Technische Universiteit Delft
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De rooster graafs Pfaffiaanse orientaties, vlak en toroïdaal

The lattice graph's Pfaffian orientations, planar and toroidal

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“De rooster graafs Pfaffiaanse orientaties, vlak en toroïdaal ”

“The lattice graph’s Pfaffian orientations, planar and toroidal ”

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Abstract

In 1961 Kasteleyn solved the dimer problem. With the use of Pfaffians he managed to find a formula to enumerate the number of perfect matchings on a lattice graph. In this thesis we take another look at the methods Kasteleyn used. Besides that, we prove that for an $m \times n$ lattice graph on the torus, where m and n are even, there does not exist a Pfaffian orientation. Instead, we prove that for an $m \times 2$ and $2 \times n$ lattice graph on the torus, where m and n are even, there does exist a Pfaffian orientation. For the $m \times n$ lattice graph on the torus, where m is even and n odd we present an orientation together with an algorithm with which we can simplify cycles. With this algorithm we prove that our orientation is Pfaffian. We will now describe the Pfaffian orientation. All horizontal edges are aimed to the right, all vertical edges switch between all going down for the first column, then up for the second column and so on. The edges that are on the border all have orientation opposite of the ongoing orientation. After calculating the Pfaffian the final formulas are as follows:

$$Z_{mn}^{(t)}(z_h, z_v) = \prod_{k=1}^{\frac{m}{2}} \prod_{l=1}^{n} 2\left(z_h^2 \sin^2 \frac{(2k-1)\pi}{m} + z_v^2 \sin^2 \frac{(2l-1)\pi}{n}\right)^{\frac{1}{2}}, n \text{ odd}, m \text{ even}$$

$$Z_{2n}^{(t)}(z_h, z_v) = \prod_{l=1}^{n} 2\left(z_h^2 + z_v^2 \sin^2(\frac{(2l-1)\pi}{n})\right)^{\frac{1}{2}}, n \text{ even}$$

$$Z_{m2}^{(t)}(z_h, z_v) = \prod_{k=1}^{\frac{m}{2}} 4\left(z_h^2 \sin^2(\frac{(2k-1)\pi}{m}) + z_v^2\right), m \text{ even}$$

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1 Introduction

Consider a normal 8×8 chessboard and a set of 32 blank dominoes. Finding a tiling of the dominoes where the entire chessboard is filled is not that difficult. Placing all dominoes horizontally would be one solution, placing them all vertically would be another. A much more difficult problem is finding all possible tilings, or at least finding how many tilings exist. The next step would then be to find the number of tilings for every $m \times n$ rectangle. In 1961, Kasteleyn, among others, solved this problem [1] by finding the number of perfect matchings of the lattice graph. This result was achieved by the use of Pfaffian orientations. Not only did Kastelyen solve this problem for the lattice graph in the plane but also for the lattice graph embedded on the torus. In this thesis we will take a look at Kasteleyn's original results. Furthermore, we will find Pfaffian orientations for specific cases of the $m \times n$ lattice on the torus.

We will start by giving some definitions and properties in Chapter 2. These will make the thesis more easily readable. In Chapter 3, we will introduce the Pfaffian together with the properties for a Pfaffian orientation. After that, in Chapter 4, we will take a look at Kasteleyn's original result for the lattice graph. We will find the same result using a different method found in [2]. Chapter 5 contains the proof that for the $m \times n$ lattice graph on the torus where both m and n are even there does not exist a Pfaffian orientation. Furthermore, we present an algorithm which will help us prove there does exist a Pfaffian orientation for the $m \times n$ lattice graph on the torus where m is even and n is odd. Using these orientations we will construct the formulas for enumerating the number of perfect matchings for the $m \times n$ lattice graph on the torus. Here we will use almost the same method Kasteleyn used for the torus, where we also fill in some gaps that were left open.

2 Definitions

In this chapter we will present some definitions that will serve as an introduction for the later chapters. Most definition will be very important in the next chapter where we will address the Pfaffian.

2.1 Perfect Matchings

This section will focus on the first of two very import aspects of the Pfaffian, which is the perfect matching. But, to know what a perfect matching is we first need to know what a matching is.

Definition 2.1.1 (Diestel [3]) *A set M of independent edges in a graph $G = (V, E)$ is called a* ***matching****.*

Definition 2.1.2 *A* ***perfect matching*** *of a graph $G = (V, E)$ is a matching where the set of independent edges M matches all vertices in V.*

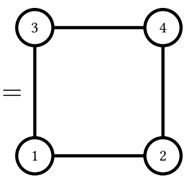
We could see a perfect matching of a graph as a partitioning of it's vertices into pairs, where every pair of vertices is connected. Since, every edge has to connect two vertices and no two edges share vertices. Knowing this, it is straightforward that a graph can only have a perfect matching if it has an even number of vertices. We will call such a graph an **even graph**. In the same way we shall call a cycle with an even number of edges an **even cycle**.

If we take the adjacency matrix $A = (a_{ij})_{i,j=1}^{2n}$ of some graph G we could take the following equation

$$\sum_{P} a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_n j_n} \tag{1}$$

Here, the sum is taken over all partitions $P = \{\{i_1, j_1\}, \ldots, \{i_n, j_n\}\}$ of the $2n$ vertices into pairs. Equation (1) will give us the number of perfect matchings of the graph G. This is because every term in the sum corresponds to a pairing of the vertices and is either 1, if all pairs in the partitioning are connected, or 0, when a pair is disconnected. Take for example the graph G with adjacency matrix A below.

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \qquad G =$$



Equation 1 would result in $a_{12}a_{34} + a_{13}a_{24} + a_{14}a_{23} = 1 + 1 + 0 = 2$. Here we see that the first two terms correspond to the two perfect matchings of G. The third term will be zero in the sum because vertices 1 and 4 are not connected and neither are vertices 2 and 3. Now, the only problem with equation 1 is that it takes a long time to calculate because we have to go through every partition. Because of this, we will be looking at the Pfaffian in the next chapter, which looks a lot like equation 1 but will be much simpler to calculate.

2.2 Orientations and Cycles

In this section we will hone in on two other aspects of the Pfaffian, those being orientations and cycles. We will denote $\vec{G}$ as an orientation of a graph G. To be able to work with orientations of graphs we will first describe the **skew adjacency matrix** of $\overrightarrow{G}$ just like in Lovasz Plummer [2].

$$A_s(\overrightarrow{G}) = (a_{ij})_{i,j=1}^n,$$

$$\text{where } a_{ij} = \begin{cases} 1, & \text{if } (u_i, u_j) \in E(\overrightarrow{G}), \\ -1, & \text{if } (u_j, u_i) \in E(\overrightarrow{G}), \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

As can be easily checked, and as the name implies, the skew adjacency matrix is skew-symmetric. This means that $A_s^T = -A_s$.

