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Aircraft responses to atmospheric turbulence. A comparative study

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Title	:	Aircraft Responses to Atmospheric Turbulence, A Comparative Study
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Abstract	:	The simulation of aircraft flying in turbulent atmospheres is performed in many ways in the world today. In this report a comparison is made between the methods as presented in references (Etkin, 1972), (Etkin, 1980), (Robinson, 1991) and (Mulder & van der Vaart, 1993). Both the Delft University of Technology (DUT) and Etkin's Four Point Aircraft (FPA) models presented herein simplify the characteristics of atmospheric turbulence greatly. The differences between the two methods appear to be small in output power spectral densities' results. The aircraft models, for both the DUT and FPA theory, are equal for most turbulence inputs. However, the approach in calculating the input power spectral densities differs considerably. The results, described by output power spectral densities, coincide over a wide frequency range.
Keyword(s)	:	Atmospheric Turbulence, Aircraft Modelling, Power Spectral Density, Frequency Domain Simulation

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List of Symbols

A	system matrix
B	input matrix
B	$\frac{b}{2L_g}$
b	wingspan
b'	$= 0.85b$ separation between points 1 and 2 in the FPA aircraft model
C	output matrix
$\bar{c}$	mean aerodynamic wing chord
$c.g.$	aircraft's centre of gravity
$C(i,j)$	element (i,j) of covariance matrix
C_L	lift coefficient
C_l	$\frac{L}{\frac{1}{2}\rho V^2 Sb}$ rolling moment coefficient
C_{l_g}	C_l due to turbulence
C_{l_p}	$\frac{\partial C_l}{\partial \frac{pb}{2V}}$
$C_{l_{p_g}}$	$\frac{\partial C_l}{\partial \frac{p_g b}{2V}}$
$C_{l_{p_w}}$	contribution of the wing to C_{l_p}
C_{l_r}	$\frac{\partial C_l}{\partial \frac{rb}{2V}}$

$C_{l_{r_{1g}}}$	$\frac{\partial C_l}{\partial \frac{r_{1g} b}{2V}}$
$C_{l_{r_{2g}}}$	$\frac{\partial C_l}{\partial \frac{r_{2g} b}{2V}}$
$C_{l_{r_v}}$	contribution of the vertical tailplane to C_{l_r}
$C_{l_{r_w}}$	contribution of the wing to C_{l_r}
C_{l_w}	contribution of the wing to C_l
$C_{l_{u_g}}$	$\frac{\partial C_{l_g}}{\partial \hat{u}_g}$
$C_{l_{\alpha_g}}$	$\frac{\partial C_{l_g}}{\partial \alpha_g}$
C_{l_β}	$\frac{\partial C_l}{\partial \beta}$
$C_{l_{\beta_g}}$	$\frac{\partial C_{l_g}}{\partial \beta_g}$
$C_{l_{\dot{\beta}}}$	$\frac{\partial C_l}{\partial \frac{\dot{\beta} b}{V}}$
$C_{l_{\dot{\beta}_g}}$	$\frac{\partial C_{l_g}}{\partial \frac{\dot{\beta}_g b}{V}}$
$C_{l_{\delta_a}}$	$\frac{\partial C_l}{\partial \delta_a}$
$C_{l_{\delta_r}}$	$\frac{\partial C_l}{\partial \delta_r}$
C_m	$\frac{M}{\frac{1}{2}\rho V^2 S\bar{c}}$ pitching moment coefficient
$C_{m_{a.c.}}$	C_m about the aerodynamic centre of the aircraft minus horizontal tail for $C_{L_w} =$
C_{m_g}	C_m due to turbulence
C_{m_h}	contribution of the horizontal tailplane to C_m
C_{m_q}	$\frac{\partial C_m}{\partial \frac{q\bar{c}}{V}}$
$C_{m_{q_g}}$	$\frac{\partial C_m}{\partial \frac{q_g\bar{c}}{V}}$

C_{m_u}	$\frac{1}{\frac{1}{2}\rho V S\bar{c}}\frac{\partial M}{\partial u}$
$C_{m_{u_g}}$	$\frac{\partial C_m}{\partial \hat{u}_g}$
$C_{m_{\dot{u}_g}}$	$\frac{\partial C_m}{\partial \frac{\dot{u}_g \bar{c}}{V}}$
C_{m_w}	contribution of the wing plus fuselage to C_m
C_{m_α}	$\frac{\partial C_m}{\partial \alpha}$
$C_{m_{\alpha_g}}$	$\frac{\partial C_{m_g}}{\partial \alpha_g}$
$C_{m_{\dot{\alpha}}}$	$\frac{\partial C_m}{\partial \frac{\dot{\alpha}\bar{c}}{V}}$
$C_{m_{\dot{\alpha}_g}}$	$\frac{\partial C_{m_g}}{\partial \frac{\dot{\alpha}_g \bar{c}}{V}}$
C_{m_δ}	$\frac{\partial C_m}{\partial \delta}$
C_n	$\frac{N}{\frac{1}{2}\rho V^2 S b}$ yawing moment coefficient
C_{n_g}	C_n due to turbulence
C_{n_p}	$\frac{\partial C_n}{\partial \frac{pb}{2V}}$
$C_{n_{p_g}}$	$\frac{\partial C_n}{\partial \frac{p_g b}{2V}}$
$C_{n_{p_w}}$	contribution of the wing plus fuselage to C_{n_p}
C_{n_r}	$\frac{\partial C_n}{\partial \frac{rb}{2V}}$
$C_{n_{r_{1_g}}}$	$\frac{\partial C_n}{\partial \frac{r_{1_g} b}{2V}}$
$C_{n_{r_{2_g}}}$	$\frac{\partial C_n}{\partial \frac{r_{2_g} b}{2V}}$
$C_{n_{r_v}}$	contribution of the vertical tailplane to C_{n_r}
$C_{n_{r_w}}$	contribution of the wing plus fuselage to C_{n_r}

$C_{n_{u_g}}$	$\frac{\partial C_{n_g}}{\partial \hat{u}_g}$
$C_{n_{\alpha_g}}$	$\frac{\partial C_{n_g}}{\partial \alpha_g}$
C_{n_β}	$\frac{\partial C_n}{\partial \beta}$
$C_{n_{\beta_g}}$	$\frac{\partial C_{n_g}}{\partial \beta_g}$
$C_{n_{\dot{\beta}}}$	$\frac{\partial C_n}{\partial \frac{\dot{\beta} b}{V}}$
$C_{n_{\dot{\beta}_g}}$	$\frac{\partial C_{n_g}}{\partial \frac{\dot{\beta}_g b}{V}}$
$C_{n_{\delta_a}}$	$\frac{\partial C_n}{\partial \delta_a}$
$C_{n_{\delta_r}}$	$\frac{\partial C_n}{\partial \delta_r}$
C_X	$\frac{X}{\frac{1}{2}\rho V^2 S}$ aerodynamic force coefficient along the aircraft's X-axis
C_{X_g}	C_X due to turbulence
C_{X_q}	$\frac{\partial C_X}{\partial \frac{q\bar{c}}{V}}$
$C_{X_{q_g}}$	$\frac{\partial C_X}{\partial \frac{q_g \bar{c}}{V}}$
C_{X_u}	$\frac{1}{\frac{1}{2}\rho V S}\frac{\partial X}{\partial u}$
$C_{X_{u_g}}$	$\frac{\partial C_X}{\partial \hat{u}_g}$
$C_{X_{\dot{u}_g}}$	$\frac{\partial C_X}{\partial \frac{\dot{u}_g \bar{c}}{V}}$
C_{X_0}	C_X in the steady flight condition
$C_{xy}(\tau)$	covariance function of x and y
C_{X_α}	$\frac{\partial C_X}{\partial \alpha}$
$C_{X_{\alpha_g}}$	$\frac{\partial C_{X_g}}{\partial \alpha_g}$

$C_{X_{\dot{\alpha}}}$	$\frac{\partial C_X}{\partial \frac{\dot{\alpha}\bar{c}}{V}}$
$C_{X_{\dot{\alpha}_g}}$	$\frac{\partial C_{X_g}}{\partial \frac{\dot{\alpha}_g\bar{c}}{V}}$
C_{X_δ}	$\frac{\partial C_X}{\partial \delta}$
C_Y	$\frac{Y}{\frac{1}{2}\rho V^2 S}$ aerodynamic force coefficient along the aircraft's Y-axis
C_{Y_g}	C_Y due to turbulence
C_{Y_p}	$\frac{\partial C_Y}{\partial \frac{pb}{2V}}$
$C_{Y_{p_g}}$	$\frac{\partial C_Y}{\partial \frac{p_g b}{2V}}$
C_{Y_r}	$\frac{\partial C_Y}{\partial \frac{rb}{2V}}$
$C_{Y_{r_{1_g}}}$	$\frac{\partial C_Y}{\partial \frac{r_{1_g} b}{2V}}$
$C_{Y_{r_{2_g}}}$	$\frac{\partial C_Y}{\partial \frac{r_{2_g} b}{2V}}$
$C_{Y_{r_v}}$	contribution of the vertical tailplane to C_{Y_r}
$C_{Y_{u_g}}$	$\frac{\partial C_{Y_g}}{\partial \hat{u}_g}$
$C_{Y_{\alpha_g}}$	$\frac{\partial C_{Y_g}}{\partial \alpha_g}$
C_{Y_β}	$\frac{\partial C_Y}{\partial \beta}$
$C_{Y_{\beta_g}}$	$\frac{\partial C_{Y_g}}{\partial \beta_g}$
$C_{Y_{\dot{\beta}}}$	$\frac{\partial C_Y}{\partial \frac{\dot{\beta} b}{V}}$
$C_{Y_{\dot{\beta}_g}}$	$\frac{\partial C_{Y_g}}{\partial \frac{\dot{\beta}_g b}{V}}$
$C_{Y_{\delta_a}}$	$\frac{\partial C_Y}{\partial \delta_a}$

$C_{Y_{\delta_r}}$	$\frac{\partial C_Y}{\partial \delta_r}$
C_Z	$\frac{Z}{\frac{1}{2}\rho V^2 S}$ aerodynamic force coefficient along the aircraft's Z-axis
C_{Z_g}	C_Z due to turbulence
C_{Z_h}	contribution of the horizontal tailplane to C_Z
C_{Z_q}	$\frac{\partial C_Z}{\partial \frac{q\bar{c}}{V}}$
$C_{Z_{q_g}}$	$\frac{\partial C_Z}{\partial \frac{q_g\bar{c}}{V}}$
C_{Z_u}	$\frac{1}{\frac{1}{2}\rho V S}\frac{\partial Z}{\partial u}$
$C_{Z_{u_g}}$	$\frac{\partial C_Z}{\partial \hat{u}_g}$
$C_{Z_{\dot{u}_g}}$	$\frac{\partial C_Z}{\partial \frac{\dot{u}_g\bar{c}}{V}}$
C_{Z_w}	contribution of the wing plus fuselage to C_Z
C_{Z_0}	C_Z in the steady flight condition
C_{Z_α}	$\frac{\partial C_Z}{\partial \alpha}$
$C_{Z_{\alpha_g}}$	$\frac{\partial C_{Z_g}}{\partial \alpha_g}$
$C_{Z_{\dot{\alpha}}}$	$\frac{\partial C_Z}{\partial \frac{\dot{\alpha}\bar{c}}{V}}$
$C_{Z_{\dot{\alpha}_g}}$	$\frac{\partial C_{Z_g}}{\partial \frac{\dot{\alpha}_g\bar{c}}{V}}$
C_{Z_δ}	$\frac{\partial C_Z}{\partial \delta}$
D	$\frac{d}{dt}$ differential operator
D_b	$\frac{b}{V}\frac{d}{dt}$ non-dimensional differential operator, asymmetric motions
D_c	$\frac{\bar{c}}{V}\frac{d}{dt}$ non-dimensional differential operator, symmetric motions
$E\{\ \}$	expectation operator

$F\{\ \}$	Fourier operator
$F_{\bar{x}}(x)$	probability distribution function
$F_{\bar{x}\bar{y}}(x,y)$	two-dimensional probability distribution function
f	longitudinal correlation function
$f_{\bar{x}}(x)$	probability density function
$f_{\bar{x}\bar{y}}(x,y)$	two-dimensional probability density function
g	lateral correlation function
g	acceleration due to gravity
$H(\omega)$	transfer function
Im	imaginary part of a complex variable or function
$I_{\hat{u}_g\hat{u}_g}(\Omega L_g, B)$	power spectral density function of $\hat{u}_g$ (asymmetric aircraft motions)
$I_{\alpha_g\alpha_g}(\Omega L_g, B)$	power spectral density function of α_g (asymmetric aircraft motions)
I_X	moment of inertia about the aircraft's X-axis
I_Y	moment of inertia about the aircraft's Y-axis
I_Z	moment of inertia about the aircraft's Z-axis
I_{XZ}	product of inertia
j	$\sqrt{-1}$
K_X	non-dimensional radius of gyration, $\mu_b K_X^2 = \frac{I_X}{\rho S b^3}$
K_Y	non-dimensional radius of gyration, $\mu_c K_Y^2 = \frac{I_Y}{\rho S \bar{c}^3}$
K_Z	non-dimensional radius of gyration, $\mu_b K_Z^2 = \frac{I_Z}{\rho S b^3}$

K_{XZ}	non-dimensional radius of gyration, $\mu_b K_{XZ} = \frac{I_{XZ}}{\rho S b^3}$
$K_{xy}(\tau)$	normalized covariance function of x and y
k_b	reduced frequency $\frac{\omega b}{V}$
k_c	reduced frequency $\frac{\omega \bar{c}}{V}$
L	aerodynamic moment about the aircraft's X-axis
L	Lift
L_g	scale of turbulence
l	wavelength of elementary flowfield
l_h	horizontal taillength
l_v	vertical taillength
l_{α_g}	
l_β	
l_{β_g}	
$l_{\dot{\beta}}$	
l_p	stability-, gust- and input derivatives in abbreviated notation
l_{p_g}	
l_r	
$l_{r_{1g}}$	
$l_{r_{2g}}$	
l_{u_g}	
l_{δ_a}	

l_{δ_r}	
M	aerodynamic moment about the aircraft's Y-axis
m	moment
m	$\frac{W}{g}$ aircraft mass
m_q	
m_{q_g}	
m_u	
m_{u_g}	
$m_{\dot{u}_g}$	
m_α	stability-, gust- and input derivatives in abbreviated notation
m_{α_g}	
$m_{\dot{\alpha}}$	
$m_{\dot{\alpha}_g}$	
m_δ	
m_θ	
N	aerodynamic moment about the aircraft's Z-axis
n	number/order of equations
n	$\frac{a_z}{g}$ normal acceleration factor at the aircraft's centre of gravity
n_{α_g}	
n_β	
n_{β_g}	

$n_{\dot{\beta}}$	
n_p	stability-, gust- and input derivatives in abbreviated notation
n_{p_g}	
n_r	
$n_{r_{1_g}}$	
$n_{r_{2_g}}$	
n_{u_g}	
n_{δ_a}	
n_{δ_r}	
p	rolling velocity about the aircraft's X-axis
p_g	$\frac{\partial w_g}{\partial y}$: gust rolling velocity about the aircraft's X-axis
$Prob$	Probability
q	pitching velocity about the aircraft's Y-axis
q_g	$\frac{\partial w_g}{\partial x}$: gust pitching velocity about the aircraft's Y-axis
r	yawing velocity about the aircraft's Z-axis
r_{1_g}	$\frac{\partial u_g}{\partial y}$: gust yawing velocity about the aircraft's Z-axis
r_{2_g}	$\frac{\partial v_g}{\partial x}$: gust yawing velocity about the aircraft's Z-axis
Re	real part of a complex variable or function
$R_{xy}(\tau)$	correlation function of x and y
S	wing area
$S_{xy}(\omega)$	power spectral density function of x and y

s_b	$\frac{Vt}{b}$ non-dimensional parameter indicating time (asymmetric aircraft motions)
$s_{\bar{c}}$	$\frac{Vt}{\bar{c}}$ non-dimensional parameter indicating time (symmetric aircraft motions)
t	time
T	sampling time interval, period
Δt	$t_k - t_{k-1}$ discretization time interval
u	component of $\underline{V}$ along the aircraft's X-axis
u_g	component of $\underline{V}_g$ along the aircraft's X_e-axis
$\hat{u}_g$	$\frac{u_g}{V}$
$\hat{u}_{g_h}$	value of $\hat{u}_g$ at the aerodynamic centre of the horizontal tailplane
$\hat{u}_{g_w}$	value of $\hat{u}_g$ at the aerodynamic centre of the wing plus fuselage
$\hat{u}_{g_x}$	value of $\hat{u}_g$ at a point with abscissa x
$\underline{u}(t)$	input vector
V	velocity of the aircraft's centre of gravity relative to the earth
V_a	velocity of the aircraft's centre of gravity relative to the surrounding air
V_{a_h}	value of V_a at the aerodynamic centre of the horizontal tailplane
V_{a_w}	value of V_a at the aerodynamic centre of the wing plus fuselage
V_g	gust velocity: difference between $V_{w_{tot}}$ and V_w
V_w	mean velocity of the air relative to the earth
$V_{w_{tot}}$	total velocity of the air relative to the earth

v	component of $\underline{V}$ along the Y-axis
v_g	component of $\underline{V}_g$ along the Y_e-axis
W	aircraft weight
W	white noise intensity matrix
W_k	$\frac{W}{\Delta t}$ discretized white noise covariance matrix
$\bar{w}(t)$	white noise input vector
w	component of $\underline{V}$ along the Z-axis
w_g	component of $\underline{V}_g$ along the Z_e-axis
X	aerodynamic force along the aircraft's X-axis
x	abscissa of the aircraft's centre of gravity in an earth-fixed frame of reference
x	abscissa of a point of the aircraft in an aircraft-fixed frame of reference
Δx	distance between two points in the turbulent atmosphere, along the aircraft's flight path
x_h	abscissa of the aerodynamic centre of the horizontal tail in an aircraft-fixed frame of reference
$\underline{x}(t)$	state vector
x_q	
x_{q_g}	
x_u	
x_{u_g}	
$x_{\dot{u}_g}$	

x_α	stability-, gust- and input derivatives in abbreviated notation
x_{α_g}	
$x_{\dot\alpha}$	
$x_{\dot\alpha_g}$	
x_δ	
x_θ	
Y	aerodynamic force along the aircraft's Y-axis
y	y-coordinate of a point of the aircraft in an aircraft-fixed frame of reference
y_{α_g}	
y_β	
y_{β_g}	
y_p	stability-, gust- and input derivatives in abbreviated notation
y_{p_g}	
y_r	
$y_{r_{1_g}}$	
$y_{r_{2_g}}$	
y_{u_g}	
$y_{\dot\beta}$	
y_{δ_a}	
y_{δ_r}	
y_φ	

$\underline{y}(t)$	output vector
Z	aerodynamic force along the aircraft's Z-axis
z	z coordinate of a point of the aircraft in an aircraft-fixed frame of reference
Z_h	contribution of the horizontal tail to the aerodynamic force Z
Z_w	contribution of the wing plus fuselage to the aerodynamic force Z
z_q	
z_{q_g}	
z_u	
z_{u_g}	
$z_{\dot{u}_g}$	
z_α	stability-, gust- and input derivatives in abbreviated notation
z_{α_g}	
$z_{\dot{\alpha}}$	
$z_{\dot{\alpha}_g}$	
z_δ	
z_θ	

Greek Symbols

α	$\frac{w}{V}$ angle of attack
α_g	$\frac{w_g}{V}$ gust angle of attack
β	$\frac{v}{V}$ gust sideslip angle
γ	flight path angle
$\delta(t)$	Dirac function
δ_a	aileron deflection
δ_e	elevator deflection
δ_r	rudder deflection
ϵ	downwash angle
θ	angle of pitch
μ	central moment
μ_b	$\frac{m}{\rho S b}$ relative aircraft mass (asymmetric aircraft motions)
μ_c	$\frac{m}{\rho S \bar{c}}$ relative aircraft mass (symmetric aircraft motions)
μ_x	mean value of x
ψ	angle of yaw
ρ	air density
ξ	seperation vector
σ	sidewash angle
σ_x^2	variance of x
$\Phi(t, t_0)$	transition matrix

φ	aircraft roll angle
Ψ	discretized input distribution matrix
τ	time constant
τ	time variable in covariance function
Ω	spatial circular frequency
ω	circular frequency
ζ	damping ratio

Abbreviations

FFT	Fast Fourier Transform
FPA	Four Point Aircraft model
DUT	Delft University of Technology model

Superscripts

T	transpose of matrix
-1	inverse of matrix
$-$	stochastic variable (scalar, vector or function)

Subscripts

a	quantities relative to surrounding airmass
$c.g.$	centre of gravity
g	gust, turbulence
h	parameters for the horizontal tailplane
k	discrete (time)
v	vertical tailplane
w	wing plus fuselage

Vectors are understriped.

Frames of Reference

Two frames of reference are used,

1. The so called 'stability frame of reference': an aircraft-fixed right-handed frame of reference $OXYZ$. The origin O lies in the aircraft's centre of gravity, the positive X-axis lies in the plane of symmetry OXZ and points forward in the direction of the undisturbed velocity V: in the undisturbed symmetric flight condition, about which the turbulence or other 'inputs' causes small deviations, the X-axis coincides with the vector of the velocity of the centre of gravity relative to the undisturbed atmosphere. The positive Y-axis points to the right, the positive Z-axis points downwards. For disturbed aircraft motions, the stability frame of reference remains fixed to the aircraft.

2. An earth-fixed, right handed frame of reference $O_eX_eY_eZ_e$. The origin O_e has a fixed position relative to the earth. In the undisturbed flight condition mentioned above, the X_e-, Y_e- and Z_e-axes coincide with the corresponding axes of the stability frame of reference $OXYZ$.

Chapter 1

Introduction

Different techniques for the calculation of aircraft responses to atmospheric turbulence are used in the world today. These techniques involve different turbulence models and different aircraft models. The turbulence model described herein originates from a general statistical description of a stochastic process. This process will be simplified in order to get a workable model of the atmosphere. It should be noted that this model is only valid for flight conditions in very specific circumstances. The Delft model simplifies the atmospheric turbulence to a two-dimensional sinus-shaped wind-field, whereas Etkin's Four Point Aircraft Model sees turbulence as a sinusoidal wave of shearing motion. In this report a description of the Four Point Aircraft Model according to Ref. (Etkin, 1980) will be presented for both the symmetrical and asymmetrical equations of motion. First of all, the relevant mathematical definitions and assumptions will be summarized in the following sections. This report investigates both the Delft University of Technology (DUT) and Four Point Aircraft (FPA) model methods and will discuss the differences.

1.1 A General Statistical Description

Atmospheric turbulence is a random process which describes the chaotic motion of the air. The random velocity vector $\underline{u}$ of the air is a function of position $\underline{r} = (x_1, x_2, x_3)^T$ and time t. The wind velocity vector has three components

$$u_g = u_g(t, x_1, x_2, x_3) \tag{1.1}$$

$$v_g = v_g(t, x_1, x_2, x_3) \tag{1.2}$$

$$w_g = w_g(t, x_1, x_2, x_3) \tag{1.3}$$

In general we can write the gust vector as:

$$u(\underline{r}, t) = (u_1, u_2, u_3)^T \tag{1.4}$$

Note: *the velocity vector* $\underline{u}$ *is random, but will not be overstriped to simplify notations. The velocity vector, as in equation (1.4), describes a multivariate* (u_1, u_2, u_3) *and multivariable* (x_1, x_2, x_3, t) *process.*

In the following we will be dealing with the relative, or separation, time τ instead of absolute times t and with the relative, or separation, distance vector $\underline{\xi} = (\xi_1, \xi_2, \xi_3)^T$ instead of the absolute positions $\underline{r} = (x_1, x_2, x_3)^T$.

The relevant correlation and spectrum functions can now be defined. The relevant definitions are summarized in the following.

The general matrix of correlation functions is:

$$C_{\underline{uu}}(\underline{r}, t; \underline{r}+\underline{\xi}, t+\tau) = E\left\{\underline{u}(\underline{r}, t) \cdot \underline{u}(\underline{r}+\underline{\xi}, t+\tau)\right\} = \tag{1.5}$$

$$= \begin{bmatrix} E\left\{u_1(\underline{r},t)\cdot u_1(\underline{r}+\underline{\xi},t+\tau)\right\} & E\left\{u_1(\underline{r},t)\cdot u_2(\underline{r}+\underline{\xi},t+\tau)\right\} & E\left\{u_1(\underline{r},t)\cdot u_3(\underline{r}+\underline{\xi},t+\tau)\right\} \\ E\left\{u_2(\underline{r},t)\cdot u_1(\underline{r}+\underline{\xi},t+\tau)\right\} & E\left\{u_2(\underline{r},t)\cdot u_2(\underline{r}+\underline{\xi},t+\tau)\right\} & E\left\{u_2(\underline{r},t)\cdot u_3(\underline{r}+\underline{\xi},t+\tau)\right\} \\ E\left\{u_3(\underline{r},t)\cdot u_1(\underline{r}+\underline{\xi},t+\tau)\right\} & E\left\{u_3(\underline{r},t)\cdot u_2(\underline{r}+\underline{\xi},t+\tau)\right\} & E\left\{u_3(\underline{r},t)\cdot u_3(\underline{r}+\underline{\xi},t+\tau)\right\} \end{bmatrix}$$

This correlation function, equation (1.5), is a 3x3 matrix where each element is an ensemble average of the product of two velocity components separated both in space and time. Equation (1.5) describes an unstationary flow, whose statistics changes with position and time. However, if the statistical properties do not vary considerably in time, the random process can be assumed to be *stationary* and only the difference in time, as described by the variable τ, is important. Furthermore, for many applications it is acceptable to model the process as *homogeneous*, with no spatial variation of statistics, therefore only the difference in space $\underline{\xi}$ is important. The covariance matrix, $C_{\underline{uu}}$, becomes a function of time and space displacements only:

$$C_{\underline{uu}}(\underline{\xi}, \tau) = E\left\{\underline{u}(\underline{r}, t) \cdot \underline{u}(\underline{r}+\underline{\xi}, t+\tau)\right\} \tag{1.6}$$

Fourier transformation of the covariance matrix $C_{\underline{uu}}(\underline{\xi}, \tau)$ yields the spectral density matrix which is also a 3x3 matrix and is independent of absolute time and space,

$$S_{\underline{uu}}(\underline{\Omega}, \omega) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} C_{\underline{uu}}(\underline{r}, t; \underline{r}+\underline{\xi}, t+\tau)\, e^{-j(\underline{\Omega}\cdot\underline{\xi}+\omega\tau)}\, d\xi_1\, d\xi_2\, d\xi_3\, d\tau \tag{1.7}$$

with Ω and ω respectively the wave-number vector $\Omega = (\Omega_x, \Omega_y, \Omega_z)^T$ and circular frequency.

An approximation almost universally accepted, see Ref. (Etkin, 1980), and quite valid for almost all cases of flight in a turbulent atmosphere is that temporal changes in the velocity field are negligible compared with the apparent temporal changes felt by the vehicle as it passes through spatial gradients. This implies that in flight at speed V on the X-axis (e.g. the stability frame of reference's X-axis), in which the actual perceived rate of change is,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + V\frac{\partial}{\partial x} \tag{1.8}$$

the partial time derivative can be neglected. This is known as the frozen field approximation. The approximation may not be valid when the airspeed V is very small in which case the first term may become dominant. It is generally assumed that the frozen field theory is valid when the following condition is satisfied,

$$\frac{V}{u} \geq \frac{1}{3}$$

The frozen field assumption results in time-independent power spectral densities. This assumption is not valid for aircraft landing in windshear conditions, V/STOL aircraft and helicopters in hover.

1.2 Correlation Functions and the Integral Scale of Turbulence

The correlation matrix $C_{\underline{uu}}(\xi)$, and the power spectral density matrix $S_{\underline{uu}}(\Omega)$, are defined by:

$$C_{\underline{uu}}(\underline{\xi}) = E\left\{\underline{u}(\underline{r}) \cdot \underline{u}(\underline{r} + \underline{\xi})\right\} \tag{1.9}$$

$$S_{\underline{uu}}(\underline{\Omega}) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} C_{\underline{uu}}(\underline{\xi})\ e^{-j(\underline{\Omega}\cdot\underline{\xi})}\ d\xi_1\ d\xi_2\ d\xi_3 \tag{1.10}$$

At this point the correlation matrix in analytical form, as defined by Batchelor, can be introduced,

$$C_{ij}(\xi) = \sigma^2\left(\frac{f(\xi) - g(\xi)}{\xi^2}\xi_i\xi_j + g(\xi)\delta_{ij}\right) \tag{1.11}$$

with,

$$\xi^2 = \xi_1^2 + \xi_2^2 + \xi_3^2$$

and,

$$\underline{\xi} = (\xi_1, \xi_2, \xi_3)^T$$

The correlation function, equation (1.11) holds for any model of homogeneous isotropic turbulence and is based on the fundamental correlation functions $f(\xi)$ and $g(\xi)$. The correlation between the velocities parallel to a connecting line between two points is termed the 'longitudinal' correlation $f(\xi)$. The correlation between velocities normal to a line connecting the two points is termed the 'lateral' correlation $g(\xi)$. The functions $f(\xi)$ and $g(\xi)$ can be found theoretically. Von Karman has derived these fundamental correlation functions by Fourier transforming his empirically found spectral densities.

Von Karman correlation functions:

longitudinal:

$$f(\xi) = \frac{2^{\frac{2}{3}}}{\Gamma(\frac{1}{3})} \left(\frac{\xi}{1.339L_g}\right)^{\frac{1}{3}} K_{\frac{1}{3}} \left(\frac{\xi}{1.339L_g}\right) \tag{1.12}$$

lateral:

$$g(\xi) = \frac{2^{\frac{2}{3}}}{\Gamma(\frac{1}{3})} \left(\frac{\xi}{1.339L_g}\right)^{\frac{1}{3}} \left[K_{\frac{1}{3}} \left(\frac{\xi}{1.339L_g}\right) - \frac{1}{2} \left(\frac{\xi}{1.339L_g}\right) K_{\frac{2}{3}} \left(\frac{\xi}{1.339L_g}\right)\right] \tag{1.13}$$

with,

$$\Gamma(z) = \int_0^{\infty} t^{z-1}\, e^{-t}\, dt \tag{1.14}$$

which may be approximated by using the following series expansion,

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!}{z(z+1)(z+2)...(z+n-1)}\, n^{z-1}$$

K denotes the modified Bessel function of the second kind,

$$K_m(z) = \frac{\pi/2}{sin\ m\pi}\left[I_{-m}(z) - I_m(z)\right] \tag{1.15}$$

In this expression, I_m denotes the modified Bessel function of the first kind of order m,

$$I_m(z) = j^{-m} J_m(jz)$$

where I_m is the standard Bessel function of the first kind, which may be approximated by,

$$J_m(z) = \left(\frac{z}{2}\right)^m \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(m+k+1)} \left(\frac{z}{2}\right)^{2k}$$

Dryden has used a different approach and has fitted the correlation functions to windtunnel data. The Dryden correlation functions are,

Dryden correlation functions:

longitudinal:

$$f(\xi) = e^{-\frac{\xi}{L_g}} \tag{1.16}$$

lateral:

$$g(\xi) = e^{-\frac{\xi}{L_g}}\left(1 - \frac{\xi}{2L_g}\right) \tag{1.17}$$

The longitudinal turbulence scale L_g and lateral scale L'_g are defined by the longitudinal and lateral correlation functions $f(\xi)$ and $g(\xi)$,

$$\text{longitudinalscale} \; : \; L_g = \int_0^\infty f(\xi)\, d\xi \tag{1.18}$$

$$\text{lateralscale} \; : \; L'_g = \int_0^\infty g(\xi)\, d\xi \tag{1.19}$$

Keep in mind that for *isotropic* turbulence, that is,

$$\sigma^2 = \overline{u_g^2} = \overline{v_g^2} = \overline{w_g^2} \tag{1.20}$$

or,

$$\sigma^2 = \sigma^2_{u_g} = \sigma^2_{v_g} = \sigma^2_{w_g}$$

the longitudinal scale L_g equals twice the lateral scale L'_g,

$$L_g = 2L'_g \tag{1.21}$$

The turbulence model can now completely be described by its scale length L_g and by its variance σ^2.

In this report the atmospheric turbulence is considered to be stationary, isotropic, homogeneous with a Gaussian distribution. The Dryden correlation functions and spectra will be used further in this report for their rational expressions. For the frequency range where the aircraft responses are calculated, the Dryden spectra and the von Karman spectra show good resemblance. Only for higher frequencies the von Karman spectra will give a better fit for the spectra of atmospheric turbulence. For rigid aircraft, however, the Dryden spectra will produce acceptable results. If flexible aircraft, or structural motions are considered, it is preferred to use the von Karman spectra since for higher frequencies the von Karman spectra resemble experimental data more accurately.

Chapter 2

Delft University of Technology Models

2.1 Introduction

When modelling turbulence inputs a distinction can be made between symmetrical and asymmetrical turbulence inputs. Symmetrical gust velocities will give rise to symmetrical aircraft responses where as asymmetrical turbulence will cause asymmetrical aircraft responses, thus assuming small disturbance theory and therefore decoupling of symmetric and asymmetric aircraft motions is allowed. In this chapter a distinction will be made between symmetrical and asymmetrical aircraft responses. The first can be regarded as a one-dimensional process, the second as a two-dimensional process. First, a two-dimensional velocity field, see figure 2.1, will be described which can easily be reduced to a one-dimensional velocity field, see figure 2.2.

2.2 Turbulence Modelling

Two-dimensional stochastic turbulence can be considered as a superposition of infinitely many elementary fields of flow, see Ref. (Gerlach & Baarspul, 1968). In each of these elementary fields, the gust velocity varies sinusoidally along the X_e-axis as well as along the Y_e-axis. The two-dimensional velocity-fields satisfy the expressions,

$$\begin{aligned} u_g(x,y) &= u_{g_{max}} Re\left\{e^{j(\Omega_x x+\Omega_y y)}\right\} \\ v_g(x,y) &= v_{g_{max}} Re\left\{e^{j(\Omega_x x+\Omega_y y)}\right\} \\ w_g(x,y) &= w_{g_{max}} Re\left\{e^{j(\Omega_x x+\Omega_y y)}\right\} \end{aligned} \tag{2.1}$$

with u_g, v_g and w_g respectively the horizontal, lateral and vertical gust components in the X_e, Y_e, Z_e frame of reference. The one-dimensional fields can be produced from the two-dimensional turbulence fields if Ω_y is set equal to 0. Equation (2.1) is then reduced to,

$$
\begin{aligned}
u_g(x) &= u_{g_{max}} Re\left\{e^{j(\Omega_x x)}\right\} \\
v_g(x) &= v_{g_{max}} Re\left\{e^{j(\Omega_x x)}\right\} \\
w_g(x) &= w_{g_{max}} Re\left\{e^{j(\Omega_x x)}\right\}
\end{aligned}
\tag{2.2}
$$

Equation (2.2) is used for the calculation of symmetrical gust inputs. In the following, another interpretation is given for the expressions in equation (2.1). As an example, the expression for u_g in equation (2.1) is therefore written as,

$$
\begin{aligned}
u_g &= u_{g_{max}} Re\left\{e^{j(\Omega_x x+\Omega_y y)}\right\} \\
&= u_{g_{max}} Re\left\{(cos\Omega_x x + j\ sin\Omega_x x)(cos\Omega_y y + j\ sin\Omega_y y)\right\} \\
&= u_{g_{max}} (cos\Omega_x x\ cos\Omega_y y - sin\Omega_x x\ sin\Omega_y y) \\
&= u_{g_1}(x,y) - u_{g_2}(x,y)
\end{aligned}
\tag{2.3}
$$

According to this expression the elementary two-dimensional turbulence field can be considered as the superposition of two separate fields in each of which the gust velocity varies sinusoidally in the X_e- as well as in the Y_e-direction, see figures 2.1 and 2.3,

$$
\begin{aligned}
u_{g_1}(x,y) &= u_{g_{max}}\ cos\Omega_x x\ cos\Omega_y y \\
u_{g_2}(x,y) &= u_{g_{max}}\ sin\Omega_x x\ sin\Omega_y y
\end{aligned}
\tag{2.4}
$$

The characteristic difference between the two fields $u_{g_1}(x,y)$ and $u_{g_2}(x,y)$ is that the first is symmetric with respect to the vertical $O_eX_eZ_e$-plane and the second is antisymmetric with respect to this plane. Quite similarly, the v_g- and w_g-fields can be separated in symmetric and antisymmetric parts,

$$
\begin{aligned}
v_{g_1}(x,y) &= v_{g_{max}} cos\Omega_x x\ cos\Omega_y y \quad \text{(antisymmetric)} \\
v_{g_2}(x,y) &= v_{g_{max}} sin\Omega_x x\ sin\Omega_y y \quad \text{(symmetric)}
\end{aligned}
\tag{2.5}
$$

$$
\begin{aligned}
w_{g_1}(x,y) &= w_{g_{max}} cos\Omega_x x\ cos\Omega_y y \quad \text{(symmetric)} \\
w_{g_2}(x,y) &= w_{g_{max}} sin\Omega_x x\ sin\Omega_y y \quad \text{(antisymmetric)}
\end{aligned}
\tag{2.6}
$$

The expressions (2.1) permit two apparently different interpretations of the way in which the gust velocity varies in the elementary field of flow. In order to gain more insight in the construction of the elementary flow field, the figures 2.1 and 2.3 are numerically superimposed. The resulting image has been plotted in figure 2.4.

In nominally steady straight flight, the aircraft's plane of symmetry usually coincides with the $O_eX_eZ_e$-plane. As a consequence the symmetric parts $u_{g_1}(x,y)$, $v_{g_2}(x,y)$ and $w_{g_1}(x,y)$

of the elementary fields can only cause symmetric deviations from steady flight. For symmetrical motions Ω_y is set to zero. According to equation (2.5), the $v_{g_2}(x,y)$-field can be omitted when considering symmetrical aircraft responses. Only the u_g and w_g turbulence fields will be considered for symmetrical aircraft responses. The three antisymmetric parts $u_{g_2}(x,y)$, $v_{g_1}(x,y)$ and $w_{g_2}(x,y)$ of the elementary turbulence fields will be used for the calculation of the asymmetrical aircraft responses.

