal{P}$ contracts an *agent* $\mathcal{A}$ to delegate the production of some good.[1] Of course, the principal has to pay the agent for the good. The salary which the principal offers to the agent for the production of $q \in \mathbb{R}_+$ products will be $t \in \mathbb{R}_+$ [euro]. The variables q and t will be called *quantity* and *transfer*, respectively. The principal draws up a *contract* in which he specifies q and t. We call this contract the (q,t)-contract. We assume that it is always the principal who draws up the contract and then presents it to the agent, who, after having studied the terms of the contract, must decide whether or not to sign it.[2] We talk about a *take-it-or-leave-it* contract, since its terms are non-negotiable.

The agent's efficiency in producing the good is determined by how much money he/she needs to produce one product. We will denote this value by θ, $\theta \in \Theta \subset \mathbb{R}_+$, and call it the agent's *marginal cost*. If $\mathcal{A}$ has the marginal cost $\theta \in \Theta$, we refer to him as an agent of type θ, or as a θ-agent. The principal does not always know the value of θ, but he does know the set Θ. The agent may pretend to be an agent of a different type. We call the agent's pretending to be an agent of a different type *mimicking*. The θ-agent announces that he is of the type $u_{\mathrm{A}} \in \mathcal{D}_{\mathrm{A}} \subset \mathbb{R}_+$. We assume $\Theta = \mathcal{D}_{\mathrm{A}}$. Then his utility ("surplus value") from signing the (q,t) - contract is

$$\mathcal{J}_{\mathrm{A}}\left(q\left(u_{\mathrm{A}}\right), t\left(u_{\mathrm{A}}\right), \theta\right) = t\left(u_{\mathrm{A}}\right) - \mathcal{C}_{\mathrm{A}}\left(q\left(u_{\mathrm{A}}\right), \theta\right) = t\left(u_{\mathrm{A}}\right) - \theta q\left(u_{\mathrm{A}}\right). \tag{7.1}$$

Here we assume that both $t = t(u_A)$ and $q = q(u_A)$ are dependent only on the agent's "announced type". Another possibility is to consider t dependent on q (see [51]).

The agent will not sign the contract if $\mathcal{J}_{\mathrm{A}}\left(q,t,\theta\right) \leq 0$. If $\mathcal{A}$ signs the contract, he will produce the demanded number of goods.

The principal's utility function is

$$\mathcal{J}_{\mathrm{P}}\left(q,t\right) = \mathcal{C}_{\mathrm{P}}(q) - t, \tag{7.2}$$

where $\mathcal{C}_{\mathrm{P}}(q)$ describes the principal's value of q products. This function is assumed to satisfy the following natural properties:

$$\frac{d\,\mathcal{C}_{\mathrm{P}}}{d\,q} > 0, \quad \frac{d^2\,\mathcal{C}_{\mathrm{P}}}{d\,q^2} < 0, \quad \mathcal{C}_{\mathrm{P}}(0) = 0.$$

The marginal value of the good for the principal is, thus, positive and strictly decreasing with respect to q on $\mathbb{R}_+$.

The situation in which both $\mathcal{P}$ and $\mathcal{A}$ know θ before $\mathcal{P}$ offers the contract is known as a situation with *complete information* (to be studied in Section 7.3), while the situation in which only the agent knows his own type before the contract is designed is called a

[1] We confine the agent to produce good only, although the formulation used in this chapter has also more general interpretation.

[2] We do not permit $\mathcal{A}$ to make a counter-offer to $\mathcal{P}$, a situation which is known as *bilateral bargaining* [55], [82].

situation with *adverse selection* (Section 7.4). The situation in which the agent can perform some unobserved actions after the contract is signed is known as *moral hazard* (see e.g. [59]). In this paper we focus mainly on the *adverse-selection-principal-agent model*.

It is usually assumed [51, 59] that $\mathcal{P}$ chooses among the two possible strategies:

- The principal will offer the contract to $\mathcal{A}$, no matter how efficient $\mathcal{A}$ is (*contract without shutdown*).
- The principal will contract $\mathcal{A}$ only if $\mathcal{A}$'s marginal cost is higher than some certain value (*contract with shutdown*), otherwise no contract will be offered.

We will consider only contracts without shutdown.

7.3 Complete-information principal-agent model

Let us first assume that the agent type is from the discrete type set $\Theta \stackrel{\text{def}}{=} \left[\underline{\theta}, \overline{\theta}\right]$. In this model the principal knows the agent's type $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$, $\underline{\theta} < \overline{\theta}$, $\underline{\theta}, \overline{\theta} \in \mathbb{R}_+$; hence, he can set up a contract slightly exceeding the agent's zero utility level and ensuring the highest possible utility for himself. The optimal q and t will be called the *first best* quantity and transfer and will be denoted by an asterisk.