We will use the skew adjacency matrix a lot when we are looking at orientations and cycles. As for the Pfaffian, cycles will be very important and more specifically the parity of cycles. What we mean by this is given a routing of the cycle, for how many edges on the cycle the routing agrees with the orientation in $\overrightarrow{G}$. Of course, such a routing is not unique since there are always two possible ways to go around a cycle. However, for an even cycle the parity of the number of edges agreeing or disagreeing with the routing will always be the same. This gives rise to the following definition.

Definition 2.2.1 *An even cycle C of G is **evenly oriented** if it has an even number of edges agreeing (or disagreeing) with the routing relative to $\overrightarrow{G}$. Otherwise, C is **oddly oriented**.*

There are two special types of cycles for which we will frequently check their parity. These are the following.

Definition 2.2.2 *Let F be a (perfect) matching of graph G. Then, an **F-alternating cycle** is a cycle whose edges alternate between edges in F and edges in $E(G) - F$.*

Definition 2.2.3 *A cycle C of G is a **nice cycle** if $G - V(C)$ contains a perfect matching*

Note that an alternating cycle is always an even cycle, since it alternates between two groups of edges. A nice cycle is not always even but we do know the following result.

Lemma 2.2.1 *An even graph can only have nice cycles which are even.*

Although, this lemma isn't a big result, it will make it easier to prove future results.

3 Pfaffian

In this chapter we will introduce the Pfaffian. We will give some examples together with some of its properties.

3.1 Definition

The Pfaffian of a skew-symmetric matrix $A = (a_{ij})_{2n\times 2n}$ is defined as

$$\mathrm{pf}(A) = \sum_P a_P$$

where we have

$$a_P = \operatorname{sgn}\begin{pmatrix} 1 & 2 & \cdots & 2n-1 & 2n \\ i_1 & j_1 & \cdots & i_n & j_n \end{pmatrix} a_{i_1,j_1}\cdots a_{i_n,j_n}$$

Here, P is a partition of the set $\{1,\ldots,2n\}$ into pairs, such that

$$P = \{\{i_1,j_1\},\ldots,\{i_n,j_n\}\}$$

with $i_k < j_k$ and $i_1 < i_2 < \cdots < i_n$. Furthermore,

$$\begin{pmatrix} 1 & 2 & \cdots & 2n-1 & 2n \\ i_1 & j_1 & \cdots & i_n & j_n \end{pmatrix}$$

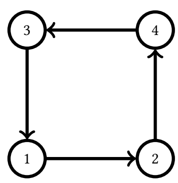
is a permutation of the elements $1,\ldots,2n$. The sign of a permutation is 1 or -1 if the permutation is even or odd respectively. As we can see, the Pfaffian is very similar to equation 1. The only difference is that we have an extra sign in the a_P terms. This might make the equation more complicated, but the following lemma is what makes the Pfaffian so useful for us.

Lemma 3.1.1 (Muir [4]) *If A is a skew symmetric matrix, then* $\det(A) = (\mathrm{pf}(A))^2$

This is also why orientations are so important for the Pfaffian. Since, for an oriented graph we can construct its skew-adjacency matrix. This way, we only need to calculate its determinant to know its Pfaffian.

As an example of the Pfaffian, lets take the following skew-symmetric matrix B of graph G with orientation $\overrightarrow{G}$

$$B = \begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix}, \qquad \overrightarrow{G} =$$



We have the following partitions, $P_1 = \{\{1,2\},\{3,4\}\}$, $P_2 = \{\{1,3\},\{2,4\}\}$ and $P_3 = \{\{1,4\},\{2,3\}\}$. This gives us permutations

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = (\,), \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} = (2\,3), \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix} = (2\,4)(4\,3)$$

Thus, we get

$$\begin{aligned}\mathrm{pf}(B) = \operatorname{sgn}\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 2 & 3 & 4\end{pmatrix} b_{12}b_{34} &+ \operatorname{sgn}\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 3 & 2 & 4\end{pmatrix} b_{13}b_{24}\\ &+ \operatorname{sgn}\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 4 & 2 & 3\end{pmatrix} b_{14}b_{23}\\ &= b_{12}b_{34} - b_{13}b_{24} + b_{14}b_{23}\\ &= -1 + 1 + 0 = 0\end{aligned}$$

Of course, this calculation is just to demonstrate. We could have far more easily calculated the determinant. However, although the calculation of the Pfaffian is a lot easier than that of Equation 1 we have now seen that the Pfaffian does not always calculate the number of perfect matchings. This is because we now deal with the signs of the permutations. It is, however, quite straightforward to realise that the absolute value of the Pfaffian is always smaller or equal to the number of perfect matchings of its corresponding graph. Therefore we have the following definition

Definition 3.1.1 *For a graph G with orientation $\overrightarrow{G}$, if the Pfaffian of $A_s(\overrightarrow{G})$ enumerates all perfect matchings of G then $\overrightarrow{G}$ is a* Pfaffian orientation. *Furthermore, we also call a graph G for which such an orientation exists Pfaffian.*

3.2 Pfaffian Orientation

In this section we give some lemma's and the theorem which states what makes an orientation Pfaffian. All of these have been gathered from Lovász-Plummer [2].

Theorem 3.2.1 (Lovász-Plummer) *Let G be any even graph and $\overrightarrow{G}$, an orientation of G. Then the following four properties are equivalent:*

1. *$\overrightarrow{G}$ is a Pfaffian orientation of G.*
2. *Every perfect matching of G has the same sign relative to $\overrightarrow{G}$.*
3. *Every nice cycle in G is oddly oriented relative to $\overrightarrow{G}$.*
4. *If G has a perfect matching, then for some perfect matching F, every F-alternating cycle is oddly oriented relative to $\overrightarrow{G}$.*

We will be using this theorem a lot in this thesis when proving an orientation is Pfaffian or not.

Lemma 3.2.2 *If $\overrightarrow{G}$ is a connected plane directed graph such that every face's boundary, except possibly the infinite face, has an odd number of lines oriented clockwise, then in every cycle the number of lines oriented clockwise is of opposite parity to the number of points of $\overrightarrow{G}$ inside the cycle. Consequently, $\overrightarrow{G}$ is Pfaffian.*

It is easy to see that a graph with the properties described in Lemma 3.2.2 would indeed be Pfaffian. As when the number of points inside of some cycle C is even, which includes all nice cycles according to Lemma 2.2.1, then C is always oddly oriented. This means condition 3 of Theorem 3.2.1 is satisfied and thus the graph is Pfaffian. This lemma will make finding a Pfaffian orientation and checking if an orientation is Pfaffian a lot simpler later on.