2.2.1 Symmetrical Turbulence Modelling

The aerodynamic forces and moment acting on the aircraft due to symmetrical turbulence are written as X, Z and M to which the subscript g, which stands for gust, is added: X_g , Z_g and M_g . These three variables are functions of the gust velocities u_g and w_g or their non-dimensional equivalents $\hat{u}_g$ and α_g and the time derivatives of them. The gust velocities affecting symmetrical aircraft motions have been derived in the foregoing $(\Omega_y = 0)$,

$$\begin{aligned} u_g(x,y) &= u_{g_{max}} \cos \Omega_x x \\ w_g(x,y) &= w_{g_{max}} \cos \Omega_x x \end{aligned} \tag{2.7}$$

or, using their non-dimensional equivalents,

$$\begin{aligned} \hat{u}_g &= \frac{u_g}{V} \\ \alpha_g &= \frac{w_g}{V} \end{aligned}$$

For calculating the aerodynamic forces and moment caused by "symmetrical gusts" we can express, for example, X_g as a series expansion,

$$\begin{aligned} X_g &= \frac{\partial X_g}{\partial \hat{u}_g}\hat{u}_g + \frac{\partial X_g}{\partial \frac{\dot{\hat{u}}_g\bar{c}}{V}}\frac{\dot{\hat{u}}_g\bar{c}}{V} + \frac{\partial X_g}{\partial \frac{\ddot{\hat{u}}_g\bar{c}^2}{V^2}}\frac{\ddot{\hat{u}}_g\bar{c}^2}{V^2} + \cdots \\ &+ \frac{\partial X_g}{\partial \alpha_g}\alpha_g + \frac{\partial X_g}{\partial \frac{\dot{\alpha}_g\bar{c}}{V}}\frac{\dot{\alpha}_g\bar{c}}{V} + \frac{\partial X_g}{\partial \frac{\ddot{\alpha}_g\bar{c}^2}{V^2}}\frac{\ddot{\alpha}_g\bar{c}^2}{V^2} + \cdots \\ &+ \frac{1}{2!}\left(\text{2nd order terms with respect to } \hat{u}_g, \alpha_g, \frac{\dot{\hat{u}}_g\bar{c}}{V}, \frac{\dot{\alpha}_g\bar{c}}{V}, \cdots\right) + \\ &+ \frac{1}{3!}(\cdots\cdots) + \cdots\cdots \text{etc.} \end{aligned} \tag{2.8}$$

Only if the expansion is extended towards an infinite number of derivatives this expression can be regarded as an exact description of the aerodynamic force due to turbulence. However, due to the assumption that $\hat{u}_g$ and α_g remain sufficiently small (i.e. of the same order of magnitude as $\hat{u}$ and α) only the linear terms in the above series expansion are maintained. Also the derivatives with respect to $\frac{\ddot{\hat{u}}_g\bar{c}^2}{V^2}$, $\frac{\ddot{\alpha}_g\bar{c}^2}{V^2}$, ... are omitted. The resulting expression for X_g then becomes,

$$X_g = \frac{\partial X_g}{\partial \hat{u}_g}\hat{u}_g + \frac{\partial X_g}{\partial \frac{\dot{\hat{u}}_g\bar{c}}{V}}\frac{\dot{\hat{u}}_g\bar{c}}{V} + \frac{\partial X_g}{\partial \alpha_g}\alpha_g + \frac{\partial X_g}{\partial \frac{\dot{\alpha}_g\bar{c}}{V}}\frac{\dot{\alpha}_g\bar{c}}{V} \tag{2.9}$$

The aerodynamic forces X_g, Z_g and moment M_g may be expressed in non-dimensional coefficients,

$$\begin{aligned} C_{X_g} &= \frac{X_g}{\frac{1}{2}\rho V^2 S} \\ C_{Z_g} &= \frac{Z_g}{\frac{1}{2}\rho V^2 S} \\ C_{m_g} &= \frac{M_g}{\frac{1}{2}\rho V^2 S\bar{c}} \end{aligned} \tag{2.10}$$

I.e., the expression for C_{X_g} reads,

$$\begin{aligned} C_{X_g} &= \frac{1}{\frac{1}{2}\rho V^2 S}\frac{\partial X_g}{\partial \hat{u}_g}\hat{u}_g + \frac{1}{\frac{1}{2}\rho V^2 S}\frac{\partial X_g}{\partial \frac{\dot{\hat{u}}_g\bar{c}}{V}}\frac{\dot{\hat{u}}_g\bar{c}}{V} + \\ &+ \frac{1}{\frac{1}{2}\rho V^2 S}\frac{\partial X_g}{\partial \alpha_g}\alpha_g + \frac{1}{\frac{1}{2}\rho V^2 S}\frac{\partial X_g}{\partial \frac{\dot{\alpha}_g\bar{c}}{V}}\frac{\dot{\alpha}_g\bar{c}}{V} \end{aligned} \tag{2.11}$$

or, in an abbreviated notation,

$$C_{X_g} = C_{X_{u_g}}\hat{u}_g + C_{X_{\dot{u}_g}}\frac{\dot{\hat{u}}_g\bar{c}}{V} + C_{X_{\alpha_g}}\alpha_g + C_{X_{\dot{\alpha}_g}}\frac{\dot{\alpha}_g\bar{c}}{V} \tag{2.12}$$

In a similar way C_{Z_g} and C_{m_g} can be written as,

$$C_{Z_g} = C_{Z_{u_g}}\hat{u}_g + C_{Z_{\dot{u}_g}}\frac{\dot{\hat{u}}_g\bar{c}}{V} + C_{Z_{\alpha_g}}\alpha_g + C_{Z_{\dot{\alpha}_g}}\frac{\dot{\alpha}_g\bar{c}}{V} \tag{2.13}$$

$$C_{m_g} = C_{m_{u_g}}\hat{u}_g + C_{m_{\dot{u}_g}}\frac{\dot{\hat{u}}_g\bar{c}}{V} + C_{m_{\alpha_g}}\alpha_g + C_{m_{\dot{\alpha}_g}}\frac{\dot{\alpha}_g\bar{c}}{V} \tag{2.14}$$

The partial derivatives, $C_{X_{u_g}}$, $C_{X_{\dot{u}_g}}$, etc. will be called *"gust derivatives"*. The gust derivatives $C_{X_{\dot{u}_g}}$, $C_{Z_{\dot{u}_g}}$ and $C_{m_{\dot{u}_g}}$ are usually neglected. All other gust derivatives can be expressed in terms of stability derivatives. The steady gust derivatives $C_{X_{u_g}}$, $C_{Z_{u_g}}$ and $C_{m_{u_g}}$ are equal to the steady stability derivatives C_{X_u}, C_{Z_u} and C_{m_u}. Furthermore, the steady gust derivatives $C_{X_{\alpha_g}}$, $C_{Z_{\alpha_g}}$ and $C_{m_{\alpha_g}}$ are equal to the steady stability derivatives C_{X_α}, C_{Z_α} and C_{m_α}. The unsteady gust derivatives $C_{X_{\dot{\alpha}_g}}$, $C_{Z_{\dot{\alpha}_g}}$ and $C_{m_{\dot{\alpha}_g}}$ are equal to respectively $C_{X_{\dot{\alpha}}} - C_{X_q}$, $C_{Z_{\dot{\alpha}}} - C_{Z_q}$ and $C_{m_{\dot{\alpha}}} - C_{m_q}$.

2.2.2 Asymmetrical Turbulence Modelling

As mentioned earlier the asymmetrical turbulence inputs are $u_{g_2}(x,y)$, $v_{g_1}(x,y)$ and $w_{g_2}(x,y)$. For simplicity they will be written as u_g, v_g and w_g. Or using their non-dimensional equivalents,

$$\begin{aligned}\hat{u}_g &= \frac{u_g}{V}\\ \beta_g &= \frac{v_g}{V}\\ \alpha_g &= \frac{w_g}{V}\end{aligned}$$

The aerodynamic sideforce C_{Y_g}, rolling moment C_{l_g} and yawing moment C_{n_g} may be written as,

$$\begin{aligned}C_{Y_g} &= C_{Y_{u_g}}\hat{u}_g + C_{Y_{\beta_g}}\beta_g + C_{Y_{\dot{\beta}_g}}\frac{\dot{\beta}_g b}{V} + C_{Y_{\alpha_g}}\alpha_g\\ C_{l_g} &= C_{l_{u_g}}\hat{u}_g + C_{l_{\beta_g}}\beta_g + C_{l_{\dot{\beta}_g}}\frac{\dot{\beta}_g b}{V} + C_{l_{\alpha_g}}\alpha_g\\ C_{n_g} &= C_{n_{u_g}}\hat{u}_g + C_{n_{\beta_g}}\beta_g + C_{n_{\dot{\beta}_g}}\frac{\dot{\beta}_g b}{V} + C_{n_{\alpha_g}}\alpha_g\end{aligned} \qquad (2.15)$$

The above parameters $C_{Y_{u_g}}$, $C_{Y_{\beta_g}}$, etc. will also be called *gust derivatives.* These partial derivatives will be discussed next.

Aerodynamic force and moments due to gust velocities parallel to the longitudinal axis (u_g)

For calculating the aerodynamic force and moments caused by the antisymmetric part of the elementary u_g-field, equation (2.4), only u_{g_2} has to be considered here,

$$u_{g_2}(x,y) = u_{g_{max}}\ sin\Omega_x x\ sin\Omega_y y$$

Next, the longitudinal gust velocity in the $O_eX_eZ_e$-plane will be referred to simply as u_g. Also the index 2 of $u_{g_2}(x,y)$ will be omitted, so,

$$u_g(x,y) = u_g\ \sin\Omega_y y \qquad (2.16)$$

with

$$u_g = u_{g_{max}}\ \sin\Omega_x x$$

Due to the variations of $u_g(x,y)$ in the Y_e-direction, a rolling and a yawing moment will act on the aircraft, see also figure 2.5. Only the contribution of the wing to the rolling moment, C_{l_g}, and the yawing moment, C_{n_g}, will be taken into account. The sideforce, C_{Y_g}, due to this turbulence velocity $u_g(x,y)$ is supposed to be negligible.

The calculation of the rolling moment acting on the wing, is based on the assumption that the additional lift due to this turbulence component can be computed by means of the strip-theory. A chordwise strip of the wing of width dy at a distance y from the plane of symmetry contributes to the rolling moment, see figure 2.5,

$$\begin{aligned} dL_g &= -c_l \frac{1}{2}\rho \left\{ [V + u_g(x,y)]^2 - V^2 \right\} \ c\ y\ dy \\ &= -c_l \frac{1}{2}\rho \left\{ 2u_g(x,y)V + (u_g(x,y))^2 \right\} \ c\ y\ dy \end{aligned} \tag{2.17}$$

assuming $(u_g(x,y))^2 \approx 0$ and using equation (2.16), equation (2.17) becomes,

$$dL_g = -\rho V\ u_{g_{max}}\ sin\Omega_x x\ c_l c\ sin\Omega_y y\ y\ dy \tag{2.18}$$

Consequently the total rolling moment due to the u_g-field is,

$$L_g = -2\rho V\ u_g \int_0^{\frac{b}{2}} c_l c\ sin\Omega_y y\ y\ dy \tag{2.19}$$

This rolling moment can be expressed by means of a non-dimensional coefficient C_{l_g}

$$C_{l_g} = \frac{L_g}{\frac{1}{2}\rho V^2 S b} = -\frac{4}{Sb}\frac{u_g}{V}\int_0^{\frac{b}{2}} c_l c\ sin\Omega_y y\ y\ dy \tag{2.20}$$

Next the non-dimensional rolling moment coefficient C_{l_g}, is written by using a newly introduced gust derivative $C_{l_{u_g}}(\Omega_y\ \frac{b}{2})$,

$$C_{l_g} = C_{l_{u_g}}(\Omega_y\ \frac{b}{2})\ \hat{u}_g \tag{2.21}$$

where,

$$\hat{u}_g = \frac{u_g}{V}$$

Hence,

$$C_{l_{u_g}}(\Omega_y\ \frac{b}{2}) = -\frac{4}{Sb}\int_0^{\frac{b}{2}} c_l c\ sin\Omega_y y\ y\ dy \tag{2.22}$$

For sufficiently small values of $\Omega_y \frac{b}{2}$, i.e. for long wavelengths of $u_g(x,y)$ in the Y_e-direction, the velocity $u_g(x,y)$ varies approximately linearly along the wingspan. For these small values of $\Omega_y \frac{b}{2}$, $u_g(x,y)$ can be approximated, by replacing $sin\Omega_y y$ by $\Omega_y y$,

$$u_g(x,y) = u_g \Omega_y y \tag{2.23}$$

This velocity distribution corresponds to the additional velocity due to a constant yawing velocity r,

$$\Delta u = -ry \tag{2.24}$$

When further elaborating the expression for $C_{l_{u_g}}$, use is made of this similarity. The rolling moment acting on the wing due to a constant yawing velocity r can also be calculated by means of the strip-theory. An element of the wing of width dy at a distance y from the plane of symmetry contributes to the rolling moment,

$$dL = c_l \frac{1}{2}\rho\, 2V\, ry\, cy\, dy \tag{2.25}$$

The total rolling moment on the wing then is,

$$L = 2\rho V r \int_0^{\frac{b}{2}} c_l c\, y^2\, dy = C_{l_{r_w}} \frac{rb}{2V}\frac{1}{2}\rho V^2 S b \tag{2.26}$$

The contribution of the wing to the stability derivative C_{l_r} is accordingly,

$$C_{l_{r_w}} = \frac{8}{Sb^2}\int_0^{\frac{b}{2}} c_l c\, y^2\, dy \tag{2.27}$$

This expression for $C_{l_{r_w}}$ can be related to $C_{l_{u_g}}$ expressed by equation (2.22),

$$C_{l_{u_g}}(\Omega_y \frac{b}{2}) = -C_{l_{r_w}} \frac{b}{2}\frac{\int_0^{\frac{b}{2}} c_l c\, sin\Omega_y y\, y\, dy}{\int_0^{\frac{b}{2}} c_l c\, y^2\, dy} \tag{2.28}$$

We will now introduce the non-dimensional function $h(\Omega_y \frac{b}{2})$,

$$h(\Omega_y \frac{b}{2}) = \frac{b}{2}\frac{\int_0^{\frac{b}{2}} c_l c\, sin\Omega_y y\, y\, dy}{\int_0^{\frac{b}{2}} c_l c\, y^2\, dy} \tag{2.29}$$

Using equation (2.29), $C_{l_{u_g}}$ can be written as,

$$C_{l_{u_g}}(\Omega_y \frac{b}{2}) = -C_{l_{r_w}}\, h(\Omega_y \frac{b}{2}) \tag{2.30}$$

It may be assumed, that the relation (2.30) between $C_{l_{u_g}}$ and $C_{l_{r_w}}$ holds true with some accuracy, also if $C_{l_{r_w}}$ is not obtained by means of the strip-theory.

In an identical manner the yawing moment acting on the wing due to the longitudinal gust velocity $u_g(x, y)$ is derived. This moment can be written as,

$$N_g = C_{n_g} \frac{1}{2}\rho V^2 S b \tag{2.31}$$

where,

$$C_{n_g} = C_{n_{u_g}}(\Omega_y \frac{b}{2})\, \hat{u}_g \tag{2.32}$$

and,

$$C_{n_{u_g}}(\Omega_y \frac{b}{2}) = -C_{n_{r_w}}\, h(\Omega_y \frac{b}{2}) \tag{2.33}$$

In the latter expression the function $h(\Omega_y \frac{b}{2})$ again is represented by equation (2.29).

Aerodynamic force and moments due to gust velocities parallel to the lateral axis (β_g)

When considering the asymmetric force and moments exerted on the aircraft by the v_g-field, only the antisymmetric part of this flowfield is of interest. Omitting the index 1, this part is presented by,

$$v_g(x,y) = v_{g_{max}}\ cos\Omega_x x\ cos\Omega_y y$$

The variation of v_g along the wingspan will be neglected. This amounts to the assumption $cos\Omega_y y = 1$.
Then v_g is written as,

$$v_g = v_{g_{max}}\ cos\Omega_x x \tag{2.34}$$

The gust velocity v_g causes a gust angle of sideslip β_g,

$$\beta_g = \frac{v_g}{V} \tag{2.35}$$

which is quite analogous to the gust angle of attack. The gust angle of sideslip causes an aerodynamic force and moments which are computed in the same way as the forces and moments due to the gust angle of attack for the symmetrical case. The results of such calculations are,

$$C_{Y_g} = \left(C_{Y_{\beta_g}} + C_{Y_{\dot{\beta}_g}} D_b\right)\beta_g \tag{2.36}$$

$$C_{l_g} = \left(C_{l_{\beta_g}} + C_{l_{\dot{\beta}_g}} D_b\right)\beta_g \tag{2.37}$$

$$C_{n_g} = \left(C_{n_{\beta_g}} + C_{n_{\dot{\beta}_g}} D_b\right)\beta_g \tag{2.38}$$

The gust derivatives $C_{Y_{\beta_g}}$, $C_{l_{\beta_g}}$ and $C_{n_{\beta_g}}$ can be expressed entirely in the stability derivatives C_{Y_β}, C_{l_β}, C_{n_β} and are equal to them. Analogous to the corresponding expressions for the symmetric gust derivative $C_{m_{\dot{\alpha}_g}}$, $C_{Y_{\dot{\beta}_g}}$ is written as,

$$C_{Y_{\dot{\beta}_g}} = C_{Y_{\dot{\beta}}} + \frac{1}{2}C_{Y_r} \tag{2.39}$$

It should be remarked, that only the contribution of the fuselage and the vertical tailplane to C_{Y_r} have to be taken into account, since only these parts of the total C_{Y_r} arise as a result of a local velocity parallel to the Y-axis. Therefore,

$$\begin{array}{lll} C_{Y_{\beta_g}} = C_{Y_\beta} & & C_{Y_{\dot{\beta}_g}} = C_{Y_{\dot{\beta}}} + \frac{1}{2}C_{Y_{r_{f+v}}} \\ C_{l_{\beta_g}} = C_{l_\beta} & & C_{l_{\dot{\beta}_g}} = C_{l_{\dot{\beta}}} + \frac{1}{2}C_{l_{r_{f+v}}} \\ C_{n_{\beta_g}} = C_{n_\beta} & & C_{n_{\dot{\beta}_g}} = C_{n_{\dot{\beta}}} + \frac{1}{2}C_{n_{r_{f+v}}} \end{array} \tag{2.40}$$

When considering aircraft having a straight wing and relatively small tailplane, the derivatives $C_{Y_{\dot{\beta}}}$, $C_{l_{\dot{\beta}}}$ and $C_{n_{\dot{\beta}}}$ may usually be neglected.

Aerodynamic force and moments due to gust velocities parallel to the vertical axis (α_g)

When considering the asymmetric force and moments exerted on the aircraft by the w_g-field, only the antisymmetric part of this field is of interest. Omitting the index 2 this part is represented by,

$$w_g(x,y) = w_{g_{max}} \, sin\Omega_x x \; sin\Omega_y y$$

This w_g-field causes a gust angle of attack α_g,

$$\alpha_g(x,y) = \frac{w_g(x,y)}{V} \tag{2.41}$$

which varies in the direction of flight as well as along the wingspan. Denoting the gust angle of attack in the $O_eX_eZ_e$-plane by α_g,

$$\alpha_g = \alpha_{g_{max}} \, sin\Omega_x x \tag{2.42}$$

it follows,

$$\alpha_g(x,y) = \alpha_g \, sin\Omega_y y \tag{2.43}$$

The calculation of the rolling and yawing moments due to the α_g-field has a great similarity to the corresponding calculation for the $\hat{u}_g$-field. The rolling moment can be written as,

$$L_g = C_{l_g} \frac{1}{2}\rho V^2 Sb \tag{2.44}$$

where,

$$C_{l_g} = C_{l_{\alpha_g}}(\Omega_y \frac{b}{2}) \, \alpha_g \tag{2.45}$$

and likewise the coefficient for the yawing moment is,

$$C_{n_g} = C_{n_{\alpha_g}}(\Omega_y \frac{b}{2}) \, \alpha_g \tag{2.46}$$

For sufficiently small values of Ω_y, i.e. for long wavelengths in the Y_e-direction, the distribution of the gust angle of attack along the span of the wing can again be approximated by replacing $sin\Omega_y y$ by $\Omega_y y$,

$$\alpha_g(x,y) = \alpha_g \, \Omega_y y \tag{2.47}$$

This distribution of the angle of attack corresponds to the additional angle of attack due to a constant rolling velocity p,

$$\Delta\alpha = \frac{p}{V} y \tag{2.48}$$

As a consequence, for sufficient small values of Ω_y the gust derivatives $C_{l_{\alpha_g}} h(\Omega_y \frac{b}{2})$ and $C_{n_{\alpha_g}} h(\Omega_y \frac{b}{2})$ can be expressed in terms of parts of the stability derivatives C_{l_p} and C_{n_p}

contributed by the wing, i.e. $C_{l_{p_w}}$ and $C_{n_{p_w}}$. After some elaboration, the following results are obtained,

$$C_{l_{\alpha_g}}(\Omega_y \frac{b}{2}) = C_{l_{p_w}} \, h(\Omega_y \frac{b}{2}) \tag{2.49}$$

and,

$$C_{n_{\alpha_g}}(\Omega_y \frac{b}{2}) = C_{n_{p_w}} \, h(\Omega_y \frac{b}{2}) \tag{2.50}$$

where the function $h(\Omega_y \frac{b}{2})$ has been given previously. For the sake of simplicity the side force due to α_g will be neglected, so,

$$C_{Y_{\alpha_g}} = 0$$

This simplification corresponds to the assumption that $C_{Y_{p_w}} = 0$.

2.2.3 Correlation Functions

Symmetrical gust

The correlation functions for the one-dimensional gust field can be written as if one only regards u_g and w_g as inputs,

$$C_{u_g u_g}(\xi) = E\{u_g(0)u_g(V\tau)\} \tag{2.51}$$
$$C_{w_g w_g}(\xi) = E\{w_g(0)w_g(V\tau)\} \tag{2.52}$$

For calculation of the correlations of the one-dimensional gust field we refer to the introduction where in equation (1.11) the following expression for the fundamental correlation function is given by,

$$C_{ij}(\xi) = \sigma^2 \left(\frac{f(\xi) - g(\xi)}{\xi^2} \xi_i \xi_j + g(\xi)\delta_{ij} \right) \tag{2.53}$$

The correlation functions C_{ij} can be calculated using the following longitudinal and lateral Dryden correlation functions,

$$f(\xi) = e^{-\frac{\xi}{L_g}}$$
$$g(\xi) = e^{-\frac{\xi}{L_g}} \left(1 - \frac{\xi}{2L_g}\right)$$

where $\xi = \sqrt{\xi_1{}^2 + \xi_2{}^2 + \xi_3{}^2}$, with $\underline{\xi} = (\xi_1, \xi_2, \xi_3)^T = (V\tau, 0, 0)^T$.
The one-dimensional correlation functions become

$$C_{u_g u_g}(\xi) = \sigma_{u_g}^2 e^{-\frac{V\tau}{L_g}}$$
$$C_{w_g w_g}(\xi) = \sigma_{u_g}^2 e^{-\frac{V\tau}{L_g}} \left(1 - \frac{V\tau}{2L_g}\right)$$

Asymmetrical gust

The autocorrelation functions for the two-dimensional case can be calculated from the following expressions if one assumes that the inputs u_g, v_g and w_g are uncorrelated,

$$\begin{aligned} C_{u_g u_g} &= E\{u_g(0,0)\, u_g(x,y)\} \\ C_{v_g v_g} &= E\{v_g(0,0)\, v_g(x,y)\} \\ C_{w_g w_g} &= E\{w_g(0,0)\, w_g(x,y)\} \end{aligned} \tag{2.54}$$

The autocorrelation functions expressed in equation (2.54) are therefore the diagonal elements of the covariance matrix of atmospheric turbulence, $C_{\underline{\bar{u}}\underline{\bar{u}}}$, equation (1.5) where the off-diagonal elements all equal zero. The three autocovariance functions will be expressed in the two basic one-dimensional autocovariance functions $f(r)$ and $g(r)$ as has been done for the one-dimensional case which was discussed before.

Considering an arbitrary point P in the $O_eX_eY_e$-plane, the velocities u_g and v_g in this plane can be resolved in components along and perpendicular to O_eP,

$$u_g = u_1\, sin\alpha + u_2\, cos\alpha = u_1\, \frac{x}{r} + u_2\, \frac{y}{r} \tag{2.55}$$

$$v_g = v_1\, cos\alpha + v_2\, sin\alpha = v_1\, \frac{y}{r} + v_2\, \frac{x}{r} \tag{2.56}$$

The expression for $C_{u_g u_g}$ then reads,

$$\begin{aligned} C_{u_g u_g} &= E\{u_g(0,0)\, u_g(x,y)\} \qquad (2.57) \\ &= E\left\{(u_1(0,0)\, \frac{x}{r} + u_2(0,0)\, \frac{y}{r})\;\; (u_1(x,y)\, \frac{x}{r} + u_2(x,y)\, \frac{y}{r})\right\} \\ &= E\left\{u_1(0,0)\, u_1(x,y)\, \left(\frac{x}{r}\right)^2 + u_2(0,0)\, u_2(x,y)\, \left(\frac{y}{r}\right)^2\right. \\ &+ \left. u_1(0,0)\, u_2(x,y)\, \left(\frac{xy}{r^2}\right) + u_2(0,0)\, u_1(x,y)\, \left(\frac{xy}{r^2}\right)\right\} \end{aligned}$$

Using these expressions for $f(r)$ and $g(r)$, the two-dimensional autocovariance functions now become,

$$\begin{aligned} C_{u_g u_g}\left(\frac{x}{L_g}, \frac{y}{L_g}\right) &= \sigma_{u_g}^2 \left\{ e^{-\frac{r}{L_g}} \left(\frac{x}{r}\right)^2 + e^{-\frac{r}{L_g}} \left(1 - \frac{1}{2}\frac{r}{L_g}\right) \left(\frac{y}{r}\right)^2 \right\} \\ C_{v_g v_g}\left(\frac{x}{L_g}, \frac{y}{L_g}\right) &= \sigma_{v_g}^2 \left\{ e^{-\frac{r}{L_g}} \left(\frac{y}{r}\right)^2 + e^{-\frac{r}{L_g}} \left(1 - \frac{1}{2}\frac{r}{L_g}\right) \left(\frac{x}{r}\right)^2 \right\} \\ C_{w_g w_g}\left(\frac{x}{L_g}, \frac{y}{L_g}\right) &= \sigma_{w_g}^2 e^{-\frac{r}{L_g}} \left(1 - \frac{1}{2}\frac{r}{L_g}\right) \end{aligned} \tag{2.58}$$

2.2.4 Input Power Spectral Density Functions

The Dryden spectra are given by

$$S_{\bar{u}_g\bar{u}_g}(\omega) = 2\sigma^2 \frac{L_g}{V} \frac{1}{1+\left(L_g\frac{\omega}{V}\right)^2} \tag{2.59}$$

$$S_{\bar{v}_g\bar{v}_g}(\omega) = \sigma^2 \frac{L_g}{V} \frac{1+3\left(L_g\frac{\omega}{V}\right)^2}{\left[1+\left(L_g\frac{\omega}{V}\right)^2\right]^2} \tag{2.60}$$

$$\tag{2.61}$$

$$S_{\bar{w}_g\bar{w}_g}(\omega) = \sigma^2 \frac{L_g}{V} \frac{1+3\left(L_g\frac{\omega}{V}\right)^2}{\left[1+\left(L_g\frac{\omega}{V}\right)^2\right]^2} \tag{2.62}$$

The spectral density function matrix can be obtained by applying the Fourier transform to the elements of the covariance matrix. The off-diagonal elements (cross power spectral densities) of the power spectral density matrix of atmospheric turbulence velocities are equal to zero, since the cross correlations are assumed to equal zero so the only three two-dimensional power spectral density functions of interest are,

$$\begin{array}{c} S_{u_gu_g}(\Omega_x L_g, \Omega_y L_g) \\ S_{v_gv_g}(\Omega_x L_g, \Omega_y L_g) \\ S_{w_gw_g}(\Omega_x L_g, \Omega_y L_g) \end{array} \tag{2.63}$$

These power spectral density functions are obtained by applying the Fourier transform for two variables to the two-dimensional autocovariance functions, see equation (2.58).

As an extension to the conventional expression for the Fourier transform, the expression for the Fourier transform $S(\Omega_x L_g, \Omega_y L_g)$ of the covariance function for two variables $\frac{x}{L_g}$ and $\frac{y}{L_g}$ reads,

$$S(\Omega_x L_g, \Omega_y L_g) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} C\left(\frac{x}{L_g}, \frac{y}{L_g}\right) e^{-j(\Omega_x x+\Omega_y y)}\, d\frac{x}{L_g} d\frac{y}{L_g} \tag{2.64}$$

Since the autocovariance functions $C\left(\frac{x}{L_g}, \frac{y}{L_g}\right)$ are all strictly even functions, see Ref. (Mulder & Vaart, 1993), we can write this as, equation (2.64),

$$S(\Omega_x L_g, \Omega_y L_g) = 4\int_{0}^{+\infty}\int_{0}^{+\infty} C\left(\frac{x}{L_g}, \frac{y}{L_g}\right) cos\Omega_x x \; cos\Omega_y y \; d\frac{x}{L_g} d\frac{y}{L_g}$$

The resulting three power spectra, equation (2.63) are obtained by substituting the autocovariance functions, equations (2.58),

$$S_{u_gu_g}(\Omega_x L_g, \Omega_y L_g) = 4\int_{0}^{+\infty}\int_{0}^{+\infty} \sigma_{u_g}^2 \left\{ e^{-\frac{r}{L_g}}\left(\frac{x}{r}\right)^2 + e^{-\frac{r}{L_g}}\left(1-\frac{1}{2}\frac{r}{L_g}\right)\left(\frac{y}{r}\right)^2 \right\}$$

$$\cos\Omega_x x \; \cos\Omega_y y \; d\frac{x}{L_g} d\frac{y}{L_g} \tag{2.65}$$

$$S_{v_g v_g}(\Omega_x L_g, \Omega_y L_g) = 4\int_0^{+\infty}\int_0^{+\infty} \sigma_{v_g}^2 \left\{ e^{-\frac{r}{L_g}}\left(\frac{y}{r}\right)^2 + e^{-\frac{r}{L_g}}\left(1-\frac{1}{2}\frac{r}{L_g}\right)\left(\frac{x}{r}\right)^2 \right\} \cos\Omega_x x \; \cos\Omega_y y \; d\frac{x}{L_g} d\frac{y}{L_g} \tag{2.66}$$

$$S_{w_g w_g}(\Omega_x L_g, \Omega_y L_g) = 4\int_0^{+\infty}\int_0^{+\infty} \sigma_{w_g}^2 e^{-\frac{r}{L_g}}\left(1-\frac{1}{2}\frac{r}{L_g}\right) \cos\Omega_x x \; \cos\Omega_y y \; d\frac{x}{L_g} d\frac{y}{L_g} \tag{2.67}$$

The above integrals can all be expressed in closed form. The results are,

$$S_{u_g u_g}(\Omega_x L_g, \Omega_y L_g) = \pi\sigma_{u_g}^2 \frac{1+\Omega_x^2 L_g^2 + 4\Omega_y^2 L_g^2}{\left(1+\Omega_x^2 L_g^2+\Omega_y^2 L_g^2\right)^{5/2}} \tag{2.68}$$

$$S_{v_g v_g}(\Omega_x L_g, \Omega_y L_g) = \pi\sigma_{v_g}^2 \frac{1+4\Omega_x^2 L_g^2 + \Omega_y^2 L_g^2}{\left(1+\Omega_x^2 L_g^2+\Omega_y^2 L_g^2\right)^{5/2}} \tag{2.69}$$

$$S_{w_g w_g}(\Omega_x L_g, \Omega_y L_g) = 3\pi\sigma_{w_g}^2 \frac{\Omega_x^2 L_g^2 + \Omega_y^2 L_g^2}{\left(1+\Omega_x^2 L_g^2+\Omega_y^2 L_g^2\right)^{5/2}} \tag{2.70}$$

There is a close relation between the above two-dimensional power spectral densities and the corresponding one-dimensional spectral density functions used to compute the symmetric motion of an aircraft. The magnitude of a one-dimensional spectral density, i.e. $S'_{u_g u_g}$ at a certain value of $\Omega_x L_g$ can be considered to express the contribution to the total power contained in the fluctuations of u_g at this particular frequency, for a frequency range $\Delta(\Omega_x L_g) = 1$. In the two-dimensional spectral density function $S_{u_g u_g}(\Omega_x L_g, \Omega_y L_g)$, at the $\Omega_x L_g$ considered, the same contribution to the power is provided by all frequencies $\Omega_y L_g$ together. The one-dimensional spectrum does not distinguish the value of $\Omega_y L_g$, at which the power is contributed. As a consequence the relation between $S'_{u_g u_g}(\Omega_x L_g)$ and $S_{u_g u_g}(\Omega_x L_g, \Omega_y L_g)$ is,

$$S'_{u_g u_g}(\Omega_x L_g) = \frac{1}{\pi}\int_0^{+\infty} S_{u_g u_g}(\Omega_x L_g, \Omega_y L_g)\; d(\Omega_y L_g) \tag{2.71}$$

Corresponding relations hold true for the power spectral density functions of v_g and w_g. If the integration of equation (2.71) is carried out, the familiar one-dimensional spectral density function results,

$$S'_{u_g u_g}(\Omega_x L_g) = 2\sigma_{u_g}^2 \frac{1}{1+\Omega_x^2 L_g^2} \tag{2.72}$$

which is indeed identical to the relevant one-dimensional Dryden power spectrum, equation (2.59). Similar results are obtained for the power spectral density functions of v_g and w_g.

2.3 Aircraft Modelling

2.3.1 Symmetrical Aircraft Motions

In this section the equations of motion for a rigid aircraft in a field of atmospheric turbulence will be presented. The equations of motion for a rigid aircraft in smooth atmospheric conditions have been derived in Ref. (Gerlach, 1981). The general aerodynamic forces and moment due to atmospheric turbulence have been derived in the previous sections. As a recapitulation the general aerodynamic forces and moment are in the frequency domain,

$$C_{X_g}(\omega) = \frac{C_{X_g}}{\hat{u}_g}(\omega)\,\hat{u}_g(\omega) + \frac{C_{X_g}}{\alpha_g}(\omega)\,\alpha_g(\omega) \tag{2.73}$$

$$C_{Z_g}(\omega) = \frac{C_{Z_g}}{\hat{u}_g}(\omega)\,\hat{u}_g(\omega) + \frac{C_{Z_g}}{\alpha_g}(\omega)\,\alpha_g(\omega) \tag{2.74}$$

$$C_{m_g}(\omega) = \frac{C_{m_g}}{\hat{u}_g}(\omega)\,\hat{u}_g(\omega) + \frac{C_{m_g}}{\alpha_g}(\omega)\,\alpha_g(\omega) \tag{2.75}$$

We have assumed, however, the aerodynamic frequency response functions may be described by,

$$C_{X_g}(\omega) = C_{X_{u_g}}\hat{u}_g(\omega) + C_{X_{\dot{u}_g}}\frac{j\omega\;\hat{u}_g(\omega)\bar{c}}{V} + C_{X_{\alpha_g}}\alpha_g(\omega) + C_{X_{\dot{\alpha}_g}}\frac{j\omega\;\alpha_g(\omega)\bar{c}}{V} \tag{2.76}$$

$$C_{Z_g}(\omega) = C_{Z_{u_g}}\hat{u}_g(\omega) + C_{Z_{\dot{u}_g}}\frac{j\omega\;\hat{u}_g(\omega)\bar{c}}{V} + C_{Z_{\alpha_g}}\alpha_g(\omega) + C_{Z_{\dot{\alpha}_g}}\frac{j\omega\;\alpha_g(\omega)\bar{c}}{V} \tag{2.77}$$

$$C_{m_g}(\omega) = C_{m_{u_g}}\hat{u}_g(\omega) + C_{m_{\dot{u}_g}}\frac{j\omega\;\hat{u}_g(\omega)\bar{c}}{V} + C_{m_{\alpha_g}}\alpha_g(\omega) + C_{m_{\dot{\alpha}_g}}\frac{j\omega\;\alpha_g(\omega)\bar{c}}{V} \tag{2.78}$$

or equivalently in the time domain, assuming constant parameters,

$$C_{X_g} = C_{X_{u_g}}\hat{u}_g + C_{X_{\dot{u}_g}}\frac{\dot{\hat{u}}_g\bar{c}}{V} + C_{X_{\alpha_g}}\alpha_g + C_{X_{\dot{\alpha}_g}}\frac{\dot{\alpha}_g\bar{c}}{V} \tag{2.79}$$

$$C_{Z_g} = C_{Z_{u_g}}\hat{u}_g + C_{Z_{\dot{u}_g}}\frac{\dot{\hat{u}}_g\bar{c}}{V} + C_{Z_{\alpha_g}}\alpha_g + C_{Z_{\dot{\alpha}_g}}\frac{\dot{\alpha}_g\bar{c}}{V} \tag{2.80}$$

$$C_{m_g} = C_{m_{u_g}}\hat{u}_g + C_{m_{\dot{u}_g}}\frac{\dot{\hat{u}}_g\bar{c}}{V} + C_{m_{\alpha_g}}\alpha_g + C_{m_{\dot{\alpha}_g}}\frac{\dot{\alpha}_g\bar{c}}{V} \tag{2.81}$$

Remember that the unsteady gust derivatives with respect to $\frac{\dot{\hat{u}}_g\bar{c}}{V}$ are usually set to zero.

The equations of motion for a rigid aircraft have been derived in Refs. (Gerlach & Baarspul, 1968) and (Etkin, 1972). These equations are valid for small deviations from steady horizontal flight. The equations are extended for flight in a turbulent atmosphere and can be written as follows (with $D_c = \frac{\bar{c}}{V}\frac{d.}{dt}$),

$$\begin{bmatrix} C_{X_u} - 2\mu_c D_c & C_{X_\alpha} & C_{Z_0} & 0 \\ C_{Z_u} & C_{Z_\alpha} + (C_{Z_{\dot\alpha}} - 2\mu_c)\, D_c & -C_{X_0} & 2\mu_c + C_{Z_q} \\ 0 & 0 & -D_c & 1 \\ C_{m_u} & C_{m_\alpha} + C_{m_{\dot\alpha}} D_c & 0 & C_{m_q} - 2\mu_c K_Y^2 D_c \end{bmatrix} \begin{bmatrix} \hat{u} \\ \alpha \\ \theta \\ \frac{q\bar{c}}{V} \end{bmatrix} =$$

$$- \begin{bmatrix} C_{X_{u_g}} & 0 & C_{X_{\alpha_g}} & 0 \\ C_{Z_{u_g}} & C_{Z_{\dot{u}_g}} & C_{Z_{\alpha_g}} & C_{Z_{\dot\alpha_g}} \\ 0 & 0 & 0 & 0 \\ C_{m_{u_g}} & C_{m_{\dot{u}_g}} & C_{m_{\alpha_g}} & C_{m_{\dot\alpha_g}} \end{bmatrix} \begin{bmatrix} \hat{u}_g \\ D_c \hat{u}_g \\ \alpha_g \\ D_c \alpha_g \end{bmatrix} \qquad (2.82)$$

The general state-space representation,

$$\dot{\underline{x}} = A\,\underline{x} + B\,\underline{u} \qquad (2.83)$$

can be obtained by rearranging equation (2.83), see Ref. (Broek & Brandt, 1984). The final result, in abbreviated notation, is,

$$\begin{bmatrix} \dot{\hat{u}} \\ \dot\alpha \\ \dot\theta \\ \frac{\dot{q}\bar{c}}{V} \end{bmatrix} = \begin{bmatrix} x_u & x_\alpha & x_\theta & 0 \\ z_u & z_\alpha & z_\theta & z_q \\ 0 & 0 & 0 & \frac{V}{\bar{c}} \\ m_u & m_\alpha & m_\theta & m_q \end{bmatrix} \begin{bmatrix} \hat{u} \\ \alpha \\ \theta \\ \frac{q\bar{c}}{V} \end{bmatrix} + \begin{bmatrix} x_{u_g} & 0 & x_{\alpha_g} & 0 \\ z_{u_g} & z_{\dot{u}_g} & z_{\alpha_g} & z_{\dot\alpha_g} \\ 0 & 0 & 0 & 0 \\ m_{u_g} & m_{\dot{u}_g} & m_{\alpha_g} & m_{\dot\alpha_g} \end{bmatrix} \begin{bmatrix} \hat{u}_g \\ \frac{\dot{\hat{u}}_g\bar{c}}{V} \\ \alpha_g \\ \frac{\dot\alpha_g\bar{c}}{V} \end{bmatrix} \qquad (2.84)$$

The definition of these symbols are recapitulated in table 2.1.

