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Depth-averaged unsteady RANS simulation of resonant shallow flows in lateral cavities using augmented WENO-ADER schemes

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Abstract

Turbulent shallow flows are characterized by the presence of horizontal large-scale vortices, caused by local variations of the velocity field. Apart from these 2D large vortices, small scale 3D turbulence, mainly produced by the interaction of the flowing water with the solid boundaries, is also present. The energy spectrum of turbulent shallow flows shows the presence of a 2D energy cascade at low wave numbers and a 3D energy cascade at high wave numbers, with a well-defined separation region between them. Horizontal flow movements (e.g. 2D large-scale vortical structures) at low wave numbers mostly determine the hydrodynamic behavior of these flows. Moreover, the generation of standing waves often occurs closely associated to the interaction of 2D horizontal flows with lateral boundaries, this is the case of seiches. To adequately reproduce these phenomena, a mathematical and numerical model able to resolve 2D turbulence is required. We herein show that depth-averaged (DA) unsteady Reynolds averaged Navier Stokes (URANS) models based on the Shallow Water Equations (SWE) are a suitable choice for the resolution of turbulent shallow flows with sufficient accuracy in an affordable computational time. The 3D small-scale vortices are modelled by means of diffusion terms, whereas the 2D large-scales are resolved. A high order numerical scheme is required for the resolution of 2D large eddies. In this work, we design a DA-URANS model based on a high order augmented WENO-ADER scheme. The mathematical model and numerical scheme are validated against observation of complex experiments in an open channel with lateral cavities that involve the presence of resonant phenomena (seiching). The numerical results evidence that the model accurately reproduces both longitudinal and transversal resonant waves and provides an accurate description of the flow field. The high order WENO-ADER scheme combined with a SWE model allows to obtain a powerful, reliable and efficient URANS simulation tool.

Keywords:

Shallow flows; turbulence; shock capturing; lateral cavities; seiching;

1. Introduction

Shallow water flows are of great interest in civil and environmental engineering. They appear in a wide variety of scenarios ranging from open channels to coastal areas [1, 2]. The water depth in these environmental flows is characterized by being much smaller than the horizontal scale. The turbulence produced in such flows is characterized by two different and coexistent scales of turbulent structures: the 2D turbulence (with length scales much larger than the water depth) and the 3D turbulence (with length scales smaller than the water depth) [3]. Local variations of the velocity field promote the generation of horizontal 2D large-scale coherent vortices. These macro vortical structures

9 play a relevant role in several aspects of hydro-morphological and biological interest: the conveyance
10 of fine sediments in suspension [4], the transport of pollutants [5] or the mass exchange of nutrients
11 between the flow and the aquatic biota [6]. In addition to 2D macro horizontal vortices, small scale
12 3D turbulent vortices, produced by the interaction of the flowing water with the solid boundaries and
13 in 2D shear layers, co-exist in environmental shallow flows. They participate in the vertical mass and
14 momentum transfer between the bottom and the flow surface.

15 The co-existence of 2D and 3D turbulent scales is reflected in the energy spectrum of the shallow
16 flow turbulence. The energy spectrum of such flows displays the presence of a 2D energy cascade at low
17 wave numbers and a 3D energy cascade at high wave numbers, with a well-defined separation region
18 between them [3, 7].

19 Besides the turbulent nature of shallow flows, another important characteristic of these flows is the
20 existence of a free surface. This free surface is the boundary between the water body and the air above
21 it. It may deform in response to physical intrinsic mechanisms that govern the flow (e.g. it is influenced
22 by the presence of turbulence, wind shear and bed topography). In free surface flows, surface gravity
23 waves usually appear and may determine the hydrodynamic pattern within the flow [8]. This additional
24 feature of shallow water turbulence involves additional energy dissipation mechanism. A particular type
25 of gravity waves are standing resonant waves that appear in bounded flows, called seiches [9]. There
26 is a wide range of spatial scales at which seiches may appear: ranging from centimeters (e.g. seiche
27 in lateral cavities of river banks [9]) to kilometers (e.g. seiche in lakes, bays or harbours [10]). The
28 understanding of seiches is relevant to determine the hydro-morphological interactions between the
29 main stream flow and the lateral cavities of river banks and the consequences of these to mass and
30 momentum transfer [11].

31 Concerning the mathematical modelling and numerical simulation of free surface shallow flows,
32 multiple approaches have been proposed by the scientific community in the past decades. The use of
33 3D models with advanced turbulence modelling techniques, such as Reynolds averaged Navier Stokes
34 (RANS) models, large eddy simulation (LES) or direct numerical simulation (DNS), has allowed com-
35 plex turbulent flows to be simulated with accuracy [12]. However, the use of 3D models is computa-
36 tionally costly and handling the presence of the free surface is not a trivial numerical task. Most numerical
37 techniques (specially for the LES approach) ignore free surface deformations and they use the so-called
38 rigid lid approximation [13]. By using this approach, gravity waves are represented as vertical pressure
39 fluctuations but the deformation of the free surface cannot be tracked, and is numerically simpler than
40 other numerical approaches that aim at computing the physical and real free surface deformation (e.g.
41 interface tracking methods and interface capturing methods [14]). A more recent hybrid approach
42 considers a two-phase LES models that accurately resolve the interface between water and air [15].

43 A rather different approach for the simulation of free surface shallow flows is the use of 2D depth-
44 averaged models. Such models, involving a significantly lower computational cost, are able to reproduce
45 the propagation of gravity waves and to account for turbulent fluctuations. They are useful for the
46 simulation of large domains, which are virtually unaffordable by means of LES methods. The shallow
47 water equations (SWE) are a hyperbolic system of conservation laws that describe the propagation of
48 nonlinear gravity waves. They can also model additional physical processes by means of extra source
49 and diffusion terms (e.g. friction, bottom topography, turbulence) [16, 17].

50 The most common approach to handle turbulence for the SWE is the RANS methodology [18,
51 19, 20, 21], which aims at the resolution of flow mean quantities by modelling the full turbulence
52 spectrum, including the large scales. The fluctuating terms are not resolved, though their effect in the
53 transport of mean quantities is included in the equations by means of a closure formulation that allows
54 to model the Reynolds stress term in terms of the mean flow. This closure formulation is normally
55 given by the so-called Boussinesq's approximation, which states that the momentum transfer caused by
56 turbulent eddies can be modelled using the concept of an eddy viscosity coefficient. Mathematically,

57 it is constructed using extra diffusion terms where the mixing intensity is controlled by such eddy
58 viscosity. In the RANS approach, the eddy viscosity coefficient accounts for both 2D large scale
59 turbulence and 3D turbulence and it is thus computed by models that consider the full structure
60 of shallow flow turbulence. Such models range from simple zero-equation models (e.g. constant eddy
61 viscosity model, parabolic eddy viscosity model or depth-averaged mixing length model), which assume
62 that turbulence is dissipated where it is generated, to the more advanced eddy-viscosity models, which
63 consider extra equations to account for the transport of turbulence (e.g. $k - \epsilon$ turbulence model) [18].

64 When using the RANS methodology, mean velocities are accurately predicted, however, velocity
65 fluctuations and phenomena such as gravity waves (e.g. seiches) cannot be represented. To circumvent
66 this limitation, the unsteady RANS (URANS) [22, 8] approach will be used in this study. The idea
67 of using the URANS approach is motivated by the particular character of shallow water turbulence
68 (i.e. the presence of two well-defined turbulent cascades) [3]. 2D large-scale vortices are normally
69 resolved by means of a high-order approximation of the numerical solution, whereas 3D small-scale
70 turbulence can be modelled by a closure equation as described before. In the literature, it is possible
71 to find a broad variety of turbulence closure models. Some of such turbulence models only account for
72 small-scale 3D turbulence and others also include the effect of unresolved 2D eddies. The shallow water
73 URANS approach can be variously called: horizontal LES (H-LES), depth averaged LES (DA-LES) or
74 DA-URANS. Nevertheless, and as stated in [22], none of these methodologies can be considered a LES
75 model in the usual sense, as they are depth averaged and 3D turbulence always has to be modelled.
76 For this reason, in this work we will always refer to them as DA-URANS.

77 Once the mathematical model is defined, an adequate numerical scheme must be selected. The tur-
78 bulence modelling requires a high order approximation of the convective fluxes to avoid extra numerical
79 diffusion and to reduce dispersion errors [23]. The non-physical diffusion at small wave numbers may
80 damp the turbulent fluctuations to be resolved and the URANS approach may ultimately be turned
81 into the traditional RANS approach. Apart from turbulence modelling, it has been widely reported in
82 the literature that an accurate resolution of shallow water waves also requires a high order of accuracy
83 [24, 25]. In this work, the design of a RANS/URANS simulation model for the SWE with bottom
84 topography and friction, based on arbitrary order augmented WENO-ADER schemes [26], is explored.
85 In particular, WENO-ADER schemes are applied to the computation of the convective fluxes and
86 source terms. Alternatively, different approaches for the discretization of the diffusive terms modelling
87 turbulent mixing are considered and assessed. Such approaches comprise a WENO-ADER reconstruc-
88 tion, a high order polynomial-ADER reconstruction and a second order discretization. Due to the
89 dominant character of the convective terms in the numerical solution, the second order discretization
90 of the diffusive terms was chosen for the sake of simplicity and computational efficiency.

91 The numerical scheme is explicit and integration in time is done in a single step thanks to the ADER
92 approach. Such method successfully allows the construction of arbitrary order schemes for systems
93 of hyperbolic conservation laws [27, 28, 29]. It is of particular interest for this work to consider the
94 application of ADER schemes for the resolution of geophysical problems [30, 31, 25], specially for the
95 resolution of the SWE [32, 25, 33, 34, 35, 36].

96 The methods are tested here up to a 3-rd order of accuracy, but they can be extended up to
97 the desired order of accuracy provided the approximation of time derivatives has been computed by
98 means of the Cauchy-Kowalevski (CK) procedure. This circumvents Butcher's barrier appearing for
99 Runge-Kutta (RK) integrators [37]. Source terms are taken into account at both cell interfaces and
100 inside cells [26, 38]. The augmented-solver approach, which ensures an exact equilibrium between
101 fluxes and sources at cell interfaces, is adopted by using the ARoe LFS solver presented in [26]. The
102 combination of this solver with a suitable integration of source terms inside cells makes the resulting
103 scheme well-balanced, preserving the quiescent equilibrium. Such combination has already been used
104 to construct high order numerical methods for the SWE with bed topography and Coriolis [38]. It must

105 be borne in mind that the bottom and friction source terms, present in the model herein proposed,
 106 have a different nature: the former is a geometric source term whereas the latter is non-geometric.
 107 Therefore, a different numerical treatment is required for each of them. In this work, the bottom
 108 source term is integrated using the augmented approach in [38], whereas the friction source term is
 109 only included inside cells using a traditional quadrature rule. This allows the construction of a well-
 110 balanced scheme that preserves the quiescent equilibrium with machine precision and converges with
 111 high order of accuracy to other solutions involving moving water.

112 The WENO reconstruction is another key feature of the numerical method herein used. It preserves
 113 the high order of accuracy in smooth regions, with low diffusion and dispersion errors [39], and is able
 114 to capture sharp gradients avoiding spurious oscillations. Furthermore, thanks to the shock-capturing
 115 character of the solvers used, the celerity of gravity waves is accurately reproduced. Seiches should be
 116 thus accurately captured.

117 The main advantages of using augmented WENO-ADER-type schemes are: a) the scheme can be
 118 extended up to the desired accuracy without upper bounds, allowing the computation of 2D turbulent
 119 scales [26, 40]; b) the scheme is explicit and fully discrete (i.e. it does not involve the computation of
 120 fluxes at different sub-steps like RK integrators and consequently, it is computationally more efficient
 121 [40]); c) the scheme allows the preservation of equilibrium states of relevance, in particular, the well-
 122 balanced equilibrium [38]; d) wave celerities are accurately computed, as a result of the combination of
 123 a shock-capturing solver with a high order scheme; and e) there is no need of a high mesh refinement
 124 making large computational domains affordable. Furthermore, the augmented WENO-ADER scheme
 125 considered in this work ensures convergence with mesh refinement at the prescribed rate even in
 126 presence of source terms, as shown in [26, 38].

127 The proposed model is herein applied to the resolution of transient problems involving resonant
 128 phenomena. In particular, we focus on the coupling between the shedding of vortices and standing
 129 gravity waves (i.e. seiches) in open channels with lateral cavities. The methods are validated using
 130 experimental data measured in a) a channel with a single cavity [8] and b) channels with multiple
 131 lateral cavities [9, 41]. A longitudinal seiche appears for case a) while a transverse seiche appears
 132 for case b). The experimental data used as a benchmark include both velocities and water depths.
 133 Thanks to the URANS approach the temporal fluctuations of the 2D vortical structures are accurately
 134 reproduced: the experimental seiching frequency is reproduced with an error lower than a 6% in all
 135 cases. On the contrary, the RANS approach proved not to be suitable for modelling resonant flows: the
 136 large amount of dissipation provided by such technique smooths artificially the flow surface oscillations
 137 leading to nonphysical results.

138 2. The mathematical model

139 2.1. System definition

140 The SWE are based on a 2D depth-averaged and hydrostatic model suitable for free-surface flows
 141 where the vertical dimension is much smaller than the longitudinal dimensions. The SWE are composed
 142 by the equation for the conservation of mass and momentum and are written in matrix form as follows:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} + \frac{\partial \mathbf{G}(\mathbf{U})}{\partial y} = \mathbf{S} + \mathbf{D}, \quad (1)$$

143 where x and y stand for the Cartesian coordinates, t is the time, $\mathbf{U} = \mathbf{U}(x, y, t) \in \mathcal{C} \subset \mathbb{R}^3$ is the vector
 144 of conserved quantities that takes values on \mathcal{C} :

$$\mathbf{U} = \begin{pmatrix} h \\ hu \\ hv \end{pmatrix}, \quad (2)$$

145 $\mathbf{F} = \mathbf{F}(\mathbf{U}) : \mathcal{C} \longrightarrow \mathbb{R}^3$ and $\mathbf{G} = \mathbf{G}(\mathbf{U}) : \mathcal{C} \longrightarrow \mathbb{R}^3$ are the physical fluxes on the coordinate directions x
 146 and y , respectively:

$$\mathbf{F} = \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ huv \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} hv \\ hvu \\ hv^2 + \frac{1}{2}gh^2 \end{pmatrix} \quad (3)$$

147 and the vectors \mathbf{S} and \mathbf{D} stand for the source and diffusion terms, which will be detailed below. Note
 148 that g is the acceleration of gravity, h is the water depth, hu is the depth averaged unitary discharge
 149 in the x direction and hv the depth averaged unitary discharge in the y direction.

150 The methods herein described are based on the assumption that the convective part of the system
 151 in Equation (1) is hyperbolic. The system in Equation (1) is said to be *hyperbolic* if the matrix
 152 $\mathcal{J}(\mathbf{U}, \mathbf{n}) \in \mathbb{R}^{3 \times 3}$ defined as:

$$\mathcal{J}(\mathbf{U}, \mathbf{n}) = \frac{\partial \mathbf{F}}{\partial \mathbf{U}} n_x + \frac{\partial \mathbf{G}}{\partial \mathbf{U}} n_y, \quad (4)$$

153 is diagonalizable with real eigenvalues for all $\mathbf{n} = (n_x, n_y) \in \mathbb{R}^2$ and for all $\mathbf{U} \in C$ with $C \subseteq \mathbb{R}^3$ the
 154 subset of physically relevant values of \mathbf{U} . If all the eigenvalues are distinct, then the system is said to
 155 be *strictly hyperbolic* [42].

156 The longitudinal and transversal velocities, u and v are depth-averaged mean components in the
 157 term of the definition of the Reynolds decomposition. The water depth h also corresponds to a mean
 158 value. The Reynolds decomposition of the instantaneous values of depth averaged velocities and water
 159 depth, hereafter denoted as U , V , and H respectively, reads:

$$U(x, y, t) = u(x, y, t) + u'(x, y, t) \quad V(x, y, t) = v(x, y, t) + v'(x, y, t) \quad (5)$$

$$H(x, y, t) = h(x, y, t) + h'(x, y, t) \quad (6)$$

160 Mean values may be obtained by several types of averaging depending on the nature of the flow
 161 [43]. The symbol ($'$) denotes turbulent fluctuation of the depth averaged values around its mean.
 162 In general, the mean values are time dependent, only under steady conditions they do not depend
 163 on time. The methods herein proposed aim at the resolution of the evolution in time of the mean
 164 properties governed by (1)–(2). The fluctuating terms are not directly resolved, though their effect in
 165 the transport of mean quantities is incorporated in the equations by means of a turbulence model.

