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Pricing Methods in a LIBOR Market Model with Stochastic Volatility

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MSc THESIS APPLIED MATHEMATICS

"Pricing Methods in a LIBOR Market Model with Stochastic Volatility"

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Abstract

The purpose of this thesis is to compare three closed-form solutions for pricing caplets under the Wu & Zhang stochastic volatility LIBOR market model. These methods can be an alternative for pricing instead of having to resort to time consuming Monte Carlo simulations. By adding a parameter to the model which should be able to mitigate the issue of increasing interest rates and still use the closed-form solutions we see that the performance of these models could worsen. Instead of adding this parameter to the model we want to reduce the variance in the simulations by using a control variate.

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Introduction

In this document we will look at three different ways to price caplets through closed-form solutions using a specific LIBOR market model with stochastic volatility. To not stray too much from our discussion the reader is assumed to have knowledge of stochastic calculus and basic finance terminology.

We all know the famous Black-Scholes-Merton (BSM) formula to price a call option (for example a caplet) through an analytical function. With a closed-form solution for the methods we will describe we do not mean that we can calculate caplet prices in the BSM way but rather the solution is given in closed-form. We still have to do numerical approximations. The BSM function is a fast and thus attractive way to price caplets but the assumption of constant or deterministic functions for volatility in a model limits it to catch volatility smiles and skews usually observed in the market. This is the reason to extend a LIBOR market model by introducing a stochastic volatility term. The price we have to pay is that to find a solution for such a model one has to resort to Monte Carlo simulations which are obviously much slower. Hence, closed-form solutions for an extended LIBOR market models could be attractive.

We will treat a specific LIBOR market model with stochastic volatility, which we call the Wu and Zhang stochastic volatility model [15]. This model lends itself to the derivation of closed-form solutions to price caplets, which will be our main focus. We will look at three different ways to derive an analytical function for the pricing of caplets. One of these methods, which we will call the Heston method, has first been discussed by Heston [7]. The other two methods are discussed by Carr and Madan [4] and are based on Fourier transforms. We will call these the Time Value and Modified Call Value methods.

An advantage of using these closed-form solutions instead of having to resort to Monte Carlo simulations is the fact that pricing becomes much faster. To arrive at these closed-form solutions, during the derivation, we will however need to make some assumptions that make use of the martingale property of the forward rates under the forward measure. By making these assumptions we will thus be approximating the price of the caplets in change for faster pricing. Having said (and noticed during simulations) that pricing caplets via the closed-form solutions is faster than a Monte Carlo simulation, we will focus on the price differences between each of these methods and the Monte Carlo method.

Furthermore we will add a parameter to the model with as objective to reduce increasing forward rates. This inclusion means that the model has to be recalibrated, however it does allow for caplets to still be priced using the closed-form solutions mentioned. We will also look at another way of reducing the variance of the forward rates by the means of a control variate.

The outline of this document is as follows. In Chapter 1 we start with an introduction of forward rates. The discussion will not only serve to get acquainted with forward rates but also with risk-neutral measure and Heath-Jarrow-Morton models all, of which will help us explain the stochastic volatility model in the next chapter. We closely follow [10]. In Chapter 2 we briefly introduce LI-BOR market models followed by an explanation of the Wu and Zhang stochastic volatility LIBOR

market model. In Chapter 3 we derive three different ways to price caplets through closed-form functions. In the next chapter we then compare the prices of each method to a Monte Carlo simulation. In Chapter 5 we add a parameter in the LIBOR market model with stochastic volatility which should reduce the variance in the Monte Carlo simulation and then present the pricing results of the implication of this "displacement" parameter. In Chapter 6 we will look at a different way to reduce the variance of the caplet prices in a Monte Carlo simulation using a control variate technique. We then end with conclusions and recommendations.

1

Introduction to Forward Rates

1.1 Why Model Forward Rates?

When buying financial products that depend on an underlying interest rate one important question we have to ask is how this rate will develop in the future. One way to model interest rates is through short rate models which assume a stochastic differential equation (SDE) that defines the process for the interest in time. Such models are well known and are used to model interest rates for a short period of time. For more on short rate models one could consult [3]. If we want to hedge a position by buying a derivative depending on an interest rate for a longer period of time we want to model forward rates. In the next section we will explain in more detail what we mean with forward rates and how we can determine these. To this end we will closely follow [10].

1.2 The Heath-Jarrow-Morton Framework

One financial product that is actually marketed and will be the basis to explain forward rates is a (zero-coupon) bond with a certain maturity time. The Heath-Jarrow-Morton (HJM) model is a term structure model which means that the interest rate on a certain period depends on zero-coupon bonds with different maturity times. We denote with P(t,T) the value at time $t \leq T$ of a zero-coupon bond which pays 1 at maturity time T, where the value 1 is also called the *notional* or *face value*. To determine what the *forward rate* is, at time $t \geq 0$ we set up a portfolio by taking a short position of size 1 in T-maturity bonds. Hence receiving income P(t,T). Then we use the amount P(t,T) to take a long position of size $\frac{P(t,T)}{P(t,T+\Delta T)}$ in $(T + \Delta T)$ -maturity bonds, where $\Delta T > 0$. At time t we set up the portfolio without any cost. At time T we have to pay 1 for the short position of the T-maturity bond in the portfolio. Later at time $T + \Delta T$ we receive $\frac{P(t,T)}{P(t,T+\Delta T)}$ from the long position in the $T + \Delta T$ -maturity bond. We have invested 1 at time T and notice that because $P(t,T) > P(t,T + \Delta T)$ we at time $T + \Delta T$ receive more than 1. When using continuously compounding, we can deduce the interest rate which we have to apply to the invested amount of size 1 to receive $\frac{P(t,T)}{P(t,T+\Delta T)}$ at time $T + \Delta T$ as follows:

$$1 \times e^{\Delta T f(t,T;\Delta T)} = \frac{P(t,T)}{P(t,T+\Delta T)}$$

or written differently

$$f(t,T;\Delta T) = \frac{1}{\Delta T} \log \frac{P(t,T)}{P(t,T+\Delta T)} = -\frac{\log P(t,T+\Delta T) - \log P(t,T)}{\Delta T}.$$
 (1.1)

1. INTRODUCTION TO FORWARD RATES

So the forward rate $f_T(t) = f(t, T; \Delta T)$ is the interest rate at time t that will be locked in at time T and is used on the interval $[T, T + \Delta T)$. Here we used continuously compounding for the interest rate and this is a flaw that causes us to introduce LIBOR market models in the next chapter. But before we stimulate any discouragement of reading this chapter let us continue our discussion. Taking the limit $\Delta T \downarrow 0$ in equation (1.1) we arrive at the definition of the forward rate at time t with maturity time T and it is thus defined as

$$f_T(t) = \lim_{\Delta T \downarrow 0} f_T(t; \Delta T) = -\lim_{\Delta T \downarrow 0} \frac{\log P(t, T + \Delta T) - \log P(t, T)}{\Delta T}$$
$$= -\frac{\partial}{\partial T} \log P(t, T).$$
(1.2)

Knowing $f_T(t)$ for $0 \le t \le T$ we can calculate P(t,T) for $0 \le t \le T$ by integrating with respect to T, hence

$$-\log P(t,T) = \int_{t}^{T} f(t,s)ds$$

and now

$$P(t,T) = e^{-\int_{t}^{T} f(t,s)ds}, \quad 0 \le t \le T.$$
(1.3)

We see that we can determine forward rates from bond prices and vice versa. However, for practical reasons determining bond prices from forward rates is easier because the integration in (1.3) is not susceptible to small changes in the forward rates while small changes in the bond prices can cause big differences when determining forward rates from (1.2). The instantaneous interest rate we can lock in at time t to borrow immediately at time t is given by

$$R(t) = f(t, t).$$

We take a time span \overline{T} , so all bonds mature at or before \overline{T} . The initial forward rate curve is given by $f_T(0), \forall T \leq \overline{T}$, and we call this the *zero curve*. The forward rate is specified by the HJM model as

$$f_T(t) = f_T(0) + \int_0^t \alpha^f(s, T) ds + \int_0^t \sigma^f(s, T) dW(s)$$
(1.4)

and in differential form

$$df_T(t) = \alpha^f(t, T)dt + \sigma^f(t, T)dW(t), \quad 0 \le t \le T,$$
(1.5)

hence t varies while the maturity time T is fixed. Here $\alpha^{f}(t,T)$ is the drift of $f_{T}(t)$, $\sigma^{f}(t,T)$ its volatility and W(s) is a Brownian motion. Everything is under the actual measure \mathbb{P} . The processes $\alpha^{f}(t,T)$ and $\sigma^{f}(t,T)$ are adapted processes in the variable t and fixed T.

We can determine the term-structure for the bond prices given in equation (1.3) by equation (1.5). It turns out that

$$dP(t,T) = P(t,T) \left(R(t) - \alpha^{P}(t,T) + \frac{1}{2} (\sigma^{P}(t,T))^{2} \right) dt - \sigma^{P}(t,T) P(t,T) dW(t)$$
(1.6)

where $\alpha^P(t,T) = \int_t^T \alpha^f(t,u) du$ and $\sigma^P(t,T) = \int_t^T \sigma^f(t,u) du$. The reader is referred to [10] for details.

To guarantee that trading in these bonds offers no opportunity for arbitrage we should find a risk-neutral probability measure $\widetilde{\mathbb{P}}$, equivalent to the measure \mathbb{P} , under which each discounted bond price

$$D(t)P(t,T) = e^{-\int_0^t R(s)ds} P(t,T), \quad 0 \le t \le T$$
(1.7)

is a martingale.

1.2. THE HEATH-JARROW-MORTON FRAMEWORK

Remark 1.2.1. Every time we mention risk-neutral we mean that there is no opportunity for an arbitrage i.e. there is no opportunity to make money out of an investment of zero. If the model offers an arbitrage opportunity the model is bad and should not be used.

We have the following important theorem.

=

Theorem 1.2.1. First fundamental theorem of asset pricing. If a market model has a risk-neutral probability measure, then it does not admit arbitrage.

So if such a risk-neutral measure exists then by the First Fundamental Theorem of Asset Pricing it is guaranteed that there will be no arbitrage in the model. Taking the differential of equation (1.7) we have

$$d(D(t)P(t,T)) = -R(t)D(t)P(t,T)dt + D(t)dP(t,T)$$
(1.8)

$$D(t)P(t,T)\left[\left(-\alpha^{P}(t,T) + \frac{1}{2}(\sigma^{P}(t,T))^{2}\right)dt - \sigma^{P}(t,T)dW(t)\right]$$
(1.9)

where we have used the discount process $D(t) = e^{-\int_0^t R(s)ds}$ and dD(t) = -R(t)D(t)dt in the first equality.

Now if we can write the term which is in the square brackets as

$$\sigma^P(t,T)[\Theta(t)dt + dW(t)]$$

then we can apply Girsanov's theorem¹ to change to a (risk-neutral) probability measure $\widetilde{\mathbb{P}}$ under which

$$\widetilde{W}(t) = \int_0^t \Theta(s) ds + W(t)$$
(1.10)

is a Brownian motion. Taking the differential we see that

$$d\widetilde{W}(t) = \Theta(t)dt + dW(t).$$
(1.11)

Now, using the Brownian motion in (1.10) we then can write equation (1.9) as

$$d(D(t)P(t,T)) = -D(t)P(t,T)\sigma^{P}(t,T)d\widetilde{W}(t)$$

Hence, D(t)P(t,T) is a martingale under the measure $\widetilde{\mathbb{P}}$. To achieve this goal we must solve

$$\left[\left(-\alpha^P(t,T) + \frac{1}{2}(\sigma^P(t,T))^2\right)dt - \sigma^P(t,T)dW(t)\right] = \sigma^P(t,T)[\Theta(t)dt + dW(t)]$$

¹One dimensional Girsanov. Let $W(t), 0 \le t \le T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{F}(t), 0 \le t \le T$, be a filtration for this Brownian motion. Under the probability measure \mathbb{P} , consider the stochastic differential equation

$$dX(t) = g(X(t))dt + \sigma(X(t))dW(t), \quad X(0) = x_0$$

where g(X(t)) and $\sigma(X(t))$ are allowed to be adapted processes. Let be given a new drift $\tilde{g}(x)$ and assume $\frac{\tilde{g}(x)-g(x)}{\sigma(x)} = \Theta(x)$ is bounded. Define the measure $\tilde{\mathbb{P}}$ by

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = Z = \exp\left\{-\frac{1}{2}\int_0^t \Theta^2(X(u))d(u) + \int_0^t \Theta(u)dW(u)\right\}.$$

Then $\widetilde{\mathbb{P}}$ is equivalent to $\mathbb{P}.$ Moreover, the process defined by

$$d\widetilde{W}(t) = -\Theta(t)dt + dW(t)$$

is a Brownian motion under $\widetilde{\mathbb{P}}$ and we also have

$$dX(t) = \tilde{g}(X(t))dt + \sigma(X(t))d\widetilde{W}(t), \quad X(0) = x_0.$$

for a certain process $\Theta(t)$. This means that a single process $\Theta(t)$ must satisfy the market price of risk equations

$$-\alpha^{P}(t,T) + \frac{1}{2}(\sigma^{P}(t,T))^{2} = \sigma^{P}(t,T)\Theta(t)$$
(1.12)

for each $T \in (0, \overline{T}]$. This process $\Theta(t)$ is called the *market price of risk* and we need as many of them to solve the market price of risk equations as there are Brownian motions in our model.

We first recall that

$$\frac{\partial}{\partial T}\alpha^P(t,T) = \alpha^f(t,T)$$

and

$$\frac{\partial}{\partial T}\sigma^P(t,T)=\sigma^f(t,T)$$

Now we differentiate the market price of risk equations with respect to T to get

$$\alpha^{f}(t,T) + \sigma^{P}(t,T)\sigma^{f}(t,T) = \sigma^{f}(t,T)\Theta(t)$$
(1.13)

Solving (1.13) for $\Theta(t)$ gives

$$\Theta(t) = \frac{\alpha^f(t,T)}{\sigma^f(t,T)} - \sigma^P(t,T), \quad 0 \le t \le T.$$
(1.14)

and we have the following theorem.

Theorem 1.2.2. Heath-Jarrow-Morton no-arbitrage condition. A term-structure model for zero-coupon bond prices of all maturities $T \in (0,\overline{T}]$ and driven by a single Brownian motion does not admit arbitrage if there exists a process $\Theta(t)$ such that (1.14) holds for all $0 \le t \le T \le \overline{T}$. The adapted processes $\alpha^f(t,T)$ and $\sigma^f(t,T)$ are respectively the drift and volatility of the forward rate and $\sigma^P(t,T)$ the volatility of the zero-coupon bond with maturity time T.

For a proof the reader is referred to [11].

1.3 The HJM Model under Risk-Neutral Measure

In the previous section we have seen that the differential of the discounted bond price under the risk-neutral measure $\widetilde{\mathbb{P}}$ can be written as

$$d(D(t)P(t,T)) = -D(t)P(t,T)\sigma^{P}(t,T)d\widetilde{W}(t)$$

where $d\widetilde{W}(t)$ is given by (1.11). Noticing that $d\frac{1}{D(t)} = \frac{R(t)}{D(t)}dt$, the undiscounted bond price process is

$$dP(t,T) = d\left(\frac{1}{D(t)} \cdot D(t)P(t,T)\right)$$
(1.15)

$$= \frac{R(t)}{D(t)}D(t)P(t,T)dt - \sigma^{P}(t,T)\frac{1}{D(t)}D(t)P(t,T)d\widetilde{W}(t)$$
(1.16)

$$= R(t)P(t,T)dt - \sigma^{P}(t,T)P(t,T)d\widetilde{W}(t).$$
(1.17)

If we assume that the HJM no-arbitrage condition (1.14) is satisfied we can write the forward rate process (1.5) as

$$df_T(t) = \alpha^F(t,T)dt + \sigma^F(t,T)dW(t) = \sigma^F(t,T)\sigma^B(t,T)dt + \sigma^F(t,T)\left(\Theta(t)dt + dW(t)\right) = \sigma^F(t,T)\sigma^B(t,T)dt + \sigma^F(t,T)d\widetilde{W}(t).$$

Altogether we have the following theorem.

Theorem 1.3.1. Term-structure evolution under risk-neutral measure. In a term-structure model satisfying the HJM no-arbitrage condition Theorem 1.2.2, the forward rates evolve according to

$$df_T(t) = \sigma^f(t, T)\sigma^P(t, T)dt + \sigma^f(t, T)d\widetilde{W}(t)$$
(1.18)

and the zero-coupon bond prices evolve according to the equation

$$dP(t,T) = R(t)P(t,T)dt - \sigma^P(t,T)P(t,T)d\widetilde{W}(t)$$
(1.19)

where $\widetilde{W}(t)$ is a Brownian motion under a risk-neutral measure $\widetilde{\mathbb{P}}$. Here $\sigma^P(t,T) = \int_t^T \sigma^f(t,u) du$ and R(t) = f(t,t) is the interest rate. Moreover, the discounted bond prices follow the process

$$d(D(t)P(t,T)) = -D(t)P(t,T)\sigma^{P}(t,T)dW(t).$$
(1.20)

The solution to (1.19) is given by

$$D(t)B(t,T) = B(0,T)e^{-\int_0^t \sigma^B(s,T)d\widetilde{W}(s) - \frac{1}{2}\int_0^t (\sigma^B(s,T))^2 ds}.$$
(1.21)

We have the following theorem to change from risk-neutral measure.