2.3.2 Asymmetrical Aircraft Motions

The non-dimensional aerodynamic force and moments due to asymmetric atmospheric turbulence are written as,

$$\begin{aligned} C_{Y_g} &= C_{Y_{u_g}}\hat{u}_g + C_{Y_{\beta_g}}\beta_g + C_{Y_{\dot\beta_g}}\frac{\dot\beta_g b}{V} + C_{Y_{\alpha_g}}\alpha_g \\ C_{l_g} &= C_{l_{u_g}}\hat{u}_g + C_{l_{\beta_g}}\beta_g + C_{l_{\dot\beta_g}}\frac{\dot\beta_g b}{V} + C_{l_{\alpha_g}}\alpha_g \\ C_{n_g} &= C_{n_{u_g}}\hat{u}_g + C_{n_{\beta_g}}\beta_g + C_{n_{\dot\beta_g}}\frac{\dot\beta_g b}{V} + C_{n_{\alpha_g}}\alpha_g \end{aligned}$$

Or, with the definition of the gust derivatives, the final expressions describing the asymmetric aerodynamic force and moments, acting on the aircraft due to atmospheric turbulence, can now be collected,

$$C_{Y_g} = C_{Y_\beta}\beta_g + C_{Y_{\dot{\beta}_g}}\frac{\dot{\beta}_g b}{V} \tag{2.85}$$

$$C_{l_g} = -C_{l_{r_w}}\hat{u}_g + C_{l_\beta}\beta_g + C_{l_{\dot{\beta}_g}}\frac{\dot{\beta}_g b}{V} + C_{l_{p_w}}\alpha_g \tag{2.86}$$

$$C_{n_g} = -C_{n_{r_w}}\hat{u}_g + C_{n_\beta}\beta_g + C_{n_{\dot{\beta}_g}}\frac{\dot{\beta}_g b}{V} + C_{n_{p_w}}\alpha_g \tag{2.87}$$

As discussed in the foregoing, the contributions of the longitudinal and vertical gust velocities to C_{l_g} and C_{n_g} in the above expressions hold true only if they are used in conjunction with the following one-dimensional estimators of the power spectral densities of $\hat{u}_g$ and α_g,

$$I_{\hat{u}_g\hat{u}_g}(\Omega_x L_g, B) = \sigma^2_{\hat{u}_g}\int_0^\infty h^2(\Omega_y\frac{b}{2})\,\frac{1+\Omega_x^2L_g^2+4\,\Omega_y^2L_g^2}{\left(1+\Omega_x^2L_g^2+\Omega_y^2L_g^2\right)^{5/2}}\,d(\Omega_y L_g) \tag{2.88}$$

approximated by,

$$I_{\hat{u}_g\hat{u}_g}(\Omega_x L_g, B) = I_{\hat{u}_g}(0,B)\,\frac{1+\tau_3^2\,\Omega_x^2L_g^2}{\left(1+\tau_1^2\,\Omega_x^2L_g^2\right)\left(1+\tau_2^2\,\Omega_x^2L_g^2\right)} \tag{2.89}$$

and,

$$I_{\alpha_g\alpha_g}(\Omega_x L_g, B) = 3\,\sigma^2_{\alpha_g}\int_0^\infty h^2(\Omega_y\frac{b}{2})\,\frac{\Omega_x^2L_g^2+\Omega_y^2L_g^2}{\left(1+\Omega_x^2L_g^2+\Omega_y^2L_g^2\right)^{5/2}}\,d(\Omega_y L_g) \tag{2.90}$$

approximated by,

$$I_{\alpha_g\alpha_g}(\Omega_x L_g, B) = I_{\alpha_g}(0,B)\,\frac{1+\tau_6^2\,\Omega_x^2L_g^2}{\left(1+\tau_4^2\,\Omega_x^2L_g^2\right)\left(1+\tau_5^2\,\Omega_x^2L_g^2\right)} \tag{2.91}$$

In the above equations the parameter B equals $\frac{b}{2L_g}$, the fraction of the aircraft's wingspan to the characteristic gust length. The power spectral density of β_g is,

$$S_{\beta_g\beta_g}(\Omega_x L_g) = \sigma^2_{\beta_g}\frac{1+3\,\Omega_x^2L_g^2}{\left(1+\Omega_x^2L_g^2\right)^2} \tag{2.92}$$

The values of $I_{\hat{u}_g}(0,B)$ and τ_1, τ_2, τ_3 in the approximated power spectral density function for the longitudinal gust velocity and of $I_{\alpha_g}(0,B)$ and τ_4, τ_5, τ_6 in the approximated power spectral density function of the vertical gust velocity are represented as functions of B in tables 2.3, 2.4 and 2.5. In the following discussion, the simplifying but not essential assumption is made, that the aircraft may be considered as a rigid body. The asymmetric motions of the aircraft, considered as small deviations from steady, symmetric and level flight are described by the following four linear differential equations, Refs. (Etkin, 1972) and (Gerlach & Baarspul, 1968) (with $D_b = \frac{b}{V}\frac{d(.)}{dt}$),

$$\begin{bmatrix} C_{Y_\beta} - 2\mu_b D_b & C_L & C_{Y_p} & C_{Y_r} - 4\mu_b \\ 0 & -\frac{1}{2}D_b & 1 & 0 \\ C_{l_\beta} & 0 & C_{l_p} - 4\mu_b K_X^2 D_b & C_{l_r} + 4\mu_b K_{XZ} D_b \\ C_{n_\beta} & 0 & C_{n_p} + 4\mu_b K_{XZ} D_b & C_{n_r} - 4\mu_b K_Z^2 D_b \end{bmatrix} \begin{bmatrix} \beta \\ \varphi \\ \frac{pb}{2V} \\ \frac{rb}{2V} \end{bmatrix} =$$

$$- \begin{bmatrix} 0 & C_{Y_\beta} & C_{Y_{\dot\beta_g}} & 0 \\ 0 & 0 & 0 & 0 \\ C_{l_{u_g}}(\Omega_y \frac{b}{2}) & C_{l_\beta} & C_{l_{\dot\beta_g}} & C_{l_{\alpha_g}}(\Omega_y \frac{b}{2}) \\ C_{n_{u_g}}(\Omega_y \frac{b}{2}) & C_{n_\beta} & C_{n_{\dot\beta_g}} & C_{n_{\alpha_g}}(\Omega_y \frac{b}{2}) \end{bmatrix} \begin{bmatrix} \hat{u}_g \\ \beta_g \\ D_b\beta_g \\ \alpha_g \end{bmatrix} \tag{2.93}$$

If these equations of motion are to be used in conjunction with the model of random atmospheric turbulence described in the previous sections, it is possible to modify the right-hand side of these equations. Provided the power spectral density estimates $I_{\hat{u}_g\hat{u}_g}$ and $I_{\alpha_g\alpha_g}$ are used for $\hat{u}_g$ and α_g, the equations (2.93) can be written as,

$$\begin{bmatrix} C_{Y_\beta} - 2\mu_b D_b & C_L & C_{Y_p} & C_{Y_r} - 4\mu_b \\ 0 & -\frac{1}{2}D_b & 1 & 0 \\ C_{l_\beta} & 0 & C_{l_p} - 4\mu_b K_X^2 D_b & C_{l_r} + 4\mu_b K_{XZ} D_b \\ C_{n_\beta} & 0 & C_{n_p} + 4\mu_b K_{XZ} D_b & C_{n_r} - 4\mu_b K_Z^2 D_b \end{bmatrix} \begin{bmatrix} \beta \\ \varphi \\ \frac{pb}{2V} \\ \frac{rb}{2V} \end{bmatrix} =$$

$$- \begin{bmatrix} 0 & C_{Y_\beta} & C_{Y_{\dot\beta_g}} & 0 \\ 0 & 0 & 0 & 0 \\ -C_{l_{r_w}} & C_{l_\beta} & C_{l_{\dot\beta_g}} & C_{l_{p_w}} \\ -C_{n_{r_w}} & C_{n_\beta} & C_{n_{\dot\beta_g}} & C_{n_{p_w}} \end{bmatrix} \begin{bmatrix} \hat{u}_g \\ \beta_g \\ D_b\beta_g \\ \alpha_g \end{bmatrix} \tag{2.94}$$

The general state-space representation, can be obtained by rearranging the equations (2.94), see Ref. (Broek & Brandt, 1984). The final result, in abbreviated notation, is,

$$\begin{bmatrix} \dot\beta \\ \dot\varphi \\ \frac{\dot{p}b}{2V} \\ \frac{\dot{r}b}{2V} \end{bmatrix} = \begin{bmatrix} y_\beta & y_\varphi & y_p & y_r \\ 0 & 0 & 2\frac{V}{b} & 0 \\ l_\beta & 0 & l_p & l_r \\ n_\beta & 0 & n_p & n_r \end{bmatrix} \begin{bmatrix} \beta \\ \varphi \\ \frac{pb}{2V} \\ \frac{rb}{2V} \end{bmatrix} + \begin{bmatrix} 0 & y_{\beta_g} & y_{\dot\beta_g} & 0 \\ 0 & 0 & 0 & 0 \\ l_{u_g} & l_{\beta_g} & l_{\dot\beta_g} & l_{\alpha_g} \\ n_{u_g} & n_{\beta_g} & n_{\dot\beta_g} & n_{\alpha_g} \end{bmatrix} \begin{bmatrix} \hat{u}_g \\ \beta_g \\ D_b\beta_g \\ \alpha_g \end{bmatrix} \tag{2.95}$$

The definition of these symbols are recapitulated in table 2.2.

	X	Z	M
u	$\frac{V}{\bar{c}}\frac{C_{X_u}}{2\mu_c}$	$\frac{V}{\bar{c}}\frac{C_{Z_u}}{2\mu_c-C_{Z_{\dot{\alpha}}}}$	$\frac{V}{\bar{c}}\frac{C_{m_u}+C_{Z_u}\frac{C_{m_{\dot{\alpha}}}}{2\mu_c-C_{Z_{\dot{\alpha}}}}}{2\mu_c K_Y^2}$
α	$\frac{V}{\bar{c}}\frac{C_{X_\alpha}}{2\mu_c}$	$\frac{V}{\bar{c}}\frac{C_{Z_\alpha}}{2\mu_c-C_{Z_{\dot{\alpha}}}}$	$\frac{V}{\bar{c}}\frac{C_{m_\alpha}+C_{Z_\alpha}\frac{C_{m_{\dot{\alpha}}}}{2\mu_c-C_{Z_{\dot{\alpha}}}}}{2\mu_c K_Y^2}$
θ	$\frac{V}{\bar{c}}\frac{C_{Z_0}}{2\mu_c}$	$\frac{V}{\bar{c}}\frac{-C_{X_0}}{2\mu_c-C_{Z_{\dot{\alpha}}}}$	$\frac{V}{\bar{c}}\frac{-C_{X_0}\frac{C_{m_{\dot{\alpha}}}}{2\mu_c-C_{Z_{\dot{\alpha}}}}}{2\mu_c K_Y^2}$
q	$\frac{V}{\bar{c}}\frac{C_{X_q}}{2\mu_c}$	$\frac{V}{\bar{c}}\frac{2\mu_c+C_{Z_q}}{2\mu_c-C_{Z_{\dot{\alpha}}}}$	$\frac{V}{\bar{c}}\frac{C_{m_q}+C_{m_{\dot{\alpha}}}\frac{2\mu_c+C_{Z_q}}{2\mu_c-C_{Z_{\dot{\alpha}}}}}{2\mu_c K_Y^2}$
δ_e	$\frac{V}{\bar{c}}\frac{C_{X_{\delta_e}}}{2\mu_c}$	$\frac{V}{\bar{c}}\frac{C_{Z_{\delta_e}}}{2\mu_c-C_{Z_{\dot{\alpha}}}}$	$\frac{V}{\bar{c}}\frac{C_{m_{\delta_e}}+C_{Z_{\delta_e}}\frac{C_{m_{\dot{\alpha}}}}{2\mu_c-C_{Z_{\dot{\alpha}}}}}{2\mu_c K_Y^2}$
u_g	$\frac{V}{\bar{c}}\frac{C_{X_{u_g}}}{2\mu_c}$	$\frac{V}{\bar{c}}\frac{C_{Z_{u_g}}}{2\mu_c-C_{Z_{\dot{\alpha}}}}$	$\frac{V}{\bar{c}}\frac{C_{m_{u_g}}+C_{Z_{u_g}}\frac{C_{m_{\dot{\alpha}}}}{2\mu_c-C_{Z_{\dot{\alpha}}}}}{2\mu_c K_Y^2}$
α_g	$\frac{V}{\bar{c}}\frac{C_{X_{\alpha_g}}}{2\mu_c}$	$\frac{V}{\bar{c}}\frac{C_{Z_{\alpha_g}}}{2\mu_c-C_{Z_{\dot{\alpha}}}}$	$\frac{V}{\bar{c}}\frac{C_{m_{\alpha_g}}+C_{Z_{\alpha_g}}\frac{C_{m_{\dot{\alpha}}}}{2\mu_c-C_{Z_{\dot{\alpha}}}}}{2\mu_c K_Y^2}$
$\dot{\alpha}_g$	$\frac{V}{\bar{c}}\frac{C_{X_{\dot{\alpha}_g}}}{2\mu_c}$	$\frac{V}{\bar{c}}\frac{C_{Z_{\dot{\alpha}_g}}}{2\mu_c-C_{Z_{\dot{\alpha}}}}$	$\frac{V}{\bar{c}}\frac{C_{m_{\dot{\alpha}_g}}+C_{Z_{\dot{\alpha}_g}}\frac{C_{m_{\dot{\alpha}}}}{2\mu_c-C_{Z_{\dot{\alpha}}}}}{2\mu_c K_Y^2}$

Table 2.1: Symbols appearing in the symmetrical equations of motion's state-space representation of the DUT Model.

	Y	L	N
β	$\frac{V}{b}\frac{C_{Y_\beta}}{2\mu_b}$	$\frac{V}{b}\frac{C_{l_\beta}K_Z^2+C_{n_\beta}K_{XZ}}{4\mu_b\left(K_X^2K_Z^2-K_{XZ}^2\right)}$	$\frac{V}{b}\frac{C_{l_\beta}K_{XZ}+C_{n_\beta}K_X^2}{4\mu_b\left(K_X^2K_Z^2-K_{XZ}^2\right)}$
φ	$\frac{V}{b}\frac{C_L}{2\mu_b}$	0	0
p	$\frac{V}{b}\frac{C_{Y_p}}{2\mu_b}$	$\frac{V}{b}\frac{C_{l_p}K_Z^2+C_{n_p}K_{XZ}}{4\mu_b\left(K_X^2K_Z^2-K_{XZ}^2\right)}$	$\frac{V}{b}\frac{C_{l_p}K_{XZ}+C_{n_p}K_X^2}{4\mu_b\left(K_X^2K_Z^2-K_{XZ}^2\right)}$
r	$\frac{V}{b}\frac{C_{Y_r}-4\mu_b}{2\mu_b}$	$\frac{V}{b}\frac{C_{l_r}K_Z^2+C_{n_r}K_{XZ}}{4\mu_b\left(K_X^2K_Z^2-K_{XZ}^2\right)}$	$\frac{V}{b}\frac{C_{l_r}K_{XZ}+C_{n_r}K_X^2}{4\mu_b\left(K_X^2K_Z^2-K_{XZ}^2\right)}$
δ_a	$\frac{V}{b}\frac{C_{Y_{\delta_a}}}{2\mu_b}$	$\frac{V}{b}\frac{C_{l_{\delta_a}}K_Z^2+C_{n_{\delta_a}}K_{XZ}}{4\mu_b\left(K_X^2K_Z^2-K_{XZ}^2\right)}$	$\frac{V}{b}\frac{C_{l_{\delta_a}}K_{XZ}+C_{n_{\delta_a}}K_X^2}{4\mu_b\left(K_X^2K_Z^2-K_{XZ}^2\right)}$
δ_r	$\frac{V}{b}\frac{C_{Y_{\delta_r}}}{2\mu_b}$	$\frac{V}{b}\frac{C_{l_{\delta_r}}K_Z^2+C_{n_{\delta_r}}K_{XZ}}{4\mu_b\left(K_X^2K_Z^2-K_{XZ}^2\right)}$	$\frac{V}{b}\frac{C_{l_{\delta_r}}K_{XZ}+C_{n_{\delta_r}}K_X^2}{4\mu_b\left(K_X^2K_Z^2-K_{XZ}^2\right)}$
u_g	0	$\frac{V}{b}\frac{C_{l_{u_g}}K_Z^2+C_{n_{u_g}}K_{XZ}}{4\mu_b\left(K_X^2K_Z^2-K_{XZ}^2\right)}$	$\frac{V}{b}\frac{C_{l_{u_g}}K_{XZ}+C_{n_{u_g}}K_X^2}{4\mu_b\left(K_X^2K_Z^2-K_{XZ}^2\right)}$
β_g	$\frac{V}{b}\frac{C_{Y_{\beta_g}}}{2\mu_b}$	$\frac{V}{b}\frac{C_{l_{\beta_g}}K_Z^2+C_{n_{\beta_g}}K_{XZ}}{4\mu_b\left(K_X^2K_Z^2-K_{XZ}^2\right)}$	$\frac{V}{b}\frac{C_{l_{\beta_g}}K_{XZ}+C_{n_{\beta_g}}K_X^2}{4\mu_b\left(K_X^2K_Z^2-K_{XZ}^2\right)}$
$\dot{\beta}_g$	$\frac{V}{b}\frac{C_{Y_{\dot{\beta}_g}}}{2\mu_b}$	$\frac{V}{b}\frac{C_{l_{\dot{\beta}_g}}K_Z^2+C_{n_{\dot{\beta}_g}}K_{XZ}}{4\mu_b\left(K_X^2K_Z^2-K_{XZ}^2\right)}$	$\frac{V}{b}\frac{C_{l_{\dot{\beta}_g}}K_{XZ}+C_{n_{\dot{\beta}_g}}K_X^2}{4\mu_b\left(K_X^2K_Z^2-K_{XZ}^2\right)}$
α_g	0	$\frac{V}{b}\frac{C_{l_{p_w}}K_Z^2+C_{n_{p_w}}K_{XZ}}{4\mu_b\left(K_X^2K_Z^2-K_{XZ}^2\right)}$	$\frac{V}{b}\frac{C_{l_{p_w}}K_{XZ}+C_{n_{p_w}}K_X^2}{4\mu_b\left(K_X^2K_Z^2-K_{XZ}^2\right)}$

Table 2.2: Symbols appearing in the asymmetrical equations of motion's state-space representation of the DUT Model.

B	$\frac{I_{\hat{u}_g}(0,B)}{\sigma^2_{\hat{u}_g}}$	$\frac{I_{\alpha_g}(0,B)}{\sigma^2_{\alpha_g}}$
0.50	0.7856621	0.5380229
0.45	0.7026423	0.4843205
0.40	0.6159092	0.4273037
0.35	0.5261885	0.3674375
0.30	0.4345357	0.3054225
0.25	0.3424367	0.2422765
0.20	0.2519903	0.1794823
0.15	0.1662091	0.1192077
0.125	0.1263831	0.0909742
0.10	0.0895637	0.0647137
0.075	0.0567340	0.0411580
0.0625	0.0422104	0.0306902
0.05	0.0292262	0.0212969
0.03125	0.0132418	0.0096887
0.015625	0.0039835	0.0029280

Table 2.3: $\frac{I_{\hat{u}_g}(0,B)}{\sigma^2_{\hat{u}_g}}$ and $\frac{I_{\alpha_g}(0,B)}{\sigma^2_{\alpha_g}}$ as a function of B.

B	τ_1	τ_2	τ_3
0.50	0.662562	2.311377	2.298718
0.45	0.607202	1.241514	1.204641
0.40	0.544252	1.016470	0.949548
0.35	0.472419	0.895606	0.793271
0.30	0.406748	0.832718	0.703821
0.25	0.346800	0.788367	0.642029
0.20	0.288690	0.747955	0.590821
0.15	0.231815	0.706023	0.545338
0.125	0.202945	0.682303	0.522628
0.10	0.172928	0.653908	0.497035
0.075	0.141145	0.618429	0.467082
0.0625	0.124455	0.596290	0.448961
0.05	0.106813	0.569551	0.427748
0.03125	0.077782	0.512936	0.383390
0.015625	0.048239	0.423350	0.312979

Table 2.4: τ_1, τ_2 and τ_3 in the approximated power spectral density function of the horizontal gust velocity, as a function of B.

B	τ_4	τ_5	τ_6
0.50	0.480764	1.492572	1.527124
0.45	0.458294	1.332911	1.358464
0.40	0.426746	1.120000	1.140000
0.35	0.386097	1.787000	0.773000
0.30	0.337007	1.589747	0.552325
0.25	0.279943	0.551119	0.482539
0.20	0.218703	0.488882	0.390730
0.15	0.162684	0.440944	0.324153
0.125	0.136627	0.417279	0.296144
0.10	0.111941	0.392720	0.271229
0.075	0.087681	0.365723	0.247885
0.0625	0.076006	0.351389	0.237504
0.05	0.064521	0.336211	0.227862
0.03125	0.047613	0.310788	0.214478
0.015625	0.033226	0.283501	0.202983

Table 2.5: τ_4, τ_5 and τ_6 in the approximated power spectral density function of the horizontal gust velocity, as a function of B.

Chapter 3

Four Point Aircraft Models

3.1 Introduction

In this chapter the Four Point Aircraft (FPA) model will be described. This FPA-model is a further development of the Linear Field Model. The Linear Field Model is a one point aircraft model, whereas the FPA-model uses a discretized aircraft on which four points are located over the aircraft's configuration. With respect to these four points, the turbulence inputs will be defined by calculating correlation functions with respect to these points. From these correlation functions the input Power Spectral Density (PSD) functions of the gust inputs will be calculated using MATLAB's Fast Fourier Transform (FFT). In this manner the resulting aerodynamic forces and moments, induced by atmospheric turbulence, are calculated using normal stability derivatives and the corresponding atmospheric disturbances' inputs.

3.2 Turbulence Modelling

3.2.1 Linear Field Models

In the linear field approximation the turbulence inputs to the equations of motion, and hence the equations of motion themselves, differ from those as developed at DUT. The gust velocities u_g, v_g and w_g are assumed to be linear functions of the ordinates x, y and z, but since most aircraft may be assumed to be planar, the gradients in Z-direction can be neglected. We are now only interested in x and y gradients of the three turbulence velocities in addition to their values at the mass centre.

The velocity vector of a gust field is a random function of time and space. The wind velocity vector $\overline{W}$ has three components,

$$\overline{W} = \begin{bmatrix} u_g \\ v_g \\ w_g \end{bmatrix} = \begin{bmatrix} u_g(t,x,y,z) \\ v_g(t,x,y,z) \\ w_g(t,x,y,z) \end{bmatrix} \tag{3.1}$$

A region is now considered, small enough to represent the turbulent field as a linear function from the origin. The windvector is considered to be only a function of place (frozen field analysis). The wind velocity vector can now be written as, see Refs. (Robinson, 1991) and (Etkin, 1980),

$$\overline{W}(r) = \overline{W}_0 + E_1\overline{r} + E_2\overline{r} + \tilde{\omega}_A\overline{r} \tag{3.2}$$

with,

$$\overline{W}_0 = \begin{bmatrix} u_{go} \\ v_{go} \\ w_{go} \end{bmatrix} \tag{3.3}$$

$$E_1 = \begin{bmatrix} u_x & 0 & 0 \\ 0 & v_x & 0 \\ 0 & 0 & w_x \end{bmatrix} \tag{3.4}$$

$$E_2 = \begin{bmatrix} 0 & \epsilon_z & \epsilon_y \\ \epsilon_z & 0 & \epsilon_x \\ \epsilon_y & \epsilon_x & 0 \end{bmatrix} \tag{3.5}$$

$$\tilde{\omega}_A = \begin{bmatrix} 0 & -r_A & q_A \\ r_A & 0 & -p_A \\ -q_A & p_A & 0 \end{bmatrix} \tag{3.6}$$

where $\overline{W}_0$ is the vector of turbulence magnitude at the centre of gravity, E_1 is the matrix of axial rate of strain, E_2 is the matrix of shear rate of strain and $\tilde{\omega}$ is the matrix of rigid body rotation. In the above are,

$$\overline{W} = \begin{bmatrix} u_g \\ v_g \\ w_g \end{bmatrix} \tag{3.7}$$

$$\overline{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{3.8}$$

$$\begin{bmatrix} p_A \\ q_A \\ r_A \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{bmatrix} \tag{3.9}$$

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \end{bmatrix} \tag{3.10}$$

Expanding equation (3.6) the following relation is found,

$$\overline{W} = \begin{bmatrix} u_{g_0} + \frac{\partial u}{\partial x}x + \frac{\partial u}{\partial y}y + \frac{\partial u}{\partial z}z \\ v_{g_0} + \frac{\partial v}{\partial x}x + \frac{\partial v}{\partial y}y + \frac{\partial v}{\partial z}z \\ w_{g_0} + \frac{\partial w}{\partial x}x + \frac{\partial w}{\partial y}y + \frac{\partial w}{\partial z}z \end{bmatrix} \tag{3.11}$$

3.2.2 Gust Gradients

If the turbulence is considered to be constant over the aircraft the terms E_1, E_2 and $\tilde{\omega}$ can be neglected. This consideration is called the point approximation and is only valid if the scale of turbulence is large compared to the dimensions of the aircraft, see figure 3.6.

In the linear field approximation the gust velocities u_g, v_g and w_g are linear functions of x, y and z. Since a conventional aircraft is considered to be planar the only interest lies with the gust gradients in $x-$ and $y-$directions. Also the gust velocities u_{g_0}, v_{g_0} and w_{g_0} at the centre of gravity are not considered at this moment. Only the spatial rate of change of atmospheric disturbances is considered.

The gust velocities are written as,

$$u_g = \frac{\partial u_g}{\partial x}\,x + \frac{\partial u_g}{\partial y}\,y \tag{3.12}$$

$$v_g = \frac{\partial v_g}{\partial x}\,x + \frac{\partial v_g}{\partial y}\,y \tag{3.13}$$

$$w_g = \frac{\partial w_g}{\partial x}\,x + \frac{\partial w_g}{\partial y}\,y \tag{3.14}$$

Of these six gradients, two of them ($\frac{\partial u_g}{\partial x}$ and $\frac{\partial v_g}{\partial y}$) represent strains with velocity fields which have small effect on the longitudinal aerodynamics and none at all on the lateral aerodynamics. The gradient $\partial u_g/\partial x$ represents the difference in u_g along its longitudinal axis. This gust penetration effect for u_g will be neglected, as has been done in the DUT-model. This because of the low-frequent buildup of aerodynamic forces and moment due to turbulence in "u_g" direction (phugoid mode excitation).
The gradient $\partial v_g/\partial y$ represents the difference in v_g along its lateral axis and has small effect on longitudinal and lateral aerodynamics, and will also be neglected.
Thus, four gust gradients remain: $\partial u_g/\partial y$, $\partial v_g/\partial x$, $\partial w_g/\partial x$ and $\partial w_g/\partial y$.

Considering a planar airplane, see figure 3.1, we see that the gust downwash at coordinate (x, y) is equal to equation (3.14). However, if the vehicle is pitching and rolling, the contribution of its rates of rotation to the wing's normal velocity at the same coordinate (x, y) is, see figure 3.2,

$$w_{wing_{normal}}(x,y) = py - qx \tag{3.15}$$

with p and q the respective rates of rotation. Thus, the wing boundary conditions (normal relative wind $w_g - w_{wing_{normal}}$) produced by equations (3.14) and (3.15) are identical if,

$$p = -\frac{\partial w_g}{\partial y}, \quad q = +\frac{\partial w_g}{\partial x}$$

and the wings pressure distributions will be the same whether it is rotating or exposed to a linear gust gradient. For this reason we can write,

$$p_g = \frac{\partial w_g}{\partial y}, \quad q_g = -\frac{\partial w_g}{\partial x} \tag{3.16}$$

and treat the net effective pitch and roll rates (insofar as aerodynamic forces and moments are concerned) as $(q - q_g)$ and $(p - p_g)$. Therefore, the aerodynamic forces and moments induced by these gust gradients may be written as (in non dimensional form, assuming small disturbance theory and, hence, decoupling of symmetric and asymmetric aircraft motions),

$$C_{X_g} = C_{X_{q_g}} \frac{q_g \bar{c}}{V} \tag{3.17}$$

$$C_{Z_g} = C_{Z_{q_g}} \frac{q_g \bar{c}}{V} \tag{3.18}$$

$$C_{m_g} = C_{m_{q_g}} \frac{q_g \bar{c}}{V} \tag{3.19}$$

and,

$$C_{Y_g} = C_{Y_{p_g}} \frac{p_g b}{2V} \tag{3.20}$$

$$C_{l_g} = C_{l_{p_g}} \frac{p_g b}{2V} \tag{3.21}$$

$$C_{n_g} = C_{n_{p_g}} \frac{p_g b}{2V} \tag{3.22}$$

Furthermore, the two remaining gust gradients are, see figure 3.3,

$$r_{1_g} = -\frac{\partial u_g}{\partial y}, \quad r_{2_g} = \frac{\partial v_g}{\partial x} \tag{3.23}$$

and, therefore, the corresponding force and moment contributions become (again assuming decoupling of symmetric and asymmetric aircraft motions),

$$C_{Y_g} = C_{Y_{r_{1_g}}} \frac{r_{1_g} b}{2V} + C_{Y_{r_{2_g}}} \frac{r_{2_g} b}{2V} \tag{3.24}$$

$$C_{l_g} = C_{l_{r_{1_g}}} \frac{r_{1_g} b}{2V} + C_{l_{r_{2_g}}} \frac{r_{2_g} b}{2V} \tag{3.25}$$

$$C_{n_g} = C_{n_{r_{1_g}}} \frac{r_{1_g} b}{2V} + C_{n_{r_{2_g}}} \frac{r_{2_g} b}{2V} \tag{3.26}$$

Note the minus sign in the q_g and r_{1_g} turbulence inputs. The reason for this notation is made clear in figures 3.2 and 3.3, showing an unswept wing and vertical tail system. The relative velocity distribution across the wing associated with $\partial u_g/\partial y$ and that associated with the yaw rate r are identical when $r = r_{1_g}$, and the normal relative velocity at the fin associated with $\partial v_g/\partial x$ is the same as that for the yaw rate r when $r = r_{2_g}$.

3.2.3 Four Point Aircraft Models

In respectively the previous chapter and previous section a general statistical description has been presented for both the DUT model and linear field approximation. One of the drawbacks of the linear field model is that this model is only valid over a limited frequency range. Etkin's four point aircraft model, which overcomes to some extent this drawback of the linear field approximation, will be introduced. As far as the aircraft models are concerned, both the four point model and the linear field model are equal. However, when the input correlation and spectral densities are concerned, they differ for both models.

For small wavelength, e.g. small λ_x, the linear field approximation exaggerates the simulated aircraft responses, see figure 3.6 For high turbulence wave lengths the linear field approximation is quite good, see for example figure 3.5. If the wave length becomes smaller then, for example, the aircraft's length or wingspan, from figures 3.5 and 3.6 it can be seen that the linear field approximation is no longer accurate. In fact, the correlation between turbulence speeds at the important aerodynamic surfaces and the centre of gravity is lost when we are using the linear field approximation. Therefore, the linear field approximation is valid up to a prescribed frequency or wave length. For longitudinal turbulence the wave length is limited to $\lambda_x \geq 10\ l_h$ and for lateral turbulence the wave length is limited to $\lambda_y \geq 10\ b$ with respectively l_h and b equal to the taillength and wingspan. See also for example figure 3.6 where the effect of aircraft size is demonstrated.

Etkin's four point aircraft model, (Etkin, 1980), overcomes the above mentioned drawbacks of the linear field approximation. This four point aircraft model is a better approximation for lower turbulence wavelengths in both the x- (λ_x) and y- (λ_y) directions. In this model, the gust velocities are considered at four points, see figure 3.7, and these four points are used to define the gust gradients introduced in the previous section. For u_g and v_g we use the values at the centre of gravity (u_{g_0} and v_{g_0}), but because the vertical turbulence velocity w_g is so important, we take it to be the average at the three wing points. By choosing the points "0", "1" and "2" on a straight line, as shown in figure 3.7, sweepback is neglected.

Contrary to the FPA/Linear-Field-model, the DUT-model accounts for every spectral component, see for example

$$I_{\alpha_g\alpha_g}(\Omega_x L_g, B) = 3\,\sigma^2_{\alpha_g} \int_0^\infty h^2(\Omega_y \frac{b}{2}) \frac{\Omega_x^2 L_g^2 + \Omega_y^2 L_g^2}{\left(1+\Omega_x^2 L_g^2 + \Omega_y^2 L_g^2\right)^{5/2}}\, d(\Omega_y L_g) \tag{3.27}$$

while for example the one-dimensional input spectrum of the linear field model is limited by the span of the aircraft, b, and the turbulence scale length, L_g, see reference (Scholtens, 1989)

$$I_{\alpha_g\alpha_g}(\Omega_x L_g, B) = \frac{1}{L_g^2} \int_0^{\frac{1}{B}} (\Omega_y L_g)^2 \frac{\Omega_x^2 L_g^2 + \Omega_y^2 L_g^2}{\left(1+\Omega_x^2 L_g^2 + \Omega_y^2 L_g^2\right)^{5/2}}\, d(\Omega_y L_g) \tag{3.28}$$

with

$$B = \frac{b}{2L_g}$$

The inputs for the Four Point Aircraft Model considered in this section are,

$$\begin{aligned}
u_g &= u_0 \\
v_g &= v_0 \\
w_g &= \frac{1}{3}(w_0 + w_1 + w_2) \\
p_g &= \frac{w_1 - w_2}{b'} \\
q_g &= \frac{w_3 - w_0}{l_h} \\
r_{1_g} &= \frac{u_2 - u_1}{b'} \\
r_{2_g} &= \frac{v_0 - v_3}{l_v}
\end{aligned} \tag{3.29}$$

The subscripts 0, 1, 2 and 3 in the turbulence speeds refer to the points depicted in figure 3.8. The parameters l_h and l_v are respectively the distance between the aerodynamic centres of the horizontal tailplane and the centre of gravity and the distance between the vertical tailplane and the centre of gravity. For the parameter b' a generally used value of 0.85 b is used with b the wingspan. This parameter, however, may change considering the spanwise lift distribution, i.e. compare the wing lift distribution for wings with or without winglets.

In conclusion, the dimensional turbulence inputs for the symmetric and asymmetric aircraft motions become,

Symmetric turbulence inputs

$$\underline{u}_{sym} = [u_g \; w_g \; q_g]^T$$

Asymmetric turbulence inputs

$$\underline{u}_{asym} = \left[v_g \; p_g \; r_{1_g} \; r_{2_g}\right]^T$$

Normally, the time rate of change with respect to these turbulence inputs are considered as well. In this case, only the rate of change of w_g (symmetric aircraft motions) and v_g (asymmetric aircraft motions) is considered,

Symmetric turbulence inputs

$$\dot{\underline{u}}_{sym} = [0 \; \dot{w}_g \; 0]^T$$

Asymmetric turbulence inputs

$$\underline{u}_{asym} = [\dot{v}_g \; 0 \; 0 \; 0]^T$$

and, therefore, the dimensional inputs for the aircraft models become,

Symmetric turbulence inputs

$$\underline{u}_{sym} = [u_g \; w_g \; q_g \; \dot{w}_g]^T$$

Asymmetric turbulence inputs

$$\underline{u}_{asym} = \left[v_g \; p_g \; r_{1_g} \; r_{2_g} \; \dot{v}_g\right]$$

3.2.4 Correlation Functions

In this section the correlation functions will be presented for each turbulence input, both symmetric as asymmetric. The correlation functions will be derived using the four point aircraft model, see figure 3.8.

As an example, consider p_g,

$$p_g = \frac{w_1 - w_2}{b'}$$

The autocorrelation function C_{ij} is,

$$\begin{aligned} C_{p_g p_g}(\tau) &= E\{p_g(t)\, p_g(t+\tau)\} \\ &= \frac{1}{b'^2}\, E\{(w_1 - w_2)(w_1' - w_2')\} \\ &= \frac{1}{b'^2}\, [E\{w_1 w_1'\} + E\{w_2 w_2'\} - E\{w_1 w_2'\} - E\{w_2 w_1'\}] \end{aligned} \tag{3.30}$$

where w_1 and w_2 are values of w_g at points "1" and "2" at time t, and w_1' and w_2' are values of w_g at the same points at time $t + \tau$. The points "0", "1", "2" and "3" in figure 3.8 respectively have the coordinates $(x_1, 0, 0)^T$, $(x_1, \frac{b'}{2}, 0)^T$, $(x_1, -\frac{b'}{2}, 0)^T$ and $(x_1 - l_v, 0, 0)^T$. The points "0' ", "1' ", "2' " and "3' " in figure 3.8 respectively have the coordinates $(x_1 + V\tau, 0, 0)^T$, $(x_1 + V\tau, \frac{b'}{2}, 0)^T$, $(x_1 + V\tau, -\frac{b'}{2}, 0)^T$ and $(V\tau x_1 - l_v, 0, 0)^T$. Therefore, in figure 3.8 the separation vectors $\underline{\xi}_{ij}$ become,

$$\underline{\xi}_{11'} = (V\tau, 0, 0)^T$$

$$\underline{\xi}_{12'} = (V\tau, -b', 0)^T$$

$$\underline{\xi}_{21'} = (V\tau, b', 0)^T$$

$$\underline{\xi}_{22'} = (V\tau, 0, 0)^T$$

Recalling the correlation matrix equation (1.11),

$$E\{u_i u_j\} = C_{ij}(\xi) = \sigma^2 \left(\frac{f(\xi) - g(\xi)}{\xi^2} \xi_i \xi_j + g(\xi) \delta_{ij} \right)$$

with,

$$\xi = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2} \tag{3.31}$$

and, ξ_i the i-th element of the vector $\underline{\xi}_{ij}$, ánd that we are dealing with the vertical gust velocity w (for p_g), in the correlation matrix equation (1.11) $i = j = 3$, the u_i and u_j both become u_3 (which corresponds to the vertical gust velocity w). Therefore, the autocorrelation $C_{p_g p_g}$, equation (3.31), becomes,

$$C_{p_gp_g}(\tau) = \frac{2}{b'^2}\left[C_{33}(V\tau,0,0) - C_{33}(V\tau,b',0)\right] \tag{3.32}$$

or using equation (1.11),

$$\frac{(b')^2}{2\sigma^2}\,C_{p_gp_g}(\tau) = g(\xi_1) - g(\xi_3) \tag{3.33}$$

with non-dimensional seperation vectors ξ equal to,

$$\frac{\xi_1}{aL_g} = \zeta_1 = \left|\frac{V\tau}{aL_g}\right| \tag{3.34}$$

$$\frac{\xi_3}{aL_g} = \zeta_3 = \sqrt{\left(\frac{V\tau}{aL_g}\right)^2 + \left(\frac{b'}{aL_g}\right)^2} \tag{3.35}$$

In a similar way, the other auto- and crosscorrelation functions for the other turbulence inputs may be derived in a similar manner. They are summarized in the following.

$$\frac{l_h^2 C_{q_gq_g}(\tau)}{\sigma^2} = 2g(\xi_1) - g(\xi_4) - g(\xi_5) \tag{3.36}$$

$$\frac{(b')^2 C_{r_{1g}r_{1g}}(\tau)}{2\sigma^2} = f(\xi_1) - K_1 f(\xi_3) - (1 - K_1)\,g(\xi_3) \tag{3.37}$$

$$\frac{l_v^2 C_{r_{2g}r_{2g}}(\tau)}{\sigma^2} = 2g(\xi_1) - g(\xi_6) - g(\xi_7) \tag{3.38}$$

$$\frac{C_{u_gu_g}(\tau)}{\sigma^2} = f(\xi_1) \tag{3.39}$$

$$\frac{C_{v_gv_g}(\tau)}{\sigma^2} = g(\xi_1) \tag{3.40}$$

$$\frac{C_{w_gw_g}(\tau)}{\sigma^2} = \frac{1}{3}g(\xi_1) + \frac{4}{9}g(\xi_2) + \frac{2}{9}g(\xi_3) \tag{3.41}$$

$$\frac{3l_h C_{w_gq_g}(\tau)}{\sigma^2} = g(\xi_5) - g(\xi_1) + 2g(\xi_8) - 2g(\xi_2) \tag{3.42}$$

$$\frac{b' C_{v_g r_{1_g}}(\tau)}{\sigma^2} = 2K_2 \left[g(\xi_2) - f(\xi_2) \right] \tag{3.43}$$

$$\frac{l_v C_{v_g r_{2_g}}(\tau)}{\sigma^2} = g(\xi_1) - g(\xi_7) \tag{3.44}$$

$$\frac{b' l_v C_{r_{1_g} r_{2_g}}(\tau)}{\sigma^2} = K_3 \left[g(\xi_9) - f(\xi_9) \right] + K_2 \left[f(\xi_2) - g(\xi_2) \right] \tag{3.45}$$

With,

$$K_1 = \frac{\left(\frac{V\tau}{L_g}\right)^2}{\left(\frac{V\tau}{L_g}\right)^2 + \left(\frac{b'}{2L_g}\right)^2} \tag{3.46}$$

$$K_2 = \frac{\left(\frac{V\tau}{L_g}\right)\left(\frac{b'}{2L_g}\right)}{\left(\frac{V\tau}{L_g}\right)^2 + \left(\frac{b'}{L_g}\right)^2} \tag{3.47}$$

$$K_3 = \frac{\left(\frac{V\tau}{L_g} - \frac{l_v}{L_g}\right)\left(\frac{b'}{2L_g}\right)}{\left(\frac{V\tau}{L_g} - \frac{l_v}{L_g}\right)^2 + \left(\frac{b'}{L_g}\right)^2} \tag{3.48}$$

$$\frac{\xi_2}{aL_g} = \zeta_2 = \sqrt{\left(\frac{V\tau}{aL_g}\right)^2 + \left(\frac{b'}{2aL_g}\right)^2} \tag{3.49}$$

$$\frac{\xi_4}{aL_g} = \zeta_4 = \left| \frac{V\tau}{aL_g} + \frac{l_h}{aL_g} \right| \tag{3.50}$$

$$\frac{\xi_5}{aL_g} = \zeta_5 = \left| \frac{V\tau}{aL_g} - \frac{l_h}{aL_g} \right| \tag{3.51}$$

$$\frac{\xi_6}{aL_g} = \zeta_6 = \left| \frac{V\tau}{aL_g} + \frac{l_v}{aL_g} \right| \tag{3.52}$$

$$\frac{\xi_7}{aL_g} = \zeta_7 = \left| \frac{V\tau}{aL_g} - \frac{l_f}{aL_g} \right| \tag{3.53}$$

$$\frac{\xi_8}{aL_g} = \zeta_8 = \sqrt{\left(\frac{V\tau}{aL_g} - \frac{l_h}{aL_g}\right)^2 + \left(\frac{b'}{2aL_g}\right)^2} \tag{3.54}$$

$$\frac{\xi_9}{aL_g} = \zeta_9 = \sqrt{\left(\frac{V\tau}{aL_g} - \frac{l_v}{aL_g}\right)^2 + \left(\frac{b'}{2aL_g}\right)^2} \tag{3.55}$$

Graphs of these correlations are given in figures 3.10 to 3.20 in which the longitudinal and lateral correlation functions according to Dryden were used ($f(\xi)$ and $g(\xi)$). They are functions of $V\tau/L_g$ and the parameters b'/L_g, l_v/L_g and l_h/L_g. The correlations are plotted in dimensional form. The kinks in some of the correlation functions are the result of the FPA representation. They occur at that value of $V\tau/L_g$ for which the tail arrives at the position occupied $\tau = \frac{l_h}{V}$ or $\tau = \frac{l_v}{V}$ seconds earlier by the centre of gravity (gust penetration effect). The results presented in figures 3.10 to 3.20 are calculated for the example aircraft, a Cessna Ce-500 Citation, see figure 3.9. The data for this aircraft is presented in table 4.1.