The principal maximizes

$$\mathcal{J}_{\mathrm{P}}\left(q(\theta), t(\theta)\right) = \mathcal{C}_{\mathrm{P}}\left(q\left(\theta\right)\right) - t\left(\theta\right)$$

subject to the agent's participation constraint

$$\mathcal{J}_{\mathrm{A}}\left(q(\theta), t(\theta), \theta\right) = t(\theta) - q(\theta)\theta > 0.$$

The principal's optimal strategy is then $u^*_{\mathrm{P}}(\theta) \stackrel{\text{def}}{=} \left(q^*(\theta), t^*(\theta)\right)$, where (with $\varepsilon \downarrow 0$)

$$t^*(\theta) = q(\theta)\theta + \varepsilon, \quad q^*(\theta) = \arg\max_{q(\theta)} \mathcal{C}_{\mathrm{P}}(q(\theta)) - q\left(\theta\right)\theta - \varepsilon, \tag{7.3}$$

with $u_{\mathrm{A}} = \theta$, $\theta \in \Theta$. Thus,

$$\frac{d}{dq} C_p(q^*) = \theta. \tag{7.4}$$

The agent will accept $\mathcal{P}$'s offer and gain $\varepsilon \downarrow 0$ utility.

Remark 7.1 In the following example we will assume that $\mathcal{A}$ is of type θ from the two-element set $\Theta = \{\underline{\theta}, \overline{\theta}\}$, $\overline{\theta} > \underline{\theta} > 0$. If

$$\theta = \begin{cases} \underline{\theta} & \text{we say, that } \mathcal{A} \text{ is } \textit{efficient}; \\ \overline{\theta} & \text{we say, that } \mathcal{A} \text{ is } \textit{inefficient}. \end{cases}$$

We will denote the transfer and the quantity offered to the $\underline{\theta}$-agent by $\underline{t} = t(\underline{\theta})$, $\underline{q} = t(\underline{\theta})$, and to the $\overline{\theta}$-agent by $\overline{t} = t(\overline{\theta})$, $\overline{q} = t(\overline{\theta})$. We will refer to the contract offered to the $\underline{\theta}$-agent as the $(\underline{q}, \underline{t})$-contract and to the contract offered to the $\overline{\theta}$-agent as the $(\overline{q}, \overline{t})$-contract.

□

Example 7.1
The employer of a factory ($\mathcal{P}$) delegates to a worker ($\mathcal{A}$) to make a certain number of products. The principal's objective function is given as $\mathcal{C}_\mathrm{P}(q(\theta)) \stackrel{\mathrm{def}}{=} \ln(q(\theta)+1)$. The agent can be only of the $\underline{\theta}$ or $\overline{\theta}$ type, where $\underline{\theta} = 0.1$ [euro], $\overline{\theta} = 0.2$ [euro]. The principal maximizes his utility function

$$\begin{aligned}\mathcal{J}_\mathrm{P}(q(\theta),t(\theta)) &= \mathcal{C}_\mathrm{P}(q(\theta)) - t(\theta)\\ &= \ln(q(\theta)+1) - t(q(\theta))\end{aligned}$$

subject to $t(\theta) - \theta q(\theta) > 0$, where $\theta \in \{\underline{\theta},\overline{\theta}\}$. From (7.3) it follows that the principal offers $t^*(\theta)$ satisfying $t^*(\theta) = \theta q^*(\theta) + \varepsilon$, where $\varepsilon \downarrow 0$, and $q^*(\theta)$ satisfies

$$q^*(\theta) = \frac{1-\theta}{\theta}.$$

Hence, $\mathcal{P}$ demands $q^*(\theta) = 9$ products for $t^*(\theta) = (0.9+\varepsilon)$ [euro], if $\theta = \underline{\theta} = 0.1$ [euro], and $\mathcal{P}$ demands $q^*(\theta) = 4$ products for $t^*(\theta) = (0.8+\varepsilon)$ [euro], if $\theta = \overline{\theta} = 0.2$ [euro]. The agent's profit is always $\varepsilon \downarrow 0$ and $\mathcal{P}$'s profit is $\ln(10) - 0.9 - \varepsilon \approx (1.4-\varepsilon)$ [euro] if $\mathcal{A}$ is efficient, and $\ln(5) - 0.9 - \varepsilon \approx (0.71-\varepsilon)$ [euro] if $\mathcal{A}$ is inefficient. □ □

7.4 Adverse-selection principal-agent model

Under the adverse selection the principal is not aware of the agent's type $\theta \in \Theta = [\underline{\theta},\overline{\theta}]$ before writing the contract, but he/she does know Θ.

The following example shows the more specific situation with $\Theta = \{\underline{\theta},\overline{\theta}\}$.