Theorem 1.3.2. Change of risk-neutral measure. Let S(t) and N(t) be the prices of two assets denominated in a common currency and let $\sigma(t) = (\sigma_1(t), \ldots, \sigma_d(t))$ and $\nu(t) = (\nu_1(t), \ldots, \nu_d(t))$ denote their respective volatility processes. Suppose we have the following SDE's

$$d(D(t)S(t)) = D(t)S(t)\sigma(t) \cdot d\widetilde{W}(t), \quad d(D(t)N(t)) = D(t)N(t)\nu(t) \cdot d\widetilde{W}(t)$$
(1.22)

where $\widetilde{W}(t) = (\widetilde{W}_1(t), \dots, \widetilde{W}_d(t))$ is a d-dimensional Brownian motion under the risk-neutral measure $\widetilde{\mathbb{P}}$. Now suppose we write S(t) in amounts of N(t) so that the price of S(t) becomes $S^{(N)}(t) = \frac{S(t)}{N(t)}$. Under a new probability measure $\widetilde{\mathbb{P}}^{(N)}$ given by

$$\widetilde{\mathbb{P}}^{(N)}(A) = \frac{1}{N(0)} \int_{A} D(T)N(T)d\widetilde{\mathbb{P}} \quad \forall A \in \mathcal{F},$$
(1.23)

the process $S^{(N)}(t)$ is a martingale. Moreover

$$dS^{(N)}(t) = S^{(N)}(t)[\sigma(t) - \nu(t)] \cdot d\widetilde{W}^{(N)}(t), \qquad (1.24)$$

where the d-dimensional Brownian motion $\widetilde{W}^{(N)}(t) = (\widetilde{W}_1^{(N)}(t), \ldots, \widetilde{W}_d^{(N)}(t))$ is provided by the multidimensional Girsanov theorem

$$\widetilde{W}_j^{(N)}(t) = -\int_0^t \nu_j(s)ds + \widetilde{W}_j(t), \quad j = 1, \dots, d.$$

The proof that $S^{(N)}(t)$ is a martingale can be found in [12].

1.4 Relations between Forward Rates, Zero-coupon Bonds and Short Rates

Term-structure models driven by Brownian motion are HJM models. Such models include forward rates for which the drift and volatility have to satisfy the conditions of Theorem 1.2.2 for a risk-neutral measure to exist. Once these conditions are satisfied the equations (1.18)-(1.20) describe the forward rates and the bonds under the risk-neutral measure. We will take a closer look at relations between the forward rate, the zero-coupon bond prices and the short rates.

From equation (1.3) we see that if we want to make a model for the bond prices we can do this in a few ways:

- We may specify the dynamics of the short rate and then try to derive bond prices using no-arbitrage arguments.
- We may directly specify the dynamics of all possible zero-coupon bonds.
- We may specify the dynamics of all possible forward rates and use (1.3) to obtain bond prices.

Now suppose we have the following dynamics:

Short rate dynamics

$$dR(t) = a(t)dt + b(t)dW(t).$$
 (1.25)

Bond price dynamics

$$dP(t,T) = P(t,T)m(t,T)dt + P(t,T)v(t,T)dW(t).$$
(1.26)

Forward rate dynamics

$$df_T(t) = \alpha^f(t, T)dt + \sigma^f(t, T)dW(t).$$
(1.27)

The Brownian motions W are allowed to be vector valued, then v(t, T) and $\sigma^f(t, T)$ are row vectors. The processes a(t) and b(t) are scalar adapted processes and $m(t, T), v(t, T), \alpha^f(t, T)$ and $\sigma^f(t, T)$ are adapted processes in the parameter T. The equations (1.26) and (1.27) are SDE's in t for each time to maturity T. With some regularity assumptions for the processes; $m(t, T), v(t, T), \alpha^f(t, T)$ and $\sigma^f(t, T)$ and $\sigma^f(t, T)$ are continuously differentiable in T and all processes allow for the interchange of order of integration, we have the following result.

Proposition 1.4.1. Relation between forward rates, bond prices and short rates.

1. If P(t,T) satisfies (1.26) then for the forward rate dynamics we have

$$df_T(t) = \alpha^f(t, T)dt + \sigma^f(t, T)dW(t)$$

where $\alpha^{f}(t,T)$ and $\sigma^{f}(t,T)$ are given by

$$\begin{cases} \alpha^{f}(t,T) &= \frac{\partial v(t,T)}{\partial T}v(t,T) - \frac{\partial}{\partial T}m(t,T) \\ \sigma^{f}(t,T) &= -\frac{\partial}{\partial T}v(t,T). \end{cases}$$

2. If $f_T(t)$ satisfies (1.27) then the short rate dynamics satisfy

$$dR(t) = a(t)dt + b(t)dW(t)$$

where a(t) and b(t) are given by

$$\left\{ \begin{array}{rll} a(t) &=& \frac{\partial}{\partial T}f(t,t)+\alpha^f(t,t)\\ b(t) &=& \sigma^f(t,t). \end{array} \right.$$

3. If $f_T(t)$ satisfies (1.27) then P(t,T) satisfies (1.6), i.e.

$$dP(t,T) = P(t,T) \left(R(t) - \alpha^{P}(t,T) + \frac{1}{2} (\sigma^{P}(t,T))^{2} \right) dt - \sigma^{P}(t,T) P(t,T) dW(t)$$

where

$$\begin{cases} \alpha^P(t,T) = \int_t^T \alpha^f(t,u) du \\ \sigma^P(t,T) = \int_t^T \sigma^f(t,u) du. \end{cases}$$

A proof of this proposition can be found in [2].

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 $\mathbf{2}$

LIBOR Market Model (with Stochastic Volatility)

In this chapter we will start by explaining the forward LIBOR in a similar way that we did in the previous chapter. Then we will give the general setup of a LIBOR market model to a get a better understanding of the model we are going to use which is a LIBOR market model with stochastic volatility. This model, which we will call the Wu & Zhang stochastic volatility model, is then explained where we will closely follow [15].

2.1 Forward LIBOR and Backset LIBOR

Suppose we set up the same portfolio as we did in section 1.2. Hence, let $0 \le t \le T$ and ΔT be given, we take a short position of size 1 in *T*-maturity zero-coupon bonds and a long position of size $\frac{P(t,T)}{P(t,T+\Delta T)}$ in $(T + \Delta T)$ -maturity zero-coupon bonds. The difference with section 1.2 is that we now instead of using continuous compounding, will use *simple interest rates*. The problem with continuous compounding is that for a certain choice of the drift function in equation (1.18) due to randomness some paths might explode immediately no matter what initial condition is used. For a brief discussion see [13]. By using simple interest rates we will get rid of the *dt* term that causes this problem. To deduce the interest rate which we have to apply to the invested amount of size 1 to receive $\frac{P(t,T)}{P(t,T+\Delta T)}$ at time $T + \Delta T$ we notice:

$$1 + \Delta T f_T(t) = \frac{P(t,T)}{P(t,T+\Delta T)}$$

or written differently

$$f_T(t) = \frac{P(t,T) - P(t,T + \Delta T)}{\Delta T P(t,T + \Delta T)}.$$
(2.1)

For $0 \le t \le T$ we call $f_T(t)$ the forward LIBOR¹ and for t = T we call it (spot) LIBOR.

When pricing caplets the interest rate that is used on payment date $T + \Delta T$ is the interest rate set on payment date T. We call this rate *backset LIBOR*. We have the following theorem for the price of a backset LIBOR on a notional of 1.

Theorem 2.1.1. Risk-neutral price of a backset LIBOR. Let $0 \le t \le T$ and $\Delta T > 0$ be given. The risk-neutral price at time t on a contract that pays backset LIBOR $f_T(T)$ on payment

 $^{^{1}}$ LIBOR stands for London Interbank Offered Rate. It is the rate at which banks are willing to lend money to and borrow money from other banks.

date $T + \Delta T$ is

$$C(t) = \begin{cases} P(t, T + \Delta T) f_T(t), & 0 \le t \le T, \\ P(t, T + \Delta T) f_T(T), & T \le t \le T + \Delta T. \end{cases}$$
(2.2)

Proof: When $T \leq t \leq T + \Delta T$ we notice that $f_T(T)$ has already been fixed (at time T). Now the value of a contract that pays 1 at time $T + \Delta T$ is $P(t, T + \Delta T)$ and hence, the value of a contract that pays backset LIBOR $f_T(T)$ at time $T + \Delta T$ is $P(t, T + \Delta T)f_T(T)$. When $0 \leq t \leq T$ we see from (2.1) that

$$P(t, T + \Delta T)f_T(t) = \frac{1}{\Delta T}(P(t, T) - P(t, T + \Delta T)).$$

So if we can show that $\frac{1}{\Delta T}(P(t,T) - P(t,T + \Delta T))$ is the value at time $0 \le t \le T$ of the backset LIBOR contract we are done. Suppose, at time $\frac{1}{\Delta T}(P(t,T) - P(t,T + \Delta T))$, we have the amount $\frac{1}{\Delta T}(P(t,T) - P(t,T + \Delta T))$ and we use this to set up a portfolio that is $long \frac{1}{\Delta T}$ zero-coupon bonds with maturity time T and short $\frac{1}{\Delta T}$ zero-coupon bonds with maturity time $T + \Delta T$. At time T we receive $\frac{1}{\Delta T}$ from our long position and we use this amount to buy $\frac{1}{\Delta TP(t,T+\Delta T)}$ zero-coupon bonds with maturity time $T + \Delta T$. The position of our portfolio is now $\frac{1}{\Delta TP(t,T+\Delta T)} - \frac{1}{\Delta T}$ in zero-coupon bonds maturing at $T + \Delta T$. So at time $T + \Delta T$ the portfolio pays

$$\frac{1}{\Delta T P(T, T + \Delta T)} - \frac{1}{\Delta T} = \frac{P(T, T) - P(T, T + \Delta T)}{\Delta T P(T, T + \Delta T)} = f_T(T).$$

We see that the amount $\frac{1}{\Delta T}(P(t,T) - P(t,T + \Delta T))$ we have used at time $0 \le t \le T$ to set up the portfolio must be the risk-neutral price at time t of the payment $f_T(T)$ at time $T + \Delta T$.

In the next section we will elaborate on the LIBOR market model and the pricing of caplets. To this end we will state the next definition and theorem. A proof of the theorem can be found in [14].

Definition 2.1.1. Forward contract. A forward contract is an agreement to pay a specified amount K at a specified payment date T, where $0 \le T \le \overline{T}$, for an asset whose price at t is given as C(t). From now on we will refer to K as the strike (rate). We call the T-forward price $For^{C}(t,T)$ of the asset at time t, where $0 \le t \le T \le \overline{T}$, the value of K that makes the forward contract have price 0 at time t. Hence, the value of the forward contract at time t is zero and there is no arbitrage.

Theorem 2.1.2. *T*-forward price. Assume there is a market for zero-coupon bonds of all maturities. Then the T-forward price $For^{C}(t,T)$ is given as

$$For^{C}(t,T) = \frac{C(t)}{P(t,T)}, \qquad 0 \le t \le T \le \overline{T}.$$
(2.3)

2.2 LIBOR Market Models

Market rate models use real market interest rates, e.g. LIBOR rates to model the evolution of interest rates and can be made to exactly fit market prices. A LIBOR market model is a term structure model which means that the forward rate for a certain period depends on zero-coupon bonds with different maturity times. In fact the initial amount grows by $(1+\Delta T_j f_j(t)), 1 \le j \le N$, where $f_j(t)$ is the forward rate used for the period $[T_j, T_{j+1})$ and $\Delta T_j = T_{j+1} - T_j$ is the tenor².

²In practice ΔT_j is not actually used but is approximated as what we call a day-count convention, denoted in year fraction. It is a way of defining the elapsed time between two dates. This elapsed time then does not necessarily have to agree with the tenor. One can for example have the day-count convention Actual/365 where one has to divide the amount of days between the two dates by the assumed 365 days in a year. This gives the elapsed time between the dates as a fraction of the year. Another day-count convention is 30/360 where one assumes each month to have 30 days and a year 360 days.

2.2. LIBOR MARKET MODELS

We notice that T_2 is the first payment date and T_{N+1} is the last. In the previous section we have seen that forward rates relate to zero-coupon bonds through

$$P(t, T_j) = (1 + \Delta T_j f_j(t)) P(t, T_{j+1}) \quad 1 \le j \le N$$
(2.4)

and we can rewrite this to see that

$$f_j(t) = \frac{1}{\Delta T_j} \left(\frac{P(t, T_j)}{P(t, T_{j+1})} - 1 \right), \quad 1 \le j \le N.$$

In Figure 2.1 we can see a depiction of the relation between forward rates and maturity times.



Figure 2.1: Forward rates and their maturity times.

An interest rate caplet pays at T_{j+1} the difference between the variable (LIBOR) rate at time T_j and the strike rate K > 0 whenever this variable rate exceeds K at time T_j i.e. $(f_j(T_j) - K)^+$. So to price a caplet on payment date T_{j+1} we need to look at the backset LIBOR $f_j(T_j)$. We consider the contract that pays $f_j(T_j)$ at time T_{j+1} whose price we determined earlier in Theorem 2.1.1 as C(t). Now suppose we want to express the price of this payment of backset LIBOR in terms of zero-coupon bonds with maturity time T_{j+1} . Hence, we want to use T_{j+1} -maturity zero-coupon bonds as numéraire³. From Theorem 2.1.1 the price of the contract that pays $f_j(T_j)$ at time T_{j+1} is in terms of the numéraire $P(t, T_{j+1})$ given as

$$\frac{C(t)}{P(t,T_{j+1})} = \begin{cases} f_j(t), & 0 \le t \le T_j, \\ f_j(T_j), & T_j \le t \le T_{j+1}. \end{cases}$$
(2.5)

From Definition 2.1.1 and Theorem 2.1.2 we see that for $0 \le t \le T_j, 1 \le j \le N$, the forward LIBOR $f_j(T_j)$ is the T_{j+1} -forward price of the contract paying backset LIBOR $f_j(T_j)$ at time T_{j+1} .

If we now build a term-structure with a single Brownian motion under the actual probability measure \mathbb{P} and satisfying the Heath-Jarrow-Morton no-arbitrage condition specified in Theorem 1.2.2, then there exists a Brownian motion $\widetilde{Z}(t)$ under a risk-neutral probability measure $\widetilde{\mathbb{P}}$ such that forward rates are given by equation (1.18) and zero-coupon bond prices by equation (1.19). From Theorem 1.3.2 we see that the risk-neutral measure associated with the numéraire $P(t, T_{j+1})$ is given by

$$\widetilde{\mathbb{P}}^{j+1}(A) = \frac{1}{P(0, T_{j+1})} \int_{A} D(T_{j+1}) d\widetilde{\mathbb{P}} \quad \forall A \in \mathcal{F}$$

$$\widetilde{\sigma}^{j+1}(\mu) = \int_{a}^{t} P(\mu, T_{j+1}) d\mu = \widetilde{\sigma}^{j}(\mu)$$
(2.6)

moreover

$$\widetilde{Z}^{j+1}(t) = \int_0^t \sigma^P(u, T_{j+1}) du + \widetilde{Z}(t)$$

is a Brownian motion under the forward measure $\widetilde{\mathbb{P}}^{j+1}$, where $\sigma^P(u, T_{j+1})$ is the volatility of the zero-coupon bond with maturity time T_{j+1} . From Theorem 1.3.2 it also follows that $\frac{C(t)}{P(t,T_{j+1})}$ is a martingale under the measure $\widetilde{\mathbb{P}}^{j+1}$ which directly implies that there must exist an adapted process $\gamma_j(t)$ for $0 \leq t \leq T_j, 1 \leq j \leq N$, such that

$$df_j(t) = \gamma_j(t)f_j(t)d\tilde{Z}^{j+1}, \quad t \in [0, T_{j+1})$$
(2.7)

³For more on the topic of numéraires, and the different kinds of numéraires commonly used in LIBOR market models, the reader is referred to [9] and [14].

where \widetilde{Z}^{j+1} is a Brownian motion under the forward measure $\widetilde{\mathbb{P}}^{j+1}$. In section 2.3 we will see that the process $\gamma_j(t)$ is related to the volatilities of zero-coupon bonds. We see in equation (2.7) that there is no dt term, which was the problematic term in equation (1.18). The solution of this equation is

$$f_j(t) = f_j(0)e^{-\frac{1}{2}\int_0^t \gamma_j^2(s)ds + \int_0^t \gamma_j(s)d\tilde{Z}^{j+1}(s)}.$$

In the case $\gamma_j(t)$ is a deterministic function f_j is lognormally distributed under the measure $\widetilde{\mathbb{P}}^{j+1}$, where $\log f_j(t)$ has mean $-\frac{\log(f_j(0))}{2} \int_0^t \gamma_j(s)^2 ds$ and variance $\int_0^t \gamma_j(s)^2 ds$.

The LIBOR market model leads to the following nice result.

Theorem 2.2.1. Black's formula for pricing a caplet. Consider a caplet that pays $(f_j(T_j) - K)^+$ at time T_{j+1} , $1 \le j \le N$, where K > 0 is the strike. With the assumption that forward LIBOR rate is given by (2.7) and $\gamma_j(t)$ is nonrandom, the price of the caplet at time 0 is given by

$$P(0, T_{j+1})[f_j(0)N(d_+) - KN(d_-)]$$

where

$$d_{\pm} = \frac{1}{\sqrt{\int_0^{T_j} \gamma^2(t, T_j) dt}} \left(\log \frac{f_{T_j}(0)}{K} \pm \frac{1}{2} \int_0^{T_j} \gamma^2(t, T_j) dt \right)$$

and $N(\cdot)$ is the standard normal distribution.