166 The term $\mathbf{S} = \mathbf{S}_z + \mathbf{S}_f$ includes the source terms, which involve the stress exerted by the bottom
 167 topography, \mathbf{S}_z , and by the bed roughness, \mathbf{S}_f . Such sources are also called bed slope and friction
 168 source terms respectively and are written as:

$$\mathbf{S}_z = \begin{pmatrix} 0 \\ -gh \frac{dz}{dx} \\ -gh \frac{dz}{dy} \end{pmatrix}, \quad \mathbf{S}_f = \begin{pmatrix} 0 \\ -c_f |\mathbf{v}| u \\ -c_f |\mathbf{v}| v \end{pmatrix}, \quad (7)$$

169 where $|\mathbf{v}| = \sqrt{u^2 + v^2}$ is the velocity magnitude, $z = z(x, y)$ represents the bottom topography, which
 170 is fixed in time, and c_f is the friction coefficient, computed using Manning's formulation as follows:

$$c_f = \frac{gn^2}{h^{1/3}}. \quad (8)$$

171 On the other hand, the term \mathbf{D} includes the molecular and the turbulent diffusion, being the latter
 172 a model of the mixing processes happening at the small turbulent scales according to the RANS and

173 URANS approach. The vector \mathbf{D} is thus expressed as the sum of the molecular and the turbulent
 174 diffusion:

$$\mathbf{D} = \begin{pmatrix} 0 \\ \left[\frac{\partial}{\partial x} (\nu h \frac{\partial u}{\partial x}) + \frac{\partial}{\partial y} (\nu h \frac{\partial u}{\partial y}) \right] + \left[-\frac{\partial \overline{hu'u'}}{\partial x} - \frac{\partial \overline{hu'v'}}{\partial y} \right] \\ \left[\frac{\partial}{\partial x} (\nu h \frac{\partial v}{\partial x}) + \frac{\partial}{\partial y} (\nu h \frac{\partial v}{\partial y}) \right] + \left[-\frac{\partial \overline{hv'u'}}{\partial x} - \frac{\partial \overline{hv'v'}}{\partial y} \right] \end{pmatrix}, \quad (9)$$

175 where ν is the dynamic viscosity of the fluid and $\overline{u'u'}$, $\overline{u'v'}$, $\overline{v'u'}$ and $\overline{v'v'}$ are the depth-averaged
 176 Reynolds turbulent stresses, which can be approximated by means of Boussinesq's assumption. Note
 177 that the overline denotes mean values of the fluctuations. This assumption states that the momentum
 178 transfer caused by turbulent eddies can be modelled using the concept of an eddy viscosity. The
 179 Reynolds stress tensor is thus proportional to the mean strain rate tensor. Following [8, 21], the
 180 following approximations are used:

$$-\overline{u'u'} = 2\nu_t \frac{\partial u}{\partial x} - \frac{2}{3}k \quad (10)$$

$$-\overline{u'v'} = -\overline{v'u'} = \nu_t \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \quad (11)$$

$$-\overline{v'v'} = 2\nu_t \frac{\partial v}{\partial y} - \frac{2}{3}k \quad (12)$$

181 where k is the mean and depth averaged turbulent kinetic energy, defined as $k = \frac{1}{2}(\overline{u'^2} + \overline{v'^2})$. Different
 182 formulations of the viscous and turbulent diffusion terms in (9) are found in the literature [19, 8, 20, 21].
 183 In this work, we consider the approach provided in [20, 21, 17], where the diffusion terms are expressed
 184 as $\mathbf{D} = \mathbf{D}_x + \mathbf{D}_y$, where:

$$\mathbf{D}_x = \frac{\partial}{\partial x} \begin{pmatrix} 0 \\ hT_{xx} \\ hT_{yx} \end{pmatrix}, \quad \mathbf{D}_y = \frac{\partial}{\partial y} \begin{pmatrix} 0 \\ hT_{xy} \\ hT_{yy} \end{pmatrix}, \quad (13)$$

185 with T_{xx} , T_{xy} , T_{yx} and T_{yy} the depth-averaged stresses, which read as in [21]:

$$T_{xx} = 2(\nu + \nu_t) \frac{\partial u}{\partial x} \quad (14)$$

$$T_{xy} = T_{yx} = (\nu + \nu_t) \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \quad (15)$$

$$T_{yy} = 2(\nu + \nu_t) \frac{\partial v}{\partial y} \quad (16)$$

186 where ν is the fluid viscosity and ν_t the turbulent (eddy) viscosity. Note that k in (10)–(12) will be
 187 neglected as a zero-equation model for ν_t will be used [21].

188 *2.2. Turbulence models for the eddy viscosity ν_t*

189 3D and 2D turbulent structures coexist in shallow water flows and it corresponds to a spectral
 190 distribution of energy as displayed in Figure 1. Both 3D and 2D vortical structures participate in the
 191 mass and momentum exchange within the flow. However, the nature of each vortical movement is
 192 different. 3D turbulence is mainly produced by the friction with the solid boundaries and comprises
 193 length scales smaller than the water depth. 2D turbulence is generated by gradients of horizontal shear
 194 stress due to gradients of the velocity field between flow regions. The length scales are typically larger
 195 than the water depth [22, 3].

196 The most common techniques to solve shallow water flows using a 2D model are based on the
 197 URANS and RANS approaches. The range of scales resolved (i.e. computed with the set of equations
 198 listed in the mathematical model) or modelled (i.e. computed by means of a closure equation) when
 199 using such methods is depicted in Figure 1. The URANS approach usually resolves the large scale
 200 motions of the flow (2D horizontal eddies) that contribute most to the turbulent transport but requires
 201 a model for the unresolved small scale turbulent processes (3D turbulence) that involve vertical mixing.
 202 The depth-averaged parabolic eddy viscosity model is one of the most preferred models to represent
 203 the small scale turbulence produced by the friction with the bottom [3]. On the other hand, the RANS
 204 approach does not resolve any turbulent scale. The 2D large scale turbulent mixing and the 3D small
 205 scale turbulent dissipation are thus modelled. The depth-averaged mixing length model is often used
 206 as the closure equation. The depth-averaged mixing length model will be hereafter considered for both
 207 RANS and URANS simulation, as it allows to model the unresolved 2D small eddies (if necessary when
 208 the resolution is not enough) and the 3D unresolved turbulent motion.

209 From the point of view of a URANS model applied to shallow water flows, the large scale 2D
 210 horizontal eddies are incorporated and considered as part of a time varying mean flow (i.e. $u(x, y, t)$,
 211 $v(x, y, t)$ and $h(x, y, t)$). From the point of view of a RANS model, all the time varying components
 212 are modelled, hence the mean fields corresponds to the time-averaged variables (i.e. $u(x, y)$, $v(x, y)$
 213 and $h(x, y)$).

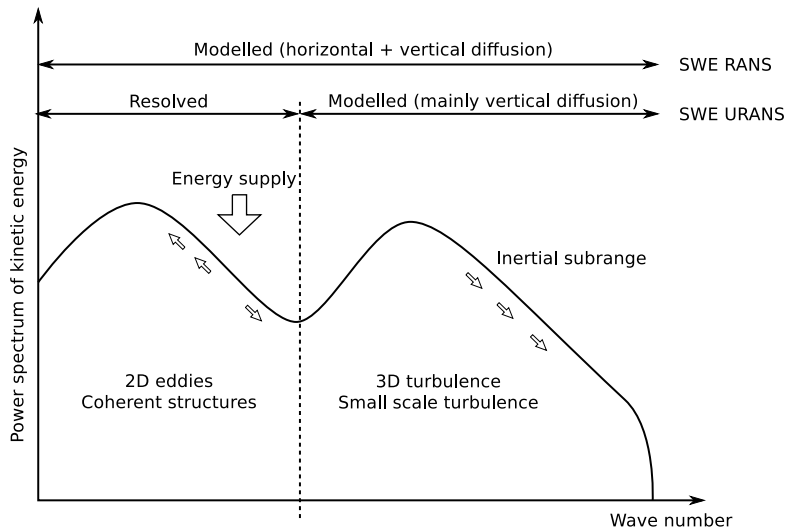


Figure 1: Schematic representation of the two-range turbulence spectrum in shallow water flows. The wave number is defined as the inverse turbulent length, which is related with the size of the eddies. The range of scales modelled/resolved by the URANS and RANS approaches is provided. This figure is adapted from [22, 3].

214 *2.3. Depth-averaged mixing length model*

215 The depth-averaged mixing length model is used to account for both the horizontal (2D) and
 216 vertical production of turbulence (3D). This model reads as:

$$\nu_t = \sqrt{(\nu_t^v)^2 + (\nu_t^h)^2} \quad (17)$$

217 where ν_t^v and ν_t^h are the vertical and horizontal eddy viscosities. The vertical component is mainly
 218 produced due to the bed friction and it is calculated as:

$$\nu_t^v = \lambda U^* h, \quad (18)$$

219 where λ is an empirical coefficient and $U^* = \sqrt{c_f(u^2 + v^2)}$ is the bed shear velocity. Note that c_f is
 220 the friction coefficient defined in Equation (8). The parameter λ can be theoretically estimated as $\kappa/6$,
 221 with κ the Von Karman's constant, though it is normally retained as a calibration parameter [21]. The
 222 model in Equation (18) corresponds to the depth-averaged parabolic eddy viscosity model.

223 On the other hand, the horizontal component of turbulence is mainly produced by horizontal
 224 velocity gradients and it is resolved as:

$$\nu_t^h = \beta l_s^2 \sqrt{2 \left(\frac{\partial u}{\partial x} \right)^2 + 2 \left(\frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2} \quad (19)$$

225 with $l_s = \kappa \min(c_m h, y_w)$ [21, 19], where c_m is an empirical coefficient (normally set to 0.267 [19]), y_w
 226 is the distance to the nearest wall and β is a calibration constant.

227 In this work, the depth-averaged mixing length model is used both for URANS and RANS simula-
 228 tions (see Figure 1). In the case of URANS approach, the horizontal eddies are mostly resolved. The
 229 strength of the horizontal turbulent mixing, β , is thus much lower than in the case of RANS simula-
 230 tions. This yields to an eddy viscosity coefficient that is mainly composed by the vertical diffusivity.
 231 Note that in the limit when $\beta = 0$, the depth-averaged parabolic eddy viscosity model is recovered (i.e.
 232 only the vertical diffusivity is modelled). On the other hand, when adopting the RANS methodology,
 233 the horizontal diffusivity may become dominant, specially in cases involving strong shearing flows. In
 234 such cases, the mixing processes are not represented as convective processes (as it is done with URANS
 235 simulation) but as purely diffusive processes, governed by ν_t .

236 It is worth pointing out that the use of the eddy viscosity in (19) for URANS simulation, in
 237 combination with the approximation of the depth-averaged Reynolds stress tensor in (14)–(16), is
 238 equivalent to the well-known Smagorinsky (SGS) sub-grid model when the coefficient βl_s^2 is of the
 239 order of magnitude of $C_s \Delta x$, where C_s is found in the interval 0.4 to 0.8 for shallow flows [44, 45]. This
 240 evidences the role of the horizontal component of the eddy viscosity in (19) as a sub-grid dissipation
 241 model for the unresolved scales.

242 **3. Numerical model: augmented WENO-ADER scheme in Cartesian grid**

243 Let us consider the system of conservation laws in (1)–(2) to compose the following Initial Boundary
 244 Value Problem (IBVP):

$$\left\{ \begin{array}{l} \text{PDEs: } \frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot (\mathbf{F}, \mathbf{G}) = \mathbf{S} + \mathbf{D} \\ \text{IC: } \mathbf{U}(\mathbf{x}, 0) = \mathring{\mathbf{U}}(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega \\ \text{BC: } \mathbf{U}(\mathbf{x}, t) = \mathbf{U}_{\partial\Omega}(\mathbf{x}, t) \quad \forall \mathbf{x} \in \partial\Omega \end{array} \right. \quad (20)$$

245 defined in the domain $\Omega \times [0, T]$, where $\Omega = [a, b] \times [c, d]$ is the spatial domain. The initial condition is
 246 given by $\bar{\mathbf{U}}(\mathbf{x})$ and the boundary condition by $\mathbf{U}_{\partial\Omega}(\mathbf{x}, t)$. The spatial domain is discretized in $N_x \times N_y$
 247 volume cells, defined as $\Omega_{ij} \subseteq \Omega$, such that $\Omega = \bigcup_{i,j=1}^N \Omega_{ij}$, with cell edges at:

$$a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \dots < x_{N_x - \frac{1}{2}} < x_{N_x + \frac{1}{2}} = b, \quad (21)$$

248 and

$$c = y_{\frac{1}{2}} < y_{\frac{3}{2}} < \dots < y_{N_y - \frac{1}{2}} < y_{N_y + \frac{1}{2}} = d, \quad (22)$$

249 Cells and cell sizes are defined as:

$$\Omega_{ij} = \left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}} \right] \times \left[y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}} \right], \quad i = 1, \dots, N_x, j = 1, \dots, N_y \quad (23)$$

250 and

$$\vartheta_{ij} = (x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}) \cdot (y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}), \quad i = 1, \dots, N_x, j = 1, \dots, N_y, \quad (24)$$

251 respectively. In the case of regular grid we have $\vartheta_{ij} = \Delta x^2$.

252 Inside each cell, at time t^n , the conserved quantities are defined as cell averages as:

$$\bar{\mathbf{U}}_{ij}^n = \frac{1}{\vartheta_{ij}} \int_{\Omega_{ij}} \mathbf{U}(\mathbf{x}, t^n) dA \quad i = 1, \dots, N_x, j = 1, \dots, N_y. \quad (25)$$

253 where $dA = dx dy$ is the differential element of surface. Let us consider again the system in (20) and
 254 integrate it over the discrete domain $\Omega_{ij} \times \Delta t$, where $\Delta t = t^{n+1} - t^n$. Application of the Gauss-
 255 Ostrogradsky theorem yields to:

$$\bar{\mathbf{U}}_{ij}^{n+1} = \bar{\mathbf{U}}_{ij}^n - \frac{1}{\vartheta_{ij}} \int_0^{\Delta t} \int_{\partial\Omega_{ij}} (\mathbf{F}, \mathbf{G}) \cdot \hat{\mathbf{n}} dl dt + \frac{1}{\vartheta_{ij}} \int_0^{\Delta t} \int_{\Omega_{ij}} \mathbf{S} dAdt + \frac{1}{\vartheta_{ij}} \int_0^{\Delta t} \int_{\Omega_{ij}} \mathbf{D} dAdt, \quad (26)$$

256 where dl is the differential length. If we consider a regular Cartesian grid with $\Delta x = \Delta y$, we obtain
 257 the following fully-discrete updating formula:

$$\bar{\mathbf{U}}_{ij}^{n+1} = \bar{\mathbf{U}}_{ij}^n - \frac{\Delta t}{\Delta x^2} \left(\mathbf{F}_{i+1/2,j}^- - \mathbf{F}_{i-1/2,j}^+ \right) - \frac{\Delta t}{\Delta x^2} \left(\mathbf{G}_{i,j+1/2}^- - \mathbf{G}_{i,j-1/2}^+ \right) + \frac{\Delta t}{\Delta x^2} \left(\bar{\mathbf{S}}_{ij} + \bar{\mathbf{D}}_{ij} \right), \quad (27)$$

258 where $\mathbf{F}_{i\mp 1/2,j}^\pm$ and $\mathbf{G}_{i,j\mp 1/2}^\pm$ are the numerical fluxes at cell interfaces and

$$\bar{\mathbf{S}}_{ij} \approx \frac{1}{\Delta t} \int_0^{\Delta t} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} \mathbf{S} dy dx d\tau, \quad \bar{\mathbf{D}}_{ij} \approx \frac{1}{\Delta t} \int_0^{\Delta t} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} \mathbf{D} dy dx d\tau. \quad (28)$$

259 are the approximation of the space-time integral of the source terms and diffusion terms inside the
 260 cell. Both approximations are explicit. The time step, Δt , is computed dynamically according to the
 261 condition in [46] to preserve the stability of the numerical solution. Such condition states that the
 262 sum of the Courant-Friedrichs-Levy (*CFL*) and Peclet (*Pe*) number must be below 0.5 for Cartesian
 263 meshes.

264 The numerical fluxes are computed as the space-time integral of the numerical fluxes over the cell
 265 edges. To construct a numerical scheme of order $(2k-1)$ -th, it is sufficient to approximate such integrals
 266 using a $(2k-1)$ -th order Gaussian quadrature, thus requiring k quadrature points. For instance, the
 267 numerical flux $\mathbf{F}_{i+1/2,j}^-$ is computed as follows:

$$\mathbf{F}_{i+1/2,j}^- = \frac{\Delta x}{2} \sum_{q=1}^k w_q \mathbf{F}_{i+1/2,j,q}^- \quad (29)$$

268 where w_q are the Gaussian weights inside the interval $[-1, 1]$ at the $q = 1, \dots, k$ quadrature points along
 269 the cell edge and $\mathbf{F}_{i+1/2,j,q}^-$ the numerical fluxes at each of these points, computed by means of the
 270 resolution of the Cauchy problem.

