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Periodic event-triggered control with a relaxed triggering condition

Aleksandra Szymanek, Gabriel de A. Gleizer and Manuel Mazo Jr.

Abstract— In networked control systems (NCSs), extensive data exchange between plants and controllers leads to an unnecessary usage of communication and computational resources. Aperiodic sample-and-hold methods such as event-triggered control (ETC) can reduce the number of transmissions, allowing more applications to operate within the same network. However, most existing event-triggering mechanisms enforce a Lyapunov function of the continuous-time closed-loop system to be (almost) always decreasing. We propose a relaxed triggering condition for periodic event-triggered control (PETC) based on bounding the Lyapunov function with an exponentially decaying reference function, which reduces the communications while guaranteeing the same decay rate as competing strategies. We provide sufficient global exponential and input-to-state stability conditions for linear time-invariant (LTI) systems under our event-based state feedback, giving explicit performance guarantees in the presence of additive disturbances. Finally, some simulation results illustrate the performance of the proposed control strategy.

Index Terms— event-triggered control, networked control systems, state feedback, disturbances, input-to-state stability

I. INTRODUCTION

In computer controlled systems, periodic control is a standard choice for implementing a feedback controller [1]. An undoubtful advantage of this approach is the existence of reliable design and analysis methods, as well as guarantees on stability and performance. However, periodic execution of control tasks may lead to unnecessary usage of resources, especially in networked control systems (NCSs), where communication is severely constrained. To reduce the number of transmissions via network, one can instead implement the controller in an aperiodic way. Some early approaches to event-based control include [2]–[4]. Shortly after that, event-triggered control (ETC) was introduced [5], followed by self-triggered control (STC) [6]. Both strategies have been developed intensively since then [7]–[14]. Most of the ETC implementations assume continuous monitoring of the triggering condition, which is difficult to achieve in computer controlled systems, thus needing the discretization of a strategy designed for continuous time. Furthermore, as discussed in [15], event-based control can result in the undesired Zeno behavior. The above-mentioned reasons motivated the concept of periodic event-triggered control (PETC) [16], in which the triggering condition is only checked at periodic measurement times, resulting in a more practical approach with a minimum inter-event time by design.

Although ETC in general generates fewer transmissions than periodic control, there are significant differences in the number of events for specific ETC and STC implementations due to the different underlying triggering conditions. Most of them (e.g., [7], [8], [11], [16]) require that the Lyapunov function of the continuous-time closed-loop system has to be monotonically decreasing; once it stops decreasing, the control action is updated. However, this is not necessary to stabilize the system, usually leading to excessive updates. One approach that aims at relaxing the triggering condition is dynamic triggering [17], that requires the Lyapunov function to be decreasing only on average. For this, it introduces a dynamic variable that acts as a buffer and can balance out the increase of the Lyapunov function to some extent. This strategy was later adapted to PETC [18]. In general, dynamic triggering proves to be less conservative than the corresponding static event-generators. All of the aforementioned control strategies have exponential stability proved with a given convergence rate through LMIs. Due to the conservativeness involved in these LMI approaches, the actual convergence of the Lyapunov function is usually much faster than prescribed. Ideally, the triggering mechanism would lead to a decay rate as close as possible from prescribed, further saving communication resources by doing so.

We achieve this through a relaxed triggering condition for PETC based on bounding the evolution of a monitored Lyapunov function by an exponentially decaying signal. It can be regarded as a PETC implementation of the triggering mechanism for STC from [9], aiming at the same performance when the system is not subject to disturbances. To achieve this, our triggering mechanism performs a one-step-ahead prediction of the Lyapunov function to avoid an actual violation. A second condition, verifying if the Lyapunov function has overpassed the reference, is aggregated for the perturbed case. These two conditions, allied with the fact that it is a (reactive) ETC, result in improved disturbance rejection when compared to the STC from [9], which solely relies on multi-step disturbance-free state projections. We prove global exponential stability of the disturbance-free closed loop, and finite $\mathcal{L}_\infty$ gain from additive disturbance to state, providing formulas for the associated gains. Our PETC yields fewer events when compared to the existing PETC strategies and achieves better disturbance rejection than STC. Hence, we believe this is a competitive PETC strategy for application in resource-constrained NCSs.

A. Notation

$\mathbb{R}^+$ denotes the set of non-negative real numbers, while $\mathbb{R}_0^+$ the same set including 0. $\mathbb{N}$ is a set of natural numbers excluding 0 and $\mathbb{N}_0$ including 0. For a vector $x \in \mathbb{R}^n$, we denote by $|x| := \sqrt{x^T x}$ its 2-norm. For a matrix $A \in \mathbb{R}^{n \times m}$, we denote by A^T its transpose. For a symmetric square matrix $P \in \mathbb{R}^{n \times n}$, we write $P \succ 0$ ($P \succeq 0$) if P is positive (semi-)definite. By $\lambda_m(P)$ and $\lambda_M(P)$ we denote the minimum and maximum eigenvalue of P, respectively. Solutions of an autonomous system with state x and initial

condition x_0 are denoted by $x_{x_0}(t)$; if it has exogenous inputs u and δ, a trajectory is denoted by $x_{x_0 u\delta}(t)$. For a signal $\delta : \mathbb{R}_0^+ \to \mathbb{R}^{n_\delta}$, its $\mathcal{L}_\infty$-norm is denoted by $||\delta||_\infty := \text{ess sup}_t |\delta(t)|$. We say $\delta \in \mathcal{L}_\infty$ if $||\delta||_\infty < \infty$. A function $\beta : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to be a $\mathcal{K}$-function if it is continuous, strictly increasing and $\beta(0) = 0$. Also, it is said to be a $\mathcal{K}_\infty$-function if it is a $\mathcal{K}$-function and $\beta(s) \to \infty$ as $s \to \infty$. For any function $f : \mathbb{R}_0^+ \to \mathbb{R}^n$ and $t \geq 0$, we use $f(t^+)$ to denote the limit $f(t^+) = \lim_{s \to t, s > t} f(s)$.