3.2.5 Input Power Spectral Density Functions

For the calculation of the input auto and cross power spectral densities, the auto and cross correlation functions are Fourier transformed. For this purpose the MATLAB routine `fft` may be used, and an estimate for the power spectral densities (periodograms) is obtained. The Fast Fourier Transform algorithm is based on,

$$I[k] = \sum_{n=0}^{N-1} C[n]\, e^{-j\frac{2\pi kn}{N}} \tag{3.56}$$

with $I[k]$ the power spectral density's estimate ("periodogram") and $C[n]$ the correlation function. See also the examples in the next chapter. The numerically calculated power spectral densities are given in figures 3.21 to 3.35. These power spectral densities are also given in dimensional form.

3.3 Aircraft Modelling

3.3.1 Symmetric Aircraft Motions

For the FPA model the aerodynamic forces and moment, induced by symmetrical gusts, are written as,

$$C_{X_g} = C_{X_{u_g}}\hat{u}_g + C_{X_{\alpha_g}}\alpha_g + C_{X_{q_g}}\frac{q_g\bar{c}}{V} \tag{3.57}$$

$$C_{Z_g} = C_{Z_{u_g}}\hat{u}_g + C_{Z_{\alpha_g}}\alpha_g + C_{Z_{q_g}}\frac{q_g\bar{c}}{V} + C_{Z_{\dot{\alpha}}}\frac{\dot{\alpha}_g\bar{c}}{V} \tag{3.58}$$

$$C_{m_g} = C_{m_{u_g}}\hat{u}_g + C_{m_{\alpha_g}}\hat{\alpha}_g + C_{m_{q_g}}\frac{q_g\bar{c}}{V} + C_{m_{\dot{\alpha}}}\frac{\dot{\alpha}_g\bar{c}}{V} \tag{3.59}$$

with,

$$\begin{aligned} C_{X_{u_g}} &= C_{X_u} \\ C_{Z_{u_g}} &= C_{Z_u} \\ C_{m_{u_g}} &= C_{m_u} \end{aligned} \tag{3.60}$$

$$\begin{aligned} C_{X_{\alpha_g}} &= C_{X_\alpha} \\ C_{Z_{\alpha_g}} &= C_{Z_\alpha} \\ C_{m_{\alpha_g}} &= C_{m_\alpha} \end{aligned} \tag{3.61}$$

$$\begin{aligned} C_{X_{q_g}} &= -C_{X_q} \\ C_{Z_{q_g}} &= -C_{Z_q} \\ C_{m_{q_g}} &= -C_{m_q} \end{aligned} \tag{3.62}$$

and $C_{Z_{\dot{\alpha}}}$ and $C_{m_{\dot{\alpha}}}$ the normal stability derivatives with respect to the acceleration along the Z_s stability frame of reference axis.

Also, remember that now q_g is defined as,

$$q_g = \frac{\partial w_g}{\partial x} \tag{3.63}$$

since the original minus sign in,

$$q_g = -\frac{\partial w_g}{\partial x} \tag{3.64}$$

is corrected for in the aerodynamic model.

Therefore, the non-dimensional turbulence inputs for the four point aircraft model are,

$$\underline{u} = \left[\hat{u}_g \; \alpha_g \; \frac{q_g\bar{c}}{V} \; \frac{\dot{\alpha}_g\bar{c}}{V}\right]^T \tag{3.65}$$

The equations for symmetric aircraft motion are extended for a flight in a turbulent atmosphere, and will be written as,

$$\begin{bmatrix} C_{X_u} - 2\mu_c D_c & C_{X_\alpha} & C_{Z_0} & 0 \\ C_{Z_u} & C_{Z_\alpha} + (C_{Z_{\dot\alpha}} - 2\mu_c) D_c & -C_{X_0} & 2\mu_c + C_{Z_q} \\ 0 & 0 & -D_c & 1 \\ C_{m_u} & C_{m_\alpha} + C_{m_{\dot\alpha}} D_c & 0 & C_{m_q} - 2\mu_c K_Y^2 D_c \end{bmatrix} \begin{bmatrix} \hat{u} \\ \alpha \\ \theta \\ \frac{q\bar{c}}{V} \end{bmatrix} =$$

$$- \begin{bmatrix} C_{X_{u_g}} & C_{X_{\alpha_g}} & C_{X_{q_g}} & C_{X_{\dot\alpha}} \\ C_{Z_{u_g}} & C_{Z_{\alpha_g}} & C_{Z_{q_g}} & C_{Z_{\dot\alpha}} \\ 0 & 0 & 0 & 0 \\ C_{m_{u_g}} & C_{m_{\alpha_g}} & C_{m_{q_g}} & C_{m_{\dot\alpha}} \end{bmatrix} \begin{bmatrix} \hat{u}_g \\ \alpha_g \\ \frac{q_g\bar{c}}{V} \\ D_c\alpha_g \end{bmatrix} \tag{3.66}$$

The general state-space representation will be written as,

$$\underline{\dot{x}} = A\,\underline{x} + B\,\underline{u} \tag{3.67}$$

or,

$$\begin{bmatrix} \dot{\hat{u}} \\ \dot\alpha \\ \dot\theta \\ \frac{\dot{q}\bar{c}}{V} \end{bmatrix} = \begin{bmatrix} x_u & x_\alpha & x_\theta & 0 \\ z_u & z_\alpha & z_\theta & z_q \\ 0 & 0 & 0 & \frac{V}{\bar{c}} \\ m_u & m_\alpha & m_\theta & m_q \end{bmatrix} \begin{bmatrix} \hat{u} \\ \alpha \\ \theta \\ \frac{q\bar{c}}{V} \end{bmatrix} + \begin{bmatrix} x_{u_g} & x_{\alpha_g} & x_{q_g} & x_{\dot\alpha} \\ z_{u_g} & z_{\alpha_g} & z_{q_g} & z_{\dot\alpha} \\ 0 & 0 & 0 & 0 \\ m_{u_g} & m_{\alpha_g} & m_{q_g} & m_{\dot\alpha} \end{bmatrix} \begin{bmatrix} \hat{u}_g \\ \alpha_g \\ \frac{q_g\bar{c}}{V} \\ \frac{\dot\alpha_g\bar{c}}{V} \end{bmatrix} \tag{3.68}$$

The definition of these symbols are recapitulated in table 3.1. The turbulence inputs are generated by calculating the correlation functions numerically and then numerically Fourier transforming them to input spectra (using MATLAB's FFT routine `fft.m`).

3.3.2 Asymmetric Aircraft Motions

For the four point aircraft model the aerodynamic forces and moment, induced by asymmetrical gusts, are written as,

$$C_{Y_g} = C_{Y_{\beta_g}}\beta_g + C_{Y_{\dot\beta}}\frac{\dot\beta_g b}{V} + C_{Y_{p_g}}\frac{p_g b}{2V} + C_{Y_{r_{1_g}}}\frac{r_{1_g} b}{2V} + C_{Y_{r_{2_g}}}\frac{r_{2_g} b}{2V} \tag{3.69}$$

$$C_{l_g} = C_{l_{\beta_g}}\beta_g + C_{l_{\dot\beta}}\frac{\dot\beta_g b}{V} + C_{l_{p_g}}\frac{p_g b}{2V} + C_{l_{r_{1_g}}}\frac{r_{1_g} b}{2V} + C_{l_{r_{2_g}}}\frac{r_{2_g} b}{2V} \tag{3.70}$$

$$C_{n_g} = C_{n_{\beta_g}}\beta_g + C_{n_{\dot\beta}}\frac{\dot\beta_g b}{V} + C_{n_{p_g}}\frac{p_g b}{2V} + C_{n_{r_{1_g}}}\frac{r_{1_g} b}{2V} + C_{n_{r_{2_g}}}\frac{r_{2_g} b}{2V} \tag{3.71}$$

$$\tag{3.72}$$

with,

$$\begin{aligned} C_{Y_{\beta_g}} &= C_{Y_\beta} \\ C_{l_{\beta_g}} &= C_{l_\beta} \\ C_{n_{\beta_g}} &= C_{n_\beta} \end{aligned} \tag{3.73}$$

$$\begin{aligned} C_{Y_{p_g}} &= C_{Y_{p_w}} \\ C_{l_{p_g}} &= C_{l_{p_w}} \\ C_{n_{p_g}} &= C_{n_{p_w}} \end{aligned} \tag{3.74}$$

$$\begin{aligned} C_{Y_{r_1}} &= 0 \\ C_{l_{r_1}} &= -C_{l_{r_w}} \\ C_{n_{r_1}} &= -C_{n_{r_w}} \end{aligned} \tag{3.75}$$

$$\begin{aligned} C_{Y_{r_2}} &= C_{Y_{r_v}} \\ C_{l_{r_2}} &= C_{l_{r_v}} \\ C_{n_{r_2}} &= C_{n_{r_v}} \end{aligned} \tag{3.76}$$

and $C_{Y_{\dot\beta}}$, $C_{l_{\dot\beta}}$ and $C_{n_{\dot\beta}}$ the normal stability derivatives with respect to the acceleration along the Y_s stability frame of reference axis.

Note that in the latter equations the subscripts w and v denote the contributions to the stability derivatives of the wing and the vertical tailplane respectively.

Also, remember that r_{1_g} is now defined as,

$$r_{1_g} = \frac{\partial u_g}{\partial y} \tag{3.77}$$

since the original minus sign in,

$$r_{1_g} = -\frac{\partial u_g}{\partial y} \tag{3.78}$$

is corrected for in the aerodynamic model.

Summarizing, the turbulence inputs for the linear field approximation become,

$$\underline{u}_{asym} = \left[\beta_g \ \frac{p_g b}{2V} \ \frac{r_{1_g} b}{2V} \ \frac{r_{2_g} b}{2V} \ \frac{\dot\beta_g b}{V}\right]^T \tag{3.79}$$

The equations for asymmetric aircraft motion are extended for a flight in a turbulent atmosphere, and will be now written as,

$$\begin{bmatrix} C_{Y_\beta} - 2\mu_b D_b & C_L & C_{Y_p} & C_{Y_r} - 4\mu_b \\ 0 & -\frac{1}{2}D_b & 1 & 0 \\ C_{l_\beta} & 0 & C_{l_p} - 4\mu_b K_X^2 D_b & C_{l_r} + 4\mu_b K_{XZ} D_b \\ C_{n_\beta} & 0 & C_{n_p} + 4\mu_b K_{XZ} D_b & C_{n_r} - 4\mu_b K_Z^2 D_b \end{bmatrix} \begin{bmatrix} \beta \\ \varphi \\ \frac{pb}{2V} \\ \frac{rb}{2V} \end{bmatrix} =$$

$$- \begin{bmatrix} C_{Y_{\beta_g}} & C_{Y_{p_g}} & C_{Y_{r_{1g}}} & C_{Y_{r_{2g}}} & C_{Y_{\dot\beta}} \\ 0 & 0 & 0 & 0 & 0 \\ C_{l_{\beta_g}} & C_{l_{p_g}} & C_{l_{r_{1g}}} & C_{l_{r_{2g}}} & C_{l_{\dot\beta}} \\ C_{n_{\beta_g}} & C_{n_{p_g}} & C_{n_{r_{1g}}} & C_{n_{r_{2g}}} & C_{n_{\dot\beta}} \end{bmatrix} \begin{bmatrix} \beta_g \\ \frac{p_g b}{2V} \\ \frac{r_{1g} b}{2V} \\ \frac{r_{2g} b}{2V} \\ \frac{\beta_g b}{V} \end{bmatrix} \tag{3.80}$$

The general state-space representation, can be obtained by rearranging these equations. The final result, in abbreviated notation, is,

$$\begin{bmatrix} \dot\beta \\ \dot\varphi \\ \frac{\dot pb}{2V} \\ \frac{\dot rb}{2V} \end{bmatrix} = \begin{bmatrix} y_\beta & y_\varphi & y_p & y_r \\ 0 & 0 & 2\frac{V}{b} & 0 \\ l_\beta & 0 & l_p & l_r \\ n_\beta & 0 & n_p & n_r \end{bmatrix} \begin{bmatrix} \beta \\ \varphi \\ \frac{pb}{2V} \\ \frac{rb}{2V} \end{bmatrix} + \begin{bmatrix} y_{\beta_g} & y_{p_g} & y_{r_{1g}} & y_{r_{2g}} & y_{\dot\beta} \\ 0 & 0 & 0 & 0 & 0 \\ l_{\beta_g} & l_{p_g} & l_{r_{1g}} & l_{r_{2g}} & l_{\dot\beta} \\ n_{\beta_g} & n_{p_g} & n_{r_{1g}} & n_{r_{2g}} & n_{\dot\beta} \end{bmatrix} \begin{bmatrix} \beta_g \\ \frac{p_g b}{2V} \\ \frac{r_{1g} b}{2V} \\ \frac{p_{2g} b}{2V} \\ \frac{\beta_g b}{V} \end{bmatrix} \tag{3.81}$$

The definition of these symbols are recapitulated in table 3.2. The turbulence inputs are generated by calculating the correlation functions numerically and then numerically Fourier transforming them to input spectra (using MATLAB's FFT routine `fft.m`).

	X	Z	M
u	$\frac{V}{\bar{c}}\frac{C_{X_u}}{2\mu_c}$	$\frac{V}{\bar{c}}\frac{C_{Z_u}}{2\mu_c - C_{Z_{\dot{\alpha}}}}$	$\frac{V}{\bar{c}}\frac{C_{m_u} + C_{Z_u}\frac{C_{m_{\dot{\alpha}}}}{2\mu_c - C_{Z_{\dot{\alpha}}}}}{2\mu_c K_Y^2}$
α	$\frac{V}{\bar{c}}\frac{C_{X_\alpha}}{2\mu_c}$	$\frac{V}{\bar{c}}\frac{C_{Z_\alpha}}{2\mu_c - C_{Z_{\dot{\alpha}}}}$	$\frac{V}{\bar{c}}\frac{C_{m_\alpha} + C_{Z_\alpha}\frac{C_{m_{\dot{\alpha}}}}{2\mu_c - C_{Z_{\dot{\alpha}}}}}{2\mu_c K_Y^2}$
$\dot{\alpha}$	$\frac{V}{\bar{c}}\frac{C_{X_{\dot{\alpha}}}}{2\mu_c}$	$\frac{V}{\bar{c}}\frac{C_{Z_{\dot{\alpha}}}}{2\mu_c - C_{Z_{\dot{\alpha}}}}$	$\frac{V}{\bar{c}}\frac{C_{m_{\dot{\alpha}}} + C_{Z_{\dot{\alpha}}}\frac{C_{m_{\dot{\alpha}}}}{2\mu_c - C_{Z_{\dot{\alpha}}}}}{2\mu_c K_Y^2}$
θ	$\frac{V}{\bar{c}}\frac{C_{Z_0}}{2\mu_c}$	$\frac{V}{\bar{c}}\frac{-C_{X_0}}{2\mu_c - C_{Z_{\dot{\alpha}}}}$	$\frac{V}{\bar{c}}\frac{-C_{X_0}\frac{C_{m_{\dot{\alpha}}}}{2\mu_c - C_{Z_{\dot{\alpha}}}}}{2\mu_c K_Y^2}$
q	$\frac{V}{\bar{c}}\frac{C_{X_q}}{2\mu_c}$	$\frac{V}{\bar{c}}\frac{2\mu_c + C_{Z_q}}{2\mu_c - C_{Z_{\dot{\alpha}}}}$	$\frac{V}{\bar{c}}\frac{C_{m_q} + C_{m_{\dot{\alpha}}}\frac{2\mu_c + C_{Z_q}}{2\mu_c - C_{Z_{\dot{\alpha}}}}}{2\mu_c K_Y^2}$
δ_e	$\frac{V}{\bar{c}}\frac{C_{X_{\delta_e}}}{2\mu_c}$	$\frac{V}{\bar{c}}\frac{C_{Z_{\delta_e}}}{2\mu_c - C_{Z_{\dot{\alpha}}}}$	$\frac{V}{\bar{c}}\frac{C_{m_{\delta_e}} + C_{Z_{\delta_e}}\frac{C_{m_{\dot{\alpha}}}}{2\mu_c - C_{Z_{\dot{\alpha}}}}}{2\mu_c K_Y^2}$
q_g	$\frac{V}{\bar{c}}\frac{C_{X_{q_g}}}{2\mu_c}$	$\frac{V}{\bar{c}}\frac{C_{Z_{q_g}}}{2\mu_c - C_{Z_{\dot{\alpha}}}}$	$\frac{V}{\bar{c}}\frac{C_{m_{q_g}} + C_{m_{\dot{\alpha}}}\frac{C_{Z_{q_g}}}{2\mu_c - C_{Z_{\dot{\alpha}}}}}{2\mu_c K_Y^2}$
u_g	$\frac{V}{\bar{c}}\frac{C_{X_{u_g}}}{2\mu_c}$	$\frac{V}{\bar{c}}\frac{C_{Z_{u_g}}}{2\mu_c - C_{Z_{\dot{\alpha}}}}$	$\frac{V}{\bar{c}}\frac{C_{m_{u_g}} + C_{Z_{u_g}}\frac{C_{m_{\dot{\alpha}}}}{2\mu_c - C_{Z_{\dot{\alpha}}}}}{2\mu_c K_Y^2}$
α_g	$\frac{V}{\bar{c}}\frac{C_{X_{\alpha_g}}}{2\mu_c}$	$\frac{V}{\bar{c}}\frac{C_{Z_{\alpha_g}}}{2\mu_c - C_{Z_{\dot{\alpha}}}}$	$\frac{V}{\bar{c}}\frac{C_{m_{\alpha_g}} + C_{Z_{\alpha_g}}\frac{C_{m_{\dot{\alpha}}}}{2\mu_c - C_{Z_{\dot{\alpha}}}}}{2\mu_c K_Y^2}$
$\dot{\alpha}_g$	$\frac{V}{\bar{c}}\frac{C_{X_{\dot{\alpha}_g}}}{2\mu_c}$	$\frac{V}{\bar{c}}\frac{C_{Z_{\dot{\alpha}_g}}}{2\mu_c - C_{Z_{\dot{\alpha}}}}$	$\frac{V}{\bar{c}}\frac{C_{m_{\dot{\alpha}_g}} + C_{Z_{\dot{\alpha}_g}}\frac{C_{m_{\dot{\alpha}}}}{2\mu_c - C_{Z_{\dot{\alpha}}}}}{2\mu_c K_Y^2}$

Table 3.1: Symbols appearing in the symmetrical equations of motion's state-space representation of the Four Point Aircraft (FPA) model.

	Y	L	N
β	$\frac{V}{b}\frac{C_{Y_\beta}}{2\mu_b}$	$\frac{V}{b}\frac{C_{l_\beta}K_Z^2+C_{n_\beta}K_{XZ}}{4\mu_b\left(K_X^2K_Z^2-K_{XZ}^2\right)}$	$\frac{V}{b}\frac{C_{l_\beta}K_{XZ}+C_{n_\beta}K_X^2}{4\mu_b\left(K_X^2K_Z^2-K_{XZ}^2\right)}$
$\dot{\beta}$	$\frac{V}{b}\frac{C_{Y_{\dot\beta}}}{2\mu_b}$	$\frac{V}{b}\frac{C_{l_{\dot\beta}}K_Z^2+C_{n_{\dot\beta}}K_{XZ}}{4\mu_b\left(K_X^2K_Z^2-K_{XZ}^2\right)}$	$\frac{V}{b}\frac{C_{l_{\dot\beta}}K_{XZ}+C_{n_{\dot\beta}}K_X^2}{4\mu_b\left(K_X^2K_Z^2-K_{XZ}^2\right)}$
φ	$\frac{V}{b}\frac{C_L}{2\mu_b}$	0	0
p	$\frac{V}{b}\frac{C_{Y_p}}{2\mu_b}$	$\frac{V}{b}\frac{C_{l_p}K_Z^2+C_{n_p}K_{XZ}}{4\mu_b\left(K_X^2K_Z^2-K_{XZ}^2\right)}$	$\frac{V}{b}\frac{C_{l_p}K_{XZ}+C_{n_p}K_X^2}{4\mu_b\left(K_X^2K_Z^2-K_{XZ}^2\right)}$
r	$\frac{V}{b}\frac{C_{Y_r}-4\mu_b}{2\mu_b}$	$\frac{V}{b}\frac{C_{l_r}K_Z^2+C_{n_r}K_{XZ}}{4\mu_b\left(K_X^2K_Z^2-K_{XZ}^2\right)}$	$\frac{V}{b}\frac{C_{l_r}K_{XZ}+C_{n_r}K_X^2}{4\mu_b\left(K_X^2K_Z^2-K_{XZ}^2\right)}$
δ_a	$\frac{V}{b}\frac{C_{Y_{\delta_a}}}{2\mu_b}$	$\frac{V}{b}\frac{C_{l_{\delta_a}}K_Z^2+C_{n_{\delta_a}}K_{XZ}}{4\mu_b\left(K_X^2K_Z^2-K_{XZ}^2\right)}$	$\frac{V}{b}\frac{C_{l_{\delta_a}}K_{XZ}+C_{n_{\delta_a}}K_X^2}{4\mu_b\left(K_X^2K_Z^2-K_{XZ}^2\right)}$
δ_r	$\frac{V}{b}\frac{C_{Y_{\delta_r}}}{2\mu_b}$	$\frac{V}{b}\frac{C_{l_{\delta_r}}K_Z^2+C_{n_{\delta_r}}K_{XZ}}{4\mu_b\left(K_X^2K_Z^2-K_{XZ}^2\right)}$	$\frac{V}{b}\frac{C_{l_{\delta_r}}K_{XZ}+C_{n_{\delta_r}}K_X^2}{4\mu_b\left(K_X^2K_Z^2-K_{XZ}^2\right)}$
p_g	$\frac{V}{b}\frac{C_{Y_{p_g}}}{2\mu_b}$	$\frac{V}{b}\frac{C_{l_{p_g}}K_Z^2+C_{n_{p_g}}K_{XZ}}{4\mu_b\left(K_X^2K_Z^2-K_{XZ}^2\right)}$	$\frac{V}{b}\frac{C_{l_{p_g}}K_{XZ}+C_{n_{p_g}}K_X^2}{4\mu_b\left(K_X^2K_Z^2-K_{XZ}^2\right)}$
r_{1_g}	$\frac{V}{b}\frac{C_{Y_{r_{1_g}}}}{2\mu_b}$	$\frac{V}{b}\frac{C_{l_{r_{1_g}}}K_Z^2+C_{n_{r_{1_g}}}K_{XZ}}{4\mu_b\left(K_X^2K_Z^2-K_{XZ}^2\right)}$	$\frac{V}{b}\frac{C_{l_{r_{1_g}}}K_{XZ}+C_{n_{r_{1_g}}}K_X^2}{4\mu_b\left(K_X^2K_Z^2-K_{XZ}^2\right)}$
r_{2_g}	$\frac{V}{b}\frac{C_{Y_{r_{2_g}}}}{2\mu_b}$	$\frac{V}{b}\frac{C_{l_{r_{2_g}}}K_Z^2+C_{n_{r_{2_g}}}K_{XZ}}{4\mu_b\left(K_X^2K_Z^2-K_{XZ}^2\right)}$	$\frac{V}{b}\frac{C_{l_{r_{2_g}}}K_{XZ}+C_{n_{r_{2_g}}}K_X^2}{4\mu_b\left(K_X^2K_Z^2-K_{XZ}^2\right)}$
β_g	$\frac{V}{b}\frac{C_{Y_\beta}}{2\mu_b}$	$\frac{V}{b}\frac{C_{l_\beta}K_Z^2+C_{n_\beta}K_{XZ}}{4\mu_b\left(K_X^2K_Z^2-K_{XZ}^2\right)}$	$\frac{V}{b}\frac{C_{l_\beta}K_{XZ}+C_{n_\beta}K_X^2}{4\mu_b\left(K_X^2K_Z^2-K_{XZ}^2\right)}$

Table 3.2: Symbols appearing in the asymmetrical equations of motion's state-space representation of the Four Point Aircraft (FPA) model.

Chapter 4

Results

4.1 Comparing DUT and Four Point Aircraft Models

The aircraft models according to the Delft University of Technology (DUT) and the Four Point Aircraft model theory, are in fact the same. The only difference involves the calculation of the input spectra. As with respect to the latter, the major difference between these two models is that the DUT model is in fact a One Point Aircraft Model, see Ref. (Mulder & Vaart, 1993), as compared to the Four Point Aircraft Model according to Ref. (Etkin, 1980).

In this chapter results will be presented for the Cessna Ce-500 Citation. The flight condition (approach) and stability and control derivatives are given in table 4.1.

4.2 Models for Symmetric Aircraft Motions

4.2.1 Aircraft Models and Input Spectra for Symmetrical Longitudinal Turbulence

In this section the aircraft and input models for symmetrical longitudinal turbulence u_g will be given.

The non-dimensional aerodynamic forces and moments due to symmetrical longitudinal turbulence u_g are according to the DUT and Four Point Aircraft Model (with $\hat{u}_g = \frac{u_g}{V}$),

$$
\begin{aligned}
C_{X_g} &= C_{X_{u_g}}\,\hat{u}_g \\
C_{Z_g} &= C_{Z_{u_g}}\,\hat{u}_g \\
C_{m_g} &= C_{m_{u_g}}\,\hat{u}_g
\end{aligned}
\tag{4.1}
$$

Also, for both the DUT and Four Point Aircraft Model, the input correlation function $C_{\hat{u}_g\hat{u}_g}(\tau)$ are equal,

$$C_{\hat{u}_g\hat{u}_g}(\tau) = \sigma^2_{\hat{u}_g}\; e^{-\frac{\xi}{L_g}}$$

Remember that both the DUT and Four Point Aircraft Model are here one point models and, therefore, the separation vector $\underline{\xi}$ equals $[V\tau, 0, 0]^T$. The absolute value equals $\xi = V\tau$, which is to be used in the definition of the correlation function. Slight differences occur in the results of the input power spectral density $S_{\hat{u}_g\hat{u}_g}(\omega)$. In figure 3.21 the input power spectral densities for $\hat{u}_g$ are given. The analytically derived PSD function according to the DUT model equals,

$$S_{\hat{u}_g\hat{u}_g}(\omega) = 2\sigma^2_{\hat{u}_g}\frac{L_g}{V}\frac{1}{1+\left(L_g\frac{\omega}{V}\right)^2}$$

The numerically obtained input PSD function for the Four Point Aircraft Model equals,

$$I_{\hat{u}_g\hat{u}_g}[k] = \sum_{n=0}^{N-1} C_{\hat{u}_g\hat{u}_g}[n]\; e^{-j\frac{2\pi kn}{N}} = \sum_{n=0}^{N-1} \sigma^2_{\hat{u}_g}\; e^{-\frac{\xi}{L_g}}\; e^{-j\frac{2\pi kn}{N}}$$

or with $\xi = V\; n\; \Delta\tau$ (with $\Delta\tau$ the sampling time interval),

$$I_{\hat{u}_g\hat{u}_g}[k] = \sum_{n=0}^{N-1} \sigma^2_{\hat{u}_g}\; e^{-\frac{V\, n\, \Delta\tau}{L_g}}\; e^{-j\frac{2\pi kn}{N}}$$

In figures 4.1 to 4.4 the output PSD functions of the state variables $\hat{u}$, α, θ and $\frac{q\bar{c}}{V}$ are given. Only for higher frequencies the spectra do not coincide, since the power spectral density estimate $I_{\hat{u}_g\hat{u}_g}[k]$ is a biased estimate for the input spectrum $S_{\hat{u}_g\hat{u}_g}(\omega)$. Over a wide frequency range the PSD functions coincide, as it should be, since both the DUT and Four Point Aircraft Model are in fact equal (both aircraft and input model).

4.2.2 Aircraft Models and Input Spectra for Symmetrical Vertical Turbulence

In this section the aircraft and input models for symmetrical longitudinal turbulence w_g will be given.

The non-dimensional aerodynamic forces and moments due to symmetrical longitudinal turbulence w_g are according to the DUT Model (with $\alpha_g = \frac{w_g}{V}$),

<u>DUT Model</u>

$$\begin{aligned} C_{X_g} &= C_{X_{\alpha_g}}\,\alpha_g + C_{X_{\dot{\alpha}_g}}\,\frac{\dot{\alpha}_g\bar{c}}{V} \\ C_{Z_g} &= C_{Z_{\alpha_g}}\,\alpha_g + C_{Z_{\dot{\alpha}_g}}\,\frac{\dot{\alpha}_g\bar{c}}{V} \\ C_{m_g} &= C_{m_{\alpha_g}}\,\alpha_g + C_{m_{\dot{\alpha}_g}}\,\frac{\dot{\alpha}_g\bar{c}}{V} \end{aligned} \tag{4.2}$$

with,

$$\begin{aligned} C_{X_{\alpha_g}} &= C_{X_\alpha} \\ C_{Z_{\alpha_g}} &= C_{Z_\alpha} \\ C_{m_{\alpha_g}} &= C_{m_\alpha} \end{aligned} \tag{4.3}$$

$$\begin{aligned} C_{X_{\dot{\alpha}_g}} &= C_{X_{\dot{\alpha}}} - C_{X_q} = 0 \\ C_{Z_{\dot{\alpha}_g}} &= C_{Z_{\dot{\alpha}}} - C_{Z_q} \\ C_{m_{\dot{\alpha}_g}} &= C_{m_{\dot{\alpha}}} - C_{m_q} \end{aligned} \tag{4.4}$$

The non-dimensional aerodynamic forces and moments due to symmetrical longitudinal turbulence w_g are according to the Four Point Aircraft Model (with $\alpha_g = \frac{w_g}{V}$ and $q_g = \frac{\partial w_g}{\partial x}$),

<u>Four Point Aircraft Model</u>

$$\begin{aligned} C_{X_g} &= C_{X_{\alpha_g}}\,\alpha_g + C_{X_{q_g}}\,\frac{q_g\bar{c}}{V} \\ C_{Z_g} &= C_{Z_{\alpha_g}}\,\alpha_g + C_{Z_{q_g}}\,\frac{q_g\bar{c}}{V} + C_{Z_{\dot{\alpha}}}\,\frac{\dot{\alpha}_g\bar{c}}{V} \\ C_{m_g} &= C_{m_{\alpha_g}}\,\alpha_g + C_{m_{q_g}}\,\frac{q_g\bar{c}}{V} + C_{m_{\dot{\alpha}}}\,\frac{\dot{\alpha}_g\bar{c}}{V} \end{aligned} \tag{4.5}$$

with,

$$\begin{aligned} C_{X_{\alpha_g}} &= C_{X_\alpha} \\ C_{Z_{\alpha_g}} &= C_{Z_\alpha} \\ C_{m_{\alpha_g}} &= C_{m_\alpha} \end{aligned} \tag{4.6}$$

$$\begin{aligned} C_{X_{q_g}} &= -C_{X_q} \\ C_{Z_{q_g}} &= -C_{Z_q} \\ C_{m_{q_g}} &= -C_{m_q} \end{aligned} \tag{4.7}$$

The input correlation functions for both the DUT and Four Point Aircraft Model are,

DUT Model

$$C_{\alpha_g\alpha_g}(\tau) = \sigma^2_{\alpha_g}\, e^{-\frac{\xi}{L_g}} \left(1 - \frac{\xi}{2L_g}\right) = \sigma^2_{\alpha_g}\, g(\xi)$$

with $\xi = V\tau$.

Four Point Aircraft Model

$$C_{w_gw_g}(\tau) = \sigma^2 \left(\frac{1}{3}g(\xi_1) + \frac{4}{9}g(\xi_2) + \frac{2}{9}g(\xi_3)\right)$$

$$C_{q_gq_g}(\tau) = \frac{\sigma^2}{l_h^2}\left(2g(\xi_1) - g(\xi_4) - g(\xi_5)\right)$$

and, since the the w_g and q_g inputs are correlated (see chapter 3),

$$C_{w_gq_g}(\tau) = \frac{\sigma^2}{3l_h}\left(g(\xi_5) - g(\xi_1) + 2g(\xi_8) - 2g(\xi_2)\right)$$

The separation vectors used in the above correlation functions are summarized in table 4.2. For the calculation of the input power spectral density functions, the correlation functions are transformed to the frequency domain using a Fast Fourier Transform (FFT). The input power spectral densities will be corrected for the non-dimensional inputs, e.g., $S_{\alpha_g\alpha_g}(\omega) = \frac{1}{V^2}S_{w_gw_g}(\omega)$, etc..

The input power spectral density of $\alpha_g = \frac{w_g}{V}$ according to the DUT model is,

$$S_{\alpha_g\alpha_g}(\omega) = \sigma^2_{\alpha_g}\frac{L_g}{V}\frac{1+3\left(L_g\frac{\omega}{V}\right)^2}{\left[1+\left(L_g\frac{\omega}{V}\right)^2\right]^2}$$

The input PSD functions for the Four Point Aircraft Model are calculated using MATLAB's Fast Fourier Transform routine **fft.m**.

In figure 3.22 the input PSD functions for α_g are given. In, respectively, figures 3.23 and 3.24 the input PSD functions for $\dot{\alpha}_g$ and q_g (not $\frac{q_g\bar{c}}{V}$) are depicted. In figure 3.25 the cross power spectral density of α_g and $\frac{q_g\bar{c}}{V}$ is given.
In figures 4.5 to 4.8 the output PSD functions of the state variables $\hat{u}$, α, θ and $\frac{q\bar{c}}{V}$ are given. The results almost coincide over a wide frequency range. Especially for higher frequencies the spectra do not coincide, since the power spectral density estimates for the four point aircraft model are biased.

4.3 Models for Asymmetric Aircraft Motions

4.3.1 Aircraft Models and Input Spectra for Asymmetrical Longitudinal Turbulence

In this section the aircraft and input models for asymmetrical longitudinal turbulence u_g will be given.

DUT Model

The correlation function for u_g is written as,

$$C_{u_gu_g}(x,y) = \sigma^2_{u_g}\left\{f(r)\left(\frac{x}{r}\right)^2 + g(r)\left(\frac{y}{r}\right)^2\right\}$$

with f and g the Dryden correlation functions. The power spectral density function belonging to u_g may be calculated analytically and equals,

$$S_{u_gu_g}(\Omega_xL_g,\Omega_yL_g) = \pi\sigma^2_{u_g}\frac{1+\Omega_x^2L_g^2+4\Omega_y^2L_g^2}{\left(1+\Omega_x^2L_g^2+\Omega_y^2L_g^2\right)^{5/2}} \tag{4.8}$$

The non-dimensional aerodynamic forces and moments caused by asymmetrical longitudinal turbulence u_g are written as (with $\hat{u}_g = \frac{u_g}{V}$, see also chapter 2),

$$C_{Y_g} = C_{Y_{u_g}}(\Omega_y \frac{b}{2})\hat{u}_g = 0 \tag{4.9}$$

$$C_{l_g} = C_{l_{u_g}}(\Omega_y \frac{b}{2})\hat{u}_g = -C_{l_{r_w}}\ h(\Omega_y \frac{b}{2})\hat{u}_g \tag{4.10}$$

$$C_{n_g} = C_{n_{u_g}}(\Omega_y \frac{b}{2})\hat{u}_g = -C_{n_{r_w}}\ h(\Omega_y \frac{b}{2})\hat{u}_g \tag{4.11}$$

with,

$$h(\Omega_y \frac{b}{2}) = \frac{b}{2} \frac{\int_0^{\frac{b}{2}} c_l c\ sin\Omega_y y\ y\ dy}{\int_0^{\frac{b}{2}} c_l c\ y^2\ dy}$$

The non-dimensional aerodynamic force and moments caused by asymmetric longitudinal turbulence u_g may now be written as,

$$C_{Y_g} = 0 \tag{4.12}$$

$$C_{l_g} = C_{l_{u_g}}(\Omega_y \frac{b}{2})\hat{u}_g \tag{4.13}$$

$$C_{n_g} = C_{n_{u_g}}(\Omega_y \frac{b}{2})\hat{u}_g \tag{4.14}$$

The input power spectral density for $\hat{u}_g$ is equation (4.8).
However, now the cumulative effect of all asymmetric longitudinal turbulence u_g in Y- (or Ω_y-) direction is taken and the periodogram $I_{\hat{u}_g\hat{u}_g}$ is defined as,

$$I_{\hat{u}_g\hat{u}_g}(\Omega_x L_g, B) = \sigma^2_{\hat{u}_g} \int_0^\infty h^2(\Omega_y \frac{b}{2}) \frac{1 + \Omega_x^2 L_g^2 + 4\Omega_y^2 L_g^2}{\left(1 + \Omega_x^2 L_g^2 + \Omega_y^2 L_g^2\right)^{5/2}} d(\Omega_y L_g)$$

and is approximated by,

$$I_{\hat{u}_g\hat{u}_g}(\Omega_x L_g, B) = I_{\hat{u}_g}(0, B) \frac{1 + \tau_3^2\ \Omega_x^2 L_g^2}{\left(1 + \tau_1^2\ \Omega_x^2 L_g^2\right)\left(1 + \tau_2^2\ \Omega_x^2 L_g^2\right)} \tag{4.15}$$

Equation (4.15) is used in conjunction with,

$$C_{Y_g} = C_{Y_{u_g}} \hat{u}_g \tag{4.16}$$

$$C_{l_g} = C_{l_{u_g}} \hat{u}_g \tag{4.17}$$

$$C_{n_g} = C_{n_{u_g}} \hat{u}_g \tag{4.18}$$

and,

$$
\begin{aligned}
C_{Y_{u_g}} &= 0 && (4.19)\\
C_{l_{u_g}} &= -C_{l_{r_w}} && (4.20)\\
C_{n_{u_g}} &= -C_{n_{r_w}} && (4.21)
\end{aligned}
$$

which are now independent of frequency Ω_y since all effects of the asymmetrical longitudinal turbulence u_g in Y- (or Ω_y-) direction, is accounted for in the effective one-dimensional power spectral density $I_{\hat{u}_g\hat{u}_g}(\Omega_x L_g, B)$. As an input periodogram, equation (4.15) is usually written as,

$$
I_{\hat{u}_g\hat{u}_g}(\omega) = \frac{L_g}{V} I_{\hat{u}_g}(0,B)\, \frac{1+\tau_3^2\left(\frac{\omega L_g}{V}\right)^2}{\left(1+\tau_1^2\left(\frac{\omega L_g}{V}\right)^2\right)\left(1+\tau_2^2\left(\frac{\omega L_g}{V}\right)^2\right)} \qquad (4.22)
$$

Values for τ_1, τ_2, τ_3 and $I_{\hat{u}_g}(0,B)$ are summarized in tables 2.4 and 2.3.

Four Point Aircraft Model

The aircraft responses to asymmetrical longitudinal turbulence u_g are now described using the input r_{1_g},

$$
r_{1_g} = \frac{\partial u_g}{\partial y} \qquad (4.23)
$$

The autocorrelation function of this input equals,

$$
C_{r_{1_g}r_{1_g}}(\tau) = \frac{2\sigma^2}{(b')^2}\left(f(\xi_1) - K_1 f(\xi_3) - (1-K_1)\, g(\xi_3)\right) \qquad (4.24)
$$

See table 4.2 for the definition of the separation vectors.

The non-dimensional aerodynamic force and moments caused by asymmetric longitudinal turbulence u_g (r_{1_g}) are written as,

$$
\begin{aligned}
C_{Y_g} &= C_{Y_{r_{1_g}}}\frac{r_{1_g} b}{2V} && (4.25)\\
C_{l_g} &= C_{l_{r_{1_g}}}\frac{r_{1_g} b}{2V} && (4.26)\\
C_{n_g} &= C_{n_{r_{1_g}}}\frac{r_{1_g} b}{2V} && (4.27)
\end{aligned}
$$

with,

$$C_{Y_{r_{1_g}}} = 0 \tag{4.28}$$
$$C_{l_{r_{1_g}}} = -C_{l_{r_w}} \tag{4.29}$$
$$C_{n_{r_{1_g}}} = -C_{n_{r_w}} \tag{4.30}$$

The subscript w denotes the contribution of the wing to the derivatives C_{l_r} and C_{n_r}.

The input power spectral density function for r_{1_g}, $S_{r_{1_g}r_{1_g}}(\omega)$, is calculated numerically from the autocorrelation function $C_{r_{1_g}r_{1_g}}(\tau)$.

In, respectively, figures 3.28 and 3.31 the input PSD functions for $\hat{u}_g$ and $\frac{r_{1_g}b}{2V}$ are given.

In figures 4.9 to 4.12 the output PSD functions of the state variables β, ϕ, $\frac{pb}{2V}$ and $\frac{rb}{2V}$ are given. The results coincide remarkably well over a wide frequency range. It must be pointed out that the aircraft models for both the DUT and Four Point Aircraft Model are equal, however, the calculation of the input power spectral densities is completely different for both models. Also, for the Four Point Aircraft Model all effects of turbulence in Ω_y direction are accounted for in the numerically obtained input power spectral density function $S_{r_{1_g}r_{1_g}}(\omega)$ which found its origin in the correlation function $C_{r_{1_g}r_{1_g}}(\tau)$.