271 The numerical fluxes are computed solving an arbitrary order approximation of the Cauchy problem
 272 at the quadrature points along cell interfaces. This is given by the so-called DRP, which is defined in
 273 the x direction for the numerical fluxes on the east and west interfaces and in the y direction for those
 274 fluxes on the north and south interfaces. It is worth noting that the source term is included in the
 275 definition of the DRP, according to [26]. The DRP $_K$ defined in the x direction, at the interface $i + 1/2$
 276 and quadrature point q , reads as [26]:

$$\begin{cases} \frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} = \bar{\mathbf{S}}_{i+1/2} \\ \mathbf{U}(x, t = 0) = \begin{cases} \mathbf{U}_{ij}(x, y_{i+1/2,j,q}) & x < 0 \\ \mathbf{U}_{i+1j}(x, y_{i+1/2,j,q}) & x > 0 \end{cases} \end{cases} \quad (30)$$

277 where $\mathbf{U}_{ij}(x, y)$ and $\mathbf{U}_{i+1j}(x, y)$ are smooth spatial reconstructions, defined using the WENO method.
 278 Such functions are evaluated at the particular location where the DRP is defined, $y = y_{i+1/2,j,q}$. On the
 279 other hand, $\bar{\mathbf{S}}_{i+1/2}$ represents the integral of the source term at cell interfaces, which only is non-zero
 280 when considering geometric source terms (e.g. bed elevation source term). It is computed as:

$$\bar{\mathbf{S}}_{i+1/2} \approx \frac{1}{\Delta t} \int_0^{\Delta t} \int_{x_{i+1/2}^-}^{x_{i+1/2}^+} \mathbf{S} \, dx \, d\tau. \quad (31)$$

281 The solution for the DRP in (30) is constructed using the flux expansion approach as:

$$\begin{aligned} \mathbf{F}_{i+1/2,j,q}^- &= \mathbf{F}_{i+1/2,j,q}^{-,(0)} + \sum_{k=1}^K \mathbf{F}_{i+1/2,j,q}^{-,(k)} \frac{\Delta t^k}{(k+1)!}, \\ \mathbf{F}_{i+1/2,j,q}^+ &= \mathbf{F}_{i+1/2,j,q}^{+,(0)} + \sum_{k=1}^K \mathbf{F}_{i+1/2,j,q}^{+,(k)} \frac{\Delta t^k}{(k+1)!}, \end{aligned} \quad (32)$$

282 where $\mathbf{F}_{i+1/2,j,q}^{-,(0)}$, $\mathbf{F}_{i+1/2,j,q}^{-,(k)}$, $\mathbf{F}_{i+1/2,j,q}^{+,(0)}$ and $\mathbf{F}_{i+1/2,j,q}^{+,(k)}$ are computed by solving the DRP $_K$. When using
 283 the LFS-ARoe solver [26], the coefficients of (32) read as:

$$\begin{aligned} \mathbf{F}_{i+1/2}^{-,(k)} &= \mathbf{F}_{i_E}^{(k)} + \sum_{m=1}^{N_\lambda} \left(\tilde{\lambda}^- \alpha^{(k)} - \beta^{-,(k)} \right)_{i+1/2}^m \tilde{\mathbf{e}}_{i+1/2}^m, \quad k = 0, K \\ \mathbf{F}_{i+1/2}^{+,(k)} &= \mathbf{F}_{(i+1)_W}^{(k)} + \sum_{m=1}^{N_\lambda} \left(\tilde{\lambda}^+ \alpha^{(k)} - \beta^{+,(k)} \right)_{i+1/2}^m \tilde{\mathbf{e}}_{i+1/2}^m, \quad k = 0, K \end{aligned} \quad (33)$$

284 where $\mathbf{F}_{i_E}^{(k)}$ and $\mathbf{F}_{(i+1)_W}^{(k)}$ are the left and right-hand limits to the cell edge of the physical flux ($k = 0$) and
 285 their k -th time derivatives, $\alpha^{(k)}$ are the wave strengths, $\beta^{-,(k)}$ the source strengths and $\tilde{\lambda}^\pm$ and $\tilde{\mathbf{e}}_{i+1/2}^m$
 286 the approximate wave celerities and eigenvectors defined using Roe's averages [47]. The computation
 287 of the aforementioned quantities is detailed in [38]. The fluxes $\mathbf{G}_{i,j+1/2,q}^-$ and $\mathbf{G}_{i,j-1/2,q}^+$, are computed
 288 analogously.

289 *3.1. Equilibrium properties*

290 When the velocity vanishes, the equation for the conservation of energy and momentum yield to:

$$\nabla(h+z) = 0 \quad (34)$$

291 which represents the hydrostatic equilibrium, also known in the literature as *lake at rest* condition. At
 292 the discrete level and considering a Cartesian grid, Equation (34) is decomposed into the Cartesian
 293 directions as:

$$\delta(h+z)_{x_2, x_1} = 0, \quad \delta(h+z)_{y_2, y_1} = 0 \quad (35)$$

294 where (x_1, y_1) and (x_2, y_2) are two different points. To construct a well-balanced scheme, the previous
 295 discrete conditions in (35) must be satisfied. This can only be achieved if the WENO reconstruction
 296 method is applied to $\eta = h+z$ and z first, and h is computed from the difference of these reconstructions
 297 as $h_{(\cdot)}^{(0)} = \eta_{(\cdot)}^{(0)} - z_{(\cdot)}^{(0)}$, where $\eta_{(\cdot)}^{(0)}$ and $z_{(\cdot)}^{(0)}$ are the reconstructed water surface elevation and bottom
 298 elevation and $h_{(\cdot)}^{(0)}$ the computed water depth. The discharges hu and hv are also reconstructed using
 299 the WENO method.

300 *3.2. Numerical approximation of source terms*

301 Apart from a suitable reconstruction procedure, source terms have to be properly integrated both
 302 at cell interfaces, using the discretization in (28), and inside cells, using (31), to construct a well
 303 balanced scheme. The numerical approximation of source terms in (28) and (31) can be expressed
 304 using a Taylor power series expansion as follows:

$$\bar{\mathbf{S}}_{ij} = \bar{\mathbf{S}}_{ij}^{(0)} + \sum_{k=1}^K \bar{\mathbf{S}}_{ij}^{(k)} \frac{\Delta t^k}{(k+1)!}, \quad \bar{\mathbf{S}}_{i+1/2} = \bar{\mathbf{S}}_{i+1/2}^{(0)} + \sum_{k=1}^K \bar{\mathbf{S}}_{i+1/2}^{(k)} \frac{\Delta t^k}{(k+1)!}, \quad (36)$$

305 where

$$\begin{aligned} \bar{\mathbf{S}}_{ij}^{(0)} &= \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} \mathbf{S}_i(x, y, 0) \, dy dx, & \bar{\mathbf{S}}_{i+1/2}^{(0)} &= \int_{x_{i+1/2}^-}^{x_{i+1/2}^+} \mathbf{S}_i(x, y, 0) \, dx, \\ \bar{\mathbf{S}}_{ij}^{(k)} &= \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} \partial_t^k \mathbf{S}(x, y, 0) \, dy dx, & \bar{\mathbf{S}}_{i+1/2}^{(k)} &= \int_{x_{i+1/2}^-}^{x_{i+1/2}^+} \partial_t^k \mathbf{S}(x, y, 0) \, dx, \end{aligned} \quad (37)$$

306 are the spatial integrals of the source term and its time derivatives at the initial time.

307 It must be noted that the first component of the source term vector will always be zero, as no
 308 source is considered for the mass conservation equation. On the other hand, the source term may be
 309 non-zero for the x and y momentum equations. The 2-nd and 3-rd components of the numerical source
 310 term, $\bar{\mathbf{S}}_{ij}$, corresponding to the x and y momentum equations, will be hereafter referred to as \bar{S}_{ij}^x and
 311 \bar{S}_{ij}^y , respectively. The same notation is used for $\bar{\mathbf{S}}_{i+1/2}$.

312 *3.2.1. Bed slope*

313 Bed slope source term must be discretized in a particular way so that the scheme preserves the
 314 well-balanced property. Since this source has a geometric nature, it is discretized both at cell interfaces
 315 and inside cells. At cell interfaces, the leading term is computed as:

$$\bar{S}_{i+1/2}^{x,(0)} = (-g\bar{h}\delta z)_{i+1/2}^{(0)}, \quad \bar{S}_{i+1/2}^{y,(0)} = 0, \quad (38)$$

316 which satisfy the steady state equilibrium condition $\left(\tilde{\lambda}^{\pm}\alpha^{(0)} - \beta^{\pm,(0)}\right)_{i+1/2}^m = 0$ in Equation (33). Note
 317 that $\bar{h}_{i+1/2}^{(0)} = 0.5(h_{i_E}^{(0)} + h_{(i+1)_W}^{(0)})$ and $\delta z_{i+1/2}^{(0)} = z_{(i+1)_W}^{(0)} - z_{i_E}^{(0)}$.
 318 Higher order terms are computed as:

$$\bar{\mathbf{S}}_{i+1/2}^{x,(k)} = \left(-g\bar{h}^{(k)}\delta z^{(0)}\right)_{i+1/2}. \quad (39)$$

319 The cell-centered discretization of the bed slope source term needs to be derived to also retain the
 320 order of accuracy of the scheme. To this end, the approach proposed in [38], based on Romberg's
 321 integration, is used for the zero-th order terms:

$$\bar{S}_{ij}^{x,(0)} = \frac{\Delta x}{2} \sum_{\beta=1}^k w_{\beta} \left\{ \bar{S}_{ij,\beta}^{x,(0)} \right\}_m^n, \quad \bar{S}_{ij}^{y,(0)} = \frac{\Delta x}{2} \sum_{\alpha=1}^k w_{\alpha} \left\{ \bar{S}_{ij,\alpha}^{y,(0)} \right\}_m^n \quad (40)$$

322 where $\left\{ \bar{S}_{ij,\beta}^{x,(0)} \right\}_m^n$ and $\left\{ \bar{S}_{ij,\alpha}^{y,(0)} \right\}_m^n$ are based on Richardson's extrapolation of the differential formulation
 323 of the integral of the source term [38]. Such extrapolation provides an arbitrary order approximation
 324 of the integral of the bed slope source term along the straight line connecting two quadrature points
 325 at opposite walls (in Cartesian directions).

326 Concerning the derivative terms, there is no need of a particular discretization to ensure the well-
 327 balanced property as time derivatives vanish under steady state. A 2D Gaussian integration is used
 328 [38]:

$$\bar{S}_{ij}^{x,(k)} = \frac{\Delta x^2}{4} \sum_{\beta=1}^k w_{\beta} \sum_{\alpha=1}^k w_{\alpha} \left(-gh^{(k)}\partial_x z\right)_{\alpha,\beta}, \quad \bar{S}_{ij}^{y,(k)} = \frac{\Delta x^2}{4} \sum_{\beta=1}^k w_{\beta} \sum_{\alpha=1}^k w_{\alpha} \left(-gh^{(k)}\partial_y z\right)_{\alpha,\beta} \quad (41)$$

329 where $h_{\alpha,\beta}^{(k)}$ is the k -th time derivative of h at the quadrature point.

3.2.2. Friction

330 The friction term is discretized here as a centered source term, which means that it is not accounted
 331 for in the definition of the DRP. Such approach does not ensure an exact equilibrium between bed and
 332 friction slope but ensures convergence with arbitrary order to this equilibrium state. The following 2D
 333 Gaussian quadrature is proposed to approximate the integral of the leading term inside the cell:
 334

$$\bar{S}_{ij}^{x,(0)} = \frac{\Delta x^2}{4} \sum_{\alpha=1}^k w_{\alpha} \sum_{\beta=1}^k w_{\beta} \left(-c_f |\mathbf{v}^{(0)}| u^{(0)}\right)_{\alpha,\beta}, \quad \bar{S}_{ij}^{y,(0)} = \frac{\Delta x^2}{4} \sum_{\alpha=1}^k w_{\alpha} \sum_{\beta=1}^k w_{\beta} \left(-c_f |\mathbf{v}^{(0)}| v^{(0)}\right)_{\alpha,\beta}, \quad (42)$$

335 with $c_f = c_f(\mathbf{U}^{(0)}, n)$ and n the Manning coefficient. To construct a Gaussian quadrature for the
 336 derivative terms of the source term, the CK procedure must be used first to provide an approximation
 337 of the time derivatives of the source at the quadrature points:

$$S_{\alpha,\beta}^{x,(k)} = \frac{\partial^k}{\partial t^k} (-c_f |\mathbf{v}| u)_{\alpha,\beta}, \quad S_{\alpha,\beta}^{y,(k)} = \frac{\partial^k}{\partial t^k} (-c_f |\mathbf{v}| v)_{\alpha,\beta}. \quad (43)$$

338 Then, we can construct the 2D Gaussian quadrature as follows:

$$\bar{S}_{ij}^{x,(k)} = \frac{\Delta x^2}{4} \sum_{\alpha=1}^k w_{\alpha} \sum_{\beta=1}^k w_{\beta} S_{\alpha,\beta}^{x,(k)}, \quad \bar{S}_{ij}^{y,(k)} = \frac{\Delta x^2}{4} \sum_{\alpha=1}^k w_{\alpha} \sum_{\beta=1}^k w_{\beta} S_{\alpha,\beta}^{y,(k)}. \quad (44)$$

339 *3.3. Numerical approximation of the diffusion terms*

340 The diffusion terms in (13) are based on Boussinesq approximation and involve the numerical
 341 approximation of spatial gradients of the velocity. Here, two different approaches to compute such
 342 spatial gradients are tested:

- 343 • Sub-cell derivative reconstruction procedure:

344 This is an arbitrary order reconstruction technique originally proposed in [48]. It is based on
 345 constructing a high order polynomial inside each cell by using as interpolating data the high
 346 order reconstructed information (provided by the WENO reconstruction or any other recon-
 347 struction method [49]) at different points inside the cell. The derivatives of such polynomial will
 348 approximate the derivatives of the variables of interest.

349 The WENO method for reconstructing the sub-cell data uses a stencil selection procedure to
 350 prevent spurious oscillations and guarantees monotonicity near strong gradients or discontinuities.
 351 Such features are normally observed in boundary layers, where the relation between the physical
 352 width of the boundary layer and the cell size is crucial for an accurate reconstruction of the
 353 boundary data. In cases where the width of the boundary layer is smaller than the cell size, the
 354 local velocity gradients are interpreted by the WENO reconstruction as a discontinuity between
 355 smooth flow domains (i.e. the limiting case would be given by two regions connected by a
 356 discontinuity as presented in Appendix A). The sub-cell derivative reconstruction thus yields
 357 to nil values of spatial derivatives in the boundary layer, which it is not consistent with the real
 358 physics and ultimately it leads to an underestimation of the diffusive terms in Equation 13 (i.e.
 359 the magnitude of the turbulent mixing modelled by the diffusive terms is underestimated).

360 A different possibility for reconstructing the sub-cell data consists of using a linear reconstruction
 361 by means of a classical high order polynomial. With a classical polynomial, non-linear limiters,
 362 such as those present in the WENO method, are avoided. Such way, the aforementioned physical
 363 inconsistency is partially overcome as the jump in the flow velocity is spread along a length
 364 corresponding to the cell size. This avoids the limitation of large sub-cell gradients, leading to
 365 larger diffusion at these regions. The overall performance of the non-limited sub-cell derivative
 366 reconstruction procedure is better than the WENO sub-cell derivative reconstruction procedure,
 367 even though the diffusive terms are not accurately discretized either. In the framework of very
 368 high order schemes, it would be worth exploring more sophisticated reconstruction techniques
 369 that are more consistent with the physics of the flow.