II. PROBLEM STATEMENT

Consider an LTI system of the form

$$\dot{x}(t) = A^p x(t) + B^p \hat{u}(t) + E\delta(t), \tag{1}$$

where $x(t) \in \mathbb{R}^{n_x}$ denotes the states of the plant, $\hat{u}(t) \in \mathbb{R}^{n_u}$ is the control input and $\delta(t) \in \mathbb{R}^{n_\delta}$ is the disturbance vector. Matrices $A^p \in \mathbb{R}^{n_x \times n_x}$, $B^p \in \mathbb{R}^{n_x \times n_u}$ and $E \in \mathbb{R}^{n_x \times n_\delta}$ are known and $\delta(t) \in \mathcal{L}_\infty$. Moreover, we assume that an upper bound on $||\delta||_\infty$ is known. A state-feedback controller under PETC implementation is described by the following:

$$\hat{u}(t) = K\hat{x}(t), \quad t \in \mathbb{R}_0^+, \tag{2}$$

with $\hat{x}(t) \in \mathbb{R}^{n_x}$ being the last measurement available to the controller. This availability is dictated by a triggering condition $\mathcal{C} : \mathbb{R}^{2n_x} \to \mathbb{R}$, which is checked by an intelligent sensory system at every instant $t_k = k\Delta$, $k \in \mathbb{N}$, where Δ is some properly chosen sampling interval. The new measurement is sent to the controller only when this condition is enabled, which, for $t \in (t_k, t_{k+1}]$, can be written as

$$\hat{x}(t) = \begin{cases} x(t_k), & \text{when } \mathcal{C}(x(t_k), \hat{x}(t_k)) > 0, \\ \hat{x}(t_k), & \text{when } \mathcal{C}(x(t_k), \hat{x}(t_k)) \leq 0. \end{cases} \tag{3}$$

Matrix K is designed such that $A^p + B^p K$ is Hurwitz.[1] Thus, there exists a Lyapunov function $V(x) = x^T P x$ such that

$$(A^p + B^p K)^T P + P(A^p + B^p K) = -Q, \tag{4}$$

with some $Q \succ 0$, which specifies the decrease rate of V. We define the *decay rate* of the Lyapunov function as the largest $\lambda_0 \in \mathbb{R}^+$ that satisfies $V(x_{x_0}(t)) \leq V(x_0)e^{-\lambda_0 t}, \forall t \in \mathbb{R}_0^+, \forall x_0 \in \mathbb{R}^{n_x}$. For the continuous-time closed-loop system and given P and Q satisfying (4), this is $\lambda_0 = \lambda_m(P^{-1}Q)$.

The triggering condition is usually dependent on the values of x and $\hat{x}$, that can be put together in one vector $\xi := \begin{bmatrix} x^T & \hat{x}^T \end{bmatrix}^T$. Moreover, a significant number of existing triggering conditions can be expressed in the quadratic form $\mathcal{C}(\xi(t_k)) := \xi^T(t_k)Q\xi(t_k)$, $Q \in \mathbb{R}^{n_\xi \times n_\xi}$, which will also be adopted in this paper.

Before stating the objective of the paper, let us define the necessary stability and performance notions.

Definition 2.1 (GES): The system (1) is said to be globally exponentially stable (GES), if there exist $\sigma \in \mathbb{R}^+$ and $\rho \in \mathbb{R}^+$, such that for any $x(0) = x_0 \in \mathbb{R}^{n_x}$ and $\delta \equiv 0$ all corresponding solutions to (1) satisfy: $|x(t)| \leq \sigma |x_0| e^{-\rho t}$ for all $t \in \mathbb{R}_0^+$.

Definition 2.2 (EISS): The system (1) is said to be exponentially input-to-state stable (EISS), if there exist $\sigma \in \mathbb{R}^+$, $\rho \in \mathbb{R}^+$ and $\gamma \in \mathcal{K}_\infty$, such that for any $x(0) = x_0 \in \mathbb{R}^{n_x}$

and $\delta \in \mathcal{L}_\infty$ all corresponding solutions to (1) satisfy: $|x(t)| \leq \sigma |x_0| e^{-\rho t} + \gamma(||\delta||_\infty)$ for all $t \in \mathbb{R}_0^+$. Furthermore, if there is a $g < \infty$ satisfying $\gamma(d) \leq gd, \forall d \in \mathbb{R}^+$, we call g the $\mathcal{L}_\infty$ gain from disturbance to state.

The main objective of this paper is to design a triggering condition that minimizes the gap between the actual and the prescribed decay rate of the monitored Lyapunov function in order to reduce the number of control updates.

III. MAIN RESULTS

This section describes the relaxed triggering condition for PETC, which is the main contribution of this paper, giving guidelines for choosing the required parameters, and presenting stability and performance analyses. In particular, we prove the closed-loop system is GES and EISS.