Lemma 3.2.3 *Let $\overrightarrow{G}$ be an arbitrary orientation of an undirected graph G. Let F_1 and F_2 be any two perfect matchings of G and let k denote the number of evenly oriented alternating cycles formed in $F_1 \cup F_2$. Then* $\operatorname{sgn}(F_1)\operatorname{sgn}(F_2) = (-1)^k$.

Here, the sign of a perfect matching is defined as the sign of its corresponding term in the Pfaffian. This lemma can be very useful because if we know the sign of a perfect matching we can then very easily find the sign of other perfect matchings by looking at their alternating cycles.

4 Dimer Problem on a Square Lattice

In this chapter we will discuss the orientation given by Kasteleyn. Using this orientation we will find a formula to calculate the number of perfect matchings of a lattice graph in the plane.

4.1 Pfaffian Orientation of the Lattice Graph

We will now construct the formula found by Kasteleyn for the number of dimer coverings of an $m \times n$ lattice or the number of perfect matchings of the $m \times n$ lattice graph. For this, we will look at two slightly different methods, which both result in the same outcome. We will look at the method Kasteleyn came up with and the method found in Lovasz Plummer [2].

Kasteleyn, in his calculations, made a distinction between vertical and horizontal dimers. He indicated the number of horizontal dimers by N_2 and the number of vertical dimers by N_2'. To not only be able to calculate the number of ways to fully cover the lattice but also calculate how many horizontal and vertical dimers are needed for each covering Kasteleyn used the configuration generating function

$$Z_{mn}(z_h, z_v) = \sum_{N_2, N_2'} g(N_2, N_2') z_h^{N_2} z_v^{N_2'}$$

where the sum runs over all combinations N_2 and N_2' such that $2(N_2 + N_2') = mn$ and z_h and z_v are two variables. Here, $g(N_2, N_2')$ is the combinatorial factor. This is the number of ways to cover the lattice with N_2 vertical and N_2' horizontal dimers. Also, because we need to have an even number of vertices for $Z_{mn}(z_h, z_v)$ to not be zero we let m be even. This means that, for example, for the 2×3 lattice we would have

$$Z_{23}(z_h, z_v) = z_h^3 + 2z_h z_v^2$$

since we have 1 covering with 3 horizontal dimers and 2 coverings with 1 horizontal and 2 vertical dimers.

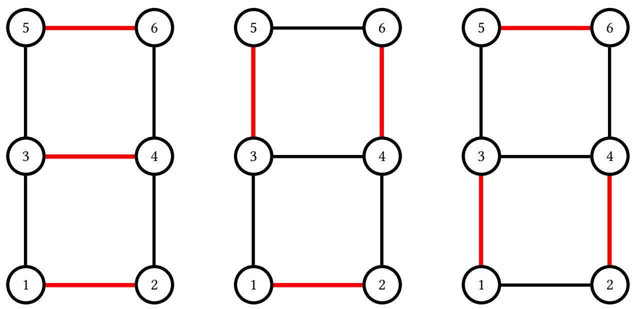


Figure 1: Coverings given by $Z_{23}(z_h, z_v)$

We can also see now that if we chose z_h and z_v to be equal to 1 we will simply get the total number of coverings.

Kasteleyn has constructed a skew adjacency matrix of the lattice. In this matrix the entries are the following

$$\begin{array}{lll} D(i,j\,;\,i+1,j) = z_h & \text{for } 1 \le i \le m-1, & 1 \le j \le n \\ D(i,j\,;\,i,j+1) = (-1)^i z_v & \text{for } 1 \le i \le m, & 1 \le j \le n-1 \\ D(i,j\,;\,i',j') \quad = -D(i',j'\,;\,i,j) & \text{for } 1 \le i,i' \le m, & 1 \le j,j' \le n \\ \text{All other entries} = 0 & & \end{array} \tag{3}$$

In the graphs corresponding to these entries we label the vertices by

$$p := (j-1)m + i \qquad \text{for } 1 \le i \le m,\ 1 \le j \le n \tag{4}$$

as we have done before. This means we label the graph left-to-right, bottom-up. The matrix D now looks like this

$$D = \begin{pmatrix} z_h Q_m & z_v F_m & & & \\ -z_v F_m & z_h Q_m & z_v F_m & & \\ & -z_v F_m & & & \\ & & & \ddots & \\ & & & & z_v F_m \\ & & & -z_v F_m & z_h Q_m \end{pmatrix}$$

where Q_m and F_m are the $m \times m$ matrices

$$Q_m = \begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & 1 & & & \\ & -1 & & & & \\ & & & \ddots & & \\ & & & & & 1 \\ & & & & -1 & 0 \end{pmatrix}, \quad F_m = \begin{pmatrix} -1 & 0 & & & & \\ 0 & 1 & 0 & & & \\ & 0 & & & & \\ & & & \ddots & & \\ & & & & -1 & 0 \\ & & & & 0 & 1 \end{pmatrix}$$

The matrices we use to represent the orientations might differ somewhat from what Kasteleyn has done. However, the Pfaffian only depends on it's partitions P so we will still get the same results. As will not be very surprising, the coefficients Kasteleyn chose correspond to a Pfaffian orientation. If we look at the graph corresponding to these coefficients, where z_h and z_v are the weights of horizontal and vertical edges respectively, it would look like figure 2 below.

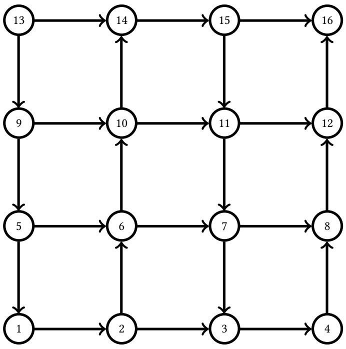


Figure 2: Lattice Graph with Orientation

Using Lemma 3.2.2 we can now easily verify that this orientation is indeed Pfaffian. Since the orientation is very repetitive we only have two kinds of faces. These faces have either 1 or 3 edges in the direction of the orientation making them oddly oriented. Thus, Lemma 3.2.2 tells us that this orientation is Pfaffian.