- 370 • Centered differences:

371 Another possibility to compute the spatial gradients is to use centered differences, departing from
 372 cell-averaged data. This approach is second order accurate and is free from limiting techniques,
 373 hence it is adequate to approximate the diffusion terms in (13). Moreover, it has lower compu-
 374 tational cost than the sub-cell derivative reconstruction procedure. By using this discretization,
 375 Equation (13) yields to:

$$\bar{\mathbf{D}}_{x,ij} = \Delta x \delta \begin{pmatrix} 0 \\ \bar{hT}_{xx} \\ \bar{hT}_{yx} \end{pmatrix}_{i-1/2,j}^{i+1/2,j}, \quad \bar{\mathbf{D}}_{y,ij} = \Delta x \delta \begin{pmatrix} 0 \\ \bar{hT}_{xy} \\ \bar{hT}_{yy} \end{pmatrix}_{i,j-1/2}^{i,j+1/2}. \quad (45)$$

376 where:

$$\bar{h}_{i+1/2,j} = \frac{h_{i+1,j} + h_{i,j}}{2}, \quad \bar{h}_{i-1/2,j} = \frac{h_{i,j} + h_{i-1,j}}{2}, \quad (46)$$

377 are the average water depths at x -cell interfaces, computed using cell averages. The average water
 378 depths at y -cell interfaces are computed analogously. More generally, $(\bar{\cdot})_{i+1/2,j} = \frac{(\cdot)_{i+1,j} + (\cdot)_{i,j}}{2}$ and
 379 $(\bar{\cdot})_{i,j+1/2} = \frac{(\cdot)_{i,j+1} + (\cdot)_{i,j}}{2}$. The approximate stresses, \bar{T}_{xx} , \bar{T}_{xy} , \bar{T}_{yx} and \bar{T}_{yy} , are approximated using
 380 central differences as follows:

$$(\bar{T}_{xx})_{i+1/2,j} = 2 \left(\nu + \bar{\nu}_{t_{i+1/2,j}} \right) \left(\frac{u_{i+1,j} - u_{i,j}}{\Delta x} \right), \quad (47)$$

$$(\bar{T}_{yx})_{i+1/2,j} = \left(\nu + \bar{\nu}_{t_{i+1/2,j}} \right) \left(\frac{1}{2\Delta x} \left(\frac{u_{i+1,j+1} + u_{i,j+1}}{2} - \frac{u_{i+1,j-1} + u_{i,j-1}}{2} \right) + \frac{v_{i+1,j} - v_{i,j}}{\Delta x} \right), \quad (48)$$

$$(\bar{T}_{xy})_{i,j+1/2} = \left(\nu + \bar{\nu}_{t_{i,j+1/2}} \right) \left(\frac{u_{i,j+1} - u_{i,j}}{\Delta x} + \frac{1}{2\Delta x} \left(\frac{v_{i+1,j+1} + v_{i+1,j}}{2} - \frac{v_{i-1,j+1} + v_{i-1,j}}{2} \right) \right), \quad (49)$$

$$(\bar{T}_{yy})_{i,j+1/2} = 2 \left(\nu + \bar{\nu}_{t_{i,j+1/2}} \right) \left(\frac{v_{i,j+1} - v_{i,j}}{\Delta x} \right). \quad (50)$$

381 The centered differences have a truncation error of order $\mathcal{O}(\Delta x^2)$, which is lower than that of the
 382 convective part. However, due to the particular application of the scheme that involve small diffusion
 383 coefficients, the presence of the $\mathcal{O}(\Delta x^2)$ term in the truncation error is far from limiting the higher
 384 order of accuracy of the scheme (i.e. the convective term dominates over the diffusive terms as the
 385 Peclet number [46] is several orders of magnitude lower than the CFL number).

386 Among the two physically consistent approaches: (i) a non-WENO polynomial and (ii) the centered
 387 derivatives, the second order centered derivatives technique is eventually chosen, as stated in the
 388 Appendix A. Although the level of accuracy is lower with the centered derivatives than with a
 389 high order polynomial reconstruction, the computational effort saved with the centered derivatives
 390 compensated such selection.

391 3.4. Boundary wall numerical treatment: the composite wall friction model

392 The friction in a 1D model of a channel is computed by means of a composite Manning coefficient,
 393 which includes the friction of the lateral walls. The Horton-Einstein equation is thus used:

$$n = \left(\frac{n_b^{3/2} b + 2n_w^{3/2} h}{b} \right)^{2/3} \quad (51)$$

394 where n_b is the roughness coefficient of the bed, n_w the roughness coefficient of the walls, b is the width
 395 of the channel and h the water depth. In the numerical model proposed here, a composite Manning
 396 coefficient is considered in those cells containing solid walls (i.e. the boundaries that enclose the water
 397 body). The roughness coefficient in such cells, n , is computed by accounting for the contribution of
 398 the bed plus the contribution of the walls. In the case of having solid walls along the x direction, we
 399 compute the x -Manning composite coefficient as:

$$n_x = \left(\frac{n_b^{3/2} \Delta y + n_w^{3/2} h}{\Delta y} \right)^{2/3} \quad (52)$$

400 and analogously when having solid walls along the y direction:

$$n_y = \left(\frac{n_b^{3/2} \Delta x + n_w^{3/2} h}{\Delta x} \right)^{2/3} \quad (53)$$

401 The proposed model accounts for the shear stress on the walls by including an extra friction
 402 component on the boundary cells, introducing all the extra friction on the first row of cells by re-
 403 calculating the Manning coefficient using the Einstein-Horton formulation. The shear momentum
 404 introduced in the first cell by the effect of the wall roughness is eventually transferred to the inner
 405 cells (in the spanwise direction, towards the center of the channel) thanks to the diffusion terms and
 406 the numerical diffusion. The size of the boundary layer developed in the numerical solution will thus
 407 depend on the wall roughness, the diffusion coefficient and ultimately, in some extent, on the mesh
 408 resolution.

409 In the present model there is still a broken link between the wall roughness and the mixing length:
 410 both play an important role on the vortical dynamics. As a future work, it will be useful to carry out
 411 a further investigation on more accurate wall friction models by including the relation between such
 412 quantities: the wall roughness and the mixing length. For instance, it will be valuable to implement a
 413 wall friction law which could dynamically span over a certain number of cells and ultimately, develop
 414 a velocity profile over the wall. The wall roughness could thus actively participate in the vortical
 415 production of the mixing layer.

416 4. Numerical results

417 4.1. Single lateral cavities

418 4.1.1. Problem configuration

419 The fundamental behavior of unsteady flow in an open channel with a lateral cavity was investi-
 420 gated, both experimentally and numerically, by Kimura and Hosoda in [8]. A schematic diagram of
 421 the flow domain is shown in Figure 2. A rectangular dead zone of variable longitudinal length (two
 422 cases are proposed with $L = 15$ and $L = 22.5$ cm) is attached to the side wall of an open channel. The
 423 length in the transverse direction, W , is 15.0 cm, and the channel width, B , is 10.0 cm. The hydraulic
 424 variables for the two laboratory tests proposed in [8], hereafter referred to as Case 1 and Case 2, are
 425 listed in Table 1. For the velocity measurements, an anemometer with a propeller with a diameter of
 426 3 mm was used for velocities over 6 cm/s, and a thermal type anemometer with a diameter of 3 mm
 427 was used for velocities below 6 cm/s. Temporal velocity variations were measured at half the depth
 428 along the interface along the $T - T'$ section in 2. One of the fundamental properties of this type of
 429 flow is the generation of seiches due to the coupling between the shedding of vortices at the opening
 430 of the cavity with a traveling gravity wave inside the cavity. A selective amplification of vortices is
 431 induced by the presence of the seiche. Such vortices are eventually transported inside the cavity and
 432 dissipated. According to [8], the analytical estimation of the period of the seiche is given by:

$$T = \frac{2L}{nc}, \quad n = 1, 2, 3, \dots \quad (54)$$

433 yielding a period of $T = 0.958$ s and $T = 1.011$ s for cases 1 and 2 respectively, which are larger than
 434 the experimental measurements, in Table 1. Note that such values are estimated considering the first
 435 harmonic, that is $n = 1$.

436 The experiments were simulated using the proposed WENO-ADER scheme. Note that the lengths
 437 L_{up} and L_{down} in Figure 2 are required to define the computational domain. Such parameters are
 438 unspecified in the original work [8] and here we choose $L_{up} = 15$ cm and $L_{down} = 20$ cm for Case

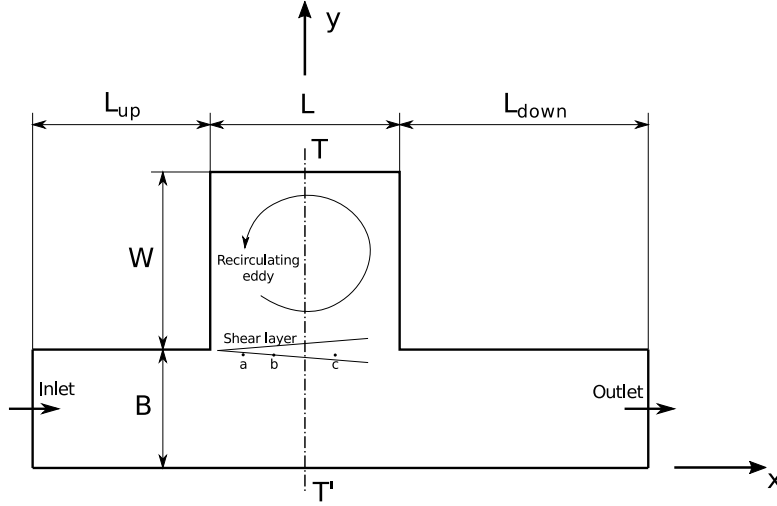


Figure 2: Sketch of the experimental setup used by [8] and detail of the main features of the flow.

Case	L (cm)	W (cm)	B (cm)	q (l/s)	h_0 (cm)	Slope	Fr	Re	T exp. (s)
1	15.0	15.0	10.0	0.255	1.00	1/500	0.81	2781	0.89
2	22.5	15	10.0	0.747	2.02	1/500	0.83	8150	0.87

Table 1: Experimental hydraulic parameters.

439 1 and $L_{up} = 15$ cm and $L_{down} = 25$ cm for Case 2. For all simulations, a stability condition of
 440 $CFL + Pe = 0.4$ is used.

4.1.2. Case 1

442 Numerical results for case 1 are presented in this section. Different sub-cases, defined by different
 443 calibration and simulation parameters, are presented. Details of the parameters for the simulations
 444 are summarized in Table 2. Runs 1 to 5 correspond to unsteady RANS (URANS) simulations whereas
 445 run 6 corresponds to a RANS simulation, which only resolves averaged quantities.

Run	Model	q_{bc} (l/s)	h_{bc} (cm)	ν (m ² /s)	λ	β	n_b	n_w	Δx (mm)	Order
1.1	URANS	0.255	1.0	1E-6	0.3	0.3	0.010	0.03	1.25	3
1.2	URANS	0.255	1.0	1E-6	0.3	0.3	0.010	0.03	2.5	3
1.3	URANS	0.255	1.0	1E-6	0.3	0.3	0.010	0.03	5	3
1.4	URANS	0.255	1.0	1E-6	0.3	0.3	0.010	0.03	2.5	1
1.5	URANS	0.255	1.0	1E-6	0.15	0.05	0.010	0.025	1.25	3
1.6	RANS	0.255	1.0	1E-6	1.5	2.0	0.010	0.025	1.25	1

Table 2: Numerical parameters for the turbulent model and the simulations.

446 The Manning coefficient has to be properly calibrated to ensure uniform flow along the channel
 447 with a water depth of $h = 1$ cm. We have selected $n_b = 0.010$ as the optimal Manning coefficient.
 448 In the original work by Kimura and Hosoda in [8], non-slip boundary conditions were imposed in
 449 combination with an adaptive mesh refinement near the walls. In this work, we consider the composite
 450 Horton-Einsten Manning formulation to account for the friction with the walls as described before. The
 451 numerical results evidence that the lower the wall roughness, the larger the velocity in the recirculation

452 inside the lateral cavity. According to the results, the optimal choice for the wall Manning coefficient
 453 should be between $n_w = 0.02$ and $n_w = 0.03$. A sensitivity analysis of the solution to the grid size was
 454 carried out in runs 1.1–1.3, where three different grids with $\Delta x = 1.25$ (run 1.1), $\Delta x = 2.5$ (run 1.2)
 455 and $\Delta x = 5$ mm (run 1.3) were used.

456 Numerical results for the time-averaged streamwise velocities along the section $T - T'$ are presented
 457 in Figure 3 (a) and results for the mean water surface evolution in time at the upstream edge of the
 458 cavity are presented in Figure 3 (b). The amplitude of the seiche decreases as the grid size increases.
 459 In addition, its period is practically not modified, as reported in [8]. It is observed that the grid
 460 resolution of $\Delta x = 5$ mm is not enough to properly capture the expected amplitude of the seiche and
 461 the measured magnitudes of velocity. On the other hand, grid sizes $\Delta x = 2.5$ and $\Delta x = 1.25$ mm
 462 provide similar results that mimic the experimental measurements and the results in [8], which means
 463 that $\Delta x = 2.5$ m is sufficient to provide the necessary accuracy.

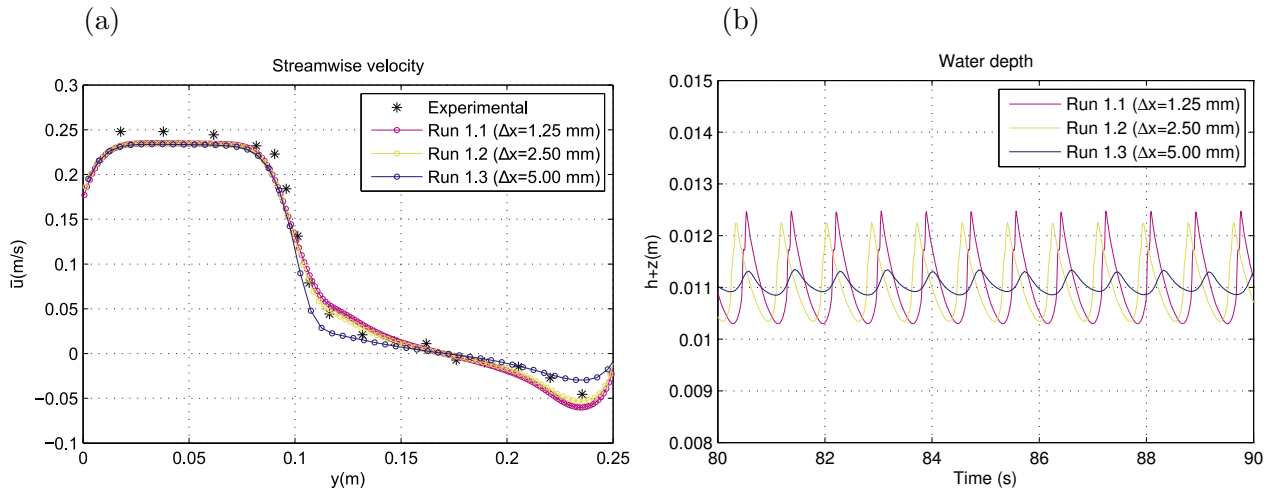


Figure 3: (a) Computed and measured time-averaged streamwise velocity along $T - T'$ for $\Delta x = 1.25$ (run 1.1), $\Delta x = 2.5$ (run 1.2) and $\Delta x = 5$ mm (run 1.3). (b) Mean water depth at the upstream edge of the cavity computed using $\Delta x = 1.25$ (run 1.1), $\Delta x = 2.5$ (run 1.2) and $\Delta x = 5$ mm (run 1.3). Time-averaged quantities are denoted by the overbar symbol ($\bar{\cdot}$).

464 The role of the order of accuracy of the numerical scheme was also investigated. A comparison of
 465 the results provided by the 3-rd order WENO-ADER scheme (run 1.2) and a 1-st order scheme (run
 466 1.4), using a $\Delta x = 2.5$ mm grid, is presented in Figure 4. The numerical results evidence that a first
 467 order of accuracy is too diffusive to reproduce the standing gravity wave, since the shear instability at
 468 the opening of the cavity cannot be captured. Additionally, it is observed that the velocities inside the
 469 cavity, along the section $T - T'$, are underestimated by the 1-st order scheme. This scheme is thus not
 470 adequate to reproduce resonant flows. Only when using the RANS approach (i.e. when using adequate
 471 values for the turbulent viscosity), the 1-st order scheme is valid.

472 In [8], the authors provided measurements of the time evolution of the streamwise velocity mag-
 473 nitude at different locations along the shear layer. In particular, they define the points a , b and c ,
 474 whose coordinates are provided in [8]. Such data allow to assess the validity of the numerical model
 475 to reproduce the vortex generation/amplification process and its coupling with the seiche.