Example 7.2
Let us assume that the principal from Example 7.1 does not know the agent's type (while knowing both the $\underline{\theta}$ and $\overline{\theta}$ values.). He designs the pair of contracts

$$\left\{\left(\frac{1-\underline{\theta}}{\underline{\theta}}, 1-\underline{\theta}+\varepsilon\right),\ \left(\frac{1-\overline{\theta}}{\overline{\theta}}, 1-\overline{\theta}+\varepsilon\right)\right\}$$

with $\underline{\theta} = 0.2$ [euro], $\overline{\theta} = 0.1$ [euro], hoping that each agent will pick the contract matching his type. If $\mathcal{A}$ is efficient, it pays for him to pretend to be an inefficient agent to obtain utility

$$\mathcal{J}_\mathrm{A}(\underline{\theta},\overline{q},\overline{t}) = t(\overline{\theta}) - \overline{q}\underline{\theta} = \left(1-\overline{\theta}+\varepsilon\right) - \frac{1-\overline{\theta}}{\overline{\theta}}\underline{\theta} = (0.4+\varepsilon) \text{ [euro]},$$

while the $\underline{\theta}$-agent's utility without mimicking is

$$\mathcal{J}_\mathrm{A}\left(\overline{\theta},\overline{q},\overline{t}\right) = \overline{t} - \overline{q}\overline{\theta} = \left(1-\overline{\theta}+\varepsilon\right) - \frac{1-\overline{\theta}}{\overline{\theta}}\overline{\theta} = \varepsilon \text{ [euro]}.$$

If the agent is inefficient, it does not pay to him to mimic the efficient agent, because

$$\begin{aligned}
\mathcal{J}_{\mathrm{A}}\left(\overline{\theta}, q(\overline{\theta}), t(\overline{\theta})\right) &= t(\overline{\theta}) - \overline{q}\overline{\theta} \\
&= \left(1 - \overline{\theta} + \varepsilon\right) - \frac{1-\overline{\theta}}{\overline{\theta}}\overline{\theta} = \varepsilon \textit{ [euro]}, \\
\mathcal{J}_{\mathrm{A}}\left(\overline{\theta}, q(\underline{\theta}), t(\underline{\theta})\right) &= t(\underline{\theta}) - \underline{q}\overline{\theta} \\
&= (1 - \underline{\theta} + \varepsilon) - \frac{1-\underline{\theta}}{\underline{\theta}}\overline{\theta} = (\varepsilon - 0.9) \textit{ [euro]}.
\end{aligned}$$

□

When $\Theta = [\underline{\theta}, \overline{\theta}]$, the principal has an *a priori* belief about the agent's type. This belief is embodied in the probability distribution f with cumulative distribution function F on Θ, which will be called a principal *prior*.

The principal offers the contract variables as mappings from Θ.

The principal offers the $(q(\cdot), t(\cdot))$-contract where $q(\cdot) : \Theta \to \mathbb{R}_+$, $t(\cdot) : \Theta \to \mathbb{R}_+$, hoping that every $\hat{\theta}-$agent ($\hat{\theta} \in \Theta$) will choose the $\left(q\left(\hat{\theta}\right), t\left(\hat{\theta}\right)\right)$-contract. Thus, q and t become functions of the agent's possible type space. These functions are known before the agent announces his type. The mechanism of announcing transfer and quantity as functions from an agent's decision space before the contract is signed is called a *direct revelation mechanism* [51].

Definition 7.1
A direct revelation mechanism is a mapping $\gamma_{\mathrm{P}}(\cdot) : \Theta \to \mathcal{D}_{\mathrm{P}}$, where $\gamma_{\mathrm{P}} = (q(\cdot), t(\cdot))$ for $\forall \theta \in \Theta$. The principal commits to offering the transfer $t(\widehat{\theta})$ and the production level $q(\widehat{\theta})$ if the agent announces $\widehat{\theta} \in \Theta$. □

For the sake of simplicity, we assume that $q(\cdot)$, $t(\cdot)$ are differentiable with respect to each $\theta \in \Theta$.

The direct revelation mechanism is said to be *truthful*, if an agent of any type from Θ does not wish to mimic an agent of a different type.

Definition 7.2
A revelation mechanism $\gamma_{\mathrm{P}}(\cdot)$ is truthful if it satisfies for every $\tilde{\theta}$, $\hat{\theta}$ from Θ, $\tilde{\theta} < \hat{\theta}$ the following incentive compatibility constraints

$$t\left(\tilde{\theta}\right) - \tilde{\theta} q\left(\tilde{\theta}\right) \geq t\left(\hat{\theta}\right) - \tilde{\theta} q\left(\hat{\theta}\right), \tag{7.5}$$

$$t\left(\hat{\theta}\right) - \hat{\theta} q\left(\hat{\theta}\right) \geq t\left(\tilde{\theta}\right) - \hat{\theta} q\left(\tilde{\theta}\right), \tag{7.6}$$

respectively. □

By adding (7.5) and (7.6) we obtain

$$\left(\hat{\theta} - \tilde{\theta}\right)\left(q(\tilde{\theta}) - q(\hat{\theta})\right) \geq 0 \tag{7.7}$$

for all $\left(\hat{\theta}, \tilde{\theta}\right) \in \Theta \times \Theta$. Because (7.7) holds for all $\tilde{\theta}$, $\hat{\theta} \in \Theta$, also

$$\frac{dq(\theta)}{d\theta} \leq 0. \tag{7.8}$$

Therefore, if the direct revelation mechanism is truthful, $q(\cdot)$ is non-increasing.