4.2 Enumerating Perfect Matchings of the Lattice Graph

Since D is a skew-symmetric matrix we can calculate its Pfaffian by finding its determinant according to Lemma 3.1.1. To do so we will use the method found in Chapter 8 of Lovasz Plummer [2] using Kronecker products. To use this method we will rewrite our matrix D somewhat by multiplying rows and columns by -1. We multiply the first column, third and fourth row, fourth and fifth column, seventh and eight row, and so on. This way the absolute value of the determinant stays the same so we don't change the Pfaffian. The changes to D result in the following matrix

$$D_1 = \begin{pmatrix} z_h A_m & z_v I_m & & & \\ -z_v I_m & z_h A_m & z_v I_m & & \\ & -z_v I_m & & & \\ & & \ddots & & \\ & & & & z_v I_m \\ & & & -z_v I_m & z_h A_m \end{pmatrix}, \text{for } m = 4r, (r = 1, 2, ...)$$

$$D_2 = \begin{pmatrix} z_h A_m & -z_v I_m & & & \\ z_v I_m & z_h A_m & -z_v I_m & & \\ & z_v I_m & & & \\ & & \ddots & & \\ & & & & -z_v I_m \\ & & & z_v I_m & z_h A_m \end{pmatrix}, \text{for } m = 4r - 2, (r = 1, 2, ...)$$

We will see that both matrices produce the same result so we can choose either one to work with. We wil continue with D_1. We can write D_1 as the sum of two Kronecker products

$$D_1 = z_h(I_n \otimes A_m) + z_v(Q_n \otimes I_m)$$

where A_m is the $m \times m$ matrix

$$A_m = \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & & & & \\ & & & \ddots & & \\ & & & & & 1 \\ & & & & 1 & 0 \end{pmatrix}$$

Because we can write D_1 in this sum of Kronecker products it is now fairly easy to find its mn eigenvalues. This is due to the following property, also shown as a slight variation in Lovasz Plummer. Let A_m have eigenvalues $\lambda_1, \ldots, \lambda_m$ with eigenvectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ and let Q_n have eigenvalues $\mu_1, \ldots, \mu_n$ with eigenvectors $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$. Then, because we have

$$\begin{aligned} D_1(\boldsymbol{q}_i \otimes \boldsymbol{a}_j) &= (z_h(I_n \otimes A_m) + z_v(Q_n \otimes I_m))(\boldsymbol{q}_i \otimes \boldsymbol{a}_j) \\ &= z_h(I_n\boldsymbol{q}_i \otimes A_m\boldsymbol{a}_j) + z_v(Q_n\boldsymbol{q}_i \otimes I_m\boldsymbol{a}_j) \\ &= z_h(\boldsymbol{q}_i \otimes \lambda_i\boldsymbol{a}_j) + z_v(\mu_j\boldsymbol{q}_i \otimes \boldsymbol{a}_j) \\ &= (z_h\lambda_i + z_v\mu_j)(\boldsymbol{q}_i \otimes \boldsymbol{a}_j) \end{aligned}$$

D_1 has eigenvectors $\boldsymbol{q}_i \otimes \boldsymbol{a}_j$ with eigenvalues $z_h\lambda_i + z_v\mu_j$. Thus, to calculate the eigenvalues of D_1 we only need to find the eigenvalues of our matrices A_m and Q_n. This is also why we could freely choose between D_1 and D_2 because if we chose D_2 we would have had $-Q_n$ which has the same eigenvalues in absolute value as Q_n.

We will start with our matrix A_m and calculate it's characteristic polynomial. We will do this by replicating a solution found in Lovasz Combinatiorial Problems and Excercises [5] for problem 1.29.

$$p_m(\lambda) = \begin{vmatrix} \lambda & -1 & & & & \\ -1 & \lambda & -1 & & & \\ & -1 & & & & \\ & & & \ddots & & \\ & & & & & -1 \\ & & & & -1 & \lambda \end{vmatrix} \tag{5}$$

$$= \lambda \underbrace{\begin{vmatrix} \lambda & -1 & & & & \\ -1 & \lambda & -1 & & & \\ & -1 & & & & \\ & & & \ddots & & \\ & & & & & -1 \\ & & & & -1 & \lambda \end{vmatrix}}_{m-1} + \underbrace{\begin{vmatrix} -1 & 0 & & & & \\ -1 & \lambda & -1 & & & \\ & -1 & & & & \\ & & & \ddots & & \\ & & & & & -1 \\ & & & & -1 & \lambda \end{vmatrix}}_{m-1} \tag{6}$$

$$= \lambda \underbrace{\begin{vmatrix} \lambda & -1 & & & & \\ -1 & \lambda & -1 & & & \\ & -1 & & & & \\ & & & \ddots & & \\ & & & & & -1 \\ & & & & -1 & \lambda \end{vmatrix}}_{m-1} - \underbrace{\begin{vmatrix} \lambda & -1 & & & & \\ -1 & \lambda & -1 & & & \\ & -1 & & & & \\ & & & \ddots & & \\ & & & & & -1 \\ & & & & -1 & \lambda \end{vmatrix}}_{m-2} \tag{7}$$

$$= \lambda p_{m-1}(\lambda) - p_{m-2}(\lambda) \tag{8}$$

We will set $p_0(\lambda) = 1$ and $p_{-1}(\lambda) = 0$ so the equation holds for all $m \geq 1$. We will solve $p_m(\lambda) = \lambda p_{m-1}(\lambda) - p_{m-2}(\lambda)$ by substituting in x^m for $p_m(\lambda)$ and dividing by x^{m-2}. This gives us the following equation

$$x^2 - \lambda x + 1 = 0$$

whose roots are

$$\vartheta_1 = \frac{\lambda + \sqrt{\lambda^2 - 4}}{2}, \quad \vartheta_2 = \frac{\lambda - \sqrt{\lambda^2 - 4}}{2},$$

Then

$$p_m(\lambda) = c_1 \vartheta_1^{m+1} + c_2 \vartheta_2^{m+1}.$$

Taking $m = -1$ and 0 we get

$$\begin{gathered} c_1 + c_2 = 0 \\ c_1 \vartheta_1 + c_2 \vartheta_2 = 1. \end{gathered}$$

Hence

$$c_1 = \frac{1}{\sqrt{\lambda^2 - 4}}, \quad c_2 = \frac{-1}{\sqrt{\lambda^2 - 4}}$$

and

$$p_m(\lambda) = \frac{1}{\sqrt{\lambda^2 - 4}} (\vartheta_1^{m+1} - \vartheta_2^{m+1}).$$

Thus if

$$p_m(\lambda) = 0,$$

then

$$\vartheta_1^{m+1} = \vartheta_2^{m+1}$$

or, equivalently,

$$\vartheta_1 = \varepsilon^2 \vartheta_2,$$

where

$$\varepsilon = e^{\frac{k\pi i}{m+1}}, \quad 0 \leq k \leq m.$$

Solving $\vartheta_1 = \varepsilon^2 \vartheta_2$ for λ we obtain

$$\lambda = \pm(\varepsilon + \frac{1}{\varepsilon}) = \pm 2 \cos \frac{k\pi}{m+1}.$$