476 Finally, and departing from the previous results, the best calibration of the URANS model was
 477 obtained and corresponds to run 1.5. The bottom and wall friction (n_b and n_w) and the turbulence
 478 parameters of the model (λ and β) were fine tuned to provide accurate results of the time-averaged
 479 velocities along the section $T - T'$ (Figure 5), assuming a realistic description of the time evolution
 480 of the velocity at the shear layer, when compared to the measurement in [8] (Figure 6). This case

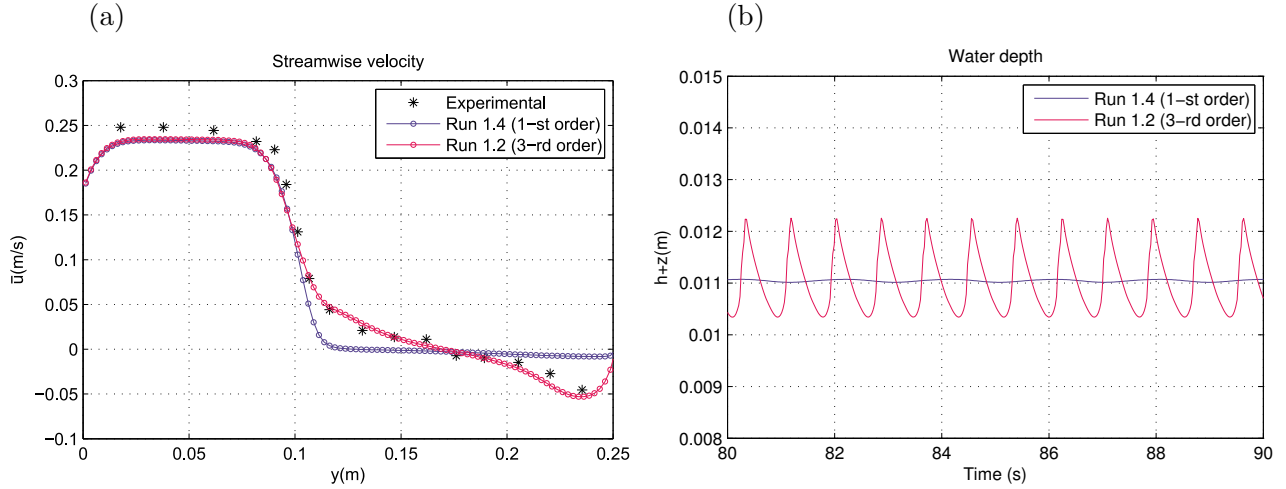


Figure 4: (a) Computed and measured time-averaged streamwise velocity along $T - T'$ for a 1-st order scheme (run 1.4) and 3-rd order scheme (run 1.2). (b) Mean water depth at the upstream edge of the cavity computed using a 1-st order scheme (run 1.4) and 3-rd order scheme (run 1.2). Time-averaged quantities are denoted by the overbar symbol ($\bar{\cdot}$).

481 corresponds to a URANS simulation where the horizontal eddies are practically resolved while small-
 482 scale dissipative effects are modelled by choosing a suitable parametrization of the turbulent viscosity.

483 The numerical results provided by the URANS simulation (run 1.5) are compared to the results
 484 provided by a pure RANS computation (run 1.6) in Figures 5 and 6. Note that the RANS approach
 485 provides a steady solution with no variation in time of the water level. The whole effect of the
 486 time fluctuating value of the water level is modelled and accounted for by the turbulence model. A
 487 comparison of the time-averaged longitudinal velocity along $T - T'$ is depicted in Figure 5 (a), showing
 488 that the URANS approach (run 1.5) provides accurate results in the main channel and in the shear
 489 layer, though it overestimates the velocity in the recirculation zone. On the other hand, the RANS
 490 approach (run 1.6) appears to be diffusive (due to excessive numerical viscosity) and inaccurate in the
 491 main channel and shear layer though it provides acceptable results inside the cavity. A comparison of
 492 the time-averaged velocity field for the URANS and RANS approaches (run 1.5 and 1.6 respectively)
 493 with the solution in [8] is presented in Figure 5 (b, c and d). It is observed that the velocity field
 494 provided by the proposed scheme when using the URANS model contains the same features than the
 495 solution in [8]. Inside the cavity, both solutions show the presence of a main vortex, referred to as S1,
 496 and three secondary eddies, denoted by S2, S3 and S4. The RANS model only shows the main vortex,
 497 S1, but the secondary eddies cannot be captured.

498 The lack of accuracy of the RANS model in the resolution of the shear layer is evidenced in Figure
 499 6: the velocity at points a ($x = 0.175$ m, $y = 0.09$ m), b ($x = 0.200$ m, $y = 0.09$ m) and c ($x = 0.250$
 500 m, $y = 0.09$ m), which is steady given the nature of RANS simulations, is lower than the time varying
 501 experimental data.

502 4.1.3. Case 2

503 In this section, numerical results for the case 2 (cavity length $L = 22.5$ m) in Table 1 are presented.
 504 The calibration chosen for Case 1 - run 1.5 is preserved to evidence the applicability of the numerical
 505 model to a different case. Details of the parameters for the simulation are summarized in Table 2. In
 506 run 2, a RANS simulation of the problem is carried out. To this end, a different configuration of the
 507 turbulence model is required in order to model all fluctuations in time. When considering the URANS
 508 approach (run 1), relatively small values of λ and β are used, so that the main coherent vortices are not
 509 damped and can be resolved by the model. For the RANS approach, such calibration constants must

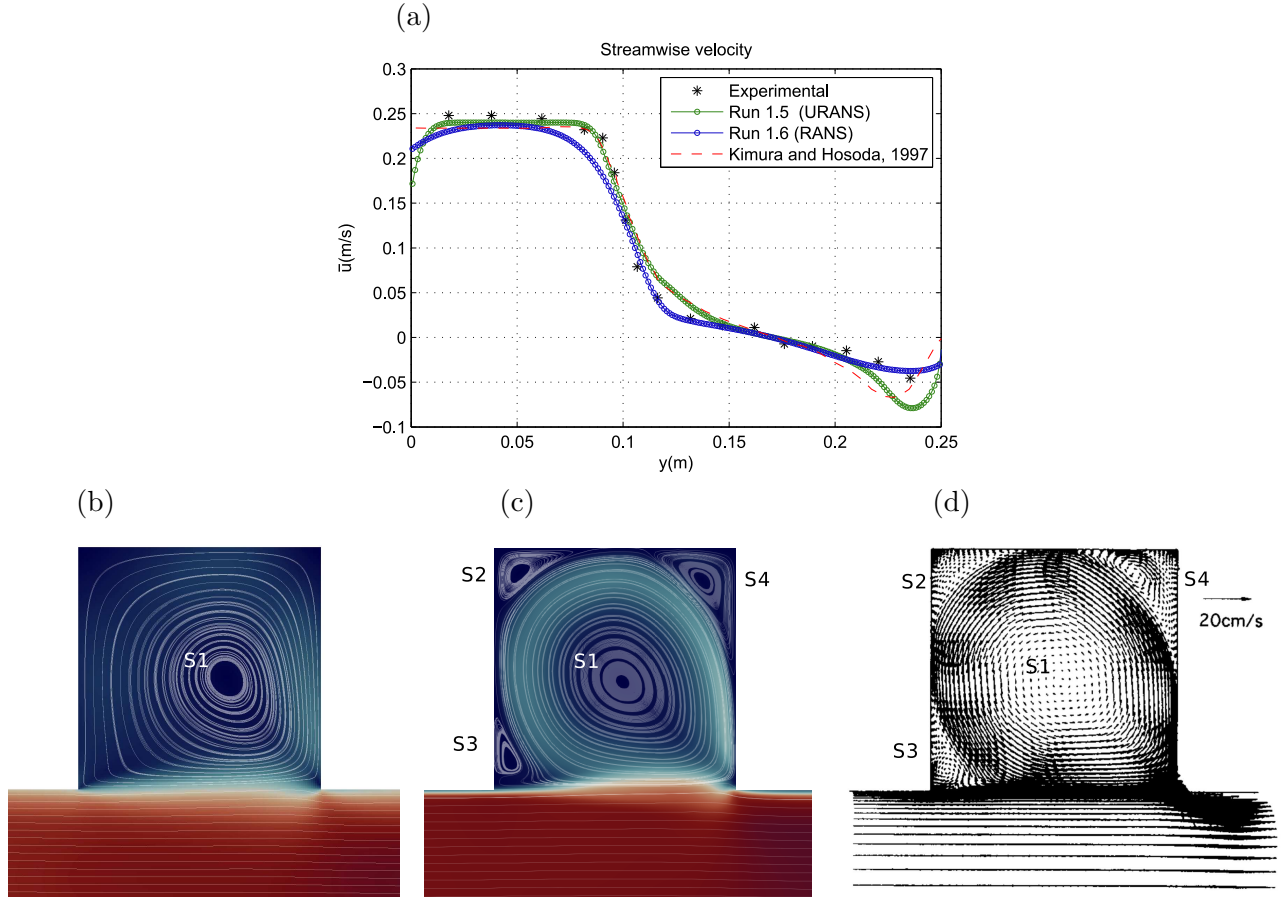


Figure 5: Computed and measured time-averaged streamwise velocity along $T - T'$ (a). Magnitude and streamlines of the time-averaged velocity provided by the RANS approach (b) and URANS approach (c). Numerical solution provided in [8] (d). The main vortical structure is denoted by S1 and the secondary eddies by S2, S3 and S4. Note that the bottom-right plot has been obtained from [8]. Time-averaged quantities are denoted by the overbar symbol ($\bar{\cdot}$).

510 be increased in order to damp any fluctuation and to model their effect as a pure diffusive process.

Run	Model	q_{bc} (l/s)	h_{bc} (cm)	ν (m^2/s)	λ	β	n_b	n_w	Δx (mm)	Order
2.1	URANS	0.747	2.02	1E-6	0.15	0.05	0.010	0.025	2.5	3
2.2	RANS	0.747	2.02	1E-6	1.5	2.0	0.010	0.025	2.5	1

Table 3: Numerical parameters for the turbulent model and the simulations.

511 In Figure 7 (a), the numerical solution for the time-averaged streamwise velocity component along
512 $T - T'$ is compared with the measured and numerical solution provided in [8]. A top-view representation
513 of the time-averaged velocity field for the RANS and URANS simulations, compared with the results
514 provided in [8], is presented in Figure 7 (b, c and d). It is observed that the numerical solution provided
515 by the proposed model matches the reference solution. The results provided by the RANS approach
516 (run 2) evidence that the velocity gradient across the shear layer is smeared and the prediction of the
517 flow field in the main channel is less accurate.

518 Figure 8 shows the evolution in time of the numerical and measured mean water surface elevation,
519 h , and u velocity at point d . As mentioned before, only when using the URANS approach (run 2.1),

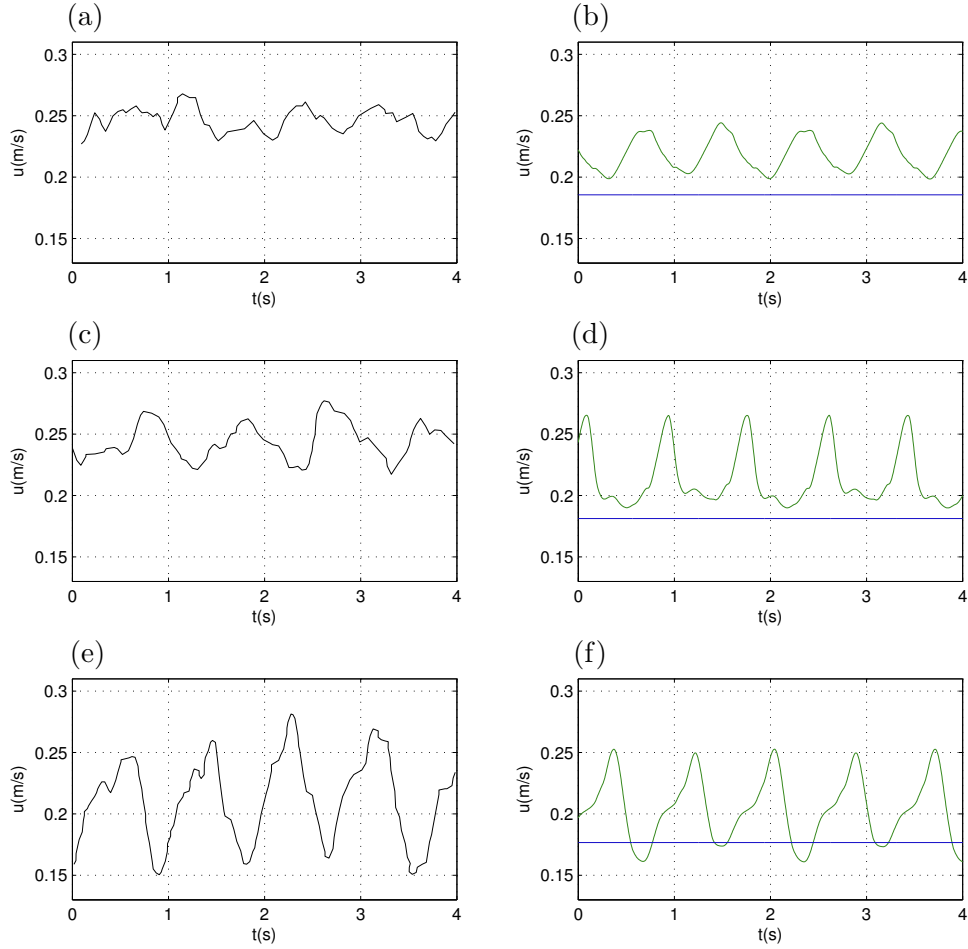


Figure 6: Measured (plots a, c and e) and numerical (plots b, d and f) evolution of the streamwise velocity in the x -direction at points a (plots a and b), b (plots d and d) and c (plots e and f). Results provided by URANS (run 1.5, green line) and RANS approaches (run 1.6, blue line).

520 temporal variations of the mean flow field are captured. As in [8], the oscillation of the predicted
 521 streamwise mean velocity, u , has a greater amplitude than in the experimental observations. This
 522 may be due to an excessive level of periodicity and coherence in the numerical solution, without extra
 523 oscillating modes, produced by an absence of vortex breakup mechanisms that would damp the periodic
 524 component associated to the seiche [50]. Figure 8 also shows that the RANS approach yields an steady
 525 solution which provides a better prediction of water depth and velocity, in terms of time-averaged
 526 values, than in Case 1.

527 The numerical results prove that the proposed WENO-ADER scheme reproduces the observed
 528 seiche, providing an accurate estimation of the frequency and amplitude of the oscillation, as well as
 529 the instantaneous and time-averaged magnitude of the velocities within an acceptable level of accuracy.
 530 Taking into account the results for Case 1 and 2 discussed above, we can conclude that the proposed
 531 numerical scheme is sufficiently valid to reproduce complex flow patterns in enclosed water bodies,
 532 involving the coupling of shear instabilities and gravity waves.

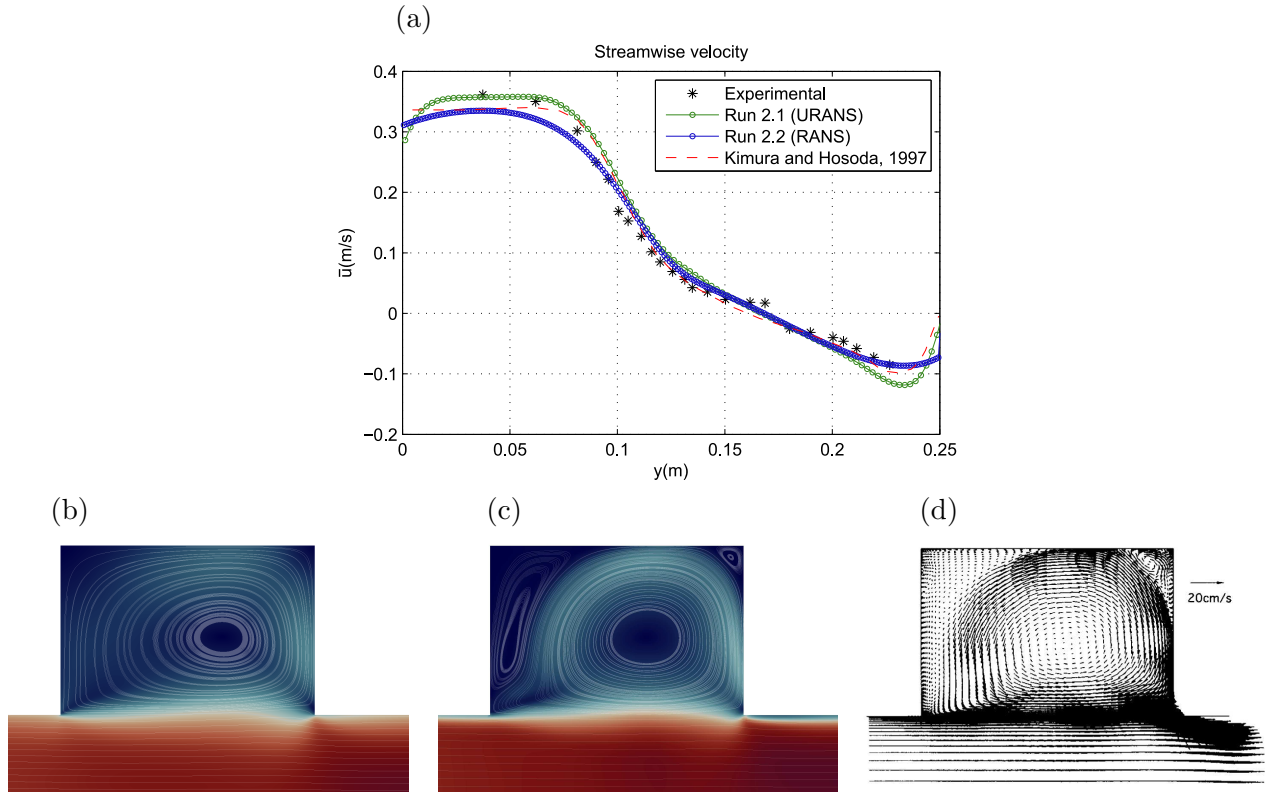


Figure 7: Computed and measured time-averaged streamwise velocity along $T - T'$ (a). Magnitude and streamlines of the time-averaged velocity field obtained by URANS (b) and RANS (c), compared to the time-averaged numerical velocity field provided in [8] (d). Time-averaged quantities are denoted by the overbar symbol ($\bar{\tau}$).