Inequality (7.5) implies that the following first-order condition for the optimal response $u_{\rm A}$ chosen by type θ is satisfied:

$$\frac{dt}{d\theta}(u_{\rm A}) - \theta \frac{dq}{d\theta}(u_{\rm A}) = 0 \tag{7.9}$$

To avoid agents' mimicking, the following equality has to be satisfied for all $\hat{\theta} \in \Theta$:

$$\frac{dt}{d\theta}(\hat{\theta}) - \hat{\theta} \frac{dq}{d\theta}(\hat{\theta}) = 0. \tag{7.10}$$

The local second-order condition

$$\frac{d^2 t}{d\theta^2}(\hat{\theta}) - \hat{\theta} \frac{d^2 q}{d\theta^2}(\hat{\theta}) \leq 0 \tag{7.11}$$

has to be satisfied as well. By differentiating (7.10), condition (7.11) can be rewritten in a simpler way as

$$-\frac{dq}{d\theta}(\hat{\theta}) \geq 0. \tag{7.12}$$

If (7.10) and (7.12) are satisfied, the θ-agent does not want to mimic an agent of an other type locally. To prevent the θ-agent from global mimicking, too, the following constraints must be fulfilled:

$$t(\theta) - \theta q(\theta) \geq t(u_{\rm A}) - \theta q(u_{\rm A}) \tag{7.13}$$

for all $(\theta, u_{\rm A}) \in \Theta \times \Theta$.

By integrating formula (7.10) we obtain

$$t(\theta) - u_{\rm A}\, q(u_{\rm A}) = t(u_{\rm A}) - \theta\, q(u_{\rm A}) + (\theta - u_{\rm A})\, q(u_{\rm A}) - \int_{u_{\rm A}}^{\theta} q(\tau) d\tau, \tag{7.14}$$

where $(\theta - u_{\rm A})\, q(u_{\rm A}) - \int_{u_{\rm A}}^{\theta} q(\tau) d\tau \geq 0$, because $q(\cdot)$ is non-increasing.

Thus, (7.9) can be extended globally. Truthful revelation mechanisms are then characterized by the two conditions (7.10) and (7.12).

We now introduce the concept of *information rents*. Under complete information introduced in Section 7.3 the principal is able to maintain all types of agents at their $\varepsilon-$utility level,

$$\mathcal{J}_{\rm A}(q^*(\theta), t^*(\theta), \theta) = t^*(\theta) - \theta q^*(\theta) = \varepsilon.$$

Under incomplete information this will be not possible anymore, at least when the principal wants all types of agents to sign the contract offered. Let us take the revelation mechanism $\gamma_{\rm P}(\cdot) = (q(\cdot), t(\cdot))$ and consider the utility that the θ-agent gains by mimicking a $u_{\rm A}$-agent, $u_{\rm A} > \theta$ (with $D = u_{\rm A} - \theta$):

$$\mathcal{J}(q(u_{\rm A}), t(u_{\rm A}), \theta) = t(u_{\rm A}) - \theta \cdot q(u_{\rm A}) = t(u_{\rm A}) - u_{\rm A}\, q(u_{\rm A}) + Dq(u_{\rm A}) \tag{7.15}$$

$$= \mathcal{J}_{\rm A}(q(u_{\rm A}), t(u_{\rm A}), u_A) + Dq(u_{\rm A}). \tag{7.16}$$

Even if the $u_{\rm A}$-agent's utility is reduced to its lowest value ε, the θ-agent benefits from an information rent $Dq(u_{\rm A})$ coming from his ability to possibly mimic a less efficient agent.

So as long as the principal insists on a positive quantity from the $\overline{\theta}$-agent, the principal must give up a possible rent to any other type of agent. The information rent is generated by the information advantage of the agent over the principal. The principal's problem is to determine the smartest way to give up the rent provided by any given incentive feasible contract. We will use the following notation: The information rents for a θ-agent will be denoted by $\mathcal{I}_\theta$, i.e., $\mathcal{I}_\theta \stackrel{\text{def}}{=} \mathcal{J}_{\text{A}}(q(\theta), t(\theta), \theta) = t(\theta) - \theta q(\theta)$.

The optimal solution of the adverse-selection-principal-agent model will be called the *second-best solution* (as opposed to the optimal solution in the situation with complete information, which is often called the *first-best solution*). This second-best solution will be denoted by SB. From (7.15) it follows that the optimal strategy for any θ-agent is to play $u_{\text{A}}^{SB} = \underline{\theta}$. The principal is aware of this.

With the use of (7.13), the local incentive constraint can be written as

$$\frac{d\,\mathcal{I}_\theta}{d\theta} = -q(\theta) + \frac{dt}{d\theta}(\theta) - \theta\frac{dq}{d\theta}(\theta) = -q(\theta)\,. \tag{7.17}$$

Thus, the principal's problem becomes

$$\max_{\{q(\cdot),t(\cdot)\}} \int_{\underline{\theta}}^{\overline{\theta}} \left(C_{\mathcal{P}}\left(q(\theta)\right) - t(\theta)\right) f(\theta)\, d\theta, \tag{7.18}$$

subject to

$$\frac{d\,\mathcal{I}_\theta}{d\theta} = -q(\theta)\,, \tag{7.19}$$

$$\frac{dq}{d\theta}(\theta) \leq 0, \tag{7.20}$$

$$\mathcal{I}_\theta > 0 \quad \text{for} \quad \forall\theta \in \Theta. \tag{7.21}$$

Equation (7.18) can also be rewritten as

$$\max_{\{q(\cdot),\mathcal{I}_{(\cdot)}\}} \int_{\underline{\theta}}^{\overline{\theta}} \left(C_{\mathcal{P}}\left(q(\theta)\right) - \theta q(\theta) - \mathcal{I}_\theta\right) f(\theta)\, d\theta \tag{7.22}$$

with the use of information rent $\mathcal{I}_\theta$.