4.3.2 Aircraft Models and Input Spectra for Asymmetrical Lateral Turbulence

In this section the aircraft and input models for asymmetrical lateral turbulence v_g will be given.

<u>DUT Model</u>

The correlation function for v_g is written as,

$$C_{v_gv_g}(x,y) = \sigma_{v_g}^2 \left\{ f(r)\left(\frac{y}{r}\right)^2 + g(r)\left(\frac{x}{r}\right)^2 \right\}$$

with f and g the Dryden correlation functions. The power spectral density belonging to v_g may be calculated analytically and equals,

$$S_{v_gv_g}(\Omega_x L_g, \Omega_y L_g) = \pi\sigma_{v_g}^2 \frac{1 + 4\Omega_x^2 L_g^2 + \Omega_y^2 L_g^2}{\left(1 + \Omega_x^2 L_g^2 + \Omega_y^2 L_g^2\right)^{5/2}} \tag{4.31}$$

When considering the asymmetric force and moments exerted on the aircraft by the v_g-field, only the antisymmetric part of this flowfield is of interest. Therefore, the v_g variations along the aircraft's lateral axis (Y-axis), or, the v_g variations in Ω_y direction, will be neglected. The gust velocity v_g now only causes a gust angle of sideslip $\beta_g = \frac{v_g}{V}$. The effective one-dimensional power spectral density for v_g is calculated from equation (4.31) by integrating this equation over all possible frequencies Ω_y,

$$S_{v_g v_g}(\Omega_x L_g) = \sigma_{v_g}^2 \frac{1 + 3\,\Omega_x^2 L_g^2}{\left(1 + \Omega_x^2 L_g^2\right)^2} \tag{4.32}$$

or, using the radial frequency ω,

$$S_{v_g v_g}(\omega) = \sigma_{v_g}^2 \frac{L_g}{V} \frac{1 + 3\,\left(\frac{\omega L_g}{V}\right)^2}{\left(1 + \left(\frac{\omega L_g}{V}\right)^2\right)^2} \tag{4.33}$$

The non-dimensional aerodynamic forces and moments caused by asymmetrical lateral turbulence v_g are written as (with $\beta_g = \frac{v_g}{V}$),

$$C_{Y_g} = C_{Y_{\beta_g}} \beta_g + C_{Y_{\dot{\beta}_g}} \frac{\dot{\beta}_g b}{V} \tag{4.34}$$

$$C_{l_g} = C_{l_{\beta_g}} \beta_g + C_{l_{\dot{\beta}_g}} \frac{\dot{\beta}_g b}{V} \tag{4.35}$$

$$C_{n_g} = C_{n_{\beta_g}} \beta_g + C_{n_{\dot{\beta}_g}} \frac{\dot{\beta}_g b}{V} \tag{4.36}$$

with,

$$C_{Y_{\beta_g}} = C_{Y_\beta} \tag{4.37}$$

$$C_{l_{\beta_g}} = C_{l_\beta} \tag{4.38}$$

$$C_{n_{\beta_g}} = C_{n_\beta} \tag{4.39}$$

and,

$$C_{Y_{\dot{\beta}_g}} = C_{Y_{\dot{\beta}}} + \frac{1}{2} C_{Y_{r_{f+v}}} \tag{4.40}$$

$$C_{l_{\dot{\beta}_g}} = C_{l_{\dot{\beta}}} + \frac{1}{2} C_{l_{r_{f+v}}} \tag{4.41}$$

$$C_{n_{\dot{\beta}_g}} = C_{n_{\dot{\beta}}} + \frac{1}{2} C_{n_{r_{f+v}}} \tag{4.42}$$

The input power spectral density for β_g is equation (4.32) divided by V^2, or, $S_{\beta_g\beta_g}(\omega) = \frac{1}{V^2}S_{v_gv_g}(\omega)$.

Four Point Aircraft Model

The aircraft responses to asymmetrical longitudinal turbulence v_g are described using also the input v_g. This input, however, is correlated with turbulence inputs $r_{1_g} = \frac{\partial v_g}{\partial y}$ and $r_{2_g} = \frac{\partial v_g}{\partial x}$.
The necessary correlation functions are,

$$C_{v_gv_g}(\tau) = \sigma^2\, g(\xi_1)$$

$$C_{v_gr_{1_g}}(\tau) = \frac{2\sigma^2 K_2}{b'}\,(g(\xi_2) - f(\xi_2))$$

$$C_{v_gr_{2_g}}(\tau) = \frac{\sigma^2}{l_v}\,(g(\xi_1) - g(\xi_7))$$

See table 4.2 for the definition of the separation vectors.

The non-dimensional aerodynamic force and moments caused by asymmetric lateral turbulence v_g are written as (with $\beta_g = \frac{v_g}{V}$, etc.),

$$C_{Y_g} = C_{Y_{\beta_g}}\beta_g + C_{Y_{\dot\beta}}\frac{\dot\beta_g b}{V} + C_{Y_{r_{1_g}}}\frac{r_{1_g}b}{2V} + C_{Y_{r_{2_g}}}\frac{r_{2_g}b}{2V} \qquad (4.43)$$

$$C_{l_g} = C_{l_{\beta_g}}\beta_g + C_{l_{\dot\beta}}\frac{\dot\beta_g b}{V} + C_{l_{r_{1_g}}}\frac{r_{1_g}b}{2V} + C_{l_{r_{2_g}}}\frac{r_{2_g}b}{2V} \qquad (4.44)$$

$$C_{n_g} = C_{n_{\beta_g}}\beta_g + C_{n_{\dot\beta}}\frac{\dot\beta_g b}{V} + C_{n_{r_{1_g}}}\frac{r_{1_g}b}{2V} + C_{n_{r_{2_g}}}\frac{r_{2_g}b}{2V} \qquad (4.45)$$

with,

$$C_{Y_{\beta_g}} = C_{Y_\beta} \qquad (4.46)$$

$$C_{l_{\beta_g}} = C_{l_\beta} \qquad (4.47)$$

$$C_{n_{\beta_g}} = C_{n_\beta} \qquad (4.48)$$

$$C_{Y_{r_{1_g}}} = 0 \qquad (4.49)$$

$$C_{l_{r_{1_g}}} = -C_{l_{r_w}} \qquad (4.50)$$

$$C_{n_{r_{1_g}}} = -C_{n_{r_w}} \qquad (4.51)$$

$$C_{Y_{r_{2g}}} = C_{Y_{r_v}} \tag{4.52}$$
$$C_{l_{r_{2g}}} = C_{l_{r_v}} \tag{4.53}$$
$$C_{n_{r_{2g}}} = C_{n_{r_v}} \tag{4.54}$$

and $C_{Y_{\dot\beta}}$, $C_{l_{\dot\beta}}$ and $C_{n_{\dot\beta}}$ the normal derivatives with respect to the acceleration along the aircraft's lateral axis. The subscripts w and v respectively denote the contribution of the wing and the vertical tailplane to the derivatives C_{l_r} and C_{n_r}.

The input auto and cross power spectral density functions, are calculated numerically from the autocorrelation functions. In, respectively, figures 3.26 and 3.27 the input PSD functions for β_g and $\dot\beta_g$ are given.

In figures 4.13 to 4.16 the output PSD functions of the state variables β, ϕ, $\frac{pb}{2V}$ and $\frac{rb}{2V}$ are given. Again, the PSD functions of the output variables correlate well over a wide frequency range for both models. In fact, both aircraft models are, again, equal. Only the input PSD functions differ for the DUT and Four Point Aircraft Model.
Also, note again that for higher frequencies the spectra do not coincide, since the power spectral density estimates for the four point aircraft model are biased.

4.3.3 Aircraft Models and Input Spectra for Asymmetrical Vertical Turbulence

In this section the aircraft and input models for asymmetrical vertical turbulence w_g will be given.

<u>DUT Model</u>

The correlation function for w_g is written as,

$$C_{w_g w_g}(x,y) = \sigma^2_{w_g} g(r)$$

with g the Dryden correlation function. The power spectral density belonging to w_g may be calculated analytically and equals,

$$S_{w_g w_g}(\Omega_x L_g, \Omega_y L_g) = 3\pi\sigma^2_{w_g} \frac{\Omega_x^2 L_g^2 + \Omega_y^2 L_g^2}{\left(1 + \Omega_x^2 L_g^2 + \Omega_y^2 L_g^2\right)^{5/2}} \tag{4.55}$$

The non-dimensional aerodynamic forces and moments caused by asymmetrical vertical turbulence w_g are written as (with $\alpha_g = \frac{w_g}{V}$),

$$
\begin{aligned}
C_{Y_g} &= C_{Y_{\alpha_g}}(\Omega_y \frac{b}{2})\alpha_g = 0 && (4.56)\\
C_{l_g} &= C_{l_{\alpha_g}}(\Omega_y \frac{b}{2})\alpha_g = C_{l_{p_w}}\ h(\Omega_y \frac{b}{2})\alpha_g && (4.57)\\
C_{n_g} &= C_{n_{\alpha_g}}(\Omega_y \frac{b}{2})\alpha_g = C_{n_{p_w}}\ h(\Omega_y \frac{b}{2})\alpha_g && (4.58)
\end{aligned}
$$

with,

$$
h(\Omega_y \frac{b}{2}) = \frac{b}{2} \frac{\int_0^{\frac{b}{2}} c_l c \ sin \Omega_y y \ y \ dy}{\int_0^{\frac{b}{2}} c_l c \ y^2 \ dy}
$$

The non-dimensional aerodynamic force and moments caused by asymmetric vertical turbulence w_g may now be written as,

$$
\begin{aligned}
C_{Y_g} &= 0 && (4.59)\\
C_{l_g} &= C_{l_{\alpha_g}}(\Omega_y \frac{b}{2})\alpha_g && (4.60)\\
C_{n_g} &= C_{n_{\alpha_g}}(\Omega_y \frac{b}{2})\alpha_g && (4.61)
\end{aligned}
$$

The input power spectral density for α_g is equation (4.55). However, now the cumulative effect of all asymmetric vertical turbulence w_g in Y- (or Ω_y-) direction is taken and the periodogram $I_{\alpha_g \alpha_g}$ is defined as,

$$
I_{\alpha_g \alpha_g}(\Omega_x L_g, B) = 3\sigma_{\alpha_g}^2 \int_0^{\infty} h^2(\Omega_y \frac{b}{2}) \frac{\Omega_x^2 L_g^2 + \Omega_y^2 L_g^2}{\left(1 + \Omega_x^2 L_g^2 + \Omega_y^2 L_g^2\right)^{5/2}} d(\Omega_y L_g)
$$

and is approximated by,

$$
I_{\alpha_g \alpha_g}(\Omega_x L_g, B) = I_{\alpha_g}(0, B) \frac{1 + \tau_6^2 \ \Omega_x^2 L_g^2}{\left(1 + \tau_4^2 \ \Omega_x^2 L_g^2\right)\left(1 + \tau_5^2 \ \Omega_x^2 L_g^2\right)} \tag{4.62}
$$

Equation (4.62) is used in conjunction with,

$$
\begin{aligned}
C_{Y_g} &= C_{Y_{\alpha_g}} \alpha_g && (4.63)\\
C_{l_g} &= C_{l_{\alpha_g}} \alpha_g && (4.64)\\
C_{n_g} &= C_{n_{\alpha_g}} \alpha_g && (4.65)
\end{aligned}
$$

and,

$$C_{Y_{\alpha_g}} = 0 \tag{4.66}$$
$$C_{l_{\alpha_g}} = C_{l_{p_w}} \tag{4.67}$$
$$C_{n_{\alpha_g}} = C_{n_{p_w}} \tag{4.68}$$

which are now independent of frequency Ω_y since all effects of the asymmetrical vertical turbulence w_g in Y- (or Ω_y-) direction, is accounted for in the effective one-dimensional power spectral density $I_{\alpha_g\alpha_g}(\Omega_x L_g, B)$. As an input periodogram, equation (4.62) is usually written as,

$$I_{\alpha_g\alpha_g}(\omega) = \frac{L_g}{V} I_{\alpha_g}(0, B) \frac{1 + \tau_6^2 \left(\frac{\omega L_g}{V}\right)^2}{\left(1 + \tau_4^2 \left(\frac{\omega L_g}{V}\right)^2\right)\left(1 + \tau_5^2 \left(\frac{\omega L_g}{V}\right)^2\right)} \tag{4.69}$$

Some values of τ_4, τ_5, τ_6 and $I_{\alpha_g}(0, B)$ are summarized in tables 2.5 and 2.3.

Four Point Aircraft Model

The aircraft responses to asymmetrical vertical turbulence w_g is now described using the input p_g,

$$p_g = \frac{\partial w_g}{\partial y} \tag{4.70}$$

The autocorrelation function of this input equals,

$$C_{p_g p_g}(\tau) = \frac{2\sigma^2}{(b')^2} \left(g(\xi_1) - g(\xi_3)\right) \tag{4.71}$$

The non-dimensional force and moments caused by asymmetric vertical turbulence w_g (p_g) are also written as,

$$C_{Y_g} = C_{Y_{p_g}} \frac{p_g b}{2V} \tag{4.72}$$
$$C_{l_g} = C_{l_{p_g}} \frac{p_g b}{2V} \tag{4.73}$$
$$C_{n_g} = C_{n_{p_g}} \frac{p_g b}{2V} \tag{4.74}$$

and,

$$C_{Y_{p_g}} = 0 \tag{4.75}$$
$$C_{l_{p_g}} = C_{l_{p_w}} \tag{4.76}$$
$$C_{n_{p_g}} = C_{n_{p_w}} \tag{4.77}$$

The subscript w denotes the contribution of the wing to the derivatives C_{l_p} and C_{l_p}.

The input power spectral density function for p_g, $S_{p_g p_g}(\omega)$, is calculated numerically from the autocorrelation function $C_{p_g p_g}(\tau)$

In, respectively, figures 3.29 and 3.30 the input PSD functions for α_g and $\frac{p_g b}{2V}$ are given.

In figures 4.17 to 4.20 the output PSD functions of the state variables β, ϕ, $\frac{pb}{2V}$ and $\frac{rb}{2V}$ are given. The results, again, coincide remarkably well over a wide frequency range. It must be pointed out that the aircraft models for both the DUT and Four Point Aircraft Model are equal, however, the calculation of the input power spectral densities is completely different for both models. Also, for the Four Point Aircraft Model all effects of turbulence in Ω_y direction are accounted for in the numerically obtained input power spectral density function $S_{p_g p_g}(\omega)$ which found its origin in the correlation function $C_{p_g p_g}(\tau)$.

$x_{c.g.}$	=	0.30	$\bar{c}$							
W	=	44291	N	V	=	59.9	m/sec	μ_b	=	15
m	=	4514	kg	h	=	3048	m	K_X^2	=	0.012
S	=	24.2	m^2	ρ	=	0.9046	kg/m^3	K_Z^2	=	0.037
$\bar{c}$	=	2.022	m	μ_c	=	102		K_{XZ}	=	0.002
b	=	13.36	m					K_Y^2	=	0.980
C_{X_0}	=	0		C_{Z_0}	=	-1.1360				
C_{X_u}	=	-0.2199		C_{Z_u}	=	-2.2720		C_{m_u}	=	0
C_{X_α}	=	0.4653		C_{Z_α}	=	-5.1600		C_{m_α}	=	-0.4300
				$C_{Z_{\dot{\alpha}}}$	=	-1.4300		$C_{m_{\dot{\alpha}}}$	=	-3.7000
C_{X_q}	=	0		C_{Z_q}	=	-3.8600		C_{m_q}	=	-7.0400
C_{X_δ}	=	0		C_{Z_δ}	=	-0.6238		C_{m_δ}	=	-1.5530
C_{Y_β}	=	-0.9896		C_{ℓ_β}	=	-0.0772		C_{n_β}	=	0.1638
C_{Y_p}	=	-0.0870		C_{ℓ_p}	=	-0.3444		C_{n_p}	=	-0.0108
C_{Y_r}	=	0.4300		C_{ℓ_r}	=	0.2800		C_{n_r}	=	-0.1930
$C_{Y_{\delta_a}}$	=	0		$C_{\ell_{\delta_a}}$	=	-0.2349		$C_{n_{\delta_a}}$	=	0.0286
$C_{Y_{\delta_r}}$	=	0.3037		$C_{\ell_{\delta_r}}$	=	0.0286		$C_{n_{\delta_r}}$	=	-0.1261

Table 4.1: Stability and control derivatives of flight condition for Cessna Ce-500 Citation.

ξ_1	=	$\lvert V\tau \rvert$
ξ_2	=	$\sqrt{(V\tau)^2 + \left(\frac{b'}{2}\right)^2}$
ξ_3	=	$\sqrt{(V\tau)^2 + (b')^2}$
ξ_4	=	$\lvert V\tau + l_h \rvert$
ξ_5	=	$\lvert V\tau - l_h \rvert$
ξ_6	=	$\lvert V\tau + l_v \rvert$
ξ_7	=	$\lvert V\tau - l_f \rvert$
ξ_8	=	$\sqrt{(V\tau - l_h)^2 + \left(\frac{b'}{2}\right)^2}$
ξ_9	=	$\sqrt{(V\tau - l_v)^2 + \left(\frac{b'}{2}\right)^2}$

Table 4.2: Definition of seperation vectors as used in auto- and crosscorrelation function calculation.

Chapter 5

Concluding Remarks

In this report two different approaches for calculating the symmetric and asymmetric aircraft responses to atmospheric turbulence have been presented. The Delft University of Technology (DUT) model, see Ref. (Mulder & Vaart, 1993), which is basically a one point aircraft model, is compared to a Four Point Aircraft Model, according to the theory of Etkin, see Ref. (Etkin, 1980). Only frequency domain simulations are possible with the latter, although the theory may be extended to a time domain approach, see appendix B.

The results of both the DUT and FPA model methods coincide remarkably well over a wide frequency range, although the approaches of setting up the aircraft model and calculating the input power spectral densities differ considerably. In fact, both aircraft models are equal, however, the calculation of input power spectral density functions differs considerably. For the FPA model one can speak of Power Spectral Density decomposition, hence, using several (correlated) inputs. Results have been presented for the Cessna Ce-500 Citation.

The FPA model is limited to a certain frequency range whereas the DUT model can be used over an unlimited frequency range. This limitation is a result of the four point linear field approximation.

It must be pointed out, that the DUT model uses analytically derived input Power Spectral Density functions, while these input spectra are calculated numerically for the FPA model. For the latter model the numerically calculated Power Spectral Density functions are derived from auto- and crosscorrelation functions, obtained from time domain simulations. Care must be taken by choosing the sampling-interval, total simulation time and, thus, the number of sample points, since important modes' identification may be cancelled out by the discretization interval.

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Appendix A

Frequency Domain Simulations

For frequency domain simulations the transfer functions of output variables to all gust inputs may be calculated using MATLAB. However, an important observation must be made; some of the inputs may be correlated and hence input cross power spectral densities may exist. The transfer functions from all inputs to outputs are calculated using the `mv2fr` routine in MATLAB:

```
MV2FR Frequency response of MIMO system
MV2FR(A,B,C,D,W)  calculates the MVFR matrix of the system:
MVFR = MultiVariable Frequency Response

.
x = Ax + Bu
y = Cx + Du

               -1
G(s) = C(sI-A) B + D

Vector W contains the frequencies, in radians, at which the
frequency response is to be evaluated. Each component matrix
of the resulting MVFR matrix has as many columns as
inputs and as many rows as outputs.

MV2FR(NUM,COMDEN,W) calculates the MVFR matrix from the
transfer function description  G(s) = NUM(s)/COMDEN(s)
where NUM is multivariable matrix of polynomial coefficients and
COMDEN is the vector containing Common Denominator polynomial
coefficients in descending powers of s.

MV2FR(A,B,C,D,W,iu) and MV2FR(NUM,COMDEN,W,iu) are provided for
compatibility with the CONTROL TOOLBOX's NYQUIST.
They produce an MVFR matrix for a single input, iu, and the results
```

```
are returned as one row per frequency.

The CONTROL TOOLBOX's BODE function is provided by
[MAG,PH]=R2P(MV2FR(A,B,C,D,W,iu))

Dr M.P. Ford 4th August 1987
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 MRN0019
```

An example is given below.

Example

A three-input/single-output may be defined by its state space definition as,

$$\begin{aligned} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \\ y &= \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{bmatrix} d_1 & d_2 & d_3 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \end{aligned} \tag{A.1}$$

Assuming a three-input/single-output system according to,

$$Y(\omega) = H_{yu_1}(\omega)U_1(\omega) + H_{yu_2}(\omega)U_2(\omega) + H_{yu_3}(\omega)U_3(\omega) = H_{yu}(\omega)\underline{u}(\omega)$$

the power spectral density of this output equals,

$$S_{yy}(\omega) = H_{yu}(\omega)\ S_{\underline{uu}}(\omega)\ H^*_{yu}(\omega)^T$$

with,

$$H_{yu}(\omega) = [H_{yu_1}(\omega)\ H_{yu_2}(\omega)\ H_{yu_3}(\omega)]$$

and,

$$S_{\underline{uu}}(\omega) = \begin{bmatrix} S_{u_1u_1}(\omega) & S_{u_1u_2}(\omega) & S_{u_1u_3}(\omega) \\ S_{u_2u_1}(\omega) & S_{u_2u_2}(\omega) & S_{u_2u_3}(\omega) \\ S_{u_3u_1}(\omega) & S_{u_3u_2}(\omega) & S_{u_3u_3}(\omega) \end{bmatrix}$$

If inputs are uncorrelated, the input cross power spectral densities equal zero and the input power spectral density matrix becomes,

$$S_{\underline{uu}}(\omega) = \begin{bmatrix} S_{u_1u_1}(\omega) & 0 & 0 \\ 0 & S_{u_2u_2}(\omega) & 0 \\ 0 & 0 & S_{u_3u_3}(\omega) \end{bmatrix}$$

Appendix B

Time Domain Simulations

Although the model presented by (Etkin, 1980) can only be used for frequency domain simulations, time domain simulations may be performed as well.

Considering that the power spectral densities of the concerning input variables are calculated numerically, there seems to be no analytical form for them. However, it is always possible to approximate these power spectral densities by prescribed analytical formulae of which the unknown parameters are to be identified. This problem may be addressed as a curve fitting problem. In this section a linear least squares technique for fitting a real set of data will be presented.

Considering that the numerical frequency data set of a power spectral density is known and let this data be described by $S(\omega_i)$ for $i = 1$ to N (N are the number of frequency points considered). The power spectral density data is accompanied by a set of frequency data called $\omega(i)$, again for $i = 1$ to N. The power spectral density data $S(\omega)$ will be approximated by the analytical PSD function $P(\omega)$, which will be described by,

$$P(\omega) = \frac{K_1 + \gamma_3\,\omega^2}{(1 + \gamma_1\,\omega^2)\,(1 + \gamma_2\,\omega^2)} \tag{B.1}$$

The analytical function $P(\omega)$ will now be defined as the fraction of two polynomials,

$$P(\omega) = \frac{N(\omega)}{D(\omega)} \tag{B.2}$$

with,

$$N(\omega) = K_1 + \gamma_3\,\omega^2 \tag{B.3}$$

$$D(\omega) = \left(1 + \gamma_1\,\omega^2\right)\left(1 + \gamma_2\,\omega^2\right) \tag{B.4}$$

Now consider the error $\epsilon(\omega)$ between the actual data $S(\omega)$ and the approximating data $P(\omega)$ by,

$$\epsilon(\omega) = S(\omega) - P(\omega) = S(\omega) - \frac{N(\omega)}{D(\omega)} \tag{B.5}$$

and defining a cost function equal to,

$$J_1 = \sum_{i=1}^{N} (\epsilon(\omega_i))^2 \tag{B.6}$$

and to solve this problem for the unknown parameters K_1, γ_1, γ_2 and γ_3 for the smallest cost function J_1, leads to a non-linear problem because of the non-linear parameters γ_1 and γ_2. However, by introducing a new error function according to,

$$\varepsilon(\omega) = S(\omega) \cdot D(\omega) - N(\omega) \tag{B.7}$$

and a new cost function,

$$J = \sum_{i=1}^{N} (\varepsilon(\omega_i))^2 \tag{B.8}$$

it leads to a linear problem in all unknown parameters K_1, γ_1, γ_2 and γ_3! A simple matrix inversion will provide the unknown parameters. Remember that the analytical power spectral density model,

$$P(\omega) = \frac{K_1 + \gamma_3\, \omega^2}{(1 + \gamma_1\, \omega^2)\,(1 + \gamma_2\, \omega^2)}$$

can also be written in the form,

$$P(\omega) = K \frac{1 + \tau_3^2\, \left(\frac{\omega L_g}{V}\right)^2}{\left(1 + \tau_1^2\, \left(\frac{\omega L_g}{V}\right)^2\right)\left(1 + \tau_2^2\, \left(\frac{\omega L_g}{V}\right)^2\right)} \tag{B.9}$$

with,

$$K = K_1 \tag{B.10}$$

$$\tau_1 = \sqrt{\gamma_1}\frac{V}{L_g} \tag{B.11}$$

$$\tau_2 = \sqrt{\gamma_2}\frac{V}{L_g} \tag{B.12}$$

$$\tau_3 = \sqrt{\frac{\gamma_3}{K_1}}\frac{V}{L_g} \tag{B.13}$$

This only holds if γ_1, γ_2 and γ_3 are non-negative.

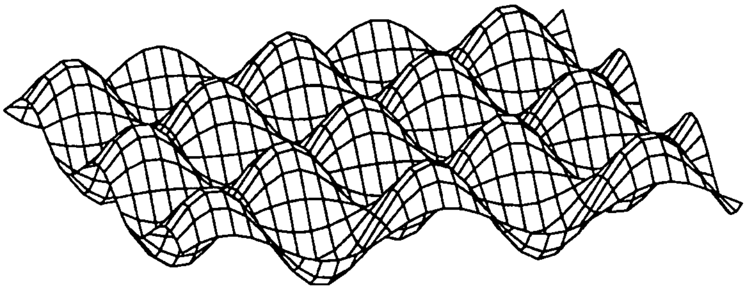

Figure 2.1: Two-dimensional symmetrical flowfield

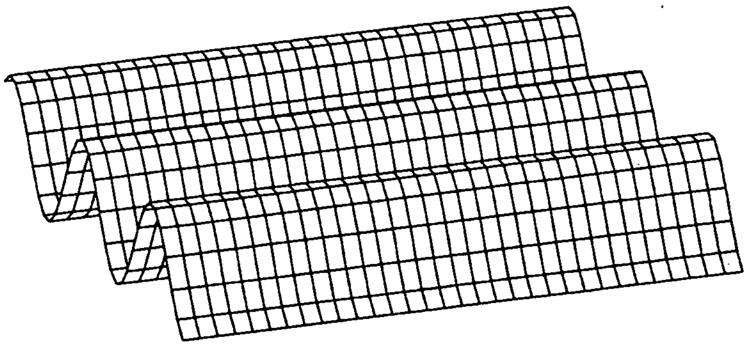

Figure 2.2: One-dimensional symmetrical flowfield

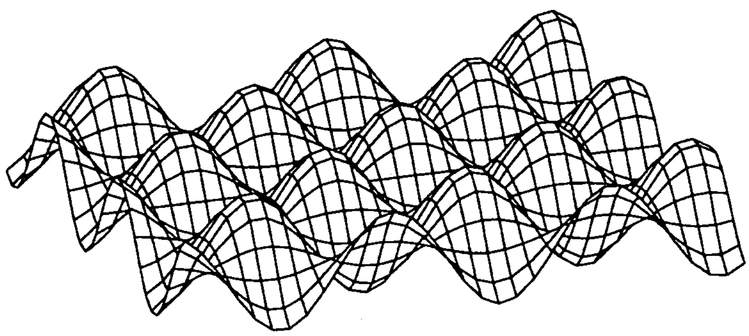

Figure 2.3: Two-dimensional anti-symmetrical flowfield

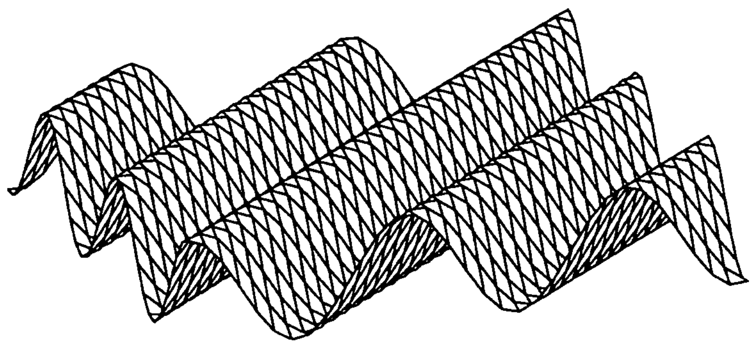

Figure 2.4: Two-dimensional flowfield resulting from superposition of symmetrical and anti-symmetrical flow fields

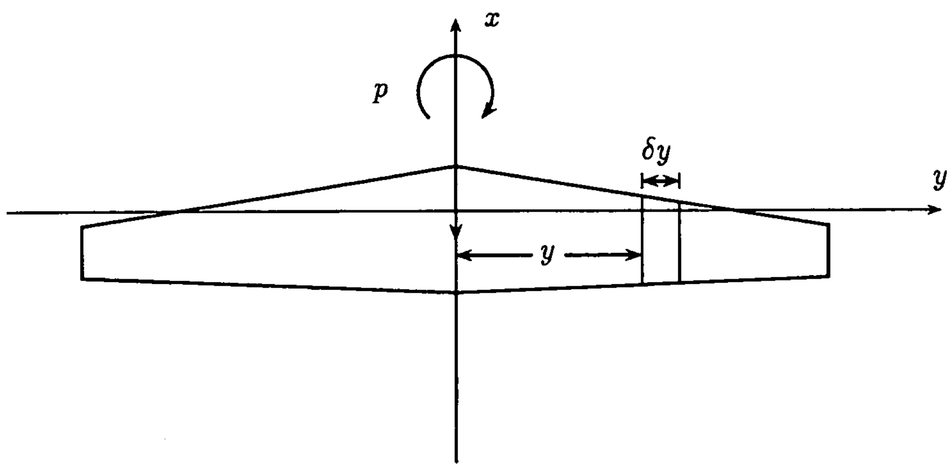

Figure 2.5: The contribution to the rolling moment by a chordwise strip of the wing.

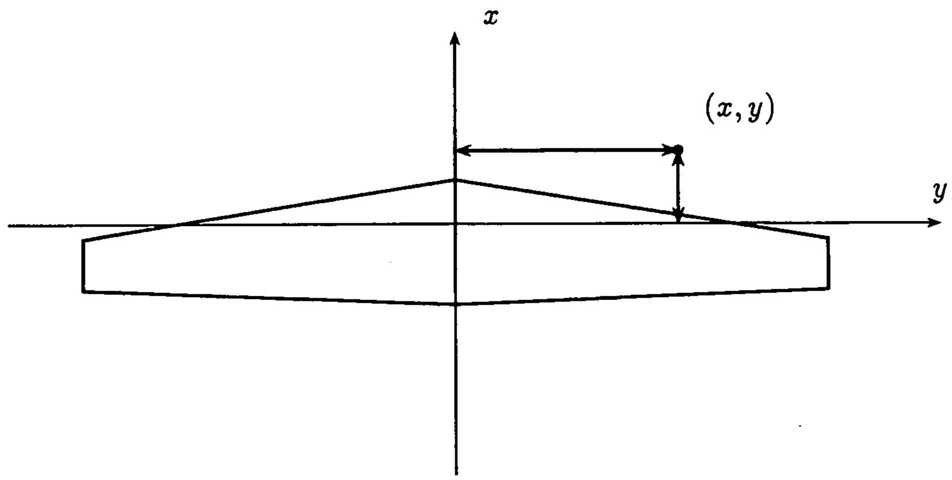

Figure 3.1: The downwash at point (x,y) of a planar aircraft

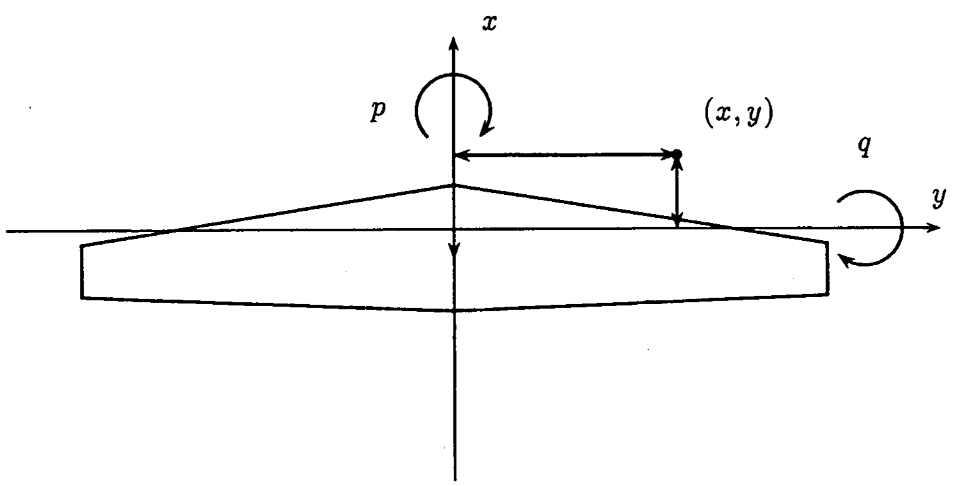

Figure 3.2: The downwash at point (x,y) of a planar aircraft due to rolling and yawing motion, with $w(x, y) = py - qx$

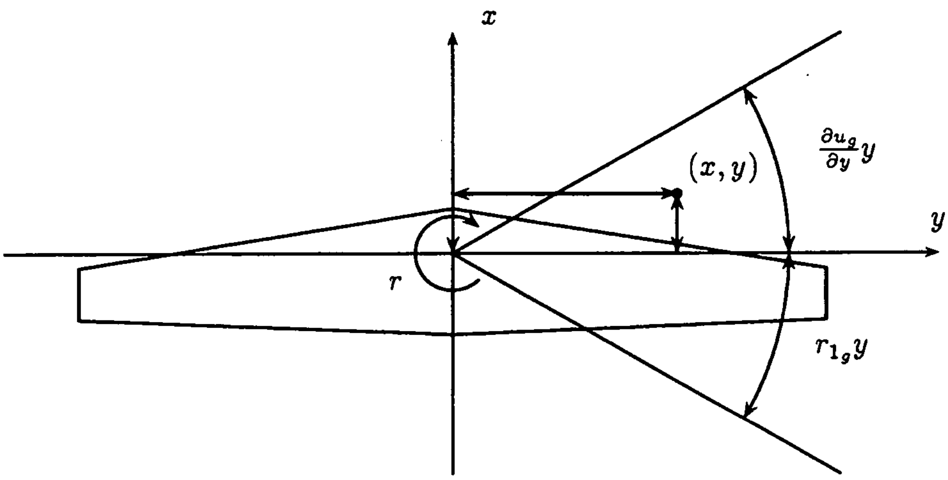

Figure 3.3: The turbulence input r_{1_g}, with $r_{1_g} = -\frac{\partial u_g}{\partial y}$

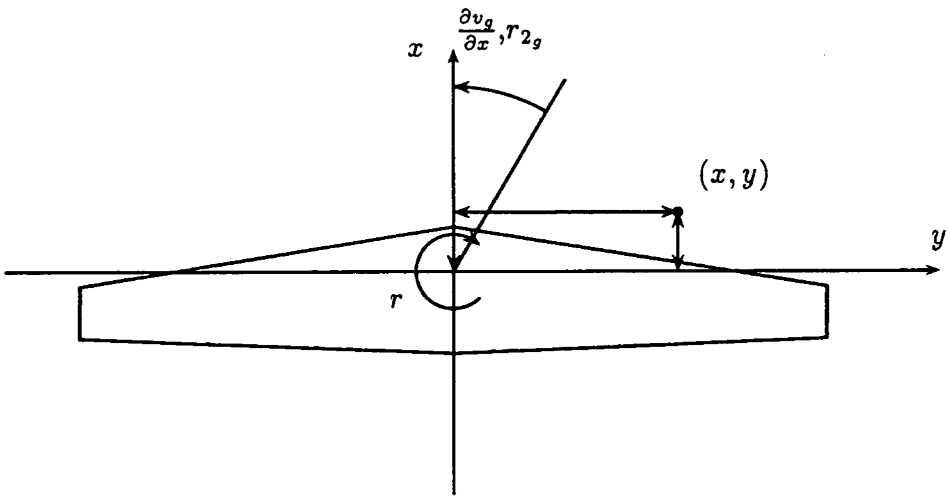

Figure 3.4: The turbulence input r_{2_g}, with $r_{2_g} = \frac{\partial v_g}{\partial x}$

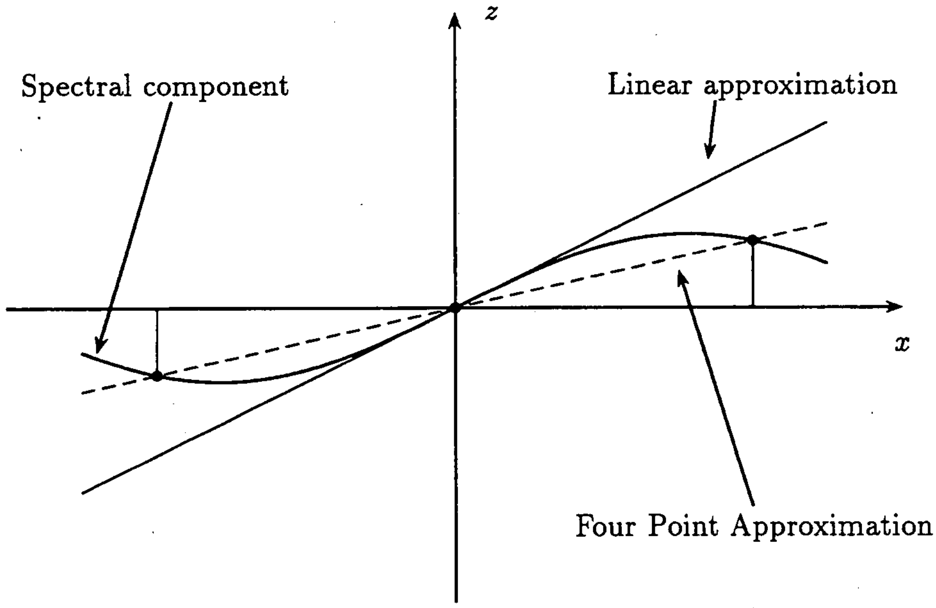

Figure 3.5: Comparison of the Linear Field Approximation and the Four Point Model