We see here that we still haven't mentioned anything about stochastic volatility, instead we have taken volatilities here to be deterministic functions and this is immediately a limitation of the model. From market data it is usually observed that caplets show implied volatility smiles and choosing the γ_j 's to be deterministic limit this model to reproduce such a smile. This justifies including stochastic volatilities in the model.

In the next section we will explain the details of a specific stochastic volatility model, namely the Wu and Zhang model [15]. This model has many similarities with a model first described by Heston [7]. For the derivation of the model in the next section we will closely follow [15].

2.3 The Wu and Zhang Stochastic Volatility Model

2.3.1 Introduction to the Model

Suppose we have a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\mathcal{F}(t), 0 \leq t \leq \overline{T}$, where \overline{T} is a fixed final time. Let P(t,T) be the price at time t < T of a zero-coupon bond with maturity time $T < \overline{T}$ and face value 1. Under the risk-neutral measure \mathbb{P} this zero-coupon bond P(t,T) is assumed to follow the lognormal process

$$dP(t,T) = P(t,T)(R(t)dt + \sigma_T(t) \cdot d\mathbf{\hat{Z}}(t)), \qquad (2.8)$$

where R(t) is the risk-free interest rate, $\sigma_T(t) = \sigma(t, T)$ is the volatility vector of P(t, T), $\widetilde{\mathbf{Z}}(t)$ is a *d*-dimensional Brownian motion under the risk-neutral measure $\widetilde{\mathbb{P}}$ and "." denotes inner product. For clarity we will explain below what we mean with this inner product.

We also assume the following regularity condition for the volatility $\sigma_T(t)$: for $\sigma_T(t)$ defined on the interval [0, t] for all t < T, $\sigma_T(t)$ satisfies

$$\widetilde{\mathbb{E}}\left[\| \sigma_T \|^2 \right] < \infty, \qquad (\forall t < T)$$

where with $\|\cdot\|$ throughout this document we will always mean the 2-norm, e.g. for the case at hand $\|\sigma_T\| = \|\sigma_T\|_2 = \sqrt{\int_0^t \sigma_T^2(s) ds}$.

In the previous section we have seen that simple interest rates relate to zero-coupon bonds through

$$f_j(t) = \frac{1}{\Delta T_j} \left(\frac{P(t, T_j)}{P(t, T_{j+1})} - 1 \right), \quad 1 \le j \le N.$$
(2.9)

Now we use Itô's lemma to derive an SDE for the forward rates:

$$df_{j}(t) = \frac{1}{\Delta T_{j}} d\left(\frac{P(t,T_{j})}{P(t,T_{j+1})} - 1\right) = \frac{1}{\Delta T_{j}} d\left(\frac{P(t,T_{j})}{P(t,T_{j+1})}\right)$$
$$= \frac{1}{\Delta T_{j}} \left[\left(\frac{1}{P(t,T_{j+1})}\right) dP(t,T_{j}) + P(t,T_{j}) d\left(\frac{1}{P(t,T_{j+1})}\right) + dP(t,T_{j}) d\left(\frac{1}{P(t,T_{j+1})}\right) \right].$$
(2.10)

Since

$$d\left(\frac{1}{P(t,T_{j+1})}\right) = -\frac{1}{P^2(t,T_{j+1})}dP(t,T_{j+1}) + \frac{1}{P^3(t,T_{j+1})}(dP(t,T_{j+1}))^2$$
(2.11)

$$= \frac{1}{P(t,T_{j+1})} \{ (-R(t) + \| \sigma_j(t) \|^2) dt - \sigma_j(t) \cdot d\widetilde{\mathbf{Z}}(t) \},$$
(2.12)

we get

$$df_{j}(t) = \frac{1}{\Delta T_{j}} \frac{P(t, T_{j})}{P(t, T_{j+1})} \left([\sigma_{j}(t) - \sigma_{j+1}(t)] \cdot d\widetilde{\mathbf{Z}}(t) + \| \sigma_{j+1}(t) \|^{2} dt - \sigma_{j}(t) \cdot \sigma_{j+1}(t) dt \right) \\ = \frac{1}{\Delta T_{j}} \frac{P(t, T_{j})}{P(t, T_{j+1})} \left[\sigma_{j}(t) - \sigma_{j+1}(t) \right] \cdot \left[d\widetilde{\mathbf{Z}}(t) - \sigma_{j+1}(t) dt \right].$$
(2.13)

Now from equation (2.9) we see that

$$\frac{P(t,T_j)}{P(t,T_{j+1})} = 1 + \Delta T_j f_j(t)$$
(2.14)

and substituting this into equation (2.13) we have

$$df_j(t) = f_j(t) \frac{1}{\Delta T_j} \frac{1 + \Delta T_j f_j(t)}{f_j(t)} \left[\sigma_j(t) - \sigma_{j+1}(t) \right] \cdot \left[d\widetilde{\mathbf{Z}}(t) - \sigma_{j+1}(t) dt \right].$$

$$= f_j(t) \gamma_j(t) \cdot \left[d\widetilde{\mathbf{Z}}(t) - \sigma_{j+1}(t) dt \right], \qquad 1 \le j \le N,$$

where

$$\gamma_j(t) = \frac{1 + \Delta T_j f_j(t)}{\Delta T_j f_j(t)} \left[\sigma_j(t) - \sigma_{j+1}(t) \right].$$
(2.15)

We see that the volatility $\gamma_j(t)$ of the forward rate $f_j(t)$ is expressed as a function of volatilities of zero-coupon bonds. In the LIBOR market model the functions $\gamma_j(t)$ are chosen first and the volatilities of the zero-coupon bonds then follow by rewriting equation (2.15) as

$$\sigma_{j+1}(t) = -\sum_{k=l(t)}^{j} \frac{\Delta T_k f_k(t)}{1 + \Delta T_k f_k(t)} \gamma_k(t) + \sigma_{T_{l(t)}}(t)$$

where l(t) is the smallest integer such that $T_{l(t)} \ge t$.

Remark 2.3.1. Inner product and factor loadings/weighings. To understand the inner product we notice that in our case we adopt two sources of randomness i.e. d = 2 and we can see the inner product as follows:

$$\gamma_j(t) \cdot d\widetilde{\mathbf{Z}}(t) = \hat{\gamma}_j(t) \left(\beta_1(t) d\widetilde{Z}_1(t) + \beta_2(t) d\widetilde{Z}_2(t) \right)$$

where the weighings $\beta_1(t)$ and $\beta_2(t)$ are such that $\beta_1^2(t) + \beta_2^2(t) = 1$ and the functions $\hat{\gamma}_j(t)$ are free to be chosen. Note that in practice this function $\hat{\gamma}_j(t)$ is the function that is predetermined in the model together with a vector of the weighings for the Brownian motions $d\tilde{Z}_1(t)$ and $d\tilde{Z}_2(t)$.

Now we consider taking stochastic volatilities. Specifically, we take a particular stochastic factor in the volatility term of our model. How this stochastic process is exactly defined will be discussed later. For now we call this process V(t). Under the risk-neutral measure $\widetilde{\mathbb{P}}$ the zero-coupon bonds P(t,T) are now assumed to follow the lognormal process

$$dP(t,T) = P(t,T)(R(t)dt + \sqrt{V(t)}\sigma(t,T) \cdot d\widetilde{\mathbf{Z}}(t)).$$
(2.16)

We can repeat the preceding steps to derive the new differential equation for the forward rates. However, we will not present the mathematical details here but just for the sake of completeness the reader can find these in Appendix A. The result of these computations is that the SDE for the forward rates becomes

$$df_j(t) = f_j(t)\sqrt{V(t)}\gamma_j(t) \cdot \left[d\widetilde{\mathbf{Z}}(t) - \sqrt{V(t)}\sigma_{j+1}(t)dt\right], \qquad 1 \le j \le N.$$

We return to our stochastic volatility process V(t) and we assume this evolves according to a Cox-Ingersoll-Ross (CIR) process⁴, namely

$$dV(t) = \kappa(\theta - V(t))dt + \epsilon \sqrt{V(t)}dW(t),$$

where κ, θ and ϵ (ϵ not necessarily a small number) are constants and W(t) a Brownian motion such that ($\mathbf{Z}(t), W(t)$) is a (d+1)-dimensional Brownian motion under the risk-neutral measure. Hence, we have the following SDE's for our LIBOR market model with stochastic volatility

$$df_j(t) = f_j(t)\sqrt{V(t)}\gamma_j(t) \cdot \left[d\widetilde{\mathbf{Z}}(t) - \sqrt{V(t)}\sigma_{j+1}(t)dt\right], \qquad 1 \le j \le N$$
(2.17)

$$dV(t) = \kappa(\theta - V(t))dt + \epsilon \sqrt{V(t)}d\widetilde{W}(t).$$
(2.18)

In this model correlations between the stochastic volatility process and the forward rates are allowed and are given by

$$\widetilde{\mathbb{E}}\left[\left(\frac{\gamma_j(t)}{\|\gamma_j(t)\|} \cdot d\widetilde{\mathbf{Z}}(t)\right) \cdot d\widetilde{W}(t)\right] = \rho_j(t)dt, \quad \text{with } \|\rho_j(t)dt\| \le 1.$$
(2.19)

2.3.2 The Model under the Forward Measure

We want to change from measure $\widetilde{\mathbb{P}}$ associated with the money market account B(t) to the measure $\widetilde{\mathbb{P}}^{j+1}$ associated with the zero-coupon bond $P(t, T_{j+1})$. We recall that because B(t) is the money market account its process is given by

$$dB(t) = r(t)B(t)dt, \qquad B(0) = 1$$

⁴The CIR process $dV(t) = \kappa(\theta - V(t))dt + \epsilon\sqrt{V(t)}d\widetilde{W}(t)$, where κ, θ and ϵ are positive constants, is a well known mean-reverting process (also to model short interest rates) with mean reversion rate κ and mean reversion level θ . When $V(t) = \theta$ the drift term vanishes. When $V(t) < \theta$ the drift term is positive which pushes V(t) back towards θ and when $V(t) > \theta$ this term is negative which pushes V(t) again towards θ . In this model V(t) does not become negative, if V(t) reaches zero the $d\widetilde{W}(t)$ disappears and the positive drift term $\kappa\theta$ makes V(t) positive again. Moreover, if $2\kappa\theta > \epsilon^2$ then V(t) is strictly positive.

and the solution to this differential equation is given by

$$B(t) = e^{\int_0^t R(s)ds}.$$

Notice that the solution of equation (2.16) is

$$P(t,T) = P(0,T) \exp\left\{\int_0^t \left(R(s) - \frac{1}{2}V(s) \parallel \sigma_T \parallel^2\right) ds + \sqrt{V(s)}\sigma_T(s) \cdot d\widetilde{\mathbf{Z}}(s)\right\}.$$

Now, restricted to a filtration $\mathcal{F}(t)$, the Radon-Nikodým derivative of $\widetilde{\mathbb{P}}^{j+1}$ with respect to $\widetilde{\mathbb{P}}$ is given by⁵

$$\frac{d\widetilde{\mathbb{P}}^{j+1}}{d\widetilde{\mathbb{P}}}\bigg|_{\mathcal{F}(t)} = \frac{P(t,T_{j+1})/P(0,T_{j+1})}{B(t)/B(0)}$$
$$= \frac{e^{\int_0^t \left(R(s) - \frac{1}{2}V(s) \|\sigma_{j+1}\|^2\right) ds + \sqrt{V(s)}\sigma_{j+1}(s) \cdot d\widetilde{\mathbf{Z}}(s)}}{e^{\int_0^t R(s) ds}}$$
$$= e^{\int_0^t - \frac{1}{2}V(s) \|\sigma_{j+1}\|^2 + \sqrt{V(s)}\sigma_{j+1}(s) \cdot d\widetilde{\mathbf{Z}}(s)}.$$

With (multidimensional) Girsanov's theorem we can then identify the market price of risk equations $\Theta(t) = (\Theta_1(t), \dots, \Theta_d(t))$ as

$$\Theta(t) = -\sqrt{V(t)}\sigma_{j+1}(t) \tag{2.20}$$

and we can use this to change measure or we can recall from our discussion in the previous chapter that a LIBOR market model is a martingale under the forward measure. Taking a quick glance at equation (2.17) it is easy to see that the following should hold:

$$\widetilde{\mathbf{Z}}^{j+1}(t) = \widetilde{\mathbf{Z}}(t) - \sqrt{V(t)}\sigma_{j+1}(t)dt$$

To deduce $\widetilde{W}^{j+1}(t)$ we use the following:

$$\begin{split} d\widetilde{W}^{j+1}(t) &= d\widetilde{W}(t) + Cov\left(d\widetilde{W}(t), \sqrt{V(t)}\sigma_{j+1} \cdot d\widetilde{\mathbf{Z}}(t)\right) \\ &= d\widetilde{W}(t) + \sqrt{V(t)}\sigma_{j+1}Cov\left(d\widetilde{W}(t), d\widetilde{\mathbf{Z}}(t)\right) \\ &= d\widetilde{W}(t) + \sqrt{V(t)}\sum_{k=1}^{j} \frac{\Delta T_{k}f_{k}(t)}{1 + \Delta T_{k}f_{k}(t)}\gamma_{k}(t)Cov\left(d\widetilde{W}(t), d\widetilde{\mathbf{Z}}(t)\right) \\ &= d\widetilde{W}(t) + \sqrt{V(t)}\sum_{k=1}^{j} \frac{\Delta T_{k}f_{k}(t)}{1 + \Delta T_{k}f_{k}(t)}\gamma_{k}(t)\frac{\parallel\gamma_{k}(t)\parallel}{\gamma_{k}(t)}Cov\left(d\widetilde{W}(t), \frac{\gamma_{k}(t)}{\parallel\gamma_{k}(t)\parallel} \cdot d\widetilde{\mathbf{Z}}(t)\right) \\ &= d\widetilde{W}(t) + \sqrt{V(t)}\sum_{k=1}^{j} \frac{\Delta T_{k}f_{k}(t)}{1 + \Delta T_{k}f_{k}(t)} \parallel\gamma_{k}(t)\parallel Cov\left(d\widetilde{W}(t), \frac{\gamma_{k}(t)}{\parallel\gamma_{k}(t)\parallel} \cdot d\widetilde{\mathbf{Z}}(t)\right) \\ &\stackrel{(*)}{=} d\widetilde{W}(t) + \sqrt{V(t)}\sum_{k=1}^{j} \frac{\Delta T_{k}f_{k}(t)}{1 + \Delta T_{k}f_{k}(t)} \parallel\gamma_{k}(t)\parallel\rho_{j}(t)dt \\ &= d\widetilde{W}(t) + \sqrt{V(t)}\xi_{j}(t)dt \end{split}$$

where $\xi_j(t) = \sum_{k=1}^j \frac{\Delta T_k f_k(t)}{1 + \Delta T_k f_k(t)} \| \gamma_k(t) \| \rho_j(t)$ and for (*) we notice

$$Cov\left(d\widetilde{W}(t), \frac{\gamma_k(t)}{\|\gamma_k(t)\|} d\widetilde{\mathbf{Z}}(t)\right) = \widetilde{\mathbb{E}}\left[\left(\frac{\gamma_j(t)}{\|\gamma_j(t)\|} \cdot d\widetilde{\mathbf{Z}}(t)\right) \cdot d\widetilde{W}(t)\right] - \widetilde{\mathbb{E}}\left[\frac{\gamma_j(t)}{\|\gamma_j(t)\|} \cdot d\widetilde{\mathbf{Z}}(t)\right] \widetilde{\mathbb{E}}\left[d\widetilde{W}(t)\right] = \rho_j(t)dt$$

⁵See for example [5] for the first equality.

Hence, in terms of the new Brownian motions $\widetilde{\mathbf{Z}}^{j+1}(t)$ and $\widetilde{W}^{j+1}(t)$ under the probability measure $\widetilde{\mathbb{P}}^{j+1}$ the market model becomes

$$df_j(t) = f_j(t)\sqrt{V(t)}\gamma_j(t) \cdot d\widetilde{\mathbf{Z}}^{j+1}(t), \qquad (2.21)$$

$$dV(t) = \left[\kappa\theta - (\kappa + \epsilon\xi_j(t)) V(t)\right] dt + \epsilon \sqrt{V(t)} d\widetilde{W}^{j+1}(t).$$
(2.22)

2.3.3 Preparing the Model for Closed-Form Solutions

We can see that the stochastic volatility is still a square-root process in the SDE for the forward rates under $\widetilde{\mathbb{P}}^{j+1}$. But now there is a dependence on $f_j(t)$ which is in the coefficient $\xi_j(t)$. This dependency will ruin any possibility to evaluate options analytically. However, we can apply a technique called 'freezing coefficients' to remove this dependency. We have the following result which we will also explain.

Proposition 2.3.1. By using 'freezing coefficients' we can remove the dependence of the stochastic volatility process V(t) on the forward rates and we have

$$dV(t) = \kappa \left[\theta - \tilde{\xi}_j(t)V(t)\right] dt + \epsilon \sqrt{V(t)} d\widetilde{W}^{j+1}(t)$$
(2.23)

where $\tilde{\xi}_j(t) = 1 + \frac{\epsilon}{\kappa} \xi_j(t)$ and $\xi_j(t) \approx \sum_{k=1}^j \frac{\Delta T_k f_k(0) \rho_k(t) \|\gamma_k(t)\|}{1 + \Delta T_k f_k(0)}$.