Here we may omit $\pm$ since $-\cos \frac{k\pi}{m+1} = \cos \frac{(m+1-k)\pi}{m+1}$. It is easily see by substitution that these numbers are roots of $p_m(\lambda)$ for $k = 1, \ldots, m$; therefore, it is not a root for $k = 0$. Thus the eigenvalues of A_m are

$$\lambda_k = 2 \cos \frac{k\pi}{m+1}, \quad k = 1, \ldots, m.$$

For the eigenvalues of our matrix Q_n we can use the same procedure giving us the eigenvalues

$$\mu_l = 2i \cos \frac{l\pi}{n+1}, \quad l = 1, \ldots, n.$$

Since the eigenvalues of our matrix D_1 are $z_h \lambda_i + z_h \mu_j$, these are equal to

$$2 \left(z_h \cos \frac{k\pi}{m+1} + i z_v \cos \frac{l\pi}{n+1} \right), \quad k = 1, \ldots, m; l = 1, \ldots, n. \tag{9}$$

The determinant of D_1 is now the product of these eigenvalues. However, what we want to know is the determinant of matrix D and to do so we need to calculate the absolute value of the determinant of D_1. Because of this, we can use the absolute values of equation 9. Giving us

$$2^{mn} \prod_{k=1}^{m} \prod_{l=1}^{n} \left(z_h^2 \cos^2 \frac{k\pi}{m+1} + z_v^2 \cos^2 \frac{l\pi}{n+1} \right)^{\frac{1}{2}}. \tag{10}$$

Hence, because of Lemma 3.1.1

$$Z_{mn}(z_h, z_v) = 2^{\frac{1}{2}mn} \prod_{k=1}^{m} \prod_{l=1}^{n} \left(z_h^2 \cos^2 \frac{k\pi}{m+1} + z_v^2 \cos^2 \frac{l\pi}{n+1} \right)^{\frac{1}{4}}. \tag{11}$$

5 Torus Extension

In this section we will find a formula for the number of perfect matchings of the lattice graph on a torus. We will again take m to be even here. Firstly we will prove that for an $m \times n$ lattice where n is even there is no Pfaffian orientation. After that, we will show an orientation that we prove to be Pfaffian for $m \times n$ lattices where m or n are equal to 2, or when n is odd. Lastly, we will calculate the respective formulas for the enumeration of the perfect matchings of these lattice graphs.

5.1 No Pfaffian Orientation for Even n

To prove a graph does not have a Pfaffian orientation we need to look at Theorem 3.2.1 More precisely we will look at condition 3 of this Theorem. This condition states that every nice cycle in the graph has to be oddly oriented relative to its orientation. This means that, to prove that the $m \times n$ lattice graph on the torus with even n and m and n greater or equal to 4 does not have a Pfaffian orientation we only need to find one nice cycle which is evenly oriented. To do this we will show a specific example where a nice cycle is evenly oriented and prove that this holds for all lattices with even n.

Consider the graph in figure 3 with the labeled edges from a to i. We will be looking at the cycles with the following edges: (a,b,c,d), (b,f,h,e), (c,g,i,f) and (a,e,h,i,g,d).

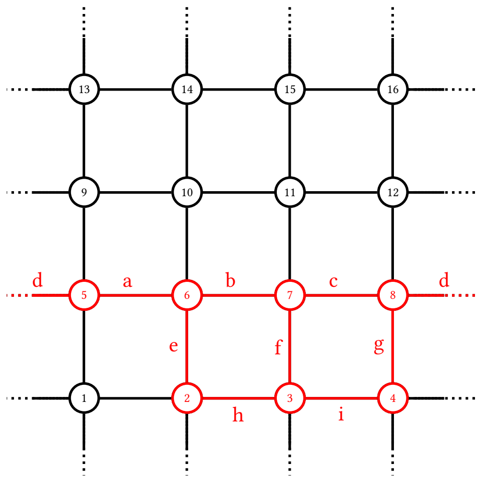


Figure 3: Torus Graph with labeled edges

Note that edge d is shown twice but represents one edge because it loops around the torus. It can be easily checked that these are nice cycles. Since they are nice cycles we would want all of them to be oddly oriented to satisfy condition 3 of Theorem 3.2.1. Let's say the edges can have a value of 0 or 1. We'll say going to the right and going down gets the value 1 where going left or up gets the value of 0. We are interested in the number of edges that go with the orientation modulo 2 since we want to know if it is even or odd. This means that if we want to know this number of edges from, for example, the cycle (b,f,h,e) we would write this as b + f + (1-h) + (1-e). This is because for h and e to go with the orientation they would get the value 0 but their terms would still need to contribute 1 to the number of edges going with the orientation. Since we want all nice cycles to be oddly oriented we want all of these sums to be equal to 1 mod 2. This means we get the following system of equations.

$$a + b + c + d = 1 \bmod 2$$
$$b + f + (1 - h) + (1 - e) = 1 \bmod 2$$
$$c + g + (1 - i) + (1 - f) = 1 \bmod 2$$
$$a + e + h + i + (1 - g) + d = 1 \bmod 2$$

This can be simplified to

$$a + b + c + d = 1 \bmod 2 \tag{12a}$$
$$b + f + h + e = 1 \bmod 2 \tag{12b}$$
$$c + g + i + f = 1 \bmod 2 \tag{12c}$$
$$a + e + h + i + g + d = 0 \bmod 2 \tag{12d}$$

If we add all these equation together we will see that this gives a contradiction. Since all edges are counted exactly twice, adding all equation gives

$$2(a + b + c + d + e + f + g + h + i) = 1 \bmod 2 \tag{13}$$

which can never be true. This means that we cannot have an orientation where all nice cycles are oddly oriented, thus we cannot have a Pfaffian orientation for this 4×4 lattice on the torus. Now, we can extend this lattice with multiples of 2 vertically and the same thing still holds true. When we do this in the horizontal direction we can look at our edge d as representing multiple edges. This would mean Equation 12a and 12d both get a multiple of 2 edges added to them. This means Equation 13 will still be of the same form and thus our result holds for all even m and n.