With the use of (7.17), the participation constraint (7.21) simplifies to $\mathcal{I}_{\overline{\theta}} > 0$. Clearly the $\overline{\theta}$-agent will obtain $\mathcal{J}\left(q(\overline{\theta}), t(\overline{\theta}), \overline{\theta}\right) = \mathcal{I}_{\overline{\theta}} = \varepsilon$. For the sake of simplicity, we will not consider the constraint (7.20) now and check if this constraint is satisfied after finding the optimal strategy for the principal.

Equation (7.19) can be rewritten as follows:

$$\mathcal{I}_{\overline{\theta}} - \mathcal{I}_\theta = -\int_{\theta}^{\overline{\theta}} q(\tau)d\tau, \tag{7.23}$$

i.e. (with $\mathcal{I}_{\overline{\theta}} = \varepsilon$),

$$\mathcal{I}_\theta = \int_{\theta}^{\overline{\theta}} q(\tau)d\tau + \varepsilon. \tag{7.24}$$

The principal's objective function becomes

$$\int_{\underline{\theta}}^{\overline{\theta}} \left(\mathcal{C}_{\mathrm{P}}\left(q\left(\theta\right)\right) - \theta q\left(\theta\right) - \int_{\theta}^{\overline{\theta}} q\left(\tau\right) d\tau - \varepsilon \right) f\left(\theta\right) d\theta \tag{7.25}$$

$$= \int_{\underline{\theta}}^{\overline{\theta}} \left(\mathcal{C}_{\mathrm{P}}\left(q\left(\theta\right)\right) - \left(\theta + \frac{F\left(\theta\right)}{f\left(\theta\right)} \right) q\left(\theta\right) - \varepsilon \right) f\left(\theta\right) d\theta. \tag{7.26}$$

Point-wise minimization of (7.26) leads to the optimal solution for the principal $q^{SB}(\cdot)$:

$$\frac{d\,\mathcal{C}_{\mathcal{P}}}{d\,q}\left(q^{SB}(\theta)\right) = \theta + \frac{F\left(\theta\right)}{f\left(\theta\right)}. \tag{7.27}$$

All the agents' types obtain a positive utility by playing $u_{\mathrm{A}}^{SB} = \overline{\theta}$:

$$\mathcal{J}_{\mathrm{A}}\left(q(\overline{\theta}), t(\overline{\theta}), \theta\right) = \int_{\theta}^{\overline{\theta}} q^{SB}(\tau) d\tau + \varepsilon. \tag{7.28}$$

If the so-called *monotone hazard property*

$$\frac{d}{d\theta}\left(\frac{F(\theta)}{f(\theta)}\right) \geq 0$$

holds for all $\theta \in \Theta$, the solution $q^{SB}(\cdot)$ satisfying (7.28) will be decreasing, and the omitted constraint (7.20) is satisfied. The monotone hazard property is satisfied for most single-peak densities [7].

7.5 Conclusions and future research

We have proposed to view the adverse-selection-principal-agent model as a special case of a one-leader-one-follower static inverse Stackelberg game. Starting from the complete-information-principal-agent model we showed that only the least efficient type of agent will gain the same profit (ε [euro]) whether signing a complete-information contract or an adverse-selection contract. Agent of any other type will be better signing an adverse-selection contract. Dynamic contracts are a subject for future research.

Chapter 8

Conclusions and Future Research

This chapter summarizes the research proposed and developed throughout this thesis. Its scope and main contributions to the current state-of-art in game theory, traffic control, electricity market theory, and theory of incentives are briefly discussed in Section 8.1. The future research possibilities and directions are discussed in Section 8.2.

8.1 Contributions to the state-of-the-art

Game theory is a widely used and investigated field. Although this field has been extensively studied and in recent years the focus has been directed more towards game theoretic applications than towards fundamental research, there are still game theoretic areas that have not been studied in a sufficient detail and, therefore, almost no theoretical results in these areas are known. One of such fields is the field of the so-called inverse Stackelberg games. In Chapter 3 of this thesis these games were defined and their properties were studied. Applications of the Stackelberg and the inverse Stackelberg games in the static optimal toll design problem, the dynamic optimal toll design problem, electricity market liberalization problem, and the theory of incentives (contracts) were studied in Chapters 4, 5, 6, and 7.

Our contributions with respect to the state-of-the-art in the main topics covered in this thesis are the following:

- **Conducted research in the field of game theory**
  We recapitulated some classical results from the field of game theory. We introduced the so-called inverse Stackelberg games, with clear focus on one-leader-one-follower and one-leader-more-followers problems. We showed a way of how to find an optimal strategy for the leader and presented situations in which the optimal strategy of the leader
  - exists and is unique;
  - exists and is nonunique;
  - does not exist;
  - is generally unknown.