To see this we write the Taylor series of $\xi_j(t)$ where the variable in this case is $f_k(t)$ and we evaluate at point $f_k(0)$ for k = 1, ..., j:

$$\xi_{j}(t) = \sum_{k=1}^{j} \frac{\Delta T_{k} f_{k}(0) \rho_{k}(t) \| \gamma_{k}(t) \|}{1 + \Delta T_{k} f_{k}(0)} \\ + \frac{\rho_{k}(t) \| \gamma_{k}(t) \| \Delta T_{k}}{(1 + \Delta T_{k} f_{k}(0))^{2}} (f_{k}(t) - f_{k}(0)) \\ - \frac{\rho_{k}(t) \| \gamma_{k}(t) \| \Delta T_{k}^{2}}{(1 + \Delta T_{k} f_{k}(0))^{3}} (f_{k}(t) - f_{k}(0))^{2} \\ + \text{ higher order terms.}$$

Since the forward rates are given in percentages even the second order terms already become very small and we expand the Taylor series up to second order to get

$$\xi_{j}(t) = \sum_{k=1}^{j} \frac{\Delta T_{k} f_{k}(0) \rho_{k}(t) \| \gamma_{k}(t) \|}{1 + \Delta T_{k} f_{k}(0)} \\ + \frac{\rho_{k}(t) \| \gamma_{k}(t) \| \Delta T_{k}}{(1 + \Delta T_{k} f_{k}(0))^{2}} (f_{k}(t) - f_{k}(0)) \\ + O\left(\rho_{k}(t) \| \gamma_{k}(t) \| \Delta T_{k}^{2} (f_{k}(t) - f_{k}(0))^{2}\right).$$

We notice that because $f_j(t)$ is a martingale we have $\mathbb{E}^{j+1}[f_j(t)|\mathcal{F}(0)] = f_j(0)$. This implies

$$\widetilde{\mathbb{E}}^{j+1}[\xi_j(t)|\mathcal{F}(0)] = \sum_{k=1}^{j} \frac{\Delta T_k f_k(0)\rho_k(t) \| \gamma_k(t) \|}{1 + \Delta T_k f_k(0)} + O(\rho_k(t) \| \gamma_k(t) \| Var(\Delta T_k(f_k(t))))$$

and

$$Var\left(\xi_{j}(t)|\mathcal{F}(0)\right) \approx O\left(\left(\rho_{k}(t) \parallel \gamma_{k}(t) \parallel\right)^{2} Var(\Delta T_{k}f_{k}(t))\right)$$

According to the model the quadratic variation accumulated by the forward rates is $[f_k, f_k](t) = (f_k(t) || \gamma_k(t) ||)^2 V(t)t$ where [., .](t) denotes quadratic variation. Taking this as an estimate for the variance of $f_k(t)$ gives $Var(\Delta T_k f_k(t)) \approx (\Delta T_k f_k(t) || \gamma_k(t) ||)^2 V(t)t$ and noting that the $f_k(t)$ are small percentages the Taylor expansion of $\xi_i(t)$ is dominated by the first term in the summation.

Option Price Valuation

In this section we present three alternative methods (apart from a Monte Carlo simulation) for pricing caplets using the Wu & Zhang stochastic volatility model. Since the first method has many similarities with the method in [7] we will call it the Heston method. The other two methods are two Fourier inversion methods which we will call Modified Call Value Method and Time Value Method. We will end this chapter by comparing pricing results of these methods to a Monte Carlo simulation.

3.1 Heston Method

From the previous chapter we have seen that by making some assumptions to keep the model analytically tractable, we retain nice equations for the LIBOR rates with stochastic volatility. We start by starting these results. Our LIBOR market model with stochastic volatility is

$$df_j(t) = f_j(t)\sqrt{V(t)}\gamma_j(t) \cdot d\widetilde{\mathbf{Z}}^{j+1}(t), \qquad 1 \le j \le N$$
(3.1)

$$dV(t) = \kappa \left[\theta - \tilde{\xi}_j(t)V(t)\right] dt + \epsilon \sqrt{V(t)} d\widetilde{W}^{j+1}(t)$$
(3.2)

where $(\widetilde{\mathbf{Z}}^{j+1}(t), \widetilde{W}^{j+1}(t))$ is a (d+1)-dimensional Brownian motions under the measure $\widetilde{\mathbb{P}}^{j+1}$. We see that the LIBOR rates are martingales under the forward measure and the square root term multiplying the volatility is a CIR process.

Let $C_{T_j}(k)$ be the price of a call on a European option with maturity time T_j and let $k = \ln(K/f_j(0))$ where K is the strike price and $f_j(0)$ is the zero forward curve for $1 \le j \le N$. The price of this option is given by

$$C_{T_{j}}(k) = P(0, T_{j+1})\Delta T_{j}\widetilde{\mathbb{E}}^{j+1} \left[(f_{j}(T_{j}) - K)^{+} | \mathcal{F}_{0} \right]$$

$$= P(0, T_{j+1})\Delta T_{j}f_{j}(0)\widetilde{\mathbb{E}}^{j+1} \left[\left(\frac{f_{j}(T_{j})}{f_{j}(0)} - \frac{K}{f_{j}(0)} \right)^{+} | \mathcal{F}_{0} \right]$$

$$= P(0, T_{j+1})\Delta T_{j}f_{j}(0)G_{T_{j}}(k), \quad \text{for } j = 1, \dots, N$$

$$(3.3)$$

where

3

$$G_{T_{j}}(k) = \widetilde{\mathbb{E}}^{j+1} \left[\left(\frac{f_{j}(T_{j})}{f_{j}(0)} - \frac{K}{f_{j}(0)} \right)^{+} | \mathcal{F}_{0} \right]$$

$$= \widetilde{\mathbb{E}}^{j+1} \left[e^{\ln(f_{j}(T_{j})/f_{j}(0))} \mathbf{1}_{f_{j}(T_{j}) > K} | \mathcal{F}(0) \right] - \frac{K}{f_{j}(0)} \widetilde{\mathbb{E}}^{j+1} \left[\mathbf{1}_{f_{j}(T_{j}) > K} | \mathcal{F}(0) \right]$$
(3.4)

and $\widetilde{\mathbb{E}}^{j+1}[\cdot]$ denotes the expectation under the forward measure $\widetilde{\mathbb{P}}^{j+1}$. The expectations in equation (4.5) can be evaluated using characteristic functions. Let $X(t) = \ln(f_j(t)/f_j(0))$, then the characteristic function of $X(T_j)$ is defined by

$$\phi_{T_j}\left(X(t), V(t), t; z\right) = \widetilde{\mathbb{E}}^{j+1}\left[e^{zX(T_j)} \left|\mathcal{F}(0)\right], \qquad z \in \mathbb{C}.$$

Using the definition of the characteristic function we can evaluate the expectations in (4.5) by the following equations

$$\widetilde{\mathbb{E}}^{j+1}\left[\mathbf{1}_{f_{j}(T_{j})>K} | \mathcal{F}(0)\right] = \frac{1}{2} + \frac{1}{\pi} \int_{0}^{\infty} \frac{\operatorname{Im}\left[e^{-iv\ln(K/f_{j}(0))}\phi_{T_{j}}(iv)\right]}{v} dv, \qquad (3.5)$$

$$\widetilde{\mathbb{E}}^{j+1}\left[e^{\ln(f_j(T_j)/f_j(0))}\mathbf{1}_{f_j(T_j)>K} | \mathcal{F}(0)\right] = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\operatorname{Im}\left[e^{-iv\ln(K/f_j(0))}\phi_{T_j}(1+iv)\right]}{v} dv.$$
(3.6)

For a derivation of these equations the reader is referred to Appendix A. Now it only rests to determine an expression for $\phi_{T_j}(z)$ and we are ready to compute the call option prices $C_{T_j}(k)$ in equation (3.3).

Assume $\phi = \phi_{T_j} = \widetilde{\mathbb{E}}^{j+1}[\eta(X(T_j), V(T_j))|X(t) = x, V(t) = V]$, which is a conditional expectation of a function of x and v at a later maturity time T_j , is twice differentiable. By Itô's lemma we have:

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial V}dV + \frac{\partial\phi}{\partial t}dt + \frac{1}{2}\left(\frac{\partial^{2}\phi}{\partial V^{2}}(dV)^{2} + 2\frac{\partial^{2}\phi}{\partial V\partial x}dVdx + \frac{\partial^{2}\phi}{\partial x^{2}}(dx)^{2}\right)$$

$$= \left[\frac{\partial\phi}{\partial t} + (\kappa\theta - \kappa\xi V)\frac{\partial\phi}{\partial V} - \frac{1}{2}\lambda^{2}V\frac{\partial\phi}{\partial x} + \frac{1}{2}\epsilon^{2}V\frac{\partial^{2}\phi}{\partial V^{2}} + \epsilon\rho\lambda V\frac{\partial^{2}\phi}{\partial V\partial x} + \frac{1}{2}\lambda^{2}V\frac{\partial^{2}\phi}{\partial x^{2}}\right]dt(3.7)$$
$$+ \sqrt{V}\lambda\frac{\partial\phi}{\partial x}d\widetilde{\mathbf{Z}} + \epsilon\sqrt{V}\frac{\partial\phi}{\partial V}d\widetilde{W}.$$

Since the forward rate f is a martingale we know that ϕ must be a martingale. Hence,

$$\widetilde{\mathbb{E}}\left[d\phi\right] = 0$$

Applying this to equation (3.8) gives the Fokker-Planck forward equation:

$$\frac{\partial\phi}{\partial t} + (\kappa\theta - \kappa\xi V)\frac{\partial\phi}{\partial V} - \frac{1}{2}\lambda^2 V\frac{\partial\phi}{\partial x} + \frac{1}{2}\epsilon^2 V\frac{\partial^2\phi}{\partial V^2} + \epsilon\rho\lambda V\frac{\partial^2\phi}{\partial V\partial x} + \frac{1}{2}\lambda^2 V\frac{\partial^2\phi}{\partial x^2} = 0$$
(3.8)

where

$$\xi = \xi_j(t), \quad \lambda = \parallel \gamma_j(t) \parallel \text{ and } \rho = \rho_j.$$

The Fokker-Planck equation can be used in many ways depending on the terminal condition. For the case at hand we should choose the terminal condition to be $\phi(x, V, T; z) = e^{zx}$ since this renders the solution to be the characteristic function of X_{T_i} .

The next step is to solve equation (3.8) with the terminal condition $\phi(x, V, T; z) = e^{zx}$. Because of the technicality and tediousness of these calculations the reader is referred to Appendix B. Here we will only state the solution in the next proposition.

Proposition 3.1.1. For equation (3.8) with terminal condition $\phi(x, V, T; z) = e^{zx}$ we consider a solution of the form $\tilde{\phi}(x, V, \tau; z) = e^{A(\tau, z) + B(\tau, z)V + zx}$ (= $\phi(x, V, t; z)$), where we make a change of variable $\tau = T - t$ which is the time to maturity. For $\epsilon \neq 0$ and piecewise constant coefficients on the intervals $\tau_j \leq \tau < \tau_{j+1}$, $j = 0, 1, \ldots, m - 1$, $\tilde{\phi}$ is a weak solution where A and B are given as

$$A(\tau, z) = A(\tau_j, z) + \frac{\kappa \theta}{\epsilon^2} \left[(a+d)(\tau - \tau_j) - 2\ln\left(\frac{1 - g_j e^{d(\tau - \tau_j)}}{1 - g_j}\right) \right] B(\tau, z) = B(\tau_j, z) + \frac{(a+d-\epsilon^2 B(\tau_j, z))(1 - e^{d(\tau - \tau_j)})}{\epsilon^2 (1 - g_j e^{d(\tau - \tau_j)})}$$

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for
$$\tau_j \le \tau < \tau_{j+1}, \quad j = 0, 1, \dots, m-1,$$

and where

$$a = \kappa \xi - \rho \epsilon \lambda z, \quad d = \sqrt{a^2 - \lambda^2 \epsilon^2 (z^2 - z)}, \quad g_j = \frac{a + d - \epsilon^2 B(\tau_j, z)}{a - d - \epsilon^2 B(\tau_j, z)}.$$

Although we are totally equipped to calculate option prices by using the analytical formulas given by (3.5)-(3.6), we will in the next section discuss two other possibilities to express option prices without having to resort to time consuming Monte Carlo simulations. Both these methods make use of Fourier (inversion) to arrive at an analytical expression for pricing and in particular can be implemented as Fast Fourier Transforms for even faster pricing.

3.2 Fourier Inversion Methods for Pricing

In this section two analytic expressions for the valuation of an option price will be discussed. These methods make use of Fourier transform and contain the characteristic function which we derived in the previous section.

3.2.1 Modified Call Value Method

This method is first developed by Carr and Madan [4] and can be used to exploit the advantages of using a Fast Fourier Transform (FFT) when doing calculations. This could be done by computing caplet prices for different strikes around zero, see for example [4]. The same applies for the Time Value method which we discuss in the next subsection. However, since we want to compare the performance of the different alternative methods including the Heston method we will not be using FFT but rather different strike rates K.

We are interested in the forward price of the same option given in section 3.1, which is given by

$$C_{T_{j}}(k) = P(0, T_{j+1})\Delta T_{j}\widetilde{\mathbb{E}}^{j+1} \left[(f_{j}(T_{j}) - K)^{+} | \mathcal{F}_{0} \right]$$

$$= P(0, T_{j+1})\Delta T_{j}f_{j}(0)\widetilde{\mathbb{E}}^{j+1} \left[\left(\frac{f_{j}(T_{j})}{f_{j}(0)} - \frac{K}{f_{j}(0)} \right)^{+} | \mathcal{F}_{0} \right]$$

$$= P(0, T_{j+1})\Delta T_{j}f_{j}(0)G_{T_{j}}(k) \quad \text{for } j = 1, \dots, N.$$

Let $q_{T_j}(s)$ denote the risk-neutral probability density function of the stochastic variable $X(T_j) = \ln(f_j(T_j)/f_j(0))$ so that we can write

$$G_{T_j}(k) \equiv \int_k^\infty \left(e^s - e^k\right) q_{T_j}(s) ds.$$

Note that $G_{T_j}(k)$ is not square integrable over $(-\infty, \infty)$ because it tends to 1 as k tends to $-\infty$. However, we will need this property if we want to regain $G_{T_j}(k)$ for "almost every" k through an inverse Fourier transform later. This can be achieved by considering the modified call price defined by:

$$g_{T_i}(k) \equiv e^{\alpha k} G_{T_i}(k) \qquad \text{for } \alpha > 0.$$

For certain positive values of α the function $g_{T_j}(k)$ is square integrable on the real line. The choice of α according to [4] must be such that $\phi_{T_j}(1 + \alpha) < \infty$ however we will take a more practical approach to this. We consider the Fourier transform of $g_{T_j}(k)$ which is given by

$$\psi_{T_j}(v) = \int_{-\infty}^{\infty} e^{ivk} g_{T_j}(k) dk$$

Once we have determined an expression for $\psi_{T_j}(v)$ (in terms of ϕ_{T_j}), we can compute the call option prices using the inverse transform:

$$G_{T_j}(k) = \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{\infty} e^{-ivk} \psi_{T_j}(v) dv = \frac{e^{-\alpha k}}{\pi} \int_0^{\infty} e^{-ivk} \psi_{T_j}(v) dv, \text{ for almost every } k.$$
(3.9)

The second equality holds due to the fact that $G_{T_j}(k)$ is real. This implies that $\psi_{T_j}(v)$ is odd in its imaginary part and even in its real part. Now let us determine the expression for the function $\psi_{T_i}(k)$:

$$\begin{split} \psi_{T_j}(v) &= \int_{-\infty}^{\infty} e^{ivk} g_{T_j}(k) dk = \int_{-\infty}^{\infty} e^{ivk} \int_{k}^{\infty} e^{\alpha k} \left(e^s - e^k \right) q_{T_j}(s) ds dk \\ &= \int_{-\infty}^{\infty} q_{T_j}(s) \int_{-\infty}^{s} \left(e^{s + \alpha k} - e^{(1 + \alpha)k} \right) e^{iuk} dk ds \\ &= \int_{-\infty}^{\infty} q_{T_j}(s) \left[\frac{e^{(1 + \alpha + iv)s}}{\alpha + iv} - \frac{e^{(1 + \alpha + iv)s}}{1 + \alpha + iv} \right] ds \\ &= \int_{-\infty}^{\infty} q_{T_j}(s) \frac{e^{(1 + \alpha + iv)s}}{(\alpha + iv)(1 + \alpha + iv)} ds \\ &= \frac{\phi_{T_j}(1 + \alpha + iv)}{(\alpha + iv)(1 + \alpha + iv)}. \end{split}$$

3.2.2 Time Value Method

In this section a different method is used where we determine the time value of an out-of-the-money option. With $k = \ln(K/f_j(0))$, where K is the strike rate, and $f_j(0)$ is the zero forward curve for $1 \le j \le N$, we let $z_{T_j}(k)$ be the T_j maturity put price of a European option when $K < f_j(0)$ and let it be the T_j maturity call price when $K > f_j(0)$. Notice that these imply k < 0 for the put option and k > 0 for the call option. We will derive an expression for the Fourier transform of $z_{T_j}(k)$ in terms of the characteristic function of X_{T_j} .