5.2 Pfaffian Orientation of the Lattice Graph on the Torus

The orientation we will prove to be Pfaffian is one that Kasteleyn partly used in his result for the dimer problem on a torus. However, in his result this orientation wasn't Pfaffian since it was for all n and all even m. The orientation is the following

$$\begin{array}{lll} D(i,j\,;\,i+1,j) = z_h & \text{for } 1 \le i \le m-1, & 1 \le j \le n \\ D(i,j\,;\,i,j+1) = (-1)^i z_v & \text{for } 1 \le i \le m, & 1 \le j \le n-1 \\ D(m,j\,;\,1,j) \quad = -z_h & \text{for} & 1 \le j \le n \\ D(i,n\,;\,i,1) \quad = (-1)^{i+1} z_v & \text{for } 1 \le i \le m & \\ D(i,j\,;\,i',j') \quad = -D(i',j'\,;\,i,j) & \text{for } 1 \le i,i' \le m, & 1 \le j,j' \le n \\ \text{All other entries} = 0 & & \end{array} \tag{14}$$

As we can see the orientation we use is the same on the torus as it was in the plane. The only thing added is that the boundaries of the graph are now connected and are oriented opposite to the ongoing orientation. If we look at the graph it looks like this

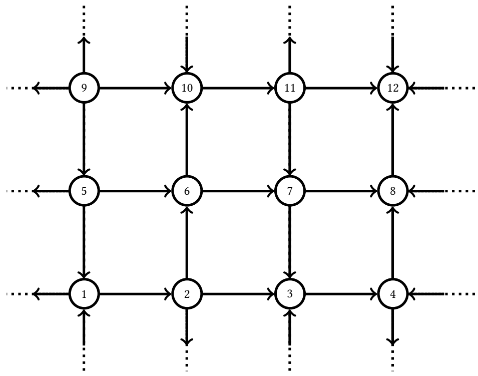


Figure 4: Orientation on the Torus

In Section 4.1 we have proven that our orientation is Pfaffian using Lemma 3.2.2. However, this Lemma stated that the graph has to be a planar graph, which our new graph clearly isn't. Despite this, we can still use Lemma 3.2.2 in smaller portions. Firstly, we know all even cycles that don't go around the torus are oddly oriented. We saw this before in section 4.1, where now the only thing that's different is the extra "border" edges. This hasn't changed the orientation of the cycles in any way and by ignoring these extra edges the graph becomes planar. This way the Lemma can still be used. In the same way this holds true for cycles that go across the "border" of the torus, but don't fully loop around it. This can be done because if we redraw our lattice by shifting everything one step to the right we get the following representation

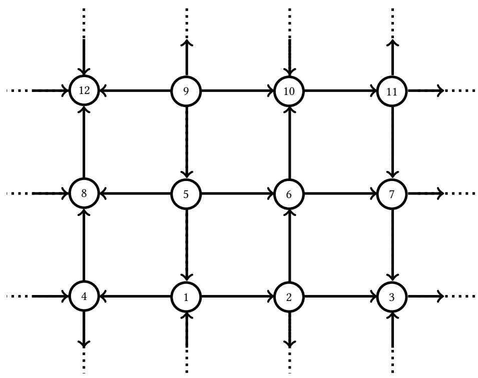


Figure 5: Alternative Drawing of Figure 4

And now, we can again use Lemma 3.2.2 for this graph by ignoring the border edges because all face's boundaries are oddly oriented. We can do this as many times as we need to, also by shifting up or down, to find that all nice cycles that don't loop around the torus are oddly oriented. This now only leaves us to check nice cycles that loop the torus horizontally, vertically or in both directions. Now, moving on we will look at these cycles for the two cases described earlier.

5.2.1 The $2 \times n$ and $m \times 2$ Case

We will start with the $2 \times n$ and $m \times 2$ cases where n and m are even. Proving an orientation is Pfaffian takes a lot more steps than proving one is not. However, for these cases the proof stays quite concise.

For this proof we will use condition 4 of Theorem 3.2.1. We will use two different perfect matchings, one for each case. We will call these F_n and F_m. Here, F_n is the perfect matching consisting of only horizontal edges and F_m the perfect matching consisting of only vertical edges. Figure 6 will clarify.

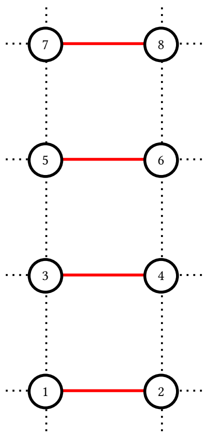


Figure 6: Perfect matchings F_n and F_m for $2 \times n$ and $m \times 2$ lattice graphs on the torus

It can be easily checked that the only F_n-alternating cycles are a square, a straight horizontal cycle and two vertical cycles that flip sides. These are shown in figure 7.

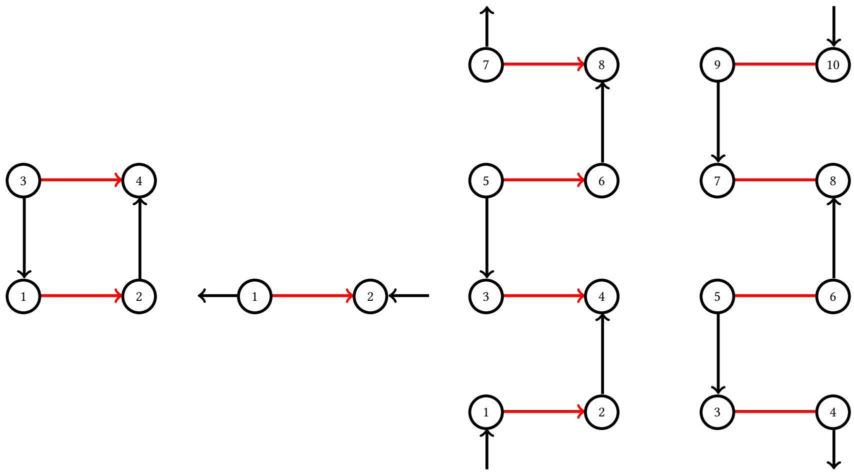


Figure 7: F_n-alternating cycles for a 2×4 lattice graph on the torus

As we can see all of these cycles are oddly oriented making the orientation Pfaffian. When we extend the graph vertically by a multiple of 2 we add an even number of edges that go with the orientation. Because of this, the orientation is Pfaffian for all $2 \times n$ lattice graphs on the torus with even n. The same result can be found for the $m \times 2$ lattice graph on the torus using perfect matching F_m.

5.2.2 The Odd n Case

As has been said before, proving an orientation is Pfaffian will take quite some steps. We will work in the same way as before, meaning we will try to satisfy condition 4 of Theorem 3.2.1. Since our graph can be much larger this time around there will be a lot more cycles that we will need to check. Because of this, we want to be able to link a larger cycle of which we don't know the parity to a simpler cycle of which we do know the parity. When checking for the parity of alternating cycles we will be using perfect matching F_n as our alternating set.

We won't have to check alternating cycles that lie in the plane since the orientation is made such that the perfect matchings in the plane all have sign 1.

First of all, let us describe this simpler cycle. We will call the following cycle in red H_0.