A. Triggering condition

Unlike most of the existing triggering conditions, in our relaxed triggering an exponentially decaying function bounds the actual Lyapunov function: conceptually, the triggering condition is $x_{x_0}(t)^T P x_{x_0}(t) > x_0^T P x_0 e^{-\lambda t}$, with $0 < \lambda < \lambda_0$ being the desired convergence rate. However, some modifications are made for PETC implementability. First, an auxiliary discrete-time state $\eta : \mathbb{N} \to \mathbb{R}^{n_x}$ is introduced:

$$\eta(0) = P^{\frac{1}{2}} x(0), \tag{5a}$$

$$\eta(t_{k+1}) = e^{-0.5\lambda\Delta} \eta(t_k^+), \tag{5b}$$

$$\eta(t_i^+) = P^{\frac{1}{2}} x(t_i), \tag{5c}$$

where $\{t_i\}_{i \in \mathbb{N}}$ are the triggering times. The intention of the auxiliary variable is such that $\eta^T \eta$ imitates the desired convergence of the Lyapunov function. Denoting $\zeta := \begin{bmatrix} x^T & \hat{x}^T & \eta^T \end{bmatrix}^T$, the sequence of triggering times t_i is obtained from $t_0 = 0$ and

$$\begin{aligned} t_{i+1} = \inf\{t > t_i | \zeta(t)^T Q_1 \zeta(t) > 0 \quad \vee \\ \zeta(t)^T Q_2 \zeta(t) > 0 \quad \vee \\ t = t_i + N_{max}\Delta, t = k\Delta, k \in \mathbb{N}\}, \end{aligned} \tag{6}$$

where $N_{max}\Delta$ is a designed maximum inter-event time and

$$Q_1 := \begin{bmatrix} P & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -I \end{bmatrix},$$

$$Q_2 := \begin{bmatrix} A^T PA & A^T PBK & 0 \\ (BK)^T PA & (BK)^T PBK & 0 \\ 0 & 0 & -Ie^{-\lambda\Delta} \end{bmatrix},$$

$$A := e^{A^p \Delta}, \quad B := \int_0^\Delta e^{A^p s} B^p ds.$$

An intuitive explanation of this triggering condition is that $\zeta(t_k)^T Q_1 \zeta(t_k) > 0$ checks if, at the current sampling time t_k, the Lyapunov function of our interest is above the bound, while $\zeta(t_k)^T Q_2 \zeta(t_k) > 0$ checks if this will happen at the next sampling time t_{k+1}. If any of these two conditions is true, the mechanism triggers. It also triggers if neither is true, but the maximum inter-event time is reached. In the disturbance-free case, the one-step-ahead prediction from the second triggering condition prevents the Lyapunov function from exceeding the bound, which improves performance.

The proposed triggering condition has three design parameters: the sampling interval Δ, a desired Lyapunov-function decay rate λ, and the maximum inter-event number of steps N_{max}. As already mentioned, λ must be smaller than the decay rate of the continuous-time closed-loop λ_0. This ensures a minimum inter-event time, which, in turn, influences the choice for Δ. This minimum inter-event time $\tau_{min} > 0$ is given by [9, Lemma 4.1]:

$$\tau_{min} = \min\{\tau \in \mathbb{R}^+ : \det M(\tau) = 0\}, \qquad (7)$$

where

$$M(\tau) := C(e^{F^T \tau} C^T P C e^{F\tau} - C^T P C e^{-\lambda \tau})C^T,$$

$$F := \begin{bmatrix} A^p + B^p K & B^p K \\ -A^p - B^p K & -B^p K \end{bmatrix}, \quad C := \begin{bmatrix} I & 0 \end{bmatrix}.$$

Once the minimum inter-event time is computed, the sampling time Δ has to be chosen such that $\Delta < \tau_{min}$. These guidelines for choosing λ and Δ are the necessary conditions for stability of the system under our PETC implementation, and N_{max} is essential for performance, as will be seen next.

B. Stability analysis

We start the stability analysis by proving GES of the PETC system with our relaxed triggering condition in the absence of disturbances. Let us define:

$$g(\Delta, N_{max}) := e^{\frac{\omega \mu \Delta}{\mu - \omega}} \left(e^{\lambda \Delta} + e^{\lambda N_{max} \Delta}\right)^{\frac{\mu}{\mu - \omega}} \left(1 + e^{\lambda N_{max} \Delta}\right)^{\frac{-\omega}{\mu - \omega}},$$

$$G = \begin{bmatrix} P^{\frac{1}{2}} A^p P^{-\frac{1}{2}} + (P^{\frac{1}{2}} A^p P^{-\frac{1}{2}})^T & P^{\frac{1}{2}} B^p K P^{-\frac{1}{2}} \\ (P^{\frac{1}{2}} B^p K P^{-\frac{1}{2}})^T & 0 \end{bmatrix}, \quad (8)$$

$$\mu = \lambda_m(G), \quad \omega = \lambda_M(G).$$

Theorem 3.1: If $\lambda < \lambda_0$ and $\Delta < \tau_{min}$, the sequence of control updates times given by (6) renders the closed loop system (1) GES with

$$\sigma = \left(\frac{\lambda_M(P)}{\lambda_m(P)}\right)^{\frac{1}{2}} \left(g(\Delta, N_{max})\right)^{\frac{1}{2}} \quad \text{and} \quad \rho = \frac{1}{2}\lambda.$$

In the presence of additive bounded disturbances we give performance guarantees in the following theorem.