First let us take a closer look at the time value of an option. The time value of an option is the option value minus the intrinsic value of the option. The intrinsic value of an option is the payoff of the option by exercising at the current time. Hence,

$$z_{T_j}(k) = \text{option value} - \text{intrinsic value}$$
$$\triangleq G_{T_j}(k) - \left(1 - \frac{K}{f_j(0)}\right)^+. \tag{3.10}$$

Now assume that $z_{T_j}(k)$ is in $L^2(\mathbb{R})$, then we can take its Fourier transform to obtain

$$\zeta_{T_j}(v) = \int_{-\infty}^{\infty} e^{ivk} z_{T_j}(k) dk.$$

When we invert this transform we get the prices of out-of-the-money options:

$$z_{T_j}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivk} \zeta_{T_j}(v) dv.$$

We may define $z_{T_i}(k)$ as

$$z_{T_j}(k) = \int_{-\infty}^{\infty} \left[\left(e^k - e^s \right) \mathbf{1}_{s>k} \mathbf{1}_{k>0} + \left(e^s - e^k \right) \mathbf{1}_{s$$

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As stated before now we take its Fourier transform to get an expression for $\zeta_{T_i}(v)$:

$$\begin{aligned} \zeta_{T_{j}}(v) &= \int_{-\infty}^{\infty} e^{ivk} z_{T_{j}}(k) dk \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\left(e^{k} - e^{s} \right) \mathbf{1}_{s > k} \mathbf{1}_{k > 0} + \left(e^{s} - e^{k} \right) \mathbf{1}_{s < k} \mathbf{1}_{k < 0} \right] q_{T_{j}}(s) ds dk \\ &= \int_{0}^{\infty} e^{ivk} \int_{k}^{\infty} \left(e^{s} - e^{k} \right) q_{T_{j}}(s) ds dk + \int_{-\infty}^{0} e^{ivk} \int_{-\infty}^{k} \left(e^{k} - e^{s} \right) q_{T_{j}}(s) ds dk \\ &= \int_{0}^{\infty} q_{T_{j}}(s) ds \int_{0}^{s} \left(e^{s} e^{ivk} - e^{(1+iv)k} \right) dk + \int_{-\infty}^{0} q_{T_{j}}(s) ds \int_{s}^{0} \left(e^{(1+iv)k} - e^{s} e^{ivk} \right) dk. \end{aligned}$$

After computing the inner integrals and noticing that we can write the outer integrals according to the definition of characteristic functions, we get the result

$$\zeta_{T_j}(v) = \frac{\phi_{T_j}(1+iv) - 1}{iv - v^2}.$$
(3.11)

To compute caplet prices later by a numerical scheme we would need $\zeta_{T_j}(0)$, but from equation (3.11) we see that v = 0 might pose a singularity problem. However, by the martingale property of X_{T_j} we see that for v = 0 we have $\phi_{T_j}(1) = 1$ and this point is a removable singularity.

From equation (3.10) we see that, excluding discounting, we could compute caplet prices by

$$G_{T_j}(k) = \left(1 - \frac{K}{f_j(0)}\right)^+ + \frac{1}{\pi} \int_0^\infty e^{-ivk} \zeta_{T_j}(v) dv.$$
(3.12)

Now we have analytical expressions for all three alternative methods presented and we can turn to numerical implementation.

3.3 Numerical Implementation

3.3.1 Evaluating the Integrals

The integral in equation (3.12) has to be computed numerically. Since it involves integrating on an infinite domain we will have to make a truncation decision for our computations. We could notice the following:

$$\left|\phi_{T_{j}}(1+iv)\right| = \left|\widetilde{\mathbb{E}}^{j+1}\left[e^{(1+iv)X_{T_{j}}}\left|\mathcal{F}(0)\right]\right| \le \widetilde{\mathbb{E}}^{j+1}\left[\left|e^{X_{T_{j}}}e^{ivX_{T_{j}}}\right|\left|\mathcal{F}(0)\right] = \widetilde{\mathbb{E}}^{j+1}\left[e^{X_{T_{j}}}\left|\mathcal{F}(0)\right]\right] = 1$$

where the last equality follows from the martingale property of X_{T_i} . Now we have

$$|\zeta_{T_j}(v)| = \left|\frac{\phi_{T_j}(1+iv) - 1}{iv - v^2}\right| \le \left|\frac{2}{\sqrt{v^4 + v^2}}\right| < \frac{2}{v^2}$$

hence,

$$\left| \int_A^\infty e^{-ivk} \zeta_{T_j}(v) dv \right| \le \int_A^\infty \frac{2}{v^2} dv = \frac{2}{A}.$$

Therefore, a truncation error in the order of one basis point is achieved by truncating the integral at $A = 10^4$. However, computational results show that we can take this bound much smaller and for both the Fourier methods we take this bound to be A = 50.

With a truncation of the integral in equation (3.12) we can now turn to a numerical integration and we choose to implement a composite trapezoidal rule. If we choose a uniform grid with N + 1 grid points (0, 1, ..., N) to approximate the function f(x) on an interval [a, b], then the approximation according to the trapezoidal rule is given by

$$\int_{a}^{b} f(x)dx \approx \frac{h}{2} \sum_{k=0}^{N-1} (f(x_{k+1}) - f(x_{k})) = \frac{h}{2} (f(x_{0}) + 2f(x_{1}) + 2f(x_{2}) + \dots + 2f(x_{N-1}) + f(x_{N}))$$
$$= h \left(\frac{f(x_{0}) + f(x_{N})}{2} + \sum_{k=1}^{N-1} f(x_{k}) \right).$$

where $h = \frac{b-a}{N}$ is the grid spacing and the order of accuracy of the trapezoidal rule is $O(h^2)$.

Hence, we make the following approximation for the integral in equation (3.12)

$$\int_0^\infty e^{-ivk} \zeta_{T_j}(v) dv \approx \left(\frac{\zeta_{T_j}(0)}{2} + \sum_{n=1}^{N-1} e^{-ivk} \zeta_{T_j}(v_n) + \frac{e^{-iv_N k} \zeta_{T_j}(v_N)}{2} \right) \Delta u,$$

where $v_n = n\Delta u$ and $\Delta u = \frac{A}{N}$.

The integral in equation (3.9) for the Modified Call Value method and the integrals (3.5)-(3.6) for the Heston method are also computed using the composite trapezoidal rule as numerical approximation. In all cases we have used A = 50 and N = 10000.

3.3.2 Implementation of Monte Carlo

In this section we want to start by describing the Monte Carlo method used in the simulations and then follow in the next section with implementation results.

For the implementation of the Monte Carlo method we use equations (2.17)-(2.18) i.e. the model under the risk neutral measure. We insert the correlation between the forward rates and the stochastic factor for the volatility in the equation;

$$\frac{df_t(t)}{f_j(t)} = -V(t)\gamma_j(t) \cdot \sigma_{j+1}(t)dt + \sqrt{V(t)}d\widetilde{\mathbf{Z}}(t)
= -V(t)\gamma_j(t) \cdot \sigma_{j+1}(t)dt + \sqrt{V(t)}\left(\sqrt{1-\rho^2}\gamma_j(t) \cdot d\widehat{\mathbf{Z}}(t) + \rho \parallel \gamma_j(t) \parallel_2 d\widetilde{W}(t)\right)$$

where $(\hat{\mathbf{Z}}(t), \widetilde{W}(t))$ is a (d+1)-dimensional Brownian motion. By Itô's lemma and using a log-Euler algorithm as numerical method we get

$$f_{j}(t + \Delta t) = f_{j}(t)e^{-V(t)(\gamma_{j}(t)\cdot\sigma_{j+1}(t) + \frac{1}{2}\|\gamma_{j}(t)\|_{2})\Delta t + \sqrt{V(t)}\sqrt{1 - \rho^{2}}\gamma_{j}(t)\cdot\Delta\hat{\mathbf{Z}}(t) + \rho\|\gamma_{j}(t)\|_{2}\Delta\widetilde{W}(t)}.$$

For the stochastic volatility term we have used a Milstein method.

Definition 3.3.1. Milstein approximation of an SDE. Consider the SDE given by

$$dV(t) = f(V(t))dt + g(V(t))dW(t), \qquad 0 \le t \le T,$$

with initial condition $V(0) = v_0$ and W(t) is a Brownian motion. The approximation of the solution of this SDE according to the Milstein method is given by

$$V_{n+1} = V_n + f(V_n)\Delta t + g(V_n)\Delta W_n + \frac{1}{2}g(V_n)g'(V_n)\left((\Delta W_n)^2 - \Delta t\right), \qquad 0 \le n \le N,$$

where the initial condition $V(0) = v_0$, $\Delta t = \frac{T}{N}$, $g'(v) = \frac{\partial g(v)}{\partial v}$ and ΔW_n are independent and identically distributed normal random variables with mean zero and variance Δt .

Remark 3.3.1. *Adjusting Milstein.* For the calculations of the stochastic volatility we have actually used

$$V_{n+1} = \max\{V_n + f(V_n)\Delta t + g(V_n)\Delta W_n + \frac{1}{2}g(V_n)g'(V_n)\left((\Delta W_n)^2 - \Delta t\right), 0\}$$

since we need to take the square root in our log-Euler scheme for the forward rates.

3.3.3 Results

To test the performance of the closed-form methods we will use two different datasets and hence also two different parameter settings. For the time step $\Delta t = 1/12$ has been taken since we want to price caplets monthly and the number of paths taken in the Monte Carlo simulations is 1000, more paths do not have a big impact especially on the mean of all the path samples which is eventually taken to be the Monte Carlo solution. We want to point out that to reduce the variance in our simulations we make use of antithetic variates¹ in our implementation.

In Figure 3.1 we can see how a Monte Carlo simulation for the forward rates is distributed along with given percentiles using as input a European Central Bank AAA (ECB-AAA) zero curve.



Figure 3.1: Depiction of the course of forward rates in 30 years time.

In Figure 3.2 we can see caplet prices for a Monte Carlo simulation with strike K = 0.01 for all paths simulated and given percentiles. In the same figure we can also see the Monte Carlo (mean) solution and the solution of all three different methods. The prices of these caplets originate from the forward rates in Figure 3.1.

We point out that the parameter α in the Modified Call Value method cannot be held constant when the strike rate is changed since this could lead to big differences between this method and the Monte Carlo price. Hence, α needs to be recalibrated accordingly. Notice that finding the optimal value for α has not been our main focus. However, the values for α have been chosen in

¹See for example [8] for more on antithetic variates and variance reduction techniques.



an area which leads to small differences e.g we can see in Figure 3.3 that a value of $\alpha \approx \frac{1}{4}$ would be most appropriate and this value has also been used in Figure 3.2. The differences between the

Figure 3.2: Caplet prices for K = 0.01 and $\alpha = 1/4$ for the Modified Call Value method.

caplet prices for each method and the caplet price that follows from a Monte Carlo simulation are minimal. Only for the Modified Call Value method we can see a bigger difference for the shorter times to maturity. This has to do with the parameter α that has to be chosen. In Figure 3.3 we see the absolute errors for the Modified Call Value method for strike K = 0.01 and different values for α . The value $\alpha = \frac{1}{4}$ in Figure 3.2 has been chosen from Figure 3.3 in an area to have small differences.



Figure 3.3: Absolute error between the Modified Call Value method and a Monte Carlo simulation for strike price is K = 0.01.

3.3. NUMERICAL IMPLEMENTATION

In Figure 3.4 the differences between the closed-form methods and the Monte Carlo (mean) solution are given when taking different strike values K = 0.01 and K = 0.02 in respectively the upper and lower picture. In Figure 3.5 the same has been done for K = 0.025 and K = 0.03. The strike values K have been chosen around the values in the zero curve.



Figure 3.4: Absolute error between the Time Value, Modified Call Value and the Heston method when comparing these to a Monte Carlo simulation for strike prices K = 0.01 and K = 0.02.



Figure 3.5: Absolute error between the Time Value, Modified Call Value and the Heston method when comparing these to a Monte Carlo simulation for strike prices K = 0.025 and K = 0.03.

3.3. NUMERICAL IMPLEMENTATION

For a different dataset, ECB-AAA Q1-2012 (and hence with different model parameters), we see in Figure 3.6 the differences between caplet prices for each method compared to the Monte Carlo price for strikes when K = 0.01 and K = 0.02 in respectively the upper and lower picture. In Figure 3.7 we can see the same for K = 0.025 and K = 0.03. The strike values K have again been chosen around the values in the zero curve.



Figure 3.6: Absolute error between the Time Value, Modified Call Value and the Heston method when comparing these to a Monte Carlo simulation for strike prices K = 0.01 and K = 0.02. (ECB-AAA data, Q1-2012)



Figure 3.7: Absolute error between the Time Value, Modified Call Value and the Heston method when comparing these to a Monte Carlo simulation for strike prices K = 0.025 and K = 0.03. (ECB-AAA data, Q1-2012)

It is apparent that the performance quality of the methods vary but the Time Value method still performs the best while the Modified Call Value method shows the biggest differences.

3.3. NUMERICAL IMPLEMENTATION

To again stress the choice of parameter α for the Modified Call value method, in Figure 3.8 we see the absolute errors between the Modified Call Value method and Monte Carlo solution for strike values K = 0.01 and K = 0.02. In each depiction we see how the differences vary for different values of α to show the choice for α that we have used in the figures above.



Figure 3.8: Absolute error between the Modified Call Value method and a Monte Carlo simulation for strike prices K = 0.01 and K = 0.02. (ECB-AAA data, Q1-2012)

While the Heston and the Modified Call Value methods sometimes show big pricing differences compared to the Monte Carlo price of a caplet, in general we conclude that the Time Value method offers a good closed-form alternative to approximate caplet prices under the Wu & Zhang stochastic volatility model. However, the circumstances under which this is valid need to be assessed. We recommend the performance to always be tested first for the given set of parameters in the model before used for pricing although bad performance has not been apparent. In the next chapter where we add a new parameter in the model we will see that performance could worsen.

Adding a Displacement Parameter to the Model

4.1 The new Model

We have seen that under the forward measure $\widetilde{\mathbb{P}}^{j+1}$ our extended market model, equations (2.21)-(2.22), is

$$df_j(t) = f_j(t)\sqrt{V(t)\gamma_j(t)} \cdot d\mathbf{Z}^{j+1}(t),$$
$$dV(t) = [\kappa\theta - (\kappa + \epsilon\xi_j(t))V(t)] dt + \epsilon\sqrt{V(t)}d\widetilde{W}^{j+1}(t).$$

Hence, the forward rates are lognormal under this measure. This feature might cause the rates to get quite high in the course of time. To try and resolve this issue we will introduce a *displacement parameter* δ in our model to reduce the rates through the volatility. Our model including the displacement parameter becomes

$$df_j(t) = (f_j + \delta)(t)\sqrt{V(t)}\gamma_j(t) \cdot d\widetilde{\mathbf{Z}}^{j+1}(t), \qquad (4.1)$$

$$dV(t) = \left[\kappa\theta - (\kappa + \epsilon\xi_j(t))V(t)\right]dt + \epsilon\sqrt{V(t)}d\widetilde{W}^{j+1}(t)$$
(4.2)

where the 'freezing coefficients' are now given by

$$\tilde{\xi}_j(t) = 1 + \frac{\epsilon}{\kappa} \xi_j(t)$$
 and $\xi_j(t) \approx \sum_{k=1}^j \frac{\Delta T_k(f_k(0) + \delta)\rho_k(t) \parallel \gamma_k(t) \parallel}{1 + \Delta T_k f_k(0)}.$

Besides being mostly a repeat of section 2.3, the mathematical details of the implications of the displacement parameter are too cumbersome to explain at the moment and therefore the reader is referred to Appendix D. For now let us look at an example of how the displacement parameter might help to dampen the forward rates.

Consider a simple stochastic differential equation for the (lognormal) forward rates (under a certain measure) given by

$$df(t) = f(t)\sigma dZ$$

with initial condition $f(0) = f_0$ and maturing at some time T. The solution to this problem is given by

$$f(t) = f_0 e^{-\frac{1}{2}\sigma^2 t + \sigma\sqrt{(t)Z}}$$

and we see that f(t) is lognormally distributed.

When we include a displacement parameter the stochastic differential equation becomes

$$df(t) = (f(t) + \delta)\sigma dZ \quad \delta > 0, \ f(0) = f_0.$$

The solution is now given by

$$f(t) = (f_0 + \delta)e^{-\frac{1}{2}\sigma^2 t + \sigma\sqrt{t}Z} - \delta.$$

$$(4.3)$$

If all the parameters are kept unchanged the distribution will become wider and our issue of increasing rates is far from resolved. However, this is of course bad practice. For the model to be consistent with market data it has to be recalibrated. Roughly, we could see that recalibrating will affect the volatility in the new model in the following way:

$$\sigma^{new} \approx \frac{f(t)}{f(t) + \delta} \sigma^{old}$$

Since $\delta > 0$ we have $\sigma^{new} < \sigma^{old}$, so the volatility in the new model has become smaller because of the displacement parameter.

The displacement parameter δ , however helpful it might be, also has its disadvantages. First, the aforementioned calibration means more computations and hence a slower pricing method. Second, equation (4.3) hints at the forward rates possibly becoming negative, and in fact they could. In the next section we will discuss the results of implementing the different pricing methods including the displacement parameter.