Both static and dynamic problems were studied. Some general statements about properties of the inverse Stackelberg games were made. Also some unresolved issues were mentioned.

- **Conducted research in the field of the optimal toll design**
We proposed the concept of the traffic-flow-dependent toll in the context of the optimal toll design problem and in various case studies we dealt with finding an optimal strategy for the road authority minimizing his/her objective function, while the drivers minimized their own (perceived) travel costs. Although the extensively studied first-best optimal toll design problem is clearly an inverse Stackelberg game[1], in the field of the second-best optimal toll design the concept of the traffic-flow-dependent toll was not introduced before. The drivers in the optimal toll design problem act as one super-player with traffic flows on alternative routes (where tolls are imposed by the road authority) being his/her decision variables. We were dealing with both static and dynamic variants of the optimal toll design problem.

  We considered two different situations according to the information that the drivers have:

  - The situation with complete information in which the drivers are aware of all traffic conditions and minimize their *actual travel costs*. In equilibrium state the deterministic user equilibrium (DUE) applies.
  - The situation with incomplete information, in which the drivers are not aware of all traffic conditions and minimize their *actual perceived travel costs*. In equilibrium state some stochastic user equilibrium (SUE) applies. As an example of such an equilibrium the logit-based stochastic equilibrium (LB-SUE) was used.

  The deterministic user equilibrium is a limiting case of the logit-based stochastic equilibrium when the so-called perception error tends to infinity. With the deterministic equilibrium the optimal toll design problem is analytically solvable, unlike in the case with the more general LB-SUE.

  Also, since DUE is a limiting case of LB-SUE, the algorithm that we have proposed for solving the optimal toll design problem with the second-best traffic-flow dependent toll and the drivers driven by LB-SUE can be used for solving the optimal toll design problem with drivers driven by DUE, too. This algorithm uses neural networks simulation and belongs to advanced heuristic methods, which can be efficiently used for solving NP-hard problems. The optimal toll design problem belongs to the class of this type of problems.

  We have shown that the use of the traffic-flow dependent toll may improve the system performance remarkably, while the traffic flow-dependent toll can never yield a worse outcome than the traffic-flow invariant toll. The choice of tolled links influence the outcome of the game remarkably.

  Theorems about the existence of the solution for the general variant of the optimal toll design problem have been stated for both the static and the dynamic situations. Case studies of various network types were presented, too.

[1]So far the first-best optimal toll design problem has not been recognized as an inverse Stackelberg game, although it is a clear example of the game of this type.

- **Conducted research in energy market liberalization problem**
  We have proposed a model of the European electricity market. Eight countries were included in the model: Belgium, Denmark, Finland, France, Germany, The Netherlands, Norway, and Sweden. The model uses real data about electricity production prices, emission factors, and electricity consumption in individual countries. Different types of games, differing in the following criteria, have been considered:

  – form of the leadership (no leader, one leader - Stackelberg game with one leader, two leaders - Stackelberg game with two leaders);

  – type of the competition among the leaders and the followers (perfect vs. imperfect);

  – role of borders (game with the cross-border electricity transmissions allowed vs. game with no cross-border electricity transmissions);

  – role of emissions (emission constraints included or no emission constraints).

  As a result of the case studies we have drawn the following conclusions:

  – The electricity prices are the highest if one of the electricity producers acts as a leader, i.e., has a monopoly in his country. The prices decrease with two competing leaders and are the lowest when none of the electricity producers acts as the leader and perfect competition takes place.

  – While perfectly competitive electricity market increases the emission factors when emission restrictions are not imposed, a right choice of emission constraints may decrease emission factors, while the electricity prices do not increase that much.

  – The electricity prices decrease if the cross-border electricity transactions are allowed.

  The outcomes of our model coincide with the experiences in the real electricity market.

- **Conducted research in theory of incentives**
  The principal-agent problem from the economical theory of incentives has been identified as an example of the inverse Stackelberg games. Various problems of this type have been solved, with a focus on the optimal strategy for the principal as the leader and interesting phenomena. The only situation, in which the principal receives positive outcome no matter how efficient the agent that the principal is contracting is, is the situation with full information. While the principal-agent theory is a classical one, we have presented it as a special case of an inverse Stackelberg game.

8.2 Future research

In this section we will discuss possible future research directions for each of the main topics addressed in this thesis.