Theorem 3.2: If $\lambda < \lambda_0$, $\Delta < \tau_{min}$ and $\delta \in \mathcal{L}_\infty$, the sequence of control update times given by (6) yields the closed loop system (1) EISS with

$$\gamma(||\delta||_\infty) = \frac{\lambda_M(P)}{\lambda_m(P)} \int_0^\Delta |e^{A^p r} E| dr \left(\frac{g(\Delta, N_{max})^{\frac{1}{2}}}{1 - e^{-\frac{1}{2}\lambda \Delta}} + 1\right) ||\delta||_\infty.$$

The proofs of the two theorems given in this section can be found in the Appendix.

IV. NUMERICAL EXAMPLE

To illustrate how the relaxed triggering condition reduces the number of communication instants we compare our approach with triggering conditions (7) and (14) from [16], where the monitored Lyapunov function is required to be decreasing from sample to sample, and with dynamic PETC [18]. As an example we take the plant from [19]:

$$\frac{d}{dt}x(t) = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \delta(t).$$

We set $x_0 = \begin{bmatrix} 10 & 0 \end{bmatrix}^T$ as initial condition and $K = \begin{bmatrix} 1 & -4 \end{bmatrix}$. The associated Lyapunov function and matrix Q satisfying (4) were chosen to be $P = \begin{bmatrix} 1 & 0.25 \\ 0.25 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 0.5 & 0.25 \\ 0.25 & 1.5 \end{bmatrix}$, which yield $\lambda_0 \approx 0.4836$. We chose the desired decay rate $\lambda = 0.3$, thus $\rho = 0.15$. The corresponding minimum inter-event time according to Lemma 1.1 is $\tau_{min} = 0.3$, and so we set $\Delta = 0.05 \le \tau_{min}$. The maximum inter-event time was taken as 2 seconds, hence $N_{max} = 40$. Using Theorems 3.1 and 3.2, the EISS parameters according to Definition 2.2 are $\sigma = 2.8974$ and $\gamma(d) = gd, g = 18.4483$.

For the triggering conditions (7) and (14) from [16] parameters σ, β, respectively, were chosen to minimise the number of events, while still ensuring the LMIs presented therein to be feasible, with $\rho = 0.3$. The values we found are $\sigma = 0.16$ and $\beta = 0.95$. Similar approach was applied in finding σ for dynamic PETC, where $\sigma = 0.15$ was found. Additionally, the value of acceptable $\mathcal{L}_2$ gain (θ) from disturbance to state had to be chosen. For a fair comparison we set it to $\theta = 20$, such that it is of comparable order of magnitude to the $\mathcal{L}_\infty$ gain resulting from our PETC.

Figure 1 shows the evolution of the monitored Lyapunov function for the four triggering conditions, with $\delta \equiv 0$. Additionally, in Fig. 1(d) we show how our imposed bound on the Lyapunov function $\eta^T \eta$ evolves in order to give an intuition on how our relaxed triggering condition works. Table I presents a comparison among the number of events from all four triggering conditions, for other desired convergence rates and $\Delta = 0.05$. Statistics on the number of communications for each case were computed based on 10 simulations where initial conditions were varied, pseudo-randomly uniformly sampled such that both state components ranged from -10 to 10. For all triggering conditions, the parameters were chosen such that the number of transmissions was small as possible. For all cases, the relaxed triggering condition yielded the fewest number of communications. The performance difference is bigger the faster the desired convergence is. Moreover, our triggering condition results in lower standard deviation compared to the PETC implementations with second and third best number of communications.

A possible disadvantage of the relaxed triggering condition are the oscillations of the monitored Lyapunov function, as can be seen in Figure 1(d); they corresponded to oscillations in the state trajectories, which is generally undesired. One possible reason is that the original controller is designed for a faster convergence rate than the one imposed by the triggering condition. Setting $\lambda = 0.45$ ($\rho = 0.225$), which is very close to λ_0, reduced the oscillations to some extent.

Figure 2(a) illustrates how the relaxed PETC deals with disturbances. Here, ρ was set to 0.225 and disturbance $\delta(t) = \sin(t)$ was present throughout the whole simulation. The total number of events in this case was 52, which was mostly because the disturbance started to dominate the dynamics at the final part of the simulation. Dynamic PETC (Fig. 2(b)) seems to be more robust, sampling-wise, at these regions, having had a total of 29 events. Bursts of events when disturbances are present and states approach the origin are common for static event-triggering mechanisms (ETMs) [13]. Our relaxed PETC can be consider as a static ETM

(a) Triggering condition (7) from [16]

(b) Triggering condition (14) from [16]