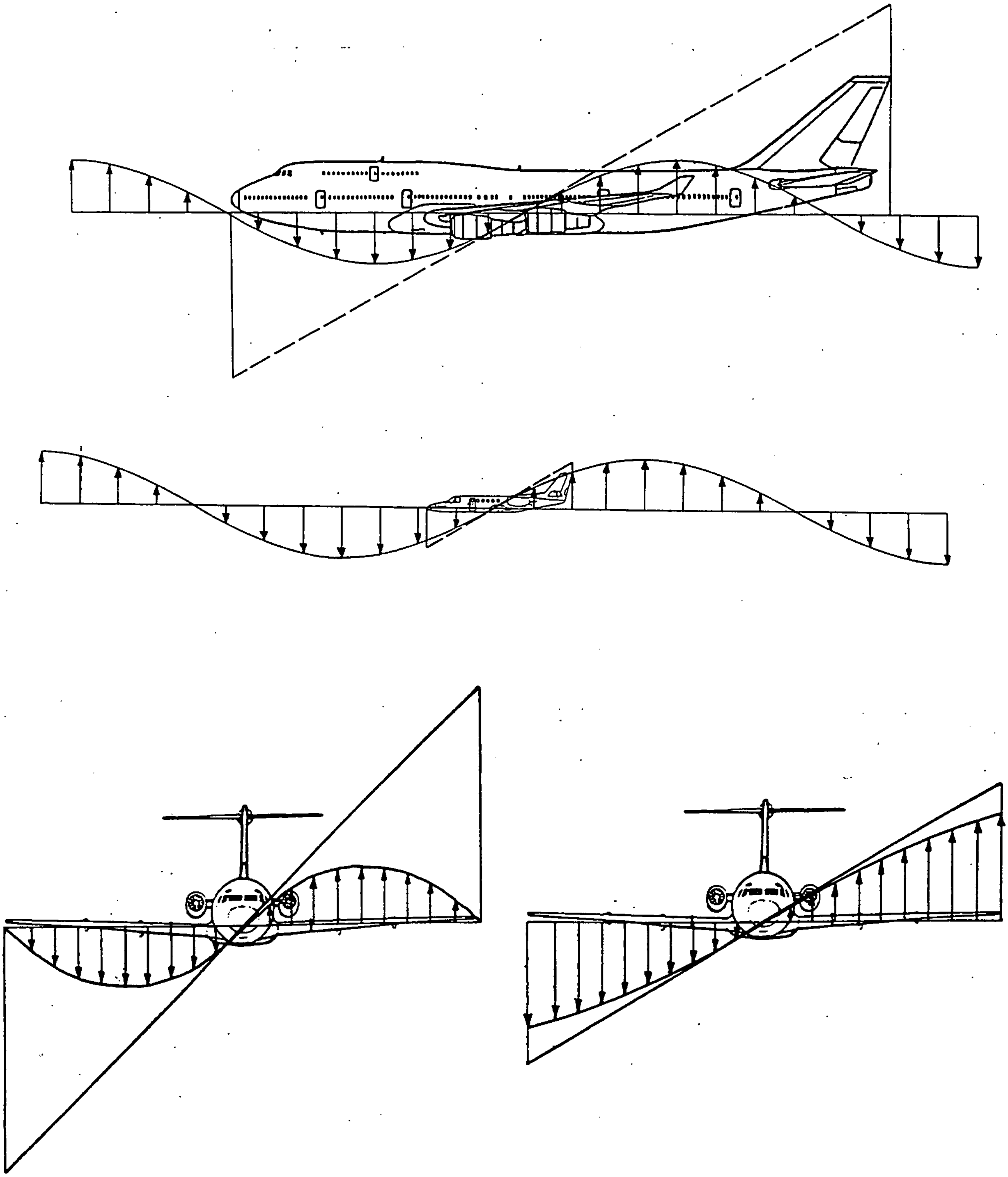

Figure 3.6: Linear Field Approximation applied to different sized aircraft

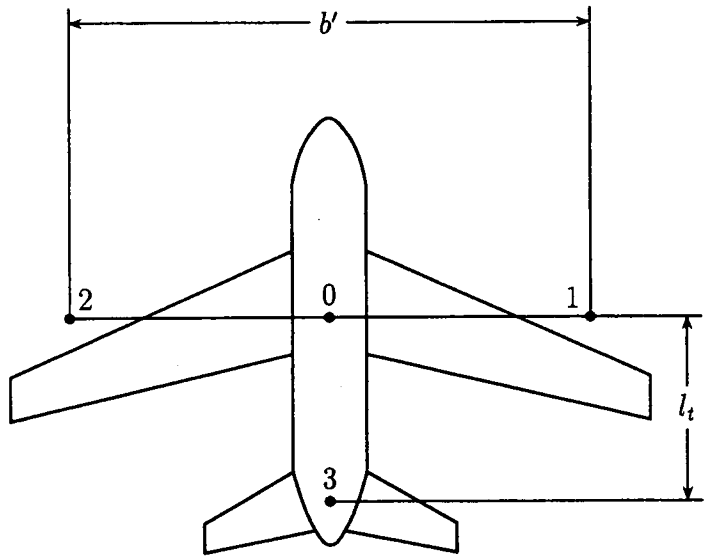

Figure 3.7: The Four Point Approximation

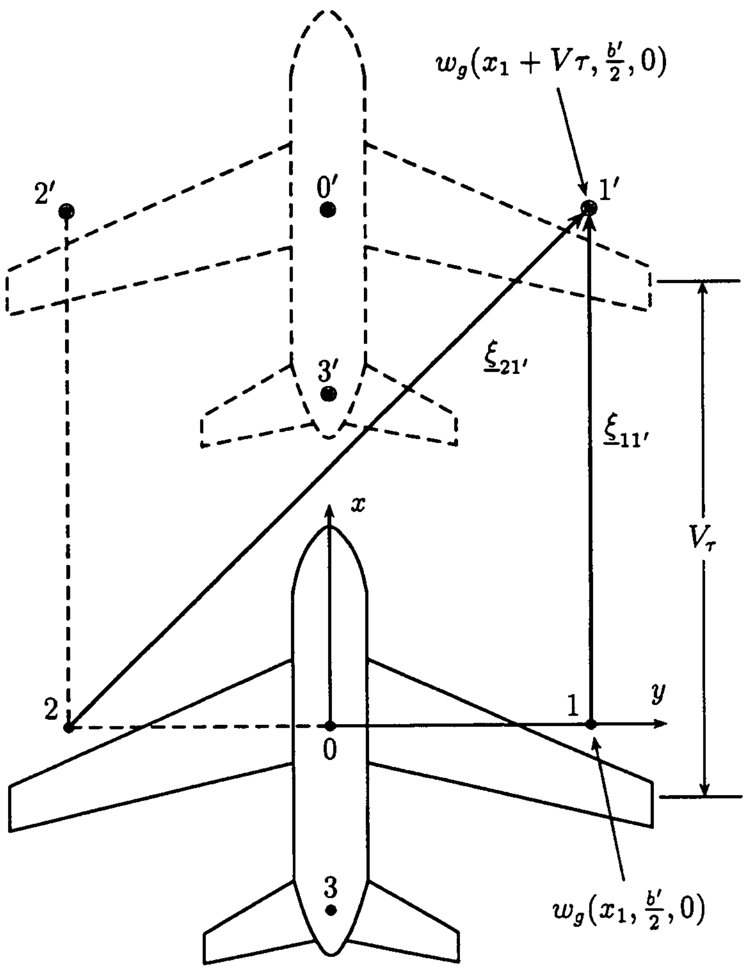

Figure 3.8: The separation vectors of the Four Point Model

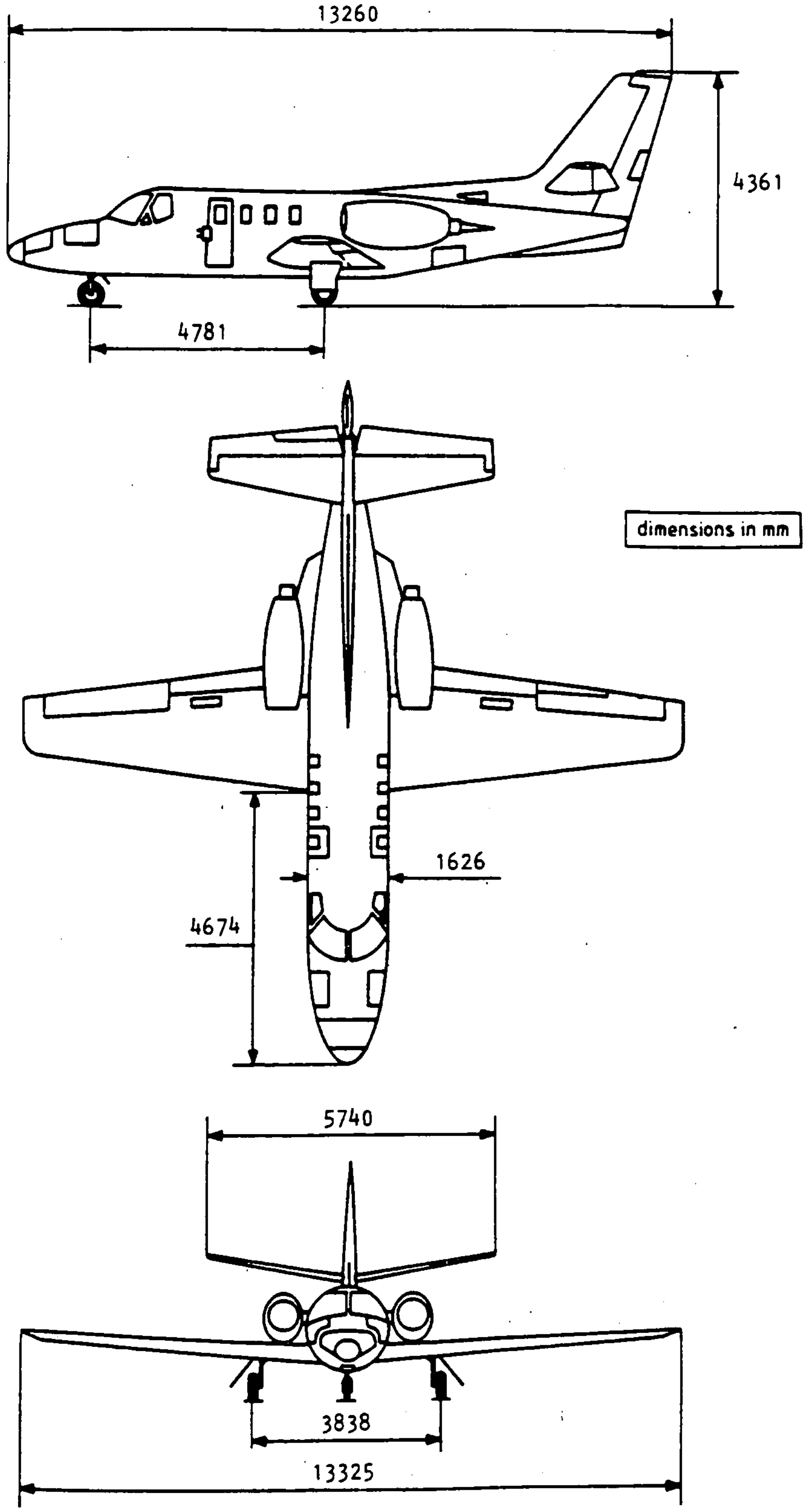

Figure 3.9: The Cessna Ce-500 Citation

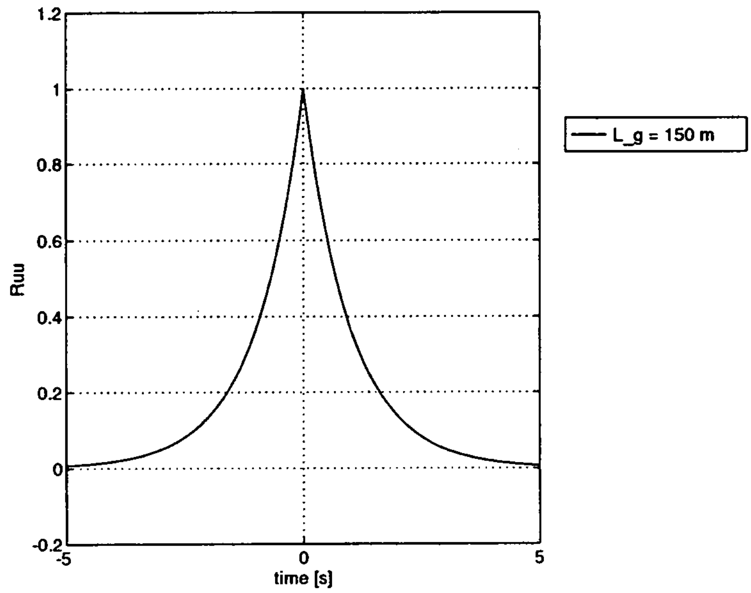

Figure 3.10: Non-dimensional autocorrelation function $C_{u_g u_g}$ for the four point aircraft aircraft model

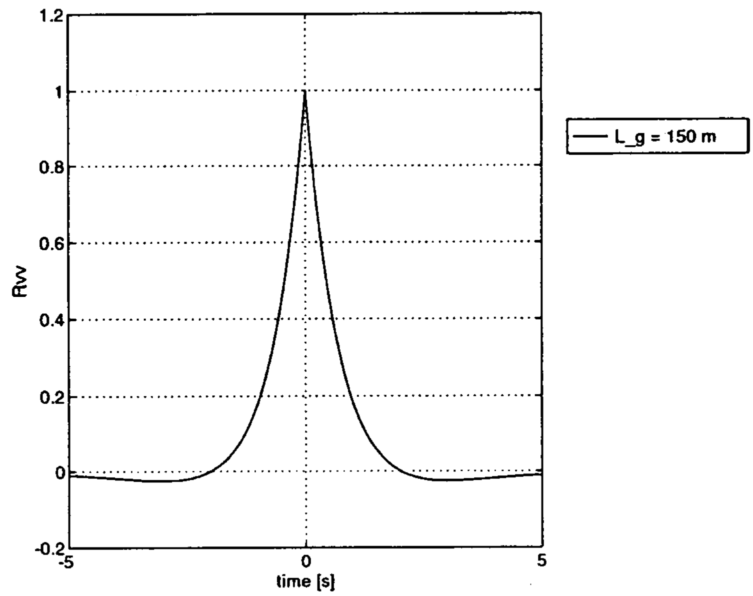

Figure 3.11: Non-dimensional autocorrelation function $C_{v_g v_g}$ for the four point aircraft model

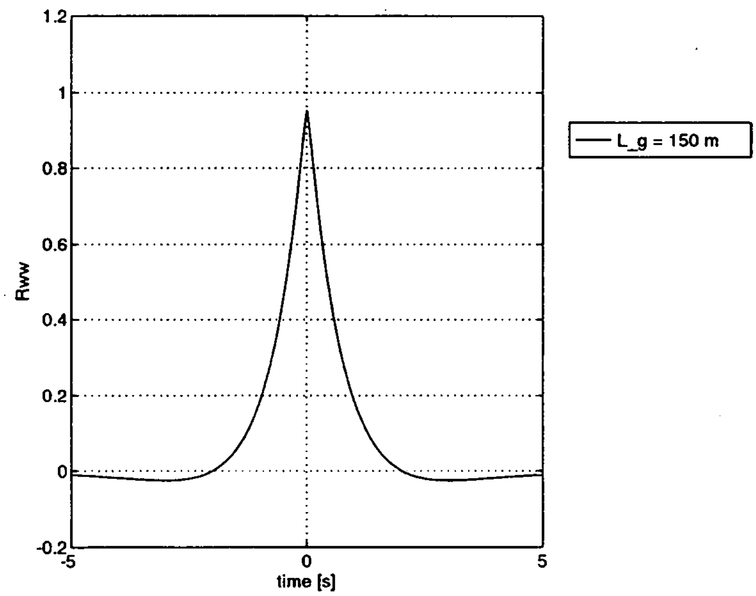

Figure 3.12: Non-dimensional autocorrelation function $C_{w_g w_g}$ for the four point aircraft model

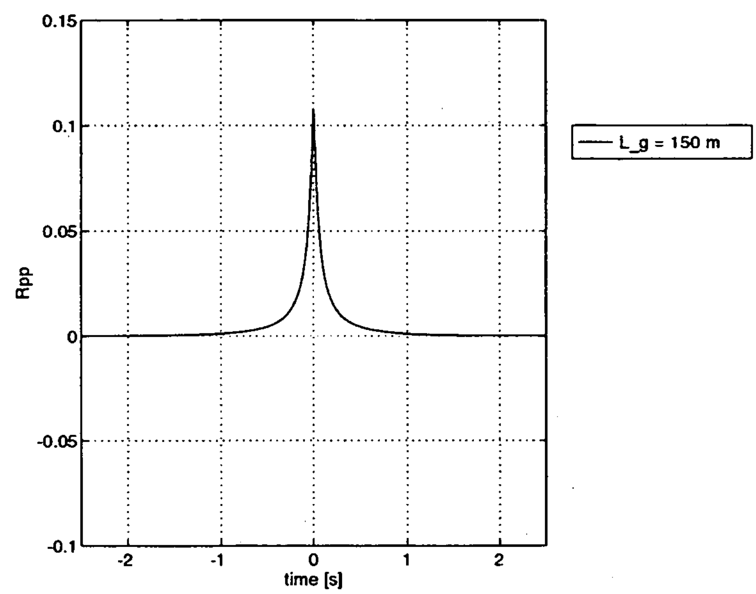

Figure 3.13: Non-dimensional autocorrelation function $C_{p_g p_g}$ for the four point aircraft model

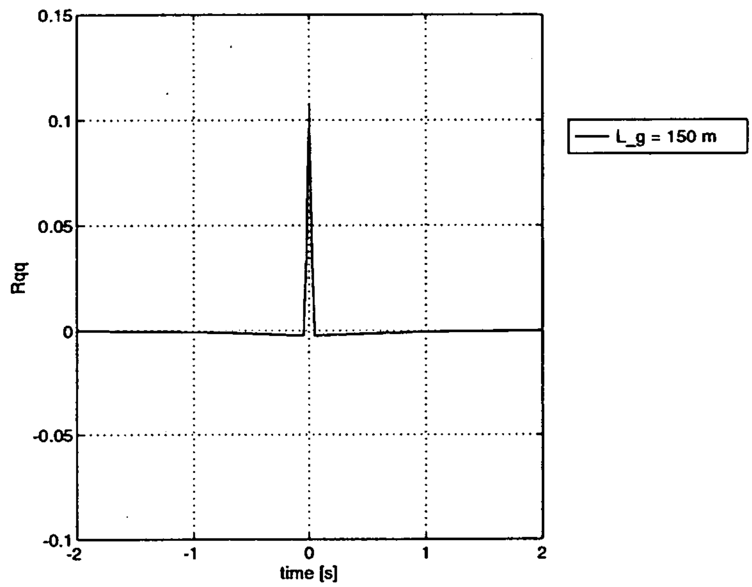

Figure 3.14: Non-dimensional autocorrelation function $C_{q_g q_g}$ for the four point aircraft model

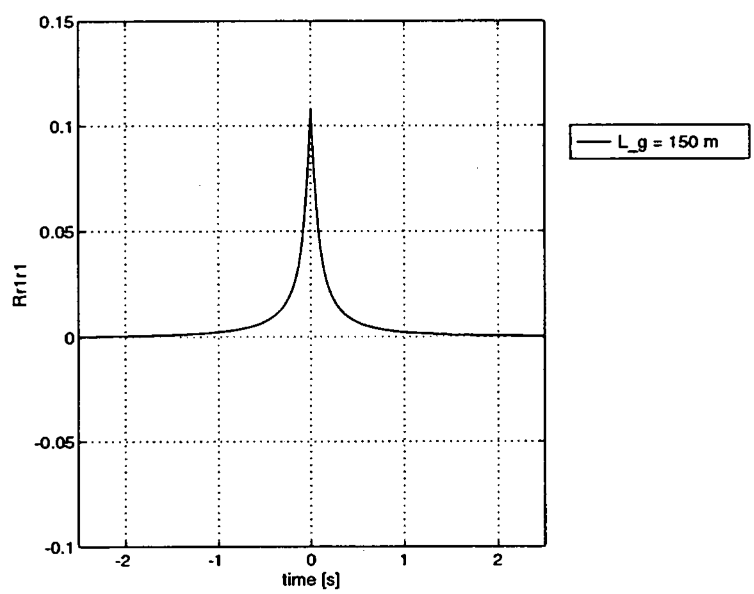

Figure 3.15: Non-dimensional autocorrelation function $C_{r_{1g} r_{1g}}$ for the four point aircraft model

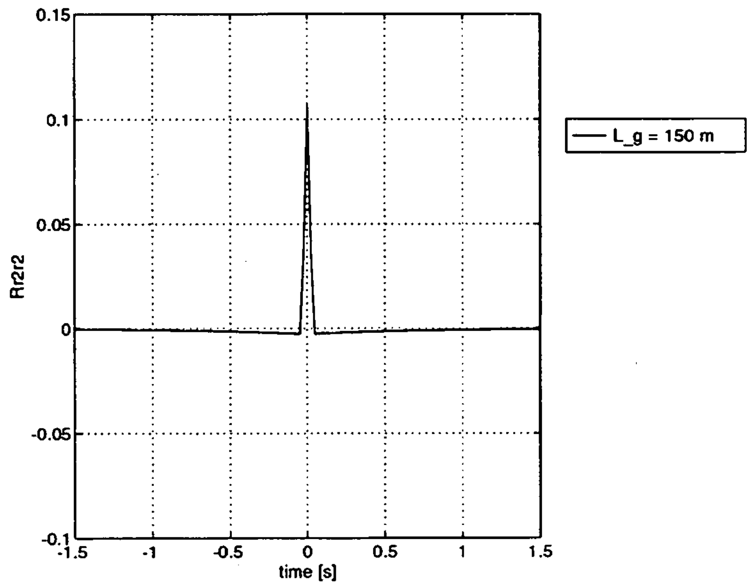

Figure 3.16: Non-dimensional autocorrelation function $C_{r_{2g}r_{2g}}$ for the four point aircraft model

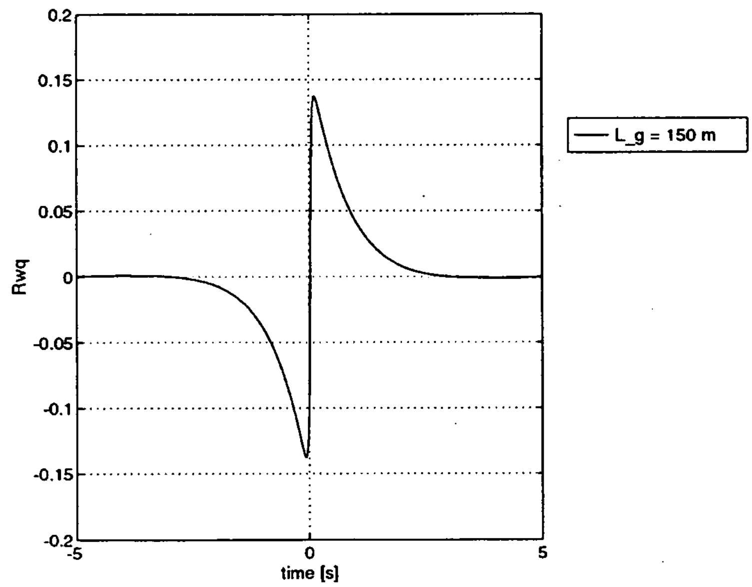

Figure 3.17: Non-dimensional autocorrelation function $C_{w_g q_g}$ for the four point aircraft model

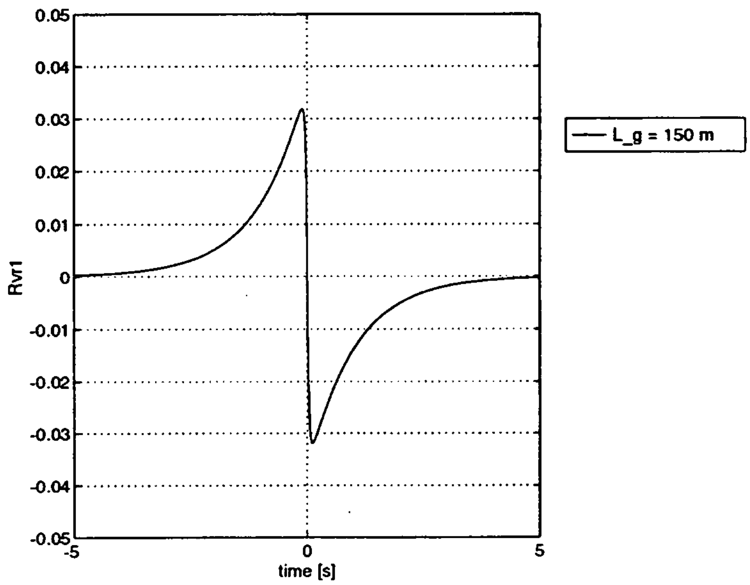

Figure 3.18: Non-dimensional autocorrelation function $C_{v_g r_{1_g}}$ for the four point aircraft model

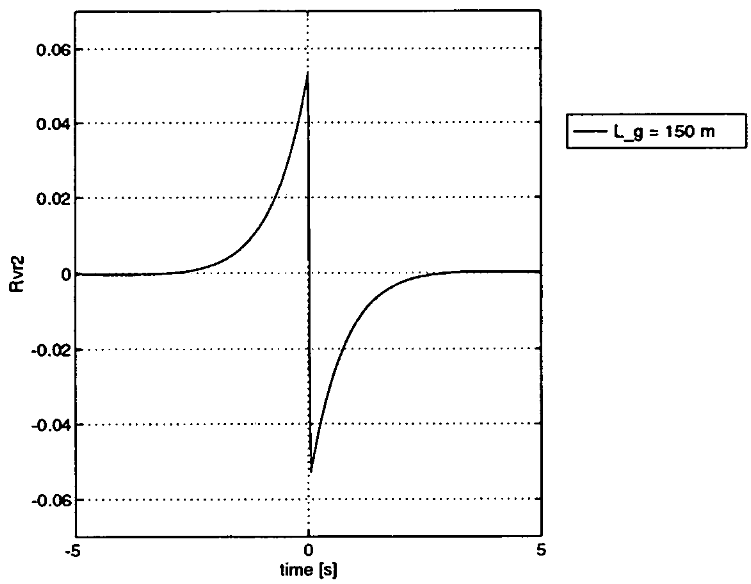

Figure 3.19: Non-dimensional autocorrelation function $C_{v_g r_{2_g}}$ for the four point aircraft model

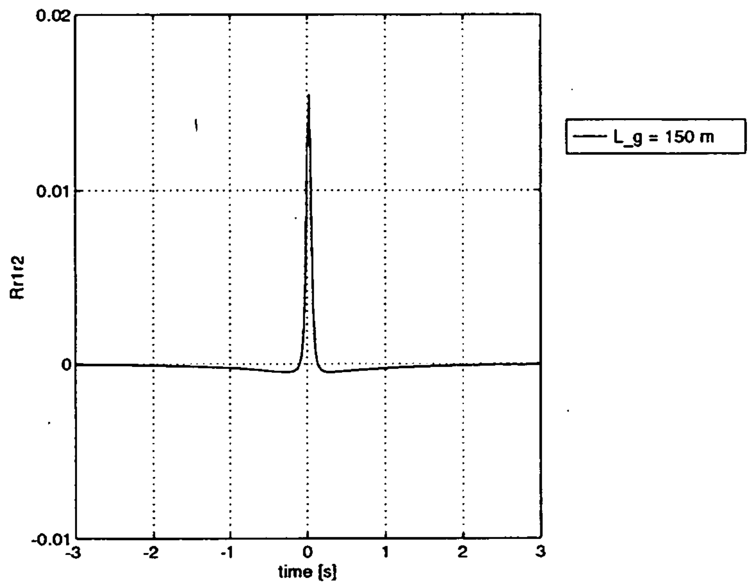

Figure 3.20: Non-dimensional autocorrelation function $C_{r_{1_g} r_{2_g}}$ for the four point aircraft model

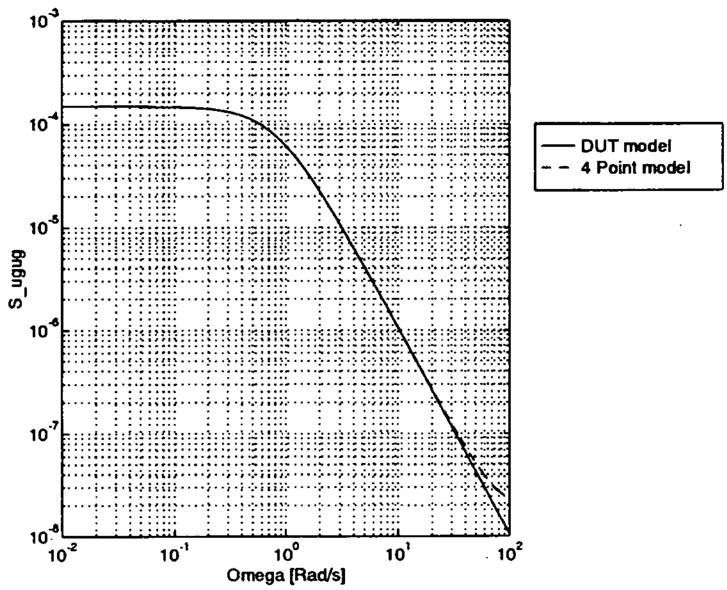

Figure 3.21: Input power spectral density of symmetrical gust $\hat{u}_g$.

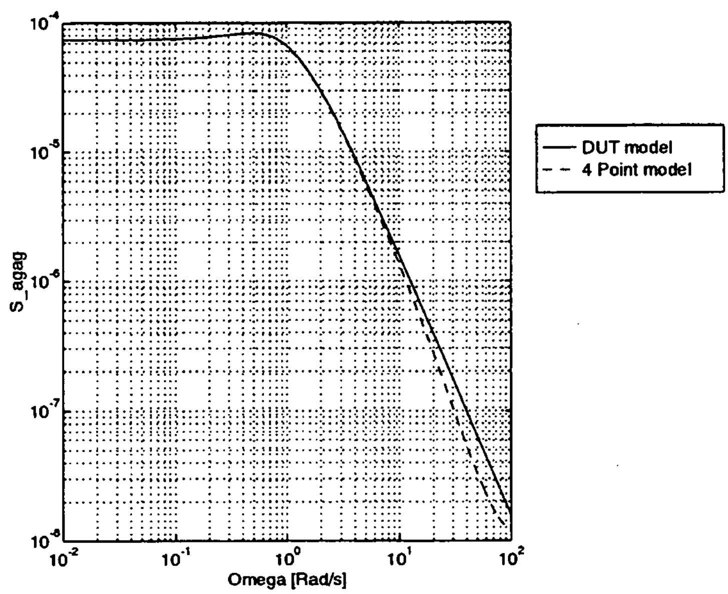

Figure 3.22: Input power spectral density of symmetrical gust α_g.

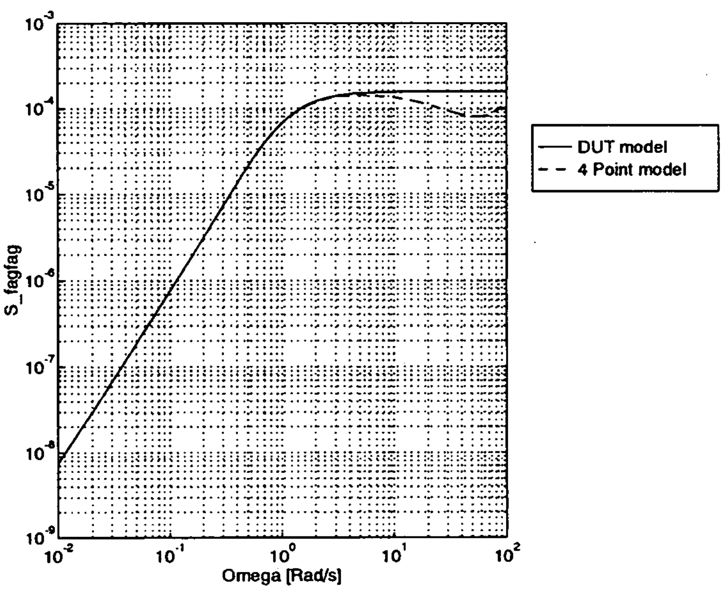

Figure 3.23: Input power spectral density of symmetrical gust $\dot{\alpha}_g$.

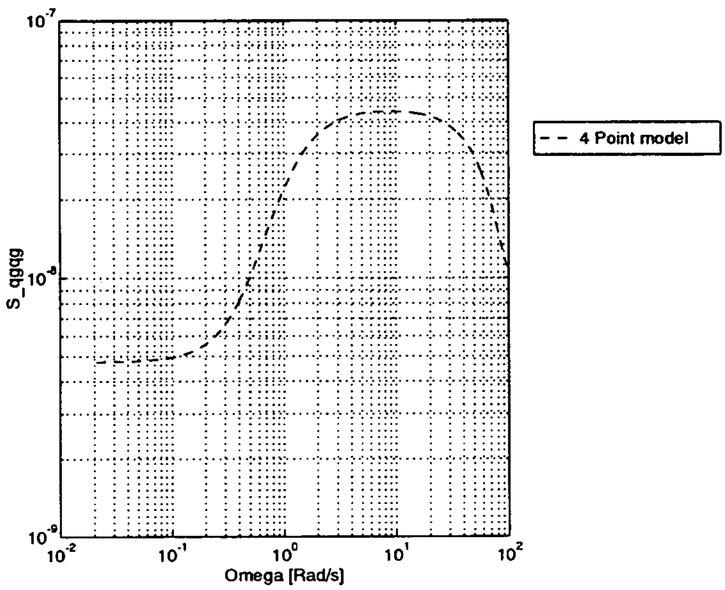

Figure 3.24: Input power spectral density of symmetrical gust $\frac{q_g \bar{c}}{V}$.

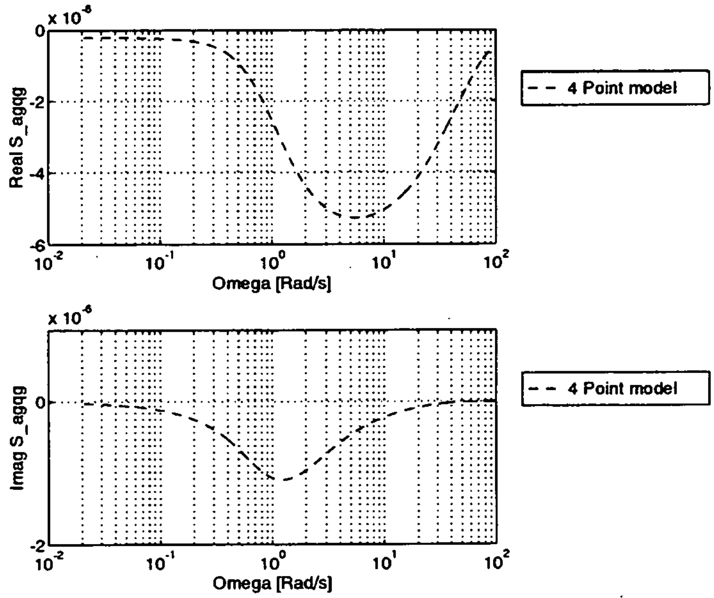

Figure 3.25: Input cross power spectral density functions of symmetrical gust $S_{\alpha_g \frac{q_g \bar{c}}{V}}$.

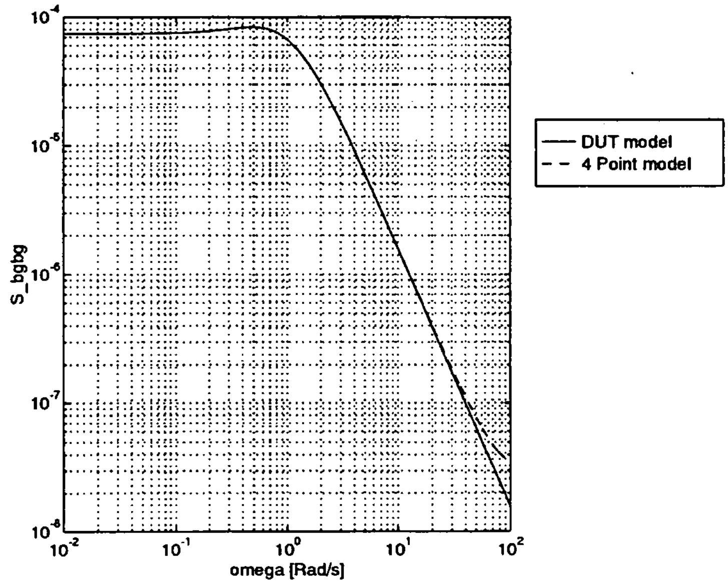

Figure 3.26: Input power spectral density of asymmetrical gust β_g.

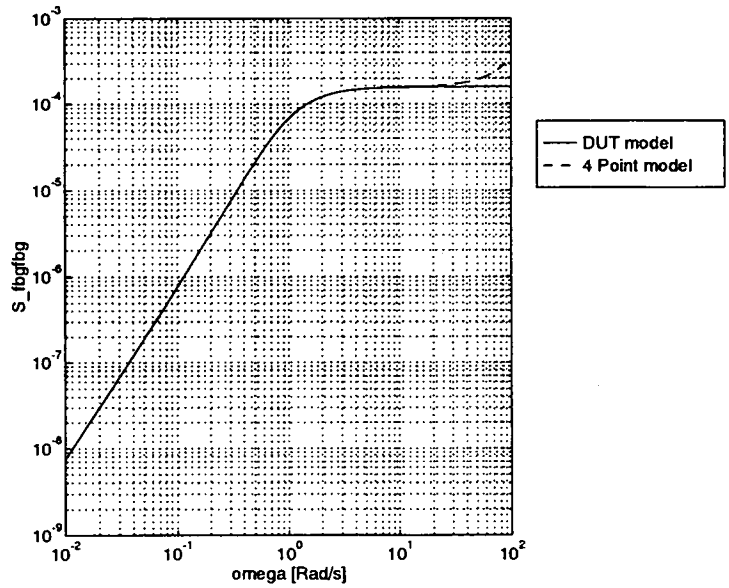

Figure 3.27: Input power spectral density of asymmetrical gust $\dot{\beta}_g$.

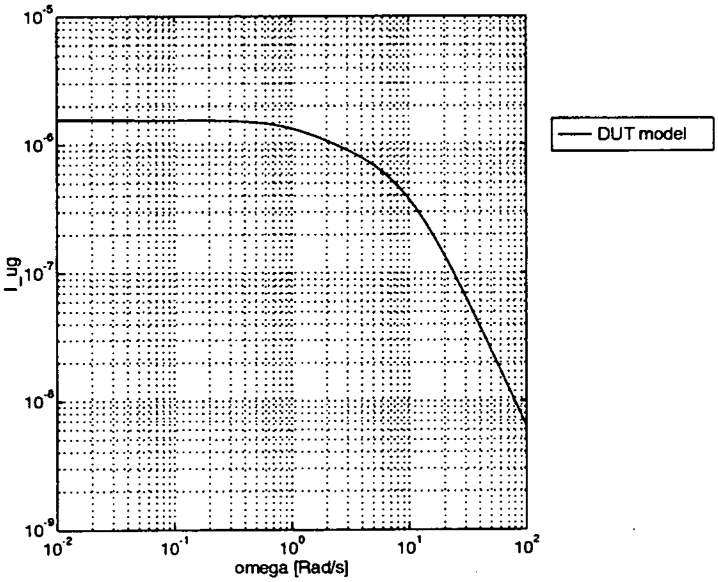

Figure 3.28: Input periodogram of asymmetrical gust $\hat{u}_g$.

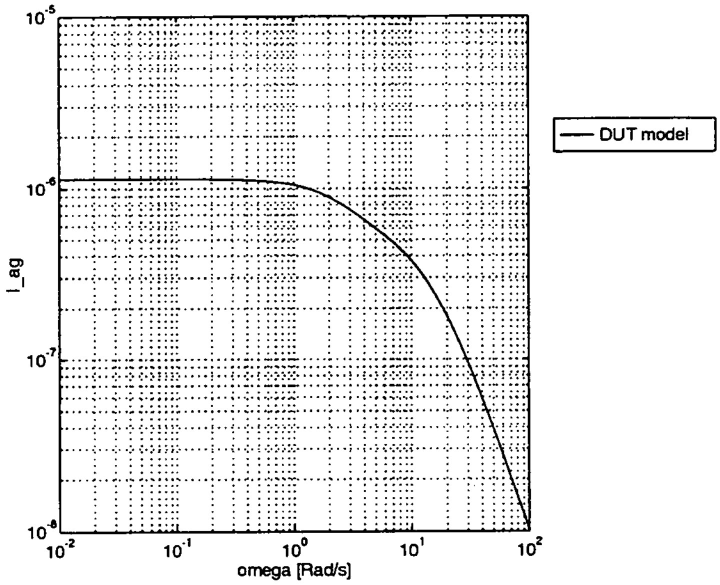

Figure 3.29: Input periodogram of asymmetrical gust α_g.

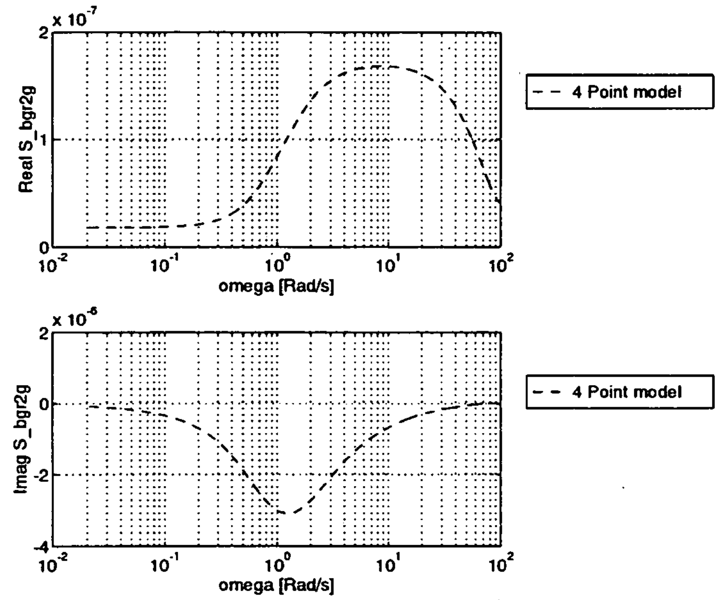

Figure 3.34: Input cross power spectral density functions $S_{\beta_g \frac{r_{2g} b}{2V}}$.