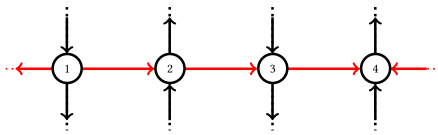


Figure 8: Cycle H_0

As we can see Cycle H_0 is just the shortest horizontal cycle around the torus. Important to note here is that H_0 is always oddly orientated with this orientation. Because of the nature of an alternating cycle, the only possible cycles that loop around the torus are cycles that loop the torus horizontally an odd amount of times and vertically an even amount of times. Here, we also see not looping around a side as an even amount of times. That these are the only possible cycles has to do with the fact that we can't loop the torus vertically once when we have an odd number of vertical vertices. It is important to remember that, since m is even and n is odd, these cycles always have an even number of horizontal edges and an even number of vertical edges.

We will construct an algorithm to reduce F_n-alternating cycles to Cycle H_0. Let C be any cycle on the lattice graph on the torus. If C surrounds a rectangle in such a way that three sides of the rectangle connect to the same side split by the cycle, then we can remove the rectangle without changing the parity of the cycle. This may sound a little confusing considering technically a cycle looping the torus doesn't split the graph. Therefore, let's look at a little example to clarify.

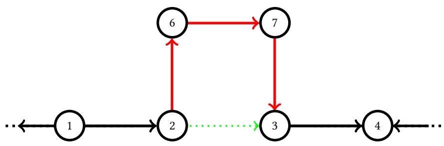


Figure 9: Cycle example

Let the black and red arrows represent our cycle. As we can see by drawing in the green arrow we create a rectangle of which its three red edges connect to a different "side" than its green edge. Again, this is technically the same side on the torus. Now, what we can do here is remove the red edges of the rectangle and connect the green edge, oriented according to the orientation, to create H_0. In doing so we remove three edges agreeing with the orientation and we add one. This way the parity of the cycle hasn't changed. Now because we have H_0, of which we know is oddly oriented, we know that our original cycle is also oddly oriented.

There are three types of rectangles for which we can use the above method. These are:

1. even $\times$ even

2. odd $\times$ odd
3. even $\times$ odd

The even$\times$even and odd$\times$odd rectangles can always be removed, without changing the parity of the cycle, no matter on what side the green edge lays. The even$\times$odd rectangles however, can only be removed when the green arrow lays on either of the odd sides. Because of the way alternating cycles are made any F_n-alternating cycle that loops the torus horizontally once can be reduced to the H_0 cycle by removing all rectangles. This means that all F_n-alternating cycles looping the torus horizontally once are oddly oriented.

When we have F_n-alternating cycles looping the torus multiple times we always end up with the same situation after reducing the cycle. Let's say we loop the torus vertically e times and horizontally o times. After all rectangles have been removed from the cycle we will see that all horizontal edges will agree with the orientation, except the border edges. This means that for the horizontal edges we always have $(m-1)*o$ edges agreeing with the orientation, which is odd. For the vertical edges, we can pair edges that go along the same direction. What we will have left is pairs on the borders which will differ in direction. This means we will have an even amount of edges agreeing with the orientation, the first pairs, together with the e edges, the second pairs. This means for the vertical edges we have an even amount of edges agreeing with the orientation. Putting the horizontal and vertical edges together we see that we always have an odd number of edges agreeing with the orientation, meaning that the original cycle was oddly oriented.

Since we now have seen that all possible F_n-alternating cycles on the $m \times n$ (odd n) lattice graph on the torus are oddly oriented, we can conclude that our orientation at Equation 14 is indeed Pfaffian.

5.3 Kasteleyn's Torus Result

As we have proven in Chapter 5.1 the $m \times n$ lattice graph on the torus with even n does not have a Pfaffian orientation. Despite this, Kasteleyn still managed to find a formula to enumerate the number of perfect matchings. The way he did this is by using four different orientations. The orientation we are using is one of those. Except, as we already saw, none of those are Pfaffian for even n. The problem with the orientations is that their Pfaffians count some perfect matchings with the wrong sign. We can check which matchings are counted with which sign by using the method we used in the previous section. Because, for even n, it is possible to loop around the torus vertically once it would be useful to use a second simpler cycle V_0. This cycle would be the shortest cycle vertically around the torus.

Now that we can loop around the torus vertically we can have any combination of vertical and horizontal loops around the torus. It can be checked that, for our orientation, all F_n-alternating cycles will still be oddly oriented except for cycles that loop the torus an odd amount of times horizontally and vertically. In that case we use the same method as before. That way we get $(m-1)*o$ horizontal edges agreeing with the orientation, which is odd. For the vertical edges we have an even number of pairs agreeing with the orientation again together with e edges. This is odd as well, meaning that the whole cycle is evenly oriented. According to Lemma 3.2.3 the perfect matchings creating these F_n-alternating cycles will be counted with a negative sign in the Pfaffian. In the same way, the other orientations Kasteleyn presents in [1] could be checked to see which perfect matchings are counted with which signs.

5.4 Enumerating Perfect Matchings on the Torus

We will now determine the formula to count the number of perfect matchings of the lattice graph on the torus for odd n. We use the following notation for our configuration generating function on the torus: $Z_{mn}^{(t)}(z_h, z_v)$. To do this we will first have a look at the skew adjacency matrix of the graph in Figure 4.

$$D^{(t)} = \begin{pmatrix} z_h Q_m^{(t)} & z_v F_m & & & z_v F_m \\ -z_v F_m & z_h Q_m^{(t)} & z_v F_m & & \\ & -z_v F_m & & & \\ & & \ddots & & \\ & & & & z_v F_m \\ -z_v F_m & & & -z_v F_m & z_h Q_m^{(t)} \end{pmatrix}$$

where $Q_m^{(t)}$ is the $m \times m$ matrix

$$Q_m^{(t)} = \begin{pmatrix} 0 & 1 & & & 0 & 1 \\ -1 & 0 & 1 & & & 0 \\ & -1 & & & & \\ & & & \ddots & & \\ 0 & & & & & 1 \\ -1 & 0 & & & -1 & 0 \end{pmatrix},$$

As we can see the only differences are the added 1 and -1 in the top-right and bottom-left corners. To calculate the Pfaffian of matrix $D^{(t)}$ we will, again, have to calculate its determinant. To do this we will use the same method Kasteleyn used to calculate the Pfaffian in the plane. However, we will present a little more details.