(c) Dynamic PETC from [18]
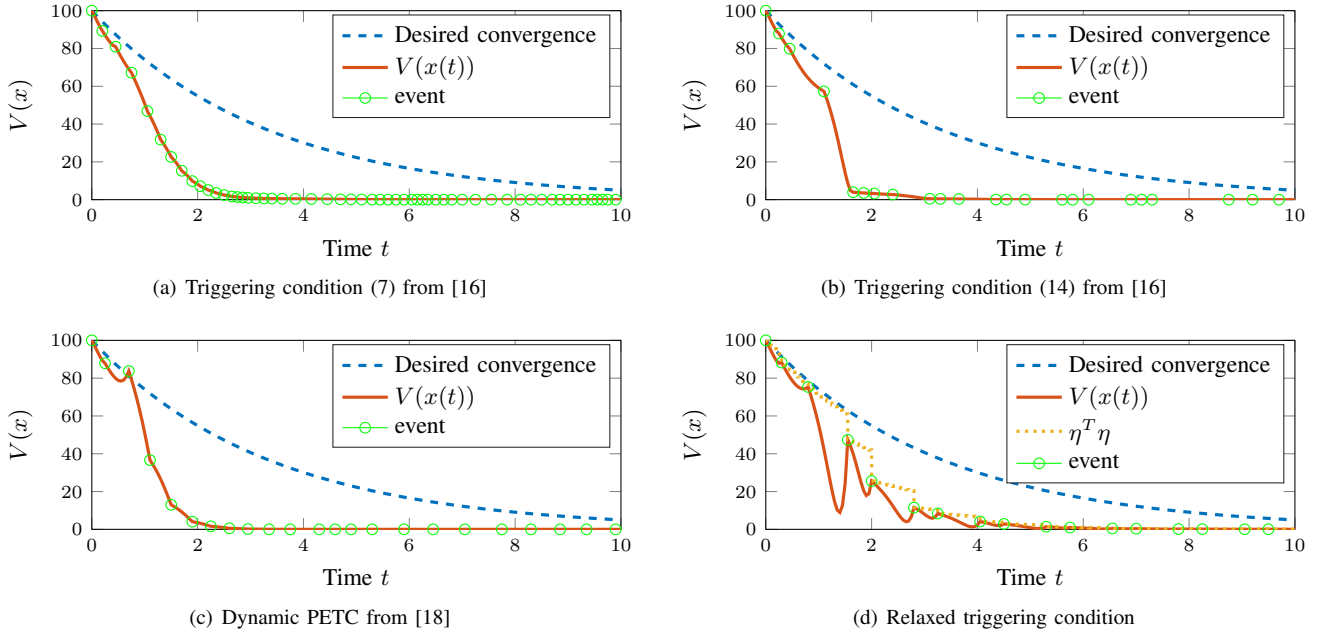

(d) Relaxed triggering condition

Fig. 1. The evolution of the monitored Lyapunov function for three different triggering conditions. Common parameters are: $\Delta = 0.05$ and $\rho = 0.15$. For (7) from [16] $\sigma = 0.16$, for (14) from [16] $\beta = 0.95$, for dynamic PETC from [18] $\sigma = 0.15$ and for our relaxed triggering condition $N_{max} = 20$.

TABLE I

AVERAGE (μ) AND STANDARD DEVIATION (σ) OF NUMBER OF COMMUNICATIONS FOR DIFFERENT TRIGGERING CONDITIONS AND DECAY RATES

ρ	This work	(7) from [16]	(14) from [16]	Dynamic PETC [18]
0.1	$\mu = \mathbf{17.6}$ $\sigma = 0.699$	$\mu = 52.8$ $\sigma = 0.422$	$\mu = 19.8$ $\sigma = 0.789$	$\mu = 22.4$ $\sigma = 1.647$
0.125	$\mu = \mathbf{17.3}$ $\sigma = 0.823$	$\mu = 53.5$ $\sigma = 0.527$	$\mu = 23.2$ $\sigma = 1.317$	$\mu = 26.3$ $\sigma = 2.71$
0.15	$\mu = \mathbf{17.6}$ $\sigma = 0.843$	$\mu = 56.6$ $\sigma = 0.516$	$\mu = 22.2$ $\sigma = 1.751$	$\mu = 28.6$ $\sigma = 3.026$
0.175	$\mu = \mathbf{17.6}$ $\sigma = 0.843$	$\mu = 58.7$ $\sigma = 0.483$	$\mu = 21.3$ $\sigma = 1.252$	$\mu = 29.2$ $\sigma = 2.3$
0.2	$\mu = \mathbf{17.6}$ $\sigma = 0.843$	$\mu = 59.8$ $\sigma = 0.422$	$\mu = 23$ $\sigma = 2.582$	$\mu = 30.3$ $\sigma = 1.636$
0.225	$\mu = \mathbf{17.7}$ $\sigma = 0.823$	$\mu = 62.9$ $\sigma = 0.316$	$\mu = 33.2$ $\sigma = 2.348$	$\mu = 32.4$ $\sigma = 0.966$

(a) Relaxed triggering condition with $N_{max} = 40$

(b) Dynamic PETC from [18]

Fig. 2. Evolution of the Lyapunov function with the relaxed triggering condition for $\Delta = 0.05$, $\rho = 0.225$ and $\delta = \sin(t)$.

because it does not take into account any past values of the states or the Lyapunov function. Nevertheless, apart from these burst the relaxed PETC does not degrade to periodic control when allowed to run for a longer time.