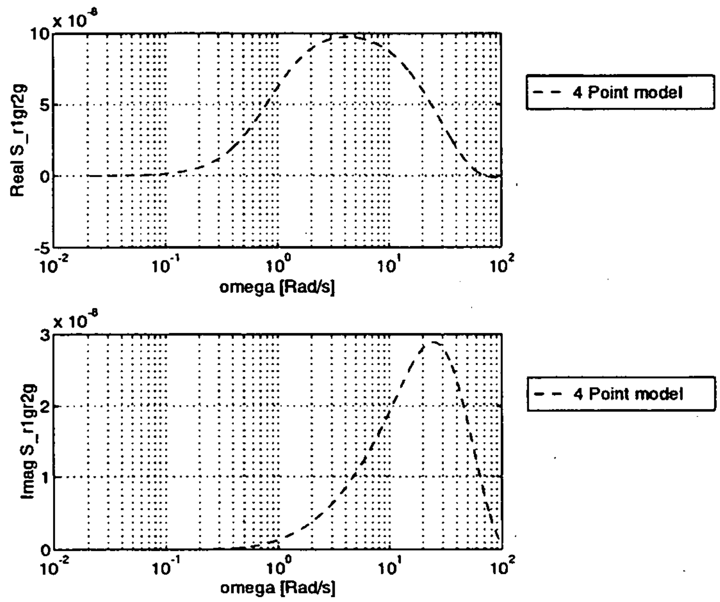

Figure 3.35: Input cross power spectral density functions $S_{\frac{r_{1g} b}{2V} \frac{r_{2g} b}{2V}}$.

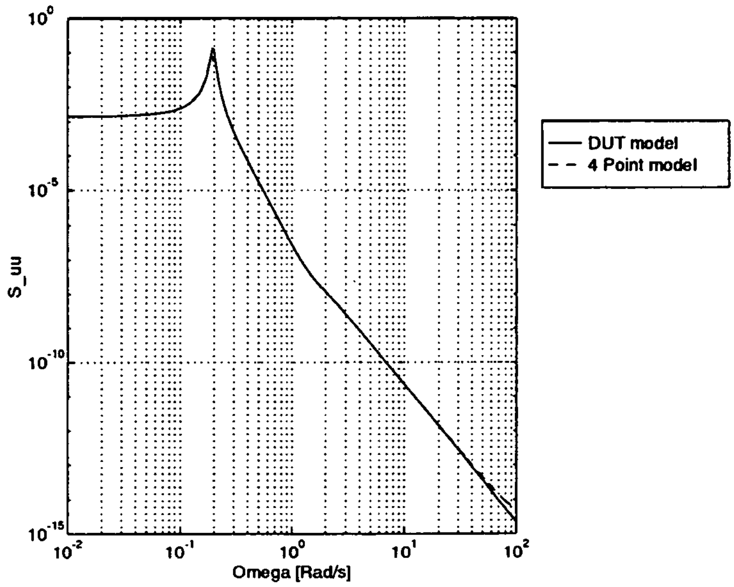

Figure 4.1: Power spectral density of $\hat{u}$ due to symmetrical gust u_g

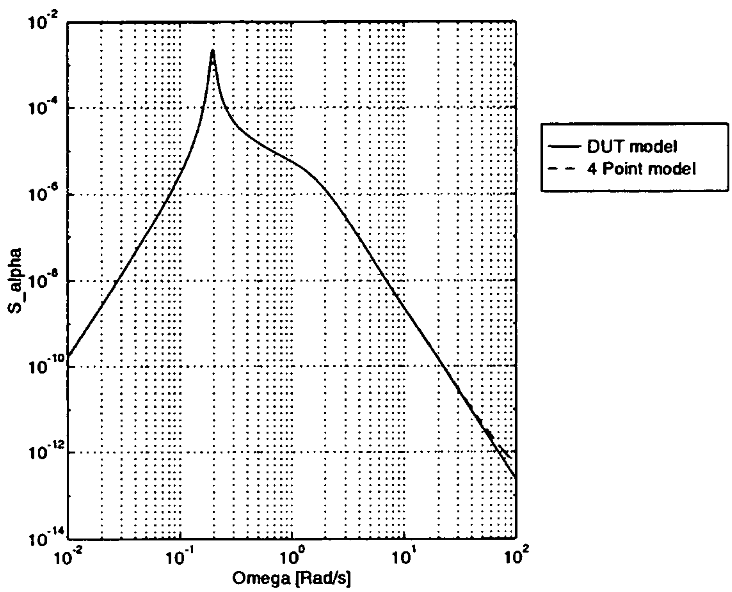

Figure 4.2: Power spectral density of α due to symmetrical gust u_g

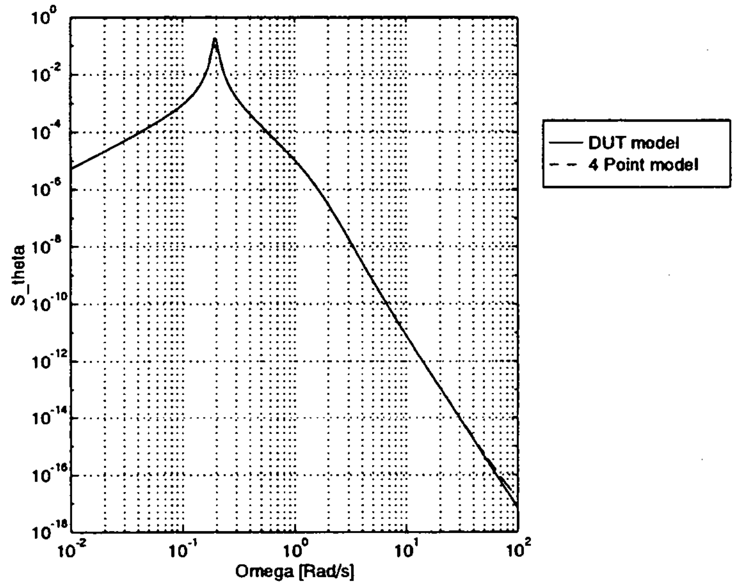

Figure 4.3: Power spectral density of θ due to symmetrical gust u_g

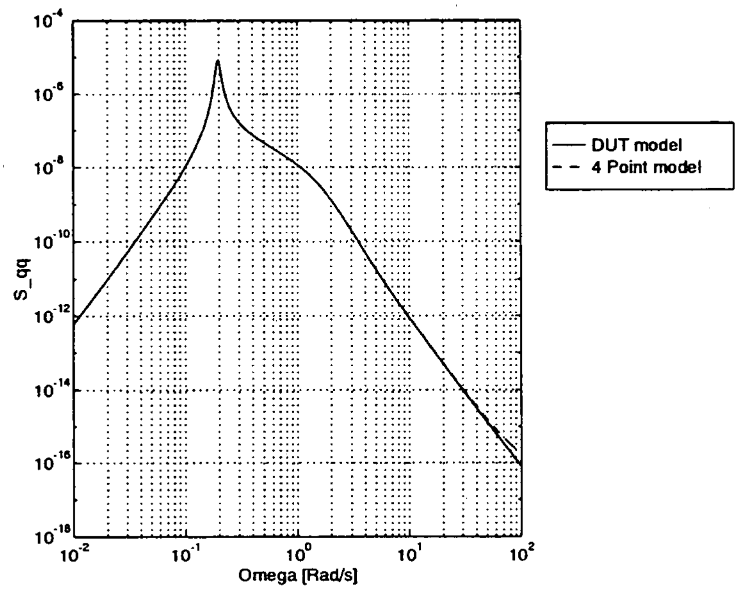

Figure 4.4: Power spectral density of $\frac{q\bar{c}}{V}$ due to symmetrical gust u_g

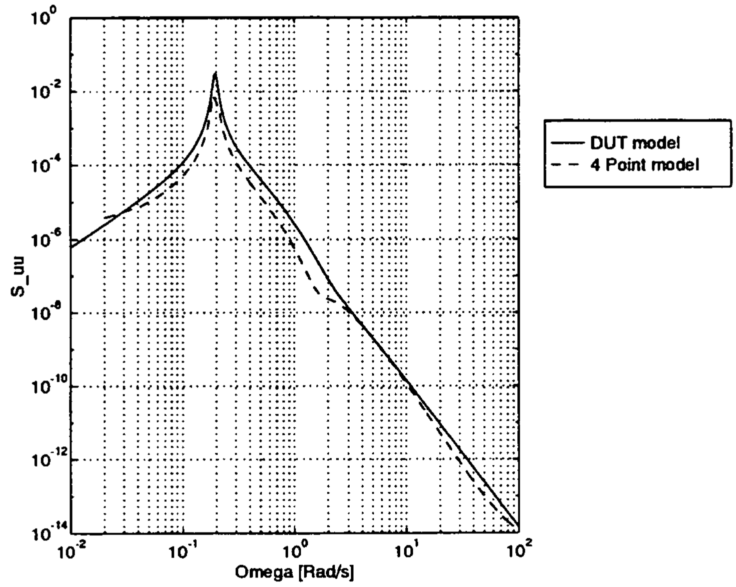

Figure 4.5: Power spectral density of $\hat{u}$ due to symmetrical gust α_g

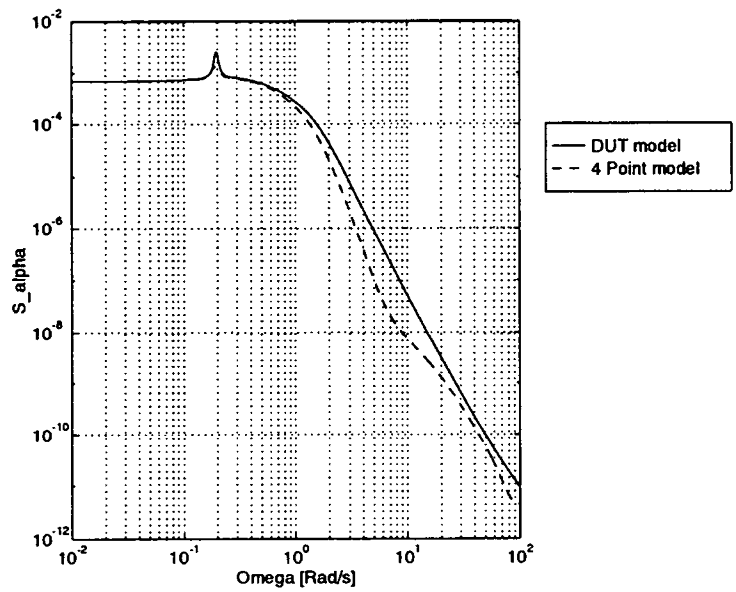

Figure 4.6: Power spectral density of α due to symmetrical gust α_g

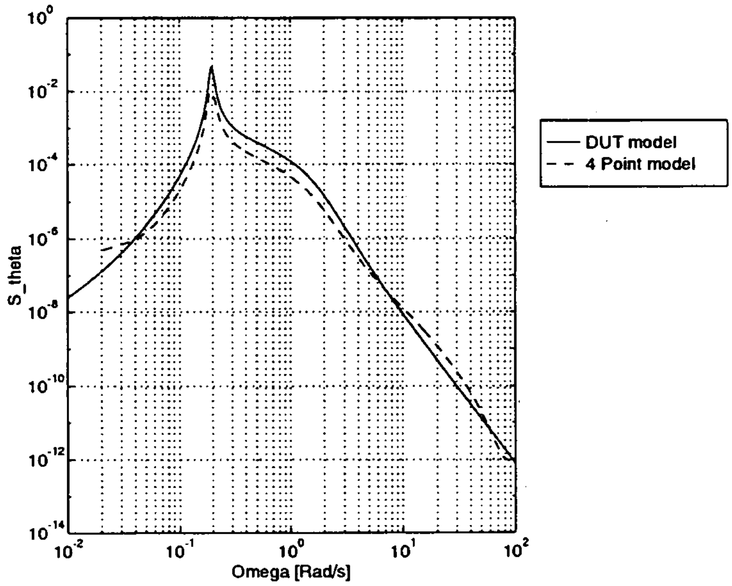

Figure 4.7: Power spectral density of θ due to symmetrical gust α_g

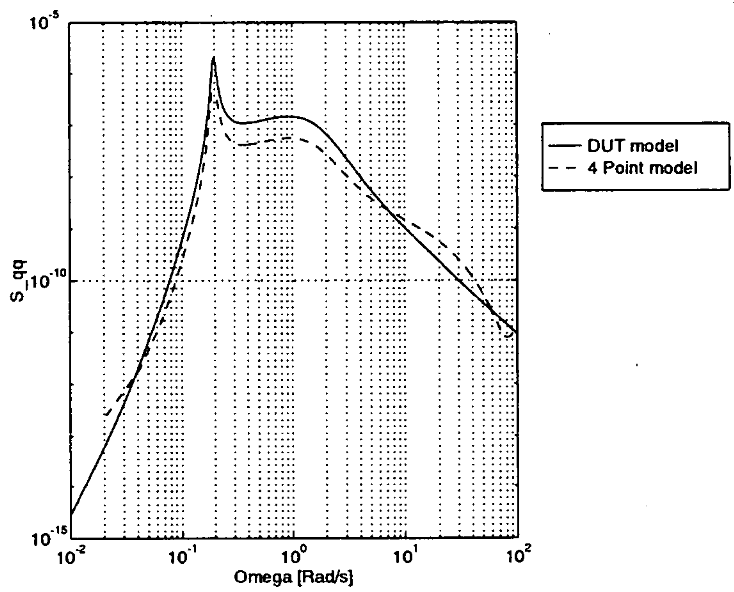

Figure 4.8: Power spectral density of $\frac{q\bar{c}}{V}$ due to symmetrical gust α_g

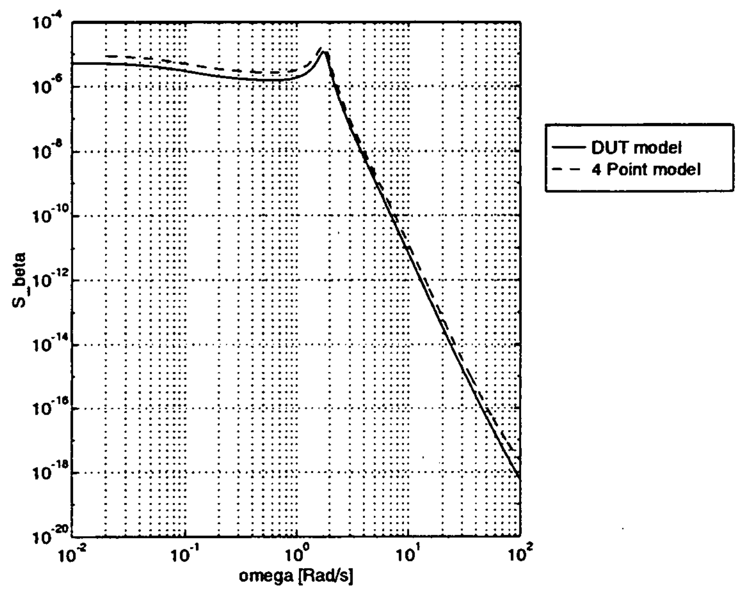

Figure 4.9: Power spectral density of β due to asymmetrical gust u_g

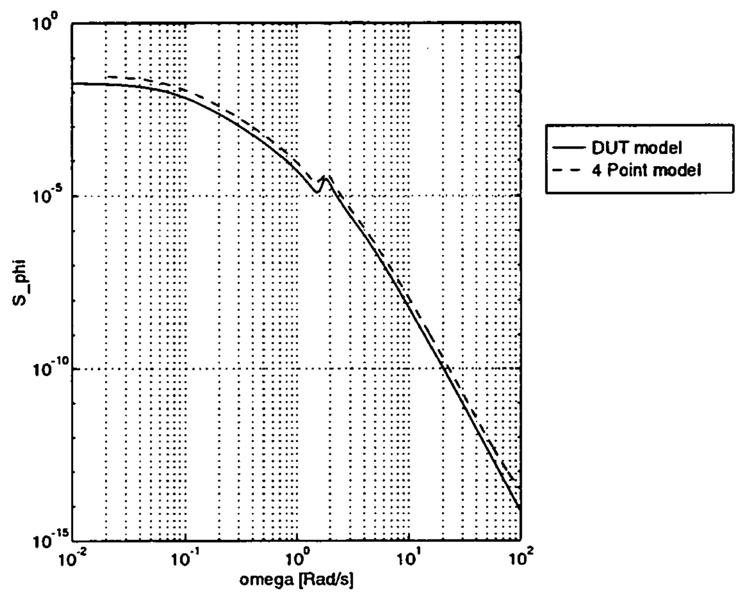

Figure 4.10: Power spectral density of ϕ due to asymmetrical gust u_g

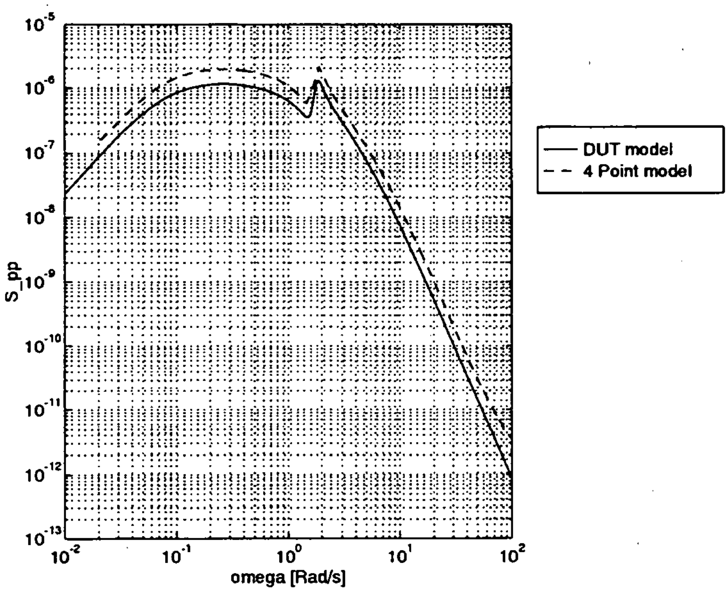

Figure 4.11: Power spectral density of $\frac{pb}{2V}$ due to asymmetrical gust u_g

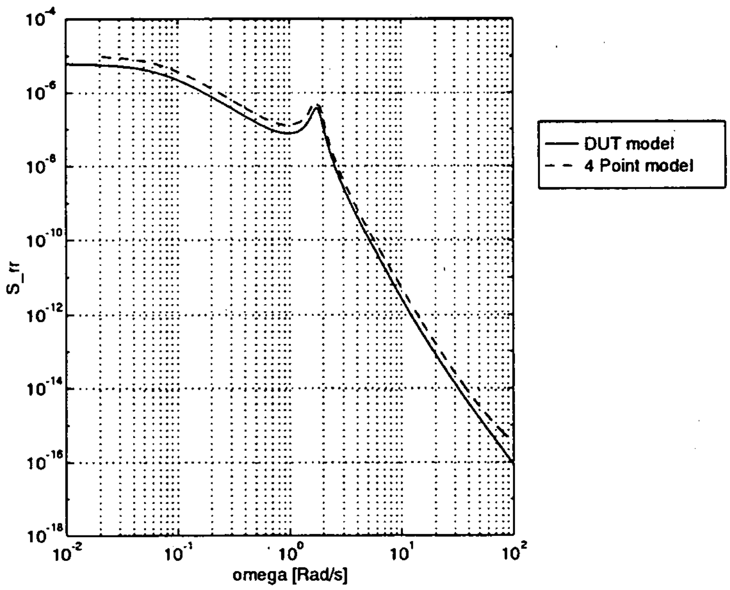

Figure 4.12: Power spectral density of $\frac{rb}{2V}$ due to asymmetrical gust u_g

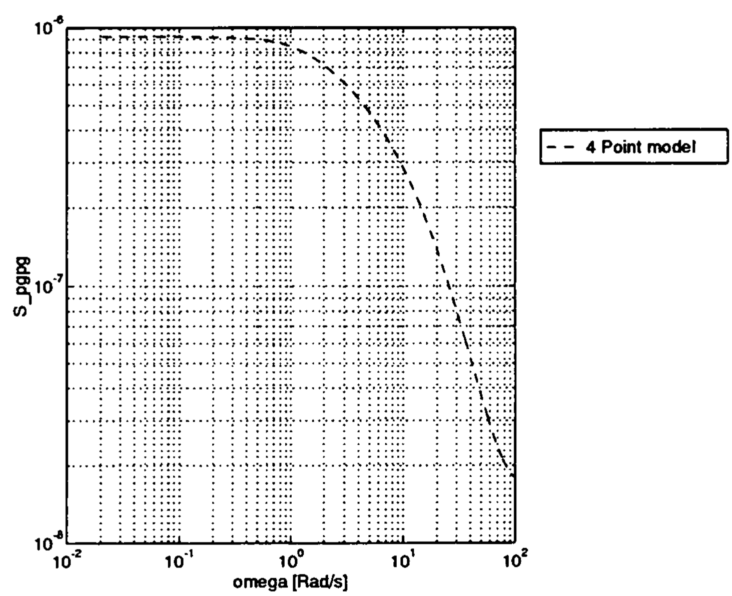

Figure 3.30: Input power spectral density functions of asymmetrical gust $\frac{p_g b}{2V}$.

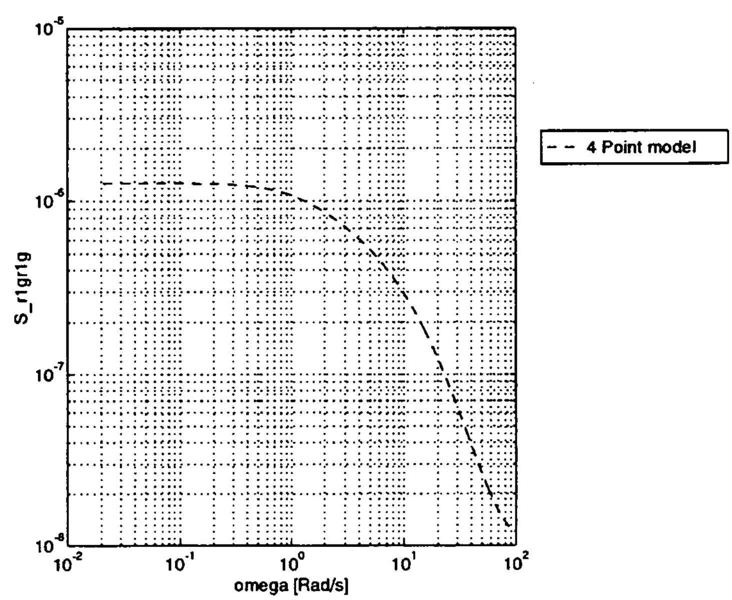

Figure 3.31: Input power spectral density functions of asymmetrical gust $\frac{r_{1_g} b}{2V}$.

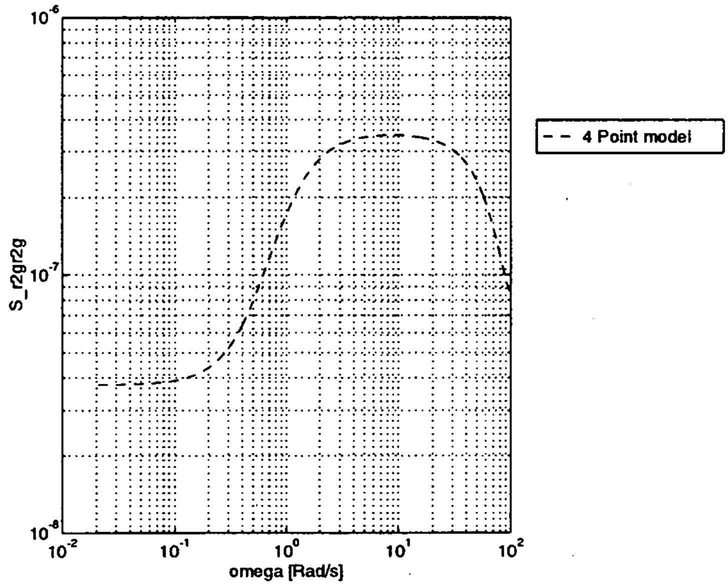

Figure 3.32: Input power spectral density functions of asymmetrical gust $\frac{r_{2_g} b}{2V}$.

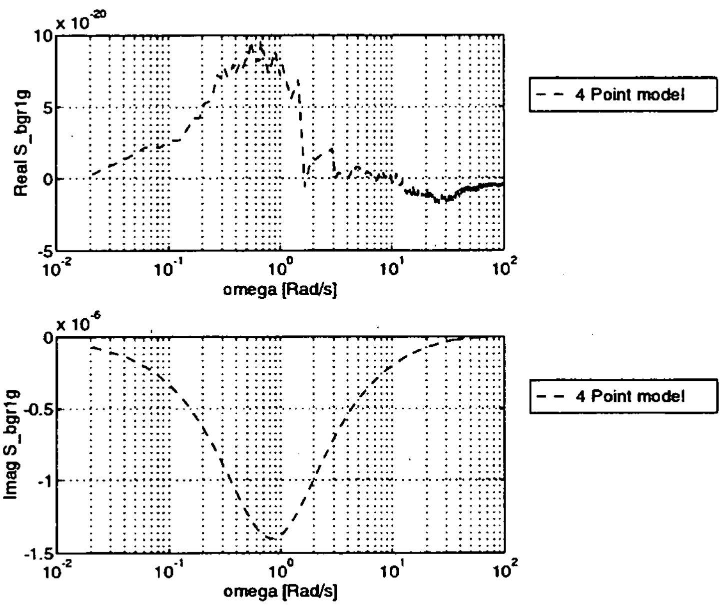

Figure 3.33: Input cross power spectral density functions $S_{\beta_g \frac{r_{1_g} b}{2V}}$.

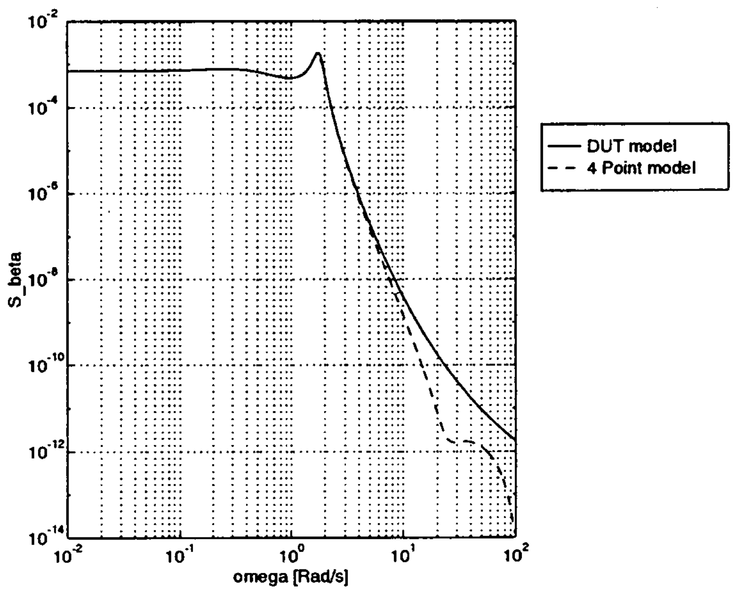

Figure 4.13: Power spectral density of β due to asymmetrical gust v_g

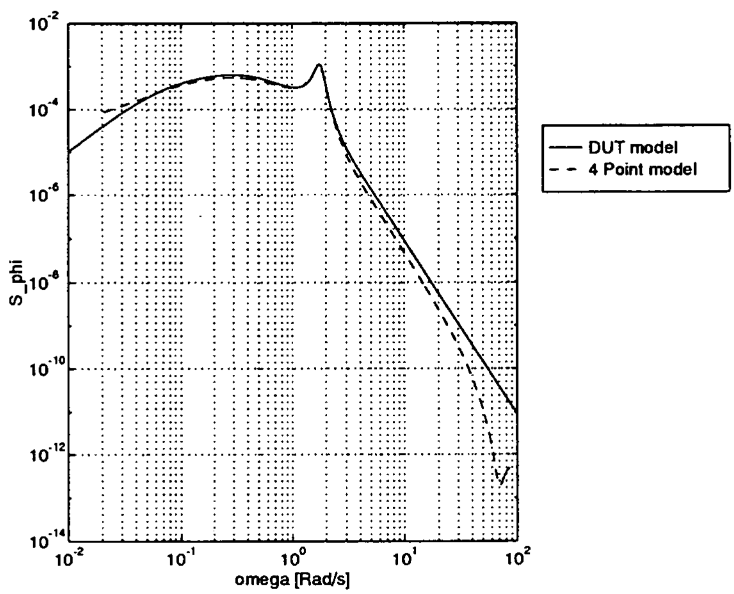

Figure 4.14: Power spectral density of ϕ due to asymmetrical gust v_g

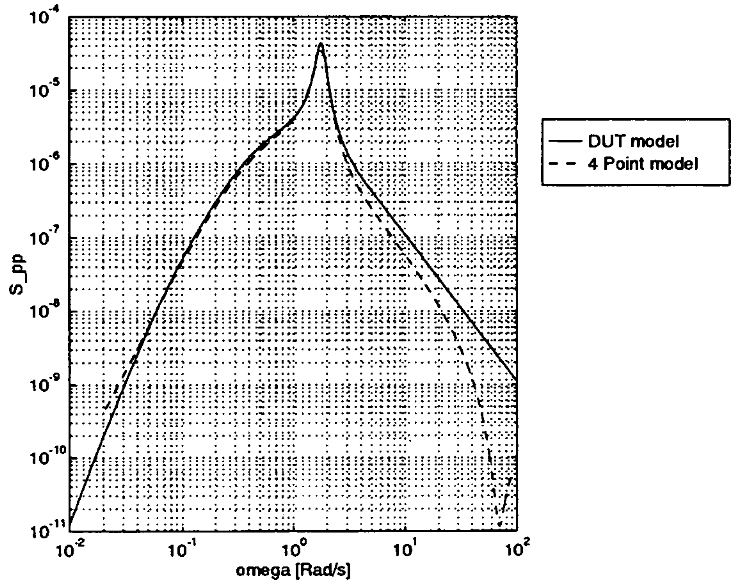

Figure 4.15: Power spectral density of $\frac{pb}{2V}$ due to asymmetrical gust v_g

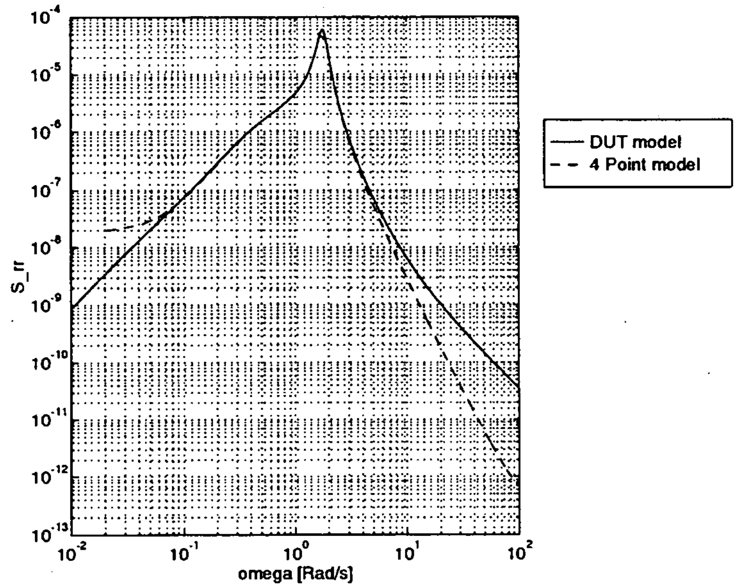

Figure 4.16: Power spectral density of $\frac{rb}{2V}$ due to asymmetrical gust v_g

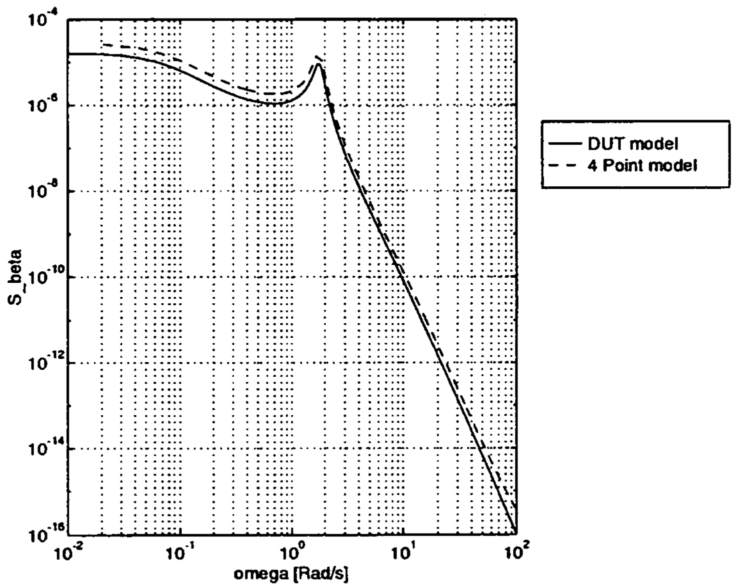

Figure 4.17: Power spectral density of β due to asymmetrical gust w_g

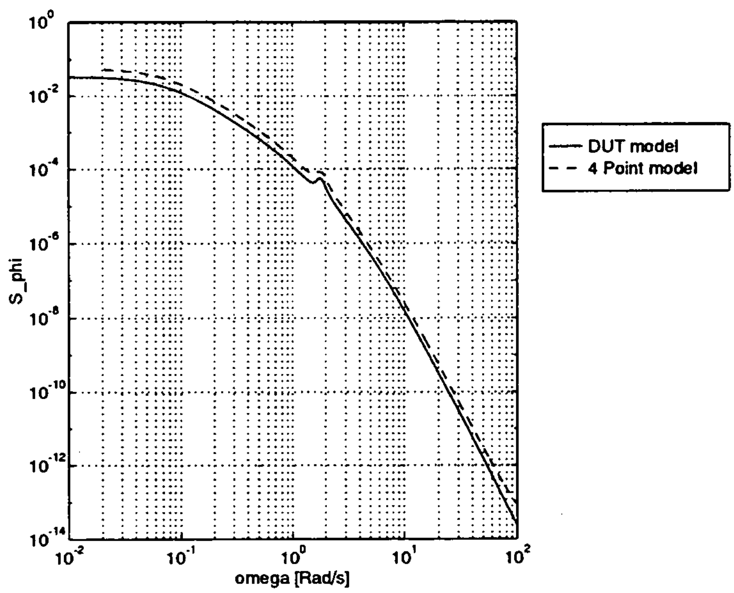

Figure 4.18: Power spectral density of ϕ due to asymmetrical gust w_g

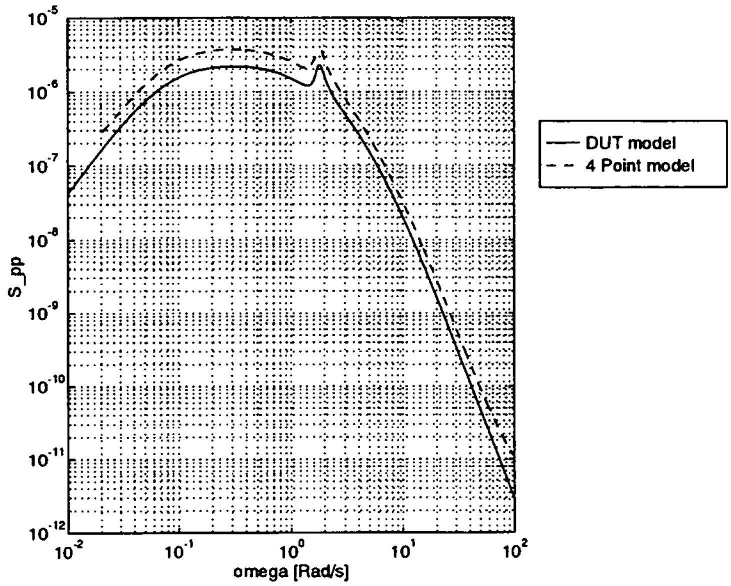

Figure 4.19: Power spectral density of $\frac{pb}{2V}$ due to asymmetrical gust w_g

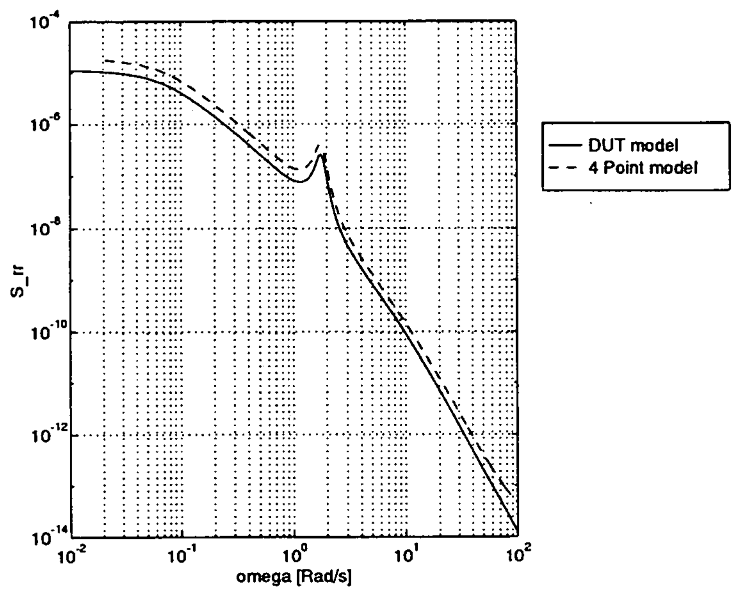

Figure 4.20: Power spectral density of $\frac{rb}{2V}$ due to asymmetrical gust w_g