533 *4.2. Multiple lateral cavities*

534 *4.2.1. Problem definition*

535 In this section, the proposed WENO-ADER scheme is applied to the resolution of the flow in
 536 channels with multiple lateral cavities. Such cavities are built in the banks of rivers to create harbors
 537 or to promote areas with hydraulic and morphological diversity that enhance habitat suitability. For
 538 instance, this is a very common measure to recover and revitalize riverine habitats that have degener-
 539 ated because of anthropogenic activities (e.g. river channelization for different purposes such as land
 540 expansion, flood protection or agriculture) [11, 51].

541 The hydrodynamic response of lateral cavities is characterized by a main recirculation region that
 542 may trap the fine sediments travelling within the flow in the main channel as defined in [9, 41, 52]. In
 543 particular conditions, a seiche appears inside the cavities, which homogenizes the small-scale fluctua-
 544 tions in the flow and promotes vertical mixing, compromising the trapping efficiency of the cavities.
 545 Unlike in geometric configurations involving a single cavity, as those in the previous section, the in-
 546 teraction of multiple cavities has an additional effect in the propagation and formation of resonant
 547 waves.

548 The complex flow structures produced by the geometric configurations herein described challenge
 549 the prediction capabilities of the numerical methods and are thus considered in this section to evaluate
 550 the performance of the proposed numerical methods. Systematic laboratory tests in an open channel
 551 equipped with bank lateral embayments were carried out by Juez et al. [9, 41] for different geometric
 552 configurations. The hydrodynamic response of the configurations tested was assessed by means of

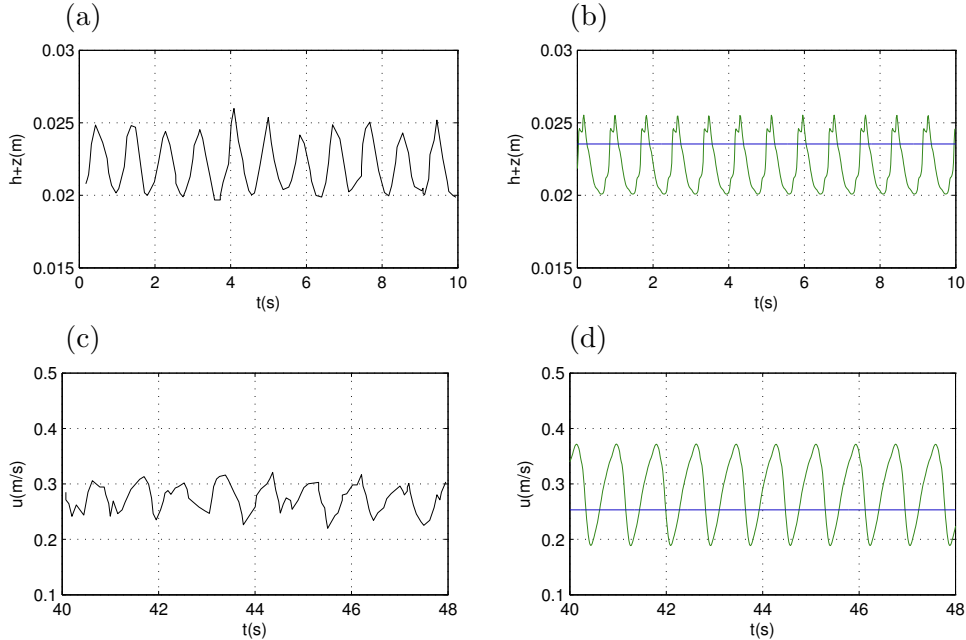


Figure 8: Experimental (a,c) and numerical (b,d) mean water surface elevation (a,b) and time evolution of mean streamwise velocity (c,d) at point d . URANS approach in green line and RANS approach in blue line.

553 velocity (surface PIV) and water surface elevation (ultrasounds probes) measurements. The experi-
 554 mental results obtained in these tests are considered here as a benchmark. In particular, the flow for
 555 the geometric configurations 2.1 and 3.1 [9, 41] will be reproduced.

556 Comparison with experimental data will allow to explore the role of the turbulence model in the
 557 prediction of the frequency and amplitude of the seiche. The effect of the numerical diffusion in the
 558 solution, which is related to the grid size, is investigated.

559 A schematic representation of a sector of a channel with lateral cavities is depicted in Figure 9.
 560 The relevant dimensions, also represented in Figure 9, are the total width of the channel (base channel
 561 and cavities), B , the width of the base channel, b , the lateral width of the cavities, $W = (B - b)/2$,
 562 the length of the cavities, l , and the distance between cavities, L . The geometric configurations 2.1
 563 and 3.1 [9, 41] depicted in Figure 10.

564 4.2.2. Geometric configuration 3.1

565 The geometric configuration 3.1 is characterized by a total channel width of $B = 1$ m, a base
 566 channel width of $b = 0.5$ m and a length of the cavities equal to their separation, that is $l = L = 0.5$
 567 m. The geometry of the channel and the location of three numerical probes, P1, P2 and P3, measuring
 568 the water depth, are depicted in Figure 10 (b). The cavity where surface PIV measurements were
 569 carried out is highlighted in blue and experimental measurement of water depth was only done at the
 570 location of P1 and P3. The slope of the channel is 0.1% and the flow was configured to be uniform,
 571 with $h = 0.05$ m and $Q = 8.5$ l/s.

572 In this particular configuration, a seiche with a period of $T = 2.84$ s was observed [41]. The
 573 analytical estimation of the period of the seiche can be calculated with $T = 2B/nc$, setting $B = 1$ m
 574 and $h = 0.05$ m, and yields is $T = 2.86$ s. This result is in good agreement with the experimental
 575 measurement.

576 All simulations will be computed using the 3-rd order WENO-ADER scheme if no other method
 577 is specified. The simulation time is $t = 300$ s, which is sufficient for the seiche to be stable and allows

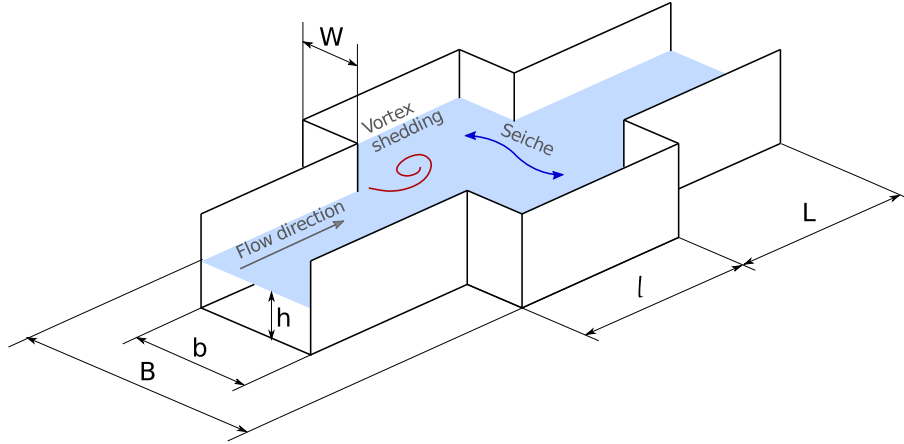


Figure 9: Representation of a sector of the channel with lateral cavities including the relevant geometric dimensions and flow features.

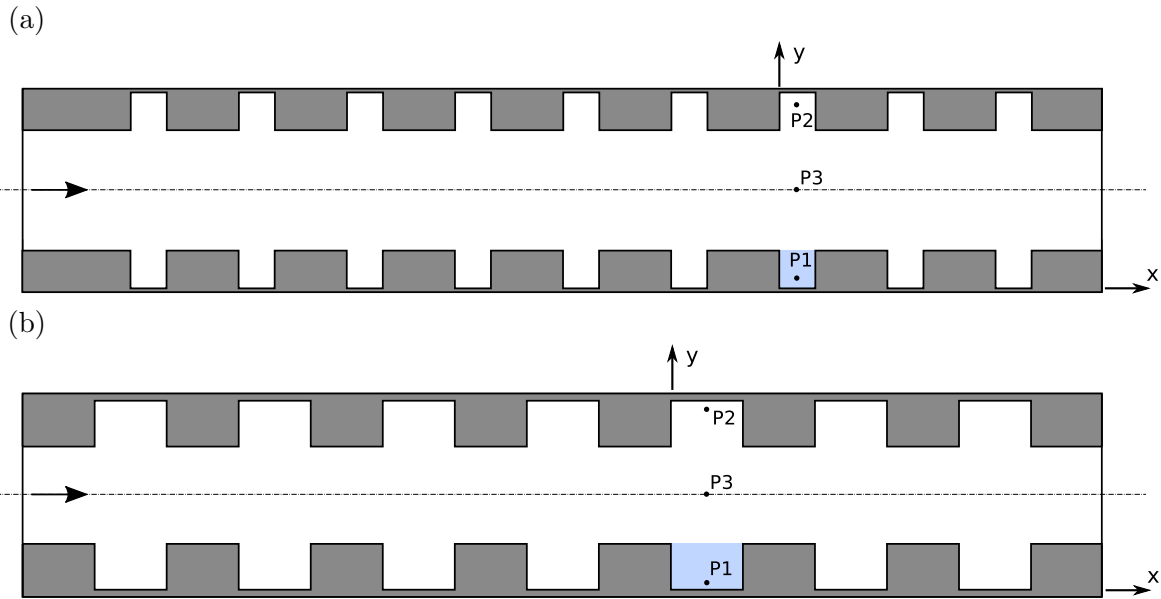


Figure 10: Representation of the channel configuration 2.1 (a) and 3.1 (b), including the location of the probes P1, P2 and P3. The cavity used in the experimental measurements is highlighted in blue.

578 to use an adequate time window for the time integration of the velocities. The computational domain
 579 is given by $\Omega = [0, 7.5] \times [0, 1]$ and four different cell sizes will be used, $\Delta x = 0.0125$ m, $\Delta x = 0.01$
 580 m, $\Delta x = 0.00625$ m and $\Delta x = 0.005$ m. The stability condition is set to $CFL + Pe = 0.45$ for all
 581 simulations. The boundary conditions are given by a constant discharge of $hu = Q/b = 0.017$ m²/s
 582 upstream and a constant water depth $h = 0.05$ m downstream. Note that the boundary conditions are
 583 imposed by means of the characteristic variables at the inlet and outlet cell interfaces.

584 The Manning coefficient on the channel bed is set to $n = 0.01$ sm^{-1/3} and on the channel walls
 585 to $n = 0.03$ sm^{-1/3} in the case of concrete (inner walls) or $n = 0.01$ sm^{-1/3} in the case of glass (side
 586 walls). See [9, 41] for a detailed description of the experimental setup. The depth-averaged mixing
 587 length turbulence model is used, if no other method is specified, considering the following calibration
 588 $\lambda = 0.15$ and $\beta = 0.05$.

589 Remark that a initial perturbation in the transverse direction must be added in order to trigger the

590 presence of the seiche. Otherwise, the numerical solution is perfectly symmetrical and only longitudinal
 591 traveling waves are observed since the grid is also symmetric with respect to the channel longitudinal
 592 axis. Figure 11 shows the evolution in time of the difference $h_{P1} - h_{P2}$ for a simulation without
 593 initial perturbation and with a initial perturbation of $v = 0.008$ m/s. Note that the visualization of
 594 the difference $h_{P1} - h_{P2}$ allows to quantify the level of symmetry of the solution. Only when such
 595 difference is below machine precision, the solution is symmetric. Figure 11 (a) shows the solution for
 596 the whole simulation time. Figure 11 (b) shows a zoom of a small region of the unperturbed solution.
 597 The grid size is $\Delta x = 0.00625$ m. It is observed that only when the initial condition is perturbed, the
 598 span-wise symmetry is broken up and a coherent oscillation is observed. This oscillation represents the
 599 seiche. On the other hand, the unperturbed solution also shows an oscillation, as shown in the zoomed
 600 plot. This oscillation has an amplitude of the magnitude of machine truncation errors. It can be thus
 601 considered numerical noise.

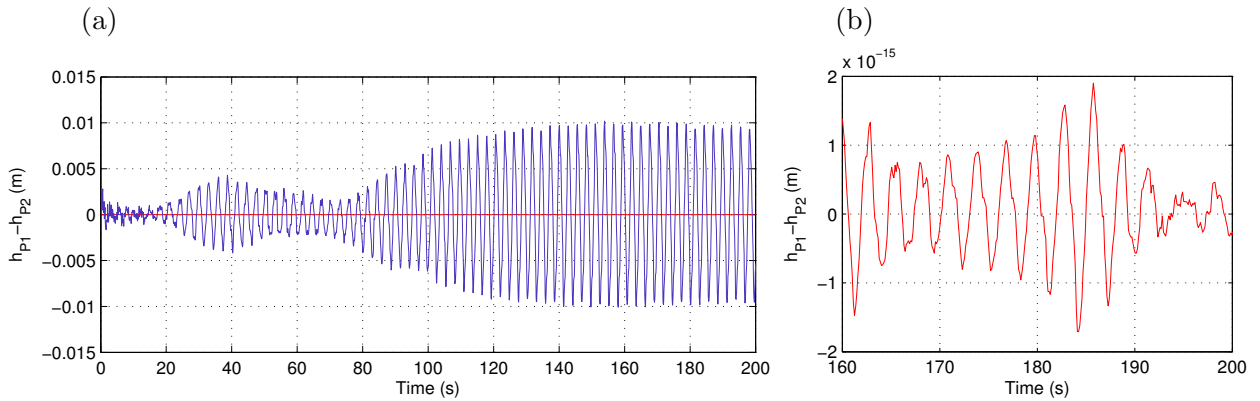


Figure 11: (a) Evolution in time of the mean water depth difference between points P1 and P2, $h_{P1} - h_{P2}$. The red line is the numerical solution when no initial perturbation is added whereas the blue line represents the numerical solution when adding a initial perturbation of $v = 0.008$ m/s. (b) Zoomed region of the unperturbed solution that shows that $h_{P1} - h_{P2}$ is below machine precision.

602 The role of the turbulence model in the numerical prediction of the seiche is now investigated. In
 603 Figure 12, the evolution in time of the numerical mean water depth at P1, P2 and P3 is depicted
 604 for the three different meshes described above. In the upper row, results obtained when using the
 605 depth-averaged mixing length turbulence model ($\lambda = 0.15$ and $\beta = 0.05$) are presented, while in
 606 the bottom row, results without turbulence model are depicted. It is observed that the use of the
 607 turbulence model in combination with a high order of accuracy in the numerical integration is crucial
 608 for a suitable prediction of the water depth oscillation. Only with the help of a turbulent model, the
 609 numerical solution can become mesh independent, as observed in Figure 12. On the other hand, if the
 610 turbulence model is not considered, the periodicity of the oscillation is destroyed as the mesh is refined
 611 due to a lack of dissipation (the numerical diffusion is highly reduced and there is no other source of
 612 diffusion). This is also evidenced in Figure 13, where the power spectrum of the signals, computed by
 613 means of an FFT algorithm, is depicted. It is observed that the analytical estimation of the period of
 614 the seiche is well captured when using the turbulence model, with independence of the grid.

615 A plot of the evolution of the peak-to-peak amplitude, $h_{pp} = 2\sqrt{2}\sigma_h$ (where σ_h is the standard
 616 deviation of h in time), with respect to the cell size is presented in Figure 14. The results are computed
 617 using the depth averaged mixing length turbulence model ($\lambda = 0.15$ and $\beta = 0.05$) and evidence that a
 618 resolution of $\Delta x = 0.00625$ m is sufficient to avoid any dependence with the grid. This can be observed
 619 as the convergence of h_{pp} at P1 to a virtually constant value. Note that the factor $2\sqrt{2}$ is the ratio
 620 between the peak-to-peak amplitude and the standard deviation of a sinusoidal signal. In this case, the

621 seiche is close to a sinusoidal fluctuation, hence h_{pp} will be an adequate estimation of the peak-to-peak
 622 amplitude.