V. CONCLUSIONS AND FUTURE WORK

In this paper, we presented a relaxed triggering condition for PETC based on bounding the evolution of the Lyapunov function with a decaying exponential. The system is guaranteed to be GES in the case without disturbances and has a finite $\mathcal{L}_\infty$ gain from disturbance to state when bounded additive disturbances are acting on the system. The proposed strategy reduces the number of communications by minimizing the gap between the desired and the actual convergence rate. Compared to other PETC strategies, where the Lyapunov function is required to be decreasing from sample to sample, our triggering condition significant reduces the
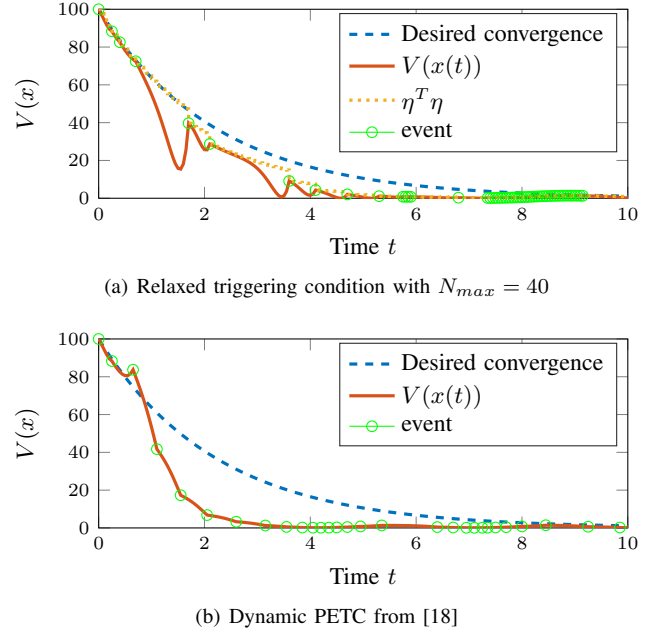
number of events, especially when the influence of disturbances is small. As follow-up work, aside from investigating the oscillations caused by our triggering mechanism, we are currently investigating if the number of events can be further reduced by predicting if the monitored Lyapunov function could cross the reference exponential bound multiple times.

APPENDIX

Proof of Theorem 3.1: Let us start by defining the monitored Lyapunov function $V(t) := x(t)^T P x(t)$, the

exponentially decaying continuous-time bound $S(t_i + \tau) := V(t_i)e^{-\lambda\tau}$ and sampling times within one inter-event time $r_n = t_i + n\Delta$, $n \in \mathbb{N}_0$, $r_n \in [t_i, t_{i+1})$. At discrete sampling instants the bound given by our PETC implementation is equal to $S(t)$, namely $\eta(r_n)^T\eta(r_n) = S(r_n)$. Therefore, in the absence of disturbances, the monitored Lyapaunov function satisfies $V(r_n) \leq S(r_n)$. The behavior is thus the same as the STC's from [9], and so is this proof, except for a small modification. To bound the evolution of the Lyapunov function in between the samples, let us start with finding the derivative $\dot{V}$ for the PETC system $\dot{x} = A^p x(t) + B^p K x(t_i)$. For $t \in [t_i, t_{i+1}]$, we have

$$\dot{V}(t) = v(t)^T G v(t), \quad v(t)^T := \left[\left(P^{\frac{1}{2}}x(t)\right)^T \quad \left(P^{\frac{1}{2}}x(t_i)\right)^T \right]. \tag{9}$$

Matrix G (8) is symmetric and hence orthogonally diagonalizable. Furthermore, it holds that $\mu = \lambda_m(G) < 0$ and $\omega = \lambda_M(G) > 0$. To prove it, let us write $G = \begin{bmatrix} H & D \\ D^T & 0 \end{bmatrix}$, with $D \neq 0$. According to the Schur complement condition, $G \succ 0$ if and only if $H \succ 0$ and $G/H = -D^T H^{-1} D \succ 0$. If we assume that $H \succ 0$, then also $H^{-1} \succ 0$. If so, there exists a non-singular matrix M such that $H^{-1} = MM^T$. We can rewrite $D^T H^{-1} D = D^T MM^T D = (M^T D)^T M^T D \succeq 0$. That gives us $H \succ 0$ and $G/H \preceq 0$, so G is not (semi-)positive definite. Similarly, we can show that G is not (semi-)negative definite (by applying the same reasoning for $-G$). Hence, G must have at least one positive and at least one negative eigenvalue.

Since $v(t)^T v(t) = V(t) + V(t_i)$, (9) can be used to lower and upper bound the derivative of the Lyapunov function:

$$\mu\big(V(t)+V(t_i)\big) \leq \dot{V}(t) \leq \omega\big(V(t)+V(t_i)\big), \quad t \in [t_i, t_{i+1}].$$

By integrating the above inequality on both sides, we can bound $V(t)$ itself by:

$$V(t + s) \leq e^{\omega s}V(t) + V(t_i)(e^{\omega s} - 1), \tag{10a}$$
$$V(t + s) \geq e^{\mu s}V(t) + V(t_i)(e^{\mu s} - 1), \tag{10b}$$

for $t + s \in [t_i, t_{i+1}]$. We know the values of the Lypaunov function at sampling instants so we use them to obtain bounds on $V(r_n + s)$, when $s \in [0, \Delta]$:

$$V(r_n + s)$$
$$\leq \begin{cases} V(r_n + s) \leq e^{\omega s}V(r_n) + V(t_i)\big(e^{\omega s} - 1\big), & s \in [0, s^*], \\ e^{\mu(s-\Delta)}V(r_{n+1}) + V(t_i)\big(e^{\mu(s-\Delta)} - 1\big), & s \in [s^*, \Delta), \end{cases}$$

where s^* is the point where branches (10a) and (10b) meet. Since the first bound is increasing and the second decreasing in s, $V(r_n + s^*)$ is the maximum of the Lyapunov function between r_n and r_{n+1}. We can find s^* by equating the two bounds, which results in:

$$s^* = \frac{1}{\omega - \mu} \log\left(\frac{V(r_{n+1}) + V(t_i)}{V(r_n) + V(t_i)}\right) + \frac{\mu\Delta}{\mu - \omega}$$