4.2 The Pricing Methods Including Displacement

To price a European call option, $C_{T_j}(k)$, with maturity time T_j let $k = \ln \frac{K+\delta}{f_j(0)+\delta}$ where K is the strike price and $\delta > 0$ is the displacement parameter. The price of this option is given by

$$C_{T_{j}}(k) = P(0, T_{j+1})\Delta T_{j}\widetilde{\mathbb{E}}^{j+1} \left[(f_{j}(T_{j}) - K)^{+} | \mathcal{F}_{0} \right]$$

$$= P(0, T_{j+1})\Delta T_{j}(f_{j}(0) + \delta)\widetilde{\mathbb{E}}^{j+1} \left[\left(\frac{f_{j}(T_{j}) + \delta}{f_{j}(0) + \delta} - \frac{K + \delta}{f_{j}(0) + \delta} \right)^{+} | \mathcal{F}_{0} \right]$$

$$= P(0, T_{j+1})\Delta T_{j}(f_{j}(0) + \delta)G_{T_{j}}(k) \quad \text{for } j = 1, \dots, N$$

$$(4.4)$$

where

$$G_{T_j}(k) = \widetilde{\mathbb{E}}^{j+1} \left[\left(\frac{f_j(T_j) + \delta}{f_j(0) + \delta} - \frac{K + \delta}{f_j(0) + \delta} \right)^+ | \mathcal{F}_0 \right].$$

This function $G_{T_j}(k)$ will be the main focus in the next subsections for pricing according to the different methods.

4.2.1 Heston Method

To price the call option via characteristic function we let $X(t) = \ln\{(f_j(t) + \delta)/(f_j(0) + \delta)\}$ and define the characteristic function of $X(T_j)$ to be

$$\phi_{T_j} = \widetilde{\mathbb{E}}^{j+1} \left[e^{zX(T_j)} \left| \mathcal{F}(0) \right], \qquad z \in \mathbb{C}.$$

With the definition of the characteristic function we evaluate the expectations in the function $G_{T_j}(k)$,

$$G_{T_j}(k) = \widetilde{\mathbb{E}}^{j+1} \left[e^{\ln\{(f_j(T_j)+\delta)/(f_j(0)+\delta)\}} \mathbf{1}_{f_j(T_j)>K} |\mathcal{F}(0) \right] - \frac{K+\delta}{f_j(0)+\delta} \widetilde{\mathbb{E}}^{j+1} \left[\mathbf{1}_{f_j(T_j)>K} |\mathcal{F}(0) \right].$$

These expectations are now given by

$$\widetilde{\mathbb{E}}^{j+1}\left[\mathbf{1}_{f_{j}(T_{j})>K}\left|\mathcal{F}(0)\right] = \frac{1}{2} + \frac{1}{\pi} \int_{0}^{\infty} \frac{\operatorname{Im}\left[e^{-iv\ln\{(K+\delta)/(f_{j}(0)+\delta)\}}\phi_{T_{j}}(iv)\right]}{v} dv,$$
$$\widetilde{\mathbb{E}}^{j+1}\left[e^{\ln(f_{j}(T_{j})+\delta)/(f_{j}(0)+\delta)}\mathbf{1}_{f_{j}(T_{j})>K}\left|\mathcal{F}(0)\right] = \frac{1}{2} + \frac{1}{\pi} \int_{0}^{\infty} \frac{\operatorname{Im}\left[e^{-iv\ln\{(K+\delta)/(f_{j}(0)+\delta)\}}\phi_{T_{j}}(1+iv)\right]}{v} dv$$

The derivation of the above equations is similar to the one of equations (3.5)-(3.6) and this is explained in Appendix B. The expression for ϕ_{T_j} remains the same as the one explained in Appendix C, with a small alteration of the freezing coefficient $\xi_j(t)$ which is now is as in the previous paragraph.

4.2.2 Modified Call Value Method

Let $q_{T_j}(s)$ be the risk-neutral probability density function of the stochastic variable $X(T_j) = \ln\{(f_j(T_j) + \delta)/(f_j(0) + \delta)\}$ and we can write

$$G_{T_j}(k) \equiv \int_k^\infty \left(e^s - e^k\right) q_{T_j}(s) ds$$

where $k = \ln(K + \delta)/(f_j(0) + \delta)$. The function $G_{T_j}(k)$ is not square integrable over $(-\infty, \infty)$ because when k tends to $-\infty G_{T_j}(k)$ tends to 1. To achieve square integrability of $G_{T_j}(k)$ we multiply by $e^{\alpha k}$ for some $\alpha > 0$ to arrive at

$$g_{T_i}(k) \equiv e^{\alpha k} G_{T_i}(k).$$

Following the same lines as was done in section 3.2.1 we have

$$G_{T_j}(k) = \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{\infty} e^{-ivk} \psi_{T_j}(v) dv = \frac{e^{-\alpha k}}{\pi} \int_{0}^{\infty} e^{-ivk} \psi_{T_j}(v) dv \text{ for almost every } k.$$

where

$$\psi_{T_j}(v) = \frac{\phi_{T_j}(1+\alpha+iv)}{(\alpha+iv)(1+\alpha+iv)}.$$

4.2.3 Time Value Method

We consider again $k = \ln(K/f_j(0))$, where K is the strike rate, and $f_j(0)$ is the rate at t = 0 for every caplet with maturity time T_j . We let $z_{T_j}(k)$ be the T_j maturity put price when $K < f_j(0)$ and let it be the T_j maturity call price when $K > f_j(0)$.

The time value of an option including the displacement factor is now given by

$$z_{T_j}(k) \triangleq G_{T_j}(k) - \left(1 - \frac{K+\delta}{f_j(0)+\delta}\right)^+.$$

Assume that $z_{T_i}(k)$ is in $L^2(\mathbb{R})$ we can take its Fourier transform to obtain

$$\zeta_{T_j}(v) = \int_{-\infty}^{\infty} e^{ivk} z_{T_j}(k) dk.$$

and taking the inverse Fourier transform we get the prices of out-of-the-money options:

$$z_{T_j}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivk} \zeta_{T_j}(v) dv.$$

We may also define $z_{T_i}(k)$ as

$$z_{T_j}(k) = \int_{-\infty}^{\infty} \left[\left(e^k - e^s \right) \mathbf{1}_{s>k} \mathbf{1}_{k>0} + \left(e^s - e^k \right) \mathbf{1}_{s$$

and taking its Fourier transform like we have done in section 3.2.2 to get an expression for $\zeta_{T_j}(v)$ we arrive at

$$\zeta_{T_j}(v) = \frac{\phi_{T_j}(1+iv) - 1}{iv - v^2}$$

Now we can calculate the function $G_{T_i}(k)$ by

$$G_{T_j}(k) = \left(1 - \frac{K + \delta}{f_j(0) + \delta}\right)^+ + \frac{1}{\pi} \int_0^\infty e^{-ivk} \zeta_{T_j}(v) dv.$$
(4.5)

4.3 Results

For implementation of the numerical schemes to compute $G_{T_j}(k)$ we again use a trapezoidal rule. For the Monte Carlo simulation we now have the following:

$$f_j(t + \Delta t) = (f_j(t) + \delta)e^{-V(t)(\gamma_j(t) \cdot \sigma_{j+1}(t) + \frac{1}{2}\|\gamma_j(t)\|_2)\Delta t + \sqrt{V(t)}\sqrt{1 - \rho^2}\gamma_j(t) \cdot \Delta \hat{\mathbf{Z}}(t) + \rho\|\gamma_j(t)\|_2\Delta \widetilde{W}(t)} - \delta$$

where $(\hat{\mathbf{Z}}(t), \widetilde{W}(t))$ is a (d+1)-dimensional Brownian motion. The stochastic volatility factor is again implemented using the adjusted Milstein scheme. For the implementation of all the methods and all the numerical schemes we have used the exact numerical settings as in the previous chapter, in particular for the Monte Carlo method the grid spacings in the simulations are $\Delta t = 1/12$ and we make 1000 simulations using antithetic variates.

The displacement parameter has been chosen small, $\delta = 0.0001$, and this proves very challenging for the methods. In Figures 4.1-4.2 we can see the differences between each of the closed-form methods when compared to the Monte Carlo solution. In this case after recalibration all the parameters in the model become small, of the same order of the displacement parameter. While the Monte Carlo solution remains practically unchanged, the performance of the closed-form methods is drastically reduced. We notice the change in scale on the Absolute Error axis compared to Chapter 3 to accommodate the differences.

In particular, when ϵ approaches zero the functions ϕ might start to not work properly. This makes clear that care needs to be taken with the parameters in the model and while the methods could be used to approximate the Monte Carlo solution, like we have seen in the previous chapter, in some cases they perform poorly.



Figure 4.1: For the model including displacement parameter this figure shows the absolute error between the Time Value, Modified Call Value and the Heston method when comparing these to a Monte Carlo simulation for strike prices K = 0.01 and K = 0.02. (ECB-AAA data, Q1-2012)



Figure 4.2: For the model including displacement parameter this figure shows the absolute error between the Time Value, Modified Call Value and the Heston method when comparing these to a Monte Carlo simulation for strike prices K = 0.025 and K = 0.03. (ECB-AAA data, Q1-2012)

$\mathbf{5}$

Variance Reduction Techniques

In the previous chapter we have seen how adding a displacement factor to the model can reduce high rates. We have also mentioned that to reduce the confidence intervals of our results we have implemented antithetic variates in our Monte Carlo simulations. Another example of a variance reduction technique¹ is the use of a *control variate*. In this chapter we will implement and show the results when we use a control variate instead of a displacement parameter to compute caplet prices under our model. But first we start by explaining what a control variate is.

5.1 A Control Variate

Suppose we have a stochastic variable X and we want to estimate $\mathbb{E}X$. If we can find a random variable Y with know expectation \mathbb{Y} that is, in a way, close to X, this could help us reduce the variance in our simulation. Indeed, we can just as well simulate the random variable

$$Z = X + \mathbb{E}Y - Y$$

because we notice that $\mathbb{Z} = \mathbb{E}X$. However, we notice that

$$Var(Z) = Var(X + \mathbb{E}X - Y) = Var(X - Y).$$

We see that performing a Monte Carlo simulation on the variable Z could help us reduce variance but for it to be helpful we need Var(X - Y) < Var(X). This is what we meant by saying that Y needs to be in a way close to X. Let us show how a control variate works with an example.

Example 5.1.1. Suppose we want to estimate $\mathbb{E}\left[\sin(\sqrt{U})\right]$ where $U \sim U(0,1)$ is uniformly distributed on [0,1]. Since $\sin(U)$ and $\sin(\sqrt{U})$ are close on the given interval we can use $\sin(U)$ as a control variate and notice that $\mathbb{E}[\sin(U)] = \int_0^1 \sin(u) du = 1 - \cos(1)$. Hence, we will sample $Z = \sin(\sqrt{U}) + 1 - \cos(1) - \sin(U)$. In Table 5.1 we can see the 95% confidence intervals for different sampling sizes N when sampling $X = \sin(\sqrt{U})$ and sampling $Z = \sin(\sqrt{U}) + 1 - \cos(1) - \sin(U)$. In the last column we can see the width ratio of the confidence intervals and the improvement of our simulation is clear.

5.2 Using Control Variate in the Model

We recall that our LIBOR market model is given by

$$df_j(t) = f_j(t)\sqrt{V(t)}\gamma_j(t) \cdot \left[d\widetilde{\mathbf{Z}}(t) - \sqrt{V(t)}\sigma_{j+1}(t)dt\right], \qquad 1 \le j \le N$$
(5.1)

$$dV(t) = \kappa(\theta - V(t))dt + \epsilon \sqrt{V(t)}dW(t).$$
(5.2)

¹For more on variance reduction techniques one could consult [8].

Μ	$X = \sin(\sqrt{U})$	$Z = \sin(\sqrt{U}) + 1 - \cos(1) - \sin(U)$	Width ratio
10^{2}	[0.1603, 0.8340]	[0.4672, 0.6935]	2.98
10^{3}	[0.1650, 0.8339]	[0.4673, 0.6935]	2.96
10^{4}	[0.1574, 0.8344]	[0.4668, 0.6935]	2.99
10^{4}	[0.1572, 0.8347]	[0.4665, 0.6934]	2.99

Table 5.1: 95% confidence intervals using standard Monte Carlo and control variate to estimate $\mathbb{E}\left[\sin(\sqrt{(U)})\right]$. In the last column the ratio of the widths is given.

What we want to achieve is to approximate the forward rates in this model with another model for which the mean of the forward rates are known. To achieve this goal we assume the stochastic volatility factor follows a Vasiček process instead of a CIR process. Hence, we then have the following model:

$$df_j^{\dagger}(t) = f_j^{\dagger}(t)\sqrt{V(t)}\gamma_j(t) \cdot \left[d\widetilde{\mathbf{Z}}(t) - \sqrt{V(t)}\sigma_{j+1}(t)dt\right], \qquad 1 \le j \le N$$
(5.3)

$$dV(t) = \kappa(\theta - V(t))dt + \epsilon d\widetilde{W}(t).$$
(5.4)

where we assume the same correlation between the forward rates and the stochastic factor i.e.

$$\widetilde{\mathbb{E}}\left[\left(\frac{\gamma_j(t)}{\|\gamma_j(t)\|} \cdot d\widetilde{\mathbf{Z}}(t)\right) \cdot d\widetilde{W}(t)\right] = \rho_j(t)dt, \quad \text{with } \|\rho_j(t)dt\| \le 1.$$

The SDE $dV(t) = \kappa(\theta - V(t))dt + \epsilon d\widetilde{W}(t)$ has nice properties. First, just as the CIR process it is also mean-reverting with the same mean reversion level and the same mean reversion rate. Second, unlike the CIR process the Vasiček process can be solved analytically. The solution of equation (5.4) is

$$V(t) = e^{-\kappa t} V(0) + \theta (1 - e^{-\kappa t}) + \epsilon e^{-\kappa t} \int_0^t e^{\kappa u} d\widetilde{W}(u).$$
(5.5)

One can verify this by taking the differential of the right-hand side of equation (5.5). It is then easy to see that V(t) is normally distributed with mean $e^{-\kappa t}V(0) + \theta(1 - e^{-\kappa t})$ and variance $\frac{\theta\epsilon^2}{2\kappa^2}(1 - e^{-2\kappa t})$. A consequence of this is that V(t) can become negative and since it is a square root term in our model this is not desirable.

Recalling the purpose of a control variate we have to be able to find the mean of the forward rates in the new model given by equations (5.3)-(5.4). Despite the fact that we now have a closedform solution (5.5) for V(t), it is still rather complicated to find the mean of the forward rates in our new model because of the Itô integral in equation (5.5). Instead we take $V^*(t) = \widetilde{\mathbb{E}}V(t)$ and we notice that with the model

$$df_j^{\dagger*}(t) = f_j^{\dagger*}(t)\sqrt{V^*(t)}\gamma_j(t) \cdot \left[d\widetilde{\mathbf{Z}}(t) - \sqrt{V^*(t)}\sigma_{j+1}(t)dt\right], \qquad 1 \le j \le N$$
(5.6)

$$V^{*}(t) = e^{-\kappa t}V(0) + \theta(1 - e^{-\kappa t})$$
(5.7)

we can use the Black-Scholes-Merton formula for time-varying, non-random interest rate and volatility to calculate $\widetilde{\mathbb{E}}[f_i^{\dagger *}(T_j)]$.

Theorem 5.2.1. Black-Scholes-Merton formula for time-varying, non-random interest rate and volatility. Consider f given by the following SDE

$$df(t) = r(t)f(t)dt + \sigma(t)dZ(t),$$

where r(t) and $\sigma(t)$ are time-varying, non-random functions and \widetilde{Z} is a Brownian motion under the risk-neutral measure $\widetilde{\mathbb{P}}$. Let a maturity time T > 0 and a strike rate K > 0 be given. We consider a European call on f which has value at time t = 0 given as

$$C(0, f(0)) = \widetilde{\mathbb{E}}\left[exp\left\{-\int_0^T r(t)dt\right\}(f(T) - K)^+\right].$$

Let

$$BSM(T, x; K, \Phi, \Sigma) = xN(d_{+}) - e^{-\Phi T}KN(d_{-})$$

where

$$d_{\pm} = \frac{1}{\Sigma\sqrt{T}} \left(\log \frac{x}{K} + (\Phi \pm \Sigma^2/2)T \right)$$

and $N(\cdot)$ is the standard normal distribution. Then the price of the European call at time zero is

$$C(0, f(0)) = BSM\left(T, f(0); K, \frac{1}{T} \int_0^T r(t)dt, \sqrt{\frac{1}{T} \int_0^T \sigma^2(t)dt}\right).$$

Notice that because of the division by K in d_{\pm} we cannot take K = 0 for our purpose but have to take the limit $K \to 0$. Notice also that in (5.6) we have two (independent) Brownian motions. However we can transform this equation using the following

$$\begin{aligned} \gamma_j(t) \cdot d\widetilde{\mathbf{Z}}(t) &= \hat{\gamma}_j(t) \left(\beta_1(t) d\widetilde{Z}_1(t) + \beta_2(t) d\widetilde{Z}_2(t) \right) \\ &= \sqrt{\hat{\gamma}_j^2(t) \left(\beta_1^2(t) + \beta_2^2(t) \right)} dZ_3(t) \\ &= \hat{\gamma}_j(t) d\widetilde{Z}_3(t) \end{aligned}$$

where $\widetilde{Z}_3(t)$ is a Brownian motion under the measure $\widetilde{\mathbb{P}}$.