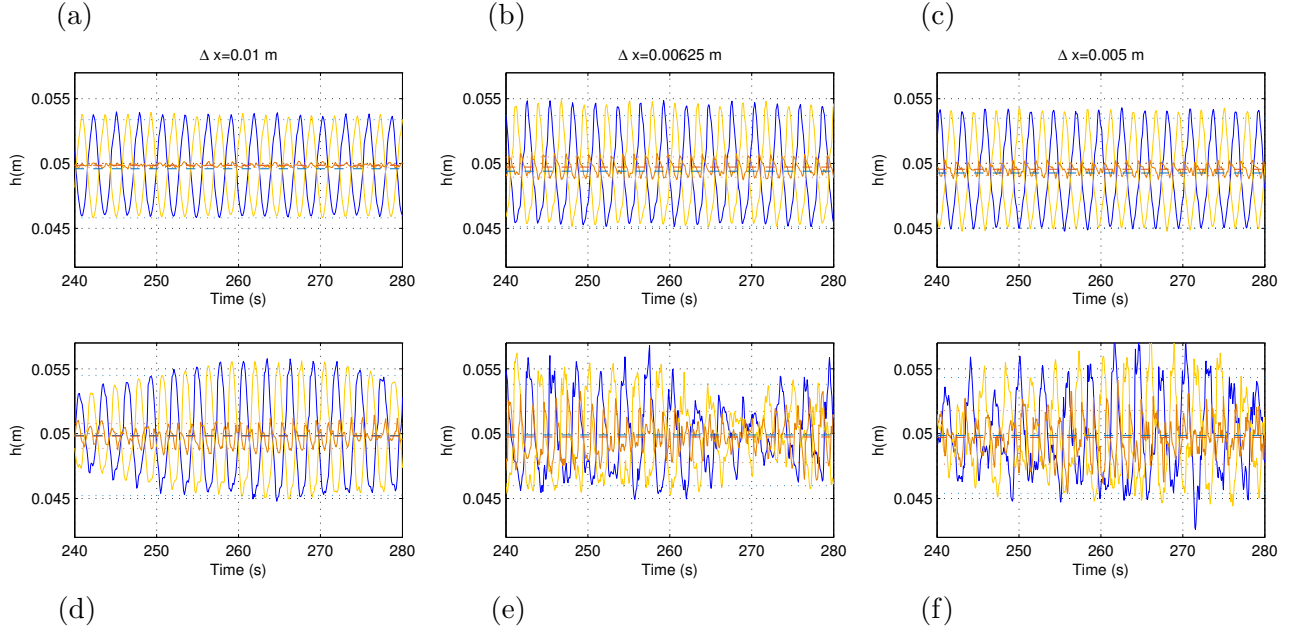


Figure 12: Mean water depth evolution in time at P1 (blue), P2 (yellow) and P3 (orange), computed by the 3-rd order WENO-ADER scheme in three different meshes with $\Delta x = 0.01$ m (a,d), $\Delta x = 0.00625$ m (b,e) and $\Delta x = 0.005$ m (c,f), using the depth-averaged mixing length turbulence model ($\lambda = 0.15$ and $\beta = 0.05$) (a,b,c) and without using any turbulence model ($\lambda = \beta = 0$) (d,e,f).

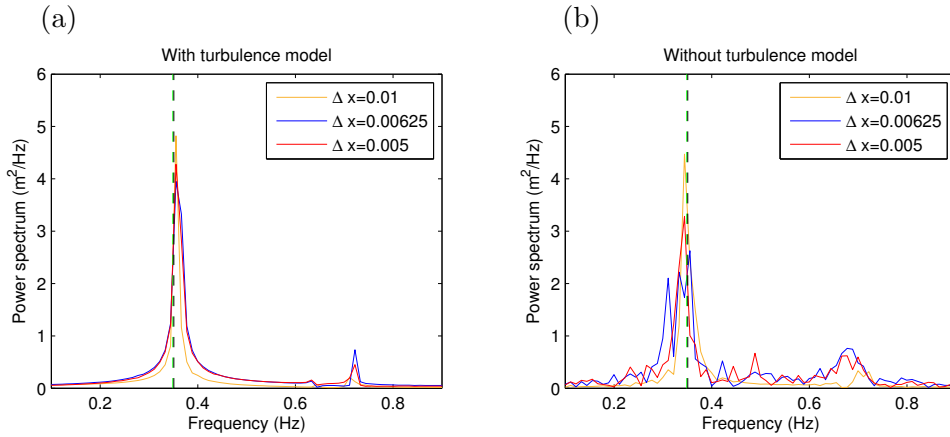


Figure 13: Power spectrum density distribution of the mean water depth fluctuation, measured at P1, for the case with turbulence model (a) and without turbulence model (b). The analytical and experimental frequency of the seiche are depicted with a black and green dashed line, respectively.

623 A comparison between the experimental water depth oscillation, measured at P1 and P3, and the
 624 numerical prediction is depicted in Figure 15. It is again observed that the numerical model correctly
 625 reproduces the frequency and amplitude of the seiche at both locations.

626 The time-averaged velocity field inside the cavity highlighted in Figure 10 was experimentally
 627 assessed using a surface PIV technique. A comparison between the experimental measurements and the
 628 numerical estimation of the velocity inside the cavity is presented in Figure 16. The numerical results

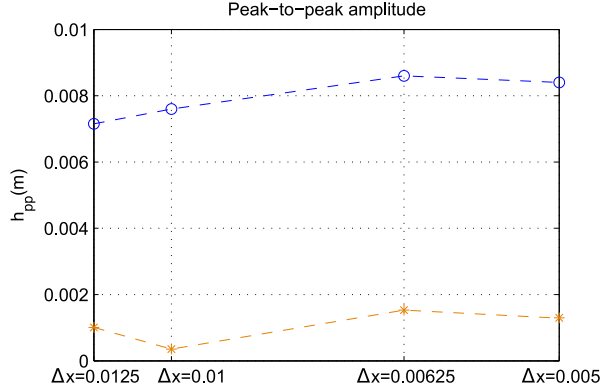


Figure 14: Plot of the peak-to-peak water depth amplitude, $h_{pp} = 2\sqrt{2}\sigma_h$, at P1 (blue) and P3 (orange), against the cell size. The solution is computed using the 3-rd order WENO-ADER scheme and the depth-averaged mixing length turbulence model.

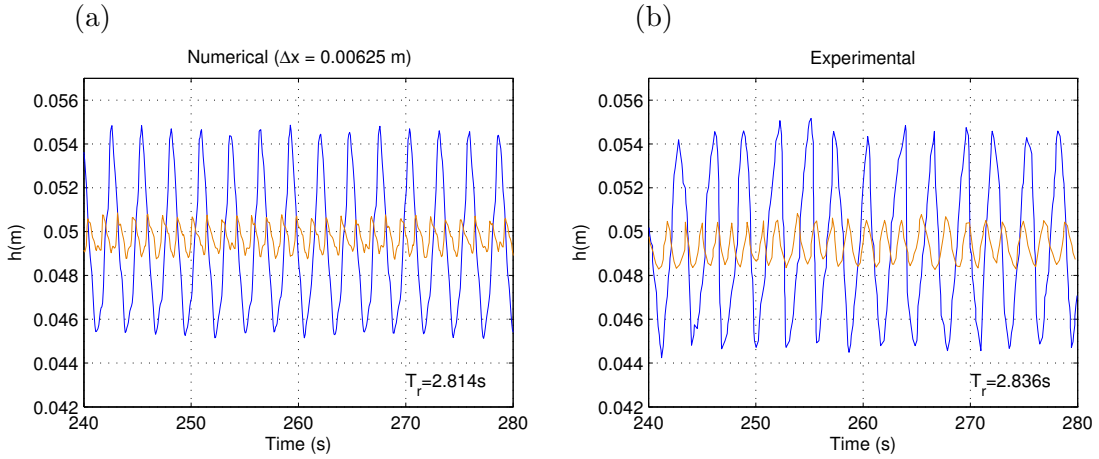


Figure 15: Numerical (a) and experimental (b) mean water depth evolution in time at P1 (blue) and P3 (orange). The numerical solution is computed using the 3-rd order WENO-ADER scheme and the depth-averaged mixing length turbulence model, using $\Delta x = 0.00625$ m. The period of the seiche is obtained using an FFT algorithm.

629 are computed using $\Delta x = 0.0625$ m. Figure 16 (a and b) shows a 2D comparison of the y component
630 of the time-averaged velocity field. It evidences that the numerical model is able to reproduce the
631 recirculating flow inside the cavity. A more quantitative comparison is presented in Figure 16 (c and
632 d), where cross sectional representations of the measured and computed time averaged velocities are
633 plotted. The time averaged streamwise velocity is plotted over the y direction at $x = 0.125, 0.25, 0.375$
634 m, whereas the time averaged spanwise velocity is plotted over the x direction at $y = 0.05, 0.125, 0.2$
635 m. It is observed that the numerical model slightly overpredicts the magnitude of the velocity in the
636 recirculation region, if compared to the surface PIV measurements.

637 4.2.3. Geometric configuration 2.1

638 The geometric configuration 2.1 is characterized by a total width of the channel $B = 1$ m, a width
639 of the base channel $b = 0.6$ m, a length of the cavities equal to $l = 0.25$ m and a separation between
640 cavities of $L = 0.5$ m. The slope of the channel is 0.1% and the flow was configured with $Q = 8.5$ l/s
641 and $h = 0.048$ m at the outlet. In this particular configuration, a periodic seiche was not reported [41].

642 The numerical solution was computed again using the 3-rd order WENO-ADER scheme. The
643 simulation time is $t = 300$ s and the computational domain is given by $\Omega = [0, 7.5] \times [0, 1]$. The

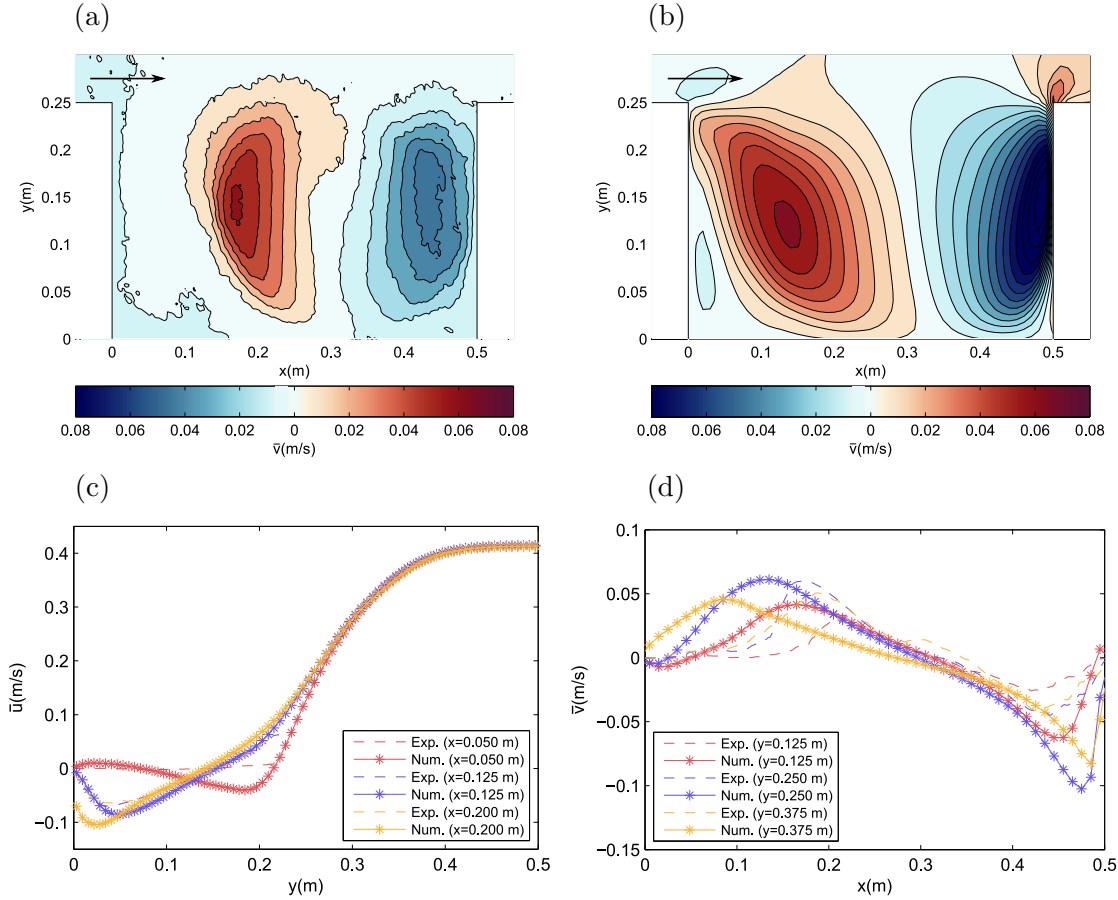


Figure 16: Measured (a) and computed (b) time-averaged spanwise velocity inside a cavity. Cross sectional representation of the measured and computed streamwise time-averaged velocity, along the y direction (c) and spanwise time-averaged velocity, along the x direction (d). Time-averaged quantities are denoted by the overbar symbol ($\bar{\cdot}$).

644 boundary conditions are given by a constant unit discharge of $hu = Q/b = 0.0141\bar{6} \text{ m}^2/\text{s}$ upstream
 645 and a constant water depth $h = 0.048 \text{ m}$ downstream.

646 The Manning coefficient on the channel bed is set to $n = 0.01 \text{ sm}^{-1/3}$ and on the channel walls to
 647 $n = 0.03 \text{ sm}^{-1/3}$ in the case of concrete (inner walls) or $n = 0.01 \text{ sm}^{-1/3}$ in the case of glass (side walls).
 648 The depth-averaged mixing length turbulence model is used with the following calibration $\lambda = 0.15$
 649 and $\beta = 0.05$.

650 As in the previous case, the time-averaged velocity field inside one of the cavities is experimentally
 651 assessed using surface PIV. A comparison between the experimental measurements and the numerical
 652 estimation of the velocity inside the cavity is presented in Figure 17. The numerical results are
 653 computed using $\Delta x = 0.005 \text{ m}$. Figure 17 (a and b) shows a 2D comparison of the y component of the
 654 time-averaged velocity. In Figure 17 (c and d), a cross sectional representation of the measured and
 655 computed time-averaged velocities are plotted. The streamwise component of the velocity is plotted
 656 over the y direction at $x = 0.0625, 0.125, 0.1875 \text{ m}$. In addition, the spanwise component of the velocity
 657 is plotted over the x direction at $y = 0.05, 0.1, 0.15 \text{ m}$. As in the geometric configuration 3.1, it is
 658 observed that the numerical results yield to a higher velocity magnitude in the recirculation region,
 659 specially near the downstream wall.

660 The calibration found in this work for the horizontal eddy viscosity coefficient in URANS simulation
 661 (i.e. $\beta = 0.05$) is consistent with previous literature as the coefficient βl_s^2 is of the order of magnitude

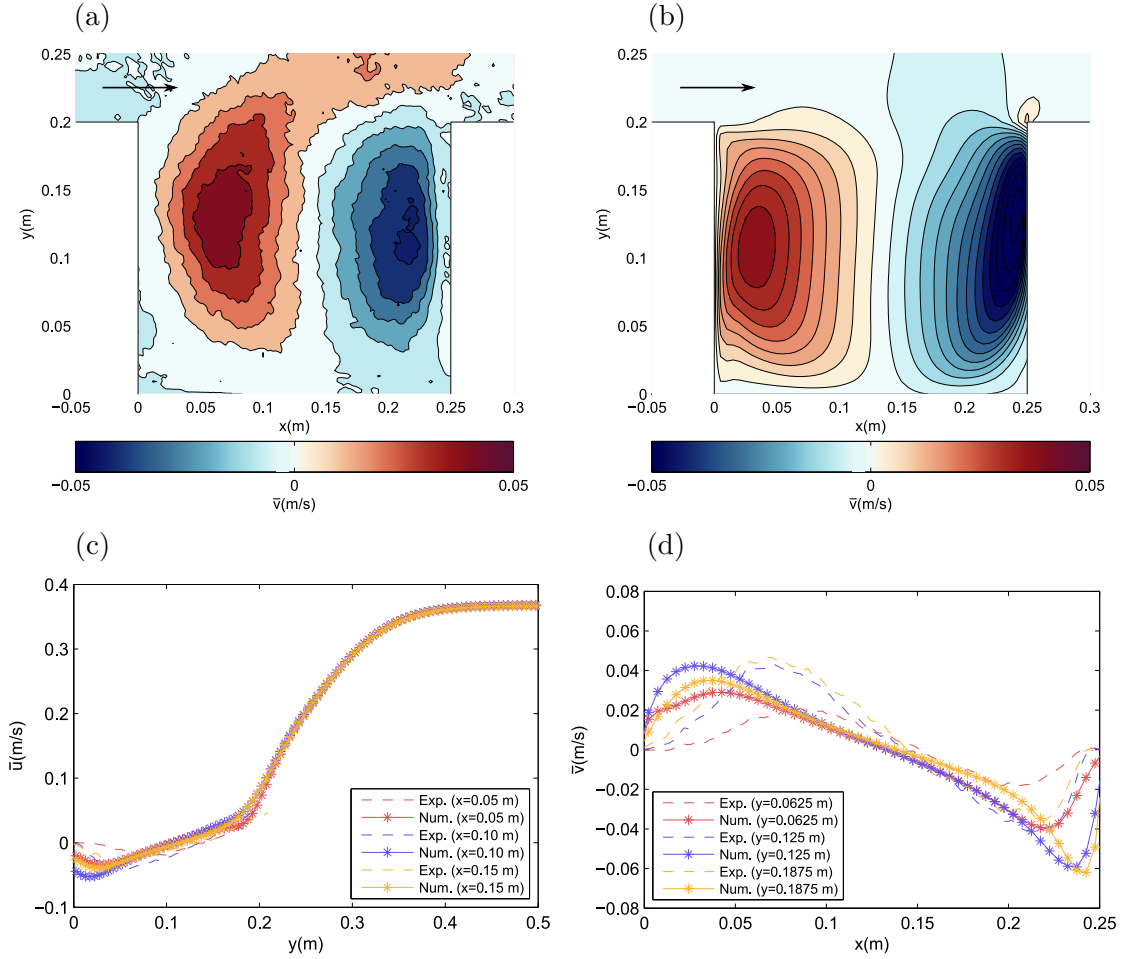


Figure 17: Measured (a) and computed (b) time-averaged spanwise velocity inside a cavity. Cross sectional representation of the measured and computed time-averaged streamwise velocity, along the y direction (c) and time-averaged spanwise velocity, along the x direction (d). Time-averaged quantities are denoted by the overbar symbol ($\bar{\cdot}$).