We substitute back the obtained expression for s^* to one of the bounds on $V(r_n + s)$ to obtain

$$V(r_n + s^*) \leq -V(t_i)$$
$$+ e^{\frac{\omega\mu\Delta}{\mu-\omega}}\left(\big(V(r_n) + V(t_i)\big)^{\frac{\mu}{\mu-\omega}}\big(V(r_{n+1}) + V(t_i)\big)^{\frac{-\omega}{\mu-\omega}}\right).$$

Using the fact that $V(r_n) \leq S(r_n)$ for all $r_n \in \mathbb{N}_0$ and dropping the first term, we have

$$V(r_n + s^*) \leq$$
$$e^{\frac{\omega\mu\Delta}{\mu-\omega}}\left(\big(S(r_n) + S(t_i)\big)^{\frac{\mu}{\mu-\omega}}\big(S(r_{n+1}) + S(t_i)\big)^{\frac{-\omega}{\mu-\omega}}\right).$$

Because $S(r_n) = e^{\lambda s^*}S(r_n + s^*)$, $S(r_{n+1}) = e^{\lambda(s^*-\Delta)}S(r_n + s^*)$ and $S(t_i) = e^{\lambda(n\Delta+s^*)}S(r_n + s^*)$, and due to the fact that $\frac{\mu}{\mu-\omega} + \frac{-\omega}{\mu-\omega} = 1$, we can factor out $S(r_n + s*)$ to obtain $V(r_n + s^*) \leq \tilde{g}(\Delta, n)S(r_n + s^*)$ with

$$\tilde{g}(\Delta, n) = e^{\frac{\omega\mu\Delta}{\mu-\omega}}\left(e^{\lambda s^*} + e^{\lambda(n\Delta+s^*)}\right)^{\frac{\mu}{\mu-\omega}}$$
$$\cdot \left(e^{\lambda(s^*-\Delta)} + e^{\lambda(n\Delta+s^*)}\right)^{\frac{-\omega}{\mu-\omega}}.$$

To make the above formula independent of time, we bound $s^* \leq \Delta$ and $n = N_{max} - 1$, since the right-hand side is increasing in both parameters:

$$\tilde{g}(\Delta, N_{max} - 1) \leq e^{\frac{\omega\mu\Delta}{\mu-\omega}}\left(e^{\lambda\Delta} + e^{\lambda N_{max}\Delta}\right)^{\frac{\mu}{\mu-\omega}}$$
$$\cdot\left(1 + e^{\lambda N_{max}\Delta}\right)^{\frac{-\omega}{\mu-\omega}} = g(\Delta, N_{max}). \tag{11}$$

$V(r_n + s^*) \leq g(\Delta, N_{max})S(r_n + s^*)$ holds for all $n \in [0, N_{max}]$, so we can write

$$V(t_i + \tau) \leq g(\Delta, N_{max})S(t_i + \tau).$$

Now we can use the fact that our PETC implementation yields $V(t_{i+1}) \leq V(t_i)e^{-\lambda\tau_i}$ to obtain

$$V(t) \leq g(\Delta, N_{max})V(t_0)e^{-\lambda t},$$

where we substituted $t = t_i + \tau$. Finally, using bounds

$$\lambda_m(P)x(t)^T x(t) \leq V(t) \leq \lambda_M(P)x(t)^T x(t)$$

we arrive at

$$|x(t)| \leq \left(\frac{\lambda_M(P)}{\lambda_m(P)}\right)^{\frac{1}{2}}\big(g(\Delta, N_{max})\big)^{\frac{1}{2}}|x(t_0)|e^{-\frac{1}{2}\lambda t}. \tag{12}$$

Choosing $t_0 = 0$ concludes the proof. ∎

Before giving the proof for Theorem 3.2, we introduce two lemmas.

Lemma 1.1 ([9, Lemma A.1]): Consider system (1) and a positive definite function $\tilde{V}(x) = (x^T P x)^{\frac{1}{2}}$, $P \succ 0$. For any given $0 \leq T < \infty$ the following bound holds:

$$\tilde{V}(x_{x_0 u \delta}(t)) \leq \tilde{V}(x_{x_0 u 0}(t)) + \gamma_{P,T}(||\delta||_\infty), \quad \forall t \in [0, T],$$

$$\gamma_{P,T}(s) := s\frac{\lambda_M(P)}{\lambda_m^{\frac{1}{2}}(P)}\int_0^T |e^{A^p r}E| dr.$$

Lemma 1.2: The Lyapunov function $\tilde{V}(x) = (x^T P x)^{\frac{1}{2}}$, $P \succ 0$ of system (1) under the proposed PETC implementation, with $\lambda < \lambda_0$, $\Delta < \tau_{min}$ and $\delta \in \mathcal{L}_\infty$, satisfies

$$\tilde{V}(t_{i+1}) \leq \tilde{V}(t_i)e^{-\frac{1}{2}\lambda\tau_i} + \gamma_{P,\Delta}(||\delta||_\infty).$$

Proof of Lemma 1.2: From our PETC implementation, if the inter-event time is $\tau_i = t_{i+1} - t_i = n_i\Delta$, then for the case if $\delta = 0$ it holds that

$$V(x(r_n)) \leq V(x(t_i))e^{-\lambda n\Delta},$$
$$\tilde{V}(x(r_n)) \leq \tilde{V}(x(t_i))e^{-\frac{1}{2}\lambda n\Delta}, \tag{13}$$

for all $n = 0, ..., n_i$. When bounded disturbances are present (13) holds only for $n = 0, ..., n_{i-1}$, due to a possible worst-case scenario when $\zeta(n_i\Delta)^T Q_1 \zeta(n_i\Delta) > 0$. It corresponds to the situation when at time $r_{n_{i-1}}$ one-step-ahead prediction of the Lyapunov function would still be below the bound, but due to disturbances at r_{n_i} it exceeded the bound. Using Lemma 1.1 we can write it as

$$\tilde{V}(x_{x(r_{n_{i-1}})u\delta}(r_{n_i})) \le \tilde{V}(x_{x(r_{n_{i-1}})u0}(r_{n_i})) + \gamma_{P,\Delta}(||\delta||_\infty).$$