- **Future research in Game theory**
  While in Chapter 3 important phenomena in inverse Stackelberg games were introduced, mainly by means of specific examples, some general properties have not been studied yet. The following topics are interesting subjects for future research:
  - Existence of the solutions to general problem of the inverse Stackelberg type.
  - An inverse Stackelberg game with a higher number of players.
  - Inverse Stackelberg game with leaders or followers being cooperative among themselves.
  - The problems with incomplete information.
- **Future research in Optimal toll design problem**
  The problems, that should be addressed in the future research, are:
  - The optimal toll design problem with traffic-flow-dependent toll with elastic demand.
  - The problems with heterogeneous user classes of the drivers.
  - The problems with the traffic-flow dependent tolls that are not polynomial functions of the traffic flows.
  - The optimal toll design problem with the drivers driven by different user equilibrium than LB-SUE. There exist more realistic models of the travelers' behavior, where the travelers are driven by the equilibria that are extensions of the LB-SUE. Also probit-based models can be used. The problem becomes difficult to solve in this case, though. We expect that also in this situation the problem the traffic-flow dependent toll brings better outcomes for the road authority.
  - The dynamic optimal toll design problem with the travelers' departure time choice.
- **Future research in Energy market liberalization problem**
  The issues that deserve future research are:
  - Dynamic model - although a possible extension of the current model to the situation with more time steps (discrete dynamic model) was discussed, case studies performed were more of academic nature, while computations with "real-size" models were not performed.
  - Incorporating more countries into the model. To be able to see the influence of the liberalization process throughout the Europe, all European countries have to be included.
  - Model with elastic electricity demand. Although this option was briefly studied when the extension of the model to the dynamic problem was discussed.
  - Game with the electricity consumers being incorporated into the model. In this case the electricity producers can be leaders in a Stackelberg or in an inverse Stackelberg game and the electricity consumers can be the followers.

  - Including net region-specific electricity losses per country and also including net losses of both the countries when electricity cross-border transmission takes place.

- **Future research in Theory of incentives**
  Important subjects for future research are:

  - Dynamic contracts. In Chapter 7 the clear emphasis was on static contracts, while the extension to the dynamic version of the problem was mentioned quite briefly.
  - Problems with moral hazard, problems combining moral hazard with adverse selection.
  - Problems with more principals and/or agents.

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Samenvatting

Over Stackelberg- en inverse Stackelbergspellen en hun toepassing in het ontwerpen van optimale tollen, de liberalisering van energiemarkten en in de theorie van aansporingen

Inverse Stackelbergspellen zijn het onderwerp geworden van recent onderzoek in speltheorie. Tot nu toe stond de theorie voor dit soort spellen slechts in de kinderschoenen en was er dus zeer weinig bekend over inverse Stackelbergspellen. In dit proefschrift wordt ingegaan op het theoretisch oplossen van zulke problemen en wordt een aantal zeer uitdagende problemen uit een variëteit aan domeinen in het raamwerk van inverse Stackelbergspellen geplaatst en opgeslost.

In Stackelbergspellen bepaalt een zogenaamde *leider* acties voor één of meer zogenaamde *volgers*. In het algemeen is het vinden van een optimale strategie voor een leider in deze spellen extreem moeilijk; in veel gevallig zelfs onmogelijk. Beginnend met eenvoudige statische problemen en daarna verdergaand met meer moeilijke dynamische problemen, wordt in dit proefschrift aangetoond hoe de optimale strategie voor een leider op een heuristische manier gevonden kan worden.

In dit proefschrift wordt de toepassing van speltheorie in de volgende drie specifieke gebieden voorgesteld: bet bepalen van optimale tollen, liberalisering van de elektriciteitssector en de theorie van aansporingen.

Het ontwerpen van een optimale tol wordt in de proefschrift beschreven als een spel van het Stackelberg type. Een wegbeheerder representeert hierbij de leider en de weggebruikers representeren de volgers. De wegbeheerder bepaalt de tol voor een aantal van de wegen in een wegennetwerk. De wegbeheerder doet dit op een zodanige manier dat zijn doelfunctie wordt gemaximaliseerd, terwijl de weggebruikers hun beslissingen maken op een zodanige manier dat hun reiskosten worden geminimaliseerd. Als de tol die de wegbeheerder bepaalt niet afhankelijk is van de verkeersstroom, dan is het op te lossen probleem een klassiek Stackelbergspel. Als de tol wel afhankelijk is van de verkeersstroom, dan is het probleem een invers Stackelbergspel. In dit proefschrift wordt een optimale stroomafhankelijke tol voor de wegbeheerder gevonden, voor zowel statische als dynamische varianten van het tolontwerpprobleem. Als het oplossingsconcept voor de weggebruikers wordt bepaald met behulp van een zogenaamd deterministisch gebruikersequilibrium, dan kan het probleem analytisch worden aangepakt. Als het zogenaamde stochastische gebruikersequilibrium wordt gebruikt, dan moeten numerieke methoden worden gebruikt om een oplossing te vinden. Aangezien dit probleem NP-moeilijk is, stellen wij voor om een oplossingsaanpak te gebruiken die gebaseerd is op neurale netwerken. We vergelijken de uitkomsten van de spellen met stroomonafhankelijke tol en stroomafhankelijke tol. We concluderen dat de stroomafhankelijke tol de prestaties van het systeem significant kan verbeteren. Daarnaast worden interessante eigenschappen van dit probleem beschreven en bediscussieerd.