Hence, equations (5.6)-(5.7) become

$$df_{j}^{\dagger*}(t) = f_{j}^{\dagger*}(t)\sqrt{V^{*}(t)}\hat{\gamma}_{j}(t) \left[d\widetilde{Z}_{3}(t) - \sqrt{V^{*}(t)}\sigma_{j+1}(t)dt \right], \qquad 1 \le j \le N$$
(5.8)

$$V^{*}(t) = e^{-\kappa t} V(0) + \theta(1 - e^{-\kappa t})$$
(5.9)

and we can find $\widetilde{\mathbb{E}}[f_i^{\dagger*}(T_j)]$ setting $K \to 0$ using the BSM formula.

Now instead of simulating forward rates as a random variable

$$X = f_j \tag{5.10}$$

we simulate

$$Z = X + \widetilde{\mathbb{E}}[f_j^{\dagger *}(T_j)] - Y, \qquad 1 \le j \le N$$
(5.11)

where $Y = f_j^{\dagger}$.

5.3 Results

To show results when using a control variate we use the same Monte Carlo schemes as in previous chapters with the same grid spacings $\Delta = 1/12$ and again 1000 simulations using antithetic variates. The data is again from ECB-AAA. In Figure 5.1 we can see the results of a (standard) Monte Carlo simulation, a simulation of X in equation (5.10). In the same figure we can a Monte Carlo simulation using control variate, a simulation of Z in equation (5.11). We see that when using the control variate the variance in our simulation has reduced i.e. we have tighter confidence intervals. In Figure (5.2) we see caplet prices that follow from our simulation for strike K = 0.01. It is apparent that we have also reduced the confidence intervals of the caplet prices.



Figure 5.1: Forward rates using a standard Monte Carlo simulation and using a control variate (CV).



Figure 5.2: Caplet prices for K = 0.01 using a standard Monte Carlo simulation and using a control variate (CV).

5. VARIANCE REDUCTION TECHNIQUES

6

Conclusions and Recommendations

6.1 On the Methods

We have looked at three alternative closed-form methods to approximate the model described in section 2.3. The advantage of these methods is that they are faster than a Monte Carlo simulation. Though, since they are approximations we have to deal with pricing differences.

We have seen for the model described in section 2.3 that the Heston method and the two Fourier inversion methods we have treated are able to approximate the Monte Carlo solution to the model well without a displacement parameter δ included. However, when we include a displacement parameter, because of the recalibration of all parameters in the model the results might be not too pleasing. Since this is true for all the alternative methods we point out that it has to do with the characteristic function ϕ . Some choices of the parameters might just not work well and the parameter values that are appropriate or not to use for these methods has to be studied.

We have also used a trapezoidal rule to approximate the integrals in all the methods. The effect of using more accurate numerical methods can be studied.

For the stochastic volatility factor V(t) in the model we used Milstein's method. We have also implemented this next to a moment matched log-normal scheme¹ for our stochastic volatility factor given by:

$$V(t + \Delta t) = \mathbb{E}[V(t + \Delta t)|\mathcal{F}(t)]e^{\frac{1}{2}\Lambda^2(t)\Delta t + \Lambda(t)\Delta\widetilde{W}(t)}$$

where

$$\Lambda^{2}(t) = \frac{1}{\Delta t} \ln \frac{\mathbb{E}[V^{2}(t + \Delta t) | \mathcal{F}(t)]}{(\mathbb{E}[V(t + \Delta t) | \mathcal{F}(t)])^{2}}$$

and

$$\begin{split} \mathbb{E}[V(t+\Delta t)|\mathcal{F}(t)]) &= \theta + (V(t)-\theta)e^{-\kappa\Delta t} \\ \mathbb{E}[V^2(t+\Delta t)|\mathcal{F}(t)] &= (1+\frac{\epsilon^2}{2\kappa\theta})(\mathbb{E}[V(t+\Delta t)|\mathcal{F}(t)])^2 - \frac{\epsilon^2}{2\kappa\theta}e^{-2\kappa\Delta t}V^2(t). \end{split}$$

This method automatically assures that the term V(t) will not become negative which is something we have to take into account in our model. Since the differences between using the Milstein scheme and a moment matched log-normal scheme are negligible we wouldn't expect drastic changes using

¹See for example [1].

other (more accurate) methods.

Overall we have seen that the Time Value method performs the best to approximate the Monte Carlo solution. However, we want to stress that care needs to be taken with the choice of parameters e.g. when the volatility term ϵ in the stochastic volatility factor is equal to zero a Monte Carlo simulation will have no problem computing a solution while the closed-form methods will break down. See Proposition 3.1.1 and the large differences in the results of Chapter 4.

The problem of the methods breaking down also appears when we take K = 0 because for all the methods we have taken $k = \ln(\frac{K}{f_j(0)})$ in the integral. For values of K near zero we have noticed from implementation results that the Heston method shows increasing differences while the Time Value method remains close to the Monte Carlo solution.

Because of increasing uncertainty in the course of time, the distribution of the forward rates will widen. Instead of adding a displacement parameter in the model we have proposed the technique of a control variate. A disadvantage of a control variate is that one still has to resort to Monte Carlo simulations with even more calculations. However, negative interest rates are less apparent then when trying to control rates with a displacement parameter. The use of control variates in this model as a variance reduction technique deserves more research.

6.2 The Cap/"Plafondrente" Option

The "Plafondrente" (or Cap) option on a mortgage is a way for a client to protect him or herself against high interest rates which have to be paid on the loan amount of the mortgage.

When issuing a mortgage a client could choose to pay the interest part of the monthly payments agreed in the contract as a fixed interest rate. Since mortgages are in general contracts that last for a substantial amount of years, the interest amounts paid could be high. The client could also choose to pay according to a variable interest rate which might depend for example on the 1-month EURIBOR rate, r_k^{EUR} . This means that each month the interest rate is redefined and set equal to the 1-month EURIBOR interest rate prevailing at the end of the previous month. By paying this short lived interest rate the client can benefit from a relatively low interest rate in comparison to the fixed rate. However, since this interest rate is variable it can also become high, maybe higher than the client would be willing to pay.

To protect oneself from (too) high interest rates the client could buy a Cap option. A maximum interest rate is then agreed upon in the contract and the client will never pay more than this maximum, r^{max} . If the interest rate which needs to be paid is lower than the cap, the client will pay the interest rate. However if the interest rate should exceed the maximum interest rate, the client will pay the maximum. This is illustrated in Figure 6.1. It is clear that whenever the interest rate should exceed the maximum the bank has a loss namely, the amount on the mortgage outstanding times the difference between the interest rate that should have been paid minus the maximum rate, $A_k(r_k^{EUR} - r^{max})$, where A_k is the amount at time k on which interest should be paid. This option is of course not free and the losses are charged to the client. The costs are computed upon issue and are translated into an interest rate amount that will become part of the interest rate that needs to be paid. This means that to hedge its position the bank will buy caplets that have a payoff with strike rate equal to the maximum rate in the client's contract. This cap is the Cap/"Plafondrente" option.

The pricing of caps is done using Black's formula which we have discussed earlier and we know that in a LIBOR market model cap prices should agree with prices using Black's formula. However, we have added stochastic volatility in the LIBOR market model to alleviate the assumption in Black's model that volatilities are nonrandom. To compare cap prices using the methods described in this



Figure 6.1: Illustration of the Cap/"Plafondrente" option.

document to prices using Black's formula one has to first calibrate the models to prices given by Black's formula. With the calibrated parameters one can then use the models for pricing. Time constraints have not allowed us to study this.

6. CONCLUSIONS AND RECOMMENDATIONS

Appendix A

Adding Stochastic Volatility to the LIBOR Market Model

Note that the process for the zero-coupon bonds is given as

$$dP(t,T) = P(t,T)(R(t)dt + \sqrt{V(t)}\sigma(t,T) \cdot d\widetilde{\mathbf{Z}}(t)).$$

From equation (2.10) and substituting for the new process for dP(t,T) we have

$$df_{j}(t) = \frac{1}{\Delta T_{j}} \left(\frac{P(t,T_{j})}{P(t,T_{j+1})} \left[R(t)dt + \sqrt{V(t)}\sigma_{j}(t) \cdot d\widetilde{\mathbf{Z}}(t) \right] + P(t,T_{j}) \left[-\frac{1}{P^{2}(t,T_{j+1})} \left\{ P(t,T_{j+1}) \left[R(t)dt + \sqrt{V(t)}\sigma_{j+1}(t) \cdot d\widetilde{\mathbf{Z}}(t) \right] \right\} + \frac{1}{P^{3}(t,T_{j+1})} P^{2}(t,T_{j+1})V(t) \| \sigma(t,T_{j+1}) \|_{2}^{2} dt \right] - \frac{P(t,T_{j})}{P(t,T_{j+1})}V(t)\sigma_{j+1}(t) \cdot \sigma_{j}(t)dt \right)$$

$$= \frac{1}{\Delta T_{j}} \frac{P(t,T_{j})}{P(t,T_{j+1})} \left([\sigma_{j}(t) - \sigma_{j+1}(t)] \sqrt{V(t)} \cdot d\widetilde{\mathbf{Z}}(t) + V(t) \| \sigma(t,T_{j+1}) \|_{2}^{2} dt - \sigma_{j}(t) \cdot \sigma_{j+1}(t)V(t)dt \right)$$

$$= \frac{1}{\Delta T_{j}} \frac{P(t,T_{j})}{P(t,T_{j+1})} \left[\sigma_{j}(t) - \sigma_{j+1}(t) \right] \cdot \left[\sqrt{V(t)} d\widetilde{\mathbf{Z}}(t) - V(t)\sigma_{j+1}(t)dt \right]. \quad (A.1)$$

Substituting (2.14) into (A.1) gives

$$df_{j}(t) = f_{j}(t)\frac{1}{\Delta T_{j}}\frac{1+\Delta T_{j}f_{j}(t)}{f_{j}(t)}\left[\sigma_{j}(t)-\sigma_{j+1}(t)\right]\cdot\left[\sqrt{V(t)}d\widetilde{\mathbf{Z}}(t)-V(t)\sigma_{j+1}(t)dt\right].$$

$$= f_{j}(t)\sqrt{V(t)}\gamma_{j}(t)\cdot\left[d\widetilde{\mathbf{Z}}(t)-\sqrt{V(t)}\sigma_{j+1}(t)dt\right], \qquad 1 \le j \le N,$$

where $\gamma_j(t)$ is unchanged and thus remains as given by equation (2.15).

56APPENDIX A. ADDING STOCHASTIC VOLATILITY TO THE LIBOR MARKET MODEL

Appendix B

Introducing Characteristic Functions

Let $X(t) = \ln(f_j(t)/f_j(0))$ and define the characteristic function of $X(T_j)$ being

$$\phi_{T_j}(z) = \widetilde{\mathbb{E}}^{j+1} \left[e^{zX(T_j)} \left| \mathcal{F}(0) \right] = \int_{-\infty}^{\infty} e^{zX(T_j)} dF(X(T_j)), \qquad z \in \mathbb{C},$$

where $F(X(T_j))$ is the distribution of the stochastic variable $X(T_j) = \ln(f_j(T_j)/f_j(0))$. Following Gil-Pelaez [6] we set $X(T_j) = x$ and note that

$$\begin{split} \lim_{\delta \downarrow 0, T \uparrow \infty} \int_{\delta}^{T} \frac{e^{iv \ln \frac{K}{f_{j}(0)}} \phi(-v) - e^{iv \ln \frac{K}{f_{j}(0)}} \phi(v)}{2\pi i v} dv \quad \stackrel{(*),(**)}{=} \quad \lim_{\delta \downarrow 0, T \uparrow \infty} \int_{\delta}^{T} \int_{-\infty}^{\infty} \frac{\sin \left(v \left[\ln \left(\frac{K}{f_{j}(0)}\right) - x\right]\right)}{\pi v} dv dF(x) \\ \stackrel{(***)}{=} \quad \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\pi}{2} \operatorname{sgn} \left(\ln \left(\frac{K}{f_{j}(0)}\right) - x\right) dF(x) \\ = \quad \frac{1}{2} \left[\int_{-\infty}^{\ln \frac{K}{f_{j}(0)}} dF(x) - \int_{\ln \frac{K}{f_{j}(0)}}^{\infty} dF(x)\right] \\ = \quad \frac{1}{2} \left[F \left(\ln \frac{K}{f_{j}(0)}\right) - \left(1 - F \left(\ln \frac{K}{f_{j}(0)}\right)\right)\right] \\ = \quad \frac{1}{2} \left[2F \left(\ln \frac{K}{f_{j}(0)}\right) - 1\right] \\ = \quad F \left(\ln \frac{K}{f_{j}(0)}\right) - \frac{1}{2}, \end{split}$$

where in the first equality we used (*) and (**), which are the definition of the characteristic function $e^{ivy}\phi(-v) - e^{-ivy}\phi(v) = \int_{-\infty}^{\infty} \left(e^{iv(y-x)} - e^{iv(x-y)}\right) dF(x)$ and $\frac{e^{iv(y-x)} - e^{iv(x-y)}}{2\pi iv} = \frac{\sin(v(y-x))}{\pi v}$ respectively. In the second equality we used (* * *) : $\int_{0}^{\infty} \frac{\sin(\alpha x)}{x} dx = \frac{\pi}{2} \operatorname{sgn} \alpha$. Since

$$\frac{e^{ivx}\phi(-v) - e^{-ivx}\phi(v)}{2\pi iv} = -\frac{1}{\pi} \operatorname{Re}\left(\phi(v)\frac{e^{-ivx}}{iv}\right)$$

we have that

$$\begin{split} F\left(\ln\frac{K}{f_j(0)}\right) &= \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \frac{\operatorname{Re}\left(\phi(v)e^{-iv\ln\frac{K}{f_j(0)}}\right)}{iv} dv \\ 1 - \widetilde{\mathbb{E}}^{j+1}\left[\mathbf{1}_{\left\{\ln\frac{f_j(T_j)}{f_j(0)} \ge \ln\frac{K}{f_j(0)}\right\}}\right] &= \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \frac{\operatorname{Re}\left(\phi(v)e^{-iv\ln\frac{K}{f_j(0)}}\right)}{iv} dv \end{split}$$

and hence

$$\widetilde{\mathbb{E}}^{j+1}\left[\mathbf{1}_{\{f_j(T_j)\geq K\}}\right] = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\operatorname{Im}\left(\phi(v)e^{-iv\ln\frac{K}{f_j(0)}}\right)}{v} dv.$$

The expectation $\widetilde{\mathbb{E}}^{j+1}\left[e^{X(T_j)}\mathbf{1}_{\{f_j(T_j)\geq K\}}\right]$ can be computed in a similar way.

Appendix C

Solution for the Characteristic Function

The characteristic functions $\phi(x, V, t; z)$ for the forward rates which we need to solve to price caplets in accordance to Heston, satisfy the Fokker-Planck partial differential equation given by

$$\frac{\partial\phi}{\partial t} + (\kappa\theta - \kappa\xi V)\frac{\partial\phi}{\partial V} - \frac{1}{2}\lambda^2 V\frac{\partial\phi}{\partial x} + \frac{1}{2}\epsilon^2 V\frac{\partial^2\phi}{\partial V^2} + \epsilon\rho\lambda V\frac{\partial^2\phi}{\partial V\partial x} + \frac{1}{2}\lambda^2 V\frac{\partial^2\phi}{\partial x^2} = 0$$
(C.1)

with terminal condition

$$\phi(x, V, T; z) = e^{zx} \tag{C.2}$$

and where the functions ξ, λ and ρ are defined as

$$\xi = \xi_j(t), \quad \lambda = \parallel \gamma_j(t) \parallel \text{ and } \rho = \rho_j.$$

To determine $\phi(x, V, T; z)$ we follow Heston [7] and consider a solution of the form

$$\tilde{\phi}(x,V,\tau;z) = e^{A(\tau,z) + B(\tau,z)V + zx}$$
(C.3)

where we have made a change of variable $\tau = T - t$ which is the time to maturity. Substituting this solution into (C.1) and (C.2) we have the following result.

Proposition C.0.1. If we consider a solution of the form (C.3) for the Fokker-Planck partial differential equation (C.1) with terminal condition given by (C.2) we get the following two differential equations

$$\frac{dA}{d\tau} = \kappa \theta B \tag{C.4}$$

$$\frac{dB}{d\tau} = \frac{1}{2}\epsilon^2 B^2 + (\rho\epsilon\lambda z - \kappa\xi)B + \frac{1}{2}\lambda^2(z^2 - z)$$
(C.5)

where the terminal condition now becomes the initial conditions

$$A(0,z) = 0,$$
 $B(0,z) = 0.$ (C.6)

Proof:

First we determine all the necessary partial derivatives of $\tilde{\phi}$ and then substitute them into (C.1).