The left-hand side becomes $\tilde{V}(t_{i+1})$ and on the right-hand side we can use (13) to bound the prediction made without taking into account the disturbances. ∎

Proof of Theorem 3.2: We start by iterating Lemma 1.2.

$$\tilde{V}(t_i) \le e^{-\frac{1}{2}\lambda(t_i-t_0)}\tilde{V}(t_0) + \gamma_{P,\Delta}(||\delta||_\infty)\sum_{k=0}^{i-1} e^{-\frac{1}{2}\lambda\tau_{min}k}$$
$$\le e^{-\frac{1}{2}\lambda(t_i-t_0)}\tilde{V}(t_0) + \gamma_{P,\Delta}(||\delta||_\infty)\sum_{k=0}^{\infty} e^{-\frac{1}{2}\lambda\Delta k}$$
$$\le e^{-\frac{1}{2}\lambda(t_i-t_0)}\tilde{V}(t_0) + \gamma_{P,\Delta}(||\delta||_\infty)\frac{1}{1-e^{-\frac{1}{2}\lambda\Delta}},$$

where we used the fact that $\tau_{min} > \Delta$. Without loss of generality, we choose $t_0 = 0$:

$$\tilde{V}(t_i) \le \tilde{V}(0)e^{-\frac{1}{2}\lambda t_i} + \frac{\gamma_{P,\Delta}(||\delta||_\infty)}{1-e^{-\frac{1}{2}\lambda\Delta}}. \qquad (14)$$

From GES of the unperturbed case, we have

$$\tilde{V}(x_{x(r_n)u0}(r_n + s)) \le g(\Delta, N_{max})^{\frac{1}{2}}\tilde{V}(t_i)e^{-\frac{1}{2}\lambda(n\Delta+s)}. \qquad (15)$$

for all $s \in [0, \Delta]$. In PETC we make only a one-step-ahead prediction and use the true value of the Lyapunov function at r_n to predict r_{n+1}. Using Lemma 1.1 and with $x(r_n)$ as the initial condition for predicting x at time $r_n + s$, we write:

$$\tilde{V}(x_{x(r_n)u\delta}(r_n + s)) \le \tilde{V}(x_{x(r_n)u0}(r_n + s)) + \gamma_{P,\Delta}(||\delta||_\infty) \qquad (16)$$

for all $s \in [0, \Delta]$. Substituting (15) in (16) gives

$$\tilde{V}(x_{x(r_n)u\delta}(r_n + s)) \le g(\Delta, N_{max})^{\frac{1}{2}}\tilde{V}(t_i)e^{-\frac{1}{2}\lambda(n\Delta+s)}$$
$$+ \gamma_{P,\Delta}(||\delta||_\infty), \quad \forall s \in [0, \Delta],$$

which holds for all $n \in [0, ..., N_{max}]$, and, with $\tau = n\Delta + s$, becomes

$$\tilde{V}(x(t_i + \tau)) \le g(\Delta, N_{max})^{\frac{1}{2}}\tilde{V}(t_i)e^{-\frac{1}{2}\lambda\tau} + \gamma_{P,\Delta}(||\delta||_\infty), \qquad (17)$$

where the extensive subscripts are dropped for compactness of expressions. By replacing (14) into (17) we arrive at

$$\tilde{V}(x(t_i+\tau)) \le g(\Delta, N_{max})^{\frac{1}{2}}\tilde{V}(t_0)e^{-\frac{1}{2}\lambda(t_i+\tau)}$$
$$+ \gamma_{P,\Delta}(||\delta||_\infty)\left(\frac{g(\Delta, N_{max})^{\frac{1}{2}}}{1-e^{-\frac{1}{2}\lambda\Delta}}e^{-\frac{1}{2}\lambda\tau} + 1\right).$$

Now, substituting $t_i + \tau = t$ and bounding $e^{-\frac{1}{2}\lambda\tau} \le 1$ gives

$$\tilde{V}(x(t)) \le g(\Delta, N_{max})^{\frac{1}{2}}\tilde{V}(t_0)e^{-\frac{1}{2}\lambda t}$$
$$+ \gamma_{P,\Delta}(||\delta||_\infty)\left(\frac{g(\Delta, N_{max})^{\frac{1}{2}}}{1-e^{-\frac{1}{2}\lambda\Delta}} + 1\right).$$

Finally, using the following bound $\lambda_m^{\frac{1}{2}}(P)|x| \le \tilde{V}(x) \le \lambda_M^{\frac{1}{2}}(P)|x|$, one can bound the evolution of states as follows:

$$|x(t)| \le \sigma|x(0)|e^{-\frac{1}{2}\lambda t}$$
$$+ \lambda_m^{-\frac{1}{2}}(P)\gamma_{P,\Delta}(||\delta||_\infty)\left(\frac{g(\Delta, N_{max})^{\frac{1}{2}}}{1-e^{-\frac{1}{2}\lambda\Delta}} + 1\right),$$

which proves that the system is EISS. ∎

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