Listings

```
% This program calculates the Correlation functions and Power Spectral
% Densities according to a 4 point model and using the Dryden correlation
% functions. This program is the same as turcor4.m except that the
% von Karman correlation functions and power spectral densities are
% omitted.

clear
clg


V      = 126.4;
Lg     = 150;
L      = Lg;
sigma  = 1;
sigma2 = sigma*sigma;


a      = 1.339;
b      = 0.85*13.36;   % m

lt     = 5.5;      % m
lf     = lt;
aL     = a*L;
bL     = b/L;
ltL    = lt/L;
lfL    = lf/L;



a1     = 2^(2/3);
a3     = 1/3;
a4     = 2/3;
b1     = gamma(a3);
a1b1   = a1/b1;

l1     = ((pi/2)/sin(a3*pi));
l2     = ((pi/2)/sin(a4*pi));

i      = 0;

dtau=0.03;

for tau=-150:dtau:150
    i=i+1
    TAU(i)=tau;
    xi1(i)=abs(V*tau);
    xi2(i)=sqrt((V*tau)^2 + (b/2)^2);
    xi3(i)=sqrt((V*tau)^2 + b^2);
    xi4(i)=abs(V*tau + lt);
    xi5(i)=abs(V*tau - lt);
    xi6(i)=abs(V*tau + lf);
    xi7(i)=abs(V*tau - lf);
    xi8(i)=sqrt((V*tau - lt)^2 + (b/2)^2);
    xi9(i)=sqrt((V*tau - lf)^2 + (b/2)^2);
%

% Specific functions f(xi) and g(xi) according to Dryden;
%
    fd1(i)=exp(-xi1(i)/L);
    gd1(i)=fd1(i)*(1-(xi1(i)/(2*L)));

    fd2(i)=exp(-xi2(i)/L);
    gd2(i)=fd2(i)*(1-(xi2(i)/(2*L)));

    fd3(i)=exp(-xi3(i)/L);
    gd3(i)=fd3(i)*(1-(xi3(i)/(2*L)));

    fd4(i)=exp(-xi4(i)/L);
    gd4(i)=fd4(i)*(1-(xi4(i)/(2*L)));
```



```
    fd5(i)=exp(-xi5(i)/L);
    gd5(i)=fd5(i)*(1-(xi5(i)/(2*L)));

    fd6(i)=exp(-xi6(i)/L);
    gd6(i)=fd6(i)*(1-(xi6(i)/(2*L)));

    fd7(i)=exp(-xi7(i)/L);
    gd7(i)=fd7(i)*(1-(xi7(i)/(2*L)));

    fd8(i)=exp(-xi8(i)/L);
    gd8(i)=fd8(i)*(1-(xi8(i)/(2*L)));

    fd9(i)=exp(-xi9(i)/L);
    gd9(i)=fd9(i)*(1-(xi9(i)/(2*L)));


    Vtau=V*tau;

    K1 = ((Vtau/L)^2) / ((Vtau/L)^2 + (b/L)^2);
    K2 = ((Vtau/L)*(b/(2*L))) / ((Vtau/L)^2 + (b/(2*L))^2);
    K3 = ((Vtau/L-lf/L)*(b/(2*L))) / (((Vtau/L)-lf/L)^2 + (b/(2*L))^2);

% Correlation functions according to Dryden

    Ruud(i)   = sigma2*fd1(i);
    Rvvd(i)   = sigma2*gd1(i);
    Rwwd(i)   = sigma2*((1/3)*gd1(i) + (4/9)*gd2(i) + (2/9)*gd3(i));
    Rqqd(i)   = (sigma2/lt^2)*(2*gd1(i) - gd4(i) - gd5(i));
    Rppd(i)   = (2*sigma2/b^2)*(gd1(i) - gd3(i));
    Rr1r1d(i) = (2*sigma2/b^2)*(fd1(i) - K1*fd3(i) - (1-K1)*gd3(i));
    Rr2r2d(i) = (sigma2/lf^2)*(2*gd1(i) - gd6(i) - gd7(i));
    Rwqd(i)   = (sigma2/(3*lt))*(gd5(i) -gd1(i) + 2*gd8(i) - 2*gd2(i));
    Rvr1d(i)  = (sigma2/b)*(2*K2*(gd2(i) - fd2(i)));
    Rvr2d(i)  = (sigma2/lf)*(gd1(i) - gd7(i));
    Rr1r2d(i) = (2*sigma2/(b*lf))*(K3*(gd9(i) - fd9(i)) + K2*(fd2(i) - gd2(i)));

end

disp('I am ready with the calculation of the Correlation functions')


R1 = [Ruud(5001:10001) Ruud(1:5000)];
R2 = [Rvvd(5001:10001) Rvvd(1:5000)];
R3 = [Rwwd(5001:10001) Rwwd(1:5000)];
R4 = [Rppd(5001:10001) Rppd(1:5000)];
R5 = [Rqqd(5001:10001) Rqqd(1:5000)];
R6 = [Rr1r1d(5001:10001) Rr1r1d(1:5000)];
R7 = [Rr2r2d(5001:10001) Rr2r2d(1:5000)];
R8 = [Rwqd(5001:10001) Rwqd(1:5000)];
R9 = [Rvr1d(5001:10001) Rvr1d(1:5000)];
R10= [Rvr2d(5001:10001) Rvr2d(1:5000)];
R11= [Rr1r2d(5001:10001) Rr1r2d(1:5000)];


Suud   = fft(R1)*dtau;
Svvd   = fft(R2)*dtau;
Swwd   = fft(R3)*dtau;
Sppd   = fft(R4)*dtau;
Sqqd   = fft(R5)*dtau;
Sr1r1d = fft(R6)*dtau;
Sr2r2d = fft(R7)*dtau;
Swqd   = fft(R8)*dtau;
Svr1d  = fft(R9)*dtau;
Svr2d  = fft(R10)*dtau;
Sr1r2d = fft(R11)*dtau;

N=length(Suud);
omega=2*pi*(0:5000)/(N*dtau);
```

```
save 1150V126.mat
```

```
clear

% INPUT TURBULENCE- AND AIRCRAFT PARAMETERS


%cit1a
cit2a

yug=0;
yag=0;

yfbg =(V/b)*CYFBG/(2*mub);
lfbg =(CLFBG*KZ2+CNFBG*KXZ)/den;
nfbg =(CLFBG*KXZ+CNFBG*KX2)/den;



% STATE- AND INPUT MATRICES
A=[yb yphi yp    yr
   0  0    2*V/b 0
   lb 0    lp    lr
   nb 0    np    nr ];



B=[yug ybg yag yfbg
   0   0   0   0
   lug lbg lag lfbg
   nug nbg nag nfbg];



om=logspace(-2,2,300);

wx=om;
wy=0;


numIug = 1+((wx*Lg).^2/V^2)+4*((wy*Lg).^2/V^2);
denIug = ( 1+((wx*Lg).^2/V^2)+((wy*Lg).^2/V^2) ).^(5/2);
%Iug    = (pi*sigmaug_V^2)*numIug./denIug;

numIag = 1+4*((wx*Lg).^2/V^2)+((wy*Lg).^2/V^2);
denIag = ( 1+((wx*Lg).^2/V^2)+((wy*Lg).^2/V^2) ).^(5/2);
%Iag    = (3*pi*sigmaag^2)*numIag./denIag;


numI  = 1+tau3^2*(om*Lg/V).^2;
denI1 = 1+tau1^2*(om*Lg/V).^2;
denI2 = 1+tau2^2*(om*Lg/V).^2;
Iug   = Iug0*numI./(denI1.*denI2);

numI  = 1+tau6^2*(om*Lg/V).^2;
denI1 = 1+tau4^2*(om*Lg/V).^2;
denI2 = 1+tau5^2*(om*Lg/V).^2;
Iag   = Iag0*numI./(denI1.*denI2);

numS    = 1+3*((Lg/V)^2)*om.^2;
denS    = (1+((Lg/V)^2)*om.^2).^2;
Svgvgd  = (Lg/V)*(sigmavg^2)*numS./denS;
Sbgbg   = Svgvgd/(V^2);
Sfbgfbg = Sbgbg.*(om.^2);

Cb   = [1 0 0 0];
Cphi = [0 1 0 0];
Cp   = [0 0 1 0];
Cr   = [0 0 0 1];
D    = [0 0 0 0];
```



```
Iug     = Iug';
Sbgbg   = Sbgbg';
Iag     = Iag';
Sfbgfbg = Sfbgfbg';

disp('Response to ug enter 1, vg enter 2 or wg enter 3')
ug = input(' Response to turbulence ;Enter 1, 2 or 3 :   ')

if ug==1,

C     = Cb;
Htemp = mv2fr(A,B,C,D,om,1);
Sbb   = Htemp.*conj(Htemp).*Iug;
C     = Cphi;
Htemp = mv2fr(A,B,C,D,om,1);
Sphi  = Htemp.*conj(Htemp).*Iug;
C     = Cp;
Htemp = mv2fr(A,B,C,D,om,1);
Spp   = Htemp.*conj(Htemp).*Iug;
C     = Cr;
Htemp = mv2fr(A,B,C,D,om,1);
Srr   = Htemp.*conj(Htemp).*Iug;


loglog(om,Sbb);title('Sbb');
pause
loglog(om,Sphi);title('Sphi');
pause
loglog(om,Spp);title('Spp');
pause
loglog(om,Srr);title('Srr');
pause


elseif ug==2

C      = Cb;
Htemp1 = mv2fr(A,B,C,D,om,2);
Htemp2 = mv2fr(A,B,C,D,om,4);
S1     = (Htemp1.*conj(Htemp2)).*(((-j*om')*(b/V)).*Sbgbg);
S2     = (Htemp2.*conj(Htemp1)).*(((j*om')*(b/V)).*Sbgbg);
S3     = (Htemp1.*conj(Htemp1)).*Sbgbg;
S4     = (Htemp2.*conj(Htemp2)).*(((b/V)^2)*Sfbgfbg);
S      = S1+S2+S3+S4;
Sbb    = S;


C      = Cphi;
Htemp1 = mv2fr(A,B,C,D,om,2);
Htemp2 = mv2fr(A,B,C,D,om,4);
S1     = (Htemp1.*conj(Htemp2)).*(((-j*om')*(b/V)).*Sbgbg);
S2     = (Htemp2.*conj(Htemp1)).*(((j*om')*(b/V)).*Sbgbg);
S3     = (Htemp1.*conj(Htemp1)).*Sbgbg;
S4     = (Htemp2.*conj(Htemp2)).*(((b/V)^2)*Sfbgfbg);
S      = S1+S2+S3+S4;
Sphi   = S;

C      = Cp;
Htemp1 = mv2fr(A,B,C,D,om,2);
Htemp2 = mv2fr(A,B,C,D,om,4);
S1     = (Htemp1.*conj(Htemp2)).*(((-j*om')*(b/V)).*Sbgbg);
S2     = (Htemp2.*conj(Htemp1)).*(((j*om')*(b/V)).*Sbgbg);
S3     = (Htemp1.*conj(Htemp1)).*Sbgbg;
S4     = (Htemp2.*conj(Htemp2)).*(((b/V)^2)*Sfbgfbg);
S      = S1+S2+S3+S4;
Spp    = S;

C      = Cr;
Htemp1 = mv2fr(A,B,C,D,om,2);
```

```
Htemp2 = mv2fr(A,B,C,D,om,4);
S1     = (Htemp1.*conj(Htemp2)).*(((-j*om')*(b/V)).*Sbgbg);
S2     = (Htemp2.*conj(Htemp1)).*(((j*om')*(b/V)).*Sbgbg);
S3     = (Htemp1.*conj(Htemp1)).*Sbgbg;
S4     = (Htemp2.*conj(Htemp2)).*(((b/V)^2)*Sfbgfbg);
S      = S1+S2+S3+S4;
Srr    = S;


loglog(om,Sbb);title('Sbb');
pause
loglog(om,Sphi);title('Sphi');
pause
loglog(om,Spp);title('Spp');
pause
loglog(om,Srr);title('Srr');
pause

else

C     = Cb;
Htemp = mv2fr(A,B,C,D,om,3);
Sbb   = Htemp.*conj(Htemp).*Iag;
C     = Cphi;
Htemp = mv2fr(A,B,C,D,om,3);
Sphi  = Htemp.*conj(Htemp).*Iag;
C     = Cp;
Htemp = mv2fr(A,B,C,D,om,3);
Spp   = Htemp.*conj(Htemp).*Iag;
C     = Cr;
Htemp = mv2fr(A,B,C,D,om,3);
Srr   = Htemp.*conj(Htemp).*Iag;



loglog(om,Sbb);title('Sbb');
pause
loglog(om,Sphi);title('Sphi');
pause
loglog(om,Spp);title('Spp');
pause
loglog(om,Srr);title('Srr');
pause


end
```

```
clear


%citls
cit2s

g=9.81;
sigmaug = sigma;
sigmawg = sigma;



% STATE- AND INPUT MATRICES
A=[xu xa xt 0
   zu za zt zq
   0  0  0  V/c
   mu ma mt mq   ];


B=[xd xag xfag
   zd zag zfag
   0  0   0
   md mag mfag];

B1=[xug;
    zug;
    0;
    mug];


Cn=(V/g)*[-zu -za -zt (V/c)-zq];
Dn=(V/g)*[-zd -zag -zfag];
Dn1=(V/g)*[-zug];



om=logspace(-2,2,300);

numS   = 1;
denS   = 1+((Lg/V)^2)*om.^2;
Sugugd = 2*(Lg/V)*(sigmaug^2)*numS./denS;
Sugug  = Sugugd/(V^2);

numS   = 1+3*((Lg/V)^2)*om.^2;
denS   = (1+((Lg/V)^2)*om.^2).^2;
Swgwgd = (Lg/V)*(sigmawg^2)*numS./denS;
Sagag  = Swgwgd/(V^2);

Sfagfag=Sagag.*(om.^2);

Cu = [1 0 0 0];
Ca = [0 1 0 0];
Ct = [0 0 1 0];
Cq = [0 0 0 1];
D  = [0 0 0];



ug = input(' Response to horizontal turbulence enter 1, else 0!   ');

if ug==1,

B = B1;
D = 0;

Sugug = Sugug';
C     = Cu;
Htemp = mv2fr(A,B,C,D,om,1);
Suu   = Htemp.*conj(Htemp).*Sugug;
```



```
C     = Ca;
Htemp = mv2fr(A,B,C,D,om,1);
Saa   = Htemp.*conj(Htemp).*Sugug;
C     = Ct;
Htemp = mv2fr(A,B,C,D,om,1);
Stt   = Htemp.*conj(Htemp).*Sugug;
C     = Cq;
Htemp = mv2fr(A,B,C,D,om,1);
Sqq   = Htemp.*conj(Htemp).*Sugug;
C     = Cn;
D     = Dn1;
Htemp = mv2fr(A,B,C,D,om,1);
Snn   = Htemp.*conj(Htemp).*Sugug;

loglog(om,Suu);title('Suu');
pause
loglog(om,Saa);title('Saa');
pause
loglog(om,Stt);title('Stt');
pause
loglog(om,Sqq);title('Sqq');
pause
loglog(om,Snn);title('Snn');
pause

else

Htemp1  = mv2fr(A,B,Cu,D,om,2);
Htemp2  = mv2fr(A,B,Cu,D,om,3);
H2temp1 = Htemp1.*conj(Htemp1);
H2temp2 = Htemp2.*conj(Htemp2);

S1  = (Htemp1.*conj(Htemp2)).*(-j*((om*c/V).*Sagag))';
S2  = (Htemp2.*conj(Htemp1)).*(j*((om*c/V).*Sagag))';
S3  = H2temp1.*Sagag';
S4  = ( (c/V)^2 )*H2temp2.*Sfagfag';
S   = S1+S2+S3+S4;
Suu = S;

Htemp1  = mv2fr(A,B,Ca,D,om,2);
Htemp2  = mv2fr(A,B,Ca,D,om,3);
H2temp1 = Htemp1.*conj(Htemp1);
H2temp2 = Htemp2.*conj(Htemp2);

S1  = (Htemp1.*conj(Htemp2)).*(-j*((om*c/V).*Sagag))';
S2  = (Htemp2.*conj(Htemp1)).*(j*((om*c/V).*Sagag))';
S3  = H2temp1.*Sagag';
S4  = ( (c/V)^2 )*H2temp2.*Sfagfag';
S   = S1+S2+S3+S4;
Saa = S;


Htemp1  = mv2fr(A,B,Ct,D,om,2);
Htemp2  = mv2fr(A,B,Ct,D,om,3);
H2temp1 = Htemp1.*conj(Htemp1);
H2temp2 = Htemp2.*conj(Htemp2);

S1  = (Htemp1.*conj(Htemp2)).*(-j*((om*c/V).*Sagag))';
S2  = (Htemp2.*conj(Htemp1)).*(j*((om*c/V).*Sagag))';
S3  = H2temp1.*Sagag';
S4  = ( (c/V)^2 )*H2temp2.*Sfagfag';
S   = S1+S2+S3+S4;
Stt = S;


Htemp1  = mv2fr(A,B,Cq,D,om,2);
Htemp2  = mv2fr(A,B,Cq,D,om,3);
H2temp1 = Htemp1.*conj(Htemp1);
H2temp2 = Htemp2.*conj(Htemp2);
```

```
S1  = (Htemp1.*conj(Htemp2)).*(-j*((om*c/V).*Sagag))';
S2  = (Htemp2.*conj(Htemp1)).*(j*((om*c/V).*Sagag))';
S3  = H2temp1.*Sagag';
S4  = ( (c/V)^2 )*H2temp2.*Sfagfag';
S   = S1+S2+S3+S4;
Sqq = S;


Htemp1  = mv2fr(A,B,Cn,Dn,om,2);
Htemp2  = mv2fr(A,B,Cn,Dn,om,3);
H2temp1 = Htemp1.*conj(Htemp1);
H2temp2 = Htemp2.*conj(Htemp2);

S1  = (Htemp1.*conj(Htemp2)).*(-j*((om*c/V).*Sagag))';
S2  = (Htemp2.*conj(Htemp1)).*(j*((om*c/V).*Sagag))';
S3  = H2temp1.*Sagag';
S4  = ( (c/V)^2 )*H2temp2.*Sfagfag';
S   = S1+S2+S3+S4;
Snn = S;


loglog(om,Suu);title('Suu');
pause
loglog(om,Saa);title('Saa');
pause
loglog(om,Stt);title('Stt');
pause
loglog(om,Sqq);title('Sqq');
pause
loglog(om,Snn);title('Snn');
pause


end
```

```
clear

%cit1a     % If used, change .MAT file to 1150V126
cit2a     % If used, change .MAT file to 1150V60


ypg = 0;
lpg = (CLPW*KZ2+CNPW*KXZ)/den;
npg = (CLPW*KXZ+CNPW*KX2)/den;


yr1 = 0;
lr1 = (-CLRW*KZ2-CNRW*KXZ)/den;
nr1 = (-CLRW*KXZ-CNRW*KX2)/den;

yr2 = (V/b)*CYR/(2*mub);
lr2 = (CLR*KZ2+CNR*KXZ)/den;
nr2 = (CLR*KXZ+CNR*KX2)/den;


yug = 0;
yag = 0;

yfb = (V/b)*CYFB/(2*mub);
lfb = (CLFB*KZ2+CNFB*KXZ)/den;
nfb = (CLFB*KXZ+CNFB*KX2)/den;


% STATE- AND INPUT MATRICES
A=[yb yphi yp    yr
   0  0    2*V/b 0
   lb 0    lp    lr
   nb 0    np    nr ];



B=[yb ypg yr1 yr2 yfb
   0  0   0   0   0
   lb lpg lr1 lr2 lfb
   nb npg nr1 nr2 nfb];

%load 1150V126
load 1150V60


Spgpg   = Sppd(1:5001)*( (b/(2*V))^2 );
Sr1gr1g = Sr1r1d(1:5001)*( (b/(2*V))^2 );
Sr2gr2g = Sr2r2d(1:5001)*( (b/(2*V))^2 );
Sbgbg   = Svvd(1:5001)/(V^2);
Sbgr1g  = Svr1d(1:5001)*( b/(2*(V^2)));
Sbgr2g  = Svr2d(1:5001)*( b/(2*(V^2)));
Sr1gr2g = Sr1r2d(1:5001)*( (b/(2*V))^2 );
Sfbgfbg = Sbgbg.*(w.^2);

Spgpg   = Spgpg';
Sr1gr1g = Sr1gr1g';
Sr2gr2g = Sr2gr2g';
Sbgbg   = Sbgbg';
Sbgr1g  = Sbgr1g';
Sbgr2g  = Sbgr2g';
Sr1gr2g = Sr1gr2g';
Sfbgfbg = Sfbgfbg';


Cb   = [1 0 0 0];
Cphi = [0 1 0 0];
Cp   = [0 0 1 0];
Cr   = [0 0 0 1];
D    = [0 0 0 0 0];
```



```
%-----------------------------------------------------------------
%
% Response to the different turbulence inputs
%
%-----------------------------------------------------------------



disp('Response to r1g (ug) enter 1, pg (wg) enter 2, vg (etkin) enter 3')
disp('or vg (dut) enter 4')
disp(' ');
ug = input(' Response to turbulence ;Enter 1, 2, 3 or 4 :   ')

om=w';

if ug==1,

%----------------------------------------------------------------
% Response to r1g (ug)
%----------------------------------------------------------------

C     = Cb;
Htemp = mv2fr(A,B,C,D,om,3);
Sbb   = Htemp.*conj(Htemp).*Sr1gr1g;
C     = Cphi;
Htemp = mv2fr(A,B,C,D,om,3);
Sphi  = Htemp.*conj(Htemp).*Sr1gr1g;
C     = Cp;
Htemp = mv2fr(A,B,C,D,om,3);
Spp   = Htemp.*conj(Htemp).*Sr1gr1g;
C     = Cr;
Htemp = mv2fr(A,B,C,D,om,3);
Srr   = Htemp.*conj(Htemp).*Sr1gr1g;


loglog(om,Sbb);
pause
loglog(om,Sphi);
pause
loglog(om,Spp);
pause
loglog(om,Srr);
pause

elseif ug==2

%---------------------------------------------------------
% Response to pg (wg)
%---------------------------------------------------------

C     = Cb;
Htemp = mv2fr(A,B,C,D,om,2);
Sbb   = Htemp.*conj(Htemp).*Spgpg;
C     = Cphi;
Htemp = mv2fr(A,B,C,D,om,2);
Sphi  = Htemp.*conj(Htemp).*Spgpg;
C     = Cp;
Htemp = mv2fr(A,B,C,D,om,2);
Spp   = Htemp.*conj(Htemp).*Spgpg;
C     = Cr;
Htemp = mv2fr(A,B,C,D,om,2);
Srr   = Htemp.*conj(Htemp).*Spgpg;


loglog(om,Sbb);
pause
loglog(om,Sphi);
```

```
pause
loglog(om,Spp);
pause
loglog(om,Srr);
pause

elseif ug==3

%-----------------------------------------------------------
% Response to vg (Etkin,bg)
%-----------------------------------------------------------

C       = Cb;
Htemp1 = mv2fr(A,B,C,D,om,1);
Htemp2 = mv2fr(A,B,C,D,om,5);
Htemp3 = mv2fr(A,B,C,D,om,3);
Htemp4 = mv2fr(A,B,C,D,om,4);
S1      = (Htemp1.*conj(Htemp2)).*(((-j*om)*(b/V)).*Sbgbg);
S2      = (Htemp1.*conj(Htemp3)).*Sbgr1g;
S3      = (Htemp1.*conj(Htemp4)).*Sbgr2g;
S4      = (Htemp2.*conj(Htemp1)).*(((j*om)*(b/V)).*conj(Sbgbg));
S5      = (Htemp2.*conj(Htemp3)).*(((j*om)*(b/V)).*Sbgr1g);
S6      = (Htemp2.*conj(Htemp4)).*(((j*om)*(b/V)).*Sbgr2g);
S7      = (Htemp3.*conj(Htemp1)).*conj(Sbgr1g);
S8      = (Htemp3.*conj(Htemp2)).*(((-j*om)*(b/V)).*conj(Sbgr1g));
S9      = (Htemp3.*conj(Htemp4)).*Sr1gr2g;
S10     = (Htemp4.*conj(Htemp1)).*conj(Sbgr2g);
S11     = (Htemp4.*conj(Htemp2)).*(((-j*om)*(b/V)).*conj(Sbgr2g));
S12     = (Htemp4.*conj(Htemp3)).*conj(Sr1gr2g);
S13     = (Htemp1.*conj(Htemp1)).*Sbgbg;
S14     = (Htemp2.*conj(Htemp2)).*(((b/V)^2)*Sfbgfbg);
S15     = (Htemp3.*conj(Htemp3)).*Sr1gr1g;
S16     = (Htemp4.*conj(Htemp4)).*Sr2gr2g;
S       = S1+S2+S3+S4+S5+S6+S7+S8+S9+S10+S11+S12+S13+S14+S15+S16;
Sbb     = S;
disp('ready with the calculation of Sbb')

Stemp1 = S2+S5+S7+S8+S9+S12+S15;


C       = Cphi;
Htemp1 = mv2fr(A,B,C,D,om,1);
Htemp2 = mv2fr(A,B,C,D,om,5);
Htemp3 = mv2fr(A,B,C,D,om,3);
Htemp4 = mv2fr(A,B,C,D,om,4);
S1      = (Htemp1.*conj(Htemp2)).*(((-j*om)*(b/V)).*Sbgbg);
S2      = (Htemp1.*conj(Htemp3)).*Sbgr1g;
S3      = (Htemp1.*conj(Htemp4)).*Sbgr2g;
S4      = (Htemp2.*conj(Htemp1)).*(((j*om)*(b/V)).*conj(Sbgbg));
S5      = (Htemp2.*conj(Htemp3)).*(((j*om)*(b/V)).*Sbgr1g);
S6      = (Htemp2.*conj(Htemp4)).*(((j*om)*(b/V)).*Sbgr2g);
S7      = (Htemp3.*conj(Htemp1)).*conj(Sbgr1g);
S8      = (Htemp3.*conj(Htemp2)).*(((-j*om)*(b/V)).*conj(Sbgr1g));
S9      = (Htemp3.*conj(Htemp4)).*Sr1gr2g;
S10     = (Htemp4.*conj(Htemp1)).*conj(Sbgr2g);
S11     = (Htemp4.*conj(Htemp2)).*(((-j*om)*(b/V)).*conj(Sbgr2g));
S12     = (Htemp4.*conj(Htemp3)).*conj(Sr1gr2g);
S13     = (Htemp1.*conj(Htemp1)).*Sbgbg;
S14     = (Htemp2.*conj(Htemp2)).*(((b/V)^2)*Sfbgfbg);
S15     = (Htemp3.*conj(Htemp3)).*Sr1gr1g;
S16     = (Htemp4.*conj(Htemp4)).*Sr2gr2g;
S       = S1+S2+S3+S4+S5+S6+S7+S8+S9+S10+S11+S12+S13+S14+S15+S16;
Sphi    = S;
disp('ready with the calculation of Sphi')

Stemp2 = S2+S5+S7+S8+S9+S12+S15;


C       = Cp;
```



```
Htemp1 = mv2fr(A,B,C,D,om,1);
Htemp2 = mv2fr(A,B,C,D,om,5);
Htemp3 = mv2fr(A,B,C,D,om,3);
Htemp4 = mv2fr(A,B,C,D,om,4);
S1      = (Htemp1.*conj(Htemp2)).*(((-j*om)*(b/V)).*Sbgbg);
S2      = (Htemp1.*conj(Htemp3)).*Sbgr1g;
S3      = (Htemp1.*conj(Htemp4)).*Sbgr2g;
S4      = (Htemp2.*conj(Htemp1)).*(((j*om)*(b/V)).*conj(Sbgbg));
S5      = (Htemp2.*conj(Htemp3)).*(((j*om)*(b/V)).*Sbgr1g);
S6      = (Htemp2.*conj(Htemp4)).*(((j*om)*(b/V)).*Sbgr2g);
S7      = (Htemp3.*conj(Htemp1)).*conj(Sbgr1g);
S8      = (Htemp3.*conj(Htemp2)).*(((-j*om)*(b/V)).*conj(Sbgr1g));
S9      = (Htemp3.*conj(Htemp4)).*Sr1gr2g;
S10     = (Htemp4.*conj(Htemp1)).*conj(Sbgr2g);
S11     = (Htemp4.*conj(Htemp2)).*(((-j*om)*(b/V)).*conj(Sbgr2g));
S12     = (Htemp4.*conj(Htemp3)).*conj(Sr1gr2g);
S13     = (Htemp1.*conj(Htemp1)).*Sbgbg;
S14     = (Htemp2.*conj(Htemp2)).*(((b/V)^2)*Sfbgfbg);
S15     = (Htemp3.*conj(Htemp3)).*Sr1gr1g;
S16     = (Htemp4.*conj(Htemp4)).*Sr2gr2g;
S       = S1+S2+S3+S4+S5+S6+S7+S8+S9+S10+S11+S12+S13+S14+S15+S16;
Spp     = S;
disp('ready with the calculation of Spp')

Stemp3 = S2+S5+S7+S8+S9+S12+S15;


C=Cr;
Htemp1 = mv2fr(A,B,C,D,om,1);
Htemp2 = mv2fr(A,B,C,D,om,5);
Htemp3 = mv2fr(A,B,C,D,om,3);
Htemp4 = mv2fr(A,B,C,D,om,4);
S1      = (Htemp1.*conj(Htemp2)).*(((-j*om)*(b/V)).*Sbgbg);
S2      = (Htemp1.*conj(Htemp3)).*Sbgr1g;
S3      = (Htemp1.*conj(Htemp4)).*Sbgr2g;
S4      = (Htemp2.*conj(Htemp1)).*(((j*om)*(b/V)).*conj(Sbgbg));
S5      = (Htemp2.*conj(Htemp3)).*(((j*om)*(b/V)).*Sbgr1g);
S6      = (Htemp2.*conj(Htemp4)).*(((j*om)*(b/V)).*Sbgr2g);
S7      = (Htemp3.*conj(Htemp1)).*conj(Sbgr1g);
S8      = (Htemp3.*conj(Htemp2)).*(((-j*om)*(b/V)).*conj(Sbgr1g));
S9      = (Htemp3.*conj(Htemp4)).*Sr1gr2g;
S10     = (Htemp4.*conj(Htemp1)).*conj(Sbgr2g);
S11     = (Htemp4.*conj(Htemp2)).*(((-j*om)*(b/V)).*conj(Sbgr2g));
S12     = (Htemp4.*conj(Htemp3)).*conj(Sr1gr2g);
S13     = (Htemp1.*conj(Htemp1)).*Sbgbg;
S14     = (Htemp2.*conj(Htemp2)).*(((b/V)^2)*Sfbgfbg);
S15     = (Htemp3.*conj(Htemp3)).*Sr1gr1g;
S16     = (Htemp4.*conj(Htemp4)).*Sr2gr2g;
S       = S1+S2+S3+S4+S5+S6+S7+S8+S9+S10+S11+S12+S13+S14+S15+S16;
Srr     = S;
disp('ready with the calculation of Srr')

Stemp4 = S2+S5+S7+S8+S9+S12+S15;


loglog(om,Sbb);
pause
loglog(om,Sphi);
pause
loglog(om,Spp);
pause
loglog(om,Srr);
pause


elseif ug==4

%-----------------------------------------------------------
% Response to vg (dut)
```

```
%-----------------------------------------------------------

C      = Cb;
Htemp1 = mv2fr(A,B,C,D,om,1);
Htemp2 = mv2fr(A,B,C,D,om,5);
Htemp3 = mv2fr(A,B,C,D,om,4);
S1     = (Htemp1.*conj(Htemp2)).*(((-j*om)*(b/V)).*Sbgbg);
S2     = (Htemp1.*conj(Htemp3)).*Sbgr2g;
S3     = (Htemp2.*conj(Htemp1)).*(((j*om)*(b/V)).*Sbgbg);
S4     = (Htemp2.*conj(Htemp3)).*(((j*om)*(b/V)).*Sbgr2g);
S5     = (Htemp3.*conj(Htemp1)).*conj(Sbgr2g);
S6     = (Htemp3.*conj(Htemp2)).*(((-j*om)*(b/V)).*conj(Sbgr2g));
S7     = (Htemp1.*conj(Htemp1)).*Sbgbg;
S8     = (Htemp2.*conj(Htemp2)).*(((b/V)^2)*Sfbgfbg);
S9     = (Htemp3.*conj(Htemp3)).*Sr2gr2g;
S      = S1+S2+S3+S4+S5+S6+S7+S8+S9;
Sbb    = S;
disp('ready with the calculation of Sbb')


C      = Cphi;
Htemp1 = mv2fr(A,B,C,D,om,1);
Htemp2 = mv2fr(A,B,C,D,om,5);
Htemp3 = mv2fr(A,B,C,D,om,4);
S1     = (Htemp1.*conj(Htemp2)).*(((-j*om)*(b/V)).*Sbgbg);
S2     = (Htemp1.*conj(Htemp3)).*Sbgr2g;
S3     = (Htemp2.*conj(Htemp1)).*(((j*om)*(b/V)).*conj(Sbgbg));
S4     = (Htemp2.*conj(Htemp3)).*(((j*om)*(b/V)).*Sbgr2g);
S5     = (Htemp3.*conj(Htemp1)).*conj(Sbgr2g);
S6     = (Htemp3.*conj(Htemp2)).*(((-j*om)*(b/V)).*conj(Sbgr2g));
S7     = (Htemp1.*conj(Htemp1)).*Sbgbg;
S8     = (Htemp2.*conj(Htemp2)).*(((b/V)^2)*Sfbgfbg);
S9     = (Htemp3.*conj(Htemp3)).*Sr2gr2g;
S      = S1+S2+S3+S4+S5+S6+S7+S8+S9;
Sphi   = S;
disp('ready with the calculation of Sphi')


C      = Cp;
Htemp1 = mv2fr(A,B,C,D,om,1);
Htemp2 = mv2fr(A,B,C,D,om,5);
Htemp3 = mv2fr(A,B,C,D,om,4);
S1     = (Htemp1.*conj(Htemp2)).*(((-j*om)*(b/V)).*Sbgbg);
S2     = (Htemp1.*conj(Htemp3)).*Sbgr2g;
S3     = (Htemp2.*conj(Htemp1)).*(((j*om)*(b/V)).*conj(Sbgbg));
S4     = (Htemp2.*conj(Htemp3)).*(((j*om)*(b/V)).*Sbgr2g);
S5     = (Htemp3.*conj(Htemp1)).*conj(Sbgr2g);
S6     = (Htemp3.*conj(Htemp2)).*(((-j*om)*(b/V)).*conj(Sbgr2g));
S7     = (Htemp1.*conj(Htemp1)).*Sbgbg;
S8     = (Htemp2.*conj(Htemp2)).*(((b/V)^2)*Sfbgfbg);
S9     = (Htemp3.*conj(Htemp3)).*Sr2gr2g;
S      = S1+S2+S3+S4+S5+S6+S7+S8+S9;
Spp    = S;
disp('ready with the calculation of Spp')


C      = Cr;
Htemp1 = mv2fr(A,B,C,D,om,1);
Htemp2 = mv2fr(A,B,C,D,om,5);
Htemp3 = mv2fr(A,B,C,D,om,4);
S1     = (Htemp1.*conj(Htemp2)).*(((-j*om)*(b/V)).*Sbgbg);
S2     = (Htemp1.*conj(Htemp3)).*Sbgr2g;
S3     = (Htemp2.*conj(Htemp1)).*(((j*om)*(b/V)).*conj(Sbgbg));
S4     = (Htemp2.*conj(Htemp3)).*(((j*om)*(b/V)).*Sbgr2g);
S5     = (Htemp3.*conj(Htemp1)).*conj(Sbgr2g);
S6     = (Htemp3.*conj(Htemp2)).*(((-j*om)*(b/V)).*conj(Sbgr2g));
S7     = (Htemp1.*conj(Htemp1)).*Sbgbg;
S8     = (Htemp2.*conj(Htemp2)).*(((b/V)^2)*Sfbgfbg);
S9     = (Htemp3.*conj(Htemp3)).*Sr2gr2g;
```



```
S      = S1+S2+S3+S4+S5+S6+S7+S8+S9;
Srr    = S;
disp('ready with the calculation of Srr')




loglog(om,Sbb);
pause
loglog(om,Sphi);
pause
loglog(om,Spp);
pause
loglog(om,Srr);
pause


end
```

```
clear

%cit1s   % If used, change .MAT file to 1150V126
cit2s   % If used, change .MAT file to 1150V60

g=9.81;


% CALCULATION OF AIRCRAFT SYMMETRIC STABILITY DERIVATIVES
zfa = (V/c)*(CZfa/(twmuc-CZfa));
mfa = (V/c)*((Cmfa+CZfa*Cmfa/(twmuc-CZfa))/(twmuc*KY2));
xqg = 0;
zqg = (V/c)*(CZq/(twmuc-CZfa));
mqg = (V/c)*((Cmq+Cmfa*CZq/(twmuc-CZfa))/(twmuc*KY2));

% STATE- AND INPUT MATRICES

A=[xu xa xt 0
   zu za zt zq
   0  0  0  V/c
   mu ma mt mq ];

B=[xu xa xqg 0
   zu za zqg zfa
   0  0  0  0
   mu ma mqg mfa];


Cn = (V/g)*[-zu -za -zt (V/c)-zq];
Dn = (V/g)*[-zu -za -zqg -zfa];


%load 1150V126
load 1150V60

w=omega;

Sugug   = Suud(1:5001)/(V^2);
Sagqg   = Swqd(1:5001)*(c/V^2);
Sagag   = Swwd(1:5001)/(V^2);
Sqgqg   = Sqqd(1:5001)*(c/V)^2;
Sfagfag = Sagag.*(w.^2);

N=5001;

Cu = [1 0 0 0];
Ca = [0 1 0 0];
Ct = [0 0 1 0];
Cq = [0 0 0 1];
D  = [0 0 0 0];


%------------------------------------------------------------------
%
% Response to ug put variable ug equal to 1 else to 0
%
%------------------------------------------------------------------


ug = input(' Response to horizontal turbulence enter 1, else 0!   ');

if ug==1,

Sugug = Sugug';

C     = Cu;
Htemp = mv2fr(A,B,C,D,w,1);
Suu   = Htemp.*conj(Htemp).*Sugug;
C     = Ca;
```



```
Htemp = mv2fr(A,B,C,D,w,1);
Saa   = Htemp.*conj(Htemp).*Sugug;
C     = Ct;
Htemp = mv2fr(A,B,C,D,w,1);
Stt   = Htemp.*conj(Htemp).*Sugug;
C     = Cq;
Htemp = mv2fr(A,B,C,D,w,1);
Sqq   = Htemp.*conj(Htemp).*Sugug;
C     = Cn;
D     = Dn;
Htemp = mv2fr(A,B,C,D,w,1);
Snn   = Htemp.*conj(Htemp).*Sugug;

loglog(w,Suu);
pause
loglog(w,Saa);
pause
loglog(w,Stt);
pause
loglog(w,Sqq);
pause
loglog(w,Snn);
pause

else



C      = Cu;
D      = D;
Htemp1 = mv2fr(A,B,C,D,w,2);
Htemp2 = mv2fr(A,B,C,D,w,3);
Htemp3 = mv2fr(A,B,C,D,w,4);

S1  = (Htemp1.*conj(Htemp2)).*Sagqg';
S2  = (Htemp1.*conj(Htemp3)).*((-j*w')*(c/V).*conj(Sagag)');
S3  = (Htemp2.*conj(Htemp1)).*conj(Sagqg');
S4  = (Htemp2.*conj(Htemp3)).*((-j*w')*(c/V).*conj(Sagqg'));
S5  = (Htemp3.*conj(Htemp1)).*((j*w')*(c/V).*Sagag');
S6  = (Htemp3.*conj(Htemp2)).*((j*w')*(c/V).*Sagqg');
S7  = Htemp1.*conj(Htemp1).*Sagag';
S8  = Htemp2.*conj(Htemp2).*Sqgqg';
S9  = Htemp3.*conj(Htemp3).*Sfagfag'.*(c/V)^2;
clear Htemp1 Htemp2 Htemp3

Suu = S1+S2+S3+S4+S5+S6+S7+S8+S9;

C      = Ca;
D      = D;
Htemp1 = mv2fr(A,B,C,D,w,2);
Htemp2 = mv2fr(A,B,C,D,w,3);
Htemp3 = mv2fr(A,B,C,D,w,4);

S1  = (Htemp1.*conj(Htemp2)).*Sagqg';
S2  = (Htemp1.*conj(Htemp3)).*((-j*w')*(c/V).*Sagag');
S3  = (Htemp2.*conj(Htemp1)).*conj(Sagqg');
S4  = (Htemp2.*conj(Htemp3)).*((-j*w')*(c/V).*conj(Sagqg'));
S5  = (Htemp3.*conj(Htemp1)).*((j*w')*(c/V).*conj(Sagag)');
S6  = (Htemp3.*conj(Htemp2)).*((j*w')*(c/V).*Sagqg');
S7  = Htemp1.*conj(Htemp1).*Sagag';
S8  = Htemp2.*conj(Htemp2).*Sqgqg';
S9  = Htemp3.*conj(Htemp3).*Sfagfag'.*(c/V)^2;
clear Htemp1 Htemp2 Htemp3

Saa = S1+S2+S3+S4+S5+S6+S7+S8+S9;

C      = Ct;
D      = D;
Htemp1 = mv2fr(A,B,C,D,w,2);
```

```
Htemp2 = mv2fr(A,B,C,D,w,3);
Htemp3 = mv2fr(A,B,C,D,w,4);

S1 = (Htemp1.*conj(Htemp2)).*Sagqg';
S2 = (Htemp1.*conj(Htemp3)).*((-j*w')*(c/V).*Sagag');
S3 = (Htemp2.*conj(Htemp1)).*conj(Saqqg');
S4 = (Htemp2.*conj(Htemp3)).*((-j*w')*(c/V).*conj(Saqqg'));
S5 = (Htemp3.*conj(Htemp1)).*((j*w')*(c/V).*conj(Sagag)');
S6 = (Htemp3.*conj(Htemp2)).*((j*w')*(c/V).*Saqqg');
S7 = Htemp1.*conj(Htemp1).*Sagag';
S8 = Htemp2.*conj(Htemp2).*Sqgqg';
S9 = Htemp3.*conj(Htemp3).*Sfagfag'.*(c/V)^2;
clear Htemp1 Htemp2 Htemp3

Stt=S1+S2+S3+S4+S5+S6+S7+S8+S9;


C      = Cq;
D      = D;
Htemp1 = mv2fr(A,B,C,D,w,2);
Htemp2 = mv2fr(A,B,C,D,w,3);
Htemp3 = mv2fr(A,B,C,D,w,4);

S1  = (Htemp1.*conj(Htemp2)).*Sagqg';
S2  = (Htemp1.*conj(Htemp3)).*((-j*w')*(c/V).*Sagag');
S3  = (Htemp2.*conj(Htemp1)).*conj(Sagqg');
S4  = (Htemp2.*conj(Htemp3)).*((-j*w')*(c/V).*conj(Sagqg'));
S5  = (Htemp3.*conj(Htemp1)).*((j*w')*(c/V).*conj(Sagag)');
S6  = (Htemp3.*conj(Htemp2)).*((j*w')*(c/V).*Sagqg');
S7  = Htemp1.*conj(Htemp1).*Sagag';
S8  = Htemp2.*conj(Htemp2).*Sqgqg';
S9  = Htemp3.*conj(Htemp3).*Sfagfag'.*(c/V)^2;
clear Htemp1 Htemp2 Htemp3

Sqq = S1+S2+S3+S4+S5+S6+S7+S8+S9;


C      = Cn;
D      = Dn;
Htemp1 = mv2fr(A,B,C,D,w,2);
Htemp2 = mv2fr(A,B,C,D,w,3);
Htemp3 = mv2fr(A,B,C,D,w,4);

S1  = (Htemp1.*conj(Htemp2)).*Sagqg';
S2  = (Htemp1.*conj(Htemp3)).*((-j*w')*(c/V).*Sagag');
S3  = (Htemp2.*conj(Htemp1)).*conj(Sagqg');
S4  = (Htemp2.*conj(Htemp3)).*((-j*w')*(c/V).*conj(Sagqg'));
S5  = (Htemp3.*conj(Htemp1)).*((j*w')*(c/V).*conj(Sagag)');
S6  = (Htemp3.*conj(Htemp2)).*((j*w')*(c/V).*Sagqg');
S7  = Htemp1.*conj(Htemp1).*Sagag';
S8  = Htemp2.*conj(Htemp2).*Sqgqg';
S9  = Htemp3.*conj(Htemp3).*Sfagfag'.*(c/V)^2;
clear Htemp1 Htemp2 Htemp3

Snn = S1+S2+S3+S4+S5+S6+S7+S8+S9;

loglog(w,Suu);
pause
loglog(w,Saa);
pause
loglog(w,Stt);
pause
loglog(w,Sqq);
pause
loglog(w,Snn);
pause


end
```