De liberalisering van de elektriciteitsmarkten is in dit proefschrift gedefinieerd als een competatief spel tussen elektriciteitsproducenten in 8 Europese landen. Elektriciteitsvraag wordt hierbij als gegeven beschouwd. De producenten maken een keuze uit de inzet van beschikbare middelen voor elektriciteitsproductie en de hoeveelheid te produceren elektriciteit op een zodanige manier dat hun winst wordt gemaximaliseerd. Verschillende spelscenario's worden beschouwd: Perfecte competitie, een spel met één leidende producent per land en een spel met twee leidende producenten per land (waarbij de leiders onderling een zogenaamde Nashstrategie gebruiken). De uitwisseling van elektriciteit tussen naburige landen is toegestaan en beperkingen op emissies worden meegenomen. Een numeriek model, gebruikmakend van realistische data, wordt voorgesteld om het probleem op te lossen. Onze resultaten suggeren dat de liberalisatie van de elektriciteitsmarkten tot een daling in de prijs voor elektriciteit kan leiden.

Ten slotte behandelen wij zogenaamde *principal-agent* modellen uit de theorie van aansporingen als een speciale groep van inverse Stackelbergspellen. In dit geval is de *principal* de leider an de *agent* de volger. De leider contracteert de volger met het doel om een bepaald aantal goederen te produceren. De mate van effectiviteit van de volger kan variëren. Deze effectiviteit is onbekend bij de leider. Het probleem van het vinden van een optimale strategie voor de leider wordt behandeld. Interessante fenomenen in dit spel worden gepresenteerd en een optimale strategie voor de leider wordt afgeleid.

Kateřina Staňková

Summary

On Stackelberg and Inverse Stackelberg Games and their Applications in the Optimal Toll Design Problem, the Energy Markets Liberalization Problem, and in the Theory of Incentives

Inverse (or reverse) Stackelberg games have become the subject of recent game theory research, as a special type or as an extension of Stackelberg games. So far, only very little theory about inverse Stackelberg games is available and the available theory is still in its infancy. In this thesis we focus on theoretically solving such problems and we propose to treat several challenging problems in various fields inside this framework.

In Stackelberg games a so-called *leader* determines actions for one or more so-called *followers*. The problem of finding an optimal strategy for the leader in these games is in general extremely hard to solve, and often even completely unsolvable. Starting from simple static problems and proceeding to more difficult dynamic ones, we show how to find the optimal strategy for the leader in a heuristic manner.

In this thesis, the application of game theory is proposed in the following domains: The optimal toll design problem, the electricity markets liberalization problem, and the theory of incentives.

The optimal toll design problem is a game of the Stackelberg type in which a road authority acts as the leader and drivers in the road network act as the followers. The road authority sets tolls on some of the links in the network in order to maximize its objective function, while the drivers make their travel decisions in order to minimize their perceived travel costs. If the toll that the road authority sets is traffic-flow invariant, the problem is the "classical" Stackelberg game; if the toll is traffic-flow dependent, the problem is of the inverse Stackelberg type. We determine the optimal traffic-flow dependent toll for the road authority for both static and dynamic variants of the problem. If the solution concept for the drivers' behavior is the deterministic user equilibrium, the problem can be dealt with analytically. If the stochastic user equilibrium applies, numerical methods have to be applied to find a solution. As the problem is NP-hard, we use a neural-networks based solution approach to solve the problem. We compare outcomes of the games with traffic-flow invariant and traffic-flow dependent toll and conclude that the traffic-flow dependent toll can improve the system performance remarkably. Interesting phenomena in this problem and its properties are discussed, too.

The electricity markets liberalization problem is defined in this thesis as a noncooperative game among electricity producers in eight European countries, in which the electricity demand is exogenous. The producers choose among available means of electricity productions and quantities to produce in order to maximize their profit. Different game scenarios are considered: Perfect competition, a game with one leading producer per each country, and a game with two leading producers, playing Nash among themselves, for each country. The transmission of electricity between neighboring countries is allowed and emission constraints are considered. A numerical model, using real data, is developed in order to solve the problem. Our results suggest that liberalization of electricity markets leads to electricity price decrease.

Finally, we deal with so-called principal-agent models from the theory of incentives as a specific group of inverse Stackelberg problems. Here the principal as a leader contracts an agent as a follower in order to produce certain goods. The agent can be of different efficiency, often unknown to the principal. The problem of finding the optimal strategy for the principal is dealt with. Interesting phenomena in this game are presented and an optimal strategy for the leader is derived.

Kateřina Staňková

Curriculum vitae

Kateřina Staňková was born on May 12, 1981 in Ostrava, Czech Republic. She finished her pre-university education in 1999 at Gymnasium Matiční, Ostrava, Czech Republic. After this, Kateřina Staňková started her studies in Applied Mathematics at VŠB-TU Ostrava, Ostrava, Czech Republic. She received the title of *inženýr* (comparable with Master of Science) in Applied mathematics from this university in 2004. For her graduation project, she performed research on active set-based algorithms for bound constrained quadratic programming. The research involved in this project was supervised by Prof. RNDr. Zdeněk Dostál, CSc.

Since 2005, Kateřina Staňková has been working on her PhD project at the Delft Institute of Applied Mathematics at Delft University of Technology, The Netherlands. The research of her PhD project has been on inverse Stackelberg games with applications to optimal toll design, electricity market liberalization, and theory of incentives, and has been supervised by Prof.dr. Geert Jan Olsder and dr. Michiel C.J. Bliemer.