662 of $C_s \Delta x$. For instance, $\beta l_s^2 \sim 10^{-6}$ for cases 2.1 and 3.1, which is in good agreement with $C_s \Delta x$
 663 for the selected grids (i.e. $\Delta x = 0.005$ m and $\Delta x = 0.00625$ m for cases 2.1 and 3.1, respectively) and for
 664 C_s inside the expected range, according to [45].

665 5. Concluding remarks

666 A depth averaged 2D URANS hydrodynamic solver based on an arbitrary order augmented WENO-
 667 ADER scheme for the SWE is designed for the resolution of turbulent shallow flows. A large extent of
 668 the large-scale 2D turbulence spectrum is thus resolved thanks to the high accuracy of the numerical
 669 scheme. Conversely, depth averaged calculations do not resolve 3D turbulence and the modelling of the
 670 unresolved subdepth-scale of 3D turbulence is required. Part of the dissipation produced by this effect
 671 is accounted for by means of friction losses, using a friction source term in the momentum equations.
 672 However, this was not sufficient hence an eddy viscosity model, related to the friction velocity and
 673 water depth, is used. The proposed model is also able to compute RANS simulations by means of
 674 including an extra contribution in the eddy viscosity due to local variations of the horizontal velocity.

675 The depth-averaged mixing length model is used for the evaluation of the eddy viscosity both for
 676 the RANS and URANS approaches. When using the RANS approach, both 3D and 2D turbulence is

677 modelled. When using the URANS methodology, 3D turbulence is modelled as in the RANS approach
678 and 2D turbulence is mostly resolved. Only the smallest horizontal scales (sub-grid scales) cannot be
679 resolved and have to be modelled using a horizontal eddy viscosity coefficient. In URANS simulation,
680 such approach can be regarded as a sub-grid model for the unresolved horizontal 2D scales.

681 Concerning the numerical scheme, a WENO-ADER method is used. It can be regarded as an
682 arbitrary order extension of the first order Godunov's method. The WENO reconstruction is used
683 to provide an arbitrary order of accuracy in space, avoiding Gibbs oscillations. The ADER approach
684 allows to extend the updating scheme to arbitrary order in time without needing extra sub-steps.
685 Thanks to the use of WENO-ADER schemes, the numerical diffusion and dispersion are reduced to
686 the desired level so that the scheme is able to resolve small-scale features of the flow. This makes such
687 schemes suitable for the resolution of 2D turbulence, as it is possible to control to which extent the
688 energy spectrum is resolved or modelled. As a result, the numerical scheme shows convergence with
689 mesh refinement even in cases with transient propagation of gravity waves and shedding of vortices.

690 Different approaches for the numerical approximation of the turbulent diffusion terms are assessed.
691 The use of a WENO-based reconstruction of the derivatives is not suitable due to the non-linear limiting
692 of sharp gradients provided by the WENO method when the length scales of the velocity gradients are
693 lower than the cell size, making the solution not consistent with physics. On the contrary, the linear
694 version of such method circumvents this problem and provides a more accurate solution. For the sake
695 of simplicity and efficiency, a second order discretization using centered differences is chosen.

696 Bed variations are also considered in the SWE in order to account for a complex bathymetry. The
697 thrust exerted by the bed slope is accounted for as a source term in the momentum equations. The
698 ARoe solver, in combination with a particular integration of the source terms of arbitrary order, is
699 used to satisfy the well-balanced property and preserve the lake-at-rest equilibrium state. The DRP
700 is solved by means of the ARL solver, which is a high order extension of the ARoe solver.

701 The performance of the proposed model is assessed by comparing with experimental data. Four
702 benchmark cases, two of them involving a channel with a single lateral cavity, and other two involving
703 a channel with multiple lateral cavities, are used. The numerical results evidence that the URANS
704 model accurately reproduces both longitudinal and transversal seiche waves and provides an accurate
705 description of the flow field. In certain cases, the magnitude of the time-averaged velocity in the
706 recirculation zones is slightly overestimated. This is presumably due to the non-purely 2D nature
707 of the flow, thus it has been more clearly noticed in cases where the ratio between the horizontal
708 dimensions of the cavity and the flow depth is low. When considering the RANS approach, the scheme
709 reproduces the measured time-averaged velocity field, but fluctuations in time and seiches cannot be
710 reproduced.

711 It is observed that the amplitude and shape of the seiche is, in general, sensitive to the parameters
712 of the turbulence model. However, the frequency keeps a quasi-constant value regardless the changes in
713 the calibration of the turbulence model. The numerical frequency is systematically in good agreement
714 with the theoretical/experimental estimation. Furthermore, it is evidenced that there is no need to
715 use very fine meshes thanks to the WENO reconstruction technique and the ADER time stepping. A
716 number of 40 to 60 cells along the width of the cavity has been reported in all tests to be sufficient to
717 reproduce the experimental data.

718 The proposed model offers a suitable solution for the computation of turbulent shallow flows with
719 high order of accuracy, allowing to resolve a large extent of the 2D turbulence spectrum, while modelling
720 the 3D small scale turbulence. The model is explicit and fully discrete, and preserves the fundamental
721 equilibrium states of interest (e.g. lake-at-rest) thanks to a robust integration of the source terms that
722 ensure high order of accuracy without losing stability. The framework for the construction of arbi-
723 trary order schemes has been presented, allowing to find a good compromise between the accuracy and
724 the computational cost of the scheme. The model is suitable for coarse computational meshes, thanks

725 to (a) the high order of accuracy and (b) to the accurate modelling of the sub-grid unresolved physics
 726 (i.e. wall and bottom friction and small scale turbulence), using high order sub-cell reconstructed
 727 data. This makes the proposed model a useful tool for realistic scenarios, comprising large spatial
 728 and temporal scales, with an eventual application to more complex phenomena (e.g. geomorphological
 729 applications). As a future work, the development of a more accurate wall friction model that relates
 730 the wall roughness and the mixing length will be considered. The design of a wall friction law that
 731 spans over the cells within the boundary layer will be investigated in the framework of the proposed
 732 model.

733

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739 **Appendix A. Details on the numerical approximation of the diffusion terms**

740 To show the numerical performance of the two different approaches for the reconstruction of the
 741 spatial gradients outlined in the text, a numerical experiment involving a shear flow is presented. Let
 742 us consider a pure shear flow with constant water depth depth in the streamwise direction, x , defined
 743 in the semi-infinite domain $\Omega = [-\infty, \infty] \times [0, 1]$. The transverse velocity, v , is nil and the streamwise
 744 velocity, u , is equal to -0.001 if $0 \leq y \leq 0.5$ and 0.001 if $0.5 < y \leq 1$. Bed slope and bed friction are
 745 not considered. All variations in x are nil. This problem can be reduced to a 1D problem in the y
 746 direction and Equation (1)–(2) becomes:

$$\frac{\partial(hu)}{\partial t} + \frac{\partial(huv)}{\partial y} = \nu h \frac{\partial^2 u}{\partial y^2} \quad (\text{A.1})$$

747 For the conditions of the problem, the solution of (A.1) is given by:

$$hu(y, t) = 5 \cdot 10^{-4} \operatorname{erf} \left(\frac{y - 0.5}{2\sqrt{\nu t}} \right) \quad (\text{A.2})$$

748 and it leads to $v = 0$. In discrete form, Equation (A.1) becomes:

$$(hu)_j^{n+1} = (hu)_j^n + \frac{\Delta t}{\Delta x} \left((huv)_{j+1/2}^* - (huv)_{j-1/2}^* \right) + \frac{1}{\Delta x} \bar{D} \quad (\text{A.3})$$

749 where

$$\bar{D} \approx \int_{t^n}^{t^{n+1}} \int_{y_{i-1/2}}^{y_{i+1/2}} \nu h \frac{\partial^2 u}{\partial y^2} dy dt \quad (\text{A.4})$$

750 As in the analytical solution, when $v = 0$, the numerical flux for the shear momentum provided by
 751 the ARoe solver, $(huv)_{j+1/2}^*$, is nil. This is straightforward to prove by analyzing the third component
 752 of the y -version of (33), which involves the numerical flux for the shear momentum:

$$(huv)_{j+1/2}^* = (huv)_{j+1} - \tilde{v} \left(\frac{\tilde{u}\delta h}{2} - \frac{\tilde{v}\delta h - \delta(hv)}{2\tilde{c}} \tilde{u} + \delta(hu) - \tilde{u}\delta h \right) - \tilde{c} \left(\frac{\tilde{u}\delta h}{2} - \frac{\tilde{v}\delta h - \delta(hv)}{2\tilde{c}} \tilde{v} \right), \quad (\text{A.5})$$

753 If inserting $\tilde{v} = \delta h = \delta(hv) = 0$ in Equation (A.5), it yields to:

$$(huv)_{j+1/2}^* = (huv)_{j+1} = 0 \quad (\text{A.6})$$

754 hence, no shear momentum will be transferred across the interface in the numerical solution due to
 755 convective transport. Therefore, Equation (A.3) becomes:

$$(hu)_j^{n+1} = (hu)_j^n + \frac{1}{\Delta x} \bar{D} \quad (\text{A.7})$$

756 The term \bar{D} must be numerically approximated by using one of the options described before. If
 757 choosing the first option, based on the sub-cell derivative reconstruction procedure in [48], \bar{D} yields to:

$$\bar{D} = \Delta t \nu h \left(\left(\partial_y u + \sum_{k=1}^K \partial_t^k (\partial_y u) \frac{\Delta t^k}{(k+1)!} \right)_{j_N} - \left(\partial_y u + \sum_{k=1}^K \partial_t^k (\partial_y u) \frac{\Delta t^k}{(k+1)!} \right)_{j_S} \right) \quad (\text{A.8})$$

758 where the subscript N and S stand for north and south, with reference to the cell center. For instance,
 759 $(\partial_y u)_{j_N}$ is the sub-cell spatial derivative reconstruction at $y = y_{j_N}$. Note that this numerical sub-cell
 760 derivative can be reconstructed departing from data provided either by a WENO reconstruction or by
 761 an optimal polynomial reconstruction.

762 On the other hand, if choosing centered differences in Equations (45)–(50), \bar{D} yields to:

$$\bar{D} = \Delta t \nu h \left(\frac{u_{j+1} - u_j}{\Delta x} - \frac{u_j - u_{j-1}}{\Delta x} \right) \quad (\text{A.9})$$

763 Numerical results for the problem configuration detailed above, comparing the performance of the
 764 different approximations of the diffusion terms, are presented in Figure A.18. Four different tests, listed
 765 in Table A.4, are carried out. The first test does not consider any diffusion, hence the numerical scheme
 766 must preserve the discrete equilibrium, according to Equation (A.3). This is evidenced in Figure A.18
 767 (a), which shows that the exact initial equilibrium is maintained and that there is no mixing across
 768 the shear layer.

769 On the other hand, tests 2, 3 and 4 do involve a nonzero viscosity. Test 2 is based on the sub-cell
 770 WENO derivative reconstruction in (A.8) and, as expected, the solution in Figure A.18 (c) evidences
 771 that such method prevents the diffusive mixing across the discontinuity. The initial equilibrium is
 772 maintained since the reconstructed spatial gradients (and their higher order time derivatives) are nil
 773 in the whole domain, $(\partial_y u)_{j(\cdot)} = 0$, with independence of the order of accuracy. Therefore, the method
 774 is unable to reproduce the analytical solution.

775 Test 3 is based on the same sub-cell derivative reconstruction approach, but using an optimal
 776 polynomial reconstruction rather than the WENO reconstruction. In this case, the derivatives at the
 777 discontinuities are properly approached, hence the method is more adequate to compute the diffusion
 778 terms. Numerical results for a 3-rd and 5-th order scheme are presented in Figure A.18 (d).

779 In Test 4, the diffusion terms are computed by means of centered differences in (A.9). As in Test
 780 3, this method is able to accurately reproduce the analytical solution as depicted in Figure A.18 (b).

781 The numerical results evidence that the centered differences approach provides a lower accuracy
 782 than the the sub-cell derivative reconstruction method. However, the differences in accuracy of such
 783 approaches for the particular application of this methods will be negligible when compared to the
 784 numerical error of the convective terms. Furthermore, carrying out a sub-cell derivative reconstruction
 785 based on the optimal polynomial reconstruction supposes an added computational expense. Therefore,
 786 the centered differences approach is chosen for the particular applications herein considered.

Test	ν	Approximation of diffusion terms	Δx	Reconstruction order
1	0	None	0.02	-
2	0.001	Sub-cell derivative reconstruction (WENO limiting)	0.02	3-rd, 5-th
3	0.001	Sub-cell derivative reconstruction (optimal rec.)	0.05	3-rd, 5-th
4	0.001	Centered differences	0.05	2-nd

Table A.4: Numerical tests to assess the validity of the approximation of the diffusion term.

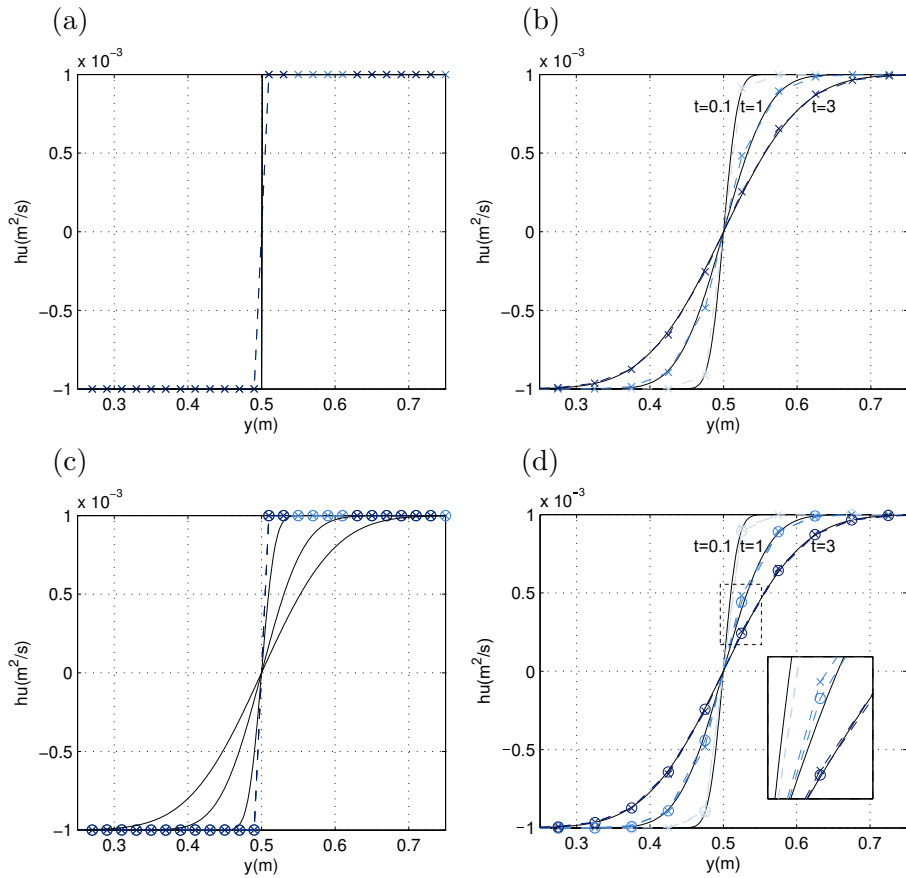


Figure A.18: Cross sectional representation of the exact (solid line) and numerical hu in the y direction for test 1 (a), test 2: sub-cell derivative reconstruction with WENO limiting (c), test 3: sub-cell derivative reconstruction with optimal reconstruction (d) and test 4: centered differences (b). The numerical solution is presented at $t = 0.1, 1, 3$ s. Crosses and circles stand for 3-rd and 5-th order solutions, respectively.

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