For the sake of readability we omit the function variables in brackets.

$$\begin{split} \frac{\partial \tilde{\phi}}{\partial t} &= -\frac{\partial \tilde{\phi}}{\partial \tau} \quad = \quad -\left(\frac{dA}{d\tau} + V\frac{dB}{d\tau}\right)e^{A+BV+zx} \\ \frac{\partial \tilde{\phi}}{\partial V} &= \quad Be^{A+BV+zx} \\ \frac{\partial \tilde{\phi}}{\partial x} &= \quad ze^{A+BV+zx} \\ \frac{\partial^2 \tilde{\phi}}{\partial V^2} &= \quad B^2 e^{A+BV+zx} \\ \frac{\partial^2 \tilde{\phi}}{\partial V \partial x} &= \quad zBe^{A+BV+zx} \\ \frac{\partial^2 \tilde{\phi}}{\partial x^2} &= \quad z^2 e^{A+BV+zx}. \end{split}$$

Now substituting the above equations we have

$$\left(-\left(\frac{dA}{d\tau}+V\frac{dB}{d\tau}\right)+(\kappa\theta-\kappa\xi V)B-\frac{1}{2}\lambda^2 Vz+\frac{1}{2}\epsilon^2 VB^2+\epsilon\rho\lambda VzB+\frac{1}{2}\lambda^2 Vz^2\right)e^{A+BV+zx}=0.$$

This implies

$$-\frac{dA}{d\tau} - V\frac{dB}{d\tau} + (\kappa\theta - \kappa\xi V)B - \frac{1}{2}\lambda^2 Vz + \frac{1}{2}\epsilon^2 VB^2 + \epsilon\rho\lambda VzB + \frac{1}{2}\lambda^2 Vz^2 = 0$$

or

$$\frac{dA}{d\tau} + V\frac{dB}{d\tau} = \kappa\theta B + V\left(\frac{1}{2}\epsilon^2 B^2 + (\rho\epsilon\lambda z - \kappa\xi)B + \frac{1}{2}\lambda^2(z^2 - z)\right).$$

Now we choose $\frac{dA}{d\tau}$ and $\frac{dB}{d\tau}$ accordingly to arrive at

$$\begin{aligned} \frac{dA}{d\tau} &= \kappa \theta B \\ \frac{dB}{d\tau} &= \frac{1}{2} \epsilon^2 B^2 + (\rho \epsilon \lambda z - \kappa \xi) B + \frac{1}{2} \lambda^2 (z^2 - z). \end{aligned}$$

For the boundary condition we have

$$\tilde{\phi}(x, V, 0; z) = e^{A(0, z) + B(0, z)V + zx} = e^{zx} = \phi(x, V, T; z)$$

which implies

$$A(0,z) = B(0,z) = 0.$$

Now it rests to determine the functions $A(\tau, z)$ and $B(\tau, z)$. The differential equation for $B(\tau, z)$ is a Ricatti equation, which has no analytical solution for general coefficients. However, the coefficients are piecewise constant on intervals between maturity times and in this case an analytical solution does exist. Hence, we could recursively solve this equation for $B(\tau)$, $\tau \in [\tau_j, \tau_{j+1})$ and $j = 0, 1, \ldots, N-1$. We have the following proposition.

Proposition C.0.2. For piecewise constant coefficients and $\epsilon \neq 0$, equations (C.4) and (C.5) with initial conditions (C.6) have a solution of the form

$$A(\tau, z) = A(\tau_j, z) + \frac{\kappa \theta}{\epsilon^2} \left[(a+d)(\tau - \tau_j) - 2\ln\left(\frac{1 - g_j e^{d(\tau - \tau_j)}}{1 - g_j}\right) \right]$$

$$B(\tau, z) = B(\tau_j, z) + \frac{(a+d-\epsilon^2 B(\tau_j, z))(1 - e^{d(\tau - \tau_j)})}{\epsilon^2 (1 - g_j e^{d(\tau - \tau_j)})}$$

for
$$\tau_j \le \tau < \tau_{j+1}, \quad j = 0, 1, \dots, m-1,$$

where

$$a = \kappa \xi - \rho \epsilon \lambda z, \quad d = \sqrt{a^2 - \lambda^2 \epsilon^2 (z^2 - z)}, \quad g_j = \frac{a + d - \epsilon^2 B(\tau_j, z)}{a - d - \epsilon^2 B(\tau_j, z)}.$$

Proof:

First we solve the Ricatti equation, the differential equation for B.

$$\begin{aligned} \frac{dB}{d\tau} &= \frac{1}{2}\epsilon^2 B^2 + \underbrace{(\rho\epsilon\lambda z - \kappa\xi)}_{-a} B + \frac{1}{2}\lambda^2(z^2 - z) \\ &= \frac{1}{2}\epsilon^2 (B - \frac{a}{\epsilon^2})^2 - \frac{a^2}{2\epsilon^2} + \frac{1}{2}\lambda^2(z^2 - z) \\ &= \frac{1}{2}\epsilon^2 (B - \frac{a}{\epsilon^2})^2 - \frac{1}{2\epsilon^2} \underbrace{(a^2 - \epsilon^2\lambda^2(z^2 - z))}_{d^2} \\ &= \frac{1}{2}\epsilon^2 (B - \frac{a}{\epsilon^2})^2 - \frac{1}{2\epsilon^2} d^2 \\ &= \frac{\epsilon^2}{2} \left(\left(B - \frac{a}{\epsilon^2} \right)^2 - \frac{d^2}{\epsilon^4} \right) \\ &= \frac{\epsilon^2}{2} \left(B - \frac{a + d}{\epsilon^2} \right) \left(B - \frac{a - d}{\epsilon^2} \right) \end{aligned}$$

and after rearranging terms we get

$$\frac{dB}{\left(B - \frac{a+d}{\epsilon^2}\right)\left(B - \frac{a-d}{\epsilon^2}\right)} = \frac{\epsilon^2}{2}d\tau.$$

Now we rewrite the left-hand side and integrate on each interval $\tau_j \leq \tau < \tau_{j+1}$ to arrive at

$$\int_{\tau_j}^{\tau} \frac{\epsilon^2}{2d} \left(\frac{1}{B - \frac{a+d}{\epsilon^2}} - \frac{1}{B - \frac{a-d}{\epsilon^2}} \right) dB = \int_{\tau_j}^{\tau} dd\tau.$$

After integration we get

$$\left[\ln\left|\frac{B-\frac{a+d}{\epsilon^2}}{B-\frac{a-d}{\epsilon^2}}\right|\right]_{\tau_j}^{\tau} = [d\tau]_{\tau_j}^{\tau} + C.$$

Now we work out this equation and solve for B.

$$\ln \left| \frac{B(\tau, z) - \frac{a+d}{\epsilon^2}}{B(\tau, z) - \frac{a-d}{\epsilon^2}} \right| - \ln \left| \frac{B(\tau_j, z) - \frac{a+d}{\epsilon^2}}{B(\tau_j, z) - \frac{a-d}{\epsilon^2}} \right| = d(\tau - \tau_j) + C$$
$$\ln \left| \left(\frac{B(\tau, z) - \frac{a+d}{\epsilon^2}}{B(\tau, z) - \frac{a-d}{\epsilon^2}} \right) \left(\frac{B(\tau_j, z) - \frac{a-d}{\epsilon^2}}{B(\tau_j, z) - \frac{a+d}{\epsilon^2}} \right) \right| = d(\tau - \tau_j) + C$$
$$\ln \left| \left(\frac{B(\tau, z) - \frac{a+d}{\epsilon^2}}{B(\tau, z) - \frac{a-d}{\epsilon^2}} \right) \frac{1}{g_j} \right| = d(\tau - \tau_j) + C$$

where $g_j = \frac{a+d-\epsilon^2 B(\tau_j,z)}{a-d-\epsilon^2 B(\tau_j,z)}$. We now take the exponential of both sides and get

$$\left(\frac{B(\tau,z) - \frac{a+d}{\epsilon^2}}{B(\tau,z) - \frac{a-d}{\epsilon^2}}\right)\frac{1}{g_j} = e^{d(\tau-\tau_j)+C}.$$

Solving for $B(\tau, z)$ gives

$$B(\tau, z) = \frac{(a+d) - (a-d)g_j e^{d(\tau-\tau_j)+C}}{\epsilon^2 (1-g_j e^{d(\tau-\tau_j)+C})}.$$

For $0 \leq \tau < \tau_1$ we can determine the constant C using the initial condition B(0, z) = 0. This leads to

$$B(0,z) = (a+d) - (a-d)\left(\frac{a+d}{a-d}\right)e^{C} = 0$$

and note that this amounts to C = 0. Hence, we have

$$B(\tau, z) = \frac{(a+d) - (a-d)g_j e^{d(\tau-\tau_j)}}{\epsilon^2 (1-g_j e^{d(\tau-\tau_j)})}.$$
 (C.7)

Using the definition of g_j we can write $(a + d) = (a - d - \epsilon^2 B(\tau_j, z))g_j + \epsilon^2 B(\tau_j, z)$ and $(a - d) = \frac{(a + d - \epsilon^2 B(\tau_j, z))}{g_j} + \epsilon^2 B(\tau_j, z)$. Substituting these into (C.7) gives

$$\begin{split} B(\tau,z) &= \frac{(a-d-\epsilon^2 B(\tau_j,z))g_j + \epsilon^2 B(\tau_j,z) - \left((a+d-\epsilon^2 B(\tau_j,z)) + \epsilon^2 B(\tau_j,z)g_j\right)e^{d(\tau-\tau_j)}}{\epsilon^2(1-g_je^{d(\tau-\tau_j)})} \\ &= \frac{(a-d-\epsilon^2 B(\tau_j,z))g_j + \epsilon^2 B(\tau_j,z) - \left((a-d-\epsilon^2 B(\tau_j,z))g_j + \epsilon^2 B(\tau_j,z)g_j\right)e^{d(\tau-\tau_j)}}{\epsilon^2(1-g_je^{d(\tau-\tau_j)})} \\ &= \frac{(a-d-\epsilon^2 B(\tau_j,z))g_j(1-e^{d(\tau-\tau_j)}) + \epsilon^2 B(\tau_j,z)(1-g_je^{d(\tau-\tau_j)})}{\epsilon^2(1-g_je^{d(\tau-\tau_j)})} \\ &= B(\tau_j,z) + \frac{(a+d-\epsilon^2 B(\tau_j,z))\left(1-e^{d(\tau-\tau_j)}\right)}{\epsilon^2(1-g_je^{d(\tau-\tau_j)})}. \end{split}$$

Now we turn our attention to the differential equation for A. We first integrate and notice that

$$\int_{\tau_j}^{\tau} \frac{dA}{d\tau} d\tau = \int_{\tau_j}^{\tau} \kappa \theta B d\tau$$
$$A(\tau, z) - A(\tau_j, z) = C + \kappa \theta \int_{\tau_j}^{\tau} B(\tau, z) d\tau.$$

Which leads to

$$\begin{split} A(\tau,z) &= C + A(\tau_j,z) + \kappa \theta \int_{\tau_j}^{\tau} B(\tau,z) d\tau \\ &= C + A(\tau_j,z) + \frac{\kappa \theta}{\epsilon^2} \left(\epsilon^2 B(\tau_j,z)(\tau - \tau_j) + (a + d - \epsilon^2 B(\tau_j,z)) \int_{\tau_j}^{\tau} \frac{1 - e^{d(\tau - \tau_j)}}{1 - g_j e^{d(\tau - \tau_j)}} d\tau \right) \\ &= C + A(\tau_j,z) + \frac{\kappa \theta}{\epsilon^2} \left(\epsilon^2 B(\tau_j,z)(\tau - \tau_j) + (a + d - \epsilon^2 B(\tau_j,z)) \left[(\tau - \tau_j) - \int_{\tau_j}^{\tau} \frac{(1 - g_j) e^{d(\tau - \tau_j)}}{1 - g_j e^{d(\tau - \tau_j)}} d\tau \right] \right) \\ &= C + A(\tau_j,z) + \frac{\kappa \theta}{\epsilon^2} \left(\epsilon^2 B(\tau_j,z)(\tau - \tau_j) + (a + d - \epsilon^2 B(\tau_j,z)) \left[(\tau - \tau_j) - \frac{1}{d} \int_{1}^{e^{d(\tau - \tau_j)}} \frac{(1 - g_j)}{1 - g_j u} du \right] \right) \\ &= C + A(\tau_j,z) + \frac{\kappa \theta}{\epsilon^2} \left(\epsilon^2 B(\tau_j,z)(\tau - \tau_j) + (a + d - \epsilon^2 B(\tau_j,z)) \left[(\tau - \tau_j) + \frac{(1 - g_j)}{dg} \ln \frac{1 - g_j e^{d(\tau - \tau_j)}}{1 - g_j} \right] \\ &= C + A(\tau_j,z) + \frac{\kappa \theta}{\epsilon^2} \left[(a + d)(\tau - \tau_j) - 2 \ln \left(\frac{1 - g_j e^{d(\tau - \tau_j)}}{1 - g_j} \right) \right]. \end{split}$$

Using the condition A(0, z) = 0 implies C = 0 and we get the desired result in Proposition 3.1.1.

Appendix D

Implications of Including a Displacement Parameter

We recall the relation between simple forward rates and zero-coupon bonds:

$$f_j(t) = \frac{1}{\Delta T_j} \left(\frac{P(t, T_j)}{P(t, T_{j+1})} - 1 \right), \quad 1 \le j \le N$$

and the process followed by the zero-coupon bonds is the lognormal process

$$dP(t,T) = P(t,T)(R(t)dt + \sqrt{V(t)}\sigma(t,T) \cdot d\widetilde{\mathbf{Z}}(t)).$$

By Itô's lemma and including a displacement parameter we now have

$$df_{j}(t) = \frac{1}{\Delta T_{j}} \frac{P(t,T_{j})}{P(t,T_{j+1})} \left[\sigma_{j}(t) - \sigma_{j+1}(t)\right] \cdot \left[\sqrt{V(t)}d\widetilde{\mathbf{Z}}(t) - V(t)\sigma_{j+1}(t)dt\right]$$

$$= (f_{j}(t) + \delta) \frac{1}{\Delta T_{j}} \frac{1 + \Delta T_{j}f_{j}(t)}{f_{j}(t) + \delta} \left[\sigma_{j}(t) - \sigma_{j+1}(t)\right] \cdot \left[\sqrt{V(t)}d\widetilde{\mathbf{Z}}(t) - V(t)\sigma_{j+1}(t)dt\right].$$

$$= (f_{j}(t) + \delta)\sqrt{V(t)}\gamma_{j}(t) \cdot \left[d\widetilde{\mathbf{Z}}(t) - \sqrt{V(t)}\sigma_{j+1}(t)dt\right], \qquad 1 \le j \le N,$$

where $\gamma_j(t)$ is given by

$$\gamma_j(t) = \frac{1 + \Delta T_j f_j(t)}{\Delta T_j(f_j(t) + \delta)} \left[\sigma_j(t) - \sigma_{j+1}(t)\right].$$

Since our stochastic volatility factor remains the same we have the following LIBOR market model with stochastic volatility:

$$df_j(t) = (f_j(t) + \delta)\sqrt{V(t)}\gamma_j(t) \cdot \left[d\widetilde{\mathbf{Z}}(t) - \sqrt{V(t)}\sigma_{j+1}(t)dt\right],$$
(D.1)

$$dV(t) = \kappa(\theta - V(t))dt + \epsilon \sqrt{V(t)}d\widetilde{W}(t).$$
(D.2)

Now changing from the money market account measure to the forward measure and following section 2.3 we see that

$$d\widetilde{\mathbf{Z}}^{j+1}(t) = d\widetilde{\mathbf{Z}}(t) - \sqrt{V(t)}\sigma_{j+1}(t)dt$$

and

$$d\widetilde{W}^{j+1}(t) = d\widetilde{W}(t) + \sqrt{V(t)} \sum_{k=1}^{j} \frac{\Delta T_k(f_k(t) + \delta)}{1 + \Delta T_k f_k(t)} \| \gamma_k(t) \| \rho_j(t) dt$$
$$= d\widetilde{W}(t) + \sqrt{V(t)} \xi_j(t) dt$$

where $\xi_j(t) = \sum_{k=1}^j \frac{\Delta T_k(f_k(t)+\delta)}{1+\Delta T_k f_k(t)} \parallel \gamma_k(t) \parallel \rho_j(t)$ and $\widetilde{\mathbb{E}}\left[\left(\frac{\gamma_j(t)}{\|\gamma_j(t)\|} \cdot d\widetilde{\mathbf{Z}}(t)\right) \cdot d\widetilde{W}(t)\right] = \rho_j(t)dt$, with $\|\rho_j(t)dt\| \leq 1$.

Hence, in terms of the new Brownian motions $\widetilde{\mathbf{Z}}^{j+1}(t)$ and $\widetilde{W}^{j+1}(t)$ under the probability measure $\widetilde{\mathbb{P}}^{j+1}$ the market model becomes

$$df_j(t) = (f_j(t) + \delta)\sqrt{V(t)}\gamma_j(t) \cdot d\widetilde{\mathbf{Z}}^{j+1}(t),$$
$$dV(t) = [\kappa\theta - (\kappa + \epsilon\xi_j(t))V(t)] dt + \epsilon\sqrt{V(t)}d\widetilde{W}^{j+1}(t).$$

Again we want to regain analytical tractability by using freezing coefficients. By Proposition 2.3.1 this gives the extended LIBOR market model with displacement coefficient

$$df_j(t) = (f_j(t) + \delta)\sqrt{V(t)}\gamma_j(t) \cdot d\widetilde{\mathbf{Z}}^{j+1}(t),$$
(D.3)

$$dV(t) = \kappa \left[\theta - \tilde{\xi}_j(t)V(t) \right] dt + \epsilon \sqrt{V(t)} d\widetilde{W}^{j+1}(t)$$
(D.4)

where $\tilde{\xi}_j(t) = 1 + \frac{\epsilon}{\kappa} \xi_j(t)$ and $\xi_j(t) \approx \sum_{k=1}^j \frac{\Delta T_k(f_k(0) + \delta)\rho_k(t) \|\gamma_k(t)\|}{1 + \Delta T_k f_k(0)}$.

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