In Chapter 4 we could multiply some rows and columns to be able to write our matrix as a sum of kronecker products containing two identity matrices. This time around however, the terms that were added now prevent us from performing this trick. Therefore, we have to use a different method. We will, however, write $D^{(t)}$ as a sum of Kronecker products again like so

$$D^{(t)} = z_h(I_n \otimes Q_m^{(t)}) + z_v(Q_n^{(t)} \otimes F_m). \tag{15}$$

What we will now do is try to get $D^{(t)}$ into a more suitable form $\widetilde{D}^{(t)}$ to calculate its determinant. We will do this by diagonalizing the matrices $Q_m^{(t)}$ and $Q_n^{(t)}$ with the following matrix

$$V_n(v, h) = \frac{1}{\sqrt{n}} \exp \frac{v(2h-1)\pi i}{n}.$$

The following result from [6] will show that $D^{(t)}$ is almost diagonalizable by $V_n \otimes V_m$

$$\begin{aligned} \widetilde{D}^{(t)} &= (V_n^{-1} \otimes V_m^{-1}) D^{(t)} (V_n \otimes V_m) && (16) \\ &= (V_n^{-1} \otimes V_m^{-1}) \left(z_h(I_n \otimes Q_m^{(t)}) + z_v(Q_n^{(t)} \otimes F_m) \right) (V_n \otimes V_m) && (17) \\ &= z_h(V_n^{-1} I_n V_n \otimes V_m^{-1} Q_m^{(t)} V_m) + z_v(V_n^{-1} Q_n^{(t)} V_n \otimes V_m^{-1} F_m V_m) && (18) \\ &= z_h(\widetilde{I}_n \otimes \widetilde{Q}_m^{(t)}) + z_v(\widetilde{Q}_n^{(t)} \otimes \widetilde{F}_m) && (19) \end{aligned}$$

$D^{(t)}$ is almost diagonalizable because although $Q_m^{(t)}$, $Q_n^{(t)}$ and I_n are diagonalizable F_m is not. However, we will see that this won't bring up any major issues.

We will write out the new matrices.

$$\widetilde{Q}_m^{(t)} = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_m \end{pmatrix}, \quad \widetilde{Q}_n^{(t)} = \begin{pmatrix} \mu_1 & & & 0 \\ & \mu_2 & & \\ & & \ddots & \\ 0 & & & \mu_n \end{pmatrix} \tag{20}$$

$$\widetilde{F}_m = \begin{pmatrix} 0 & & & -1 \\ & & -1 & \\ & \iddots & & \\ -1 & & & 0 \end{pmatrix}, \quad \widetilde{I}_n = I_n \tag{21}$$

Writing out $\widetilde{Q}_m^{(t)}$ and $\widetilde{Q}_n^{(t)}$ we can easily see that their eigenvalues are equal to $\mathrm{e}^{\frac{(2k-1)\pi}{m}} - \mathrm{e}^{-\frac{(2k-1)\pi}{m}}$ and $\mathrm{e}^{\frac{(2l-1)\pi}{n}} - \mathrm{e}^{-\frac{(2l-1)\pi}{n}}$, thus

$$\lambda_k = 2i \sin \frac{(2k-1)\pi}{m} \quad (k = 1, 2, \dots, m), \tag{22}$$

$$\mu_l = 2i \sin \frac{(2l-1)\pi}{n} \quad (l = 1, 2, \dots, n). \tag{23}$$

If we write out $\widetilde{D}^{(t)}$ using equation 19 we find that it consist of n nonzero $m \times m$ blocks on the diagonal. These are the following

$$\begin{bmatrix} z_h\lambda_1 & & & z_v\mu_l & & \\ & \ddots & & & \ddots & \\ & & z_h\lambda_{\frac{m}{2}} & & & z_v\mu_l \\ z_v\mu_l & & & z_h\lambda_{\frac{m}{2}+1} & & \\ & \ddots & & & \ddots & \\ & & z_v\mu_l & & & z_h\lambda_m \end{bmatrix}, \quad \text{for } l = 1, 2, ..., n. \tag{24}$$

We can switch two rows and two columns of $\widetilde{D}^{(t)}$ to create a 2×2 block on the diagonal. This way the determinant of $\widetilde{D}^{(t)}$ doesn't change. We can do this such that the entire matrix will be made up out of the following 2×2 blocks.

$$\begin{bmatrix} z_h\lambda_k & z_v\mu_l \\ z_v\mu_l & z_h\lambda_{k+\frac{m}{2}} \end{bmatrix} \tag{25}$$

The determinant of a block diagonal matrix is the product of the determinants of the blocks. Thus, we now finally get a simple to calculate equation. After filling in the eigenvalues of $\widetilde{Q}_m^{(t)}$ and $\widetilde{Q}_n^{(t)}$ and using the fact that $\sin(x + \pi) = -\sin(x)$ we get the following equation

$$\det(\widetilde{D}^{(t)}) = \prod_{k=1}^{\frac{m}{2}} \prod_{l=1}^{n} 4 \left(z_h^2 \sin^2 \frac{(2k-1)\pi}{m} + z_v^2 \sin^2 \frac{(2l-1)\pi}{n} \right) \tag{26}$$

After applying Lemma 3.1.1 we find our desired equation

$$Z_{mn}^{(t)}(z_h, z_v) = \prod_{k=1}^{\frac{m}{2}} \prod_{l=1}^{n} 2 \left(z_h^2 \sin^2 \frac{(2k-1)\pi}{m} + z_v^2 \sin^2 \frac{(2l-1)\pi}{n} \right)^{\frac{1}{2}}, \text{for odd } n \tag{27}$$

For the cases where m or n are equal to 2 we use the same steps except our matrix $D^{(t)}$ will look a little different. By replacing $Q_m^{(t)}$ or $Q_n^{(t)}$ with

$$\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

in Equation 15, depending on which is equal to 2, we get the correct matrix. This will result in the following two equations.

$$Z_{2n}^{(t)}(z_h, z_v) = \prod_{l=1}^{n} 2 \left(z_h^2 + z_v^2 \sin^2 (\frac{(2l-1)\pi}{n}) \right)^{\frac{1}{2}}, \text{ for } n \text{ even} \tag{28}$$

$$Z_{m2}^{(t)}(z_h, z_v) = \prod_{k=1}^{\frac{m}{2}} 4 \left(z_h^2 \sin^2 (\frac{(2k-1)\pi}{m}) + z_v^2 \right) \tag{29}$$

6 Conclusion and Future Research

Using previous works from Kasteleyn and Lovász and Plummer we managed to create the formula to enumerate the number of perfect matchings of lattice graphs on the torus. To get to this result we made sure to fill in a lot of the things that Kasteleyn left out in his proofs. We also used some variatons and different methods to get to our end results. We hope this has made the results more straightforward to get to. Although this result is not a brand new thing, we hope that the algorithm we created can help to find more Pfaffian orientations. Not only for the lattice but also for the triangular and hexagonal lattices.

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